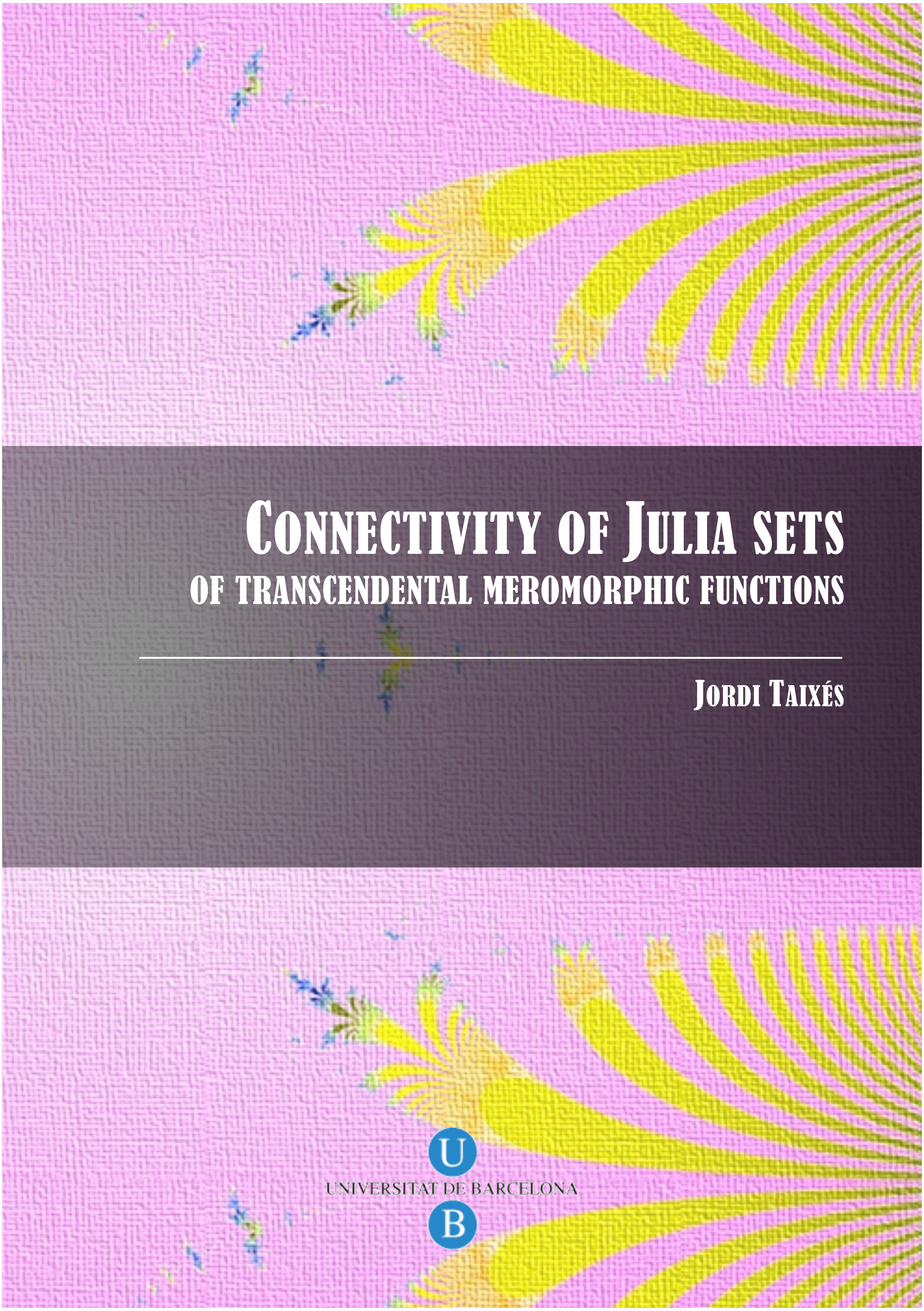
Connectivity of Julia sets of transcendental meromorphic functions

Jordi Taixés i Ventosa



Connectivity of Julia sets of transcendental meromorphic functions

Jordi Taixés

Universitat de Barcelona

Universitat de Barcelona
Facultat de Matemàtiques
Dept. Matemàtica Aplicada i Anàlisi

Programa de doctorat de Matemàtiques
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Connectivity of Julia sets of transcendental meromorphic functions

Tesi doctoral de
Jordi Taixés i Ventosa

Els directors

Dra. Núria Fagella i Rabionet ....

Dr. Xavier Jarque i Ribera ....

El candidat

Sr. Jordi Taixés i Ventosa ....

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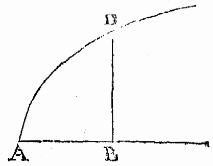
DE ANALYSI

Per Æquationes Numero Terminorum INFINITAS.

Methodum generalem, quam de Curvarum quantitate per Infinitam terminorum Seriem mensurandam, olim excogitaveram, in sequentibus breviter explicatam potius quam accuratè demonstratam habes.



BASI AB Curvæ alicujus AD, sit Applicata BD perpendicularis: Et vocetur $AB = x$, $BD = y$, & sint a, b, c, &c. Quantitates datæ, & m, n, Numeri Integri. Deinde,

Curvarum Simplicium Quadratura.

REGULA I.

Si $ax^{\frac{m}{n}} = y$; *Erit* $\frac{an}{m+n}x^{\frac{m+n}{n}}$ = *Areæ ABD.*

Res Exemplo patebit.

1. Si x^2 ($= 1x^{\frac{2}{1}}$) $= y$, hoc est, $a = 1 = n$, & $m = 2$; Erit $\frac{1}{3}x^3$ = ABD.

A

2. Si

"No day but today."
Jonathan Larson (1960–1996), *Rent*

Preface

Besides setting quite a significant personal milestone, this PhD Thesis concludes a long period of research on Complex Dynamics at the Universitat de Barcelona. Back in the autumn of 2003, when I was at the very beginning of my PhD, I spent a few months at the Institut Henri Poincaré (Paris) in the framework of a Trimestre en Systèmes Dynamiques organised by the great, inspirational, father-of-so-much Adrien Douady, among others. The topic of this Thesis came out there as a result of conversations between Xavier Buff, Mitsuhiro Shishikura and my advisors, after a lecture on connectivity of Julia sets of rational functions given by Shishikura. At that time, the mere words "quasiconformal surgery" sounded like magic to me — today, it is *only* its powerfulness that remains but unbelievable.

During these years, many people have helped me with this project in one way or another, mathematically or not, and I am grateful to all of them because this Thesis is also the ultimate result of their contributions. Still, I would like to express further gratitude to some of them.

First of all — and above all —, to my advisors Núria Fagella and Xavier Jarque for their immense support, for the uncountable discussions on the subject and for all the time dedicated. You have been like parents to me at times, and you know this Thesis also belongs to you.

In 2004 and 2006, I spent two periods of six months at the Mathematics Institute of the University of Warwick under a Marie Curie programme. I want to thank Sebastian van Strien, Adam Epstein and Lasse Rempe for giving me this opportunity and for all the useful conversations.

Likewise, I want to thank Walter Bergweiler, Xavier Buff and Arnaud Chéritat for their hospitality during my visits at the Christian-Albrechts-Universität (Kiel) and at the Université Paul Sabatier (Toulouse), and for all the discussions held during those visits — and in many other occasions.

Out of the many wonderful people I have got to know while working on this project, I would like to express special gratitude to Christian Henriksen, Philip Rippon, Gwyneth Stallard and Toni Garijo, from whom I have learnt many valuable things.

To all the complex dynamicists with whom I have shared good times at conferences, workshops and other events of the such, and to *els joves del Departament*, with whom I have shared courses, coffees, meals and times of all kinds on the rest of the days.

To my beloved *Choir* and to my theatre fellows for the amazing, crazy life outside the academia, and for having helped me develop also an artistic side. I do thank you for this, guys. Without that, this Thesis would surely not have been possible in this form.

To the Institute of Geomatics for their support during the final stage of my PhD.

Finalment, vull donar les gràcies molt especialment als meus, pel seu suport incondicional en la decisió de començar (i d'acabar) un projecte personal de la mida d'una tesi doctoral, i, també, pel seu suport incondicional en moltes d'altres decisions. Perquè faci el que faci sé que us tinc allà. Perquè us estimo.

Jordi Taixés
July 2011

Resum

Es defineix el *mètode de Newton* associat a una funció en una variable complexa f com el sistema dinàmic

$$N_f(z) = z - \frac{f(z)}{f'(z)}\,.$$

Com a algoritme per trobar arrels de funcions, una qüestió fonamental és entendre la dinàmica de N_f al voltant dels seus punts fixos, ja que corresponen a les arrels de la funció f. En altres paraules, volem entendre les *conques d'atracció* de N_f, és a dir, aquells conjunts de punts que convergeixen a les arrels de f sota la iteració de N_f.

Per altra banda, les conques d'atracció només són un tipus de component estable o component del *conjunt de Fatou* $\mathcal{F}(f)$, que es defineix com el conjunt de punts $z \in \widehat{\mathbb{C}}$ per als quals la família $\{f^n\}_{n\geq 1}$ està definida i és normal en un entorn de z. El *conjunt de Julia* o conjunt de caos és el seu complementari, $\mathcal{J}(f) = \widehat{\mathbb{C}} \setminus \mathcal{F}(f)$. (En aquestes definicions i a partir d'ara, $\widehat{\mathbb{C}}$ es refereix a l'*esfera de Riemann*, és a dir, la superfície de Riemann compacta $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.)

L'estudi de la topologia d'aquests dos conjunts és un dels temes centrals de la Dinàmica Holomorfa. Per al cas particular del mètode de Newton, Feliks Przytycki [35] va demostrar que, donada qualsevol arrel d'un polinomi P, la seva conca d'atracció com a punt fix de N_P és simplement connexa. Hans-Günter Meier [33] va demostrar que el conjunt de Julia del mètode de Newton d'un polinomi de grau 3 és connex, i més tard Tan Lei [43] va generalitzar aquest resultat a polinomis de grau superior. L'any 1990, Mitsuhiro Shishikura [40] va demostrar el resultat que és de fet la base d'aquest treball: Si P és un polinomi no constant, llavors $\mathcal{J}(N_P)$ és connex (o, equivalentment, totes les components de $\mathcal{F}(N_P)$ són simplement connexes). De fet, Shishikura va demostrar aquest resultat com a conseqüència d'un teorema molt més general sobre funcions racionals, que enunciem tot seguit.

Teorema A (Shishikura [40]). *Si el conjunt de Julia d'una funció racional R és no connex, llavors R té almenys dos punts fixos feblement repulsors.*

Denotem per *punt fix feblement repulsor* un punt fix que és o bé repulsor o bé parabòlic de multiplicador 1. És un resultat de Pierre Fatou [25] que tota funció racional té almenys un punt d'aquest tipus.

Vegem ara com d'aquest resultat més general es dedueix l'anterior sobre el mètode de Newton. Si P és un polinomi, llavors N_P és una funció racional que té per punts fixos exactament les arrels de P, més el punt ∞. D'aquests, tots els punts fixos finits resulten ser atractors (o bé fins i tot superatractors en cas que com a arrel de P siguin simples), i ∞ és l'únic punt fix repulsor. Per tant, si les funcions racionals que provenen d'aplicar el mètode de Newton a un polinomi només tenen un punt fix feblement repulsor, forçosament pel Teorema A el seu conjunt de Julia ha de ser no connex.

El nostre objectiu és donar les versions transcendents naturals dels resultats de Shishikura sobre funcions racionals i polinomis, és a dir, demostrar la conjectura següent.

Conjectura A. *Si el conjunt de Julia d'una funció meromorfa transcendent f és no connex, llavors f té almenys un punt fix feblement repulsor.*

Per entendre bé aquesta afirmació, és important observar que una singularitat essencial d'una funció meromorfa transcendent f es troba sempre en el seu conjunt de Julia, de manera que ∞ pot connectar components connexes de $\mathcal{J}(f) \cap \mathbb{C}$ que d'altra manera serien no connexes.

Per altra banda, cal observar també que el resultat de Fatou sobre punts fixos feblement repulsors és específic de les funcions racionals, i que en les funcions transcendents la singularitat essencial juga d'alguna manera el seu paper. Amb un raonament sobre els punts fixos del mètode de Newton anàleg al d'abans es dedueix que el mètode de Newton d'una funció entera transcendent no té cap punt fix feblement repulsor, de manera que utilitzant la Conjectura A s'obté aquest corol·lari.

Conjectura B (Corol·lari). *El conjunt de Julia del mètode de Newton d'una funció entera transcendent és connex.*

L'estratègia per demostrar la Conjectura A és la següent: Com que el conjunt de Julia és el complementari del conjunt de Fatou, la connexitat de $J(f)$ està directament relacionada amb la connexitat simple de $\mathcal{F}(f)$. Més concretament, el conjunt de Julia de f és no connex si, i només si, alguna component connexa del seu conjunt de Fatou és no simplement connexa. Com veurem tot seguit, el nombre de possibles components de Fatou no simplement connexes és prou petit com perquè separar la demostració del resultat global en diferents casos particulars segons les components de Fatou sigui una estratègia viable.

S'entén que quan parlem de *component de Fatou* ens referim a una component del conjunt de Fatou, és a dir, a un domini de normalitat dels iterats de f maximal. La vora de cadascuna de les components de Fatou pertany al conjunt de Julia, mentre que, en el seu interior, les òrbites dels punts es comporten de manera similar. La rigidesa de l'estructura complexa en les funcions holomorfes i meromorfes fa que el nombre de possibles comportaments asimptòtics dels punts en un domini maximal sigui petit, i això permet fer una classificació completa dels tipus de components de Fatou.

Definició A. Sigui f una funció en una variable complexa i U una component de Fatou de f. Diem que U és *preperiòdica* si existeixen enters $n > m \geq 0$ tals que $f^n(U) = f^m(U)$. Diem que U és *periòdica* si $m = 0$, i que és *fixa* si $n = 1$. S'anomena *domini errant* a una component de Fatou que no sigui preperiòdica.

Notació. Direm que una component de Fatou és *n-periòdica* si és periòdica de període mínim n.

A la seva vegada, les components de Fatou periòdiques es classifiquen com segueix. Aquesta classificació va ser donada essencialment per Fatou i Hubert Cremer, i es troba per primera vegada en aquesta forma a [6].

Teorema B. *Sigui U una component de Fatou p-periòdica d'una funció en una variable complexa f. Llavors, U és un dels casos següents.*

Conca d'atracció immediata. *U conté un punt atractor p-periòdic z_0 tal que $\lim_{n\to\infty} f^{np}(z) = z_0$ per a qualsevol $z \in U$.*

Conca parabòlica *o* **domini de Leau.** *∂U conté un punt q-periòdic z_0, amb $q|p$, tal que $\lim_{n\to\infty} f^{nq}(z) = z_0$ per a qualsevol $z \in U$. A més, es té que $(f^p)'(z_0) = 1$.*

Disc de Siegel. *Existeix un homeomorfisme holomorf $\phi\colon U \to \mathbb{D}$ tal que $(\phi \circ f^p \circ \phi^{-1})(z) = e^{2\pi i\theta}z$, per a algun $\theta \in \mathbb{R} \setminus \mathbb{Q}$.*

Anell de Herman. *Existeix un real $r > 1$ i existeix un homeomorfisme holomorf $\phi\colon U \to \{1 < |z| < r\}$ tal que $(\phi \circ f^p \circ \phi^{-1})(z) = e^{2\pi i\theta}z$, per a algun $\theta \in \mathbb{R} \setminus \mathbb{Q}$.*

Domini de Baker. *∂U conté un punt z_0 tal que $\lim_{n\to\infty} f^{np}(z) = z_0$ per a qualsevol $z \in U$, però la imatge $f^p(z_0)$ no està definida.*

En el nostre cas ens interessen només les components de Fatou no simplement connexes, que d'entrada exclouen el cas del disc de Siegel. Per altra banda, si una component de Fatou preperiòdica caigués en un cicle de components periòdiques també no simplement connexes, llavors el cas quedaria automàticament reduït a algun dels casos de la classificació de les components de Fatou periòdiques. D'aquesta manera, n'hi ha prou amb considerar només el cas d'aquelles components de Fatou (preperiòdiques) que tinguin per imatge una component simplement connexa.

Tenint en compte aquestes observacions, podem reescriure la Conjectura A com segueix, tot utilitzant la classificació de les components de Fatou que acabem de descriure.

Conjectura C. *Sigui f una funció meromorfa transcendent. Llavors,*

1. *si f té una conca d'atracció immediata no simplement connexa; o bé*
2. *si f té una conca parabòlica no simplement connexa; o bé*
3. *si f té un anell de Herman; o bé*
4. *si f té un domini de Baker no simplement connex; o bé*
5. *si f té un domini errant no simplement connex; o bé*
6. *si f té una component de Fatou no simplement connexa U tal que $f(U)$ és simplement connexa,*

llavors, f té almenys un punt fix feblement repulsor.

Cal dir que el cas 5 dels dominis errants va ser demostrat per Walter Bergweiler i Norbert Terglane [9] com a eina per trobar solucions de certes equacions diferencials sense dominis errants. La seva demostració es basa en la tècnica que utilitza Shishikura per demostrar el cas racional.

Dels cinc restants, en aquesta Tesi demostrem els casos 1, 2 i 6 (vegeu també [23, 24]), i donem una idea per a la demostració del cas 3. La demostració completa del cas dels anells de Herman i el cas dels dominis de Baker queden, doncs, com a treball en curs i per a un futur projecte. La demostració dels casos 1, 2, i 6 és, per tant, el resultat central d'aquesta Tesi.

Teorema Principal. *Sigui f una funció meromorfa transcendent amb o bé una conca d'atracció immediata no simplement connexa, o bé una conca parabòlica no simplement connexa, o bé una component de Fatou no simplement connexa amb imatge simplement connexa. Llavors, f té almenys un punt fix feblement repulsor.*

Passem ara a donar una idea de la demostració d'aquest Teorema Principal, que es basa fonamentalment en dues tècniques: la *cirurgia quasiconforme* i l'estudi de l'existència de *punts fixos virtualment repulsors* mitjançant un teorema de Xavier Buff, entre d'altres resultats. De la definició d'aquests punts i d'aquests resultats en parlarem després d'una breu introducció a la cirurgia quasiconforme.

El que avui dia es coneix en la literatura de Dinàmica Holomorfa amb el nom de 'cirurgia quasiconforme' és una tècnica per construir funcions holomorfes que tinguin una certa dinàmica prefixada. El terme 'cirurgia' suggereix que una part important del procés consistirà en retallar i cosir certs espais i certes funcions per tal d'aconseguir aquest comportament desitjat. Aquest primer pas es coneix amb el nom de *cirurgia topològica*. Per altra banda, l'adjectiu 'quasiconforme' indica que la funció que construïrem en aquest primer pas és òbviament no holomorfa, ja que en el procés de retallar i cosir funcions n'obtindrem una de regularitat inferior. El segon pas del procés consisteix, doncs, en trobar una funció *conjugada* a aquesta funció de regularitat inferior (és a dir, que tingui la mateixa dinàmica que ella), i això s'aconsegueix fent servir el cèlebre Teorema de l'Aplicació de Riemann Mesurable. Aquest segon pas es coneix amb el nom de *suavització holomorfa*.

Les aplicacions quasiconformes van ser introduïdes en la Dinàmica Complexa el 1981 per DENNIS SULLIVAN [42] en un seminari a l'Institut des Hautes Études Scientifiques de París, i molt aviat va ser reconeguda pels dinamicistes com una eina remarcable. Com a exemple, ADRIEN DOUADY i JOHN HUBBARD van desenvolupar tota la teoria d'*aplicacions quasi-polinòmiques* (vegeu [19]) fent servir aplicacions quasiconformes, i més tard SHISHIKURA va donar un gran impuls a la cirurgia quasiconforme tot trobant-ne noves aplicacions a les funcions racionals (vegeu [39]). Així és com es defineixen.

Definició B. Siguin U i V conjunts oberts de $\mathbb{C}$. Diem que un homeomorfisme $\phi\colon U \to V$ és K-*quasiconforme* si té derivades febles de quadrat integrable localment, i la funció

$$\mu_\phi(z) := \frac{\partial\phi/\partial\overline{z}}{\partial\phi/\partial z}(z)$$

satisfà que

$$|\mu_\phi(z)| \leq \frac{K-1}{K+1} < 1$$

gairebé a tot arreu.

La funció mesurable μ_ϕ representa, de fet, un camp d'el·lipses mesurable, i la condició $|\mu_\phi(z)| \leq (K-1)/(K+1) < 1$ vol dir que l'el·lipticitat del camp és fitada. La pròpia definició d'aplicació quasiconforme ens mostra que tot homeomorfisme quasiconforme indueix un camp d'el·lipses mesurable amb el·lipticitat fitada, però aquest és un concepte que també es pot definir independentment de cap aplicació "auxiliar" ϕ.

Definició C. Sigui U un conjunt obert de $\mathbb{C}$. Diem que una funció mesurable $\mu\colon U \to \mathbb{C}$ definida gairebé a tot arreu és un k-*coeficient de Beltrami* d'U si

$$||\mu||_\infty = k := \frac{K-1}{K+1} < 1.$$

Havent vist que tota aplicació quasiconforme ϕ defineix un coeficient de Beltrami μ_ϕ, és natural demanar-se si el recíproc és també cert. És a dir: Donat un coeficient de Beltrami μ i l'anomenada *equació de Beltrami*

$$\frac{\partial\phi}{\partial\overline{z}} = \mu(z)\frac{\partial\phi}{\partial z},$$

podem trobar una aplicació quasiconforme ϕ tal que $\mu_\phi \equiv \mu$? El teorema següent, demostrat per CHARLES MORREY, BOGDAN BOJARSKI, LARS AHLFORS i LIPMAN BERS respon aquesta pregunta afirmativament.

Teorema C. *Sigui μ un k-coeficient de Beltrami de $\mathbb{C}$ (resp. de $\widehat{\mathbb{C}}$ o de $U \cong \mathbb{D}$). Llavors, existeix una aplicació K-quasiconforme $\phi\colon \mathbb{C} \to \mathbb{C}$ (resp. $\phi\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ o $\phi\colon U \to \mathbb{D}$) tal que $\mu_\phi = \mu$, on $K = (1+k)/(1-k)$. A més, ϕ és única llevat de postcomposició amb aplicacions conformes de $\mathbb{C}$ (resp. de $\widehat{\mathbb{C}}$ o de $\mathbb{D}$).*

Tota la tècnica de la cirurgia quasiconforme es basa en aquest potent resultat. Vegem doncs com s'aplica a la Dinàmica Complexa.

De la mateixa manera que una funció holomorfa és una funció que és localment conforme excepte en un nombre discret de punts, diem que una aplicació és *quasiregular* si és localment quasiconforme excepte en un nombre discret de punts. Sigui $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ una aplicació quasiregular la dinàmica de la qual voldríem veure realitzada per una funció holomorfa de $\widehat{\mathbb{C}}$. El lema següent ens mostra que n'hi ha prou amb saber construir el coeficient de Beltrami adequat.

Lema A. *Sigui μ un coeficient de Beltrami de $\mathbb{C}$ i $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ una aplicació quasiregular tal que $f^*\mu = \mu$. Llavors, existeix una funció holomorfa $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ que és quasiconformement conjugada a f. És a dir, existeix un homeomorfisme quasiconforme $\phi\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ tal que la funció $g := \phi \circ f \circ \phi^{-1}$ és holomorfa.*

Aquí, f^* representa el functor contravariant $f^*\colon L^\infty(\widehat{\mathbb{C}}) \to L^\infty(\widehat{\mathbb{C}})$ induït per $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ i definit per

$$f^*\mu := \frac{\partial f/\partial \bar{z} + (\mu \circ f)\overline{(\partial f/\partial z)}}{\partial f/\partial z + (\mu \circ f)\overline{(\partial f/\partial \bar{z})}},$$

que essencialment traslllada (per l'acció de f) el camp d'el·lipses definit per μ (en l'espai tangent de l'espai d'arribada de f) a l'espai tangent de l'espai de sortida de f.

La manera en què nosaltres aplicarem aquests resultats al nostre problema es resumeix d'aquesta manera: Tenim una funció meromorfa transcendent f amb una component de Fatou no simplement connexa U. Si fóssim capaços de convertir f en una funció racional ja hauríem acabat, ja que llavors el resultat de Fatou ens donaria automàticament el punt fix feblement repulsor que volem. Malgrat tot, en les funcions transcendents la singularitat essencial (i, per tant, la dinàmica extremadament caòtica de f al voltant d'aquest punt) fa que la nostra funció estigui lluny de ser racional. Ara bé, fent servir que U és no simplement connexa i que, per tant, existeixen almenys dues components connexes del seu complementari, substituïrem el comportament caòtic de f en la component connexa del complementari de U que contingui la singularitat essencial pel d'alguna funció que sigui senzilla però que coincideixi amb f allà on es produeixi el canvi de funció. A més, construïrem un camp d'el·lipses com el que demana el Lema A per tal que aquest ens doni una funció racional que s'assembla molt a f dinàmicament. Sabem que aquesta funció racional tindrà un punt fix feblement repulsor, i no és difícil veure que aquest punt n'indueix un d'anàleg en f gràcies a la semblança dinàmica.

Pel que fa a l'existència de punts fixos virtualment repulsors, en primer lloc veurem una extensió força intuïtiva del resultat de FATOU per a funcions racionals: Si una funció qualsevol es comporta localment com una funció racional, llavors també aquesta té almenys un punt fix feblement repulsor. Per a això, definim primer aquest concepte de "comportar-se localment com una funció racional" i tot seguit enunciem el resultat, que és del mateix BUFF.

Definició D. Una *aplicació quasi-racional* és una funció holomorfa pròpia $f\colon U \to V$ de grau $d \geq 2$, on U i V són subconjunts oberts i connexos de $\widehat{\mathbb{C}}$ amb característica d'Euler finita que satisfan $\overline{U} \subset V$.

Teorema D (Buff [12]). *Tota aplicació quasi-racional té almenys un punt fix feblement repulsor.*

Recordem que una aplicació $f\colon X \to Y$ és *pròpia* si, per a qualsevol compacte $K \subset Y$, l'antiimatge $f^{-1}(K) \subset X$ és també compacta. Malgrat que les nostres funcions meromorfes transcendents són de grau infinit i per tant no són pas pròpies globalment, sí que poden ser pròpies (i fins i tot quasi-racionals) quan es restringeixen a dominis adequats.

Lema B. *Siguin f una funció meromorfa transcendent, $Y \subset \mathbb{C}$ un conjunt obert i connex, i X una component connexa fitada de $f^{-1}(Y)$. Llavors, la restricció $f|_X\colon X \to Y$ és pròpia. Si, a més, el complementari de Y té un nombre finit de components connexes i $\overline{X} \subset Y$, llavors $f|_X\colon X \to Y$ és quasi-racional.*

Ara bé, la definició d'aplicació quasi-racional conté una hipòtesi molt forta, i és que U ha d'estar compactament contingut en V. Com veurem, el teorema clau de BUFF ens allibera d'aquesta hipòtesi, a canvi que V sigui simplement connex, que és una situació que en molts casos ens serà més fàcil de detectar que la d'aplicació quasi-racional. El resultat del teorema és en realitat l'existència d'un punt fix virtualment repulsor, que és una propietat lleugerament més forta que la de ser feblement repulsor i, per tant, és perfectament aplicable al nostre cas.

El concepte de punt fix virtualment repulsor té el seu origen en els treballs d'ADAM EPSTEIN. La seva definició es basa en la de l'índex holomorf d'un punt fix, que recordem a continuació.

Definició E. L'*índex holomorf* d'un punt fix z d'una funció complexa f és el residu

$$\iota(f,z) := \frac{1}{2\pi i}\oint_z \frac{dw}{w-f(w)}\,.$$

En el cas que el punt fix sigui simple (és a dir, que el seu multiplicador sigui $\lambda(z) \neq 1$), l'índex ve donat per

$$\iota(f,z) = \frac{1}{1-\lambda(z)}\,.$$

El punt fix z s'anomena *virtualment repulsor* si es té que

$$\mathrm{Re}\,(\iota(f,z)) < \frac{m}{2}\,,$$

on $m \geq 1$ denota la multiplicitat del punt fix z.

Com hem dit, el punt clau d'aquesta discussió sobre punts fixos virtualment repulsors és que, en particular, són feblement repulsors. En efecte, si el punt fix és simple (de multiplicitat $m = 1$), llavors

$$\mathrm{Re}\left(\frac{1}{1-\lambda(z)}\right) < \frac{1}{2} \iff |\lambda(z)| > 1,$$

mentre que si és múltiple ($m > 1$), llavors el seu multiplicador és exactament $\lambda(z) = 1$. En qualsevol dels dos casos, el punt fix és també feblement repulsor.

Per altra banda, la propietat de ser virtualment repulsor no és preservada sota conjugació topològica, ja que l'índex holomorf d'un punt només es manté sota conjugació analítica (vegeu [34]). Vegeu també [40] per a una demostració sobre el fet que la propietat de ser feblement repulsor sí que és preservada sota conjugació topològica.

Estem ja en condicions d'enunciar el teorema principal de Buff, sobre punts fixos virtualment repulsors.

Teorema E (Buff [12]). *Siguin $U \subset \mathbb{D}$ un conjunt obert i $f\colon U \to \mathbb{D}$ una aplicació holomorfa pròpia de grau $d \geq 2$. Si $|f(z) - z|$ es manté allunyat de zero quan $z \in U$ tendeix a ∂U, llavors f té almenys un punt fix virtualment repulsor.*

Ja hem dit que si demanem que U estigui compactament contingut en $\mathbb{D}$, llavors f és una aplicació quasi-racional. Si, a més, U és simplement connex, llavors f és una aplicació quasi-polinòmica (vegeu [19]). En aquest cas, pel Teorema de Rectificació, f és híbridament equivalent (i, en particular, quasiconformement conjugada) a un polinomi P en U. Se segueix directament del resultat de Fatou que f té almenys un punt fix feblement repulsor en U.

Finalment, recordem que les nostres funcions no són pas racionals, de manera que ens cal adaptar aquest resultat a la nostra situació. El corol·lari següent serà el resultat que habitualment farem servir en les nostres demostracions.

Corol·lari A. *Siguin $U \subset V \subset \mathbb{C}$ conjunts oberts i sigui $f\colon U \to V$ una funció holomorfa pròpia. Suposem que V és simplement connex i que la vora ∂V és localment connexa en $\widehat{\mathbb{C}}$. Si $|f(z) - z|$ es manté allunyat de zero (en la mètrica esfèrica) quan $z \in U$ tendeix a ∂U, llavors f té almenys un punt fix feblement repulsor en U.*

Recordem que es diu que un espai topològic és *localment connex* si cada punt admet una base d'entorns oberts i connexos. En el nostre cas, és necessari que controlem bé la topologia d'alguns dels conjunts que construïrem, ja que no és trivial, en la presència d'una singularitat essencial, que els conjunts conservin la seva regularitat en ser iterats per una funció transcendent. En particular, voldrem utilitzar el Teorema de Carathéodory per veure que unes certes aplicacions de Riemann extenen a la vora de forma contínua, i precisament això passa quan la vora del seu domini de definició és localment connexa.

