Appendix A

Weather fluctuations in the forcing

The water motions in the morphodynamic model described in chapters 2 and 2 are forced by wind stress and alongshore gradients in the mean free surface elevation. The sea bottom evolution which is described by the model takes place during a very long time scale as the result of the cumulative effect of many episodic storms. On such a long time scale, the forcing is essentially time dependent mainly due to the sequence stormy weather-calm weather. As a first step, the present thesis has been based on taking an averaged forcing and neglecting the effects of fluctuations. A detailed description of these latter effects are considered beyond the purpose of the present thesis. Nevertheless, it is worthwhile to give a sketch of the main ideas of such a "statistical" model. This yields insight into the limitations of the present "fully averaged model". Also, some of the implications of the statistical model can already be foreseen. This is the aim of this appendix.

All the variables can be decomposed into a mean plus a fluctuation, where the mean refers to an average over a time period T much larger than the typical time between two consecutive storms, say $T = O(10yr)$. For instance, for the velocity, $\boldsymbol{v} = \bar{\boldsymbol{v}} + \boldsymbol{v}'$, where $\bar{\boldsymbol{v}}$ and $\boldsymbol{v}'$ describe the mean and fluctuation, respectively. The governing equations can then be averaged as

$$\bar{\boldsymbol{v}} \cdot \boldsymbol{\nabla} \bar{\boldsymbol{v}} + \boldsymbol{f}_c \times \bar{\boldsymbol{v}} + g \boldsymbol{\nabla} \bar{z}_s - \frac{\bar{\boldsymbol{\tau}}_s - \bar{\boldsymbol{\tau}}_b}{\rho \bar{D}} = -\overline{\boldsymbol{v}' \cdot \boldsymbol{\nabla} \boldsymbol{v}'}$$

$$\boldsymbol{\nabla} \cdot (\bar{D} \bar{\boldsymbol{v}}) = 0 \qquad , \qquad \frac{\partial \bar{z}_b}{\partial t} + \boldsymbol{\nabla} \cdot \bar{\boldsymbol{q}} = 0$$

where the hypotheses assumed in section 2.1, namely, the quasi-steady approximation and small Froude number have been already taken into account. Note that depth fluctuations donot show up in this model which is a consequence of the rigid lid approximation (free surface effects are neglected) and the fact that fluctuations on the topography due to storm events can be discarded.

Now, the problem is how to parametrize bottom shear stress, $\bar{\boldsymbol{\tau}}_b$, sediment transport, $\bar{\boldsymbol{q}}$, and the "Reynolds stress" term, $\overline{\boldsymbol{v}' \cdot \boldsymbol{\nabla} \boldsymbol{v}'}$. In order to parametrize the first two, it is convenient to consider the instantaneous velocity near the bottom

$$\boldsymbol{v}_{ib} = \bar{\boldsymbol{v}} + \boldsymbol{v}' + \boldsymbol{v}''$$

where $\boldsymbol{v}''$ is the velocity which is not explicitly described by the shallow water model (contribution due to waves, turbulence, etc.). Here we assume that v'' is the wave orbital velocity near the bottom, being oscillatory with a period much smaller than the time scale of weather events and $\overline{\boldsymbol{v}''} = 0$. Under the assumption that the mean current is weak in comparison to fluctuations and to wave orbital velocity, $|\bar{\boldsymbol{v}}| \ll |\boldsymbol{v}' + \boldsymbol{v}''|$, and that the instantaneous bottom shear stress is given by $\boldsymbol{\tau}_b = \rho c_d |\boldsymbol{v}_{ib}| \boldsymbol{v}_{ib}$, the mean bottom stress can be evaluated as

$$\bar{\boldsymbol{\tau}}_b = \rho c_d \overline{|\boldsymbol{v}''| \boldsymbol{v}'} + [\mathbf{R}] \cdot \bar{\boldsymbol{v}} \tag{A.1}$$

where the second order tensor $[\mathbf{R}]$ has components:

