Universitat Autònoma de Barcelona
Departament d'Economia i d'Història Econòmica

Three Essays on First-Price Auctions

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Preface

When we [Leon Tolstoi, his family and G. Kennan] *had risen from the table he* [L. Tolstoi] *produced and proceeded to sell at auction to the highest bidder a richly embroidered towel, the work of a peasant woman, which, he said, had been brought to him as a present, but which he was unwilling to accept because the giver was very poor and really in need of the money that the towel represented. Amid general laughter Count Tolstoi's son and I, who were the principal bidders, ran the price up by successive offers of five kopeks more to two roubles and a half, when the auctioneer, with non-professional candor, declared that that was too much; that the American traveler in the course of the bidding had offered two roubles, which was about what the towel was worth, and that consequently it was his duty to award it to him. Young Tolstoi, with mock indignation, protested against the unfairness of that sort of an auction, but his motion for a new trial was overruled on the novel ground that the towel belonged to the auctioneer, who therefore had an unquestionable right to knock it down to any bidder whom he chose. His son laughingly acquiesced in the ruling, and the merry group which had gathered about the auctioneer dispersed.*

George Kennan, *A Visit to Count Tolstoi*

As we see, auctions can be held under any imaginable rules the auctioneer wants to impose.

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Chapter 1

Introduction

In his seminal paper of 1961, Vickrey established an early version of the revenue equivalence theorem. He analyzed a one-object auction in which the bidders' valuations were independent draws form a uniform distribution. He showed that the English and the Dutch auctions are efficient and revenue equivalent, and he proposed a new auction mechanism: the second-price sealed-bid auction (also called Vickrey auction) whose equilibrium in dominant strategies induces the bidders to directly reveal their valuation for the object. Moreover, he pointed out that when there are asymmetries the Dutch auction (as the strategically equivalent first-price auction) is no longer efficient. To illustrate the difficulties of analyzing asymmetric first-price auctions, he solved for the equilibrium in a 2-bidder auction where one of the bidders' valuations was fixed, while the other bidder drew it from the uniform distribution on [0,1]. Vickrey also considered auctions of multiple units of identical objects where bidders have unit-demands. He described the simultaneous uniform-price auction, which induces the bidders to truthfully reveal their valuations, and results in an efficient allocation of the objects. Moreover, he analyzed a sequential second-price auction with two objects where the bidders' valuations were uniformly distributed, and noticed that the allocation and the expected price in each round were identical to those in the uniform-price auction. Vickrey went further still, by warning the reader that when bidders have multi-unit demands, his uniform-price auction would neither induce truthful revealing nor achieve an efficient allocation, since a bid for subsequent units may be the one that fixes the price for the first units as well. Three decades later, some reputed economists still recommended the uniform-price auction to sell Treasury bonds, with the flawed argument that sincere bidding was a dominant strategy.[1] Given this, it is not surprising that William Vickrey was awarded the Nobel prize in 1996.

In this dissertation we analyze three first-price auction mechanisms that in one way or another are related to Vickrey's paper. As in his paper, we assume private and independent valuations. In our first paper, a variant of the first-price sequential auction is analyzed under the unit-demand assumption. Weber (1983) generalized Vickrey's analysis of sequential auctions by showing that when bidders have unit-demands and private and independent valuations, the ascending and descending-price sequential auctions are efficient and revenue equivalent, and that the path of prices follows a martingale. In the auction we analyze, in contrast, the price can only decrease, and the allocation of the objects is inefficient. In the second paper, we generalize Vickrey's analysis of an asymmetric auction to an n-bidder auction where one of the bidders' valuation is common knowledge, while the rest of the bidders draw their valuations from a general (common) distribution function.

[1] See Ausubel and Cramton (2002) who also analyze the "demand reduction" phenomenon already outlined by Vickrey.

Griesmer, Levitan and Shubik (1967) generalized Vickrey's two bidder analysis to uniformly distributed valuations with different supports. But curiously, after that one, very few analyses of asymmetric first-price auctions were published before the nineties.[2] To finish, we analyze a sequential first-price auction where the bidders have multi-unit demands. One of the features that cause the complexity of these auctions are the asymmetries that arise after the first auction: the winner of the first object will typically reveal information about his preferences, in a way that makes the second round asymmetric. For this reason, most of the studies in sequential auctions consider 2-object second-price auctions. Since in the second stage truthful revealing is a dominant strategy, asymmetries are avoided. In one of the scenarios we analyze, we make use of the conclusions we obtained in the previous paper, i.e. we assume that if a separationg pure strategy equilibrium exists, the winner of the first auction will reveal his valuation. We also analyze the case where a "buyer's option" allows the winner of the first object to obtain the second unit at the same price.

1.1 Multi-Unit Descending-Price Auctions Where the Clock Never Comes Back

In the second chapter we study a multi-unit auction where the objects are sold sequentially by descending-price auctions. While in the "standard" sequential auction analyzed by Weber (1983) the price returns to a high level each time an object is sold, in our auction the price can only decrease: after the first object has been assigned to the highest

[2]Maskin and Riley were pioneers in the study of asymmetric first-price auctions in the early eighties. However, (part of) their studies have been only recently published (see Makin and Riley 1996, 2000a, and 2000b).

bidder, the rest of the objects are offered to the other bidders at the same price. If the objects fail to be sold at that price, the auction is resumed letting the price descend again, and not starting again at a high price.

The analysis of this auction in a continuous valuations set-up is involved, and we are not able to prove existence of an equilibrium. However, we are able to state the conditions that the bidding functions must satisfy in a symmetric, monotone equilibrium. If such an equilibrium exists, the outcome of the auction is inefficient with positive probability. Applying the revenue equivalence theorem we conclude that the auction cannot maximize the seller's expected revenue. The analysis, however, suggests that the dispersion of prices could be lower in this auction than it is in the standard sequential descending auction. We then analyze a discrete-valuation model in order to be able to compare the average expected prices and variances. We show that the average expected prices are lower in our auction, and that so is the variance of the seller's expected revenue. This may make our auction attractive for a risk averse seller. We give an example of a family of von Neumann-Morgenstern utility functions under which the seller's expected utility may be higher in each of the auctions depending on the value of a parameter α.

1.2 First-Price Auctions Where One of the Bidders' Valuations Is Common Knowledge

In the third chapter we analyze an asymmetric first-price auction where the valuation of one of the bidders is common knowledge. We show that no pure strategy equilibrium exists and we characterize a mixed strategy equilibrium in which the bidder whose valuation is common knowledge randomizes his bid while the other bidders play a (monotone) pure strategy. There are two kinds of equilibria depending on the common-knowledge valuation. If it is very high, the maximal bid is the same for all bidders. The equilibrium is then similar to the one described in Vickrey (1961). However, if the valuation which has been revealed is low, the rival bidders will bid above the random bid support with positive probability. The problem is then more involved, since the random bid's support is not easily determined.

The outcome of the auction is inefficient with positive probability, and the expected profit of the bidder whose valuation is common knowledge is lower than in a standard auction in which her valuation is private knowledge. However, it is not clear that the other bidders are better off: the fact that one of the bidders plays a mixed strategy has an effect on the other bidders comparable to that caused by a random reserve price. This may force all them to bid more aggressively than they would in the standard auction. The effect on the seller's expected revenue is also ambiguous. In an example with the uniform distribution, we compare the expected profits of seller and buyers in this auction with those in a standard symmetric private valuation model. In our example, with 2 and 3 bidders, the seller's expected revenue is higher in the asymmetric auction than in a standard auction.

1.3 Sequential First-Price Auctions with Multi-Unit Demand

In chapter 4 we review the literature in sequential auctions with multi-unit demand, and we analyze a sequential first-price auction with and without a buyer's option. The existence of a buyer's option, which allows the winner of an auction to buy additional units at the price he pays for the first one, has been proposed as an explanation for the so-called "declining price anomaly" reported, among others, by Ashenfelter (1989), and McAfee and Vincent (1993). The analysis of a sequential auction with a buyer's auction was first addressed by Black and de Meza (1992) by means of a sequential second-price auction. We use the same model in order to compare the first-price auction with the second-price auction. As they do, we analyze the sequential auction under two different assumptions on the bidders' valuations for the second unit. The first assumption consists of one-dimensional preferences of the bidders, which in the case of the sequential first-price auction without a buyer's option implies non-existence of a pure strategy equilibrium. When the buyer's option is introduced, an equilibrium exists where the winner of the first auction always exercises the option to buy. These two results differ from those found by Black and de Meza for the second-price auction, which implied an efficient equilibrium in the auction without the buyer's option, and an equilibrium where the option was only exercised under certain conditions in the model with the option. The second assumption on the bidders' distribution of valuations consists of a stochastic valuation for the second unit. In this case the first-price auction without the buyer's option is equivalent to the second-price auction. Both auctions are efficient, and the path of expected prices is increasing. To finish, we analyze the stochastic-valuation model with the buyer's option. We are not able to characterize an equilibrium to this auction (even in the simplest case of two bidders with uniformly-distributed valuations), but it is easy to see that if a symmetric equilibrium exists the path of prices in the two-bidder auction must be decreasing.

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Chapter 2

Multi-Unit Descending-Price Auctions Where the Clock Never Turns Back

2.1 Introduction

Sequential auctions in which the bidders have a positive valuation for only one unit of the goods have been widely studied in economic literature.[1] Under the classical assumptions (private and independent valuations, symmetry, monotonicity of the bidding functions) these auctions are efficient, and as an extension of the revenue equivalence theorem for the multi-unit case, it can be concluded that any efficient way of auctioning the objects is equivalent, that is, the seller's and buyers' expected rents are identical. More-

[1] See for example Weber (1983) and Milgrom and Weber (1982).

over, any efficient auction combined with an appropriate reserve price maximizes the seller's expected revenue (Engelbrecht-Wiggans, 1988, Maskin and Riley, 1989).

The sequential descending-price auction, which under the above assumptions is efficient, works as follows: the auctioneer announces a very high price and then lowers it continuously until some bidder accepts to pay the current price and buys the first object. Then, the price returns to its former high level and the next round starts. Throughout this paper, we will refer to this mechanism as to the "standard (sequential descending) auction".

There is, however, a variant of the descending auction that has been neglected by the literature. This auction is different to the one described above in that after each sale, the price does not return to its former high level, but continues its way downwards. Our auction is as follows: the seller announces a high price and lowers it continuously until some bidder accepts it. This buyer wins the first object and then the seller asks whether someone else is willing to buy at the same price. If somebody does, she gets one of the objects left. If more than one bidder wants to exercise the option, they are assigned one object each. In case there are more buyers than objects, the latter are allocated at random. Therefore, there is some kind of buyer's option addressed to all the bidders.[2] If there are objects left after this process, the auction is resumed from the price at which the last object(s) has (have) been sold. This procedure is repeated until there are no goods left.

In this paper we analyze this auction mechanism. Since it is widely used to sell fish along the Valencian coast in East Spain, we will refer to it as the "Valencian auction". Its use, however, is also common in other areas. Cassady (1967, pp. 60-62) explains how in Dutch auctions the winner chooses the amount of goods he wants, and then "the balance

[2]In fact this sould be viewed as a mechanism to avoid inexistence of equilibrium in the continuous case.

[of the lot] is ordinarily made available to other would-be buyers at the same price". He also suggests that in some cases "the balance of a lot would be sold by continuing the price downward". Note that if after having offered the option to buy to all bidders there are still goods left, it does not make any sense to auction them starting at a high price again (doing it would be equivalent to letting the price continue downwards, since no bidder would accept to pay a higher price than the one at which he had refused to exercise the option).

The Valencian auction is here analyzed in two frameworks: a discrete-valuation model, and a continuous model. The reason to do this is that, while it is difficult to study the bidding strategies in the continuous model, this set-up has characteristics that we think are worth to be discussed. We give now some insights into each of the models.

The continuous model: In a standard descending sequential auction the path of expected prices follows a martingale. In each round, after the sale of an object, the bidders "update" their bids, taking into account that now there is one less bidder but also one less object. In the standard auction this can lead to an increase of the price (with respect to the previous rounds prices).

In the Valencian auction, since the price cannot increase, this can lead to rationing (if there are more bidders willing to buy at that price than objects left) and therefore to inefficiencies. If rationing takes place with positive probability, then some bidders can win an object against a higher valuation rival. This fact could increase the first rounds bids (in comparison with those in the standard auction) since high bidders may be willing to pay more in order to secure themselves an object. On the other hand, in the Valencian auction, when a bidder considers stopping the countdown, she should take into account not only that she has the highest valuation (which she has in the standard sequential auction too), but also that the expected price of the second object is strictly lower than the price she will pay, something that is not true in the standard auction. Although we are not able to compare these prices in general, it is easy to show that the seller's expected revenue will be lower in the Valencian auction. Myerson (1981) states that an optimal (one-object) auction is one in which the object is assigned to the highest-valuation bidder provided his valuation is higher than a specific level. The same logic applies to multi-unit auctions. In the Valencian auction the highest-valuation bidder wins the first object, but the second object is not assigned to the second highest valuation bidder with probability one. This allocation is not only inefficient, but also non-optimal in order to maximize the seller's expected revenues.

In many cases (i.e., when the option to buy is exercised) the realized prices are identical for both objects, and therefore, we can expect that the dispersion of prices should be lower in the Valencian auction. If the seller(s) were risk averse, this property could make the Valencian auction more attractive than the standard one. Because this is not easy to analyze this in a continuous-valuation set-up, we analyze a discrete-valuation model.

The discrete model: Of course, in a discrete-valuation context the revenue equivalence theorem does not hold.[3] The discrete-type model is useful, however, in order to compare the seller's expected revenues and the dispersion of prices.

We show that no pure strategy equilibrium exists in any of the auctions we analyze and we characterize a symmetric mixed strategy equilibrium in each of them. We compute numerically the expected prices and the variances in the two auctions and compare them.

[3] The revenue equivalence theorem is based in the revelation principle, which requires that the players' types be drawn from an atomless, strictly increasing distribution.

We find that both the expected prices and the variances are lower in the Valencian auction than in the standard one. This holds for the variance of the prices in each stage and also for the variance of the seller's total revenue. We also provide an example of von Neumann-Morgenstern preferences displaying constant absolute risk aversion and compute the expected utility of a risk averse seller with such preferences. We find the values of the parameters for which the Valencian auction would be preferred by the seller.

In section 2.2 we introduce the discrete model and characterize the equilibrium for the Valencian and for the standard auction. In section 2.3 we provide some insights into the continuous set-up. Conclusions are found in section 2.4.

2.2 The discrete model

There are 3 potential buyers and 2 identical objects for sale. Each bidder i has a valuation v_i for one unit of the goods, which takes a value $\overline{v}$ with probability $p \in [0, 1]$ and a "low" value $\underline{v}$, with probability $1 - p$, where $\overline{v}$, $\underline{v} \in [0, 1]$, and $\overline{v} > \underline{v}$. The valuation for a second unit is zero. Bidders are risk neutral.

Given her valuation, in the standard auction each bidder has to choose: 1) the price at which she will stop the first auction if nobody has done it before, and 2) the price at which she stops the second auction given that another bidder has stopped the first auction at price P^*. In the Valencian auction, each bidder decides: 1) the price at which she stops the auction in the first round if nobody else has done it yet; 2) whether to exercise the option or not, when another bidder has won the first object at price P^*, and 3) the price at which she stops the auction if it has been restarted from the winner's price P^* and if her rival has not done so yet.

Notice first that in both auctions the low valuation bidders will always obtain a zero profit: the standard Bertrand argument shows that the minimal equilibrium price must be $\underline{v}$. Therefore, and as usual, we assume that while ties among bidders with the same valuation are broke at random, ties between different types are solved in favor of the highest-valuation bidder.[4]

Second, if a symmetric (separating) pure strategy equilibrium exists, in the first auction the bidders will learn with certainty their rivals' valuations. When the losers of the first auction have both a high valuation, this fact reduces their expected profits (since the unique equilibrium implies bidding $\overline{v}$). In both auction mechanisms, the optimal bid of the high valuation bidders should be high enough to avoid a profitable upward deviation which would secure the deviant an object, and low enough so that bidding $\underline{v}$, which implies a lower probability of winning but also a lower price, is not profitable. Proposition 1 states that no bid satisfies such conditions.

Proposition 1 *No equilibrium in pure strategies exists, neither in the Valencian auction nor in the standard auction.*

Proof. See Appendix 2.5. ■

2.2.1 Mixed strategy equilibria

In this section we characterize a mixed strategy equilibrium for each of the two auction formats described above. We analyze first the Valencian auction.

[4]Hence, if a high-valuation bidder ties with a low-valuation one, the former gets the object (note that the low type is indifferent at that point, since she gets a profit of zero in both cases).

Proposition 2 *In the Valencian auction an equilibrium in mixed strategies is:*

If $v_i = \underline{v}$ *bidder i bids* $\underline{v}$. *If she loses the first auction, she bids the same in the second auction.*

If $v_i = \overline{v}$ *and bidder i loses in the first auction, in the second auction she exercises the buyer's option always. In the first auction she bids randomly in the interval* $[\underline{v},\ \overline{v} - \exp^p(1-p)(\overline{v} - \underline{v})]$ *according to a differentiable distribution function F defined by:*[5]

$$\begin{aligned} & \ln(1 - p[1 - F(x)]) + \frac{1-p}{1 - p(1 - F(x))} + \ln(\overline{v} - x) - \ln(\overline{v} - \underline{v}) - \ln(1-p) - 1 \\ = \ & 0 \ \forall x \in [\underline{v}, \overline{v} - e^p(1-p)(\overline{v} - \underline{v})] \end{aligned} \tag{2.1}$$

Proof. See Appendix 2.5. ■

Notice that the bid in the first auction releases information about the probability that the rival bidders have a high valuation. For example, if the winner bid of the first auction is very close to $\underline{v}$, then the probability that one of the other bidders has a high valuation is very low. After the first auction, the losers update their beliefs about their rival's valuation. A low winning bid implies that the probability of facing a high rival in the second auction is low, but also makes the exercise of the option "cheaper". We have seen that at equilibrium, given the beliefs about the rival's valuation induced by the winner's bid, it is optimal to exercise the option always.

We now consider the standard auction.

Proposition 3 *In the standard auction an equilibrium in mixed strategies is:*

[5] Differentiating (2.1) with respect to x it is easy to see that $F(x)$ is a distribution function on $[\underline{v}, \overline{v} - e^p(1-p)(\overline{v} - \underline{v})]$.

If $v_i = \underline{v}$ *bidder i bids* $\underline{v}$*. If she loses the first auction, she bids the same in the second auction.*

If $v_i = \overline{v}$ *bidder i bids randomizes her bid in the following way:*

In the first auction, she bids in the interval $[\underline{v}, \underline{v} + p^2(\overline{v} - \underline{v})]$ *according to the distribution function*

$$F(b) = \frac{(1-p)}{p} \frac{(b-\underline{v}) + \sqrt{(b-\underline{v})(\overline{v}-\underline{v})}}{(\overline{v}-b)} \tag{2.2}$$

If she has lost the first auction, in the second auction her bid depends on the winning bid of the first auction. Denoting it by b_w*, bidder i randomizes her bid in the interval* $[\underline{v},\ \overline{v} - \frac{(\overline{v}-\underline{v})(\overline{v}-b_w)}{(\overline{v}-\underline{v})+\sqrt{(\overline{v}-\underline{v})(b_w-\underline{v})}}]$ *according to the distribution function*

$$H(B; b_w) = \frac{(\overline{v}-b_w)}{(b_w-\underline{v}) + \sqrt{(b_w-\underline{v})(\overline{v}-\underline{v})}} \frac{B-\underline{v}}{\overline{v}-B} \tag{2.3}$$

Proof. See Appendix 2.5. ■

Note that the maximal bid in the second auction depends on the winning bid in the first auction: if b_w is maximal, then it is $\underline{v} + p(\overline{v} - \underline{v})$. If b_w is minimal ($b_w = \underline{v}$) then the bidder know she faces a low-valuation rival, and bids $\underline{v}$.