La manera com utilitzarem aquests resultats sobre punts fixos feblement repulsors i punts fixos virtualment repulsors és clara, ja que en alguns casos serà possible trobar un conjunt $V \subset \widehat{\mathbb{C}}$ que contingui una antiimatge U d'ell mateix a dins seu, i veient que aquests conjunts es troben en les hipòtesis dels resultats que acabem de veure, obtindrem el punt fix feblement repulsor que volem.

Les tècniques de cirurgia quasiconforme i punts fixos virtualment repulsors es van alternant indistintament arreu dels casos de components de Fatou que demostrem, segons les hipòtesis que anem afegint en cada subcas, mentre que la demostració de Bergweiler i Terglane del cas dels dominis errants i la idea de la demostració del cas dels anells de Herman utilitzen només la cirurgia proposada per Shishikura en el cas racional.

Segurament aquestes tècniques també eliminaran una bona part de subcasos del cas dels dominis de Baker, però també pot molt ben ser que només amb elles no n'hi hagi prou per completar la demostració i s'hagi de fer servir encara algun procediment diferent. El seu estudi queda obert per a un projecte futur.

Contents

1 Introduction

In the year 1669, a young fellow of the Trinity College of the University of Cambridge presented a treatise on quadrature of simple curves and on resolution of equations. Concerning the latter, he wrote: "Because the whole difficulty lies in the resolution, I shall first illustrate the method I use in a numeral equation," and the procedure he described next became the germ of possibly the most powerful root-finding algorithm used today. The young fellow was Isaac Newton and the treatise was *De analysi per æquationes numero terminorum infinitas*, one of his most celebrated works.

Using the "numeral equation" $y^3 - 2y - 5 = 0$, Newton then illustrates his resolution method as follows: He proposes the number 2 as an initial guess of the solution which differs from it by less than a tenth part of itself. Calling p this *small* difference between 2 and the solution y, he writes $2 + p = y$ and substitutes this value in the equation, which gives a new equation to be solved: $p^3+6p^2+10p-1 = 0$. Since p is small, the higher order terms $p^3 + 6p^2$ are quite smaller relatively, therefore they can be neglected to give $10p - 1 = 0$, from where $p = 0.1$ may be taken as an initial guess for the solution of the second equation. Now, it is clear how the algorithm continues, since, writing $0.1 + q = p$ and substituting this value in the second equation, a third equation $q^3 + 6.3q^2 + 11.23q + 0.061 = 0$ is obtained, and so on.

Using this method, Newton constructs a sequence of polynomials, plus a sequence of root approximations that converge to 0 and add up to the solution of the original equation.

A couple of decades later, Joseph Raphson discussed Newton's recurrence and improved the method by using the concept of derivative of a polynomial. It was in 1740 that Thomas Simpson described the algorithm as an iterative method for solving general nonlinear equations using fluxional calculus, essentially obtaining the well-known formula $x_{n+1} = x_n - f(x_n)/f'(x_n)$ for finding the roots of a function f. In the same publication, Simpson also gave the generalisation to systems of two equations and noted that the method can be used for solving optimisation problems. Today, the so-called *Newton's method* (or *Newton-Raphson method*) is probably the most common — and usually efficient — root-finding algorithm.

As in the previous example, Newton's method is frequently used to solve problems of real variable — either in dimension one or greater —, although the plane of complex numbers is often the natural environment provided that the functions to be dealt with do have a certain regularity. Already in 1879, Arthur Cayley applied Newton's method to complex polynomials and tried to identify the basins of attraction of its roots. Cayley did provide a neat solution for this problem in the case of quadratic polynomials, but the cubic case appeared to be far more difficult — and a few years later he finally gave only partial results. Today, it is enough to see the pictures of a few cubic polynomials' dynamical planes to understand why Cayley was never able to work out such a complex structure with the mathematical tools of 125 years ago.

Newton's method associated to a complex holomorphic function f is then defined by the dynamical system

$$N_f(z) = z - \frac{f(z)}{f'(z)}\,.$$

A natural question is what kind of properties we might be interested in or, put more generally, what kind of study we want to make of it. From the dynamical point of view, and given the purpose of any root-finding algorithm, a fundamental question is to understand the dynamics of N_f about its fixed points, as they correspond to the roots of the function f; in other words, we would like to understand the *basins of attraction* of N_f, the sets of points that converge to a root of f under the iteration of N_f.

Basins of attraction are actually just one type of *stable* component or component of the *Fatou set* $\mathcal{F}(f)$, the set of points $z \in \widehat{\mathbb{C}}$ for which $\{f^n\}_{n\geq 1}$ is defined and normal in a neighbourhood of z (recall $\widehat{\mathbb{C}}$ stands for the *Riemann sphere*, the compact Riemann surface $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$). The *Julia set* or *set of chaos* is its complement, $J(f) := \widehat{\mathbb{C}} \setminus \mathcal{F}(f)$. These two sets are named after the French mathematicians Pierre Fatou and Gaston Julia, whose work began the study of modern Complex Dynamics at the beginning of the 20th century.

At first, one could think that if the fixed points of N_f are exactly the roots of f, then Newton's method is a neat algorithm in the sense that it will always

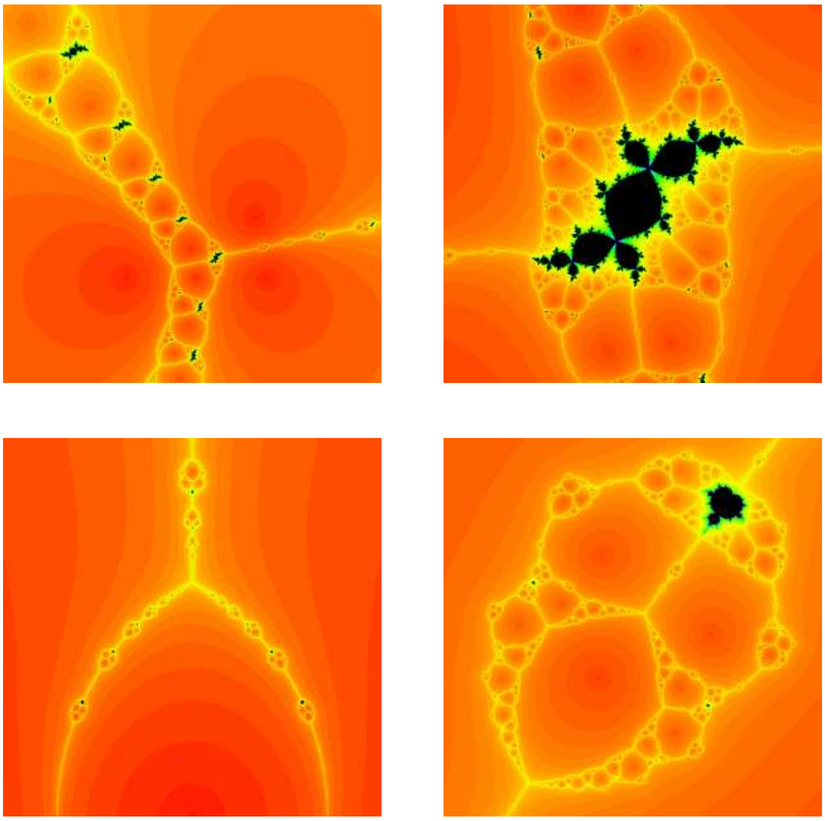


Figure 1.1: The two images above are the dynamical plane of $f_a(z)$ for $a = 0.913+0.424\,i$, and the images below are the parameter space of this family. The black regions on the right-hand pictures (magnifications of the other two) indicate the values of non-convergence. The parameter a has been chosen so that there exists an attracting periodic orbit of period 6.

converge to one of the roots. But notice that not every stable component is a basin of attraction; even not every attracting behaviour is suitable for our purposes: Basic examples like Newton's method applied to cubic polynomials of the form $f_a(z) = z(z-1)(z-a)$, for certain values of $a \in \mathbb{C}$, lead to open sets of initial values converging to attracting periodic cycles. Actually, also the set of such parameters $a \in \mathbb{C}$, for this family of functions, is an open set of the corresponding parameter space (see [15, 19]). Figure 1.1 shows both phenomena in coloured complex planes. Different colours represent different rates of convergence towards the roots of f_a, while black means either convergence somewhere else or non-convergence.

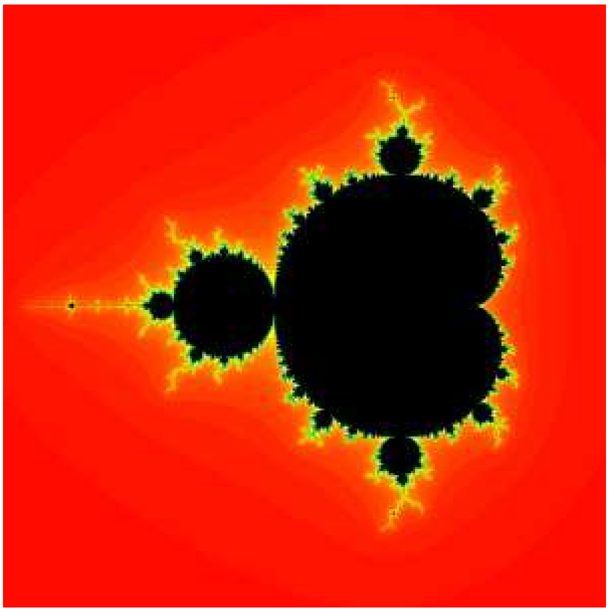


Figure 1.2: The Mandelbrot set.

These facts suggest a division between two directions of dynamical study: On the one hand, given a certain function f, we can try to understand the general behaviour of points under iterates of f, that is to say, the study of its stable and chaos sets — the *dynamical plane*. On the other hand, if we have a family of functions depending on one or several parameters, we might then be interested in knowing for which values of the parameter(s) a certain property occurs — the *parameter space*. A well-known example of this division is given by the family of quadratic polynomials $f_c(z) = z^2 + c$, $c \in \mathbb{C}$, for which the dichotomy between connected and totally disconnected Julia sets has been proved. In this case, the parameter space shows the *Mandelbrot set*, the locus of polynomials $f_c(z)$ with connected Julia set (see Figure 1.2).

The fixed points of N_f are the roots of the function f and the poles of f', since

$$N_f(z) = z \iff z - \frac{f(z)}{f'(z)} = z \iff \frac{f(z)}{f'(z)} = 0\,.$$

When the method is applied to a polynomial, infinity becomes a fixed point as well, whereas if N_f is transcendental, this point is an essential singularity. In Lemma 1.1 we will see when this case occurs.

As for their behaviour, if we compute Newton's method's derivative we have

$$N_f' = 1 - \frac{(f')^2 - f \cdot f''}{(f')^2} = \frac{f \cdot f''}{(f')^2}\,,$$

which means that simple roots of f are superattracting fixed points of N_f. This is an extraordinary property from the point of view of root-finding algorithms, as it is equivalent to say that, in a neighbourhood of such points, N_f is conjugate to $z \mapsto z^k$, for some $k > 1$, for which local convergence is very fast.

Multiple roots of f are attracting fixed points of N_f, but no longer superattracting. In fact, their multiplier is $(m-1)/m$, where m is the multiplicity of the root, so in this case the rate of attraction is linear.

When Newton's method is applied to a polynomial P of degree d, the point at infinity has multiplier $N_P'(\infty) = d/(d-1)$, so it is repelling — in particular, weakly repelling.

Notice that the critical points of N_f are the simple roots of f, as well as its inflection points $\{z \in \widehat{\mathbb{C}} : f''(z) = 0\}$. Of course, every simple root of f is both a critical point and a fixed point of N_f, but inflection points of f become *free* critical points of N_f, which can lead to undesirable Fatou components (as mentioned earlier). From the root-finding point of view, some tools have been developed to cope with this kind of situations: Given a polynomial P, one can find explicitly a finite set of points such that, for every root of P, at least one of the points will converge to this root under N_P (see [29]).

Now let us focus our attention on the case in which f is transcendental. We have the following result (see [8]).

Lemma 1.1. *If a complex function f is transcendental, then so is N_f, except when f is of the form $f = R\,e^P$, with R rational and P a polynomial. In this case, N_f is a rational function.*

The dynamical system N_f for functions of the form $f = R\,e^P$ has also been investigated, especially when f is entire, i.e., of the form $f = Pe^Q$, where P and Q are polynomials. Mako Haruta [28] proved that, if $\deg Q \geq 3$, the area of the basins of attraction of the roots of f is finite. Figen Çilingir and Xavier Jarque [14] studied the area of the basins of attraction of the roots of f in the case $\deg Q = 1$, and Antonio Garijo and Jarque [26] extended the previous results in the cases $\deg Q = 1$ and $\deg Q = 2$. For yet another reference on the subject, see also [30].

It is worth saying that there exist a number of variations of Newton's method, which can improve its efficiency in some cases. One of the most usual versions is the *relaxed Newton's method*, which consists in the iteration of the map $N_{f,h} = \mathrm{id} - h \cdot f/f'$, where h is a fixed complex parameter. In general, for certain choices of rational functions R and parametres h, the method has additional attractors, which causes the algorithm not to work reliably. Nevertheless, it has been proved in [44] that, for almost all rational functions R, the additional attractors vanish if h is chosen sufficiently small.

A lot of literature concerning Newton's method's Julia and Fatou sets has been written, above all when applied to algebraic functions. Feliks Przytycki [35] showed that every root of a polynomial P has a simply connected immediate basin of attraction for N_P. Hans-Günter Meier [33] proved the connectedness of the Julia set of N_P when $\deg P = 3$, and later Tan Lei [43] generalised this result to higher degrees of P. In 1990, Mitsuhiro Shishikura [40] proved the result that actually sets the basis of the present work: For any non-constant polynomial P, the Julia set of N_P is connected (or, equivalently, all its Fatou components are simply connected). In fact, he obtained this result as a corollary of a much more general theorem for rational functions. We denote by a *weakly repelling fixed point* a fixed point which is either repelling or parabolic of multiplier 1 (see Subsection 2.1.1). It was proven by Fatou that every rational function has at least one weakly repelling fixed point (see Theorem 2.6).

Theorem 1.2 (Shishikura [40]). *If the Julia set of a rational function R is disconnected, then R has at least two weakly repelling fixed points.*

Let us see how this applies to Newton's method. If P is a polynomial, then N_P is a rational function whose fixed points are exactly the roots of the polynomial P, plus the point at infinity. The finite fixed points are all attracting, even superattracting if, as roots of P, they are simple. The point at infinity, instead, is always repelling. Hence, rational functions arising from Newton's methods of polynomials have exactly one weakly repelling fixed point and, in view of Theorem 1.2, their Julia set must be connected.

This Thesis, however, deals with Newton's method applied to transcendental maps. In the same direction, in 2002 Sebastian Mayer and Dierk Schleicher [32] extended Przytycki's theorem by showing that every root of a transcendental entire function f has a simply connected immediate basin of attraction for N_f. This work has been recently continued by Johannes Rückert and Schleicher in [38], where they study Newton maps in the complement of such Fatou components. Our long-term goal is to prove the natural transcendental versions of Shishikura's results — although this Thesis covers just part of it —, which can be conjectured as follows.

Conjecture 1.3. *If the Julia set of a transcendental meromorphic function f is disconnected, there exists at least one weakly repelling fixed point of f.*

It is important to notice that essential singularities are always in the Julia set of a transcendental meromorphic function f and therefore infinity can connect two unbounded connected components of $\mathcal{J}(f) \cap \mathbb{C}$ otherwise disconnected.

Observe that Fatou's theorem on weakly repelling fixed points only applies to rational maps. For transcendental maps, the essential singularity at infinity plays the role of the weakly repelling fixed point, and therefore no such point must necessarily be present for an arbitrary map. From this fact, and from the discussion above about Newton's method, it follows that transcendental meromorphic functions that come from applying Newton's method to transcendental entire functions happen to have no weakly repelling fixed points at all, so the next result is obtained forthwith.

Conjecture 1.4 (Corollary). *The Julia set of the Newton's method of a transcendental entire function is connected.*

Recall that the Julia set (closed) is the complement of the Fatou set (open). Hence, as it was already mentioned, the connectivity of the Julia set is equivalent to the simple connectivity of the Fatou set. Because of this fact, a possible proof of Conjecture 1.3 splits into several cases, according to different Fatou components (see Section 3.2). In this Thesis we will see three of such cases (see [23, 24]), which, together, give raise to the following result.

Main Theorem 1.5. *Let f be a transcendental meromorphic function with either a multiply-connected attractive basin, or a multiply-connected parabolic basin, or a multiply-connected Fatou component with simply-connected image. Then, there exists at least one weakly repelling fixed point of f.*

Notice how this theorem actually connects with the result of Mayer and Schleicher mentioned above.

In order to prove this theorem, we use mainly two tools: the method of *quasiconformal surgery* and a theorem of Xavier Buff on *virtually repelling fixed points*. On the one hand, quasiconformal surgery (see Section 2.4) is a powerful tool that allows to create holomorphic maps with some prescribed dynamics. One usually starts glueing together — or *cutting* and *sewing*, this is why this procedure is called 'surgery' — several functions having the required dynamics; in general, the map f obtained is not holomorphic. However, if certain conditions are satisfied, the Measurable Riemann Mapping Theorem, due to Charles Morrey, Bogdan Bojarski, Lars Ahlfors and Lipman Bers, can be applied to find a holomorphic map g, conjugate to the original function g. On the other hand, Buff's theorem states that, under certain local conditions, a map possesses a virtually repelling fixed point. These conditions are a generalization of the polynomial-like setup and the property of being a virtually repelling fixed point is only slightly stronger than that of weakly repelling. Hence in those cases where we can apply Buff's theorem, the result follows in a very direct way.

Structure of the Thesis. This Introduction puts the subject of the Thesis into historical context and gives a little state of the art about the study of the topology of the Fatou and Julia sets of the dynamical system generated by applying Newton's method to polynomials and transcendental entire functions. In particular, it gives Shishikura's main result and our 'transcendental' conjectures and Main

Theorem. Chapter 2 provides us with some background tools from various topics in Holomorphic Dynamics, to be used in the following chapters. These topics range from pure Dynamical Systems stuff, such as basics on iteration theory or the classification of Fatou components, to concepts coming from other fields, like quasiconformal surgery from Analysis or local connectivity from Topology. In these 'borrowed stuff' cases we will see how such concepts are adapted to Holomorphic Dynamics and become actual tools in our context. Sections 3, 4 and 5 contain our proof for our Main Theorem, separated by type of Fatou component. Thus, Section 3 is dedicated to the proof for the case of immediate attractive basins, Section 4 to parabolic basins and Section 5 to preperiodic Fatou components. Also, what actually opens Section 3 is a preamble with Shishikura's proof for the attractive rational case plus an introduction to the general transcendental case that tells how our main conjecture splits into the different Fatou-component cases. Finally, Section 6 rounds up our global case-by-case discussion with a collection of results and ideas about wandering domains, Herman rings and Baker domains, for completeness. The section concludes with some remarks about future projects and further work on the subject.

2
Preliminaries and tools

In this chapter we provide some general background in various topics in holomorphic dynamics, to be used in the following chapters of the Thesis. The first section contains some of the basics on iteration theory and holomorphic dynamics; the second section gives a general summary to the technique of quasiconformal surgery, a powerful tool in the field of Complex Dynamics that allows to construct holomorphic maps having some prescribed dynamics; in the third section we find a few words on the topological concept of local connectivity; finally, the last section is devoted to state some theorems that guarantee, for a holomorphic function, to have a repelling or weakly repelling fixed point.

2.1 Background on holomorphic dynamics

In this section we give the basic concepts in holomorphic dynamics that we will be using all the time later on, such as iteration, multiplier of a periodic point or Fatou and Julia sets, to give only a few examples. Also, we will give the formal definitions for many of the concepts that already appeared in Chapter 1. For detailed introductions to holomorphic dynamics we refer to the books [34, 13, 7], to mention only a few.

This section is in turn divided into three subsections, where the topics are basics on iteration theory, the Fatou and Julia sets, and the relationship between Fatou components and singular values.

2.1.1 Basics on iteration theory

We shall work with three types of maps or functions: rational, i.e., holomorphic on the *Riemann sphere* $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, transcendental entire and transcendental meromorphic.

Definition 2.1 (**Transcendental function).** A complex function f is *transcendental* if it has at least one essential singularity. By an *entire (transcendental)* map we mean functions which are holomorphic in $\mathbb{C}$ and have an essential singularity at infinity. We denote by *meromorphic (transcendental)* maps with an essential singularity at infinity, having at least one pole which is not an ommitted value.

We refer to [8] for a general discussion on transcendental maps. In order to avoid repetition, we will use the term *complex function* or *complex map* to refer to a map in any of the classes above.

We write f^n for the nth iterate of f, that is, $f^0(z) = z$ and $f^n(z) = f(f^{n-1}(z))$ when $n \geq 1$. Our maps are in general non-invertible. Hence when we write f^{-n}, we mean it in the sense of sets, that is, $f^{-n}(A)$ denotes the set of points whose nth image belongs to the set A. Sometimes, however, we might use $(f^n)^{-1}$ to denote some particular local inverse branch of f^n.

For a given point z, the sequence

$$O^+(z) := \{z, f(z), f^2(z), f^3(z), \ldots\}$$

is called the *(forward) orbit* of the point z. The *backward orbit* of z, $O^-(z)$, is given by the set

$$O^-(z) = \bigcup_{n \geq 0} f^{-n}(z).$$

We say that $z \in \widehat{\mathbb{C}}$ is *exceptional* if $O^-(z)$ is finite. It is not difficult to see that a complex function f has at most two exceptional points. If f is rational, its exceptional points must belong to the Fatou set. If f is transcendental entire, then infinity is always one of them, so f has at most one more exceptional point, finite, and it can belong to either Fatou or Julia sets (an example of this is $z = 0$ for the exponential map). Finally, if f is transcendental meromorphic, then infinity does have preimages at the poles of f, therefore f has at most two finite exceptional points.

Among all points z in the domain of definition of the complex function f, the periodic points play an important role in the study of f as a dynamical system.

Definition 2.2 (**Periodic point).** Let f be a complex function. For $p \geq 1$, we say that z is a p-periodic point (or a periodic point of minimal period p) if $f^p(z) = z$ and $f^k(z) \neq z$ for $k < p$. We say that z is periodic if it is p-periodic for some p. If $p = 1$, we call z a *fixed point*. We say that z is (strictly) preperiodic if $f^k(z)$ is a periodic point, for some $k > 1$, but z itself is not.

Definition 2.3 (**Multiplier**)**.** The *multiplier* of a p-periodic point z of a complex function f is the value

$$\lambda = \begin{cases} (f^p)'(z) & \text{if } z \neq \infty \\ (h \circ f^p \circ h^{-1})'(0) & \text{if } z = \infty, \text{ where } h(z) = 1/z\,. \end{cases}$$

Of course the case $z = \infty$ applies just in the case where $f(\infty)$ is defined, *i.e.*, the rational case.

According to the multiplier, the behaviour of a p-periodic point is classified as follows:

- if $|\lambda| < 1$, z is called *attracting* (*superattracting* if $\lambda = 0$) and for all $w \in U$, a sufficiently small neighborhood of z, we have $f^{pk}(w) \to z$, when $k \to \infty$;
- if $|\lambda| = 1$, z is called *indifferent* (*parabolic* if $\lambda = e^{2\pi i\theta}$, with $\theta \in \mathbb{Q}$);
- if $|\lambda| > 1$, z is called *repelling* and for all $w \in U$, a sufficiently small neighborhood of z, we have $f^{-pk}(w) \to z$, where f^{-p} denotes an appropiate branch of the inverse fixing z.

Note that if z is p-periodic, all other points in the forward orbit of p are also p-periodic, with the same multiplier as z (by the chain rule). We call λ the multiplier of the periodic cycle, and all the statements above apply to each point in the periodic cycle.

Invariant sets are very important in dynamical systems in general and in holomorphic ones in particular.

Definition 2.4 (**Invariant set**)**.** A subset $\mathcal{S} \in \mathbb{C}$ (or $\widehat{\mathbb{C}}$) is called *forward invariant* if $f(S) \subset S$, *backward invariant* if $f^{-1}(S) \subset S$, and *(completely) invariant* if it is both forward and backward invariant.

For instance, the orbit of a point is forward invariant but not backward invariant since, in general, a point has more than one preimage.

As we have already mentioned, in this Thesis, weakly repelling fixed points will play a crucial role.

Definition 2.5 (**Weakly repelling fixed point**)**.** A fixed point is said to be *weakly repelling* if it is either repelling or parabolic with multiplier 1.

The following result guarantees the existence of at least one weakly repelling fixed point for rational maps of degree at least two.

Theorem 2.6 (**Fatou [25]**)**.** *Any rational map of degree greater than one has, at least, one weakly repelling fixed point.*

The rest of this subsection is dedicated to prove this theorem. The proof is based in the Holomorphic Fixed Point Formula and the Rational Fixed Point Theorem. The proof we present is extracted from [34].

The multiplicity of a finite fixed point w of a rational map f ($f(w) = w$) of degree $d \geq 0$ is defined to be the unique integer $m \geq 1$ for which the power series expansion of the function $f(z) - z$ about w has the form

$$f(z) - z = a_m(z-w)^m + a_{m+1}(z-w)^2 + \dots \quad a_m \neq 0$$

We claim that $m \geq 2$ if and only if $f'(w) \neq 1$. To see the claim we just take $g(z) = f(z) - z$ and consider its power series expansion about w, that is,

$$g(z) = g'(w)(z-w) + \frac{1}{2}g''(w)(z-w)^2 + \dots \ .$$

If the fixed point is at infinity we can define the multiplicity similarly by introducing the new coordinates $\eta = 1/z$.

Lemma 2.7 (**Fixed point count**). *If f is a rational function of degree $d \geq 0$ and $f \neq \mathrm{Id}$ then f has exactly $d+1$ fixed points, counting multiplicity.*

Proof. Conjugating, if necessary, by a fractional linear automorphism we may assume that $z = \infty$ is not a fixed point. Then $f(z) = p(z)/q(z)$ where p and q are two polynomials which have no common factors and satisfy $\deg(p) \leq deg(q) = d$. Of course the equation $f(z) = z$ has $d+1$ solutions, counting multiplicity. □

Let $f\colon U \to \mathbb{C}$ be a holomorphic function defined in a connected open set $U \in \mathbb{C}$. Assume there is an isolated $w \in U$ such that $f(w) = w$. The *residue fixed point index* of f at w is defined as

$$\iota(f, w) = \frac{1}{2\pi i} \oint_w \frac{dz}{z - f(z)}\,,$$

where we integrate along a small loop around the fixed point w (in the positive direction).

Lemma 2.8. *Let $f\colon U \to \mathbb{C}$ be a holomorphic function. If w is a fixed point of f with multiplier $\lambda := f'(w) \neq 1$, then*

$$\iota(f, w) = \frac{1}{1-\lambda} \neq 0\,.$$

Proof. Take $w = 0$ and expand f as a power series around 0:

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \dots$$

Since $\lambda \neq 1$, it follows that $z - f(z) = (1-\lambda)z \cdot (1 + O(z))$. Hence

$$\frac{1}{z - f(z)} = \frac{1 + O(z)}{(1-\lambda)z} = \frac{1}{(1-\lambda)z} + O(1)\,.$$

Integrating around the small circle $|w| = \varepsilon$ and using the Residue's Theorem we have

$$\oint_w \frac{dz}{z - f(z)} = \oint_w \frac{dz}{(1-\lambda)z} = \frac{2\pi i}{1-\lambda}\,,$$

as desired. □

Since the residue fixed point index $\iota(f, w)$ is a local concept (around the fixed point w), if we have a (global) rational map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ we can compute the index for the associated local map $z \to f(z)$. It can be proven that $\iota(f, w)$ does not depend on any particular choice of local coordinates.

Theorem 2.9 (Rational Fixed Point Theorem). *For any rational map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, $f \neq \mathrm{id}$, we have the relation*

$$\sum_{\{w = f(w)\}} \iota(f, w) = 1,$$

where the sum runs over all fixed points of f.

Proof. Conjugating, if necessary, by a fractional linear automorphism, we may assume that $f(\infty) \neq \{0, \infty\}$. Then,

$$\frac{1}{z - f(z)} - \frac{1}{z} = \frac{f(z)}{z\,(z - f(z))} \underset{z \to \infty}{\sim} \frac{f(\infty)}{z^2}$$

Integrating along the loop $|w| = r$, it is clear from the previous computations that

$$\oint_w \left(\frac{1}{z - f(z)} - \frac{1}{z}\right) dz = 0$$

if r is sufficiently large. Consequently, it follows from the Residue's Theorem that

$$\frac{1}{2\pi i} \oint_w \frac{dz}{z - f(z)} = \frac{1}{2\pi i} \oint_w \frac{dz}{z} = 1\,.$$

Since, for r sufficiently large, the first term is equal to the sum of the residues $\iota(f, w_k)$ over all fixed points, the result follows. □

Once we know that the sum of the residues over all fixed points in $\widehat{\mathbb{C}}$ of a rational map is 1, in order to prove Theorem 2.6 it will be enough to know the role of the non- weakly repelling fixed points in the sum.

Lemma 2.10. *Let w be a fixed point of f with multiplier $\lambda \neq 1$, and let $\iota(f, w)$ be its residue fixed point index. Then,*

1. *w is attracting if and only if* $\operatorname{Re}(\iota(f, w)) > \frac{1}{2}$*;*

2. *w is indifferent if and only if* $\operatorname{Re}(\iota(f, w)) = \frac{1}{2}$*;*

Proof. For 1, using Lemma 2.8 it suffices to show that w is attracting if and only if $\operatorname{Re}\left(\frac{1}{1-\lambda}\right) > \frac{1}{2}$. We have

$$\operatorname{Re}\left(\frac{1}{1-\lambda}\right) > \frac{1}{2} \iff \frac{1}{1-\lambda} + \frac{1}{1-\overline{\lambda}} > 1 \iff \lambda\overline{\lambda} < 1 \iff |\lambda| < 1,$$

where the second equivalence follows from multiplying both sides of the expression by $(1-\lambda)(1-\overline{\lambda}) > 0$.

For 2, it suffices to change strict inequalities by equalities in the computation above. $\square$

We are now in a position to prove Fatou's theorem.

Proof of Theorem 2.6. If there were no fixed points of multiplier $\lambda = 1$, then there must exist $d + 1$ distinct fixed points $\{w_k\}_k$. If these were all attracting or indifferent, then each of their indexes would satisfy $\operatorname{Re}(\iota(f, w_k)) \geq \frac{1}{2}$ and hence their sum would have real part no smaller than $\frac{d+1}{2} > 1$, a contradiction. $\square$

2.1.2 The Fatou and Julia sets

As explained in Chapter 1, the main goal of Complex Dynamics (and, more generally, of discrete Dynamical Systems) is to have a deep understanding of the asymptotic behaviour of all possible orbits generated by the iterates of a map. As it turns out, the phase portrait of a complex function splits into two totally invariant sets, very different dinamically: the set of initial conditions whose orbit is tame (the Fatou set), and its complement, formed by chaotic orbits (the Julia set).

The right notion to deal with this dichotomy is that of normality of the sequence of iterates, which is deeply related to equicontinuity.

Definition 2.11 (Normal family). Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a family of complex functions. We say that $\mathcal{F}$ is *normal* at a point $z \in \widehat{\mathbb{C}}$ if there exists a neighbourhood U of z such that $\{f_i|_U\}_{i\in I}$ is *equicontinuous*, that is to say, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f_i(z) - f_i(w)| < \varepsilon$ if $|z - w| < \delta$, for all $z, w \in U$ and for all $i \in I$.