$$R_{ij} = c_d \overline{|\boldsymbol{v}''|} \delta_{ij} + c_d \overline{\frac{v_i'' v_j''}{|\boldsymbol{v}''|}}$$

with δ_{ij} the Kronecker delta symbol and indices i and j can have the values 1 or 2. In a similar way, if the instantaneous sediment flux is

$$\boldsymbol{q} = \nu_0 |\boldsymbol{v}_{ib}|^b \left(\frac{\boldsymbol{v}_{ib}}{|\boldsymbol{v}_{ib}|} - \gamma \boldsymbol{\nabla} z_b\right)$$

where $b > 1$ and ν_0 are specified by using parameterizations which are discussed in e.g. Van Rijn (1993) the mean sediment transport can be calculated as

$$\bar{\boldsymbol{q}} = \nu_0 \overline{|\boldsymbol{v}''|^{b-1}\boldsymbol{v}'} + [\mathbf{Q}] \cdot \bar{\boldsymbol{v}} - \nu_0 \gamma \overline{|\boldsymbol{v}''|^b} \boldsymbol{\nabla} \bar{z}_b \tag{A.2}$$

where the components of the second order tensor $[\mathbf{Q}]$ are given by:

$$Q_{ij} = \nu_0 \overline{|\boldsymbol{v}''|^{b-1}} \delta_{ij} + \nu_0 (b-1) \overline{|\boldsymbol{v}''|^{b-3} v_i'' v_j''} \tag{A.3}$$

By using (A.1) and (A.2), the basic state equations (2.6) can be rewritten as

$$-f_c V + g \frac{\mathrm{d}\xi}{\mathrm{d}x} = \frac{\bar{\tau}_{sx}}{\rho \bar{D}} - c_d \frac{\overline{|\boldsymbol{v}''| v_x'}}{\bar{D}} - R_{12} \frac{V}{\bar{D}} - \overline{v_x' \frac{\partial v_x'}{\partial x}} \tag{A.4}$$

$$gs = \frac{\bar{\tau}_{sy}}{\rho \bar{D}} - c_d \frac{\overline{|\boldsymbol{v}''| v_y'}}{\bar{D}} - R_{22} \frac{V}{\bar{D}} - \overline{v_x' \frac{\partial v_y'}{\partial x}} \tag{A.5}$$

Notice that all derivatives with respect to y vanish as there is alongshore uniformity in the basic state. If we now define

$$\tau_{sx}^* = \bar{\tau}_{sx} - \rho c_d \overline{|\boldsymbol{v}''| v_x'} \qquad \tau_{sy}^* = \bar{\tau}_{sy} - \rho c_d \overline{|\boldsymbol{v}''| v_y'}$$

$$gs^* = gs + \overline{v_x' \frac{\partial v_y'}{\partial x}} \qquad g\xi^* = g\xi + \frac{1}{2}\overline{v_x'^2}$$

equations (A.4)-(A.5) read

$$-f_c V + g \frac{\mathrm{d}\xi^*}{\mathrm{d}x} = \frac{\tau_{sx}^*}{\rho \bar{D}} - R_{12} \frac{V}{\bar{D}} \qquad gs^* = \frac{\tau_{sy}^*}{\rho \bar{D}} - R_{22} \frac{V}{\bar{D}} \tag{A.6}$$

Given the forcing $\boldsymbol{\tau}_{sy}^*$ and s^* the second of these equations has the same structure than the second of (2.6) and can be solved for V in the same manner (recall that the influence of the free surface elevation $\bar{\xi}$ on the total depth $\bar{D}$ is neglected, in accordance with the small Froude number assumption). Once $V(x)$ is known, the first equation can be solved for ξ^*, from where the free surface elevation, ξ, can be obtained. Even though the first equality in (A.6) has two additional terms, τ_{sx}^* and $R_{12}V$, this only produces an effect on the free surface elevation, but not on the basic alongshore current. The mass conservation equation is satisfied identically while the sediment conservation yields

