Universitat Autònoma de Barcelona

Convergence and integrability of Fourier transforms

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Chapter 1

Introduction

In this dissertation we are mostly interested in two different topics. The first one is the convergence of Fourier transforms, with special emphasis on uniform convergence. We are going to solve these problems through variational methods, according to recent substantial advances that have been done in this topic. More precisely, we consider the general monotone functions, which we shall now briefly describe, and comprehensively discuss in Chapter 2. The second problem we consider is the weighted integrability of generalized Fourier transforms. In more detail, we study integral operators that generalize the Fourier transform and study necessary and sufficient conditions on weights for these operators to be bounded between weighted Lebesgue spaces. We proceed to briefly expose the main concepts and problems considered in the present dissertation.

Let us introduce some basic definitions. An integral transform is an expression of the form

$$Tf(y) = \int_X K(x,y) f(x)\, dx, \tag{1.1}$$

where X is an appropriate domain of integration (in this dissertation, X will always be a subset of $\mathbb{R}^n$), $f : X \to \mathbb{C}$ is a measurable function, and $K : X \times Y \to \mathbb{C}$ (also measurable) is the kernel of the transform T. The transform of f, Tf, is a function defined on the points $y \in Y$ where the integral (1.1) converges.

Let us list a few of the well known and widely used integral transforms.

- Fourier transform:
$$Tf(y) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot y} f(x)\, dx,$$
where $x \cdot y$ denotes the scalar product of x and y;

- Sine/Cosine transform:
$$T_\varphi f(y) = \int_0^\infty \varphi(xy) f(x)\, dx,$$
where either $\varphi(x) = \sin x$ or $\varphi(x) = \cos x$;

- Hankel transform of order $\alpha \geq -1/2$ [126]:
$$T_\alpha f(y) = \int_0^\infty x^{2\alpha+1} f(x) j_\alpha(xy)\, dx,$$

where j_α is the normalized Bessel function of order α (cf. [1, 47, 136]). This transform is related to the Fourier transform and will be discussed with more detail in Chapter 4;

- The $\mathscr{H}_\alpha$ transform, with $\alpha > -1/2$ [109, 110, 136]:

$$T_\alpha f(y) = \int_0^\infty (xy)^{1/2} f(x) \mathbf{H}_\alpha(xy)\, dx,$$

where $\mathbf{H}_\alpha$ is the Struve function of order α (cf. [1, 47, 136]). This transform is also related to the Hankel transform, and will be discussed in more detail in Chapter 5;

- Laplace transform [120]:

$$Tf(y) = \int_0^\infty f(x) e^{-xy}\, dx;$$

- Stieltjes transform of order $\lambda > 0$ [6, 119, 138]:

$$T_\lambda f(y) = \int_0^\infty \frac{f(x)}{(x+y)^\lambda}\, dx;$$

There are several other integral transforms we could mention here, as the Dunkl transform [38, 111], or the Mellin transform [136], but the list would be too long. We refer the reader to the books [26], [37], [117], and [136] for several other examples of integral transforms, and to the book [48] for a comprehensive list of functions and their respective integral transforms.

Recall that if $1 \le p \le \infty$ and $f : X \to \mathbb{C}$ is a measurable function, we say that $f \in L^p(X)$ if $\|f\|_{L^p(X)} < \infty$, where

$$\|f\|_{L^p(X)} := \left(\int_X |f(x)|^p\, dx \right)^{1/p}, \qquad \text{if } p < \infty,$$

and

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in X} |f(x)| := \inf \big\{ C \ge 0 : |f(x)| \le C \text{ for almost every } x \in X \big\}.$$

The functional $\|\cdot\|_p$ is a norm for every $1 \le p \le \infty$ and $L^p(X)$ are Banach spaces (cf. [112]).

Usually, depending on the properties of K, one assumes different properties on f in order to ensure the well-definiteness of (1.1). For example, if K is bounded on $X \times Y$, a usual assumption is that $f \in L^1(X)$. In particular, this is the case of the Fourier transform (with $X = \mathbb{R}^n$).

Several problems involving integral transforms (1.1) can be considered. Chapters 3 and 4 are devoted to study the uniform convergence of certain types of Fourier transforms, concretely the sine and Hankel transforms. In Chapter 5 we study weighted norm inequalities, i.e., inequalities of the type

$$\|u \cdot Tf\|_{L^q(Y)} \le C \|v \cdot f\|_{L^p(X)}, \tag{1.2}$$

where $C > 0$ is independent of the choice of f and u and v are weights (i.e., nonnegative locally integrable functions) defined on Y and X respectively.

We recall the notion of uniform convergence.

Definition 1.1. We say that a family of functions $\{f_s\}_{s\in\mathbb{R}_+}$, $f_s : X \to \mathbb{C}$, converges uniformly to the function f on a set $E \subset X$ if for every $\varepsilon > 0$ there exists $s_0 \in \mathbb{R}_+$ such that

$$|f_s(x) - f(x)| < \varepsilon,$$

for every $s \geq s_0$ and every $x \in E$.

Remark 1.2. Uniform convergence can also be defined for a different set of parameters than $\mathbb{R}_+$. For instance, if the set of parameters s is taken to be $\mathbb{N}$, then we obtain a sequence of functions. For a more general setting we refer to [145].

The notion of uniform convergence is desirable, as shown in undergraduate courses, because in such case the limit function inherits properties that the family of functions may possess. We recall, for instance, that the limit function of a family of continuous functions that converge uniformly is continuous, or that the integral of the limit function equals the limit of the integrals of the functions from the family.

There is an analogue to the usual Cauchy criterion for uniform convergence that we will use repeatedly (cf. [145]).

Theorem 1.3. *A necessary and sufficient condition for $\{f_s\}_{s\in\mathbb{R}_+}$ to converge uniformly on $E \subset X$ is that for every $\varepsilon > 0$, there exists $s_0 \in \mathbb{R}_+$ such that*

$$|f_{s_1}(x) - f_{s_2}(x)| < \varepsilon, \qquad \text{for every } s_1, s_2 \geq s_0 \text{ and every } x \in E,$$

or equivalently,

$$\lim_{t,s\to\infty} \sup_{x\in E} |f_t(x) - f_s(x)| = 0.$$

A concept that is central in this work is general monotonicity. As the name indicates, it generalizes classical monotonicity, and it was introduced by S. Tikhonov in [131, 135] for sequences. Since then, several authors have considered problems involving general monotone sequences and functions (see the references in Chapter 2).

For a nonnegative function β defined on $(0,\infty)$, the class of β-*general monotone* (GM) functions is defined to be the set of functions $f : (0,\infty) \to \mathbb{C}$ that are locally of bounded variation on $(0,\infty)$, and for which there exists a constant $C > 0$ such that

$$\int_x^{2x} |f'(t)|\, dt \leq C\beta(x).$$

Here we also assume f is differentiable for simplicity, a more accurate and general definition will be given in Section 2.3.

As observed in [86], since monotone functions f satisfy

$$f(x) \leq 2\int_{x/2}^{x} \frac{f(t)}{t}\, dt \qquad \text{for all } x > 0,$$

and

$$\int_x^{2x} |f'(t)|\, dt \leq f(x) \qquad \text{for all } x > 0,$$

then it is natural to consider

$$\beta(x) = |f(x)|, \qquad \text{or } \beta(x) = \int_{x/2}^{x} \frac{|f(t)|}{t}\, dt$$

if we wish to generalize monotone functions. In fact, one of the classes of GM functions that is most considered in the existing literature is that given by

$$\beta(x) = \int_{x/\lambda}^{\lambda x} \frac{|f(t)|}{t}\, dt,$$

where $\lambda > 1$ is a constant chosen conveniently.

It is clear that different choices of β give rise to different classes of functions. We will give the known embeddings between these classes, and moreover we will construct a larger class (Section 2.5) than those considered before, which is very general but whose functions still inherit desirable properties that monotone functions satisfy.

Another important observation about GM functions is that in the above choices of β we have put absolute value bars on f. We implicitly meant that we do not only consider nonnegative functions, as in the monotone case, but real-valued or even complex-valued functions. In fact, some of the problems whose solutions are known for monotone functions, have the same solution for *real-valued* functions of certain GM classes, see Sections 2.4, 3.1, and 4.5, as well as the papers [41, 46, 50].

We emphasize that in order to generalize monotonicity, instead of looking at the values of the function itself, as for instance the quasi-monotone functions, i.e., those f such that $x^\tau f(x)$ is monotone for some $\tau < 0$ (cf. [121]), it is also convenient to look at its variation. For monotone functions f vanishing at infinity, one has

$$\int_x^\infty |f'(t)|\, dt = f(x).$$

One may then obtain generalizations to monotone functions by manipulating the above equality, as for instance L. Leindler did in [83], where he introduced the sequences with *rest of bounded variation*, whose analogue for functions reads as

$$\int_x^\infty |f'(t)|\, dt \le C|f(x)|, \tag{1.3}$$

for all $x > 0$ and some absolute constant $C > 0$. This class is rather small, since condition (1.3) is very restrictive.

We will also deal with general monotone functions of two variables (cf. Section 2.6).

The first problem where we apply GM functions is the study of uniform convergence of sine integrals, or in other words, Fourier transforms of odd functions. Such a problem was first considered by Chaundy and Jolliffe back in 1916 [25], where they showed that if $\{a_n\}$ is nonnegative and decreasing, then $\sum_{n=1}^\infty a_n \sin nx$ converges uniformly on $[0, 2\pi)$ if and only if $\lim_{n\to\infty} na_n = 0$.

In the context of functions, F. Móricz [96] showed that if f is decreasing and such that $tf(t) \in L^1(0,1)$, then

$$\int_0^\infty f(t) \sin ut\, dt \tag{1.4}$$

converges uniformly on $[0,\infty)$ if and only if $xf(x) \to 0$ as $x \to \infty$. Note that for functions f decreasing to zero, there holds

$$xf(x) = x\int_x^\infty |f'(t)|\, dt,$$

and in fact, M. Dyachenko, E. Liflyand, and S. Tikhonov proved in [40] that a sufficient condition for (1.4) to converge uniformly is that

$$x\int_x^\infty |f'(t)|\,dt \to 0 \qquad \text{as } x\to\infty,$$

without any monotonicity assumption on f, provided that $tf(t)\in L^1(0,1)$.

Our main result in this direction is the following: for any real-valued f from the GM class we construct in Section 2.5, we prove that the uniform convergence of (1.4) is equivalent to $xf(x)\to 0$, provided that the integrals $\int_x^{2x}|f(t)|\,dt$ are bounded as $x\to\infty$. This latter hypothesis is shown to be sharp and is needed to have some control on f, since the GM class we use is too wide, and otherwise we cannot obtain useful estimates for f. For smaller GM classes, the fact that $\int_x^{2x}|f(t)|\,dt$ is bounded at infinity follows from the condition $xf(x)\to 0$ as $x\to\infty$. Summarizing, we generalize the known results in two ways: first, we consider a class of GM functions containing all the previously considered ones, and secondly, we solve the problem for real-valued functions. To this end, we adapt the technique of L. Feng, V. Totik, and S. P. Zhou from [50] to the context of functions. In particular, the necessity of the condition $xf(x)\to 0$ as $x\to\infty$ for the uniform convergence of (1.4) was only proved for nonnegative f.

We also study the uniform convergence of double sine transforms

$$\int_0^\infty\int_0^\infty f(x,y)\sin ux\sin vy\,dx\,dy. \tag{1.5}$$

In the two-dimensional case there are various types of convergence of double integrals. We are mostly concerned about *regular* convergence, but we also partially discuss *Pringsheim* convergence. See the beginning of Section 3.2 for the precise definitions. Here we just mention that regular convergence is stronger than Pringsheim convergence.

Analogously as in the one-dimensional case, we show that in general, if

$$xy\int_y^\infty\int_x^\infty |d_{11}f(s,t)|\,ds\,dt\to 0 \qquad \text{as } x+y\to\infty, \tag{1.6}$$

then (1.5) converges uniformly in the regular sense, where $d_{11}f$ denotes the "mixed differences" of f (the terms $ds\,dt$ in (1.6) are meaningless save for the order of integration, cf. Remark 2.26). It is also shown that if f is from a certain class of two-dimensional general monotone functions, then (1.6) is equivalent to

$$xyf(x,y)\to 0 \qquad \text{as } x+y\to\infty,$$

and moreover this condition is also necessary for the uniform convergence of (1.5) in the regular sense whenever $f\ge 0$.

Chapter 4 is devoted to study the uniform convergence of weighted Hankel transforms

$$\mathcal{L}^\alpha_{\nu,\mu}f(r) = r^\mu\int_0^\infty (rt)^\nu f(t)j_\alpha(rt)\,dt, \quad \alpha\ge -1/2, \quad \nu,\mu\in\mathbb{R}, \quad r\ge 0, \tag{1.7}$$

where j_α is the normalized Bessel function of order α. Its basic properties are discussed in Section 4.1.

The family of transforms (1.7) was recently introduced by L. De Carli in [27], where she studied necessary and sufficient conditions on the parameters α,ν, and μ, and the weights u and v (chosen to be power functions), for the inequality (1.2) to hold.

The Hankel transform of order $\alpha \geq -1/2$, one of the most important transforms of the type (1.7), is defined as

$$H_\alpha f(r) = \mathcal{L}^\alpha_{2\alpha+1,-(2\alpha+1)} f(r) = \int_0^\infty t^{2\alpha+1} f(t) j_\alpha(rt)\, dt,$$

and, as is well known, represents the Fourier transform of radial functions defined on $\mathbb{R}^n$ whenever $\alpha = n/2 - 1$ (cf. [126]). In fact, if $n = 1$, the Hankel transform of order $-1/2$ is just the cosine transform, so that H_α may be viewed as a generalization of the cosine transform.

The sine transform can also be written as a transform of the form (1.7), namely it corresponds to the choice of parameters $\alpha = 1/2$, $\nu = 1$, and $\mu = 0$. We will divide the transforms (1.7) satisfying $0 \leq \mu + \nu \leq \alpha + 3/2$ into two types, depending on the choice of the parameters, and according to their uniform convergence criteria. The first type consists on those transforms $\mathcal{L}^\alpha_{\nu,\mu} f$ with $\mu = -\nu$, or equivalently, $\mu + \nu = 0$ (such as the cosine or Hankel transforms). We call them cosine-type transforms. The second type of transforms $\mathcal{L}^\alpha_{\nu,\mu} f$ we study are those satisfying $0 < \mu + \nu \leq \alpha + 3/2$, which we call sine-type transforms. Of course, the sine transform is one of these.

First of all we give rather rough sufficient conditions on f for the pointwise and uniform convergence of $\mathcal{L}^\alpha_{\nu,\mu} f$, without any restriction on the parameters, and show that in general we cannot have uniform convergence of $\mathcal{L}^\alpha_{\nu,\mu} f$ on $\mathbb{R}_+$ if $\mu + \nu < 0$ or $\mu + \nu > \alpha + 3/2$.

The main result concerning cosine-type transforms states that if f satisfies

$$M^{\nu+1} f(M) \to 0 \text{ as } M \to \infty,$$
$$M^{\alpha+3/2} \int_M^\infty t^{\nu-\alpha-1/2} |df(t)| \to 0 \text{ as } M \to \infty,$$

then $\mathcal{L}^\alpha_{\nu,-\nu} f$ converges uniformly if and only if $\int_0^\infty t^\nu f(t)\, dt$ converges. In the case of sine-type transforms, the uniform convergence follows from the conditions

$$M^{1-\mu} f(M) \to 0 \quad \text{as } M \to \infty,$$
$$M^{\alpha+3/2-\mu-\nu} \int_M^\infty t^{\nu-\alpha-1/2} |df(t)| \to 0 \quad \text{as } M \to \infty.$$

In both cases we are able to rewrite the variational conditions in terms of conditions on the function f itself when general monotonicity is assumed.

In the case of sine-type transforms we also discuss the sharpness of the obtained results.

To conclude Chapter 4, we study the equivalence between conditions that guarantee the uniform convergence and the boundedness of Hankel transforms of functions from a concrete GM class. Namely, we show that the uniform convergence of $H_\alpha f$ is equivalent to the boundedness of the function $H_\alpha f(r)$, and also equivalent to the convergence of $\int_0^\infty t^{2\alpha+1} f(t)\, dt$.

In Chapter 5 we study weighted norm inequalities of integral transforms. More precisely, we investigate necessary and sufficient conditions on the weights u and v for (1.2) to hold for different choices of the transform T.

We first consider generalized Fourier transforms [136], i.e., those of the form

$$\int_0^\infty s(x) f(x) K(x, y)\, dx,$$

where s is locally integrable, and the kernel $K : \mathbb{R}_+^2 \to \mathbb{C}$ satisfies the estimate

$$|K(x,y)| \leq \begin{cases} 1, & \text{if } xy \leq 1, \\ C\big(s(x)w(y)\big)^{-1}, & \text{if } xy > 1, \end{cases}$$

for some $C > 0$, where w is a function for which there exist constants $C_1 > C_2 > 0$ such that $C_2 s(x) \leq w(1/x) \leq C_1 s(x)$ for every $x \geq 0$. The Hankel transform is an example of such transforms.

It is important to remark that in our approach we are interested on necessary and sufficient conditions on the weights u and v themselves, and not on their decreasing rearrangements, contrarily as in several previous investigations.

We also deal with transforms of the form

$$Tf(y) = y^{c_0} \int_0^\infty x^{b_0} f(x) K(x,y)\, dx, \tag{1.8}$$

that have kernels of power type, i.e., that satisfy the estimate

$$|K(x,y)| \leq \begin{cases} Cx^{b_1}y^{c_1}, & \text{if } xy \leq 1, \\ Cx^{b_2}y^{c_2}, & \text{if } xy > 1, \end{cases}$$

as for instance Hankel and sine transforms. Another example is the so-called $\mathscr{H}_\alpha$ transform, defined as

$$\mathscr{H}_\alpha f(y) = \int_0^\infty (xy)^{1/2} f(x) \mathbf{H}_\alpha(xy)\, dx, \qquad \alpha > -1/2,$$

where $\mathbf{H}_\alpha$ is the Struve function of order α (see Section 5.2).

In order to study transforms with kernels of power type we reduce ourselves to the cases when u and v are power functions, so that we obtain all necessary and sufficient conditions involved in terms of the powers and the parameters b_i and c_i ($i = 0, 1, 2$).

In Section 5.5 we use an idea of Sadosky and Wheeden [114] to relax the sufficient conditions that guarantee that the inequality

$$\|x^{-\beta} Tf\|_q \leq C\|x^\gamma f\|_p, \qquad 1 < p \leq q < \infty, \tag{1.9}$$

holds, where T is of the form (1.8) and has a kernel of power type. They showed, in particular, that the sufficient conditions that guarantee (1.9) can be relaxed in the case of the Fourier transform provided that $\int_{-\infty}^\infty f(x)\, dx = 0$. We show that this can be done for any transform whose kernel $K(x,y)$ admits a representation by power series (the sine, cosine, Bessel, and Struve functions are examples of those).

Finally, we use an approach based on knowing upper estimates for the antiderivative of the kernel $K(x,y)$ (as a function of x) to obtain sufficient conditions for (1.9) to hold whenever f satisfies general monotonicity conditions, for transforms T of the form (1.8). This approach yields sharp sufficient conditions that are already known in some cases (concretely, for the sine and Hankel transforms), and previously unknown ones, as in the case of the $\mathscr{H}_\alpha$ transform.

Let $\varphi(x)$ and $\psi(x)$ be nonnegative functions. Through the sequel we will use the notation $\varphi(x) \lesssim \psi(x)$ if there exists $C > 0$ such that $\varphi(x) \leq C\psi(x)$ for all x in a given domain. Likewise, we denote $\varphi(x) \gtrsim \psi(x)$ if there exists $C > 0$ such that $\varphi(x) \geq C\psi(x)$. We denote $\varphi(x) \asymp \psi(x)$ if $\varphi(x) \lesssim \psi(x)$ and $\varphi(x) \gtrsim \psi(x)$ simultaneously.

Also, the notation

$$\varphi(x) = o\big(\psi(x)\big) \qquad \text{as } x \to x_0,$$

with $x_0 \in \mathbb{R} \cup \{-\infty, \infty\}$, means that

$$\frac{\varphi(x)}{\psi(x)} \to 0 \qquad \text{as } x \to x_0,$$

and

$$\varphi(x) = O\big(\psi(x)\big) \qquad \text{as } x \to x_0$$

means that there exists $C > 0$ such that

$$\frac{\varphi(x)}{\psi(x)} \leq C \qquad \text{as } x \to x_0.$$

Chapter 2

General Monotonicity

This chapter is devoted to present and discuss the concept of *general monotonicity*, which, as the name indicates, extends the notion of usual monotonicity for sequences and functions.

A nonnegative sequence $\{a_n\}$ is said to be monotone (or nonincreasing) if $a_{n+1} \leq a_n$ for every $n \in \mathbb{N}$, or equivalently, if the sequence $\Delta a_n := a_n - a_{n+1}$ is nonnegative.

A nonnegative function f defined on an interval $I \subset \mathbb{R}$ is said to be monotone (or nonincreasing) if for every $x, y \in I$, $x \leq y$ implies $f(y) \leq f(x)$.

Monotonicity condition plays a fundamental role in many problems of analysis. More specifically, in the theory of trigonometric series, for the general case, it is often relatively easy to obtain (sometimes trivial) partial results based on rough hypotheses. However, the monotonicity condition of a sequence $\{a_n\}$ often allows us to **characterize** properties of series in terms of the rate of decay or summability of their coefficients a_n. Let us mention some problems where this goal has been achieved: the uniform convergence [25], the degree of approximation and smoothness [91], the asymptotic behaviour at the origin [61], the L^1 convergence [51], and the L^p convergence with $1 < p < \infty$ [62]. Of course, these problems also have their counterparts in the case of Fourier integrals, and we shall see some of them in the sequel.

Thus, monotonicity is a desirable property, but it is also very restrictive in applications. In the first place, if we analyse a trigonometric series

$$\sum_{n=0}^{\infty} a_n \varphi(nx),$$

where either $\varphi(t) = \sin t$ or $\varphi(t) = \cos t$, it is necessary to assume that $a_n \to 0$ if we wish to have convergence at any fixed x. This implies that if $\{a_n\}$ is monotone, it cannot vary its sign. Another restriction is that either a finite number or all of the terms are strictly positive, since $a_m = 0$ implies $a_n = 0$ for all $n \geq m$.

2.1 Extensions of monotone sequences/functions

One of the most popular generalizations of monotonicity is the *bounded variation* condition. We say that a complex sequence $\{a_n\}$ is of bounded variation if

$$\sum_{n=1}^{\infty} |\Delta a_n| < \infty, \qquad \Delta a_n := a_n - a_{n+1}.$$

It is well known that a sequence of bounded variation can be written as a difference of two decreasing sequences, namely if

$$b_n = \sum_{k=n}^{\infty} |\Delta a_k|, \qquad c_n = \sum_{k=n}^{\infty} |\Delta a_k| - a_n,$$

then b_n and c_n are clearly decreasing, and $b_n - c_n = a_n$.

The definition of a function of bounded variation is more delicate. For $I = [a, b] \subset \mathbb{R}$, a *partition* P of the interval I is a finite set of points

$$P := \{a = x_0 < x_1 < \cdots < x_n = b\}.$$

For a function $f : I \to \mathbb{C}$, we define the *variation* of f associated to the partition P as

$$V(f, P) := \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|.$$

Then, the *variation of* f over the interval I is

$$V(f; I) := \sup_{P \in \mathcal{P}(I)} V(f, P), \tag{2.1}$$

where $\mathcal{P}(I)$ denotes the set of all partitions of I. If the supremum in (2.1) is finite, we say that f is a function of bounded variation on I, and we write $f \in BV(I)$. Such condition is defined for closed intervals, but it can be extended to intervals not necessarily closed (as for instance, (a, b), $[a, b)$), as follows. For any interval $J \subset \mathbb{R}$ and $f : J \to \mathbb{C}$, if the supremum

$$V(f; J) := \sup_{[a,b] \subset J} V(f; [a, b]),$$

is finite, we say that f is of bounded variation on J, written $f \in BV(J)$. Of course, this definition can also be used to define the bounded variation property on $\mathbb{R}$ if we choose $J = \mathbb{R}$.

Remark 2.1. It follows easily that if a function f is of bounded variation on a compact interval, then it is bounded on such interval. Moreover, it has derivative almost everywhere in the interval and such derivative is locally integrable, see [73, Ch. 9].

A useful tool related to functions of bounded variation is the so-called *Riemann-Stieltjes integral* (see [108], for instance), which is a generalization of the usual Riemann integral. Let $f, g : [a, b] \subset \mathbb{R} \to \mathbb{C}$, and for $n \in \mathbb{N}$, let $P = \{x_k\}_{k=0}^{n}$ be a partition of $[a, b]$. Let $\{\xi_k\}_{k=0}^{n-1}$ be such that $x_k \le \xi_k \le x_{k+1}$ for $k = 0, \dots, n-1$. Denote $\|P\| = \max_k \{x_{k+1} - x_k\}$. If the limit

$$\lim_{\|P\| \to 0} \sum_{k=0}^{n-1} g(\xi_k)\big(f(x_{k+1}) - f(x_k)\big) \tag{2.2}$$

exists, then it is called the *Riemann-Stieltjes integral of* f *with respect to* g over the interval $[a, b]$, and it is denoted by

$$\int_a^b g(t)df(t). \tag{2.3}$$

It is known [21] that a sufficient condition for such an integral to exist is that f is continuous on $[a,b]$ and f is of bounded variation on the same interval. Likewise, the Riemann-Stieltjes integral can be defined in the improper sense for non-closed or unbounded intervals.

Observe that if $f(x)=x$ and the limit (2.2) exists, it equals the Riemann integral of g over $[a,b]$.

We will use the following property repeatedly: if f is of bounded variation on $[a,\infty)$ for some $a\in\mathbb{R}$, then

$$\int_b^\infty df(t) = -f(b) + \lim_{x\to\infty} f(x), \tag{2.4}$$

for any $b\geq a$. Indeed, to prove (2.4), it suffices to take $g\equiv 1$ in (2.2) and compute the resulting telescopic sum, taking into account that it converges absolutely in the limit, since f is of bounded variation. Note that if $f(x)\to 0$ as $x\to\infty$, then the right-hand side of (2.4) reduces to $f(b)$. The estimate

$$|f(b)|\leq \int_b^\infty |df(t)|$$

will be used repeatedly throughout this work.

Remark 2.2. If f is differentiable on (a,b), it follows easily by applying the mean value theorem on (2.2) that

$$\int_a^b g(t)df(t) = \int_a^b g(t)f'(t)\,dt,$$

whenever the limit in (2.2) exists.

The Riemann-Stieltjes integral is closely related to functions of bounded variation. In fact, this is a starting point of the variational approaches we use in the following chapters. It is well known that the variation of a function over an interval can be written as a Riemann-Stieltjes integral (with absolute value on the integrand). That is, defining

$$\int_a^b |df(t)| := \lim_{\|P\|\to 0}\sum_{k=0}^{n-1}|f(x_{k+1})-f(x_k)|,$$

where $P=\{x_k\}_{k=0}^n\in\mathcal{P}([a,b])$, there holds

$$\int_a^b |df(t)| = V(f;[a,b]),$$

for any $f\in BV([a,b])$.

Likewise, for a continuous function w one can define the integral

$$\int_a^b w(t)|df(t)| \tag{2.5}$$

as the *weighted variation of f with respect to w* over the interval $[a,b]$ by putting w in place of g in (2.2) and incorporating the absolute value bars inside the sum. The function w is the weight function. If f is of bounded variation on a compact interval $[a,b]$, the integral in (2.5) converges (cf. [21]). If $b=\infty$, the integral is defined as the limit of integrals over finite intervals.

2.2 General monotone sequences

Although the sequences of bounded variation have shown to be a suitable generalization for monotone sequences, the bounded variation condition is too mild to guarantee monotonicity type properties. In fact, as mentioned in [86], none of the problems stated at the beginning of this chapter can be solved within the framework of all sequences of bounded variation. In general we can only say that such sequences satisfy

$$\sum_{k=n}^{\infty} |\Delta a_k| = o(1) \qquad \text{as } n \to \infty,$$

whilst we sometimes need stronger conditions on the rate of decay of the above series. Thus, the next step is to consider quantitative characteristics of the sequences of bounded variation, and this is where general monotonicity comes into play.

The concept of general monotone sequence (GMS) was first introduced by S. Tikhonov [131, 135].

Definition 2.3. We say that a sequence $\{a_n\} \subset \mathbb{C}$ is general monotone (written $\{a_n\} \in GMS$) if there exists a constant $C > 0$ such that the inequality

$$\sum_{k=n}^{2n} |\Delta a_k| \le C|a_n| \tag{2.6}$$

holds for every $n \in \mathbb{N}$.

It was also proved in [135] that $\{a_n\} \in GMS$ if and only if the following conditions hold:

$$|a_k| \le C|a_n| \quad \text{for } n \le k \le 2n;$$
$$\sum_{k=n}^{N} |\Delta a_k| \le C\Big(|a_n| + \sum_{k=n+1}^{N} \frac{|a_k|}{k}\Big) \quad \text{for } N > n.$$

Although the GMS generalizes monotone sequences, we find that condition (2.6) is still somehow restrictive. For instance, general monotone sequences, like monotone ones, satisfy the property that if $a_m = 0$ for some m, then $a_n = 0$ for $n \ge m$. One can go even further and consider a wider class of sequences (see [130]), namely GMS_2, being those for which there exist constants $C, \lambda > 1$ such that

$$\sum_{k=n}^{2n} |\Delta a_k| \le C \sum_{k=n/\lambda}^{\lambda n} \frac{|a_k|}{k}. \tag{2.7}$$

It is proved in [130] that $GMS \subset GMS_2$, and moreover, as noted in [86, Example 2.3], for any positive sequence $a \in GMS$ and any $n > 2$, there holds $\{a_1, \dots, a_{n-1}, 0, a_{n+1}, \dots\} \in GMS_2 \backslash GMS$.

Other classes of general monotone sequences may be defined as follows: given $\{a_n\} \subset \mathbb{C}$ and a nonnegative sequence $\{\beta_n\}$, we say that a is a general monotone sequence with majorant β (written $(a, \beta) \in \mathcal{GMS}$) if there exists $C > 0$ such that

$$\sum_{k=n}^{2n} |\Delta a_k| \le C\beta_n, \qquad \text{for all } n. \tag{2.8}$$

We refer the reader to [45, 77, 130], which contain examples of the sequence β.

Usually the sequence $\{\beta_n\}$ will depend on $\{a_n\}$, as for instance in (2.7). However, abusing of notation, we will write $a \in GMS(\beta)$ for simplicity. A typical assumption when studying $GMS(\beta)$ sequences is $\beta_n \to 0$ as $n \to \infty$. However, it is sometimes interesting to consider $GMS(\beta)$ sequences such that $\beta_n \to \infty$ as $n \to \infty$, see e.g. [44].

The $GMS(\beta)$ classes have been widely used recently by several authors in order to solve some the aforementioned problems and different ones, and they have also been object of generalization in order to further study such problems, see for instance [18, 41, 43, 44, 45, 50, 59, 69, 77, 85, 90, 128, 129, 130, 132, 135].

In this dissertation we are concerned about general monotonicity properties for functions, thus we will not go into further details of GMS and its generalizations.

2.3 General monotone functions

We can define the general monotonicity property for functions similarly as for sequences in a natural way, that is, replacing the sum of increments in (2.8) by the variation of the function (cf. [86, 88]).

Definition 2.4. Let $f : \mathbb{R}_+ \to \mathbb{C}$ be locally of bounded variation, and let $\beta : \mathbb{R}_+ \to \mathbb{R}_+$. We say that the couple $(f, \beta) \in \mathcal{GM}$ (or f is a general monotone function with majorant β) if there exists $C > 0$ such that for every $x > 0$

$$\int_x^{2x} |df(t)| \leq C\beta(x). \tag{2.9}$$

The majorant β will typically depend on the function f. For example, the counterpart to the class of sequences GMS (Definition 2.3) is given by $\beta(x) = |f(x)|$. Abusing of notation, we may write $f \in GM(\beta)$ when the expression of β in terms of f is given.

We shall now present some examples of choices of β in (2.9) giving rise to different classes of general monotone functions that will be referred to throughout the sequel.

Examples.

- $\beta_1(x) = |f(x)|$;
- $\beta_2(x) = \displaystyle\int_{x/\lambda}^{\lambda x} \frac{|f(t)|}{t}\, dt$, where $\lambda > 1$;
- $\beta_3(x) = \displaystyle\frac{1}{x} \sup_{s \geq x/\lambda} \int_s^{2s} |f(t)|\, dt$, where $\lambda > 1$.

Remark 2.5. Since for any fixed constant $\lambda > 1$ there holds

$$\int_{x/\lambda}^{\lambda x} \frac{|f(t)|}{t}\, dt \asymp \frac{1}{x} \int_{x/\lambda}^{\lambda x} |f(t)|\, dt,$$

we may use any of these two expressions for β_2.

Remark 2.6. The appearance of $\lambda > 1$ in β_3 is essential if we wish that the class $GM(\beta_3)$ contains the class of monotone functions. Indeed, if we took $\lambda = 1$, then rapidly decreasing functions such as $f(x) = e^{-x}$ would not satisfy $f \in GM(\beta_3)$. In fact, monotone functions f satisfy the inequality

$$|f(x)| \le \left(1 - \frac{1}{\lambda}\right)^{-1} \frac{1}{x} \int_{x/\lambda}^{x} |f(t)|\, dt$$

for any $\lambda > 1$, thus such parameter makes sure that all decreasing functions fall into the scope of GM functions.

Remark 2.7. Throughout the sequel we will use the fact that if $f \in GM(\beta_2)$, then $t^\nu f(t) \in GM(\beta_2)$ for every $\nu \in \mathbb{R}$ (see [86]). Taking into account the abuse of notation made when writing $f \in GM(\beta)$, the property $t^\nu f(t) \in GM(\beta_2)$ should be understood as

$$\int_x^{2x} |d(t^\nu f(t))| \lesssim \int_{x/\lambda}^{\lambda x} t^{\nu-1} |f(t)|\, dt, \qquad x > 0.$$

If we denote by M the class of monotone functions, then

$$M \subsetneq GM(\beta_1) \subsetneq GM(\beta_2) \subsetneq GM(\beta_3),$$

see [45, 86, 135].

To conclude this section, we give a basic estimate of GM functions as well as an estimate for certain weighted variations that we will use repeatedly. Its proof (with more general statements) was first given in the survey paper about general monotone sequences and functions [86]. For the sake of completeness, we also include the proof here.

Proposition 2.8. *Let $f \in GM(\beta)$. Then, for every $x > 0$ and every $u \in [x, 2x]$,*

$$|f(u)| \le \frac{1}{x} \int_x^{2x} |f(t)|\, dt + C\beta(x),$$

where C is the constant from (2.9).

Proof. For any $u, v \in [x, 2x]$, we have

$$|f(u)| - |f(v)| \le |f(u) - f(v)| \le \int_x^{2x} |df(t)| \le C\beta(x).$$

Integrating with respect to v over $[x, 2x]$, we obtain

$$x|f(u)| \le \int_x^{2x} |f(t)|\, dt + xC\beta(x),$$

and the result follows. $\square$

Remark 2.9. Note that if β is such that $\beta(x) \ge \frac{C'}{x} \int_x^{2x} |f(t)|\, dt$, it follows from Proposition 2.8 that for any $x > 0$ and any $u \in [x, 2x]$,

$$|f(u)| \le (C + C')\beta(x).$$

Remark 2.10. If $f \in GM(\beta_2)$, we can assume without loss of generality that $\lambda \geq 2$. Therefore, in view of Remark 2.9, we can deduce that if $f \in GM(\beta_2)$, then

$$|f(x)| \leq C \int_{x/\lambda}^{\lambda x} \frac{|f(t)|}{t}\, dt.$$

From now on we will always assume $\lambda \geq 2$ when dealing with the class $GM(\beta_2)$.

Proposition 2.11. *Let $f \in GM(\beta)$ and $x > 0$. Let $w : [x/2, \infty) \to \mathbb{R}_+$ be such that $w(y) \asymp w(z)$ for all $y, z \in [u, 2u]$ and all $u > x/2$. Then*

$$\int_x^\infty w(t)|df(t)| \lesssim \int_{x/2}^\infty \frac{w(t)}{t}\beta(t)\, dt$$

Proof. First of all note that

$$\int_x^\infty w(t)\,|df(t)| \lesssim \int_{x/2}^\infty \frac{1}{t}\int_t^{2t} w(s)\,|df(s)|\, dt,$$

see e.g. [87]. Since $w(s) \asymp w(t)$ for all $s \in [t, 2t]$, by the GM condition of f, we have

$$\int_{x/2}^\infty \frac{1}{t}\int_t^{2t} w(s)\,|df(s)|\, dt \asymp \int_{x/2}^\infty \frac{w(t)}{t}\int_t^{2t} |df(s)|\, dt \lesssim \int_{x/2}^\infty \frac{w(t)}{t}\beta(t)\, dt,$$

as desired. ∎

As an example of a function w satisfying the hypotheses of Proposition 2.11, consider $w(t) = t^\gamma$ with $\gamma \in \mathbb{R}$. In particular, this was proved in [88] for $w(t) = 1$.

Note that if $f \in GM(\beta_2)$, then Proposition 2.11 allows us to deduce

$$\int_x^\infty w(t)|df(t)| \lesssim \int_{x/2}^\infty \frac{w(t)}{t^2}\int_{t/\lambda}^{\lambda t} |f(s)|\, ds\, dt \lesssim \int_{x/(2\lambda)}^\infty \frac{w(t)}{t}|f(t)|\, dt, \tag{2.10}$$

i.e., we can obtain conditions on the weighted variation of f from its weighted integrability conditions (provided that w is defined on $[x/(2\lambda), \infty)$ and satisfies the required assumption in this latter interval). We will make use of this fact in Chapter 4, where conditions on the weighted variation of f play a fundamental role. For future reference, we state (2.10) as a corollary:

Corollary 2.12. *Let $f \in GM(\beta_2)$ and $x > 0$. Let $w : [x/(2\lambda), \infty) \to \mathbb{R}_+$ be such that $w(y) \asymp w(z)$ for all $y, z \in [u, 2u]$ and all $u > x/(2\lambda)$. Then*

$$\int_x^\infty w(t)|df(t)| \lesssim \int_{x/(2\lambda)}^\infty \frac{w(t)}{t}|f(t)|\, dt.$$

2.4 Abel-Olivier test for real-valued $GM(\beta_2)$ functions

The goal of this section is to give a version of the well-known Abel-Olivier test for convergence of integrals in the case of real-valued functions from the class $GM(\beta_2)$. Such a test states that *if $f : \mathbb{R}_+ \to \mathbb{R}_+$ is monotone and the integral $\int_0^\infty f(t)\, dt$ converges, then*

$$tf(t) \to 0 \qquad \text{as } t \to \infty.$$

Its proof is rather trivial, since it follows from the inequality $xf(x) \le 2\int_{x/2}^{x} f(t)\,dt$, valid for all monotone f.

Taking into account the properties of $GM(\beta)$ functions, it is not difficult to extend Abel-Olivier test to the framework of nonnegative functions from $GM(\beta)$ classes, for reasonable choices of β (for instance, $\beta = \beta_2$, cf. [90]). Recent works [33, 41, 46, 50] show that several properties easily derived for nonnegative $GM(\beta_2)$ functions also hold in the case of $f \in GM(\beta_2)$ having non-constant sign. In other words, some of the problems that have been solved in the case of monotone functions can also be solved in the case of real-valued $GM(\beta_2)$ functions. It is worth mentioning that the class $GM(\beta_3)$ is too wide for this purpose, and additional essential hypoteses may come into play in such a case, as we shall see in Chapter 3 (see also [33, 41]). Of course, the analogue for the class GMS_2 is also true. We also cover that case in the present section.

Before proving Abel-Olivier's test for real-valued $GM(\beta_2)$ functions, we prove two auxiliary results, namely Lemmas 2.13 and 2.14, that are analogue to those proved in [46] for the class of sequences GMS_2.

Throughout this section we assume without loss of generality that the parameter λ in the definition of the class $GM(\beta_2)$ is of the form $\lambda = 2^\nu$, with $\nu \in \mathbb{N}$, and C will denote the constant from the GM condition (cf. Definition 2.4). For any function $f : \mathbb{R}_+ \to \mathbb{C}$ and any $n \in \mathbb{N}$, we define

$$A_n := \sup_{2^n \le t \le 2^{n+1}} |f(t)|,$$
$$B_n := \sup_{2^{n-2\nu} \le t \le 2^{n+2\nu}} |f(t)|.$$

For $n \in \mathbb{N} \cup \{0\}$, we say that n is a *good* number if either $n = 0$ or $B_n \le 2^{4\nu} A_n$. The rest of natural numbers consists of *bad* numbers.

To illustrate such definitions we give a couple of examples. On the one hand, if $f(t) = 1/t^2$ for $t \ge 1$, since

$$A_n = \frac{1}{2^{2n}}, \qquad B_n = \frac{1}{2^{2n-4\nu}},$$

then $B_n = 2^{4\nu} A_n$, and all natural numbers n (associated to f) are good. On the other hand, if $f(t) = 1/t^3$ for $t \ge 1$, since

$$A_n = \frac{1}{2^{3n}}, \qquad B_n = \frac{1}{2^{3n-6\nu}},$$

then $B_n = 2^{6\nu} A_n \not\le 2^{4\nu} A_n$, thus all natural numbers n are bad. More generally, if f decreases rapidly enough (faster than $1/t^2$, as for instance $1/t^3$ or e^{-t}), then all numbers $n \ne 0$ associated to f are bad.

Lemma 2.13. *Let $f \in GM(\beta_2)$ be real-valued. For any good number $n > 0$, there holds*

$$|E_n| := \left| \left\{ x \in [2^{n-\nu}, 2^{n+\nu}] : |f(x)| > \frac{A_n}{8C2^{2\nu}} \right\} \right| \ge \frac{2^n}{8C2^{5\nu}}, \tag{2.11}$$

where $|E|$ denotes the Lebesgue measure of E.

Proof. Assume (2.11) does not hold for $n > 0$. Let us define $D_n := [2^{n-\nu}, 2^{n+\nu}]\backslash E_n$. Then, since n is good,

$$\int_{2^{n-\nu}}^{2^{n+\nu}} \frac{|f(t)|}{t}\,dt = \int_{D_n} \frac{|f(t)|}{t}\,dt + \int_{E_n} \frac{|f(t)|}{t}\,dt$$
$$\leq \frac{2^{n+\nu}A_n}{8C2^{2\nu}2^{n-\nu}} + \frac{2^n B_n}{8C2^{5\nu}2^{n-\nu}} = \frac{B_n}{8C2^{4\nu}} + \frac{A_n}{8C} \leq \frac{A_n}{4C}.$$

Since $f \in GM(\beta_2)$, for any $x \in [2^n, 2^{n+1}]$,

$$|f(x)| \geq A_n - \int_{2^n}^{2^{n+1}} |df(t)| \geq A_n - C\int_{2^{n-\nu}}^{2^{n+\nu}} \frac{|f(t)|}{t}\,dt \geq A_n - \frac{A_n}{4} > \frac{A_n}{2},$$

which contradicts our assumption. □

Before stating the next lemma, let us introduce the following notation.

$$E_n^+ := \{x \in E_n : f(x) > 0\}, \qquad E_n^- := \{x \in E_n : f(x) \leq 0\}.$$

Lemma 2.14. *Let $f \in GM(\beta_2)$ be real-valued. For any good number $n > 0$, there is an interval $(\ell_n, m_n) \subset [2^{n-\nu}, 2^{n+\nu}]$ such that at least one of the following is true:*

1. *for any $x \in (\ell_n, m_n)$, there holds $f(x) \geq 0$ and*
$$|E_n^+ \cap (\ell_n, m_n)| \geq \frac{2^n}{256C^3 2^{15\nu}};$$
2. *for any $x \in (\ell_n, m_n)$, there holds $f(x) \leq 0$ and*
$$|E_n^- \cap (\ell_n, m_n)| \geq \frac{2^n}{256C^3 2^{15\nu}}.$$

Proof. First of all, note that by Lemma 2.13 one has that either $|E_n^+| \geq \dfrac{2^n}{16C2^{5\nu}}$ or $|E_n^-| \geq \dfrac{2^n}{16C2^{5\nu}}$. We assume the former, and prove that item 1. holds.

Let us cover the set E_n^+ by a union of intervals $\{I_j = [r_j, s_j]\}_{j=1}^{p_n}$ in $\left[2^{n-\nu}, 2^{n+\nu} + \varepsilon_n\right]$, with the property that the I_j's intersect with each other at most at one point. The number ε_n will be conveniently chosen later. We proceed as follows. Let $r_1 = \inf E_n^+$, and

$$\zeta_1 = \inf\{x \in [r_1, 2^{n+\nu}] : f(x) \leq 0\}.$$

If such ζ_1 does not exist, then we simply let $s_1 = 2^{n+\nu}$ and finish the process. Contrarily, we define

$$s_1 = \zeta_1 + \varepsilon_n.$$

Once we have the first interval $I_1 = [r_1, s_1]$, if $|E_n^+\backslash I_1| > 0$, we let $r_2 = \inf E_n^+\backslash I_1$, and define ρ_2 similarly as above, thus obtaining a new interval $I_2 = [r_2, s_2] = [r_2, \zeta_2 + \varepsilon_n]$. We continue this process until our collection of intervals is such that

$$|E_n^+\backslash(I_1 \cup I_2 \cup \cdots \cup I_{p_n})| = 0,$$

so that $E_n^+ \subset \cup_{j=1}^{p_n} I_j$.

By construction, for any $1 \le j \le p_n - 1$, we can find $y_j \in [r_j, \zeta_j]$ such that $y_j \in E_n^+$, and $z_j \in [\zeta_j, s_j]$ such that $f(z_j) \le 0$. Thus,

$$\int_{I_j} |df(t)| = \int_{r_j}^{s_j} |df(t)| \ge f(y_j) - f(z_j) \ge f(y_j) > \frac{A_n}{8C2^{2\nu}}.$$

Hence,

$$\int_{2^{n-\nu}}^{2^{n+\nu}} |df(t)| \ge \sum_{j=1}^{p_n-1} \int_{I_j} |df(t)| \ge (p_n - 1)\frac{A_n}{8C2^{2\nu}}.$$

On the other hand, the fact that n is good together with the assumption $f \in GM(\beta_2)$ imply that

$$\begin{aligned}\int_{2^{n-\nu}}^{2^{n+\nu}} |df(t)| &\le C2\nu \int_{2^{n-2\nu}}^{2^{n+2\nu}} \frac{|f(t)|}{t}\, dt \le C2\nu B_n \int_{2^{n-2\nu}}^{2^{n+2\nu}} \frac{1}{t}\, dt \\ &= C2\nu B_n \log 2^{4\nu} \le C2^{4\nu} 8\nu^2 A_n \log 2 \le C2^{7\nu} A_n.\end{aligned}$$

We can deduce from the above estimates that

$$p_n \le 8C^2 2^{9\nu} + 1 \le 8C^2 2^{10\nu}.$$

Since $E_n^+ \subset \cup_{j=1}^{p_n} I_j$, and the I_j's intersect with each other at most at one point, we have

$$\frac{2^n}{16C2^{5\nu}} \le |E_n^+| = \left| E_n^+ \cap \left(\bigcup_{j=1}^{p_n} I_j \right) \right| = \left| \bigcup_{j=1}^{p_n} (E_n^+ \cap I_j) \right| \le p_n \max_{1\le j\le p_n} |E_n^+ \cap I_j|,$$

so we can deduce that there exists $1 \le j_0 \le p_n$ such that

$$|E_n^+ \cap I_{j_0}| \ge \frac{2^n}{128C^3 2^{15\nu}}.$$

Taking $\varepsilon_n = 2^n/(256C^3 2^{15\nu})$, the interval $(\ell_n, m_n) = (r_{j_0}, s_{j_0} - \frac{2^n}{256C^3 2^{15\nu}}) = (r_{j_0}, \zeta_{j_0}) \subset [2^{n-\nu}, 2^{n+\nu}]$, has the desired properties by construction, and the proof is complete. □

We are in a position to prove Abel-Olivier's test for real-valued $GM(\beta_2)$ functions.

Theorem 2.15. *Let $f \in GM(\beta_2)$ be real-valued and vanishing at infinity. If the integral*

$$\int_0^\infty f(t)\, dt \tag{2.12}$$

converges, then

$$tf(t) \to 0 \qquad \text{as } t \to \infty.$$

Proof. Recall that the convergence of (2.12) is equivalent to the condition

$$\left| \int_M^N f(t)\, dt \right| \to 0 \qquad \text{as } N > M \to \infty.$$

We distinguish two cases, namely if there are finitely or infinitely many good numbers. Assume first there are infinitely many.

For any good number $n > 0$, it follows from Lemma 2.14 that

$$2^n A_n \frac{1}{2048 C^4 2^{17\nu}} < \left| \int_{\ell_n}^{m_n} f(t)\, dt \right|,$$

and moreover $f(x) \geq 0$ (or $f(x) \leq 0$) for all $x \in (\ell_n, m_n) \subset [2^{n-\nu}, 2^{n+\nu}]$. Since the integrals $\int_{\ell_n}^{m_n} f(t)\, dt$ vanish as $n \to \infty$ (by the convergence of (2.12)) we deduce that

$$2^n A_n \to 0 \qquad \text{as } n \to \infty,\ n \text{ good}. \tag{2.13}$$

We now prove that $2^n A_n$ also vanishes as $n \to \infty$ whenever n is bad. If n is a bad number, then $A_n < 2^{-4\nu} B_n$, and $B_n = A_{s_1}$, for some s_1 satisfying $|n - s_1| \leq 2\nu$.

