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Descent in Lawson homology and morphic cohomology

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Contents

Introduction

The traditional way to study algebraic cycles on an algebraic variety X, is via the Chow groups $\mathrm{CH}_k(X)$. In the beginnings of the 90's, Blaine Lawson and Eric Friedlander developed a different way to study algebraic cycles on varieties over $\mathbb{C}$. Instead of taking discrete groups of cycles on X modulo some equivalence relation, they use the Chow varieties to provide the spaces of algebraic cycles $\mathbf{Z}_k(X)$ with a topology, and they use homotopy invariants of those cycle spaces as geometric invariants of X.

In particular, the homotopy groups of those spaces produce a bigraded family of abelian groups $\mathrm{L}_k\mathrm{H}_n(X)$ called Lawson homology that behave like a Borel-Moore homology. They also define a contravariant version $\mathrm{L}^q\mathrm{H}^n(X)$ called morphic cohomology, that behaves like a cohomology theory. Both theories are related by duality in the case X is smooth. Lawson homology extends Chow groups modulo algebraic equivalence in the same way as motivic cohomology extends Chow groups modulo rational equivalence.

The main interest of such a theory is that, in a sense, it takes an approch to motivic invariants opposed to what Voevodsky does. Voevodsky constructs its motivic cohomology invariants from the top down, embedding algebraic varieties in the right category, namely complexes of presheaves with transfers, and then forcing the desired properties into the theory: Nisnevich descent, homotopy invariance, cdh descent, etc. This approach is very general, has excel·lent structural properties, and its power is demonstrated by Voevodsky's proof of the Bloch-Kato conjecture.

On the other hand, the semi-topological ideas of Friedlander and Lawson are a bottom-up approach. They are not as general, since it is restricted to varieties over $\mathbb{C}$, and only sees cycles modulo algebraic equivalence, but it starts with a very concrete, and very natural construction, which is close to the geometry of the variety X, and surprisingly they are able to prove quite a lot of the expected structural properties. Enough to produce a reasonable theory and establish comparison results with Voevodsky's theory.

It was expected that it could deliver results complementary to Voevodsky's techniques. However, at this point, this promise has not been completely fulfilled. For example, most results in the semi-topological world of Friedlander and Lawson can be reached via Voevodsky techniques. However, it is the believe of the author that there is still a lot of unexplored ground in the semi-topological direction, and the possibility of using particularities of the geometry of cycle spaces to produce results or computations of motivic invariants seems quite compelling. This thesis attempts to cover a tiny bit of ground in this direction.

We now briefly describe the individual chapters of this thesis.

In Chapter 1, we collect the basics of the semi-topological theories for algebraic cycles and cocycles, developed mainly by Friedlander and Lawson. This is an exposition of the state of the art, and we do not claim any results. We cover the topology on spaces of cycles, Lawson homology and morphic cohomology, including the bivariant cocycle spaces, and end with the duality theorem relating both theories.

There is a lot of literature about a semi-topological version of K-theory, that recently has found an application by Blanc in constructing topological K-theory of noncommutative spaces [4], however this is out of the scope of this thesis.

In Chapter 2 we reformulate a descent theorem of Guillén and Navarro [35] in the modern language of $(\infty, 1)$-categories, following the work of Joyal and particularly Lurie [57]. The aim is to produce a variant of the main theorem in [35] with a much simpler proof. Our main theorem is

Theorem 2.3.3. Let $(\mathcal{C}, \tau)$ be an ∞-category together with a Grothendieck topology, and let $\mathcal{C}_0 \subset \mathcal{C}$ be a full subcategory, such that for any $X \in \mathcal{C}$, and any covering sieve $U \in \mathrm{Cov}(X)$, the inclusion $U \cap \mathcal{C}_0 \subset \mathcal{U}$ is cofinal. Let $\mathcal{D}$ be a complete ∞-category $\mathcal{D}$. Then, the restriction functor ε_* induces an equivalence

$$\mathbf{Sh}_\tau(\mathcal{C}, \mathcal{D}) \xrightarrow{\simeq} \mathbf{Sh}_\tau(\mathcal{C}_0, \mathcal{D}) \tag{0.1}$$

As a corollary of this, we obtain a theorem analogous to the extension theorem in [35]. We say analogous, because neither theorem implies the other.

In Chapter 3 we study several flavours of descent for the Lawson homology and morphic cohomology. Our main contributions are as follows:

We define a refined Gysin maps extending the definition from Friedlander Gabber [17], following Fulton's refined Gysin maps for Chow groups. Then we prove an excess intersection formula for those refined Gysin maps, and use it to prove a blow-up formula for the blow-up of a variety X along a regularly embedded center. We do not assume smoothness, extending results of Hu in [41] to the singular case.

Theorem 3.3.1. Let X be an algebraic variety, with a regularly embedded subvariety Y of codimension c, fitting in the following abstract blow-up square

$$\begin{tikzcd}
\widetilde{Y} \arrow[r, "j"] \arrow[d, "p"] & \widetilde{X} \arrow[d, "q"] \\
Y \arrow[r, "i"] & X
\end{tikzcd}$$

Let $E = p^* N_Y X / N_{\widetilde{Y}} \widetilde{X}$, which is a vector bundle of rank $e = c - 1$ on $\widetilde{Y}$. Then, we have a split short exact sequence

$$0 \longrightarrow \mathrm{L}_k\mathrm{H}_n(Y) \xrightarrow{a} \mathrm{L}_k\mathrm{H}_n(\widetilde{Y}) \oplus \mathrm{L}_k\mathrm{H}_n(X) \xrightarrow{b} \mathrm{L}_k\mathrm{H}_n(\widetilde{X}) \longrightarrow 0 \tag{0.2}$$

where

$$\begin{aligned}
a(\alpha) &= (c_e(E) \cap p^*(\alpha), i_*(\alpha)) \\
b(\alpha, \beta) &= j_*(\alpha) - q^!(\beta).
\end{aligned}$$

After that, we collect several flavours of descent properties know for the semi-topological theories. In particular, we prove cdh descent for the bivariant complexes of cocycles $\mathcal{M}_k(X,Y)$ with respect to the second variable. Not much is known regarding descent properties in the variable X on the general case. Not even descent for the Zariski topology.

We conclude the chapter by proving a generalized duality theorem relating complexes of algebraic cocycles, and extending the results in [19]. In particular, we prove

Theorem 3.4.5. Let X, Y, W be algebraic varieties, with X normal and quasi-projective, Y projective and W smooth projective. Then, the duality map

$$\mathcal{M}_k^*(X \times W, Y) \longrightarrow \mathcal{M}_k^*(X, W \times Y) \tag{0.3}$$

is a homotopy equivalence.

Finally, in Chapter 4 we study morphic cohomology in the particular case of toric varieties. Toric varieties are a class of rational varieties which have a very explicit and rich combinatorial description. We prove Zariski descent for morphic cohomology of toric varieties, with respect to torus equivariant open sets.

Then we develop a spectral sequence that computes morphic cohomology for toric varieties.

Theorem 4.2.16. Let $X(\Delta)$ be a toric variety associated to a fan Δ and $\mathcal{F}^*$ a bounded above cochain complex of sheaves. There is a convergent spectral sequence

$$E_1^{r,s} = \operatorname{Ext}^s(\check{\mathcal{C}}_r(\Delta, \mathbb{Z}), \mathcal{F}^*) \Longrightarrow \mathbb{H}^{r+s}(X(\Delta), \mathcal{F}^*). \tag{0.4}$$

Moreover, if $\mathcal{F}^*$ has homotopy invariant cohomology,

$$E_1^{r,s} \cong \bigoplus_{\sigma \in \Delta^{(r)}} \mathbb{H}^s(T_\sigma, \mathcal{F}^*), \tag{0.5}$$

and the differentials on the first page $d_1 : E_1^{r,s} \to E_1^{r+1,s}$ are given by

$$d_1 = \sum_{\substack{\sigma \in \Delta^{(r)} \\ \tau \in \Delta^{(r+1)} \\ \tau \leq \sigma}} \epsilon(\tau, \sigma) r^*_{\tau,\sigma} \tag{0.6}$$

This very explicit computation allows us to prove cdh descent for toric varieties and prove Suslin's conjecture in this case.

Toric varieties are a very special class of varieties, of course, but the interest of those computations lay in the fact that very few computations of morphic cohomology for singular varieties are known.

1 Semi-topological theories

In this chapter we will review the basic material regarding the Friedlander-Lawson approach to algebraic cycles and Chow groups, that began in [49] and [13].

In the classical approach to algebraic cycles, as beautifully exposed by Fulton in [29], one attempts to tame the study of algebraic cycles by identifying cycles that "continuously deform" one to another, the same way we identify homologous singular cycles in singular homology. This way, we obtain the so called Chow groups of algebraic cycles. Of course, the algebraic picture is much more subtle than the topological picture, in the sense that there are different precise meanings for the words "continuously deform", which lead to different flavours of Chow groups.

On the other hand, in the Friedlander-Lawson approach, instead of using an equivalence relation on the abelian group of algebraic cycles, we make use of the fact that the set of algebraic cycles has itself a topology, then we can compute topological invariants of these spaces, namely its homotopy groups, as geometrical invariants of X analogous to Chow groups. These Friedlander-Lawson invariants are closely related, to the Chow groups of algebraic cycles modulo algebraic equivalence, and appear to be at the core of several deep conjectures on algebraic cycles.

For us, an algebraic variety, will mean an equidimensional integral scheme defined over $\mathbb{C}$. Although in its foundational paper [13], Friedlander develops the theory for varieties over an algebraically closed field using étale homotopy types. In this work we will assume our base field is $\mathbb{C}$.

A good survey for some of the material on this chapter is [50]. Specific references to the original papers will be provided on the individual results.

1.1 Chow varieties and spaces of algebraic cycles

Let's start by reviewing some classical constructions on algebraic cycles.

1.1.1 The group of algebraic cycles

Definition 1.1.1. Let X be an algebraic variety. An **algebraic k-cycle** on X is a finite formal sum

$$\alpha = \sum_i n_i[V_i]$$

where $n_i \in \mathbb{Z}$ and $V_i \subset X$ are reduced and irreducible subvarieties of dimension k. The set of k-cycles in X together with the addition operation forms an abelian group that we will denote by $\mathbf{Z}_k(X)$.

Definition 1.1.2. A k-cycle is said to be **effective** if $n_i \geq 0$. The set of effective k-cycles on X together with the addition operation forms a monoid and we will denote it by $\mathbf{C}_k(X)$.

Remark 1.1.3. Sometimes it will be useful to refer to cycles by codimension. On an equidimensional variety X of dimension d, we will use the following notation:

$$\mathbf{Z}^k(X) = \mathbf{Z}_{d-k}(X),$$
$$\mathbf{C}^k(X) = \mathbf{C}_{d-k}(X).$$

Definition 1.1.4. Let X be a projective variety. The **degree** of an effective algebraic cycle $c = \sum_i n_i[V_i]$ is given by

$$\deg c = \sum_i n_i \deg V_i,$$

where $\deg V_i$ means the degree of the projective variety V_i, namely the intersection number with a sufficiently generic codimension k linear variety inside $\mathbb{P}^N$. The set of effective k-cycles of degree d on X will be denoted by $\mathbf{C}_{k,d}(X)$.

The nilpotent functions in an algebraic scheme may be interpreted as multiplicities in the following way.

Definition 1.1.5. Let X be an algebraic scheme of dimension n, with irreducible components $X_1, \dots, X_r$. Let η_i be the generic point corresponding to X_i. The **fundamental cycle** $[X] \in \mathbf{Z}_n(X)$, is defined by

$$[X] = \sum_i \operatorname{length}(\mathcal{O}_{X,\eta_i})[X_i].$$

There are two possible functorialities we can define on the sets of cycles we have just defined, one covariant and one contravariant.

Definition 1.1.6. Let $f : X \to Y$ be a proper morphism of algebraic varieties. The **push-forward** of cycles

$$f_* : \mathbf{Z}_k(X) \to \mathbf{Z}_k(Y)$$

is the group homomorphism defined on the generators by

$$f_*([V]) = \begin{cases} \deg(V/f(V))[f(V)], & \text{if } \dim V = \dim f(V) \\ 0, & \text{otherwise.} \end{cases}$$

where the degree $\deg(V/f(V))$ is the number of pre-images of a sufficiently generic point of $f(V)$.

Definition 1.1.7. Let $f : X \to Y$ be a flat morphism of algebraic varieties of relative dimension r (i.e. $r = \dim X - \dim Y$). Then we have a **pull-back** of cycles

$$f^* : \mathbf{Z}_k(Y) \to \mathbf{Z}_{k+r}(X)$$

which is the abelian group homomorphism defined on generators by

$$f^*([V]) = [X \times_Y V]$$

where $X \times_Y V$ is the fibered product as schemes of X and V over Y.

Remark 1.1.8. Since $X \times_Y V$ in Definition 1.1.7 may have nilpotents, the irreducible components of this fibered product may have nontrivial multiplicities.

Finally, there is an exterior, or cartesian product on algebraic cycles, as follows

Definition 1.1.9. Let X, Y be algebraic varieties. There is a morphism of abelian groups

$$\cdot \times \cdot \colon \mathbf{Z}_k(X) \otimes \mathbf{Z}_l(X) \to \mathbf{Z}_{k+l}(X \times Y),$$

defined on generators by

$$[V] \times [W] = [V \times W]$$

and extended as a multilinear map.

1.1.2 The Chow variety of a projective variety

Our aim is to provide the spaces $\mathbf{Z}_k(X)$ with a topology. To achieve this, Friedlander and Lawson use the classical construction of the Chow variety, that provides the space of effective cycles of bounded degree with the structure of an algebraic variety. See [60], Chapter 21 in [38] or Section 1.3 in Kollár's book [47] for a more sophisticated approach.

The classical construction of the Chow variety goes as follows. To every k-cycle on a projective variety X, we associate a divisor on another algebraic variety Q_k. There are different constructions, involving different choices of Q_k. In the approach we describe, Q_k will be a product of projective spaces.

Let $X \subset \mathbb{P}^N$ be an irreducible projective variety. Let's take $Q_k = \mathbb{P}^{N\vee} \times \cdots \times \mathbb{P}^{N\vee}$, a product of $k+1$ copies of the dual projective space $\mathbb{P}^{N\vee}$ which parametrizes hyperplanes in $\mathbb{P}^N$.

Now we pick a cycle of the form $c = [V]$ for an irreducible subvariety V of dimension k. Consider now the incidence relation $\Theta_V \subset X{\times}Q_k$ defined by

$$\Theta_V = \{(x, h_0, \dots, h_k) | x \in h_0 \cap \cdots \cap h_k \cap V\}$$

This incidence relation Θ_V is an irreducible subvariety of $X{\times}Q_k$. Moreover,

$$\dim \Theta_V = k + (n-1)(k+1) = n(k+1) - 1.$$

Now, the projection $\pi_2 \colon X \times Q_k \to Q_k$ is proper and generically injective when restricted to Θ_V, because the generic intersection of $k+1$ hyperplanes and V is empty. We then define

$$\Phi_V = \pi_{2*}(\Theta_V),$$

which, by properness and (generic) injectivity of π_2, is a codimension 1 irreducible subvariety of Q_k.

Now let's fix all h_i but one. Since V has degree d, a generic pencil of hyperplanes on the remaining $\mathbb{P}^{N\vee}$ meets Φ_V at exactly d points, so Φ_V has multi-degree $(d, \dots, d)$. This means that the divisor Φ_V is defined, up to a constant by a global section

$$s_V \in \Gamma((\mathbb{P}^{N\vee})^{k+1}, \mathcal{O}(d) \otimes \cdots \otimes \mathcal{O}(d)).$$

Definition 1.1.10. Let $X \subset \mathbb{P}^N$ be a projective variety. There is a map

$$\Phi\colon \mathbf{C}_{k,d}(X) \to \mathbb{P}\Gamma((\mathbb{P}^{N\vee})^{k+1}, \mathcal{O}(d)^{\otimes(k+1)})$$

defined as follows

$$\Phi(n_1[V_1] + \cdots + n_r[V_r]) = s_{V_1}^{n_1} \cdots s_{V_r}^{n_r}.$$

Theorem 1.1.11. The map Φ above is injective, and its image is a closed algebraic subvariety.

Proof. See Proposition 1.1 [13] and [60]. Consider the incidence relation

$$\Psi = \{(x, h_0, \dots, h_k) \mid x \in h_0 \cap \cdots \cap h_k\} \subset X \times Q_k$$

and denote by $\Psi_x \subset Q_k$ the fibers of Ψ over $x \in X$.

Then, we can recover the irreducible variety V from the divisor $D = \Phi(V)$ since $x \in V$ if, and only if $\Psi_x \subset D$. This, together with unicity of the decomposition of a divisor as sum of irreducibles, gives the injectivity in general. ●

This theorem allows us to regard $\mathbf{C}_{k,d}(X)$ as a projective variety, called the Chow variety of k-cycles of degree d. We will be interested in the complex topology of these varieties. Let's look at some examples:

Example 1.1.12 (0-cycles)**.** The case of 0-cycles is particularly simple: an effective 0-cycle of degree d in X is just a set of d points in X counted with multiplicities. This is just a symmetric product

$$\mathbf{C}_{0,d}(X) = X \times \cdots \times X / S_d.$$

When $X = \mathbb{P}^1$, the symmetric product on the right is isomorphic to $\mathbb{P}^d$ by the map from $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \to \mathbb{P}\mathbb{C}[x,y]_d$ to the projectivization of degree d homogeneous polynomials in two variables.

$$[\alpha_1 : \beta_1], \dots, [\alpha_d : \beta_d] \mapsto \prod_i (\alpha_i x - \beta_i y)$$

If X is a more general smooth curve the symmetric product will still be a smooth variety, however, if X is a higher dimensional variety, the symmetric products will become singular.

Example 1.1.13 (Divisors on $\mathbb{P}^n$)**.** The degree d effective divisors on $\mathbb{P}^n$ correspond bijectively to regular sections of the line bundle $\mathcal{O}(d)$ up to scalar multiple. This way, we can make the following identification

$$\mathbf{C}_{n-1,d}(\mathbb{P}^n) = \mathbb{P}H^0(\mathcal{O}(d)).$$

Example 1.1.14 (conic curves in $\mathbb{P}^3$)**.** Let C be a conic curve on $\mathbb{P}^3$. The curve C is either a plane conic or it is the disjoint union of two non-itersecting lines.

$$\mathbf{C}_{1,2}(\mathbb{P}^3) = A \cup B,$$

where A is the set of pairs of lines on $\mathbb{P}^3$ and B is the set of plane conics on $\mathbb{P}^3$.

On one hand, the lines in $\mathbb{P}^3$ are parametrized by the grasssmanian $\mathrm{Grass}_{\mathbb{C}}(2,4)$ so $A = \mathrm{Grass}_{\mathbb{C}}(2,4)^2/S_2$.

On the other hand, plane conics on $\mathbb{P}^3$ can be parametrized as follows: First pick a plane in $\mathbb{P}^3$, which belongs to the Grassmanian $\mathrm{Grass}_{\mathbb{C}}(3,4) \simeq \mathbb{P}^3$, and then pick a plane conic on that plane. Because of Example 1.1.13 above, degree 2 effective divisor on $\mathbb{P}^2$ correspond to points in $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(2))) \simeq \mathbb{P}^5$.

In conclusion, $\mathbf{C}_{1,2}(\mathbb{P}^3)$ has two irreducible components, one isomorphic to the Grassmannian $\mathrm{Grass}_{\mathbb{C}}(2,4)^2/S_2$, and the other isomorphic to $\mathbb{P}^3 \times \mathbb{P}^5$, that meet at the locus of degenerate plane conics.

In general though, Chow varieties are very difficult to deal with explicitly. Even counting its irreducible components is a very nontrivial matter. Besides, they are usually badly singular, and not equidimensional. The fundamental insight of Friedlander and Lawson, was that even though the Chow varieties are very difficult, and appears to be hopeless to obtain a well-behaved homology theory of algebraic variaties out of them. Once its homotopy type is suitably stabilized by group completion, they do indeed produce such a homology theory. This will be the topic of the sections that follow.

1.1.3 The space of algebraic cycles

We can use the Chow variety to provide the spaces of cycles $\mathbf{Z}_k(X)$ with a topology. Observe that

$$\mathbf{C}_k(X) = \{0\} \sqcup \bigsqcup_{d>0} \mathbf{C}_{k,d}(X)$$

is a disjoint union of projective varieties. If we take all Chow varieties together in this manner, it becomes a monoid with the addition of cycles.

$$\mathbf{C}_{k,d}(X) \times \mathbf{C}_{k,d'}(X) \to \mathbf{C}_{k,d+d'}(X).$$

Proposition 1.1.15. Let X be a projective algebraic variety. Then

1. The space $\mathbf{C}_k(X)$ together with the addition of cycles is a topological monoid, with an operation given by an algebraic map.
2. The space $\mathbf{C}_k(X)$ is independent of the projective embedding of X, up to homeomorphism.

Proof. 1) The sum of algebraic cycles corresponds to the product of homogeneous polynomials according to the map Φ in Theorem 1.1.11. But the coefficients of a product of polynomials depend algebraically on the coefficients of the factors.

2) See Proposition 1.7 in [13]. ●

Algebraically, the monoid $\mathbf{C}_k(X)$ is free and, it has the cancellation property: if $c + d = c + d'$, then $d = d'$. This allows us to describe $\mathbf{Z}_k(X)$ as a group completion of $\mathbf{C}_k(X)$ in the following way

$$\mathbf{Z}_k(X) = \mathbf{C}_k(X)^+ = \mathbf{C}_k(X) \times \mathbf{C}_k(X)/\sim, \tag{1.1}$$

The right hand side represents the positive and negative part of the cycle, and the equivalence relation $\sim$ is such that $(c_+, c_-) \sim (c_+ + d, c_- + d)$ for any $d \in \mathbf{C}_k(X)$. It is easy to check that the group operation is continuous with respect to the quotient topology on $\mathbf{Z}_k(X)$, making it a topological abelian group.

Definition 1.1.16. Let X be a projective variety. The **space of algebraic k-cycles** on X, denoted by $\mathbf{Z}_k(X)$, is the free abelian group generated by the reduced and irreducible k-dimensional subvarieties of X, together with the quotient topology inherited through isomorphism (1.1).

Remark 1.1.17. Although we are mainly interested in the homotopy type of $\mathbf{Z}_k(X)$, this space is, in a certain sense, "almost" an algebraic object. It is a quotient of an algebraic (albeit infinite-dimensional) variety $\mathbf{C}_k(X) \times \mathbf{C}_k(X)$, by an algebraic equivalence relation.

For some results, it will be useful to move matters from the category of topological abelian groups, to that of chain complexes of abelian groups. To do this, we take singular chains on $\mathbf{Z}_k(X)$ as follows

Definition 1.1.18. The **complex of algebraic cycles** on a variety X is the complex of abelian groups $\mathcal{Z}_{k,*}(X)$ such that

$$\mathcal{Z}_{k,n}(X) = \mathrm{Sing}_{n-2k}\,\mathbf{Z}_k(X) = \mathrm{Map}(\Delta^{n-2k}, \mathbf{Z}_k(X)),$$

and the differentials $\partial\colon \mathcal{Z}_{k,n}(X) \to \mathcal{Z}_{k,n-1}(X)$ is the usual alternating sum of face maps.

The Dold-Kan theorem tells us that

$$H_n \mathcal{Z}_{*,k}(X) \cong \pi_{n-2k}\mathbf{Z}_k(X).$$

There is a different way to introduce a topology on the space of algebraic cycles $\mathbf{Z}_k(X)$, developed by Lima-Filho in [54], using families of cycles parametrized by an other variety S. We can use different types of families (flat families or equidimensional families). It turns out that all three approaches, the Chow topology as described above and the two topologies defined by families coincide.

Let S be an algebraic variety. Let

$$\sigma = \sum_i n_i[\Gamma_i]$$

be an algebraic cycle on $S \times X$ which is flat and of relative dimension k over S (i.e. every Γ_i has this property). This cycle induces naturally a map

$$f_\sigma\colon S \to \mathbf{Z}_k(X)$$

defined by

$$f_\sigma(s) = \sum_i n_i[\{s\} \times_S \Gamma_i] \in \mathbf{Z}_k(\{s\} \times X).$$

Definition 1.1.19. The flat topology on the space of cycles, that we will denote $\mathbf{Z}_k^{\mathrm{fl}}$ is the finest topology that makes continuous all the maps $f_\sigma\colon S \to \mathbf{Z}_k(X)$ for every S and σ flat and equidimensional over S.

We can make an analogous definition with some small variations in the case of equidimensional families. In this case it is convenient to restrict oneself to consider only smooth varieties S, and change the pullback fibre used to build f_σ out of the cycle σ by an intersection theoretic fibre. In this way, given σ an algebraic cycle on $S \times X$ equidimensional over S but non necessarily flat, we can define

$$f_\sigma \colon S \to \mathbf{Z}_k(X)$$

by

$$f_\sigma(s) = \sigma \cdot \{s\} \times X \in \mathbf{Z}_k\{s\} \times X,$$

where the notation $\sigma \cdot \{s\} \times X$ denotes the intersection cycle of σ and the slice $\{s\} \times X$.

Definition 1.1.20. The **equidimensional topology** on the space of algebraic cycles $\mathbf{Z}_k(X)$ is the finest one making all the maps $f_\sigma \colon S \to \mathbf{Z}_k(X)$ continuous, where S is smooth and σ is a cycle on $S \times X$ equidimensional over S. We will denote this topology by $\mathbf{Z}_k^{\mathrm{eq}}(X)$.