Definition 2.12 (Fatou set and Julia set). The *Fatou set* (or *stable set*) of a complex function f is defined by

$$\mathcal{F}(f) = \{z \in \widehat{\mathbb{C}} : \{f^n\}_{n\geq 1} \text{ is defined and normal in a neighbourhood of } z\},$$

and the *Julia set* (or *chaotic set*) is its complement, $\mathcal{J}(f) = \widehat{\mathbb{C}} \setminus \mathcal{F}(f)$.

In order to avoid checking whether the sequence of iterates $\{f^n\}_n$ is equicontinuous at each point, the next theorem due to Paul Montel is a useful criterion to see when a certain set of points belongs to the Fatou set.

Theorem 2.13 (Montel's Theorem). *Let U be an open set of $\widehat{\mathbb{C}}$ and let $\mathcal{F} = \{f_n\colon U \to \widehat{\mathbb{C}}\}_{n\geq 1}$ be a family of holomorphic functions with at least three points which never occur as values. In other words, $f^n(z) \notin \{a, b, c\}$ for any $z \in U$, any $n \geq 1$ and any three different points $a, b, c \in \widehat{\mathbb{C}}$. Then, $\mathcal{F}$ is normal on U.*

In such a situation, we have $U \subset \mathcal{F}(f)$. For instance, if we take $g(z) = z^2$ we have that $\mathcal{F}(g) = \widehat{\mathbb{C}} \setminus \mathbb{S}^1$ and $\mathcal{J}(g) = \widehat{\mathbb{C}} \setminus \mathcal{F}(g) = \mathbb{S}^1$. This is a straightforward consequence of Montel's Theorem 2.13, since the sets $\mathbb{D}$, $\mathbb{S}^1$ and $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ are (completely) invariant, and 0 and ∞ have no preimages other than themselves.

The Fatou and Julia sets possess many interesting dynamical properties, which we summarise in the following lemma.

Lemma 2.14 (Properties of $\mathcal{F}(f)$ and $\mathcal{J}(f)$). *Let f be a complex function of degree $d \geq 2$. Then, the following statements hold.*

1. *$\mathcal{F}(f)$ is open and $\mathcal{J}(f)$ is closed.*
2. *$\mathcal{F}(f)$ and $\mathcal{J}(f)$ are both completely invariant.*
3. *$\mathcal{J}(f)$ is non-empty and* perfect *(that is, it does not contain isolated points). Furthermore, if f is transcendental, then infinity belongs to $\mathcal{J}(f)$, since it is an essential singularity. In particular, if f is transcendental meromorphic, then*
$$\mathcal{J}(f) = \overline{\bigcup_{k\geq 0} f^{-k}(\infty)}\,.$$
4. *$\mathcal{J}(f) = \overline{\bigcup_{k\geq 0} f^{-k}(z)}$ for any non-exceptional $z \in \mathcal{J}(f)$ (and there are at most two exceptional points).*
5. *$\mathcal{J}(f)$ is the closure of the set of repelling periodic points of f.*
6. *Either $\mathcal{J}(f)$ is $\widehat{\mathbb{C}}$ or it has empty interior.*

Sketch of the proof. Statements 1 and 2 follow from the definitions. For 3, the fact that the Julia set of a rational map is non-empty (it is actually infinite) follows from assuming the opposite and then taking $\phi := \lim_{k\to\infty} f^{n_k}$. On one hand, ϕ must be a rational map (defined in $\widehat{\mathbb{C}}$), and, on the other hand, ϕ must be of infinite degree — a contradiction. If f is entire the proof is more elaborated. It is not difficult to see that every transcendental entire function has infinitely many periodic points of all periods greater than 1; therefore, replacing f by f^2 we have that f^2 has infinitely many fixed points. Now it is well known that $\mathcal{J}(f) = \mathcal{J}(f^2)$, so if infinitely many of the fixed points of f^2 are in $\mathcal{J}(f^2)$, we are done. Otherwise, we may assume there exist two fixed points of f^2, p and q, in $\mathcal{F}(f)$. One can see that they cannot belong to the same component of $\mathcal{F}(f)$, so any path connecting p and q should cross $\mathcal{J}(f)$ and therefore $\mathcal{J}(f)$ is an infinite set. Alternatively, Alexandre Erëmenko [20] proved that the Julia set of an entire map is non-empty by showing that its *escaping set* (the set of points whose orbit tends to infinity) is non-empty and has a non-empty intersection with the Julia set. If f is meromorphic it is easy to see that $f^{-3}(\infty)$ is infinite by using Picard's Theorem. Statement 4 follows from Theorem 2.13. Statement 5 for rational functions was first proved by Fatou and Julia independently (and using different approaches). For transcendental functions the proof uses the Five Island Ahlfors's Theorem (see [5]). For 6, it is clear from Theorem 2.13 that if the Julia set contains an open set in $\mathbb{C}$, then $\mathcal{J}(f) = \widehat{\mathbb{C}}$. □

Finally, we observe that the case $\mathcal{J}(f) = \widehat{\mathbb{C}}$ is actually possible: It is easy to prove that any rational map having all its critical points pre-periodic has an empty Fatou set (see, for instance, [7]). The function

$$f(z) = \frac{(z-2)^2}{z^2}$$

is an example of such a rational map. As for transcendental functions, examples of $\mathcal{J}(f) = \widehat{\mathbb{C}}$ are provided by the entire family $f_\lambda(z) = \lambda e^z$ (first proven by Michał Misiurewicz for $\lambda = 1$) (see Figure 2.1) and the meromorphic family $f_\lambda(z) = \lambda \tan z$, for suitable values of the parameter λ.

2.1.3 Fatou components and singular values

As mentioned, points in the Fatou set correspond to tame orbits. This means that points which are close to each other have the same asymptotic behaviour when iterated. Therefore, it is not surprising that the Fatou set (when non-empty) is formed by the union of domains or components called *Fatou components*, which correspond to orbits with a similar behaviour. In fact, Fatou components are maximal domains of normality of the iterates of f. Because of the rigidity of complex functions, there are only a few possible asymptotic behaviours of points

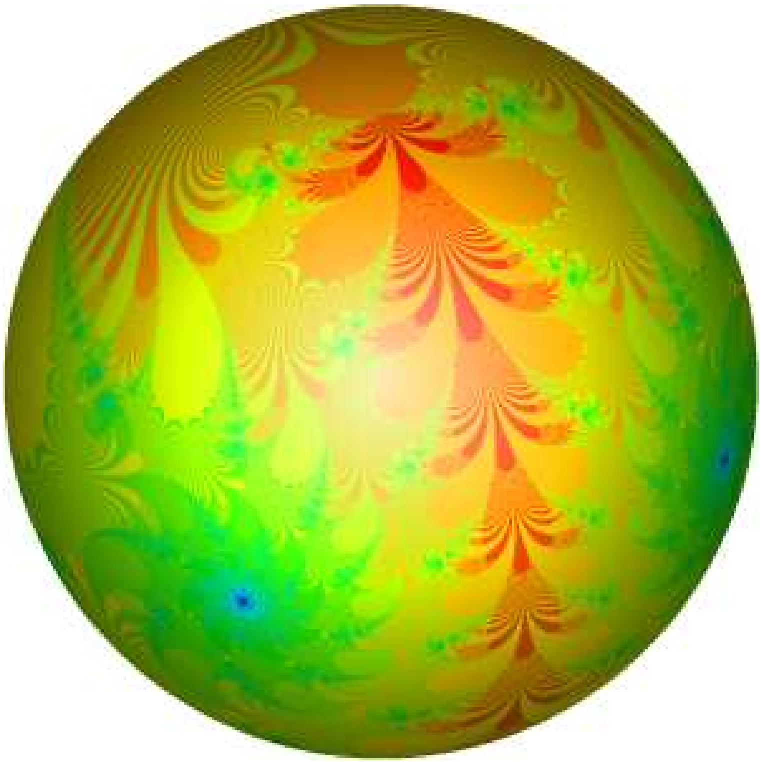


Figure 2.1: Dynamics of the function $f(z) = 0.5\,e^z$ on the Riemann sphere, a case where $\mathcal{J}(f) = \widehat{\mathbb{C}}$. Different colours denote different rates of escape towards infinity.

in a domain of normality, and this makes it possible to give a complete classification of all possible asymptotic behaviours of a Fatou component.

The classification of the Fatou components (together with its close relationship with the singularities of the inverse function) is one of the cornerstones of Holomorphic Dynamics, and it is the subject of this section.

Definition 2.15 (**Types of Fatou components**). Let f be a complex function and U a (connected) component of $\mathcal{F}(f)$; U is called *preperiodic* if there exist integers $n > m \geq 0$ such that $f^n(U) = f^m(U)$. We say that U is *periodic* if $m = 0$, and *fixed* if $n = 1$. A Fatou component is said to be a *wandering domain* if it fails to be preperiodic.

The next classification of periodic Fatou components is essentially due to Fatou and Hubert Cremer, and was first stated in this form in [6].

Theorem 2.16 (Classification of periodic Fatou components). *Let U be a p-periodic Fatou component of a complex function f. Then, U is one of the following.*

- (Immediate) attractive basin*: U contains an attracting p-periodic point z_0 and $f^{np}(z) \to z_0$, as $n \to \infty$, for all $z \in U$.*
- Parabolic basin *or* Leau domain*: ∂U contains a unique p-periodic point z_0 and $f^{np}(z) \to z_0$, as $n \to \infty$, for all $z \in U$. Moreover, $(f^p)'(z_0) = 1$.*
- Siegel disc*: There exists a holomorphic homeomorphism $\phi\colon U \to \mathbb{D}$ such that $(\phi \circ f^p \circ \phi^{-1})(z) = e^{2\pi i\theta} z$, for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$.*
- Herman ring*: There exist $r > 1$ and a holomorphic homeomorphism $\phi\colon U \to \{1 < |z| < r\}$ such that $(\phi \circ f^p \circ \phi^{-1})(z) = e^{2\pi i\theta} z$, for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$.*
- Baker domain*: ∂U contains a point z_0 such that $f^{np}(z) \to z_0$, as $n \to \infty$, for all $z \in U$, but $f^p(z_0)$ is not defined.*

Remark 2.17 (Connectedness of the Julia set). Observe that $\mathcal{J}(f)$ is connected if, and only if, either $\mathcal{F}(f)$ is empty or each one of its connected components is simply connected.

A natural question that arises from this classification is how many Fatou components there are for a given complex function f, and how they are distributed. A key tool to investigate the number and distribution of Fatou components is the study of the singularities of the inverse function f^{-1}.

Definition 2.18 (Critical point and critical value). Let f be a complex function. The point c is a *critical point* if $f'(c) = 0$. Its image $v = f(c)$ is then a *critical value*. We denote the set of critical values by $\mathcal{CR}(f)$.

If f is rational, the critical values are the only possible singularities of the inverse function, since f is a local homeomorphism around every non-critical point of $\widehat{\mathbb{C}}$. That is not the case of transcendental functions, where certain branches of the inverse function might not be defined at points where f^{-1} is unbounded, as the following definition shows.

Definition 2.19 (Asymptotic value). Let f be a complex function. A point $z \in \mathbb{C}$ is a *(finite) asymptotic value* if there exists a curve α such that

$$\lim_{|\alpha| \to \infty} f(\alpha) = \infty\,.$$

We denote the set of asymptotic values of f by $\mathcal{A}(f)$.

Definition 2.20 (**Singular value**)**.** A *singular value* (or *singularity of the inverse function* f^{-1}) is a point that belongs to the set

$$\operatorname{sing}(f^{-1}) := \overline{\mathcal{CR}(f) \cup \mathcal{A}(f)}\,.$$

The set $\operatorname{sing}(f^{-1})$ plays a crucial role in Holomorphic Dynamics since, roughly speaking, every cycle of Fatou components has an "associated" singular value, as the following theorem claims. This result was proved by Fatou for rational maps, and his proof extends naturally to the transcendental case.

Theorem 2.21 (**Fatou components and singular values**)**.** *Let f be a complex function and let $U = \{U_1, \ldots, U_p\}$ be a periodic cycle of Fatou components of f.*

- *If U is a cycle of immediate attractive basins or parabolic basins, then*

  $$U_k \cap \operatorname{sing}(f^{-1}) \neq \emptyset$$

  for some $1 \leq k \leq p$.

- *If U is a cycle of Siegel discs or Herman rings, then*

  $$\partial U_k \subset \overline{O^+(\operatorname{sing}(f^{-1}))}$$

  for all $1 \leq k \leq p$.

Remark 2.22 (**Finite type maps**)**.** Rational functions and transcendental entire functions of *finite type* (that is, with a finite number of singularities of the inverse function) do not have wandering domains nor Baker domains. The absence of wandering domains was proved by Dennis Sullivan [41, 42] for rational functions, and by Erëmenko and Mikhail Lyubich [21, 22] and Lisa Goldberg and Linda Keen [27] for entire maps.

2.2 Quasiconformal surgery

What is known today in Holomorphic Dynamics as *quasiconformal surgery* is a technique to construct holomorphic maps with some prescribed dynamics. The term 'surgery' suggests that certain spaces and maps will be cut and sewed in order to construct the desired behaviour. This is usually the first step of the process and is known as *topological surgery*. On the other hand, the adjective 'quasiconformal' indicates that the map one constructs in this first step is not holomorphic, but of lesser regularity. The second step is then to find a conjugate map (that means a map with the same dynamics) which is holomorphic, and this is done using the celebrated Measurable Riemann Mapping Theorem, the powerful tool which makes this technique possible. This second step is called *holomorphic smoothing*.

Quasiconformal mappings were first introduced in Complex Dynamics in 1981 by Sullivan [42] in a seminar at the Institut des Hautes Études Scientifiques (Paris), and very soon adopted by mathematicians in the field as a remarkable tool. As an example, Adrien Douady and John Hubbard developed the well-known theory of polynomial-like mappings (see [19]) using quasiconformal mappings, and Shishikura gave a great impulse to quasiconformal surgery by finding new applications to rational functions (see [39]).

Excellent references on quasiconformal mappings include [1, 31, 3] among others, while quasiconformal surgery as a technique is treated in [11]. From these sources we now extract a brief introduction to the basic concepts and the main results.

It is well known that conformal maps are $\mathbb{C}$-differentiable homeomorphisms which have the property of preserving angles between curves. This can also be seen as their differential (from the real point of view) being $\mathbb{C}$-linear, and therefore mapping infinitesimal circles (in the tangent space at the point z) to infinitesimal circles (in the tangent space at the image of z). Very roughly speaking, quasiconformal mappings are homeomorphisms which will happen to be differentiable almost everywhere, with non-zero differential almost everywhere, with the property of distorting angles in a bounded fashion. As before, this can be seen as their differential (whenever defined) mapping infinitesimal circles to infinitesimal ellipses in the corresponding tangent spaces, so that all ellipses in this field (defined almost everywhere) have ellipticity bounded by a certain constant. We shall see that some extra conditions will be necessary, but the geometrical idea is as described above.

To make this definitions precise we need to introduce some concepts and terminology. Since the differentials are always $\mathbb{R}$-linear maps, we start by discussing those first.

Let $\mathbb{C}_{\mathbb{R}}$ denote the complex plane viewed as the 2-dimensional oriented euclidean $\mathbb{R}$-vector space with the orthonormal positively oriented standard basis $\{1, i\}$. In $\mathbb{C}_{\mathbb{R}}$ we shall use coordinates either (x, y) or $(z, \bar{z})$. Any $\mathbb{R}$-linear map $L\colon \mathbb{C}_{\mathbb{R}} \to \mathbb{C}_{\mathbb{R}}$ which is invertible and orientation preserving can be written as

$$L(z) = az + b\bar{z},$$

with $a, b, z \in \mathbb{C}$ and $|a| > |b|$. Let us define the *Beltrami coefficient* of L as

$$\mu \equiv \mu_L := \left|\frac{b}{a}\right| e^{2\theta i},$$

for some $\theta \in [0, \pi)$. Then one can check that $L^{-1}(\mathbb{S}^1)$ consists of an ellipse $E(L)$ whose minor axis has argument θ and whose *ellipticity* — i.e., the ratio between its axes — equals $K_L = (1 - |\mu_L|)/(1 + |\mu_L|)$. Observe that if L is $\mathbb{C}$-linear (i.e., conformal), then $b = 0$ and $E(L)$ is a circle.

Now let $U, V \subset \mathbb{C}$ be open sets and suppose $\phi\colon U \to V$ is a map in the class $D^+(U,V)$ of orientation-preserving maps which are $\mathbb{R}$-differentiable almost everywhere, with non-zero differential almost everywhere on their domain, and with the differential $D_u f\colon T_u U \to T_{f(u)} V$ depending measurably on u. Using the infinitesimal coordinates dz and $d\bar{z}$, the differential can be written as

$$D_u f = \partial_z f(u) dz + \partial_{\bar{z}} f(u) d\bar{z}\,,$$

where

$$\partial_z f = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \qquad \text{and} \qquad \partial_{\bar{z}} f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Based on the discussion above, notice that $D_u f$ defines an infinitesimal ellipse in $T_u(U)$ with Beltrami coefficient equal to

$$\mu_f(u) = \frac{\partial_{\bar{z}} f(u)}{\partial_z f(u)}.$$

The *dilatation* of this ellipse can be written as

$$K_f(u) \equiv K_{D_u f} := \frac{1 + |\mu_f(u)|}{1 - |\mu_f(u)|}.$$

Observe that if f is conformal at u then $\partial_{\bar{z}} f(u) = 0$ and hence the ellipse is a circle. In view of this discussion, bounded angle distortion will correspond to the field of ellipses induced by $D_u f$ having bounded ellipticity. In the definition of quasiconformal mappings, the existence of the differential in the usual sense is not assumed (it will be in fact a consequence), although the condition of distortion takes the form described above.

Definition 2.23 (**K-quasiconformal map).** Let U and V be open sets in $\mathbb{C}$; a homeomorphism $\phi\colon U \to V$ is said to be *K-quasiconformal* if it has locally square integrable weak derivatives and the function

$$\mu_\phi(z) := \frac{\partial\phi/\partial\bar{z}}{\partial\phi/\partial z}(z)$$

satisfies that

$$|\mu_\phi(z)| \leq \frac{K-1}{K+1} < 1$$

in L^2_{loc}, i.e., almost everywhere. The notation $k := \frac{K-1}{K+1}$ is standard.

We now list here some standard properties of quasiconformal maps which will be useful for our purposes.

Proposition 2.24 (Properties of quasiconformal maps). *Let $\phi\colon U \to V$ be a K-quasiconformal homeomorphism. Then,*

1. *ϕ is differentiable almost everywhere in the regular sense;*
2. *ϕ^{-1} is K-quasiconformal;*
3. *$\varphi \circ \phi$ is $K \cdot K'$-quasiconformal for every K'-quasiconformal homeomorphism $\varphi\colon V \to W$.*

We have seen that a quasiconformal homeomorphism ϕ induces a measurable field of infinitesimal ellipses (defined up to multiplication by a real number) with bounded ellipticity, which in turn can be coded by a measurable function $\mu_\phi(z)$ with modulus bounded by a constant $k < 1$. All these concepts can be defined on their own, detached from the original map ϕ.

Definition 2.25 (k-Beltrami coefficient). Let $U \subset \mathbb{C}$ be an open set; a measurable function $\mu\colon U \to \mathbb{C}$ defined almost everywhere is called a *k-Beltrami coefficient* of U if $||\mu||_\infty = k < 1$.

By the infinity norm $||\mu||_\infty$ we actually mean the essential supremum

$$\operatorname*{ess\,sup}_{z \in U} |\mu(z)|\,,$$

that is to say, the supremum taken over the set where the function μ is defined. If the function is defined everywhere, the essential supremum does equal the infinity norm.

To every k-Beltrami coefficient μ of U, we can associate an *almost complex structure* σ, that is, a measurable field of (infinitesimal) ellipses in the tangent bundle TU, defined up to multiplication by a positive real constant. More precisely: for almost every point $u \in U$, we can define an ellipse in T_uU whose minor axis has argument $\arg(\mu(u))/2$, and whose ellipticity equals $K(u) := (1+|\mu(u)|)/(1-|\mu(u)|)$. Notice that this value is bounded between 1 and $K := (1+k)/(1-k) < \infty$ almost everywhere. The *standard almost complex structure* is the one defined by circles almost everywhere, or, equivalently, the one induced by the Beltrami coefficient $\mu_0 \equiv 0$.

Now with this terminology, every K-quasiconformal mapping ϕ induces a k-Beltrami coefficient (where $k = (K-1)/(K+1)$) or, equivalently, it induces an almost complex structure σ_ϕ with dilatation bounded by K.

In the same way that a holomorphic map is a map which is locally conformal at all but a discrete number of points, we define a *quasiregular map* as one which is locally quasiconformal at all but a discrete number of points. Therefore a quasiregular map is not required to be a homeomorphism. One can check that a quasiregular map is the composition of a holomorphic function and a quasiconformal homeomorphism.

Another important notion is the concept of *pull-back*. To fix ideas, let us first note that a quasiconformal (or quasiregular) homeomorphism pulls back the Beltrami coefficient $\mu_0 = 0$ to μ_f, or equivalently, the field of infinitesimal circles in TV, to the field of infinitesimal ellipses in TU induced by ϕ by means of its differential. The precise and general definition is as follows.

Definition 2.26 (Pull-back). Let U and V be open sets in $\mathbb{C}$. A quasiregular map $\phi\colon U \to V$ induces a contravariant functor $\phi^*\colon L^\infty(V) \to L^\infty(U)$ defined by

$$\phi^*\mu := \frac{\partial\phi/\partial\bar{z} + (\mu\circ\phi)\overline{(\partial\phi/\partial z)}}{\partial\phi/\partial z + (\mu\circ\phi)\overline{(\partial\phi/\partial\bar{z})}}\,.$$

Notice that if $\mu\colon V \to \mathbb{C}$ is a Beltrami coefficient, then so is its pull-back $\phi^*\mu\colon U \to \mathbb{C}$. Moreover, if ϕ is a holomorphic map, then $||\phi^*\mu||_\infty = ||\mu||_\infty$.

In geometrical terms, the field of ellipses σ in TV is pulled back to a field of ellipses $\phi^*\sigma$ on TU by means of the differential maps wherever defined.

When the Beltrami coefficient μ is defined in terms of a quasiregular map ψ as above ($\mu \equiv \mu_\psi$), one can check that $\phi^*\mu_\psi = \mu_{\psi\circ\phi}$.

An important result in quasiconformal surgery is Weyl's Lemma, since it gives the key to show that maps are holomorphic using only the functor they induce.

Theorem 2.27 (Weyl's Lemma). *If $\phi\colon U \to V$ is quasiconformal (resp. quasiregular) and preserves the standard almost complex structure, that is, $\phi^*\mu_0 = \mu_0$. Then, ϕ is conformal (resp. holomorphic).*

Up to this point we have defined all concepts in open subsets of the complex plane. Using charts, all definitions and results extend to Riemann surfaces and, in particular, to the Riemann sphere $\widehat{\mathbb{C}}$, the natural domain of rational maps.

We have seen how a quasiconformal map ϕ defines a Beltrami coefficient μ_ϕ, and we now turn to the study of the converse problem. More precisely: given a Beltrami coefficient μ and the so-called *Beltrami equation*

$$\frac{\partial\phi}{\partial\bar{z}} = \mu(z)\frac{\partial\phi}{\partial z}\,,$$

can we find an actual quasiconformal map ϕ such that $\mu_\phi \equiv \mu$? The celebrated Measurable Riemann Mapping Theorem, proven by Morrey, Bojarski, Ahlfors and Bers answers this question positively (see [2] or [17]).

Theorem 2.28 (Measurable Riemann Mapping Theorem). *Let μ be a k-Beltrami coefficient of $\mathbb{C}$ (resp. of $\widehat{\mathbb{C}}$ or of $U \cong \mathbb{D}$). Then, there exists a K-quasiconformal map $\phi\colon \mathbb{C} \to \mathbb{C}$ (resp. $\phi\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ or $\phi\colon U \to \mathbb{D}$) such that $\mu_\phi = \mu$, where $K = (1+k)/(1-k)$. Moreover, ϕ is unique up to post-composition with conformal maps of $\mathbb{C}$ (resp. of $\widehat{\mathbb{C}}$ or of $\mathbb{D}$).*

As a consequence, in the case of $\mathbb{C}$ it is enough to fix the image of two points to ensure the unicity of ϕ. In the case of $\widehat{\mathbb{C}}$ we need to use three points.

The whole technique of quasiconformal surgery is based on this powerful result. Let us see then how it applies to Complex Dynamics.

Suppose that $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasiregular map whose dynamics we would like to see realised by a holomorphic map of $\widehat{\mathbb{C}}$. We say that a Beltrami coefficient μ is f*-invariant* if $f^*\mu = \mu$. Likewise, we say that an almost complex structure σ is f*-invariant* if $f^*\sigma = \sigma$, i.e., if the infinitesimal field of ellipses remains invariant after it is pulled back by the map f.

Lemma 2.29 (Key Lemma of quasiconformal surgery). *Let μ be a Beltrami coefficient of $\mathbb{C}$ and $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ a quasiregular map such that $f^*\mu = \mu$. Then, f is quasiconformally conjugate to a holomorphic map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. That is, there exists a quasiconformal homeomorphism $\phi\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $g := \phi \circ f \circ \phi^{-1}$ is holomorphic.*

Proof. Applying the Measurable Riemann Mapping Theorem to μ, there exists a quasiconformal map ϕ with $\mu = \phi^*\mu_0$. Now, let us define $g := \phi \circ f \circ \phi^{-1}$; we just need to see that g is indeed holomorphic. To that end, observe that the standard almost standard structure is g-invariant. Indeed,

$$g^*\mu_0 = (\phi f \phi^{-1})^*\mu_0 = (\phi^{-1})^* f^* \phi^* \mu_0 = (\phi^{-1})^* f^* \mu = (\phi^{-1})^* \mu = \mu_0.$$

On the other hand, g is quasiconformal since it is the composition of quasiconformal maps with a holomorphic one. It then follows from Weyl's Lemma that g is holomorphic. □

2.3 Local connectivity

In this section we give just a few words on the topological concept of *local connectivity*, to be used at some point in our proofs in order to show that some sets have "nice" boundaries. More precisely, we need to show that the Riemann maps we use extend continuously to the boundary.

For a couple of comprehensive references on local connectivity particularly focused on Holomorphic Dynamics, see [34, 36].

Definition 2.30 (Locally connected set). We say that a topological space X is *locally connected* at x if for every open neighbourhood U of x there exists a connected, open set V with $x \in V \subset U$. The space X is said to be locally connected if it is locally connected at x, for all $x \in X$.

Equivalently — using topology terminology —, a topological space is locally connected if every point admits a neighbourhood basis of open connected sets.

If we do not require the neighbourhood V of x to be open, then we speak of *weak local connectivity*. Of course a space which is locally connected at x is weakly locally connected at x, but the converse does not hold. However, it is equally clear that a locally connected space is weakly locally connected, and here it turns out that the converse is true.

Now the following well-known result gives the desired relationship between continuity of maps and local connectivity.

Theorem 2.31 (Carathéodory's Theorem). *Let $\varphi\colon \mathbb{D} \to U \subset \widehat{\mathbb{C}}$ be a conformal isomorphism. Then, φ extends to a continuous map from the closed disc $\overline{\mathbb{D}}$ onto $\overline{U}$ if and only if the boundary ∂U is locally connected, or if and only if the complement $\widehat{\mathbb{C}} \setminus U$ is locally connected.*

Finally, a useful lemma to understand the topology of those points where a given space is not locally connected. This result will be key for our discussion, and the idea for its proof is due to Christian Henriksen.

Lemma 2.32. *Let K be a continuum in $\widehat{\mathbb{C}}$. Then, the set of points where K is not locally connected contains no isolated points.*

Proof. Suppose K is not locally connected at z_0. Let $U = B(z_0, 2\varepsilon)$ be given, where ε is chosen sufficiently small so that $U \cap K$ contains no connected neighbourhood of z_0. Let C_α, $\alpha \in A$, denote the components of $K \cap U$, indexed so that C_0 is the component that contains z_0. Notice that each C_α is closed relative to U, so for $\alpha \neq 0$ the distance $d(z_0, C_\alpha)$ is positive. Now, $\inf d(z_0, C_\alpha)$ has to be zero, because otherwise C_0 would be a connected neighbourhood of z_0. Since K is connected, the closure of each C_α must meet ∂U. It follows that there exist infinitely many components C'_α which meet the circle $\partial B(z_0, \varepsilon)$. Since the circle is compact, there must exist a point z_1 such that each neighbourhood of z_1 meets infinitely many of the components.

It follows that K cannot be locally connected at z_1. Indeed, a connected neighbourhood of z_1 of $K \cap U$ would have to be a subset of the C_α that contains z_1, but points from others components accumulate on z_1. □

2.4 On rational-like maps and virtually repelling fixed points

We know that Fatou's Theorem 2.6 provides the existence of a weakly repelling fixed point for every rational map of degree $d \geq 2$. To prove the main theorems in Chapters 3, 4 and 5, which involve transcendental meromorphic maps, we need to ensure the existence of a weakly repelling fixed point under more general situations. We will do this in two steps following Buff [12].

The first step corresponds to the case of rational-like maps, and we show how this can be adapted to the transcendental setting as long as the meromorphic map restricted to a certain subset of $\mathbb{C}$ is proper.

Definition 2.33 (**Rational-like map).** A *rational-like map* is a proper holomorphic map $f: U \to V$ of degree $d \geq 2$, where U and V are connected open subsets of $\widehat{\mathbb{C}}$ with finite Euler characteristic and $\overline{U} \subset V$.

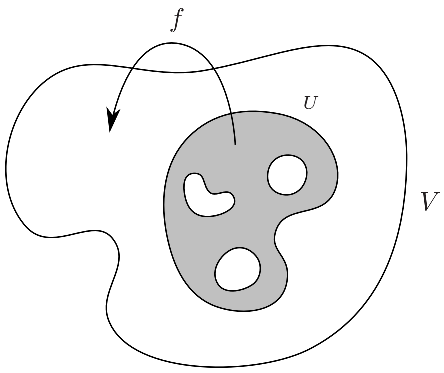


Figure 2.2: A rational-like map in the special case of V simply connected.

Theorem 2.34 (**Buff [12] on rational-like maps).** *If $f: U \to V$ is a rational-like map, then it has at least one weakly repelling fixed point.*

Recall that a map $f: X \to Y$ is *proper* if the preimage set $f^{-1}(K) \subset X$ is compact for any compact set $K \subset Y$. Although our transcendental meromorphic maps are of infinite degree and therefore they fail to be proper globally, they still may be proper — and even rational-like — when restricted to appropriately chosen domains.

Lemma 2.35 (**Proper maps).** *Let f be a transcendental meromorphic function, $Y \subset \mathbb{C}$ a connected open set and X a bounded connected component of $f^{-1}(Y)$. Then, the restriction $f|_X: X \to Y$ is a proper map. If moreover Y is finitely connected and $\overline{X} \subset Y$, then $f|_X: X \to Y$ is rational-like.*

Proof. Let K be a compact set of Y, so $\infty \notin K$. Also, $f^{-1}(K) \subset X$ is bounded, so f is locally ($f|_{f^{-1}(K)}: f^{-1}(K) \to K$) holomorphic and, therefore, the preimage set $f^{-1}(K)$ of the compact set $K \subset Y$ is also compact. Therefore, $f|_X$ is proper and hence of finite degree.