Let us first assume that $s_1 < n$ and find the largest good number m which is smaller than n. If there is a good number in the set $[s_1, n-1] \cap \mathbb{N}$, we just choose m to be the largest good number from such a set and conclude the procedure. On the contrary, s_1 is a bad number. Then there exists s_2 satisfying $|s_1 - s_2| \leq 2\nu$ such that $A_{s_1} < 2^{-4\nu} B_{s_1} = 2^{-4\nu} A_{s_2}$. Also, note that $s_2 < s_1$; the opposite is not possible. Indeed, if $s_2 > s_1$, taking into account that $s_2 \leq n + 2\nu$,

$$A_{s_1} < 2^{-4\nu} B_{s_1} = 2^{-4\nu} A_{s_2} \leq 2^{-4\nu} B_n = 2^{-4\nu} A_{s_1},$$

which is absurd. Similarly as before, if there is any good number in the set $[s_2, s_1 - 1] \cap \mathbb{N}$, we choose m to be the largest good number from such a set and conclude the procedure.

Repeating this process, we arrive at a finite sequence $n = s_0 > s_1 > \cdots > s_{j-1} > s_j$, $j \geq 1$, where all the numbers in the set $[s_{j-1}, s_0 - 1] \cap \mathbb{N}$ are bad, and there is at least one good number $m \in [s_j, s_{j-1} - 1] \cap \mathbb{N}$ (fix it to be the largest from $[s_j, s_{j-1} - 1] \cap \mathbb{N}$). Note that $A_{s_k} < 2^{-4\nu} A_{s_{k+1}}$ and $|s_k - s_{k+1}| \leq 2\nu$ for any $0 \leq k \leq j-1$, thus $n \leq s_j + 2j\nu$. Also, the number m obtained by this procedure tends to infinity as $n \to \infty$, since there are infinitely many good numbers and m is the largest good number that is smaller than n. Since m is good, we deduce

$$2^n A_n < 2^{n-4\nu} A_{s_1} < \cdots < 2^{n-4j\nu} A_{s_j} \leq 2^{s_j - 2j\nu} A_{s_j} \leq 2^{2\nu} 2^m A_m \to 0$$

as $n \to \infty$, by (2.13).

Suppose now that $n < s_1$, and assume $A_n > 0$ (if this n does not exist, our assertion follows trivially). Then either s_1 is a good number, or there exists $s_1 < s_2$ such that $|s_2 - s_1| \leq 2\nu$ and $A_{s_1} < 2^{-4\nu} A_{s_2}$ (note that the case $s_2 < s_1$ is not possible, since it leads to a contradiction as above). Similarly as before, we iterate the procedure until we find a finite sequence $n = s_0 < s_1 < \cdots < s_{j-1} < s_j$, where $|s_k - s_{k+1}| \leq 2\nu$ and s_k are bad for $0 \leq k \leq j-1$, i.e., $A_{s_k} < 2^{-4\nu} A_{s_{k+1}}$. Note that such a good number s_j can always be found. Indeed, if we assume the contrary, that is, we obtain an infinite sequence

$$n = s_0 < s_1 < \cdots < s_j < \cdots,$$

where all numbers s_k are bad, then $A_{s_{k+1}}/A_{s_k} > 2^{4\nu}$ for all k. Since $A_n > 0$, we obtain that $A_{s_k} \to \infty$ as $k \to \infty$, which contradicts the hypothesis of f vanishing at infinity. We can now deduce from the inequality $n < s_j$ that

$$2^n A_n < 2^{n-4\nu} A_{s_1} < \cdots < 2^{n-4j\nu} A_{s_j} < 2^{s_j} A_{s_j} \to 0$$

as $n \to \infty$, by (2.13). This completes the part of the proof where we assume there are infinitely many good numbers.

Assume now there are finitely many good numbers n. Assume that $N \in \mathbb{N}$ is such that $m \le N$ for all good numbers m. If $n > N$, then n is a bad number, hence $A_n < 2^{-4\nu} B_n$, and $B_n = A_{s_1}$ for some s_1 satisfying $|n - s_1| \le 2\nu$.

If $s_1 < n$, one can find, in a similar way as above, a sequence $n = s_0 > s_1 > \cdots > s_{j-1} > s_j$, where $s_0, s_1, \ldots, s_{j-1}$ are bad and s_j is good, and moreover $n \le s_j + 2j\nu$. Since $s_j \le N$,

$$j \ge \frac{n - s_j}{2\nu} \ge \frac{n - N}{2\nu}, \tag{2.14}$$

and we deduce

$$2^n A_n < 2^{n-4\nu} A_{s_1} < \cdots < 2^{n-4j\nu} A_{s_j} \le 2^{s_j - 2j\nu} A_{s_j} \le 2^{N-2j\nu} \max_{0 \le k \le N} A_k.$$

The latter vanishes as $n \to \infty$, since in such case $j \to \infty$, by (2.14).

Finally, we are left to investigate the case $s_1 > n$. We actually show that this case is not possible. Let n be such that $A_n > 0$ (if this n does not exist, our assertion follows trivially). If $s_1 > n$, then there is an infinite sequence of bad numbers $n = s_0 < s_1 < s_2 < \cdots$ such that $A_{s_j} < 2^{-4\nu} B_{s_j} = 2^{-4\nu} A_{s_{j+1}}$ for every $j \ge 0$. Hence,

$$\frac{A_{s_{j+1}}}{A_{s_j}} > 2^{4\nu} \qquad \text{for all } j \ge 0,$$

i.e., the sequence A_{s_k} does not vanish as $k \to \infty$. This contradicts the hypothesis of f vanishing at infinity, showing that the case $s_1 > n$ is not possible and thus completing the proof. $\square$

We now derive some corollaries of Theorem 2.15. Let us first improve Theorem 2.15 to an "if and only if" statement. A multidimensional analogue of the following is obtained in [36] for the so-called weak monotone sequences (see also [90]).

Corollary 2.16. *Let $\nu \in \mathbb{R}\backslash\{0\}$. Let $f \in GM(\beta_2)$ be real-valued, vanishing at infinity, and such that $t^\nu f(t) \to 0$ as $t \to 0$. Then $\int_0^\infty t^{\nu-1} f(t)\,dt$ converges if and only if*

$$t^\nu f(t) \to 0 \text{ as } t \to \infty \qquad \text{and} \qquad \int_0^\infty t^\nu df(t) \text{ converges,}$$

and moreover,

$$\int_0^\infty t^{\nu-1} f(t)\,dt = -\frac{1}{\nu} \int_0^\infty t^\nu df(t).$$

Proof. First we note that if $f \in GM(\beta_2)$, then $t^\nu f(t) \in GM(\beta_2)$ for every $\nu \in \mathbb{R}$ (cf. [86]). Integration by parts along with the condition $t^\nu f(t) \to 0$ as $t \to 0$ implies that, for any $N \in \mathbb{R}_+$,

$$\int_0^N t^{\nu-1} f(t)\,dt = \frac{1}{\nu} N^\nu f(N) - \frac{1}{\nu} \int_0^N t^\nu df(t),$$

see [21]. Letting $N \to \infty$ yields the desired result, where we apply Theorem 2.15 to prove the "only if" part and the equality given in the statement. $\square$

The analogue of Theorem 2.15 for the class GMS_2 reads as follows:

Corollary 2.17. *Let $\{a_n\} \in GMS_2$ be real-valued. If the series $\sum_{n=0}^{\infty} a_n$ converges, then*

$$na_n \to 0 \qquad \text{as } n \to \infty.$$

Proof. Let

$$f(t) = a_n, \qquad t \in (n, n+1],\, n \in \mathbb{N} \cup \{0\}.$$

It is clear that $f \in GM(\beta_2)$ if and only if $\{a_n\} \in GMS_2$. Moreover, the convergence of $\sum_{n=0}^{\infty} a_n$ is equivalent to the convergence of (2.12) and implies that the function f vanishes at infinity. Applying Theorem 2.15, we derive that $tf(t) \to 0$ as $t \to \infty$, or in other words, $na_n \to 0$ as $n \to \infty$. □

With Corollary 2.17 in hand one can prove the analogue of Corollary 2.16 for the class of sequences GMS_2, which we do not state for the sake of brevity.

2.5 A generalization of the class $GM(\beta_3)$

Our next goal is to introduce a class of general monotone functions that extends the class $GM(\beta_3)$ (see p. 13 or (2.15) below).

In order to construct a new $GM(\beta)$ class we need to give an expression of the majorant β. We take as a starting point the class $GM(\beta_3)$; recall that

$$\beta_3(x) = \frac{1}{x} \sup_{s \geq x/\lambda} \int_s^{2s} |f(t)|\, dt, \qquad \lambda > 1. \tag{2.15}$$

If we denote $I(x) = I(f;x) := \int_x^{2x} |f(t)|\, dt$, then we can rewrite

$$\beta_3(x) = \frac{1}{x} \sup_{s \geq x/\lambda} I(s), \qquad \lambda > 1.$$

Let us now denote, for a function $\varphi : (0,\infty) \to \mathbb{R}_+$, $B(x,\varphi) = \sup\limits_{s \geq x/\lambda} \varphi(s)$. Then it is clear that $\beta_3(x) = x^{-1} B(x, I)$. This motivates our next definition.

Definition 2.18. Let $\mathcal{M}_+$ be the space of nonnegative functions defined on $(0,\infty)$. We say that an operator $B : \mathcal{M}_+ \to \mathcal{M}_+$ is admissible if for any $\varphi \in \mathcal{M}_+$, the function $B(\cdot, \varphi)$ satisfies the following properties:

(i) if φ vanishes at infinity, then $B(\cdot,\varphi)$ vanishes at infinity;

(ii) if φ is bounded at infinity, then $B(\cdot,\varphi)$ is bounded at infinity;

(iii) for every $x > 0$, $\varphi(x) \leq B(x,\varphi)$;

(iv) the function $B(x,\varphi)$ is monotone in x.

We readily observe that $B(x,\varphi) = \sup\limits_{s \geq x/\lambda} \varphi(s)$ satisfies all the properties listed in Definition 2.18. We now introduce the generalization of the class $GM(\beta_3)$, obtained by means of such admissible operators.

Definition 2.19. We say that $f \in GM_{\text{adm}}$ if there exists an admissible operator B such that $f \in GM(\beta)$, where

$$\beta(x) = \frac{1}{x} B(x, I).$$

In fact, the β from Definition 2.19 is obtained by just replacing the supremum in (2.15) by an operator satisfying similar properties. Obviously, one has $GM(\beta_3) \subset GM_{\text{adm}}$. The proper inclusion $GM(\beta_3) \subsetneq GM_{\text{adm}}$ is proved below in Proposition 2.22. However, there are some observations to be done before. In the first place, note that if we define, for an admissible operator B,

$$\beta_B(x) = \frac{1}{x} B(x, I), \tag{2.16}$$

then

$$GM_{\text{adm}} = \bigcup_{B \text{ admissible}} GM(\beta_B).$$

Secondly, properties (iii) and (iv) from Definition 2.18 are useful in terms of calculations, and for any given operator B satisfying properties (i) and (ii) from Definition 2.18, we can construct an admissible operator $\widetilde{B}$ (i.e., satisfying properties (i)–(iv)). Indeed, if we define, for $\varphi \in \mathcal{M}_+$,

$$\widetilde{B}(x, \varphi) = \sup_{y \geq x} \max \big\{ \varphi(y), B(y, \varphi) \big\},$$

then $\widetilde{B}$ meets all conditions from Definition 2.18. Moreover, if we denote

$$\beta(x) = \frac{1}{x} B(x, I), \qquad \tilde{\beta}(x) = \frac{1}{x} \widetilde{B}(x, I),$$

it is clear that $GM(\beta) \subset GM\big(\tilde{\beta}\big)$.

For $f \in GM_{\text{adm}}$, let B be an admissible operator such that $f \in GM(\beta_B)$ with β_B defined by (2.16). The class GM_{adm} is the widest known class for which the conditions $xf(x) \to 0$ as $x \to \infty$ and $x\beta_B(x) \to 0$ as $x \to \infty$ are equivalent. Indeed, one direction of this equivalence follows from property (i), whilst the other direction follows from Proposition 2.8 and property (iii) of B (see also Remark 2.9). Both conditions $xf(x) \to \infty$ as $x \to \infty$ and $x\beta_B(x) \to 0$ as $x \to \infty$ are key in the study of the uniform convergence of sine transforms, see Theorem 3.4 below.

We now list some examples of admissible operators. Note that for some of the operators B_j appearing here we only assume that the functions $B_j(\cdot, \varphi)$, $\varphi \in \mathcal{M}_+$, satisfy properties (i)–(ii) of Definition 2.18; afterwards, one may construct an admissible operator (i.e., satisfying (i)–(iv) of Definition 2.18) $\widetilde{B}_j$ following the process just described above.

(1) $B_1(x, \varphi) = \varphi(x)$;

(2) $B_2(x, \varphi) = \varphi(x)^\alpha$, where $\alpha > 0$;

(3) $B_3(x, \varphi) = \int_{x/\lambda}^{\lambda x} \varphi(t)/t \, dt$, where $\lambda > 1$;

(4) $B_4(x, \varphi) = x^\alpha \int_{x/\lambda}^{\infty} \varphi(t)/t^{\alpha+1}$, where $\lambda > 1$ and $\alpha > 0$;

(5) $B_5(x, \varphi) = \sup_{s \geq x/\lambda} \varphi(s)$, where $\lambda > 1$;

(6) $B_6(x, \varphi) = \sup_{s \geq \psi(x)} \varphi(s)$, where ψ is increasing to infinity (see [76, 77, 132]);

(7) The composition of two admissible operators is an admissible operator, i.e., if D_1 and D_2 are admissible, it follows readily from the definition that the function

$$B_7(x,\varphi) = D_1(x, D_2(\cdot,\varphi))$$

satisfies properties (i)–(iv).

Remark 2.20. We cannot allow $\alpha = 0$, in B_4, since the operator would not be admissible. Indeed, if

$$\varphi(x) = \begin{cases} 0, & \text{if } x < 2, \\ (\log x)^{-1}, & \text{otherwise,} \end{cases}$$

then φ clearly vanishes at infinity, but for any $x > 2\lambda$, one has

$$\int_{x/\lambda}^{\infty} \frac{\varphi(t)}{t}\, dt = \int_{x/\lambda}^{\infty} \frac{1}{t\log t}\, dt = \infty,$$

and therefore $B_4(\cdot,\varphi)$ does not satisfy condition (ii) of Definition 2.18.

Using Proposition 2.8, Remark 2.9, and property (iii) of admissible operators, we have the following estimate for $f \in GM_{\text{adm}}$.

Corollary 2.21. *If $f \in GM_{adm}$, for any $x > 0$ and any $t \in [x, 2x]$,*

$$|f(t)| \le \frac{(C+1)}{x} B(x, I), \tag{2.17}$$

where C is the constant from (2.9).

To conclude this section, we show that the class GM_{adm} is strictly larger than $GM(\beta_3)$.

Proposition 2.22. *The proper inclusion $GM(\beta_3) \subsetneq GM_{adm}$ holds.*

Proof. As mentioned above, the inclusion is clear. Thus, we only need to find a function $f \in GM_{\text{adm}} \backslash GM(\beta_3)$ and a suitable admissible operator B. Let us define $n_j = 4^j$ and

$$f(x) := \begin{cases} n_j^{-\frac{1}{1-\alpha}}, & \text{if } n_j \le x \le n_j + 1, \quad j \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$$

and for $0 < \alpha < 1$, we define the admissible operator $B_\alpha(x,\varphi) := \sup_{s \ge x/\lambda} \varphi(s)^\alpha$, with $\lambda > 1$.

For any $x \in (n_{j-1}+1, n_j+1]$, one has

$$\int_x^{2x} |df(t)| \le 2n_j^{-\frac{1}{1-\alpha}}.$$

Moreover, since $n_{j+1} = 4n_j$, for j large enough, (recall that $I(x) = \int_x^{2x} |f(t)|\, dt$)

$$\frac{1}{x} B_\alpha(x, I) \ge \frac{1}{n_j+1} B_\alpha(x, I) \asymp n_j^{-1-\frac{\alpha}{1-\alpha}} = n_j^{-\frac{1}{1-\alpha}} \gtrsim \int_x^{2x} |df(t)|,$$

so that $f \in GM(\beta_\alpha)$, where $\beta_\alpha(x) = x^{-1}B_\alpha(x, I)$, and therefore $f \in GM_{\mathrm{adm}}$. On the other hand,

$$\frac{1}{n_j} \sup_{s \geq n_j/\lambda} \int_s^{2s} |f(t)|\, dt \asymp n_j^{-1-\frac{1}{1-\alpha}} = n_j^{-\frac{2-\alpha}{1-\alpha}}.$$

However, if the inequality

$$\int_{n_j}^{2n_j} |df(t)| = 2n_j^{-\frac{1}{1-\alpha}} \lesssim n_j^{-\frac{2-\alpha}{1-\alpha}}$$

is true, then $2 - \alpha \leq 1$, i.e., $\alpha \geq 1$, which cis a contradiction. Thus, $f \notin GM(\beta_3)$, and our claim follows. $\square$

2.6 General monotonicity in two variables

Before defining the concept of general monotonicity for functions of two variables, it is necessary that we introduce the variation of a function in two dimensions. We also take this opportunity to briefly discuss the concept of bounded variation for functions of two variables. C. R. Adams and J. R. Clarkson collected several different definitions in [3] and studied the relationship between them. They also investigated the properties of functions of bounded variation in two dimensions [2, 4].

Here we are interested in the so-called Hardy bounded variation condition. Let us introduce the following notation. For any nonnegative increasing sequences $\{x_n\}, \{y_n\}$, a function $f : \mathbb{R}_+^2 \to \mathbb{C}$, and $x, y \in \mathbb{R}_+$, we put

$$\begin{aligned}
\Delta_{10} f(x_j, y) &:= f(x_j, y) - f(x_{j+1}, y),\\
\Delta_{01} f(x, y_k) &:= f(x, y_k) - f(x, y_{k+1}),\\
\Delta_{11} f(x_j, y_k) &:= \Delta_{01}\big(\Delta_{10} f(x_j, y_k)\big) = \Delta_{10}\big(\Delta_{01} f(x_j, y_k)\big)\\
&= f(x_j, y_k) - f(x_{j+1}, y_k) - f(x_j, y_{k+1}) + f(x_{j+1}, y_{k+1}).
\end{aligned}$$

There are alternative definitions for the mixed differences $\Delta_{11} f$ (cf. [3]), as for instance

$$\Delta f(x_j, y_k) := f(x_{j+1}, y_{k+1}) - f(x_j, y_k).$$

We, however, disregard those alternative definitions, since the differences $\Delta_{11} f$ are the most appropriate in the topics treated in this dissertation. Indeed, they appear naturally when one applies Abel's transformation, or integration by parts once in each variable.

For a compact rectangle $J := [a, b] \times [c, d] \subset \mathbb{R}_+^2$, we define the *Hardy variation of* f *over* J (also referred to as the *Vitali variation*, cf. [3]) as

$$HV_J(f) := \sup_{P \in \mathcal{P}(J)} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} |\Delta_{11} f(x_j, y_k)|,$$

where $P = \{a = x_0 < x_1 < \cdots < x_m = b\} \times \{c = y_0 < y_1 < \cdots < y_n = d\}$ is a partition of J and $\mathcal{P}(J)$ denotes the set of all partitions of J.

Definition 2.23. We say that $f : \mathbb{R}_+^2 \to \mathbb{C}$ is of Hardy bounded variation on J ($f \in HBV(J)$) if $HV_J(f) < \infty$ and, in addition, the marginal functions $f(x_0, \cdot)$ and $f(\cdot, y_0)$ are of bounded variation on $[c, d]$ and $[a, b]$, respectively, for some $x_0 \in [a, b]$ and $y_0 \in [c, d]$.

Likewise, we can define the concept of bounded variation on the whole $\mathbb{R}^2_+$ (or on any non-compact rectangle) taking the variation to be the supremum of the variation over compact rectangles:

Definition 2.24. We say that $f : \mathbb{R}^2_+ \to \mathbb{C}$ is of Hardy bounded variation on $\mathbb{R}^2_+$ ($f \in HBV(\mathbb{R}^2_+)$) if the marginal functions $f(x_0, \cdot)$, $f(\cdot, y_0)$ are of bounded variation on $\mathbb{R}_+$ for any $x_0, y_0 \in \mathbb{R}_+$, and $HV_J(f) < \infty$ for any compact rectangle $J \subset \mathbb{R}^2_+$, and furthermore

$$HV_{\mathbb{R}^2_+}(f) := \sup_{J\subset\mathbb{R}^2_+} HV_J(f) < \infty.$$

Remark 2.25. Originally in the literature, Definition 2.23 was (a priori) more restrictive; it required that

$$f_2^{x_0} := f(x_0, \cdot) \in BV([c, d]), \qquad f_1^{y_0} := f(\cdot, y_0) \in BV([a, b]), \tag{2.18}$$

for every $x_0 \in [a, b]$ and every $y_0 \in [c, d]$, respectively. However, W. H. Young proved in [143] that such condition is redundant; it is enough to assume that (2.18) holds for only one $x_0 \in [a, b]$ and one $y_0 \in [c, d]$, respectively. Then (2.18) follows for all $x_0 \in [a, b]$ and all $y_0 \in [c, d]$, respectively, provided that $HV_J(f) < \infty$. We refer the reader to [67, §254] for further details.

Like in the one-dimensional case, we can show that if $J = [a, b] \times [c, d]$ and $HV_J(f)$ is finite, it coincides with the double Riemann-Stieltjes integral

$$\int_c^d \int_a^b |d_{11}f(s, t)|\, ds\, dt := \lim_{\|P\|\to 0} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} |\Delta_{11}f(x_j, y_k)|, \tag{2.19}$$

where $P = \{x_j\}_{j=0}^m \times \{y_k\}_{k=0}^n \in \mathcal{P}(J)$, and

$$\|P\| := \max\left\{ \max_j\{x_{j+1} - x_j\}, \max_k\{y_{k+1} - y_k\}\right\}.$$

Remark 2.26. In the two-dimensional Stieltjes integral (see (2.19)) we are including the terms "$ds\, dt$" with the sole purpose of specifying the order of integration, since their incorporation is not formally correct. We will keep this notation in single Stieltjes integrals of functions of two variables (i.e., integrals with $d_{10}f$ and $d_{01}f$ defined right below).

Whenever $z \geq 0$ is fixed, the one-dimensional variation of f with respect to the first or second variable on $[a, b] \subset \mathbb{R}_+$ is defined as the variation of the function of one variable $f_z(x) = f(x, z)$ or $f_z(y) = f(z, y)$ over $[a, b]$ respectively, and it is denoted by

$$\int_a^b |d_{10}f(s, z)|\, ds \quad \text{or} \quad \int_a^b |d_{01}f(z, t)|\, dt,$$

respectively.

We can also define the Riemann-Stieltjes integral of $g : \mathbb{R}^2_+ \to \mathbb{C}$ with respect to f over a compact rectangle $J = [a, b] \times [c, d]$ as follows. For a partition $P = \{x_j\}_{j=0}^m \times \{y_k\}_{k=0}^n \in \mathcal{P}(J)$ of J and sequences $\{\xi_j\}_{j=0}^{m-1}$, $\{\zeta_k\}_{k=0}^{n-1}$ satisfying $x_j \leq \xi_j \leq x_{j+1}$ for $j = 0, \dots, m-1$ and $\zeta_k \leq y_k \leq \zeta_{k+1}$ for $k = 0, \dots, n-1$, we write

$$\int_c^d \int_a^b g(s, t) d_{11}f(s, t)\, ds\, dt = \lim_{\|P\|\to 0} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} g(\xi_j, \zeta_k)\Delta_{11}f(x_j, y_k), \tag{2.20}$$

whenever the limit exists. Likewise, we can define the weighted variation of f with respect to a continuous function g by incorporating the absolute value inside the sum in (2.20), analogously as done in the one dimensional case. For noncompact rectangles J, (2.20) is defined as the limit of the corresponding integrals over compact rectangles "approximating" J.

Similarly as in the one-dimensional case, we will use the estimate

$$|f(x,y)| \leq \int_y^\infty \int_x^\infty |d_{11}f(s,t)|\, ds\, dt \tag{2.21}$$

valid for all f of Hardy bounded variation on $[x,\infty)\times[y,\infty)$ and vanishing as $x+y\to\infty$. To prove (2.21), it suffices to apply (2.4) to each variable. This estimate will be used repeatedly in Section 3.2.

One can also consider the Riemann-Stieltjes integral in the improper sense whenever the rectangle is a product of not necessarily compact intervals. In particular, if $HV_{\mathbb{R}^2_+}(f) < \infty$, then

$$HV_{\mathbb{R}^2_+}(f) = \int_0^\infty \int_0^\infty |d_{11}f(s,t)|\, ds\, dt.$$

It is worth emphasizing that in order to be able to write the Hardy variation of f over a rectangle J it is not necessary that f is of Hardy bounded variation on J, but only $HV_J(f) < \infty$.

Note that if $f \in HBV(\mathbb{R}^2_+)$, then f is necessarily bounded, since all the marginal functions are of bounded variation on $\mathbb{R}_+$. Thus, it is a rather restrictive property. Since in this work we are particularly interested in double sine integrals

$$F(u,v) = \int_0^\infty \int_0^\infty f(x,y)\, \sin ux\, \sin vy\, dx\, dy, \tag{2.22}$$

where f is assumed to be of bounded variation. However, we are also interested in certain functions f that are unbounded near the origin and decay fast at infinity, so that we can expect a good behaviour of the integral (2.22) in terms of convergence. As an example, consider

$$f(x,y) = \begin{cases} (xy)^{-1}, & \text{if } x,y<1, \\ e^{-(x+y)}, & \text{otherwise.} \end{cases} \tag{2.23}$$

This motivates us to consider a slightly more general bounded variation condition, namely to restrict it to some subset of $\mathbb{R}^2_+$ not containing the origin, instead of the whole $\mathbb{R}^2_+$.

Definition 2.27. Let $c>0$. We say that f is of Hardy bounded variation on $\mathbb{R}^2_+\backslash[0,c)^2$, written $f\in HBV\big(\mathbb{R}^2_+\backslash[0,c)^2\big)$, if

$$\sup_{J\subset\mathbb{R}_+\times[c,\infty)} HV_J(f), \qquad \sup_{J\subset[c,\infty)\times\mathbb{R}_+} HV_J(f),$$

are finite, and moreover, the marginal functions $f(x_0,\cdot)$, $f(\cdot,y_0)$ are of bounded variation on $\mathbb{R}_+$ for any $x_0,y_0\geq c$, respectively.

Under this definition, functions are only required to be bounded on $\mathbb{R}^2_+\backslash[0,c)^2$, and allows us to consider examples such as (2.23).

A similar concept is that of *local bounded variation*. We say that a function $f:\mathbb{R}^2_+\to\mathbb{C}$ is locally of bounded variation (in the sense of Hardy) on a set $S\subset\mathbb{R}^2_+$ (written $f\in HBV_{\rm loc}(S)$) if it is of Hardy bounded variation on every compact rectangle $J\subset S$.

We are now ready define general monotone functions of two variables:

Definition 2.28. Let $f:\mathbb{R}_+^2\to\mathbb{C}$, $f\in HBV_{\mathrm{loc}}(\mathbb{R}_+^2)$, and $\beta:(0,\infty)^2\to\mathbb{R}_+$. We say that f is a general monotone function with majorant β (written $(f,\beta)\in\mathcal{GM}^2$), if there exists a constant $C>0$ such that

$$\int_y^{2y}\int_x^{2x}|d_{11}f(s,t)|\,ds\,dt\leq C\beta(x,y),\qquad \text{for all } x,y>0. \tag{2.24}$$

As in the one-dimensional case, the majorant β will typically depend on the function f. In this case we also write $f\in GM^2(\beta)$, abusing of notation.

The superscript 2 in $\mathcal{GM}^2$ and GM^2 stands for the dimension.

The concept of general monotonicity in several dimensions was first introduced by S. Tikhonov and M. Dyachenko for the two-dimensional case in [42], and for multivariate case in [43]. We denote, for $d>1$, $\mathbf{n}=(n_1,\dots,n_d)\subset\mathbb{N}^d$.

Definition 2.29. Let $\beta=\{\beta_{\mathbf{n}}\}$ be a nonnegative sequence. We say that a complex-valued sequence $\{a_{\mathbf{n}}\}$ is β-general monotone ($\{a_{\mathbf{n}}\}\in GMS_{\#}^d(\beta)$) if there exists a constant $C>0$ such that for every $\mathbf{n}\in\mathbb{N}^d$,

$$\sum_{\mathbf{k}=\mathbf{n}}^{\infty}\left|\Delta^{1,\dots,1}a_{\mathbf{k}}\right|\leq C\beta_{\mathbf{n}},$$

where the operator $\Delta^{1,\dots,1}$ is defined as follows: $\Delta^{1,\dots,1}=\prod_{j=1}^d\Delta^j$, and $\Delta^j a_{\mathbf{n}}=a_{\mathbf{n}}-a_{n_1,n_2,\dots,n_{j-1},n_j+1,n_{j+1},\dots,n_d}$.

Here $\Delta^{1,\dots,1}$ is the mixed difference (see [105]).

The subscript "#" from $GMS_{\#}^d$ in Definition 2.29 is to distinguish between alternative definitions of general monotonicity that have been used in the literature, which we will discuss.

In [80, 84], the authors consider a different definition of general monotone functions of two variables. Namely they required that there exist β^1,β^2 such that

$$\int_x^{2x}|d_{10}f(s,y)|\,ds\leq C\beta^1(x,y),\qquad \int_y^{2y}|d_{01}f(x,t)|\,dt\leq C\beta^2(x,y). \tag{2.25}$$

However, instead of choosing these β^1,β^2 arbitrarily, we use the following intrinsic expressions that follow from (2.24): if $f\in HBV_{\mathrm{loc}}(\mathbb{R}_+^2)$ and

$$f(x,y)\to 0\qquad \text{as } x+y\to\infty,$$

we can formally write for any $x,y>0$,

$$\int_x^{2x}|d_{10}f(s,y)|\,ds=\int_x^{2x}\left|\int_y^{\infty}d_{11}f(s,t)\,dt\right|ds\leq\int_y^{\infty}\int_x^{2x}|d_{11}f(s,t)|\,ds\,dt, \tag{2.26}$$

$$\int_y^{2y}|d_{01}f(x,t)|=\int_y^{2y}\left|\int_x^{\infty}d_{11}f(s,t)\,ds\right|dt\leq\int_y^{2y}\int_x^{\infty}|d_{11}f(s,t)|\,ds\,dt. \tag{2.27}$$

Defining

$$\beta^1(x,y):=\sum_{k=0}^{\infty}\beta(x,2^ky),\qquad \beta^2(x,y):=\sum_{j=0}^{\infty}\beta(2^jx,y),$$

it is clear that (2.25) holds. It is important to stress that such definitions of β^1 and β^2 is formal, as the series representing them may be divergent if β does not decay fast enough.

Note also that (2.26) and (2.27) require that f is of Hardy bounded variation on $(0,\infty)^2$, although we only use this condition to motivate the definitions of β^1, β^2, since they are then defined only in terms of the Hardy variation over rectangles.

The two-dimensional analogue of the function class $GM(\beta_2)$ defined in Section 2.3 was considered by P. Kórus and F. Móricz in [80], where they dealt with the $GM^2(\beta)$ class given by

$$\beta(x,y)=\frac{1}{xy}\int_{y/\lambda}^{\lambda y}\int_{x/\lambda}^{\lambda x}|f(s,t)|\,ds\,dt, \tag{2.28}$$

for some $\lambda>1$, and also chose β^1 and β^2 arbitrarily to be

$$\beta^1(x,y)=\frac{1}{x}\int_{x/\lambda}^{\lambda x}|f(s,y)|\,ds \qquad \beta^2(x,y)=\frac{1}{y}\int_{y/\lambda}^{\lambda y}|f(x,t)|\,dt.$$

They call such functions of *mean value bounded variation*, and denote their corresponding class by $MVBVF^2$. It is also worth mentioning that in [80], the authors deal with locally absolutely continuous functions on $(0,\infty)$ $\big(f\in AC_{\rm loc}\big((0,\infty)^2\big)\big)$. In Definition 2.28 we require the condition $f\in HBV_{\rm loc}(\mathbb{R}_+^2)$ (in the sense of Hardy, cf. Definition 2.23). It is known that $AC_{\rm loc}(\mathbb{R}^2_+)\subsetneq HBV_{\rm loc}(\mathbb{R}^2_+)$ (see [15]). However, the classes $AC_{\rm loc}\big((0,\infty)^2\big)$ and $HBV_{\rm loc}\big(\mathbb{R}^2_+\big)$ are not comparable, thus we cannot rigorously compare $GM^2(\beta)$ classes with the class $MVBVF^2$. Nonetheless, the conditions $f\in AC_{\rm loc}\big((0,\infty)^2\big)$ and $f\in HBV_{\rm loc}\big(\mathbb{R}^2_+\big)$ are rather technical, thus we are more concerned in finding whether there exists f with sufficiently good properties (for instance, $f\in HBV_{\rm loc}\big((0,\infty)^2\big)$), such that (2.24) does not hold with β from (2.28), but holds with another choice of β instead.

Similarly as in the one-dimensional case, we want to introduce a new $GM^2(\beta)$ class that will "contain" (in the above sense) all the previously known ones. In fact, this new class will be analogue to the class $GM_{\rm adm}$ defined in Section 2.5.

We first define the concept of admissible operator in two dimension analogous to that of Definition 2.18.

Definition 2.30. Let $\mathcal{M}^n_+$ denote the space of nonnegative functions defined on $(0,\infty)^n$. We say that an operator $B:\mathcal{M}^2_+\to\mathcal{M}^1_+$ is admissible if for any $\varphi\in\mathcal{M}^2_+$, the function $B(\cdot,\cdot,\varphi)$ satisfies the following properties:

(i) if $\varphi(x_0,y_0)\to 0$ as $x_0+y_0\to\infty$, then $B(x,y,\varphi)\to 0$ as $x+y\to\infty$;

(ii) for all $x,y>0$, there holds $\varphi(x,y)\le B(x,y,\varphi)$.

Note that there is difference between this definition and its one-dimensional analogue (Definition 2.18). Indeed, conditions (ii) and (iv) in Definition 2.18 do not have analogues in Definition 2.30. On the one hand. condition (ii) from Definition 2.18 is needed to prove a result that we could not extend to the two-dimensional framework, (namely the necessity part of Theorem 3.7 below), so we can disregard it. On the other hand, condition (iv) from Definition 2.18 is just a technical condition that may not be assumed, as it only simplifies calculations (compare Corollary 2.21 with Lemma 2.33 and Remark 2.34 below).

Before proceeding, let us introduce the following notation. For a function f defined on $\mathbb{R}^2_+$ and $x,y>0$, we define

$$I_{12}(f;x,y)=I_{12}(x,y):=\int_y^{2y}\int_x^{2x}|f(s,t)|\,ds\,dt.$$

We are now ready to introduce an analogue of the $GM_{\rm adm}$ class in two dimensions.

Definition 2.31. We say that a function $f:\mathbb{R}_+^2\to\mathbb{C}$, $f\in HBV_{\mathrm{loc}}(\mathbb{R}_+^2)$, belongs to the class GM^2_{adm} if there exists an admissible operator B such that $f\in GM^2(\beta)$, where

$$\beta(x,y)=\frac{1}{xy}B\big(x,y,I_{12}\big). \tag{2.29}$$

Several observations are natural analogues to those of the one-dimensional setting: if for an admissible operator B we define

$$\beta_B(x,y)=\frac{1}{xy}B\big(x,y,I_{12}\big),$$

then

$$GM^2_{\mathrm{adm}}=\bigcup_{B\text{ admissible}} GM^2(\beta_B).$$

Also, condition (ii) of Definition 2.30 does not make us lose generality in terms of GM^2 classes, i.e., if B is an operator satisfying condition (i) of Definition 2.30, and we define, for $\varphi\in\mathcal{M}_+^2$,

$$\widetilde{B}\big(x,y,\varphi\big)=\max\big\{\varphi(x,y),B\big(x,y,\varphi\big)\big\},$$

then $\widetilde{B}$ is admissible, and if we denote

$$\beta(x,y)=\frac{1}{xy}B\big(x,y,I_{12}\big),\qquad \tilde{\beta}(x,y)=\frac{1}{xy}\widetilde{B}\big(x,y,I_{12}\big),$$

then $GM^2(\beta)\subset GM^2\big(\tilde{\beta}\big)$. Thus, we are essentially interested in operators B satisfying condition (i) of Definition 2.30. Examples of such operators are the following:

(1) $B_1\big(x,y,\varphi\big)=\varphi(x,y)$;

(2) $B_2\big(x,y,\varphi\big)=\varphi(x,y)^\alpha$, where $\alpha>0$;

(3) $B_3\big(x,y,\varphi\big)=\int_{x/\lambda}^{\lambda x}\int_{y/\lambda}^{\lambda y}\varphi(s,t)/(st)\,ds\,dt$, where $\lambda>1$;

(4) $B_4\big(x,y,\varphi\big)=(xy)^\alpha\int_{x/\lambda}^{\infty}\int_{y/\lambda}^{\infty}\varphi(s,t)/(st)^{\alpha+1}\,ds\,dt$, where $\lambda>1$ and $\alpha>0$;

(5) $B_5\big(x,y,\varphi\big)=\sup_{s+t\geq(x+y)/\lambda}\varphi(s,t)$, where $\lambda>1$;

(6) $B_6\big(x,y,\varphi\big)=\sup_{s+t\geq\psi(x+y)}\varphi(s,t)$, where $\psi:\mathbb{R}_+\to\mathbb{R}_+$ is increasing to infinity (see also [75]);

(7) The composition of admissible operators is admissible. That is, if D_1 and D_2 are admissible operators, then

$$B_7\big(x,y,\varphi\big)=(D_1\circ D_2)\big(x,y,\varphi\big)=D_1\big(x,y,D_2\big(\cdot,\cdot,\varphi\big)\big)$$

is admissible.

Remark 2.32. Similarly as in the one-dimensional case, we cannot allow $\alpha=0$ in B_4, since the operator would not be admissible. To see this, it suffices to consider

$$\varphi(x,y)=\begin{cases}0, & \text{if } x \text{ or } y<2,\\ (\log x\log y)^{-1}, & \text{otherwise,}\end{cases}$$

and proceed as in Remark 2.20.

We can easily find a function f for which there exists an admissible operator B such that

$$\int_y^{2y}\int_x^{2x}|d_{11}f(s,t)|\,ds\,dt\le \frac{C}{xy}B\big(x,y,I_{12}\big),$$

holds for some absolute constant $C>0$, but

$$\int_y^{2y}\int_x^{2x}|d_{11}f(s,t)|\,ds\,dt\le \frac{D}{xy}\int_{y/\lambda}^{\lambda y}\int_{x/\lambda}^{\lambda x}|f(s,t)|\,ds\,dt,$$

does not hold for any absolute constants $\lambda, D>0$, as follows: since if $f(x,y)=g(x)h(y)$, one has that

$$\int_c^d\int_a^b|d_{11}f(s,t)|\,ds\,dt=\bigg(\int_a^b|dg(s)|\bigg)\bigg(\int_c^d|dh(t)|\bigg),$$

it suffices to reduce to the one-dimensional case and consider the counterexample from Proposition 2.22. This would allow us to write the inclusion $MVBVF^2\subsetneq GM^2_{\text{adm}}$ if we disregarded the technical conditions that the functions must satisfy in the definition of each class, namely $AC_{\text{loc}}\big((0,\infty)^2\big)$ and $f\in HBV_{\text{loc}}\big(\mathbb{R}^2_+\big)$, respectively.

To conclude this chapter, we present an estimate of GM^2_{adm} analogue to (2.17). To this end, let us introduce the notation

$$I_1(f;x,y)=I_1(x,y):=\int_x^{2x}|f(s,y)|\,ds,\quad I_2(f;x,y)=I_2(x,y):=\int_y^{2y}|f(x,t)|\,dt.$$

Lemma 2.33. *Let $f\in GMF^2_{adm}$ be such that*

$$f(x,y)\to 0\qquad \textit{as } x+y\to\infty. \tag{2.30}$$

Assume f is of Hardy bounded variation on $[c_1,\infty)\times[c_2,\infty)$ for some $c_1,c_2\ge 0$. If $x,y>0$ are such that $x\ge c_1$ and $y\ge c_2$, then for any $u\in[x,2x]$ and $v\in[y,2y]$,

$$|f(u,v)|\le\frac{3C+1}{xy}\sum_{j=0}^\infty\sum_{k=0}^\infty\frac{1}{2^j2^k}B\big(2^jx,2^ky,I_{12}\big), \tag{2.31}$$

where C is the constant from (2.24).

Remark 2.34. If we assume $B(x,y,I_{12})$ is monotone in each variable, it follows from (2.31) that if $x,y>0$ are such that $x\ge c_1$ and $y\ge c_2$, then for every $u\in[x,2x]$ and $v\in[y,2y]$, there holds

$$|f(u,v)|\le\frac{12C+4}{xy}B\big(x,y,I_{12}\big),$$

which is a two-dimensional version of (2.17) (recall that in the one-dimensional case we assumed $B(x,I)$ was monotone, whilst in the two-dimensional case such assumption was not considered; compare Definitions 2.18 and 2.30).

Proof of Lemma 2.33. Let $u_1,u_2\in[x,2x]$ and $v_1,v_2\in[y,2y]$. It is clear that

$$\begin{aligned}|f(u_1,v_1)|-|f(u_2,v_1)|-|f(u_1,v_2)|-|f(u_2,v_2)|&\le|\Delta_{11}f(u_1,v_1)|\\&\le\int_y^{2y}\int_x^{2x}|d_{11}f(s,t)|\,ds\,dt.\end{aligned}$$

Hence,

$$|f(u_1,v_1)| \le \frac{C}{xy} B\big(x,y,I_{12}\big) + |f(u_2,v_1)| + |f(u_1,v_2)| + |f(u_2,v_2)|.$$

Integrating both sides with respect to u_2 over $[x,2x]$, and v_2 over $[y,2y]$, we obtain

$$xy|f(u_1,v_1)| \le CB\big(x,y,I_{12}\big) + xI_2(u_1,y) + yI_1(x,v_1) + I_{12}(x,y).$$

Thus, property (ii) of the operator B (cf. Definition 2.30),

$$|f(u_1,v_1)| \le \frac{C+1}{xy} B\big(x,y,I_{12}\big) + \frac{1}{y} I_2(u_1,y) + \frac{1}{x} I_1(x,v_1). \tag{2.32}$$

Finally, we estimate the terms $I_2(u_1,y)/y$ and $I_1(x,v_1)/x$. The GM^2 condition, the fact that f is of Hardy bounded variation on $[c_1,\infty)\times[c_2,\infty)$, and (2.30) yield

$$\begin{aligned}
\frac{1}{y} I_2(u_1,y) &= \frac{1}{y}\int_y^{2y} |f(u_1,w)|\,dw = \frac{1}{y}\int_y^{2y} \left| \int_{u_1}^{\infty} d_{10} f(s,w)\,ds \right| dw \\
&\le \frac{1}{y}\int_y^{2y}\int_x^{\infty} |d_{10} f(s,w)|\,ds\,dw = \frac{1}{y}\int_y^{2y} \left(\int_x^{\infty} \left| \int_w^{\infty} d_{11} f(s,t)\,dt \right| ds \right) dw \\
&\le \frac{1}{y}\int_y^{2y} \left(\int_y^{\infty}\int_x^{\infty} |d_{11} f(s,t)|\,ds\,dt \right) dw = \int_y^{\infty}\int_x^{\infty} |d_{11} f(s,t)|\,ds\,dt \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \int_{2^k y}^{2^{k+1} y}\int_{2^j x}^{2^{j+1} x} |d_{11} f(s,t)|\,ds\,dt \le C \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \beta\big(2^j x, 2^k y\big) \\
&= \frac{C}{xy}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{2^j 2^k} B\big(2^j x, 2^k y, I_{12}\big).
\end{aligned} \tag{2.33}$$

Similarly, we can get the same upper estimate for $I_1(x,v_1)/x$. Therefore, combining (2.32) and (2.33), we arrive at (2.31). $\square$

Chapter 3

Uniform convergence of sine transforms

In this chapter we study the uniform convergence of one-dimensional sine transforms

$$\int_0^\infty f(t)\sin ut\,dt, \qquad u\in\mathbb{R}_+, \tag{3.1}$$

and two-dimensional sine transforms

$$\int_0^\infty\int_0^\infty g(x,y)\sin ux\sin vy\,dx\,dy, \qquad u,v\in\mathbb{R}_+. \tag{3.2}$$

Our approach is in general based on variational assumptions on f and g, so that whenever f and g satisfy general monotonicity conditions we can characterize the uniform convergence of (3.1) and (3.2) in terms of the magnitude of f and g at infinity, as is usually done in this topic. In both cases, we will give a comprehensive description of the known results and then generalize them. To this end, we will assume $f\in GM_{\mathrm{adm}}$ (cf. Section 2.5) and $g\in GM^2_{\mathrm{adm}}$ (cf. Section 2.6) in the one and two-dimensional case, respectively. The main results from this chapter have been published in [32, 33].

Before proceeding further, if we denote by $\hat{f}$ the Fourier transform of f (cf. Introduction), we note that

$$\hat{f}(u)=2\int_0^\infty f(t)\cos 2\pi ut\,dt, \qquad \text{if } f \text{ is even},$$
$$\hat{f}(u)=2i\int_0^\infty f(t)\sin 2\pi ut\,dt, \qquad \text{if } f \text{ is odd},$$

that is, the sine and cosine transforms are partial cases of the Fourier transform. Likewise, the Hankel transform (which is introduced and discussed in Chapter 4) appears naturally when studying the Fourier transform of radial functions in the multivariate case, so it is also a partial case of the Fourier transform in some cases. In fact, the cosine transform is a particular case of Hankel transform.

3.1 One-dimensional sine transform

In this section we are interested in studying the uniform convergence of the sine transforms (3.1), where f is a locally integrable function such that $tf(t)\in L^1(0,1)$. This condition will be assumed everywhere in this section. As usual in this topic, we also assume $f(t) \to 0$ as $t \to \infty$.

We start with a brief survey on the topic of uniform convergence of sine series, and continue with the known results relating to the uniform convergence of sine integrals.

We will see that the (general) monotonicity property plays a fundamental role in the study of this topic.

3.1.1 Known results

The first result we should mention is due to Chaundy and Jolliffe [25], [146, V. I, p. 182] (1916). They characterized the nonnegative monotone sequences $\{a_n\}$ whose sine series converge uniformly.

Theorem 3.1. *Let $a_n \geq 0$ be monotonically decreasing to zero. Then, the series*

$$\sum_{n=1}^{\infty} a_n \sin nx$$

converges uniformly in $x \in [0, 2\pi)$ if and only if $na_n \to 0$.

For the sake of completeness, we include the corresponding statement for the cosine series.

Theorem 3.2. *Let $a_n \geq 0$. Then, the series*

$$\sum_{n=0}^{\infty} a_n \cos nx$$

converges uniformly in $x \in [0, 2\pi)$ if and only if $\sum_{n=0}^{\infty} a_n < \infty$.

Theorem 3.2 is rather trivial due to the nonnegativity of a_n and the convergence at $x = 0$ (no monotonicity assumption is needed). Thus, an interesting problem concerning the uniform convergence of cosine series is when $a_n \not\geq 0$. We discuss this problem for sequences of the class GMS_2 in Section 4.5. See also [39, 41, 44].

In 2007, Tikhonov proved that the statement of Theorem 3.1 holds true if we replace the condition $\{a_n\}$ is monotone by $\{a_n\} \in GMS$, see [135]; in fact, the sufficiency part does not even require $a_n \geq 0$ (recall that GMS may be complex-valued). It is worth mentioning that several other authors generalized Theorem 3.1 before the latter appeared by assuming different conditions on $\{a_n\}$ that generalize monotonicity. For instance, Nurcombe proved the statement for quasi-monotone sequences (QMS) [100], i.e., those nonnegative sequences $\{a_n\}$ for which there exists $\tau > 0$ such that $n^{-\tau} a_n$ is decreasing. See also [124, 141], where the problem was studied for an extension of QMS, and [116], where some problems on Fourier series were studied through an application of interpolation theory to QMS. Leindler also proved a version of Theorem 3.1 for sequences with *rest of bounded variation*, i.e., those for which there exists $C > 0$ such that

$$\sum_{k=n}^{\infty} |a_k - a_{k+1}| \leq C|a_n|,$$

see [83]. In any case, all the classes of sequences we have just mentioned are subclasses of GMS. For further details, we refer the reader to [135], which contains detailed descriptions on how those classes relate to each other.

Theorem 3.1 was further generalized by Tikhonov for sequences of the class GMS_2 [130], by Dyachenko and Tikhonov for the sequence class which is an analogue of $GM(\beta_3)$ [45], i.e., the $GMS(\beta)$ class with

$$\beta_n = \frac{1}{n} \sup_{k\geq n/\lambda} \sum_{s=k}^{2k} |a_s|, \qquad \text{for some } \lambda > 1, \tag{3.3}$$

and by Kórus [77] for the $GMS(\beta)$ given by

$$\beta_n = \frac{1}{n} \sup_{k\geq c_n} \sum_{s=k}^{2k} |a_s|,$$

where c_n is some appropriate sequence increasing to infinity. Note that in (3.3), $c_n = n/\lambda$.

Finally, Dyachenko, Mukanov, and Tikhonov [41] constructed the class GMS_{adm}, analogously as we defined the function class GM_{adm} in Section 2.5, and proved Theorem 3.1 for sequences from such a class. In fact, those classes of sequences and functions were considered simultaneously in the papers [33] and [41].