Theorem 1.1.21. Let X be an algebraic variety. Then the identity morphism on $\mathbf{Z}_k(X)$ induces homeomorphisms between the equidimensional and flat topologies $\mathbf{Z}_k^{\mathrm{fl}}(X) \cong \mathbf{Z}_k^{\mathrm{eq}}(X)$. If in addition X is projective, this topology is homeomorphic to the topology induced by Chow varieties

$$\mathbf{Z}_k^{\mathrm{ch}}(X) \simeq \mathbf{Z}_k^{\mathrm{fl}}(X) \simeq \mathbf{Z}_k^{\mathrm{eq}}(X).$$

Proof. The comparison between flat and Chow is Theorem 5.8 in [54]. The comparison between flat and equidimensional is Theorem 3.1 in [54]. ●

Finally, we can define a family of compact subsets of $\mathbf{Z}_k(X)$ as follows

$$K_l = \pi\Big(\bigsqcup_{d,d' \leq l} \mathbf{C}_{k,d}(X) \times \mathbf{C}_{k,d'}(X)\Big). \tag{1.2}$$

The subsets $K_l \subset \mathbf{Z}_k(X)$ define a compact filtration of the space of cycles, and the topology on $\mathbf{Z}_k(X)$ is generated by the topology of this filtration, in the sense that $U \subset \mathbf{Z}_k(X)$ is open if, and only if it is open for every K_l.

Proposition 1.1.22. Let $X \subset \mathbb{P}^N$ be a projective algebraic variety. The space of cycles $\mathbf{Z}_k(X)$ is a Hausdorff topological abelian group, with the homotopy type of a countable CW-complex, and compactly generated by the subspaces K_l.

Proof. See Theorem 4.7 in [54]. ●

Remark 1.1.23. Observe that, using this topology, we no longer require the projective hypothesis.

It turns out that the constructions from Section 1.1.1 are compatible with the topology on $\mathbf{Z}_k(X)$. In particular

Theorem 1.1.24. Let X, Y be algebraic varieties and $f \colon X \to Y$ a morphism between them.

1. Assume f is proper. Then, the push-forward of cycles $f_*\colon \mathbf{Z}_k(X) \to \mathbf{Z}_k(Y)$ is continuous.
2. Assume f is flat. Then, the pull-back of cycles $f^*\colon \mathbf{Z}_k(X) \to \mathbf{Z}_k(Y)$ is continuous.
3. The exterior product of cycles $\cdot \times \cdot\colon \mathbf{Z}_k(X) \otimes \mathbf{Z}_k(Y) \to \mathbf{Z}_{k+l}(X \times Y)$ is continuous.

Proof. See Proposition 2.9 in [54]. ●

Theorem 1.1.25. Let X be an algebraic variety, and Y a closed subvariety with open complement U. The exact sequence of groups

$$\mathbf{Z}_k(Y) \longrightarrow \mathbf{Z}_k(X) \longrightarrow \mathbf{Z}_k(U) \tag{1.3}$$

is a fibration sequence of topological abelian groups. In particular it induces a long exact sequence on homotopy groups.

Proof. See Theorem 4.9 in [54]. ●

Definition 1.1.26. Let $X \subset \mathbb{P}^N$ be a projective variety. Pick a linear embedding $\mathbb{P}^N \subset \mathbb{P}^{N+1}$ and a point $p_\infty \in \mathbb{P}^{N+1} \setminus \mathbb{P}^N$. The **Lawson suspension** of X, that we will denote by ΣX is the set of points in $\mathbb{P}^{N+1}$ contained on a line which meets both p_∞ and X. That is

$$\Sigma X = \left\{ sx + tp_\infty \mid x \in X, [s:t] \in \mathbb{P}^1 \right\}.$$

Taking suspension of cycles, we get a map $\Sigma_*\colon \mathbf{Z}_k(X) \to \mathbf{Z}_{k+1}(\Sigma X)$, defined as follows

$$\Sigma_*([V]) = [\Sigma V]. \tag{1.4}$$

Theorem 1.1.27 (Lawson suspension theorem)**.** Let X be a projective variety. The map induced by the suspension on spaces of cycles

$$\Sigma\colon \mathbf{Z}_k(X) \to \mathbf{Z}_{k+1}(\Sigma X),$$

is a homotopy equivalence.

Proof. See Theorem 3 in [49]. ●

Finally, there is one important construction, arising from the fact that $\mathbf{Z}_k(X)$ is itself an algebraic object (composed of Chow varieties), and which amounts to taking "cycles on the space of cycles". Friedlander calls this the **graphing construction**.

Theorem 1.1.28. Let X be an n-dimensional algebraic variety, and $\xi \in \mathbf{Z}^{n-k}(\mathbf{C}_{k,d}(X) \times X)$ be the universal cycle associated to the chow variety $\mathbf{C}_{k,d}(X)$. There is a continuous map $\Gamma\colon \mathbf{Z}_k(\mathbf{C}_{l,d}(X)) \to \mathbf{Z}_{k+l}(X)$, which on irreducible varieties is defined by

$$\Gamma([V]) = \pi_{2*}([(V \times X) \times_{\mathbf{C}_{k,d}(X)\times X} \xi])$$

Proof. See [16]. ●

Using this construction, Friedlander defines a notion of correspondence, which acts on the space of cycles strictly, i.e. without the need to use any moving lemma, or construction "up to homotopy".

Definition 1.1.29. Let X, Y be algebraic varieties. A **k-correspondence** from X to Y is an algebraic map $c\colon X \to \mathbf{C}_{k,d}(Y)$ for some $d > 0$. We denote by $\mathbf{Corr}_k(X, Y)$ the space of such correspondences.

Remark 1.1.30. Because correspondences are a certain kind of cycles on the product variety $X \times \mathbf{C}_{k,d}(Y)$, they inherit the topology from the Chow variety.

Theorem 1.1.31. The action by correspondences defines a continuous map $\mathbf{Corr}_k(X, Y) \times \mathbf{Z}_l(X) \to \mathbf{Z}_{k+l}(Y)$.

Proof. See [11]. ●

1.2 Lawson homology

In this section, we will see how the homotopy groups of the cycle spaces $\mathbf{Z}_k(X)$ serve as invariants of X, that know a lot about the algebraic geometry of X.

1.2.1 Definition of Lawson homology

Definition 1.2.1. Let X be an algebraic variety. The **Lawson homology groups** are the abelian groups defined by

$$\mathrm{L}_k\mathrm{H}_n(X) = \pi_{n-2k}\mathbf{Z}_k(X).$$

Remark 1.2.2. The homotopy groups $\pi_{n-2k}\mathbf{Z}_k(X)$ are represented by homotopy classes of maps from a $(n-2k)$-dimensional sphere to the space of cycles, that is

$$\mathrm{L}_k\mathrm{H}_n(X) = \pi_{n-2k}(\mathbf{Z}_k(X)) = [S^{n-2k}, \mathbf{Z}_k(X)].$$

This way, the Lawson homology group can be understood as representing cycles in X with n degrees of freedom where $2k$ of them are algebraic, and the remaining ones are topological.

Example 1.2.3. In the $k = 0$ case, the space of degree d effective k-cycles $\mathbf{C}_{k,d}(X)$ happens to be the d-fold symmetric product X^d/S_d. A classical theorem of Dold and Thom [9] asserts that the homotopy groups of the infinite symmetric product coincides with the Borel-Moore singular homology of X

$$\mathrm{L}_0\mathrm{H}_n(X) \cong \mathrm{H}_n^{\mathrm{BM}}(X, \mathbb{Z}).$$

Recall that the Borel-Moore homology is a variation of singular homology that satisfies Poincaré duality for non-compact varieties. It can be defined via infinite locally finite singular chains, or alternatively, via a compactificatification of X as follows. Let $\overline{X}$ be a compactification of X with complement X^∞. Then

$$\mathrm{H}_n^{\mathrm{BM}}(X, \mathbb{Z}) = \mathrm{H}_n(\overline{X}, X^\infty, \mathbb{Z}).$$

Example 1.2.4. An other interesting case is the following.

$$L_kH_{2k}(X) \cong CH_k^{alg}(X),$$

where $CH_k^{alg}(X)$ is the group of algebraic cycles modulo algebraic equivalence. See [13]. From this we conclude that the homotopy type of $\mathbf{Z}_k(X)$ encodes interesting geometrical information about X.

1.2.2 Functorialities and localization

The functorialities on the space of cycles induce the following functorialities on Lawson homology.

Theorem 1.2.5. Let X, Y be algebraic varieties, and $f\colon X \to Y$ a morphism between them.

1. If f is proper, there are push-forward maps

$$f_*\colon L_kH_n(X) \to L_kH_n(Y).$$

2. If f is flat of relative dimension r, there are pull-back maps

$$f^*\colon L_kH_n(Y) \to L_{k+r}H_{n+2r}(X).$$

Proof. These are just the maps induced on homotopy groups by the pull-back and push-forward on cycle spaces in Theorem 1.1.24. ●

Theorem 1.2.6. Let X be an algebraic variety, and $Y \subset X$ a closed subvariety, with open complement U. There is a long exact sequence of Lawson homology groups as follows

$$\cdots \longrightarrow L_kH_n(Y) \longrightarrow L_kH_n(X) \longrightarrow L_kH_n(U) \longrightarrow L_kH_{n-1}(Y) \longrightarrow \cdots$$

Proof. It follows from Theorem 1.1.25. ●

Corollary 1.2.7. Lawson homology has the following Mayer-Vietoris propeties

1. Let $X = U_0 \cup U_1$ be the union of two Zariski open subvarieties. Then there is a long exact sequence

$$\cdots L_kH_n(X) \longrightarrow L_kH_n(U_0) \oplus L_kH_n(U_1) \longrightarrow L_kH_n(U_0 \cap U_1) \longrightarrow L_kH_{n-1}(X) \cdots$$

2. Let $X = X_0 \cup X_1$ be the union of two Zariski closed subvarieties. Then there is a long exact sequence

$$\cdots L_kH_n(X_0 \cap X_1) \longrightarrow L_kH_n(X_0) \oplus L_kH_n(X_1) \longrightarrow L_kH_n(X) \longrightarrow L_kH_{n-1}(X_0 \cap X_1) \cdots$$

Proof. Both are consequences of the localization theorem 1.2.6. ●

1.2.3 Homotopy invariance and the s-map

Lawson's suspension theorem 1.1.27 translates into a homotopy-invariance property on Lawson homology.

Theorem 1.2.8. Let X be an algebraic variety, and $E \to X$ a vector bundle of rank r over X. Then the flat pullback

$$f^* : \mathrm{L}_k\mathrm{H}_n(X) \longrightarrow \mathrm{L}_{k+r}\mathrm{H}_{n+2r}(E),$$

is an isomorphism.

Proof. See Proposition 2.3 in [17]. ●

Remark 1.2.9. This theorem gives a homotopy invariance property for Lawson homology. In the particular case $E = X \times \mathbb{A}^1$, we get

$$\mathrm{L}_k\mathrm{H}_n(X) \cong \mathrm{L}_{k+1}\mathrm{H}_{n+2}(X \times \mathbb{A}^1).$$

There is a shift of indices because Lawson homology is a Borel-Moore type theory, and $\mathbb{A}^1$ is not acyclic, but has the same Borel-Moore homology as a 2-sphere.

Let X be an algebraic variety. Consider the diagram

$$\begin{tikzcd} \mathbf{Z}_k(X) \times \mathbb{P}^1 \arrow[r] \arrow[d] & \mathbf{Z}_k(X \times \mathbb{P}^1) \arrow[d] & \\ \mathbf{Z}_k(X) \wedge \mathbb{P}^1 \arrow[r] & \mathbf{Z}_k(X \times \mathbb{A}^1) & \mathbf{Z}_{k-1}(X) \arrow[l, "\simeq"'] \end{tikzcd} \tag{1.5}$$

where the top horizontal map sends $(\alpha, p) \mapsto \alpha \times \{p\}$.

Definition 1.2.10. Let X be an algebraic variety, the **s-map** is the continuous map

$$\mathbf{Z}_k(X) \longrightarrow \Omega^2\mathbf{Z}_{k-1}(X)$$

adjoint to the lower horizontal map in (1.5).

Proposition 1.2.11. The following diagram commutes

$$\begin{tikzcd} \mathrm{L}_k\mathrm{H}_{2k}(X) \arrow[d, "s^k"'] \arrow[dr, "\mathrm{cyc}"] & \\ \mathrm{L}_0\mathrm{H}_{2k}(X) \arrow[r, "\cong"] & \mathrm{H}^{\mathrm{BM}}_{2k}(X) \end{tikzcd} \tag{1.6}$$

Proof. See Proposition 6.4 in [22]. ●

The s-map breaks the cycle map to singular homology into steps, where in each step we "topologize" one algebraic degree of freedom. In general, the s-map is neither injective nor surjective. We can use it to produce filtrations either at the target or at the source, which are related to deep conjectures in algebraic geometry, in various ways.

Definition 1.2.12. The **s-filtration** on Lawson homology is the increasing filtration $S^i \subset \mathrm{L}_k\mathrm{H}_n(X)$ given by the kernel of iterates of the s-map

$$S^i = \ker(s^i).$$

The **topological filtration** on Lawson homology is the decreasing filtration on Lawson homology given by the images of iterates of the s-map

$$T^i = \mathrm{Img}(s^i).$$

Conjecture 1.2.13 (Generalized Hodge)**.** Let X be a smooth projective variety over $\mathbb{C}$. The piece $T^i \subset \mathrm{L}_0\mathrm{H}_{2k}(X, \mathbb{Q})$ of the topological filtration with rational coefficients coincides with the largest Hodge structure contained in $H_{k-i,k+i}(X, \mathbb{Q}) \oplus \cdots \oplus H_{k+i,k-i}(X, \mathbb{Q})$.

Remark 1.2.14. In particular, this conjecture implies the generalized Hodge conjecture as corrected by Grothendieck in [33].

Conjecture 1.2.15 (Suslin)**.** Let X be a smooth projective variety of dimension d over $\mathbb{C}$. The cycle map

$$s^k \colon \mathrm{L}_k\mathrm{H}_n(\mathrm{X}) \longrightarrow \mathrm{L}_0\mathrm{H}_k(\mathrm{X})$$

is an isomorphism for $n \leq k + d$ and is injective for $n = k + d - 1$.

Remark 1.2.16. One one hand, this conjecture is a Lawson analogue of the Beilinson-Lichtenbaum conjecture for motivic cohomology, which is now a theorem after Voevodsky's proof of Bloch-Kato. In a sense, Suslin's conjecture is a version of Beilinson-Lichtenbaum with integral coefficients instead of torsion. On the other hand Beilinson has proved in a recent paper [1] that a weak form of Suslin's conjecture 1.2.15 implies Grothendieck's Standard Conjectures. This means that the integral part of the conjecture is a very deep result about algebraic cycles.

1.2.4 Some computations

In this section we collect some useful computations of Lawson homology.

It is convenient to extend the definition of Lawson homology $L_kH_n(X)$ for $k < 0$. This can be done as a result of homotopy invariance.

Definition 1.2.17. Let X be an algebraic variety. Let $k, n \in \mathbb{Z}$. We define

$$L_kH_n(\mathrm{X}) = \begin{cases} \pi_{n-2k}\mathbf{Z}_{k+r}(\mathrm{X} \times \mathbb{A}^r) & \text{if } n \geq 2k, \\ 0 & \text{if } n < 2k. \end{cases}$$

where r is an nonnegative integer such that $r + k \geq 0$.

It is an easy consequence of the homotopy invariance for Lawson homology that this is well defined, i.e. it does not depend on the r chosen. Moreover, for $k > 0$ it coincides with the old definition. In fact, for $k < 0$ it holds

$$\begin{aligned} L_kH_n(X) &= \pi_{n-2k}\mathbf{Z}_0(X \times \mathbb{A}^{|k|}) \\ &= L_0H_{n-2k}(X) \\ &= H^{\mathrm{BM}}_{n-2k}(X \times \mathbb{A}^{|k|}) \\ &= H^{\mathrm{BM}}_{n}(X) \\ &= L_0H_n(X). \end{aligned}$$

Remark 1.2.18. At first thought we could have defined $L_kH_n(X) = 0$ for $k < 0$ but this would not have been contravariantly functorial. For example pick $\mathbb{A}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \to \mathbb{P}^2$ and compute L_1H_k.

Now we describe a couple of standard computations. First a computation of Lawson homology for $\mathbb{P}^n$.

Proposition 1.2.19. The Lawson homology of $\mathbb{P}^n$ is computed as follows

$$\mathrm{L}_k\mathrm{H}_n(\mathbb{P}^d) = \begin{cases} \mathbb{Z} & \text{if } n = 2k, 2k+2 \dots 2d, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Use induction on d, and the localization exact sequence from 1.2.6.

$$\cdots \mathrm{L}_k\mathrm{H}_n(\mathbb{P}^{d-1}) \longrightarrow \mathrm{L}_k\mathrm{H}_n(\mathbb{P}^d) \longrightarrow \mathrm{L}_k\mathrm{H}_n(\mathbb{A}^d) \longrightarrow \mathrm{L}_k\mathrm{H}_{n-1}(\mathbb{P}^{d-1}) \cdots$$

●

An other accessible computation is in the case of divisors.

Theorem 1.2.20. Let X be a smooth algebraic variety of dimension d. There is a fibration sequence

$$\mathbb{P}^\infty \longrightarrow \mathbb{Z}_{d-1}(X) \longrightarrow \mathrm{Pic}(X)$$

that gives

$$\mathrm{L}_{d-1}\mathrm{H}_n(X) = \begin{cases} \mathrm{NS}(X) & \text{n = 2d-2,} \\ H_1(\mathrm{Pic}_0(X)) & \text{n = 2d-1,} \\ \mathbb{Z} & \text{n = 2d,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. See Theorem 4.6 in [13]. ●

Finally, Hu proves the following blow-up formula for Lawson homology of smooth varieties

Theorem 1.2.21. Let X be a smooth projective variety and Y a smooth subvariety of codimension r. Let's denote by $\widetilde{X}$ the blow up of X along Y. Then we have the following formula for Lawson homology

$$\mathrm{L}_k\mathrm{H}_n(\widetilde{X}) \cong \bigoplus_{i=1}^{r-1} \mathrm{L}_{k-i}\mathrm{H}_{n-2i}(Y) \oplus \mathrm{L}_k\mathrm{H}_n(X).$$

Proof. See Theorem 3.1 in [42]. ●

1.3 Morphic cohomology

In this section we will describe a theory related to Lawson homology by Poincaré duality. As we will see, this dual theory, is considerably more difficult to work with. It was developed by Friedlander and Lawson in [21], [19].

To motivate the definition, observe that $\mathbf{Z}_0(\mathbb{A}^k)$ has the same homotopy type as an Eilenberg MacLane space $K(\mathbb{Z}, 2k)$. This follows from Theorem 1.2.8. So, singular homology can be written as

$$\mathrm{H}^k(X, \mathbb{Z}) = \pi_0\, \mathbf{Map}(X, \mathbf{Z}_0(\mathbb{A}^k)).$$

Now it seems natural to attempt to define a semi-topological version of singular cohomology by replacing the space **Map** of continuous maps with a space **Hom** of algebraic morphisms. The details of this are not immeadiate, because $\mathbf{Z}_0(\mathbb{A}^k)$ is not really an algebraic variety, but a definition can be made, as we will see in this section. Along the way, $\mathbb{A}^k$ is replaced by an algebraic variety Y producing, in fact, a bivariant theory.

1.3.1 The spaces of algebraic cocycles

Friedlander and Lawson construct a theory of algebraic cocycles in [21] and [12].

Let X and Y be algebraic varieties, and $U \subset X$ a subvariety. We define the subspace $\mathbf{Z}_k^{\mathrm{equi}/U}(X \times Y) \subset \mathbf{Z}_{k+\dim(X)}(X \times Y)$ of cycles equidimensional over U as follows

$$\mathbf{Z}_k^{\mathrm{equi}/U}(X \times Y) = \{n_1[W_1] + \cdots + n_r[W_r] \mid W_i|_U \to U \text{ is equidimensional of rel. dim. } k\}.$$

Definition 1.3.1. Let X be a normal quasi-projective variety of dimension d, with projective closure $\overline{X}$ and closed complement X^∞. Let Y be an other quasi-projective with closure $\overline{Y}$ and closed complement Y^∞. The spaces of **algebraic cocycles**, or morphic cohomology spaces $\mathbf{M}_k(X, Y)$ are

$$\mathbf{M}_k(X, Y) = \mathbf{Z}_{d+k}^{\mathrm{equi}/X}(\overline{X} \times \overline{Y}) / (\mathbf{Z}_{d+k}(X^\infty \times \overline{Y}) + \mathbf{Z}_{d+k}(\overline{X} \times Y^\infty)).$$

together with the quotient topology.

Remark 1.3.2. The hypothesis of normality is needed to have contravariant functioriality for the cocycle spaces on the first variable, since this functoriality is a consequence of Theorem 1.3.4 above.

As in the case of spaces of cycles, it is useful to use the complexes of singular chains on the spaces of cocycles.

Definition 1.3.3. The **complex of algebraic cocycles** associated to X and Y is the cochain complex of abelian groups $\mathcal{M}_k^*(X,Y)$ defined as follows

$$\mathcal{M}_k^n(X,Y) = \mathrm{Sing}_{2k-n}\mathbf{M}_k(X,Y) = \mathrm{Map}(\Delta^{2k-n}, \mathbf{M}_k(X,Y)).$$

The differential is given by the alternating sum of the face maps.

By the Dold-Kan theorem, we know that

$$H^n\mathcal{M}_k^*(X,Y) \cong \pi_{2k-n}\mathbf{M}_k(X,Y) \tag{1.7}$$

The spaces $\mathbf{M}_k(X,Y)$ have an alternative interpretation as function spaces.

Theorem 1.3.4. Let X be a normal quasi-projective variety and Y a projective variety. The graph map

$$\Gamma : \mathbf{Hom}(X, \mathbf{C}_k(Y))^+ \longrightarrow \mathbf{M}^k(X,Y)$$

is a homeomorphism of topological spaces, where the topology on $\mathbf{Hom}(X, \mathbf{C}_k(Y))$ is such that $\{\alpha_n\}_n \to \alpha$ in $\mathbf{Hom}(X, \mathbf{C}_k(Y))$ if, and only if

1. $\{\alpha_n\}_n \to \alpha$ for the compact-open topology.
2. For some closure $\overline{X}$ of X, the closures of the graphs $\{\Gamma(\alpha_n)\}_n$ have degree uniformly bounded by some $N \geq 0$.

Proof. See Proposition 1.9 in [12]. ●

1.3.2 Definition of morphic cohomology

Having developed the bivariant spaces $\mathbf{M}_k(X,Y)$, there is a natural way to define a semi-topological cohomology: take algebraic maps to some algebraic model of an Eilenberg-Maclane space.

From the homotopy invariance theorem 1.2.8, we see that

$$\pi_l\mathbf{Z}_k(\mathbb{A}^r) = \begin{cases} \mathbb{Z} & \text{if } l = 2r-2k, \\ 0 & \text{otherwise.} \end{cases}$$

We can use this to define morphic cohomology, as follows.

Definition 1.3.5. Let X be a normal quasi-projective variety. The **morphic cohomology spaces** are defined as

$$\mathbf{M}^q(X) = \mathbf{M}_0(X, \mathbb{A}^q). \tag{1.8}$$

with associated complexes

$$\mathcal{M}^{q,n}(X) = \pi_{2q-n}\mathbf{M}_0(X, \mathbb{A}^q). \tag{1.9}$$

Definition 1.3.6. The **morphic cohomology groups** are defined by

$$\mathrm{L}^q\mathrm{H}^n(X) = \pi_{2q-n}\mathbf{M}^q(X).$$

There is a cup product in morphic cohomology. Let X, X' be varieties over $\mathbb{C}$. The projections from $X \times X'$ to the factors induce an exterior product

$$\mathrm{L}^q\mathrm{H}^n(X) \otimes \mathrm{L}^{q'}\mathrm{H}^{n'}(X') \longrightarrow \mathrm{L}^{q+q'}\mathrm{H}^{n+n'}(X \times X'). \tag{1.10}$$

Theorem 1.3.7. Composing the exterior product (1.10) with the diagonal embedding defines a cup product in morphic cohomology

$$\mathrm{L}^q\mathrm{H}^n(X) \otimes \mathrm{L}^{q'}\mathrm{H}^{n'}(X) \longrightarrow \mathrm{L}^{q+q'}\mathrm{H}^{n+n'}(X). \tag{1.11}$$

which is graded commutative: $a \cdot b = (-1)^{nn'} b \cdot a$ for $a \in \mathrm{L}^q\mathrm{H}^n(X)$, $b \in \mathrm{L}^{q'}\mathrm{H}^{n'}(X)$.

Proof. See Corollary 6.2 in [21]. ●

Remark 1.3.8. Let us denote by LH the morphic cohomology ring of a point. As stated in Proposition 1.3.17 below, LH is a graded ring concentrated in cohomological degree 0, where the grading comes from the q-index. Then, the structure map $X \to \operatorname{Spec} \mathbb{C}$ provides $\mathrm{L}^*\mathrm{H}^n(X)$ with the structure of a graded LH-module.

1.3.3 The duality map

The main technical tool to prove the duality theorem is the follwing result.

Theorem 1.3.9. Let X be a projective variety of dimension d, over $\mathbb{C}$. Let $r, s, e \geq 0$ with $r + s > d$. Then, there exist a Zariski open set $U \subset \mathbb{P}^1$ such that $0 \in U$, and a continuous algebraic map

$$\Psi = (\Psi^+, \Psi^-)\colon \mathbf{C}_s(X) \times U \longrightarrow \mathbf{C}_s(X)^2$$

such that

1. For any cycle $\alpha \in \mathbf{C}_s(X)$,
$$\alpha = \Psi^+(\alpha, 0) - \Psi^-(\alpha, 0).$$
2. For any $p \in U$, $\Psi^\pm(-, p)$ are morphisms of groups.
3. For any effective cycle $\alpha \in \mathbf{C}_s(X)$, the restriction $\Psi(\alpha, -)$ produces a rational equivalence from α to $\Psi^+(\alpha, p) - \Psi^-(\alpha, p)$.
4. Let α, β in X, of dimensions r, s and degree $\leq e$, and let $p \in U \setminus \{0\}$. Then, any component in the scheme-theoretic intersections $\alpha \cap \Psi^+(\beta, p)$ or $\alpha \cap \Psi^-(\beta, p)$, which intersect non-properly (i.e. in dimension $> r + s - d$), belongs to the singular locus of X.