We have seen that the maximal bid in the first stage of the Valencian auction is $\overline{v} - e^p(1-p)(\overline{v} - \underline{v})$, while in the standard auction it is $\underline{v} + p^2(\overline{v} - \underline{v})$. The former is greater than the latter if and only if $1 - e^p(1-p) < p^2$, which is never the case given that $p \in [0, 1]$. Therefore, the maximal bid is always higher in the standard auction. In the second stage the maximal bid of the standard auction can be even higher than in the first auction, while in the Valencian auction the price cannot increase. Because the distribution function of the random bid is not explicitly defined in the Valencian auction, we cannot compute explicitly the expected prices generated by each of the auction formats. However, without lost of generality, we can compare the revenues by doing the simplifying assumption $\overline{v} = 1$ and $\underline{v} = 0$.[6]

We compute the variances and expected prices in each stage of each auction format, and the total variance for different values of p, and we find that all these functions are lower in the Valencian auction. (Note that we do not discuss the possible existence of other equilibria.)

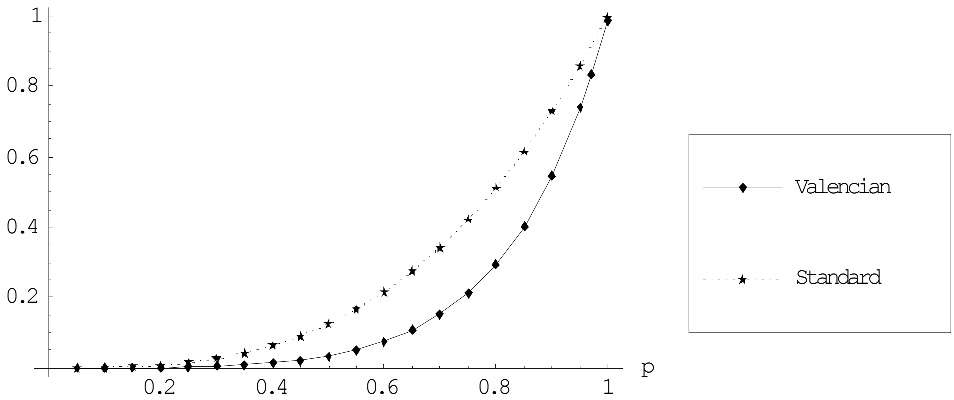

Figure 2.1: Expected prices in the standard and in the Valencian auction depending on p, the probability of having a valuation equal to 1.

Figures 2.1 and 2.2 show the total expected prices and variances in each auction as a function of p. To give a more complete analysis of at least one case we also compute the expected prices and variances in each stage of each auction format, and the total expected prices and variances for the case $p = \frac{1}{2}$. This can be seen in table 2.1. Note that, as we said, in the first auction the maximal (possible) price is higher in the standard auction, while in

[6]The comparison for the general case follows by rescaling.

the second auction it depends on the price realization of the first auction. In the standard auction the second auction price can be higher than that of the first auction. If the latter were the maximum (0.25) then in the second auction the price could be as high as 0.5. Of course, this only happens when the realizations of the winner bids are precisely the maximal in both auctions. In the Valencian auction, since the price cannot increase, the maximum price in the second auction is the same as in the first one. As we already said, the expected prices and variances in each auction are lower in the Valencian auction, as well as the total expected price and variance.

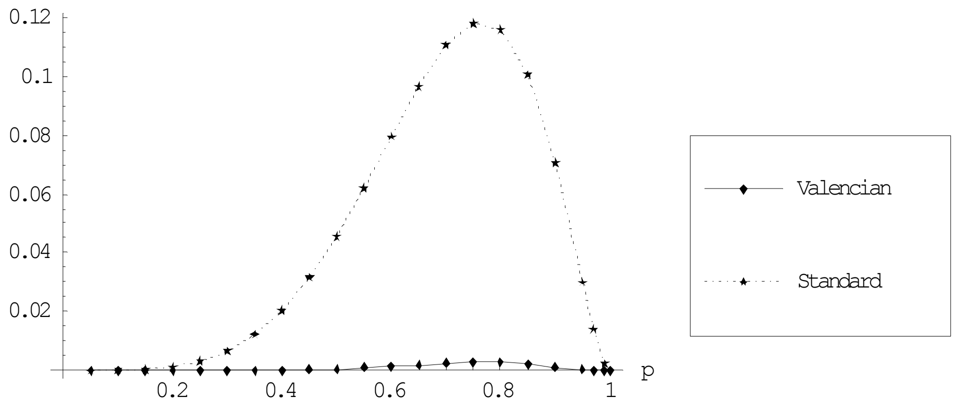

Figure 2.2: Expected variances in the standard and in the Valencian auctions.

Given that the variance is lower in the Valencian auction, we can expect that, if the seller is risk averse, he may prefer this auction rather than the standard one. We use a family of von Neumann-Morgenstern utility functions displaying constant absolute risk aversion and we compute for which parameters the Valencian auction will be superior. The utility function we use is $u(x) = -e^{-\alpha x}$ and we find that for $\alpha > 4.5$ (approximately) the Valencian auction yields a greater expected utility than the standard auction does. Table

2.2 shows the seller's expected utility under risk neutrality and under risk aversion for some arbitrary values of α when his utility function is $u(x) = -e^{-\alpha x}$. The expected utility is computed for the total revenue of the auctions.

Table 2.1: expected prices and variances in the standard and Valencian auctions when $\underline{v} = 0$, $v = 1$, and $p = 0.5$.

	Standard	**Valencian**
Max P first stage	0.25	0.1756
Max P second stage	$\frac{\sqrt{P_1}+P_1}{1+\sqrt{P_1}}$, (0.5)	P_1, (0.1756)
E(P) in first stage	0.125	0.0833
E(P) in second stage	0.125	0.0553
Variance in first stage	0.0072	0.0035
Variance in second stage	0.0241	0.0015
Total expected price	0.125	0.0693
Total variance	0.0456	0.0012

Table 2.2: Seller's expected utility under risk neutrality and risk aversion when the utility function is $u(x) = -e^{-\alpha x}$

	Standard	**Valencian**
Risk Neutrality, $\alpha = 1$	-0.7788	-0.8704
Risk Aversion, $\alpha = 1$	-0.8353	-0.8758
Risk Neutrality, $\alpha = 4.5$	-0.3279	-0.5430
Risk Aversion, $\alpha = 4.5$	-0.6048	-0.6046
Risk Neutrality, $\alpha = 7$	-0.1738	-0.3785
Risk Aversion, $\alpha = 7$	-0.5559	-0.4971

2.3 The continuous model

In this section we analyze the Valencian auction in a continuous set-up. The only difference with the previous section's model is that now the bidders' valuations are independent realizations of a continuous random variable with distribution function F and density f, that takes positive values in the interval $[0, 1]$.

In the previous section we saw that in the Valencian auction a high valuation bidder always exercised the option if she lost the first auction. In the continuous model this will not always be the case. Assuming the existence of a symmetric, monotone equilibrium, the increase of competition after the first auction is not enough to make the exercise of the option attractive for the losers: the winner's price must also be low enough from the point of view of the second bidder, or, in other words, the difference between the two highest valuations must be small enough.

A strategy of this game is a function which assigns, to each valuation v_i and to each price in the first round of the auction P^*, a vector $(b_f(v_i), b_s(v_i), b_t(v_i, P^*))$ which defines 1) the price at which bidder i stops the auction in the first round if nobody else has done it yet; 2) the maximum price she is willing to pay when another bidder has stopped the auction and the seller offers the second object at the winner's price, and 3) the price at which she stops the auction if it has been restarted from the winner's price P^* and if her rival has not done so yet.

We look for a symmetric, monotone equilibrium. Therefore, the highest-valuation bidder will win the first object. Notice that in the second stage the two remaining bidders have to decide whether or not to accept the second item at the winner's price. A bidder with valuation v_i will exercise this option if the price is lower than $b_s(v_i)$. Thus, $b_s(v_i)$ is some kind of cut-off point rather than a bid.

We will see that $b_s(v_i) > b_t(v_i, P^*) > b_f(v_i)$.[7] This implies that a bidder is willing to pay her maximal price in the moment that the option to buy is offered to her, that is, in the second stage of the game. In the first stage, a bidder with valuation v_i will wait for the price to drop until $b_f(v_i)$ before she stops the auction. However, if one of her rivals stops the auction before, bidder i, updating her beliefs, will be ready to pay more than

[7] We will see that $b_t(v_i, P^*)$ does not actually depend on P^*.

$b_f(v_i)$: when the highest bidder has stopped the auction there is only one object left, and the third bidder may exercise the option as well, so that exercising the option herself may be the last chance to win an object. If nobody exercises the buyer's option, the losers of the first auction update their beliefs again: if the rival has not exercised the option, her valuation cannot be "so" high. In the third stage, the bidders submit a new bid, $b_t(v_i)$, which is less than the price at which they would have exercised the option, but more than what they were willing to pay in the first stage. To illustrate this, suppose for a moment that $v_1 > v_2 > v_3$. Bidder 1 wins the first object at price $b_f(v_1) = P^*$. Bidder 2 exercises the option if $b_s(v_2) \geq P^*$. Therefore, the buyer's option will be exercised if and only if $b_f(v_1) \leq b_s(v_2)$, or $v_1 \leq b_f^{-1}(b_s(v_2))$. Moreover, if $v_1 \leq b_f^{-1}(b_s(v_3))$, then bidder 3 will also exercise the option, and the second object will be assigned to her with probability 0.5, resulting in an inefficient allocation. If v_1 is high enough, that is, if $v_1 > b_f^{-1}(b_s(v_2))$, then nobody exercises the option and the price descends again. Then, bidder 2 will not stop the auction at price $b_s(v_2)$, but at a lower price. Note also that $b_s(v_i) < b_f(v_i)$ is a necessary condition for the option to be exercised with positive probability.

We find the optimality conditions by backward induction. First, we compute $b_t(v_i, P^*)$, the bidding strategy when nobody has exercised the buyer's option. At this point there is a single object left, and the price descends from P^*, the winner's price. Then, b_t is the optimal strategy in a one-object standard descending auction with two buyers with valuations in the range $[0, b_s^{-1}(P^*)]$, that is,

$$b_t(v_i, P^*) = v_i - \frac{\int_0^{v_i} F(x)dx}{F(v_i)}. \tag{2.4}$$

Observe that b_t does not depend on P^*. This is a consequence of the fact that the remaining bidders are still symmetric after the first auction: all they have learn about each other is that they both have a valuation below $b_s^{-1}(P^*)$.[8] Therefore, they both bid the expected valuation of their rival conditional on it being less than their own, which is the RHS of equation (2.4). To simplify notation, hereinafter we write $b_t(v_i)$.

Now we derive two necessary conditions that the functions $b_f(v_i)$ and $b_s(v_i)$ must satisfy if a monotone symmetric equilibrium exists. To simplify notation, we define $s(v_i) = b_f^{-1}(b_s(v_i))$. Thus, $s(v_i)$ is the highest type of the winner of the first round that does not discourage bidder i from exercising the buyer's option. Conversely, we have $s^{-1}(v_i) = b_s^{-1}(b_f(v_i))$.

Since $b_s(v_i)$ is the maximum price which player i would pay to buy the second object at the winner's price, at this price she should be indifferent between buying the object or not. In the second case she will win the item and pay less for it only if her rival has a lower valuation than hers, which implies that the auction continues and the price goes down again. Thus, we must have

$$\frac{F(v_i)}{F(s(v_i))}\left[v_i - b_t(v_i)\right] = \left\{\frac{F(v_i)}{F(s(v_i))} + \frac{1}{2}\left(1 - \frac{F(v_i)}{F(s(v_i))}\right)\right\}\left[v_i - b_s(v_i)\right] \ \forall v_i$$

The first term is the buyer expected profit when she does not exercise the buyer's option. In this case, she wins the object when her valuation is higher than her rival's (conditional on both valuations being under the winner's) and pays $b_t(v_i)$ for it. In the right hand side we have the expected rents of exercising the buyer's option. If player i's valuation is above her rival's, the last will not exercise the buyer's option (since i is indifferent between doing so or not). Otherwise, her rival will do so and will win the item with a probability of $\frac{1}{2}$. Then,

[8] The fact that (under the private and independent valuations assumption) the bidding function does not depend on previous winning bids is a standard result in sequential auctions, and was already remarked by Weber (1983).

the probability that agent i gets the object is the one in curly brackets and her expected profit, that probability times her valuation minus the price she pays, $b_s(v_i)$. Notice that this condition, which defines $b_s(v_i)$, implies $b_t(v_i) < b_s(v_i)$. Substituting $b_t(v_i)$ by its value in (2.4), simplifying, and rearranging, we obtain

$$\int_0^{v_i} F(x)dx = \frac{1}{2}[F(v_i) + F(s(v_i))](v_i - b_s(v_i)) \ \forall v_i \tag{2.5}$$

The second condition that $b_f(v_i)$ and $b_s(v_i)$ must satisfy comes from the bidder's initial maximization problem: ex ante, the expected profit for a bidder with value v_i is:

$$\begin{aligned} E[\pi|v_i] &= F(v_i)^2[v_i - b_f(v_i)] + 2\ [1 - F(s(v_i))]\ F(v_i)\ [v_i - b_t(v_i)] + \\ &+2\int_{v_i}^{s(v_i)} \left[\frac{1}{2}\{F(x) - F[s^{-1}(x)]\} + F(s^{-1}(x))\right] [v_i - b_f(x)]f(x)dx \end{aligned}$$

The first term is the probability of having the highest valuation times the profit in that case. The second term is the probability that the winner has such a high value that player i does not want to exercise the buyer's option, that is, the second term considers the situation in which v_j is higher than $s(v_i)$, where j is the winner. In this case, player i lets the auction continue and wins the second object at price $b_t(v_i)$ if and only if the third bidder has a value under v_i. The last term is the probability that player i wants to exercise the option once some of her rivals has won the first object. In this case i will win the item with probability one half when the other player competes with her, and with a probability of one when her rival does not want to exert the buyer's option.

The expected profits of a player with valuation v_i that behaves as if it were type z are:

$$E[\pi|\ z; v_i] = F(z)^2[v_i - b_f(z)] + 2\{1 - F[s(v_i)]\}\ F(v_i)[v_i - b_t(v_i)] +$$

$$+2\int_{z}^{s(v_i)}\{\frac{1}{2}[F(x)-F\{s^{-1}(x)\}]+F[s^{-1}(x)]\ [v_i-b_f(x)]\}\ f(x)dx.$$

For $b_f(v_i)$ to be incentive compatible, the derivative with respect to z evaluated at v_i must be zero. Differentiating and rearranging we obtain the condition

$$b_f'(v_i)F(v_i)^2=\{F(v_i)-F[s^{-1}(v_i)]\}f(v_i)[v_i-b_f(v_i)]\ \forall v_i. \tag{2.6}$$

Equation (2.6) states that the marginal cost of increasing the first bid, $b_f(v_i)$, must be equal to the marginal benefit of doing so: on the one hand, an increase in the bid implies paying more when i is the highest bidder (left), but with probability $2\left\{F(v_i)-F[s^{-1}(v_i)]\right\}f(v_i)$ somebody was beating her and the other buyer was exercising the option (at the same time as bidder i). In half of the cases, bidder i did not win any object. In that case, by increasing her bid, i wins the first object instead of losing both. She obtains a profit of her valuation minus her bid.

Equations (2.5) and (2.6) define $b_f(v_i)$ and $b_s(v_i)$ as a function of each other, involving a differential equation and inverse functions (through the function s). This makes difficult to raise conclusions about their shape or properties and to compare the expected prices with those of a standard auction. However, it is easy to prove that, as we said above, $b_f(v_i)<b_t(v_i)$. Differentiating (2.4) we have $b_t'(v_i)=\frac{(v_i-b_t(v_i))f(v_i)}{F(v_i)}$. Combining this with equation (2.6) it is easy to see that if $b_t(v_i)=b_f(v_i)$, then $b_t'(v_i)>b_f'(v_i)$. Hence, since $b_f(0)=b_t(0)=0$, we must have that $b_t(v_i)>b_f(v_i)$.[9]

Comparison with the bids in a standard auction.

Weber (1983), describes the unique symmetric equilibrium bidding strategy of the

[9]This comes from a lemma which is often used in the auction theory literature (see for example Burguet, 2000, pp 22).

sequential first-price auction, which is strategically equivalent to the oral, descending-price auction. In equilibrium, in each round each potential buyer bids the expected valuation of the highest bidder who will not win any object, conditional on her own valuation being the highest one (which it will be if the bidder wins the auction).

Denote the l'th round optimal bidding function in the standard (efficient) auction by $b_{le}(v_i)$. In the second round, a bidder with valuation v_i bids the expected valuation of her rival conditional on it being below v_i, that is, $b_{2e}(v_i) = b_t(v_i)$.[10] When the option is not exercised, the price in the second auction will be the same under both auction formats. Also, we know that $b_{1e}(v_i) < b_{2e}(v_i)$ and we have seen that $b_f(v_i) < b_t(v_i) < b_s(v_i)$.[11] We cannot compare $b_f(v_i)$ with $b_{1e}(v_i)$ (that is, we do not know in which auction the first object price is highest) but it is easy to see that the second object's price can be higher or lower in each auction format depending on the specific realization of the valuations. The highest price that a buyer with valuation v_i can pay in the Valencian auction is $b_s(v_i)$. This happens if a bidder type $s(v_i)$ wins the first auction and bidder i exercises the option and wins the second object. On the other hand, the lowest price a bidder i can pay in the second stage of the Valencian auction is $b_f(v_i)$. This will happen if there is a tie between the two highest bidders, and bidder i, being one of them, loses the first object but wins the second by exercising the option. As we saw, $b_f(v_i) < b_t(v_i)$ so in this case the second buyer pays in Valencian auction less than she would have paid in the standard one. Therefore, when the second buyer exercises the option, she will pay for it more or less than in the standard

[10] Denote by P the price paid for the first object. In the Valencian auction it is common knowledge that both bidders have a valuation below $b_s^{-1}(P)$. In the standard auction it is common knowledge that they both have a valuation below $b_{1e}^{-1}(P)$. But when we computed $b_t(v_i)$ we already pointed out that it is independent of that information.

[11] As we said $b_{1e}(v_i)$ is the 2^{nd} order statistic of 2 draws of F conditional on being below v_i, while $b_{2e}(v_i)$ is the expectation of a draw conditional on being below v_i. Therefore $b_{1e}(v_i) < b_{2e}(v_i)$.

auction depending on the valuation of the winner of the first auction. Notice that the fact that the third bidder can also win the object if her valuation is high enough (and also she exercises the option) does not affect this analysis.

Also, we can say something about the seller's expected revenues generated by this equilibrium.

Proposition 4 *The seller's expected revenues are greater in the standard auction than in the Valencian auction.*

Proof. From the revenue equivalence theorem we know that the seller's expected revenue depends only on how the auction assigns the objects and on the payoff of the lowest type.[12] It as been shown that an optimal auction is one which assigns the objects with probability one to the highest virtual valuation bidders, that is, one such that

$$J(v_i) = v_i - \frac{1 - F(v_i)}{f(v_i)}$$

is the highest. Assuming that $J(v_i)$ is increasing, the revenue is maximized by using a mechanism which assigns the objects to the highest bidders. In case a reserve price can be fixed in order to maximize the seller's expected revenues, it should be the one allowing the highest bidders to win an object only when their virtual valuation is positive, that is, those bidders with valuation $v \geq v^*$, where v^* is such that $J(v^*) = 0$.[13] While the standard auction assigns the objects to the bidders with the highest valuations, the Valencian auction assigns the second one with positive probability to the third bidder. Therefore, the Valencian auction cannot maximize the seller's expected profit. ■

[12] Which in this case is zero in both the Valencian and the standard auction.

[13] This comes from the generalization to multi-unit auctions of Myerson (1981). A simple exposition of the seller's maximization problem can be found in Burguet (2000).