$$\frac{d}{dx}\left(Q_{12} V + \nu_0 \overline{|\boldsymbol{v}''|^{b-1} v_x'} - \bar{\gamma} \frac{d\bar{z}_b}{dx}\right) = 0$$

with $\bar{\gamma} = \nu_0 \gamma \overline{|\boldsymbol{v}''|^b}$. This equation would give the equilibrium topographic profile of the inner shelf, $\bar{z}_b(x)$. Observations indicate that to a first approximation this profile is characterized by a constant slope and this is what has been assumed in section 2.1. Therefore, a mean basic state similar to that considered in the thesis can still be assumed even in the presence of fluctuations.

An interesting issue of this "statistical" formulation, already at this simple stage, is the prediction of a decrease of the effective wind stress τ_{sy}^* with respect to the mean value, $\bar{\tau}_{sy}$. This can be understood as follows. Let us consider a dominance of wind along the positive y-direction, so that $\bar{\tau}_{sy} > 0$ and $V > 0$. The longshore current fluctuation will be positive during storms and negative during fair weather, but due to the fact that fair weather occurs more often than stormy weather, the magnitude of the fluctuation is larger during storms than during calm weather (because of $\overline{\tau_{sy}'} = 0$). Then, since the wave orbital velocity $|\boldsymbol{v}''|$ is much larger during storms, i.e., when $v_y' > 0$, there is a net contribution in the positive y-direction so that $\rho c_d \overline{|\boldsymbol{v}''| v_y'} > 0$. At the same time, the effective sea surface slope, s^*, is not significantly smaller or larger than its mean value. This can be argued as follows. The cross-shore velocity fluctuation, v'_x, has not any preferred direction during the entire storm duration but will change sign from the initial stage to the final stage of the event. According to the balance between surface shear stress and bottom shear stress considered in section 2.1, the wind driven current tends to be cross-shore uniform. Thus, the term $\partial v'_y/\partial x$ will be very small. Therefore, the correlation $\overline{v'_x \partial v'_y/\partial x}$ may be expected to be negligible and $s^* \simeq s$.

The fact that $|\tau^*_{sy}| < |\bar{\tau}_{sy}|$ while $s^* \simeq s$ causes an increase of the a parameter in the model. In other words, even if wind stress may be dominant during storms, the sea surface slope is much more effective in driving the long term mean alongshore current. Some simple computations have been done by considering the following forcing sequence: 1 day of constant wind stress (always in the same direction), 9 days of calm weather. The sea surface slope has been assumed to be present all the time. According to observations off the central Dutch coast, this distribution is not unrealistic. Then, the computations predict an increase in parameter a of the order 40%.

Once the statistical formulation and the basic state have been established, the next step would be to write down the linear stability equations. Now, this requires to evaluate the first order variation of the fluctuation on the flow, $\boldsymbol{v}'$, due to the presence of the small topographic perturbation, h. This will not be further pursued in the present thesis.

Appendix B

Solution procedure: Steady model

B.1 Equations

The three linear differential equations (2.12) can be reduced to a single equation for the cross–shore velocity component of the form:

$$\mathcal{U}_2 u_{xx} + \mathcal{U}_1 u_x + \mathcal{U}_0 u = \mathcal{H}_1 h_x + \mathcal{H}_0 h \tag{B.1}$$

where

$$\begin{aligned}
\mathcal{U}_2 =& \beta_2 \\
\mathcal{U}_1 =& \beta_{2x} + \beta_3 - \hat{f}\alpha_2 \\
\mathcal{U}_0 =& \beta_{3x} - \hat{f}\alpha_3 + \frac{r_1}{H} + \mathrm{i}kV \\
\mathcal{H}_1 =& -\beta_1 \\
\mathcal{H}_0 =& -\beta_{1x} + \hat{f}\alpha_1
\end{aligned}$$

with the α and β coefficients given below. Solving (B.1) for u as a function of h, under the boundary conditions $u = 0$ at $x = 0$ and for $x \to \infty$, defines the linear operator $\mathbf{U}$. Then, back substitution into (2.12) yields for the linear operators $\mathbf{V}$ and $\mathbf{E}$:

$$\mathbf{V} = \alpha_1 \mathbf{I} + \alpha_2 \mathbf{U}_x + \alpha_3 \mathbf{U} \tag{B.2a}$$

$$\mathbf{E} = \beta_1 \mathbf{I} + \beta_2 \mathbf{U}_x + \beta_3 \mathbf{U} \tag{B.2b}$$

where $\mathbf{I}$ denotes the identity operator.