It is worth emphasizing that in almost all generalizations of Theorem 3.1 we mentioned above, the sufficiency part is relatively easy to prove even for complex sequences, rather than nonnegative ones. In fact, it was shown in [45] that a sufficient condition for the uniform convergence of the series $\sum_{n=1}^{\infty} a_n \sin nx$ on $[0, 2\pi)$ is that

$$n\sum_{k=n}^{2n} |a_k - a_{k+1}| \to 0 \qquad \text{as } n\to\infty, \tag{3.4}$$

for any $\{a_n\} \subset \mathbb{C}$. In other words, (3.4) is equivalent to $\{a_n\} \in GMS(\beta)$ with $n\beta_n \to 0$ as $n \to \infty$.

Proving the necessity part of Theorem 3.1 for real-valued sequences $\{a_n\}$ is more complicated. Very recently, in [50], Feng, Totik, and Zhou achieved such goal for sequences of the class GMS_2 (cf. (2.7), p. 12). This is in part what motivates the work presented in Section 3.1. The result of Feng, Totik, and Zhou was generalized in [41], where the authors proved the necessity part of Theorem 3.1 for sequences of the class GM_{adm}.

A survey of the results in this line for two-dimensional sine series is presented in Section 3.2.

It is also worth mentioning that the technique the authors used to prove the necessity part of Theorem 3.1 for the GMS_2 class in [50] also motivated other authors to develop other techniques to deal with the GMS_2 class that allow to solve problems in the framework of real sequences, instead of nonnegative ones. For example, see the recent work [46], where Dyachenko and Tikhonov prove, among other results, that if

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx, \qquad g(x) = \sum_{n=0}^{\infty} a_n \cos nx,$$

$0 < \alpha < 1$, and $\{a_n\} \in GMS_2$ is a real-valued sequence, then $f, g \in \text{Lip } \alpha$ if and only if $n^{1+\alpha} a_n \to 0$ as $n \to \infty$. Here Lip α denotes the space of functions defined by

$$\text{Lip } \alpha = \left\{ f : [0, 2\pi) \to \mathbb{C} : \sup_{x,y\in[0,2\pi]} |f(x) - f(y)| \leq C|x - y|^{\alpha} \right\}.$$

The latter result for monotone sequences is a classical statement due to Lorentz [91] (see also [8, Ch. X]), and it was generalized by considering nonnegative $GMS(\beta)$ sequences in place of monotone ones in [45, 130, 133, 134]. This problem was first solved within the framework of real-valued GMS_2 sequences in [46].

There are different generalizations of the class GMS than those discussed here [86, 128] for which an analogue statement to that of Theorem 3.1 holds, see [68, 128] and the references therein.

On the side of sine integrals, let us first observe that, unlike in the case of series, we have to guarantee the convergence of the integrals

$$\int_0^\delta f(t)\sin ut\,dt, \qquad \delta>0.$$

For example, the above integral is infinite for any $\delta>0$ if $f(t)=t^{-2}$. In order to avoid such scenarios, we will assume that $tf(t)\in L^1(0,1)$, so that for $0<\delta<1$,

$$\int_0^\delta f(t)\sin ut\,dt\le u\int_0^\delta t|f(t)|\,dt\le Cu.$$

Móricz [96] proved the following criterion for functions in the $GM(\beta_2)$ class:

Theorem 3.3. *Let $f\in GM(\beta_2)$, $f\ge 0$, be such that $tf(t)\in L^1(0,1)$. Then, the sine integral*

$$\int_0^\infty f(t)\sin ut\,dt \tag{3.5}$$

converges uniformly on $\mathbb{R}_+$ if and only if

$$tf(t)\to 0 \qquad \text{as } t\to\infty.$$

By saying that the sine integral (3.5) converges uniformly we mean that the partial integrals $\int_0^N f(t)\sin ut\,dt$ converge uniformly as $N\to\infty$. It is also worth mentioning that the sufficiency part of Theorem 3.3 was proved for complex-valued functions.

Later on, Dyachenko, Liflyand, and Tikhonov [40] generalized the statement of Theorem 3.3 for functions of the class $GM(\beta_3)$. What is more, they obtained a more general sufficient condition for complex-valued functions. More precisely, they proved the following.

Theorem 3.4. *Let $f:\mathbb{R}_+\to\mathbb{C}$ be such that $tf(t)\in L^1(0,1)$.*

(i) *If $f\in GM(\beta)$ and $x\beta(x)\to 0$ as $x\to\infty$, or equivalently, if*

$$x\int_x^{2x}|df(t)|\to 0 \qquad \text{as } x\to\infty, \tag{3.6}$$

then $\int_0^\infty f(t)\sin ut\,dt$ converges uniformly on $\mathbb{R}_+$.

(ii) *If $f\in GM(\beta_3)$ and $f\ge 0$, the uniform convergence of $\int_0^\infty f(t)\sin ut\,dt$ implies*

$$tf(t)\to 0 \qquad \text{as } t\to\infty.$$

It is clear that if $f\in GM(\beta_3)$ and $tf(t)\to 0$ as $t\to\infty$, then $x\beta_3(x)\to 0$ as $x\to\infty$, in which case we can write an "if and only if" statement.

Corollary 3.5. *Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be such that $f\in GM(\beta_3)$ and $tf(t)\in L^1(0,1)$. Then $\int_0^\infty f(t)\sin ut$ converges uniformly on $\mathbb{R}_+$ if and only if*

$$tf(t)\to 0 \qquad \text{as } t\to\infty.$$

For the sake of completeness, we write the corresponding result for the cosine transform from [40], although we are not going to discuss it further in this chapter. We deal with cosine (or more generally, Hankel) transforms of functions from the class $GM(\beta_2)$ in Section 4.5.

Theorem 3.6. *Let $f\in GM(\beta_3)$ be such that $f\in L^1(0,1)$, and assume (3.6) holds. Then, the cosine transform*

$$\int_0^\infty f(t)\cos ut\,dt$$

converges uniformly on $\mathbb{R}_+$ if and only if $\int_0^\infty f(t)\,dt$ converges.

Note that in Theorem 3.6 we do not need the assumption $f\geq 0$.

To conclude this section, we make the following observation concerning condition (3.6) that will be useful later: note that it is equivalent to

$$x\int_x^\infty |df(t)|\to 0 \qquad \text{as } x\to\infty.$$

More generally, for any $\gamma>0$, one has

$$x^\gamma\int_x^\infty |df(t)|\to 0 \quad \text{as } x\to\infty \quad \text{if and only if} \quad x^\gamma\int_x^{2x} |df(t)|\to 0 \quad \text{as } x\to\infty. \tag{3.7}$$

Indeed, one direction is clear. To prove the other direction, since $x^\gamma\int_x^{2x}|df(t)|=o(1)$ as $x\to\infty$, we write

$$x^\gamma\int_x^\infty |df(t)| = x^\gamma\sum_{k=0}^\infty\int_{2^kx}^{2^{k+1}x}|df(t)| = x^\gamma\sum_{k=0}^\infty o\left(\frac{1}{(2^kx)^\gamma}\right)=\sum_{k=0}^\infty\frac{1}{2^{k\gamma}}o(1)=o(1),$$

as $x\to\infty$.

3.1.2 New results

The purpose of this subsection is to generalize the statement of Corollary 3.5 in two different ways: first, to prove it for a class of functions that contains all other classes that have been considered before, namely GM_{adm} (recall that $GM(\beta_3)\subsetneq GM_{\mathrm{adm}}$, cf. Proposition 2.22). Secondly, we also wish to prove the necessity part of Corollary 3.5 for real-valued functions instead of nonnegative ones. To this end, we use the technique from [50] (which deals with sequences from the class GMS_2) adapted to the framework of functions.

The main statement of this section reads as follows.

Theorem 3.7. *Let $f\in GM_{adm}$ be a real-valued function such that $tf(t)\in L^1(0,1)$. Assume $I(x)=I(f,x)=\int_x^{2x}|f(t)|\,dt$ is bounded at infinity. Then, a necessary and sufficient condition for the uniform convergence of*

$$\int_0^\infty f(t)\sin ut\,dt$$

on $\mathbb{R}_+$ is that

$$tf(t) \to 0 \qquad \text{as } t \to \infty. \tag{3.8}$$

Proof. We first prove the sufficiency part. Since $f \in GM_{\text{adm}}$, there exists a constant $C > 0$ and an admissible operator B such that

$$\int_x^{2x} |df(t)| \le C\frac{B(x,I)}{x}, \qquad \text{for all } x > 0.$$

Condition (3.8) implies that $I(x) \to 0$ as $x \to \infty$, so that $B(x,I) \to 0$ as $x \to \infty$, by property (i) of $B(x,I)$ (cf. Definition 2.18), and the result follows by the first part of Theorem 3.4.

In order to prove the necessity part, it is enough to show that $I(x) \to 0$ as $x \to \infty$. Once this is done, the result will simply follow by property (i) of $B(x,I)$ and the estimate from Corollary 2.21.

For any $x > 0$, let

$$A(x) := \big\{t \in [x,2x] : |f(t)| \ge I(x)/2x\big\}.$$

By definition of $A(x)$ and the estimate from Corollary 2.21, we have

$$\begin{aligned} I(x) &= \left(\int_{[x,2x]\setminus A(x)} |f(t)|\,dt + \int_{A(x)} |f(t)|\,dt\right) \le \frac{I(x)}{2} + \int_{A(x)} |f(t)|\,dt \\ &\le \frac{I(x)}{2} + C_0|A(x)|\frac{B(x,I)}{x}, \end{aligned}$$

where $|A(x)|$ denotes the Lebesgue measure of the set $A(x)$. We can now deduce

$$|A(x)| \ge \frac{x}{2C_0}\cdot\frac{I(x)}{B(x,I)},$$

provided that $B(x,I) > 0$, and consequently,

$$\int_{A(x)} |f(t)|\,dt \ge |A(x)|\frac{I(x)}{2x} \ge \frac{1}{4C_0}\cdot\frac{I(x)^2}{B(x,I)}. \tag{3.9}$$

Since the integral $\int_0^\infty f(t)\sin ut\,dt$ converges uniformly, for a fixed $\varepsilon > 0$ we can find $\xi > 0$ such that

$$\left|\int_{\xi_1}^{\xi_2} f(t)\sin ut\,dt\right| < \varepsilon, \quad \text{if } \xi \le \xi_1 \le \xi_2, \quad u \in \mathbb{R}_+. \tag{3.10}$$

Now we can choose $x \ge \xi$ such that $I(x) > 0$ (note that in this case $B(x,I) > 0$, by property (iii) of B, cf. Definition 2.18). We can always make this choice of x, since if it did not exist, then $I(x) = 0$ for all $x \ge \xi$, and our assertion would be trivial. Also note that due to Corollary 2.21 and property (ii) of $B(x,I)$, $f(x)$ is bounded at infinity. Thus, there exists $\delta = \delta(\varepsilon, x)$ such that

$$\int_w^{w+\delta} |f(t)|\,dt \le \varepsilon, \qquad \text{for all } w \ge x. \tag{3.11}$$

For example, take $\delta = \min\{\delta', x\}$, where $\delta' = \varepsilon/\sup_{t\ge x}|f(t)|$.

Our next goal is to cover the set $A(x)$ by almost disjoint intervals S_j. More precisely, we construct a collection of intervals $\{S_j\}_{j=1}^n$ (with $n = n(x)$) such that $|S_j \cap S_k| = 0$ whenever $j \neq k$ and $\big|A(x)\backslash\big(S_1 \cup \ldots \cup S_n\big)\big| = 0$, or in other words,

$$A(x) \subset \left(\bigcup_{j=1}^{n} S_j \right) \cup E(x),$$

where $|E(x)| = 0$. Note that for such a collection, one trivially has

$$\int_{A(x)} |f(t)|\, dt \leq \sum_{j=1}^{n} \int_{S_j} |f(t)|\, dt.$$

The construction of the intervals $S_j = [v_j, \nu_j]$ is done as follows: first let $v_1 = \inf A(x)$.

(1) If there exists $v_1 < y_1 \leq 2x$ such that f has constant sign[1] in $(v_1, y_1]$, and $|f(t)| > I(x)/4x$ for every $t \in (v_1, y_1)$, while $|f(y_1)| \leq I(x)/4x$, then we define $\nu_1 = y_1 + \delta$, with δ as above.

(2) If there is no $y_1 \in (v_1, 2x]$ satisfying all the properties described in case (1), let

$$z_1 = \inf\{\xi \in [v_1, 2x] : f(v_1)f(\xi) \leq 0\}.$$

If such an infimum exists, we define $\nu_1 = z_1 + \delta$, with δ as above.

(3) If neither y_1 nor z_1 described in cases (1) and (2) exist, we put $\nu_1 = 2x$.

After finding ν_1, we set $S_1 = [v_1, \nu_1]$, and if $A(x)\backslash S_1$ has positive measure, we define $v_2 = \inf A(x)\backslash S_1$. By the same procedure, we find ν_2 and define $S_2 = [v_2, \nu_2]$, and so on until we reach n such that

$$\big|A(x)\backslash\big(S_1 \cup \ldots \cup S_n\big)\big| = 0.$$

Let $1 \leq j < n$. We now prove that

$$\int_{v_j}^{\nu_j} |df(t)| \geq \frac{I(x)}{4x},$$

which will eventually allow us to obtain an upper estimate for n.

(1) If ν_j was chosen by case (1). Note that there exists[2] $w \in [v_j, y_j)$ such that $|f(w)| \geq I(x)/2x$, whilst $|f(y_j)| \leq I(x)/4x$. Thus,

$$\int_{v_j}^{\nu_j} |df(t)| \geq |f(w) - f(y_j)| = |f(w)| - |f(y_j)| \geq \frac{I(x)}{4x}.$$

[1] We say that f has constant sign in a set X if and only if $f(x_1)f(x_2) > 0$ for all $x_1, x_2 \in X$.

[2] By definition, for any set $A \subset \mathbb{R}$, we have that $m = \inf A$ if and only if (a) m is a lower bound of A, and (b) for every $m' > m$ there exists $x \in A$ such that $x < m'$. In our case, we can find $w \geq v_j$ with $w \in A(x)$, i.e., $|f(w)| \geq I(x)/2x$.

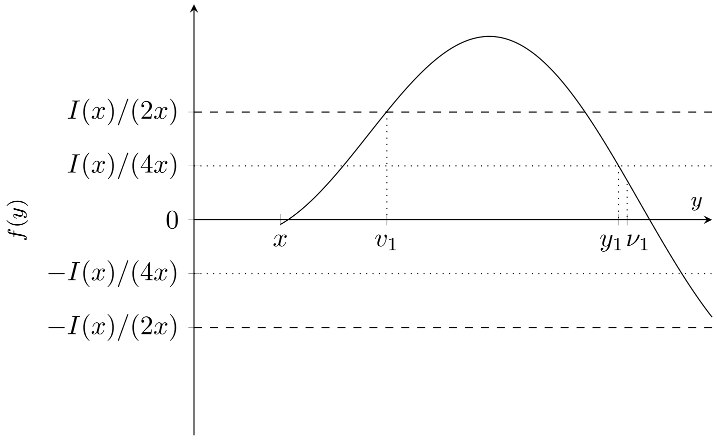

Figure 3.1: Choice of ν_1 given by case (1). Note that the variation over the interval $[v_1, \nu_1]$ is greater than or equal to $I(x)/(4x)$.

(2) If ν_j was chosen by case (2), similarly as in case (1), there exists $w \in [v_j, v_j + \delta)$ such that $|f(w)| \geq I(x)/2x$. Since

$$z_j = \inf\{\xi \in [v_j, 2x] : f(v_j)f(\xi) \leq 0\},$$

there must exist $z \in [v_j, z_j + \delta)$ such that $f(z)f(w) \leq 0$. Indeed, if this z does not exist, then $f(y)f(w) > 0$ for all $y \in [v_j, z_j + \delta)$, and in particular, $f(w)f(v_j) > 0$. This implies that $f(y)f(v_j) > 0$ for all $y \in [v_j, z_j + \delta)$, or in other words, f has constant sign in the latter interval. Hence,

$$\inf\{\xi \in [v_j, 2x] : f(v_j)f(\xi) \leq 0\} > z_j,$$

which is a contradiction. Therefore, we conclude

$$\int_{v_j}^{\nu_j} |df(t)| \geq |f(w) - f(z)| \geq \frac{I(x)}{2x} \geq \frac{I(x)}{4x}.$$

Finally, it is only left to remark that if ν_j is chosen by case (3), then $j = n$. We can now proceed estimating n from above (when $n > 1$). By the GM_{adm} condition, property (iv) of $B(x, I)$ (monotonicity on x), and the fact that $\delta \leq x$, we have

$$\begin{aligned} 2\frac{C}{x}B(x,I) &\geq \frac{C}{2x}B(2x,I) + \frac{C}{x}B(x,I) \geq \int_{2x}^{4x} |df(t)| + \int_{x}^{2x} |df(t)| \\ &\geq \int_{x}^{2x+\delta} |df(t)| \geq \sum_{j=1}^{n-1} \int_{v_j}^{\nu_j} |df(t)| \geq (n-1)\frac{I(x)}{4x}. \end{aligned}$$

Thus,

$$n \leq \frac{8Cx}{x}\frac{B(x,I)}{I(x)} + 1 \leq 9C\frac{B(x,I)}{I(x)}. \tag{3.12}$$

If $n = 1$, inequality (3.12) is trivially true.

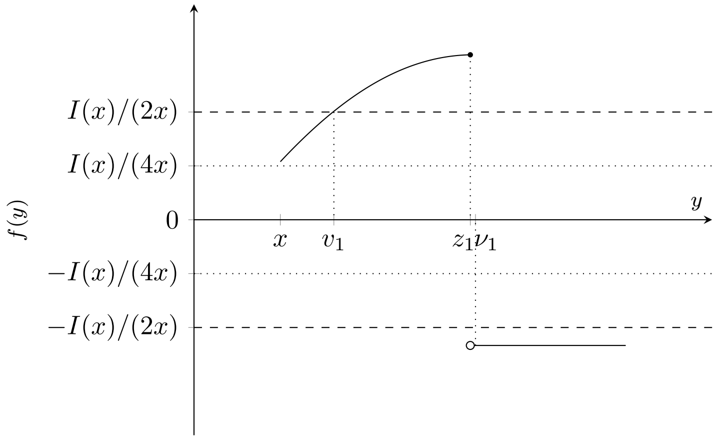

Figure 3.2: Choice of ν_1 given by case (2). Note that the variation over the interval $[v_1, \nu_1]$ is greater than or equal to $I(x)/(4x)$ (in fact, greater than or equal to $I(x)/(2x)$).

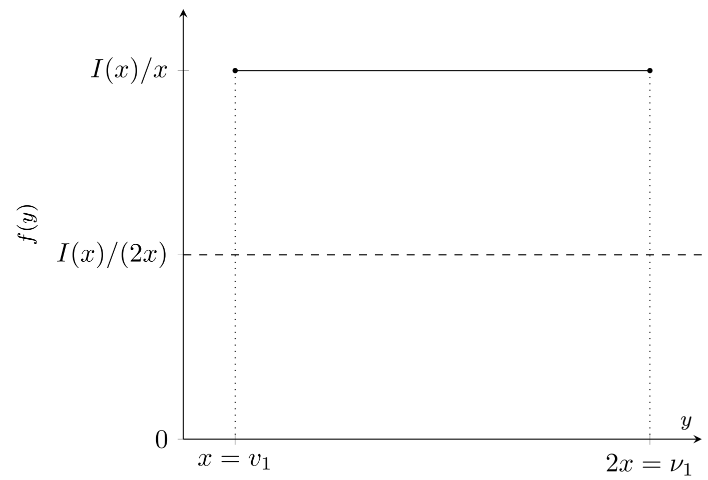

Figure 3.3: Choice of ν_1 given by case (3). Note that in this "degenerate" case we cannot give a nontrivial lower bound for the variation of f over the interval $[v_1, 2x]$.

Now let $u = \pi/8x$. Then $ut \leq \pi/2$ for all $t \in [x, 4x]$, so that $\sin ut \geq 1/4$ on the latter interval. Since f has constant sign on $(v_j, \nu_j - \delta)$ (by construction of the S_j's), it follows from (3.10) and (3.11) that

$$\begin{aligned} \frac{1}{4}\int_{v_j}^{\nu_j} |f(t)|\,dt &= \frac{1}{4}\left(\int_{v_j}^{\nu_j-\delta} |f(t)|\,dt + \int_{\nu_j-\delta}^{\nu_j} |f(t)|\,dt\right) \\ &\leq \left|\int_{v_j}^{\nu_j-\delta} f(t)\sin ut\,dt\right| + \frac{\varepsilon}{4} < \varepsilon + \frac{\varepsilon}{4}. \end{aligned}$$

Therefore, for any $1 \leq j \leq n$,

$$\int_{v_j}^{\nu_j} |f(t)|\,dt < 5\varepsilon. \tag{3.13}$$

Since $\left|A(x)\backslash\big(S_1 \cup \ldots \cup S_n\big)\right| = 0$, summing up on j the integrals in (3.13), it follows from (3.12) that

$$\int_{A(x)} |f(t)|\,dt \leq \sum_{j=1}^{n} \int_{v_j}^{\nu_j} |f(t)|\,dt < 5n\varepsilon \leq 45C\frac{B(x,I)}{I(x)}\varepsilon. \tag{3.14}$$

Finally, combining (3.9) and (3.14), we obtain

$$\frac{1}{4C_0}\cdot\frac{I(x)^2}{B(x,I)} \leq 45C\frac{B(x,I)}{I(x)}\varepsilon; \qquad \frac{I(x)^3}{B(x,I)^2} \leq 180CC_0\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we deduce that $I(x)^3/B(x,I)^2$ vanishes as $x \to \infty$. Moreover, since $B(x,I)$ is bounded for x large enough (by property (ii) of $B(x,I)$, cf. Definition 2.18), then $I(x)^3 \to 0$ as $x \to \infty$, which is what we wanted to prove. $\square$

Remark 3.8. It is worth to comment about the appearance of δ defined in (3.11) in the proof of Theorem 3.7. We strongly make use of this small value in the construction of the intervals S_j. Let us disregard δ for a moment, and assume we are in the situation given by Figure 3.2. If we define

$$\nu_1 = z_1 = \min\big\{\xi \in [v_1, 2x] : f(v_1)f(\xi) \leq 0\big\},$$

then we have $v_2 = \inf A(x)\backslash[v_1, z_1] = z_1$. However, according to the picture we are once again in case (2), and moreover, $\nu_2 = z_2 = z_1$, so that $S_2 = \{z_1\}$. At this point it is clear we enter in an infinite loop (unless $\left|A(x)\backslash[v_1, z_1]\right| = 0$, which in general should not be true). By taking $\nu_1 = z_1 + \delta$, we avoid these pathological situations, whilst at the same time we have a sufficiently small upper estimate for the integrals $\int_w^{w+\delta} |f(t)|\,dt$.

Remark 3.9. Note that if the function f from Theorem 3.7 satisfies $f(x) \geq 0$ for every $x > x_0$, we do not need to assume the boundedness of $I(x)$ at infinity. In fact, if $u = \pi/4x$ in (3.10), we have

$$\varepsilon > \left|\int_x^{2x} f(t)\sin\left(\frac{\pi}{4x}t\right)dt\right| \geq \frac{1}{2}\left|\int_x^{2x} f(t)\,dt\right| = \frac{1}{2}\int_x^{2x} |f(t)|\,dt = \frac{I(x)}{2}.$$

In this case $I(x) \to 0$ as $x \to \infty$, so that the proof of the necessity part of Theorem 3.7 becomes much easier if we assume $f \geq 0$.

As a corollary of Theorem 3.7 combined with Remark 3.9, we write the corresponding "if and only if" statement for nonnegative GM_{adm} functions:

Corollary 3.10. *Let $f \in GM_{adm}$ be a nonnegative function such that $tf(t) \in L^1(0,1)$. Then, $\int_0^\infty f(t)\sin ut\,dt$ converges uniformly on $\mathbb{R}_+$ if and only if (3.8) holds.*

Before proceeding further, let us remark that any function f that is locally of bounded variation on $\mathbb{R}_+$ and such that $I(x)$ is unbounded at infinity, then formally $f \in GM(\beta_3) \subset GM_{\text{adm}}$.

Comparing Corollary 3.5 and Theorem 3.7, we see that if we want to consider real-valued instead of nonnegative ones in Theorem 3.7, we need the extra assumption "$I(x)$ bounded at infinity". Our next concern is whether such hypothesis is essential in Theorem 3.7. We give an affirmative answer to this question by finding a function such that formally $f \in GM(\beta_3)$ such that $I(x)$ is unbounded at infinity (thus we only require $f \in BV_{\text{loc}}(\mathbb{R}_+)$), and for which $\int_0^\infty f(t)\sin ut\,dt$ converges uniformly although $tf(t) \not\to 0$ as $t \to \infty$. We also mention that the situation is similar for the case of sine series with coefficients in the GMS_{adm} class, see [41].

In order to prove our assertion, we make use of the so-called *Rudin-Shapiro sequence* [71, 113, 122]. The following is a well known result [113], often referred to as Rudin-Shapiro's lemma.

Lemma 3.11. *There exists a sequence $\{\varepsilon_n\}_{n=0}^\infty$, $\varepsilon_n = \pm 1$, such that*

$$\left| \sum_{n=0}^{m} \varepsilon_n e^{inx} \right| < 5\sqrt{m+1} \tag{3.15}$$

for all $x \in [0, 2\pi)$ and all $m \in \mathbb{N}$.

We call the sequence $\{\varepsilon_n\}_{n=0}^\infty$ from Lemma 3.11 the *Rudin-Shapiro sequence*, and the trigonometric polynomials on the left-hand side of (3.15) are known as *Rudin-Shapiro polynomials*. We need the following analogue of Lemma 3.11 for integrals.

Lemma 3.12. *[Rudin-Shapiro's lemma for Fourier integrals] There exists a function $h : [0,\infty) \to \{-1, 1\}$ such that*

$$\left| \int_0^M h(t) e^{iut}\,dt \right| < 6\sqrt{M} \tag{3.16}$$

for all $u \in \mathbb{R}$ and all $M > 0$.

Proof. We define the function h by means of the Rudin-Shapiro sequence: for $n \in \mathbb{N} \cup \{0\}$,

$$h(t) = \varepsilon_n, \qquad \text{if } t \in [n, n+1).$$

The assertion is trivial if $M \le 1$, so we can assume $M > 1$, and we can also restrict ourselves to the case $u \ge 0$. Put $N = [M]$, where $[\cdot]$ denotes the floor function. First, for $u = 0$, it follows from Lemma 3.11 that

$$\left| \int_0^M h(t)\,dt \right| \le \left| \sum_{n=0}^{N-1} \varepsilon_n \right| + \left| \int_N^M h(t)\,dt \right| < 6\sqrt{M}.$$

Secondly, if $u > 0$,

$$\int_0^N h(t)e^{iut}\,dt = \sum_{n=0}^{N-1}\int_n^{n+1} h(t)e^{iut}\,dt = \frac{e^{iu}-1}{iu}\sum_{n=0}^{N-1}\varepsilon_n e^{inu}.$$

Thus, by Lemma 3.11,

$$\left|\int_0^N h(t)e^{iut}\,dt\right| < 5\sqrt{N}\left|\frac{e^{iu}-1}{u}\right| \le 5\sqrt{N},$$

and therefore

$$\left|\int_0^M h(t)e^{iut}\,dt\right| \le \left|\int_0^N h(t)e^{iut}\,dt\right| + \left|\int_N^M h(t)e^{iut}\,dt\right| < 6\sqrt{M},$$

which completes the proof. □

Remark 3.13. The constant on the right-hand side of (3.16) is not optimal. It can be improved, for instance, by considering sharper estimates of (3.15). Putting $C\sqrt{m+1}$ on the right-hand side of (3.15), it is known that the optimal C lies between $\sqrt{6}$ and $\big(2+\sqrt{2}\big)\sqrt{3/5}$ (cf. [115] and the references therein).

We are in a position to prove the sharpness of Theorem 3.7 with respect to the condition "$I(x)$ bounded at infinity", or in other words, that such an assumption cannot be dropped from the statement of Theorem 3.7.

Theorem 3.14. *There exists a uniformly converging sine integral $\int_0^\infty f(t)\sin ut\,dt$ such that*

$$tf(t)\to\infty \qquad \text{as } t\to\infty,$$

and

$$\int_x^{2x}|f(t)|\,dt\to\infty \qquad \text{as } x\to\infty.$$

Proof. Let $c_n = n^{-2}2^{-n/2}$, and let h be the Rudin-Shapiro function (i.e., the function defined in Lemma 3.12). We prove that the Fourier integral given by

$$\int_0^1 h(t)e^{iut}\,dt + \sum_{n=1}^{\infty} c_n\int_{2^{n-1}}^{2^n} h(t)e^{iut}\,dt = \int_0^\infty f(t)e^{iut}\,dt, \tag{3.17}$$

converges uniformly, where $f(t) = c_n h(t)$ for $t\in[2^{n-1},2^n)$ with $n\ge 1$, and $f(t) = h(t)$ for $t\in[0,1)$. If $n\ge 1$ and $2^{n-1}\le z_1 < z_2\le 2^n$, it follows from Lemma 3.12 that

$$\left|\int_{z_1}^{z_2} f(t)e^{iut}\,dt\right| \le c_n\left(\left|\int_0^{z_1} h(t)e^{iut}\,dt\right| + \left|\int_0^{z_2} h(t)e^{iut}\,dt\right|\right) \le n^{-2}2^{-n/2}12\cdot 2^{n/2} = 12n^{-2}.$$

Hence, for arbitrary $\xi_1<\xi_2$,

$$\left|\int_{\xi_1}^{\xi_2} f(t)e^{iyt}\,dt\right| \le 12\sum_{k=n_1}^{n_2}\frac{1}{k^2}\to 0 \qquad \text{as } \xi_2>\xi_1\to\infty,$$

where $n_1 = \max\{n \in \mathbb{N} : 2^n \le \xi_1\}$ and $n_2 = \min\{n \in \mathbb{N} : 2^n \ge \xi_2\}$. Thus, the uniform convergence of (3.17) follows, and so does that of

$$\int_0^\infty f(t) \sin ut \, dt.$$

However, the integrals $\int_x^{2x} |f(t)| \, dt$ are not bounded at infinity. To prove this claim, fix $x > 1$ and let $n \in \mathbb{N} \cup \{0\}$ be such that $2^n \le x < 2^{n+1}$. Then,

$$\int_x^{2x} |f(t)| \, dt \ge x c_{n+2} \gtrsim n^{-2} 2^{n/2} \to \infty \qquad \text{as } n \to \infty.$$

Finally, with n and x as above, we also have

$$x|f(x)| \ge 2^n c_{n+2} = n^{-2} 2^{n/2} \to \infty \qquad \text{as } n \to \infty,$$

as desired. □

As mentioned before, a version of Theorem 3.3 was proved for real-valued sequences of the class GMS_2 in [50]. Combining the proof of [50, Theorem 3.1] with that of Theorem 3.7, we can obtain the following.

Theorem 3.15. *Let $f \in GM(\beta_2)$ be a real-valued function such that $tf(t) \in L^1(0,1)$. Then, condition (3.8) is necessary and sufficient for the uniform convergence of the integral $\int_0^\infty f(t) \sin ut \, dt$ on $\mathbb{R}_+$.*

The proof of Theorem 3.15 is not included, since it is essentially a combination of those of [50, Theorem 3.1] and Theorem 3.7.

Note that in Theorem 3.15, dealing with the class $GM(\beta_2)$, we do not assume that $I(x)$ (or $\int_{x/\lambda}^{\lambda x} |f(t)| \, dt$) is bounded at infinity, since it already follows from the uniform convergence of $\int_0^\infty f(t) \sin ut \, dt$ (see the proof of [50, Theorem 3.1]).

One may also wonder if the condition

$$tf(t) \to 0 \qquad \text{as } t \to \infty$$

may be replaced by "$tf(t)$ bounded at infinity", and still obtain the conclusion of Theorem 3.7. This is known not to be true, as shown by the example $f(t) = t^{-1}$ (cf. [40]).

3.2 Two-dimensional sine transform

This section is devoted to the study of the uniform convergence of double sine transforms

$$\int_0^\infty \int_0^\infty g(x,y) \sin ux \sin vy \, dx \, dy, \qquad u, v \in \mathbb{R}_+. \tag{3.18}$$

Throughout this section, $g : \mathbb{R}_+^2 \to \mathbb{C}$ is assumed to satisfy $g(x,y) \to 0$ as $\max\{x,y\} \to \infty$, or equivalently, as $x + y \to \infty$.

The integral (3.18) can be viewed as the Fourier transform of functions with an "odd-type" symmetry, i.e., such that $g(x,y) = -g(-x,y) = -g(x,-y)$ for all $x, y \in \mathbb{R}$. In this case there also holds $g(x,y) = g(-x,-y)$.

It is necessary to review some concepts about convergence of double (or multiple) integrals first. We start by mentioning a few historical facts about convergence of multiple series. Pringsheim [106] realised that such convergence depends on how partial sums are ordered, contrarily as in the one-dimensional case, and he investigated many different ways of ordering those partial sums in [106]. Moreover, he noticed that for series with nonnegative terms all types of convergence were equivalent (see also [107]). One of the most natural ways of arranging partial sums is the rectangular one, often referred to as *Pringsheim's summation* (which had already been introduced by Stolz in [127]). Convergence of the partial sums under this arrangement is called *Pringsheim convergence*. A more restrictive type of convergence, called *regular convergence*, was introduced by Hardy in [60], and its counterpart for double integrals, introduced below, will be the main type of convergence that we study.

Given $g \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$, we say that the double integral

$$\int_0^\infty \int_0^\infty g(s,t)\,ds\,dt, \tag{3.19}$$

converges in the sense of Pringsheim if the partial rectangular integrals

$$I(g;x,y) = \int_0^x \int_0^y g(s,t)\,ds\,dt$$

converge to a finite limit as $x, y \to \infty$, or equivalently, as $\min\{x,y\} \to \infty$. In this case, the Cauchy convergence criterion holds, or in other words, a necessary and sufficient condition for the convergence of (3.19) in the sense of Pringsheim is that for every $\varepsilon > 0$ there exists $z = z(\varepsilon) > 0$ such that

$$\big|I(g;x_1,y_1) - I(g;x_2,y_2)\big| < \varepsilon, \qquad \text{if } \min\{x_1,x_2,y_1,y_2\} > z.$$

We say that (3.19) converges in the regular sense if

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} g(s,t)\,ds\,dt \to 0 \qquad \text{as } x_1 + y_1 \to \infty, \quad x_2 > x_1, \quad y_2 > y_1.$$

Note that if $x_1, y_1 \geq 0$, then $x_1 + y_1 \to \infty$ is equivalent to $\max\{x_1, y_1\} \to \infty$. We will always write $x_1 + y_1 \to \infty$, but keeping in mind this equivalence.

It is useful to observe [95] the that convergence in the sense of Pringsheim of (3.19) is equivalent to the fulfilment of the following conditions:

$$\begin{aligned}
&\int_{x_1}^{x_2} \int_0^y g(s,t)\,ds\,dt \to 0 \qquad \text{as } x_1 \to \infty, \quad x_2 > x_1, \quad y \to \infty,\\
&\int_0^x \int_{y_1}^{y_2} g(s,t)\,ds\,dt \to 0 \qquad \text{as } y_1 \to \infty, \quad y_2 > y_1, \quad x \to \infty,
\end{aligned}$$

whilst regular convergence of (3.19) is equivalent to

$$\begin{aligned}
&\int_{x_1}^{x_2} \int_0^y g(s,t)\,ds\,dt \to 0 \qquad \text{as } x_1 \to \infty, \quad x_2 > x_1, \quad y \in \mathbb{R}_+,\\
&\int_0^x \int_{y_1}^{y_2} g(s,t)\,ds\,dt \to 0 \qquad \text{as } y_1 \to \infty, \quad y_2 > y_1, \quad x \in \mathbb{R}_+.
\end{aligned}$$

From this observation it readily follows that convergence in the regular sense implies convergence in the sense of Pringsheim. The contrary is not true, as shown by the following example from [94]:

$$g(x,y)=\begin{cases} k, & \text{if } (x,y)\in(k+2,k+3]\times(0,1] \text{ or } (0,1]\times(k+2,k+3],\ k\in\mathbb{N},\\ -k, & \text{if } (x,y)\in(k+2,k+3]\times(1,2] \text{ or } (1,2]\times(k+2,k+3],\ k\in\mathbb{N},\\ 0, & \text{otherwise.}\end{cases}$$

For every $x,y\geq 2$, one can easily see that $I(g;x,y)=0$, so that the integral (3.19) converges to 0 in the sense of Pringsheim, despite g being unbounded. However, (3.19) cannot converge in the regular sense, since for any $k\in\mathbb{N}$,

$$\int_0^1\int_{k+2}^{k+3} g(s,t)\,ds\,dt=k,$$

which clearly does not vanish as $k\to\infty$.

We refer the reader to [94] for a more detailed discussion between these two types of convergence (also presented for double series) and further examples. We also mention that a generalization of Fubini's theorem [145, Ch. 11] is also proved in [94]. More precisely, it is shown that the order of integration on (3.19) may be switched if the integral converges regularly, instead of requiring the stronger condition $g\in L^1\big(\mathbb{R}^2_+\big)$.

We are now in a position to define the uniform convergence in both the regular and Pringsheim's sense of the double sine transform (3.18); for $M,N>0$, define the partial double integral of (3.18) as

$$S_{M,N}(u,v)=S_{M,N}(g;u,v):=\int_0^N\int_0^M g(x,y)\sin ux\sin vy\,dx\,dy. \tag{3.20}$$

On the one hand, according to the Cauchy criterion, we say that (3.18) converges uniformly in the sense of Pringsheim if

$$|S_{M,N}(u,v)-S_{M',N'}(u,v)|\to 0, \qquad \text{as } \min\{M,M',N,N'\}\to\infty,$$

uniformly in u,v. On the other hand, we say that (3.18) converges uniformly in the regular sense if

$$\int_N^{N'}\int_M^{M'} g(x,y)\sin ux\sin vy\,dx\,dy\to 0 \qquad \text{as } M+N\to\infty, \quad M'>M,\ N'>N,$$

uniformly in u,v.

3.2.1 Known results

Before presenting our statements, let us give a summary of the known results related to our framework. In this subsection we assume that the double sequences $\{a_{m,n}\}$ we consider satisfy $a_{m,n}\to 0$ as $m+n\to\infty$. The problem of uniform convergence of double sine series was first considered by Žak and Šneĭder in 1966 [144]. Denote

$$\Delta_{10}a_{m,n}=a_{m,n}-a_{m+1,n}, \qquad \Delta_{01}a_{m,n}=a_{m,n}-a_{m,n+1},$$
$$\Delta_{11}a_{m,n}=\Delta_{01}a_{m,n}=\Delta_{10}a_{m,n}=a_{m,n}-a_{m+1,n}-a_{m,n+1}+a_{m+1,n+1}.$$

They proved the following.

Theorem 3.16. *Let $\{a_{m,n}\}$ be a nonnegative double sequence such that $\Delta_{10}a_{m,n} \geq 0$, $\Delta_{01}a_{m,n} \geq 0$, and $\Delta_{11}a_{m,n} \geq 0$ for all $m,n \geq 1$. Then, a necessary and sufficient condition for the uniform convergence in the regular sense of*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n} \sin mx \sin ny \tag{3.21}$$

on $[0,2\pi)^2$ is that $mna_{m,n} \to 0$ as $m+n \to \infty$.

The convergence in the regular sense of double series referred to above is defined analogously as for double integrals, replacing integral by sums, and functions by sequences.

It is also worth mentioning that the uniform convergence of double cosine and sine-cosine series were studied recently. Although we are not including those results here, we refer the reader to [39, 78].

Recently, Kórus, Leindler, and Móricz gave extensions of Theorem 3.16 for certain classes of double GM sequences [75, 81, 84] (in fact, they called those classes with different names, but for our convenience we will stick to the notation introduced in Chapter 2). They considered the following alternative of Definition 2.29.

Definition 3.17. Let $\{\alpha_{m,n}\}, \{\beta_{m,n}\}, \{\gamma_{m,n}\}$ be nonnegative double sequences. We say that a sequence $\{a_{m,n}\}$ belongs to the class $GMS^2(\alpha,\beta,\gamma)$ if there exists a constant $C>0$ such that

$$\sum_{j=m}^{2m} \left|\Delta_{10}a_{j,n}\right| \leq C\alpha_{m,n}, \qquad m \geq m_1,\ n \geq n_1,\ m_1,n_1 \in \mathbb{N}, \tag{3.22}$$

$$\sum_{k=n}^{2n} \left|\Delta_{01}a_{m,k}\right| \leq C\beta_{m,n}, \qquad m \geq m_2,\ n \geq n_2,\ m_2,n_2 \in \mathbb{N}, \tag{3.23}$$

$$\sum_{j=m}^{2m}\sum_{k=n}^{2n} \left|\Delta_{11}a_{j,k}\right| \leq C\gamma_{m,n}, \qquad m \geq m_3,\ n \geq n_3,\ m_3,n_3 \in \mathbb{N}. \tag{3.24}$$

The superscript "2" in GMS^2 denotes the dimension.

The following GMS^2 classes have been considered so far:

1. In [81], the class $GMS^2\big(\alpha^1,\beta^1,\gamma^1\big)$, consisting of double sequences $\{a_{m,n}\}$ for which there exists $\lambda \geq 2$ such that (3.22)–(3.24) hold with

$$\alpha^1_{m,n} = \frac{1}{m}\sum_{j=m/\lambda}^{\lambda m} |a_{j,n}|, \qquad m \geq \lambda,\ n \geq 1,$$

$$\beta^1_{m,n} = \frac{1}{n}\sum_{k=n/\lambda}^{\lambda n} |a_{m,k}|, \qquad m \geq 1,\ n \geq \lambda,$$

$$\gamma^1_{m,n} = \frac{1}{mn}\sum_{j=m/\lambda}^{\lambda m}\sum_{k=n/\lambda}^{\lambda n} |a_{j,k}|, \qquad m,n \geq \lambda;$$

2. in [75], the class $GMS^2\big(\alpha^2,\beta^2,\gamma^2\big)$, consisting of double sequences $\{a_{m,n}\}$ for which there exist $\lambda\ge 2$ and sequences $\big\{b^1_n\big\}$, $\big\{b^2_n\big\}$ and $\big\{b^3_n\big\}$ increasing to infinity such that (3.22)–(3.24) hold with

$$\alpha^2_{m,n}=\frac{1}{m}\Bigg(\sup_{b^1_m\le M\le\lambda b^1_m}\sum_{j=M}^{2M}|a_{j,n}|\Bigg),\qquad m\ge\lambda,\ n\ge 1,$$

$$\beta^2_{m,n}=\frac{1}{n}\Bigg(\sup_{b^2_n\le N\le\lambda b^2_n}\sum_{k=N}^{2N}|a_{m,k}|\Bigg),\qquad m\ge 1,\ n\ge\lambda,$$

$$\gamma^2_{m,n}=\frac{1}{mn}\Bigg(\sup_{M+N\ge b^3_{m+n}}\sum_{j=M}^{2M}\sum_{k=N}^{2N}|a_{j,k}|\Bigg),\qquad m,n\ge\lambda;$$

3. also in [75], the class $GMS^2\big(\alpha^3,\beta^3,\gamma^3\big)$ consisting of double sequences $\{a_{m,n}\}$ for which there exist $\lambda\ge 2$ and sequences $\big\{b^1_n\big\}$, $\big\{b^2_n\big\}$ and $\big\{b^3_n\big\}$ increasing to infinity such that (3.22)–(3.24) hold with

$$\alpha^3_{m,n}=\frac{1}{m}\Bigg(\sup_{M\ge b^1_m}\sum_{j=M}^{2M}|a_{j,n}|\Bigg),\quad \text{for } m\ge\lambda,\ n\ge 1,$$

$$\beta^3_{m,n}=\frac{1}{n}\Bigg(\sup_{N\ge b^2_n}\sum_{k=N}^{2N}|a_{m,k}|\Bigg),\qquad \text{for } m\ge 1,\ n\ge\lambda,$$

$$\gamma^3_{m,n}=\frac{1}{mn}\Bigg(\sup_{M+N\ge b^3_{m+n}}\sum_{j=M}^{2M}\sum_{k=N}^{2N}|a_{j,k}|\Bigg),\qquad \text{for } m,n\ge\lambda.$$

It is proved in [75] that

$$GMS^2\big(\alpha^1,\beta^1,\gamma^1\big)\subsetneq GMS^2\big(\alpha^2,\beta^2,\gamma^2\big)\subsetneq GMS^2\big(\alpha^3,\beta^3,\gamma^3\big).$$

Note that the "monotone" sequences considered in Theorem 3.16 clearly belong to the class $GMS^2\big(\alpha^1,\beta^1,\gamma^1\big)$. Moreover, in relation with Theorem 3.16 the following result is obtained (in fact, the corresponding version is proved for the class $GMS^2\big(\alpha^1,\beta^1,\gamma^1\big)$ in [81]).

Theorem 3.18. (i) *Let $\{a_{m,n}\}\subset\mathbb{C}$ be a sequence from the class $GMS^2\big(\alpha^3,\beta^3,\gamma^3\big)$. If*

$$mna_{m,n}\to 0\qquad \text{as } m+n\to\infty, \tag{3.25}$$

then the double sine series (3.21) converges uniformly in the regular sense on $[0,2\pi)^2$.

(ii) *If $\{a_{m,n}\}\subset\mathbb{R}_+$ belongs to the class $GMS^2\big(\alpha^2,\beta^2,\gamma^2\big)$ and its corresponding double sine series (3.21) converges uniformly in the regular sense on $[0,2\pi)^2$, then (3.25) holds.*

The "if and only if" statement reads as follows:

Corollary 3.19. *Let $\{a_{m,n}\}\subset\mathbb{R}_+$ belong to the class $GMS^2\big(\alpha^2,\beta^2,\gamma^2\big)$. Then its corresponding sine series (3.21) converges uniformly in the regular sense on $[0,2\pi)^2$ if and only if (3.25) holds.*

Leindler [84] gave general sufficient conditions for the uniform convergence of (3.21) in the regular sense that involve the differences of $\{a_{m,n}\}$. These conditions are a two-dimensional analogue of (3.4).

Theorem 3.20. *Let $\{a_{m,n}\}\subset\mathbb{C}$ be a double sequence from the class $GMS^2(\alpha,\beta,\gamma)$. Assume*

$$\alpha_{m,n}=o\big(m^{-1}\big),\qquad \beta_{m,n}=o\big(n^{-1}\big),\qquad \gamma_{m,n}=o\big((mn)^{-1}\big)\qquad \textit{as } m+n\to\infty.$$

Then the double sine series (3.21) converges uniformly in the regular sense on $[0,2\pi)^2$.

Remark 3.21. If we assume $\gamma_{m,n}=o\big((mn)^{-1}\big)$ as $m+n\to\infty$, and define

$$\alpha_{m,n}=\sum_{\ell=0}^{\infty}\gamma_{m,2^\ell n},\qquad \beta_{m,n}=\sum_{\ell=0}^{\infty}\gamma_{2^\ell m,n},$$

we can show that conditions

$$\alpha_{m,n}=o\big(m^{-1}\big),\qquad \beta_{m,n}=o\big(n^{-1}\big)\qquad \text{as } m+n\to\infty$$

are redundant in Theorem 3.20. In fact, this is the reason why we defined the $GM^2(\beta)$ classes only in terms of the Hardy variation over rectangles (cf. Definition 2.28). Indeed, one has

$$\begin{aligned}\sum_{j=m}^{2m}|\Delta_{10}a_{j,n}|&=\sum_{j=m}^{2m}\left|\sum_{k=n}^{\infty}\Delta^{11}a_{j,k}\right|\le\sum_{j=m}^{2m}\sum_{k=n}^{\infty}|\Delta^{11}a_{j,k}|=\sum_{\ell=0}^{\infty}\sum_{j=m}^{2m}\sum_{k=2^\ell n}^{2^{\ell+1}n}|\Delta^{11}a_{j,k}|\\&\le\sum_{\ell=0}^{\infty}\gamma_{m,2^\ell n}=\sum_{\ell=0}^{\infty}o\left(\frac{1}{m2^\ell n}\right)=o\big((mn)^{-1}\big)=o\big(m^{-1}\big)\end{aligned}$$

as $m+n\to\infty$, and similarly for the sums $\sum_{k=n}^{2n}|\Delta_{01}a_{m,k}|$.