If moreover Y is finitely connected, then the Euler characteristic of X must be finite, since the boundary of Y has a finite number of preimages. Since X is relatively compact in Y, then $f|_X \colon X \to Y$ is rational-like, as claimed. $\square$

We shall need a generalisation of the rational-like setting in the sense that we allow U not to be compactly contained in V but, in return, we restrict to the case where V is simply connected (notice that in Theorem 2.34 it is required that $\overline{U} \subset V$). In this case, the next result of BUFF's will guarantee the existence of a virtually repelling fixed point — and, as we shall see, this will be enough for our purposes.

The concept of virtually repelling fixed point goes back to ADAM EPSTEIN. It is slightly stronger than that of weakly repelling fixed point and its definition is based on the residue fixed point index (see Section 2.1 or [12, 34]), which, in this context, is referred to as the *holomorphic index*. We recall here its definition.

Definition 2.36 (Holomorphic index and virtually repelling fixed point). The *holomorphic index* of a complex function f at a fixed point z is the residue

$$\iota(f,z) := \frac{1}{2\pi i}\oint_z \frac{dw}{w - f(w)}\,.$$

In the case of a simple fixed point (multiplier $\rho(z) \neq 1$), the index is given by

$$\iota(f,z) = \frac{1}{1-\rho(z)}\,.$$

The fixed point z is called *virtually repelling* if we have that

$$\operatorname{Re}\left(\iota(f,z)\right) < \frac{m}{2}\,,$$

where $m \geq 1$ denotes the multiplicity of the fixed point z.

Remark 2.37 (Virtual repellency vs Weak repellency).

- Virtually repelling fixed points are in particular weakly repelling. Indeed if $m > 1$ then the multiplier satisfies $\rho(z) = 1$, while in the simple case we have that
$$\operatorname{Re}\left(\frac{1}{1-\rho(z)}\right) < \frac{1}{2} \iff |\rho(z)| > 1\,.$$
- Virtual repellency, unlike weak repellency, is not preserved under topological conjugacy, since the residue index is only kept under analytic conjugacy (see [34]). See also [40] for a proof of this property for weakly repelling fixed points.

Theorem 2.38 (**Buff [12] on virtually repelling fixed points**)**.** *Let $U \subset \mathbb{D}$ be an open set and $f: U \to \mathbb{D}$ a proper holomorphic map of degree $d \geq 2$. If $|f(z)-z|$ is bounded away from zero as $z \in U$ tends to ∂U, then f has at least one virtually repelling fixed point.*

Remark 2.39. Observe that if we require U to be compactly contained in $\mathbb{D}$, then f is a rational-like mapping. If, moreover, U is simply connected then f is *polynomial-like* (see [19]). By the Straightening Theorem, f is hybrid equivalent — in particular, quasiconformally conjugate — to a polynomial P in U. It follows from Theorem 2.6 applied to P that f must have a weakly repelling fixed point in U.

Since we are not dealing with rational maps, we shall adapt Theorem 2.38 to our situation with the following version.

Corollary 2.40 (**Virtually repelling fixed points in transcendental maps**)**.** *Let $U \subset V \subset \mathbb{C}$ be open sets and suppose that $f: U \to V$ is a proper holomorphic function. Assume that V is simply connected and that ∂V is locally connected in $\widehat{\mathbb{C}}$. If $|f(z)-z|$ is bounded away from zero (in the spherical metric) as $z \in U$ tends to ∂U, then there exists at least one virtually repelling fixed point of f in U.*

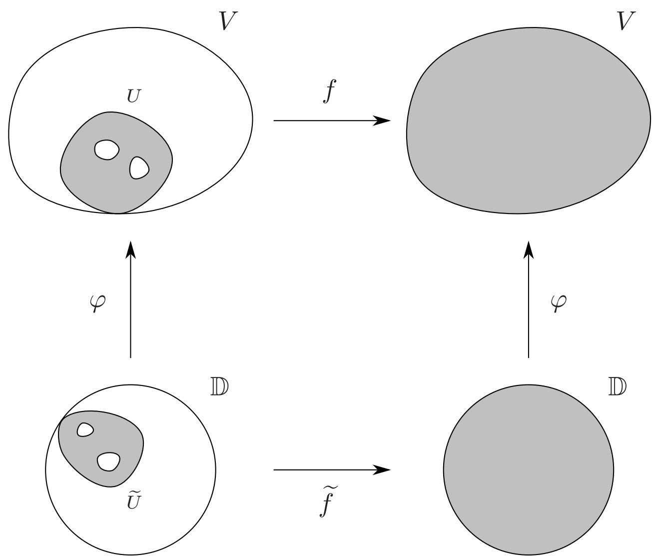


Figure 2.3: Sketch of the proof of Corollary 2.40. Observe that V or U could be unbounded.

Proof. By a change of coordinates we may assume that U and V are bounded and use the Euclidean metric. Since the set V is open, simply connected and ∂V is locally connected, we have that any Riemann mapping $\varphi\colon \mathbb{D} \to V$ extends continuously to $\partial\mathbb{D}$. Let $\widetilde{U} = \varphi^{-1}(U)$, which is a subset of $\mathbb{D}$ since U is contained in V. (See Figure 2.3.)

Let us now define the map $\widetilde{f} := \varphi^{-1} \circ f \circ \varphi$, which is a proper map conjugate to f by the conformal map φ. We now should check that $|\widetilde{f}(\widetilde{z}) - \widetilde{z}|$ is bounded away from zero as $\widetilde{z} \in \widetilde{U}$ tends to $\partial\widetilde{U}$. So let us assume that $\{\widetilde{z}_n\}_n \subset \widetilde{U}$ is a sequence of points tending to $\partial\widetilde{U}$ such that $|\widetilde{f}(\widetilde{z}_n) - \widetilde{z}_n| \to 0$ as $n \to \infty$. We may assume that this is a convergent sequence, just taking an accumulation point and some appropiate subsequence. Let $\widetilde{z}_*$ be the limit point. Since φ extends to the boundary of $\mathbb{D}$, the sequence $z_n =: \varphi(\widetilde{z}_n)$ tends to $z_* := \varphi(\widetilde{z}_*)$. Because of the assumption, we have that $\widetilde{f}(\widetilde{z}_n)$ must also converge to $\widetilde{z}_*$. Since φ conjugates f with $\widetilde{f}$ and extends to the boundary, we have that $\varphi(\widetilde{f}(\widetilde{z}_n)) = f(z_n)$ and this sequence also converges to z_*. Because both sequences have the same limit, it follows that $|f(z_n) - z_n| \to 0$ as $n \to \infty$, which is a contradiction.

Having proven this property, it follows that $\widetilde{f}$ has at least one virtually repelling fixed point $\widetilde{z}_0$ due to Theorem 2.38. Since conformal conjugacies preserve this property of fixed points, we have that there exists a virtually repelling fixed point $\varphi(\widetilde{z}_0)$ of f (in U). □

Remark 2.41. In particular, Corollary 2.40 gives the existence of a weakly repelling fixed point of f, which is the property we shall use in our arguments.

3
Attractive basins

3.1 Shishikura's rational case

Our work on connectivity of Julia sets of transcendental meromorphic functions is based on that of SHISHIKURA's for rational maps. In this chapter we would like to show the main results in his paper, as well as part of their proofs, since they also cover some very specific situations of our transcendental result. The case chosen is that concerning immediate attractive basins and it has been rearranged so that the general structure matches the discourse on transcendental functions in Section 3.2.

The following theorem and corollary, along with all the other results and proofs in this section, are due to SHISHIKURA and extracted from [40].

Theorem 3.1. *If the Julia set of a rational map f is disconnected, there exist two weakly repelling fixed points of f.*

Corollary 3.2. *The Julia set of a rational map with only one weakly repelling fixed point is connected; in other words, all its Fatou components are simply connected. In particular, the Julia set of the Newton's method of a non-constant polynomial is connected.*

Corollary 3.2 is an immediate consequence of the previous theorem, for the Newton's method of a non-constant polynomial has all its fixed points attracting except for the one fixed point at infinity, which is (weakly) repelling.

In order to prove Theorem 3.1, Shishikura uses a case-by-case approach, according to different types of Fatou component — recall that for a general complex function, these are wandering domains, preperiodic components and periodic components, the latter ones described in Theorem 2.16. For the Julia set of a rational map to be disconnected, there must exist at least one multiply-connected Fatou component; namely, an immediate attractive basin, Leau domain, Herman ring or preperiodic component, since Siegel discs cannot be multiply connected and rational maps have neither wandering domains nor Baker domains. Furthermore, the preperiodic case may be treated in a slightly special way, since preperiodic components eventually landing on multiply-connected periodic components can clearly be omitted, so the image of a preperiodic Fatou component may be assumed simply connected.

The strategy that we have only just outlined can be shaped into the following theorem.

Theorem 3.3. *Let f be a rational map of degree greater than one. Then,*

- *if f has a multiply-connected immediate attractive or parabolic basin, there exist two weakly repelling fixed points;*
- *if f has a Herman ring, there exist two weakly repelling fixed points;*
- *if f has a multiply-connected Fatou component U such that $f(U)$ is simply connected, every component of $\widehat{\mathbb{C}} \setminus U$ contains a weakly repelling fixed point.*

The next two sections contain a two-step version of part of Shishikura's proof for this result — namely, the case of the attractive basin. Thus, Section 3.1.1 deals but with fixed immediate attractive basins, while strictly periodic immediate attractive basins are left to Section 3.1.2. We refer to [40] for a complete proof of Theorem 3.3.

3.1.1 Fixed attractive basins

Let us first sketch the process that forces the existence of at least two weakly repelling fixed points, provided that the rational map f has a multiply-connected fixed immediate attractive basin. Since the basin is multiply connected, there exist at least two components of its complement — we want to show that two of them contain a weakly repelling fixed point each. Using quasiconformal surgery, we can construct a rational map g, conjugate to f where needed, with a weakly repelling fixed point in some suitable subset of the sphere so as for f to have such a point in one of the components of the complement of the basin.

Although this description applies to both fixed and periodic cases, in this section we just show the proof for the first one, that is to say: A rational map of degree greater than one with a multiply-connected fixed immediate attractive basin has, at least, two weakly repelling fixed points.

Let us call α the attracting fixed point of f contained in the multiply-connected fixed immediate attractive basin, $\mathcal{A}^*$. Take a small disc neighbourhood U_0 of α such that $\overline{f(U_0)} \subset U_0$. For each $n \geq 0$, let U_n be the connected component of $f^{-n}(U_0)$ that contains α.

From the choice of U_0, we have that

$$\mathcal{A}^* = \bigcup_{n\geq 0} U_n \,.$$

Therefore, there exists $n > 0$ such that U_n is multiply connected — otherwise, the union of the increasing simply-connected open sets U_n would be simply connected. More precisely, there exists $n_0 > 0$ such that U_{n_0} is multiply connected but U_{n_0-1} is simply connected (see Figure 3.1). Rename $U := U_{n_0}$ for simplicity of the text.

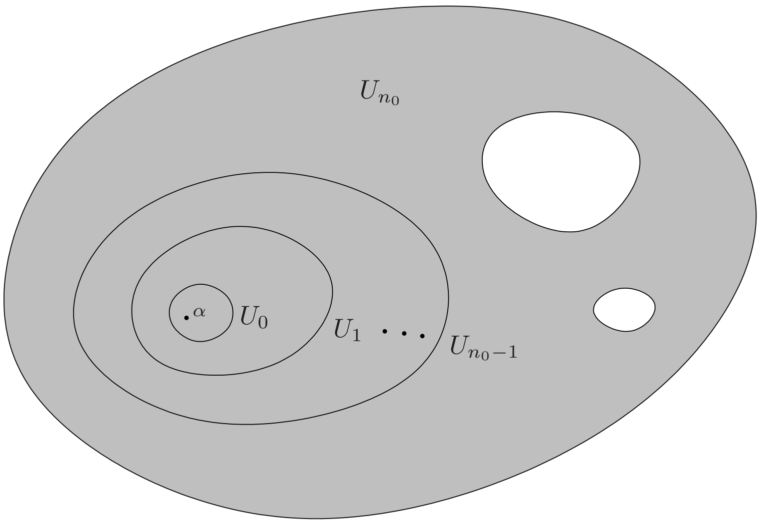


Figure 3.1: The increasing sequence of open neighbourhoods of α, where U_{n_0-1} is simply connected and U_{n_0} is multiply connected.

Since U is multiply connected, there exist at least two connected components of $\widehat{\mathbb{C}} \setminus U$; choose one of them and call it E. From the construction of U, notice that $f(U) = f(U_{n_0}) = U_{n_0-1} \subset U_{n_0} = U$ and, therefore, $f(U) \subset U \subset \mathcal{A}^*$.

Now that we have suitable sets to work with, the next step of this surgery process is the construction of some quasiregular map — with certain desired dynamics —, to which the Measurable Riemann Mapping Theorem (see Section 2.2) can be applied. The following lemma produces exactly such a function.

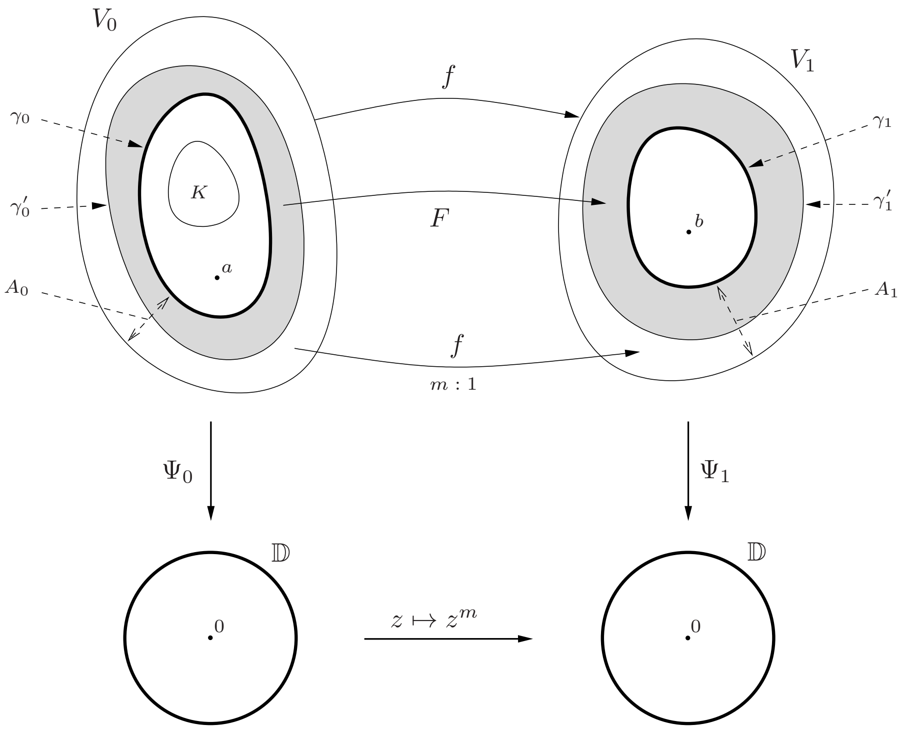


Figure 3.2: We first construct two annuli $A_0 \subset V_0 \cap N$ and $A_1 \subset V_1$, with $\partial A_i = \partial V_i \cup \gamma_i$ and $K \cap \overline{A_0} = \emptyset$, $a \notin \overline{A_0}$, $b \notin \overline{A_1}$, in such a way that the restriction $f_{|A_0}\colon A_0 \to A_1$ be a covering map of degree m and A_0 contain no critical points of f. Then we consider (conformal) Riemann mappings $\Psi_i\colon V_i \setminus \overline{A_i} \to \mathbb{D}$ such that $\Psi_0(a) = \Psi_1(b) = 0$, and define $\widetilde{f}$ on $V_0 \setminus \overline{A_0}$ as $\widetilde{f} := \Psi_1^{-1} \circ (z \mapsto z^m) \circ \Psi_0$. Thus both f and $\widetilde{f}$ are covering maps from γ_0 to γ_1 of the same degree without critical points, hence homotopic. Take $\gamma_1' \subset A_1$ and $\gamma_0' := f^{-1}(\gamma_1') \cap A_0$ as in the figure, and let F be the natural linear interpolation map defined between f on γ_0' and $\widetilde{f}$ on γ_0. Now the map $f_1\colon V_0 \to V_1$, defined as f between ∂V_0 and γ_0', F between γ_0' and γ_0, and $\widetilde{f}$ on $V_0 \setminus \overline{A_0}$, has the properties as required. The shaded regions indicate the dynamics of F.

Lemma 3.4 (Interpolation Lemma). *Let V_0 and V_1 be simply-connected open sets in $\widehat{\mathbb{C}}$, with $\#(\widehat{\mathbb{C}} \setminus V_0) \geq 1$, and f a holomorphic map from a neighbourhood N of ∂V_0 to $\widehat{\mathbb{C}}$ such that $f(\partial V_0) = \partial V_1$ and $f(V_0 \cap N) \subset V_1$; choose a compact set K in V_0 and two points $a \in V_0$ and $b \in V_1$. Then, there exists a quasiregular mapping $f_1 \colon V_0 \to V_1$ such that*

- *$f_1 = f$ in $V_0 \cap N_1$, where N_1 is a neighbourhood of ∂V_0 with $N_1 \subset N$;*
- *f_1 is holomorphic in a neighbourhood of K;*
- *$f_1(a) = b$.*

Shishikura's proof for the Interpolation Lemma is somewhat technical and can be found in [40], although Figure 3.2 offers a sketch of it.

In our situation (see Figure 3.3), we write $V_0 := \widehat{\mathbb{C}} \setminus E$ and $V_1 := f(U)$, call $K := \overline{f(U)}$ and choose $a = b \in f(U)$ arbitrarily. This way, a quasiregular mapping $f_1 \colon \widehat{\mathbb{C}} \setminus E \to f(U)$ is obtained from Lemma 3.4.

Roughly speaking, the map f_1 simplifies f outside E, where its behaviour cannot be controlled, although it still agrees with f on the boundary of this set.

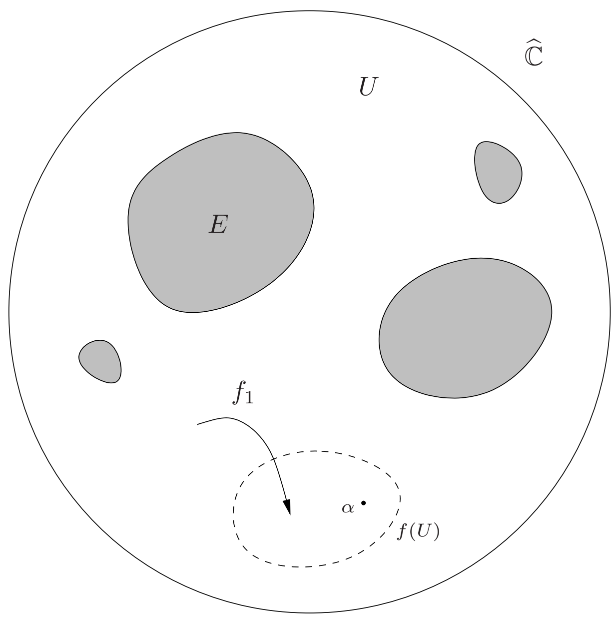


Figure 3.3: The sets U, $f(U)$ and E on the Riemann sphere. The shaded sets are connected components of $\widehat{\mathbb{C}} \setminus U$.

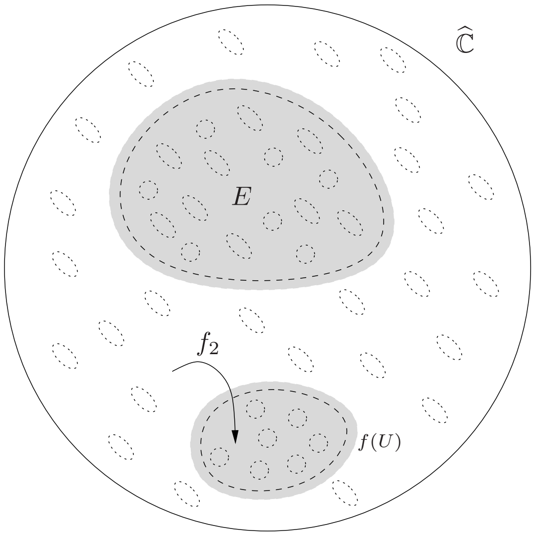


Figure 3.4: Construction of the almost complex structure σ. Recall that $U = \widehat{\mathbb{C}} \setminus E$. The grey area denotes the region where f_2 is holomorphic.

We define yet another function $f_2 \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by cutting and glueing f and f_1 where needed:

$$f_2 := \begin{cases} f & \text{on } E \\ f_1 & \text{on } \widehat{\mathbb{C}} \setminus E\,. \end{cases}$$

This function is quasiregular, since f is rational and so holomorphic, f_1 is quasiregular, and they coincide on an open annulus surrounding ∂E. Furthermore, we have — just from its definition — that f_2 is holomorphic in E and in a neighbourhood of $\overline{f(U)}$, and it has a fixed point at a, for $f_2(a) = f_1(a) = b = a$. Notice that $f_2(\widehat{\mathbb{C}} \setminus E) = f(U)$ and $\overline{f(U)} \subsetneq \widehat{\mathbb{C}} \setminus E$; hence $f(U)$ is invariant and the fixed point $a \in f(U)$ is a global attractor of f_2 in $\widehat{\mathbb{C}} \setminus E$. This concludes the topological step of the construction.

In order to apply the Measurable Riemann Mapping Theorem, it only remains to construct an appropriate f_2-invariant almost complex structure, so define

$$\sigma := \begin{cases} \sigma_0 & \text{on } f(U) \\ (f_2^n)^* \sigma_0 & \text{on } f_2^{-n}(f(U)),\ \text{for } n \in \mathbb{N} \\ \sigma_0 & \text{elsewhere.} \end{cases}$$

(See Figure 3.4.)

By construction, $f_2^* \sigma = \sigma$ almost everywhere, since σ is defined based on the dynamics of f_2. Moreover, σ has bounded ellipticity: indeed, f_2 is holomorphic everywhere except in $X := \widehat{\mathbb{C}} \setminus (E \cup \overline{f(U)})$, where it is quasiregular. But orbits pass through X at most once, since $f_2(X) \subset f(U)$ and points never leave $f(U)$ under iteration of f_2.

These are precisely the hypothesis of Lemma 2.29, so there exists a map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, holomorphic on the whole sphere — and hence rational —, which is conjugate to f_2 by some quasiconformal homeomorphism ϕ. Only for simplicity, let ψ be the inverse function of such homeomorphism, $\psi := \phi^{-1}$.

Now Fatou's Theorem 2.6 ensures the existence of a weakly repelling fixed point z_0 of g, except when $\deg g = 1$ and g is an elliptic transformation. However, notice that

$$g(\psi(\widehat{\mathbb{C}} \setminus E)) = \psi(f_2(\widehat{\mathbb{C}} \setminus E)) = \psi(f(U)) \subsetneq \psi(U) \subsetneq \psi(\widehat{\mathbb{C}} \setminus E),$$

so g is a contraction and $\psi(a)$ is an attracting fixed point of g; in other words, g can never be an elliptic transformation. Also, observe that $\psi(\widehat{\mathbb{C}} \setminus E)$ is contained in the basin of $\psi(a)$.

Besides, the family of iterates

$$\mathcal{G} := \{g^n|_{\psi(\widehat{\mathbb{C}}\setminus E)}\}_{n\geq 1}$$

omits the open set $\psi(X)$, therefore $\mathcal{G}$ is normal in $\psi(\widehat{\mathbb{C}} \setminus E)$ by Montel's Theorem, that is, $\psi(\widehat{\mathbb{C}} \setminus E) \subset \mathcal{F}(g)$. But weakly repelling fixed points belong to the Julia set, so $z_0 \in \psi(E)$. Because such points are preserved under conjugacy, also f_2 has a weakly repelling fixed point $\phi(z_0)$, in E; and so does f, since both functions coincide precisely on this set (see Figure 3.5).

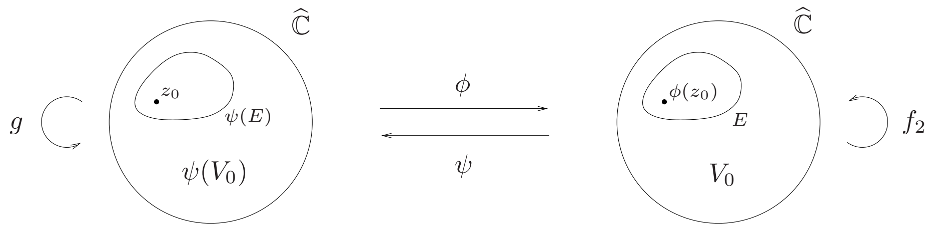


Figure 3.5: The properties of g (including the existence of a weakly repelling fixed point) are transferred to f_2 due to the conjugacy ϕ. Recall that $V_0 = \widehat{\mathbb{C}} \setminus E$.

The set E was arbitrarily chosen from at least two components of $\widehat{\mathbb{C}} \setminus U$, which means that f has at least two weakly repelling fixed points. This concludes the proof of Theorem 3.3 for fixed immediate attractive basins.

3.1.2 Periodic attractive basins

In this section, we focus our attention on the case of periodic immediate attractive basins of period greater than one. The surgery process involved here is quite similar to that for fixed immediate attractive basins (see Subection 3.1.1), so we will give the differences in detail and try to abridge the arguments when identical.

Analogously to the fixed case, let $\langle\alpha\rangle$ be the attracting cycle of f contained in the multiply-connected p-periodic immediate attractive basin, $\mathcal{A}^*$, and let $\mathcal{A}^*(\alpha)$ be the connected component of $\mathcal{A}^*$ containing α. Take a small disc neighbourhood U_0 of α such that $\overline{f^p(U_0)} \subset U_0$, and, for each $n \geq 0$, define U_n as the connected component of $f^{-n}(U_0)$ such that $U_n \cap \langle\alpha\rangle \neq \emptyset$.

As before, we can put $\mathcal{A}^*(\alpha)$ as

$$\mathcal{A}^*(\alpha) = \bigcup_{n\geq 0} U_{np}\,,$$

so, in the sequence $\{U_k\}_k$, there is a multiply-connected set U with simply-connected image. Shishikura formalises this statement with the following lemma.

Lemma 3.5. *Let f be a rational map of degree greater than one with a multiply-connected p-periodic immediate attractive basin. Then, there exists a connected open set U, contained in the basin, such that*

- *U is multiply connected and $f(U)$ is simply connected;*
- *U is a connected component of $f^{-1}(f(U))$;*
- *$\overline{f^p(U)} \subset U$.*

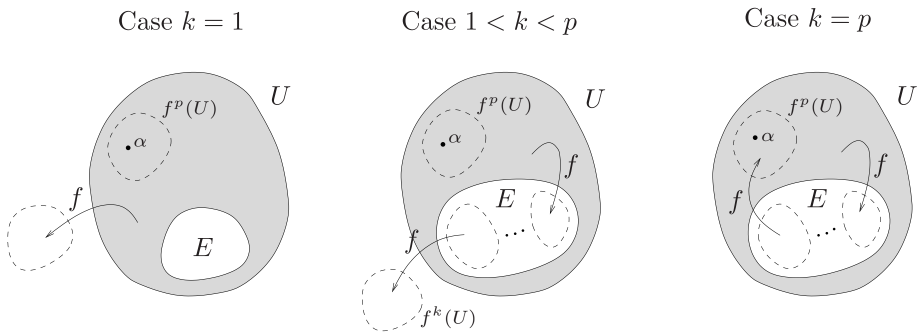


Figure 3.6: Three possible distributions — according to k — of the most relevant sets of this construction. U is shaded in grey.

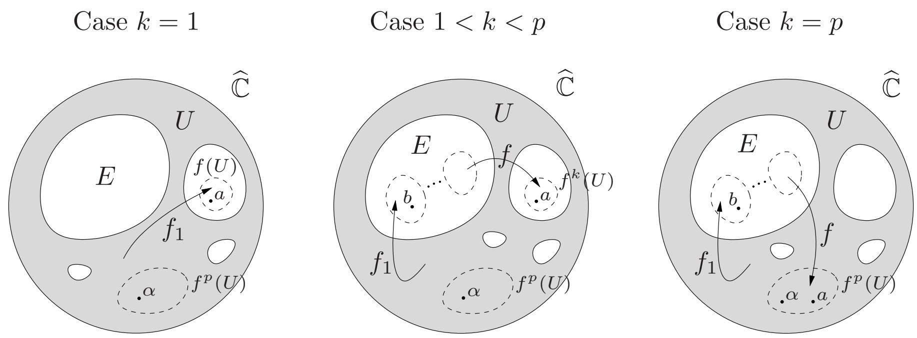


Figure 3.7: The topological surgery construction for the three possible cases, drawn on $\widehat{\mathbb{C}}$.

Next, let E be one of the connected components of the complement of U. Since $U \subset \mathcal{A}^*$ and $p > 1$, its image $f(U)$ must lie in either E or some other component of $\widehat{\mathbb{C}} \setminus U$. Then, let us assume that $k-1$ iterations of U under f belong to E and precisely the kth iteration lands outside it, with $k \in \mathbb{N}$; that is to say, $f^i(U) \subset E$, for all $0 < i < k$, and $f^k(U) \subset \widehat{\mathbb{C}} \setminus E$. (Notice that this assumption is not restrictive: Since $\overline{f^p(U)} \subset U$, necessarily k must range $0 < k \leq p$.) See Figure 3.6 for an overview of all possible cases.

In analogy to the fixed case, we will define a quasiregular map $f_2 \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ that will map $\widehat{\mathbb{C}} \setminus E$ strictly inside itself, this time after k iterations. More precisely, set $V_0 := \widehat{\mathbb{C}} \setminus E$ and $V_1 := f(U)$, which lies in either E (when $k > 1$) or $\widehat{\mathbb{C}} \setminus E$ (when $k = 1$). Set also $K := \overline{f^k(U)}$ and choose $b \in f(U)$ and $a = f^{k-1}(b) \in K$. By the Interpolation Lemma 3.4, there exists a quasiregular map $f_1 \colon \widehat{\mathbb{C}} \setminus E \to f(U)$ which agrees with f on ∂E, is holomorphic in a neighbourhood of K and satisfies $f_1(a) = b$.

Observe that if $k = 1$, then the situation is completely equal to the fixed case (see Figure 3.7).

From here on we proceed as in Section 3.1.1, setting $f_2 = f$ on E and $f_2 = f_1$ on $\widehat{\mathbb{C}} \setminus E$. This makes f_2 a quasiregular map of $\widehat{\mathbb{C}}$, holomorphic in both E and a neighbourhood of $\overline{f^k(U)}$, with a k-periodic point $f_2^k(a) = f^{k-1}(f_1(a)) = f^{k-1}(b) = a$. Observe also that $f_2^k(\widehat{\mathbb{C}} \setminus E) = f^k(U)$ and $\overline{f^k(U)} \subsetneq \widehat{\mathbb{C}} \setminus E$; it follows that f_2^k is a contraction and a a global attractor in $\widehat{\mathbb{C}} \setminus E$.

As before, we may define an almost complex structure σ by

$$\sigma := \begin{cases} \sigma_0 & \text{on } f(U) \\ (f_2^n)^* \sigma & \text{on } f_2^{-n}(f(U)), \text{ for } n \in \mathbb{N} \\ \sigma_0 & \text{elsewhere.} \end{cases}$$

Observe that $\sigma = \sigma_0$ on $\bigcup_{i=1}^{k} f^i(U)$ (see Figure 3.8).