Since the seller's expected revenue is lower in the Valencian auction as compared with the standard auction, then so must be the average expected prices. We had seen that it was also the case in the discrete model, and we had proved that in the discrete model the dispersion of the prices is lower in the Valencian auction. Although we are not able to prove it analytically, we believe that the same must happen in the continuous case. The reason is that when the option is exercised the prices are identical in both stages of the auction, and that in some case, the exercise of the option will avoid extremely high or extremely low prices in the second stage.

We think it is worth to mention here another curious variant of the descending auction called "string selling". This system is used to sell furs in London and St. Petersburg. As Cassady reports, the winner of the first lot is given "the option of buying other lots consisting of similar or identical skins at the same price; he may continue until his wants are satisfied, until the string is ended, or until challenged by a competitor". That is, at every moment another bidder can raise the price and win the right to buy additional lots in the string at the new price. Moreover, "the buyer who is bidding on a string of lots can also relinquish his position [...] Another buyer may then come in at a slightly lower price."[14] Although it does not seem to be the case in Cassady's example, it is easy to see that if the bidders wished a single unit of the goods this simple mechanism would be efficient (and therefore equivalent to the standard auction), since the price would be adjusted upwards each time demand exceeds supply and vice versa.[15]

[14]Cassady (1967, pp. 156).

[15]Of course, provided that a symmetric pure strategy equilibrium exists.

2.4 Conclusions

We have analyzed a variant of the oral descending auction in two different set-ups. In a continuous set-up, we have seen that the symmetric, pure strategy equilibrium, if it exists, can lead to inefficient outcomes. Therefore this mechanism does not maximize the seller's expected revenue and performs worse than a standard auction without reserve price. In a discrete-valuation set-up, we characterize a mixed strategy equilibrium for each of the auction formats (standard and Valencian auction) and compare them. Although the allocation of the objects is efficient, the average prices are lower in the Valencian auction than in the standard auction, as it happened in the continuous model. However, it can be shown that, under the discrete-valuation assumption, the dispersion of the prices is also lower, a fact that can make this auction preferred by risk averse sellers. We give an example of von Neumann-Morgenstern utility functions displaying CARA and compute the expected prices and variances in the two auction formats for some values of the parameters. Depending on the degree of risk aversion of the seller(s), either auction may be preferred by them.

The main limitation of our model is the assumption that bidders desire a single unit. Although in the Valencian auction the winner cannot buy less than one unit (box), it is conceivable that he can buy more.[16] Indeed, in Cassady's description of the auction it is understood that the winner can choose how much he takes, which implies that the lot for sale is divisible and the bidder chooses how many units he buys.[17]

[16] In the Valencian auction we know the bidders are final consumers and do not usually buy more than one box of fish.

[17] The merchandise is offered on the basis of a certain price per unit of weight (usually kilograms). Cassady does not specify if a lot contains all the available (homogeneous) goods or identical lots are sequentially sold (according to a standard sequential auction procedure).

It is well known that when the bidders desire more than one unit of the item the most common auction mechanisms are neither efficient nor equivalent. Moreover, it has been shown that an auction which allocates the objects efficiently does not maximize the seller's revenues, while the mechanism that would do it remains unknown. Therefore, under the multi-unit-demand assumption the ranking between the Valencian and the standard auction, both in terms of efficiency and of expected revenues, is not clear.

2.5 Appendix

Proof of Proposition 1

The proof is structured as follows: first we give the optimal bidding strategy of the bidders with a low valuation in the two auction formats. Then we address each of the auction formats separately. In both cases, however, we start proving that in a pure strategy equilibrium all high valuation bidders should submit the same bid, which must be higher than $\underline{v}$. Then, we give equilibrium conditions that an optimal bid should satisfy, and show that no bidding strategy can satisfy those conditions.

1) Low valuation bidders strategy: we assume, without loss of generality, that in the first auction all bidders with valuation $\underline{v}$ bid $\underline{v}$.[18]

2) High valuation bidders:

a) **Valencian auction**:

It is clear that a bidder type $\overline{v}$ must bid at least $\underline{v}$. It is also clear that if all bidders bid $\underline{v}$, then each of them has incentives to marginally increase her bid and win the object with probability one. Now, suppose (without loss of generality) that bidder 1 bids $B > \underline{v}$ if she has valuation $\overline{v}$. Then at least one of her rivals also bids B with positive probability, (otherwise, bidder 1 can increase her expected profits by reducing her bid: she wins with the same probability and pays less). Suppose then that bidders 1 and 2 bid B and bidder 3 bids $\underline{v}$. For each of them, it must not pay to deviate. We assume that all the high bids are observed by all the buyers and that a higher bidder has priority to exercise the option.[19]

[18] As we already mentioned, at equilibrium the lower valuation bidders must obtain a zero profit. For this, it is enough that two of the bidders bid up to their valuation.

[19] The first assumption seems reasonable in an oral descending auction, in which all the buyers would raise

Expected profit to bidders $1,2 : \overline{v} - B$

Expected profit to bidder 3: $(1-p)^2(\overline{v} - \underline{v})$

If $(\overline{v} - B) > (1-p)^2(\overline{v} - \underline{v})$, then bidder 3 is better off bidding $B + \varepsilon$.

If $(1-p)^2(\overline{v} - \underline{v}) \geq (\overline{v} - B)$, then both bidder 1 and 2 have incentives to bid $\underline{v}$ rather than B : in the second auction they can deviate and win for sure paying only $\underline{v} + \varepsilon$.

Therefore, all the high valuation bidders must submit the same bid. Denote it by B. We may have the following situations: 1) In the second auction the high valuation bidders exercise the buyer's option; 2) In the second auction the high valuation bidders exercise the buyer's option only if facing a high valuation rival. And 3) In the second auction nobody exercises the option.

We rule out now each of these possibilities:

1) A high valuation bidder has an expected profit in the whole game of:

$p^2((\overline{v} - B)\frac{2}{3}) + 2p(1-p)(\overline{v} - B) + (1-p)^2(\overline{v} - B) = (1 - \frac{1}{3}p^2)(\overline{v} - B).$

But if she marginally increases her bid in the first auction, she obtains $(\overline{v} - B - \varepsilon) > (1 - \frac{1}{3}p^2)(\overline{v} - B)$, so that it pays to deviate.[20]

2) A high valuation bidder has a total expected profit of:

$p^2((\overline{v} - B)\frac{2}{3}) + 2p(1-p)((\overline{v} - B)\frac{1}{2} + (\overline{v} - \underline{v})\frac{1}{2}) + (1-p)^2(\overline{v} - B) =$

$(1 - \frac{1}{3}p^2)\overline{v} - (1 - p + \frac{2}{3}p^2)B - p(1-p)\underline{v}$

If she deviates bidding slightly above B she gets $\overline{v} - B - \varepsilon$. Therefore, to avoid

their hands more or less simultaneously. In a first price auction, this would imply that the number of high bids is announced and then the winner is chosen at random. The second assumption implies that if there are two high bids in the first auction, each of them receives an object, or, in other words, that a high valuation bidder who bids less than her rivals in the first auction cannot exercise the option to buy. None of these assumptions is crucial for the inexistence of a pure strategy equilibrium, but they simplify the proof.

[20] This inequality is true for all $B < \overline{v}$, but it is clear that bidding $B \geq \overline{v}$ cannot be optimal, since it would yield a non-positive expected profit.

"upwards deviating" we need

$$(1-\frac{1}{3}p^2)\overline{v}-(1-p+\frac{2}{3}p^2)B-p(1-p)\underline{v}\geq\overline{v}-B \quad (2.7)$$

Suppose now one bidder deviates downwards in the first auction. The best she can do is to bid $\underline{v}$ and exercise the option only if her two rivals have bid B in the first auction. Her expected profit is $p^2(\overline{v}-B)\frac{1}{2}+(1-p^2)(\overline{v}-\underline{v})=(1-\frac{1}{2}p^2)\overline{v}-\frac{1}{2}p^2B-(1-p^2)\underline{v}$. Then we need

$$(1-\frac{1}{2}p^2)\overline{v}-\frac{1}{2}p^2B-(1-p^2)\underline{v}\leq(1-\frac{1}{3}p^2)\overline{v}-(1-p+\frac{2}{3}p^2)B-p(1-p)\underline{v} \quad (2.8)$$

Therefore, a pure strategy equilibrium bidding function should satisfy equations (2.7) and (2.8), but that's impossible as long as $p<1$ and $\underline{v}<\overline{v}$.[21]

3) An equilibrium where the option is never exercised cannot exist: if all bidders have a high valuation the only equilibrium in the second stage is to bid the maximum, but then it is better to exercise the option in the previous stage.

b) **Standard auction:**

As before, at least two of the high valuation bidders must bid $B>\underline{v}$. Suppose that at equilibrium bidder 3 bids $\underline{v}$ in the first auction. Then, in the first auction she learns her rivals' valuations, while her rivals do not learn hers. When all bidders have a high valuation no pure strategy equilibrium exists in the second stage.[22]

[21] There is still another possibility: if we did not allow a low bidder of the first auction to exercise the option in the second stage of the game, then condition (2.8) would be less restrictive, namely $(1-p^2)(\overline{v}-\underline{v})\leq(1-\frac{1}{3}p^2)\overline{v}-(1-p+\frac{2}{3}p^2)B-p(1-p)\underline{v}$ Combining this with equation (2.7) there exists a solution if and only if $p=\frac{1}{2}$, in which case a pure strategy equilibrium exists where $B=\frac{1}{4}\overline{v}+\frac{3}{4}\underline{v}$.

[22] Bidder 3 would always beat her rival if he played a pure strategy (unless he bids $\overline{v}$, which cannot be optimal, given that he can obtain a profit of $p(\overline{v}-\underline{v})$ by bidding $\underline{v}$. It is easy to see that no mixed strategy equilibrium exists either (observe that both bidders should bid in the same, continuous support, and while one bidder obtains a profit of $p(\overline{v}-\underline{v})$ bidding $\underline{v}$, the other obtains 0 at that point).

Now, suppose a pure strategy equilibrium exists in which the high valuation bidders bid B in the first auction. In the second auction everybody knows her rival's type. The only pure strategy equilibrium in the second stage is that if more than one player has a high valuation both of them bid their valuation. If only one bidder has a high valuation she gets the object paying $\underline{v}$. We have:

- The expected profit of the whole game to a high valuation player is:

$E(\pi|B) = p^2\frac{1}{3}(\overline{v} - B) + 2p(1-p)(\frac{1}{2}(\overline{v} - B) + \frac{1}{2}(\overline{v} - \underline{v})) + (1-p)^2(\overline{v} - B)$

- If she deviates and bids $\underline{v}$ in the first auction, her expected profit is at least[23]: $(1-p^2)(\overline{v} - \underline{v})$. Therefore we need $E(\pi|B) \geq (1-p^2)(\overline{v} - \underline{v})$, which requires

$$B \leq \frac{p^2}{3(1-p)+p^2}\overline{v} + \frac{3(1-p)}{3(1-p)+p^2}\underline{v} \tag{2.9}$$

- If she deviates in the first auction bidding slightly above B she gets $\overline{v} - B$. Therefore we need $E(\pi|B) \geq \overline{v} - B$, which implies

$$B \geq \frac{2p}{3-p}\overline{v} + \frac{3(1-p)}{3-p}\underline{v} \tag{2.10}$$

But the problem given by equations (2.9) and (2.10) does not have a solution as long as $p < 1$ and $\underline{v} < \overline{v}$. ■

Proof of Proposition 2

First, we prove that the bidding functions given in proposition 6 are optimal. Then, that all bids in the interval $[\underline{v}, \overline{v} - e^p(1-p)(\overline{v} - \underline{v})]$ yield the same expected revenue to a high valuation bidder. To finish, we prove that it is optimal for a high valuation bidder to exercise the option to buy always.

[23]We assume here that this bidder bids $\underline{v}$ also in the second auction. She might, however, bid more and beat her rival, since given her first auction bid, he will believe she is of type $\underline{v}$ and bid accordingly.

It is obvious no bidder can increase her expected profit by deviating from equilibrium: bidding below $\underline{v}$ yields a profit of zero. For low valuation bidders, bidding more than $\underline{v}$ implies a negative expected profit, while for high valuation bidders, bidding more than $\overline{v} - e^p(1-p)(\overline{v} - \underline{v})$ implies an expected profit of less than $e^p(1-p)(\overline{v} - \underline{v})$, which is her expected profit in equilibrium.

The high valuation bidders must be indifferent bidding any quantity in $[\underline{v}, \overline{v} - e^p(1-p)(\overline{v} - \underline{v})]$. Therefore, we must have:

$$p^2 \left[F(B)^2(\overline{v} - B) + 2\int_B^{\overline{B}} \int_{\underline{B}}^{x_1} \frac{1}{2}(\overline{v} - x_1) f(x_2) f(x_1) dx_2 dx_1 \right] +$$

$$+2p(1-p)\left[F(B)(\overline{v} - B) + \int_B^{\overline{B}} (\overline{v} - x) f(x) dx \right] + (1-p)^2 \left[\overline{v} - B \right] = \overline{v} - \overline{B} \; \forall B \in [\underline{v}, \overline{B}] \tag{2.11}$$

Where the first term in brackets is the expected profit to a bidder when her rivals also have a high valuation: with probability $F(B)^2$ she wins the first auction, and otherwise she exercises the option and gets the object with probability $\frac{1}{2}$. The second term in brackets is the equivalent when she has only a high valuation rival and the last term in the LHS refers to the case where she is confronted to low valuation rivals. In the RHS is the expected revenue of submitting the highest bid in the first auction (therefore winning with probability one). Computing the integrals, simplifying and rearranging, we have that

$$E(\pi) = 1 - \overline{B} \Longleftrightarrow F(B)^2 \left[\frac{1}{2} p^2 (\overline{v} - B) \right] + \overline{B}(1 - 2p + \frac{3}{2}p^2)) + \frac{1}{2}p^2 \int_B^{\overline{B}} F(x)^2 dx +$$

$$+2p(1-p)\int_B^{\overline{B}} F(x) dx - \frac{1}{2}p^2 \overline{v} - (1-p)^2 B = 0 \tag{2.12}$$

Differentiating equation (2.12) w.r.t. B, simplifying and rearranging we obtain the differential equation:[24]

$$f(B) = \frac{(1-p+F(B))^2}{p^2 F(B)(\overline{v}-B)} \tag{2.13}$$

with solution:

$$\frac{\ln(\overline{v}-x)}{p^2} + \frac{\ln(1-p+pF(x))}{p^2} + \frac{1-p}{p^2(1-p+pF(x))} = C_1 \tag{2.14}$$

We impose the initial condition $F(\underline{v}) = 0$ in equation (2.14).[25] This yields $C_1 = \frac{\ln(\overline{v}-\underline{v})+\ln(1-p)+1}{p^2}$ and substituting it in (2.14) we obtain equation (2.1). To find the random bid upper bound, we impose $F(\overline{B}) = 1$, which yields[26]

$$\overline{B} = \overline{v} - e^p(1-p)(\overline{v}-\underline{v}).$$

We prove now that it is optimal for a high valuation bidder to exercise the option in the second auction always. In the second stage of the game the bidders must update their beliefs about the probability that their rival has a high valuation. Depending on the price they have to pay if they exercise the option and on the probability that the rival has a high valuation, the bidders have to check if exercising the option is indeed their best strategy.

For each bidder, given that she lost the first auction, the probability that her rival in the second auction has a high valuation is $P(\overline{v}) = \frac{P(\overline{v},\overline{v})}{P(\overline{v},\overline{v})+2P(\overline{v},\underline{v})} = \frac{p^2}{p^2+2p(1-p)} = \frac{p}{2-p}$. Respectively, the probability of facing a low valuation rival is: $P(\underline{v}) = 2\frac{1-p}{2-p}$. Also, if a single

[24]Standard theorems in differential equations guarantee that a solution to equation (2.13) exists.

[25]This is a necessary condition for equilibrium: first, note that there cannot be mass at $\underline{B}$, since a bidder would increase his revenue bidding marginally above $\underline{B}$. On the other hand, if $\underline{B} > \underline{v}$ and there is no mass at $\underline{B}$, a high valuation bidder would rather bid $\underline{v}$ than $\underline{B}$, since he would win with the same probability and would pay less. This reasoning applies also for the mixed strategy equilibria in the next section: the infimum of the random bid support will be always $\underline{v}$.

[26]Notice that $e^p(1-p)$ decreases with p, and takes values from 1 to 0. Therefore $\overline{B}$ is lowest (equal to $\underline{v}$) when $p = 1$ and highest ($\overline{v}$) when $p = 0$.

bidder had a high valuation and randomized her bid in the first auction, the probability that the highest bid submitted was B is $f(B)$, while the probability that two rivals randomized their bids and the highest bid was B is $2F(B)f(B)$. Applying Bayes' rule once more we have that the probability that a bidder has a high rival in the second auction given that the winner bid B is $P(\overline{v}|B) = \frac{P(B|\overline{v})P(\overline{v})}{P(B|\overline{v})P(\overline{v})+P(B|\underline{v})P(\underline{v})} = \frac{F(B)p}{1-p[1-F(B)]}$ and the probability that she faces a low valuation rival is $P(\underline{v}|b) = \frac{1-p}{1-p[1-F(B)]}$.

Now, the expected profit of exerting the option is

$\frac{F(B)p}{1-p[1-F(B)]}\frac{1}{2}(\overline{v} - B) + \frac{1-p}{1-p[1-F(B)]}(\overline{v} - B)$.

We need to prove that this is greater than the expected profit of not exercising the option, which is $\frac{1-p}{1-p[1-F(B)]}(\overline{v} - \underline{v})$. This holds iff

$\left[\frac{1}{2}F(B)p + (1-p)\right](\overline{v} - B) \geq (1-p)(\overline{v} - \underline{v})$.

It is easy to see that at $B = \underline{v}$ the above expression holds with equality. Now, it is enough to show that the function in the LHS increases with B (since the RHS is constant). Differentiating the LHS we have:

$\frac{1}{2}f(B)p(\overline{v} - B) - \left[\frac{1}{2}F(B)p + (1-p)\right]$, which is positive iff $f(B) \geq \frac{F(B)p+2(1-p)}{p(\overline{v}-B)}$.

But it is easy to see from equation (2.13) that this holds $\forall B$. ■

Proof of Proposition 3

As in the proof of proposition **2**, it is clear that no bidder can increase her expected profit by deviating from equilibrium.

Now, we prove that in both stages of the game the high valuation bidders are indifferent about bidding at any point of the support of their random bids.

We start by the second stage. As we said, we denote the winner bid of the first auction by b_w. As in the Valencian auction, in the second stage the bidders must update their beliefs about the probability that their rival has a high valuation. Applying Bayes rule we update the probabilities of having a high or low rival given b_w. We have:

$P(\overline{v}|b_w) = \frac{F(b_w)p}{1-p[1-F(b_w)]}$, $P(\underline{v}|b_w) = \frac{1-p}{1-p[1-F(b_w)]}$.

The expected profit of bidding $\underline{v}$ is therefore $\frac{1-p}{1-p[1-F(b_w)]}(\overline{v}-\underline{v})$, while the profit of bidding $\overline{B}$ is $\overline{v}-\overline{B}$. Therefore we need

$$\overline{v}-\overline{B} = \frac{1-p}{1-p[1-F(b_w)]}(\overline{v}-\underline{v})$$

substituting $F(b_w)$ by its value in (2.2) and solving for $\overline{B}$, we obtain

$$\overline{B} = \overline{v} - \frac{(\overline{v}-\underline{v})(\overline{v}-b_w)}{(\overline{v}-\underline{v})+\sqrt{(\overline{v}-\underline{v})(b_w-\underline{v})}}$$

We denote the random bid in the second auction by B, $B \in [\underline{v}, \overline{B}]$. Its distribution function, $H(B; b_w)$, must make the (symmetric) rival indifferent between bidding the minimum, $\underline{v}$, or submitting any bid on the support $[\underline{v}, \overline{B}]$. Therefore, the following condition must be satisfied:

$[P(\overline{v}|b_w)H(B;b_w) + P(\underline{v}|b_w)](\overline{v}-B) = \frac{1-p}{1-p[1-F(b_w)]}(\overline{v}-\underline{v})$. This implies

$$H(B;b_w) = \frac{1}{F(b_w)}\frac{(1-p)}{p}\left[\frac{B-\underline{v}}{\overline{v}-B}\right] \quad \forall B \in [\underline{v}, \overline{B}]$$

Substituting $F(b_w)$ we obtain equation (2.3). It is easy to see that $H(B;b_w)$ is a distribution function.