On the side of double sine integrals, one can also find recent literature on the topic. The analogue of Theorem 3.16 for double sine integrals [95] reads as follows:

Theorem 3.22. *Let $g:(0,\infty)^2\to\mathbb{R}_+$ be such that $xyg(x,y)\in L^1_{\mathrm{loc}}\big(\mathbb{R}^2_+\big)$, and assume that*

$$\begin{aligned}\Delta_{10}g(x_1,y)&\ge 0, &&\textit{for all } x_2>x_1>0,\ y>0,\\ \Delta_{01}g(x,y_1)&\ge 0, &&\textit{for all } x>0,\ y_2>y_1>0,\\ \Delta_{11}g(x_1,y_1)&\ge 0, &&\textit{for all } x_2>x_1>0,\ y_2>y_1>0.\end{aligned}$$

Then, the double sine integral (3.18) converges uniformly in the regular sense on $[0,2\pi)^2$ if and only if

$$xyg(x,y)\to 0\qquad \textit{as } x+y\to\infty.$$

Kórus and Móricz worked with the following definition of GM functions of two variables in order to extend Theorem 3.22:

Definition 3.23. Let $\alpha,\beta,\gamma:(0,\infty)^2\to\mathbb{R}_+$. We say that a function $g:\mathbb{R}^2_+\to\mathbb{C}$, $g\in AC_{\mathrm{loc}}\big((0,\infty)^2\big)$ belongs to the class $GM^2_*(\alpha,\beta,\gamma)$ if there exist a constant $C>0$ and $x_i,y_i>0$, $i=1,2,3$, such that

$$\int_x^{2x}|d_{10}g(s,y)|\,ds\le C\alpha(x,y),\qquad \text{for all } x\ge x_1,\ y\ge y_1, \tag{3.26}$$

$$\int_y^{2y}|d_{01}g(x,t)|\,dt\le C\beta(x,y),\qquad \text{for all } x\ge x_2,\ y\ge y_2, \tag{3.27}$$

$$\int_y^{2y}\int_x^{2x}|d_{11}g(s,t)|\,ds\,dt\le C\gamma(x,y),\qquad \text{for all } x\ge x_3,\ y\ge y_3. \tag{3.28}$$

Here the subscript "$*$" in GM^2_* denotes that we are restricting ourselves to considering functions from the class $AC_{\mathrm{loc}}\big((0,\infty)^2\big)$. In contrast, the $GM^2(\beta)$ functions we consider (cf. Definition 2.28) are required to belong to the class $HBV_{\mathrm{loc}}\big(\mathbb{R}^2_+\big)$, or more generally, to the class $HBV_{\mathrm{loc}}\big(\mathbb{R}^2_+\backslash[0,c)^2\big)$ for some $c>0$. This is a rather technical assumption that is used when dealing with the Hardy variation of a function over rectangles of the form $[0,a]\times[b_1,b_2]$ (resp. $[b_1,b_2]\times[0,a]$), with $b_1\ge c$.

The $MVBVF^2$ class we mentioned in Section 2.6 (cf. [79, 80]) corresponds to the class $GM^2_*(\alpha_1,\beta_1,\gamma_1)$ consisting of functions g for which there exists $\lambda\ge 2$ such that (3.26)–(3.28) hold with

$$\alpha(x,y)=\frac{1}{x}\int_{x/\lambda}^{\lambda x}|g(s,y)|\,ds,\qquad \beta(x,y)=\frac{1}{y}\int_{y/\lambda}^{\lambda y}|g(x,t)|\,dt,$$
$$\gamma(x,y)=\frac{1}{xy}\int_{y/\lambda}^{\lambda y}\int_{x/\lambda}^{\lambda x}|g(s,t)|\,ds\,dt,$$

for all $x,y>0$. In [79, 80], the authors proved the following.

Theorem 3.24. *Let $g\in GM^2_*(\alpha_1,\beta_1,\gamma_1)$ be such that $xyg(x,y)\in L^1_{\mathrm{loc}}\big(\mathbb{R}^2_+\big)$.*

(i) *If the condition*

$$xyg(x,y)\to 0\qquad \text{as } x+y\to\infty \tag{3.29}$$

holds, then the double sine transform (3.18) converges uniformly in the regular sense on $\mathbb{R}^2_+$.

(ii) *If g is nonnegative and the double sine transform (3.18) converges uniformly in the regular sense on $\mathbb{R}^2_+$, then (3.29) holds.*

Thus, the "if and only if" statement for the class $GM^2_*(\alpha_1,\beta_1,\gamma_1)$ reads as follows.

Corollary 3.25. *Let $g:(0,\infty)^2\to\mathbb{R}_+$ belong to the class $GM^2_*(\alpha_1,\beta_1,\gamma_1)$ and be such that $xyg(x,y)\in L^1_{\mathrm{loc}}\big(\mathbb{R}^2_+\big)$. Then, the double sine transform (3.18) converges uniformly in the regular sense on $\mathbb{R}^2_+$ if and only if (3.29) holds.*

Furthermore, they obtained a necessary condition for the uniform convergence of (3.18) in the sense of Pringsheim.

Theorem 3.26. *Let $g:(0,\infty)^2\to\mathbb{R}_+$ belong to the class $GM^2_*(\alpha_1,\beta_1,\gamma_1)$ and be such that $xyg(x,y)\in L^1_{\mathrm{loc}}\big(\mathbb{R}^2_+\big)$. If the double sine transform (3.18) converges uniformly in the sense of Pringsheim on $\mathbb{R}^2_+$, then*

$$xyg(x,y)\to 0\qquad \text{as } \min\{x,y\}\to\infty. \tag{3.30}$$

3.2.2 New results

Our main result concerning double sine transforms is an analogue of Theorem 3.20, yielding sufficient conditions for the uniform convergence of (3.18) in the regular sense, without including the redundant hypotheses outlined in Remark 3.21. Let us denote

$$\mathcal{G}(u,v) := \int_0^\infty \int_0^\infty g(x,y)\sin ux \sin vy\, dx\, dy. \tag{3.31}$$

Theorem 3.27. *Let $g:\mathbb{R}_+^2\to\mathbb{C}$ be such that $g\in GM^2(\beta)$ and $g\in HBV\big(\mathbb{R}_+^2\big)$. If*

$$\beta(x,y) = o\big((xy)^{-1}\big) \quad \text{as } x+y\to\infty, \tag{3.32}$$

then (3.31) converges uniformly in the regular sense on $\mathbb{R}_+^2$, and moreover,

$$\sup_{u,v\in\mathbb{R}_+} |\mathcal{G}(u,v) - S_{M,N}(u,v)| \le 9(\varepsilon_{M,0}+\varepsilon_{0,N}+\varepsilon_{M,N}), \tag{3.33}$$

where $S_{M,N}(u,v)$ is the partial double sine integral given by (3.20), and

$$\varepsilon_{\mu,\nu} = \sup_{\substack{\mu'\ge\mu\\ \nu'\ge\nu}} \mu'\nu' \int_{\nu'}^\infty \int_{\mu'}^\infty |d_{11}g(s,t)|\, ds\, dt. \tag{3.34}$$

Remark 3.28. Condition (3.32) is equivalent to the condition

$$\int_y^\infty \int_x^\infty |d_{11}g(s,t)|\, ds\, dt = o\big((xy)^{-1}\big) \quad \text{as } x+y\to\infty.$$

The proof is similar to that of (3.7).

The proof of Theorem 3.27 goes through two lemmas; the first of them concerns rewriting the double sine integral (3.31) in terms of the variation of g.

Lemma 3.29. *Let $M,N\ge 0$ and let $g:\mathbb{R}_+^2\to\mathbb{C}$ be such that $g\in HBV\big(\mathbb{R}_+^2\big)$. Then,*

$$\int_N^\infty \int_M^\infty g(x,y)\sin ux\sin vy\, dx\, dy$$
$$= \int_N^\infty \int_M^\infty d_{11}g(s,t)\bigg(\int_M^s \sin ux\, dx\bigg)\bigg(\int_N^t \sin vy\, dy\bigg) ds\, dt.$$

Proof. Since $g\in HBV\big(\mathbb{R}_+^2\big)$, we have

$$\int_y^\infty \int_x^\infty |d_{11}g(s,t)|\, ds\, dt = o(1) \quad \text{as } x+y\to\infty.$$

Now let $M'>M$, $N'>N$, and $u,v>0$. Also, let us denote

$$\phi_u(s) := \int_M^s \sin ux\, dx, \quad \psi_v(t) := \int_N^t \sin vy\, dy.$$

For every $s,t\in\mathbb{R}_+$, there holds $|\phi_u(s)|\le 2/u$, $|\psi_v(t)|\le 2/v$. Since $g\in HBV(\mathbb{R}_+^2)$ and $g(x,y)\to 0$ as $x\to\infty$ (for any fixed $y\ge 0$), we can write

$$g(x,y) = \int_x^\infty d_{10}g(s,y)\, ds,$$

and we have the following equality:

$$\int_N^{N'}\int_M^{M'} g(x,y)\sin ux\sin vy\,dx\,dy$$
$$=\int_N^{N'}\int_M^{M'}\left(\int_x^\infty d_{10}g(s,y)\,ds\right)\sin ux\sin vy\,dx\,dy. \tag{3.35}$$

The latter integral converges absolutely for all values of $M'>M$ and $N'>N$. Thus, we can change the order of integration, and obtain that (3.35) is equal to

$$\begin{aligned}
&\int_N^{N'}\int_M^{M'} d_{10}g(s,y)\phi_u(s)\sin vy\,ds\,dy+\phi_u(M')\int_N^{N'}\int_{M'}^\infty d_{10}g(s,y)\sin vy\,ds\,dy\\
&=\int_N^{N'}\int_M^{M'}\phi_u(s)\left(\int_y^\infty d_{11}g(s,t)\,dt\right)\sin vy\,ds\,dy\\
&\quad+\phi_u(M')\int_N^{N'}\int_{M'}^\infty\left(\int_y^\infty d_{11}g(s,t)\,dt\right)\sin vy\,ds\,dy\\
&=\int_N^{N'}\int_M^{M'}\phi_u(s)\psi_v(t)\,d_{11}g(s,t)\,ds\,dt+\psi_v(N')\int_{N'}^\infty\int_M^{M'}\phi_u(s)d_{11}g(s,t)\,ds\,dt\\
&\quad+\phi_u(M')\int_N^{N'}\int_{M'}^\infty\psi_v(t)\,d_{11}g(s,t)\,ds\,dt+\phi_u(M')\psi_v(N')\int_{N'}^\infty\int_{M'}^\infty d_{11}g(s,t)\,ds\,dt.
\end{aligned}\tag{3.36}$$

It is clear that the first term on the right-hand side of (3.36) tends to

$$\int_N^\infty\int_M^\infty\phi_u(s)\psi_v(t)d_{11}g(s,t)\,ds\,dt$$
$$=\int_N^\infty\int_M^\infty d_{11}g(s,t)\left(\int_M^s\sin ux\,dx\right)\left(\int_N^t\sin vy\,dy\right)ds\,dt$$

as $M',N'\to\infty$. Therefore, the statement of Lemma 3.29 follows if we prove that the three remaining terms on the right-hand side of (3.36) vanish as $M',N'\to\infty$. For the second term, we have

$$\begin{aligned}
\left|\psi_v(N')\int_{N'}^\infty\int_M^{M'}\phi_u(s)d_{11}g(s,t)\,ds\,dt\right|&\le\frac{2}{v}\int_{N'}^\infty\int_M^{M'}|\phi_u(s)\,d_{11}g(s,t)|\,ds\,dt\\
&\le\frac{4}{uv}\int_{N'}^\infty\int_M^\infty|d_{11}g(s,t)|\,ds\,dt=\frac{1}{uv}o(1)
\end{aligned}$$

as $M+N'\to\infty$. The third term on the right-hand side of (3.36) is estimated similarly:

$$\begin{aligned}
\left|\phi_u(M')\int_N^{N'}\int_{M'}^\infty\psi_v(t)\,d_{11}g(s,t)\,ds\right|&\le\frac{2}{u}\int_N^{N'}\int_{M'}^\infty|\psi_v(t)\,d_{11}g(s,t)|\,ds\,dt\\
&\le\frac{4}{uv}\int_N^\infty\int_{M'}^\infty|d_{11}g(s,t)|\,ds\,dt=\frac{1}{uv}o(1)
\end{aligned}$$

as $M'+N\to\infty$. Finally, for the last term on the right-hand side of (3.36) we have

$$\left|\phi_u(M')\psi_v(N')\int_{N'}^\infty\int_{M'}^\infty d_{11}g(s,t)\,ds\,dt\right|\le\frac{4}{uv}\int_{N'}^\infty\int_{M'}^\infty|d_{11}g(s,t)|\,ds\,dt=\frac{1}{uv}o(1)$$

as $M'+N'\to\infty$, which concludes the proof. □

The following lemma deals with estimates of the "residual" integrals

$$R_{M,N}(u,v) = R_{M,N}(g;u,v) := \int_N^\infty \int_M^\infty g(x,y) \sin ux \sin vy \, dx\, dy.$$

Lemma 3.30. *Let $M,N \geq 0$ and let $g : \mathbb{R}_+^2 \to \mathbb{C}$, $g \in HBV\big(\mathbb{R}_+^2\big)$. Then,*

$$\sup_{u,v\in\mathbb{R}_+} R_{M,N}(u,v) \leq 9\varepsilon_{M,N},$$

where $\varepsilon_{M,N}$ is defined by (3.34).

Proof. Let $u,v \neq 0$ and $M,N \geq 0$. By Lemma 3.29, we can write

$$\int_N^\infty \int_M^\infty g(x,y) \sin ux \sin vy \, dx\, dy$$
$$= \int_N^\infty \int_M^\infty d_{11}g(s,t) \bigg(\int_M^s \sin ux \, dx \bigg) \bigg(\int_N^t \sin vy \, dy \bigg) ds\, dt.$$

We distinguish four cases.

1. If $1/u \leq M$ and $1/v \leq N$, then
$$\bigg| \int_N^\infty \int_M^\infty d_{11}g(s,t) \bigg(\int_M^s \sin ux \, dx \bigg) \bigg(\int_N^t \sin vy \, dy \bigg) ds\, dt \bigg|$$
$$\leq \frac{4}{uv} \int_N^\infty \int_M^\infty |d_{11}g(s,t)| \, ds\, dt \leq 4\varepsilon_{M,N},$$
thus in this case $R_{M,N}(u,v) \leq 4\varepsilon_{M,N}$.

2. If $1/u \leq M$, and $1/v > N$, using again the representation
$$g(x,y) = \int_x^\infty d_{10}g(s,y)\, ds,$$
we obtain
$$\bigg| \int_N^\infty \int_M^\infty g(x,y) \sin ux \sin vy \, dx\, dy \bigg|$$
$$= \bigg| \int_N^{1/v} \int_M^\infty g(x,y) \sin ux \sin vy \, dx\, dy + \int_{1/v}^\infty \int_M^\infty g(x,y) \sin ux \sin vy \, dx\, dy \bigg|$$
$$\leq \bigg| \int_N^{1/v} \int_M^\infty \int_x^\infty d_{10}g(s,y) \sin ux \sin vy \, ds\, dx\, dy \bigg| \tag{3.37}$$
$$+ \bigg| \int_{1/v}^\infty \int_M^\infty g(x,y) \sin ux \sin vy \, dx\, dy \bigg|. \tag{3.38}$$
Note first that integral (3.38) is covered by Case 1, and therefore it is bounded from above by $4\varepsilon_{M,1/v}$, and in this case $\varepsilon_{M,1/v} \leq \varepsilon_{M,N}$. An argument similar to that of Lemma 3.29 shows that (3.37) may be written as
$$\bigg| \int_N^{1/v} \int_M^\infty d_{10}g(s,y) \bigg(\int_M^s \sin ux \, dx \bigg) \sin vy \, ds\, dy \bigg|.$$

We estimate the latter from above by

$$\frac{2}{u}\int_N^{1/v}\int_M^\infty |d_{10}g(s,y)\sin vy|\,ds\,dy \\ \leq 2Mv\int_N^{1/v} y\int_y^\infty\int_M^\infty |d_{11}g(s,t)|\,ds\,dt\,dy \leq 2\varepsilon_{M,N}.$$

Collecting these estimates, we obtain

$$R_{M,N}(u,v) \leq 6\varepsilon_{M,N}. \tag{3.39}$$

3. If $1/u > M$ and $1/v \leq N$, a similar argument as in Case 2 yields (3.39) again.
4. If $1/u > M$ and $1/v > N$:

$$\begin{aligned} &\left|\int_N^\infty\int_M^\infty g(x,y)\sin ux\sin vy\,dx\,dy\right| \\ &= \left|\int_N^{1/v}\int_M^{1/u} + \int_N^{1/v}\int_{1/u}^\infty + \int_{1/v}^\infty\int_M^{1/u} + \int_{1/v}^\infty\int_{1/u}^\infty g(x,y)\sin ux\sin vy\,dx\,dy\right| \\ &= |I_1(u,v)+I_2(u,v)+I_3(u,v)+I_4(u,v)|. \end{aligned}$$

We estimate each of these four integrals separately. First of all, for I_1, we have

$$\begin{aligned} |I_1(u,v)| &\leq \int_N^{1/v}\int_M^{1/u} |g(x,y)|ux\,vy\,dx\,dy \\ &\leq uv\int_N^{1/v}\int_M^{1/u} xy\int_y^\infty\int_x^\infty |d_{11}g(s,t)|\,ds\,dt\,dx\,dy \leq \varepsilon_{M,N}. \end{aligned}$$

The upper estimate for I_2 is obtained similarly as that of (3.37) from Case 2:

$$\begin{aligned} |I_2(u,v)| &= \left|\int_N^{1/v}\int_{1/u}^\infty d_{10}g(s,y)\left(\int_{1/u}^s \sin ux\,dx\right)\sin vy\,ds\,dy\right| \\ &\leq \frac{2v}{u}\int_N^{1/v} y\int_y^\infty\int_{1/u}^\infty |d_{11}g(s,t)|\,ds\,dt\,dy \leq 2\varepsilon_{M,N}. \end{aligned}$$

The upper estimate for I_3 is analogue to that of I_2, i.e.,

$$|I_3(u,v)| \leq 2\varepsilon_{M,N}.$$

Finally, by Lemma 3.29, we can write

$$\begin{aligned} |I_4(u,v)| &= \left|\int_{1/u}^\infty\int_{1/v}^\infty d_{11}g(s,t)\left(\int_{1/u}^s \sin ux\,dx\right)\left(\int_{1/v}^t \sin vy\,dy\right)dt\,ds\right| \\ &\leq \frac{4}{uv}\int_{1/v}^\infty\int_{1/u}^\infty |d_{11}g(s,t)|\,ds\,dt \leq 4\varepsilon_{M,N}. \end{aligned}$$

Collecting all the estimates for the integrals $I_j(u,v)$, $j=1,\dots,4$, we conclude that in this case $R_{M,N}(u,v) \leq 9\varepsilon_{M,N}$. □

We are now in a position to prove Theorem 3.27.

Proof of Theorem 3.27. Recall that the uniform regular convergence of (3.31) means that

$$\int_{y_0}^{y_1}\int_{x_0}^{x_1} g(x,y)\sin ux\sin vy\,dx\,dy\to 0 \quad \text{as } x_0+y_0\to\infty,\ x_0<x_1,\ y_0<y_1,$$

uniformly in u,v.

We can use the residual integrals to write

$$\begin{aligned}
&\left|\int_{y_0}^{y_1}\int_{x_0}^{x_1} g(x,y)\sin ux\sin vy\,dx\,dy\right|\\
&=\big|R_{x_0,y_0}(u,v)-R_{x_0,y_1}(u,v)-R_{x_1,y_0}(u,v)+R_{x_1,y_1}(u,v)\big|\\
&\le\big|R_{x_0,y_0}(u,v)\big|+\big|R_{x_0,y_1}(u,v)\big|+\big|R_{x_1,y_0}(u,v)\big|+\big|R_{x_1,y_1}(u,v)\big|.
\end{aligned}\tag{3.40}$$

By Lemma 3.30, we can estimate (3.40) from above by $4\cdot 9\varepsilon_{x_0,y_0}$. By (3.32) (see also Remark 3.28), it follows that ε_{x_0,y_0} tends to zero uniformly in u,v as $x_0+y_0\to\infty$, and thus the uniform convergence of (3.31) in the regular sense follows. Finally, it remains to estimate the uniform error when approximating $\mathcal{G}(u,v)$ by $S_{M,N}(u,v)$.

$$\begin{aligned}
\sup_{u,v\in\mathbb{R}_+}|\mathcal{G}(u,v)-S_{M,N}(u,v)| &= \sup_{u,v\in\mathbb{R}_+}|R_{0,N}(u,v)+R_{M,0}(u,v)-R_{M,N}(u,v)|\\
&\le \sup_{u,v\in\mathbb{R}_+}|R_{0,N}(u,v)|+|R_{M,0}(u,v)|+|R_{M,N}(u,v)|\\
&\le 9(\varepsilon_{0,N}+\varepsilon_{M,0}+\varepsilon_{M,N}),
\end{aligned}$$

where the last inequality follows from Lemma 3.30. □

We must emphasize that Lemmas 3.29 and 3.30 are still valid under the weaker assumption $g\in HBV\big(\mathbb{R}^2_+\backslash[0,c)^2\big)$ with $c>0$, if we impose the restriction $M,N\ge c$ in their statements. Thus Theorem 3.27 is also true under this framework, provided that $xyg(x,y)\in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$, and it allows us to consider functions whose double sine transform is well defined although they are unbounded near the origin, as the example in (2.23), i.e.,

$$g(x,y)=\begin{cases}(xy)^{-1}, & \text{if } x,y<1,\\ e^{-(x+y)}, & \text{otherwise.}\end{cases}$$

Recall that if $g\in HBV\big(\mathbb{R}^2_+\big)$ then it is bounded, so that in contrast with Theorem 3.24, we do not need to assume $xyg(x,y)\in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$, since it already follows from the boundedness of g.

As a corollary of Theorem 3.27, we get the following.

Corollary 3.31. *Let $g\in GM^2_{adm}$ be such that*

$$xyg(x,y)\to 0 \quad \text{as } x+y\to\infty.$$

Then (3.31) converges uniformly in the regular sense on $\mathbb{R}^2_+$, and moreover,

$$\sup_{u,v\in\mathbb{R}_+}|\mathcal{G}(u,v)-S_{M,N}(u,v)|\le 9C(\delta_{M,0}+\delta_{0,N}+\delta_{M,N}),\tag{3.41}$$

where[3]

$$\delta_{\mu,\nu} = \sup_{\substack{\mu' \geq \mu \\ \nu' \geq \nu}} B\big(\mu', \nu', I_{12}\big),$$

and C is the constant from the GM^2 condition given by (2.24).

Proof. The condition $xyg(x, y) \to 0$ as $x + y \to \infty$ implies that

$$\begin{aligned} I_{12}(g; x, y) &= \int_y^{2y} \int_x^{2x} |g(s,t)|\, ds\, dt \\ &\leq xy \sup_{(s,t)\in[x,2x]\times[y,2y]} |g(s,t)| \to 0 \quad \text{as } x + y \to \infty. \end{aligned} \tag{3.42}$$

Since the operator B is admissible, by property (i) of Definition 2.30 and (3.42), we have that

$$B\big(x, y, I_{12}\big) \to 0 \quad \text{as } x + y \to \infty. \tag{3.43}$$

By definition (cf. Definitions 2.28 and 2.30), (3.43) precisely means that

$$\beta(x,y) = o\left(\frac{1}{xy}\right) \quad \text{as } x + y \to \infty.$$

Hence, we are under the hypotheses of Theorem 3.27, and the uniform convergence of (3.31) in the regular sense follows, and moreover (3.33) holds. Finally, since $g \in GM^2_{\mathrm{adm}}$, combining (2.29) and (3.34), we have that for any $M, N > 0$,

$$\varepsilon_{M,N} \leq C \sup_{\substack{M' \geq M \\ N' \geq N}} \frac{M'N'}{M'N'} B\big(M', N', I_{12}\big) = C\delta_{M,N}.$$

Thus, substituting the latter inequality into (3.33), we obtain (3.41). Note also that (3.42) implies $\delta_{M,N} \to 0$ as $M + N \to \infty$. ☐

To conclude this section, we show necessary conditions for the uniform convergence of (3.31) both in the regular and Pringsheim's sense.

Theorem 3.32. *Let $g \in GM^2_{adm}$ be nonnegative. Then condition (3.29) is necessary for the uniform convergence of (3.31) in the regular sense.*

Proof. Let $\varepsilon > 0$. From the uniform convergence of (3.31) in the regular sense it follows that there exists $z > 0$ such that

$$\left| \int_{y_0}^{y_1} \int_{x_0}^{x_1} g(x,y) \sin ux \sin vy\, dx\, dy \right| < \varepsilon, \quad \forall u, v \in \mathbb{R}_+,$$

whenever $x_0 + y_0 > z$, and $x_0 < x_1$, $y_0 < y_1$. If $x_0, y_0 > 0$, setting $x_1 = 2x_0$, $y_1 = 2y_0$, and choosing $u = \pi/4x_0$, $v = \pi/4y_0$, we obtain

$$\begin{aligned} \varepsilon &> \left| \int_{y_0}^{2y_0} \int_{x_0}^{2x_0} g(x,y) \sin ux \sin vy\, dx\, dy \right| \\ &\geq \frac{4}{\pi^2} \int_{y_0}^{2y_0} \int_{x_0}^{2x_0} g(x,y) \frac{\pi x}{4x_0} \frac{\pi y}{4y_0}\, dx\, dy \geq \frac{1}{4} \int_{y_0}^{2y_0} \int_{x_0}^{2x_0} g(x,y)\, dx\, dy \\ &= \frac{1}{4} I_{12}(x_0, y_0). \end{aligned}$$

[3] Since $B\big(x, y, I_{12}\big)$ is not defined for $x = 0$ or $y = 0$, then we define $\delta_{\mu,\nu}$ as the supremum over $\mu' > 0$ or $\nu' > 0$, whenever $\mu = 0$ or $\nu = 0$.

Since $\varepsilon > 0$ is arbitrary, we deduce that $I_{12}(x,y) \to 0$ as $x+y \to \infty$, and consequently, by property (i) of the operator B (cf. Definition 2.30),

$$B\big(x,y,I_{12}\big) \to 0 \quad \text{as } x+y \to \infty.$$

Finally, the result follows by Lemma 2.33, since

$$\begin{aligned} xy|g(x,y)| &\leq C \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{2^j 2^k} B\big(2^j x, 2^k y, I_{12}\big) \\ &\leq 4C \sup_{\substack{j\geq 0 \\ k\geq 0}} B\big(2^j x, 2^k y, I_{12}\big) \to 0 \quad \text{as } x+y \to \infty, \end{aligned}$$

as desired. □

The "if and only if" statement for nonnegative GM^2_{adm} functions follows from Corollary 3.31 and Theorem 3.32.

Theorem 3.33. *Let $g \in GM^2_{adm}$ be nonnegative. Then, condition (3.29) is necessary and sufficient for the uniform convergence of the double sine integral (3.31) in the regular sense.*

Regarding uniform convergence in the sense of Pringsheim, we have the following.

Theorem 3.34. *Let $g \in GM^2_{adm}$ be nonnegative, and suppose that the operator B satisfies, instead of property (i) of Definition 2.30, the following weaker condition: if $I_{12}(x,y) \to 0$ as $\min\{x,y\} \to \infty$, then $B\big(x,y,I_{12}\big) \to 0$ as $\min\{x,y\} \to \infty$. Then, a necessary condition for the uniform Pringsheim convergence of (3.31) is that*

$$xyg(x,y) \to 0 \quad \text{as } \min\{x,y\} \to \infty.$$

Proof. From the uniform convergence of (3.31) in the sense of Pringsheim, applying the Cauchy criterion, it follows that for every $\varepsilon > 0$ there exists $z > 0$ such that

$$\left| \int_{y_0}^{y_1} \int_{x_0}^{x_1} g(x,y) \sin ux \sin vy \, dx \, dy \right| < \varepsilon, \quad \forall u,v \in \mathbb{R}_+,$$

whenever $\min\{x_0, y_0\} > z$, and $x_0 < x_1$, $y_0 < y_1$. The rest of the proof is the same as for Theorem 3.32, replacing $x+y \to \infty$ by $\max\{x,y\} \to \infty$, and using the weaker property of B stated in Theorem 3.34, instead of property (i) of Definition 2.30. □

Since if $g \in GM^2_{\mathrm{adm}}$, then (3.29) guarantees the uniform convergence of (3.31) in the regular sense (Corollary 3.31), the convergence in the sense of Pringsheim also follows from (3.29). However, this condition may be too strong taking into account that Pringsheim convergence is much more relaxed than regular one. On the other hand, one may wonder if (3.30) is sufficient in this case, although in this context it is a mild condition; in fact, a function satisfying (3.30) does not even need to vanish as $x+y \to \infty$, thus it may not be enough to guarantee the uniform convergence of (3.31) in the sense of Pringsheim.

Chapter 4

Uniform convergence of weighted Hankel transforms

The purpose of this chapter is to study the uniform (and also pointwise) convergence of the *Hankel transform*

$$H_\alpha f(r) = \frac{2\pi^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^\infty t^{2\alpha+1} f(t) j_\alpha(2\pi r t)\, dt, \qquad r \geq 0, \qquad \alpha \geq -1/2, \tag{4.1}$$

or more generally, of the *weighted Hankel transform*

$$\mathcal{L}^\alpha_{\nu,\mu} f(r) = r^\mu \int_0^\infty (rt)^\nu f(t) j_\alpha(rt)\, dt, \qquad r \geq 0, \qquad \alpha \geq -1/2, \tag{4.2}$$

where $\nu, \mu \in \mathbb{R}$ are such that $0 \leq \mu + \nu \leq \alpha + 3/2$ (this restriction on the parameters will be disregarded when studying pointwise convergence). Here j_α denotes the normalized Bessel function of order α [47], defined as

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^\infty \frac{(-1)^n (z/2)^{2n}}{n!\Gamma(n+\alpha+1)}, \tag{4.3}$$

and for which several basic properties are given in the next section (here and from now on, Γ denotes the Euler gamma function $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx$, see e.g. [1]). Note that the re-scaling and multiplicative constants from (4.1) are disregarded in (4.2), since they are unimportant in terms of convergence. Throughout this chapter, the function f in (4.2) is taken to be locally of bounded variation on $(0,\infty)$. In this case, it follows that f is bounded on any compact set $K \subset (0,\infty)$, thus it is also locally integrable on $(0,\infty)$.

The Hankel transform (4.1) of order $\alpha = n/2 - 1$ represents the Fourier transform of a radial function defined on $\mathbb{R}^n$. More precisely, if $F(x) = f_0(|x|)$ is a radial function of n variables, then $\widehat{F}$ is also a radial function, and moreover

$$\widehat{F}(y) = H_\alpha f_0(y) = |\mathbb{S}^{n-1}| \int_0^\infty t^{n-1} f_0(t) j_\alpha(2\pi |y| t)\, dt, \quad \alpha = \frac{n}{2} - 1, \quad n \geq 2, \tag{4.4}$$

where $|\mathbb{S}^{n-1}|$ denotes the area of the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, see [126].

The transforms (4.2) arose in a recent paper by De Carli [27], where she considered the class of operators

$$\overline{\mathcal{L}} = \{\overline{\mathcal{L}}^\alpha_{\mu,\nu} : \alpha \geq -1/2,\ \mu, \nu \in \mathbb{R}\},$$

where

$$\overline{\mathcal{L}}^{\alpha}_{\nu,\mu} f(r) = r^{\mu} \int_0^{\infty} (rt)^{\nu} f(t) J_{\alpha}(rt)\, dt,$$

and studied their $L^p - L^q$ mapping properties (that will be discussed in Chapter 5). The function $J_\alpha(z)$ is the Bessel function of order α, related to its normalized version j_α by the identity

$$j_\alpha(z) = \Gamma(\alpha+1)\left(\frac{z}{2}\right)^{-\alpha} J_\alpha(z).$$

Certain operators from $\overline{\mathcal{L}}$ (up to re-scaling and multiplicative constants) appear as the Fourier transform of a radial function multiplied by a spherical harmonic, see [28, 30, 126, 142]. Note also that operators from $\overline{\mathcal{L}}$ relate to those from (4.2) by means of the equality

$$\mathcal{L}^{\alpha}_{\nu,\mu} f = 2^{\alpha}\Gamma(\alpha+1)\overline{\mathcal{L}}^{\alpha}_{\nu-\alpha,\mu} f.$$

Let us now give some examples of integral transforms written in terms of (4.2). Here we denote by F a radial function of n variables and $F(x) = f_0(|x|)$.

1. Since $j_{1/2}(z) = \sin z/z$, the sine transform $\int_0^\infty f(t)\sin rt\, dt$ equals $\mathcal{L}^{1/2}_{1,0} f(r)$, where $r \in \mathbb{R}_+$.

2. Since $j_{-1/2}(z) = \cos z$, the cosine transform $\int_0^\infty f(t)\cos rt\, dt$ coincides with $\mathcal{L}^{-1/2}_{0,0} f(r)$, where $r \in \mathbb{R}_+$.

3. The Fourier transform of a radial function with $n \geq 2$ (cf. (4.4)) can be written as

$$\widehat{F}(y) = |\mathbb{S}^{n-1}|\mathcal{L}^{n/2-1}_{n-1,-(n-1)} f_0(2\pi|y|), \qquad y \in \mathbb{R}^n$$

4. The classical Hankel transform H_α given by (4.1) satisfies

$$H_\alpha f(r) = \frac{2\pi^{\alpha+1}}{\Gamma(\alpha+1)}\mathcal{L}^{\alpha}_{2\alpha+1,-(2\alpha+1)} f(2\pi r), \qquad r \in \mathbb{R}_+.$$

5. If $n \geq 2$ and ψ_k is a solid spherical harmonic of degree k, then

$$\widehat{\psi_k F}(y) = \psi_k(y)\cdot\frac{2\pi^{n/2}}{\Gamma(\alpha+1)}\left(\frac{\pi}{i}\right)^k \mathcal{L}^{\alpha}_{2\alpha+1,-(2\alpha+1)} f_0(2\pi|y|), \qquad y \in \mathbb{R}^n.$$

   with $\alpha = (n+2k-2)/2$, see [126, Ch. IV].

6. For $n \geq 2$, let $\mathscr{D}_k$ denote the Dunkl transform, defined by means of a root system $R \subset \mathbb{R}^n$, a reflection group $G \subset O(n)$, and a G-invariant multiplicity function $k : R \to \mathbb{R}$. Then

$$\mathscr{D}_k F = H_{n/2-1+\langle k\rangle} f_0,$$

   where $\langle k\rangle = \frac{1}{2}\sum_{x\in R} k(x)$ (cf. [38, 111] and the references therein). We also refer the reader to [11], where a generalization of the Dunkl transform is introduced, and [55], where uncertainty principle relations for the generalized Dunkl transform are obtained.

By definition, the uniform convergence of $\mathcal{L}^{\alpha}_{\nu,\mu} f$ means that the family of partial integrals

$$r^{\mu} \int_0^N (rt)^{\nu} f(t) j_{\alpha}(rt)\, dt, \qquad N > 0,$$

converges uniformly in r as $N \to \infty$, or equivalently,

$$r^{\mu} \int_M^N (rt)^{\nu} f(t) j_{\alpha}(rt)\, dt \to 0 \qquad \text{as } N > M \to \infty, \tag{4.5}$$

uniformly in r.

Before proceeding further, let us make the following observation. Trivially, if a function f satisfies $tf(t) \in L^1(0,1)$ and $f \in L^1(1,\infty)$, it follows that its sine transform (3.1) converges uniformly (and similarly, $f \in L^1(\mathbb{R})$ implies the uniform convergence of its cosine transform). Note that these conditions also imply their absolute convergence, so that their uniform convergence follows from the same integrability conditions that imply their absolute convergence. However, this is not true in general for the transforms (4.2). Since

$$|\mathcal{L}^{\alpha}_{\nu,\mu} f(r)| \lesssim r^{\mu+\nu} \int_0^1 t^{\nu} |f(t)|\, dt + r^{\mu+\nu-\alpha-1/2} \int_1^{\infty} t^{\nu-\alpha-1/2} |f(t)|\, dt,$$

which follows from the upper estimate (4.11) for the Bessel function presented in the following section, we note that the absolute convergence of $\mathcal{L}^{\alpha}_{\nu,\mu} f$ follows from the conditions

$$t^{\nu} f(t) \in L^1(0,1), \qquad t^{\nu-\alpha-1/2} f(t) \in L^1(1,\infty). \tag{4.6}$$

Conditions (4.6) do not imply the uniform convergence of $\mathcal{L}^{\alpha}_{\nu,\mu} f$ in general because the kernel

$$K^{\alpha}_{\nu,\mu}(t,r) = r^{\mu} (rt)^{\nu} j_{\alpha}(rt) \tag{4.7}$$

of $\mathcal{L}^{\alpha}_{\nu,\mu}$ need not be uniformly bounded, unlike in the case of the sine and cosine transforms. To illustrate our claim, we consider the example $f(t) = 1/t$ with $\mu = -1$ and $0 < \nu < \alpha + 1/2$. Conditions (4.6) clearly hold but the estimate (4.9) of j_{α} (see p. 63) implies that

$$r^{\mu} \int_{1/(2r)}^{1/r} (rt)^{\nu} f(t) j_{\alpha}(rt)\, dt \asymp \frac{1}{r} \int_{1/(2r)}^{1/r} t^{-1}\, dt = \frac{\log 2}{r}$$

does not vanish as $r \to 0$, thus $\mathcal{L}^{\alpha}_{\nu,\mu} f$ does not converge uniformly (cf. (4.5)).

Nevertheless, there is a special case when the uniform convergence of $\mathcal{L}^{\alpha}_{\nu,\mu} f$ follows from (4.6) (as shown in Proposition 4.9 below), namely when

$$\mu + \nu = \alpha + 1/2.$$

In particular, the sine and cosine transforms satisfy this identity, since they are represented by an operator of the form $\mathcal{L}^{\alpha}_{\nu,\mu}$ with $\alpha = 1/2, \nu = 1, \mu = 0$ and $\alpha = -1/2, \nu = \mu = 0$, respectively.

According to the relationship between the parameters, we will subdivide the operators $\mathcal{L}^{\alpha}_{\nu,\mu}$ into two main classes: first, those satisfying $\mu + \nu = 0$ (as for example, the cosine transform), and secondly, those satisfying $0 < \mu + \nu \leq \alpha + 3/2$ (as for example, the sine transform). We will call the former *cosine-type* transforms, and the latter

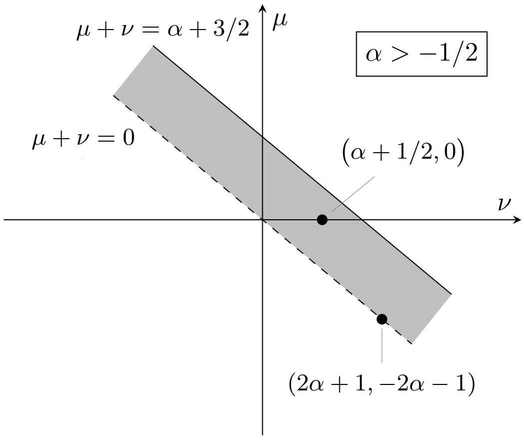

Figure 4.1: Relation between the parameters of cosine-type transforms ($\mu+\nu=0$) and sine-type transforms ($0<\mu+\nu\leq\alpha+3/2$), for a given $\alpha>-1/2$.

sine-type transforms. As we shall see in Sections 4.3 and 4.4, this nomenclature is motivated by the uniform convergence criteria for each case, which are generalizations of Theorems 3.4 and 3.6.

Figure 4.1 shows the range of the parameters μ and ν, for which $\mathcal{L}^{\alpha}_{\nu,\mu}$ is a sine or cosine-type transform, for a fixed $\alpha>-1/2$. Every point lying on the dashed line $\mu+\nu=0$ (or equivalently, $\mu=-\nu$) corresponds to a cosine-type transform. Among those, we highlight the point $(2\alpha+1,-2\alpha-1)$, which corresponds to the Hankel transform of order α (4.1). The area between the dashed line $\mu=-\nu$ and the line $\mu+\nu=\alpha+3/2$ corresponds to sine-type transforms. The point $(\alpha+1/2,0)$ corresponds to the sine transform whenever $\alpha=1/2$, and whenever $\alpha>-1/2$, to transforms whose sufficient condition for their uniform convergence is the same as in Theorem 3.4, as we will see. Note that the latter yields the cosine transform when $\alpha=-1/2$. For every point of the plane outside the grey strip, as mentioned earlier, we will give (rather rough) sufficient conditions on f to guarantee pointwise convergence of $\mathcal{L}^{\alpha}_{\nu,\mu}f$, as well as uniform convergence on certain subintervals of $\mathbb{R}_+$. We also show that whenever the parameters ν and μ lie outside the strip $0\leq\mu+\nu\leq\alpha+3/2$, the aforementioned sufficient conditions that imply the uniform convergence on certain subintervals of $\mathbb{R}_+$ do not yield uniform convergence on the whole $\mathbb{R}_+$, in general.

As we will see, several statements in this chapter require assumptions on the weighted variations of functions. Thus, it is also interesting to study the case of functions f of certain $GM(\beta)$ class in order to replace conditions on the weighted variation of f by weighted integrability conditions on f (cf. Proposition 2.11), or its magnitude at infinity, as done in Theorem 3.4 and Corollary 3.5. To this end, we restrict ourselves to consider functions from the class $GM(\beta_2)$; in fact, if $f\in GM(\beta_3)$ (see (2.15)) satisfies certain weighted integrability conditions, we may still have no useful information on the variations $\int_x^{2x}|df(t)|$ if the integrals $\int_x^{2x}|f(t)|\,dt$ are unbounded as $x\to\infty$.

Unless otherwise specified, all functions f considered in this chapter are complex-valued and defined on $\mathbb{R}_+$.

The main results of this chapter are given in [31, 34].

4.1 Basic properties of the normalized Bessel function

Here we list several properties of the normalized Bessel function $j_\alpha(z)$, which can be found in [47, Chapter VII]. In what follows we assume $z \in \mathbb{R}_+$. Recall that $j_\alpha(z)$ can be represented by the power series in (4.3). This series converges uniformly and absolutely on any bounded interval. For all $z > 0$, there holds $|j_\alpha(z)| \le j_\alpha(0) = 1$. If $z \le 1$,

$$1 - j_\alpha(z) \le Cz^2, \tag{4.8}$$

with $C < 1$ (see Lemma 4.2 below), and therefore

$$j_\alpha(z) \asymp 1, \qquad z \le 1. \tag{4.9}$$

Moreover, we have the following asymptotic estimate (cf. [126]):

$$j_\alpha(z) = \frac{C_\alpha}{z^{\alpha+1/2}} \cos\left(z - \frac{\pi(\alpha+1/2)}{2}\right) + O\big(z^{-\alpha-3/2}\big), \quad z \to \infty. \tag{4.10}$$

Combining (4.9) and (4.10), we obtain

$$|j_\alpha(z)| \lesssim \min\left\{1, \frac{1}{z^{\alpha+1/2}}\right\} \quad \text{for all } z > 0. \tag{4.11}$$

More precisely, if $z \ge 1$ we have

$$|j_\alpha(z)| \le S_\alpha z^{-\alpha-1/2}, \tag{4.12}$$

where

$$S_\alpha := \sup_{z \ge 1} z^{\alpha+1/2} |j_\alpha(z)|.$$

Remark 4.1. Should the reader be interested on the asymptotic behaviour of S_α as $\alpha \to \infty$, we refer to [103], where sharp upper bounds for S_α are obtained. To give a brief overview, it is known that for $\alpha > 1/2$, S_α is strictly increasing to infinity as a function of α and the supremum is attained at the first maximum of the function $z^\alpha j_\alpha(z)$ (see [82]). Also, it is shown in [103] that

$$\lim_{\alpha\to\infty} \frac{S_\alpha}{\alpha^{1/6} 2^\alpha \Gamma(\alpha+1)} = 0.6748\ldots$$

Finally, we have the following property concerning the derivatives of j_α:

$$\frac{d}{dz}\Big(z^{2\alpha+2} j_{\alpha+1}(z)\Big) = (2\alpha+2) z^{2\alpha+1} j_\alpha(z), \quad \alpha \ge -1/2, \tag{4.13}$$

from which we deduce

$$\frac{d}{dz} j_{\alpha+1}(z) = \frac{2\alpha+2}{z}\big(j_\alpha(z) - j_{\alpha+1}(z)\big), \quad \alpha \ge -1/2. \tag{4.14}$$

Let us now proceed to prove some other properties of j_α that we will need. On the first place we have upper and lower estimates for $j_\alpha(z)$ whenever z is small, tighter than those given above.

Lemma 4.2. *Let $\alpha \geq -1/2$. For every $z \leq 2\sqrt{\alpha+1}$ and every $m \in \mathbb{N} \cup \{0\}$ there holds*

$$\Gamma(\alpha+1) \sum_{n=0}^{2m+1} \frac{(-1)^n (z/2)^{2n}}{n!\Gamma(n+\alpha+1)} \leq j_\alpha(z) \leq \Gamma(\alpha+1) \sum_{n=0}^{2m} \frac{(-1)^n (z/2)^{2n}}{n!\Gamma(n+\alpha+1)}.$$

Proof. The proof relies on the fact that for every alternating series $\sum_{n=0}^{\infty}(-1)^n a_n$ with terms a_n decreasing to zero, the estimate

$$\sum_{n=0}^{2m+1} (-1)^n a_n \leq \sum_{n=0}^{\infty} (-1)^n a_n \leq \sum_{n=0}^{2m} (-1)^n a_n \tag{4.15}$$

holds for every $m \in \mathbb{N} \cup \{0\}$. Thus, the result follows if we prove that for any fixed $z \leq 2\sqrt{\alpha+1}$, the terms of the series (4.3) decrease to zero in absolute value. That is equivalent to say

$$\frac{(z/2)^{2n}}{n!\Gamma(n+\alpha+1)} - \frac{(z/2)^{2n+2}}{(n+1)!\Gamma(n+\alpha+2)} \geq 0.$$

Routine simplifications show that the above inequality is equivalent to

$$z \leq 2\sqrt{(n+1)(n+\alpha+1)}. \tag{4.16}$$

Clearly, (4.16) holds for every $n \in \mathbb{N} \cup \{0\}$ if and only if it holds for $n = 0$, i.e., if and only if $z \leq 2\sqrt{\alpha+1}$. $\square$

Another important tool for us is an upper estimate for the primitive function of $t^\nu j_\alpha(rt)$, which will also be used in Chapter 5. To this end, we first observe that since j_α is continuous, if we denote

$$g^\nu_{\alpha,r}(t) := \begin{cases} \displaystyle\int_0^t s^\nu j_\alpha(rs)\, ds, & \text{if } \nu \geq \alpha+1/2 \text{ and } \alpha > -1/2, \text{ or } \nu > \alpha+1/2, \\ \displaystyle -\int_t^\infty s^\nu j_\alpha(rs)\, ds, & \text{if } \nu < \alpha+1/2, \\ \displaystyle\frac{\sin rt}{r}, & \text{if } \nu = 0 \text{ and } \alpha = -1/2, \end{cases} \tag{4.17}$$

then it follows from the fundamental theorem of calculus and estimate (4.11) that $g^\nu_{\alpha,r}$ is the primitive function of $t^\nu j_\alpha(rt)$ with null additive constant. Indeed, note that for any $0 < t_0 < t_1 < \infty$, we have

$$\int_{t_0}^{t_1} s^\nu j_\alpha(rs)\, ds = g^\nu_{\alpha,r}(t_1) - g^\nu_{\alpha,r}(t_0).$$

Since $j_\alpha(z) \asymp 1$ for $z \leq 1$, if $\nu \geq \alpha+1/2$ and $\alpha > -1/2$, or $\nu > \alpha+1/2$, we have that

$$\lim_{t_0 \to 0} g^\nu_{\alpha,r}(t_0) = 0.$$

On the other hand, since $|j_\alpha(z)| \lesssim z^{-\alpha-1/2}$ for $z > 1$, for every $\nu < \alpha+1/2$,

$$\lim_{t_1 \to \infty} g^\nu_{\alpha,r}(t_1) = 0,$$

which was to be proved.

Remark 4.3. Note that

$$\int_0^t s^\nu j_\alpha(rs)\,ds = \frac{t^{\nu+1}}{\nu+1}\,{}_1F_2\left(\frac{1}{2}(\nu+1); \frac{1}{2}(\nu+3), \alpha+1; -\frac{(rt)^2}{4}\right), \quad \nu > -1,$$

where ${}_pF_q$ denotes the generalized hypergeometric function (see [93, Ch. 6]).

We start by rewriting $\int_M^N t^\nu j_\alpha(rt)\,dt$ in terms of higher order Bessel functions.

Lemma 4.4. *Let $\alpha \geq -1/2$, $r > 0$ and $0 < M < N$. Then, for any $n \geq 1$ and $\nu \in \mathbb{R}$ such that $\nu \neq 2(\alpha+\ell)+1$ with $\ell = 0, \dots, n-1$, one has*

$$\begin{aligned}\int_M^N t^\nu j_\alpha(rt)\,dt = &\sum_{k=1}^{n} C_{k,\nu,\alpha}\big(N^{\nu+1} j_{\alpha+k}(rN) - M^{\nu+1} j_{\alpha+k}(rM)\big)\\ &+ C'_{n,\nu,\alpha}\int_M^N t^\nu j_{\alpha+n}(rt)\,dt, \end{aligned} \tag{4.18}$$

where the constants $C_{k,\nu,\alpha}$, $C'_{n,\nu,\alpha}$ are nonzero.

Proof. The proof is done by induction on n. For $n = 1$, we can rewrite the integral on the left-hand side of (4.18) as $\int_M^N t^{\nu-2\alpha-1} t^{2\alpha+1} j_\alpha(rt)\,dt$, and the result follows after integrating by parts together with (4.13). In this case we have $C_{1,\nu,\alpha} = \dfrac{1}{2\alpha+2}$ and $C'_{1,\nu,\alpha} = -\dfrac{\nu-2\alpha-1}{2\alpha+2}$, which are nonzero constants due to the choice of α and ν.

If (4.18) holds for some $n \geq 1$, since

$$C'_{n,\nu,\alpha}\int_M^N t^\nu j_{\alpha+n}(rt)\,dt = C'_{n,\nu,\alpha}\int_M^N t^{\nu-2(\alpha+n)-1} t^{2(\alpha+n)+1} j_{\alpha+n}(rt)\,dt,$$

the result follows similarly as before; in this case we obtain $C_{n+1,\nu,\alpha} = \dfrac{C'_{n,\nu,\alpha}}{2(\alpha+n)+2}$ and $C'_{n+1,\nu,\alpha} = -C'_{n,\nu,\alpha}\dfrac{\nu-2(\alpha+n)-1}{2(\alpha+n)+2}$, which are nonzero constants due to the choice of α and ν. □

Lemma 4.5. *Under the assumptions of Lemma 4.4, we have, for any $\nu \in \mathbb{R}$ such that $\nu = 2(\alpha+\ell)+1$ with some $\ell \in \mathbb{N}\cup\{0\}$,*

$$\int_M^N t^\nu j_\alpha(rt)\,dt = \sum_{k=1}^{\ell+1} C_{k,\nu,\alpha}\big(N^{\nu+1} j_{\alpha+k}(rN) - M^{\nu+1} j_{\alpha+k}(rM)\big), \tag{4.19}$$

where all the constants $C_{k,\nu,\alpha}$ coincide with those of Lemma 4.4.