Proof. See Theorem 3.1 in [20]. ●

Remark 1.3.10. Theorem 1.3.9 above says that given a couple of families of cycles on X of degree bounded by e, there is a homotopy in $\mathbf{C}_k(X)$ which moves one family so that any pair of cycles intersect in proper dimension. This is the key to prove an analogue of Poincaré duality between Lawson and morphic homologies.

Definition 1.3.11. Let X be a quasi-projective variety of dimension d. The **duality map**, is the morphism of abelian groups

$$D\colon \mathrm{L}^q\mathrm{H}^n(X) \longrightarrow \mathrm{L}_{d-q}\mathrm{H}_{2d-n}(X)$$

induced by the chain of continuous maps of spaces of cycles

$$\mathbf{M}^q(X) \longleftarrow \mathbf{Z}_d^{\mathrm{equi}/X}(X \times \mathbb{A}^q) \longrightarrow \mathbf{Z}_d(X \times \mathbb{A}^q) \xrightarrow{\simeq} \mathbf{Z}_{d-q}(X)$$

Theorem 1.3.12. Let X be a smooth quasi-projective variety. Then the duality map from Definition 1.3.11 is an isomorphism. In particular

$$\mathrm{L}^q\mathrm{H}^n(X) \cong \mathrm{L}_{d-q}\mathrm{H}_{2d-n}(X).$$

Proof. See Theorem 3.3 in [19] for the projective case and Theorem 5.2 in [12] for the quasi-projective case. ●

1.3.4 Functoriality and the Mayer-Vietoris property

For $g\colon Y' \to Y$ be a proper morphism, there is an induced homomorphism $g_*\colon \mathbf{M}_k(X, Y') \to \mathbf{M}_k(X, Y)$ given by the push-forward of cycles.

Theorem 1.3.13. Let $f\colon X' \to X$ be a morphism of normal quasi-projective varieties and $g\colon Y' \to Y$ a morphism of quasi-projective varieties. Then

1. There is a continuous pull-back $f^*\colon \mathbf{M}_k(X, Y) \to \mathbf{M}_k(X', Y)$.
2. If g is proper, there is a continuous push-forward $g_*\colon \mathbf{M}_k(X, Y') \to \mathbf{M}_k(X, Y)$.
3. If g is flat, there is a continuous pull-back $g^*\colon \mathbf{M}_k(X, Y) \to \mathbf{M}_k(X, Y')$.

This makes the cocycle spaces $\mathbf{M}_k(X, Y)$ into a covariant functor for proper morphisms on Y, and contravariant for morphisms of normal varieties on X.

Proof. See Propositions 3.1 and 3.3 in [12]. ●

These bivariant cocycle spaces, behave like cycle spaces on Y and like function spaces on X. In particular, we have a localization exact sequence for Y.

Theorem 1.3.14. Let X, Y be quasi-projective varieties with X normal. Let $\overline{Y}$ be a projective closure with closed complement Y^∞. Then there is a triangle in the derived category of chain complexes of abelian groups $\mathbf{D}(\mathbb{Z})$

$$\mathcal{M}_k(X, Y^\infty) \longrightarrow \mathcal{M}_k(X, \overline{Y}) \longrightarrow \mathcal{M}_k(X, Y) \longrightarrow \mathcal{M}_k(X, Y^\infty)[1]$$

Proof. See Proposition 2.2 [12]. ●

From this, it follows a Mayer-Vietoris property for Zariski covers of Y, and for closed covers, the same way as for cycle spaces.

On the other hand, it is not known whether the Mayer-Vietoris property for the Zariski topology on X holds in general. We have an exact sequence of abelian groups

$$\mathbf{M}_k(X,Y) \longrightarrow \mathbf{M}_k(U,Y) \oplus \mathbf{M}_k(V,Y) \longrightarrow \mathbf{M}_k(U \cap V, Y)$$

However, the available techniques do not seem to be enough to prove that this is a fibration or, at least, that it produces long exact sequences.

1.3.5 Homotopy invariance

The spaces of cocycles $\mathbf{M}_k(X,Y)$ have homotopy invariance property with respect to both variables.

Theorem 1.3.15. Let X, Y be quasi-projective varieties, with X normal.

1. The projection $p\colon X \times \mathbb{A}^1 \to X$ induces a homotopy equivalence
$$p^* : \mathbf{M}_k(X,Y) \longrightarrow \mathbf{M}_k(X \times \mathbb{A}^1, Y)$$
2. The projection $p\colon Y \times \mathbb{A}^1 \to Y$ induces a homotopy equivalence
$$q^* : \mathbf{M}_k(X,Y) \longrightarrow \mathbf{M}_k(X, Y \times \mathbb{A}^1)$$

Proof. See Proposition 3.5 in [12]. ●

Remark 1.3.16. The homotopy invariance with respect to X is somewhat surprising, having in mind that the Mayer-Vietoris property is unknown. Even worse, if one uses descent techniques to force the Mayer-Vietoris property, like Friedlander does in [14], the homotopy invariance property is lost.

1.3.6 Computations

There are essentially two methods to perform computations in morphic cohomology, and both require the varieties to be smooth. The first method consists in using the duality theorem and performing the computation for Lawson homology. In other words, for smooth varieties, every computation of Lawson homology leads to a computation for morphic cohomology.

The second method uses comparison maps with motivic and singular homology. In particular, one can use Bloch-Kato's theorem to obtain computations of morphic cohomology on a certain range of indices. This technique is used, for example in [68].

Both of this methods, however, require the smoothness hypothesis. For singular varieties, there are virtually no nontrivial computations available. Part of the problem, is that morphic cohomology for singular varieties is much more subtle than Lawson homology. In particular it is not known whether the Mayer-Vietoris property for the Zariski topology holds in the singular case.

Now we collect some basic computations

Proposition 1.3.17.

1. For $k \geq 0$, $\mathrm{L}^*\mathrm{H}^*(\mathbb{A}^k) \cong \mathbb{Z}[s]$, where s is a free generator of bidegree $(1,0)$ (degree 1 with respect to the q-grading).
2. For $k \geq 0$, $\mathrm{L}^*\mathrm{H}^*(\mathbb{P}^k) \cong \mathbb{Z}[s,h]/(h^{k+1})$, where s has bidegree $(1,0)$ and h has bidegree $(1,2)$.
3. $\mathrm{L}^*\mathrm{H}^*(\mathbb{G}_m) \cong \mathbb{Z}[s,e]/(e^2)$, where s is a generator of bidegree $(1,0)$ and e is a generator of bidegree $(1,1)$.

Proof. 1) and 2) follow from duality and the computation of Lawson homology of $\mathbb{P}^k$.

3) Take the open cover of $\mathbb{P}^1$ by two affine spaces. Then we have the following piece of Mayer-Vietoris sequence

$$\mathrm{L}^*\mathrm{H}^n(\mathbb{P}^1) \longrightarrow \mathrm{L}^*\mathrm{H}^n(\mathbb{A}^1)^{\oplus 2} \longrightarrow \mathrm{L}^*\mathrm{H}^n(\mathbb{G}_m) \longrightarrow \mathrm{L}^*\mathrm{H}^{n+1}(\mathbb{P}^1) \longrightarrow \mathrm{L}^*\mathrm{H}^{n+1}(\mathbb{A}^1)^{\oplus 2}$$

which, using 1) and 2) for the computations of $\mathbb{P}^1$ and $\mathbb{A}^1$ gives the result. ●

Remark 1.3.18. Note that, in particular, $\mathrm{LH} \cong \mathbb{Z}[s]$. Then, by Remark 1.3.8 the morphic cohomology groups $\mathrm{L}^*\mathrm{H}^n(X)$ are $\mathbb{Z}[s]$-modules. The action by s on $\mathrm{L}^*\mathrm{H}^n(X)$ corresponds to the s-maps in morphic cohomology.

As this structure of LH-module in morphic cohomology is functorial, the exterior product (1.10) factors through

$$\mathrm{L}^*\mathrm{H}^*(X) \otimes_{\mathrm{LH}} \mathrm{L}^*\mathrm{H}^*(Y) \longrightarrow \mathrm{L}^*\mathrm{H}^*(X \times Y). \tag{1.12}$$

Remark 1.3.19. The Künneth homomorphism (1.12) is not an isomorphism in general.

2 Descent and the extension theorem

In this section we discuss a proof of a version of the main theorem in [35] using techniques of ∞-categories. The virtue of this approach is that it simplifies the proof, at least conceptually, and makes the theorem applicable in new contexts.

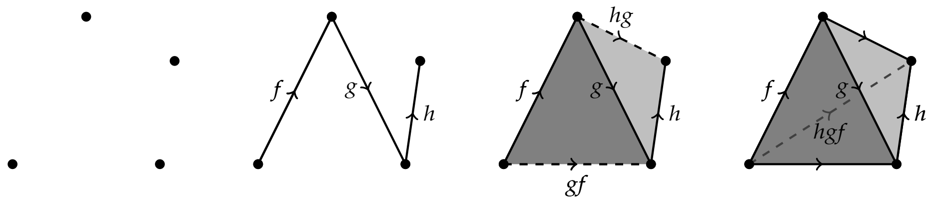
2.1 Preliminaries on infinity-sheaves

2.1.1 Infinity-categories

For us, ∞-category will mean $(\infty, 1)$-category, in the sense that all k-morphisms for $k > 1$ are invertible. Following Lurie's work [57], we will take quasi-categories as our model for ∞-categories.

Definition 2.1.1. An **∞-category** is a simplicial set that has the lifting propery for all inner horns $\Lambda^n_k \hookrightarrow \Delta^n$ with $0 < k < n$.

The 0-cells correspond to objects, the 1-cells to morphisms and the filling of 2-cells give weak compositions of morphisms. Finally the filling for higher cells represent a tower of coherent weak associativities.



Remark 2.1.2. An ∞-category which also, satisfies the lifting property for the exterior horns, i.e. a Kan complex, is an ∞-groupoid: the lifting for external 2-cells produces homotopy inverses for all 1-morphisms.

Once we believe that Kan complexes model ∞-groupoids, which is, after all, an old idea going back to Grothendieck [34], we expect an ∞-category to be a category weakly enriched in Kan complexes. The most naïve model for this are simplicially enriched categories, where the enrichment is strict.

Both, quasi-categories and simplicially enriched categories are equivalent presentations for the theory of $(\infty, 1)$-categories, in the sense that there are model category structures on both, which happen to be Quillen equivalent. See [3] for more details.

To any simplicially enriched category, we can assign its homotopy coherent nerve $N\mathcal{C}$. This construction has a left adjoint $\mathfrak{G}$, that produces a simplicially enriched category out of a simplicial set.

$$\begin{tikzcd} \mathbf{sSet} \arrow[r, bend left, "\mathfrak{G}"] & \mathbf{sCat} \arrow[l, bend left, "N"] \end{tikzcd} \tag{2.1}$$

In a simplicially enriched category $\mathcal{C}$, we have a mapping space $\operatorname{Map}_{\mathcal{C}}(a, b)$ between any two objects a, b. Also, we define its homotopy category $h_0\mathcal{C}$ as the category with the same objects as $\mathcal{C}$, and $\operatorname{Hom}_{h_0\mathcal{C}}(a, b) = \pi_0 \operatorname{Map}_{\mathcal{C}}(a, b)$. Note that $h_0\mathcal{C}$ is a plain 1-category.

Now, simplicially enriched categories have a model structure [2] for which weak equivalences are simplicial functors that induce categorical equivalences on homotopy categories, and weak homotopy equivalences on mapping spaces. These equivalences can be transported to **sSet** via the functor $\mathfrak{G}$. Then, there is a model structure on **sSet** with such morphisms as weak equivalences, and whose fibrant objects are the quasi-categories (unpublished) and [57] Theorem 2.2.5.1. With these two model structures on **sSet** and **sCat**, the adjunction (2.1) becomes a Quillen equivalence ([57] Theorem 2.2.5.1).

Example 2.1.3. Let $\mathcal{C}$ be a 1-category, together with a class of morphisms $\mathcal{W}$. From this, we can construct the Dwyer-Kan [10] simplicial localization $L_{\mathcal{W}}\mathcal{C}$, a simplicially enriched category which we interpret as an ∞-category via the homotopy coherent nerve.

In the particular case where $(\mathcal{C}, \mathcal{W})$ has the structure of a simplicial model category, we have a more convenient route to produce an ∞-category. We take $\mathcal{C}_{cf}$ the subcategory of cofibrant-fibrant objects in $\mathcal{C}$ with the usual mapping spaces as the resulting ∞-category.

Example 2.1.4. The ∞-category of spaces **Spc** is the one obtained from the usual model category structure on spaces. For concreteness, we use the simplicial category of Kan complexes, or its corresponding quasi-category as a model for **Spc**.

In this language, let $\mathcal{C}, \mathcal{D}$ be quasi-categories. We can easily construct the **functor category** $\mathbf{Func}(\mathcal{C}, \mathcal{D})$ as the usual mapping space in **sSet**.

$$\mathbf{Func}(\mathcal{C}, \mathcal{D}) = \operatorname{Map}_{\mathbf{sSet}}(\mathcal{C}, \mathcal{D}),$$

which happens to be a quasi-category ([57] Proposition 1.2.7.3). This is particularly interesting, as this notion of functor is inherently weak (the higher simplices constitute an infinite chain of coherence diagrams), yet it is defined in terms of a mapping space of simplicial sets, an object we are very familiar with.

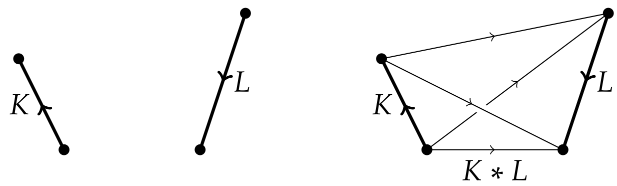
2.1.2 Limits and colimits

Limits and colimits are defined in the context of ∞-categorgies, by an universal property. Let's look at one particularly simple type of limits first:

Definition 2.1.5. An object $X \in \mathcal{C}$ is a **final** object if for any $A \in \mathcal{C}$, the mapping spaces $\operatorname{Map}(A, X)$ is contractible. Dually, X is said to be **initial** if for any $A \in \mathcal{C}$, the mapping spaces $\operatorname{Map}(X, A)$ are contractible.

Definition 2.1.6. The **join** of two simplicial sets K, L is a new simplicial set denoted by $K * L$ and defined as follows:

$$
\begin{aligned}
(K * L)_0 &= K_0 \sqcup L_0, \\
(K * L)_1 &= K_1 \sqcup L_1 \sqcup K_0 \times L_0, \\
(K * L)_n &= K_n \sqcup L_n \sqcup \bigsqcup_{r+s=n-1} K_r \times K_s.
\end{aligned}
$$



Let $p\colon L \to \mathcal{C}$ be a map of ∞-categories. There exist ∞-categories $\mathcal{C}_{/p}$ and $\mathcal{C}_{p/}$ satisfying the following properties

$$\mathrm{Hom}(K, \mathcal{C}_{/p}) = \mathrm{Hom}_p(K * L, \mathcal{C}) \tag{2.2}$$

$$\mathrm{Hom}(K, \mathcal{C}_{p/}) = \mathrm{Hom}_p(L * K, \mathcal{C}) \tag{2.3}$$

Where the right hand side refers to the subset of morphisms $f \in \mathrm{Hom}(K * L, \mathcal{C})$ such that $f|L = p$.

Definition 2.1.7. Let $p\colon L \to \mathcal{C}$ be a diagram in $\mathcal{C}$. A **limit** of p is an initial object in $\mathcal{C}_{/p}$. A **colimit** of p is a final object in $\mathcal{C}_{/p}$.

We denote by $\mathbf{Func}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ the full subcategory of $\mathbf{Func}(\mathcal{C}, \mathcal{D})$ generated by functors that preserve limit diagrams.

Remark 2.1.8. There is an ∞-categorical notion of adjoint functors. In essence, an adjoint pair of functors between $\mathcal{C}$ and $\mathcal{D}$ consists of a fibration

$$\mathcal{M} \to \Delta^1,$$

that satisfies a cartesian and cocartesian fibration conditions, together with identifications of $\mathcal{C} \cong \mathcal{M}_0$ and $\mathcal{D} \cong \mathcal{M}_1$. With this notion of adjoint-ness and certain accessibility assumptions on $\mathcal{C}$ and $\mathcal{D}$, there is an adjoint function theorem stating that limit-preserving functors coincide with the ones having a left adjoint.

Since we will not use the details, we refrain from deepening into the technicalities of this construction. It can be found in Section 5.2.2 in [57].

2.1.3 Space-valued sheaves

The role played by **Set** in the category **Cat** of 1-categories is played by the ∞-category **Spc** in the category $\mathbf{Cat}_\infty$ of ∞-categories.

Definition 2.1.9. Let $\mathcal{C}$ be an ∞-category. The category of **presheaves** on $\mathcal{C}$ is the ∞-category of functors $\mathbf{pSh}(\mathcal{C}) = \mathbf{Func}(\mathcal{C}^{\mathrm{op}}, \mathbf{Spc})$.

The mapping space functor $\mathrm{Map}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Spc}$ gives, by adjunction, a Yonneda embedding ([57] Section 5.1.3)

$$h\colon \mathcal{C} \hookrightarrow \mathbf{pSh}(\mathcal{C})$$

This Yonneda embedding realizes $\mathbf{pSh}(\mathcal{C})$ as a free cocompletion of $\mathcal{C}$ in the following sense.

Proposition 2.1.10. Let $\mathcal{C}$, $\mathcal{D}$ be ∞-categories. Then, the Yonneda embedding induces an equivalence of ∞-categories

$$\mathbf{Func}^{\mathrm{L}}(\mathbf{pSh}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathbf{Func}(\mathcal{C}, \mathcal{D}) \tag{2.4}$$

Proof. See [57] Theorem 5.1.5.6. ●

In order to define sheaves, we need to recall the notion of a Grothendieck topology in the context of ∞-categories ([57] Definition 6.2.2.1).

Definition 2.1.11. Let $\mathcal{C}$ be an ∞-category. A **sieve** on an object X is a subcategory $U \subset \mathcal{C}_{/X}$ closed by precomposition, in the sense that

- For every $f \in \mathrm{Hom}_{\mathcal{C}_{/X}}(A, B)$ and $g \in \mathrm{Hom}_U(B, C)$, the composition $g \circ f$ belongs to $\mathrm{Hom}_U(A, C)$.

A **Grothendieck topology** on $\mathcal{C}$, is a choice of a distinguished class $\mathrm{Cov}(X)$ of sieves (the covering sieves) for any object $X \in \mathcal{C}$, such that

1. $\mathcal{C}_{/X} \in \mathrm{Cov}(X)$.
2. for every $f\colon Y \to X$ and $U \in \mathrm{Cov}(X)$, we have $f^*U \in \mathrm{Cov}(Y)$.
3. Let U be a sieve on X and $V \in \mathrm{Cov}(X)$. If $f^*(U) \in \mathrm{Cov}(Z)$ for any $Z \in V$ and any morphism $f \in \mathrm{Hom}_V(Z, X)$, then $U \in \mathrm{Cov}(X)$.

Remark 2.1.12. The notion of sieve has an alternative point of view, which sometimes may be more convenient. To any sieve $U \subset \mathcal{C}_{/X}$ we can assign a subfunctor $h_U \hookrightarrow h_X$ as follows

$$h_U(Y) = \mathrm{Hom}_U(Y, X) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, X).$$

Then, the set of sieves on an object $X \in \mathcal{C}$ is in bijection with the set of subfunctors of the representable functor h_X ([57] Proposition 6.2.2.5).

Remark 2.1.13. Some care must be taken with the notion of subobject in an ∞-category. A morphism $f\colon Y \to Z$ in $\mathcal{C}$ is a monomorphism when, for every object X, the continuous map $\mathrm{Map}(X, Y) \to \mathrm{Map}(X, Z)$ is split, in the sense that $\mathrm{Map}(X, Y)$ is homotopy equivalent to a union of connected components in $\mathrm{Map}(X, Z)$.

Definition 2.1.14. Let $\mathcal{C}$ be an ∞-category together with a Grothendieck topology τ. The ∞-category of sheaves on $\mathcal{C}$ is the full subcategory of $\mathbf{pSh}(\mathcal{C})$ generated by the presheaves $\mathcal{F}$ such that the map

$$\mathrm{Map}(h_X, \mathcal{F}) \xrightarrow{\simeq} \mathrm{Map}(h_U, \mathcal{F}) \tag{2.5}$$

is a weak homotopy equivalence for any object X and any covering sieve $U \in \mathrm{Cov}(X)$.

Remark 2.1.15. The ∞-category of sheaves $\mathbf{Sh}_\tau(\mathcal{C})$ is a Bousfield localization of $\mathbf{pSh}(\mathcal{C})$ for the class of covering sieve inclusions $h_U \hookrightarrow h_X$ ([57] Lemma 6.2.2.7). The sheaves are the local objects with respect to this class of morphisms. In particular, there is a left adjoint $\mathrm{sh}_\tau \colon \mathbf{pSh}(\mathcal{C}) \to \mathbf{Sh}_\tau(\mathcal{C})$ which commutes with finite limits. This is an ∞-categorical analogue of the associated sheaf functor.

Remark 2.1.16. Let $U \in \mathrm{Cov}(X)$ be a covering sieve. Then, h_U is a colimit of representables as follows

$$h_U = \operatorname*{colim}_{Z \in U} h_Z.$$

It then follows that, for any sheaf $\mathcal{F}$, we have

$$\begin{aligned}\mathrm{Map}(U, \mathcal{F}) &\simeq \mathrm{Map}(\operatorname*{colim}_{Z \in U} h_Z, \mathcal{F}) \\ &\simeq \lim_{Z \in U} \mathcal{F}(Z).\end{aligned}$$

So, the sheaf condition (2.5) says exactly that $\mathcal{F}$ sends the colimit cones $\{h_Z \to h_X\}_{Z \in U}$ to limit cones in **Spc**.

2.1.4 The internal homotopy

The categories of sheaves have an internal homotopy theory, in the sense that there is a notion of homotopy sheaves, that can be used to detect weak equivalences of objects, in the same way homotopy groups detect weak homotopy equivalences in **Spc**.

Definition 2.1.17. Let $(\mathcal{C}, \tau)$ be an ∞-category together with a Grothendieck topology. We can associate to any sheaf $\mathcal{F} \in \mathbf{Sh}_\tau(\mathcal{C})$, its **homotopy sheaves** $\pi_n \mathcal{F} \in \mathbf{Sh}_\tau(h_0\mathcal{C})$, which are the set-valued sheaves on $h_0\mathcal{C}$ defined as

$$\pi_n \mathcal{F} = \mathrm{sh}_\tau(U \mapsto \pi_n \mathcal{F}(U)). \tag{2.6}$$

One natural question that comes to mind is whether we can detect weak equivalences in $\mathcal{C}$ using these homotopy sheaves. This would be an abstract version of Whitehead's theorem in classical homotopy theory. The answer is "no" in general, there may be morphisms inducing isomorphisms on all homotopy sheaves, which are not weak equivalences.

Definition 2.1.18. Let $\mathcal{C}$ be an ∞-topos. If the class of weak equivalences $\mathcal{W}$ (i.e. the morphisms which become isomorphisms on $h_0\mathcal{C}$) coincides with the morphisms inducing isomorphisms on homotopy sheaves, then the topos $\mathcal{C}$ is called **hypercomplete**.

Example 2.1.19.
1. Any ∞-topos with enough points is hypercomplete (Remark 6.5.4.7 in [57]).

2. An ∞-topos locally of bounded homotopical dimension is hypercomplete (Corollary 7.2.1.12 in [57]).

Proposition 2.1.20. Let $(\mathcal{C}, \tau)$, be an ∞-category, together with a hypercomplete Grothendieck topology. Let W_τ be the class of morphisms in $\mathbf{pSh}(\mathcal{C})$ that are sent to equivalences by sh_τ. Then, $f : \mathcal{F} \to \mathcal{G}$ belongs to W_τ if, and only if $\pi_n f : \pi_n \mathcal{F} \to \pi_n \mathcal{G}$ are isomorphisms of sheaves of sets, for every $n \geq 0$.

Proof. Because $\mathbf{Sh}_\tau(\mathcal{C})$ is hypercomplete, Proposition 6.5.2.14 in [57], tells us that $\mathbf{Sh}_\tau(\mathcal{C})$ is equivalent to the ∞-category underlying the Joyal model structure on simplicial presheaves. Since weak equivalences in the Joyal model structure are the ones inducing isomorphisms on homotopy sheaves, we are done. ●

2.1.5 $\mathcal{D}$-valued sheaves

The following definition in [55] Definition 1.1.9, is inspired by Remark 2.1.16 above.

Definition 2.1.21. Let $\mathcal{C}$ be an ∞-category, with a Grothendieck topology τ and let $\mathcal{D}$ be a complete ∞-category. The category of **$\mathcal{D}$-valued sheaves** $\mathbf{Sh}_\tau(\mathcal{C}, \mathcal{D})$ is the full subcategory of functors $\mathcal{F} \in \mathbf{Func}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ such that, for any $X \in \mathcal{C}$ and $U \in \mathrm{Cov}(X)$, the induced map

$$\mathcal{F}(X) \xrightarrow{\simeq} \lim_{Z \in U} \mathcal{F}(Z) \tag{2.7}$$

is an equivalence in $\mathcal{D}$.