We now address the first stage. For any bid b that a high valuation bidder submits in the first stage of the game, the total expected profit must be equal to the expected profit of bidding $\underline{v}$ at each stage, (since $\underline{v}$ is in the support of the random bid in both stages). This is $\left(1-p^2\right)(\overline{v}-\underline{v})$. Hence:

$$E(\pi)=p^2\left[F(b)^2(\overline{v}-b)+\int_b^{\underline{v}+p^2(\overline{v}-\underline{v})}\int_{\underline{v}}^{x_1}2f(x_2)dx_2\frac{1-p}{1-p[1-F(x_1)]}f(x_1)dx_1(\overline{v}-\underline{v})\right]+$$
$$+2p(1-p)\left[F(b)(\overline{v}-b)+\int_b^{\underline{v}+p^2(\overline{v}-\underline{v})}\frac{1-p}{1-p[1-F(x)]}f(x)dx(\overline{v}-\underline{v})\right]+$$
$$+(1-p)^2(\overline{v}-b)=(1-p^2)(\overline{v}-\underline{v})\ \forall b\in[\underline{v},\underline{v}+p^2(\overline{v}-\underline{v})]$$

Computing the integrals, simplifying and rearranging we get:

$$p^2F(b)^2(\overline{v}-b)-2p\,(1-p)\,F(b)(b-\underline{v})-(1-p)^2(b-\underline{v})=0\ \forall b\in[\underline{v},\underline{v}+p^2(\overline{v}-\underline{v})] \quad (2.15)$$

Which solving for $F(b)$ yields equation (2.2).

It is easy to see that $F(\overline{b})=1$ and that F is increasing in b, i.e., that $F(x)$ is a distribution function. ■

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Chapter 3

First-Price Auctions Where One of the Bidders' Valuations Is Common Knowledge

3.1 Introduction

In this paper we study a first-price private-value auction in which the valuation of one of the bidders is common knowledge. There are many situations in which some or all the players of a game may have information about the private valuation of other player(s). For example, an auction may be held in order to renew a license to run a business (from university cafeterias to radio licences). Information about the current incumbent (sometimes himself a winner of a previous auction) may be available to his rivals and/or to the seller. Actually, sequential auctions are a source of information revelation: in first-price auctions, the seller can learn information about the buyers' preferences through the losing bids. On the other hand, when the buyers have multi-unit demand, the bid of the winner of the first auction may reveal information about his preferences for the rest of the objects. Of course, the use of this information in subsequent stages of a game will affect the strategic behavior of the players in the first stage. While it is out of the scope of this paper to analyze these multi-stage games, we think that studying this auction may be useful to understand the costs and/or possible advantages of this information release.

In his 1961 seminal paper Vickrey compared first-price and second-price auctions of a single object and stated an early version of the revenue equivalence theorem. To give an example of an asymmetric case, he also studied an auction where two asymmetric bidders competed for an object. One of the bidders' valuations was fixed, while the other one was uniformly distributed. Vickrey solved for the first-price and second-price sealed-bid auctions and compared the seller's expected revenue.

While in the second-price auction a dominant strategy is (as usual) to bid one's own valuation, the equilibrium in the first-price auction is more involved. The bidder with known valuation must randomize her bid, while her rival uses a strictly increasing bidding function. The difference in the seller's revenue across auctions depends on the value of the fixed valuation: while for low valuations the second-price auction is superior, for high valuations the converse holds.

Already in 2000, Kaplan and Zamir (henceforth, KZ) analyze a two-bidder model in which the seller has information about the bidders' valuations, and can exploit it or not depending on his capacity to commit himself to some specific revelation policy. In this framework they analyze Vickrey's two-bidder problem for a general distribution of the unknown valuation.

We generalize Vickrey's analysis to an arbitrary number of bidders and a general distribution function. We will see that extending the equilibrium analysis is not trivial: while our equilibrium shares some features with the two-bidder equilibrium, most of the techniques used to compute the latter are not useful when more players are involved. We show that no pure strategy equilibrium exists in the auction we analyze and we characterize an equilibrium in which the bidder whose valuation is common knowledge plays a mixed strategy and the other bidders play pure strategies. Our equilibrium shares these features with the one described in Vickrey's example (and in KZ), and the outcome of the auction is, like in theirs, inefficient. The random bid of one of the bidders operates as a random reserve price from the point of view of the other bidders.

The expected profit of the bidder whose valuation is known is lower than in a standard auction, while the effect on the other players' expected profits is ambiguous: on the one hand they benefit of an informational advantage, but on the other hand they face a bidder (the one whose valuation is common knowledge) who may bid more aggressively than she would in a standard private-value auction. This may force all of her rivals to bid more aggressively too, but this will not always be the case. The bidders will bid more or less aggressively than in the standard auction depending on the valuation which has been revealed. If it is low, they will bid less: it is easy to see that in the limit, when the valuation which is revealed is zero, the rival bidders will bid as in a standard auction with one less bidder, that is, less aggressively than if all valuations were private knowledge. By contrast, if the valuation which has been revealed is high, all bidders will bid more aggressively than they would in a standard auction.[1]

After the analysis of the asymmetric auction, we address the question of the seller's (and buyers') expected revenues in this auction compared to those in a standard, symmetric one. In particular, we would like to know if the seller has incentives to announce one of the bidders' valuations when he has this information. Because the answer to this question is ambiguous, we analyze an example with the uniform distribution. The analysis we undertake differs slightly from those of Vickrey and KZ. Vickrey focuses on seller's revenue comparisons between first-price and second-price auctions when the valuation of one of the bidders is fixed. KZ assume that the seller knows either both valuations or their rankings and explore the seller's best strategy given restrictions on his capacity to convey informative signals and on his commitment power. In that framework, they analyze the case in which the seller reveals the highest of the two valuations, and compute the seller's and the buyer's expected profits for the case in which both valuations are uniformly distributed.

In our model we assume that, while ex-ante all valuations are identically and uniformly distributed, the seller can commit himself to announce one of them (which is not necessarily the highest one). We compute the seller's expected revenue for the case of two and three bidders and we find out that in both cases the seller's expected revenue is higher if he follows the policy of announcing one of the bidders' valuations. Note that if we drop the ex-ante symmetry assumption, and assume that it is likely that the valuation which is revealed is high, the seller will have surely incentives to find out and announce that

[1] Actually, we will see that in equilibrium the low-valuation bidders bid in our auction more than they bid in a standard auction. However, they win with probability zero, so their bids are not relevant in order to discuss the seller's expected revenue.

valuation. This could be the case, for example, if an authority holds an auction to renew a licence to run some business, and the current incumbent is taking part of it (the incumbent could be supposed to have a high valuation because of know-how, no entry costs, etc.) The authority could order an auditing of the firm and make the results public.

With this analysis, this paper contributes to the scarce literature on asymmetric first-price auctions under the private-value assumption. Indeed, after Vickrey's analysis in the early sixties and Griesmer, Levitan, and Shubik's (1967) generalization to asymmetric uniform distributions (i.e. different supports), little work had been published before the nineties. Plum (1992) characterizes Nash equilibria in a particular asymmetric sealed-bid auction. Marshall, Meurer, Richard, and Stromquist (1994) propose an algorithm to solve some class of asymmetric auctions and provide some numerical analysis. Lebrun (1996) studies asymmetric first-price auctions with an arbitrary number of bidders and in (1999) he proves existence and uniqueness of equilibrium when the distributions of the valuations have the same support. Maskin and Riley (1996 and 2000a) study existence and uniqueness of equilibrium in asymmetric first-price auctions and in (2000b) they explore the optimal bidding strategies, and the seller's expected revenues for several two-asymmetric-bidder examples. Li and Riley (1999) generalize some of those results to an arbitrary number of bidders using numerical methods. To finish, Landsberger, Rubinstein, Wolfstetter, and Zamir (2001) study a two-bidder auction where the ranking of valuations is common knowledge. They find that the lower valuation bidder bids more aggressively than his rival, and that, in spite of the induced inefficiency, the seller's expected revenue is higher than in the standard auction.

In section 3.2 we present the model and characterize the equilibrium. In section 3.3 we analyze an example where the bidders' valuations are uniformly distributed, and compare the seller's expected revenue with that in the standard auction. Conclusions are found in section 3.4.

3.2 The Model.

There are $n \geq 3$ buyers with a positive valuation for one object which is to be sold in a first-price auction or in an oral descending auction. We assume that each bidder draws his valuation from a twice differentiable distribution function F with support in $[0, 1]$, which is identical for each bidder. All players are risk neutral. We assume that the valuation of one of the bidders is common knowledge. For simplicity, we will refer to this bidder as bidder 1, and in the feminine, and to the rest of the bidders, in the masculine, by using the subindex i, where $i = 2, ..., n$. In this section we characterize an equilibrium to this game. First, we show that any equilibrium necessarily involves mixed strategies.

3.2.1 (Non existence of) a pure strategy equilibrium

Suppose bidder 1 plays a pure strategy. Her valuation is common knowledge, so that her rivals anticipate her bid. Denote this bid by r. Then her rivals' best response will be to bid as in an auction with $n - 1$ bidders and a common knowledge reserve price equal to r. It is well known that the equilibrium of this auction is symmetric and unique.[2] Here we show that if the bidders $2, ..., n$ bid according to this equilibrium, bidder 1 has incentives to deviate from her pure strategy bid.

[2]A proof of the uniqueness of equilibrium can be found in Lebrun (1999).

Proposition 5 *There is no pure strategy equilibrium in undominated strategies in the game described above.*

Proof. The equilibrium bidding strategy in a first-price auction with $n-1$ bidders and a reservation price of r is[3]

$$b(v_i) = v_i - \frac{\int_r^{v_i} F(x)^{n-2}dx}{F(v_i)^{n-2}} \quad \forall v_i \geq r \tag{3.1}$$

It is easy to see that $lim_{+\ v_i \to r} b'(v_i) \mid_r \ = 0$, which implies that the rate at which a bidder with valuation r is willing to increase his bid in return for a greater probability of winning is zero (although increasing in v_i). But as long as $v_1 > r$, bidder 1's willingness to increase her bid is strictly positive, so that she has an incentive to deviate upwards. Formally, given (3.1), bidder 1's maximization problem is

$$Max_{\ r}\ P[b(v_i) \leq r]^{n-1}(v_1 - r) \tag{3.2}$$

In order to be an optimal solution to the above problem, $\overline{r}$ must satisfy:

$$\overline{r} = ArgMax_{\ r \geq \overline{r}}\ F[b^{-1}(r)]^{n-1}(v_1 - r) \tag{3.3}$$

Substituting r by $b(z)$ in (3.3), we can rewrite condition (3.3) as

$$\overline{r} = ArgMax_{\ z \geq b^{-1}(\overline{r})}\ F(z)^{n-1}(v_1 - b(z)) \tag{3.4}$$

The marginal benefit of bidding $b(z)$ above $\overline{r}$ in (3.4) must be non positive, thus

$$(n-1)F(z)^{n-2}f(z)(v_1 - b(z)) - b'(z)F(z)^{n-1} \mid_{z=r} \ \leq 0. \tag{3.5}$$

[3]This bidding strategy can be easily computed by standard methods, and can be found, for example, in Riley and Samuelson (1981).

Since $lim_{+\ z \to r} b'(z)\,|_r\ = 0$, the term above is positive unless $v_1 - r \leq 0$, which would leave a non positive profit to bidder 1. But she can obtain a positive profit bidding under v_1. Therefore, condition (3.5) cannot be satisfied.[4]. ■

3.2.2 A mixed strategy equilibrium

In this section we characterize an equilibrium in which, given v_1, player 1 randomizes her bid in some interval $[\underline{b}, \overline{b}]$, while the other players bid according to a strictly increasing bidding function $b(v_i)$. Bidders 2 to n with valuation below $\underline{b}$ do not have any chance to win the auction, and we assume that they bid their own valuation. We can distinguish two kinds of equilibria depending on the value of v_1. There is a cut-off point, $\widehat{v}_1$ such that if $v_1 > \widehat{v}_1$ the bidders i with valuation $v_i \geq \underline{b}$ bid in the same support as bidder 1 does. By contrast, if $v_1 \leq \widehat{v}_1$, the highest valuation bidders will bid above the support of bidder 1's random bid. We denote by $y(\overline{b})$ the valuation of a bidder i who bids $\overline{b}$, that is, $y(\overline{b}) = b^{-1}(\overline{b})$. If $v_1 \leq \widehat{v}_1$, then $y(\overline{b}) < 1$. Figure 3.1 illustrates the mixed strategy equilibrium. In proposition 6 we give the equilibrium bidding strategies. We now define the conditions that the points $\widehat{v}_1, \overline{b}$, and $y(\overline{b})$ must satisfy, and a condition that $\underline{b}$ must satisfy in *any* mixed strategy equilibrium.[5]

Lemma 1 *The infimum of the support of bidder 1's random bid, $\underline{b}$, must satisfy the following*

[4]A pure strategy equilibrium where bidder 1 bids v_1 could be sustained if she won with probability zero bidding below that point, that is, if one (or more) of her rivals bid always at least v_1. Of course, this "overbidding" is a dominated strategy for the rival(s).

[5]Notice that we have already ruled out any pure strategy equilibrium to this game, and, moreover, any equilibrium in which bidder 1 plays a pure strategy. Therefore, condition (3.6) is very general, and must be satisfied in any equilibrium of the game.

condition:

$$v_1 = \underline{b} + \frac{F(\underline{b})}{(n-1)f(\underline{b})} \tag{3.6}$$

Proof. See Appendix 3.5. ■

To guarantee that equation (3.6) has a unique solution, we assume that $v_i + \frac{F(v_i)}{(n-1)f(v_i)}$ is increasing in v_i.

We now define some points:

Let $\widehat{v}_1$ be the bidder 1's valuation at which

$$\widehat{v}_1 = \frac{(n-2) + \underline{b}F(\underline{b})^{n-1}}{(n-2) + F(\underline{b})^{n-1}} \tag{3.7}$$

Notice that $\underline{b}$ depends on v_1, so that here we are defining a fixed point, where $\underline{b}$ must satisfy equation (3.6) with $v_1 = \widehat{v}_1$. As we said $\widehat{v}_1$ is the maximal valuation of bidder 1 such that her rivals bid above the support of her random bid.

If $v_1 \leq \widehat{v}_1$, let $\overline{b}$ and $y(\overline{b})$ be the points satisfying the following conditions[6]:

$$F(y(\overline{b}))^{n-1}(v_1 - \overline{b}) = F(\underline{b})^{n-1}(v_1 - \underline{b}) \tag{3.8}$$

$$(n-2)\left(y(\overline{b}) - \overline{b}\right) = (n-1)(v_1 - \overline{b}) \tag{3.9}$$

Where, as we said, $y(\overline{b})$ is the valuation of a bidder i who submits a bid of $\overline{b}$.

If $v_1 > \widehat{v}_1$, let $\overline{b} = v_1 - (v_1 - \underline{b})F(\underline{b})^{n-1}$ and $y(\overline{b}) = 1$.

As long as $v_1 > 0$, equations (3.6) to (3.9) imply $0 < \underline{b} < \overline{b} < v_1 < y(\overline{b}) \leq 1$.[7]

[6] Notice that the points defined in these two equations depend again on v_1, not only directly, but also through $\underline{b}$.

[7] If $v_1 = 0$ we have $\underline{b} = \overline{b} = y(\overline{b}) = 0$.

Proposition 6 *There exists an equilibrium where bidder 1 randomizes her bid in* $[\underline{b}, \overline{b}]$ *with positive density, and bidder* $i = 2, ..., n$ *bids*

$$b(v_i) = \begin{cases} v_i & v_i \leq \underline{b} \\ v_1 - \frac{(v_1 - \underline{b})F(\underline{b})^{n-1}}{F(v_i)^{n-1}} & \underline{b} < v_i \leq y(\overline{b}) \\ \frac{\int_{y(\overline{b})}^{v_i} x(n-2)F(x)^{n-3}f(x)dx}{F(v_i)^{n-2}} + \frac{\overline{b}F(y(\overline{b}))^{n-2}}{F(v_i)^{n-2}} & v_i > y(\overline{b}) \end{cases}$$

Notice that if $v_1 > \widehat{v}_1$ the bidding strategy of bidders 2 to n consists only of the two first regions above: $y(\overline{b}) = 1$, so that no bidders have valuation above $y(\overline{b})$. The equilibrium is then very similar to the one described by Vickrey (1961), and Kaplan and Zamir (2000) for the two-bidder case. By contrast, if $v_1 < \widehat{v}_1$ the bidders with valuations higher than $y(\overline{b})$ bid above the support of bidder 1's random bid, competing for the good among themselves, rather than with bidder 1. This cannot happen when there are only two bidders.

To prove proposition 6 we proceed as follows: first, we characterize the bidding strategy of bidders $2, ..., n$ assuming they behave symmetrically, and given that bidder 1 randomizes her bid in the interval $[\underline{b}, \overline{b}]$. The next step is to compute $H(.)$, the probability distribution of bidder 1's random bid. Then, we derive the conditions given above that the values $\overline{b}$ and $y(\overline{b})$ must satisfy, and derive the value of $\widehat{v}_1$. To finish, we prove global optimality of the bidding functions.

As we have seen, the bidding strategy of bidders 2 to n, $b(v_i)$, can be divided in two or three different regions of values v_i depending on v_1:

(1) We assume that the bidders with valuation $v_i \in [0, \underline{b}]$ bid their own valuation. They

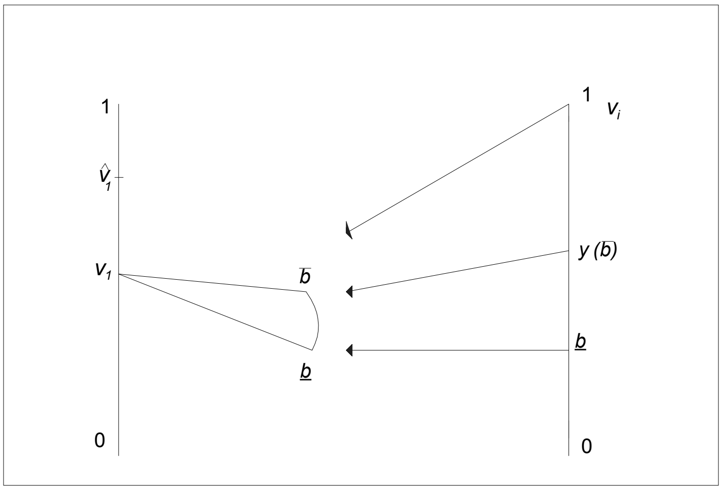

Figure 3.1: The mixed strategy equilibrium when $v_1 < \widehat{v}_1$: bidder 1 randomizes her bid in the interval $[\underline{b}, \overline{b}]$. Among her rivals, those with low valuations bid below $\underline{b}$, those with "intermediate" valuations bid in the same interval as bidder 1, and the rest bid above $\overline{b}$.

have no chance of winning, since player 1 bids at least $\underline{b}$. Therefore, to bid v_i is a best response for them.