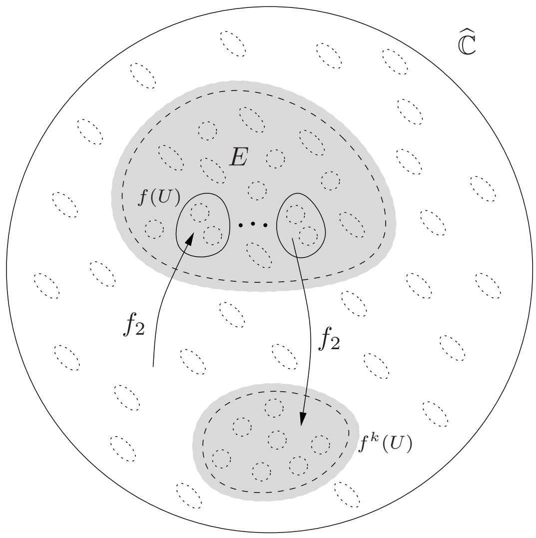


Figure 3.8: Construction of the almost complex structure σ. In grey we find the region where f_2 is holomorphic.

Furthermore, σ is f_2-invariant by construction and has bounded distortion, since orbits pass through $\widehat{\mathbb{C}} \setminus (E \cup \overline{f^k(U)})$ (the set where f_2 is not holomorphic) at most once.

With this setting — and following the fixed case —, Lemma 2.29 and Theorem 2.6 guarantee the existence of a weakly repelling fixed point of f in E, which is exactly what we wanted to prove.

3.2 The transcendental case

Shishikura's Theorem 3.1 inspires the analogous result for trascendental maps, that is to say, our Conjecture 1.3 on connectivity of Julia sets of transcendental meromorphic functions and its relationship to the existence of weakly repelling fixed points.

Following Shishikura, we can use the classification in Theorem 2.16 to individualise the main statement according to Fatou components. Notice for a start that Siegel discs can never be multiply connected, whereas the case of preperiodic Fatou components that will eventually fall on a multiply-connected periodic component is automatically proven by the other cases. Taking these into account, the statement is as follows.

Conjecture 3.6. *Let f be a transcendental meromorphic function. Then,*

- *if f has a multiply-connected immediate attractive or parabolic basin, Baker domain or wandering domain, or*
- *if f has a Herman ring, or*
- *if f has a multiply-connected Fatou component U such that $f(U)$ is simply connected,*

there exists at least one weakly repelling fixed point of f.

Remark 3.7. The case of the multiply-connected wandering domain was already proven by WALTER BERGWEILER and NORBERT TERGLANE [9] in a different context, namely, in the search of solutions of certain differential equations with no wandering domains.

Our Main Theorem 1.5 deals with the cases of immediate attractive basins, parabolic basins and preperiodic Fatou components. Now this section contains the proof for the first case, rewritten as the following theorem, while the proofs for the other two cases can be found in Chapters 4 and 5.

Theorem 3.8 (Attractive basins case). *Let f be a transcendental meromorphic function with a multiply-connected p-periodic immediate attractive basin $\mathcal{A}^*$. Then, there exists at least one weakly repelling fixed point of f.*

We use two quite different strategies in order to prove this theorem. The first one is based on SHISHIKURA's surgery construction and applies when either $\mathcal{A}^*$ is bounded, or preimages of a sufficiently small neighbourhood of the attractive point in $\mathcal{A}^*$ do not behave too wildly. The second technique, used in the rest of the cases, involves BUFF's results on rational-like maps and virtually repelling fixed points (see Section 2.4).

Let us first assume that $\mathcal{A}^*$ is bounded. In this very particular case we can also assume the existence of a connected open set $U \subset \mathcal{A}^*$ such as Lemma 3.5 gives — that is to say, multiply connected and such that $f(U)$ is simply connected, U is a connected component of $f^{-1}(f(U))$ and $f^p(\overline{U}) \subset U$ —, since the basin has no accesses to infinity and therefore preimages of compact sets (in the construction of U) keep compact.

We have $U \subset \mathcal{A}^* \subset \mathcal{F}(f)$, so the essential singularity must be contained in the complement $\widehat{\mathbb{C}} \setminus U$. Moreover, since U is multiply connected, there exists at least one connected component E of $\widehat{\mathbb{C}} \setminus U$ which does not contain the singularity. As in the rational (periodic) case (see Subsection 3.1.2), we assume that the iterations of U under f do not jump outside E until the kth one, and proceed analogously to find a function f_2 that preserves f on E but has attracting dynamics (interpolation function f_1) on $\widehat{\mathbb{C}} \setminus E$.

Notice that f_2 is indeed quasiregular: On $\widehat{\mathbb{C}} \setminus E$, the map f_1 is quasiregular and infinity is no longer an essential singularity; on E, now f sends the poles to the (non-special) point at infinity — as f is meromorphic, f_2 is holomorphic on E as a map defined on the Riemann sphere —; by definition of f_1, the functions f and f_1 agree on the neighbourhood $V_0 \cap N_1$, so the glueing is continuous.

At this point, the topological step of the surgery process is done. The further holomorphic smoothing and end of the proof goes on exactly as in Section 3.1.2, therefore f has a weakly repelling fixed point in E.

As for the unbounded case, we cannot apply the previous surgery construction in general, since the existence of asymptotic values and Fatou components with the essential singularity on their boundary can lead to unbounded preimages of bounded sets, while trying to construct U. Instead, we will use this very property to force the situation described in Buff's Theorem 2.34 or in Corollary 2.40.

So let us assume from now on that $\mathcal{A}^*$ is unbounded. The cases of the fixed basin ($p = 1$) and the (strictly) periodic basin ($p > 1$) are next treated separately.

3.2.1 Fixed attractive basins

In this case, the immediate attractive basin $\mathcal{A}^*$ consists of a single (fixed) Fatou component. Let $\alpha \in \mathcal{A}^*$ be its one attracting fixed point. We first construct a nested sequence of open sets containing α as follows: Let U_0 be a neighbourhood of α such that $\overline{f(U_0)} \subset U_0$, that is, put $U_0 := \varphi^{-1}(\Delta)$, where φ is the linearisation map of the fixed point α and Δ is a disc in its linearisation coordinates; and define U_n as the connected component of $f^{-n}(U_0)$ that contains α, for all $n \in \mathbb{N}$. Notice that $U_0 \subset U_1 \subset \ldots$ because of the choice of the initial neighbourhood U_0.

Since $\mathcal{A}^*$ is multiply connected, there exists $n_0 \in \mathbb{N}$ such that $U_0, \ldots, U_{n_0-1}$ are simply connected and U_{n_0} is multiply connected. This implies that the complement of U_{n_0} have at least one bounded connected component, since its fundamental group is $\pi_1(U_{n_0}) \neq \{0\}$. In view of this, let E be one of the bounded connected components of $\widehat{\mathbb{C}} \setminus U_{n_0}$ (see Figure 3.9).

As Figure 3.9 suggests, at some point the sets $\{U_k\}_k$ might become unbounded, so further preimages of such sets could have poles and prepoles on their boundaries. The actual condition for this fact to happen can be written in terms of the intersection set $\partial E \cap \mathcal{J}(f)$ and is specified in the following lemma.

Lemma 3.9. *Let f be a transcendental meromorphic function with an unbounded multiply-connected fixed immediate attractive basin $\mathcal{A}^*$, and let $\{U_k\}_{k=0}^{n_0}$ and E be as above. Then, the following are equivalent:*

(1). $U_0, \ldots, U_{n_0-1}$ are all bounded;

(2). $\partial E \cap \mathcal{J}(f) = \emptyset$;

(3). ∂E contains no poles.

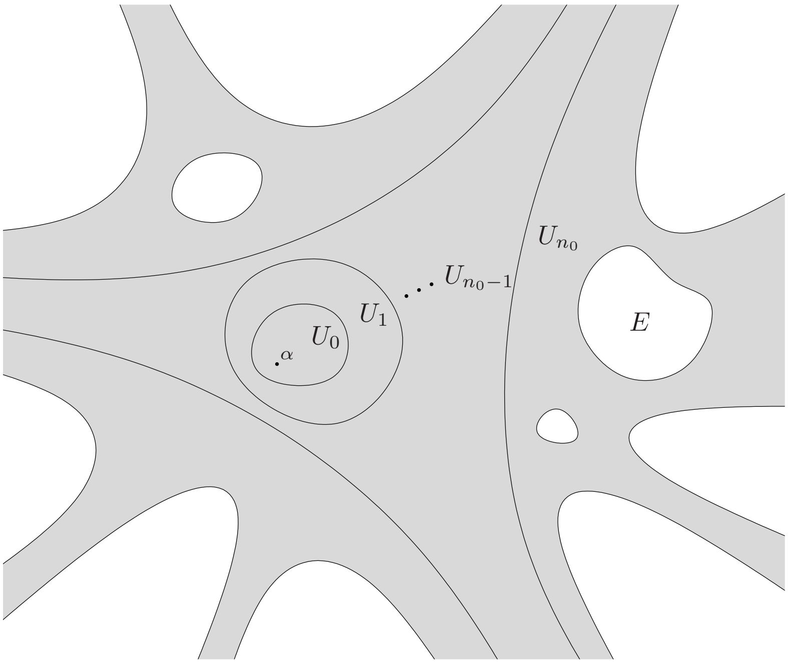


Figure 3.9: The sequence $\{U_k\}_k$ and the bounded set E. In grey, the multiply-connected set U_{n_0}.

Proof. Let us first see how *(1)* implies *(2)*. The boundaries of $U_0, \dots, U_{n_0-1}$ belong to the Fatou set and are bounded. Since ∂E is mapped onto ∂U_{n_0-1}, it follows that $\partial E \cap \mathcal{J}(f) = \emptyset$. Statement *(2)* trivially gives *(3)*. For *(3)* implies *(1)*, suppose there exists $k \in \mathbb{N}$, with $0 < k < n_0$, such that U_k is unbounded. Since this is an increasing sequence, $U_k, U_{k+1}, \dots$ are all unbounded and in particular so is U_{n_0-1}. But $\partial U_{n_0-1} \subset f(\partial E)$, because U_{n_0-1} is simply connected, and the set E is bounded. Then ∂E must contain at least one pole, which contradicts *(3)*. $\square$

Therefore, in the case where ∂E never meets $\mathcal{J}(f)$, the set U_{n_0} can be renamed U and we have the following situation: U is multiply connected and $f(U) = f(U_{n_0}) = U_{n_0-1}$ is simply connected; U is a connected component of $f^{-1}(f(U)) = f^{-1}(U_{n_0-1})$, by definition; $f(\overline{U}) \subset \overline{U_{n_0-1}} \subset U_{n_0} = U$, since U_{n_0-1} is bounded and U open. Now this situation is but the setting we had in the case of $\mathcal{A}^*$ bounded, with $p = 1$ (see Figure 3.10). Surgery can thus be applied in the same fashion (see Subsection 3.1.1) to obtain a quasiregular map that send $\widehat{\mathbb{C}} \setminus E$ to U_{n_0-1} and equal

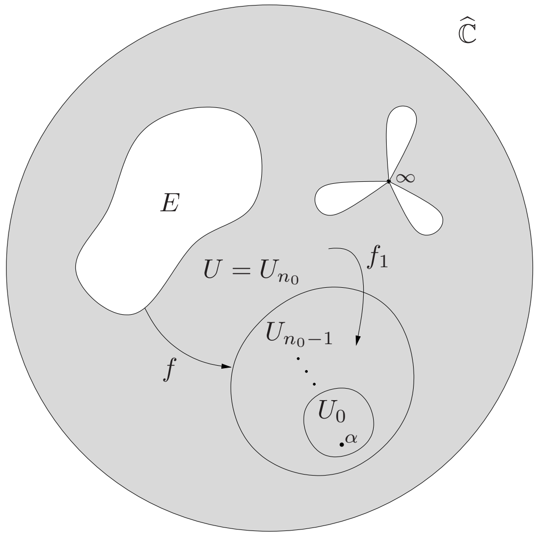


Figure 3.10: Sketch of the case where ∂E never meets the Julia set, on the Riemann sphere. The shaded set represents U. Surgery can be applied as in the case where $\mathcal{A}^*$ is bounded and $p = 1$; compare with Figure 3.3.

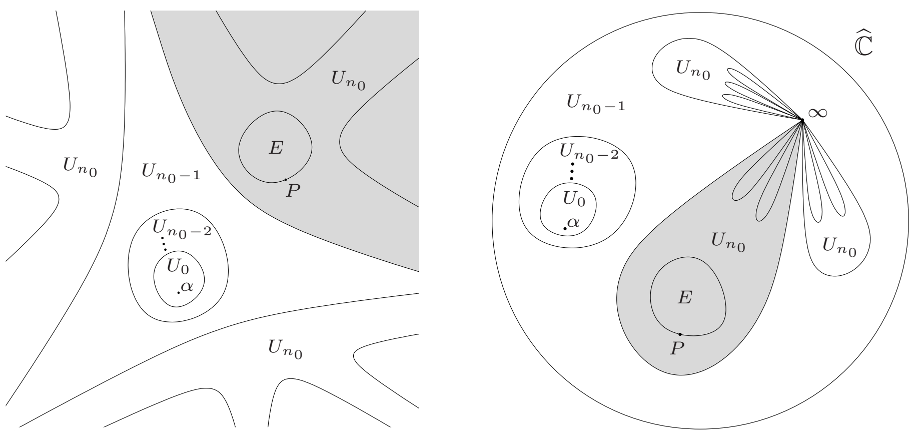


Figure 3.11: The increasing sequence of open sets $\{U_k\}_k$ and the decreasing one $\{V_k\}_k$. In this example, U_{n_0-1} is the first unbounded set in the sequence and, consequently, $V_n = \widehat{\mathbb{C}} \setminus U_n$ for all $n < n_0 - 1$. The shaded set corresponds to V_{n_0-1}, while $V_{n_0} = E$. The same situation has been drawn on the plane and on the Riemann sphere.

f on E. Observe that the essential singularity is no longer there and, therefore, the holomorphic map that we obtain from the surgery procedure is a rational map. This gives the desired weakly repelling fixed point in E.

A completely different situation arises when ∂E does intersect $\mathcal{J}(f)$. In this case Lemma 3.9 asserts the existence of at least one pole P in ∂E. From now on, this is the situation we deal with.

As mentioned, in this case we no longer use quasiconformal surgery, but Theorem 2.34 and Corollary 2.40 — in other words, we want to find an open subset of $\widehat{\mathbb{C}}$ that contains a preimage of itself and whose boundary does not share fixed points with the boundary of such preimage. (We shall see it suffices that infinity not be on the preimage's boundary.)

Let us first construct a (shrinking) nested sequence of sets, in the complement of the open sets $\{U_k\}_k$, by defining V_n to be the connected component of $\widehat{\mathbb{C}} \setminus U_n$ that contains E, for all $0 \leq n \leq n_0$. Notice that the closed sets $V_0, \ldots, V_{n_0-1}$ are all unbounded, for U_{n_0} is the first multiply-connected set of its sequence, and $V_{n_0} = E$ is bounded by definition. Notice also that this component containing E is simply connected (since U_n is connected) and indeed unique, and that $V_0 \supset V_1 \supset \ldots \supset V_{n_0} = E$, since $U_0 \subset U_1 \subset \ldots$ and all the $\{V_k\}_k$ must contain E (see Figure 3.11).

From Lemma 3.9 and from the fact that U_0 is bounded, there exists $n_1 \in \mathbb{N}$, with $0 < n_1 < n_0$, such that $U_0, \ldots, U_{n_1-1}$ are bounded and $U_{n_1}, U_{n_1+1}, \ldots$ are unbounded. Moreover, since the preimage of an unbounded set may contain poles on its boundary, we can assume there exists $n_2 \in \mathbb{N}$, with $0 < n_1 < n_2 \leq n_0$, such that $P \notin \partial V_0, \ldots, \partial V_{n_2-1}$ and $P \in \partial V_{n_2}$. The following lemma shows that, in this case, $P \in \partial V_n$ for all $n_2 \leq n \leq n_0$.

Lemma 3.10. *Suppose there exists $k < n_0$ such that $P \in \partial V_k$. Then, $P \in \partial V_j$, for all $k \leq j \leq n_0$.*

Proof. It is clear that $P \in \partial V_{n_0}$, given that $E = V_{n_0}$. Now, suppose there exists $k < j < n_0$ such that $P \notin \partial V_j$.

By definition, $E \subset V_j$ and therefore $P \in \mathring{V}_j$. However, on the other hand, since $V_j \subset \widehat{\mathbb{C}} \setminus U_j$, we have that $U_k \subset U_j \subset \widehat{\mathbb{C}} \setminus V_j$. It follows that $\overline{U_k} \subset \overline{U_j} \subset \overline{\widehat{\mathbb{C}} \setminus V_j}$ and hence $P \in \overline{\widehat{\mathbb{C}} \setminus V_j}$, given that $P \in \partial U_k$. But we assumed that $P \notin \partial V_j$, so we deduce that $P \in \operatorname{int}(\widehat{\mathbb{C}} \setminus V_j)$. This contradicts the fact that $P \in \mathring{V}_j$. $\square$

If $n_2 = n_0$, the first set V_k which contains P on its boundary is E itself (see Figure 3.12). As V_{n_0-1} is unbounded, there exists some connected component X of $f^{-1}(V_{n_0-1})$ such that $P \in \partial X$. Furthermore, the preimage X must be contained in E, since points immediately outside E belong to U_{n_0} (whose image under f is U_{n_0-1}), and hence cannot be preimage of points in $V_{n_0-1} \subset \widehat{\mathbb{C}} \setminus U_{n_0-1}$. Call $\widetilde{V}$ the connected component of $\mathring{V}_{n_0-1}$ that contains E, and $\widetilde{X}$ a connected component of

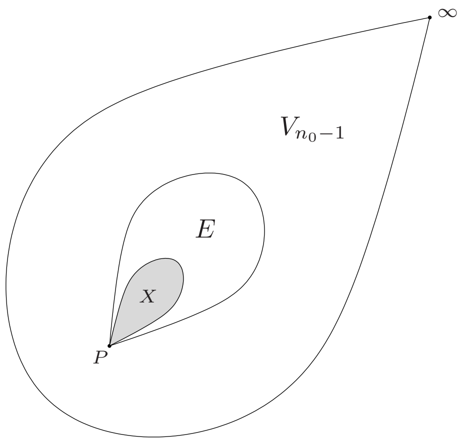


Figure 3.12: The situation where $n_2 = n_0$, i.e., the first set V_k that contains the pole P on its boundary is $V_{n_0} = E$ itself. Then, a preimage X of V_{n_0-1} must exist in E.

$f^{-1}(\widetilde{V})$ in E. The boundaries $\partial\widetilde{V}$ and $\partial\widetilde{X}$ do not have any common fixed point because $|f(z) - z|$ is bounded away from zero as $z \in \widetilde{X}$ tends to $\partial\widetilde{X}$, so the map $f \colon \widetilde{X} \to \widetilde{V}$ satisfies the hypotheses of Corollary 2.40 and therefore f has a weakly repelling fixed point.

The most general case is that where $0 < n_1 < n_2 < n_0$. One example of this situation is given by Figure 3.13, namely when $n_2 = n_1 + 1$ and $n_0 = n_2 + 2$.

Observe that, in this case, the interior of the sets $\{V_k\}_k$ with $k \geq n_2$ might have more than one connected component (as shown in the example of Figure 3.13). In order to avoid this, in our setting we define yet another sequence $\{W_k\}_k$, where each W_n is *the* unbounded connected component of $\mathring{V}_n$, for all $n_2 \leq n < n_0$. Such an unbounded component must be indeed unique and simply connected, since the sets $\{V_k\}_k$ are all simply connected (because the sets $\{U_k\}_k$ are all connected) and they have "nice" boundaries, as the following lemma shows (see Figure 3.14).

Lemma 3.11. *In the situation described hitherto, ∂W_n, ∂V_n and ∂U_n are all locally connected in $\widehat{\mathbb{C}}$.*

Proof. First notice that, by construction of the sets W_n and V_n (and their relative positions), we have the inclusions $\partial W_n \subset \partial V_n \subset \partial U_n$ and it suffices to prove that ∂U_n is locally connected for any $n \geq 0$.

Let $z \in \partial U_n$. If $z \notin \bigcup_{0<k<n} f^{-k}(\infty)$, then f^n is a local homeomorphism between a sufficiently small neighbourhood Θ of z and a neighbourhood Θ' of $f^n(z) \in \partial U_0$. But U_0 was chosen to be a small disc neighbourhood of the point α, so $\partial U_0 \cap \Theta'$ is an arc through $f^n(z)$ and ∂U_0 is locally connected at $f^n(z)$.

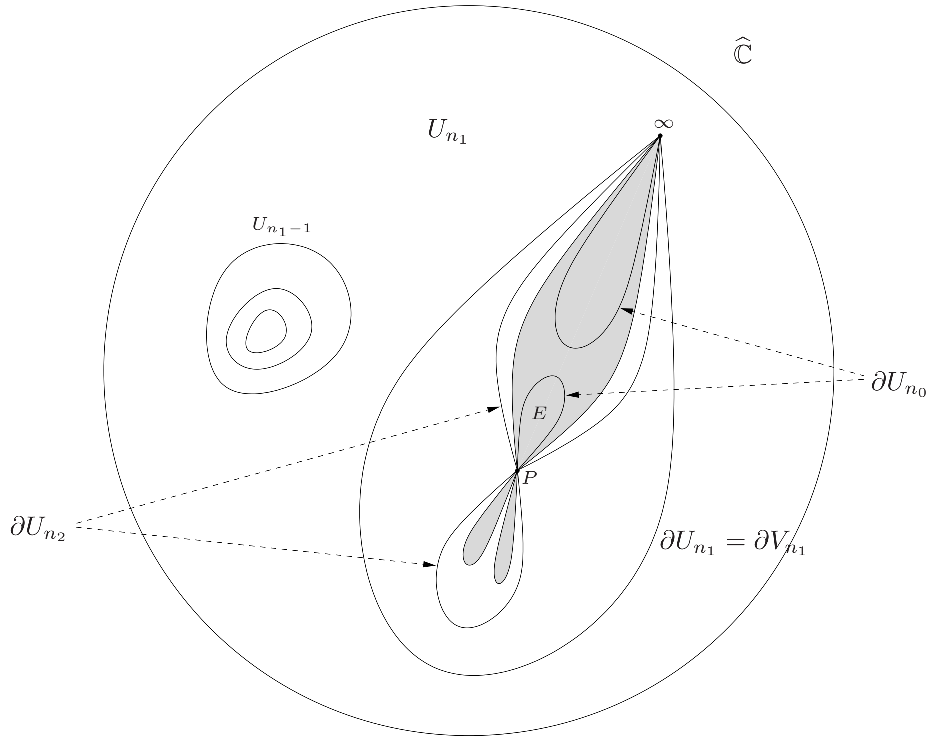


Figure 3.13: A possible distribution of the sets $U_1, \ldots, U_{n_0}$, with $0 < n_1 < n_2 < n_0$, and more precisely $n_2 = n_1 + 1$ and $n_0 = n_2 + 2$. To simplify, the sets $\{U_k\}_k$ have been drawn only with one access to infinity. Observe that $\mathring{V}_{n_2}$ and $\mathring{V}_{n_2+1}$ have two and three connected components, respectively. The shaded area represents V_{n_2+1}.

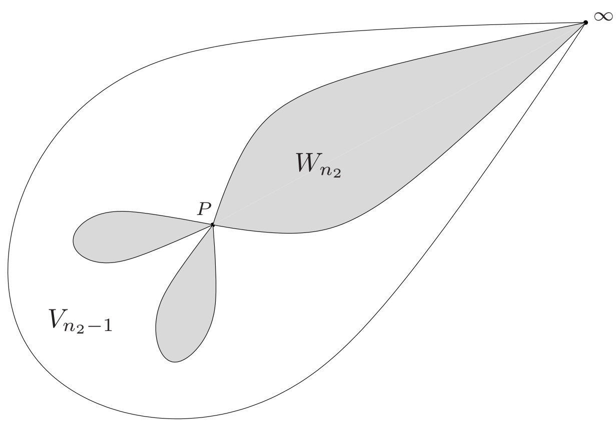


Figure 3.14: The open set W_{n_2} is the unbounded component of the interior of the (shaded) set V_{n_2}.

Since f^n is a local homeomorphism between Θ and Θ', $\partial U_n \cap \Theta$ is a collection of d arcs through z, where d is the local degree of f^n around z, and ∂U_n is locally connected at z. Now it just remains to check if ∂U_n is locally connected also at the points of the set $\bigcup_{0<k<n} f^{-k}(\infty)$. However, notice that this set is either finite or countable and, in any case, its points are isolated, so Lemma 2.32 provides the local connectivity we wanted. □

With these tools, our proof will continue as follows: For every $n_2 \leq n < n_0$, we will first consider the preimage sets of W_n attached to P. If any connected component of $f^{-1}(W_n)$ happens to be bounded, then Corollary 2.40 can be applied and the proof will finish, as we will show in Lemma 3.12. But if all of them were unbounded, then it is clear both W_n and each of its preimages would have infinity as a fixed point (of the restricted map) on their boundaries, contradicting the hypotheses of Corollary 2.40. In this case we will jump to the next step and repeat the procedure with W_{n+1}. We will now make this argument precise.

As boundedness of preimages plays quite an important role, for clarity's sake we define for $n_2 \leq n < n_0$ the families of sets

$$\mathcal{X}_n := \{X \subset \widehat{\mathbb{C}} \text{ bounded connected component of } f^{-1}(W_n) : P \in \partial X\}.$$

In other words, $\mathcal{X}_n$ is the set of bounded connected components of $f^{-1}(W_n)$ with P on their boundary. Now the following lemma proves the key point of our iterative process.

Lemma 3.12. *Fix $n^* \in \mathbb{N}$ such that $n_2 \leq n^* < n_0$ and suppose $\mathcal{X}_n = \emptyset$, for all $n_2 \leq n < n^*$, but $\mathcal{X}_{n^*} \neq \emptyset$. Then, there exists at least one weakly repelling fixed point of f.*

Proof. Let $X \in \mathcal{X}_{n^*}$. It is clear that $X \subset V_{n^*+1} \subset V_{n^*} \subset V_{n^*-1}$, where the first inclusion follows from the fact that $V_{n^*} \setminus V_{n^*+1} \subset U_{n^*+1}$ and its points never fall in W_{n^*} under iteration of f (see Figure 3.15). If $X \subset W_{n^*}$, then the map $f\colon X \to W_{n^*}$ satisfies the hypotheses of Corollary 2.40 (by construction and using Lemma 3.11), which provides a weakly repelling fixed point of f. Otherwise, X is contained in one of the bounded components B of $\mathring{V}_{n^*}$ (see Figure 3.16). Consider preimages of W_{n^*-1}, that is to say, connected components of $f^{-1}(W_{n^*-1})$; since $W_{n^*} \subset W_{n^*-1}$, there exists a preimage Y of W_{n^*-1} such that $X \subset Y$. But also $Y \subset V_{n^*}$ (for the same reason that $X \subset V_{n^*+1}$), which means that $Y \subset B$ by continuity. This makes Y bounded, since so is B, therefore $Y \in \mathcal{X}_{n^*-1}$ and $\mathcal{X}_{n^*-1} \neq \emptyset$, contradicting our initial assumption. $\square$

Using this result, the end of the proof becomes straightforward: For every $n \in \mathbb{N}$ such that $n_2 \leq n < n_0$, check whether $\mathcal{X}_n \neq \emptyset$. As it turns out, the last family of sets of the sequence $\{\mathcal{X}_k\}_k$ always has this property, $\mathcal{X}_{n_0-1} \neq \emptyset$, since preimages of W_{n_0-1} with P on their boundary lie in $V_{n_0} = E$, which is bounded by definition. Therefore, take the smallest n for which $\mathcal{X}_n \neq \emptyset$ holds, and Lemma 3.12 gives a weakly repelling fixed point of f.

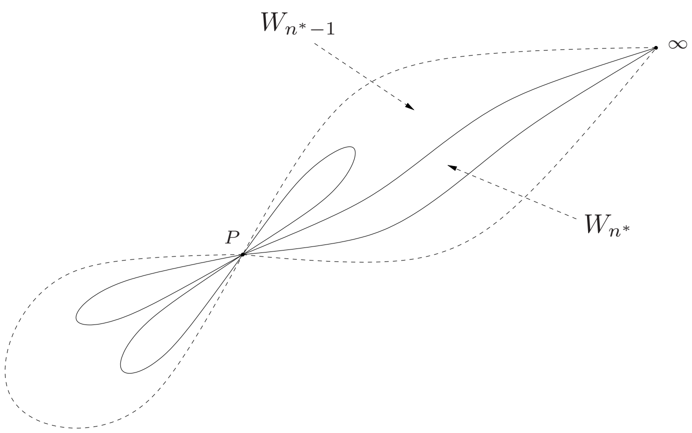


Figure 3.15: A bounded preimage X of W_{n^*-1} containing P on its boundary must be always in W_{n^*} and hence in W_{n^*-1}. Corollary 2.40 gives then a weakly repelling fixed point. Here the dashed lines represent V_{n^*-1}, while the continuous ones correspond to V_{n^*}.

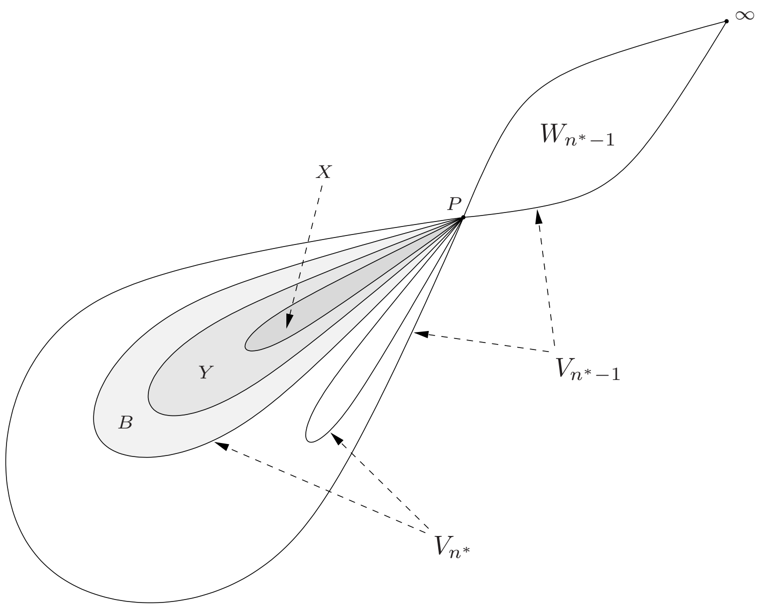


Figure 3.16: In the situation where X lies in one of the bounded components B of $\mathring{V}_{n^*}$, there exists a preimage Y of W_{n^*-1} such that $X \subset Y \subset B$.

3.2.2 Periodic attractive basins

This case begins with the same setting as the fixed basin, although it soon becomes much simpler. Let $\mathcal{A}^*$ be the multiply-connected p-periodic immediate attractive basin of f and $\langle\alpha\rangle \subset \mathcal{A}^*$ be its attracting p-periodic cycle. As before, we define U_0 to be a suitable neighbourhood of α, so that $f^p(U_0) \subset U_0$, and U_n as the connected component of $f^{-n}(U_0)$ that intersects $\langle\alpha\rangle$, for all $n \in \mathbb{N}$. Analogously to the fixed case, we have that $U_l \subset U_{p+l} \subset U_{2p+l} \subset \ldots$, for all $0 \leq l < p$.