Proof. If $\ell = 0$, the result immediately follows from (4.13). If $\ell > 0$, we can apply Lemma 4.4 with $\nu' = 2(\alpha+\ell-1)+1$ in place of ν, and then by (4.13),

$$C'_{\ell,\nu,\alpha}\int_M^N t^{2(\alpha+\ell)+1} j_{\alpha+\ell}(rt)\,dt = C_{\ell+1,\nu,\alpha}\big(N^{\nu+1} j_{\alpha+\ell+1}(rN) - M^{\nu+1} j_{\alpha+\ell+1}(rM)\big),$$

where $C_{\ell+1,\nu,\alpha} = \dfrac{C'_{\ell,\nu,\alpha}}{2(\alpha+\ell)+1}$. □

Remark 4.6. We can allow $M = 0$ in Lemmas 4.4 and 4.5 whenever $\nu > -1$, as the integral $\int_0^N t^\nu j_\alpha(t)\,dt$ converges in such case, due to (4.9).

The following lemma yields an upper estimate for $\big|\int_M^N t^\nu j_\alpha(rt)\,dt\big|$, and relies on Lemmas 4.4 and 4.5.

Lemma 4.7. *Let $\alpha \geq -1/2$, $r > 0$ and $0 < M < N$. For any $\nu \in \mathbb{R}$ and any $n \geq 1$ such that $\nu \neq \alpha + n - 1/2$, we have*

$$\left| \int_M^N t^\nu j_\alpha(rt)\,dt \right| \lesssim \frac{1}{r^{\alpha+1/2}} \sum_{k=1}^{n} \frac{1}{r^k} \big(N^{\nu-k-\alpha+1/2} + M^{\nu-k-\alpha+1/2}\big). \tag{4.20}$$

Proof. If ν is as in Lemma 4.5, then (4.20) follows by just applying the estimate $|j_\alpha(z)| \lesssim z^{-\alpha-1/2}$, $z > 0$ (cf. (4.11)), to all the terms on the right-hand side of (4.19). On the contrary, if ν is as in Lemma 4.4, we estimate the sum of (4.18) from above in a similar way, whilst since $\nu - \alpha - n - 1/2 \neq -1$,

$$\begin{aligned}\left| \int_M^N t^\nu j_{\alpha+n}(rt)\,dt \right| &\lesssim \frac{1}{r^{n+\alpha+1/2}} \int_M^N t^{\nu-\alpha-n-1/2}\,dt \\ &\lesssim \frac{1}{r^{n+\alpha+1/2}} \big(N^{\nu-n-\alpha+1/2} + M^{\nu-n-\alpha+1/2}\big),\end{aligned}$$

which coincides precisely with the n-th term of the sum on the right-hand side of (4.20). $\square$

We can finally obtain the desired upper bound for (4.17).

Lemma 4.8. *For any $\alpha \geq -1/2$, $\nu \in \mathbb{R}$, and $r, t > 0$, the estimate*

$$|g^\nu_{\alpha,r}(t)| \lesssim \frac{t^{\nu-\alpha-1/2}}{r^{\alpha+3/2}} \tag{4.21}$$

holds.

Proof. We distinguish two cases: $\nu \neq \alpha + 1/2$, or $\nu = \alpha + 1/2$. In the first case, estimate (4.21) follows readily by applying Lemma 4.7 with $n = 1$ and letting $M \to 0$ or $N \to \infty$ if $\nu > \alpha + 1/2$ or $\nu < \alpha + 1/2$, respectively.

If $\nu = \alpha + 1/2$, and $\alpha = -1/2$, (4.21) follows immediately from (4.17). For $\alpha > -1/2$, we can apply Lemma 4.4 with $n = 2$ and $M = 0$ (see also Remark 4.6) to obtain

$$\begin{aligned}|g^{\alpha+1/2}_{\alpha,r}(t)| &= \left| C_1 t^{\alpha+3/2} j_{\alpha+1}(rt) + C_2 t^{\alpha+3/2} j_{\alpha+2}(rt) + C_3 \int_0^t s^{\alpha+1/2} j_{\alpha+2}(rs)\,ds \right| \\ &\lesssim \frac{1}{r^{\alpha+3/2}} + \frac{1}{tr^{\alpha+5/2}} + \int_0^t s^{\alpha+1/2} |j_{\alpha+2}(rs)|\,ds.\end{aligned}$$

It follows from (4.11) that

$$\int_0^t s^{\alpha+1/2} |j_{\alpha+2}(rs)|\,ds \lesssim \int_0^{1/r} s^{\alpha+1/2}\,ds + \frac{1}{r^{\alpha+5/2}} \int_{1/r}^\infty s^{-2}\,ds \lesssim \frac{1}{r^{\alpha+3/2}}.$$

Collecting the above estimates, we deduce

$$|g^{\alpha+1/2}_{\alpha,r}(t)| \lesssim \frac{1}{r^{\alpha+3/2}} + \frac{1}{tr^{\alpha+5/2}},$$

which implies (4.21) whenever $t \geq 1/r$.

Finally, if $t < 1/r$, using (4.17) together with (4.9) we obtain

$$|g^{\nu}_{\alpha,r}(t)| \asymp \int_0^t s^{\alpha+1/2}\,ds \asymp t^{\alpha+3/2} < \frac{1}{r^{\alpha+3/2}},$$

which completes the proof. □

4.2 Pointwise and uniform convergence of $\mathcal{L}^{\alpha}_{\nu,\mu}f$: first approach

This section is devoted to finding sufficient conditions on f that guarantee the pointwise convergence of $\mathcal{L}^{\alpha}_{\nu,\mu}f$. If $t^{\nu}f(t) \in L^1(0,1)$, the convergence of $\mathcal{L}^{\alpha}_{\nu,\mu}f$ at $r_0 \in \mathbb{R}_+$ means that

$$\lim_{M\to\infty} \left| r_0^{\mu+\nu} \int_0^M t^{\nu} f(t) j_{\alpha}(r_0 t)\,dt \right| < \infty.$$

In contrast with the criteria for uniform convergence (see Theorems 4.16 and 4.23 in the following sections), here we do not impose restrictions on the parameters. The analysis of convergence at the origin is rather simple:

(i) if $\mu+\nu<0$, then $\mathcal{L}^{\alpha}_{\nu,\mu}f(0)$ is not defined;

(ii) if $\mu+\nu=0$, the convergence of $\mathcal{L}^{\alpha}_{\nu,\mu}f(0)$ is equivalent to $\left|\int_0^\infty t^{\nu}f(t)\,dt\right| < \infty$;

(iii) if $\mu+\nu>0$, then $\mathcal{L}^{\alpha}_{\nu,\mu}f(0)=0$.

We proceed to study the pointwise convergence of $\mathcal{L}^{\alpha}_{\nu,\mu}f(r)$ at $r>0$. When possible, we also give sufficient conditions for the uniform convergence on subintervals of $\mathbb{R}_+$. The statements in this section can be subdivided into two categories, depending on their hypotheses. First, we have those relying on the integrability of f, and secondly, those involving conditions on the weighted variation of f.

4.2.1 Integrability conditions

We begin with the results involving integrability conditions of f.

Proposition 4.9. *Let f be such that $t^{\nu}f(t) \in L^1(0,1)$ and $t^{\nu-\alpha-1/2}f(t) \in L^1(1,\infty)$. Then $\mathcal{L}^{\alpha}_{\nu,\mu}f(r)$ converges at any $r>0$. Moreover,*

1. *if $\mu+\nu-\alpha-1/2<0$, then $\mathcal{L}^{\alpha}_{\nu,\mu}f$ converges uniformly on any interval $[\varepsilon,\infty)$ with $\varepsilon>0$;*
2. *if $\mu+\nu-\alpha-1/2>0$, then $\mathcal{L}^{\alpha}_{\nu,\mu}f$ converges uniformly on any interval $[0,\varepsilon]$ with $\varepsilon>0$;*
3. *if $\mu+\nu-\alpha-1/2=0$, then $\mathcal{L}^{\alpha}_{\nu,\mu}f$ converges uniformly on $\mathbb{R}_+$.*

Proof. It is clear that the pointwise convergence of $\mathcal{L}^{\alpha}_{\nu,\mu}f(r)$ at $r>0$ is equivalent to

$$\int_M^\infty t^{\nu} f(t) j_{\alpha}(rt)\,dt = o(1) \qquad \text{as } M\to\infty,$$

which holds by simply applying the estimate (4.11) and the fact that $t^{\nu-\alpha-1/2}\in L^1(1,\infty)$.

Let us now prove the statement concerning uniform convergence. For each of the three cases, since $t^{\nu-\alpha-1/2}f(t)\in L^1(1,\infty)$, it follows from the estimate (4.11) that

$$r^{\mu+\nu}\int_M^\infty t^\nu f(t)j_\alpha(rt)\,dt \le \varepsilon^{\mu+\nu-\alpha-1/2}\int_M^\infty t^{\nu-\alpha-1/2}|f(t)|\,dt = o(1) \qquad \text{as } M\to\infty,$$

i.e., the integrals (4.5) vanish uniformly in r (on each corresponding interval) as $M\to\infty$. □

Proposition 4.9 allows us to easily derive sufficient conditions for the uniform convergence of $\mathcal{L}^{\alpha}_{\nu,\mu}f$ on $\mathbb{R}_+$ whenever $0\le\mu+\nu\le\alpha+1/2$.

Corollary 4.10. *Let $0\le\mu+\nu\le\alpha+1/2$. If $t^\nu f(t)\in L^1(\mathbb{R}_+)$, then $\mathcal{L}^{\alpha}_{\nu,\mu}f$ converges uniformly on $\mathbb{R}_+$.*

Proof. First, if $0\le\mu+\nu<\alpha+1/2$, note that since $\alpha\ge -1/2$, $t^\nu f(t)\in L^1(\mathbb{R}_+)$ implies $t^{\nu-\alpha-1/2}f(t)\in L^1(1,\infty)$, so we can apply Proposition 4.9 to deduce that $\mathcal{L}^{\alpha}_{\nu,\mu}f$ converges uniformly on any interval $[\varepsilon,\infty)$ with $\varepsilon>0$, whilst the uniform convergence on the interval $[0,\varepsilon]$ follows from

$$r^{\mu+\nu}\int_M^\infty t^\nu f(t)j_\alpha(rt)\,dt \le \varepsilon^{\mu+\nu}\int_M^\infty t^\nu|f(t)|\,dt\to 0 \qquad \text{as } M\to\infty.$$

Secondly, if $\mu+\nu=\alpha+1/2$, then $t^{\nu-\alpha-1/2}f(t)=t^{-\mu}f(t)$, and therefore $t^\nu f(t)\in L^1(\mathbb{R}_+)$ implies $t^{-\mu}f(t)\in L^1(1,\infty)$ (since $\nu\ge-\mu$ for every $\alpha\ge-1/2$), and the result follows by Proposition 4.9. □

4.2.2 Variational conditions

The statements of this subsection involve conditions on the variation of f. Note that if $f\in GM(\beta_2)$, we can derive these variational conditions from integrability conditions of f (see Proposition 2.11 and Corollary 2.12). Similarly as above, we also give sufficient conditions for the uniform convergence of $\mathcal{L}^{\alpha}_{\nu,\mu}f$ on subintervals of $\mathbb{R}_+$ when possible.

Proposition 4.11. *Let f be such that $t^\nu f(t)\in L^1(0,1)$. Assume that*

$$\int_1^\infty t^{\nu-\alpha-1/2}|df(t)|<\infty \quad \text{and} \quad M^{\nu-\alpha-1/2}|f(M)|\to 0 \quad \text{as } M\to\infty, \tag{4.22}$$

then $\mathcal{L}^{\alpha}_{\nu,\mu}f(r)$ converges at any $r>0$. Moreover, for any $\varepsilon>0$,

1. *if $\mu+\nu-\alpha-3/2>0$, the convergence is uniform on any interval $[0,\varepsilon]$;*
2. *if $\mu+\nu-\alpha-3/2<0$, the convergence is uniform on any interval $[\varepsilon,\infty)$;*
3. *if $\mu+\nu-\alpha-3/2=0$, the convergence is uniform on $\mathbb{R}_+$.*

Remark 4.12. In the case $\nu\ge\alpha+1/2$, if f vanishes at infinity the convergence of $\int_1^\infty t^{\nu-\alpha-1/2}|df(t)|$ implies that $M^{\nu-\alpha-1/2}f(M)\to 0$ as $M\to\infty$. Indeed,

$$M^{\nu-\alpha-1/2}|f(M)|\le M^{\nu-\alpha-1/2}\int_M^\infty |df(t)|\le \int_M^\infty t^{\nu-\alpha-1/2}\,|df(t)|,$$

and the right-hand side of the latter vanishes as $M\to\infty$. Thus, in this case we only need to assume the convergence of $\int_1^\infty t^{\nu-\alpha-1/2}|df(t)|$ in Proposition 4.11.

Proof of Proposition 4.11. Fix $r>0$. Since $t^\nu f(t)\in L^1(0,1)$, the convergence of (4.2) is equivalent to

$$\lim_{M\to\infty}\left|\int_1^M t^\nu f(t) j_\alpha(rt)\,dt\right|<\infty.$$

Integrating by parts, we have

$$\int_1^M t^\nu f(t) j_\alpha(rt)\,dt = g^\nu_{\alpha,r}(M)f(M)-g^\nu_{\alpha,r}(1)f(1)-\int_1^M g^\nu_{\alpha,r}(t)\,df(t), \tag{4.23}$$

where $g^\nu_{\alpha,r}(t)$ is given by (4.17). Now we estimate from above each term on the right-hand side of (4.23) (note that $g^\nu_{\alpha,r}(1)f(1)$ is bounded, since f is bounded on any compact set). It follows from (4.21) and (4.22) that

$$|g^\nu_{\alpha,r}(M)f(M)|\lesssim \frac{M^{\nu-\alpha-1/2}}{r^{\alpha+3/2}}|f(M)|\to 0 \quad \text{as } M\to\infty.$$

Finally, by (4.21),

$$\int_1^M |g^\nu_{\alpha,r}(t)\,df(t)|\lesssim \frac{1}{r^{\alpha+3/2}}\int_1^M t^{\nu-\alpha-1/2}\,|df(t)|.$$

Hence, by letting $M\to\infty$ we find that the convergence of $\int_1^\infty t^{\nu-\alpha-1/2}|df(t)|$ implies that of $\int_1^\infty |g^\nu_{\alpha,r}(t)\,df(t)|$. This concludes the part concerning pointwise convergence.

The assertion related to uniform convergence is easily proved by simply applying estimates (4.11) and (4.21):

$$\begin{aligned} r^\mu\left|\int_M^N (rt)^\nu f(t) j_\alpha(rt)\,dt\right| &= r^{\mu+\nu}\left|g^\nu_{\alpha,r}(N)f(N)-g^\nu_{\alpha,r}(M)f(M)-\int_M^N g^\nu_{\alpha,r}(t)\,df(t)\right| \\ &\lesssim r^{\mu+\nu-\alpha-3/2}\bigg(N^{\nu-\alpha-1/2}|f(N)|+M^{\nu-\alpha-1/2}|f(M)| \\ &\qquad + \int_M^N t^{\nu-\alpha-1/2}\,|df(t)|\bigg). \end{aligned}$$

Thus, the latter expression vanishes

1. uniformly in $r\in[0,\varepsilon]$ if $\mu+\nu-\alpha-3/2>0$;
2. uniformly in $r\in[\varepsilon,\infty)$ if $\mu+\nu-\alpha-3/2<0$;
3. uniformly in $r\in\mathbb{R}_+$ if $\mu+\nu-\alpha-3/2=0$,

as $N>M\to\infty$. □

For functions $f\in GM(\beta_2)$, we can derive a version of Proposition 4.11 depending on integrability conditions of f, which are less restrictive than those from Proposition 4.9.

Corollary 4.13. *Let $f\in GM(\beta_2)$ be such that $t^\nu f(t)\in L^1(0,1)$. If $t^{\nu-\alpha-3/2}f(t)\in L^1(1,\infty)$, all the statements of Proposition 4.11 hold.*

Proof. If $f\in GM(\beta_2)$, the condition $t^{\nu-\alpha-3/2}f(t)\in L^1(1,\infty)$ implies that $t^{\nu-\alpha-1/2}f(t)$ vanishes at infinity (see Remark 2.10). Furthermore, by Corollary 2.12, we have that all hypotheses of Proposition 4.11 are satisfied, and the result follows. □

The last statement of this subsection is a combination of Propositions 4.9 and 4.11.

Corollary 4.14. *Let f be such that $t^{\nu}f(t)\in L^1(0,1)$. Assume that $\alpha+1/2\le\mu+\nu<\alpha+3/2$. If the conditions in (4.22) hold, and if $t^{\nu-\alpha-1/2}f(t)\in L^1(1,\infty)$, then $\mathcal{L}^{\alpha}_{\nu,\mu}f$ converges uniformly on $\mathbb{R}_+$.*

Note that except for the case $\alpha=-1/2$ and $\mu+\nu=0$, the parameters for which Corollary 4.14 can be applied correspond to sine-type transforms.

4.2.3 Examples

Let us discuss an application of Proposition 4.11, closely related to the following classical statement [146, Ch. I, Theorem 2.6] (see also [8, Ch. I, §30]) concerning pointwise convergence of trigonometric series: *Let $\varphi(x)$ be either $\sin x$ or $\cos x$. If $a_n\to 0$ and $\{a_n\}$ is of bounded variation, or equivalently,*

$$\sum_{n=N}^{\infty}|a_n-a_{n+1}|=o(1)\quad \textit{as } N\to\infty,$$

then $\sum_{n=0}^{\infty}a_n\varphi(nx)$ converges pointwise in $x\in(0,2\pi)$, and the convergence is uniform on any interval $[\varepsilon,2\pi-\varepsilon]$, $\varepsilon>0$.

A version of the latter statement for the sine and cosine transforms follows from Proposition 4.11 (see item 2 of the latter, and observe that for the sine and cosine transforms both conditions $\mu+\nu-\alpha-3/2<0$ and $\nu-\alpha-1/2=0$ hold).

Corollary 4.15. *Let $f,g:\mathbb{R}_+\to\mathbb{C}$ be vanishing at infinity and such that $f\in L^1(0,1)$ and $tg(t)\in L^1(0,1)$. Assume that f and g are of bounded variation on $[\delta,\infty)$ for some $\delta>0$. Then,*

$$\int_0^{\infty}f(t)\cos rt\,dt\qquad \textit{and}\qquad \int_0^{\infty}g(t)\sin rt\,dt$$

converge for every $r>0$, and the convergence is uniform on every interval $[\varepsilon,\infty)$, with $\varepsilon>0$.

Finally, we show that we cannot guarantee the uniform convergence of $\mathcal{L}^{\alpha}_{\nu,\mu}f$ on $\mathbb{R}_+$ outside the range of parameters $0\le\mu+\nu\le\alpha+3/2$, whenever f satisfies both conditions from (4.22). The case $\mu+\nu<0$ is clear, since in this case $\mathcal{L}^{\alpha}_{\nu,\mu}f(0)$ is not even defined. The case $\mu+\nu>\alpha+3/2$ is more involved. We proceed to constructing a counterexample.

Let

$$f(t)=\begin{cases} t^{-\nu}, & \text{if } t<2,\\ \dfrac{t^{-\nu+\alpha+1/2}}{(\log t)^2}, & \text{if } t\ge 2.\end{cases}$$

On the one hand, since for any $\nu\in\mathbb{R}$ and $\alpha\ge-1/2$ one has

$$f'(t)=(-\nu+\alpha+1/2)\frac{t^{-\nu-1/2+\alpha}}{(\log t)^2}-2\frac{t^{-\nu-1/2+\alpha}}{(\log t)^3},\qquad t>2,$$

it is clear that

$$\int_1^{\infty}t^{\nu-\alpha-1/2}|df(t)|\lesssim 1+\int_2^{\infty}t^{\nu-\alpha-1/2}|f'(t)|\,dt\lesssim\int_2^{\infty}\frac{1}{t(\log t)^2}\,dt<\infty.$$

On the other hand, for $t \geq 2$

$$t^{\nu-\alpha-1/2} f(t) = \frac{1}{(\log t)^2} \to 0 \qquad \text{as } t \to \infty,$$

and hence f satisfies both conditions from (4.22). Let us now prove that $\mathcal{L}^{\alpha}_{\nu,\mu} f$ does not converge uniformly on $\mathbb{R}_+$ (although it does on any interval $[0,\varepsilon]$ for any $\varepsilon > 0$, according to Proposition 4.11). Let $2 < M < N$. Integration by parts along with property (4.13) of j_α yields

$$\begin{aligned} r^{\mu+\nu}\left|\int_M^N t^\nu f(t) j_\alpha(rt)\, dt\right| \leq r^{\mu+\nu} \frac{1}{2\alpha+2}&\left|\left[\frac{t^{\alpha+3/2}}{(\log t)^2} j_{\alpha+1}(rt)\right]_M^N\right| \\ &+ \frac{\alpha+1/2}{2\alpha+2}\left|\int_M^N \frac{t^{\alpha+1/2}}{(\log t)^2} j_{\alpha+1}(rt)\, dt\right| \\ &+ \frac{2}{2\alpha+2}\left|\int_M^N \frac{t^{\alpha+1/2}}{(\log t)^3} j_{\alpha+1}(rt)\, dt\right| =: a_0 + b_0 + c_0. \end{aligned}$$

First,

$$a_0 = r^{\mu+\nu}\left|\frac{N^{\alpha+3/2}}{(\log N)^2} j_{\alpha+1}(rN) - \frac{M^{\alpha+3/2}}{(\log M)^2} j_{\alpha+1}(rM)\right|.$$

If we choose $r = (\log M)^{2/(\mu+\nu-\alpha-3/2)}$ and M so that $j_{\alpha+1}(rM) \asymp (rM)^{-\alpha-3/2}$ (such M can be found through the expansion (4.10)), we obtain by letting $N \to \infty$,

$$a_0 \asymp \frac{r^{\mu+\nu-\alpha-3/2}}{(\log M)^2} = 1.$$

We now prove that both terms b_0 and c_0 vanish as $N > M \to \infty$ (for this particular choice of r). Then it follows that $\mathcal{L}^{\alpha}_{\nu,\mu} f$ does not converge uniformly on $\mathbb{R}_+$. Let us proceed to estimate b_0 from above first. Again, integration by parts and (4.13) yield

$$\begin{aligned} r^{\mu+\nu}\left|\int_M^N \frac{t^{\alpha+1/2}}{(\log t)^2} j_{\alpha+1}(rt)\, dt\right| \leq r^{\mu+\nu} \frac{1}{2\alpha+4}&\left|\left[\frac{t^{\alpha+3/2}}{(\log t)^2} j_{\alpha+2}(rt)\right]_M^N\right| \\ &+ \frac{\alpha+5/2}{2\alpha+4}\left|\int_M^N \frac{t^{\alpha+1/2}}{(\log t)^2} j_{\alpha+2}(rt)\, dt\right| \\ &+ \frac{2}{2\alpha+4}\left|\int_M^N \frac{t^{\alpha+1/2}}{(\log t)^3} j_{\alpha+2}(rt)\, dt\right| =: a_1 + b_1 + c_1. \end{aligned}$$

By (4.11), it is clear that

$$a_1 \lesssim r^{\mu+\nu-\alpha-5/2}\left(\frac{1}{M(\log M)^2} + \frac{1}{N(\log N)^2}\right) \lesssim \frac{(\log M)^{2\left|\frac{\mu+\nu-\alpha-5/2}{\mu+\nu-\alpha-3/2}\right|}}{M} \to 0$$

as $N > M \to \infty$, as for b_1 and c_1, we have

$$\begin{aligned} b_1 + c_1 &\lesssim r^{\mu+\nu} \int_M^N \frac{t^{\alpha+1/2}}{(\log t)^2} |j_{\alpha+2}(rt)|\, dt \lesssim r^{\mu+\nu-\alpha-5/2} \int_M^N \frac{1}{t^2(\log t)^2}\, dt \\ &\leq \frac{r^{\mu+\nu-\alpha-5/2}}{M} \leq \frac{(\log M)^{2\left|\frac{\mu+\nu-\alpha-5/2}{\mu+\nu-\alpha-3/2}\right|}}{M} \to 0 \end{aligned}$$

as $N > M \to \infty$. Let us now investigate the term c_0. Integration by parts, (4.11), and (4.13) yield

$$r^{\mu+\nu}\left|\int_M^N \frac{t^{\alpha+1/2}}{(\log t)^3} j_{\alpha+1}(rt)\, dt\right| \lesssim r^{\mu+\nu}\left(\left|\left[\frac{t^{\alpha+3/2}}{(\log t)^3} j_{\alpha+2}(rt)\right]_M^N\right| + \int_M^N \frac{t^{\alpha+1/2}}{(\log t)^3}|j_{\alpha+2}(rt)|\, dt\right),$$

and it can be shown similarly as above that the latter vanishes as $N > M \to \infty$. Therefore, we conclude that $\mathcal{L}^\alpha_{\nu,\mu} f$ does not converge uniformly.

4.3 Uniform convergence of $\mathcal{L}^\alpha_{\nu,\mu} f$ with $\mu+\nu=0$ (cosine-type transforms)

This section is devoted to the study of the uniform convergence of cosine-type transforms, i.e., those of the form $\mathcal{L}^\alpha_{\nu,\mu} f$ with $\mu+\nu=0$. Recall that the classical Hankel transform of order α belongs to this case ($\nu = 2\alpha+1 = -\mu$).

It is usual when studying Fourier integrals to assume that the function f in the integrand vanishes at infinity. Here we do not always need this assumption, since the kernel (4.7) of $\mathcal{L}^\alpha_{\nu,\mu}$ vanishes as $t\to\infty$ for certain choices of the parameters and fixed $r\in\mathbb{R}_+$. For this reason, we present two results as the main ones of this section (Theorems 4.16 and 4.17), one for functions that vanish at infinity, and another (more general) one that does not include such an assumption. We will finish this section by obtaining two more results: one for continuous functions (Theorem 4.20), and another one for functions from the class $GM(\beta_2)$ (Theorem 4.22).

The first result of this section reads as follows.

Theorem 4.16. *Let $\nu\in\mathbb{R}$ and $\mu=-\nu$. Let f be such that $t^\nu f(t)\in L^1(0,1)$, and*

$$f(M) = o\big(M^{-\nu-1}\big) \qquad \text{as } M\to\infty, \tag{4.24}$$

$$\int_M^\infty t^{\nu-\alpha-1/2}|df(t)| = o\big(M^{-\alpha-3/2}\big) \qquad \text{as } M\to\infty. \tag{4.25}$$

Then, a necessary and sufficient condition for $\mathcal{L}^\alpha_{\nu,\mu} f$ to converge uniformly on $\mathbb{R}_+$ is that

$$\left|\int_0^\infty t^\nu f(t)\, dt\right| < \infty. \tag{4.26}$$

Secondly, we have the corresponding result for functions vanishing at infinity.

Theorem 4.17. *Let $\nu\in\mathbb{R}$ and $\mu=-\nu$. Let f be vanishing at infinity and such that $t^\nu f(t)\in L^1(0,1)$. Assume that*

$$\int_M^\infty |df(t)| = o\big(M^{-\nu-1}\big) \quad \text{as } M\to\infty, \qquad \text{if } \nu<\alpha+1/2 \text{ and } \nu>-1, \tag{4.27}$$

$$\int_M^\infty t^{\nu-\alpha-1/2}|df(t)| = o\big(M^{-\alpha-3/2}\big) \quad \text{as } M\to\infty, \qquad \text{if } \nu\geq\alpha+1/2 \text{ or } \nu\leq -1. \tag{4.28}$$

Then (4.26) is necessary and sufficient to guarantee the uniform convergence of $\mathcal{L}^\alpha_{\nu,\mu} f$ on $\mathbb{R}_+$.

Before proving the above theorems, let us make some observations.

Remark 4.18. (i) Theorem 3.6, which is a uniform convergence criterion for the cosine transform, is a particular case of Theorem 4.17, namely when $\nu = \mu = 0$ and $\alpha = -1/2$. This is why we call operators $\mathcal{L}^{\alpha}_{\nu,\mu}$ with $\nu = -\mu$ "cosine-type transforms".

(ii) If we assume $f \geq 0$ in Theorems 4.16 and 4.17, the uniform convergence of $\mathcal{L}^{\alpha}_{\nu,\mu} f$ is trivially equivalent to $t^{\nu} f(t) \in L^1(\mathbb{R}_+)$, since $j_\alpha(0) = 1$ and $|j_\alpha(z)| \leq j_\alpha(0)$ for all $z > 0$.

(iii) The criterion for the uniform convergence of the classical Hankel transform (4.1) can be derived by letting $\nu = 2\alpha + 1$ in Theorem 4.16, i.e., *if* (4.24) *and* (4.25) *hold, then $H_\alpha f$ converges uniformly on $\mathbb{R}_+$ if and only if*

$$\left| \int_0^\infty t^{2\alpha+1} f(t)\, dt \right| < \infty.$$

Also, note that the cosine transform corresponds to the Hankel transform of order $\alpha = -1/2$ and is therefore covered by the latter statement.

(iv) If f vanishes at infinity, then (4.25) implies (4.24), which means that for certain choice of parameters, condition (4.24) may be redundant (more precisely, this is the case when $\nu \geq -1$). This is the reason we also present Theorem 4.17, for functions vanishing at infinity, which has simpler hypotheses than Theorem 4.16.

Let us now prove Theorems 4.16 and 4.17.

Proof of Theorem 4.16. The necessity part follows from the convergence at $r = 0$ and the fact that $j_\alpha(0) = 1$.

In order to prove the sufficiency part, we show that the integrals (4.5) vanish uniformly in r as $N > M \to \infty$.

Let $0 < M < N$. If $r \geq 1/M$, integration by parts yields

$$\int_M^N t^{\nu} f(t) j_\alpha(rt)\, dt = \left[f(t) g^{\nu}_{\alpha,r}(t) \right]_M^N - \int_M^N g^{\nu}_{\alpha,r}(t)\, df(t). \tag{4.29}$$

It follows from (4.21) and (4.29) that

$$\begin{aligned} \left| \int_M^N t^{\nu} f(t) j_\alpha(rt)\, dt \right| &\lesssim \frac{1}{r^{\alpha+3/2}} \sup_{x \geq M} x^{\nu-\alpha-1/2} |f(x)| + \frac{1}{r^{\alpha+3/2}} \int_M^N t^{\nu-\alpha-1/2}\, |df(t)| \\ &\leq \sup_{x \geq M} x^{\nu+1} |f(x)| + M^{\alpha+3/2} \int_M^\infty t^{\nu-\alpha-1/2}\, |df(t)|, \end{aligned}$$

and both terms vanish as $M \to \infty$, by (4.24) and (4.25).

If $r < 1/M$, we write

$$\int_M^N t^{\nu} f(t) j_\alpha(rt)\, dt = \left(\int_M^{1/r} + \int_{1/r}^N \right) t^{\nu} f(t) j_\alpha(rt)\, dt.$$

On the one hand, the integral $\int_{1/r}^{N} t^{\nu} f(t) j_\alpha(rt)\, dt$ can be estimated as above. On the other hand, by (4.8),

$$\begin{aligned}\left|\int_M^{1/r} t^\nu f(t) j_\alpha(rt)\,dt\right| &\le \sup_{x>M}\left|\int_M^x t^\nu f(t)\,dt\right| + \left|\int_M^{1/r} t^\nu f(t)\big(1-j_\alpha(rt)\big)\,dt\right| \\ &\le \sup_{x>M}\left|\int_M^x t^\nu f(t)\,dt\right| + r\int_M^{1/r} t^{\nu+1}|f(t)|rt\,dt \\ &\le \sup_{x>M}\left|\int_M^x t^\nu f(t)\,dt\right| + \left(\sup_{x\ge M} x^{\nu+1}|f(x)|\right)\int_M^{1/r} r\,dt \\ &\le \sup_{x>M}\left|\int_M^x t^\nu f(t)\,dt\right| + \sup_{x\ge M} x^{\nu+1}|f(x)|. \qquad (4.30)\end{aligned}$$

The first term of (4.30) vanishes as $M\to\infty$ by (4.26), whilst the second term also vanishes as $M\to\infty$, by (4.24). □

Proof of Theorem 4.17. First of all, since f is vanishing at infinity, we have that $|f(M)| \le \int_M^\infty |df(t)|$ for all $M>0$.

Let us first consider the case $\nu<\alpha+1/2$. On the one hand,

$$M^{\nu+1}|f(M)| \le M^{\nu+1}\int_M^\infty |df(t)| \to 0 \qquad \text{as } M\to\infty.$$

On the other hand,

$$M^{\alpha+3/2}\int_M^\infty t^{\nu-\alpha-1/2}|df(t)| \le M^{\nu+1}\int_M^\infty |df(t)| \to 0 \qquad \text{as } M\to\infty,$$

or in other words, (4.27) implies both hypotheses (4.24) and (4.25) of Theorem 4.16, and the result follows.

If $\nu\ge\alpha+1/2$,

$$M^{\nu+1}|f(M)| \le M^{\nu+1}\int_M^\infty |df(t)| \le M^{\alpha+3/2}\int_M^\infty t^{\nu-\alpha-1/2}|df(t)| \to 0 \qquad \text{as } M\to\infty,$$

i.e., we are under the conditions of Theorem 4.16, and the result follows (notice that in this case (4.25) implies (4.24)).

Finally, if $\nu\le -1$, since f vanishes at infinity, condition (4.24) is automatically satisfied, and the result follows, since condition (4.28) is precisely (4.25). □

Remark 4.19. If $\nu<\alpha+1/2$, (4.28) implies (4.27), and if $\nu\ge\alpha+1/2$ then (4.27) implies (4.28). In Theorem 4.17 we assume either (4.27) or (4.28) but in any case the assumption is the less restrictive from those.

The following criterion relies on conditions of a function f itself, rather than on its variation. Recall that if f is continuous and we write

$$F_\nu(x) = -\int_x^\infty t^\nu f(t)\,dt$$

as an improper Riemann integral, then (4.26) implies that F_ν is well defined and moreover $F_\nu'(x) = x^\nu f(x)$, in virtue of the fundamental theorem of calculus.

Theorem 4.20. *Let $\nu \in \mathbb{R}$ and $\mu = -\nu$. Let $f \in C(1,\infty)$ be such that $t^\nu f(t) \in L^1(0,1)$. Assume that $\alpha > 1/2$, and*

$$f(M) = o\big(M^{-\nu-1}\big) \qquad \text{as } M \to \infty. \tag{4.31}$$

Then the transform $\mathcal{L}^\alpha_{\nu,\mu} f$ converges uniformly on $\mathbb{R}_+$ if and only if (4.26) is satisfied.

Observe that the range of α for which Theorem 4.20 is valid is reduced compared to the one of Theorems 4.16 and 4.17 (where we deal with the full range $\alpha \geq -1/2$).

Remark 4.21. If f vanishes at infinity, then for $\nu > -1$ and $\nu < \alpha+1/2$, (4.27) obviously implies (4.31), and for $\nu \geq \alpha + 1/2$, (4.28) implies (4.31), since $|f(M)| \leq \int_M^\infty |df(t)|$, so that $M^{\nu+1}|f(M)| \leq M^{\alpha+3/2}\int_M^\infty t^{\nu-\alpha-1/2}|df(t)|$. However, the converse is not true. Indeed, consider $f(t) = t^{-\nu-2}\sin t$, for $t > 1$. It is clear that (4.31) holds, and thus $\mathcal{L}^\alpha_{\nu,\mu} f$ converges uniformly (with $\nu = -\mu$), but $f'(t) = -(\nu+2)t^{-\nu-3}\sin t + t^{-\nu-2}\cos t$, and therefore one has for $M > 1$,

$$M^{\nu+1}\int_M^\infty |f'(t)|\,dt \asymp M^{\nu+1}\int_M^\infty t^{-\nu-2}\,dt \asymp 1,$$

if $\nu < \alpha + 1/2$ and $\nu > -1$, and

$$M^{\alpha+3/2}\int_M^\infty t^{\nu-\alpha-1/2}|f'(t)|\,dt \asymp M^{\alpha+3/2}\int_M^\infty t^{-\alpha-5/2}\,dt \asymp 1,$$

if $\nu \geq \alpha + 1/2$, i.e., (4.31) does not imply (4.27) nor (4.28) (in each respective case). Therefore, Theorem 4.20 can be seen as a complement to Theorems 4.16 and 4.17.

Proof of Theorem 4.20. The necessity part is clear, due to the convergence at $r = 0$.

Now we proceed to prove the sufficiency part. Let us denote

$$F_\nu(x) := -\int_x^\infty t^\nu f(t)\,dt.$$

First of all, since for $\alpha \geq 1/2$ one has $(d/dt)j_\alpha(rt) = (2\alpha+2)(j_{\alpha-1}(rt) - j_\alpha(rt))/t$ (see (4.14)), it follows from the estimate (4.11) that

$$\left|\frac{d}{dt}j_\alpha(rt)\right| \lesssim \frac{1}{t^{\alpha+1/2}r^{\alpha-1/2}}, \tag{4.32}$$

whenever $rt \geq 1$, or equivalently, $r \geq 1/t$. We proceed to estimate the integral

$$\int_M^\infty t^\nu f(t) j_\alpha(rt)\,dt,$$

which is equivalent to estimate the integrals (4.5) as $N \to \infty$. On the one hand, if $r \geq 1/M$, we integrate by parts and obtain

$$\begin{aligned}
\left|\int_M^\infty t^\nu f(t) j_\alpha(rt)\,dt\right| &\leq \big|j_\alpha(rM)F_\nu(M)\big| + \left|\int_M^\infty F_\nu(t)\left(\frac{d}{dt}j_\alpha(rt)\right)dt\right| \\
&\leq \max_{N\geq M}|F_\nu(N)| + \max_{N\geq M}|F_\nu(N)|\int_M^\infty \left|\frac{d}{dt}j_\alpha(rt)\right| dt \\
&\lesssim \max_{N\geq M}|F_\nu(N)|\left(1 + \frac{1}{r^{\alpha-1/2}}\int_M^\infty \frac{1}{t^{\alpha+1/2}}\,dt\right) \\
&\leq \max_{N\geq M}|F_\nu(N)|\left(1 + M^{\alpha-1/2}\int_M^\infty \frac{1}{t^{\alpha+1/2}}\,dt\right) \\
&\asymp \max_{N\geq M}|F_\nu(N)|,
\end{aligned}$$

where we have applied (4.32) and used the fact that $\alpha>1/2$. Since F_ν vanishes at infinity whenever (4.26) is satisfied, the above estimate vanishes as $M\to\infty$. On the other hand, if $r<1/M$, we write

$$\int_M^\infty t^\nu f(t) j_\alpha(rt)\,dt=\left(\int_M^{1/r}+\int_{1/r}^\infty\right)t^\nu f(t)j_\alpha(rt)\,dt,$$

and estimate $\int_{1/r}^\infty t^\nu f(t)j_\alpha(rt)\,dt$ as above. Similarly as in the proof of Theorem 4.16, estimate (4.8) yields

$$\begin{aligned}\left|\int_M^{1/r} t^\nu f(t)j_\alpha(rt)\,dt\right| &\le \max_{x>M}\left|\int_M^x t^\nu f(t)\,dt\right| + r\int_M^{1/r} t^{\nu+1}|f(t)|\,dt\\ &\le \max_{x>M}\left|\int_M^x t^\nu f(t)\,dt\right| + \max_{x\ge M} x^{\nu+1}|f(x)|,\end{aligned}$$

which vanishes as $M\to\infty$. $\square$

Finally, we give a criterion for real-valued functions from the class $GM(\beta_2)$.

Theorem 4.22. *Let $f\in GM(\beta_2)$ be real-valued and such that $t^\nu f(t)\in L^1(0,1)$. Then $\mathcal{L}^\alpha_{\nu,-\nu}f$ converges uniformly if and only if $\int_0^\infty t^\nu f(t)\,dt$ converges.*

Proof. The necessity part is clear from the convergence at $r=0$ and $j_\alpha(0)=1$.

In order to prove the sufficiency part, we use Abel-Olivier's test for real-valued $GM(\beta_2)$ functions (Theorem 2.15). Since $f\in GM(\beta_2)$, it follows that $t^\nu f(t)\in GM(\beta_2)$ for every $\nu\in\mathbb{R}$ (cf. [86]). Therefore, the convergence of $\int_0^\infty t^\nu f(t)\,dt$ implies that $t^{\nu+1}f(t)\to 0$ as $t\to\infty$, by Theorem 2.15, which is precisely condition (4.24).

To conclude the proof, we show that if $f\in GM(\beta_2)$, then (4.24) implies (4.25), and the result will follow by Theorem 4.16. Indeed, since $\alpha\ge -1/2$,

$$\begin{aligned}\int_M^\infty t^{\nu-\alpha-1/2}|df(t)| &\lesssim \int_{M/2}^\infty \frac{1}{t}\int_t^{2t} s^{\nu-\alpha-1/2}|df(s)| \lesssim \int_{M/2}^\infty t^{\nu-\alpha-5/2}\int_{t/\lambda}^{\lambda t}|f(s)|\,ds\\ &\lesssim \int_{M/2}^\infty \left(\sup_{t/\lambda\le x\le \lambda t} x^{\nu+1}|f(x)|\right)t^{-\alpha-5/2}\,dt\\ &\lesssim \left(\sup_{x\ge M/(2\lambda)} x^{\nu+1}|f(x)|\right)M^{-\alpha-3/2}.\end{aligned}$$

Thus, by (4.24),

$$M^{\alpha+3/2}\int_M^\infty t^{\nu-\alpha-1/2}|df(t)|\lesssim \sup_{x\ge M/(2\lambda)} x^{\nu+1}|f(x)|\to 0\qquad \text{as } M\to\infty,$$

i.e., (4.25) holds. This completes the proof. $\square$

We will discuss Theorem 4.22 in more detail and give an extended version of it in the case of the Hankel transform ($\nu=2\alpha+1$) in Section 4.5. In particular, we also discuss the boundedness of $\mathcal{L}^\alpha_{2\alpha+1,-(2\alpha+1)}f(r)$ as a function of r.

4.4 Uniform convergence of $\mathcal{L}^{\alpha}_{\nu,\mu}f$ with $0<\mu+\nu\leq\alpha+3/2$ (sine-type transforms)

In this section we study the uniform convergence of sine-type transforms, or equivalently, the transforms $\mathcal{L}^{\alpha}_{\nu,\mu}f$ with $0<\mu+\nu\leq\alpha+3/2$. In general we will give only sufficient conditions for the uniform convergence of these transforms on $\mathbb{R}_+$. For nonnegative functions from the class $GM(\beta_2)$ we also obtain necessary conditions. Recall that $f\in GM(\beta_2)$ are those satisfying

$$\int_x^{2x}|df(t)|\lesssim\frac{1}{x}\int_{x/\lambda}^{\lambda x}|f(t)|\,dt$$

for some $\lambda\geq 2$ and all $x>0$ (see Section 2.3).

For the same reason as in Subsection 4.3, we have two main results in this part, namely a general one, and a simplified version where functions are assumed to vanish at infinity.

Theorem 4.23. *Let $\nu,\mu\in\mathbb{R}$ be such that $0<\mu+\nu\leq\alpha+3/2$. Let f be such that $t^{\nu}f(t)\in L^1(0,1)$. If the conditions*

$$f(M)=o\big(M^{\mu-1}\big)\qquad \text{as } M\to\infty, \tag{4.33}$$

$$\int_M^{\infty}t^{\nu-\alpha-1/2}|df(t)|=o\big(M^{\mu+\nu-\alpha-3/2}\big)\qquad \text{as } M\to\infty \tag{4.34}$$

are satisfied, then $\mathcal{L}^{\alpha}_{\nu,\mu}f$ converges uniformly on $\mathbb{R}_+$.

Observe that conditions (4.24) and (4.25) from Theorem 4.16 are the same as (4.33) and (4.34), respectively, for the particular case $\mu=-\nu$. However, note that in Theorem 4.16 the convergence of the integral $\int_0^{\infty}t^{\nu}f(t)\,dt$ is also required, whilst in Theorem 4.23 it is not.

Remark 4.24. In the extremal case $\mu+\nu=\alpha+3/2$, the conditions in (4.22) from Proposition 4.11 are equivalent to (4.33) and (4.34). In other words, the statements of Proposition 4.11 and Theorem 4.23 coincide in this extremal case.

The statement for functions vanishing at infinity reads as follows.

Theorem 4.25. *Let $\nu,\mu\in\mathbb{R}$ be such that $0<\mu+\nu\leq\alpha+3/2$, and let f be vanishing at infinity and such that $t^{\nu}f(t)\in L^1(0,1)$. Assume that*

$$\int_M^{\infty}|df(t)|=o\big(M^{\mu-1}\big)\quad \text{as } M\to\infty,\qquad \text{if } \nu<\alpha+1/2 \text{ and } \mu<1, \tag{4.35}$$

$$\int_M^{\infty}t^{\nu-\alpha-1/2}|df(t)|=o\big(M^{\mu+\nu-\alpha-3/2}\big)\quad \text{as } M\to\infty,\qquad \text{if } \nu\geq\alpha+1/2 \text{ or } \mu\geq 1. \tag{4.36}$$

Then $\mathcal{L}^{\alpha}_{\nu,\mu}f$ converges uniformly on $\mathbb{R}_+$.

Remark 4.26. (1) Note also that part (i) of Theorem 3.4 is the particular case of Theorem 4.25 with $\alpha=1/2$, $\nu=1$ and $\mu=0$ (see also (3.7)).

(2) Let us observe an interesting property of the operator $\mathcal{L}^{\alpha}_{\alpha+1/2,0}$, with $\alpha>-1/2$ (if $\alpha=-1/2$, such operator corresponds to the cosine transform). Its kernel $K_{\alpha}(r,t)=K_{\alpha}(rt):=(rt)^{\alpha+1/2}j_{\alpha}(rt)$ is uniformly bounded and does not vanish at infinity in

any of the variables r nor t (for any fixed α, this is the only kernel of the type (4.7) with this property). Moreover, K_α vanishes at the origin. Thus, such kernel has a similar behaviour as the kernel $K_{1/2}(rt)=\sin rt$ of the sine transform. In fact, the sufficient condition that guarantees the uniform convergence of $\mathcal{L}^{\alpha}_{\alpha+1/2,0}f$ and that of the sine transform of f is the same, namely (cf. Theorem 4.23)

$$\int_M^\infty |df(t)| = o(1/M) \qquad \text{as } M\to\infty.$$

We now proceed to prove Theorems 4.23 and 4.25.

Proof of Theorem 4.23. We show that the integrals (4.5) vanish uniformly in r as $N>M\to\infty$. Let $0<M<N$, and assume that $1/r\le M$. Integration by parts together with (4.17) and estimate (4.21) yield

$$\begin{aligned}
r^{\mu+\nu}\int_M^N t^\nu f(t) j_\alpha(rt)\,dt &= r^{\mu+\nu}\Big(\big[f(t)g^{\nu}_{\alpha,r}(t)\big]_M^N - \int_M^N g^{\nu}_{\alpha,r}(t)\,df(t)\Big)\\
&\lesssim r^{\mu+\nu}\Big(\frac{N^{\nu-\alpha-1/2}}{r^{\alpha+3/2}}|f(N)| + \frac{M^{\nu-\alpha-1/2}}{r^{\alpha+3/2}}|f(M)|\\
&\qquad + \frac{1}{r^{\alpha+3/2}}\int_M^N t^{\nu-\alpha-1/2}\,|df(t)|\Big)\\
&\lesssim \sup_{x\ge M} x^{1-\mu}|f(x)| + M^{\alpha+3/2-\mu-\nu}\int_M^\infty t^{\nu-\alpha-1/2}\,|df(t)|.
\end{aligned}$$

Both terms on the right-hand side vanish as $M\to\infty$, by the assumptions (4.33) and (4.34).

If $1/r>M$, we write

$$r^{\mu+\nu}\int_M^N t^\nu f(t) j_\alpha(rt)\,dt = r^{\mu+\nu}\Big(\int_M^{1/r}+\int_{1/r}^N\Big)t^\nu f(t) j_\alpha(rt)\,dt,$$

and estimate the integral $\int_{1/r}^N t^\nu f(t) j_\alpha(rt)\,dt$ as above. Moreover, since $\mu+\nu>0$, we have

$$\begin{aligned}
\Big|r^\mu\int_M^{1/r}(rt)^\nu f(t) j_\alpha(rt)\,dt\Big| &\le r^\mu\int_M^{1/r}(rt)^\nu|f(t)|\,dt = r^{\mu+\nu}\int_M^{1/r} t^\nu|f(t)|\,dt\\
&= r^{\mu+\nu}\int_M^{1/r} t^{\mu+\nu-1}t^{1-\mu}|f(t)|\,dt\\
&\le \Big(\sup_{x\ge M} x^{1-\mu}|f(x)|\Big) r^{\mu+\nu}\int_0^{1/r} t^{\mu+\nu-1}\,dt \asymp \sup_{x\ge M} x^{1-\mu}|f(x)|,
\end{aligned}$$

which vanishes as $M\to\infty$, by (4.33). □

Proof of Theorem 4.25. We prove that our hypotheses imply those of Theorem 4.23, and the result will follow. Consider first the case $\mu<1$ and $\nu<\alpha+1/2$. Then

$$M^{1-\mu}|f(M)| \le M^{1-\mu}\int_M^\infty |df(t)| \to 0 \qquad \text{as } M\to\infty,$$

and

$$M^{\alpha+3/2-\mu-\nu}\int_M^\infty t^{\nu-\alpha-1/2}|df(t)|\leq M^{1-\mu}\int_M^\infty |df(t)|\to 0\qquad \text{as } M\to\infty,$$

i.e., (4.33) and (4.34) hold.