(2) For $v_i > \underline{b}$, bidder $i \neq 1$ should bid at least $\underline{b}$, since bidding below $\underline{b}$ yields an expected profit of zero. Assume that bidders 2 to n bid symmetrically. Denote by $b_s(v_i)$ the bidding function of these players that would make bidder 1 indifferent about bidding any quantity in $[\underline{b}, b_s(1)]$. Its inverse, b_s^{-1}, must satisfy the following equation:

$$F[b_s^{-1}(x)]^{n-1}(v_1 - x) = F(\underline{b})^{n-1}(v_1 - \underline{b}) \quad \forall x \in [\underline{b}, b_s(1)], \tag{3.10}$$

Differentiating (3.10), applying the inverse function theorem, and rearranging, we obtain the following differential equation:

$$b_s'(v_i) = \frac{(n-1)f(v_i)(v_1 - b_s(v_i))}{F(v_i)} \tag{3.11}$$

with initial condition $b_s(\underline{b}) = \underline{b}$ the solution is[8]

$$b_s(v_i) = v_1 - \frac{(v_1 - \underline{b})F(\underline{b})^{n-1}}{F(v_i)^{n-1}} \tag{3.12}$$

Of course, the optimal bidding strategy for bidders 2 to n depends on the probability distribution of bidder 1's random bid. Therefore, we will choose that distribution function in order to make bidding according to $b_s(v_i)$ optimal for bidder i with valuation $v_i \in [\underline{b}, y(\overline{b})]$, so that $y(\overline{b}) = b_s^{-1}(\overline{b})$.

(3) As we have said, if v_1 is low enough, bidders with a high valuation will bid above $\overline{b}$ (for example, it is easy to see that this will be the case if $v_1 = 0$). Now, provided v_1 is low enough (namely provided that $v_1 < \widehat{v}_1$) there will be a third region of valuations, $v_i \in [y(\overline{b}), 1]$, where player i bids above $\overline{b}$, so that he knows that he will not be beaten by player 1 and that he is competing against only $n-2$ bidders. A necessary condition for equilibrium in this region is:

$$v_i = \arg\max{}_z \; F(z)^{n-2}(v_i - b_t(z)) \tag{3.13}$$

And also, $b_t(y(\overline{b})) = \overline{b}$.

Then, we must have:

$$b_t(v_i) = \frac{\int_{y(\overline{b})}^{v_i} x(n-2)F(x)^{n-3}f(x)dx}{F(v_i)^{n-2}} + \frac{\overline{b}F(y(\overline{b}))^{n-2}}{F(v_i)^{n-2}}. \tag{3.14}$$

[8]Note that b_s corresponds to bidder i's optimal bidding function in the second region of valuations (the subindex "s" stands for "second"). For exponential purposes, it is useful to refer to this function in particular (instead of to the bidding function in proposition 2). Also, since b_s is the way bidders 2 to n must bid in order to make bidder 1 indifferent, it is defined in a wider interval of valuations, $[\underline{b}, 1]$ than the one in which it is the optimal bidding function. Something similar happens for the function b_t below, where "t" stands for "third" (region of valuations).

We now study how bidder 1 must randomize her bid to make b_s optimal for her rivals.

Lemma 2 *Denote the distribution function of bidder 1's random bid by H, and assume it has a derivative, which we denote by h. The function H must satisfy the following differential equation:*

$$h\left[b_s(v_i)\right] = \left[\frac{1}{(v_i - b_s(v_i))} - \frac{(n-2)}{(n-1)(v_1 - b_s(v_i))}\right] H\left[b_s(v_i)\right] \quad (3.15)$$

given the initial condition $H(\overline{b}) = 1$.[9]

Proof. In order to sustain the mixed strategy equilibrium, bidders 2 to n with valuations in $[\underline{b}, y(\overline{b})]$ must bid according to the function $b_s(v_i)$. These players, who bid in the interval $[\underline{b}, \overline{b}]$ face the following maximization problem:

$$Max_z H(b_s(z))F(z)^{n-2}(v_i - b_s(z)) \; \forall z \in [\underline{b}, y(\overline{b})] \quad (3.16)$$

where b_s is the bidding function given in (3.12). In equilibrium the derivative with respect to z, evaluated at v_i must be zero. Substituting $b_s'(v_i)$ by its value in (3.11), simplifying and rearranging, we obtain equation (3.15). ■

The FOC of the maximization problem (3.16) states the typical requirement that the marginal profit of increasing the bid at point $b_s(v_i)$ must be equal to the marginal cost of doing so. Instead of computing the optimal bidding strategy as usual, here we compute the adequate H in order to make our function b_s optimal.

[9] The Cauchy-Peano theorem on differential equations, among others, guarantees existence of a solution to equation (3.15).

Note that we cannot used the initial condition $H(\underline{b}) = 0$ to find a particular solution of equation (3.15): given that $b_s(\underline{b}) = \underline{b}$, the RHS of the equation is not continuous at that point, and one of the necessary conditions for existence of a solution is violated. Instead, as we have seen, we use the condition $H(\overline{b}) = 1$.

The following lemma establishes the conditions that $\overline{b}$ and $y(\overline{b})$ must satisfy:

Lemma 3 *When $v_1 \leq \widehat{v}_1$, $\overline{b}$ and $y(\overline{b})$ are the points at which:*

(i) $b_s(y(\overline{b})) = b_t(y(\overline{b})) = \overline{b}$

(ii) $b_s'(y(\overline{b})) = b_t'(y(\overline{b}))$

Proof. See Appendix 3.5. ■

Conditions (i) and (ii) in lemma 3 are equivalent, respectively, to equations (3.8) and (3.9). Loosely speaking, they require continuity (i) and "smoothness" (ii) of the bidding function of bidders 2 to n.

There is an interesting interpretation to these conditions: when a bidder considers increasing his bid, he must compare how much more he pays with how much more likely he is to win the auction. While b_s is the way bidders 2 to n must bid in order to sustain the mixed strategy equilibrium, b_t, is the way they would actually bid if bidder 1 did not participate in the auction, given an initial condition $b_t(y(\overline{b})) = \overline{b}$. As we saw in proposition 5, if bidder 1 played a pure strategy, bidders 2 to n optimal bidding strategy would be given by (3.1), which is identical to $b_t(v_i)$ when the initial condition is $b_t(\underline{b}) = \underline{b}$. The slope of b_t at that point is zero: the bidders are bidding up to their own valuation, and therefore it is very costly for them to increase their bid in order to increase their probability of winning. Since condition (i) requires that the initial condition for b_t belong to the curve b_s, we move from $(\underline{b}, \underline{b})$ along the b_s curve selecting pairs $(v_i, b_s(v_i))$ as possible candidates to be the point at which bidder i's bidding function switches from b_s to b_t (that is, as candidates to satisfy condition (ii)). As v_i increases, so does the willingness of bidder i to increase his bid. By contrast, for bidder 1, the converse happens: when she bids $\underline{b}$ she has the highest "profit" margin, $v_1 - \underline{b}$. As her bid increases, that margin decreases, making raising her bid more and more costly. Condition (ii) states that $\overline{b}$ is the point at which the willingness to increase their bids in return for a marginal increase in the probability of winning is the same for all bidders.[10]

It may happen that at $v_i = 1$ bidder i's willingness to increase his bid is still lower than bidder 1's or, in other words, that the slope of b_s evaluated at $v_i = 1$ is greater than the slope of b_t at the same point (given the initial condition $b_t(1) = b_s(1)$). This is the case when $v_1 > \widehat{v}_1$, as we see in next lemma.

Lemma 4 *Bidders 2, 3,..., n bid above the support of bidder 1's random bid if and only if* $v_1 < \widehat{v}_1$.

Proof. Deriving the first order condition from (3.13) yields

$$b_t'(v_i) = \frac{(n-2)f(v_i)(v_i - b_t(v_i))}{F(v_i)} \tag{3.17}$$

Equating the slopes of b_s and b_t given by (3.11) and (3.17), and simplifying we obtain:

$$(n-1)\,(v_1 - b_s(v_i)) = (n-2)\,(v_i - b_t(v_i))$$

Imposing $b_t(v_i) = b_s(v_i)$, substituting $b_s(v_i)$ by its value from equation (3.12), simplifying,

[10] For "well behaved" distribution functions, for example, the uniform distribution, we could summarize by saying that, while b_s is concave and its slope decreases (bidder 1's willingness to increase her bid diminishes as b increases), the slope of b_t at $y(\overline{b})$ increases as $y(\overline{b})$ increases (since bidder i's margin, $y(\overline{b}) - \overline{b}$ increases too). See figure 2 in section 4, where we compute the equilibrium for the case of 3 bidders and the uniform distribution and we graph b_s and b_t.

and rearranging we have:

$$v_i = v_1 + \frac{(v_1 - \underline{b})F(\underline{b})^{n-1}}{(n-2)F(v_i)^{n-1}} \tag{3.18}$$

Of course, all we have done is to impose the conditions of lemma 3. Therefore, the point v_i satisfying equation (3.18), if it exists, is the point at which bidder i's bidding function switches from b_s and b_t, that is, our $y(\overline{b})$. Observe that the LHS of equation (3.18) increases with v_i while the RHS decreases. Hence, when $y(\overline{b})$ exists, it is unique. On the other hand, it is clear that if v_1 is high (for example, if $v_1 = 1$) equation (3.18) does not have a solution on $[0, 1]$. This will happen when evaluating equation (3.18) at $v_i = 1$ the RHS is still greater than the LHS, that is, when $1 < v_1 + \frac{(v_1 - \underline{b})F(\underline{b})^{n-1}}{(n-2)}$, or, rearranging, when $v_1 > \widehat{v}_1$. ■

When $v_1 > \widehat{v}_1$ the equilibrium is very similar to the one described in Vickrey (1961). In this case $\overline{b}$ is the maximal bid for all players, and to compute it we just need to equate bidder 1's profits when she bids $\underline{b}$ and when she bids $\overline{b}$ (in which case she wins with probability one). When there are more than two bidders we cannot use this method because, unless v_1 is high enough, it will be optimal for the highest bidders to bid more than $\overline{b}$, and therefore, bidder 1 won't win with probability one when she submits her highest bid. Instead, to compute $\overline{b}$, we need to find the points satisfying the conditions stated in lemma 3.

It is clear that bidder 1 maximizes her expected profits by bidding on $[\underline{b}, \overline{b}]$. However, we have constructed the equilibrium bidding strategy of the other bidders by imposing only the first order conditions for each interval of valuations. In the next lemma, we state that this bidding strategy is indeed optimal for them.

Lemma 5 *For every v_i, bidding according to $b(v_i)$ in proposition 6 is a global maximizer* *of bidder i's maximization problem.*

Proof. See Appendix 3.5. ■

Now we can try to compare this auction with a standard one (without reserve price, and in which all valuations are private knowledge) from the point of view of the bidders and the seller.

First, notice that our equilibrium can lead to inefficient outcomes: as long as bidder 1 randomizes her bid, it is not guarantied anymore that the highest-valuation bidder will win the object. Second, while it is obvious that bidder 1´s expected profit is lower in this auction, it is not clear whether the rest of the bidders are worse or better off.[11] Indeed, on the one hand they have an advantage over bidder 1, which should allow them to perform better, but on the other hand, player 1 may bid more aggressively, due to the disadvantage she suffers.[12] Moreover, her bid has a similar effect on her rivals as a random reserve price, which can force them to bid more aggressively than they would in the standard auction. Which of these two effects is stronger is not obvious. Since the bidding is more or less aggressive depending on the valuation of bidder 1, it is not clear either whether the seller's expected revenue in this auction is higher or lower than that in the standard auction.

[11]If bidder 1 bids $\underline{b}$ in the standard auction she wins with a higher probability than in the asymmetric auction, since her rivals never bid up to their valuation. Therefore, the expected profits to this bidder in the standard auction cannot be less than in the asymmetric auction.

[12]Without further conditions on the distribution of the bidders' valuations it is not easy to compare bidder 1's bid in the asymmetric auction with that of the standard auction. However, in the next section we show that if the bidders' valuations are uniformly distributed, bidder 1 submits a higher bid in the asymmetric auction with probability one.

3.3 Comparison with the standard auction

In this section we consider the case of three bidders whose valuations are drawn from the uniform distribution on [0,1]. With this example we illustrate the equilibrium described above, and compare the expected profits of the seller and the buyers in the auction we have analyzed with those in a standard auction. We obtain the following:

$$\begin{aligned}
\underline{b} &= \frac{2v_1}{3} \\
\widehat{v}_1 &= 0.89413 \\
\overline{b} &= 0.88157v_1,\ y(\overline{b}) = 1.1184v_1\ \forall v_1 \leq \widehat{v}_1 \\
\overline{b} &= v_1 - \frac{4v_1^3}{27},\ y(\overline{b}) = 1\ \forall v_1 > \widehat{v}_1
\end{aligned}$$

$$\begin{aligned}
b_s(v_i) &= v_1 - \frac{4v_1^3}{27v_i^2} \\
b_t(v_i) &= \frac{v_i}{2} + \frac{0.3605v_1^2}{v_i}
\end{aligned}$$

From (3.15), using the change of variable $x = b_s(v_i)$, and rearranging we obtain the differential equation:

$$\frac{dH(x)}{dx} = \left[\frac{1}{(\frac{2}{9(v_1-x)}\sqrt{3\,(v_1-x)\,v_1}v_1 - x)} - \frac{1}{2(v_1-x)}\right] H(x) \tag{3.19}$$

An interesting property of the uniform distribution is that the lower bound of bidder 1's random bid, $\underline{b}$, is precisely what bidder 1 would bid in the standard auction, that is $\underline{b} = \frac{n-1}{n}v_1$. This implies that when the valuations are uniformly distributed, bidder 1 will indeed bid more aggressively than she would in a standard auction.

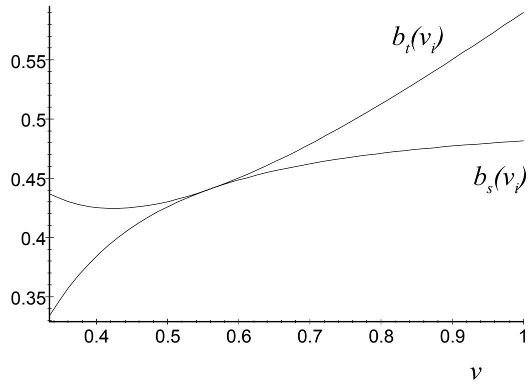

Figure 3.2: $b_s(v_i)$ and $b_t(v_i)$ where $v_1 = 0.5$.

In figure 3.2 we draw $b_s(v_i)$ and $b_t(v_i)$ as defined in the interval $[\underline{b}, 1]$, for the particular case where $v_1 = 0.5$. In that case we have $\overline{b} = 0.44079$ and $y(\overline{b}) = 0.5592$. Note that $b_s(v_i)$ and $b_t(v_i)$ satisfy the tangency condition stated in lemma 3. Figures 3.3 and 3.4 show, respectively, $H(x)$ and $h(x)$ when $v_1 = 0.5$.

Now we can compute the expected profits of the buyers and seller. We do it for the cases $n = 2$ and $n = 3$. We compute the expected profits to the participants in this kind of auction ex-ante, that is, before they learn any valuations, and we compare these results with those of a standard auction, in which all bidders' valuations are private information. The results of these computations are given in the tables below. In columns 2 and 3 we write, respectively, the ex ante expected profits of participating in our asymmetric auction and a standard auction. In the fourth column we write the difference between the second and the third one, that is, the gains to each agent of participating in our asymmetric auction instead of the standard one. In the fifth and sixth rows we write, respectively, the rents that the seller extracts from bidder 1 and from each of the other bidders.

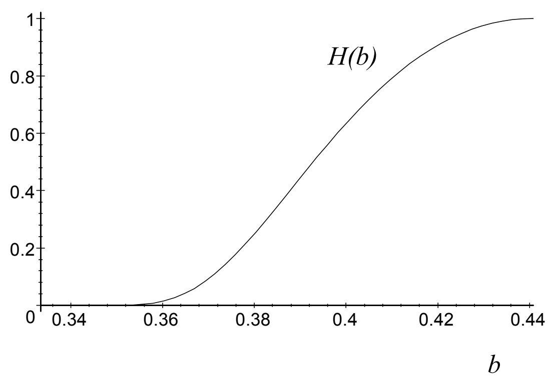

Figure 3.3: $H(b)$ where $v_1 = 0.5$.

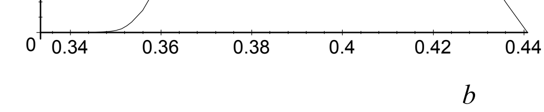

Figure 3.4: $h(b)$ where $v_1 = 0.5$.

Table 3.1: Expected profits in the case where $n = 2$.

Expected profit	Asymmetric Auction	Symmetric A.	Gains
Player 1	0.0833	0.1667	-0.0833
Player 2	0.2007	0.1667	0.034
Seller	0.3694	0.3333	0.0361
Revenue from bidder 1	0.1875	0.1667	0.0208
Revenue from bidder 2	0.1819	0.1667	0.0152
Social Surplus	0.6534	0.6667	-0.0132

Table 3.2: Expected profits in the case where $n = 3$.

Expected profit	Asymmetric Auction	Symmetric A.	Gains
Player 1	0.03704	0.0833	-0.0463
Player $i = 2, 3$	0.0985	0.0833	0.0151
Seller	0.5101	0.5	0.0101
Revenue from bidder 1	0.1426	0.1667	-0.0241
Revenue from bidders $2, 3$	0.1837	0.1667	0.0170
Social Surplus	0.7440	0.75	-0.0060

As we see, with $n = 2$ the player whose valuation has not been revealed benefits from the revelation of information less than the seller does. When $n = 3$ the opposite happens. However, in both cases all the agents but bidder 1 have a higher expected profit in the asymmetric auction than in the standard one. As for the rents that the seller extracts from the agents, we see that, with $n = 2$ both rents are higher in the asymmetric auction than in the standard one, while with $n = 3$ the rents extracted from bidder 1 are lower than those in the standard auction. The loss of efficiency is not very large compared to the seller's increase in expected profits. Hence, an authority interested in both efficiency and maximal revenue may still prefer this kind of mechanism to the standard auction when he knows the valuation of one of the bidders. Comparing our results with those of KZ we observe that, in the particular case of the uniform distribution and two bidders, the seller's expected revenues are slightly higher when the seller reveals the ranking of the valuations than when he announces one of them (0.3696 instead of 0.3694). To finish, it may be interesting to point out that the expected revenue to the seller will be higher in the asymmetric auction (compared to the standard one) if $v_1 > 0.43$ when $n = 2$ and if $v_1 > 0.47$ when $n = 3$.[13]

3.4 Conclusions

When one of the bidders' valuations is common knowledge no equilibrium in pure strategies exists. We have characterized a mixed strategy equilibrium in which the bidder whose valuation is common knowledge randomizes her bid, while the other players play pure strategies.

When the valuation which is revealed is low, the bidders with high valuations compete for the object among themselves rather than with bidder 1, whom they will beat for sure. As we have seen, when a bidder i has the same valuation as bidder 1, he bids in the support of bidder 1's random bid. This implies that in this auction the inefficiencies go in both senses: player 1 can be beaten by a lower rival but can also beat a higher rival.

We have seen that we cannot conclude anything about the seller's revenue: on the one hand, revealing the valuation of a bidder can result in more aggressive bidding from all the players if that valuation is high. On the other hand, when the valuation which is revealed is low, the effect can be reversed. We compute the ex-ante expected profits for the particular case of the uniform distribution and two and three bidders and we observe that in these cases it is in the interest of the seller to reveal one of the bidder's valuation if he has that information. The bidder whose valuation has been revealed is, as expected, worse off, while her rivals are better off.

[13] Approximately. The first value was already given by Vickrey (1961).

Note that if a second-price auction took place under our assumptions, it would still be equivalent to a symmetric second-price auction (given that in both cases truthful revealing is optimal for the bidders). Since first-price and second-price auctions are equivalent in the symmetric case, the comparison of the first-price asymmetric auction with a standard auction applies too to the (asymmetric) second-price auction, i.e. although the effect on the seller's expected revenue is ambiguous, we know that at least in some cases (the uniform distribution with 2 or 3 bidders) the first-price auction yields more revenues than the second-price auction does.