Again, there exists $n_0 \in \mathbb{N}$ such that $U_0, \ldots, U_{n_0-1}$ are simply connected and U_{n_0} is multiply connected, for so is $\mathcal{A}^*$. Call $U = U_{n_0}$ and let E be one of the bounded connected components of $\widehat{\mathbb{C}} \setminus U$ (see Figure 3.17).

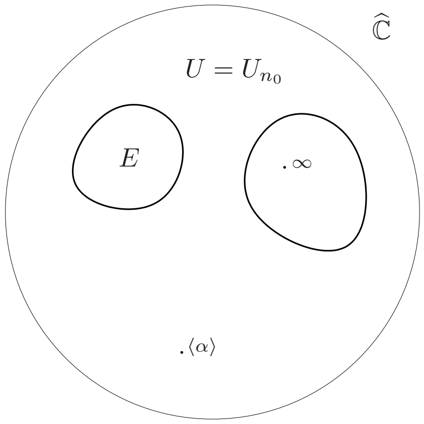


Figure 3.17: U is a multiply-connected subset of $\mathcal{A}^*$ such that $f(U)$ is simply connected. If U were unbounded, the point at infinity would be in $\partial(\widehat{\mathbb{C}} \setminus (U \cup E))$ (see Figure 3.18).

Remark 3.13. Notice the impossibility to use Lemma 3.9 to separate the different cases, as we did in the previous section. Indeed, in this periodic case the sequence $\{U_k\}_k$ is no longer nested so our proof cannot be extended beyond fixed basins.

When ∂E has no poles — analogously to the previous case — we will apply the periodic-case surgery described in Section 3.1.2 to find a weakly repelling fixed point of f. First notice the curve $f(\partial E)$ is bounded, since ∂E is bounded by definition and has no poles by hypothesis. It follows that $f(\partial E) = \partial U_{n_0-1}$, because $f(\partial E)$ is at least one of its connected components and U_{n_0-1} is simply connected. We conclude that U_{n_0-1} must be bounded, since so is $f(\partial E)$.

Now this means we can use the Interpolation Lemma 3.4 to obtain a quasiregular map $f_1 \colon \widehat{\mathbb{C}} \setminus E \to U_{n_0-1} = f(U)$, as in the previous cases, and the surgery process goes on and finishes as it did in the rational periodic case.

When ∂E does contain a pole P, the image $f(U)$ must be unbounded and, therefore, contained in one of the unbounded connected components of $\widehat{\mathbb{C}} \setminus U$. Consider a simply-connected, unbounded, closed set $V \subset \widehat{\mathbb{C}}$, containing U but not its image $f(U)$ (see Figure 3.18) — this is always possible because we are in the case $p > 1$. Notice that also $E \subset V$ by construction of V (which is simply

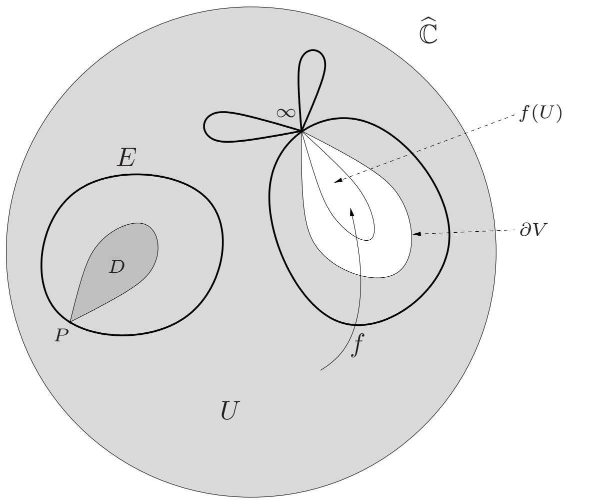


Figure 3.18: If there exists a pole P on ∂E, then there exists a set $D \subset E$ such that $f(D) = V$, where V is an unbounded simply-connected set that contains U but not $f(U)$. The thick lines correspond to ∂U, while the sets D and V appear dark- and light-shaded, respectively.

connected) and boundedness of E. Now there exists a preimage D of V, with $P \in \partial D$, and $D \subset E$ since points immediately outside E are in U and thus mapped to $f(U) \subset \widehat{\mathbb{C}} \setminus V$. Moreover, we have $D \subset E \subsetneq V$, so $\partial D \cap \partial V = \emptyset$ and Theorem 2.34 gives a weakly repelling fixed point of f.

This step concludes the periodic immediate attractive case and, with it, the proof of Theorem 3.8.

4
Parabolic basins

In this chapter we prove our Main Theorem 1.5 for the case of multiply-connected parabolic basins. Let us recall here this result, rewritten as a separate theorem, while we introduce some notation.

Theorem 4.1 (**Parabolic basins case).** *Let f be a transcendental meromorphic function with a multiply-connected p-periodic parabolic basin B. Then, there exists at least one weakly repelling fixed point of f.*

Its proof involves two quite different techniques. The first one is based upon Shishikura's proof and applies when preimages of certain sets do not behave too wildly in the presence of the essential singularity. For the second one, the assumption of a pole of f allows us to construct some sets where the hypotheses of Corollary 2.40 are met.

Recall that by *p-periodic parabolic basin B* we mean a connected component of the Fatou set such that there exists a q-periodic point $\alpha \in \partial B$, $q|p$, with $\lim_{n\to\infty} f^{nq}(z) = \alpha$ for all $z \in B$ and, in particular, $(f^p)'(\alpha) = 1$ (i.e., the immediate basin associated to a one *petal* attached to a q-periodic parabolic point). Notice that p is the period of B, not of α, so $B, f(B), \ldots, f^{p-1}(B)$ are pairwise disjoint. Also, p/q gives the number of petals sharing α as a boundary point.

First notice that if $p = 1$ (and so $q = 1$) then there exists a fixed point $\alpha \in \partial B$ such that $f'(\alpha) = 1$, i.e., there exists a weakly repelling fixed point of f and we are done. So let us assume from now on that $p > 1$.

Let $\langle\alpha\rangle$ be the cycle of points generated by the iteration of the q-periodic parabolic point α. We want to construct a sequence of open sets $\{U_k\}$, starting with a simply connected one, such that $\langle\alpha\rangle \cap \partial U_k \neq \emptyset$ and $f(U_{k+1}) = U_k$ for all $k \geq 0$.

In the following we use the so-called *Fatou coordinates*, see e.g. [34]. Without loss of generality we can assume that $\alpha = 0$ by a coordinate change, and f^p to be in normal form $f^p(z) = z(1 + az^\nu + \mathcal{O}(z^{\nu+1}))$, for some $a \in \mathbb{C}$ and $\nu = p/q$. Let $U_0 \subset B$ be the pull-back $U_0 := H^{-1}(\{w : \operatorname{Re} w > L\})$, where $H(z) := -1/\nu a z^\nu$ and $L > 0$ is large and to be precised later. It is easy to check that H is an actual conjugacy between f^p and

$$T(w) := (H \circ f^p \circ H^{-1})(w) = w + 1 + \mathcal{O}(w^{-1/\nu})\,,$$

hence we can choose L large enough so that f^p is injective on U_0 (see Figure 4.1). Also, notice that $f^p(\overline{U_0}) \subset U_0 \cup \{\alpha\}$ because of the action of T.

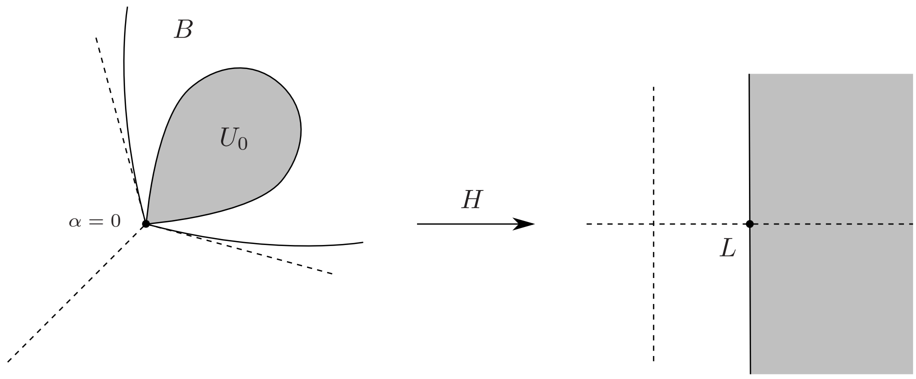


Figure 4.1: Construction of U_0 as an H-pull-back of the half-plane $\{w : \operatorname{Re} w > L\}$ (example with $\nu = 3$, so $p = 3q$). Notice that $\overline{U_0}$ contains no critical points, since $f^p|_{U_0}$ is injective. Furthermore, we can choose L in such a way that $\partial U_0 \setminus \{\alpha\}$ does not meet the *postcritical set* (forward orbits of the critical points).

Now define $\{U_k\}_{k\in\mathbb{N}}$ by pulling back U_0 under f, namely, U_k is the connected component of $f^{-k}(U_0)$ such that $\partial U_k \cap \langle\alpha\rangle \neq \emptyset$. Notice that $U_j \subset U_{p+j} \subset U_{2p+j} \subset \ldots$ and $f^j(B) = \bigcup_{k\geq 1} U_{kp-j}$, for all $0 \leq j < p$. Because B is multiply connected, there exists a (minimal) $n_0 \in \mathbb{N}$ such that $U := U_{n_0}$ is also multiply connected. Call E one of the bounded connected components of $\widehat{\mathbb{C}} \setminus U$. Notice that E is compact and full, and $\mathring{E}$ need not be connected.

Now preimages of compact sets under transcendental meromorphic maps might become unbounded and eventually contain poles and prepoles. This fact will be an obstacle to follow Shishikura's proof of the rational case, as we will show later; so, at this point, we split the proof according to the nature of ∂E.

Case 1: ∂E contains at least one pole

If ∂E contains at least one pole P, then, since $\partial E \subset \partial U$, $f(U)$ is unbounded. Because $p > 1$, $U \cap f(U) = \emptyset$ and so $f(U)$ is contained in some unbounded connected component of $\widehat{\mathbb{C}} \setminus U$. Let $V \subset \widehat{\mathbb{C}}$ be a connected simply-connected unbounded open set such that $U \subset V$ but $f(U) \subset \widehat{\mathbb{C}} \setminus \overline{V}$. For example, V could be the connected component of $\widehat{\mathbb{C}} \setminus \overline{f(U)}$ containing E. In this case we have $E \subset V$ because V is simply connected and E is bounded. Now since E is unbounded, there exists a connected component $\widetilde{U}$ of $f^{-1}(V)$ such that $P \in \partial\widetilde{U}$. Moreover, by definition we must have $\widetilde{U} \subset E$ because points immediately outside E are in U, and $f(U) \subset \widehat{\mathbb{C}} \setminus V$ (see Figure 4.2). Now, by construction of the two sets, V is connected and simply connected, and $\widetilde{U}$ is bounded and relatively compact in V, since $\widetilde{U} \subset \overline{E} \subset V$. Using Lemma 2.35 and Corollary 2.40, it follows that $f|_{\widetilde{U}} : \widetilde{U} \to V$ is indeed a rational-like map and f has a weakly repelling fixed point.

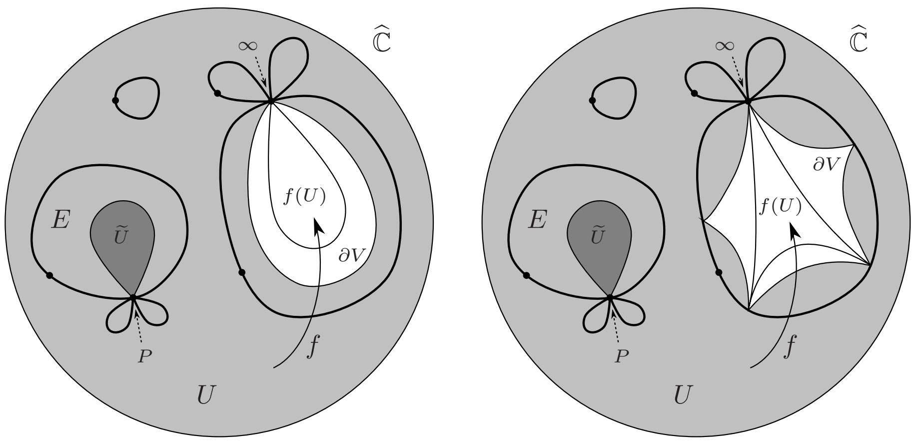


Figure 4.2: If there exists a pole P on ∂E, then there exists a set $\widetilde{U} \subset E$ such that $f(\widetilde{U}) = V$, where V is an unbounded simply-connected set that contains U but not $f(U)$. The thick lines correspond to ∂U, while the sets $\widetilde{U}$ and V appear dark- and light-shaded, respectively. The non-labelled points represent the different places where α can lie. On the right, a case where $\partial U \cap \partial f(U) \neq \emptyset$.

Case 2: ∂E contains no poles

Now if ∂E contains no poles, $f(U)$ is bounded (and simply connected by construction) therefore no other component of $\widehat{\mathbb{C}} \setminus U$ can have poles on its boundary. (Still, further images of U might be unbounded, for example, if ∂E contains prepoles.) Let us assume, without loss of generality, that $f(U), \dots, f^{k-1}(U) \subset E$ and $f^k(U) \subset \widehat{\mathbb{C}} \setminus E$, for some $1 \leq k \leq p$. In that case we will use the quasiconformal surgery technique, but must be careful with the set of preimages of α, that might intersect ∂U and make the whole process somewhat laborious.

In fact, a key point during the surgery process is the construction of an interpolating map between two different functions on two disjoint closed curves. If such curves are to touch at preimages of α or at α itself, this interpolation cannot be performed and an extra step previous to surgery will be done. Since we are focusing our attention on boundary intersections here, we shall still subdivide this case into two finer subcases as follows.

<u>Case 2.1</u>: $k < p$, or $k = p$ but $\partial f^p(U) \cap \partial E = \emptyset$

First notice that if $k < p$ then $\partial f^k(U) \cap \partial E = \emptyset$: Because, by construction, $f^p(U)$ is the first iterate to come back inside U, $f^k(U)$ is in some complementary component of U. The iterates $f^j(U)$, $j < k$, all lie inside E, but $f^k(U)$ is not in E so it is in a different component of $\widehat{\mathbb{C}} \setminus U$. Since two different components of $\widehat{\mathbb{C}} \setminus U$ cannot form a connected set we conclude that $\partial f^k(U) \cap \partial E = \emptyset$.

Now we apply quasiconformal surgery as follows: Define a quasiregular map $f_2 \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ that, after k iterations, maps $\widehat{\mathbb{C}} \setminus E$ strictly inside itself. More precisely, set $V_0 := \widehat{\mathbb{C}} \setminus E$ and $V_1 := f(U)$. Then, when $k > 1$, V_1 lies in E, and when $k = 1$, it lies in $\widehat{\mathbb{C}} \setminus E$. Set also $K := \overline{f^k(U)}$ and choose $b \in f(U)$ and $a = f^{k-1}(b) \in K$ (see Figure 4.3).

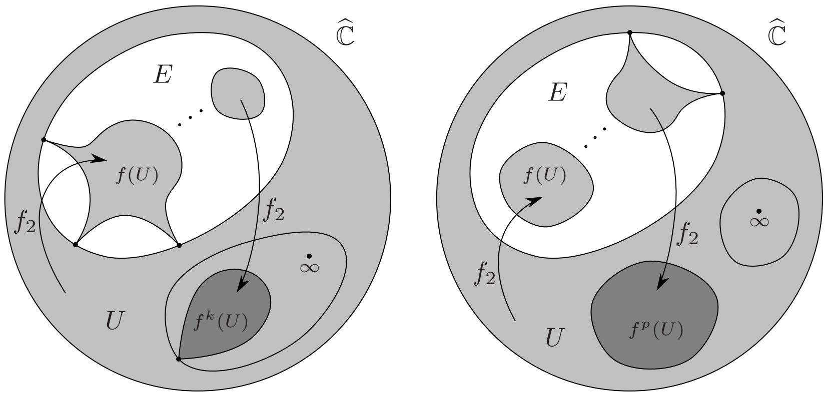


Figure 4.3: In this case, intersection between $\partial f^k(U)$ and ∂E never occurs, which is crucial for Lemma 3.4 to be applied in our case. We have drawn the cases $1 < k < p$ (left) and $k = p$ (right). In both of them, the cycle $\widehat{\mathbb{C}} \setminus E, f(U), \dots, f^k(U) \subset \widehat{\mathbb{C}} \setminus E$ appears in grey.

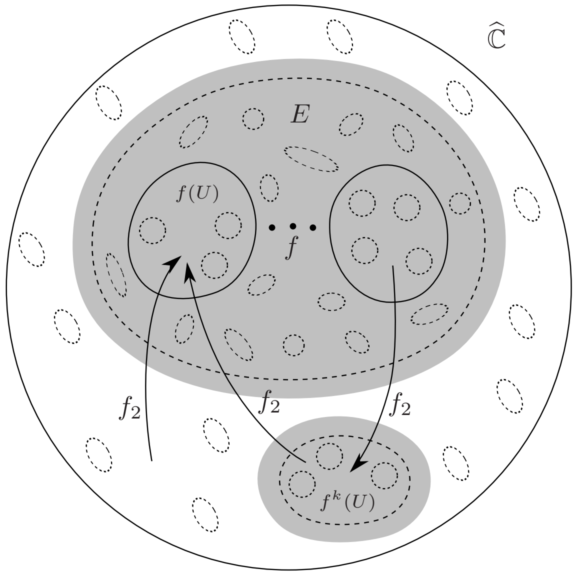


Figure 4.4: Construction of the almost complex structure σ. In grey we find the region where f_2 is holomorphic. Orbits pass through $\widehat{\mathbb{C}} \setminus (E \cup \overline{f^k(U)})$ at most once.

Now we will use the Interpolation Lemma 3.4. Applied to our case, it provides us with a quasiregular map $f_1 \colon \widehat{\mathbb{C}} \setminus E \to f(U)$ which agrees with f on ∂E, is holomorphic in a neighbourhood of $f^k(U)$ and satisfies $f_1(a) = b$.

Now we construct a map f_2 by setting $f_2 = f$ on E and $f_2 = f_1$ on $\widehat{\mathbb{C}} \setminus E$, which makes it a quasiregular map of $\widehat{\mathbb{C}}$, holomorphic in both a neighbourhood of E and a neighbourhood of $\overline{f^k(U)}$, with a k-periodic point, given that $f_2^k(a) = f^{k-1}(f_1(a)) = f^{k-1}(b) = a$. Observe also that $f_2^k(\widehat{\mathbb{C}} \setminus E) = f^k(U)$ and $\overline{f^k(U)} \subset \widehat{\mathbb{C}} \setminus E$; it follows that f_2^k is a contraction and a a global attractor for f_2^k in $\widehat{\mathbb{C}} \setminus E$.

We may define an almost complex structure σ by

$$\sigma := \begin{cases} \sigma_0 & \text{on } f(U) \\ (f_2^n)^* \sigma_0 & \text{on } f_2^{-n}(f(U)), \text{ for } n \in \mathbb{N} \\ \sigma_0 & \text{elsewhere.} \end{cases}$$

Observe that $\sigma = \sigma_0$ on $\bigcup_{i=1}^k f^i(U)$ (see Figure 4.4).

Furthermore, σ is f_2-invariant by construction and has bounded distortion, since orbits pass through $\widehat{\mathbb{C}} \setminus (E \cup \overline{f^k(U)})$ (the set where f_2 is not holomorphic) at most once.

Remark 4.2. At this point, notice the importance of the fact that f_2 be defined to be holomorphic on a neighbourhood of $\overline{f^k(U)}$, which was only possible because $f^k(U)$ is a relatively compact subset of $\widehat{\mathbb{C}} \setminus E$.

These are precisely the hypotheses of Lemma 2.29, so there exists a map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, holomorphic on the whole sphere — and hence rational —, which is conjugate to f_2 by some quasiconformal homeomorphism ϕ.

Now a theorem of Fatou ensures the existence of a weakly repelling fixed point z_0 of g, except when $\deg g = 1$ and g is an elliptic transformation. But $\phi(a)$ is an attracting k-periodic point of g, so this can never be the case.

Besides, the family $\mathcal{G} = \{g^n|_{\phi(\widehat{\mathbb{C}}\setminus E)}\}_{n\geq 1}$ omits the open set $\phi(\widehat{\mathbb{C}} \setminus (E \cup \overline{f^k(U)}))$, therefore $\mathcal{G}$ is normal in $\phi(\widehat{\mathbb{C}} \setminus E)$ by Montel's Theorem, that is, $\phi(\widehat{\mathbb{C}} \setminus E) \subset \mathcal{F}(g)$. But weakly repelling fixed points belong to the Julia set, so $z_0 \in \phi(E)$. Because such points are preserved under conjugacy, also f_2 has a weakly repelling fixed point $\phi^{-1}(z_0)$, in E; and so does f, since both functions coincide precisely on this set.

Case 2.2: $k = p$ and $\partial f^p(U) \cap \partial E \neq \emptyset$

For this case, let us first rename the elements of the periodic orbit and shift the sequence $\{U_k\}$ so that $\alpha \in \partial U \equiv \partial U_{n_0}$, i.e., so that $p|n_0$. More precisely, it is clear that there exists $0 \leq l < p$ such that $U \subset f^l(B)$; then, rename $B \equiv f^l(B)$, $\alpha \equiv f^l(\alpha)$, $U_0 \equiv f^l(U_0)$ and define the sets $U_1, \ldots, U_{l-1}$ accordingly. Notice that $U_0, \ldots, U_l$ are all simply connected by construction, but $U_{n_0} \equiv U$ is multiply connected. Since p divides n_0, we can define $c := n_0/p \in \mathbb{N}$, that is, the number of f^p-cycles from U_0 to U_{n_0} (see Figure 4.5).

Also, the sets $U_{kp+1}, \ldots, U_{(k+1)p-1} \subset E$ are necessarily bounded, so only those in the subsequence $U_0, U_p, U_{2p}, \ldots$ might become unbounded from a certain one on. In particular, only the sets of the form U_{kp+1} can have poles on their boundaries, and only the maps of the form $f|_{U_{kp}}\colon U_{kp} \to U_{kp-1}$ can be of infinite degree.

Furthermore, notice that if some intersection $\partial U_{k_1} \cap \partial U_{k_2}$ contains a preimage of some pole, then the sets U_{k_1} and U_{k_2} necessarily belong to the same subsequence $U_j \subset U_{p+j} \subset U_{2p+j} \subset \ldots$, that is, $k_1 \equiv k_2 \pmod p$ and either $U_{k_1} \subset U_{k_2}$ or $U_{k_2} \subset U_{k_1}$. In particular, only if this is the case can ∂U_{k_1} and ∂U_{k_2} share infinitely many preimages of α. This will be a key point in later arguments.

We have seen that the fact that $\partial f^p(U)$ and ∂E did not share any contact point was crucial for the quasiconformal surgery construction of Case 2.1 (see Remark 4.2). Now, the condition $\partial f^p(U) \cap \partial E \neq \emptyset$ is exactly given by hypotheses, so some extra work must be done, in the sense of modifying slightly some sets, in order to start the surgical process proper.

On the other hand, notice that the sets $\{U_k\}$ are in some sense arbitrary, since they were constructed by repeatedly pulling-back U_0, chosen arbitrarily. Also, notice that once these sets (and E) have been defined and during the process

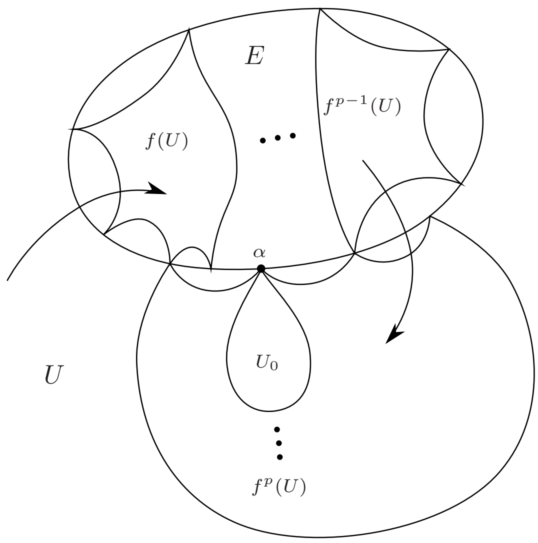


Figure 4.5: The shifted sequence $\{U_k\}$. From now on, this is the primary situation we should always bear in mind. The sets $U, f(U), \ldots, f^p(U)$ are the only ones in $\{U_k\}$ that will later play a role during the quasiconformal surgery process. Their cyclic dynamics under the action of f is also shown here.

of quasiconformal surgery (that is to say, from the construction of the auxiliar quasiregular maps on), the only sets in this sequence with a role to play are $U, f(U), \ldots, f^k(U)$ (or rather $U, f(U), \ldots, f^p(U)$ for the current case).

Thus, it seems that we can modify these sets $U, f(U), \ldots, f^p(U)$ slightly and only close to the odd contact points, so that their boundaries share as little points as possible — the following result provides us with such modification. Its proof is rather technical and will be given separately, in Section 4.1.

Proposition 4.3. *In the situation described hitherto, there exists a connected multiply-connected set* $\mathcal{U} \subset U$ *such that* $f^p(\mathcal{U})$ *is simply connected,* $\overline{f^p(\mathcal{U})} \subset \mathcal{U} \cup \{\alpha\}$ *and* $\partial f^p(\mathcal{U}) \cap \mathcal{J}(f) = \{\alpha\}$.

Now call $\mathcal{E}$ the bounded component of $\widehat{\mathbb{C}} \setminus \mathcal{U}$ that contains E. The point α need not be on $\partial\mathcal{E}$, so it could happen that $\partial f^p(\mathcal{U}) \cap \partial\mathcal{E} = \emptyset$. Were that the case, notice that $\overline{f^p(\mathcal{U})} \subset \widehat{\mathbb{C}} \setminus \mathcal{E}$ and therefore we could just repeat the surgery process of Case 2.1 — replacing U and E by their respective modifications — to find a weakly repelling fixed point of f.

Otherwise, we have $\partial f^p(\mathcal{U}) \cap \partial\mathcal{E} = \{\alpha\}$ and, as there seems to be no neat way to separate $\mathcal{E}$ from α, we will just work with a small extension of $\mathcal{E}$ whose interior contains α. More precisely, we first define $V_0 := \widehat{\mathbb{C}} \setminus \mathcal{E}$ and $V_1 := f(\mathcal{U}) \subset E \subset \mathcal{E}$, and use the Interpolation Lemma 3.4 to find a quasiregular map $f_1 \colon \widehat{\mathbb{C}} \setminus \mathcal{E} \to f(\mathcal{U})$, as usual — however, notice that we marked no compact set K nor points a and b, since now f_1 need not be holomorphic in any subset of $\widehat{\mathbb{C}} \setminus \mathcal{E}$. Also, recall that f_1 actually agrees with f in a neighbourhood N_1 of $\partial\mathcal{E}$ — call $\mathcal{N} := \mathcal{E} \cup N_1$, a neighbourhood of $\mathcal{E}$.

Lemma 4.4. *There exist a sufficiently small neighbourhood of α in $f^p(\mathcal{U})$, $\mathcal{W}^*$, an open neighbourhood $\mathcal{E}^*$ of $\mathcal{E} \cup \overline{f^p(\mathcal{W}^*)}$ in $\mathcal{N}$, and a quasiconformal map $h \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that*

- $\overline{\mathcal{W}^*} \subset \mathcal{N}$*;*
- $\overline{f^p(\mathcal{W}^*)} \subset \mathcal{E}^*$ *and* $\overline{\mathcal{E}^* \cap \mathcal{W}^*} \setminus \partial f^p(\mathcal{U}) \subset \mathcal{W}^*$*;*
- $h = \mathrm{id}$ *in* $\mathcal{E}^*$ *and* $h(f^p(\mathcal{U})) \subset \mathcal{W}^*$*.*

Roughly speaking, the map h pushes the points in $f^p(\mathcal{U})$ towards $\mathcal{E}$, but will leave points there untouched so that the action of any post-composed map be preserved entirely (see Figure 4.6).

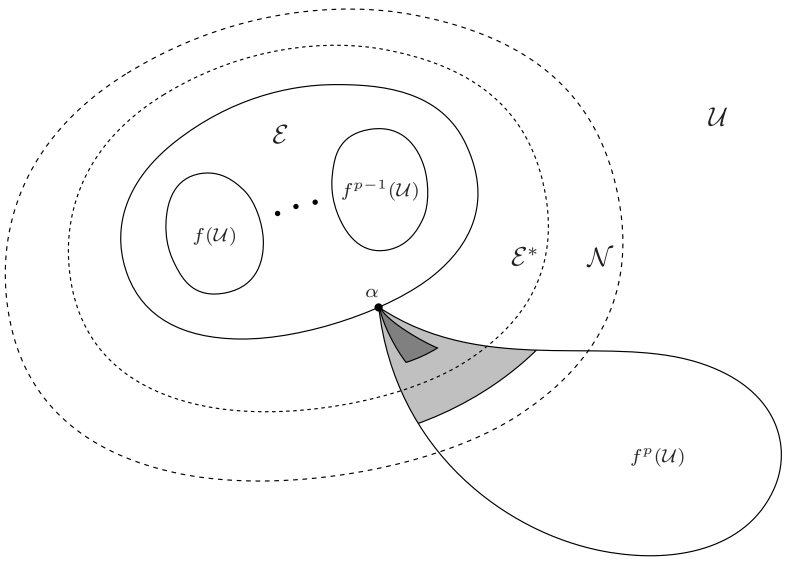


Figure 4.6: The case where $\partial f^p(\mathcal{U}) \cap \partial\mathcal{E} = \{\alpha\}$, with the sets $\mathcal{N}$, $\mathcal{W}^*$ (light-shaded), $f^p(\mathcal{W}^*)$ (dark-shaded) and $\mathcal{E}^*$. Notice that points in $f^p(\mathcal{W}^*)$ will never leave $\mathcal{E}^*$ under the action of f.

Proof. We define the set $\mathcal{W}^*$ as the connected component of $f^{-(c-1)p}(W_R)$ in $f^p(\mathcal{U})$ that has α on the boundary, with R so large as for $\overline{\mathcal{W}^*} \subset \mathcal{N}$ (see the construction of $\mathcal{W}$ in Section 4.1). By construction, it is a neighbourhood of α in $f^p(\mathcal{U})$, i.e., $\alpha \notin \overline{f^p(\mathcal{U}) \setminus \mathcal{W}^*}$, and $f^p(\overline{\mathcal{W}^*}) \subset \mathcal{W}^* \cup \{\alpha\}$. In particular, the existence of one such $\mathcal{E}^*$ follows from the latter.

Now let S be the simply-connected open set $f^p(\mathcal{U}) \setminus \overline{\mathcal{E}^*}$ with a marked boundary segment at $l := \partial S \cap \partial \mathcal{E}^*$. There exists a (conformal) Riemann map $\varphi\colon S \to Q$ that sends l to one of the sides of the open unit square Q. Consider a (quasiconformal) homothetic transformation $\tilde{h}_0\colon \overline{Q} \to \tilde{h}_0(\overline{Q})$ such that $\tilde{h}_0|_{\varphi(l)} = \mathrm{id}$ and $\tilde{h}_0(Q) \cap \varphi(S \cap \partial\mathcal{W}^*) = \emptyset$.