Theorem 2.1.22. The Yonneda embedding induces a commutative diagram of ∞-categories

$$\begin{tikzcd} \mathbf{Func}^{\mathrm{L}}(\mathbf{pSh}(\mathcal{C}), \mathcal{D}) \arrow[r, "\simeq"] & \mathbf{Func}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \\ \mathbf{Func}^{\mathrm{L}}(\mathbf{Sh}_\tau(\mathcal{C}), \mathcal{D}) \arrow[r, "\simeq"] \arrow[u, hook] & \mathbf{Sh}_\tau(\mathcal{C}, \mathcal{D}) \arrow[u, hook] \end{tikzcd} \tag{2.8}$$

where the horizontal maps are equivalences. In particular

$$\mathbf{Sh}_\tau(\mathcal{C}, \mathcal{D}) \simeq \mathbf{Func}^{\mathrm{L}}(\mathbf{Sh}_\tau(\mathcal{C}), \mathcal{D}) \tag{2.9}$$

Proof. This is [55], Proposition 1.1.12. The proof goes as follows. Because of the universal property of $\mathbf{pSh}(\mathcal{C})$ (Proposition 2.1.10), the top functor is an equivalence. Now $\mathbf{Sh}_\tau(\mathcal{C}, \mathcal{D})$ can be regarded as a full subcategory of $\mathbf{Func}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ spanned by those functors $\mathcal{F} \in \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ such that, for any covering sieve $U \in \mathrm{Cov}(X)$

$$\mathcal{F}(X) \simeq \lim_{Z \in U} \mathcal{F}(Z). \tag{2.10}$$

But this condition translates, via the top equivalence, to the fact that the corresponding functor on the left $\overline{\mathcal{F}}$: $\mathbf{Func}^{\mathrm{L}}(\mathbf{pSh}(\mathcal{C}), \mathcal{D})$ keeps being left-exact when restricted to $\mathbf{Sh}_\tau(\mathcal{C})$. ●

Remark 2.1.23. Equation (2.9) decouples the sheaf condition, which depends on the topology, from the target category $\mathcal{D}$. This fact will let us transfer results from $\mathbf{Sh}_\tau(\mathcal{C})$ to $\mathbf{Sh}_\tau(\mathcal{C}, \mathcal{D})$.

2.2 Some Grothendieck topologies on schemes

In this section we collect the definitions of some Grothendieck topologies we will use. The first section describes Voevodsky's notion of cd-structures, a way to describe Grothendieck topologies via commutative squares [66].

2.2.1 cd-structures and topologies defined by squares

Definition 2.2.1. Let $\mathcal{C}$ be a category with an initial object 0. A **cd-structure** on $\mathcal{C}$ is a class $\mathcal{P}$ of commutative squares in $\mathcal{C}$ closed under isomorphism of squares. The squares in $\mathcal{P}$ will often be called **distinguished squares**.

A cd-structure $\mathcal{P}$ on $\mathcal{C}$ gives an associated Grothendieck topology $\tau_{\mathcal{P}}$, the coarsest one for which every commutative square

$$\begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

in $\mathcal{P}$ gives a covering $\{A \to X, Y \to X\}$ for $\tau_{\mathcal{P}}$.

Definition 2.2.2. Let $\mathcal{P}$ be a cd-structure on $\mathcal{C}$ and $\tau_{\mathcal{P}}$ the associated Grothendieck topology. The class of simple coverings is the smallest class of coverings of $\tau_{\mathcal{P}}$ such that

1. The isomorphisms are simple coverings.
2. Every distinguished square

$$\begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

   in $\mathcal{P}$ gives a simple covering $\{A \to X, Y \to X\}$.
3. The composition of simple coverings is simple.

Remark 2.2.3. The finite simple coverings are the coverings obtained composing a finite number of coverings coming from distinguished squares.

Voevodsky defines three important properties for the topologies coming from cd-structures: completeness, regularity and boundedness. What matters to our discussion is that complete and regular cd-topologies are nice because the sheaf condition can be checked only on distinguished squares.

Theorem 2.2.4. Let $\mathcal{P}$ be a complete and regular cd-structure on $\mathcal{C}$. Then, a presheaf $\mathcal{F} \in \mathbf{pSh}(\mathcal{C})$ is a sheaf for the associated topology $\tau_{\mathcal{P}}$ if, and only if

1. $\mathcal{F}(\emptyset) = \mathrm{pt}$.

2. Every distinguished square

$$\begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

induces a pull-back diagram

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r] \arrow[d] & \mathcal{F}(Y) \arrow[d] \\ \mathcal{F}(A) \arrow[r] & \mathcal{F}(B) \end{tikzcd}$$

Then $\mathcal{F} \in \mathbf{Sh}_{\tau_{\mathcal{P}}}$.

Proof. This is Corollary 2.17 in [66]. ●

The boundedness condition on cd-structure guarantees boundedness of sheaf cohomology

Theorem 2.2.5. Let $\mathcal{P}$ be a complete, regular and bounded cd-structure on $\mathcal{C}$. Then, for any object $X \in \mathcal{C}$ and any sheaf of abelian groups on $\mathcal{C}$, the cohomology groups $H^n(X, \mathcal{F})$ are bounded.

Proof. See Theorem 2.27 in [66]. ●

2.2.2 Completely decomposed topologies

Now we restrict to the category $\mathcal{C} = \mathbf{Sch}_k$ of finite-dimensional schemes over a field k.

Definition 2.2.6. A **splitting sequence** for a morphism of schemes $f\colon U \to X$ is a finite filtration

$$\emptyset = X_0 \subset X_1 \subset \cdots \subset X_r = X$$

of X such that

1. Every morphism $X_i \to X_{i+1}$ is a closed embedding
2. The morphism $U \times_X X_i \to X_i$ admits a section over $X_i \setminus X_{i-1}$ for every i, i.e. there are morphisms $\sigma_i\colon X_i \setminus X_{i-1} \to U \times_X X_i$ such that $f\sigma_i = \mathrm{id}_{X_i \setminus X_{i-1}}$.

Definition 2.2.7. A morphism $f\colon U \to X$ is **completely decomposed** if every k-rational point $x\colon \operatorname{Spec} k \to X$ admits a lifting $\tilde{x}\colon \operatorname{Spec} k \to U$ such that

1. The diagram

$$\begin{tikzcd} & U \arrow[d, "f"] \\ \operatorname{Spec} k \arrow[ur, "\tilde{x}"] \arrow[r, "x"] & X \end{tikzcd}$$

is commutative,

2. f induces an isomorphism on residue fields $k(x) \to k(\tilde{x})$.

Proposition 2.2.8. A morphism of schemes $f : U \to X$ has a splitting sequence if, and only if it is completely decomposed.

Proof. See Lemma 2.15 in [67]. ●

Remark 2.2.9. In particular, in a completely decomposed morphism there is always a Zariski open $V \subset X$ that factors through $U \to X$.

2.2.3 The Nisnevich topology

The Nisnevich topology is a Grothendieck topology (strictly) in between the Zariski and the étale topologies. In a sense, it has the good properties of both topologies avoiding the bad ones. For example, We will see that the Nisnevich topology is generated by distinguished squares, like the Zariski topology, and is fine enough to allow an easy local description of closed embeddings of smooth schemes. They are, Nisnevich locally, like the zero section embedding of a vector bundle.

Definition 2.2.10. The **étale topology** on $\mathbf{Sch}_k$ is the Grothendieck topology given by the covering families $\{U_i \to X\}$ such that

1. The morphisms $U_i \to X$ are étale.
2. The morphism $U = \coprod U_i \to X$ is an epimorphism.

Definition 2.2.11. The **Nisnevich topology** on $\mathbf{Sch}_k$ is the Grothendieck topology given by the covering families $\{U_i \to X\}$ such that

1. The morphisms $U_i \to X$ are étale.
2. The morphism $U = \coprod U_i \to X$ is an epimorphism.
3. The morphism $U = \coprod U_i \to X$ is completely decomposed.

Because of the completely decomposed condition, the Nisnevich topology can be described by a cd-structure.

Definition 2.2.12. We say that a diagram of schemes

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Z \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

is a **Nisnevich distinguished square** if

1. it is pull-back diagram,
2. the map $U \to X$ is an open embedding,
3. the map $Z \to X$ is an étale morphism which induces an isomorphism $(Z \setminus W)_{\mathrm{red}} \to (X \setminus U)_{\mathrm{red}}$ with the reduced scheme structures.

The Nisnevich distinguished squares form a cd-structure on $\mathbf{Sch}_k$.

Proposition 2.2.13. Every Nisnevich covering in $\mathbf{Sch}_k$ admits a finite simple refinement for the Nisnevich cd-structure.

Proof. See Proposition 2.16 in [67]. ●

Theorem 2.2.14. The Nisnevich cd-structure on $\mathbf{Sch}_k$ is complete regular and bounded.

Proof. See Theorem 2.2 in [67]. ●

Remark 2.2.15. In particular, sheaves are detected on distinguished squares, and every sheaf of abelian groups is cohomologically bounded by the dimension of the variety.

2.2.4 The cdh topology

Definition 2.2.16. A morphism of schemes $f : Y \to X$ is called a **topological epimorphism** if it is a quotient map for the underlying topological spaces.

If this property is stable by base change through any morphism $X' \to X$, we will say f is a **universal topological epimorphism**.

Definition 2.2.17. The **h-topology** on $\mathbf{Sch}_k$ is the Grothendieck topology given by the finite covering families $\{U_i \to X\}$ such that $\coprod U_i \to X$ is a universal topological epimorphism.

The **qfh-topology** on $\mathbf{Sch}_k$ is the Grothendieck topology given by the h-coverings $\{U_i \to X\}$ which, in addition are quasi-finite.

Theorem 2.2.18. Let X be a reduced Noetherian scheme over k, and $\{U_i \to X\}$ a covering for the h-topology. Then it has a refinement $\left\{V_j \to X\right\}$ such that every map $p_j \colon V_j \to X$ factors as a composition

$$V_j \xrightarrow{i} V'_j \xrightarrow{p} \widetilde{X}_Y \xrightarrow{q} X$$

where the left map is a Zariski open embedding, the middle map is finite, and the one at the right hand side is a blow-up along a closed subscheme in X.

Proof. See Theorem 3.1.9 in [65]. ●

Remark 2.2.19. In case $\{U_i \to X\}$ is a qfh-covering, the right hand side map in the factorization from Theorem 2.2.18 is the identity.

Definition 2.2.20. The **cdh-topology** on $\mathbf{Sch}_k$ is the Grothendieck topology given by covering families $\{U_i \to X\}$ such that

1. $\{U_i \to X\}$ is a covering for the h-topology.
2. The map $\coprod U_i \to X$ is completely decomposed.

Theorem 2.2.21. Let X be a reduced Noetherian scheme over k and $\{U_i \to X\}$ a covering for the cdh topology. Then it admits a refinement $\left\{V_j \to X\right\}$ such that every map $p_j \colon V_j \to X$ factors as a composition of a Zariski open embedding and an abstract blow-up of X along a closed center.

Proof. See Proposition 5.9 [62]. ●

Definition 2.2.22. An **abstract blow-up** square is a commutative diagram

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "i'"] \arrow[d, "i"] & \widetilde{X} \arrow[d, "q"] \\ Y \arrow[r, "p"] & X \end{tikzcd}$$

such that

1. It is a pull-back square.
2. q is proper.
3. i is a closed embedding.
4. q induces an isomorphism $q \colon \widetilde{X} \setminus \widetilde{Y} \to X \setminus Y$.

The family of abstract blow-ups define a cd-structure on $\mathbf{Sch}_k$. We will refer to the Grothendieck topology it generates as the **abstract blow-up topology**.

Proposition 2.2.23. Every abstract blow-up (resp. cdh) covering $\{U_i \to X\}$ admits a finite simple refinement in the abstract blow-up (resp. cdh) cd-structure.

Proof. See Proposition 2.17 in [67]. ●

Theorem 2.2.24. The abstract blow-up (resp. cdh) squares generate a complete regular and bounded cd-structure on the category $\mathbf{Sch}_k$.

Proof. This is Theorem 2.2 in [66]. What we call "abstract blow-up topology" is called lower-cd-structure by Voevodsky. ●

The main point of the abstract blow-up topology is that we can build smooth coverings for this topology via Hironaka's resolution of singularities. It is needed a strong version of this theorem, as follows

Theorem 2.2.25 (Hironaka)**.** Let X be an algebraic variety defined over a field k of characteristic zero. Then, there exist a sequence of varieties

$$X_r \xrightarrow{p_r} X_{r-1} \xrightarrow{p_{r-1}} \cdots \xrightarrow{p_1} X_0 = X$$

such that

1. X_r is smooth.
2. The maps $p_i\colon X_i \to X_{i-1}$ are blow-ups along a smooth center Z_{i-1}
3. The centers Z_i are contained in the singular locus of X_i.

Proof. See Theorem 3.36 in [46]. ●

An other source of coverings for the abstract blow-up topology is the following lemma due to Chow.

Lemma 2.2.26 (Chow)**.** Let X be an algebraic variety. There exists a quasi-projective variety X', a proper map $p\colon X' \to X$ and a dense open subset $U \subset X$ such that $p\colon p^{-1}(U) \to U$ is an isomorphism.

Proof. See Theorem 5.6.1 in [32]. ●

Proposition 2.2.27. If k admits resolution of singularities (for example if $\operatorname{char} k = 0$), every abstract blow-up covering $\{U_i \to X\}$ of a smooth scheme X by smooth schemes admits a finite simple refinement in the abstract blow-up topology.

Proof. By the proposition 2.2.23 it only remains to check that a covering $\{Y \to X, Z \to X\}$ coming from an abstract blow-up square can be refined to a covering obtained by composition of blow-up's of smooth schemes with smooth centers. But this follows from resolution of singularities. ●

2.2.5 Cubical hyperresolutions

Cubical hyperresolutions is a technique introduced in [36], that produces smooth coverings for the abstract blow-up topology, which are particularly nice to use for computing Cech-style cohomologies, because not only the elements in the covering, but all the fibered products involved in Cech-style complexes are made smooth.

Let $\square^1 = \{1 \to 0\}$ be the category with two objects and a single morphism between them. We denote by $\square^n = \square^1 \times \cdots \times \square^1$ the indexing category for cubical diagrams.

Definition 2.2.28. Let $X_\bullet\colon J \to \mathbf{Sch}_k$ be a finite diagram of schemes. A **cubical hyperresolution** of $X_\bullet$ is a diagram $\widetilde{X}_{\bullet\bullet}\colon \square^n \times J \to \mathbf{Sch}_k$ of schemes such that

1. $\widetilde{X}_{0\ldots 0,\bullet} = X_\bullet$,
2. There exists a diagram of schemes $\widetilde{Y}_{\bullet,\bullet}\colon \square^1 \times J \to \mathbf{Sch}_k$ together with a factorization of $\widetilde{X}_{\bullet,\bullet}$ like this

$$\begin{tikzcd}
\widetilde{X}_{I'1,j} \arrow[r, "p"] \arrow[d, "q"] & \widetilde{Y}_{1,j} \arrow[r, "i"] \arrow[d, "q"] & \widetilde{X}_{0\ldots 01,j} \arrow[d, "q"] \\
\widetilde{X}_{I'0,j} \arrow[r, "p"] & \widetilde{Y}_{0,j} \arrow[r, "i"] & \widetilde{X}_{0\ldots 00,j}
\end{tikzcd} \tag{2.11}$$

for every $I' \neq (0\dots0) \in \square^{n-1}$ and $j \in J$, where i are closed embeddings of codimension at least 1, and p, q are proper morphisms.

3. The induced maps $\widetilde{X}_{0\dots01,j} \setminus \widetilde{Y}_{1,j} \to \widetilde{X}_{0\dots00,j} \setminus \widetilde{Y}_{0,j}$ are isomorphisms.
4. The right square in 2.11 is a pull-back diagram.
5. The left square in 2.11 is a cubical hyperresolution of the diagram $\widetilde{Y}\colon \square^1 \times J \to \mathbf{Sch}_k$.

Remark 2.2.29. Note that the Definition 2.2.28 of cubical hyperresolution is recursive with respect to the dimension of the cubes.

Theorem 2.2.30. Assume k is a field of characteristic 0. Let $X_\bullet : J \to \mathbf{Sch}_k$ be a finite diagram of finite dimensional schemes. Then, there exists a cubical hyperresolution $\widetilde{X}_{\bullet,\bullet} : \square^n \times J \to \mathbf{Sch}_k$ such that

1. $\widetilde{X}_{I,j}$ is smooth whenever $I \neq (0\dots0)$.
2. $\dim \widetilde{X}_{I,j} \leq \dim X_j - |I| + 1$.

Proof. See Theorem 2.15 in [36]. ●

Remark 2.2.31. A smooth cubical hyperresolution $\widetilde{X}_\bullet$ of a scheme X, gives a finite cover $\{U_i = \widetilde{X}_{0\dots1\dots0} \to X\}$ for the topology of abstract blow-ups, with the following additional properties:

1. All the fibered products in the covering $U_{i_1} \times \cdots \times U_{i_k}$ are refined by a corresponding scheme in the cubical diagram $\widetilde{X}_\bullet$ which is smooth.
2. There is a bound on the size of the covering by the dimension of X.

For this reason, smooth hyperresolutions are particularly useful to build Cech-style spectral sequences.

2.3 The extension theorem

2.3.1 An abstract extension theorem

Let $(\mathcal{C}, \tau)$ be an ∞-category together with a Grothendieck topology, and let $\mathcal{C}_0 \subset \mathcal{C}$ be a full subcategory.

Theorem 2.3.1. The restriction functor $\varepsilon^{p*} : \mathbf{pSh}(\mathcal{C}_0) \to \mathbf{pSh}(\mathcal{C})$ admits a left adjoint ε^p_*.

$$\begin{tikzcd} \mathbf{pSh}(\mathcal{C}_0) \arrow[r, bend left, "\varepsilon^{p*}"] & \mathbf{pSh}(\mathcal{C}) \arrow[l, bend left, "\varepsilon^p_*"] \end{tikzcd} \tag{2.12}$$

Proof. See [57], Corollary 4.3.2.14 for the existence of left Kan extensions, and Proposition 4.3.2.17 for the characterization of a left Kan extensions as a left adjoint. ●

This, in turn induces an adjoint pair on the respective categories of sheaves

$$\begin{tikzcd} \mathbf{Sh}_\tau(\mathcal{C}_0) \arrow[r, bend left, "\varepsilon^*"] & \mathbf{Sh}_\tau(\mathcal{C}) \arrow[l, bend left, "\varepsilon_*"] \end{tikzcd} \qquad \begin{aligned} \varepsilon_* &= \varepsilon^p_* \\ \varepsilon^* &= \mathrm{sh}_\tau\, \varepsilon^{p*} \end{aligned} \tag{2.13}$$

The categories of sheaves are localizations of the respective categories of presheaves, by a class of morphisms $\mathcal{W}$. Let $\mathcal{W}$ (resp. $\mathcal{W}_0$) be the class of morphisms in $\mathbf{pSh}(\mathcal{C})$ (resp. $\mathbf{pSh}(\mathcal{C}_0)$) that become equivalences after applying the sheafification functor sh_τ.

Theorem 2.3.2. Let $(\mathcal{C}, \tau)$ be an ∞-category with a hypercomplete Grothendieck topology. Let $\mathcal{C}_0$ be a full subcategory, such that the following condition holds:

- For every covering sieve $U \in \mathrm{Cov}(X)$, the subcategory $U_0 = U \cap \mathcal{C}_0$ generated by the objects in $\mathcal{C}_0$ is cofinal with U.

Then a morphism $f : \mathcal{F} \to \mathcal{G}$ in $\mathbf{pSh}(\mathcal{C})$ belongs to $\mathcal{W}$ if, and only if $\varepsilon_* f \in \mathcal{W}_0$.

Proof. Because of Proposition 2.1.20 we can detect morphisms in $\mathcal{W}$ using the internal homotopy theory in $\mathbf{Sh}_\tau(\mathcal{C})$. Concretely, $f \in \mathcal{W}$ if, and only if $\pi_n f : \mathrm{sh}_\tau\, \pi_n \mathcal{F} \to \mathrm{sh}_\tau\, \pi_n \mathcal{G}$ is an isomorphism of sheaves for every $n \geq 0$. The same happens for $\mathcal{W}_0$.

Now, we only need to prove that $\pi_n f$ is an isomorphism of sheaves if, and only if $\pi_n \varepsilon_*(f)$ is. For this, we use the fact that $\pi_n \mathcal{F}$ and $\pi_n \mathcal{G}$ are pre-sheaves of sets. In this case we can write the global sections of the associated sheaf over X as follows

$$\mathrm{sh}_\tau\, \pi_n \mathcal{F}(X) = ((\pi_n \mathcal{F})^+)^+,$$

where

$$\begin{aligned} \mathcal{H}^+ &= \operatorname*{colim}_{U \in \mathrm{Cov}(X)} \mathrm{Hom}_{\mathcal{C}}(h_U, \mathcal{H}) \\ &= \operatorname*{colim}_{U \in \mathrm{Cov}(X)} \mathrm{Hom}_{\mathcal{C}_0}(h_{U_0}, \mathcal{H}). \end{aligned}$$

This means that the value of $\mathcal{H}^+$ is determined by the value of $\mathcal{H}$ on the objects of $\mathcal{C}_0$. It follows then, that $\pi_n f$ is an isomorphism if, and only if $\pi_n \varepsilon_*(f)$ is, and we are done. ●

Theorem 2.3.3. Let $(\mathcal{C}, \tau)$ be an ∞-category together with a Grothendieck topology, and let $\mathcal{C}_0 \subset \mathcal{C}$ be a full subcategory, satisfying the conditions of Theorem 2.3.2. Let $\mathcal{D}$ be a complete ∞-category $\mathcal{D}$. Then, the restriction functor ε_* induces an equivalence

$$\mathbf{Sh}_\tau(\mathcal{C}, \mathcal{D}) \xrightarrow{\simeq} \mathbf{Sh}_\tau(\mathcal{C}_0, \mathcal{D}) \tag{2.14}$$

Proof. First observe that the theorem holds in the case $\mathcal{D} = \mathbf{Spc}$. The unit of the adjunction $(\varepsilon_*, \varepsilon^*)$ factors as

$$\mathcal{F} \longrightarrow \varepsilon^p_* \varepsilon^{*p} \mathcal{F} \longrightarrow \varepsilon^p_*\, \mathrm{sh}_\tau\, \varepsilon^{*p} \mathcal{F} \tag{2.15}$$

The left arrow is an equivalence of presheaves because $\varepsilon^{*p}\mathcal{F}$ is a Kan extension along a fully faithful inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ (see section 4.3.2 in [57]). The second arrow in (2.15), is a weak equivalence, because of Theorem 2.3.2, and the fact that $\varepsilon^{*p}\mathcal{F} \to \mathrm{sh}_\tau\, \varepsilon^{*p}$ is an equivalence.

As for the counit

$$c\colon \varepsilon^*\varepsilon_*\mathcal{F} \longrightarrow \mathcal{F} \tag{2.16}$$

Because of Theorem 2.3.2, the morphism c in (2.16) is an equivalence if, and only if its restriction $\varepsilon_* c$ is. But then, the unit applied to $\varepsilon_*\mathcal{F}$ gives an equivalence

$$\varepsilon_*\mathcal{F} \longrightarrow \varepsilon_*\varepsilon^*\varepsilon_*\mathcal{F} \tag{2.17}$$

which is right inverse to $\varepsilon_* c$, so (2.16) is also an equivalence.

Now, for a general $\mathcal{D}$, because of Theorem 2.1.22, we can factor ε_* through the following chain of equivalences

$$\begin{aligned}
\mathbf{Sh}_\tau(\mathcal{C},\mathcal{D}) &\simeq \mathbf{Func}^{\mathrm{L}}(\mathbf{Sh}_\tau(\mathcal{C}),\mathcal{D}) \\
&\simeq \mathbf{Func}^{\mathrm{L}}(\mathbf{Sh}_\tau(\mathcal{C}_0),\mathcal{D}) \\
&\simeq \mathbf{Sh}_\tau(\mathcal{C}_0,\mathcal{D})
\end{aligned}$$

●

Remark 2.3.4. The hypothesis of Theorem 2.3.3, namely that any covering sieve $U \in \mathrm{Cov}(X)$, the subcategory $U_0 = U \cap \mathcal{C}_0$ generated by the objects in $\mathcal{C}_0$ is cofinal with U, is fullfilled when any object X in $\mathcal{C}$ admits a covering by objects in $\mathcal{C}_0$.

2.3.2 An extension theorem on the category of schemes

We now formulate an analogue of Guillén-Navarro's extension theorem 2.1.5 in [35].

Theorem 2.3.5. Assume k is a field of characteristic zero, and let $F\colon \mathbf{Sm}_k^{\mathrm{op}} \to \mathcal{D}$ be a functor from the category of smooth schemes over k to an ∞-category $\mathcal{D}$, such that $F() = \mathrm{pt}$ and for any blow-up square on $\mathbf{Sm}_k$

$$\begin{tikzcd}
\widetilde{Y} \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] \\
Y \arrow[r] & X
\end{tikzcd}$$

The corresponding image by F is

$$\begin{tikzcd}
F(X) \arrow[r] \arrow[d] & F(\widetilde{X}) \arrow[d] \\
F(Y) \arrow[r] & F(\widetilde{Y})
\end{tikzcd}$$

is a pull-back diagram in $\mathcal{D}$.

Then, there exists a functor $F^d\colon \mathbf{Sch}_k^{\mathrm{op}} \to \mathcal{D}$ extending $\mathcal{F}$ such that any abstract blow-up square in $\mathbf{Sch}_k$ is sent to a pull-back diagram in $\mathcal{D}$.