As we said, when the bidder whose valuation is revealed is the winner of a previous auction or the incumbent in a market, it is likely that her valuation is high.[14] In this case, our numerical computation is not useful, since we were assuming ex-ante symmetric distributions of valuations. However, under this assumption, the policy of announcing the incumbent's valuation will indeed increase in the seller's expected profit.

Notice also that the random bid of bidder 1 operates, from the point of view of the other bidders, as a random (or secret) reserve price. Hence, our analysis can help to analyze such scenarios.

[14] If we consider a first-price sequential auction with two objects, in which the bidders' preferences are determined by a single parameter, and we assume that in the second stage of the game they play according to the equilibrium described above, it can be proved that in the first stage no symmetric, pure strategy equilibrium exists. Therefore, the winner of the first auction would not necessarily be the highest valuation bidder. However we can still assume that the incumbent had not been chosen using an auction, or that the distribution of valuations (or costs) changes through time.

3.5 Appendix

Proof of Lemma 1

Notice that in this lemma we are not discussing the optimality (from the point of view of bidder 1) of bidding $\underline{b}$. What we do is to rule out any equilibrium where the lower bound of her random bid is other than $\underline{b}$, where

$$\underline{b} = ArgMax_b \, F(b)^{n-1}(v_1 - b), \tag{3.20}$$

whose FOC yields (3.6). We show it by contradiction.

Suppose that an equilibrium exists where the lower bound of bidder 1's random bid is $b^* \neq \underline{b}$. The optimal response for a bidder i with valuation $v_i > b^*$ implies bidding at least b^*, since bidding less yields a zero expected profit. Therefore, bidder 1 will win the auction if and only if all her rivals have valuations less than or equal to b^*. Her expected profit is $F(b^*)^{n-1}(v_1 - b^*)$.

In undominated strategies bidders $2, ..., n$ do not bid above their valuation, so $b(v_i) \leq v_i$. By bidding $\underline{b}$, bidder 1 obtains an expected profit of $F(b^{-1}(\underline{b}))^{n-1}(v_1 - \underline{b}) \geq F(\underline{b})^{n-1}(v_1 - \underline{b}) > F(b^*)^{n-1}(v_1 - b^*)$. Hence, bidding b^* cannot be optimal for bidder 1. ■

Proof of Lemma 3

i) Suppose that $b_t(y(\overline{b})) > b_s(y(\overline{b}))$, so that there is a gap in bidder i's bidding function. Then, the first (lowest) bidder that bids according to b_t can increase his expected profit by undercutting his bid: he wins with the same probability and pays less. Suppose now that $b_t(y(\overline{b})) < b_s(y(\overline{b}))$. Then $\exists \; v_i', v_i''$ s.t. $v_i' < y(\overline{b}) < v_i''$ and $b_s(v_i') = b' > b'' = b_t(v_i'')$. We show that bidder i's bidding function has to be monotone, so that $b_t(y(\overline{b})) < b_s(y(\overline{b}))$ is not possible. Denote by $G(b)$ the probability that b is the highest bid in the auction. If b' is the optimal bid of a bidder type v_i' and b'' is the optimal bids of a bidder type v_i'', it must be true that $G(b')(v_i' - b') \geq G(b'')(v_i' - b'')$ and $G(b'')(v_i'' - b'') \geq G(b')(v_i'' - b')$. Therefore $\frac{G(b')(v_i'-b')}{G(b')(v_i''-b')} \geq \frac{G(b'')(v_i'-b'')}{G(b'')(v_i''-b'')}$. Simplifying and rearranging we obtain $(v_i' - v_i'')b' \geq (v_i' - v_i'')b''$. Since $v_i' < v_i''$ this implies $b'' \geq b'$. Contradiction.

The second equality, $b_s(y(\overline{b})) = \overline{b}$, comes from the definition of $\overline{b}$.

ii) Suppose that $b_s'(y(\overline{b})) > b_t'(y(\overline{b}))$. Since $b_s(y(\overline{b})) = b_t(y(\overline{b}))$ and both functions are continuous, we must have $b_s(x) > b_t(x)$ in a neighborhood at the right of $y(\overline{b})$. Suppose that bidder 1 bids $b' > \overline{b}$. Her expected profits are

$$[P(b' > b_t(v_i))]^{n-1}(v_1 - b') = F[b_t^{-1}(b')]^{n-1}(v_1 - b')$$

which are higher than $F[b_s^{-1}(b')]^{n-1}(v_1 - b')$, her expected profits of submitting any bid $b \in [\underline{b}, \overline{b}]$. Therefore, it would be in the interest of player 1 to deviate bidding above $\overline{b}$. Remember that b_s was defined as the way bidders 2 to n should bid to make bidder 1 indifferent between all her possible bids. As long as these bidders bid below b_s when their valuations are above $y(\overline{b})$ bidder 1 will be strictly better off bidding above $\overline{b}$.

Take now the functions b_s and b_t in all the interval $[\underline{b}, 1]$. Suppose that $b_s'(y(\overline{b})) < b_t'(y(\overline{b}))$. Deriving the first order conditions to (3.13) and (3.16) we will show that $b_s'(y(\overline{b})) < b_t'(y(\overline{b}))$ implies $h(y(\overline{b})) < 0$, which is impossible, since h is a density function. First, from (3.13) we have that b_t must satisfy

$$b_t'(v_i)F(v_i) = (n-2)f(v_i)(v_i - b_t(v_i))$$

From the first order condition of the maximization problem (3.16) we have:

$$b_s'(v_i)F(v_i) = \frac{(n-2)f(v_i)H(b_s(v_i))(v_i - b_s(v_i))}{H(b_s(v_i)) - h(b_s(v_i))(v_i - b_s(v_i))}$$

Now, evaluating both equations at $y(\overline{b})$ and taking into account that $b_s(y(\overline{b})) = b_t(y(\overline{b}))$ we have that $b_s'(y(\overline{b})) < b_t'(y(\overline{b}))$ implies

$$\frac{(n-2)f(y(\overline{b}))H(b_s[y(\overline{b})])}{H(b_s[y(\overline{b})]) - h(b_s[y(\overline{b})])\left[y(\overline{b}) - b_s[y(\overline{b})]\right]} < (n-2)f(y(\overline{b}))$$

simplifying and rearranging this is equivalent to:

$$h(b_s[y(\overline{b})])) \left[y(\overline{b}) - b_s[y(\overline{b})]\right] < 0$$

which cannot hold unless $h(b_s[y(\overline{b})]) < 0$, since $b_s(x) < x \ \forall x$. ■

Proof of Lemma 5

We have described the optimal strategy of bidders $i = 2, .., n$ but we have not proved that our solution is a global maximum, and we have not checked the second order conditions. We proceed by regions:

1) Bidders with valuation $v_i \leq \underline{b}$: A bidder has probability zero of winning and hence, an expected profit of zero. Bidding above $\underline{b}$ implies negative expected profit, since there is a positive probability of winning paying a price higher than one's own valuation. Thus, there are no incentives to bid above $\underline{b}$, while a player is indifferent bidding any quantity below it.

2) Bidders with valuation $v_i \in (\underline{b}, y(\overline{b})]$: Bidding below $\underline{b}$ implies zero expected profit, while bidder i can do positive profits by bidding according to $b_s(v_i)$, thus, there are not incentives to deviate bidding in the first region. We also need to show that bidding

according to $b_t(z)$ with $z > y(\overline{b})$ is not optimal for bidders in this region. To see that, it is enough to show that $\frac{dE(\pi_{v_i}(z))}{dz} < 0 \; \forall z > y(\overline{b})$, that is, the expected profit of a player type $v_i < y(\overline{b})$ who acts as if she were of type $z \geq y(\overline{b})$ decreases with z. We have $E(\pi_{v_i}(z)|z > y(\overline{b})) = F(z)^{n-2}(v_i - b_t(z))$. Differentiating w.r.t. z, substituting $b_t(z)$ by its value in (3.14), and $b_t'(z)$ by its derivative, and simplifying we obtain $E'(\pi(z)) = v_i - z$, which is negative $\forall z > y(\overline{b}) > v_i$. If v_1 is high enough so that bidders i to n bid only in two regions, it is easy to see that bidding above $\overline{b}$ cannot be optimal: by bidding $\overline{b}$ a player obtains the good with probability one, while bidding above only implies paying a higher price for it.

On the other hand it is necessary to check that bidding according to $b_s(v_i)$ is indeed the best strategy for players with valuation $v_i \in [\underline{b}, y(\overline{b})]$, that is, that $b_s(v_i)$ is a maximum of the objective function and not a minimum.

From the FOC of problem (3.16) we have

$$\frac{dE(\pi_{v_i}(z))}{dz} = -b_s'(z)H(b_s(z))F(z)^{n-2} +$$

$$+ \left[h(b_s(z))b_s'(z)F(z)^{n-2} + (n-2)F(z)^{n-3}f(z)H(b_s(z))\right](v_i - b_s(z)) \qquad (3.21)$$

which is zero evaluated at $z = v_i \; \forall v_i \in [\underline{b}, y(\overline{b})]$. Now, suppose a bidder type v_i behaves as if he were of type $z < v_i$. If v_i in the equation above was equal to z, we would have that expression equal to zero. But since v_i is greater than z the term $(v_i - b_s(z))$ is higher, and since both that term and the parenthesis which it multiplies are positive, the expression above must be positive, which implies that the derivative of the expected profit is positive, that is, the expected profit increases with z and it is in the interest of such a buyer to increase her bid as long as $z < v_i$. Conversely, if she acts as if she were of type $z > v_i$, the term $(v_i - b_s(z))$ decreases and so does the second term in (3.21), which implies that the derivative is negative, and therefore it is in the bidder's interest to decrease her bid. Hence, bidding $b_s(v_i)$ is optimal for a bidder of type $v_i \in [\underline{b}, y(\overline{b})]$.

3) Bidders with valuation $v_i > y(\overline{b})$: Again, bidding under $\underline{b}$ implies zero expected profit, while bidding above bidder i can obtain a positive profit. Second, we need to show that bidding in the interval $[\underline{b}, \overline{b}]$ is not optimal for bidders in this region. As before, introducing in (3.21) the valuation of a bidder in this region we have that for all $z \in [\underline{b}, y(\overline{b})]$ the derivative of the expected profit w.r.t. z is positive, that is, it is in the interest of player i to increase his bid. Finally, we can use the same reasoning to show that our $b_t(v_i)$ is a maximizer of the problem (3.13) and not a minimum: differentiating the expression in (3.13) we get the derivative of the expected profit w.r.t. v_i. If i behaves as if he were type $z < v_i$ this derivative is positive, and if he behaves as type $z > v_i$ it is negative. Indeed, our $b_t(v_i)$ is a standard bidding function given the initial condition $b_t(y(\overline{b})) = \overline{b}$. ■

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Chapter 4

Sequential First-Price Auctions with Multi-Unit Demand

4.1 Introduction

In two pioneering papers on sequential auctions Weber (1983) and Milgrom and Weber (1982) analyzed unit-demand multi-unit auctions of identical objects and showed that, under the independent, private valuation assumption and under risk neutrality, the path of expected prices in first-price and second-price sequential auctions follows a martingale, and that both auctions are efficient and revenue equivalent.[1] However, many works have reported that, in reality, the observed trend of prices is usually decreasing (for example Ashenfelter (1989) and McAfee and Vincent (1993) in wine auctions or Ashenfelter and

[1]A key assumption for this result is that the bidders have a positive valuation for only one unit of the goods. The private and independent valuations assumption is also crucial: Milgrom and Weber (1982) also analyze the case of affiliated valuations, and show that at equilibrium the path of expected prices is increasing, due to the release of information throughout the auction which softens the "winner's course".

Genesove (1992) in real state auctions). Over the last years, several explanations for the decrease in prices in sequential auctions have been provided, along with analyses addressing other important issues, like efficiency or revenue comparisons.[2]

Releasing the unit-demand assumption, Black and de Meza (1992) analyze a two-object second-price sequential auction, and show that the existence of a buyer's option which allows the winner of the first auction to acquire the second object for the same price he has paid for the first one, may lead to a decrease of prices.

In this paper we use the model of Black and de Meza to analyze the sequential first-price auction, and to study the impact of a buyer's option. We do not know any theoretical analysis of first-price auctions with a buyer's option although, in reality, the buyer's option is present in many oral descending-bid auctions. For example, the author was present in oral descending fish auctions where the buyer's option was often exercised, while Cassady (1967, pp. 194), and van den Berg, van Ours, and Pradhan (2001), among others, also report the existence of a buyer's auction in Dutch auctions. Note that in sequential auctions the equivalence between first-price auctions and descending (or Dutch) auctions holds generally (since the information released in both formats is identical). In this paper we refer to both of them indistinctly.[3]

In the rest of this section we review the papers studying sequential auctions where bidders have a positive valuation for more than one unit, i.e., where the unit-demand

[2]For example, under the unit-demand assumption, risk aversion (McAfee and Vincent,1993), participation costs (von der Fehr, 1994), heterogeneity of the goods (Gale and Hausch, 1994, Bernhardt and Scoones, 1994, Engelbrecht-Wiggans, 1994, Beggs and Grady, 1997), supply uncertainty (Jeitschko, 1999), and delegated, non-strategical bids (Ginsburgh, 1998).

[3]In contrast, the equivalence between second-price auctions and English or Japanese auctions does not usually hold (unless there are only two stages) since the losers reveal their valuations for the next rounds, which can be strategically used by their rivals.

assumption is released.

Black and de Meza (1992) study a second-price sequential auction where $n \geq 2$ bidders with downward-sloping demand-curves compete for two objects. In this auction, contrary to the empirical results, the path of expected prices is increasing. In the first auction, the optimal bid is the one making a player indifferent between winning or losing in case of tie. But the probability of tying is zero, and since the winner only pays the second highest bid (lower than his own) the price in the first auction is lower (in expectation) than the second auction price. The allocation generated by the unique pure strategy equilibrium is efficient.[4] Then it is analyzed the case where a buyer's option is introduced. Under this assumption, the path of prices can be decreasing: with a buyer's option, bidding more aggressively in the first auction not only increases the probability of winning, but also makes the exercise of the option more costly for the rival when the own bid fixes the price that the winner pays. Hence, raising one's bid can have the effect of blocking the exercise of the option by a rival bidder. Moreover, the option will only be exercised when the expected price of the second auction is above the first auction's price. Therefore, the buyer's option not only increases the first auction price, but also depresses the second auction's, reversing the price tendency of the sequential auction without a buyer's option. Another important finding is that the buyer's option can increase the seller's revenues even though it produces inefficiencies.

Burguet and Sákovics (1997) explain the declining price anomaly by supply uncertainty: if the bidders are not sure whether a second auction will be held, they will bid

[4] In one of the scenarios analyzed the equilibrium is not unique if $n = 2$, but the other pure strategy equilibrium has similar characteristic: the outcome is also efficient (and therefore revenue equivalent), and induces increasing prices.

more aggressively in the first one.[5] Supply uncertainty may be motivated by the existence of a buyer's option. Hence, they analyze a second-price auction where one of the bidders has (the same) positive valuation for the two units on sale, while the rest of the bidders have a unit demand. After the first auction, the second object is offered to the winner of the first one at the same price. If the bidder with positive valuation for two units wins the first auction, he will always exercise the option: in the second auction, the optimal bidding strategy it to bid one own's valuation, and the first auction bid will never be greater than that. When the winner of the first auction is a unit-demand one, a second auction will take place, and in this case the expected prices are decreasing. Therefore, it is shown that the "blocking component" of the aggressive bidding described in Black and de Meza is not necessary to obtain a decreasing path of prices. The existence of a two-unit demand bidder, together with the buyer's option, makes all bidders more aggressive in the first auction (as they are when there is uncertainty about a second auction taking place).

Katzman (1999) analyzes the sequential second-price auction (without a buyer's option), using a model very similar to that of Black and the Meza. Then he compares his results (identical to those of Black and de Meza) with those of the complete information case, which he also analyzes. Assuming there are only two bidders, he solves for the unique subgame perfect equilibrium for each of four cases (depending on the ranking of the bidders' realized valuations). The expected prices are either constant or decreasing, in contrast with the incomplete information case, where the path of prices is increasing in expectation. This apparent inconsistency is explained by the asymmetries induced by the

[5]This argument works under the unit-demand assumption, and has been independently reported by other authors (Pezanis-Christou, 1997, Jeitschko, 1999, Kittsteiner, Nikutta, and Winter, 2002).

complete information case: it is precisely in the most asymmetric cases (when one of the bidders has the two highest valuations) that the expected path of prices is decreasing and the allocation is inefficient (otherwise, the expected prices are constant and the allocation is efficient). Katzman analyzes then the sequential auction in an asymmetric framework (i.e. asymmetric distributions of the valuations) and gives an example of a class of distribution functions for which an asymmetric pure strategy equilibrium exists with the same characteristics of those of the (most) asymmetric complete information case. In other words, when the ex-ante asymmetries are strong enough, the induced (asymmetric) pure strategy equilibrium "converges" to the complete information equilibrium, and the path of expected prices is also decreasing. Hence, asymmetries in bidders' beliefs may explain the decreasing price anomaly.

Menezes and Monteiro (1997) analyze a sequential auction with stochastically independent objects (i.e. each bidder draws a new valuation for each object), and where bidders have a positive valuation for more than one unit. Before each auction starts each bidder learns his valuation for the object to be auctioned and decides if he participates in the auction. If he does, he incurs in a participation cost. This cost does not affect his bidding strategy in the auction, since once the decision of entering the auction has been made, it becomes a sunk cost (in fact, there is an equilibrium in which the bidders bid their true valuation in each auction). When a bidder decides not to enter an auction, he is not allowed to bid in subsequent auctions. Since the number of participants can only decrease, so do the prices in expectation.[6]

Branco (1997) gives an example where synergies between the goods generate de-

[6]The same logic works under the unit-demand assumption, as von der Fehr (1994) shows.

creasing prices. The reason is that in the first auction the potential buyers bid aggressively to take into account the synergies of obtaining both objects. In the second auction only the winner of the first auction does so.

Menezes and Monteiro (1998) try to unify Branco and Black and de Meza's results. In a model allowing for both positive and negative synergies in the bidder's valuations, they find that the expected prices are increasing with negative synergies and decreasing with positive ones. The introduction of a buyer's option has no effect in the case of positive synergies because it is never exercised in equilibrium. Unfortunately, in the case of negative synergies, they cannot go further than Black and de Meza did.[7] Menezes and Monteiro also give an example of a two-bidder model where one of the bidders has positive synergies and the other one negative synergies. The expected prices, in this case, decline. Therefore, as in Katzman (1999), asymmetries may lead to a price decline.

There are fewer works on first-price, sequential auctions. In a pioneering work, Ortega-Reichert (1968) studies a two-bidder, two-object, first-price sequential auction. Each bidder has a positive valuation for each of the goods (which is only revealed to him before each auction). All the valuations are independent draws from a common distribution function with an unknown parameter, w. After the first auction, the winner and the two bids submitted are announced. Each time a bidder learns a valuation (his own or his rival's) he updates his beliefs about the unknown parameter w. It is shown that submitting a high bid in the first auction induces the rival to bid more aggressively in the second auction, since he infers he will face a stronger rival in it. As a result, in equilibrium, both bidders submit

[7]Specifically, the problem is that it is not possible to characterize a symmetric equilibrium bidding function. However, they are able to show, as Black and de Meza did, that if a symmetric equilibrium exists the path of prices may be either increasing or decreasing.

a lower bid in the first auction than they would if they were playing a one-shot game.[8]

Robert (1996) examines a k-object, n-bidder, first-price sequential auction with multi-unit demand. Each bidder draws m positive valuations for the goods from a common distribution function, where m follows a Poisson process and is particular to each bidder and private knowledge to him. This treatment implies symmetry in each stage of the game. Robert characterizes a symmetric equilibrium where the bids in each stage depend only on the bidders' highest valuations at that stage, and finds that the path of expected prices follows a submartingale. Moreover, he analyzes the second-price sequential auction in the same framework and obtains similar results. Both auctions are efficient and revenue equivalent.[9]

Jeitschko and Wolfstetter (1998) examine a two-bidder, two-object auction with discrete valuations. Under economies (diseconomies) of scale, the winner of the first auction will have a higher (lower) probability of drawing a high valuation for the second object. Comparing first-price and second-price sequential auctions, they conclude that in the second-price auction the expected prices are increasing when synergies are negative and decreasing when they are positive (similar conclusions to those of Branco, 1997, and Black and de Meza, 1992). In a first-price auction the same happens under positive synergies. However, the expected path of prices for the first-price auction with negative synergies depends on the parameters and, for some values, it is decreasing. This last result is in conflict with Robert's, since in his (continuous-valuation) model, which assumed decreasing

[8]Note that the modeling in Ortega-Reichert implies affiliated valuations. However, as we will see, his results are related to ours in one of the cases we analyze.