The coefficients in (B.1) and (B.2) are

$$\begin{aligned}
\alpha_1 =& \frac{V}{H} \\
\alpha_2 =& \frac{\mathrm{i}}{k} \\
\alpha_3 =& \frac{\mathrm{i}}{k}\frac{H_x}{H} \\
\beta_1 =& \frac{-V^2}{H} + \frac{\mathrm{i}}{k}\left(r_2\frac{V}{H^2} - \frac{\delta}{H}\right) \\
\beta_2 =& \frac{-r_2}{k^2 H} - \frac{\mathrm{i}}{k}V \\
\beta_3 =& \frac{-r_2 H_x}{k^2 H^2} - \frac{\mathrm{i}}{k}\left(V\frac{H_x}{H} - V_x - \hat{f}\right)
\end{aligned}$$

Once the operators $\mathbf{U}$ and $\mathbf{V}$ are known, operator $\mathbf{B}$ can be computed as

$$\begin{aligned}\mathbf{B} = -|V|^{m-1}\Big\{ & \frac{(m-1)}{V} V_x \mathbf{U} + \mathbf{U}_x + \mathrm{i}km\mathbf{V} \\ & -\hat{\gamma}|V| \left(\frac{m}{V} V_x \frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}^2}{\mathrm{d}x^2} - k^2 \right) \mathbf{I} \Big\}\end{aligned} \quad \text{(B.3)}$$

Then, the solution of (2.15) with (B.3) yields the eigenvalues and eigenfunctions.

B.2 x–discretization

The diferential equations are solved by means of spectral method based upon truncated expansions in Chebyshev polynomials, and are solved at collocations points (see Canuto *et al.* 1988 and Gottlieb & Orszag 1977). Here a brief outline of the spectral numerical method used is gived. In order to use Chebyshev polynomials, the $[0, \infty)$ interval is transformed into the $(-1, 1]$ interval. The both intervals are related by the maps:

$$(-1, 1] \xrightarrow{\psi(z)=z'} (-1, 1] \xrightarrow{\phi(z')=x} [0, \infty)$$

$$\psi(z) = az^3 + (1-a)z \qquad \phi(z') = l\frac{(1-z')}{(1+z')},$$

and N Gaus-Lobatto nodes are chosen as collocations points

$$z_i = cos(\frac{\pi\, i}{N}) \qquad i = 0 \div N\,.$$

At the transformation ϕ, the l parameter is the distance where half of the collocation points are located. The tranformation ψ approach the collocation points to 0 in the $(-1, 1]$ interval and, therefore, to l in the $[0, \infty)$ interval. Note that $x_0 = 0$ and $x_N = \infty$.

The aproximate solution $f(x)$ of a function $F(x)$ on $[0, \infty)$ is expanded as a truncated Chebyshev series

$$f(x) = \sum_{j=0}^{N} \hat{f}_j T_j(z)\,,$$

Here N is the order of expansions, $T_j(z)$ the Chebyschev polynomials of first kind and order j (ie. $T_j(z) = cos(j cos^{-1} z)$) and $\hat{f}_j$ the projection of the $F(x)$ function in $T_j(z)$.