If $\nu\geq\alpha+1/2$,

$$M^{1-\mu}|f(M)|\leq M^{1-\mu}\int_M^\infty |df(t)|\leq M^{\alpha+3/2-\mu-\nu}\int_M^\infty t^{\nu-\alpha-1/2}|df(t)|\to 0$$

as $M\to\infty$, or in other words, in this case (4.36) (which is precisely (4.34)) implies (4.33).

Finally, if $\mu\geq 1$, since f vanishes at infinity, (4.33) holds, and the hypotheses of Theorem 4.23 are met. □

For functions $f\in GM(\beta_2)$ we can give an alternative statement to Theorem 4.23, namely with a sufficient condition that only depends on the magnitude of f at infinity. Furthermore, in this case we can obtain a criterion for nonnegative $GM(\beta_2)$ functions, which can be seen as an extension of Corollary 3.5 (to more general transforms, but with a more restrictive GM condition).

Theorem 4.27. *Let $\nu,\mu\in\mathbb{R}$ be such that $0<\mu+\nu<\alpha+3/2$. Let $f\in GM(\beta_2)$ be such that $t^\nu f(t)\in L^1(0,1)$.*

1. *If*
$$f(M)=o\big(M^{\mu-1}\big)\qquad \text{as } M\to\infty, \tag{4.37}$$
then $\mathcal{L}^\alpha_{\nu,\mu}f$ converges uniformly on $\mathbb{R}_+$.

2. *If $f\geq 0$ and $\mathcal{L}^\alpha_{\nu,\mu}f$ converges uniformly on $\mathbb{R}_+$, then (4.33) holds.*

The "if and only if" statement reads as follows:

Corollary 4.28. *Let $f\in GM(\beta_2)$ be nonnegative, $\alpha\geq -1/2$, and $\nu,\mu\in\mathbb{R}$ be such that $0<\mu+\nu<\alpha+3/2$. Then $\mathcal{L}^\alpha_{\nu,\mu}f$ converges uniformly on $\mathbb{R}_+$ if and only if (4.37) holds.*

Proof of Theorem 4.27. Since $f\in GM(\beta_2)$, (4.33) implies (4.34) whenever $\mu+\nu<\alpha+3/2$. Indeed,

$$\begin{aligned}\int_M^\infty t^{\nu-\alpha-1/2}|df(t)| &\lesssim \int_{M/2}^\infty \frac{1}{t}\int_t^{2t} s^{\nu-\alpha-1/2}|df(s)|\lesssim \int_{M/2}^\infty t^{\nu-\alpha-5/2}\int_{t/\lambda}^{\lambda t}|f(s)|\,ds\\ &\lesssim \int_{M/2}^\infty\Big(\sup_{t/\lambda\leq x\leq\lambda t} x^{1-\mu}|f(x)|\Big)t^{\mu+\nu-\alpha-5/2}\,dt\\ &\lesssim \Big(\sup_{x\geq M/(2\lambda)} x^{1-\mu}|f(x)|\Big)M^{\mu+\nu-\alpha-3/2},\end{aligned}$$

where in the latter inequality we have used the fact that $\mu+\nu<\alpha+3/2$, and $\lambda\geq 2$ is the $GM(\beta_2)$ constant. Thus, we deduce that

$$M^{\alpha+3/2-\mu-\nu}\int_M^\infty t^{\nu-\alpha-1/2}\,|df(t)|\lesssim \sup_{x\geq M/(2\lambda)} x^{1-\mu}|f(x)|\to 0\qquad \text{as } M\to\infty,$$

so that the result follows by applying Theorem 4.23. This completes the first part of the proof.

For the second part, the uniform convergence of $\mathcal{L}^{\alpha}_{\nu,\mu}f$ implies

$$r^{\mu}\int_{1/(2\lambda r)}^{\lambda/r}(rt)^{\nu}f(t)j_{\alpha}(rt)\,dt\asymp r^{\mu}\int_{1/(2\lambda r)}^{\lambda/r}f(t)\,dt\to 0\qquad\text{as } r\to 0.$$

By Remark 2.10, we have

$$f(1/r)\lesssim r\int_{1/(2\lambda r)}^{\lambda/r}f(t)\,dt=r^{1-\mu}r^{\mu}\int_{1/(2\lambda r)}^{\lambda/r}f(t)\,dt,$$

and we deduce that $r^{\mu-1}f(1/r)\to 0$ as $r\to 0$, or equivalently, $t^{1-\mu}f(t)\to 0$ as $t\to\infty$. □

Note that in Theorem 4.27 we exclude the case $\mu+\nu=\alpha+3/2$. The proof of Theorem 4.27 consists on showing that if $f\in GM(\beta_2)$, then condition (4.33) implies (4.34). This does not happen in general in the extremal case $\mu+\nu=\alpha+3/2$, as we show now.

Proposition 4.29. *Let $\nu\in\mathbb{R}$ and $\mu<1$ be such that $\mu+\nu=\alpha+3/2$. If $f\in GM(\beta_2)$ vanishes at infinity, the condition*

$$\int_M^{\infty}t^{1-\mu}|df(t)|=\int_M^{\infty}t^{\nu-\alpha-1/2}|df(t)|=o(1)\qquad\textit{as } M\to\infty$$

is equivalent to $t^{-\mu}f(t)\in L^1(1,\infty)$.

It is clear that condition (4.33) does not imply $t^{-\nu}f(t)\in L^1(1,\infty)$, even for monotone functions. For a simpler version of Proposition 4.29, see [90].

Proof of Proposition 4.29. Since f vanishes at infinity and $\mu<1$, the estimate

$$\int_1^{\infty}t^{-\mu}|f(t)|\,dt\leq\int_1^{\infty}t^{-\mu}\int_t^{\infty}|df(s)|=\int_1^{\infty}|df(s)|\int_1^{s}t^{-\mu}\,dt\lesssim\int_1^{\infty}t^{1-\mu}|df(t)|$$

proves one direction of the statement. Note we have not used the GM so far. For the other direction, since $f\in GM(\beta_2)$, we have

$$\begin{aligned}\int_1^{\infty}t^{1-\mu}|df(t)|&=\sum_{k=0}^{\infty}\int_{2^k}^{2^{k+1}}t^{1-\mu}|df(t)|\lesssim\sum_{k=0}^{\infty}2^{k(1-\mu)}\frac{1}{2^k}\int_{2^k/\lambda}^{\lambda 2^k}|f(t)|\,dt\\&\asymp\sum_{k=0}^{\infty}\int_{2^k/\lambda}^{\lambda 2^k}t^{-\mu}|f(t)|\,dt\lesssim\int_{1/\lambda}^{\infty}t^{-\mu}|f(t)|\,dt,\end{aligned}$$

which completes the proof. Note that the above inequality holds for any $\mu\in\mathbb{R}$. □

The condition $f\in GM(\beta_2)$ in the sufficiency part of Theorem 4.27 (and therefore also in Corollary 4.28) cannot be dropped, as shown by Proposition 4.30 below. To prove it, we construct functions that play an analogous role to that of lacunary sequences. A nonnegative sequence $\{n_k\}$ is said to be lacunary if there exists $q>1$ such that $n_{k+1}/n_k\geq q$ for all k. It is easy to construct a uniformly converging sine series

$$\sum_{n=1}^{\infty}a_n\sin nx$$

with $na_n \not\to 0$ as $n \to \infty$ using lacunary sequences, thus showing that the monotonicity assumption in Theorem 3.1 is essential. Indeed, let n_k be a lacunary sequence, and

$$a_n = \begin{cases} n^{-1}, & \text{if } n = n_k,\ k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the sine series $\sum_{n=1}^\infty a_n \sin nx = \sum_{k=1}^\infty n_k^{-1} \sin n_k x$ converges uniformly on $[0, 2\pi)$ since $\sum n_k^{-1}$ converges, but $na_n = 1$ for all $n = n_k$.

Proposition 4.30. *Let $\mu, \nu \in \mathbb{R}$ be such that $0 < \mu + \nu < \alpha + 3/2$. There exists $f \notin GM(\beta_2)$ such that condition (4.37) does not hold, but $\mathcal{L}^\alpha_{\nu,\mu} f$ converges uniformly on $\mathbb{R}_+$.*

Proof of Proposition 4.30. We first construct f in a general setting and then we subdivide the proof into two parts, namely for the case $0 < \mu + \nu \le \alpha + 1/2$, and for the case $\alpha + 1/2 < \mu + \nu < \alpha + 3/2$.

Let c_n be an increasing nonnegative sequence and $\varepsilon_n > 0$ such that $\varepsilon_n < c_{n+1} - c_n$ and $\varepsilon_n \le c_n$ for every n. Define

$$f(t) = \begin{cases} t^{\mu-1}, & \text{if } t \in [c_n, c_n + \varepsilon_n],\ n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

For such a function, it is clear that $t^{1-\mu} f(t) \not\to 0$ as $t \to \infty$. We are now going to find those choices of c_n and ε_n for which $f \notin GM(\beta_2)$ and $\mathcal{L}^\alpha_{\nu,\mu} f$ converges uniformly on $\mathbb{R}_+$. Note that for any c_n and ε_n, $t^\nu f(t) \in L^1(0,1)$, since $\mu + \nu > 0$.

Let us first consider the case $0 < \mu + \nu \le \alpha + 1/2$. According to Corollary 4.10, the uniform convergence of $\mathcal{L}^\alpha_{\nu,\mu} f$ on $\mathbb{R}_+$ follows from $t^\nu f(t) \in L^1(\mathbb{R}_+)$, which in this case is equivalent to

$$\sum_{n=1}^\infty \varepsilon_n c_n^{\nu+\mu-1} < \infty. \tag{4.38}$$

Choosing $c_n = 2^n$ and $\varepsilon_n = 2^{-n\beta}$ with $\beta > \nu + \mu - 1$, it can be easily proved that $f \notin GM(\beta_2)$. Moreover, the series (4.38) converges, and the uniform convergence of $\mathcal{L}^\alpha_{\nu,\mu} f$ on $\mathbb{R}_+$ follows.

Consider now the case $\alpha + 1/2 < \mu + \nu < \alpha + 3/2$. According to Corollary 4.14, the uniform convergence of $\mathcal{L}^\alpha_{\nu,\mu} f$ follows from the conditions

$$t^{\nu-\alpha-1/2} f(t) \to 0 \text{ as } t \to \infty, \qquad t^{\nu-\alpha-1/2} f(t) \in L^1(1,\infty), \qquad \int_1^\infty t^{\nu-\alpha-1/2} |df(t)| < \infty.$$

Since $\mu + \nu < \alpha + 3/2$, $t^{\nu-\alpha-1/2} f(t) \to 0$ as $t \to \infty$. Also,

$$\int_1^\infty t^{\nu-\alpha-1/2} f(t)\, dt = \sum_{n=1}^\infty \varepsilon_n c_n^{\mu+\nu-\alpha-3/2}, \tag{4.39}$$

and

$$\int_1^\infty t^{\nu-\alpha-1/2} |df(t)| \lesssim \sum_{n=1}^\infty \left(c_n^{\mu+\nu-\alpha-3/2} + (c_n + \varepsilon_n)^{\mu+\nu-\alpha-3/2} \right) \lesssim \sum_{n=1}^\infty c_n^{\mu+\nu-\alpha-3/2}. \tag{4.40}$$

Choosing $c_n = 2^n$ and $\varepsilon_n = 1$, it is easy to prove that $f \notin GM(\beta_2)$, and the series on the right-hand sides of (4.39) and (4.40) are convergent, so that $\mathcal{L}^\alpha_{\nu,\mu} f$ converges uniformly on $\mathbb{R}_+$, by Corollary 4.14. □

We conclude this part by discussing the optimality of Theorems 4.23 and 4.25, and their independence with respect to Corollary 4.14, which also gives sufficient conditions for the uniform convergence of sine-type transforms, and is based on integrability conditions of f. We will always assume that $0<\mu+\nu<\alpha+3/2$.

Sharpness. We first show that the conclusions of Theorems 4.23 and 4.25 do not hold in general if we replace o by O in conditions (4.33) and (4.34), or (4.35) and (4.36).

1. Case $\mu<1$. In this case, we do not discuss sharpness of Theorem 4.23, since condition (4.33) implies that f vanishes at infinity, and therefore we are in the situation of Theorem 4.25. We prove the sharpness of the latter whenever we replace o by O in either (4.35) or (4.36).

Consider the function $f(t)=t^{1-\mu}$ and $\mu+\nu<\alpha+3/2$. It is clear that neither (4.35) nor (4.36) hold, but they are satisfied if we replace o by O (in each respective case). Since $\mu+\nu>0$, we have, for any $r>0$,

$$r^{\mu+\nu}\int_{1/(2r)}^{1/r} t^{\nu}f(t)j_{\alpha}(rt)\,dt\asymp r^{\mu+\nu}\int_{1/(2r)}^{1/r} t^{\mu+\nu-1}\,dt\asymp 1, \tag{4.41}$$

and therefore $\mathcal{L}^{\alpha}_{\nu,\mu}f$ does not converge uniformly on $\mathbb{R}_+$, (the integrals (4.5) do not vanish as $N>M\to\infty$).

2. Case $\mu=1$. Note that in this case the statements of Theorems 4.23 and 4.25 are equivalent, since (4.33) precisely means that f vanishes at infinity. If $f(t)=1$, it is clear that (4.33) does not hold, but holds with O in place of o, whilst (4.34) trivially holds. In this case, we also have that the integral on the right-hand side of (4.41) (with $\mu=1$) does not vanish as $r\to 0$, thus $\mathcal{L}^{\alpha}_{\nu,\mu}f$ does not converge uniformly on $\mathbb{R}_+$.

3. Case $\mu>1$. On the one hand, the example $f(t)=t^{1-\mu}$ shows that Theorem 4.23 does not hold if we replace o by O in (4.33) and (4.34), since we are exactly in the same situation as in the case $\mu<1$ (in the sense that the right-hand side of (4.41) with $\mu>1$ does not vanish as $r\to 0$). On the other hand, the examples $f(t)=t^{\mu-2}\sin t$ and $f(t)=1$ show that in general, conditions (4.33) and (4.34) do not imply each other.

Independence of Theorem 4.23 and Corollary 4.14. Let us prove that the conditions of Theorem 4.23 do not imply those of Corollary 4.14 and vice versa. In other words, these two results complement each other.

On the one hand, let $f(t)=t^{\mu-2}\sin t$ for $t>1$ and $\alpha+1/2\le\mu+\nu<\alpha+3/2$. Since

$$f'(t)=(\mu-2)t^{\mu-3}\sin t+t^{\mu-2}\cos t,$$

we have that

$$M^{\alpha+3/2-\mu-\nu}\int_M^{\infty} t^{\nu-\alpha-1/2}|f'(t)|\,dt\asymp M^{\alpha+3/2-\mu-\nu}\int_M^{\infty} t^{\mu+\nu-\alpha-5/2}\,dt\asymp 1,$$

where we have used the following property of the sine function: from the fact that $|\sin t|\ge 1/\sqrt{2}$ for $t\in[(2k+1)\pi/4,(2k+3)\pi/4]$, $k\in 0,1,2,\ldots$, it follows that, for $\gamma\in\mathbb{R}$ and M large enough,

$$\int_M^{\infty} t^{\gamma}\,dt\asymp\sum_{k:(2k+1)\pi\ge M}\int_{\frac{(2k+1)\pi}{4}}^{\frac{(2k+3)\pi}{4}} t^{\gamma}\,dt\asymp\int_M^{\infty} t^{\gamma}|\sin t|\,dt\le\int_M^{\infty} t^{\gamma}\,dt.$$

Hence, (4.34) does not hold, and the hypotheses of Theorem 4.23 are not satisfied. Nevertheless, note that $t^{\nu-\alpha-1/2}f(t) \in L^1(0,1)$ (and $t^{-\mu}f(t) \in L^1(1,\infty)$ if $\mu+\nu = \alpha+1/2$), and moreover conditions (4.22) hold. Hence, Corollary 4.14 implies the uniform convergence of $\mathcal{L}^{\alpha}_{\nu,\mu}f$ on $\mathbb{R}_+$.

On the other hand, let $\alpha = 1/2$, $\nu = 1$ and $\mu = 0$ (recall that this choice of parameters corresponds to the sine transform). If $f(t) = \dfrac{1}{t\log t}$ for $t > 2$, then clearly $f(t) \notin L^1(2,\infty)$. On the other hand, since (4.34) holds, the uniform convergence of $\mathcal{L}^{1/2}_{1,0}f$ on $\mathbb{R}_+$ follows by Theorem 4.23 (also by Theorem 4.27, or even by Corollary 3.10).

Independence of Theorem 4.25 and Corollary 4.14. We also prove that the hypotheses of Theorem 4.25 and those of Corollary 4.14 do not imply each other.

Let us first give examples of functions that do not satisfy the assumption of Theorem 4.25, but still fall under the scope of Corollary 4.14. Consider again $f(t) = t^{\mu-2}\sin t$, with $\mu < 2$. We have already seen in the above example that

$$M^{\alpha+3/2-\mu-\nu}\int_M^\infty t^{\nu-\alpha-1/2}|f'(t)|\,dt \asymp 1,$$

and that, additionally to $t^{\nu-\alpha-1/2}f(t) \in L^1(1,\infty)$, conditions (4.22) hold. Thus, in the case $\nu \geq \alpha+1/2$ or $\mu \geq 1$, we cannot apply Theorem 4.25, but we can apply Corollary 4.14 instead to deduce the uniform convergence of $\mathcal{L}^{\alpha}_{\nu,\mu}f$ on $\mathbb{R}_+$. On the other hand, it is easy to see that if $\nu < \alpha+1/2$ and $\mu < 1$, the hypotheses of Corollary 4.14 hold, although those of Theorem 4.25 do not, since

$$M^{1-\mu}\int_M^\infty |f'(t)|\,dt \asymp M^{1-\mu}\int_M^\infty t^{\mu-2}\,dt \asymp 1.$$

We now show examples of functions that do not satisfy the assumptions of Corollary 4.14, but those of Theorem 4.25. Let $\nu \geq \alpha+1/2$ and $\alpha+1/2-\nu \leq \mu < 1$. If $f(t) = \dfrac{t^{\alpha-\nu-1/2}}{\log t}$ for $t \geq 2$, then f vanishes at infinity but $t^{\nu-\alpha-1/2}f(t) = \dfrac{1}{t\log t} \notin L^1(2,\infty)$. However,

$$M^{\alpha+3/2-\mu-\nu}\int_M^\infty t^{\nu-\alpha-1/2}|f'(t)|\,dt \lesssim M\int_M^\infty \frac{1}{t^2\log t}\,dt \lesssim \frac{1}{\log M} \to 0 \quad \text{as } M\to\infty,$$

so that (4.36) holds. In the case $\mu \geq 1$, note that $\nu < \alpha+1/2$ (since we are assuming $\mu+\nu < \alpha+3/2$), and hence the inequality $\mu+\nu \geq \alpha+1/2$ implies that $\alpha-1/2 \leq \nu$. Thus, choosing $f(t) = \dfrac{t^{\alpha-\nu-1/2}}{\log t}$ for $t \geq 2$ again, we have that f vanishes at infinity, and also satisfies (4.36), whilst $t^{\nu-\alpha-1/2}f(t) \notin L^1(2,\infty)$. Finally, consider the case $\nu < \alpha+1/2$ and $\mu < 1$. Let $f(t) = \dfrac{t^{\mu-1}}{\log t}$. The inequality $\mu+\nu \geq \alpha+1/2$ implies that

$$t^{\nu-\alpha-1/2}f(t) = \frac{t^{\mu+\nu-\alpha-3/2}}{\log t} \geq \frac{1}{t\log t} \notin L^1(2,\infty),$$

hence f is not under the hypotheses of Corollary 4.14. However, since f is monotone,

$$M^{1-\mu}\int_M^\infty |f'(t)|\,dt = M^{1-\mu}f(M) = \frac{1}{\log M} \to 0 \qquad \text{as } M\to\infty,$$

and $\mathcal{L}^{\alpha}_{\nu,\mu}f$ converges uniformly on $\mathbb{R}_+$, in virtue of Theorem 4.25 (or also by Theorem 4.27).

Remark 4.31. Note that in the extremal case $\mu+\nu=\alpha+3/2$ (which has not been considered above), the assumptions (4.22) we use in Corollary 4.14 are equivalent to the assumptions of Theorem 4.23, namely (4.33) and (4.34). The conclusion is also the same.

4.5 Hankel transforms of functions from $GM(\beta_2)$: equivalence theorems

In the existing literature where the problem of uniform convergence of cosine transforms or cosine series is studied [40, 41, 45, 130, 135], the problem of studying the boundedness of $\sum_{n=0}^\infty a_n \cos nx$ or $\int_0^\infty f(t)\cos rt\,dt$ has not attracted much attention. In fact, if we deal with nonnegative functions (or sequences), the problem is trivial. On the other hand, the tools to deal with functions (or sequences) with non-constant sign are being discovered nowadays. In this section we study the Hankel transforms of real-valued functions from the class $GM(\beta_2)$, making use of the Abel-Olivier test we proved in Section 2.4. We also cover the case of cosine series.

If $f\geq 0$, it is clear that

$$\widehat{f}_{\cos}(r)=\int_0^\infty f(t)\cos rt\,dt \tag{4.42}$$

converges uniformly in $\mathbb{R}_+$ and is bounded if and only if $\int_0^\infty f(t)\,dt$ converges, or equivalently, if $f\in L^1(\mathbb{R}_+)$. Thus, for nonnegative f the boundedness and uniform convergence of (4.42) are equivalent, since $|\widehat{f}_{\cos}(r)|\leq \int_0^\infty f(t)\,dt$. However, in the general case the uniform convergence and boundedness of (4.42) are not equivalent, as we shall now show. More precisely, we discuss the interrelation between the following conditions:

(i) convergence of $\int_0^\infty f(t)\,dt$;

(ii) boundedness of $\int_0^\infty f(t)\cos rt\,dt$;

(iii) uniform convergence of $\int_0^\infty f(t)\cos rt\,dt$ on $\mathbb{R}_+$.

For now, we consider functions f that are integrable over any compact interval $[0,N]\subset\mathbb{R}_+$.

First of all, since $\widehat{f}_{\cos}(0)=\int_0^\infty f(t)\,dt$, the convergence of the integral $\int_0^\infty f(t)\,dt$ is necessary for the (pointwise, and therefore also uniform) convergence and boundedness of $\widehat{f}_{\cos}$, but not sufficient. Indeed, consider (cf. [48, pp. 7–8])

$$\widehat{f}_{\cos}(r)=\int_0^\infty t^{-1/2}\cos t\cos rt\,dt=\frac{\sqrt{\pi}}{2\sqrt{2}}\left(\frac{1}{\sqrt{r+1}}+\frac{1}{\sqrt{r-1}}\right),\qquad r>1.$$

In this case $\widehat{f}_{\cos}(r)$ tends to infinity as $r\to 1^+$, although, as is well known, the integral $\int_0^\infty t^{-1/2}\cos t\,dt$ converges (it is the Fresnel cosine integral, see [1, pp. 300–301]).

We also note that uniform convergence of $\widehat{f}_{\cos}$ on $\mathbb{R}_+$ implies its boundedness, but not vice-versa. Certainly, if $\widehat{f}_{\cos}$ converges uniformly, for a fixed $\varepsilon>0$ we can find $N\in\mathbb{R}_+$ such that

$$\left|\int_{M_1}^{M_2} f(t)\cos rt\,dt\right|<\varepsilon,\qquad \text{if } N\leq M_1<M_2,$$

uniformly on $\mathbb{R}_+$, and hence,

$$\left|\int_0^\infty f(t)\cos rt\,dt\right|\leq \int_0^N |f(t)|\,dt+\varepsilon<\infty,$$

since f is integrable on $[0,N]$. To see that the contrary is not true, we just take $f(t) = \chi_{[0,1]}(t)$, where χ_E is the characteristic function of the set E. Then, by the Dini criterion for the Fourier transform [104, Corollary 2.3.4], if we write $g(r) = \widehat{f}_{\cos} f(r)$, then

$$\widehat{g}_{\cos}(t) \begin{cases} \pi/2, & \text{if } t \in [0,1), \\ \pi/4, & \text{if } t = 1, \\ 0, & \text{if } t > 1. \end{cases}$$

Therefore, $\widehat{g}_{\cos}$ is bounded, but the partial integrals $\int_0^N g(r)\cos rt\,dr$ converge to a discontinuous function as $N \to \infty$, thus the convergence cannot be uniform.

On the side of cosine series, the situation is similar. The convergence of $\sum_{n=0}^{\infty} a_n$ is necessary but not sufficient to guarantee the uniform convergence and boundedness of $\sum_{n=0}^{\infty} a_n \cos nx$. Indeed, the necessity part is trivial (just take $x = 0$), as for the sufficiency part let us consider $a_0 = 0$, $a_n = n^{-1}\cos n$, $n \geq 1$. By the well-known Dirichlet test for series [72], it follows that $\sum_{n=1}^{\infty} n^{-1}\cos n$ converges. However, the cosine series

$$\sum_{n=1}^{\infty} n^{-1} \cos n \cos nx$$

diverges at $x = 1$, thus it is not bounded, neither its partial sums converge uniformly. Indeed, since $\cos^2 n = (1+\cos 2n)/2$, one has

$$\sum_{n=1}^{N} \frac{\cos^2 n}{n} = \sum_{n=1}^{N} \frac{1}{2n} + \sum_{n=1}^{N} \frac{\cos 2n}{2n}.$$

The series $\sum_{n=1}^{\infty} \cos 2n/(2n)$ converges, by the Dirichlet test for series, and $\sum_{n=1}^{\infty} 1/(2n)$ diverges. Thus, our claim follows.

The fact that the uniform convergence of a cosine series on $[0,2\pi)$ implies its boundedness may be proved in the same way as the counterpart for cosine integrals we showed above. One may also apply the following argument: since the partial sums are continuous, the limit function $g(x) = \sum a_n \cos nx$ is continuous (and periodic), and therefore bounded. Finally, we prove that the boundedness of a cosine series does not imply that its partial sums converge uniformly. Indeed, consider the cosine series

$$G(x) := \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \cos\big((2n-1)x\big).$$

This series is the Fourier series of the function

$$g(x) = \begin{cases} 1, & \text{if } x \in [\pi/2, 3\pi/2], \\ 0, & x \in [0,\pi/2) \cup (3\pi/2, 2\pi), \end{cases}$$

see [146] for basic definitions and theory of Fourier series. By the well-known Dini criterion for convergence of Fourier series [104, Theorem 1.2.24], we have

$$G(x) = \begin{cases} \pi/4, & \text{if } x \in (\pi/2, 3\pi/2), \\ \pi/8, & \text{if } x = \pi/2 \text{ or } x = 3\pi/2, \\ 0, & \text{if } x \in [0,\pi/2) \cup (3\pi/2, 2\pi). \end{cases}$$

Hence, G is bounded, but the continuous partial sums

$$\sum_{n=1}^{N}\frac{(-1)^n}{2n-1}\cos\big((2n-1)x\big)$$

converge to a discontinuous function as $N\to\infty$, thus the convergence cannot be uniform.

We shall now see that in the case of Hankel transforms H_α (recall that $H_{-1/2}$ is the cosine transform), analogues of properties (i), (ii), and (iii) stated above are equivalent, provided that $f\in GM(\beta_2)$ is real-valued. We also give an upper estimate for $H_\alpha f$, depending on α. All the functions f we consider in this section will be vanishing at infinity.

Let us define, for any $f:\mathbb{R}_+\to\mathbb{C}$ and $\nu\in\mathbb{R}$,

$$M_\nu(f):=\sup_{t\in\mathbb{R}_+}t^\nu|f(t)|.$$

The following statement is needed to obtain our main result.

Lemma 4.32. *Let $f\in GM(\beta_2)$ be real-valued and $\alpha\in\mathbb{R}$. If $t^{2\alpha+1}f(t)\in L^1(0,1)$ and $\int_0^\infty t^{2\alpha+1}f(t)\,dt$ converges, then $M_{2\alpha+2}(f)=\sup_{t\in\mathbb{R}_+}t^{2\alpha+2}|f(t)|<\infty$.*

Observe that Lemma 4.32 is stated for all $\alpha\in\mathbb{R}$, though we only need the case $\alpha\geq -1/2$ when dealing with Hankel transforms.

Proof of Lemma 4.32. On the one hand, the fact that $t^{2\alpha+2}f(t)\to 0$ as $t\to 0$ follows from $t^{2\alpha+1}f(t)\in L^1(0,1)$ and the estimate given in Remark 2.10. On the other hand, $t^{2\alpha+2}f(t)\to 0$ as $t\to\infty$ follows from the convergence of $\int_0^\infty t^{2\alpha+1}f(t)\,dt$ and Theorem 2.15 (recall that $t^{2\alpha+1}f(t)\in GM(\beta_2)$ provided that $f\in GM(\beta_2)$, cf. [86]). Finally, since f is locally of bounded variation (which is a property we require in Definition 2.4 when we introduce general monotonicity), $t^{2\alpha+2}|f(t)|$ is bounded on any compact set, which yields the desired result. $\square$

Theorem 4.33. *Let $\alpha\geq -1/2$. Let $f\in GM(\beta_2)$ be real-valued and such that $t^{2\alpha+1}f(t)\in L^1(0,1)$. The following are equivalent:*

(i) *the integral $\int_0^\infty t^{2\alpha+1}f(t)\,dt$ converges;*

(ii) *the Hankel transform $H_\alpha f(r)=\int_0^\infty t^{2\alpha+1}f(t)j_\alpha(rt)\,dt$ converges uniformly on $\mathbb{R}_+$;*

(iii) *the Hankel transform $H_\alpha f(r)$ is bounded on $\mathbb{R}_+$.*

Moreover, in any of those cases, $M_{2\alpha+2}(f):=\sup_{t\in\mathbb{R}_+}t^{2\alpha+2}|f(t)|$ is finite, and for every $N\in\mathbb{R}_+$, the estimate

$$\begin{aligned}\left|\int_0^N t^{2\alpha+1}f(t)j_\alpha(rt)\,dt\right|\leq\left|\int_0^N t^{2\alpha+1}f(t)\,dt\right|&+\frac{1}{\alpha+1}N^{2\alpha+2}|f(N)|\\&+\frac{C\lambda(2\lambda)^{2\alpha+2}}{2\alpha+2}\left(\frac{\lambda^4}{2(\alpha+2)}+\frac{S_\alpha}{\alpha+3/2}\right)M_{2\alpha+2}(f)\\&+\sup_{0\leq a<b\leq\infty}\left|\int_a^b\frac{t^{2\alpha+2}}{2\alpha+2}df(t)\right|\end{aligned}\tag{4.43}$$

holds, where $S_\alpha=\sup_{x\geq 1}x^{\alpha+1/2}|j_\alpha(x)|$ (see Remark 4.1).

Note that for any $0 \le a < b$, the boundedness of the integral $\int_a^b t^{2\alpha+2} df(t)$ in (4.43) follows from the convergence of $\int_0^\infty t^{2\alpha+1} f(t)\, dt$ and Corollary 2.16.

We emphasize that the boundedness of $H_\alpha f$ follows easily from its uniform convergence, as we showed above, and estimate (4.43) is not needed to prove $H_\alpha f$ is bounded. The purpose of estimate (4.43) is to show the dependence of our upper bound for $H_\alpha f$ in terms of α. The upper estimate for $H_\alpha f$ is obtained by letting $N \to \infty$ in (4.43), and taking into account that $N^{2\alpha+2} f(N) \to 0$ as $N \to \infty$ whenever $\int_0^\infty t^{2\alpha+1} f(t)\, dt$ converges and $f \in GM(\beta_2)$ is real-valued, by Theorem 2.15.

Corollary 4.34. *Under the assumptions of Theorem 4.33, the estimate*

$$|H_\alpha f(r)| \le \left| \int_0^\infty t^{2\alpha+1} f(t)\, dt \right| + \frac{C\lambda(2\lambda)^{2\alpha+2}}{2\alpha+2} \left(\frac{\lambda^4}{2(\alpha+2)} + \frac{S_\alpha}{\alpha+3/2} \right) M_{2\alpha+2}(f) + \sup_{0\le a<b\le\infty} \left| \int_a^b \frac{t^{2\alpha+2}}{2\alpha+2} df(t) \right|$$

holds.

Proof of Theorem 4.33. First of all, note that under our assumptions $M_{2\alpha+2}(f)$ is finite, provided that the integral $\int_0^\infty t^{2\alpha+1} f(t)\, dt$ converges, by Lemma 4.32.

The fact that (i) and (ii) are equivalent is just Theorem 4.22.

We now prove (i) and (iii) are equivalent. Clearly, (iii) implies (i), since if $H_\alpha f(r)$ is bounded, then $H_\alpha f(0) = \int_0^\infty t^{2\alpha+1} f(t)\, dt$ converges. So we are left to prove that if $f \in GM(\beta_2)$, then (i) implies (iii). It suffices to prove estimate (4.43), and the claim follows by letting $N \to \infty$.

Let $r > 0$ (the case $r = 0$ is trivial, since $j_\alpha(0) = 1$). First of all, we write

$$\left| \int_0^N t^{2\alpha+1} f(t) j_\alpha(rt)\, dt \right| \le \left| \int_0^N t^{2\alpha+1} f(t)\, dt \right| + \left| \int_0^N t^{2\alpha+1} f(t)(1 - j_\alpha(rt))\, dt \right|. \tag{4.44}$$

We now integrate by parts the last integral of (4.44). By the property (4.13) of the derivative of the Bessel function, and the fact that $|j_\alpha(z)| \le 1$ for all $z \ge 0$, we get

$$\begin{aligned} \left| \int_0^N t^{2\alpha+1} f(t)(1 - j_\alpha(rt))\, dt \right| &\le \frac{1}{2\alpha+2} \left| \left[t^{2\alpha+2}(1 - j_{\alpha+1}(rt)) f(t) \right]_0^N \right| \\ &\quad + \left| \int_0^N \frac{t^{2\alpha+2}}{2\alpha+2} (1 - j_{\alpha+1}(rt)) df(t) \right| \\ &\le \frac{1}{\alpha+1} N^{2\alpha+2} |f(N)| + \left| \int_0^N \frac{t^{2\alpha+2}}{2\alpha+2} (1 - j_{\alpha+1}(rt)) df(t) \right|, \end{aligned}$$

where in the last inequality we have used that $t^{2\alpha+2} f(t) \to 0$ as $t \to 0$, which follows from the estimate

$$x^{2\alpha+2} |f(x)| \lesssim \int_{x/\lambda}^{\lambda x} t^{2\alpha+1} |f(t)|\, dt$$

for $GM(\beta_2)$ functions given in Remark 2.10 and the fact that $t^{2\alpha+1} f(t) \in L^1(0,1)$.

Assume now that $r \le 1/N$. By Lemma 4.2, we have $j_{\alpha+1}(rt) \ge 1 - (rt)^2/(4(\alpha+2))$ for $t \le N$, and therefore

$$\left| \int_0^N \frac{t^{2\alpha+2}}{2\alpha+2} (1 - j_{\alpha+1}(rt)) df(t) \right| \le \frac{1}{B_\alpha N^2} \int_0^N t^{2\alpha+4} |df(t)|,$$

where $B_\alpha = 4(\alpha+2)(2\alpha+2)$.

Let $n_0 = \min\{k \in \mathbb{Z} : 2^k \geq N\}$. Since $f \in GM(\beta_2)$, we have

$$\begin{aligned}
\frac{1}{B_\alpha N^2}\int_0^N t^{2\alpha+4}|df(t)| &\leq \frac{2^{2(1-n_0)}}{B_\alpha}\sum_{k=-\infty}^{n_0} 2^{(2\alpha+4)k}\int_{2^{k-1}}^{2^k}|df(t)| \\
&\leq \frac{C2^{2(1-n_0)}}{B_\alpha}\sum_{k=-\infty}^{n_0} 2^{(2\alpha+4)k}\int_{2^{k-1}/\lambda}^{\lambda 2^{k-1}}\frac{|f(t)|}{t}\,dt \\
&\leq \frac{C2^{2(1-n_0)}(2\lambda)^{(2\alpha+4)}}{B_\alpha}\sum_{k=-\infty}^{n_0}\int_{2^{k-1}/\lambda}^{\lambda 2^{k-1}} t^{2\alpha+3}|f(t)|\,dt \\
&\leq \frac{C2^{2(1-n_0)}(2\lambda)^{(2\alpha+5)}}{2B_\alpha}M_{2\alpha+2}(f)\int_0^{\lambda 2^{n_0-1}} t\,dt \\
&= \frac{C(2\lambda)^{(2\alpha+7)}}{16B_\alpha}M_{2\alpha+2}(f). \qquad (4.45)
\end{aligned}$$

For $r > 1/N$, we write

$$\left|\int_0^N \frac{t^{2\alpha+2}}{2\alpha+2}(1-j_{\alpha+1}(rt))df(t)\right| = \left|\left(\int_0^{1/r}+\int_{1/r}^N\right)\frac{t^{2\alpha+2}}{2\alpha+2}(1-j_{\alpha+1}(rt))df(t)\right|.$$

On the one hand, using (4.45) we get

$$\left|\int_0^{1/r}\frac{t^{2\alpha+2}}{2\alpha+2}(1-j_{\alpha+1}(rt))df(t)\right| \leq \frac{C(2\lambda)^{(2\alpha+7)}}{16B_\alpha}M_{2\alpha+2}(f).$$

On the other hand,

$$\left|\int_{1/r}^N\frac{t^{2\alpha+2}}{2\alpha+2}(1-j_{\alpha+1}(rt))df(t)\right| \leq \left|\int_{1/r}^N\frac{t^{2\alpha+2}}{2\alpha+2}df(t)\right| + \int_{1/r}^N\left|\frac{t^{2\alpha+2}}{2\alpha+2}j_{\alpha+1}(rt)df(t)\right|.$$

First, we have

$$\left|\int_{1/r}^N\frac{t^{2\alpha+2}}{2\alpha+2}df(t)\right| \leq \sup_{0\leq a<b\leq\infty}\left|\int_a^b\frac{t^{2\alpha+2}}{2\alpha+2}df(t)\right|,$$

which is finite due to the convergence of $\int_0^\infty t^{2\alpha+2}df(t)$, by Corollary 2.16. Secondly, let $n_1 = \max\{k \in \mathbb{Z} : 2^k \leq 1/r\}$. Using the $GM(\beta_2)$ condition and the estimate (4.12), we derive

$$\begin{aligned}
\left|\int_{1/r}^N\frac{t^{2\alpha+2}}{2\alpha+2}j_{\alpha+1}(rt)df(t)\right| &\leq \frac{S_{\alpha+1}}{r^{\alpha+3/2}}\int_{1/r}^\infty\frac{t^{\alpha+1/2}}{2\alpha+2}|df(t)| \\
&\leq \frac{2^{(n_1+1)(\alpha+3/2)}S_{\alpha+1}}{2\alpha+2}\sum_{k=n_1}^\infty 2^{(k+1)(\alpha+1/2)}\int_{2^k}^{2^{k+1}}|df(t)| \\
&\leq \frac{C2^{(n_1+1)(\alpha+3/2)}S_{\alpha+1}}{2\alpha+2}\sum_{k=n_1}^\infty 2^{(k+1)(\alpha+1/2)}\int_{2^k/\lambda}^{\lambda 2^k}\frac{|f(t)|}{t}\,dt.
\end{aligned}$$

Since

$$\begin{aligned}\sum_{k=n_1}^{\infty} 2^{(k+1)(\alpha+1/2)} \int_{2^k/\lambda}^{\lambda 2^k} \frac{|f(t)|}{t}\,dt &\le \lambda(2\lambda)^{\alpha+1/2} \int_{2^{n_1}/\lambda}^{\infty} t^{\alpha-1/2}|f(t)|\,dt \\ &\le \lambda(2\lambda)^{\alpha+1/2} M_{2\alpha+2}(f) \int_{2^{n_1}/\lambda}^{\infty} t^{-\alpha-5/2}\,dt \\ &= \frac{2^{\alpha+1/2}\lambda^{2\alpha+3}}{\alpha+3/2} 2^{-n_1(\alpha+3/2)} M_{2\alpha+2}(f),\end{aligned}$$

we conclude

$$\left| \int_{1/r}^{N} \frac{t^{2\alpha+2}}{2\alpha+2} j_{\alpha+1}(rt) df(t) \right| \le \frac{C\lambda(2\lambda)^{2\alpha+2} S_\alpha}{(\alpha+3/2)(2\alpha+2)} M_{2\alpha+2}(f).$$

Collecting the above estimates, we arrive at (4.43). □

For the sake of completeness, we state the particular case of Theorem 4.33 corresponding to the cosine transform, i.e., when $\alpha = -1/2$.

Corollary 4.35. *Let $f \in GM(\beta_2)$ be real-valued and such that $f \in L^1(0,1)$. The following are equivalent:*

(i) *the integral $\int_0^\infty f(t)\,dt$ converges;*

(ii) *the cosine transform $\widehat{f}_{\cos}(r) = \int_0^\infty f(t)\cos rt\,dt$ converges uniformly on $\mathbb{R}_+$;*

(iii) *the cosine transform $\widehat{f}_{\cos}(r)$ is bounded on $\mathbb{R}_+$.*

Moreover, in any of those cases, $\sup_{t\in\mathbb{R}_+} t|f(t)|$ is finite, and the estimate

$$\begin{aligned}\left|\int_0^N f(t)\cos rt\,dt\right| \le \left|\int_0^N f(t)\,dt\right| + 2N|f(N)| + 2C\lambda^2\left(\frac{\lambda^4}{3}+1\right)\sup_{t\in\mathbb{R}_+} t|f(t)| \\ + \sup_{0\le a<b\le\infty}\left|\int_a^b t\,df(t)\right|\end{aligned}$$

holds.

To conclude this part, we discuss the corresponding result for cosine series, or in other words, the discrete version of Corollary 4.35. In [130], Tikhonov mentioned the following.

Theorem 4.36. *Let $\{a_n\} \in GMS_2$ be such that $na_n \to 0$. Then the cosine series*

$$\sum_{n=0}^{\infty} a_n \cos nx$$

converges uniformly on $[0, 2\pi)$ if and only if the series $\sum_{n=0}^\infty a_n$ converges.

Remark 4.37. In the proof of Theorem 4.36, the hypothesis $na_n \to 0$ is used to prove the "if" part, whilst the "only if" part only requires the convergence of $\sum_{n=0}^\infty a_n$.

Abel-Olivier's test for real-valued sequences from the class GMS_2 allows us to improve Theorem 4.36 by dropping the hypothesis $na_n \to 0$. The precise statement reads as follows.

Theorem 4.38. *Let $\{a_n\} \in GMS_2$ be real-valued. The following are equivalent:*

(i) *the series $\sum_{n=0}^{\infty} a_n$ converges;*

(ii) *the cosine series $\sum_{n=0}^{\infty} a_n \cos nx$ converges uniformly on $[0, 2\pi)$;*

(iii) *the cosine series $\sum_{n=0}^{\infty} a_n \cos nx$ is bounded on $[0, 2\pi)$.*

Proof. Since $\{a_n\} \in GMS_2$ is real-valued, the convergence of $\sum_{n=0}^{\infty} a_n$ implies $na_n \to 0$ as $n \to \infty$, by Corollary 2.17. Therefore, according to Remark 4.37, the equivalence of (i) and (ii) follows by applying Theorem 4.36, since we have shown the hypothesis $na_n \to 0$ is redundant in order to prove the "if" part. Also, as we observed at the beginning of the present section, the uniform convergence of the partial sums $\sum_{n=0}^{N} a_n \cos nx$ implies the boundedness of the limit function $\sum_{n=0}^{\infty} a_n \cos nx$, i.e., (ii) implies (iii), even in the general case. Finally, (iii) trivially implies (i) by choosing $x = 0$. $\square$

Chapter 5

Weighted norm inequalities for Fourier-type transforms

In this chapter we study norm inequalities between weighted Lebesgue spaces for certain integral transforms. We also state some necessary conditions in Lorentz spaces. This chapter is based on the results from [35].

5.1 Definitions and known results

We have already defined in the Introduction the Lebesgue space of functions $L^p(X)$, where $X \subset \mathbb{R}^n$. Let us now give some more definitions that will be used throughout this chapter.

For $0 < p < 1$, we may also use the notation

$$\|f\|_{L^p(X)} = \left(\int_X |f(x)|^p \, dx \right)^{1/p}.$$

However, in this case the functional $\|\cdot\|_{L^p(X)}$ does not define a norm, but a quasi-norm.

A weight u defined on X is a nonnegative function that is locally integrable on X. For a weight u, the weighted Lebesgue space $L^p(X,u)$ is defined as the space of measurable functions $f : X \to \mathbb{C}$ such that

$$\|f\|_{L^p(X,u)} := \left(\int_X u(x)|f(x)|^p \, dx \right)^{1/p} = \|u^{1/p} f\|_{L^p(X)} < \infty, \qquad p \geq 0.$$

Note that if $u \equiv 1$, then $L^p(X,1) \equiv L^p(X)$. Naturally, $\|\cdot\|_{L^p(X,u)}$ defines a norm for $p \geq 1$.

For $Y \subset \mathbb{R}^n$, and an integral transform

$$Tf(y) = \int_X f(x)K(x,y) \, dx, \qquad y \in Y,$$

with $K : X \times Y \to \mathbb{C}$, we are interested in studying necessary and sufficient conditions on weights $u : Y \to \mathbb{R}_+$ and $v : X \to \mathbb{R}_+$ such that the *weighted norm inequality*

$$\|Tf\|_{L^q(Y,u)} \leq C\|f\|_{L^p(X,v)}, \qquad 1 < p \leq q < \infty, \tag{5.1}$$

holds for all $f \in L^p(X,v)$, where C is independent of the choice of f. Note that a problem to find a sharp constant C in (5.1) for specific p, q and weight functions was studied in [9, 27, 55, 56, 142] for the Fourier and related transforms.

Let us start by reviewing some of the most important results for the Fourier transform.

5.1.1 Weighted norm inequalities for Fourier transforms

Recall that the Fourier transform of an integrable function $f : \mathbb{R}^n \to \mathbb{C}$ is defined as

$$\widehat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot y}\, dx.$$

As a basic example of a norm inequality, we mention the Hausdorff-Young inequality, which states that if $1 \le p \le 2$, then

$$\|\widehat{f}\|_{L^{p'}(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)}, \tag{5.2}$$

see [7, 9]. Note that the Hausdorff-Young inequality for the case $p = 1$ corresponds to the trivial estimate of the Fourier transform, and for $p = 2$ it is actually an equality, known as Plancherel's theorem [126]. Note that inequality (5.2) carries no weights.

In the 1980s, Heinig [64], Jurkat-Sampson [70], and Muckenhoupt [97, 99] proved independently that if $1 < p \le q < \infty$ and $u, v : \mathbb{R}^n \to \mathbb{C}$ are such that

$$\sup_{r>0} \left(\int_0^r u^*(x)\, dx \right)^{1/q} \left(\int_0^{1/r} (1/v)^*(x)\, dx \right)^{1/p'} < \infty, \tag{5.3}$$

where u^* denotes the decreasing rearrangement of the function u, then the inequality

$$\|\widehat{f}\|_{L^q(\mathbb{R}^n,u)} \le C\|f\|_{L^p(\mathbb{R}^n,v)} \tag{5.4}$$

holds for every $f \in L^p(\mathbb{R}^n, v)$. Conversely, if $u(x) = u_0(|x|)$ and $v = v_0(|x|)$ (i.e., u and v are radial), u_0 decreases and v_0 increases as a function of $|x|$ on $\mathbb{R}_+$, and inequality (5.4) holds for every $f \in L^p(\mathbb{R}^n, v)$, then (5.3) is satisfied, see also [29].

Typical examples of weights u and v are power functions. If $u(x) = |x|^{-\beta q}$ and $v(x) = |x|^{\gamma p}$, the inequality

$$\|\widehat{f}\|_{L^q(\mathbb{R}^n,u)} = \||x|^{-\beta}\widehat{f}\|_{L^q(\mathbb{R}^n)} \le C\||x|^{\gamma} f\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n,v)} \tag{5.5}$$

is known as the classical Pitt's inequality (see [13, 125]), and it holds if and only if

$$\beta = \gamma + n\left(\frac{1}{q} - \frac{1}{p'}\right), \qquad \max\left\{0, n\left(\frac{1}{q} - \frac{1}{p'}\right)\right\} \le \beta < \frac{n}{q}. \tag{5.6}$$

As particular cases of Pitt's inequality, we have Hausdorff-Young inequality (5.2), or Hardy-Littlewood inequality (see [136])

$$\||x|^{n(1-2/p)}\widehat{f}\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}, \qquad 1 < p \le 2.$$

Applications of weighted norm inequalities include the study of uncertainty principle relations (cf. [10]) or restriction inequalities [29, 49, 137]. Inequality (5.4) and its variants have been extensively studied, see [5, 9, 12, 13, 70, 99] and the references therein.

Another interesting problem is to investigate whether the sharp range for β in (5.6) can be extended if we assume regularity conditions on f when studying Pitt's inequality (cf. [57, 87, 114]). For instance, it is known that if f is a radial function, $f(x) = f_0(|x|)$, inequality (5.5) holds if and only if

$$\beta = \gamma + n\left(\frac{1}{q} - \frac{1}{p'}\right), \qquad \frac{n}{q} - \frac{n-1}{2} + \max\left\{0, \frac{1}{q} - \frac{1}{p'}\right\} \le \beta < \frac{n}{q}. \tag{5.7}$$

If, additionally, $f_0 \in GM(\beta_2)$, the range for β in (5.7) can be further extended to

$$\frac{n}{q} - \frac{n+1}{2} < \beta < \frac{n}{q},$$

and it is sharp, as shown in [57]. More results on weighted norm inequalities for general monotone functions can be found in [28, 87, 88, 89].