[9]The version I know of Robert's paper contains few intuitions for the results and is clearly unfinished. However, the same theoretical model can be found in joint papers with other authors (see Donald, Paarsch, and Robert, 2002, and Robert and Montmarquette, 1999).

marginal valuations, the path of prices was increasing in expectation.

With the exception of Robert (1996), and Menezes and Monteiro (1998), whose models preserve symmetry in each stage of the game, all above works (which assume non-unit demand) analyze two-object, second-price auctions.[10] Given that in the second stage truth-revealing is a dominant strategy, problems associated to asymmetries in the second stage are avoided. Another common feature is that none of the papers considering a buyer's option prove the existence of equilibrium under that assumption.[11]

In first-price auctions the information revealed in the first stage of the game can affect the beliefs of the bidders in the second auction and change their strategic behavior. These asymmetries complicate the analysis. We compare the results of our analysis of the sequential first-price auction with those that Black and de Meza obtained for second-price auctions. As we will see, these auctions are equivalent only in one of the scenarios we analyze.

4.2 The model

There are $n \geq 3$ buyers with positive valuations for two units of a good, and two identical units to be sold in two sequential first-price auctions. The valuation for the first unit is an independent draw from a distribution function F with support [0,1] which is identical for each buyer $i = 1, 2, ...n$. We make two different assumptions about the valuation

[10] We refer here to the papers considering multi-unit demands. In case the bidders only want/can acquire one object, there are numerous studies considering k (≥ 2) sequential auctions. See for example Weber (1983), Milgrom and Weber (1982), and Engelbrecht-Wiggans (1994).

[11] Black and de Meza and Burguet and Sákovics give the conditions that a symmetric, monotone bidding function should satisfy in equilibrium, and also give examples with the uniform distribution where such equilibrium does exist.

for the second unit: 1) The valuation for the second unit is equal to α times the valuation for the first one, where $\alpha < 1$. Hence, player i has valuation v_i for the first unit and αv_i for the second one. 2) The second valuation is a draw of the same distribution function, conditional on being below the valuation for the first unit. Therefore, v_{2i} follows a distribution $G_{v_{1i}}$ on $[0, v_{1i}]$ with $G_{v_{1i}}(x) = \frac{F(x)}{F(v_{1i})}$, $g_{v_{1i}}(x) = \frac{f(x)}{F(v_{1i})}$.[12]

We consider four scenarios: for each of the two assumptions, we try to analyze the sequential auction with and without the buyer's option. In the case where the valuation for the second unit is stochastic we assume that a bidder only learns it after the first auction has taken place. This implies symmetry in the second auction, which simplifies the analysis very much.[13]

4.2.1 Unidimensional preferences

In this section we analyze the case in which the bidders' valuations are determined by a single parameter. As described above, we assume that, for each bidder, the valuation for the second item is α times the valuation for the first one. After submission of the bids, the highest bid is announced and the object is allocated to the bidder who has made it, while the other bids are not revealed. At the moment, we don't introduce a buyer's option.

Note that if a monotone pure strategy equilibrium exists in the first stage of our game, the highest bidder's valuation will be revealed in the first auction. Hence, in the second auction the losers of the first one know with certainty the valuation of the winner (α

[12]These conditional distribution and density functions are a particular case of those in Black and de Meza's model.

[13]If we allow for a more general framework, like for example, that the bidders know their valuations for the two objects before the first auction takes place, we have asymmetries in the second auction, with one bidder having a different conditional distribution of valuation than the others, a fact that complicates the analysis very much. See Maskin and Riley (2000).

times the inverse of his first auction bid). In the previous chapter we analyzed this game, that is, an auction in which one of the bidders valuations has been revealed. We showed that no equilibrium in pure strategies exists and we characterized an equilibrium in which the bidder whose valuation has been revealed randomizes his bid while the other bidders play a pure strategy. We use this result to analyze the first stage of our game, and we show that no equilibrium in pure strategies exists in the first stage of the game either.

(Non existence of) a pure strategy equilibrium

In this section we don't make assumptions on how the equilibrium in the second auction looks like, but we do use the fact that if v, the valuation of one of the bidders, is common knowledge, he must play a mixed strategy and that the lower bound of his random bid, which we denote by $\underline{b}_v$, must satisfy condition (3.6) of the previous chapter. We denote by $\overline{b}_v$ the upper bound of the random bid submitted by a bidder with type v whose valuation is common knowledge.

Lemma 6: *Suppose that bidder* 1 *with valuation* v_1 *has induced wrong beliefs in his rivals, who think his valuation is* z*. The bidders compete for an object in a first-price auction. Then, if* $v_1 < z$ *bidder* 1*'s expected profit is not lower than it would be if his rivals knew his valuation. If* $v_1 > z$ *bidder* 1*'s optimal strategy is to bid some quantity* $b^* \geq \overline{b}_z$*, and his expected profit is strictly greater than if his rivals knew his actual valuation.*[14]

Proof. See Appendix 4.4. ■

Lemma 6 implies that the beliefs induced by a "false" bid in the first auction do

[14]The optimal bidding strategy in the second auction is similar to that in Ortega-Reichert in that downwards deviating in the first auction makes optimal a higher bid in the second auction and increases the expected profit in it.

not hurt the deviant's expected profits in the second auction. Moreover, the optimal bid and his expected profit in the second auction depend on the direction in which he deviates in the first auction. Given this result, it follows:

Proposition 7 *No symmetric monotone pure strategy equilibrium exists in the first auction.*

Proof. See Appendix 4.4. ■

The intuition for the non existence of equilibrium is the following: a bidder in the first auction must consider the effect of his bid in the second auction. If he wins the first auction, the higher he has bid, the more aggressive his rivals will bid in the second auction. Hence, his optimal bid in the first auction should be lower than the optimal bid if there were a single auction. Suppose instead that he bids as if there were no second auction. What does he lose? Nothing, since in the second auction he can always bid according to $\underline{b}(\alpha v_i)$ and obtain the same revenue that he would expect if he had not deviated. But bidding as if there were not a second auction is not an equilibrium either, because then the bidders have incentives to slightly undercut their bid in the first auction in order to induce wrong beliefs of their rivals and benefit from it in the second auction.

The buyer's option

Suppose now that the seller introduces a buyer's option which allows the winner of the first auction to buy the second object for the price he has paid for the first one.

Proposition 8 *A cut-off point α^* exists such that $\forall \alpha > \alpha^*$ a pure strategy equilibrium exists in which the bidders bid according to the bidding function* $B(v_i) = \frac{(1+\alpha)}{2} \frac{\int_0^{v_i} x(n-1)F(x)^{n-2}f(x)dx}{F(v_i)^{n-1}}$, *and the winner of the first auction always exercises the buyer's option. For low values of α such an equilibrium does not exist.*

Proof. See Appendix 4.4. ■

Note that when an equilibrium exists in which the buyer's option is always exercised, the first-price auction is equivalent to a one-shot auction where a bundle with the two objects is sold. It is well known that, although it leads to inefficient allocations, bundling can increase the seller's revenues when competition is weak.[15] Suppose that the objects are sold together in a one-shot auction. When α is small, the bidders are forced to buy the two objects to be able to obtain a positive profit, even if they attach little value to the second unit. This may lead them to bid more aggressively than they would in a sequential auction (where each bidder would usually acquire a single unit).[16] As α increases, this effect softens, and the probability of an inefficient allocation diminishes. On the other hand, as n increases, the competition for each unit increases, and the advantage of bundling fades out.[17] Therefore, when n is low enough and the above equilibrium exists, introducing a buyer's option can be a good strategy for the seller.[18]

Black and de Meza compute the seller's expected revenue in the sequential second-price auction when the bidders' valuations are uniformly distributed in $[0, 1]$, and $\alpha = 0.5$. They consider the cases of $n = 2$ and $n = 3$. In their examples the seller's expected profit

[15]See Palfrey (1983) and Chakraborty (1999).

[16]To give a simple example, suppose that in the model above we have $n = 2$ and α is zero. In a sequential auction each bidder would bid zero in each auction. By bundling, the seller's expected revenue is the second order statistic of two draws of F.

[17]Back to our example, if $\alpha = 0$ the expected profit of the seller will be the second order statistic of n draws with a buyer's option and two times the third order statistic without the buyer's option. It is obvious than as n increases, the auction without the option (efficient) will yield a higher expected revenue than the auction of the bundle does.

[18]It is precisely when α is high and n low when it is more likely that the pure strategy equilibrium described above exists.

when the buyer's option is present are higher than when it is not. Therefore, it would be interesting to compare the seller's expected revenue in Black and de Meza's example with those in a first-price auction with the buyer's option. Unfortunately, for the case if $\alpha = 0.5$ no pure strategy equilibrium exists in the sequential first-price auction. Moreover, Black and de Meza don't prove the existence of an equilibrium in general, neither provide an initial condition for the differential equation defining the equilibrium bidding functions (except for the case $\alpha = 0.5$). For this reason we are not able to compute the seller's expected revenue in their auction for higher values of α. Hence, although we know that the seller's expected revenue may be higher with the buyer's option than without it in both auction formats, we are not able to compare first-price with second-price sequential auction in these terms.[19]

4.2.2 Stochastic valuation for the second unit

We analyze now the case in which the valuation for the second unit is stochastic. As we said, we assume that v_{1i}, the valuation for the first unit, is a random variable with distribution F on $[0,1]$. After the first auction has taken place, the winner draws a new valuation for the second unit, v_{2i}, which follows a distribution $G_{v_{1i}}$ on $[0, v_{1i}]$ with $G_{v_{1i}}(x) = \frac{F(x)}{F(v_{1i})}$, $g_{v_{1i}}(x) = \frac{f(x)}{F(v_{1i})}$. These assumptions imply symmetry in both stages of the game: in the second auction all bidders have a valuation distributed according to F, conditional on being below the valuation of the winner of the first auction. Therefore, the bidding strategy in the second auction is the standard one:

$b(v_i) = v_i - \frac{\int_0^{v_i} F(x)^{n-1}dx}{F(v_i)^{n-1}}$.

[19] Note that the esller's expected revenue of bundling with $n = 2$ and $\alpha = 0.5$ is the second order statistic of 2 draws of F times $(1+\alpha)$, i.e., equal to $\frac{1}{3}(1+0.5) = 0.5$. The expected profit of the second-price auction with the buyer's option computed by Black and de Meza is 0.474. Therefore, in this case, bundling yields a higher expected profit to the seller.

where v_i is the bidder's valuation for the object (in case of the winner of the first auction, it is his valuation for the second unit).[20] Assuming that a symmetric pure strategy equilibrium with increasing bidding functions exists, we can write the total expected profit of the game for a bidder type v_{1i} who pretends to be type z as:

$$E(\pi|z) = F(z)^{n-1}(v_{1i} - b(z)) + \frac{1}{F(v_{1i})}\int_0^{v_{1i}}\int_0^x F(y)^{n-1}dyf(x)dx - (n-1)\ln F(z)\int_0^{v_{1i}} F(x)^{n-1}dx \tag{4.1}$$

The second term in the RHS above is the probability of winning the first object times the expected profit in the second auction conditional on having won the first one: taking into account the optimal bidding strategy in the second auction, we have:

$$F(z)^{n-1}E(\pi_2|win) = F(z)^{n-1}\int_0^{v_{1i}} \frac{F(x)^{n-1}}{F(z)^{n-1}}\left[x - (x - \frac{\int_0^x F(z)^{n-1}dz}{F(x)^{n-1}})\right] g_{v_{1i}}(x)dx$$

which, given that $g_{v_{1i}}(x) = \frac{f(x)}{F(v_{1i})}$, simplifies to the above expression. The last term in the RHS of equation (4.1) is the probability that a rival bidder wins the first auction, times the expected profit of participating in the second auction conditional on having lost the first one. Denoting by j the winner of the first auction, this is:

$\int_z^1 (n-1)f(v_{1j})F(v_{1i})^{n-2}\frac{F(v_{1i})}{F(v_{1j})}(v_{1i} - b(v_{1i}))dv_{1j}$.

Substituting $b(v_{1i})$ by its value and computing the integral, we obtain the term above.

As usual, in equilibrium the derivative of the profit function w.r.t. z evaluated at v_{1i} has to be zero. Therefore

$(n-1)F(v_{1i})^{n-2}f(v_{1i})(v_{1i} - b(v_{1i})) - b'(v_{1i})F(v_{1i})^{n-1} - (n-1)\int_0^{v_{1i}} F(x)^{n-1}dx\frac{f(v_{1i})}{F(v_{1i})} = 0$.

Integrating, we obtain the optimal bidding function:

$$b(v_{1i}) = v_{1i} - \frac{\int_0^{v_{1i}} F(x)^{n-1}dx}{F(v_{1i})^{n-1}} - (n-1)\int_0^{v_{1i}} \frac{f(y)}{F(y)}\frac{\int_0^y F(x)^{n-1}dx}{F(v_{1i})^{n-1}}dy \tag{4.2}$$

[20] See also note (9) in chapter 2.

which is the standard bid in a one-object first-price auction minus a term that takes into account that by losing the first auction the bidder can still make a profit by participating in the second auction.

Proposition 9 *The outcome of the sequential first-price auction is efficient and revenue equivalent to the sequential second-price auction. The path of expected prices is increasing. Moreover, the expected price in each stage of the auction are the same in both formats.*

Proof. Differentiating equation (4.2) it is easy to see that the equilibrium bidding function in the first stage is strictly increasing. Since the bid in the second stage is also increasing, and since the auction is symmetric in each stage, the outcome is efficient. Therefore revenue equivalence holds. Moreover, since the same bidder wins the first object, the second auction is identical through auction formats: in the case of the first-price auction the game is symmetric and the bidding functions are the usual ones, while in the second-price auction truthful-revealing is, as usual, a dominant strategy. Therefore, the expected prices in the second stage of the game must be the same in both auction formats. It follows that the expected prices must also be the same in the first stage. Hence, as in the sequential auction, the path of prices must be increasing in expectation. ■

Note that of the four scenarios we analyze, this is the only one in which the first-price and second-price auctions are equivalent.

The buyer's option

To analyze the impact of a buyer's option in this model is very complex, due to the asymmetries that arise in the second stage of the game. To illustrate the difficulties we consider the case of only two bidders whose valuations are uniformly distributed. We use the results of Griesmer, Levitan, and Shubik (1967) in order to analyze the second auction when the winner of the first one does not exercise the buyer's option. First, we adapt Griesmer et al. results for an asymmetric one-object auction. Suppose bidder 1 has valuation $v_1 \in [0, l]$ and bidder 2 valuation $v_2 \in [0, h]$. The optimal bidding strategies are[21]

$$b_1 = \frac{lh}{v_1\left(h^2 - l^2\right)}\left(lh - \sqrt{\left(l^2h^2 - v_1^2h^2 + v_1^2l^2\right)}\right) \tag{4.3}$$

$$b_2 = \frac{lh}{v_2\left(l^2 - h^2\right)}\left(lh - \sqrt{\left(l^2h^2 - v_2^2l^2 + v_2^2h^2\right)}\right) \tag{4.4}$$

The expected profit to bidder 1 with valuation v_1 is

$$E(\pi\,|v_1) = P(b(v_2) < b(v_1))(v_1 - b(v_1)) = \frac{lh - \sqrt{\left(l^2h^2 - v_1^2h^2 + v_1^2l^2\right)}}{h^2 - l^2} \tag{4.5}$$

and, ex ante, the expected profit of participating in the auction is

$$E(\pi_1) = \int_0^l \frac{lh - \sqrt{\left(l^2h^2 - v_1^2h^2 + v_1^2l^2\right)}}{h^2 - l^2} dv_1$$

Now suppose there are two bidders who compete for two objects which are to be auctioned sequentially. As before, each bidder has a positive valuation for two units, but he only learns the valuation for the second one after the first auction. The valuation for the

[21] We require $h \neq l$. Otherwise we are in the symmetric standard case.

second unit for bidder i follows a uniform distribution on $[0, v_{1i}]$, where the notation is the same as in the previous sections.

The first auction is symmetric, and each potential buyer submits a bid which depends only on his valuation for the first object (the only one he knows). After the first auction, the highest bid is announced and the winner gets the first object. Then he learns his second valuation and decides whether to exercise the buyer's option or not. It is obvious that if his valuation for the second unit is lower than his previous bid, he will not. On the other hand, if the option is never exercised, the auction is equivalent to the one examined in the previous section, which had increasing expected prices. But if the expected price in the second auction is higher than that of the first auction, it must be optimal to exercise the option at least in some cases.

Our conjecture is that for each winner bid in the first stage of the auction, there will be a cut-off point l^* such that the winner of the first auction will exercise the option if and only if his valuation for the second unit is more than or equal to l^*.

Assume that a symmetric, monotone equilibrium exists in the first stage (auction) of the game, and denote the equilibrium bidding strategy by $b(.)$. Suppose, without loss of generality, that bidder 1 has the highest valuation for the first unit, and denote his valuation by h. In the first auction he bids $b(h)$ and wins the first object. We denote by $l(h)$ the cut-off point such that he is indifferent whether to exercise the option or not. If bidder 1 does not exercise the buyer's option, bidder 2 will update his beliefs about the distribution function of bidder 1's valuation for the second unit, which will be now uniform on $[0, l(h)]$. Therefore, the second auction, if it takes place, is asymmetric (note that bidder 2's valuation is now distributed on $[0, h]$). Denoting by v_1 bidder 1's valuation for the (second) object, and by v_2, bidder 2's valuation, the resulting auction is the one we described at the beginning of this section, and the bidders' equilibrium bidding strategies are given by equations (4.3) and (4.4).

As we saw, bidder 1's expected profits in that auction were given by (4.5). To find the cut-off point, $l(h)$, bidder 1 must be indifferent whether to exercise the option or not when his valuation is $l(h)$. Therefore, $l(h)$ must satisfy:

$$l(h) - b(h) = \frac{l(h)h - \sqrt{(l^2(h)h^2 - l^2(h)h^2 + l^2(h)l^2(h))}}{h^2 - l^2(h)} = \frac{l(h)}{h + l(h)} \tag{4.6}$$

which implies

$$b(h) = l(h) - \frac{l(h)}{h + l(h)} \tag{4.7}$$

It is difficult to analyze the first stage of the auction maintaining our notation for the bidders' valuations: note that each bidder can have the highest valuation, while we had denoted by h bidder 1's valuation for the first unit. We return to the previous notation, denoting by v_{1i} bidder i's valuation for his first unit. Observe that if a bidder with valuation v_{1i} has won the first auction and he has not exercised the option, the "updated" upper bounds of the bidders' valuations are $h(v_{1i}) = v_{1i}$ in the case of the loser of the first auction, and $l(v_{1i})$ in the case of the winner of the first auction.