Finally, define the conjugate map $h_0 := \varphi^{-1} \circ \tilde{h}_0 \circ \varphi\colon S \to h_0(S)$, which is quasiconformal (see Figure 4.7). Notice that $h_0|_l = \mathrm{id}$, so we can define

$$h := \begin{cases} h_0 & \text{on } S = f^p(\mathcal{U}) \setminus \overline{\mathcal{E}^*} \\ \mathrm{id} & \text{on } \overline{\mathcal{E}^*} \end{cases}$$

and extend it quasiconformally to a map $h\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

□

Now consider the quasiregular map $f_2\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined as

$$f_2 := \begin{cases} f \circ h & \text{on } \mathcal{E}^* \\ f_1 \circ h & \text{on } \widehat{\mathbb{C}} \setminus \mathcal{E}^* \end{cases} = \begin{cases} f & \text{on } \mathcal{E}^* \\ f_1 \circ h & \text{on } \widehat{\mathbb{C}} \setminus \mathcal{E}^*. \end{cases}$$

Also, consider the (shrinking) f_2-cycle $C := f(\mathcal{W}^*) \cup \ldots \cup f^p(\mathcal{W}^*) \subset \mathcal{E}^*$. Indeed, it is cyclic because $f_2(C) = f(C) \subset C$ (see Figure 4.8).

Setting $X := \widehat{\mathbb{C}} \setminus \mathcal{E}^*$, orbits of f_2 pass through X at most twice, since

$$\begin{aligned} \cdots \xrightarrow{f_2} f_2^{-1}(X) \xrightarrow{f_2} X \xrightarrow{h} X \subset \widehat{\mathbb{C}} \setminus \mathcal{E} \xrightarrow{f_1} f(\mathcal{U}) \xrightarrow{f_2^{p-1}} f^p(\mathcal{U}) \\ \xrightarrow{h} \mathcal{W}^* \xrightarrow{f} f(\mathcal{W}^*) \subset C \xrightarrow{f_2} C \xrightarrow{f_2} \cdots \subset \widehat{\mathbb{C}} \setminus X. \end{aligned}$$

Define the almost complex structure

$$\sigma := \begin{cases} \sigma_0 & \text{on } C \\ (f_2^n)^*\sigma & \text{on } f_2^{-n}(f(\mathcal{W}^*)),\ n \in \mathbb{N} \\ \sigma_0 & \text{elsewhere,} \end{cases}$$

which clearly is f_2-invariant by definition, and has bounded dilatation since f_2 fails to be holomorphic only at most twice. Therefore we can use Lemma 2.29 to find a rational map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ conjugate to f_2 by a quasiconformal homeomorphism ϕ.

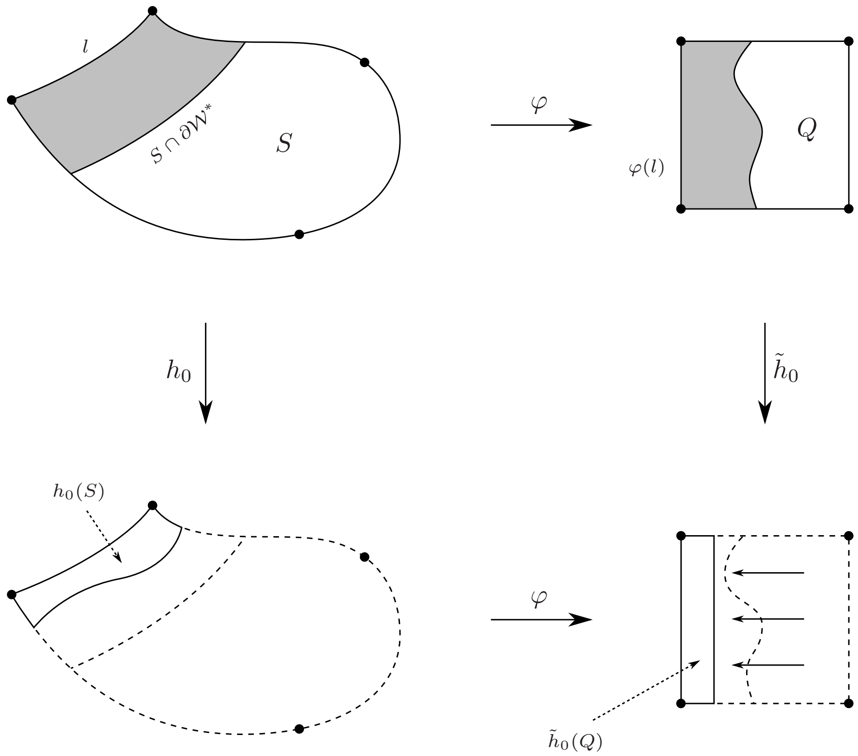


Figure 4.7: For the construction of the map $h\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, we first define an auxiliary map $h_0\colon S \to h_0(S)$ as a conjugation of a quasiconformal map on Q, where it is easy to define the desired local dynamics. In grey we find $S \cap \mathcal{W}^*$ (and its φ-image), the subset where we want $h_0(S)$ to end up.

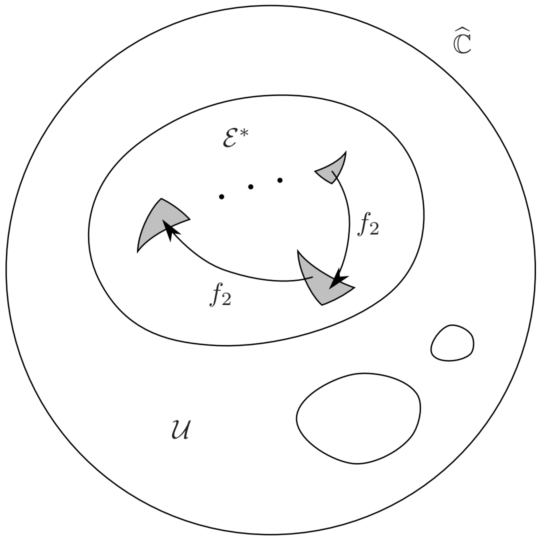


Figure 4.8: The action of f_2 on the cycle C, shaded. Notice that $f^p(\mathcal{W}^*) \subset \mathcal{W}^*$, so its f_2-image falls again in $f(\mathcal{W}^*)$.

Thus,

$$\overline{g^{p+1}(\phi(X))} = \overline{\phi(f_2(f_2^p(X)))} \subset \overline{\phi(f_2(f^p(\mathcal{U})))} \subset \overline{\phi(C)} \subset \phi(C) \cup \langle\phi(\alpha)\rangle \,,$$

so $\phi(X)$ is contained in the basin of an attracting or parabolic point. By Fatou's theorem, g has a weakly repelling fixed point in $\phi(\widehat{\mathbb{C}} \setminus X) = \phi(\mathcal{E}^*)$, hence f has a weakly repelling fixed point in $\mathcal{E}^*$.

4.1 Proof of Proposition 4.3

When removing points of $\partial f^p(U) \cap \partial E$, there is a particular point we cannot ignore — that is α itself: Because its attracting dynamics in a whole petal contained in the parabolic basin (Fatou coordinates about a parabolic point), if we redefined U as some $\widetilde{U}$ in such a way that α were not on its boundary, then points close to α would become even closer under the action of f^p, and the condition $f^p(\widetilde{U}) \subset \widetilde{U}$ would be lost (see Figure 4.9).

Rather, for an appropriate construction of one such $\mathcal{U}$ we need to modify the sets $U, f(U), \ldots, f^p(U)$ close to the contact points between their boundaries except those in the cycle $\langle\alpha\rangle$ (see Figure 4.10).

When doing so, it is clear that if the point α does not lie on the intersection $\partial f^p(\mathcal{U}) \cap \partial E$ (Figure 4.10, left), the situation is then identical to that of Case 2.1,

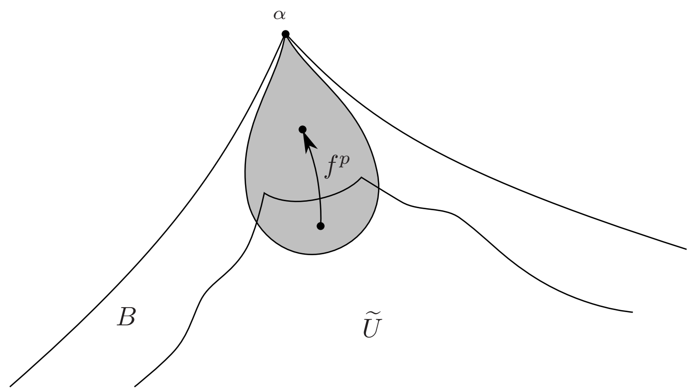


Figure 4.9: If the new set $\widetilde{U}$ left out some neighbourhood of α, there would be points in it stepping outside it under f^p. The shaded set represents the attracting petal attached to α given by the Fatou coordinates.

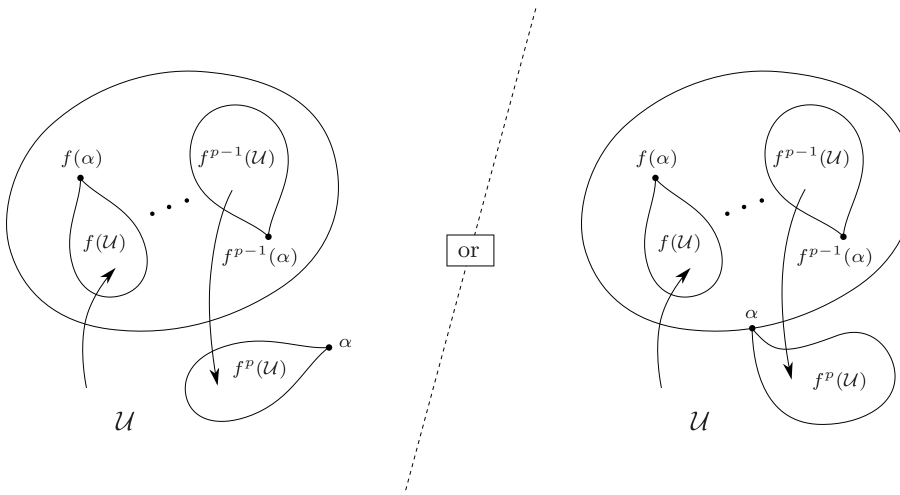


Figure 4.10: The situation we want, with the points in $\langle\alpha\rangle$ marked. Notice that a priori we do not know whether α is on ∂E or not, since the set E was chosen arbitrarily as one of the bounded connected components of the complement of U; in particular, α could even happen to be on the boundary of the unbounded component of the complement of $\mathcal{U}$. It is clear that surgery cannot be used just as in Case 2.1 when $\partial f^p(\mathcal{U}) \cap \partial E$ remains nonempty (right-hand side figure).

and therefore we can conclude the case following an analogous surgical procedure. In case α does belong to $\partial f^p(\mathcal{U}) \cap \partial E$ (Figure 4.10, right), we must define another auxiliary map before we can proceed. The end of the proof then follows with a different quasiconformal surgery argument.

Let us now construct the modification of $U, f(U), \ldots, f^p(U)$. The idea is the following: Since the ultimate aim of such modification is to eliminate contact points between $\partial f^p(U)$ and ∂E, it suffices to modify only the set $U_{n_0-p} \equiv f^p(U)$ and redefine the sets $U_{n_0-p+1} \equiv f^{p-1}(U), \ldots, U_{n_0} \equiv U$ by repeatedly pulling-back this first modification, appropriately. Of course if the changes on these sets are arbitrarily small, and, therefore, the new sets are arbitrarily close to the original ones, their respective connectivities are also to be preserved (see Figure 4.11).

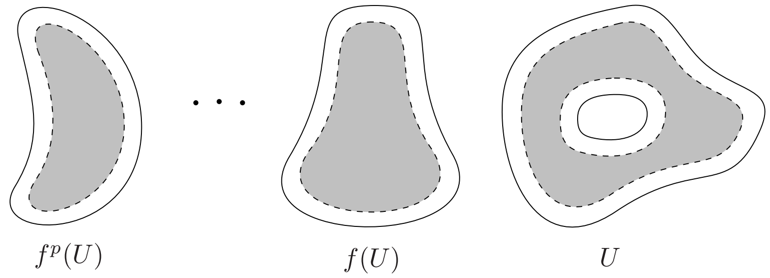


Figure 4.11: The set U is multiply connected and so is its modification (shaded here) if this one differs little from U. Similarly, the sets $f(U), \ldots, f^p(U)$ are simply connected and so are their modifications.

Following such reasoning, one could think that the modification of $f^p(U)$, which we can call $\mathcal{V}$, could simply be obtained by removing from $f^p(U)$ a disc of arbitrarily small radius centered at every contact point between $\partial f^p(U)$ and ∂E (see Figure 4.12).

But of course we want to keep the property $f^p(U) \subset U$ for the subsequent surgical work, and, if we just removed those discs taking no control whatsoever over their preimages, such inclusion could be lost: Consider a point $a \in \mathcal{A} := \partial f^p(U) \cap \partial E \setminus \{\alpha\} \subset \mathcal{J}(f)$ — for instance some $a \in \mathcal{O}^-(\alpha)$ — with some preimage $b \in f^{-p}(a)$ on the same set $\mathcal{A}$. Suppose we were to remove discs $B_\varepsilon(a)$ and $B_\varepsilon(b)$ of small radius ε centered at the points $a, b \in \mathcal{A}$ in defining $\mathcal{V}$. If the preimage of $B_\varepsilon(a)$ under f^p were to become big enough to contain points in the complement of $B_\varepsilon(b)$, then there would be points $z_0 \in (f^{-p}(B_\varepsilon(a)) \cap f^p(U)) \setminus B_\varepsilon(b)$ such that $f^p(z_0) \in B_\varepsilon(a) \subset \widehat{\mathbb{C}} \setminus \mathcal{V}$, that is to say, $z_0 \notin f^{-p}(\mathcal{V})$. Then we would have $z_0 \in \mathcal{V} \setminus f^{-p}(\mathcal{V}) \neq \emptyset$, which is precisely what we want to avoid. (See Figure 4.13.)

This very description of the problem with the preimages of points we remove from $f^p(U)$ for the construction of $\mathcal{V}$ also provides us with a hint about how to solve it, since, in the previous example, it would have been enough to take

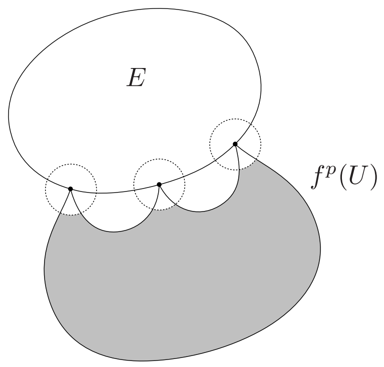


Figure 4.12: A first attempt towards the construction of $\mathcal{V}$, shaded.

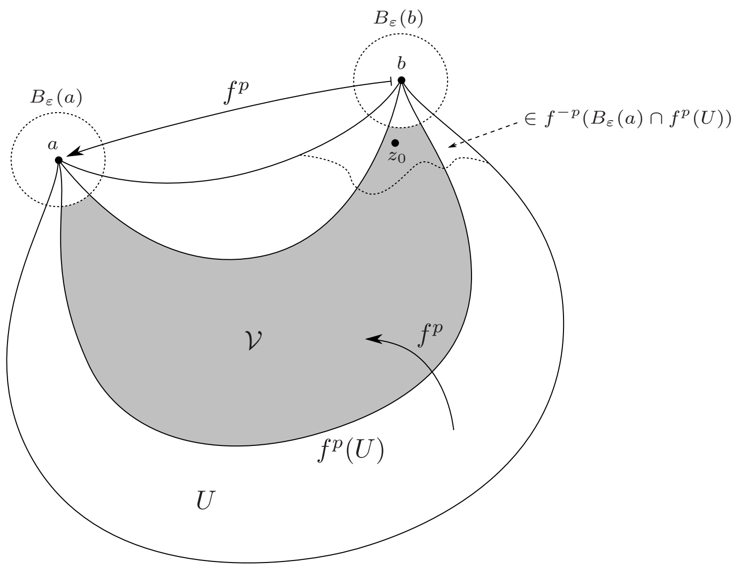


Figure 4.13: We want to keep $f^p(U) \subset U$ after the modification, i.e., we want a set $\mathcal{V}$ such that $\mathcal{V} \subset f^{-p}(\mathcal{V})$. However, if we defined it as the shaded set in this figure, there would exist points $z_0 \in \mathcal{V} \setminus f^{-p}(\mathcal{V})$ — so we need to take some control over the preimages of the discs we remove from $f^p(U)$.

$f^{-p}(B_\varepsilon(a)) \cap f^p(U)$ instead of $B_\varepsilon(b)$ so as to avoid points like z_0.

In other words, we must also exclude from $\mathcal{V}$ all the points in $f^p(U)$ whose f^p-image falls on points we "already" removed from $f^p(U)$. In fact, this generates, in turn, more points whose preimage need be controlled; and so on. Regardless of what may be expected, this is not an endless recurrent process. We have $f^p(U) \equiv U_{n_0-p} \subset f^{-n_0+p}(U_0)$ and, therefore, after $n_0 - p$ iterations all the points in $f^p(U)$ happen to be close to α — precisely in U_0. We will see that we can make $U_0 \subset \mathcal{V}$ provided that ε is chosen small enough (see Figure 4.14). At the same time, we need to be careful when taking all these preimages, since they could become so big as to impede the construction of $\mathcal{V}$.

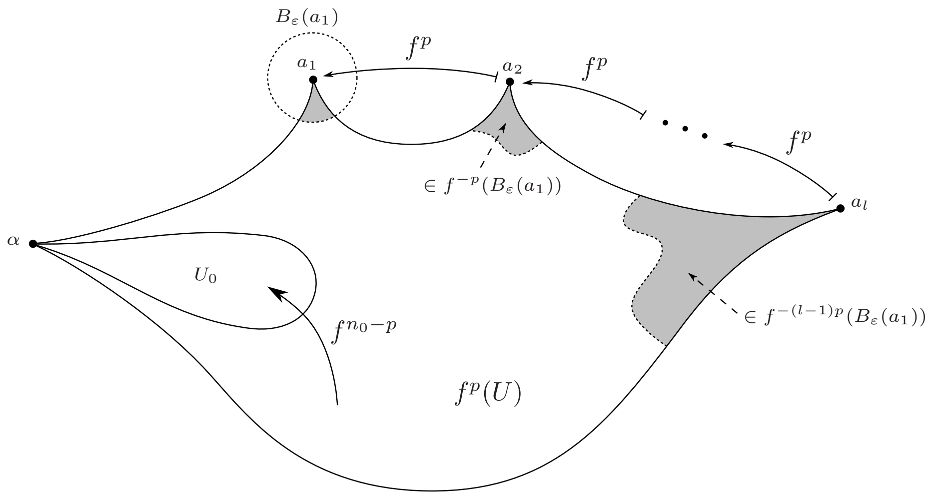


Figure 4.14: Because $f^p(U) \equiv U_{n_0-p} = U_{(c-1)p}$ and $f^p(\overline{U_0}) \subset U_0 \cup \{\alpha\}$, $\partial f^p(U)$ cannot contain preimages of higher order. Thus, given a sequence of points $a_l \mapsto \cdots \mapsto a_2 \mapsto a_1$ of $\mathcal{A}$, with $1 \le l < c$, the points $z_0 \in f^{-(l-1)p}(B_\varepsilon(a_1))$ will eventually fall inside $U_0 \subset \mathcal{V}$ and we need not worry about their preimages any more.

For all $\varepsilon > 0$, let

$$V_\varepsilon := f^p(U) \setminus \bigcup_{a \in \mathcal{A}} \bigcup_{k=0}^{c-2} f^{-kp}(B_{2^k\varepsilon}(a)).$$

The remaining part of the proof consists on showing that, for ε small enough, the set $\mathcal{V} \equiv V_\varepsilon$ is exactly the one we want.

First of all notice that $f^p(V_\varepsilon) \subset V_\varepsilon$ by definition. Now we will show that the preimages $f^{-kp}(B_{2^k\varepsilon}(a))$ can be controlled in such a way that none of them reaches the point α, otherwise excluded from V_ε. The following lemma gives sufficient conditions for this not to happen.

Lemma 4.5. *In the situation described hitherto, there exists* $\varepsilon_0 > 0$ *such that* $\alpha \in \partial V_\varepsilon$ *for all* $\varepsilon < \varepsilon_0$.

Before its proof, we define two sets which, because of their importance, will be used also beyond this result. These sets are both neighbourhoods (in $f^p(U)$) of α and provide useful information about the dynamics of f^p close to this point.

The first set to be constructed, $\mathcal{C}$, is a neighbourhood of α whose boundary contains no points of $\mathcal{A}$. For this, notice that $\mathcal{A}$ consists only of points of $\mathcal{O}^-(\alpha)$ and $\mathcal{O}^-(\infty)$, since $\mathcal{A} \subset \partial f^p(U) \cap \mathcal{J}(f)$ and, by construction of the sequence $\{U_k\}$, we have $f^{(c-1)p}(\partial f^p(U)) = \partial U_0 \subset \mathcal{F}(f) \cup \{\alpha\}$. More precisely,

$$\mathcal{A} \subset \bigcup_{1 \leq k < c} (f^{-kp}(\alpha) \cup f^{-(k-1)p}(\infty))$$

or, simply,

$$\mathcal{A} \subset f^{-(c-1)p}(\alpha) \cup \bigcup_{k=0}^{c-2} f^{-kp}(\infty)$$

if we take into account that α is q-periodic and so p-periodic. In particular, the set $\mathcal{A}$ finds its accumulation points only in $\bigcup_{k=0}^{c-2} f^{-kp}(\infty)$, and the points in $f^{-(c-1)p}(\alpha) \cap \mathcal{A}$ are all isolated in $\mathcal{A}$ (since $f^{-(c-1)p}(\alpha) \cap \bigcup_{k=0}^{c-2} f^{-kp}(\infty) = \emptyset$ because α is a periodic point). In the same way, since α is not an accumulation point of $\mathcal{A}$, there exists a simply-connected open sector $\mathcal{C} \subset f^p(U)$ such that $\alpha \in \partial \mathcal{C}$, $\alpha \notin \overline{f^p(U) \setminus \mathcal{C}}$ and $\overline{\mathcal{C}} \cap \mathcal{A} = \emptyset$ (see Figure 4.15). Actually, we can still shrink it slightly so that $\mathcal{A}$ does not meet a whole (sufficiently small) neighbourhood of $\overline{\mathcal{C}}$ — we will use this later, in order to see some technical detail.

On the other hand, we want to construct another neighbourhood of α in $f^p(U)$, to be called $\mathcal{W}$, with dynamics similar to that of U_0 in the sense that $f^p(\overline{\mathcal{W}}) \subset \mathcal{W} \cup \{\alpha\}$; in other words, the set $\mathcal{W}$ will control those points in $f^p(U)$ that happen to be already close to the point α. Notice that we cannot take U_0 itself as $\mathcal{W}$ because U_0 need not be a neighbourhood of α in $f^p(U)$, that is, $\alpha \in \overline{f^p(U) \setminus U_0}$ in general; but the construction of U_0 does inspire the use of Fatou coordinates in order to provide $\mathcal{W}$ with its same dynamics. More precisely, we will construct a subset of U_0 in a very similar fashion and then define $\mathcal{W}$ as an appropriate preimage of it in $f^p(U)$.

In fact, for all $R > L$, let

$$W_R := H^{-1}(\{w \in \mathbb{C} : \operatorname{Re} w > L, \operatorname{Re} w + |\operatorname{Im} w| > R\}) \subset U_0,$$

where recall that $H(z) = -1/\nu a z^\nu$ conjugates the maps f^p and $T(w) = w + 1 + O(w^{-1/\nu})$, and $L > 0$ is large enough for f^p to be injective on U_0 (see Figure 4.16).

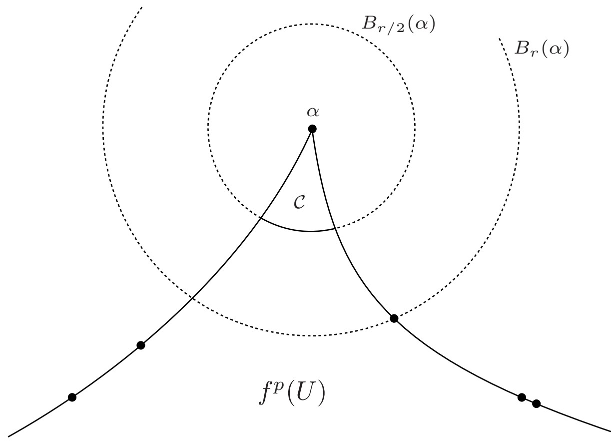


Figure 4.15: The non-labelled points represent the set $\mathcal{A}$. Since they never accumulate on α, there certainly exists such an open set $\mathcal{C}$, as shown. Furthermore, because α is a parabolic point, in a sufficiently small neighbourhood of it $f^p(U)$ is essentially a wedge like that of an attracting petal, so we can even take $\mathcal{C}$ as $B_r(\alpha) \cap f^p(U)$ with r so small as for $\mathcal{C}$ to be connected and $\overline{\mathcal{C}} \cap \mathcal{A} = \emptyset$. Even more, taking $\mathcal{C} = B_{r/2}(\alpha) \cap f^p(U)$ we ensure not only its closure but also a whole neighbourhood of $\overline{\mathcal{C}}$ free from points of $\mathcal{A}$.

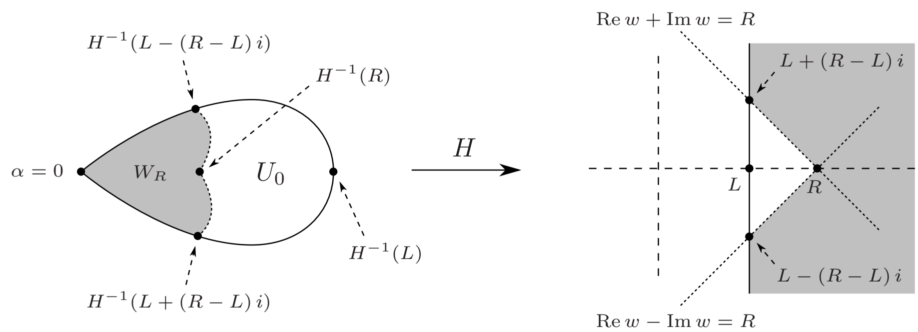


Figure 4.16: Using the same Fatou coordinates setting as in the construction of U_0, we can define W_R as a subset of it in such a way that f^p keeps its injectivity also in the subset. By taking R sufficiently large, W_R can be embedded in any (arbitrarily small) neighbourhood of α.

It is clear that since we took L so large as for $T(w) \approx w+1$ and $f^p(\overline{U_0}) \subset U_0 \cup \{\alpha\}$, then, for any $R > L$, also $f^p(\overline{W_R}) \subset W_R \cup \{\alpha\}$ holds. Moreover, W_R is a neighbourhood of α in U_0 (i.e., $\alpha \notin \overline{U_0 \setminus W_R}$), since $H(\alpha) = \infty$ and $H(\overline{U_0 \setminus W_R}) = \{w \in \mathbb{C} : \operatorname{Re} w \geq L, \operatorname{Re} w + |\operatorname{Im} w| \leq R\}$, which is a compact set.

Consider now the connected component of the preimage $f^{-(c-1)p}(W_R)$ in $f^p(U)$ that has α on the boundary (or, equivalently, contains W_R). If R were close to L, then W_R would be close to U_0 and its preimage close to $f^p(U)$, so the character of neighbourhood of α would be lost. Let us show, then, that we can choose a sufficiently large R in such a way that this preimage lies even inside the just-constructed neighbourhood $\mathcal{C}$: Consider the image set $f^{(c-1)p}(\mathcal{C}) \subset U_0$; notice that $\alpha \notin \overline{U_0 \setminus f^{(c-1)p}(\mathcal{C})}$ since, by construction of $\mathcal{C}$, there are no preimages of α on $\overline{\mathcal{C}}$. Therefore, there exists $R_0 > L$ such that $W_R \subset f^{(c-1)p}(\mathcal{C}) \subset U_0$ for any $R > R_0$ (see Figure 4.17). Define $\mathcal{W}$ as the connected component of $f^{-(c-1)p}(W_R)$ in $f^p(U)$ that has α on the boundary, for $R > R_0$, and thus $\mathcal{W} \subset \mathcal{C}$. It follows that $f^p(\overline{\mathcal{W}}) \subset \mathcal{W} \cup \{\alpha\}$ and $\alpha \notin \overline{f^p(U) \setminus \mathcal{W}}$, since, once again, $\overline{\mathcal{C}} \cap \mathcal{A} = \emptyset$.

This concludes the construction of the sets $\mathcal{C}$ and $\mathcal{W}$, so we are now in a position to prove Lemma 4.5.

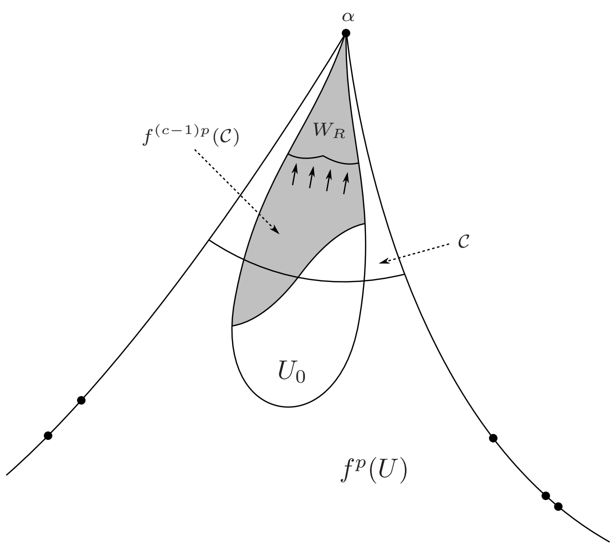


Figure 4.17: Since $\alpha \notin \overline{U_0 \setminus f^{(c-1)p}(\mathcal{C})}$, the set W_R can be shrunk arbitrarily until $W_R \subset f^{(c-1)p}(\mathcal{C})$. Notice that $f^{(c-1)p}(\mathcal{C})$, shaded here, need not be contained in $\mathcal{C}$, so $\mathcal{C}$ itself or even an image of it cannot serve as W_R.

Proof of Lemma 4.5. Consider one of the preimages $f^{-kp}(B_{2^k\varepsilon}(a))$ and suppose that $\alpha \in f^{-kp}(B_{2^k\varepsilon}(a))$. If this were the case, and because $\alpha \in \partial\mathcal{W}$, we would have that $f^{-kp}(B_{2^k\varepsilon}(a)) \cap \mathcal{W} \neq \emptyset$; so let $z_0 \in f^{-kp}(B_{2^k\varepsilon}(a)) \cap \mathcal{W}$. Then,

$$\begin{array}{ccccccc} \mathcal{C} \supset & \mathcal{W} & \supset & f^p(\mathcal{W}) & \supset \cdots \supset & f^{kp}(\mathcal{W}) \\ & \Cup & & \Cup & & \Cup \\ & z_0 & & f^p(z_0) & & f^{kp}(z_0) \end{array}$$

that is, $f^{kp}(z_0) \in \mathcal{C}$. On the other hand, from the fact that $z_0 \in f^{-kp}(B_{2^k\varepsilon}(a)) \cap \mathcal{W}$ it also follows that $f^{kp}(z_0) \in B_{2^k\varepsilon}(a)$; therefore, the point $f^{kp}(z_0)$ would belong to both sets: $f^{kp}(z_0) \in \mathcal{C} \cap B_{2^k\varepsilon}(a)$.