5.1.2 Integral transforms of Fourier type

From now on, we set $X = Y = \mathbb{R}_+$, and use the simplified notation

$$\|f\|_{p,v} := \|f\|_{L^p(\mathbb{R}_+,v)}, \qquad \|f\|_p := \|f\|_{p,1}.$$

Following [58], for a complex-valued function f defined on $\mathbb{R}_+$, we denote

$$Ff(y) = \int_0^\infty s(x)f(x)K(x,y)\,dx, \qquad y > 0, \tag{5.8}$$

where K is a continuous kernel and s is a nonnegative nondecreasing function, $s \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, and such that

$$s(2x) \lesssim s(x), \qquad x > 0. \tag{5.9}$$

Furthermore, we assume that there exists a nonnegative nondecreasing function w satisfying

$$s(x)w(1/x) \asymp 1, \qquad x > 0, \tag{5.10}$$

and such that the estimate

$$|K(x,y)| \lesssim \min\left\{1, (s(x)w(y))^{-1/2}\right\}, \qquad x, y > 0, \tag{5.11}$$

holds. Moreover, we will assume that

$$\int_0^1 s(x)|f(x)|\,dx + \int_1^\infty s(x)^{1/2}|f(x)|\,dx < \infty, \tag{5.12}$$

so that $Ff(y)$ converges pointwise on $(0,\infty)$. This is easily derived by applying the upper bound for K (5.11) on (5.8). What is more, the estimate

$$|Ff(y)| \lesssim \int_0^{1/y} s(x)|f(x)|\,dx + w(y)^{-1/2}\int_{1/y}^\infty s(x)^{1/2}|f(x)|\,dx \tag{5.13}$$

holds. We remark that the weight s could be incorporated into the kernel K; however, it is worth considering it separately, as it appears as one of the two factors in the estimate (5.11). Another reason to separate s from K is to stay close to the framework of the so-called *Fourier-type* transforms, also referred to as F-transforms (see [58, 63, 136, 140]), i.e., those satisfying (5.9)–(5.11), and for which there exists $C > 0$ such that if $f \in L^2(\mathbb{R}_+, s)$, (or in other words, $\|f\|_{2,s} < \infty$), then

$$\|Ff\|_{2,w} \le C\|f\|_{2,s}. \tag{5.14}$$

Inequality (5.14) is known as *weighted Bessel's inequality.* A classical examples of a Fourier-type transform is the Hankel transform. It is worth mentioning that in general, conditions (5.9)–(5.12) do not imply (5.14).

Weighted norm inequalities of transforms with such kernels have been studied in detail in [58], where the authors obtained sufficient conditions that guarantee inequalities of the type

$$\|Ff\|_{q,u} \lesssim \|f\|_{p,v}, \qquad 1 < p \leq q < \infty,$$

where F is a transform of Fourier type. Let us denote by $v_* = [(1/v)^*]^{-1}$, and a' the dual exponent of $1 \leq a \leq \infty$. In [58], the authors proved the following.

Theorem 5.1. *Let $1 < p \leq q < \infty$, $1 < a \leq 2$, $(p,q,a) \neq (2,2,2)$. Let F be a Fourier-type transform and let u, v be weights satisfying*

$$\sup_{r>0}\left(\int_0^{1/r} u^*(x)\,dx\right)^{1/q}\left(\int_0^r v_*(x)^{1-p'}\,dx\right)^{1/p'} < \infty, \tag{5.15}$$

$$\sup_{r>0}\left(\int_{1/r}^{\infty} x^{-q/a'}u^*(x)\,dx\right)^{1/q}\left(\int_r^{\infty} x^{-p/a'}v_*(x)^{1-p'}\,dx\right)^{1/p'} < \infty. \tag{5.16}$$

Then, the following inequality holds:

$$\|w^{1/a'}Ff\|_{q,u} \lesssim \|s^{1/a}f\|_{p,v}. \tag{5.17}$$

If $(p,q,a) = (2,2,2)$ and u, v are weights satisfying

$$\sup_{r>0}\left(\int_0^{1/r} u^*(x)\,dx\right)\left(\int_0^r v_*(x)^{1-p'}\,dx\right) < \infty,$$

the following inequality holds:

$$\|w^{1/2}Ff\|_{2,u} \lesssim \|s^{1/2}f\|_{2,v}.$$

Moreover, in [58] the authors show that condition (5.16) is redundant in the case $a' < \max\{q, p'\}$.

We are interested in studying necessary and sufficient conditions on the weights u, v for inequality (5.17) to hold, for a certain kind of transforms more general than the Fourier-type ones, namely those of the form (5.8) for which estimate (5.11) holds. Unlike in Theorem 5.1, our goal is to obtain necessary and sufficient conditions in terms of the weights u and v, instead of their decreasing rearrangements.

5.1.3 Integral transforms with power-type kernel

We define the transforms with *kernels of power type* (or power-type kernels) as those of the form

$$Ff(y) = y^{c_0}\int_0^{\infty} x^{b_0}f(x)K(x,y)\,dx,$$

where

$$|K(x,y)| \lesssim \min\{x^{b_1}y^{c_1}, x^{b_2}y^{c_2}\},$$

with $b_j, c_j \in \mathbb{R}$ for $j = 0, 1, 2$. It is clear that every transform of the form (5.8) satisfying the estimate (5.11) with $s(x) = x^{\delta}$, $\delta \in \mathbb{R}$, is a transform with power-type kernel, but the converse is not true. Note that the sine and Hankel transforms are of power-type kernel.

In order for $Ff(y)$ to converge pointwise on $(0,\infty)$, we assume that f satisfies

$$\int_0^1 x^{b_0+b_1}|f(x)|\,dx+\int_1^\infty x^{b_0+b_2}|f(x)|\,dx<\infty.$$

In the context of power-type transforms, we are interested in obtaining necessary and sufficient conditions on power-type weights $u(x)=x^{-\beta}$ and $v(x)=x^{\gamma}$ for the inequality

$$\|x^{-\beta}Ff\|_q\lesssim\|x^{\gamma}f\|_p,\qquad 1<p\le q<\infty,$$

to hold. These conditions, like in Pitt's inequality (5.5), will be written in terms of the involved powers, and will be derived from the results obtained for generalized Fourier-type transforms.

In general, kernels of the type $K(x,y)\asymp\min\{x^{b_1}y^{c_1},x^{b_2}y^{c_2}\}$ differ from the kernels satisfying *Oinarov's condition* [102], i.e., for some $d>0$,

$$d^{-1}(K(t,u)+K(u,v))\le K(t,v)\le d(K(t,u)+K(u,v)),\qquad 0<v\le u\le t<\infty. \tag{5.18}$$

In the case $b_j=c_j=0$, $j=1,2$, it is clear that $K(x,y)\asymp 1$ implies (5.18). However, this case is of no interest for us, as our main result for transforms with power-type kernel is not applicable (see Corollary 5.12 below). However, if

$$K(x,y)\asymp\begin{cases}1, & \text{if } xy\le 1,\\ (xy)^{-\delta}, & \text{if } xy>1,\end{cases}$$

with $\delta>0$, and we set $t=N^{\alpha}$, $u=N^{\beta}$, $v=N^{-(\alpha+\beta)/2}$, with N large enough, and $\alpha>\beta>0$, then (5.18) reads as

$$1\lesssim N^{\delta\frac{\alpha-\beta}{2}}\lesssim 1,$$

which is clearly not true.

5.2 The $\mathscr{H}_\alpha$ transform: definition and main properties

An important example of a transform with a kernel of power type and that is not of Fourier type is the so-called $\mathscr{H}_\alpha$ transform, defined as

$$\mathscr{H}_\alpha f(y)=\int_0^\infty (xy)^{1/2}f(x)\mathbf{H}_\alpha(xy)\,dx,\qquad \alpha>-1/2, \tag{5.19}$$

see [109, 136]. Here $\mathbf{H}_\alpha$ is the Struve function of order α [47, §7.5.4], given by the series

$$\mathbf{H}_\alpha(x)=\left(\frac{x}{2}\right)^{\alpha+1}\sum_{k=0}^\infty\frac{(-1)^k(x/2)^{2k}}{\Gamma(k+3/2)\Gamma(k+\alpha+3/2)},\qquad x\ge 0. \tag{5.20}$$

The function $\mathbf{H}_\alpha$ is continuous and satisfies the estimate

$$|\mathbf{H}_\alpha(x)|\lesssim\begin{cases}\min\{x^{\alpha+1},x^{-1/2}\}, & \alpha<1/2,\\ \min\{x^{\alpha+1},x^{\alpha-1}\}, & \alpha\ge 1/2.\end{cases} \tag{5.21}$$

Moreover, $\mathbf{H}_\alpha$ is related to the Bessel function of the first kind J_α in the following way: $\mathbf{H}_\alpha$ is the solution of the non-homogeneous Bessel differential equation

$$x^2\frac{d^2f}{dx^2}+x\frac{df}{dx}+(x^2-\alpha^2)f=\frac{4(x/2)^{\alpha+1}}{\sqrt{\pi}\Gamma(\alpha+1/2)}, \tag{5.22}$$

whilst J_α is the solution of the homogeneous differential equation corresponding to (5.22), with the property that it is bounded at the origin for nonnegative α.

The $\mathscr{H}_\alpha$ transform was extensively studied by Heywood and Rooney, see [65, 66, 109, 110] and the references therein.

Let us now discuss some of the properties satisfied by the Struve function, that will be useful later. On the first place, the derivatives satisfy the rule

$$\frac{d}{dx}\big(x^\alpha\mathbf{H}_\alpha(x)\big)=x^\alpha\mathbf{H}_{\alpha-1}(x). \tag{5.23}$$

The estimate (5.21) can be improved for values of x near the origin. More precisely, one has

$$\mathbf{H}_\alpha(x)\asymp x^{\alpha+1},\qquad x\le 1,\qquad \alpha>-1/2. \tag{5.24}$$

Let us show (5.24). In view of (5.20), (5.24) is equivalent to

$$\sum_{k=0}^\infty\frac{(-1)^k(x/2)^{2k}}{\Gamma(k+3/2)\Gamma(k+\alpha+3/2)}\asymp 1,\qquad x\le 1. \tag{5.25}$$

Note that the terms of the series (5.25) are decreasing in absolute value if $x\le 1$. Hence, applying the estimate (4.15) for alternating series, we obtain, for any $x\le 1$,

$$\begin{aligned}\frac{1}{2\Gamma(3/2)\Gamma(\alpha+3/2)}&\le\frac{1}{\Gamma(3/2)\Gamma(\alpha+3/2)}\left(1-\frac{x^2}{10(\alpha+5/2)}\right)\\&\le\sum_{k=0}^\infty\frac{(-1)^k(x/2)^{2k}}{\Gamma(k+3/2)\Gamma(k+\alpha+3/2)}\le\frac{1}{\Gamma(3/2)\Gamma(\alpha+3/2)},\end{aligned}$$

which was to be shown.

For large x, the asymptotic expansion

$$\mathbf{H}_\alpha(x)=\left(\frac{\pi x}{2}\right)^{-1/2}(\sin(x-\alpha\pi/2-\pi/4))+\frac{(x/2)^{\alpha-1}}{\Gamma(\alpha+1/2)\Gamma(1/2)}(1+O(x^{-2})), \tag{5.26}$$

holds (see [139, p. 332]), which eventually allows us to deduce (5.21).

Remark 5.2. It is worth mentioning that for $\alpha\ge 1/2$ and $x>0$, $\mathbf{H}_\alpha(x)$ is nonnegative [139, p. 337], and moreover, it easily follows from (5.26) that if $\alpha>1/2$, then there is $x_0>1$ such that

$$\mathbf{H}_\alpha(x)\asymp x^{\alpha-1},\qquad x>x_0.$$

Hence, if $\alpha>1/2$, one has $\mathbf{H}_\alpha(x)\asymp\min\{x^{\alpha+1},x^{\alpha-1}\}$ (see (5.24)).

It is useful to know upper estimates for the primitive function of $x^\nu\mathbf{H}_\alpha(xy)$, as we saw in the case of the weighted Bessel function $x^\nu j_\alpha(rx)$ in Chapter 4 (Lemma 4.8). We now obtain those estimates, which will be useful in Section 5.6. Let us denote

$$h^\nu_{\alpha,y}(x)=\int_0^x t^\nu\mathbf{H}_\alpha(ty)\,dt,\qquad \nu\ge-\alpha-1,\quad \alpha>-1/2.$$

Then it follows from (5.21) that $x^\nu \mathbf{H}_\alpha(xy) \to 0$ as $x \to 0$ (with $y \geq 0$ fixed). Therefore, by the fundamental theorem of calculus, we have

$$\frac{d}{dx} h^\nu_{\alpha,y}(x) = x^\nu \mathbf{H}_\alpha(xy),$$

Lemma 5.3. *For any $\nu \geq 1/2$ and $\alpha > -1/2$, there holds*

$$|h^\nu_{\alpha,y}(x)| \lesssim y^{-1} x^\nu \min\{(xy)^{\alpha+2}, (xy)^\alpha\}, \qquad x, y \geq 0.$$

Proof. By definition of $h^\nu_{\alpha,y}$,

$$h^\nu_{\alpha,y}(x) = \int_0^x t^\nu \mathbf{H}_\alpha(ty)\, dt = \frac{1}{y^{\nu+1}} \int_0^{xy} z^{\nu-\alpha-1} z^{\alpha+1} \mathbf{H}_\alpha(z)\, dz.$$

If $\nu = \alpha + 1$, then we simply have $h^{\alpha+1}_{\alpha,y}(x) = y^{-1} x^{\alpha+1} \mathbf{H}_{\alpha+1}(xy)$, by (5.23), and therefore, by Remark 5.2,

$$h^{\alpha+1}_{\alpha,y}(x) \asymp y^{-1} x^{\alpha+1} \min\{(xy)^{\alpha+2}, (xy)^\alpha\}.$$

If $\nu \neq \alpha + 1$, integration by parts along with (5.23) yields

$$|h^\nu_{\alpha,y}(x)| \leq \frac{1}{y} x^\nu \mathbf{H}_{\alpha+1}(xy) + \frac{|\nu - \alpha - 1|}{y^{\nu+1}} \int_0^{xy} z^{\nu-1} \mathbf{H}_{\alpha+1}(z)\, dz =: A + B.$$

Let us now estimate A and B from above. On the one hand, by Remark 5.2

$$A \asymp \frac{1}{y} x^\nu \min\{(xy)^{\alpha+2}, (xy)^\alpha\}.$$

On the other hand, we consider two cases in order to estimate B. If $xy \leq 1$, we have

$$B \asymp \frac{1}{y^{\nu+1}} \int_0^{xy} z^{\nu+\alpha+1}\, dz \asymp \frac{(xy)^{\nu+\alpha+2}}{y^{\nu+1}} = \frac{1}{y} x^\nu (xy)^{\alpha+2}.$$

If $xy > 1$,

$$B \lesssim \frac{1}{y^{\nu+1}} \int_0^{xy} z^{\nu+\alpha-1}\, dz \asymp \frac{1}{y} x^\nu (xy)^\alpha.$$

Collecting the above estimates, we conclude

$$|h^\nu_{\alpha,y}(x)| \lesssim A + B \lesssim \frac{1}{y} x^\nu \min\{(xy)^{\alpha+2}, (xy)^\alpha\},$$

as desired. □

As we mentioned above, the operator $\mathscr{H}_\alpha$ corresponds to a transform with power-type kernel. We emphasize that if we write it in the form (5.8), condition (5.11) does not hold in general.

Following [109], the $\mathscr{H}_\alpha$ transform can be defined for a wider range of α than $\alpha > -1/2$, but for our purpose we need to restrict ourselves to the indicated range.

5.3 Weighted norm inequalities for generalized Fourier-type transforms

In this section we give necessary and sufficient conditions on the weights u, v for the weighted norm inequality

$$\|w^{1/a'} Ff\|_{q,u} \lesssim \|s^{1/a} f\|_{p,v}, \qquad 1 \leq a \leq \infty, \qquad 1 < p \leq q < \infty, \tag{5.27}$$

to hold, where F is a generalized Fourier-type transform, i.e., of the form

$$Ff(y) = \int_0^\infty s(x) f(x) K(x,y)\, dx, \qquad y \in \mathbb{R}_+,$$

and such that

$$|K(x,y)| \lesssim \begin{cases} 1, & \text{if } xy \leq 1, \\ \big(s(x)w(y)\big)^{-1}, & \text{if } xy > 1. \end{cases}$$

We also assume $s, w \in L^1_{\text{loc}}(\mathbb{R}_+)$.

5.3.1 Sufficiency results

Our main tool is Hardy's inequality (cf. [20]). If $p = 1$, $q = \infty$ or $p = q = \infty$, the result holds under the usual modification of L^p norms.

Lemma 5.4. *Let $1 \leq p \leq q \leq \infty$. There exists $B > 0$ such that the inequality*

$$\left(\int_0^\infty u(y) \left(\int_0^y |g(x)|\, dx \right)^q dy \right)^{1/q} \leq B \left(\int_0^\infty v(x) |g(x)|^p\, dx \right)^{1/p}$$

holds for every measurable g if and only if

$$\sup_{r>0} \left(\int_r^\infty u(x)\, dx \right)^{1/q} \left(\int_0^r v(x)^{1-p'}\, dx \right)^{1/p'} < \infty.$$

Also, there exists $B > 0$ such that the inequality

$$\left(\int_0^\infty u(y) \left(\int_y^\infty |g(x)|\, dx \right)^q dy \right)^{1/q} \leq B \left(\int_0^\infty v(x) |g(x)|^p\, dx \right)^{1/p}$$

holds for every measurable g if and only if

$$\sup_{r>0} \left(\int_0^r u(x)\, dx \right)^{1/q} \left(\int_r^\infty v(x)^{1-p'}\, dx \right)^{1/p'} < \infty.$$

Sufficient conditions for (5.27) to hold are given by the following result.

Theorem 5.5. *Let $1 < p \leq q < \infty$ and $1 \leq a \leq \infty$. Assume tat the weights u and v are such that*

$$\sup_{r>0} \left(\int_0^{1/r} u(x) w(x)^{q/a'}\, dx \right)^{1/q} \left(\int_0^r v(x)^{1-p'} s(x)^{p'/a'}\, dx \right)^{1/p'} < \infty, \tag{5.28}$$

$$\sup_{r>0} \left(\int_{1/r}^\infty u(x) w(x)^{q(1/a'-1/2)}\, dx \right)^{1/q} \left(\int_r^\infty v(x)^{1-p'} s(x)^{p'(1/a'-1/2)}\, dx \right)^{1/p'} < \infty. \tag{5.29}$$

Then the weighted norm inequality (5.27) holds for every $f \in L^p(\mathbb{R}_+, vs^{p/a})$.

Note that the sufficient conditions in Theorem 5.5 depend both on the parameter a and on the weights s and w. However, conditions (5.15) and (5.16) from Theorem 5.1 do not depend on the weights s, w, but only on the parameter a. In fact, we can consider the weights $\overline{u} = w^{q/a'}u$ and $\overline{v} = s^{p/a}v$ in place of u and v respectively in Theorem 5.5, so that the parameter a can be omitted. However, we prefer to keep it in the formulation of our results in order to stay close to the framework of [58].

Also note that whenever s and w are increasing (as in the case of Fourier-type transforms), condition (5.15) always implies (5.28), by Hardy-Littlewood rearrangement inequality (cf. [14, Ch. II]), given by $\int_0^t u(x)\,dx \le \int_0^t u^*(x)\,dx$ for all $t > 0$ and measurable u.

Proof of Theorem 5.5. It follows from (5.13) and the change of variables $y \to 1/y$ that

$$\begin{aligned}
\|w^{1/a'}Ff\|_{q,u} \lesssim & \left(\int_0^\infty u(1/y)w(1/y)^{q/a'}y^{-2}\left(\int_0^y s(x)|f(x)|\,dx\right)^q dy\right)^{1/q} \\
&+ \left(\int_0^\infty u(1/y)w(1/y)^{q(1/a'-1/2)}y^{-2}\left(\int_y^\infty s(x)^{1/2}|f(x)|\,dx\right)^q dy\right)^{1/q} \\
&=: I_1 + I_2.
\end{aligned}$$

We proceed to estimate I_1 and I_2 from above. Applying Lemma 5.4 with $g(x) = s(x)f(x)$, we obtain

$$\begin{aligned}
I_1 &= \left(\int_0^\infty u(1/y)w(1/y)^{q/a'}y^{-2}\left(\int_0^y s(x)|f(x)|\,dx\right)^q dy\right)^{1/q} \\
&\lesssim \left(\int_0^\infty v(x)s(x)^{p/a}|f(x)|^p\,dx\right)^{1/p},
\end{aligned}$$

provided that

$$\sup_{r>0}\left(\int_r^\infty u(1/x)w(1/x)^{q/a'}x^{-2}\,dx\right)^{1/q}\left(\int_0^r v(x)^{1-p'}s(x)^{p'/a'}\,dx\right)^{1/p'} < \infty,$$

or equivalently, if (5.28) holds. Finally, if (5.29) holds, or, equivalently, if

$$\sup_{r>0}\left(\int_0^r u(1/x)w(1/x)^{q(1/a'-1/2)}x^{-2}\,dx\right)^{1/q}\left(\int_r^\infty v(x)^{1-p'}s(x)^{p'(1/a'-1/2)}\,dx\right)^{1/p'} < \infty,$$

applying Lemma 5.4 with $g(x) = s(x)^{1/2}f(x)$, we get

$$\begin{aligned}
I_2 &= \left(\int_0^\infty u(1/y)w(1/y)^{q(1/a'-1/2)}y^{-2}\left(\int_y^\infty s(x)^{1/2}|f(x)|\,dx\right)^q dy\right)^{1/q} \\
&\lesssim \left(\int_0^\infty v(x)s(x)^{p/a}|f(x)|^p\,dx\right)^{1/p},
\end{aligned}$$

which establishes inequality (5.27). □

In contrast with Theorem 5.1, although Theorem 5.5 can be applied to a larger number of operators than just the Fourier-type transforms, we see that it has some limitations (that can be avoided using an approach that relies on Calderón estimates for rearrangements [58]). For instance, if $s(x) \asymp w(x) \asymp 1$, it readily follows that we can get no sufficient conditions whenever u, v are power weights, since (5.28) and (5.29) cannot hold simultaneously. This already excludes the cosine transform from the scope of Theorem 5.5, among others.

Using a so-called "gluing lemma" (see [52]), it is possible to write conditions (5.28) and (5.29) as one different condition. Let us now state a generalization of gluing lemma [52, Lemma 2.2].

Lemma 5.6. *Let $f, g \geq 0$, $\alpha, \beta > 0$ and let φ, ψ be nonnegative and nonincreasing. Assume $\varphi(x)^\alpha \asymp \psi(x)^\beta$ for $x > 0$. Then, the conditions*

$$\sup_{r>0} \left(\int_0^r g(x)\, dx \right)^\beta \left(\int_r^\infty \varphi(x) f(x)\, dx \right)^\alpha < \infty \tag{5.30}$$

and

$$\sup_{r>0} \left(\int_0^r f(x)\, dx \right)^\alpha \left(\int_r^\infty \psi(x) g(x)\, dx \right)^\beta < \infty \tag{5.31}$$

hold simultaneously if and only if

$$\sup_{r>0} \left(\int_0^r g(x)\, dx + \frac{1}{\psi(r)} \int_r^\infty \psi(x) g(x)\, dx \right)^\beta \left(\varphi(r) \int_0^r f(x)\, dx + \int_r^\infty \varphi(x) f(x)\, dx \right)^\alpha < \infty. \tag{5.32}$$

Proof. Clearly, (5.32) is equivalent to the boundedness of

$$\begin{aligned} \sup_{r>0} \Bigg[&\varphi(r)^\alpha \left(\int_0^r g(x)\, dx \right)^\beta \left(\int_0^r f(x)\, dx \right)^\alpha + \left(\int_0^r g(x)\, dx \right)^\beta \left(\int_r^\infty \varphi(x) f(x)\, dx \right)^\alpha \\ &+ \frac{\varphi(r)^\alpha}{\psi(r)^\beta} \left(\int_r^\infty \psi(x) g(x)\, dx \right)^\beta \left(\int_0^r f(x)\, dx \right)^\alpha \\ &+ \frac{1}{\psi(r)^\beta} \left(\int_r^\infty \psi(x) g(x)\, dx \right)^\beta \left(\int_r^\infty \varphi(x) f(x)\, dx \right)^\alpha \Bigg]. \end{aligned} \tag{5.33}$$

It is obvious that (5.32) implies both (5.30) and (5.31), since $\varphi(r)^\alpha / \psi(r)^\beta \asymp 1$. In order to prove the converse, note that the second and third terms of (5.33) correspond to (5.30) and (5.31), respectively, since $\varphi(r)^\alpha/\psi(r)^\beta \asymp 1$. Thus, it is only left to prove the boundedness of the first and fourth terms of (5.33). For $r > 0$, let $b(r) \in (0, r)$ be the number such that $\int_0^{b(r)} f(x)\, dx = \int_{b(r)}^r f(x)\, dx$. Using the monotonicity of φ and ψ, and the equivalence $\varphi(r)^\alpha \asymp \psi(r)^\beta$, we obtain

$$\begin{aligned} &\varphi(r)^\alpha \left(\int_0^r g(x)\, dx \right)^\beta \left(\int_0^r f(x)\, dx \right)^\alpha \\ &\asymp \varphi(r)^\alpha \left(\int_0^{b(r)} g(x)\, dx \right)^\beta \left(\int_0^r f(x)\, dx \right)^\alpha + \psi(r)^\beta \left(\int_{b(r)}^r g(x)\, dx \right)^\beta \left(\int_0^r f(x)\, dx \right)^\alpha \\ &\leq \left(\int_0^{b(r)} g(x)\, dx \right)^\beta \left(\int_{b(r)}^r \varphi(x) f(x)\, dx \right)^\alpha + \left(\int_{b(r)}^r \psi(x) g(x)\, dx \right)^\beta \left(\int_0^{b(r)} f(x)\, dx \right)^\alpha \\ &\leq \sup_{r>0} \left(\int_0^r g(x)\, dx \right)^\beta \left(\int_r^\infty \varphi(x) f(x)\, dx \right)^\alpha + \sup_{r>0} \left(\int_r^\infty \psi(x) g(x)\, dx \right)^\beta \left(\int_0^r f(x)\, dx \right)^\alpha, \end{aligned}$$

and the latter is bounded by hypotheses. Similarly, for $r \in (0,\infty)$, let $c(r) \in (r,\infty)$ be such that $\int_r^{c(r)} \psi(x)g(x)\,dx = \int_{c(r)}^{\infty} \psi(x)g(x)\,dx$. We have

$$\begin{aligned}
&\frac{1}{\psi(r)^\beta}\left(\int_r^\infty \psi(x)g(x)\,dx\right)^\beta\left(\int_r^\infty \varphi(x)f(x)\,dx\right)^\alpha\\
&\asymp \frac{1}{\psi(r)^\beta}\left(\left(\int_r^\infty \psi(x)g(x)\,dx\right)^\beta\left(\int_r^{c(r)} \varphi(x)f(x)\,dx\right)^\alpha\right.\\
&\quad\left.+\left(\int_r^\infty \psi(x)g(x)\,dx\right)^\beta\left(\int_{c(r)}^\infty \varphi(x)f(x)\,dx\right)^\alpha\right)\\
&\asymp \frac{1}{\psi(r)^\beta}\left(\left(\int_{c(r)}^\infty \psi(x)g(x)\,dx\right)^\beta\left(\int_r^{c(r)} \varphi(x)f(x)\,dx\right)^\alpha\right.\\
&\quad\left.+\left(\int_r^{c(r)} \psi(x)g(x)\,dx\right)^\beta\left(\int_{c(r)}^\infty \varphi(x)f(x)\,dx\right)^\alpha\right)\\
&\leq \left(\int_{c(r)}^\infty \psi(x)g(x)\,dx\right)^\beta\left(\int_r^{c(r)} f(x)\,dx\right)^\alpha+\left(\int_r^{c(r)} g(x)\,dx\right)^\beta\left(\int_{c(r)}^\infty \varphi(x)f(x)\,dx\right)^\alpha\\
&\leq \sup_{r>0}\left(\int_r^\infty \psi(x)g(x)\,dx\right)^\beta\left(\int_0^r f(x)\,dx\right)^\alpha+\sup_{r>0}\left(\int_0^r g(x)\,dx\right)^\beta\left(\int_r^\infty \varphi(x)f(x)\,dx\right)^\alpha,
\end{aligned}$$

and the latter is bounded, as desired. □

In order to combine Theorem 5.5 with Lemma 5.6 we need to restrict ourselves to the case $a = 1$.

Corollary 5.7. *Let $1 < p \leq q < \infty$. Assume that (5.10) holds and that s and w are nondecreasing. If the weights u and v are such that*

$$\begin{aligned}
\sup_{r>0}\Bigg[&\left(\int_0^r v(x)^{1-p'}\,dx + s(r)^{p'/2}\int_r^\infty v(x)^{1-p'}s(x)^{-p'/2}\,dx\right)^{1/p'}\\
&\times\left(w(1/r)^{q/2}\int_{1/r}^\infty u(x)w(x)^{-q/2}\,dx + \int_0^{1/r} u(x)\,dx\right)^{1/q}\Bigg] < \infty, \qquad (5.34)
\end{aligned}$$

then the weighted norm inequality $\|Ff\|_{q,u} \lesssim \|sf\|_{p,v}$ holds for every $f \in L^p(\mathbb{R}_+, vs^p)$.

Proof of Corollary 5.7. We first rewrite conditions (5.28) and (5.29) (with $a = 1$) as

$$\sup_{r>0}\left(\int_r^\infty u(1/x)x^{-2}\,dx\right)^{1/q}\left(\int_0^r v(x)^{1-p'}\,dx\right)^{1/p'} < \infty, \qquad (5.35)$$

$$\sup_{r>0}\left(\int_0^r u(1/x)w(1/x)^{-q/2}x^{-2}\,dx\right)^{1/q}\left(\int_r^\infty v(x)^{1-p'}s(x)^{-p'/2}\,dx\right)^{1/p'} < \infty. \qquad (5.36)$$

Putting

$$f(x) = u(1/x)w(1/x)^{-q/2}x^{-2}, \qquad g(x) = v(x)^{1-p'},$$

together with $\varphi(x) = w(1/x)^{q/2}$, $\psi(x) = s(x)^{-p'/2}$, $\alpha = 1/q$ and $\beta = 1/p'$, it is clear that (5.35) and (5.36) are the same as (5.30) and (5.31) respectively. Furthermore, (5.10) is equivalent to $\varphi(x)^\alpha \asymp \psi(x)^\beta$. Hence, we are under the hypotheses of Lemma 5.7, and we can deduce that the joint fulfilment of (5.28) and (5.29) is equivalent to

$$\sup_{r>0}\Bigg[\bigg(\int_0^r v(x)^{1-p'}\,dx + s(r)^{p'/2}\int_r^\infty v(x)^{1-p'}s(x)^{-p'/2}\,dx\bigg)^{1/p'} \\ \times\bigg(w(1/r)^{q/2}\int_0^r u(1/x)w(1/x)^{-q/2}x^{-2}\,dx + \int_r^\infty u(1/x)x^{-2}\,dx\bigg)^{1/q}\Bigg] < \infty,$$

which is precisely (5.34). □

5.3.2 Necessity results in weighted Lebesgue spaces

Let us prove necessary conditions for (5.27) to hold. We consider the following assumptions on the weights u and v:

$$uw^{q/a'} \in L^1_{\mathrm{loc}}(\mathbb{R}_+), \qquad v^{1-p'}s^{p'/a'} \in L^1_{\mathrm{loc}}(\mathbb{R}_+).$$

Theorem 5.8. *Let $1 < p, q < \infty$ and $1 \le a \le \infty$. Assume that inequality (5.27) holds for every $f \in L^p(\mathbb{R}_+, vs^{p/a})$, where*

$$Ff(y) = \int_0^\infty s(x)f(x)K(x,y)\,dx.$$

(i) *If the kernel $K(x,y)$ satisfies*

$$K(x,y) \asymp 1, \qquad 0 < xy \le 1, \tag{5.37}$$

then (5.28) is valid;

(ii) *if the kernel $K(x,y)$ satisfies*

$$K(x,y) \asymp (s(x)w(y))^{-1/2}, \qquad xy > 1,$$

then (5.29) is valid.

Proof. For the first part, define

$$f_r(x) = v(x)^{1-p'}s(x)^{(1-p')(p/a-1)}\chi_{(0,r)}(x), \qquad r > 0.$$

It follows from (5.37) and the equality $1 + (1-p')(p/a-1) = p'/a'$ that, for $y \le 1/r$,

$$|Ff_r(y)| = \int_0^\infty f_r(x)K(x,y)s(x)\,dx \asymp \int_0^r v(x)^{1-p'}s(x)^{p'/a'}\,dx.$$

On the one hand, note that $f_r \in L^p(\mathbb{R}_+, vs^{p/a})$, since

$$\left\|s^{1/a}f_r\right\|_{p,v} = \left(\int_0^r v(x)^{1-p'}s(x)^{p'/a'}\,dx\right)^{1/p}, \tag{5.38}$$

and $v^{1-p'}s^{p'/a'} \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. On the other hand,

$$\left\|w^{1/a'}Ff\right\|_{q,u} \ge \left(\int_0^{1/r} u(y)w(y)^{q/a'}\,dy\right)^{1/q}\left(\int_0^r v(x)^{1-p'}s(x)^{p'/a'}\,dx\right). \tag{5.39}$$

Combining (5.27) with (5.38) and (5.39), we derive

$$\left(\int_0^{1/r} u(y)w(y)^{q/a'}\,dy\right)^{1/q}\left(\int_0^{r} v(x)^{1-p'}s(x)^{p'/a'}\,dx\right) \lesssim \left(\int_0^{r} v(x)^{1-p'}s(x)^{p'/a'}\,dx\right)^{1/p},$$

i.e., (5.28) holds.

We omit the proof of the second part, as it is essentially a repetition of that of the first part. In this case, one should consider the function

$$f_r(x) = v(x)^{1-p'}s(x)^{p'(1/a'-1/2)-1/2}\chi_{(r,\infty)}(x), \qquad r>0,$$

and proceed analogously as above. □

Theorem 5.8 shows that condition (5.28) is best possible for some classical transforms, such as the Hankel (or the cosine) transform, since $j_\alpha(xy)\asymp 1$ whenever $xy\le 1$ for every $\alpha\ge -1/2$ (i.e., (5.37) is satisfied).

Theorem 5.8 is not limited to Fourier-type transforms. For example, let us consider the Laplace transform,

$$\mathscr{L}f(y) = \int_0^\infty f(x)e^{-xy}\,dx.$$

Since $K(x,y)=e^{-xy}\asymp 1$ for $xy\le 1$ (for convenience we choose $s(x)\equiv 1$), Theorem 5.8 implies that condition (5.28) (with $a=1$) is necessary for the inequality

$$\|\mathscr{L}f\|_{q,u}\lesssim \|f\|_{p,v}$$

to hold, as proved by Bloom in [16].

Combining Theorems 5.5 and 5.8 we can get an “if and only if” statement.

Corollary 5.9. *Let the kernel K from (5.8) satisfy $K(x,y)\asymp \min\{1,(s(x)w(y))^{-1/2}\}$. Then the weighted norm inequality (5.27) holds for every $f\in L^p(\mathbb{R}_+,vs^{p/a})$ if and only if (5.28) and (5.29) are satisfied.*

As a simple example of a kernel satisfying the hypotheses of Corollary 5.9, consider

$$K(x,y)=\begin{cases}1, & \text{if } xy\le 1,\\ (s(x)w(y))^{-1/2}, & \text{if } xy>1,\end{cases}$$

so that

$$Ff(y) = \int_0^{1/y} s(x)f(x)\,dx + w(y)^{-1/2}\int_{1/y}^\infty s(x)^{1/2}f(x)\,dx.$$

5.3.3 Necessity results in weighted Lorentz spaces

To conclude the part dealing with necessary conditions for (5.27) to hold, we present a generalization of a result due to Benedetto and Heinig [13, Theorem 2], related to weighted Lorentz spaces (introduced in [92]; see also [23]). For a weight u defined on $\mathbb{R}_+$ and $1\le p<\infty$, the *weighted Lorentz space* $\Lambda^p(u)$ is defined to be the set of functions $f:\mathbb{R}_+\to\mathbb{C}$ such that

$$\|f\|_{\Lambda^p(u)} := \left(\int_0^\infty u(x)f^*(x)\,dx\right)^{1/p}.$$

Lorentz showed [92] that $\Lambda^p(u)$ is a normed linear space with norm $\|\cdot\|_{\Lambda^p(u)}$ if and only if u is nonincreasing on $(0,\infty)$.

We are not studying sufficient conditions on weighted Lorentz spaces, since the theory to deal with them is out of the scope of this work. We refer the reader to [13, 19, 101, 123] and the references therein for recent advances in the theory of Fourier inequalities in Lorentz spaces.

Recall that for a measure space (X,μ) with $X\subset\mathbb{R}^n$ and f a complex μ-measurable function defined on X, the distribution function of f is

$$D_f(t)=\mu\{x\in X:|f(x)|>t\},\quad t\in[0,\infty).$$

Note that D_f is nonnegative. Moreover, for $0<p<\infty$ (see, e.g., [14]),

$$\int_X |f(x)|^p\,d\mu(x)=p\int_0^\infty t^{p-1}D_f(t)\,dt.$$

Theorem 5.10. *Let $1<p,q<\infty$. Assume that the kernel $K(x,y)$ from (5.8) satisfies $K(x,y)\asymp 1$ for $xy\le 1$. If u and v are weights such that the inequality*

$$\bigg(\int_0^\infty (Ff)^*(x)^q u(x)\,dx\bigg)^{1/q}\le C\bigg(\int_0^\infty f^*(x)^p v(x)\,dx\bigg)^{1/p}, \tag{5.40}$$

i.e., $\|Ff\|_{\Lambda^q(u)}\lesssim\|f\|_{\Lambda^p(v)}$, holds for every $f\in\Lambda^p(v)$, then

$$\sup_{r>0}\bigg(\int_0^{1/r}u(x)\,dx\bigg)^{1/q}\bigg(\int_0^r v(x)\,dx\bigg)^{-1/p}\bigg(\int_0^r s(x)\,dx\bigg)<\infty.$$

Proof. The argument is similar to that of [13, Theorem 2]. Let $f(x)=\chi_{(0,r)}(x)$. It is clear that $f^*=f$, and the right-hand side of (5.40) is equal to $C\big(\int_0^r v(x)\,dx\big)^{1/p}$. Moreover,

$$Ff(y)=\int_0^\infty s(x)f(x)K(x,y)\,dx=\int_0^r s(x)K(x,y)\,dx.$$

If we denote $c=\min_{xy\le 1}K(x,y)$, then by hypotheses $c>0$, and for $y\le 1/r$, one has

$$Ff(y)>\frac{c}{2}\int_0^r s(x)\,dx=:A_r. \tag{5.41}$$

For any $r>0$, the following estimate holds:

$$\begin{aligned}\bigg(\int_0^\infty (Ff)^*(x)^q u(x)\,dx\bigg)^{1/q}&\ge\bigg(\int_0^{1/r}(Ff)^*(x)^q u(x)\,dx\bigg)^{1/q}\\&=\bigg(q\int_0^\infty t^{q-1}\bigg(\int_{\{x\in(0,1/r):(Ff)^*(x)>t\}}u(x)\,dx\bigg)dt\bigg)^{1/q}\\&=\bigg(q\int_0^\infty t^{q-1}\bigg(\int_0^{\min\{D_{Ff}(t),1/r\}}u(x)\,dx\bigg)dt\bigg)^{1/q}.\end{aligned} \tag{5.42}$$

where in the last step we have used that $\{x:(Ff)^*(x)>t\}=\{x:D_{Ff}(t)>x\}$. Also note that for $t<A_r$, (5.41) implies

$$(0,1/r)\subset\{y>0:Ff(y)>A_r\}\subset\{y>0:Ff(y)>t\}.$$

Thus, for $t < A_r$,

$$D_{Ff}(t) = \int_{\{x>0:|Ff(x)|>t\}} dz \geq \int_0^{1/r} dx = \frac{1}{r}.$$

In view of the latter inequality, we deduce that if $t < A_r$, then $\min\{D_{Ff}(t), 1/r\} = 1/r$. Combining such observation with (5.42), we get

$$\begin{aligned}\left(\int_0^\infty (Ff)^*(x)^q u(x)\,dx\right)^{1/q} &\geq \left(q\int_0^{A_r} t^{q-1}\left(\int_0^{1/r} u(x)\,dx\right)dt\right)^{1/q}\\ &= A_r\left(\int_0^{1/r} u(x)\,dx\right)^{1/q}.\end{aligned}$$

Finally, it follows from (5.40) and the above estimates that

$$\begin{aligned}\frac{c}{2}\left(\int_0^{1/r} u(x)\,dx\right)^{1/q}&\left(\int_0^{r} v(x)\,dx\right)^{-1/p}\left(\int_0^{r} s(x)\,dx\right)\\ &\leq \left(\int_0^\infty (Ff)^*(x)^q u(x)\,dx\right)\left(\int_0^{r} v(x)\,dx\right)^{-1/p}\\ &\leq C\left(\int_0^{r} v(x)\,dx\right)^{1/p}\left(\int_0^{r} v(x)\,dx\right)^{-1/p} = C,\end{aligned}$$

which completes the proof. □

5.4 Weighted norm inequalities for transforms with power-type kernels

In this section we study necessary and sufficient conditions for the weighted norm inequality

$$\|x^{-\beta}Ff\|_q \lesssim \|x^\gamma f\|_p$$

to hold, where F is a transform with power-type kernel (cf. Subsection 5.1.3). The main results of this section are just consequences of those of Section 5.3, obtained by taking $s(x) = x^\delta$ with $\delta > 0$ and assuming u and v are power weights.

5.4.1 Sufficient conditions

For the sake of generality, we first assume u and v are piecewise power weights. Those weights have been considered in the study of weighted restriction Fourier inequalities [17, 29], and, moreover, they play a fundamental role in the study of weighted norm inequalities for the Jacobi transform in [58] (see also [74]). For any real numbers α_1 and α_2, we denote $\overline{\alpha} = (\alpha_1, \alpha_2)$, $\overline{\alpha}' = (\alpha_2, \alpha_1)$, and

$$x^{\overline{\alpha}} := \begin{cases} x^{\alpha_1}, & \text{if } x \leq 1,\\ x^{\alpha_2}, & \text{if } x > 1.\end{cases}$$

Theorem 5.11. *Let $\beta_i, \gamma_i \in \mathbb{R}$, $i = 1, 2$, with $\beta_1 - \gamma_1 = \beta_2 - \gamma_2$. Let*

$$Ff(y) = \int_0^\infty x^\delta f(x)K(x,y)\,dx, \qquad \delta > 0,$$

and assume

$$|K(x,y)| \lesssim \begin{cases} 1, & \text{if } xy \le 1, \\ (xy)^{-\delta/2}, & \text{if } xy > 1. \end{cases}$$

For $1 < p \le q < \infty$*, the inequality*

$$\big\| x^{-\overline{\beta}'} F f \big\|_q \le C \big\| x^{\overline{\gamma}+\delta} f \big\|_p$$

holds for any $f \in L^p\big(\mathbb{R}_+, x^{(\overline{\gamma}+\delta)p}\big)$*, provided that*

$$\beta_i = \gamma_i + \frac{1}{q} - \frac{1}{p'}, \qquad i = 1, 2, \tag{5.43}$$

and

$$\frac{1}{q} - \frac{\delta}{2} < \beta_i < \frac{1}{q}, \qquad i = 1, 2. \tag{5.44}$$

Proof. Let us verify that if we define $u(x)^{1/q} = x^{-\overline{\beta}'}$ and $v(x)^{1/p} = x^{\overline{\gamma}}$, conditions (5.43) and (5.44) imply (5.28) and (5.29) with $a = 1$. On the one hand, it is clear that the integrals in (5.28) converge if and only if

$$\beta_2 < \frac{1}{q} \qquad \text{and} \qquad \gamma_1 < \frac{1}{p'}.$$

On the other hand, the integrals in (5.29) converge if and only if

$$\beta_1 > \frac{1}{q} - \frac{\delta}{2} \qquad \text{and} \qquad \gamma_2 > \frac{1}{p'} - \frac{\delta}{2}.$$

Thus, (5.43) and (5.44) together with $\beta_1 - \gamma_1 = \beta_2 - \gamma_2$ imply that all the integrals involved in (5.28) and (5.29) converge. We now show that the suprema from (5.28) and (5.29) are finite. It suffices to check their finiteness for $r < 1/2$ or $r > 2$. We start with (5.28). If $r < 1/2$,

$$\begin{aligned}\left(\int_0^{1/r} u(x)\,dx\right)^{1/q} \left(\int_0^{r} v(x)^{1-p'}\,dx\right)^{1/p'} &\asymp r^{-\gamma_1 + 1/p'} \left(C + \int_1^{1/r} x^{-\beta_1 q}\,dx\right)^{1/q} \\ &\asymp r^{-\gamma_1+1/p'} \max\{1, r^{\beta_1 - 1/q}\} \\ &= \max\{r^{-\gamma_1+1/p'}, r^{\beta_1-\gamma_1+1/p'-1/q}\},\end{aligned}$$

which is bounded for $r < 1/2$ if and only if

$$\beta_1 - \gamma_1 \ge 1/q - 1/p'. \tag{5.45}$$

If $r > 2$,

$$\begin{aligned}\left(\int_0^{1/r} u(x)\,dx\right)^{1/q} \left(\int_0^{r} v(x)^{1-p'}\,dx\right)^{1/p'} &\asymp r^{\beta_2 - 1/q} \left(C + \int_1^{r} x^{\gamma_2 p(1-p')}\,dx\right)^{1/p'} \\ &\asymp r^{\beta_2-1/q} \max\{1, r^{-\gamma_2+1/p'}\} \\ &= \max\{r^{\beta_2-1/q}, r^{\beta_2-\gamma_2+1/p'-1/q}\}.\end{aligned}$$

The latter is bounded for $r > 2$ if and only if

$$\beta_2 - \gamma_2 \leq 1/q - 1/p'. \tag{5.46}$$

The joint fulfilment of conditions (5.45) and (5.46) together with $\beta_1 - \gamma_1 = \beta_2 - \gamma_2$ is equivalent to (5.43).

Finally, we are left to verify (5.29). First, if $r < 1/2$,

$$\begin{aligned}\bigg(\int_{1/r}^\infty u(x)x^{-q\delta/2}\,dx\bigg)^{1/q}&\bigg(\int_r^\infty v(x)^{1-p'}x^{-p'\delta/2}\,dx\bigg)^{1/p'}\\ &\asymp r^{\beta_1+\delta/2-1/q}\bigg(C+\int_r^1 x^{-p'(\gamma_1+\delta/2)}\,dx\bigg)^{1/p'}\\ &\asymp \max\{x^{\beta_1+\delta/2-1/q}, r^{\beta_1-\gamma_1+1/p'-1/q}\},\end{aligned}$$

which is bounded for $r < 1/2$ if and only if (5.45) holds. Secondly, for $r > 2$,

$$\begin{aligned}\bigg(\int_{1/r}^\infty u(x)x^{-q\delta/2}\,dx\bigg)^{1/q}&\bigg(\int_r^\infty v(x)^{1-p'}x^{-p'\delta/2}\,dx\bigg)^{1/p'}\\ &\asymp r^{-\gamma_2-\delta/2+1/p'}\bigg(C+\int_{1/r}^1 x^{-q(\beta_2+\delta/2)}\,dx\bigg)^{1/q}\\ &\asymp \max\{r^{-\gamma_2-\delta/2+1/p'}, r^{\beta_2-\gamma_2+1/p'-1/q}\}.\end{aligned}$$

Since the latter is bounded for $r > 2$ if and only if (5.46) holds, (5.29) follows, which completes the last part of the proof. □

Our next result applies to transforms with power-type kernel, and is equivalent to Theorem 5.11 with non-mixed power weights (note that Theorem 5.11 applies to transforms with kernel of power type, namely those for which $|K(x,y)| \lesssim \min\{1, x^{-\delta/2}\}$).

Corollary 5.12. *Let $1 < p \leq q < \infty$, and $b_i, c_i \in \mathbb{R}$ for $i = 0,1,2$. Assume $c_1 - c_2 = b_1 - b_2 > 0$. If*

$$Ff(y) = y^{c_0}\int_0^\infty x^{b_0}f(x)K(x,y)\,dx, \tag{5.47}$$

with $|K(x,y)| \lesssim \min\{x^{b_1}y^{c_1}, x^{b_2}y^{c_2}\}$, the inequality

$$\|x^{-\beta}Ff\|_q \lesssim \|x^\gamma f\|_p \tag{5.48}$$

holds for every $f \in L^p(\mathbb{R}_+, x^{\gamma p})$ with

$$\beta = \gamma + c_0 - b_0 + c_1 - b_1 + \frac{1}{q} - \frac{1}{p'}, \qquad \frac{1}{q} + c_0 + c_2 < \beta < \frac{1}{q} + c_0 + c_1. \tag{5.49}$$

Additionally to the Fourier-type transforms with $s(x) = x^\delta$ and $\delta > 0$ (e.g., the Hankel transform of order $\alpha > -1/2$), Corollary 5.12 can be applied to any kind of transform as long as its kernel satisfies upper estimates given by power functions (e.g., the sine or $\mathscr{H}_\alpha$ transforms). We remark that although the sine transform is not of Fourier-type itself (since $|\sin xy| \leq \min\{xy, 1\}$ if $xy > 0$, and therefore it does not satisfy estimate (5.11)), it can be written as a weighted Hankel transform, as done in Chapter 4.