Proof. Observe that functors $\mathcal{F} \in \mathbf{Func}(\mathbf{Sm}_k^{\mathrm{op}}, \mathcal{D})$ with the hypothesis of the theorem are exactly the sheaves for the abstract blow-up topology on the category of smooth schemes. Then, Theorem 2.3.3 establishes an equivalence

$$\mathbf{Sh}(\mathbf{Sm}_k, \mathcal{D}) \simeq \mathbf{Sh}(\mathbf{Sch}_k, \mathcal{D}),$$

in particular, there is a unique object on the right matching F, this is the extension F^d. ●

Remark 2.3.6. This theorem is an analogue of Theorem 2.1.5 in [35]. Neither one of those implies the other. Here is a list of differences

1. The theorem in [35] starts with a functor that takes values on a homotopy category $h\mathcal{D}$, with the rectificability hypothesis on cubical diagrams. On the other hand, we require an ∞-functor with values in the ∞-category $\mathcal{D}$. Our hypothesis is still weaker than asking a strict functor with values in $\mathcal{D}$ (assuming $\mathcal{D}$ has a strict model). In a sense, we require a functor which is weak up to coherent homotopies.
2. The functors in [35] take values in a descent category (analogue to our $\mathcal{D}$). A descent category is a concept defined in that paper, that encodes a category with weak equivalences, together with a simple functor $s\colon \mathrm{Func}(\square^n, \mathcal{D}) \to \mathcal{D}$, that plays the role of the total complex in diagrams of chain complexes. On the other hand, in our formulation the simple functor is replaced by limit in the ∞-category $\mathcal{D}$.

3 Descent for semi-topological theories

3.1 Intersections in Lawson homology

In this section we will describe an intersection pairing in Lawson homology. The material in this section is taken from [17], and directly inspired by the intersection pairing in Chow groups developed in [29].

In order to construct intersections of cycles, one needs to be able to move the cycles in their equivalence class so that they meet properly (the codimension of the intersection is the sum of the codimensions of the cycles). This is classically achieved by a moving lemma, or by a deformation to the normal cone due to Fulton.

To construct intersections in Lawson homology, Friedlander and Gabber follow Fulton's approach. The novelty of this approach is that the actual moving of cycles is encoded in homotopy equivalences of spaces of cycles.

3.1.1 Intersection with a Cartier divisor

As a first step, we will construct intersections with a Cartier divisor. Let X be an algebraic variety, and $i\colon D \to X$ the inclusion of a Cartier divisor, with open complement U. Let $\mathcal{O}(D)$ be the line bundle associated to the divisor $[D]$, with total space L. Let $N_D X$ denote the total space of the normal bundle of D in X, which coincides with $\mathcal{O}(D)|_D$. There is a section $s_D\colon X \to L$ that picks D amongst the linear equivalence class of divisors represented by L, that is, we have $D = \{x \in X | s_D(x) = s_0(x)\}$.

Lemma 3.1.1.

$$\begin{tikzcd}
\mathbf{Z}_k(X) \arrow[d, dashed, "\sigma_D"'] \arrow[dr, "s_{D*}"] & & \\
\mathbf{Z}_k(N_D X) \arrow[r, "j_*"'] & \mathbf{Z}_k(L) \arrow[r, "p^*"'] & \mathbf{Z}_k(L|_U)
\end{tikzcd}$$

Proof. This is Theorem 2.4 in [17].

We need to prove the existence of the dashed arrow σ_D. Since the 3-step row is a fibration sequence because of the localization theorem 1.2.6. It is enough to prove that the composition $p^* \circ s_{D*}$ is homotopy equivalent to the zero map.

●

Remark 3.1.2. The map σ_D in Lemma 3.1.1 is the Lawson incarnation of Fulton's **specialization to the normal cone**.

This way, for any Cartier divisor $i\colon D \to X$, we have a Gysin map on Lawson homology

$$\begin{tikzcd} \mathrm{L}_k\mathrm{H}_n(X) \arrow[rr, "i^!"] \arrow[dr, "\sigma_{D*}"'] & & \mathrm{L}_{k-1}\mathrm{H}_n(D) \arrow[dl, "\simeq"] \\ & \mathrm{L}_k\mathrm{H}_n(N_Y X) & \end{tikzcd} \tag{3.1}$$

Remark 3.1.3. To construct the Gysin map we go backwards through the homotopy equivalence $\mathbf{Z}_{k-1}(X) \to \mathbf{Z}_k(N_Y X)$. In particular this means that we do not have canonical representative of the Gysin map as a strict map on the level of cycle spaces. This is to be expected though, and is a reflection of the need to move cycles so they meet at proper dimension.

3.1.2 Deformation to the normal cone

This construction is described Chapter 5 of Fulton's book [29], and provides the fundamental "moving tool" to produce intersections of algebraic cycles.

Let X be an algebraic scheme and Y a closed subscheme, defined by a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$.

First some notation:

$$\begin{aligned} \widetilde{X} &= \mathrm{Bl}_Y X = \mathbf{Proj} \bigoplus \mathcal{I}^i && \text{blow-up of } X \text{ along } Y, \\ C_Y X &= \mathbf{Spec} \bigoplus \mathcal{I}^i/\mathcal{I}^{i+1} && \text{normal cone,} \\ \widetilde{Y} &= \mathbf{Proj}\, C_Y X && \text{exceptional divisor,} \\ \overline{C_Y X} &= \mathbf{Proj}(C_Y X \oplus \mathcal{O}_X) && \text{projective completion of the normal cone} \\ M_Y X &= \mathrm{Bl}_{Y \times \{\infty\}}(X \times \mathbb{P}^1) && \text{deformation space} \end{aligned}$$

Then, there are the following closed embeddings, inside the deformation space $M_Y X$.

$$\begin{aligned} \widetilde{Y} &\to \widetilde{X} && \text{as the exceptional divisor,} \\ \widetilde{Y} &\to \overline{C_Y X} && \text{as the points at infinity,} \\ Y \times \mathbb{P}^1 &\to M_Y X && \\ \widetilde{X} \times \{\infty\} &\to M_Y X && \\ \overline{C_Y X} \times \{\infty\} &\to M_Y X && \end{aligned}$$

such that $\widetilde{X}$ and $\overline{C_Y X}$ intersect along $\widetilde{Y}$.

$$\begin{tikzcd}
Y \times \{\infty\} \arrow[rr] \arrow[dr] \arrow[ddr] & & Y \times \mathbb{P}^1 \arrow[dr] \arrow[ddr] & & Y \times \mathbb{A}^1 \arrow[ll] \arrow[dr] \arrow[ddr] & \\
& \overline{C_Y X} \cup \widetilde{X} \arrow[rr] \arrow[d] & & M_Y X \arrow[d] & & X \times \mathbb{A}^1 \arrow[ll] \arrow[d] \\
& \{\infty\} \arrow[rr] & & \mathbb{P}^1 & & \mathbb{A}^1 \arrow[ll]
\end{tikzcd} \tag{3.2}$$

Theorem 3.1.4. Let X be a an algebraic variety and $i\colon Y \to X$ a regular embedding of codimension c. Then there is a commutative diagram of continuous maps as follows

$$\begin{tikzcd}
& \mathbf{Z}_k(X) \arrow[r] \arrow[dl, dashed, "\sigma_Y"'] \arrow[d] & \mathbf{Z}_k(X \times \mathbb{A}^1) \arrow[d, "\cong"] \\
\mathbf{Z}_k(C_Y X) \arrow[r] & \mathbf{Z}_k(M_Y X \setminus \widetilde{X}) \arrow[r] & \mathbf{Z}_k(M_Y X \setminus p^{-1}(\infty))
\end{tikzcd}$$

Proof. See Proposition 3.3 in [17]. ●

Remark 3.1.5. The map σ_Y defined in Theorem 3.1.4 above is a Lawson homology version of Fulton's **specialization to the normal cone**, and lets us define a Gysin map as follows.

Definition 3.1.6. Let X be an algebraic variety and $i\colon Y \to X$ a regular embedding of codimension c. Then, there is a **Gysin map** $i^!$,

$$\begin{tikzcd}
\mathrm{L}_k\mathrm{H}_n(X) \arrow[rr, "i^!"] \arrow[dr, "\sigma_{Y*}"] & & \mathrm{L}_{k-c}\mathrm{H}_{n-2c}(Y) \arrow[dl, "\simeq"] \\
& \mathrm{L}_k\mathrm{H}_n(N_Y X) &
\end{tikzcd} \tag{3.3}$$

Remark 3.1.7. The Gysin map for a regular embedding coincides with the one defined for a divisor described before.

3.1.3 Action by Chern classes

As sketched in [17] after Proposition 2.5, we can do Chern classes in Lawson homology essentialy the same way Fulton defines them in [29] for Chow groups.

The Gysin map $i^!\colon \mathrm{L}_k\mathrm{H}_n(X) \to \mathrm{L}_{k-1}\mathrm{H}_{n-2}(D)$ for a divisor, can be interpreted as acting by the first Chern class $c_1(\mathcal{O}(D))$.

As for the higher Chern classes, we proceed by defining the total Segre class

Definition 3.1.8. Let $p\colon E \to X$ be a vector bundle of rank r, with associated projective bundle $\overline{p}\colon \mathbb{P}E \to X$. Its **Segre class** is

$$s_i(E) \cap \alpha = p_*(c_1(\mathcal{O}(1))^{r-1+i} \cap p^*(\alpha)).$$

From this, we define the Chern classes as

Definition 3.1.9. Let $p\colon E \to X$ be a rank r vector bundle on X. The **Chern classes** $c_i(E)$ are the operations $\mathrm{L}_k\mathrm{H}_n(X) \to \mathrm{L}_{k-i}\mathrm{H}_{n-2i}(X)$ such that

$$1 + c_1(E)t + c_2(E)t^2 + \cdots = (1 + s_1(E)t + s_2(E)t^2 + \cdots)^{-1}$$

as $\mathrm{End}(\mathrm{L}_*\mathrm{H}_*(X))$-valued power series in the formal variable t.

3.1.4 Some intersection formulas

First of all, using intersectons by divisors, we can prove the following projective bundle formula

Theorem 3.1.10. Let $p\colon E \to X$ be a rank r vector bundle over an algebraic variety X. Then for every $k \geq 0$ the map

$$\phi\colon \bigoplus_{i=0}^{r-1} \mathrm{L}_{k-i}\mathrm{H}_{n-2i}(X) \longrightarrow L_kH_n(\mathbb{P}(E))$$

given by

$$\phi(\alpha_0, \dots, \alpha_{r-1}) = \sum_{i=0}^{r-1} c_1(\mathcal{O}_{\mathbb{P}(E)}(1))^i \cap p^*(\alpha_i).$$

is an isomorphism.

Proof. See Theorem 2.5 in [17]. ●

The Gysin map satisfies the usual commutativity formulas in relation to flat pull-backs and proper push-forwards

Theorem 3.1.11. Consider a pull-back diagram

$$\begin{tikzcd} Y' \arrow[r, "i'"] \arrow[d, "p"] & X' \arrow[d, "q"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

where both i and i' are regular embeddings of codimension c. Then,

1. If q is proper, $i^! q_* = p_* i'^!$.
2. If q is flat, $i'^! q^* = p^* i^!$.

Proof. See Theorem 3.4 in [17]. ●

3.2 Refined Gysin maps

In this section we generalize the Friedlander-Gabber Gysin maps in Lawson homology, to construct refined Gysin homomorphisms, as developed in Section 6.2 of Fulton's book [29]. We prove an excess intersection formula for the refined Gysin map, and use it to produce a blow-up formula for Lawson homology of a regular embedding of possibly singular varieties.

Consider a pull-back diagram of varieties, where i is a regular embedding of codimension c.

$$\begin{tikzcd} Y' \arrow[r, "i'"] \arrow[d, "p"] & X' \arrow[d, "q"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

Definition 3.2.1. The **refined Gysin** maps $i^!\colon L_kH_n(X') \to L_{k-c}H_{n-2c}(Y')$ is defined by the following composition

$$L_kH_n(X') \xrightarrow{\sigma_{Y*}} L_kH_n(N_{Y'}X') \longrightarrow L_kH_n(p^*N_YX) \xrightarrow{\cong} L_{k-c}H_{n-2c}(Y')$$

Where the first map is the specialization to the normal cone defined in Theorem 3.1.4, the second map is the push-forward along the closed embedding of normal cones

$$C_{Y'}X' \to C_YX = N_YX$$

and the third map is a homotopy inverse of the flat pull-back induced by the bundle map $p^*N_YX \to Y'$.

Remark 3.2.2. Note that, the same as in Fulton's refined Gysin for Chow groups, the refined Gysin $i^!$ on X' is not the same as the refined Gysin $i'^!$. In particular, $i^!$ decreases the dimension by $\dim X - \dim Y$, while $i'^!$ decreases it by $\dim X' - \dim Y'$, and the latter may be smaller than the former. The connection between $i'^!$ and $i^!$ is given by the excess intersection formula.

3.2.1 Excess intersection formula

Theorem 3.2.3 (Excess intersection formula)**.** Consider a diagram of varieties as follows

$$\begin{tikzcd} Y'' \arrow[r, "i''"] \arrow[d, "p_2"] & X'' \arrow[d, "q_2"] \\ Y' \arrow[r, "i'"] \arrow[d, "p_1"] & X' \arrow[d, "q_1"] \\ Y \arrow[r, "i"] & X \end{tikzcd} \tag{3.4}$$

Where i and i' are regular embeddings of codimension c and c', and normal bundles N and N' respectively. Let

$$E = p_1^* N/N'$$

which is a vector bundle of rank $e = c - c'$ on Y'. Then, for any Lawson cycle $\alpha \in \mathrm{L}_k\mathrm{H}_n(X'')$ we have the following excess intersection formula relating refined Gysin maps for i and i'

$$i^!(\alpha) = c_e(p_2^* E) \cap (i')^!(\alpha). \tag{3.5}$$

Proof. Consider the diagram

$$\begin{tikzcd}
\mathrm{L}_k\mathrm{H}_n(X'') \arrow[r, "\sigma_{Y''}"] \arrow[d, "\cong"] & \mathrm{L}_k\mathrm{H}_n(N_{Y''}X'') \arrow[r] \arrow[d, "\cong"] & \mathrm{L}_k\mathrm{H}_n(p_2^* N_{Y'}X') \arrow[r] \arrow[d] & \mathrm{L}_{k-c'}\mathrm{H}_{n-2c'}(Y'') \arrow[d, "c_e(E)\cap"] \\
\mathrm{L}_k\mathrm{H}_n(X'') \arrow[r, "\sigma_{Y''}"] & \mathrm{L}_k\mathrm{H}_n(N_{Y''}X'') \arrow[r] & \mathrm{L}_k\mathrm{H}_n((p_2 p_1)^* N_Y X) \arrow[r] & \mathrm{L}_{k-c}\mathrm{H}_{n-2c}(Y'')
\end{tikzcd}$$

where the rows represent $i'^!$ and $i^!$ respectively.

We need to prove commutativity on the right square. Consider the projective bundles

$$P = \mathbb{P}(p_1^* N_Y X \oplus \mathcal{O}_{Y''})$$
$$P' = \mathbb{P}(p_1^* N_{Y'} X' \oplus \mathcal{O}_{Y''})$$

P' is embedded in P, and we have a projection $q\colon P' \to Y''$. Let ξ and ξ' be the canonical quotient bundles on Q and Q', of ranks c and c' respectively. We have

$$0 \longrightarrow \xi' \longrightarrow \xi \longrightarrow q^* E \longrightarrow 0$$

By the Whitney formula for Chern classes we have

$$c_c(\xi) = c_{c'}(\xi') c_e(q^* E),$$

Since the inverse of the isomorphism $\mathrm{L}_{k-c}\mathrm{H}_{n-2c}(Y'') \to \mathrm{L}_k\mathrm{H}_n((p_2 p_1)^* N_Y X)$ is realized by acting with $c_c(\xi)$, we are done.

●

3.2.2 Gysin map for a local complete intersection morphism

The Gysin map can be used to produce pull-back maps for more general morphism than regular embeddings.

Definition 3.2.4. A morphism $f\colon X \to Y$ is called **local complete intersection** of codimension c if it factors as a composition of a closed regular embedding i of codimension e, followed by a smooth morphism p of relative dimension $e - c$.

$$X \xrightarrow{i} P \xrightarrow{p} Y$$

We can use this decomposition to produce refined Gysin maps along such local complete intersection morphisms

Definition 3.2.5. Let $f: X \to Y$ be a local complete intersection morphism of codimension c, that factorizes as $f = pi$. Consider a pullback diagram

$$\begin{tikzcd}
X' \arrow[r, "i'"] \arrow[d] & P' \arrow[r, "p'"] \arrow[d] & Y' \arrow[d] \\
X \arrow[r, "i"] & P \arrow[r, "p"] & Y
\end{tikzcd}$$

Then the associated refined Gysin map is the composition

$$\mathrm{L}_k\mathrm{H}_n(Y') \xrightarrow{p'^*} \mathrm{L}_{k+e-c}\mathrm{H}_{n+2e-2c}(P') \xrightarrow{i^!} \mathrm{L}_{k-c}\mathrm{H}_{n-2c}(X')$$

Remark 3.2.6. The Gysin map decreases dimension if $\dim X < \dim Y$ and increases it if $\dim X > \dim Y$ (negative codimension).

3.3 Descent theorems for Lawson homology and morphic cohomology

3.3.1 Blow-ups of regular embeddings

In [41] Hu proves a blow-up formula for Lawson homology on smooth varieties. We now use the intersection theory on Lawson homology developed by Friedlander-Gabber to prove a blow-up formula for a blow-up of possibly singular varieties along a regularly embedded center.

Compare this result with the blow-up formulas for Chow groups [29] Theorem 6.7 and algebraic K-theory [63] Theorem 2.1, which are also formulated for a regularly embedded center of possibly singular varieties.

Theorem 3.3.1. Let X be an algebraic variety, with a regularly embedded subvariety Y of codimension c, fitting in the following abstract blow-up square

$$\begin{tikzcd}
\widetilde{Y} \arrow[r, "j"] \arrow[d, "p"] & \widetilde{X} \arrow[d, "q"] \\
Y \arrow[r, "i"] & X
\end{tikzcd}$$

Let $E = p^* N_Y X / N_{\widetilde{Y}} \widetilde{X}$, which is a vector bundle of rank $e = c - 1$ on $\widetilde{Y}$. Then, we have a split short exact sequence

$$0 \longrightarrow \mathrm{L}_k\mathrm{H}_n(Y) \xrightarrow{a} \mathrm{L}_k\mathrm{H}_n(\widetilde{Y}) \oplus \mathrm{L}_k\mathrm{H}_n(X) \xrightarrow{b} \mathrm{L}_k\mathrm{H}_n(\widetilde{X}) \longrightarrow 0 \tag{3.6}$$

where

$$a(\alpha) = (c_e(E) \cap p^*(\alpha), i_*(\alpha))$$
$$b(\alpha, \beta) = j_*(\alpha) - q^!(\beta).$$

Proof. Consider the piece of the localization exact sequence

$$\begin{tikzcd}
\cdots \mathrm{L}_k\mathrm{H}_n(\widetilde{Y}) \arrow[r, "j"] \arrow[d, "p"] & \mathrm{L}_k\mathrm{H}_n(\widetilde{X}) \arrow[r] \arrow[d, "q"] & \mathrm{L}_k\mathrm{H}_n(U) \arrow[r] \arrow[d, "\cong"] & \mathrm{L}_k\mathrm{H}_{n-1}(\widetilde{Y}) \cdots \arrow[d, "p"] \\
\cdots \mathrm{L}_k\mathrm{H}_n(Y) \arrow[r, "i"] & \mathrm{L}_k\mathrm{H}_n(X) \arrow[r] & \mathrm{L}_k\mathrm{H}_n(U) \arrow[r] & \mathrm{L}_k\mathrm{H}_n(Y) \cdots
\end{tikzcd}$$

Since $Y \subset X$ is a regular embedding, we know that $\widetilde{Y} = \mathbb{P}N_Y X$ is the total space of the projectivization of the normal bundle. Then by the projective bundle theorem 3.1.10,

$$p_*(c_e(E) \cap p^*(\alpha)) = \alpha.$$

This gives the injectivity of a, and the map

$$r\colon \mathrm{L}_k\mathrm{H}_n(\widetilde{Y}) \oplus \mathrm{L}_k\mathrm{H}_n(X) \longrightarrow \mathrm{L}_k\mathrm{H}_n(Y)$$

given by

$$r(\alpha, \beta) = p_*(\alpha).$$

is a right inverse.

Now we can relate p^* with $q^!$ via the excess intersection formula

$$q^!(\alpha) = c_e(E) \cap p^*(\alpha),$$

and since $q^!$ and i_* commute by 3.1.11, we conclude that $ab = 0$.

A left inverse for b is given by

$$s\colon \mathrm{L}_k\mathrm{H}_n(\widetilde{X}) \longrightarrow \mathrm{L}_k\mathrm{H}_n(\widetilde{Y}) \oplus \mathrm{L}_k\mathrm{H}_n(X)$$

where

$$s(\gamma) = (0, q_*(\gamma))$$

●

3.3.2 Nisnevich and cdh descent

The localization Theorem 1.3.14 has the following immediate corollary

Corollary 3.3.2. Lawson homology satisfies Nisnevich descent. That is, every Nisnevich distinguished square

$$\begin{tikzcd} W \arrow[r, "j"] \arrow[d, "p"] & V \arrow[d, "q"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

induces a long exact sequence of Lawson homology

$$\cdots \mathrm{L}_k\mathrm{H}_n(X) \longrightarrow \mathrm{L}_k\mathrm{H}_n(U) \oplus \mathrm{L}_k\mathrm{H}_n(V) \longrightarrow \mathrm{L}_k\mathrm{H}_n(W) \cdots$$

Proof. This follow immediately from using the localization exact 1.3.14 sequences on the rows. ●

An analogous result holds for abstract blow-up squares

Corollary 3.3.3. Lawson homology satisfies descent for abstract blow-ups. That is, every abstract blow-up square

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "j"] \arrow[d, "p"] & \widetilde{X} \arrow[d, "q"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

induces a long exact sequence of Lawson homology

$$\cdots \mathrm{L}_k\mathrm{H}_n(\widetilde{Y}) \longrightarrow \mathrm{L}_k\mathrm{H}_n(Y) \oplus \mathrm{L}_k\mathrm{H}_n(\widetilde{X}) \longrightarrow \mathrm{L}_k\mathrm{H}_n(X) \cdots$$

Proof. Follows immediately by applying the localization theorem on the rows. ●

Remark 3.3.4. Observe that Lawson homology, being a Borel-Moore type theory is covariantly functorial for proper maps but contravariantly functorial for étale morphisms. So, in a sense, Lawson homology satisfies cdh descent, but with different functorialities on the flat, and the proper parts of the cdh topology.

As a consequence of the descent for blow-ups, we have a spectral sequence as follows

Corollary 3.3.5. Let X be an algebraic variety. and $X_\bullet$ a cubical hyperresolution. Then, there is a a convergent spectral sequence

$$E^1_{p,q} = \bigoplus_{|I|=q} \mathrm{L}_k\mathrm{H}_p(X_I) \Longrightarrow \mathrm{L}_k\mathrm{H}_{p+q}(X).$$

Proof. This is the spectral sequence associated to the double complex

$$C_{p,q} = \bigoplus_{|I|=p} \mathcal{Z}_k^q(X_I).$$

●

Theorem 3.3.6. The complexes of cocycles $\mathcal{M}_k(X, Y)$ satisfy descent for abstract blow-ups with respect to push-forwards on the variable Y.

Proof. This follows for the localization exact triangle in Theorem 1.3.14. ●

It is not known whether the bivariant complexes satisfy descent with respect to the variable X in general. For smooth varieties, the duality theorem gives the following result

Theorem 3.3.7. The morphic complexes $\mathcal{M}_k(X, Y)$ satisfy descent with respect to the cdh topology on the category of smooth varieties.

Proof. Because of the duality Theorem 3.4.5, we know that in the case X is smooth

$$H^n \mathcal{M}_k(X, Y) \cong H_{n+d}\mathcal{M}_{d+k}(\mathrm{pt}, X \times Y) \cong \mathrm{L}_{n+d}\mathrm{H}_{d+k}(X \times Y).$$

From the blow-up formula 3.3.1 we deduce descent with respect to abstract blow-ups, and from the localization theorem 1.1.25 we obtain descent for Nisnevich distinguished squares. ●

Then, we can use the extension theorem 2.3.5 to the functor

$$\mathcal{M}_k(\text{-}, Y) \colon \mathbf{Sm}_k \to \mathbf{D}(\mathbb{Z}).$$

Theorem 3.3.8. There exist unique extension of of the bivariant cocycle complex

$$\mathcal{M}_k(\text{-}, Y) \to \mathcal{M}_k^{\mathrm{cdh}}(\text{-}, Y)$$

such that

1. $\mathcal{M}_k^{\mathrm{abs}}(\text{-}, Y)$ satisfies descent for the abstract blow-up topology,
2. $\mathcal{M}_k(X, Y) \simeq \mathcal{M}_k^{\mathrm{abs}}(X, Y)$ for all smooth varieties X.

In addition, $\mathcal{M}_k^{\mathrm{abs}}$ is unique, up to homotopy, with such properties.