However, since $\mathcal{A}$ does not meet some neighbourhood of $\overline{\mathcal{C}}$, it is clear that there exists $\varepsilon_0 > 0$ such that $\mathcal{C} \cap B_{2^k\varepsilon}(a) = \emptyset$ for any $\varepsilon < \varepsilon_0$ and $a \in \mathcal{A}$. Therefore, it suffices to take $\varepsilon < \varepsilon_0$ to obtain $f^{kp}(z_0) \notin \mathcal{C} \cap B_{2^k\varepsilon}(a) = \emptyset$ and $\alpha \notin f^{-kp}(B_{2^k\varepsilon}(a))$. But α does belong to $\partial f^p(U)$ so, right from the definition of V_ε, we have $\alpha \in \partial V_\varepsilon$ for all $\varepsilon < \varepsilon_0$. ☐

Remark 4.6. Notice that the key point of this proof lies in the fact that the preimages $f^{-kp}(B_{2^k\varepsilon}(a))$ are considered only up to order $k = c - 2$. Of course, if we were to take preimages of $B_{2^k\varepsilon}(a)$ indefinitely, we would surely end up meeting $\mathcal{C}$ because $B_{2^k\varepsilon}(a)$ is a neighbourhood of a point $a \in \mathcal{A} \subset \mathcal{J}(f)$; but, then, also preimages of α would accumulate on α itself so the construction of one such $\mathcal{C}$ would never be possible.

The next step towards the construction of $\mathcal{V}$ is to insure that $\mathcal{U}$ will keep multiple connectivity. This is precisely what the following lemma does.

Lemma 4.7. *In the situation described hitherto, there exists $\varepsilon_1 > 0$ such that $f^{-p}(V_\varepsilon)$ has a multiply-connected component in U that separates E and the unbounded connected component of $\widehat{\mathbb{C}} \setminus U$, for all $\varepsilon < \varepsilon_1$.*

Proof. Since U is multiply connected, let $\gamma \subset U$ be a generator path of its fundamental group (as a topological space) such that E and the unbounded connected component of $\widehat{\mathbb{C}} \setminus U$ sit in different connected components of $\widehat{\mathbb{C}} \setminus \gamma$ (see Figure 4.18).

Consider now the images $\{f^{kp}(\gamma)\}_{1 \le k < c}$ in $f^p(U)$. Because γ does not accumulate on points of $\mathcal{J}(f)$, neither do the curves $f^{kp}(\gamma)$ accumulate on points of $\mathcal{A}$, and, therefore, there exist $\{\varepsilon_{1,k} > 0\}_{1 \le k < c}$ such that, for each $1 \le k < c$, $f^{kp}(\gamma) \cap B_{2^k\varepsilon}(a) = \emptyset$ for any $\varepsilon < \varepsilon_{1,k}$ and $a \in \mathcal{A}$ (see Figure 4.19).

In this way, if $\varepsilon < \varepsilon_1$, where

$$\varepsilon_1 := \min_{1 \le k < c} \varepsilon_{1,k}\,,$$

then $f^{kp}(\gamma) \cap B_{2^k\varepsilon}(a) = \emptyset$ for any $1 \le k < c$ and $a \in \mathcal{A}$. Let us show that it follows from here that $\gamma \subset f^{-p}(V_\varepsilon)$ for all $\varepsilon < \varepsilon_1$: If it were otherwise, $\gamma \nsubseteq$

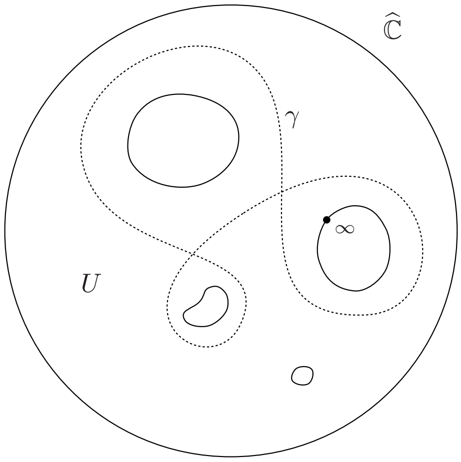


Figure 4.18: One such generator path γ, as seen on the Riemann sphere. Notice that it need not separate all the connected components of $\widehat{\mathbb{C}} \setminus U$ pairwise, although it might separate components other than E and the unbounded one.

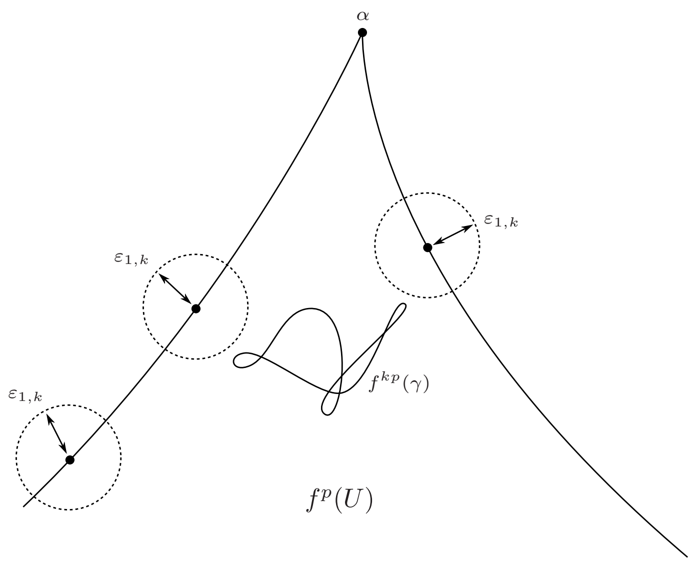


Figure 4.19: For each $1 \leq k < c$, the radius $\varepsilon_{1,k}$ can be chosen in such a way that $f^{kp}(\gamma) \cap B_{2^k\varepsilon}(a) = \emptyset$ for any $\varepsilon < \varepsilon_{1,k}$ and $a \in \mathcal{A}$. Here, the set $\mathcal{A}$ is again represented by the non-labelled points, and we show just one step $1 \leq k < c$ for the sake of clarity.

$f^{-p}(V_\varepsilon)$, then we would have $f^p(\gamma) \not\subseteq V_\varepsilon$ and, since $f^p(\gamma) \subset f^p(U)$ and $V_\varepsilon = f^p(U) \setminus \bigcup_{a\in\mathcal{A}} \bigcup_{k=0}^{c-2} f^{-kp}(B_{2^k\varepsilon}(a))$, there would exist $0 \le k \le c-2$ and $a \in \mathcal{A}$ for which $f^p(\gamma) \cap f^{-kp}(B_{2^k\varepsilon}(a)) \neq \emptyset$. So let $z_0 \in f^p(\gamma) \cap f^{-kp}(B_{2^k\varepsilon}(a))$; taking f^{kp}-images we would have $f^{kp}(z_0) \in f^{(k+1)p}(\gamma) \cap B_{2^k\varepsilon}(a)$ for some $0 \le k \le c-2$, that is, $f^{kp}(\gamma) \cap B_{2^k\varepsilon}(a) \neq \emptyset$ for some $1 \le k < c$, which is in contradiction with the construction of ε_1.

Finally, from the fact that $\gamma \subset f^{-p}(V_\varepsilon)$ for all $\varepsilon < \varepsilon_1$ and from the choice of $\gamma \subset U$, the lemma follows straightforwardly. □

Last, and in a similar spirit to that of the previous lemma, we also want to control the topology of V_ε itself, since it might happen to consist of more than one connected component due to the removal of the preimages $f^{-kp}(B_{2^k\varepsilon}(a))$ (see Figure 4.20).

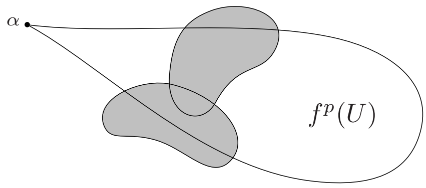


Figure 4.20: When removing the preimages $f^{-kp}(B_{2^k\varepsilon}(a))$ (shaded) from $f^p(U)$, the resulting set might be disconnected.

This will pose no problem if we focus only on the connected component of V_ε that has α on its boundary, V_ε^*; but we do have to make sure that the f^p-preimage of such component will generate a multiply-connected set, as expected.

Lemma 4.8. *In the situation described hitherto, there exists $\varepsilon_2 > 0$ such that $f^{-p}(V_\varepsilon^*)$ has a component like that of the previous lemma, for all $\varepsilon < \varepsilon_2$.*

Proof. The construction here is very similar to the proof of Lemma 4.7. In fact, consider $f^p(\gamma) \subset f^p(U)$, where $\gamma \subset U$ is that path which separates E and the unbounded connected component of $\widehat{\mathbb{C}} \setminus U$. Since $f^p(U)$ is simply connected and, in particular, path-connected, there exists a (continuous) path

$$\xi\colon [0,1] \to f^p(U) \cup \{\alpha\}$$

such that $\xi(0) = \alpha$ and $\xi(1) \in f^p(\gamma)$ (see Figure 4.21).

Consider now the images $\{f^{kp}(\xi)\}_{0\le k\le c-2}$ in $f^p(U)$. Because ξ does not accumulate on points of $\mathcal{J}(f) \setminus \{\alpha\}$, neither do the curves $f^{kp}(\xi)$ accumulate on points of $\mathcal{A}$, and, therefore, there exist $\{\varepsilon_{2,k} > 0\}_{0\le k\le c-2}$ such that, for each

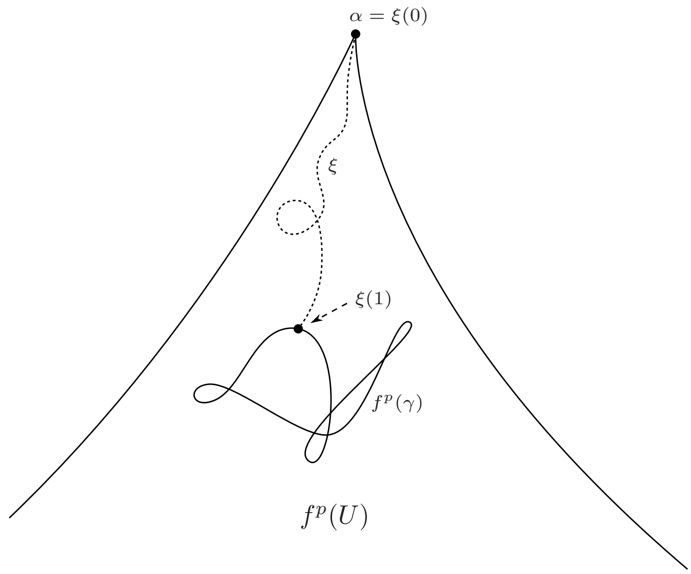


Figure 4.21: We can connect α and $f^p(\gamma)$ by a path ξ in $f^p(U) \cup \{\alpha\}$.

$0 \leq k \leq c-2$, $f^{kp}(\xi) \cap B_{2^k\varepsilon}(a) = \emptyset$ for any $\varepsilon < \varepsilon_{2,k}$ and $a \in \mathcal{A}$. In this way, it is clear that if $\varepsilon < \varepsilon_2$, where

$$\varepsilon_2 := \min_{0 \leq k \leq c-2} \varepsilon_{2,k}\,,$$

then $\xi \cap f^{-kp}(B_{2^k\varepsilon}(a)) = \emptyset$ for any $0 \leq k \leq c-2$ and $a \in \mathcal{A}$, that is to say, $\xi \subset V_\varepsilon$ and therefore $f^p(\gamma) \subset V_\varepsilon^*$.

Using an identical argument to that of the proof of Lemma 4.7 the result follows. □

This completes the construction of the modification of $f^p(U)$, since now it just remains to define $\mathcal{V} \subset f^p(U)$ as V_ε^* for some $\varepsilon < \varepsilon_0, \varepsilon_1, \varepsilon_2$, and $\mathcal{U} \subset U$ as the multiply-connected component of $f^{-p}(\mathcal{V})$ that separates E and the unbounded connected component of $\widehat{\mathbb{C}} \setminus U$ given by Lemma 4.8.

5 Preperiodic Fatou components

We have so far closed the cases of the attractive basin and the parabolic basin. Notice that our proof became specially laborious in those situations where we were unable to apply quasiconformal surgery techniques, in other words, when we could not find a multiply-connected open set with simply-connected image.

However, the case we will deal with in this section starts exactly with and is actually defined by this very hypothesis, so it is no surprise that the preperiodic case shall be proven using only surgery — in fact, using surgery in a fashion very similar to that of Shishikura's for the rational (preperiodic) case. We want to prove the following.

Theorem 5.1 (**Preperiodic Fatou components case**). *Let f be a transcendental meromorphic function with a multiply-connected (strictly preperiodic) Fatou component U such that $f(U)$ is simply connected. Then, there exists at least one weakly repelling fixed point of f.*

It is clear that U is a connected component of $f^{-1}(f(U))$, since U is a Fatou component itself. Let E be one of the bounded components of $\widehat{\mathbb{C}} \setminus U$ (one such component always exists because U is multiply connected).

In analogy to the rational case, let us focus our attention on the sequence of iterations $\{f^k(U)\}_{k\in\mathbb{N}}$. Notice that, in the preperiodic case, such iterations will not necessarily eventually abandon E because they will never come back to U. This fact gives raise to two quite different situations, depicted in Figure 5.1.

Notice that Case (b) is exactly the situation we already treated in the attractive case, so an analogous procedure gives a global quasiregular map f_2, with its

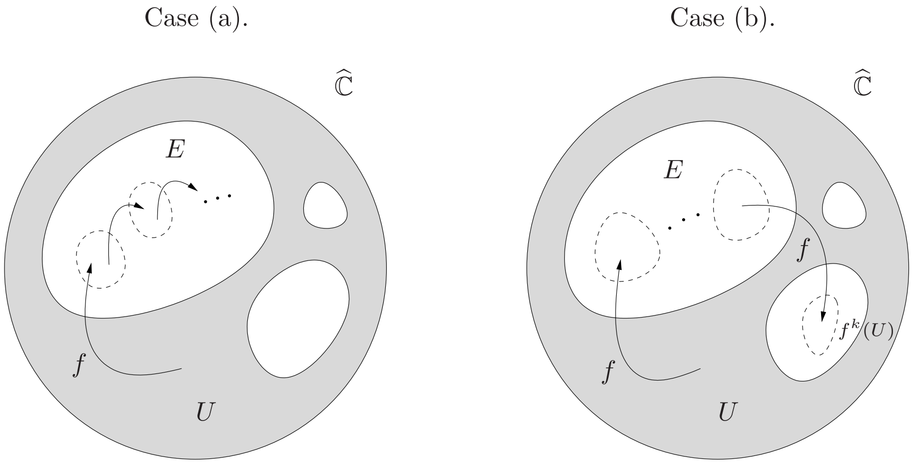


Figure 5.1: The two possible situations. In (a), the iterations of U always stay in E, $f^k(U) \subset E$ for all $k \in \mathbb{N}$; whereas in (b), there exists $k \in \mathbb{N}$ such that $f^i(U) \subset E$ for all $0 < i < k$ and $f^k(U) \subset \widehat{\mathbb{C}} \setminus E$.

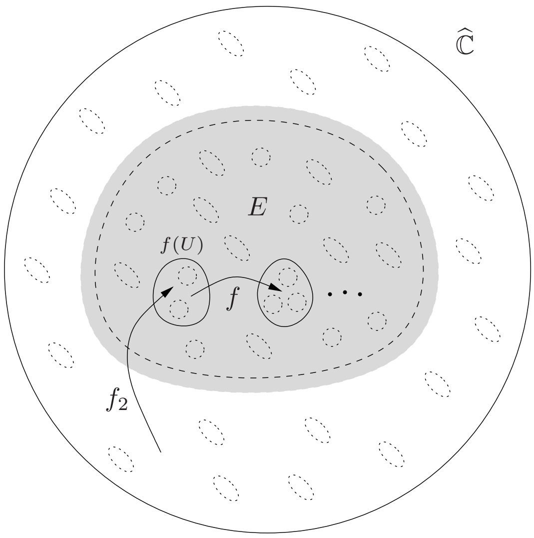


Figure 5.2: The new almost complex structure σ.

conjugate rational function g, plus the subsequent weakly repelling fixed point of f in E.

For Case (a) we define a quasiregular map $f_2 \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ in exactly the same way, i.e., via $f_1 \colon \widehat{\mathbb{C}} \setminus E \to f(U)$. However, in this case we define our f_2-invariant almost complex structure as

$$\sigma := \begin{cases} \sigma_0 & \text{on } f^n(U), \text{ for } n \in \mathbb{N} \\ f_2^* \sigma_0 & \text{on } \widehat{\mathbb{C}} \setminus E \\ (f_2^n)^* \sigma_0 & \text{on } f_2^{-n}(\widehat{\mathbb{C}} \setminus E), \text{ for } n \in \mathbb{N} \\ \sigma_0 & \text{elsewhere.} \end{cases}$$

(See Figure 5.2.)

Therefore, we have that $f_2^* \sigma = \sigma$ almost everywhere, by construction, and that σ has bounded ellipticity, since f_2 is holomorphic everywhere except in $\widehat{\mathbb{C}} \setminus E$, where it is quasiregular but orbits clearly pass at most once through.

As usual, a rational map $g \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ conjugate to f_2 is obtained from Corollary 2.29 and f inherits from it a weakly repelling fixed point in E.

6

Other Fatou components and further results

For completeness, in this chapter we give some further results and state-of-the-art ideas about those Fatou-component cases of Conjecture 3.6 not covered by our Main Theorem 1.5. Section 6.1 gives an idea of Bergweiler and Terglane's proof [9] for the case of multiply-connected wandering domains, and we refer to their paper for the details. Section 6.2 gives a partial proof for the case of Herman rings, namely it proves the case of fixed Herman rings. Finally, Section 6.3 provides a few definitions and results on Baker domains for their better understanding, which may be helpful for a proof of the case of multiply-connected Baker domains.

With these tools, notice that the case of wandering domains is thus closed. The case of Herman rings is only partially closed, but we believe Shishikura's surgery construction holds for transcendental meromorphic functions — although this is a subject for a future project. And the case of Baker domains remains open — and another subject for a future project —, since we need a deeper understanding of this Fatou component to close the case and, with it, Conjectures 1.3 and 1.4.

6.1 On wandering domains

Bergweiler and Terglane's proof [9] for the multiply-connected wandering domain case uses the surgery results of Shishikura's that we have been using in the previous chapters. This outline of their proof is thus similar in spirit to what we have been doing in this Thesis so far, and the general strategy is a case-by-case surgery approach, according to the configuration of the wandering domains. The result is the following.

Theorem 6.1 (**Bergweiler-Terglane [9]**). *Let f be a transcendental meromorphic function and suppose that f has a multiply-connected wandering domain. Then f has at least one weakly repelling fixed point.*

Idea of the proof. Suppose that f has a multiply-connected wandering domain U. Since $\mathcal{J}(f) = \overline{O^{-}(\infty)}$ (see Lemma 2.14 or [5]), we can choose a simple closed curve $\gamma \subset U$ such that $\mathcal{J}(f) \cap \operatorname{int}(\gamma) \neq \emptyset$ — recall that the Jordan Curve Theorem states that the complement of a simple closed curve $\gamma \subset \mathbb{C}$ consists of exactly two connected components: a bounded one, the *interior* $\operatorname{int}(\gamma)$, and an unbounded one, the *exterior* $\operatorname{ext}(\gamma)$. Moreover, there exists a minimal $n \geq 1$ such that $f^n(\operatorname{int}(\gamma))$ contains a pole of f, so $f^n(\gamma)$ contains a simple closed curve σ such that f has a pole in $\operatorname{int}(\sigma)$, and we may replace γ by σ if necessary.

If $f(\gamma) \subset \operatorname{ext}(\gamma)$ and either ∞ and γ are in the same component of $\widehat{\mathbb{C}} \setminus f(\gamma)$ or ∞ and γ are in different components of $\widehat{\mathbb{C}} \setminus f(\gamma)$ but $f(\gamma)$ contains a simple closed curve τ such that $\gamma \subset \operatorname{int}(\tau)$ and $f(\tau) \subset \operatorname{ext}(\tau)$, then with some work and using Shishikura's surgery technique one can show that f has a weakly repelling fixed point (in $\operatorname{int}(\gamma)$ or in $\operatorname{int}(\tau)$, respectively).

Otherwise, we have $f(\gamma) \subset \operatorname{int}(\gamma)$ or there exists a simple closed curve $\tau \subset f(\gamma)$ such that $f(\tau) \subset \operatorname{int}(\tau)$, so without loss of generality we may assume that $f(\gamma) \subset \operatorname{int}(\gamma)$.

We now suppose that $\operatorname{int}(\gamma)$ does not contain a weakly repelling fixed point and seek a contradiction.

Using surgery again, one can build a sequence of simply-connected domains $\{V_0^{(k)}\}_{k \geq 0}$ such that

1. $\infty \in V_0^{(0)}$ and $\overline{V_0^{(k)}} \subset V_0^{(k+1)}$;

2. for all $k \geq 0$, f has a pole in $\widehat{\mathbb{C}} \setminus \overline{V_0^{(k)}}$;

3. for all $\varepsilon > 0$, $\operatorname{sph\,diam}(\partial V_0^{(k)}) < \varepsilon$ for sufficiently large k; and

4. $f(\partial V_0^{(k)}) \subset \widehat{\mathbb{C}} \setminus \overline{V_0^{(k)}}$.

Using 2 and 3 we can achieve $f(\partial V_0^{(k)})$ to be contained in an arbitrarily small neighbourhood of ∞ (contained in $V_0^{(k)}$, by 1) by choosing k large and ε small, which is a contradiction with 4. $\square$

6.2 On Herman rings

Recall that a p-periodic Fatou component U of a complex function f is called a *Herman ring* if there exist $r > 1$ and a holomorphic homeomorphism $\phi\colon U \to \{1 < |z| < r\}$ such that $(\phi \circ f^p \circ \phi^{-1})(z) = e^{2\pi i\theta} z$, for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

One of the characteristics of Herman rings — as well as of Siegel discs — is that the whole Fatou component is foliated by p-invariant curves that spin following an irrational rotation.

In his proof for the rational case, Shishikura uses a cycle of p-invariant curves (in the p-periodic Herman ring) to construct the sets to which surgery will be applied. These curves are present also in the transcendental case, and we believe the same surgery construction holds for transcendental meromorphic functions.

In this section we give a proof for the case when the Herman ring is fixed ($p = 1$). More precisely, we show the following.

Theorem 6.2 (**Fixed Herman rings case**). *Let f be a transcendental meromorphic function with a fixed Herman ring. Then, there exists at least one weakly repelling fixed point of f.*

Proof. Let us suppose f has an invariant Herman ring and let γ be an invariant curve in it. By definition, there exists a holomorphic homeomorphism $h\colon \gamma \to \mathbb{S}^1$ such that $(h \circ f \circ h^{-1})(z) = e^{2\pi i\theta} z$, for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and all $z \in \gamma$.

Now call E the bounded connected component of the complement of γ. We can extend h to a quasi-conformal map $H\colon \widehat{\mathbb{C}} \setminus E \to \overline{\mathbb{D}}$ in such a way that $H|_\gamma \equiv h$ (see [10, 18]) (see Figure 6.1).

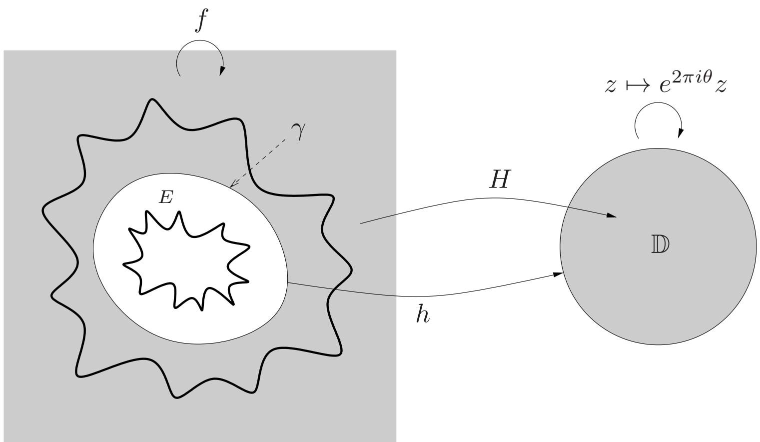


Figure 6.1: Construction of the quasi-conformal map H.

Now define

$$f_1 := \begin{cases} f & \text{on } E \\ H^{-1} \circ (z \mapsto e^{2\pi i\theta} z) \circ H & \text{on } \widehat{\mathbb{C}} \setminus E. \end{cases}$$

Notice that f_1 is well-defined, continuous at $\partial E = \gamma$, and holomorphic in E. In order to obtain a rational map realising such dynamics we need to construct an appropriate almost complex structure, so consider $\sigma := H^*\sigma_0$ on $\widehat{\mathbb{C}} \setminus E$ and spread it by the dynamics of f_1. Thus, σ is f_1-invariant by construction, and it has bounded ellipticity since the map $H^{-1} \circ (z \mapsto e^{2\pi i\theta} z) \circ H$ does not distort ellipses.

Lemma 2.29 gives a rational map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ conjugate to f_1 by a quasiconformal homeomorphism ϕ. Let ψ be the inverse $\psi := \phi^{-1}$. The rational map g has a weakly repelling fixed point z_0 (Fatou's Theorem 2.6) and a Siegel disc that contains $\psi(\widehat{\mathbb{C}} \setminus E)$, so $z_0 \in \mathcal{J}(g) \subset \psi(E)$ and therefore f_1 (and hence f) has a weakly repelling fixed point in E. □

6.3 On Baker domains

Recall that a p-periodic Fatou component U of a transcendental function f is called a *Baker domain* (or a *domain at infinity*) if ∂U contains a point z_0 such that $f^{np}(z) \to z_0$, as $n \to \infty$, for all $z \in U$, but $f^p(z_0)$ is not defined, that is, z_0 is a prepole of order k, with $1 \leq k \leq p$.

The following result is a collection of direct consequences from the definition, and is stated in this form in [8].

Proposition 6.3. *Let f be a meromorphic function, and let $\{U_0, U_1, \dots U_{p-1}\}$ be a periodic cycle of Baker domains of f. Denote by z_j the limit corresponding to U_j, and define $z_p \equiv z_0$. Then, $z_j \in \bigcup_{n=0}^{p-1} f^{-n}(\infty)$ for all $j \in \{0, 1, \dots, p-1\}$, and $z_j = \infty$ for at least one $j \in \{0, 1, \dots, p-1\}$. If $z_j = \infty$, then z_{j+1} is an asymptotic value of f.*

For a deep discussion about the existence and distribution of Baker domains of a transcendental function (either entire or meromorphic), we again refer to [8].

The first example of a transcendental entire function with a Baker domain is due to Fatou [25], who considered

$$f(z) = z + 1 + e^{-z}.$$

He observed that $\lim_{n\to\infty} f^n(z) = \infty$ when $\mathrm{Re}\,(z) > 0$, so the right half plane is contained in an invariant Baker domain. (That is the case $p = 1$ and $z_0 = \infty$ in our previous definition of Baker domain.)

Now the first example of a Baker domain of period greater than 1 was given in [6] with the transcendental meromorphic function

$$f(z) = \frac{1}{z} - e^z.$$

It has a 2-periodic cycle of Baker domains $\{U_0, U_1\}$ such that $\lim_{n\to\infty} f^{2n}(z) = \infty$ when both $z \in U_0$ and $z \in U_1$.

It is worth mentioning two important theorems for Baker domains of transcendental entire maps. For the first one, we say that a transcendental entire map f is in *class* $\mathcal{B}$ if $\operatorname{sing}(f^{-1})$ is a bounded set (see Subsection 2.1.3 for a few words about the set $\operatorname{sing}(f^{-1})$).

Theorem 6.4 (**Eremenko-Lyubich [22]**). *Let $f \in \mathcal{B}$. Then, f has no Baker domains.*

Theorem 6.5 (**Baker [4]**). *Let f a transcendental entire map. If U is a Baker domain of f, then U is simply connected.*

As for transcendental meromorphic maps, the general situation is a bit more involved, although there are some particular cases where the results for entire maps above can be extended.

It is clear that Theorem 6.4 is not true for meromorphic maps since the aforementioned function $f(z) = \frac{1}{z} - e^z$ provides a counterexample. However, we can define other classes of functions for which the results will hold.

We say that a transcendental meromorphic function f is in *class* $\mathcal{S}$ if $\operatorname{sing}(f^{-1})$ is finite. Just as for entire maps, we say that f is in *class* $\mathcal{B}$ if $\operatorname{sing}(f^{-1})$ is bounded. Furthermore, we say that f is in *class* $\mathcal{B}_n$ if the set of points $S_n(f)$ is bounded, where

$$S_n(f) = \bigcup_{k=0}^{n-1} f^k(\operatorname{sing}(f^{-1}) \setminus A_k(f))$$

and $A_k(f) = \{z \in \widehat{\mathbb{C}} : f^k \text{ is not analytic at } z\}$.

Theorem 6.6. *Let $f \in \mathcal{S}$. Then, f has no Baker domains.*

Theorem 6.7 (**Rippon-Stallard [37]**). *If f is in class $\mathcal{B}_n$, then f has no Baker domains of period n.*

As for multiply-connected Baker domains, an example was given in [16] with the (meromorphic) function

$$f(z) = z + 2 + e^{-z} + \frac{1+\pi i}{z - 10^{-2}},$$

which has an unbounded multiply-connected fixed Baker domain. In particular, this example shows that Theorem 6.5 does not hold for meromorphic functions.

Nonetheless, Mayer and Schleicher showed in [32] — the paper mentioned in the Introduction — that there are some interesting classes of functions for which Baker domains are always simply connected. As we shall see, this is the case of the Newton's method of transcendental entire functions, one of the classes of meromorphic functions this Thesis deals with (see Chapter 1).

Definition 6.8. Let U be an N_f-invariant domain in $\mathbb{C}$. An open subset $A \subset U$ is called an *absorbing set* (of U) if the following hold.

1. A is simply connected.
2. $N_f(\overline{A} \setminus \{\infty\}) \subset A$.
3. For every $z \in U$, there is a $k \geq 0$ such that $N_f^k(z) \in A$.

Definition 6.9. A domain $U \in \mathbb{C}$ is called a *virtual immediate basin* if it is maximal with respect to the following properties.

1. $\lim\limits_{n\to\infty} N_f^n(z) = \infty,$ for all $z \in U$.
2. U contains an absorbing set.

Theorem 6.10 (Mayer-Schleicher [32]). *Virtual immediate basins are simply connected.*

Notice that this theorem does not imply that every virtual immediate basin is an actual Fatou component. But if this were the case, then the Fatou component would be a Baker domain.

In view of this, a reasonable approach to our problem in the case of multiply-connected Baker domains would be to investigate first whether (general) transcendental meromorphic functions with a multiply-connected Baker domain that contains an absorbing set have a weakly repelling fixed point.

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