If a bidder deviates in the first auction and wins, this affects the upper bounds of the bidders' distributions of valuations in the second auction, l and h.[22] Hence, when

[22]More exactly, it will affect the beliefs that each bidder has about the distribution of the valuation of his rival.

considering the optimal bid, we have to take this into account. Assuming a symmetric monotone pure strategy equilibrium exists, a bidder will maximize:

$$Max_z\ F(z)\left[v_{1i}-b(z)\right]+F(z)\int_{l(z)}^{v_{1i}}(v_{2i}-b(z))f(v_{2i})dv_{2i}+$$
$$F(z)\int_0^{l(z)}\frac{l(z)z-\sqrt{\left(l(z)^2z^2-v_{2i}^2z^2+v_{2i}^2l^2(z)\right)}}{z^2-l(z)^2}f(v_{2i})dv_{2i}+$$
$$\int_z^1\int_0^{l(v_{1j})}\frac{v_{1j}l(v_{1j})-\sqrt{\left(v_{1j}^2l^2(v_{1j})-v_{1i}^2l^2(v_{1j})+v_{1i}^2v_{1j}^2\right)}}{l^2(v_{1j})-v_{1j}^2}f(v_{2j})dv_{2j}f(v_{1j})dv_{1j}$$

where the terms are, respectively[23]: 1) the probability that he wins the first auction times his profit, 2) the probability that he wins the first auction and he exercises the buyer's option times the profit for the second unit, 3) the probability that he wins the first auction but does not exercise the option times the expected profit of participation in the second auction, and 4) the probability that his rivals beats him in the first auction but he does not exercise the option times the expected profit of participating in the second auction. Taking into account that we are assuming that the distribution of valuations is uniform, this is equal to:

$$Max_z\ z(v_{1i}-b(z))+\frac{z}{v_{1i}}\int_{l(z)}^{v_{1i}}(v_{2i}-b(z))dv_{2i}+$$
$$\frac{z}{v_{1i}}\int_0^{l(z)}\frac{l(z)z-\sqrt{\left(l(z)^2z^2-v_{2i}^2z^2+v_{2i}^2l^2(z)\right)}}{z^2-l(z)^2}dv_{2i}+$$
$$\int_z^1 l(v_{1j})\frac{v_{1j}l(v_{1j})-\sqrt{\left(v_{1j}^2l^2(v_{1j})-v_{1i}^2l^2(v_{1j})+v_{1i}^2v_{1j}^2\right)}}{l^2(v_{1j})-v_{1j}^2}\frac{1}{v_{1j}}dv_{1j}$$

In equilibrium, the derivative of the expression above w.r.t. z evaluated at v_i must be equal to zero. In order to compute the optimal bidding strategy in the first auction, we should combine the first order condition of the maximization problem above with equation (4.7).

[23]Note that $f(v_{1i})=1$ and $f(v_{2i})=\frac{1}{F(v_{1i})}$.

Unfortunately, we don't think this problem can be solved analytically. However, it is easy to see that if an equilibrium exists satisfying the above conditions, the path of prices will be decreasing: it is obvious that if the winner of the first auction does not exercise the buyer's option, he will never bid in the second auction more than he has bid in the first one. Since both bidders must bid in the same support, the loser of the first auction will also bid below that point. Therefore, if the second auction takes place, the price will be strictly lower than that of the first auction. Note, however, that this does not necessary apply when there are more than two bidders. In that case, although the winner of the first auction will bid below his previous bid, his rivals could bid above (i.e., above the support of his bid).[24] Also, it may be worth to point out that the first object is efficiently assigned, and that the second object can be inefficiently assigned if: a) the winner of the first auction exercises the option and wins against a higher rival, or b) the winner does not exercise the option but he wins in the asymmetric auction. Since in the second auction the "weak" bidder is the one who won the first auction, and since the weak bidder bids more aggressively than the strong one, it is easy to see that the inefficiencies always go in the sense of giving the second object to the winner of the first one.[25]

4.3 Conclusions

We have compared the sequential first-price auction with the sequential second-price auction and we have seen that these auctions are only revenue equivalent under certain

[24]The same logic of the mixed strategy equilibrium described in the previous chapter could apply here. However, we are not aware of any theoretical paper addressing this problem with more than 2 bidders.

[25]It is a well known result that "weak" bidders bid more aggressivey than "strong" bidders do. In our case this can be seen in the bidding strategies. See also, for example Maskin and Riley (2000).

conditions. We have shown that when the valuations are unidimensional no monotone symmetric pure strategy equilibrium exists, which implies that the outcome of the auction cannot be efficient (in contrast with the outcome of the sequential second-price auction). When the winner of the first auction is allowed to buy the second unit at the price paid for first one, there exists a pure strategy equilibrium for some values of the parameters of the model. In this case the buyer's option is always exercised, leading again to a different allocation than that of the sequential second-price auction.

When the valuation for the second unit is stochastic, we have seen that without a buyer's option, first-price and second-price auctions are efficient and revenue equivalent. This result coincides with that of Robert (1996), although, like his, it relies on both models preserving symmetry in each stage. Introducing a buyer's option in this model makes it analytically intractable even for the simplest case of the uniform distribution and 2 bidders. However, if a symmetric equilibrium exists, the expected prices will be decreasing. Also, as long as symmetry is maintained (as it is in the first stage of the game) the first object will be efficiently assigned. In the second stage the inefficiencies go only in one direction: the winner of the first auction will win the second object in some occasions where his valuation is not highest. An interesting extension would be to undertake a numerical analysis of the equilibrium of this last model.

4.4 Appendix

Proof of Lemma 6

As we saw, a bidder whose valuation is common knowledge randomizes his bid in some interval. Although in the previous chapter we did not show uniqueness of the equilibrium, we did prove that the lower bound of the interval where this bidder randomizes his bid is $\underline{b}$, where $\underline{b}$ satisfies condition (3.6) in the previous chapter:

$$v_1 = \underline{b} + \frac{F(\underline{b})}{(n-1)f(\underline{b})}.$$

The proof is structured as follows: 1) we prove that $\underline{b}$, the lower bound of bidder 1's random bid is increasing in his valuation, v_1. 2) We prove that if $v_1 > z$ the optimal bid is more than $\overline{b}_z$. 3) We prove that if $v_1 > z$ the expected profit of bidder 1 is strictly higher than if his rivals knew his true type. 4) We prove that if $v_1 < z$ the expected profit is at least as large as if his rivals knew his true type.

1) $\underline{b}$ is strictly increasing in v_1.

Suppose there exist $v_1' > v_1$ such that $\underline{b}' = \underline{b}(v_1') < \underline{b}(v_1) = \underline{b}$. Then it must be true that:

$F(\underline{b}')^{n-1}(v_1' - \underline{b}') \geq F(\underline{b})^{n-1}(v_1' - \underline{b})$

$F(\underline{b}')^{n-1}(v_1 - \underline{b}') \leq F(\underline{b})^{n-1}(v_1 - \underline{b})$

Dividing the former inequality by the latter:

$\frac{v_1' - \underline{b}'}{v_1 - \underline{b}'} \geq \frac{v_1' - \underline{b}}{v_1 - \underline{b}} \Leftrightarrow (v_1' - \underline{b}')(v_1 - \underline{b}) \geq (v_1' - \underline{b})(v_1 - \underline{b}')$

$\Leftrightarrow v_1'(\underline{b}' - \underline{b}) \geq v_1(\underline{b}' - \underline{b})$. But then $v_1' \geq v_1 \Leftrightarrow \underline{b}' \geq \underline{b}$, a contradiction.

2) If $v_1 > z$ the optimal bid of the bidder is higher or equal than $\overline{b}_z$.

Suppose that in an auction a bidder has valuation z, and that this is common knowledge. We denote by $G_z(x)$ the probability that, in equilibrium, this bidder wins the auction when he submits a bid equal to x. We denote by S_z the support of the random bid of this bidder. To sustain the mixed strategy equilibrium, we need G to satisfy the following conditions:

$$G_z(x)(z - x) = F(\underline{b}_z)^{n-1}(z - \underline{b}_z) \quad \forall x \in S_{v_1} \tag{4.8}$$

$$G_z(x)(z - x) \leq F(\underline{b}_z)^{n-1}(z - \underline{b}_z) \quad \forall x \notin S_{v_1} \tag{4.9}$$

The first condition above requires that the expected profit of bidding at any point of the support of the random bid yields the same expected profit than bidding at the lower bound.[26] The second condition requires that bidding outside the support yield a expected profit less than or equal to that of bidding in the support.

Now suppose that the real valuation of the bidder is $v_1 > z$. We want to show that in this case his optimal bid is not less than $\overline{b}_z$. It is enough to show that if $v_1 > z$, then

$$G_z(x)(v_1 - x) < G_z(\overline{b}_z)(v_1 - \overline{b}_z) \; \forall x < \overline{b}_z \tag{4.10}$$

that is, that the expected profit of bidding below $\overline{b}_z$ is strictly lower than that of bidding $\overline{b}_z$. From equations (4.8) and (4.9) we have:

$$\frac{G_z(x)}{G_z(\overline{b}_z)} \leq \frac{(z - \overline{b}_z)}{(z - x)}$$

Note that if $x < \overline{b}_z$, then $\frac{z-\overline{b}_z}{z-x}$ is increasing in z, which implies $\frac{v_1-\overline{b}_z}{v_1-x} > \frac{z-\overline{b}_z}{z-x} \; \forall v_1 > z$. Therefore we have $\frac{G_z(x)}{G_z(\overline{b}_z)} \leq \frac{z-\overline{b}_z}{z-x} < \frac{v_1-\overline{b}_z}{v_1-x}$, and (4.10) holds.

[26] As we saw in the previous chapter, when bidder 1 bids at the lower bound of his random bid support, he only wins if his bid is higher than the valuation all his rivals (the rivals know that bidding below bidder 1's random bid support implies a probability zero of winning).

3) The expected profit of bidder 1 when $v_1 > z$ is strictly higher than if his rivals knew his true type: if this were the case, the bidder's expected profit would be $F(\underline{b}_{v_1})^{n-1}(v - \underline{b}_{v_1})$. If his rivals think that he is of type z, those with type $\underline{b}_z$ will bid their valuation, but the bidders with valuation above $\underline{b}_z$ will bid less than their valuation (since otherwise they obtain a non-positive expected profit). Since $z < v_1$, we must have $\underline{b}_z < \underline{b}_{v_1}$. Therefore, by bidding $\underline{b}_{v_1}$, bidder 1 will win with some probability greater than $F(\underline{b}_{v_1})^{n-1}$, so that his expected profit is higher than $F(\underline{b}_{v_1})^{n-1}(v - \underline{b}_{v_1})$, that is, higher than if he had not deviated in the first auction. Moreover, we have seen that the bidder's expected profit increases with his bid, at least as long as it is less than $\overline{b}_z$.

4) If $v_1 < z$ the expected profit is at least as large as if his rivals knew his true type. We assume that no bidder bids more than his valuation (that is a dominated strategy). If the bidder had not deviated, his expected profit would be $F(\underline{b}_{v_1})^{n-1}(v - \underline{b}_{v_1})$. Now, if he has deviated, he can still bid $\underline{b}_{v_1}$ and win at least the same. ■

Proof of Proposition 7

Suppose that a symmetric, monotone, pure strategy equilibrium exists, and denote by $B(v_i)$ bidder i's optimal bidding strategy. The winner of the first auction will randomize his bid in the second auction. We can write the expected profits of the game for a bidder type v_i who in the first stage pretends to be type z as:

$$E(\pi_z) = E(\pi_f) + E(\pi_{lz}) + E(\pi_{wz}),$$

where the first term is the expected profit of participating in the first auction, the second term is the expected profit in the second auction in case some bidder beats player i in the first one, and the third term is the expected profit to bidder i in the second auction when he won the first one bidding as if he were of type z. In lemma 6 we saw that, when the bidder wins the first auction, upwards deviating in the first auction does not decrease the expected profits in the second auction, and downwards deviation increases them. Therefore, we can write

$$\frac{d}{dz}E(\pi_{wz}) \geq 0\,\forall z \geq v_i \quad (4.11)$$

$$\frac{d}{dz}E(\pi_{wz}) < 0\,\forall z < v_i \quad (4.12)$$

The derivatives w.r.t. z of the other terms in $E(\pi_z)$ do not depend on z being greater or lower than v_i.[27] In equilibrium, the derivative w.r.t. z evaluated at v_i must satisfy:

$$E'(\pi_z)\,|_{v_i} \leq 0\,\forall z \geq v_i \quad (4.13)$$

$$E'(\pi_z)\,|_{v_i} \geq 0\,\forall z \leq v_i \quad (4.14)$$

But then, given (4.11), for condition (4.13) to hold we need $E'(\pi_f) + E'(\pi_{lz}) \leq 0\,\forall z \geq v_i$, and given (4.12), for condition (4.14) to hold we need $E'(\pi_f)+E'(\pi_{lz}) > 0\,\forall z < v_i$. Contrary to $E(\pi_{wz})$, the functions $E(\pi_f)$ and $E(\pi_{lz})$ are continuous, so that conditions (4.13) and (4.14) cannot hold simultaneously. ■

Proof of Proposition 8

Consider the following maximization problem:

$$Max_z F(z)^{n-1}(v_i + \alpha v_i - 2B(z)),$$

[27] $E(\pi_f) = F(z)^{n-1}(v_i - B(z))$, while $E(\pi_{lz})$ could be written as $P(win|\exists v > z)(v_i - b^*(v_i))$. Both of these functions are continuous in z, i.e., the sign of their derivations must be independent of z being higher or lower than v_i.

which is to be solved by an agent of type v_i who behaves as if he were type z in a symmetric equilibrium of a game in which the option is always exercised. Differentiating w.r.t. z, and evaluating at v_i we get the FOC that a symmetric monotone equilibrium bidding function must satisfy:

$$2B'(v_i)F(v_i) + 2B(v_i)f(v_i) = f(v_i)(v_i + \alpha v_i).$$

Integrating, and assuming that a bidder with valuation zero bids zero, we obtain

$$B(v_i) = \frac{(1+\alpha)}{2} \frac{\int_0^{v_i} x(n-1)F(x)^{n-2}f(x)dx}{F(v_i)^{n-1}} \tag{4.15}$$

To prove that bidding according to B and exercising the buyer's option can be an equilibrium (at least for some range of values of α) we must define the beliefs out of equilibrium. One of the simplest beliefs we can impose are the following: when the winner of the first auction decides not to exert the buyer's option his rivals' beliefs are that his type is the highest possible one: that is, that his type is $v_i = 1$, so that $\alpha v_i = \alpha$.[28] As we saw before, when the rivals' beliefs are that he has a higher valuation than he actually has, the best strategy of bidder i is to bid the $\underline{b}$ corresponding to his real valuation, which yields a expected profit of $F(\underline{b})^{n-1}(\alpha v_i - \underline{b})$. Now, if a bidder decides to deviate, he must first choose his optimal bid in the first auction. This bidder chooses b^* in order to maximize his expected profit, that is

$$Max_{b^*} F(B^{-1}(b^*))^{n-1}(v_i - b^*) + F(\underline{b})^{n-1}(\alpha v_i - \underline{b})$$

Note that the first auction bid has no effect in the second's auction expected profit, since the beliefs are independent from b^*.

With a change of variable $z = B^{-1}(b^*)$ we can rewrite:

[28]Note that these are the most "favorable" beliefs that we can impose in order to facilitate the existence of equilibrium: the higher the type my rivals think I am, the more aggressive they will bid in the second auction and the lower my expected revenue in the second auction, that is, the higher my incentives to exercise the buyer's option if I win the first auction. Therefore, it is enough to show the non existence of equilibrium given these beliefs.

$$Max_z F(z)^{n-1}(v_i - B(z)) + F(\underline{b})^{n-1}(\alpha v_i - \underline{b})$$

Differentiating w.r.t. z and equating to 0, the optimal z must satisfy:

$$B'(z) = \frac{(n-1)f(z)(v_i - B(z))}{F(z)}$$

On the other hand, we had $B(v_i) = \frac{(1+\alpha)}{2}\frac{\int_0^{v_i} z(n-1)F(z)^{n-2}f(z)dz}{F(v_i)^{n-1}} = \frac{(1+\alpha)}{2}b(v_i)$, where $b(v_i)$ is the standard bidding strategy in a one-object auction. Recall that $b(v_i)$ satisfies: $b'(v_i) = \frac{(n-1)f(v_i)(v_i - b(v_i))}{F(v_i)}$. Therefore $B'(v_i) = \frac{(1+\alpha)}{2}\frac{(n-1)f(v_i)(v_i-b(v_i))}{F(v_i)}$; Substituting $B'(.)$ in the FOC above we have:

$$\frac{(1+\alpha)}{2}\frac{(n-1)f(z)(z-b(z))}{F(z)} = \frac{(n-1)f(z)(v_i - B(z))}{F(z)}$$

Therefore

$$z = \begin{cases} \frac{2v_i}{(1+\alpha)} \ \forall v_i \leq \frac{(1+\alpha)}{2} \\ 1 \ \forall v_i > \frac{(1+\alpha)}{2} \end{cases}$$

Given his optimal strategy, the deviant's expected profit is

$$\begin{cases} F\left(\frac{2v_i}{(1+\alpha)}\right)^{n-1}\left(v_i - B(\frac{2v_i}{(1+\alpha)})\right) + F(\underline{b})^{n-1}(\alpha v_i - \underline{b}) \ \forall v_i \leq \frac{(1+\alpha)}{2} \\ v_i - B(\frac{2v_i}{(1+\alpha)}) + F(\underline{b})^{n-1}(\alpha v_i - \underline{b}) \ \forall v_i > \frac{(1+\alpha)}{2} \end{cases}$$

while the expected profit of the game when one does no deviate is

$$F(v_i)^{n-1}\left[(1+\alpha)v_i - \frac{(1+\alpha)\int_0^{v_i} z(n-1)F(z)^{n-2}f(z)dz}{F(v_i)^{n-1}}\right].$$

Therefore the pure strategy equilibrium above exists if and only if for every valuation v_i the expected profit of bidding according to B above and exercising the option (i.e., the expected profit of not deviation) are at least as large as those of deviating, that is, if the two following conditions hold:

$$F\left(\frac{2}{(1+\alpha)}v_i\right)^{n-1}\left(v_i - \frac{(1+\alpha)}{2}\frac{\int_0^{\frac{2v_i}{(1+\alpha)}} z(n-1)F(z)^{n-2}f(z)dz}{F(\frac{2v_i}{(1+\alpha)})^{n-1}}\right) + F(\underline{b})^{n-1}(\alpha v_i - \underline{b}) \leq$$

$$F(v_i)^{n-1}\left[(1+\alpha)v_i - \frac{(1+\alpha)\int_0^{v_i} z(n-1)F(z)^{n-2}f(z)dz}{F(v_i)^{n-1}}\right] \;\forall v_i \leq \frac{(1+\alpha)}{2}$$

and

$$\left(v_i - \frac{(1+\alpha)}{2}\int_0^1 z(n-1)F(z)^{n-2}f(z)dz\right) + F(\underline{b})^{n-1}(\alpha v_i - \underline{b}) \leq$$

$$F(v_i)^{n-1}\left[(1+\alpha)v_i - \frac{(1+\alpha)\int_0^{v_i} z(n-1)F(z)^{n-2}f(z)dz}{F(v_i)^{n-1}}\right] \;\forall v_i > \frac{(1+\alpha)}{2}$$

It is easy to see that in the extreme case where $\alpha = 1$ these conditions hold: the optimal strategy by deviating implies bidding according to B in the first auction (that is, the same as no deviating), while the fact of not exercising the option reduces the expected rents in the second auction. When $\alpha < 1$, a deviant increases his expected profit in the first auction while he decreases it in the second auction (both in comparison with no deviating), but the lower α, the less important is the loss in the second auction. On the other hand, as α decreases, exercising the buyer's option becomes more expensive (in the extreme, when $\alpha = 0$, exercising the option implies a negative profit). Since both the expected profit of deviating and not deviating are continuous, there must be a cut-off point α^*, such that if $\alpha > \alpha^*$ bidding according to B and always exercising the option is an equilibrium (and respectively, a α', such that if $\alpha < \alpha'$ our equilibrium does not exists).[29] ■

[29]Of course, if the expected profit functions of deviating and not deviating cross only once, then $\alpha^* = \alpha'$.

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