Proof. Here we regard $\mathcal{M}_k^{\mathrm{abs}}(\text{-}, Y)$ as a functor taking values on an ∞-enhancement of the derived category of abelian groups $\mathbf{D}(\mathbb{Z})$. The functor $\mathcal{M}_k(\mathrm{hi}, Y)$ satisfies descent for abstract blow-ups of smooth varieties because of 3.3.7. Then, Theorem 2.3.5 applies. ●

And finally, we have the following corollary

Corollary 3.3.9. The extended theory $\mathcal{M}_k^{\mathrm{abs}}(X, Y)$ satisfies Nisnevich descent on the variable X. In particular

$$\mathcal{M}_k^{\mathrm{æbs}}(X, Y) \simeq \mathcal{M}_k^{\mathrm{cdh}}.$$

Proof. Consider the extended functor $\mathcal{M}_k^{\mathrm{cdh}}(\text{-}, Y)$ with respect to the cdh topology. The natural map

$$\mathcal{M}_k^{\mathrm{abs}}(\text{-}, Y) \to \mathcal{M}_k^{\mathrm{cdh}}(\text{-}, Y),$$

induces a comparison of spectral sequences

$$\begin{tikzcd} E_1^{r,s} = \bigoplus_{|I|=r} \mathrm{H}^s \mathcal{M}_{k,*}^{\mathrm{abs}}(X_I, Y) \arrow[r, Rightarrow] \arrow[d] & \mathrm{H}^{r+s} \mathcal{M}_{k,*}^{\mathrm{abs}}(X, Y) \arrow[d] \\ E_1'^{r,s} = \bigoplus_{|I|=r} \mathrm{H}^s \mathcal{M}_{k,*}^{\mathrm{cdh}}(X_I, Y) \arrow[r, Rightarrow] & \mathrm{H}^{r+s} \mathcal{M}_{k,*}^{\mathrm{cdh}}(X, Y) \end{tikzcd}$$

Since by construction, when X is smooth we have

$$\mathcal{M}_k(X, Y) \simeq \mathcal{M}_k^{\mathrm{abs}}(X, Y) \simeq \mathcal{M}_k^{\mathrm{abs}}(X, Y).$$

the left vertical maps are isomorphisms, inducing isomorphisms on the abutment of the spectral sequences. ●

3.4 A generalized duality theorem

The duality theorem 1.3.12 tells us that that when all varieties are smooth in $\mathcal{M}_k(X, Y)$, the variety X can jump to the right $\mathcal{M}_k(pt, X \times Y)$, up to homotopy. In this section we prove a generalized duality map, valid for quasi-projective X, a general variety Y and a smooth variety W that jumps from left to right, namely, we prove that the natural map

$$\mathcal{M}_k(X \times W, Y) \to \mathcal{M}_k(X, W \times Y),$$

is a homotopy equivalence. From the motivic point of view it is no surprise a result like this: the motive of a smooth variety is dualizable. Having a result like this for the down-to-earh spaces of cycles, instead of as a consequence of an abstract construction has the value of connecting the abstract theory with the spaces of cocycles, very geometric in nature.

3.4.1 Topological tools

Here we recall a result to detect weak homotopy equivalences from Section 3 in [19].

Definition 3.4.1. Let T be a topological space. A filtration $T_0 \subset T_1 \subset \cdots$ is said to be a **good filtration** if whenever $f : K \to T$ is a continuous map from a compact space K, f factors through some T_i in the filtration.

Definition 3.4.2. Let S, T filtered topological spaces and $f : S \to T$ a filtration preserving continuous map. Then f is a **very weak deformation retract** if for every $e \geq 0$ there are maps $\alpha_e : S_e \times I \to S$, $\beta_e : T_e \times I \to T$, $\lambda_e : T_e \to S$ such that the following diagrams commute:

$$\begin{tikzcd} S_e \times \{0\} \arrow[r] \arrow[d, "\mathrm{id}\times i_0"'] & S \\ S_e \times I \arrow[ur, "\alpha_e"'] & \end{tikzcd} \qquad \begin{tikzcd} T_e \times \{0\} \arrow[r] \arrow[d, "\mathrm{id}\times i_0"'] & T \\ T_e \times I \arrow[ur, "\beta_e"'] & \end{tikzcd}$$

$$
\begin{tikzcd}
S_e \times I \arrow[r, "\alpha_e"] \arrow[d, "f_e\times\mathrm{id}"'] & S \arrow[d, "f"] \\
T_e \times I \arrow[r, "\beta_e"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
S_e \times \{1\} \arrow[r, "\mathrm{id}\times i_1"] \arrow[d, "f_e"'] & S_e \times I \arrow[r, "\alpha_e"] & S \arrow[d, "f"] \\
T_e \times \{1\} \arrow[urr, "\lambda_e"] \arrow[r, "\mathrm{id}\times i_1"'] & T_e \times I \arrow[r, "\beta_e"'] & T
\end{tikzcd}
$$

where $i_0\colon \{0\} \to I$ and $i_1\colon \{1\} \to I$ are the obvious inclusions.

Remark 3.4.3. This somewat opaque definition, is saying that, over any finite step in the filtration, we have homotopies on S_e and T_e which are compatible via f, so that on one end of the homotopy they are the identity, and on the other end, the map $T_e \times \{1\} \to T$ factors through f.

The usefulness of this definition comes from the following

Lemma 3.4.4. A very weak deformation retract $f\colon S \to T$ between topological spaces with good filtrations is a weak homotopy equivalence.

Proof. See Lemma 3.2 in [19]. ●

3.4.2 generalized duality

Theorem 3.4.5. Let X, Y, W be algebraic varieties, with X normal and quasi-projective, Y projective and W smooth projective. Then, the duality map

$$
\mathcal{M}_k^*(X \times W, Y) \xrightarrow{D} \mathcal{M}_k^*(X, W \times Y) \tag{3.7}
$$

is a homotopy equivalence.

Proof. First, assume Y is smooth. If X were also smooth, the proof would be the same as the one for Theorem 1.3.12 given in [19]. If now X is normal, quasi-projective, we can represent

$$
\mathbf{M}_k(X \times W, Y) = \mathbf{Hom}(X \times W, \mathbf{C}_k(Y))^+ = \mathbf{Hom}(X, \mathbf{Hom}(W, \mathbf{C}_k(Y)))^+.
$$

The aim is to produce a homotopy equivalence that moves any cycle in $W \times Y$ to a position equidimensional over W, so that it can be represented by a morphism to a Chow variety, like the proof of the original theorem. The new ingredient is moving the entire family of cycles parametrized by X.

To realize this idea, consider the diagram

$$
\begin{tikzcd}
\mathbf{M}_k(X \times W, Y) \arrow[r] \arrow[d] & \mathbf{Hom}(X, \mathbf{Hom}(W, \mathbf{C}_k(Y)))^+ \arrow[d] \\
\mathbf{M}_{k+d}(X, W \times Y) \arrow[r] & \mathbf{Hom}(X, \mathbf{C}_{k+d}(W \times Y))^+
\end{tikzcd} \tag{3.8}
$$

All those spaces have injective maps to $\mathbf{Z}_{k+d+\dim(X)}(X \times W \times Y)$ and there we have a good filtration by compacts K_e where

$$K_e = \{\alpha = \alpha^+ - \alpha^- \mid \deg \alpha^+ + \deg \alpha^- \leq e\}\,.$$

Then let

$$\begin{aligned} S_e &= K_e \cap \mathbf{Hom}(X, \mathbf{Hom}(W, \mathbf{C}_k(Y)))^+ \\ T_e &= K_e \cap \mathbf{Hom}(X, \mathbf{C}_{k+d}(W \times Y))^+. \end{aligned}$$

Both are good filtrations, since a compact family of algebraic cocycles has bounded degree, and the vertical duality maps are compatible with those filtrations.

At this point, we use Friedlander-Lawson's moving lemma 1.3.9, to produce homotopies

$$\begin{aligned} \alpha_e \colon S_e \times I &\to \mathbf{Hom}(X, \mathbf{Hom}(W, \mathbf{C}_k(Y)))^+, \\ \beta_e \colon T_e \times I &\to \mathbf{Hom}(X, \mathbf{C}_{k+d}(W \times Y))^+, \end{aligned}$$

so that $\alpha_0 = \mathrm{id}$, $\beta_0 = \mathrm{id}$ and β_t for all $t > 0$ factors through the duality map

$$\mathbf{Hom}(X, \mathbf{Hom}(W, \mathbf{C}_k(Y)))^+ \to \mathbf{Hom}(X, \mathbf{C}_{k+d}(W \times Y))^+.$$

We have constructed a very weak deformation retract hence, by Lemma 3.4.4 the vertical duality map in 3.8 is a weak homotopy equivalence. This proves the duality theorem in the case Y is smooth.

Now we extend the result to possibly singular varieties Y. To do so, take a smooth cubical hyperresolution $Y_\bullet$ of Y. Since the bivariant complexes of cocycles satisfy cdh descent on the sencond variable by Theorem 3.3.6, we have a comparison diagram of spectral sequences

$$\begin{array}{ccc} E_1^{r,s} = \bigoplus_{|I|=r} H^s M_k(X \times W, Y_I) & \Longrightarrow & H^{r+s} M_k(X \times W, Y) \\ \downarrow & & \downarrow \\ E_1'^{r,s} = \bigoplus_{|I|=r} H^s M_k(X, W \times Y_I) & \Longrightarrow & H^{r+s} M_k(X, W \times Y) \end{array}$$

where the vertical maps are the duality morphisms. We have already proved that on the left they are isomorphisms, as Y_I are all smooth for $I \neq (0 \ldots 0)$. We conclude that the duality map on the abutement is also an isomorphism, proving the result. ●

4 Morphic cohomology of toric varieties

4.1 Toric varieties

4.1.1 Definitions and notation

First we set the notation following [30]. Let $N \cong \mathbb{Z}^n$ be a free $\mathbb{Z}$-module of rank n, and $M = \mathrm{Hom}(N, \mathbb{Z})$ its dual $\mathbb{Z}$-module. We will denote by $N_{\mathbb{R}} = N \otimes \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes \mathbb{R}$. In this way, N and M are to be thought as lattices on $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$. Moreover, there is a duality pairing $\langle u, v\rangle$ for $u \in N_{\mathbb{R}}$ and $v \in M_{\mathbb{R}}$.

Definition 4.1.1. A **convex polyhedral cone** in $N_{\mathbb{R}}$ is a set of the form $\lambda_1 v_1 + \cdots + \lambda_k v_k$ for some vectors $v_i \in N$ and positive scalars $\lambda_i \geq 0$. A **fan** Δ is a collection of convex polyhedral cones, closed under taking faces (i.e. making some λ_i's equal to zero) and taking intersections.

For any cone $\sigma \in \Delta$ we build an affine scheme X_σ such that

$$X_\sigma = \mathrm{Spec}\, \mathbb{C}[\sigma^\vee \cap M],$$

where $\sigma^\vee = \{v \in M_{\mathbb{R}} \mid \langle u, v\rangle \geq 0,\ \forall u \in \sigma\}$ is the dual cone.

If $\tau \leq \sigma$, then $X_\tau \subset X_\sigma$. We denote by $i_{\tau,\sigma} \colon X_\tau \hookrightarrow X_\sigma$ this inclusion, which is induced by the morphism of rings $\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[\tau^\vee \cap M]$.

Moreover, inside X_σ there is a distinguished closed subvariety T_σ such that

$$T_\sigma = \mathrm{Spec}\, \mathbb{C}[\sigma^\perp \cap M],$$

where $\sigma^\perp = \{v \in M_{\mathbb{R}} \mid \langle u, v\rangle = 0,\ \forall u \in \sigma\}$ is the orthogonal cone. Observe that $\sigma^\perp$ is a vector space of dimension $\mathrm{codim}\, \sigma$. This means that $T_\sigma \cong \mathbb{G}_m^{\mathrm{codim}\, \sigma}$ is an algebraic torus. In fact, when we globalize this construction, the torus T_0 will act on the entire toric variety. Then, the torus T_σ is the lowest dimensional orbit for this action contained in X_σ.

For any cone σ, the closed embedding $j_\sigma \colon T_\sigma \hookrightarrow X_\sigma$ is induced by the morphism of rings $\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[\sigma^\perp \cap M]$ which sends the lattice vectors in $\sigma^\perp$ to themselves and the others to 0.

Finally, there is a retraction $r_\sigma \colon X_\sigma \to T_\sigma$ induced by the monomorphism of rings $\mathbb{C}[\sigma^\perp \cap M] \to \mathbb{C}[\sigma^\vee \cap M]$ which includes $\sigma^\perp$ into $\sigma^\vee$.

Theorem 4.1.2. For any fan Δ, The collection of schemes X_σ together with the inclusions $i_{\sigma,\tau}$ glue together into a global scheme $X(\Delta)$.

Proof. See Section 1.4 in [30]. ●

A variety $X(\Delta)$ constructed from a fan in this fashion is called a **toric variety**, and the collection $\{X_\sigma\}_{\sigma\in\Delta}$ is an open affine cover of $X(\Delta)$. We will denote these open embeddings by $i_\sigma\colon X_\sigma \hookrightarrow X(\Delta)$.

Proposition 4.1.3. Let $X(\Delta)$ be a toric variety and σ a cone in Δ. There is a morphism

$$h\colon X_\sigma \times \mathbb{A}^1_{\mathbb{C}} \longrightarrow X_\sigma$$

such that $h(-,1) = \mathrm{id}$, $h(-,0) = j_\sigma r_\sigma$ and $h(-,t)$ restricts to the identity on T_σ for every t. That is, the morphism h gives an algebraic homotopy equivalence between X_σ and T_σ.

Proof. Recall that $X_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$, $T_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\perp \cap M]$ and the inclusion $T_\sigma \to X_\sigma$ is given by the quotient $\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[\sigma^\perp \cap M]$, which is the identity on $\sigma^\perp$ and sends any element $v \in \sigma^\vee$ not in $\sigma^\perp$ to $0 \in \mathbb{C}[\sigma^\perp \cap M]$.

Pick $u_0 \in \sigma$ such that $\sigma^\perp = \sigma^\vee \cap u_0^\perp$. Then define

$$h^*\colon \mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[\sigma^\vee \cap M] \otimes \mathbb{C}[t],$$

by $h^*(v) = v \otimes t^{\langle u_0, v\rangle}$ for every $v \in \sigma^\vee$. This gives a morphism of schemes $h\colon X_\sigma \times \mathbb{A}^1_{\mathbb{C}} \to X_\sigma$ with the desired properties. ●

We will use the notation $\Delta^{(k)}$ for the set of all cones of codimension k in Δ.

Definition 4.1.4. An **orientation of a cone** σ is an orientation of the vector spaces $\mathbb{R}\sigma$. An **orientation of a fan** Δ will be a choice of an orientation for every cone in Δ.

We will always use fans with a fixed orientation.

Remark 4.1.5. Recall that any face $\tau \leq \sigma$ is given as $\tau = \sigma \cap u^\perp$ for some $u \in M_{\mathbb{R}}$. Let $\tau \leq \sigma$ be a face of codimension 1, given as $\tau = \sigma \cap u^\perp$. Then there exists $v \in \sigma$ such that $\langle v, u\rangle > 0$ and

$$\mathbb{R}\sigma = \mathbb{R}v + \mathbb{R}\tau \tag{4.1}$$

as subspaces of $N_{\mathbb{R}}$. This last identity allows us to transfer the orientation of σ to τ as follows: the orientation induced on τ by σ is the one compatible with the identity (4.1) and taking the orientation on $\mathbb{R}v$ given by the vector v.

Alternatively, one may think of toric varieties as categorical quotients of certain Zariski open subsets of $\mathbb{A}^m$ modulo a torus action, the same way as $\mathbb{P}^n$ is the quotient of $\mathbb{A}^{m+1}$ modulo $\mathbb{G}_m$. We describe this construction following [8].

Let $X(\Delta)$ be a toric variety. Let E be the free $\mathbb{Z}$-module generated by the 1-dimensional cones in Δ, and let $E_{\mathbb{C}} = E \otimes \mathbb{C}$. There is a linear map $E \to N$ that sends basis elements $e_\tau \in E$ to the generator of the 1-dimensional cone τ in N. This map extends to a morphism of tori with kernel T, as follows

$$1 \longrightarrow T \longrightarrow E \otimes \mathbb{C}^* \longrightarrow N \otimes \mathbb{C}^* \longrightarrow 1 \tag{4.2}$$

Now, for every 1-dimensional cone $\tau \in \Delta_{(1)}$, we associate a coordinate x_τ in the polynomial ring on $E_\mathbb{C}$, and to every cone $\sigma \in \Delta$, a monomial defined as

$$\hat{x}_\sigma = \prod_{\substack{\tau \in \Delta_{(1)} \\ \tau 6 \subset \sigma}} x_\tau .$$

Let $Z = \{\hat{x}_\sigma = 0 \mid \sigma \in \Delta\}$ be the associated variety. Observe that there is a canonical action of T on $E_\mathbb{C}$, which descends to $E_\mathbb{C} \setminus Z$. Then,

Theorem 4.1.6. The toric variety $X(\Delta)$ is isomorphic to the categorical quotient of $E_\mathbb{C} \setminus Z$ by the action of T.

Proof. See Theorem 2.1 in [8]. ●

4.1.2 Morphic cohomology of an algebraic torus

Now we compute the morphic cohomology ring of an algebraic torus. As we will need this computation for subtori of a toric variety, it will be useful to have a canonical description of this ring in terms of the lattice defining the toric variety.

Let N be a lattice of rank n, and $L_\mathbb{R} \subset M_\mathbb{R} = N_\mathbb{R}^\vee$ a subspace of dimension r generated by vectors in the lattice M. Consider the rank r sublattice $L = L_\mathbb{R} \cap M$, and its associated torus $T_L = \operatorname{Spec} \mathbb{C}[L]$.

We introduce a bit of notation. Let $\mathrm{LH} = \mathrm{L}^*\mathrm{H}^*(\mathrm{pt})$ be the morphic cohomology ring of a point. It is isomorphic to $\mathbb{Z}[s]$ with the free generator $s \in \mathrm{L}^1\mathrm{H}^0(\mathrm{pt})$. Let K be a graded LH-module. We denote by $K[l]_q$ the graded LH-module obtained from K by shifting it l steps into the increasing direction for the q-degree, that is, $(K[l]_q)_i = K_{i-l}$.

It follows from Proposition 1.3.17 that the piece of homological degree 1 $\mathrm{L}^*\mathrm{H}^1(\mathbb{G}_m)$, is isomorphic, as an LH-module, to $\mathrm{LH}[1]_q$. This is a free graded LH-module with one generator in q-degree 1, we called this generator e in Proposition 1.3.17. It corresponds, by duality, to a radial Borel-Moore chain joining 0 and ∞ in $\mathbb{G}_m$. Now, any $v \in L$ defines a character $\chi_v \colon T_L \to \operatorname{Spec} \mathbb{C}[v, v^{-1}] = \mathbb{G}_m$. Then, we define a graded morphism of rings

$$\varphi\colon \textstyle\bigoplus_{n\geq 0} (\bigwedge^n L \otimes \mathrm{LH})[n]_q \longrightarrow \mathrm{L}^*\mathrm{H}^*(T_L) \tag{4.3}$$

by

$$\varphi(v \otimes 1) = \chi_v^*(e),$$

for $v \in L$, and extended in the obvious way to the exterior algebra because $\mathrm{L}^*\mathrm{H}^*(T_L)$ is a graded commutative algebra (Theorem 1.3.7).

Proposition 4.1.7. Let X be a smooth quasi-projective variety. The Künneth homomorphism

$$\mathrm{L}^*\mathrm{H}^*(X) \otimes_{\mathrm{LH}} \mathrm{L}^*\mathrm{H}^*(\mathbb{G}_m) \longrightarrow \mathrm{L}^*\mathrm{H}^*(X \times \mathbb{G}_m). \tag{4.4}$$

is an isomorphism.

Proof. Let $i\colon \mathrm{pt} \to \mathbb{A}^1$ be the inclusion of a point and $j\colon \mathbb{G}_m \to \mathbb{A}^1$ its open complement. We have the following commutative diagram of long exact sequences

$$\begin{tikzcd}
\cdots \mathrm{L}^*\mathrm{H}^*(X) \otimes_{\mathrm{LH}} \mathrm{LH} \arrow[r, "id\otimes i_!"] \arrow[d, "\cong"] & \mathrm{L}^*\mathrm{H}^*(X) \otimes_{\mathrm{LH}} \mathrm{LH} \arrow[r, "id\otimes j^*"] \arrow[d, "\cong"] & \mathrm{L}^*\mathrm{H}^*(X) \otimes_{\mathrm{LH}} \mathrm{L}^*\mathrm{H}^*(\mathbb{G}_m) \cdots \arrow[d] \\
\cdots \mathrm{L}^*\mathrm{H}^*(X \times \mathrm{pt}) \arrow[r, "(id\times i)_!"] & \mathrm{L}^*\mathrm{H}^*(X \times \mathbb{A}^1) \arrow[r, "(id\times j)^*"] & \mathrm{L}^*\mathrm{H}^*(X \times \mathbb{G}_m) \cdots
\end{tikzcd}$$

The vertical maps are the Künneth morphisms, and $i_!$ is the Gysin map defined by duality (Theorem 1.3.12) as $i_! = \Gamma^{-1} i_* \Gamma$. The exactness of the rows comes, by duality, from the localization theorem 1.2.6. For the first row, we also need the computation in 1.3.17 to ensure that all LH-modules in the exact sequence

$$\cdots \longrightarrow \mathrm{L}^*\mathrm{H}^*(\mathrm{pt}) \xrightarrow{i_!} \mathrm{L}^*\mathrm{H}^*(\mathbb{A}^1) \xrightarrow{j^*} \mathrm{L}^*\mathrm{H}^*(\mathbb{G}_m) \longrightarrow \cdots$$

are flat, so that when tensoring with $\mathrm{L}^*\mathrm{H}^*(X)$ the exactness is preserved.

Now the first two vertical maps are isomorphisms. The first by definition, while the second as a consequence of homotopy invariance Theorem 1.3.15. Then a standard application of the five lemma proves the desired isomorphism. ●

Theorem 4.1.8. The morphism φ is an isomorphism.

Proof. We argue by induction on the rank of L. The isomorphism is clear when $\operatorname{rank} L = 1$ by the computation in 1.3.17. Let $L = L_0 \oplus \mathbb{Z}v$. This gives a product decomposition $T_L = T_{L_0} \times \mathbb{G}_m$. Now, because the Künneth isomorphism in 4.1.7 preserves the cup product, we get a commutative diagram

$$\begin{tikzcd}
(\bigwedge^n L \otimes \mathrm{LH})[n]_q \arrow[r] \arrow[d, "\varphi_n"] & \bigoplus_{r+s=n} (\bigwedge^r L_0 \otimes \mathrm{LH})[r]_q \otimes_{\mathrm{LH}} (\bigwedge^s \mathbb{Z}v \otimes \mathrm{LH})[s]_q \arrow[d] \\
\mathrm{L}^*\mathrm{H}^n(T_L) \arrow[r] & \bigoplus_{r+s=n} \mathrm{L}^*\mathrm{H}^r(T_{L_0}) \otimes_{\mathrm{LH}} \mathrm{L}^*\mathrm{H}^s(\mathbb{G}_m)
\end{tikzcd}$$

The upper row is an isomorphism by multilinear algebra results, while the lower row is an isomorphism by the Künneth isomorphism 4.1.7. The right vertical map is a sum of tensor products of φ's corresponding to lower dimensional tori, so are isomorphisms by induction hypothesis. We conclude then that the left vertical map is an isomorphism. ●

4.1.3 Mayer-Vietoris for morphic cohomology of toric varieties

Definition 4.1.9. Let X be an algebraic variety acted on by an algebraic group G. A categorical quotient of X by G is a map $\pi\colon X \to X/G$ to some variety X/G such that this map is equivariant with respect to the trivial G-action on X/G, and universal amongst all maps $X \to Z$ having this property.

Theorem 4.1.10. Let X be a normal quasi-projective algebraic variety acted on by an algebraic group G. Let X/G be the categorical quotient for this action. Then, the pull-back by the quotient π induces an isomorphism

$$\mathcal{M}^{*,q}(X/G) \cong \mathcal{M}^{*,q}(X)^G,$$

where the right hand side represents the space of invariants by the induced G-action on the morphic complex.

Proof. Take a singular chain $\alpha\colon \Delta^l \to \mathbf{M}^q(X/G)$. The pull-back by π produces a chain $\pi^*(\alpha)\colon \Delta^l \to \mathbf{M}^q(X)$, which happens to be invariant by G. So this map is injective.

Reciprocally, given a chain $\alpha\colon \Delta^l \to \mathbf{M}^q(X)$ invariant with respect to G, it factors through the subspace of invariants $\alpha\colon \Delta^l \to \mathbf{M}^q(X)^G$.

Thanks to Theorem 1.3.4 The cochain α is represented by a map

$$(\alpha^+, \alpha^-)\colon \Delta^l \to \mathbf{Hom}(X, \mathbf{C}_0(\mathbb{A}^q)) \times \mathbf{Hom}(X, \mathbf{C}_0(\mathbb{A}^q))$$

corresponding to the positive and negative parts in the group completion. Since α is fixed by G, both components α^+ and α^- must be fixed by G, and by the universal property of the categorical quotient, we see that $\alpha_\pm$ factors through

$$(\alpha'^+, \alpha'^-)\colon \Delta^l \to \mathbf{Hom}(X/G, \mathbf{C}_0(\mathbb{A}^q)) \times \mathbf{Hom}(X/G, \mathbf{C}_0(\mathbb{A}^q))$$

which represents a morphic cocycle $\alpha' \in \mathcal{M}^{l,q}(X/G)$, with the property that $\alpha = \pi^*(\alpha')$. This proves that π^* is an isomorphism of complexes. ●

We now prove that morphic cohomology satisfies Zariski descent for equivariant covers of a toric variety.