Proof of Corollary 5.12. The proof is essentially based on changing variables in Theorem 5.11. Let us define $d = c_1 - c_2$ and

$$\widetilde{K}(x,y) = x^{-b_1} y^{-c_1} K(x,y).$$

Then,

$$|\widetilde{K}(x,y)| \lesssim \min\{1, (xy)^{-d}\}, \qquad d > 0.$$

Define the auxiliary integral transform

$$Gf(y) = \int_0^\infty x^{2d} f(x) \widetilde{K}(x,y)\, dx,$$

which satisfies the hypotheses of Theorem 5.11 (with $\delta = 2d$). For $g(x) := x^{b_0+b_1-2d} f(x)$, we have

$$y^{c_0+c_1} Fg(y) = Gf(y),$$

and therefore, in virtue of Theorem 5.11, the weighted norm inequality

$$\|y^{-c_0-c_1-\beta'} Gf\|_q = \|y^{-\beta'} Fg\|_q \lesssim \|x^{\gamma'+2d} g\|_p = \|x^{\gamma'+b_0+b_1} f\|_p, \qquad 1 < p \le q < \infty,$$

holds with $\beta' = \gamma' + 1/q - 1/p'$ and $1/q - d < \beta' < 1/q$. In other words, if we set $\beta = \beta' + c_0 + c_1$ and $\gamma = \gamma' + b_0 + b_1$, then inequality (5.48) holds if $b_1 - c_1 = b_2 - c_2$ and both conditions in (5.49) are satisfied. $\square$

We now state the sufficient conditions for (5.48) to hold derived from Corollary 5.12 whenever F is the sine, Hankel, or $\mathscr{H}_\alpha$ transform. To this end, we use the estimates (4.11), (5.21), (also recall that $|\sin xy| \le \min\{xy, 1\}$ for $x, y > 0$). Those sufficient conditions read as follows:

- Sine transform: $\beta = \gamma + 1/q - 1/p'$ and
$$\frac{1}{q} < \beta < 1 + \frac{1}{q}.$$
- Hankel transform of order $\alpha > -1/2$: $\beta = \gamma - 2\alpha - 1 + 1/q - 1/p'$ and
$$\frac{1}{q} - \alpha - \frac{1}{2} < \beta < \frac{1}{q}.$$
- $\mathscr{H}_\alpha$ transform of order $\alpha > -1/2$: $\beta = \gamma + 1/q - 1/p'$ and
$$\begin{aligned} \frac{1}{q} < \beta < \frac{1}{q} + \alpha + \frac{3}{2}, & \qquad \text{if } \alpha < 1/2, \\ \frac{1}{q} + \alpha - \frac{1}{2} < \beta < \frac{1}{q} + \alpha + \frac{3}{2}, & \qquad \text{if } \alpha \ge 1/2. \end{aligned}$$

Note that the above conditions are not optimal in the case of the sine and Hankel transforms. If F is the sine transform, it is known [58] that (5.48) holds if and only if $\beta = \gamma + 1/q - 1/p'$ and

$$\max\left\{0, \frac{1}{q} - \frac{1}{p'}\right\} \le \beta < 1 + \frac{1}{q},$$

and if F is the Hankel transform of order $\alpha \geq -1/2$, (5.48) holds if and only if (see [27]) $\beta = \gamma - 2\alpha - 1 + 1/q - 1/p'$ and

$$\max\left\{0, \frac{1}{q} - \frac{1}{p'}\right\} - \alpha - \frac{1}{2} \leq \beta < \frac{1}{q}.$$

Finally, if F is the $\mathscr{H}_\alpha$ transform, Rooney proved in [109] that (5.48) holds if $\beta = \gamma + 1/q - 1/p'$ and

$$\beta \geq \max\left\{0, \frac{1}{q} - \frac{1}{p'}\right\} \qquad \text{and} \qquad \begin{array}{ll} \dfrac{1}{q} + \alpha - \dfrac{1}{2} < \beta < \dfrac{1}{q} + \alpha + \dfrac{3}{2}, & \text{if } \alpha < 1/2, \\ \dfrac{1}{q} + \alpha - \dfrac{1}{2} < \beta < \dfrac{1}{q} + \alpha + \dfrac{3}{2}, & \text{if } \alpha \geq 1/2. \end{array} \tag{5.50}$$

Note that whenever $\alpha > 1/2$, the sufficient conditions for the $\mathscr{H}_\alpha$ transform coincide with those given by Corollary 5.12, and moreover they are optimal, as we prove in the following subsection.

5.4.2 Necessary conditions

Here we are concerned on what conditions follow from (5.48). The following result goes along the same lines as Theorem 5.8.

Theorem 5.13. *Let $1 < p, q < \infty$. Assume that inequality (5.48) holds for all $f \in L^p(\mathbb{R}_+, x^{\gamma p})$, with F given by (5.47) (with $b_0, c_0 \in \mathbb{R}$).*

(i) *If the kernel $K(x,y)$ satisfies*

$$K(x,y) \asymp x^{b_1} y^{c_1}, \qquad xy \leq 1, \quad b_1, c_1 \in \mathbb{R},$$

then

$$\beta = \gamma + c_0 - b_0 + c_1 - b_1 + \frac{1}{q} - \frac{1}{p'}, \qquad \beta < \frac{1}{q} + c_0 + c_1;$$

(ii) *if the kernel $K(x,y)$ satisfies*

$$K(x,y) \asymp x^{b_2} y^{c_2}, \qquad xy > 1, \quad b_2, c_2 \in \mathbb{R},$$

then

$$\beta = \gamma + c_0 - b_0 + c_2 - b_2 + \frac{1}{q} - \frac{1}{p'}, \qquad \beta > \frac{1}{q} + c_0 + c_2.$$

Proof. For $r > 0$, let $f_r(x) = x^{-b_0 - b_1 + d}\chi_{(0,r)}(x)$, where $d > -1$ is such that $\gamma - b_0 - b_1 + d > -1/p$ for a given $\gamma \in \mathbb{R}$. Then

$$\|x^\gamma f_r\|_p = \left(\int_0^r x^{p(\gamma - b_0 - b_1 + d)}\, dx\right)^{1/p} \asymp r^{\gamma - b_0 - b_1 + d + 1/p}.$$

If $y \leq 1/r$, one has

$$F f_r(y) = y^{c_0} \int_0^r x^{-b_1 + d} K(x,y)\, dx \asymp r^{d+1} y^{c_0 + c_1},$$

Then, it follows from inequality (5.48) and the fact that $x^\gamma f_r(x) \in L^p(\mathbb{R}_+)$ that

$$r^{\gamma-b_0-b_1+d+1/p} \asymp \|x^\gamma f_r\|_p \gtrsim \|x^{-\beta} F f_r\|_q \geq \left(\int_0^{1/r} x^{-\beta q} |F f_r(x)|^q \, dx \right)^{1/q}$$
$$\asymp r^{d+1} \left(\int_0^{1/r} x^{q(-\beta+c_0+c_1)} \, dx \right)^{1/q} \asymp r^{\beta-c_0-c_1-1/q+d+1},$$

Note that the boundedness of the integral $\int_0^{1/r} x^{q(-\beta+c_0+c_1)}\,dx$ is equivalent to $\beta < 1/q + c_0 + c_1$. Moreover, the inequality $r^{\beta-c_0-c_1-1/q+d+1} \lesssim r^{\gamma-b_0-b_1+d+1/p}$ holds uniformly in $r > 0$ if and only if $\beta = \gamma + c_0 - b_0 + c_1 - b_1 + 1/q - 1/p'$. This completes the proof of the first part.

The proof of the second part is omitted, as it is analogous to that of the first part. In this case it suffices to consider the function

$$f_r(x) = x^{-b_0-b_2-d} \chi_{(r,\infty)}(x),$$

where $d > 1$ is such that $\gamma - b_0 - b_2 - d < -1/p$ for a given $\gamma \in \mathbb{R}$. $\square$

Remark 5.14. If the kernel $K(x,y)$ of (5.47) is such that

$$K(x,y) \asymp \min\{x^{b_1} y^{c_1}, x^{b_2} y^{c_2}\}, \tag{5.51}$$

with $b_1 - b_2 = c_1 - c_2 > 0$, then the sufficient conditions of Corollary 5.12 are also necessary. An example of a transform satisfying such property is the $\mathscr{H}_\alpha$ transform with $\alpha > 1/2$ (cf. Remark 5.2). This proves that Corollary 5.12 is sharp with respect to the conditions on parameters, although in general it does not give the sharp sufficient conditions for inequality (5.48) to hold whenever F has an oscillating kernel, as in the case of the Hankel or sine transforms.

Another example of a kernel $K(x,y)$ satisfying (5.51) is that of the (modified) Stieltjes transform

$$\mathcal{S}_\lambda f(y) = \int_0^\infty \frac{f(x)}{(x+1/y)^\lambda} \, dx, \qquad \lambda > 0, \qquad y > 0. \tag{5.52}$$

The kernel of (5.52) satisfies $K(x,y) \asymp y^\lambda$ whenever $x \leq 1/y$ and $K(x,y) \asymp x^{-\lambda}$ whenever $x > 1/y$. Hence, we can deduce from Corollary 5.12 and Theorem 5.13 (with $b_0 = c_0 = b_1 = c_2 = 0$, and $c_1 = -b_2 = \lambda$, to obtain that the inequality

$$\|x^\beta \mathcal{S}_\lambda f\|_q \lesssim \|x^\gamma f\|_p, \qquad 1 < p \leq q < \infty, \tag{5.53}$$

holds for all $f \in L^p(\mathbb{R}_+, x^{\gamma p})$ if and only if

$$\beta = \gamma - \lambda + \frac{1}{q} - \frac{1}{p'}, \qquad \frac{1}{q} < \beta < \frac{1}{q} + \lambda.$$

In fact, a more general approach which will be considered in future work, incorporating weights u and v (not necessarily power functions) in (5.53). A wider range of transforms will be considered, as for instance the classical Stieltjes transform $\mathcal{S}_\lambda f(1/y)$ (cf. [6, 119]), or the Hardy-Bellman operators

$$Hf(y) = \frac{1}{y} \int_0^y |f(x)| \, dx, \qquad Bf(y) = \int_y^\infty \frac{|f(x)|}{x} \, dx.$$

Those operators have been extensively studied, and are being investigated nowadays. See the papers [20, 22, 53, 54, 98, 118] and the references therein.

5.5 Integral transforms with kernel represented by a power series and functions with vanishing moments

This section is motivated by the well-known result due to Sadosky and Wheeden [114]. They proved that the sufficient conditions (5.6) that guarantee Pitt's inequality can be relaxed, provided that f has vanishing moments.

Theorem 5.15 ([114]). *Let $f \in L^p(\mathbb{R}_+, x^{\gamma p})$ be such that*

$$\int_{-\infty}^{\infty} x^j f(x)\,dx = 0, \qquad j = 0, \dots, n-1, \quad n \in \mathbb{N}.$$

Then, the weighted norm inequality

$$\left(\int_{\mathbb{R}} |x|^{-\beta q} |\widehat{f}(x)|^q\,dx\right)^{1/q} \leq C \left(\int_{\mathbb{R}} |x|^{\gamma p} |f(x)|^p\,dx\right)^{1/p} \tag{5.54}$$

holds with $\beta = \gamma + 1/q - 1/p'$ and

$$\frac{1}{q} < \beta < n + \frac{1}{q}, \qquad \beta \neq \frac{1}{q} + j,\ j = 1, \dots, n-1.$$

Before proceeding with the generalization of Theorem 5.15, let us make a few observations. First, although Theorem 5.15 is stated for the one-dimensional Fourier transform, the multidimensional analogue is also obtained in [114]. See also [24], where a similar problem with nonradial weights is considered.

Secondly, note that in comparison with the classical Pitt's inequality (cf. (5.5) and (5.6)), if f has vanishing moments then Pitt's inequality also holds for some $\beta > 1/q$. We also emphasize that Theorem 5.15 does not hold for $\beta = 1/q + j$, with $j \in \mathbb{N}$.

The proof of Theorem 5.15 relies on the fact that the kernel of the Fourier transform, the exponential function, can be written as the power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad x \in \mathbb{R}.$$

We now obtain a generalization of Theorem 5.15 for integral transforms with a power-type kernel that allow a representation by power series, following the idea of Sadosky and Wheeden used in [114].

Theorem 5.16. *Let $1 < p \leq q < \infty$ and let the integral transform F be as in (5.47). For $b_1, c_1 \in \mathbb{R}$, let*

$$K(x,y) = x^{b_1} y^{c_1} \sum_{m=0}^{\infty} a_m (xy)^{km}, \qquad k \in \mathbb{N}, \qquad a_m \in \mathbb{C}, \qquad x, y > 0, \tag{5.55}$$

with $\sum_{m=0}^{\infty} |a_k| = A < \infty$. Assume the series defining K converges for every $x, y > 0$, and moreover

$$|K(x,y)| \lesssim x^{b_2} y^{c_2} \qquad \text{for } xy > 1,$$

where $b_2, c_2 \in \mathbb{R}$, and $c_1 - c_2 = b_1 - b_2 \geq 0$. If $f \in L^p(\mathbb{R}_+, x^{\gamma p})$ is such that

$$\int_0^{\infty} x^{b_0 + b_1 + \ell k} f(x)\,dx = 0, \qquad \ell = 0, \dots, n-1, \qquad n \in \mathbb{N}, \tag{5.56}$$

then the inequality $\|x^{-\beta}Ff\|_q \le C\|x^\gamma f\|_p$ *holds with*

$$\beta = \gamma + c_0 - b_0 + c_1 - b_1 + \frac{1}{q} - \frac{1}{p'}, \qquad \frac{1}{q} + c_0 + c_1 < \beta < \frac{1}{q} + c_0 + c_1 + n\ell,$$

and $\beta \neq 1/q + c_0 + c_1 + jk$, $j = 1, \dots, n-1$.

Proof. First of all note that since $\sum |a_m| < \infty$, one has $|K(x,y)| \lesssim x^{b_1}y^{c_1}$ whenever $xy \le 1$.

By (5.56), we can write, for any $\ell = 1, \dots, n$,

$$Ff(y) = y^{c_0+c_1}\int_0^\infty x^{b_0+b_1}f(x)\left(x^{-b_1}y^{-c_1}K(x,y) - \sum_{m=0}^{\ell-1} a_m(xy)^{km}\right)dx$$

Defining

$$G_\ell(x,y) = x^{-b_1}y^{-c_1}K(x,y) - \sum_{m=0}^{\ell-1} a_m(xy)^{km} = \sum_{m=\ell}^{\infty} a_m(xy)^{km},$$

it clearly follows that for $xy \le 1$ one has

$$|G_\ell(x,y)| \le A(xy)^{k\ell}.$$

If $xy > 1$, since $x^{-b_1}y^{-c_1}|K(x,y)| \lesssim (xy)^{c_2-c_1}$ and $c_2 - c_1 \le 0$, then $|G_\ell(x,y)| \lesssim (xy)^{k(\ell-1)}$. In conclusion,

$$|G_\ell(x,y)| \lesssim \begin{cases} (xy)^{k\ell}, & xy \le 1, \\ (xy)^{k(\ell-1)}, & xy > 1, \end{cases}$$

or equivalently,

$$|G_\ell(x,y)| \lesssim \min\left\{(xy)^{k\ell}, (xy)^{k(\ell-1)}\right\}.$$

Hence, by Corollary 5.12, the transform $\mathcal{G}_\ell$ defined by

$$\mathcal{G}_\ell f(y) = y^{c_0+c_1}\int_0^\infty x^{b_0+b_1}f(x)G_\ell(x,y)\,dx$$

satisfies the inequality

$$\|x^{-\beta}\mathcal{G}_\ell f\|_q \lesssim \|x^\gamma f\|_p,$$

provided that $\beta = \gamma + c_0 - b_0 + c_1 - b_1 + 1/q - 1/p'$ and

$$\frac{1}{q} + c_0 + c_1 + k(\ell-1) < \beta < \frac{1}{q} + c_0 + c_1 + k\ell.$$

Since the latter holds for every $\ell = 1, \dots, n$, our assertion follows. □

Comparing Corollary 5.12 and Theorem 5.16, we see that for kernels of the form (5.55), inequality (5.48) holds for functions with certain moments vanishing at some values $\beta > 1/q + c_0 + c_1$, thus Theorem 5.16 extends the range of β given by Corollary 5.12. Moreover, the assertion is not true in general for $\beta = 1/q + c_0 + c_1$, as we show in Proposition 5.22 below.

Examples of transforms whose kernels satisfy the assumptions of Theorem 5.16 are the sine, Hankel, and $\mathscr{H}_\alpha$ transforms. We now write the corresponding statements for each of these transforms. Let us first make an important observation.

Remark 5.17. In contrast with Corollary 5.12, in Theorem 5.16 we can allow $b_1 = b_2$, $c_1 = c_2$. This is because in order to prove Theorem 5.16 we apply Corollary 5.12 to the transform $\mathcal{G}_\ell$, whose kernel satisfies $|G_\ell(x,y)| \lesssim \min\left\{(xy)^{k\ell}, (xy)^{k(\ell-1)}\right\}$, thus it always satisfies the hypothesis of Corollary 5.12, namely $b_1 = c_1 = k\ell > k(\ell-1) = b_2 = c_2$. In particular, Theorem 5.16 can be applied for the cosine transform, whilst Corollary 5.12 cannot.

In the case of the Hankel transform of order $\alpha \geq -1/2$ (4.1), we have the representation of j_α by power series (4.3). Thus, applying Theorem 5.16 with $b_1 = c_0 = c_1 = 0$, $b_0 = 2\alpha+1$ and $k = 2$, we obtain the following.

Corollary 5.18. *Let $1 < p \leq q < \infty$ and let $f \in L^p(\mathbb{R}_+, x^{\gamma p})$ be such that*

$$\int_0^\infty x^{2\alpha+1+2\ell} f(x)\,dx = 0, \qquad \ell = 0, \dots, n-1, \quad n \in \mathbb{N}.$$

Then the inequality

$$\|x^{-\beta} H_\alpha f\|_q \leq C \|x^\gamma f\|_p$$

holds with $\beta = \gamma - 2\alpha - 1 + 1/q - 1/p'$ and

$$\frac{1}{q} < \beta < \frac{1}{q} + 2n, \qquad \beta \neq \frac{1}{q} + 2\ell,\ \ell = 1, \dots, n-1.$$

Remark 5.19. Let us compare Theorem 5.15 and Corollary 5.18. As mentioned above, it is known that if $\int_{\mathbb{R}} f(x)\,dx = 0$, then inequality (5.54) does not necessarily hold for $\beta = 1 + 1/q$ (cf. [114]). However, it follows from Corollary 5.18 with $\alpha = -1/2$ (i.e., for the cosine transform) that *if $\int_{\mathbb{R}} f(x)\,dx = 0$ and moreover f is even, then inequality* (5.54) *holds for $\beta = 1 + 1/q$.*

Let us now state a version of Theorem 5.16 for the sine transform. Denote $\widehat{f}_{\sin}(y) = \int_0^\infty f(x) \sin xy\,dx$. Since

$$\sin xy = xy \sum_{m=0}^\infty \frac{(-1)^m}{(2m+1)!}(xy)^{2m},$$

Theorem 5.16 with $b_0 = c_0 = 0$, $b_1 = c_1 = 1$, and $k = 2$ yields the following.

Corollary 5.20. *Let $1 < p \leq q < \infty$ and let $f \in L^p(\mathbb{R}_+, x^{\gamma p})$ be such that*

$$\int_0^\infty x^{2\ell+1} f(x)\,dx = 0, \qquad \ell = 0, \dots, n-1, \quad n \in \mathbb{N}.$$

Then the inequality

$$\|x^{-\beta} \widehat{f}_{\sin}\|_q \leq C \|x^\gamma f\|_p$$

holds with $\beta = \gamma + 1/q - 1/p'$ and

$$\frac{1}{q} + 1 < \beta < \frac{1}{q} + 2n + 1, \qquad \beta \neq \frac{1}{q} + 2\ell + 1,\ \ell = 1, \dots, n-1.$$

Finally, we present the statement corresponding to the $\mathscr{H}_\alpha$ transform. In view of (5.20) and (5.21), we apply Theorem 5.16 with $b_0 = c_0 = 1/2$, $b_1 = c_1 = \alpha + 1$ and $k = 2$.

Corollary 5.21. *Let $1 < p \le q < \infty$ and $\alpha > -1/2$. Let $f \in L^p(\mathbb{R}_+, x^{\gamma p})$ be such that*

$$\int_0^\infty x^{\alpha+3/2+2\ell} f(x)\,dx = 0, \qquad \ell = 0,\dots,n-1, \quad n \in \mathbb{N}.$$

Then the inequality

$$\|x^{-\beta}\mathscr{H}_\alpha f\|_q \le C\|x^\gamma f\|_p$$

holds with $\beta = \gamma + 1/q - 1/p'$ and

$$\frac{1}{q} + \alpha + \frac{3}{2} < \beta < \frac{1}{q} + \alpha + \frac{3}{2} + 2n, \qquad \beta \ne \frac{1}{q} + \alpha + \frac{3}{2} + 2\ell,\ \ell = 1,\dots,n-1.$$

We conclude this section by showing that Theorem 5.16 does not necessarily hold with $\beta = 1/q + c_0 + c_1$ (or equivalently, with $\gamma = 1/p' + b_0 + b_1$) whenever $\int_0^\infty x^{b_0+b_1} f(x)\,dx = 0$.

Proposition 5.22. *Let $0 < q \le \infty$ and $1 < p < \infty$. Let the transform F be as in (5.47), with the kernel $K(x,y)$ of the form (5.55) and continuous in x, satisfying $|a_0| > 0$ and $\sum |a_m| = A < \infty$. Assume there is $C > 0$ such that*

$$|K(x,y)| \le \begin{cases} Cx^{b_1}y^{c_1}, & \text{if } xy \le 1,\\ Cx^{b_2}y^{c_2}, & \text{if } xy > 1,\end{cases}$$

where $b_j, c_j \in \mathbb{R}$, $j = 1,2$. Furthermore, suppose there exists $\nu \in \mathbb{R}$ and $G_y^\nu(x)$ such that $(d/dx)G_y^\nu(x) = x^\nu K(x,y)$, and that there exists $C' > 0$ for which

$$|G_y^\nu(x)| \le C' x^b y^c, \qquad b, c \in \mathbb{R}, \qquad xy \ge 1, \tag{5.57}$$

holds with $b - b_1 - \nu < 1$. Then, if $u \not\equiv 0$, the inequality

$$\left(\int_0^\infty u(x)|Ff(x)|^q\,dx\right)^{1/q} \lesssim \left(\int_0^\infty x^{p(1/p'+b_0+b_1)}|f(x)|^p\,dx\right)^{1/p} \tag{5.58}$$

does not hold for all $f \in L^p(\mathbb{R}_+, x^{\gamma p})$ satisfying $\int_0^\infty x^{b_0+b_1} f(x)\,dx = 0$.

Remark 5.23. Note that the examples we presented above (sine, Hankel, or $\mathscr{H}_\alpha$ transforms) satisfy all the hypotheses of Proposition 5.22. For example, in the case of the $\mathscr{H}_\alpha$ transform ($\alpha > -1/2$), we have $b_1 = \alpha + 1$, $b_2 = \alpha - 1$, and for any $\nu \ge 1/2$, $b = \alpha + \nu$ (cf. Lemma 5.3). The continuity of $K(x,y)$ on x, which has not been used before, is also satisfied by those examples.

Proof of Proposition 5.22. Define, for $N \in \mathbb{N}$,

$$f_N(x) = \frac{1}{x^{b_0+b_1+1}}\big(\chi_{(1/N,1)}(x) - \chi_{(1,N)}(x)\big).$$

Then

$$\int_0^\infty x^{b_0+b_1} f_N(x)\,dx = \int_{1/N}^1 \frac{1}{x}\,dx - \int_1^N \frac{1}{x}\,dx = 0,$$

and

$$\left(\int_0^\infty x^{p(1/p'+b_0+b_1)}|f_N(x)|^p\,dx\right)^{1/p} = \left(\int_{1/N}^N \frac{1}{x}\,dx\right)^{1/p} = (2\log N)^{1/p},$$

so that $f \in L^p(\mathbb{R}_+, x^{\gamma p})$ for all $N \in \mathbb{N}$. Now let $y \in (0,\infty)$ and assume N is such that $1/N < 1/y < N$. We have

$$\begin{aligned}
y^{-c_0}|Ff(y)| &= \left| \int_{1/N}^{1} \frac{1}{x^{b_1+1}} K(x,y)\,dx - \int_1^N \frac{1}{x^{b_1+1}} K(x,y)\,dx \right| \\
&\geq \left| \int_{1/N}^{1/y} \frac{1}{x^{b_1+1}} K(x,y)\,dx \right| - 2\left| \int_{1/y}^{1} \frac{1}{x^{b_1+1}} K(x,y)\,dx \right| \\
&\quad - \left| \int_{1/y}^{N} \frac{1}{x^{b_1+1}} K(x,y)\,dx \right| =: I_1 - I_2 - I_3.
\end{aligned}$$

Let us now estimate I_1 from below, and I_2, I_3 from above. First,

$$\begin{aligned}
I_1 &= \left| \int_{1/N}^{1/y} \frac{K(x,y) - x^{b_1}y^{c_1}a_0 + x^{b_1}y^{c_1}a_0}{x^{b_1+1}}\,dx \right| \\
&\geq \left| \int_{1/N}^{1/y} \frac{x^{b_1}y^{c_1}a_0}{x^{b_1+1}}\,dx \right| - \left| \int_{1/N}^{1/y} \frac{K(x,y) - x^{b_1}y^{c_1}a_0}{x^{b_1+1}}\,dx \right|.
\end{aligned}$$

Since

$$\left| \int_{1/N}^{1/y} \frac{a_0 x^{b_1} y^{c_1}}{x^{b_1+1}}\,dx \right| = |a_0| y^{c_1} \left| \int_{1/N}^{1/y} \frac{1}{x}\,dx \right| \geq y^{c_1}|a_0| \log N - y^{c_1}|a_0 \log y|,$$

and

$$\begin{aligned}
\left| \int_{1/N}^{1/y} \frac{K(x,y) - a_0 x^{b_1} y^{c_1}}{x^{b_1+1}}\,dx \right| &\leq y^{c_1} \int_{1/N}^{1/y} x^{-1} \sum_{m=1}^{\infty} |a_m| (xy)^{mk}\,dx \leq A y^{c_1+k} \int_{1/N}^{1/y} x^{k-1}\,dx \\
&\leq A y^{c_1},
\end{aligned}$$

we obtain

$$I_1 \geq y^{c_1}|a_0| \log N - y^{c_1}|a_0 \log y| - A y^{c_1} =: y^{c_1}|a_0| \log N - \eta_1(y).$$

We now proceed to estimate I_2 from above. Here we distinguish two cases:

- if $1/y < 1$ in the following we set $j = 1$,
- and if $1/y \geq 1$ we set $j = 2$.

We have

$$\begin{aligned}
I_2 &= 2\left| \int_{1/y}^{1} \frac{1}{x^{b_1+1}} K(x,y)\,dx \right| \leq 2C y^{c_j} \left| \int_{1/y}^{1} x^{b_j - b_1 - 1}\,dx \right| \\
&\leq 2C y^{c_j} \max\{1, 1/y\} \max\left\{1, y^{b_1+1-b_j}\right\} \leq 2C y^{c_j} \max\left\{1, 1/y, y^{b_1-b_j}, y^{b_1+1-b_j}\right\} \\
&=: \eta_2(y).
\end{aligned}$$

Finally, integration by parts together with the identity $(d/dx)G_y^\nu(x) = x^\nu K(x,y)$ and estimate (5.57) yield

$$\begin{aligned}I_3 &= \left| \int_{1/y}^N \frac{1}{x^{b_1+1+\nu}} x^\nu K(x,y)\,dx \right| \le N^{-b_1-1-\nu}|G_y^\nu(N)| + y^{b_1+1+\nu}|G_y^\nu(1/y)| \\ &\quad + (b_1+\nu+1)\int_{1/y}^N \frac{1}{x^{b_1+2+\nu}}|G_y^\nu(x)|\,dx \\ &\le C' y^c N^{b-b_1-1-\nu} + C' y^{c-b+b_1+1+\nu} + C'(b_1+1+\nu)y^c \int_{1/y}^N x^{b-b_1-2-\nu}\,dx \\ &\le C' y^c + C' y^{c-b+b_1+1+\nu} + C' y^c \left| \frac{b_1+1+\nu}{b-b_1-2-\nu}\right| (1+y^{-b+b_1+1+\nu}) =: \eta_3(y).\end{aligned}$$

Collecting all estimates, we obtain

$$y^{-c_0-c_1}|Ff(y)| \ge |a_0|\log N - y^{-c_1}(\eta_1(y)+\eta_2(y)+\eta_3(y)).$$

Since $u \not\equiv 0$, we can find $0 < t_1 < t_2 < \infty$ such that $\int_{t_1}^{t_2} u(x)\,dx > 0$. Choosing N so large that for every $y \in (t_1,t_2)$ there holds

$$y^{-c_0-c_1}|Ff(y)| \ge |a_0|\log N - y^{-c_1}(\eta_1(y)+\eta_2(y)+\eta_3(y)) > \frac{|a_0|}{2}\log N,$$

it can be deduced from inequality (5.58) (with the usual modification if $q=\infty$) that

$$\begin{aligned}\frac{|a_0|}{2}\log N \left(\int_{t_1}^{t_2} x^{q(c_0+c_1)}u(x)\,dx\right)^{1/q} &\le \left(\int_0^\infty |Ff(x)|^q u(x)\,dx\right)^{1/q} \\ &\lesssim \left(\int_0^\infty x^{p(1/p'+b_0+b_1)}|f_N(x)|^p\,dx\right)^{1/p} = (2\log N)^{1/p},\end{aligned}$$

which is a contradiction, since $p>1$ and N is arbitrarily large. □

5.6 Weighted norm inequalities for general monotone functions

In this section we consider functions from the class $GM(\beta_2)$ (cf. Section 2.3), and obtain sufficient conditions for the weighted norm inequality

$$\|Ff\|_{q,u} \lesssim \|f\|_{p,v}, \qquad 1 < p \le q < \infty, \tag{5.59}$$

to hold whenever $f \in GM(\beta_2)$ and F is a transform with power-type kernel (i.e., of the form (5.47) and whose kernel satisfies an upper estimate given by power functions). We will assume the kernel K is continuous in the variable x. In particular, we study whether we can relax the sufficient conditions provided by Corollary 5.12 when u and v are power weights, under the assumption $f \in GM(\beta_2)$.

Let us assume that

$$|K(x,y)| \lesssim x^{b_1}y^{c_1}, \qquad xy \le 1, \tag{5.60}$$

with $b_1, c_1 \in \mathbb{R}$. Let $G(x,y)$ be such that

$$\frac{d}{dx}G(x,y) = x^{b_0}K(x,y), \tag{5.61}$$

where the additive constant of G is taken to be zero (such G exists due to the continuity of K in the variable x). We moreover suppose that $G(x, y)$ satisfies the estimate

$$|G(x, y)| \lesssim x^b y^c, \qquad xy > 1, \tag{5.62}$$

with $b, c \in \mathbb{R}$. In this section we suppose that the functions f we consider satisfy

$$\int_0^1 x^{b_0+b_1} |f(x)|\, dx + \int_1^\infty x^{b-1} |f(x)|\, dx < \infty. \tag{5.63}$$

First we obtain straightforward upper estimates for the function G from (5.61) that follow from the upper estimates for K. This will provide an expression for b and c in (5.62) in the general case.

Proposition 5.24. *Let K satisfy (5.60), and assume that $|K(x, y)| \lesssim x^{b_2} y^{c_2}$ for $xy > 1$. Let G be given by the relation (5.61).*

(i) *If $b_0 + b_1 > 0$ and $b_0 + b_2 \neq -1$, then*

$$|G(x, y)| \lesssim \begin{cases} y^{c_1} x^{b_0+b_1+1}, & \text{if } xy \le 1, \\ y^{c_2} x^{b_0+b_2+1} + y^{c_1-b_0-b_1-1} + y^{c_2-b_0-b_2-1}, & \text{if } xy > 1; \end{cases}$$

(ii) *if $b_0 + b_2 < -1$ and $b_0 + b_1 \neq -1$, then*

$$|G(x, y)| \lesssim \begin{cases} y^{c_1} x^{b_0+b_1+1} + y^{c_1-b_0-b_1-1} + y^{c_2-b_0-b_2-1}, & \text{if } xy \le 1, \\ y^{c_2} x^{b_0+b_2+1}, & \text{if } xy > 1. \end{cases}$$

Proof. (i) Since $b_0 + b_1 > 0$, we can write $G(x, y) = \int_0^x t^{b_0} K(t, y)\, dt$, by the fundamental theorem of calculus. For $x \le 1/y$,

$$|G(x, y)| \lesssim y^{c_1} \int_0^x t^{b_0+b_1}\, dt \lesssim y^{c_1} x^{b_0+b_1+1}, \tag{5.64}$$

whilst for $x > 1/y$, using (5.64) and the estimate $|K(x, y)| \lesssim x^{b_2} y^{c_2}$, we obtain

$$|G(x, y)| \lesssim y^{c_1-b_0-b_1-1} + \int_{1/y}^x t^{b_0} |K(t, y)|\, dt \lesssim y^{c_1-b_0-b_1-1} + y^{c_2-b_0-b_2-1} + y^{c_2} x^{b_0+b_2+1}.$$

(ii) Since $b_0 + b_2 < -1$, we can write $G(x, y) = \int_x^\infty t^{b_0} K(t, y)\, dt$, again by the fundamental theorem of calculus. For $x > 1/y$,

$$|G(x, y)| \lesssim y^{c_2} \int_x^\infty t^{b_0+b_2}\, dt \asymp y^{c_2} x^{b_0+b_2+1}. \tag{5.65}$$

For $x \le 1/y$, using (5.65) and the estimate $|K(x, y)| \lesssim x^{b_2} y^{c_2}$,we obtain

$$|G(x, y)| \lesssim \int_x^{1/y} t^{b_0} |K(t, y)|\, dt + y^{c_2-b_0-b_2-1} \lesssim y^{c_1} x^{b_0+b_1+1} + y^{c_1-b_0-b_1-1} + y^{c_2-b_0-b_2-1},$$

as desired. □

Remark 5.25. Observe that the upper estimates for $|G(x,y)|$ given in Proposition 5.24 are rather rough, and they are not optimal for oscillating kernels $K(x,y)$, such as $K(x,y) = j_\alpha(xy)$. However, those estimates are useful for kernels satisfying

$$K(x,y) \asymp \begin{cases} x^{b_1}y^{c_1}, & \text{if } xy \le 1, \\ x^{b_2}y^{c_2}, & \text{if } xy > 1. \end{cases}$$

In fact, the Struve function $\mathbf{H}_\alpha$ with $\alpha > 1/2$ is such an example, and it can be easily checked that in this case the result given by Proposition 5.24 coincides with that of Lemma 5.3. For oscillating kernels it is more convenient to obtain upper estimates for G by using an iterated integration by parts, as done in Section 4.1 for the Bessel function, or in Lemma 5.3 for the Struve function (in both cases the estimates are sharp).

We will need the following pointwise upper estimate for Ff.

Lemma 5.26. *Let $f \in GM(\beta_2)$. Assume (5.60) holds, and that $G(x,y)$ defined by (5.61) satisfies (5.62). Then, the transform*

$$Ff(y) = y^{c_0}\int_0^\infty x^{b_0}f(x)K(x,y)\,dx$$

satisfies the pointwise estimate

$$|Ff(y)| \lesssim y^{c_0+c_1}\int_0^{1/y}|f(x)|x^{b_0+b_1}\,dx + y^{c+c_0}\int_{1/(\lambda y)}^\infty x^{b-1}|f(x)|\,dx, \tag{5.66}$$

where $\lambda \ge 2$ is the constant from the class $GM(\beta_2)$.

Proof. In view of (5.60), we have

$$|Ff(y)| \lesssim y^{c_0+c_1}\int_0^{1/y} x^{b_0+b_1}|f(x)|\,dx + \left| y^{c_0}\int_{1/y}^\infty f(x)x^{b_0}K(x,y)\,dx\right| =: I_1 + |I_2|.$$

Partial integration on I_2 yields the estimate

$$|I_2| \le y^{c_0}|f(x)G(x,y)|\Big|_{1/y}^{\infty} + y^{c_0}\int_{1/y}^\infty |G(x,y)\,df(x)|.$$

First, since $f \in GM(\beta_2)$, then $x^{b-1}f(x) \in GM(\beta_2)$ (cf. [86]). Thus, it follows from (5.63) and the estimate given by Remark 2.10 that $x^b f(x) \to 0$ as $x \to \infty$. Hence, by (5.62),

$$\lim_{x\to\infty}|f(x)G(x,y)| \lesssim y^c \lim_{x\to\infty} x^b|f(x)| = 0.$$

Secondly, we deduce from Remark 2.10 and (5.62) that

$$\begin{aligned} y^{c_0}|f(1/y)G(1/y,y)| &\lesssim y^{c+c_0-b}|f(1/y)| \lesssim y^{c+c_0}\int_{1/(\lambda y)}^{y/\lambda} x^{b-1}|f(x)|\,dx \\ &\le y^{c+c_0}\int_{1/(\lambda y)}^\infty x^{b-1}|f(x)|\,dx. \end{aligned}$$

Finally, we use Corollary 2.12 and the estimate (5.62) to obtain

$$y^{c_0}\int_{1/y}^\infty |G(x,y)\,df(x)| \lesssim y^{c+c_0}\int_{1/y}^\infty x^b|df(x)| \lesssim y^{c+c_0}\int_{1/(\lambda y)}^\infty x^{b-1}|f(x)|\,dx,$$

and therefore (5.66) is established. □

Remark 5.27. If $b-c-1=b_0+b_1-c_1$, one may take $\lambda=1$ in (5.66), since

$$y^{c+c_0}\int_{1/(\lambda y)}^{1/y} x^{b-1}|f(x)|\,dx \asymp y^{c_0+c_1}\int_{1/(\lambda y)}^{1/y} x^{b_0+b_1}|f(x)|\,dx \le y^{c_0+c_1}\int_0^{1/y} x^{b_0+b_1}|f(x)|\,dx$$

We are now in a position to prove sufficient conditions for the inequality (5.59) to hold.

Theorem 5.28. *Let $1<p\le q<\infty$. Let the transform F be given by (5.47). Assume (5.60) holds, and $G(x,y)$ defined by (5.61) satisfies the estimate (5.62). Then, inequality (5.59) holds for all $f\in L^p(\mathbb{R}_+,v)$ from the class $GM(\beta_2)$ if*

$$\sup_{r>0}\left(\int_0^{1/r} u(x)x^{(c_0+c_1)q}\,dx\right)^{1/q}\left(\int_0^r v(x)^{1-p'}x^{(b_0+b_1)p'}\,dx\right)^{1/p'}<\infty, \tag{5.67}$$

$$\sup_{r>0}\left(\int_{1/(\lambda r)}^{\infty} u(x)x^{(c+c_0)q}\,dx\right)^{1/q}\left(\int_r^\infty v(x)^{1-p'}x^{(b-1)p'}\,dx\right)^{1/p'}<\infty, \tag{5.68}$$

where $\lambda\ge 2$ is the $GM(\beta_2)$ constant.

Remark 5.29. Note that under certain assumptions on the parameters c, b, c_i, and b_i ($i=0,1$), we can use the gluing lemma (Lemma 5.6) to rewrite conditions (5.67) and (5.68) as one single condition, similarly as done with Theorem 5.5 and Corollary 5.7.

Proof of Theorem 5.28. Using estimate (5.66), we can write

$$\begin{aligned}\left(\int_0^\infty u(x)|Ff(x)|^q\,dx\right)^{1/y} &\lesssim \left(\int_0^\infty u(y)\left(y^{c_0+c_1}\int_0^{1/y} x^{b_0+b_1}|f(x)|\,dx\right)^q dy\right)^{1/q}\\ &\quad+\left(\int_0^\infty u(y)\left(y^{c+c_0}\int_{1/(\lambda y)}^\infty x^{b-1}|f(x)|\,dx\right)^q dy\right)^{1/q}\\ &=: I_1+I_2.\end{aligned}$$

On the one hand, by Lemma 5.4 and the change of variables $y\to 1/y$, the inequality

$$\begin{aligned}I_1&=\left(\int_0^\infty \frac{u(1/y)}{y^{2+(c_0+c_1)q}}\left(\int_0^y x^{b_0+b_1}|f(x)|\,dx\right)^q dy\right)^{1/q}\lesssim\left(\int_0^\infty v(x)|f(x)|^p\,dx\right)^{1/p}\\ &=\|f\|_{p,v}\end{aligned}$$

holds if

$$\sup_{r>0}\left(\int_r^\infty \frac{u(1/x)}{x^{2+(c_0+c_1)q}}\,dx\right)^{1/q}\left(\int_0^r v(x)^{1-p'}x^{(b_0+b_1)p'}\,dx\right)^{1/p'}<\infty,$$

or equivalently, if (5.67) is satisfied. On the other hand, again by Lemma 5.4 and the change of variables $y\to 1/y$, the inequality

$$I_2\asymp\left(\int_r^\infty \frac{u\big((\lambda y)^{-1}\big)}{y^{2+(c+c_0)q}}\left(\int_y^\infty x^{b-1}|f(x)|\,dx\right)^q dy\right)^{1/q}\lesssim\left(\int_0^\infty v(x)|f(x)|^p\,dx\right)^{1/p}=\|f\|_{p,v}$$

holds provided that

$$\sup_{r>0}\left(\int_0^r \frac{u\big((\lambda x)^{-1}\big)}{x^{2+(c+c_0)q}}\,dx\right)^{1/q}\left(\int_r^\infty v(x)^{1-p'}x^{(b-1)p'}\,dx\right)^{1/p'}<\infty,$$

or equivalently, if (5.68) holds. $\square$

The sufficient conditions for inequality (5.59) to hold whenever u and v are power weights read as follows.

Corollary 5.30. *Let $1 < p \le q < \infty$ and $f \in GM(\beta_2)$. Let F be given by (5.47). Assume (5.60) holds, and $G(x,y)$ defined by (5.61) satisfies (5.62) with $c < c_1$. Then, inequality (5.48) holds for all $f \in L^p(\mathbb{R}_+, x^{\gamma p})$ from the class $GM(\beta_2)$ if*

$$\beta = \gamma + c_0 - b_0 + c_1 - b_1 + \frac{1}{q} - \frac{1}{p'}, \qquad \frac{1}{q} + c_0 + c < \beta < \frac{1}{q} + c_0 + c_1.$$

The proof of Corollary 5.30 is omitted, as it is essentially an application of Theorem 5.28 with $u(x) = x^{-\beta q}$, $v(x) = x^{\gamma p}$, cf. the proof of Theorem 5.11.

Remark 5.31. Let us compare the conditions for β in Corollaries 5.12 and 5.30. We cannot formally compare those two results, since the assumptions on K are different, namely in Corollary 5.30 we assume $K(x,y)$ is continuous in x, whilst in Corollary 5.12 we do not. However, in practice, all the examples of integral transforms we have considered throughout this chapter have kernels K continuous in each variable, thus it makes sense to compare Corollaries 5.12 and 5.30 in our context. On the one hand, we observe that in both statements the condition $\beta < 1/q + c_0 + c_1$ is required. On the other hand, Corollary 5.12 requires that $\beta > 1/q + c_0 + c_2$, whilst Corollary 5.30 requires $\beta > 1/q + c_0 + c$. Therefore, in order for Corollary 5.30 to yield a nontrivial result we need to assume $c < c_2$.

Let us present sufficient conditions for inequality (5.48) to hold in each of the aforementioned cases, under the assumption $f \in GM(\beta_2)$. Some of the following results are already known, some others are new. It is worth noting that in all examples we show below, the conditions on the parameters $c < c_1$, and $b - c - 1 = b_0 + b_1 - c_1$ hold. All functions considered here satisfy (5.63).

1. The sine transform (for which $K(x,y) = \sin xy$ and $G(x,y) = -y^{-1}\cos xy$) satisfies $b_1 = c_1 = 1$, $b = b_0 = c_0 = 0$ and $c = -1$, thus, if $f \in GM(\beta_2)$, the sufficient conditions for the inequality

$$\|x^{-\beta}\widehat{f}_{\sin}\|_q \lesssim \|x^{\gamma} f\|_p$$

   to hold are $\beta = \gamma + 1/q - 1/p'$ and $-1 + 1/q < \beta < 1 + 1/q$. These conditions are also necessary, as shown in [87].

2. The classical Hankel transform of order $\alpha \ge -1/2$ (4.1) has kernel $K(x,y) = j_\alpha(xy)$ satisfying $j_\alpha(xy) \asymp 1$ for $xy \le 1$, and moreover

$$|G(x,y)| \lesssim y^{-\alpha-3/2} x^{\alpha+1/2}, \qquad x, y \in \mathbb{R}_+,$$

   cf. Lemma 4.8. Thus, by applying Corollary 5.30 with $b_0 = 2\alpha + 1$, $b_1 = c = c_1 = 0$, $b = \alpha + 1/2$ and $c = -\alpha - 3/2$, we get that the inequality

$$\|x^{-\beta} H_\alpha f\|_q \lesssim \|x^{\gamma} f\|_p$$

   holds with $\beta = \gamma - 2\alpha - 1 + 1/q - 1/p'$ and $1/q - \alpha - 3/2 < \beta < 1/q$. These sufficient conditions are also necessary, as proved in [28]. This includes the cosine transform ($\alpha = -1/2$), see also [87].

3. The $\mathscr{H}_\alpha$ transform with $\alpha > -1/2$ (5.19) has kernel $K(x,y) = \mathbf{H}_\alpha(xy)$ satisfying $\mathbf{H}_\alpha(xy) \asymp (xy)^{\alpha+1}$ for $xy \leq 1$. By Lemma 5.3, we have

$$|G(x,y)| \lesssim y^{\alpha-1}x^{\alpha+1/2}, \qquad xy > 1.$$

Hence, applying Corollary 5.30 with $b_0 = c_0 = 1/2$, $b_1 = c_1 = \alpha+1$, $b = \alpha + 1/2$ and $c = \alpha - 1$, we get that if $f \in GM(\beta_2)$, the inequality

$$\|x^{-\beta}\mathscr{H}_\alpha f\|_q \lesssim \|x^\gamma f\|_p \tag{5.69}$$

holds with $\beta = \gamma + 1/q - 1/p'$ and $1/q + \alpha - 1/2 < \beta < 1/q + \alpha + 3/2$. Notice that for $\alpha \geq 1/2$, this yields no improvement with respect to the general case (cf. (5.50)), but for $-1/2 < \alpha < 1/2$, the hypothesis $f \in GM(\beta_2)$ allows us to drop the condition $\beta \geq \max\{0, 1/q - 1/p'\}$.

To conclude, we prove that the range of β for which (5.69) holds given by Corollary 5.30 is sharp.

Theorem 5.32. *Let $1 < p \leq q < \infty$. Inequality (5.69) holds for all $f \in L^p(\mathbb{R}_+, x^{\gamma p})$ from the class $GM(\beta_2)$ if and only if*

$$\beta = \gamma + 1/q - 1/p', \qquad 1/q + \alpha - 1/2 < \beta < 1/q + \alpha + 3/2. \tag{5.70}$$

Proof. We only need to prove that if (5.69) holds for every f satisfying our hypotheses, then (5.70) holds, since the "if" part is just the example we just discussed above. For $\alpha > -1/2$ and $r > 0$, consider the function $f_r(x) = x^{\alpha+1/2}\chi_{(0,r)}(x)$. Note that f satisfies all hypotheses. By [48, §11.2 (2)], one has

$$\mathscr{H}_\alpha f_r(y) = r^{\alpha+1}y^{-1/2}\mathbf{H}_{\alpha+1}(ry).$$

On the one hand,

$$\|x^\gamma f\|_p = \left(\int_0^r x^{p(\gamma+\alpha+1/2)}\,dx\right)^{1/p} \asymp r^{\gamma+\alpha+1/2+1/p},$$

provided that $\gamma + \alpha + 1/2 > -1/p$. On the other hand,

$$\|x^{-\beta}\mathscr{H}_\alpha f_r\|_q = r^{\alpha+1}\left(\int_0^\infty x^{-q(\beta+1/2)}|\mathbf{H}_{\alpha+1}(rx)|^q\,dx\right)^{1/q}.$$

Since $\mathbf{H}_{\alpha+1}(rx) \asymp (rx)^{\alpha+2}$ whenever $rx \leq 1$, the latter integral is convergent near the origin if and only if $\beta < 1/q + \alpha + 3/2$. Since $\mathbf{H}_{\alpha+1}(rx) \asymp (rx)^\alpha$ whenever rx is large enough (cf. Remark 5.2), the integral converges near infinity if and only if $\beta > 1/q + \alpha - 1/2$. In order to conclude the proof, we note that

$$\|x^{-\beta}\mathscr{H}_\alpha f_r\|_q \geq r^{2\alpha+3}\left(\int_0^{1/r} x^{q(-\beta+\alpha+3/2)}\,dx\right)^{1/q} \asymp r^{\alpha+3/2+\beta-1/q}.$$

Combining the latter with inequality (5.69) and the equivalence $\|x^\gamma f\|_p \asymp r^{\gamma+\alpha+1/2+1/p}$, we get that $r^{\alpha+3/2+\beta-1/q} \lesssim r^{\gamma+\alpha+1/2+1/p}$ for every $r > 0$, i.e., $\beta = \gamma + 1/q - 1/p'$. □

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