Theorem 4.1.11. Let $X(\Delta)$ be a toric variety, and $U_1, U_2 \subset X(\Delta)$ a Zariski open cover equivariant with respect to the torus action on $X(\Delta)$. Then it induces a long exact sequence in morphic cohomology

$$\cdots \mathrm{L}^k\mathrm{H}^n(U_1 \cap U_2) \longrightarrow \mathrm{L}^k\mathrm{H}^n(U_1) \oplus \mathrm{L}^k\mathrm{H}^n(U_2) \longrightarrow \mathrm{L}^k\mathrm{H}^n(X) \cdots$$

Proof. First observe that the equivariant Zariski open sets U_1 and U_2 are themselves toric varieties, $U_1 = X(\Delta_1)$, $U_2 = X(\Delta_2)$ where the fans $\Delta_1, \Delta_2 \subset \Delta$ are subfans of Δ. Now we consider the free module E generated by the 1-dimensional cones in Δ, and T the torus as in Theorem 4.1.6.

Now consider the open sets $V_\sigma \subset E$ given by

$$V_\sigma = \{\hat{x}_\sigma \neq 0\}.$$

Then,

$$E \setminus Z_i = \bigcup_{\sigma\in\Delta_i} V_\sigma,$$

moreover, from the fact that V_σ are defined by monomials, it follows that

$$V_\sigma \cap V_\tau = V_{\sigma\cap\tau}.$$

Assume for the moment, that $\Delta_1 \cap \Delta_2$ contain all 1-dimensional cones in Δ. Using Theorem 4.1.6 we have

$$\begin{aligned} U_1 &= (E \setminus Z_1)/T \\ U_2 &= (E \setminus Z_2)/T \\ U_1 \cap U_2 &= (E \setminus Z_1 \cup Z_2)/T \\ X(\Delta) &= (E \setminus Z_1 \cap Z_2)/T. \end{aligned}$$

If the assumption does not hold, that is, Δ_1 or Δ_2 may not contain all 1-cones in Δ, it means that the free modules E_1 and E_2 associated to Δ_1 and Δ_2 have lower rank. There are canonical projections $E \to E_i$, that send the basis elements corresponding to the missing 1-dimensional cones to zero. Then, we define the following tori associated to Δ_1 and Δ_2

$$T_i = \ker(E \otimes \mathbb{C}^* \to E_i \otimes \mathbb{C}^* \to N \otimes \mathbb{C}^*).$$

Now, in this setting, the formulas above generalize as follows

$$\begin{aligned} U_1 &= (E \setminus Z_1)/T_1 \\ U_2 &= (E \setminus Z_2)/T_2 \\ U_1 \cap U_2 &= (E \setminus Z_1 \cup Z_2)/T_1T_2 \\ X(\Delta) &= (E \setminus Z_1 \cap Z_2)/T_1 \cap T_2. \end{aligned}$$

Observe that under the assumption above, $T_1 = T_2 = T$ and we recover the formulas above.

Since $E \setminus Z_i$ are smooth Zariski open subsets of E, we have a Mayer-Vietoris short exact sequence for morphic cohomology complexes

$$0 \longrightarrow \mathcal{M}^{*,q}(E \setminus Z_1 \cap Z_2) \longrightarrow \mathcal{M}^{*,q}(E \setminus Z_1) \oplus \mathcal{M}^{*,q}(E \setminus Z_2) \longrightarrow \mathcal{M}^{*,q}(E \setminus Z_1 \cup Z_2) \longrightarrow 0$$

Taking invariants by the corresponding tori, we get a short exact sequence

$$0 \longrightarrow \mathcal{M}^{*,q}(E \setminus Z_1 \cap Z_2)^{T_1\cap T_2} \longrightarrow \mathcal{M}^{*,q}(E \setminus Z_1)^{T_1} \oplus \mathcal{M}^{*,q}(E \setminus Z_2)^{T_2} \longrightarrow \mathcal{M}^{*,q}(E \setminus Z_1 \cup Z_2)^{T_1T_2} \longrightarrow 0$$

The operation of taking G-invariants is left-exact in general. In this case, we have exactness on the right, because $E \setminus Z_1 \cup Z_2$ is Zariski dense in both $E \setminus Z_1$ and $E \setminus Z_2$, so if we have a cocycle in $E \setminus Z_1 \cup Z_2$ invariant by T_1T_2 which extends to the middle spot, it must necessarily be invariant by the extended actions.

$$\begin{array}{ccccc} \mathcal{M}^{*,q}(X(\Delta)) & \longrightarrow & \mathcal{M}^{*,q}(U_1) \oplus \mathcal{M}^{*,q}(U_2) & \longrightarrow & \mathcal{M}^{*,q}(U_1 \cap U_2) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}^{*,q}(E \setminus Z_1 \cap Z_2)^{T_1\cap T_2} & \longrightarrow & \mathcal{M}^{*,q}(E \setminus Z_1)^{T_1} \oplus \mathcal{M}^{*,q}(E \setminus Z_2)^{T_2} & \longrightarrow & \mathcal{M}^{*,q}(E \setminus Z_1 \cup Z_2)^{T_1T_2} \end{array}$$

Where all the vertical maps are isomorphisms by Theorem 4.1.10. We conclude that the top row is a short exact sequence of complexes, which proves the Mayer-Vietoris property we were looking for. ●

4.2 Spectral sequence associated to a toric variety

Let $X(\Delta)$ be a toric variety of dimension n, R a ring and $\mathcal{F}^*$ a cochain complex of sheaves of R-modules on X. As usual, the hypercohomology of $\mathcal{F}^*$ is

$$\mathbb{H}^n(X(\Delta), \mathcal{F}^*) = H^n\Gamma(X(\Delta), \mathcal{I}^*)$$

where $\mathcal{I}^*$ is a K-injective resolution $\mathcal{F}^* \to \mathcal{I}^*$.

In this section we will write down a spectral sequence converging to the hypercohomology $\mathbb{H}^n(X(\Delta), \mathcal{F}^*)$ whose E_2 page is computable in terms of the combinatorics of the toric variety, and the hypercohomology of $\mathcal{F}^*$ on algebraic tori. The spectral sequence comes from the identification

$$\mathbb{H}^n(X(\Delta), \mathcal{F}^*) = \mathrm{Ext}^n(R_X, \mathcal{F}^*),$$

and the fact that the hyper-ext can be computed resolving either variable. We will chose to resolve the constant sheaf R_X producing a Čech-like resolution $\check{\mathcal{C}}_*(\Delta, R) \to R_X$ from the combinatorics of the toric variety.

A similar idea, applied to singular homology and cohomology, was exploited in the thesis [44].

4.2.1 Resolution associated to a fan

Let $X(\Delta)$ be a toric variety defined by a fan Δ, and let R be a commutative ring.

Definition 4.2.1. Let $\check{\mathcal{C}}_k(\Delta, R)$ for $k \geq 0$ be the sequence of sheaves of R-modules on $X(\Delta)$ given by

$$\check{\mathcal{C}}_k(\Delta, R) = \bigoplus_{\sigma \in \Delta^{(k)}} i_{\sigma!} i_\sigma^* R_X,$$

where R_X is the constant sheaf on $X(\Delta)$ and $i_\sigma \colon X_\sigma \to X(\Delta)$ is the inclusion of X_σ.

Moreover, we define, a sequence of morphisms $d_k \colon \check{\mathcal{C}}_k(\Delta, R) \to \check{\mathcal{C}}_{k-1}(\Delta, R)$ given by

$$d_k = \bigoplus_{\substack{\sigma \in \Delta^{(k-1)} \\ \tau \in \Delta^{(k)} \\ \tau \leq \sigma}} \epsilon(\tau, \sigma) \mu_{\tau, \sigma}$$

where $\mu_{\tau,\sigma} \colon i_{\tau!} i_\tau^* R_X \to i_{\sigma!} i_\sigma^* R_X$ is the natural inclusion of sheaves inducing the identity on the nonzero stalks, and $\epsilon(\tau, \sigma) = \pm 1$ according to whether the orientation induced by σ on τ coincides or not with the fixed orientation in τ.

Definition 4.2.2. Given a fan Δ and a cone $\sigma \in \Delta$ of codimension k, there is a fan Δ_σ defined on the lattice $N_\sigma = N/(\mathbb{R}\sigma \cap N)$ of dimension k, whose cones are the projection of cones in Δ having σ as a face.

This way, the cones in Δ_σ correspond bijectively with the cones $\tau \in \Delta$ having σ as a face.

Remark 4.2.3. Let $x \in X(\Delta)$ be a point. We denote by $\sigma(x)$ the unique cone in Δ such that $x \in T_{\sigma(x)}$. Observe that

$$(i_{\tau!}i_\tau^* R_X)_x = \begin{cases} R & \text{if } \sigma(x) \leq \tau, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the stalk $(i_{\tau!}i_\tau^* R_X)_x$ is nonzero exactly for the cones $\tau \in \Delta$ which represent cones in $\Delta_{\sigma(x)}$.

Proposition 4.2.4. Let $X(\Delta)$ be a toric variety associated to a fan Δ.

1. There is a canonical isomorphism

$$\check{\mathcal{C}}_k(\Delta, R)_x \cong \bigoplus_{\substack{\sigma \in \Delta^{(k)} \\ \sigma(x) \leq \sigma}} R$$

   and the morphism induced on this stalk by d_k is given by

$$d_{k,x}([\tau]) = \sum_{\substack{\sigma \in \Delta^{(k-1)} \\ \tau \leq \sigma}} \epsilon(\tau, \sigma)[\sigma],$$

2. The sequence of sheaves $\check{\mathcal{C}}_k(\Delta, R)$ together with the morphisms d_k form a chain complex of sheaves of R-modules.

3. Let $\Delta \subset \overline{\Delta}$ be an inclusion of fans on the same lattice, giving an open embedding $u\colon X(\Delta) \hookrightarrow X(\overline{\Delta})$. Then, there is a canonical isomorphism of complexes of sheaves $\check{\mathcal{C}}_*(\Delta, R) \cong u^*\check{\mathcal{C}}_*(\overline{\Delta}, R)$.

Proof. 1) This follows from the definition 4.2.1 and Remark 4.2.3.

2) For this, we need only to check that $(d_{k-1}d_k) = 0$ on stalks. Using the identification of these stalks in 1), we see that for $[\tau]$ an element of the basis of $\check{\mathcal{C}}_k(\Delta, R)_x$, we have

$$d_{k-1,x}d_{k,x}([\tau]) \quad = \sum_{\tau < \sigma < \eta} \epsilon(\tau, \sigma)\epsilon(\sigma, \eta)[\eta]$$

Then, for fixed τ and η there are exactly two faces in between, giving opposite signs. One can see this by looking at the image of η in $N_{\mathbb{R}}/\mathbb{R}\tau$. This image is a two-dimensional cone that obviously has exactly two faces with opposite orientation. This shows that $d_{k-1,x}d_{k,x} = (d_{k-1}d_k)_x = 0$.

3) It follows from the following computation

$$\begin{aligned} u^*\check{\mathcal{C}}_k(\overline{\Delta}, R) &= u^*\Big(\bigoplus_{\overline{\sigma} \in \overline{\Delta}^{(k)}} i_{\overline{\sigma}!}i_{\overline{\sigma}}^* R_{X(\overline{\Delta})}\Big) \\ &= u^*\Big(\bigoplus_{\sigma \in \Delta^{(k)}} (ui_\sigma)_!(ui_\sigma)^* R_{X(\overline{\Delta})}\Big) \\ &= \bigoplus_{\sigma \in \Delta^{(k)}} u^*u_!i_{\sigma!}i_\sigma^* u^* R_{X(\overline{\Delta})} \\ &\cong \bigoplus_{\sigma \in \Delta^{(k)}} i_{\sigma!}i_\sigma^* R_{X(\Delta)} = \check{\mathcal{C}}_k(\Delta, R). \end{aligned}$$

●

Definition 4.2.5. Let $a\colon \check{\mathcal{C}}_*(\Delta, R) \to R_X$ be the augmentation morphism induced by the morphisms $i_{\sigma!} i_\sigma^* R_X \to R_{X(\Delta)}$.

We will prove that $a\colon \check{\mathcal{C}}_*(\Delta, R) \to R_X$ is a quasi-isomorphism. To do so, we will relate the stalk complexes $\check{\mathcal{C}}_*(\Delta, R)_x$ with the cellular homology complex of a cellular decomposition on a ball of dimension $\operatorname{codim} \sigma(x)$.

Let Δ be a fan on a lattice N. We define

$$
\begin{aligned}
B(\Delta) &= \{p \in N_{\mathbb{R}} \mid \|p\| \leq 1\}, \\
S(\Delta) &= \{p \in N_{\mathbb{R}} \mid \|p\| = 1\}.
\end{aligned}
$$

Pick $x \in X(\Delta)$. Then, the space $S(\Delta_{\sigma(x)})$ is a sphere of dimension $\operatorname{codim} \sigma(x) - 1$, and every non-zero cone $\sigma \in \Delta_{\sigma(x)}$, gives a cell of dimension $\dim \sigma - 1$ on $S(\Delta_{\sigma(x)})$, defined by $e_\sigma = \sigma \cap S(\Delta_{\sigma(x)})$. The set $\{e_\sigma\}_{\sigma \in \Delta_{\sigma(x)}}$, together with the entire ball, gives a cellular decomposition of $B(\Delta_{\sigma(x)})$. However, we are interested in a dual cellular decomposition $e_\sigma^\vee$ which we proceed to describe now.

Definition 4.2.6. To any complete fan Δ we associate an abstract simplicial complex $K(\Delta)$ as follows:

1. The vertices in $K(\Delta)$ correspond to the cones in Δ.
2. The k-simplexes in $K(\Delta)$ are the sets of vertices belonging to flags in Δ of length k, that is, sequences of strictly included cones

$$
\tau_0 < \tau_1 < \cdots < \tau_k.
$$

Remark 4.2.7. If we had omitted the cone 0 in the definition of $K(\Delta)$ we would have obtained a combinatorial model of the barycentric subdivision of the fan Δ.

For every 1-dimensional cone $\tau \in \Delta^{(n-1)}$ let $u_\tau \in N_{\mathbb{R}}$ be the unique unit vector generating it. Then, for any non-zero cone $\sigma \in \Delta$, let v_σ be the vector

$$
v_\sigma = \sum_{\substack{\tau \in \Delta^{(n-1)} \\ \tau \leq \sigma}} u_\tau.
$$

Definition 4.2.8. For every k-simplex $(\tau_0, \ldots, \tau_k) \in K(\Delta)$ given by a flag of cones $\tau_0 < \cdots < \tau_k$, we define a subset $d_{(\tau_0, \ldots, \tau_k)} \subset B(\Delta)$ as follows,

$$
d_{(\tau_0,\ldots,\tau_k)} = \begin{cases} \{0\} & \text{if } \tau_0 = 0 \text{ and } k = 0, \\ \mathbb{R}_{\geq 0} \left\langle v_{\tau_1}, \ldots, v_{\tau_k} \right\rangle \cap B(\Delta) & \text{if } \tau_0 = 0 \text{ and } k > 0, \\ \mathbb{R}_{\geq 0} \left\langle v_{\tau_0}, \ldots, v_{\tau_k} \right\rangle \cap S(\Delta) & \text{if } \tau_0 \neq 0. \end{cases} \tag{4.5}
$$

Proposition 4.2.9. Let Δ be a complete fan. The subsets $d_{(\tau_0,\ldots,\tau_k)} \subset B(\Delta)$ are homeomorphic to closed balls of dimension k. Together form a cellular decomposition of the ball $B(\Delta)$, giving a geometric realization of the abstract simplicial complex $K(\Delta)$.

Proof. Let $(\tau_0, \dots, \tau_k) \in K(\Delta)$. Because the vectors v_{τ_i} all belong to the cone τ_k, the subsets $\mathbb{R}_{\geq 0}\left\langle v_{\tau_0}, \dots, v_{\tau_k}\right\rangle$ are strictly convex cones, so in either case of the definition 4.2.8, the resulting set $d_{(\tau_0,\dots,\tau_k)}$ is a cell: it is either a connected convex subset of $B(\Delta)$, or a connected and geodesically convex subset of $S(\Delta)$. The statement about the dimension of $d_{(\tau_0,\dots,\tau_k)}$ follows from the linear independence of the vectors v_{τ_i} associated to the flag $0 \neq \tau_0 < \cdots < \tau_k$.

Finally, observe that the boundary of a cell $d_{(\tau_0,\dots,\tau_k)}$ is formed by the cells resulting from removing one cone in the flag, all of lower dimension. This proves that the cells $d_{(\tau_0,\dots,\tau_k)}$ give a cellular decomposition of the ball $B(\Delta)$. ●

Definition 4.2.10. For every cone $\sigma \in \Delta$, let $e^\vee_\sigma \subset B(\Delta)$ be the subset defined by

$$e^\vee_\sigma = \bigcup_{\substack{k\geq 0 \\ (\tau_0,\dots,\tau_k)\in K(\Delta) \\ \sigma\leq\tau_0}} d_{(\tau_0,\dots,\tau_k)} \tag{4.6}$$

Proposition 4.2.11. Let Δ be a complete fan. The subsets $e^\vee_\sigma \subset B(\Delta)$ are homeomorphic to closed balls of dimension $\operatorname{codim}\sigma$ and form a cellular decomposition of the ball $B(\Delta)$. The ball together with this decomposition will be denoted by $B(\Delta)^\vee$.

Proof. Observe that $e^\vee_\sigma$ is a geometric realization of a subcomplex of $K(\Delta)$ which is isomorphic to $K(\Delta_\sigma)$ (follows directly from the definitions). Now, Proposition 4.2.9 applied to the simplicial complex $K(\Delta_\sigma)$ realizes $K(\Delta_\sigma)$ as a $(\operatorname{codim}\sigma)$-dimensional ball $B(\Delta_\sigma)$. So, $e^\vee_\sigma$ is homeomorphic to this ball.

The cells $e^\vee_\sigma$ cover all the ball $B(\Delta)$ by completeness of the fan, and they are attached properly because the $d_{(\tau_0,\dots,\tau_k)}$ are. ●

Proposition 4.2.12. Let Δ be a complete fan. There is a canonical isomorphism of chain complexes

$$\check{\mathcal{C}}_*(\Delta, R)_x \cong \mathrm{C}^{\mathrm{cell}}_*(B(\Delta_{\sigma(x)})^\vee, R).$$

Proof. There is a canonical isomorphism of R-modules

$$\check{\mathcal{C}}_k(\Delta, R)_x \cong \mathrm{C}^{\mathrm{cell}}_k(B(\Delta_{\sigma(x)})^\vee, R),$$

as both are generated by the cones in $\Delta_{\sigma(x)}$ of codimension k (see Proposition 4.2.4).

It only remains to check that the differentials in $\check{\mathcal{C}}_*(\Delta, R)_x$ coincide with the cellular ones. Note that the attaching maps $f_\tau \colon \partial e^\vee_\tau \to \mathrm{Sk}_{\operatorname{codim}\tau-1} B(\Delta_{\sigma(x)})^\vee$ are homeomorphisms with the image. So, for any lower dimensional cell $e^\vee_\sigma$ on the boundary of $e^\vee_\tau$, the corresponding matrix element in the cellular differential is a sign, according to the relative orientation of the cells $e^\vee_\sigma$ and $e^\vee_\tau$. This is exactly the differential in $\check{\mathcal{C}}_*(\Delta, R)_x$. ●

Corollary 4.2.13. Let Δ be an arbitrary fan. Then, the augmentation $a \colon \check{\mathcal{C}}_*(\Delta, R) \to R_X$ is a quasi-isomorphism.

Proof. First Take $\Delta \subset \overline{\Delta}$ a completion of the fan Δ. Because of 3 in Proposition 4.2.4, it is enough to check that $\check{\mathcal{C}}_*(\overline{\Delta}, R) \to R_{X(\overline{\Delta})}$ is a quasi-isomorphism for the complete fan $\overline{\Delta}$.

Now, Proposition 4.2.12 tells us that the stalk complex $\check{\mathcal{C}}_*(\overline{\Delta}, R)_x$ is isomorphic to the cellular complex associated to the cellular decomposition of the ball $B(\overline{\Delta}_{\sigma(x)})^\vee$, so its homology is

$$\mathrm{H}_k\check{\mathcal{C}}_*(\overline{\Delta}, R) \cong \mathrm{H}_k\mathrm{C}_*^{\mathrm{cell}}(B(\overline{\Delta}_{\sigma(x)})^\vee, R) = \begin{cases} R & \text{for } k = 0, \\ 0 & \text{for } k > 0. \end{cases}$$

And we conclude that the augmentation $a_x \colon \check{\mathcal{C}}_*(\overline{\Delta}, R)_x \to R$ on the stalks is a quasi-isomorphism. As quasi-isomorphisms of complexes of sheaves are detected on stalks, we are done. ●

4.2.2 The spectral sequence

Let $X(\Delta)$ be a toric variety and $\mathcal{F}^*$ be a complex of sheaves on X. We describe a spectral sequence converging to the hypercohomology $\mathbb{H}^n(X(\Delta), \mathcal{F}^*)$.

Definition 4.2.14. A complex of sheaves $\mathcal{F}^*$ is said to have **homotopy invariant cohomology** if for every variety X the projection $p \colon X \times \mathbb{A}^1 \to X$ induces an isomorphisms in hypercohomology $\mathbb{H}^n(X, \mathcal{F}^*) \cong \mathbb{H}^n(X \times \mathbb{A}^1, \mathcal{F}^*)$.

Remark 4.2.15. The complex of sheaves $\mathcal{M}^{q,*}$ defining morphic cohomology has homotopy invariant cohomology by Theorem 1.3.15.

Theorem 4.2.16. Let $X(\Delta)$ be a toric variety associated to a fan Δ and $\mathcal{F}^*$ a bounded above cochain complex of sheaves. There is a convergent spectral sequence

$$E_1^{r,s} = \mathrm{Ext}^s(\check{\mathcal{C}}_r(\Delta, \mathbb{Z}), \mathcal{F}^*) \Longrightarrow \mathbb{H}^{r+s}(X(\Delta), \mathcal{F}^*). \tag{4.7}$$

Moreover, if $\mathcal{F}^*$ has homotopy invariant cohomology,

$$E_1^{r,s} \cong \bigoplus_{\sigma \in \Delta^{(r)}} \mathbb{H}^s(T_\sigma, \mathcal{F}^*), \tag{4.8}$$

and the differentials on the first page $d_1 \colon E_1^{r,s} \to E_1^{r+1,s}$ are given by

$$d_1 = \sum_{\substack{\sigma \in \Delta^{(r)} \\ \tau \in \Delta^{(r+1)} \\ \tau \leq \sigma}} \epsilon(\tau, \sigma) r^*_{\tau,\sigma} \tag{4.9}$$

where

$$r_{\tau,\sigma} \colon T_\tau = \operatorname{Spec} \mathbb{C}[\tau^\perp] \longrightarrow T_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\perp],$$

are the morphisms induced by the natural inclusion $\sigma^\perp \to \tau^\perp$.

Proof. Let $\mathcal{F}^* \to \mathcal{I}^*$ be a K-injective resolution of $\mathcal{F}^*$. Let $\check{\mathcal{C}}_*(\Delta, \mathbb{Z}) \to \mathbb{Z}_X$ be the resolution of the constant sheaf $\mathbb{Z}_X$ from Corollary 4.2.13. We build a double complex

$$C^{r,s} = \mathrm{Hom}(\check{\mathcal{C}}_r(\Delta, \mathbb{Z}), \mathcal{I}^s),$$

with the induced differentials (going in the increasing direction of r and s). The homology of this double complex in the s direction is $\mathrm{Ext}^s(\check{\mathcal{C}}_r(\Delta, \mathbb{Z}), \mathcal{F}^*)$, giving the spectral sequence

$$E_1^{r,s} = \mathrm{Ext}^s(\check{\mathcal{C}}_r(\Delta, \mathbb{Z}), \mathcal{F}^*) \Longrightarrow \mathbb{H}^{r+s}(X(\Delta), \mathcal{F}^*).$$

As for the convergence, the complex of sheaves $\mathcal{I}^*$ is bounded above, and the schemes X_σ have finite cohomological dimension. Using the hypercohomology spectral sequence we conclude that $\mathbb{H}^k(X_\sigma, \mathcal{F}^*)$ vanishes for large k. In other words, the first page is bounded above in the s direction. By construction, it is bounded (from both sides) in the r direction, and this is enough to establish the convergence.

If $\mathcal{F}^*$ is homotopy invariant, as the immersion $T_\sigma \to X_\sigma$ are algebraic homotopy equivalences we get the isomorphism (4.8).

Finally, the differentials on the first page are induced by the r-differentials in the double complex $C^{r,s}$, which are given by the formula

$$d_1 = \sum_{\substack{\sigma \in \Delta^{(r)} \\ \tau \in \Delta^{(r+1)} \\ \tau \leq \sigma}} \epsilon(\tau, \sigma) i^*_{\tau,\sigma}$$

where $i_{\tau,\sigma}\colon X_\tau \to X_\sigma$ is the inclusion. The formula (4.9) follows from the equation $r^*_{\tau,\sigma} = j^*_\tau i^*_{\tau,\sigma} r^*_\sigma$ and the fact that j^*_τ and r^*_σ are mutually inverse isomorphisms giving the identification $\mathbb{H}^s(T_\sigma, \mathcal{F}^*) \cong \mathbb{H}^s(X_\sigma, \mathcal{F}^*)$. ●

We have a rather explicit description of the first page and differentials of the spectral sequence in 4.2.16. Together with the computation in Theorem 4.1.8 of the morphic cohomology of a torus we can make it still more explicit.

Corollary 4.2.17. Let $\mathcal{M}^{q,*}$ be the complex defining morphic cohomology. Then, the first page of the spectral sequence in 4.2.16 is

$$E_1^{r,s} \cong \bigoplus_{\sigma \in \Delta^{(r)}} (\bigwedge^s (\sigma^\perp \cap M) \otimes \mathrm{LH})[s]_q \tag{4.10}$$

and the differentials $d_1^r\colon E_1^{r,s} \to E_1^{r+1,s}$ are given by

$$d_1^r(\sum_{\sigma \in \Delta^{(r)}} x_\sigma v_{1,\sigma} \wedge \cdots \wedge v_{s,\sigma}) = \sum_{\sigma \in \Delta^{(r)}} x_\sigma \sum_{\substack{\tau \in \Delta^{(r+1)} \\ \tau \leq \sigma}} \epsilon(\tau, \sigma) v_{1,\sigma} \wedge \cdots \wedge v_{s,\sigma}$$

Proof. Follows from Theorem 4.2.16 and the computation 4.1.8. ●

Finally, using an idea from [44] which can be traced back to [64] we show that this spectral sequence degenerates rationally.

Theorem 4.2.18. The spectral sequence from Corollary 4.2.17 degenerates on the second page when tensored with $\mathbb{Q}$.

Proof. Let $\mathrm{LH}_{\mathbb{Q}} = \mathrm{LH} \otimes_{\mathbb{Z}} \mathbb{Q}$. The toric variety $X(\Delta)$ admits an $\mathbb{N}$-action. Let $m \in \mathbb{N}$, then $[m]\colon X(\Delta) \to X(\Delta)$ is the morphism which on the open sets $X_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$ is defined through the ring homomorphism $[m]^*\colon \mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[\sigma^\vee \cap M]$ given by $[v] \mapsto [mv]$ (see [64] for details).

The $\mathbb{N}$-action on $X(\Delta)$ induces an $\mathbb{N}$-action on the spectral sequence from Corollary 4.2.17. As the rational morphic cohomology of a torus $T_L = \operatorname{Spec} \mathbb{C}[L]$ is

$$\mathrm{L}^*\mathrm{H}^s(T_L)_{\mathbb{Q}} = (\bigwedge^s L \otimes \mathrm{LH}_{\mathbb{Q}})[s]_q,$$

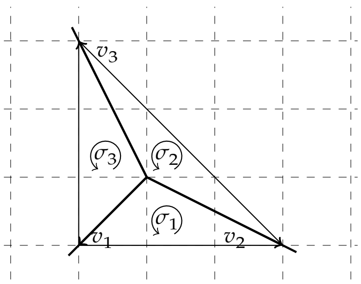
The $\mathbb{N}$-action on on the page $E_1^{r,s}$ is just multiplication by m^s. As the next pages $E_k^{r,s}$ of the spectral sequence are subquotients of $E_1^{*,*}$, the action on those pages is also given by m^s. On the other hand the differentials go $d_k\colon E_k^{r,s} \to E_k^{r+k,s+1-k}$ and the $\mathbb{N}$-action commutes with them, so

$$m^s d_k(x) = d_k(m^s x) = d_k([m]x) = [m]d_k(x) = m^{s+1-k} d_k(x),$$

where $x \in E_k^{r,s}$. Rationally, this implies $d_k(x) = 0$ when $k \geq 2$. ●

4.2.3 An example and an application

As an example of how the spectral sequence works, we give a sample computation. Consider the following fan Δ in $\mathbb{Z}^2$ as pictured in Figure 4.1.



$$\begin{aligned} v_1 &= (-1,-1), & v_1^\perp &= \langle (1,-1) \rangle, \\ v_2 &= (2,-1), & v_2^\perp &= \langle (1,2) \rangle, \\ v_3 &= (-1,2), & v_3^\perp &= \langle (-2,-1) \rangle. \end{aligned}$$

Figure 4.1: Fan Δ.

The associated toric variety $X(\Delta)$ is the quotient $\mathbb{P}^2/\mu_3$, where the action of a cubic root of unity $\zeta \in \mu_3$ is given by $\zeta[x : y : z] = [x : \zeta y : \zeta^2 z]$.

Let $R = \mathrm{LH} \cong \mathbb{Z}[s]$. Then, the spectral sequence is represented in Figures 4.2 and 4.3. The differentials on the first page are given by the matrices

$$d_1^{00} : \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \qquad d_1^{10} : \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \qquad d_1^{11} : \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & -1 \end{pmatrix}$$

So, the convergence of the spectral sequence tells us that

$$\mathrm{L}^*\mathrm{H}^n(X) = \begin{cases} R[k]_q & \text{for } n = 2k \text{ and } k \in \{0,1,2\}, \\ R/3[1]_q & \text{for } n = 3, \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{tikzcd}
s & & & \\
2 & 0 \arrow[r] & 0 \arrow[r] & R \\
1 & 0 \arrow[r] & R^3 \arrow[r, "d_1^{11}"] & R^2 \\
0 & R^3 \arrow[r, "d_1^{00}"] & R^3 \arrow[r, "d_1^{10}"] & R \\
 & 0 & 1 & 2 & r
\end{tikzcd}$$

Figure 4.2: Page $E_1^{r,s}$.

$$\begin{tikzcd}
s & & & \\
2 & 0 \arrow[drr] & 0 & R \\
1 & 0 \arrow[drr] & R & R/3 \\
0 & R & 0 & 0 \\
 & 0 & 1 & 2 & r
\end{tikzcd}$$

Figure 4.3: Page $E_2^{r,s}$.

Now we describe an application to the Suslin conjecture. Let $\varepsilon\colon \mathbf{Top} \to \mathbf{qProj}_{\mathbb{C}}$ be the morphism of sites, with the usual topology in $\mathbf{Top}$ and the Zariski topology on $\mathbf{qProj}_{\mathbb{C}}$. Let $\mathbf{R}\varepsilon_*\mathbb{Z}$ be the derived push-forward of the constant sheaf $\mathbb{Z}$ on $\mathbf{Top}$ to the Zariski site $\mathbf{qProj}_{\mathbb{C}}$. There is a natural map $\mathcal{M}^{q,*} \to \mathbf{R}\varepsilon_*\mathbb{Z}$ which, on smooth varieties, factors as

$$\mathcal{M}^{q,*} \longrightarrow \tau_{\leq q}\mathbf{R}\varepsilon_*\mathbb{Z}. \tag{4.11}$$

See [18] for details.

There is the following conjecture, a morphic analogue of the Beilinson-Lichtenbaum conjecture in the motivic world.

Conjecture 4.2.19 (Suslin)**.** The comparison morphism (4.11) above is a quasi-isomorphism on smooth varieties.

This conjecture is proved for the class of smooth linear varieties (which include smooth toric varieties) in [18] Theorem 7.14.

The spectral sequence 4.2.16 has the following Corollary.

Corollary 4.2.20. The Suslin conjecture holds for all quasi-projective toric varieties (not necessarily smooth).

Proof. First of all, we have to check that $\mathcal{M}^{q,*}|_{X(\Delta)}$ is exact above degree q, in order to have a factorization $\mathcal{M}^{q,*}|_{X(\Delta)} \to \tau_{\leq q}\mathbf{R}\varepsilon_*\mathbb{Z}|_{X(\Delta)}$ as in (4.11). This is a local statement on $X(\Delta)$, so we can restrict to an open X_σ. Now the inclusion $j_\sigma\colon T_\sigma \to X_\sigma$ is an algebraic homotopy equivalence, and they induce isomorphisms on hypercohomology

$$\mathbb{H}^n(X_\sigma, \mathcal{M}^{q,*}|_{X_\sigma}) \xrightarrow{\cong} \mathbb{H}^n(T_\sigma, \mathcal{M}^{q,*}|_{T_\sigma}),$$

so the natural map $\mathcal{M}^{q,*}|_{X_\sigma} \to \mathbf{R}j_{\sigma *}\mathcal{M}^{q,*}|_{T_\sigma}$ is a quasi-isomorphism. As T_σ is smooth, its cohomology vanishes above q, and we have the desired factorization.

Now, $\tau_{\leq q}\mathbf{R}\varepsilon_*\mathbb{Z}$ has homotopy invariant cohomology, because $\mathbf{R}\varepsilon_*\mathbb{Z}$ does, and the truncation preserves the homotopy invariance of the cohomology sheaves. We can apply Theorem 4.2.16 and get a spectral sequence converging to $\mathbb{H}^n(X(\Delta), \tau_{\leq q}\mathbf{R}\varepsilon_*\mathbb{Z})$. Moreover, the comparison map (4.11) gives a morphism of spectral sequences

$$\begin{tikzcd}
E_1^{r,s} = \bigoplus_{\sigma\in\Delta^{(r)}} \mathbb{H}^s(T_\sigma, \mathcal{M}^{q,*}) \arrow[r, Rightarrow] \arrow[d] & \mathbb{H}^{r+s}(X(\Delta), \mathcal{M}^{q,*}) \arrow[d] \\
E_1^{\prime r,s} = \bigoplus_{\sigma\in\Delta^{(r)}} \mathbb{H}^s(T_\sigma, \tau_{\leq q}\mathbf{R}\varepsilon_*\mathbb{Z}) \arrow[r, Rightarrow] & \mathbb{H}^{r+s}(X(\Delta), \tau_{\leq q}\mathbf{R}\varepsilon_*\mathbb{Z})
\end{tikzcd}$$

which is an isomorphism on the first page by Theorem 7.14 in [18], so it gives an isomorphism on the right, as claimed. ●

4.2.4 cdh descent for morphic cohomology

Finally, we prove that morphic cohomology satisfies cdh descent for toric varieties.

Let

$$\mathcal{M}^{q,*}(X) \longrightarrow \mathcal{M}^{q,*}_{\mathrm{cdh}}(X)$$

be the natural transformation to the version of morphic cohomology with cdh descent, as produced by Theorem 3.3.8.

Theorem 4.2.21. The natural transformation (4.2.4) is a quasi-isomorphism.

Proof. We apply Theorem 4.2.16 both complexes of sheaves $\mathcal{M}^{q,*}$ and $\mathcal{M}^{q,*}_{\mathrm{cdh}}$. Then, the comparison morphism (4.2.4) gives a morphism of spectral sequences

$$\begin{tikzcd}
E_1^{r,s} = \bigoplus_{\sigma\in\Delta^{(r)}} \mathbb{H}^s(T_\sigma, \mathcal{M}^{q,*}) \arrow[r, Rightarrow] \arrow[d] & \mathbb{H}^{r+s}(X(\Delta), \mathcal{M}^{q,*}) \arrow[d] \\
E_1^{\prime r,s} = \bigoplus_{\sigma\in\Delta^{(r)}} \mathbb{H}^s(T_\sigma, \mathcal{M}^{q,*}_{\mathrm{cdh}}) \arrow[r, Rightarrow] & \mathbb{H}^{r+s}(X(\Delta), \mathcal{M}^{q,*}_{\mathrm{cdh}})
\end{tikzcd}$$

Because on smooth varieties, $\mathcal{M}^{q,*}(X) \simeq \mathcal{M}^{q,*}_{\mathrm{cdh}}(X)$, we get isomorphism on the first page so the vertical map on the right is also an isomorphism. ●

Corollary 4.2.22. Morphic cohomology satisfies cdh descent on the class of toric varieties.

Proof. Immediate from 4.2.21, since $\mathcal{M}^{q,*}_{\mathrm{cdh}}$ satisfies cdh descent by construction. ●

Bibliography

[1] Alexander Beilinson. “Remarks on Grothendieck’s standard conjectures”. In: **Regulators**. Vol. 571. Contemp. Math. Amer. Math. Soc., Providence, RI, 2012, pp. 25–32. DOI: 10.1090/conm/571/11319. arXiv: 1006.1116.

[2] Julia E. Bergner. “A model category structure on the category of simplicial categories”. In: **Trans. Amer. Math. Soc.** 359.5 (2007), pp. 2043–2058. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-06-03987-0. arXiv: math/0406507.

[3] Julia E. Bergner. “A survey of $(\infty, 1)$-categories”. In: **Towards higher categories**. Vol. 152. IMA Vol. Math. Appl. Springer, New York, 2010, pp. 69–83. DOI: 10.1007/978-1-4419-1524-5_2. arXiv: math/0610239.

[4] Anthony Blanc. “Topological K-theory of complex noncommutative spaces”. 2015.

[5] Spencer Bloch and Arthur Ogus. “Gersten’s conjecture and the homology of schemes”. In: **Ann. Sci. École Norm. Sup. (4)** 7 (1974), 181–201 (1975). ISSN: 0012-9593.

[6] Chenghao Chu. “Morphic cohomology and singular cohomology of motives over the complex numbers.” English. In: **Topology Appl.** 156.17 (2009), pp. 2812–2831. ISSN: 0166-8641. DOI: 10.1016/j.topol.2009.08.017.

[7] Guillermo Cortiñas et al. “Cyclic Homology, Cdh-Cohomology and Negative K-Theory”. In: **Annals of Mathematics** 167.2 (2008). DOI: 10.4007/annals.2008.167.549. arXiv: math/0502255.

[8] David A. Cox. “The homogeneous coordinate ring of a toric variety”. In: **J. Algebraic Geom.** 4.1 (1995), pp. 17–50. ISSN: 1056-3911. arXiv: alg-geom/9210008.

[9] Albrecht Dold and René Thom. “Quasifaserungen und unendliche symmetrische Produkte”. German. In: **Ann. Math. (2)** 67 (1958), pp. 239–281. ISSN: 0003-486X; 1939-8980/e. DOI: 10.2307/1970005.

[10] W. G. Dwyer and D. M. Kan. “Simplicial localizations of categories”. In: **J. Pure Appl. Algebra** 17.3 (1980), pp. 267–284. DOI: 10.1016/0022-4049(80)90049-3.

[11] E. M. Friedlander and B. Mazur. “Correspondence homomorphisms for singular varieties”. In: **Ann. Inst. Fourier (Grenoble)** 44.3 (1994), pp. 703–727. ISSN: 0373-0956.

[12] Eric M. Friedlander. “Algebraic cocycles on normal, quasi-projective varieties”. In: **Compositio Math.** 110.2 (1998), pp. 127–162. ISSN: 0010-437X.

[13] Eric M. Friedlander. “Algebraic cycles, Chow varieties, and Lawson homology”. In: **Compositio Math.** 77.1 (1991), pp. 55–93. ISSN: 0010-437X.

[14] Eric M. Friedlander. “Bloch-Ogus properties for topological cycle theory”. In: **Ann. Sci. École Norm. Sup. (4)** 33.1 (2000), pp. 57–79. ISSN: 0012-9593.

[15] Eric M. Friedlander. "Filtrations on algebraic cycles and homology." English. In: **Ann. Sci. Éc. Norm. Supér. (4)** 28.3 (1995), pp. 317–343. ISSN: 0012-9593.

[16] Eric M. Friedlander. "Relative Chow correspondences and the Griffiths groups." English. In: **Ann. Inst. Fourier** 50.4 (2000), pp. 1073–1098. ISSN: 0373-0956; 1777-5310/e. DOI: 10.5802/aif.1785.

[17] Eric M. Friedlander and Ofer Gabber. "Cycle spaces and intersection theory". In: **Topological methods in modern mathematics (Stony Brook, NY, 1991)**. Houston, TX: Publish or Perish, 1993, pp. 325–370.

[18] Eric M. Friedlander, Christian Haesemeyer, and Mark E. Walker. "Techniques, computations, and conjectures for semi-topological K-theory". In: **Math. Ann.** 330.4 (2004), pp. 759–807. ISSN: 0025-5831.

[19] Eric M. Friedlander and H. Blaine Lawson. "Duality relating spaces of algebraic cocycles and cycles". In: **Topology** 36.2 (1997), pp. 533–565. ISSN: 0040-9383.

[20] Eric M. Friedlander and H. Blaine Lawson. "Moving algebraic cycles of bounded degree". In: **Invent. Math.** 132.1 (1998), pp. 91–119. ISSN: 0020-9910. DOI: 10.1007/s002220050219.

[21] Eric M. Friedlander and H. Blaine Lawson Jr. "A theory of algebraic cocycles". In: **Ann. of Math. (2)** 136.2 (1992), pp. 361–428. ISSN: 0003-486X.

[22] Eric M. Friedlander and Barry Mazur. "Filtrations on the homology of algebraic varieties". In: **Mem. Amer. Math. Soc.** 110.529 (1994). With an appendix by Daniel Quillen, pp. x+110. ISSN: 0065-9266.

[23] Eric M. Friedlander and Vladimir Voevodsky. "Bivariant cycle cohomology". In: **Cycles, transfers, and motivic homology theories**. Vol. 143. Ann. of Math. Stud. Princeton, NJ: Princeton Univ. Press, 2000, pp. 138–187.

[24] Eric M. Friedlander and Mark E. Walker. "Comparing K-theories for complex varieties". In: **Amer. J. Math.** 123.5 (2001), pp. 779–810. ISSN: 0002-9327.

[25] Eric M. Friedlander and Mark E. Walker. "Function spaces and continuous algebraic pairings for varieties". In: **Compositio Math.** 125.1 (2001), pp. 69–110. ISSN: 0010-437X. DOI: 10.1023/A:1002464407035.

[26] Eric M. Friedlander and Mark E. Walker. "Rational isomorphisms between K-theories and cohomology theories". In: **Invent. Math.** 154.1 (2003), pp. 1–61. ISSN: 0020-9910.

[27] Eric M. Friedlander and Mark E. Walker. "Semi-topological K-theory". In: **Handbook of K-theory. Vol. 1, 2**. Berlin: Springer, 2005, pp. 877–924.

[28] Eric M. Friedlander and Mark E. Walker. "Semi-topological K-theory using function complexes". In: **Topology** 41.3 (2002), pp. 591–644. ISSN: 0040-9383.

[29] William Fulton. **Intersection theory**. Second. Vol. 2. Ergebnisse der Mathematik und ihrer Grenzgebiete. Berlin: Springer-Verlag, 1998, pp. xiv+470. ISBN: 3-540-62046-X; 0-387-98549-2.

[30] William Fulton. **Introduction to toric varieties**. Vol. 131. Annals of Mathematics Studies. The William H. Roever Lectures in Geometry. Princeton, NJ: Princeton University Press, 1993, pp. xii+157. ISBN: 0691000492.

[31] Paul G. Goerss and John F. Jardine. **Simplicial homotopy theory**. Vol. 174. Progress in Mathematics. Basel: Birkhäuser Verlag, 1999, pp. xvi+510. ISBN: 376436064X.

[32] A. Grothendieck. "Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes". In: **Inst. Hautes Études Sci. Publ. Math.** 8 (1961), p. 222. ISSN: 0073-8301.

[33] A. Grothendieck. "Hodge's general conjecture is false for trivial reasons". In: **Topology** 8 (1969), pp. 299–303. ISSN: 0040-9383.

[34] Alexander Grothendieck. "Pursuing Stacks". 1983.

[35] Francisco Guillén and Vicente Navarro Aznar. "Un critère d'extension des foncteurs définis sur les schémas lisses". In: **Publ. Math. Inst. Hautes Études Sci.** 95 (2002), pp. 1–91. ISSN: 0073-8301. DOI: 10.1007/s102400200003.

[36] F. Guillén et al. **Hyperrésolutions cubiques et descente cohomologique**. Vol. 1335. Lecture Notes in Mathematics. Papers from the Seminar on Hodge-Deligne Theory held in Barcelona, 1982. Springer-Verlag, Berlin, 1988, pp. xii+192. ISBN: 3540500235.

[37] Christian Haesemeyer. "Descent Properties of Homotopy K-Theory". In: **Duke Math. J.** 125.3 (Dec. 2004), pp. 589–619. DOI: 10.1215/S0012-7094-04-12534-5.

[38] Joe Harris. **Algebraic Geometry: A First Course**. Vol. 133. Graduate Texts in Mathematics. Springer, 1992. ISBN: 9780387977164.

[39] Robin Hartshorne. **Algebraic geometry**. Vol. 52. Graduate Texts in Mathematics. Springer, 1977. ISBN: 9780387902449.

[40] Jeremiah Heller and Mircea Voineagu. "Equivariant semi-topological invariants, Atiyah's KR-theory, and real algebraic cycles". In: **Trans. Amer. Math. Soc.** 364.12 (2012), pp. 6565–6603. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-2012-05603-0.

[41] Wenchuan Hu. "Birational invariants defined by Lawson homology." English. In: **Mich. Math. J.** 60.2 (2011), pp. 331–354. ISSN: 0026-2285. DOI: 10.1307/mmj/1310667980.

[42] Wenchuan Hu. "Birational invariants defined by Lawson homology". In: **Michigan Math. J.** 60.2 (July 2011), pp. 331–354. DOI: 10.1307/mmj/1310667980.

[43] Wenchuan Hu and Li Li. "Lawson homology, morphic cohomology and Chow motives." English. In: **Math. Nachr.** 284.8-9 (2011), pp. 1024–1047. ISSN: 0025-584X; 1522-2616/e. DOI: 10.1002/mana.200810283.

[44] A. Jordan. "Homology and cohomology of toric varieties". PhD thesis. Universität Konstanz, 1998.

[45] Roy Joshua. "Algebraic K-theory and higher Chow groups of linear varieties". In: **Math. Proc. Cambridge Philos. Soc.** 130.1 (2001), pp. 37–60. ISSN: 0305-0041. DOI: 10.1017/S030500410000476X.

[46] János Kollár. **Lectures on resolution of singularities**. Vol. 166. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2007, pp. vi+208. ISBN: 978-0-691-12923-5; 0-691-12923-1.

[47] János Kollár. **Rational Curves on Algebraic Varieties**. Vol. 32. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, 1995. ISBN: 9783540601685.

[48] János Kollár. **Rational curves on algebraic varieties**. Vol. 32. Ergebnisse der Mathematik. Berlin: Springer-Verlag, 1996, pp. viii+320. ISBN: 3540601686.

[49] H. Blaine Lawson Jr. "Algebraic cycles and homotopy theory". In: **Ann. of Math. (2)** 129.2 (1989), pp. 253–291. ISSN: 0003-486X.

[50] H. Blaine Lawson Jr. “Spaces of algebraic cycles”. In: **Surveys in differential geometry, Vol. II (Cambridge, MA, 1993)**. Int. Press, Cambridge, MA, 1995, pp. 137–213.

[51] Paulo Lima-Filho. “Completions and fibrations for topological monoids”. In: **Trans. Amer. Math. Soc.** 340.1 (1993), pp. 127–147. ISSN: 0002-9947.

[52] Paulo Lima-Filho. “Lawson homology for quasiprojective varieties”. In: **Compositio Math.** 84.1 (1992), pp. 1–23. ISSN: 0010-437X.

[53] Paulo Lima-Filho. “The topological group structure of algebraic cycles”. In: **Duke Math. J.** 75.2 (1994), pp. 467–491. ISSN: 0012-7094.

[54] Paulo Lima-Filho. “Topological properties of the algebraic cycles functor”. In: **Transcendental aspects of algebraic cycles**. Vol. 313. London Math. Soc. Lecture Note Ser. Cambridge: Cambridge Univ. Press, 2004, pp. 75–119.

[55] Jacob Lurie. “Derived Algebraic Geometry 5: structured spaces”. 2011.

[56] Jacob Lurie. “Higher Algebra”. 2011.

[57] Jacob Lurie. **Higher topos theory**. Vol. 170. Annals of Mathematics Studies. Princeton University Press, 2008. ISBN: 9780691140490.

[58] Carlo Mazza, Vladimir Voevodsky, and Charles Weibel. **Lecture notes on motivic cohomology**. Vol. 2. Clay Mathematics Monographs. Providence, RI: American Mathematical Society, 2006, pp. xiv+216. ISBN: 978-0-8218-3847-1; 0-8218-3847-4.

[59] Abdó Roig-Maranges. “Morphic cohomology of toric varieties”. In: **Homology Homotopy Appl.** 14.1 (2012), pp. 113–132. ISSN: 1532-0073. DOI: 10.4310/HHA.2012.v14.n1.a6. arXiv: 1001.3106.

[60] Pierre Samuel. **Méthodes d’algèbre abstraite en géométrie algébrique**. Seconde édition, corrigée. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 4. Berlin: Springer-Verlag, 1967, pp. xii+133.

[61] N. Spaltenstein. “Resolutions of unbounded complexes”. In: **Compositio Math.** 65.2 (1988), pp. 121–154. ISSN: 0010-437X.

[62] Andrei Suslin and Vladimir Voevodsky. “Bloch-Kato conjecture and motivic cohomology with finite coefficients”. In: **The arithmetic and geometry of algebraic cycles (Banff, AB, 1998)**. Vol. 548. NATO Sci. Ser. C Math. Phys. Sci. Kluwer Acad. Publ., Dordrecht, 2000, pp. 117–189.

[63] R. W. Thomason. “Les K-groupes d’un schéma éclaté et une formule d’intersection excédentaire”. In: **Invent. Math.** 112.1 (1993), pp. 195–215. ISSN: 0020-9910. DOI: 10.1007/BF01232430.

[64] Burt Totaro. “Chow groups, Chow cohomology, and linear varieties”. English. In: **Forum Math. Sigma** 2 (2014), p. 25. ISSN: 2050-5094/e. DOI: 10.1017/fms.2014.15.

[65] V. Voevodsky. “Homology of schemes”. In: **Selecta Math. (N.S.)** 2.1 (1996), pp. 111–153. ISSN: 1022-1824. DOI: 10.1007/BF01587941.

[66] Vladimir Voevodsky. “Homotopy Theory of Simplicial Sheaves in Completely Decomposable Topologies”. In: **Journal of Pure and Applied Algebra** 214.8 (2010), pp. 1384–1398. DOI: 10.1016/j.jpaa.2009.11.004.

[67] Vladimir Voevodsky. “Unstable Motivic Homotopy Categories in Nisnevich and Cdh-Topologies”. In: **Journal of Pure and Applied Algebra** 214.8 (2010), pp. 1399–1406. DOI: 10.1016/j.jpaa.2009.11.005.

[68] Mircea Voineagu. “Semi-topological K-theory for certain projective varieties”. In: **J. Pure Appl. Algebra** 212.8 (2008), pp. 1960–1983. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2008.01.004. arXiv: math/0601008.