Expansions for first and second order derivaties at the collocation points from the values of the funtion at these points, $f(x_j) = f_j$, are

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_i} = \sum_{j=0}^{N} D^1_{ij} f_j \qquad \left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_{x=x_i} = \sum_{j=0}^{N} D^2_{ij} f_j$$

The derivaties operators D^1_{kl} and D^2_{kl} follow from the derivaties of the Chebyschev polynomials and the map $z \mapsto x$, and read

$$D^1_{kl} = \frac{1}{\mathrm{d}x/\mathrm{d}z}\bar{D}_{kl} \qquad D^2_{kl} = \frac{1}{(\mathrm{d}x/\mathrm{d}z)^2}\left(\bar{D}^2_{kl} - \frac{\mathrm{d}^2x/\mathrm{d}z^2}{\mathrm{d}x/\mathrm{d}z}\bar{D}_{kl}\right)$$

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{\mathrm{d}\psi}{\mathrm{d}z}\frac{\mathrm{d}\phi}{\mathrm{d}z'} \qquad \frac{\mathrm{d}^2x}{\mathrm{d}z^2} = \frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}\frac{\mathrm{d}\phi}{\mathrm{d}z'} + \left(\frac{\mathrm{d}\psi}{\mathrm{d}z}\right)^2\frac{\mathrm{d}^2\phi}{\mathrm{d}z'^2}\,.$$

The elements $\bar{D}_{kl}$ are defined as

$$\bar{D}_{00} = \frac{1}{6}(2N^2+1) \qquad \bar{D}_{NN} = \frac{-1}{6}(2N^2+1)$$
$$\bar{D}_{kk} = \frac{-z_k}{2(1-z_k^2)} \qquad k = 1 \div N-1$$
$$\bar{D}_{kl} = \frac{\bar{c}_k}{\bar{c}_l}\frac{(-1)^{k+l}}{z_j - z_l} \qquad k,l = 0 \div N, j \neq k$$
$$\bar{c}_0 = \bar{c}_N = 2$$
$$\bar{c}_j = 1 \qquad j = 1 \div N-1$$

and the elements $\bar{D}^2_{kl}$ as

$$\bar{D}^2_{kl} = \sum_{m=0}^{N} \bar{D}_{km}\bar{D}_{ml} \qquad k,l = 0 \div N\,.$$

In case that a funtion $H(x)$ verifies boundary conditions

$$H(x=0) = 0 \qquad H(x \to \infty) \to 0\,,$$

a linear combination of Chebyschev polynomials, which verifies the same boundary conditions, is used. That read as:

$$g_j(z) = T_j(z) - \frac{1}{2}\left\{(1+(-1)^j)T_0(z) + (1+(-1)^{j+1})T_1(z)\right\}.$$

The aproximation of H at the collocation points, h_i, is

$$h_i = \sum_{j=2}^{N} G^0_{ij}\hat{h}_j \qquad G^0_{ij} = g_j(z_i) \qquad i = 0 \div N, j = 2 \div N\,.$$

Expansions for first and second order derivaties, at the collocation points, are

$$\left.\frac{dh}{dx}\right|_{x=x_i} = \sum_{j=2}^{N} G^1_{ij}\hat{h}_j \qquad \left.\frac{d^2h}{dx^2}\right|_{x=x_i} = \sum_{j=2}^{N} G^2_{ij}\hat{h}_j$$
$$G^1_{ij} = \sum_{k=0}^{N} D^1_{ik}G^0_{kj} \qquad G^2_{ij} = \sum_{k=0}^{N} D^2_{ik}G^0_{kj}\,.$$

Note that at the last expansions, values at the collocation points come from the projections of $H(x)$ in the basis $g_j(z)$.

B.3 Discretizated equations

B.3.1 FOT-problem

Application of the collocation method to the equation for u, equation (B.1), yields

$$\sum_{j=1}^{N-1}\Big(\mathcal{U}_2(x_i)D^2_{ij} + \mathcal{U}_1(x_i)D^1_{ij} + \mathcal{U}_0(x_i)\delta_{ij}\Big)u_j =$$
$$\sum_{j=2}^{N}\Big(\mathcal{H}_1(x_i)G^1_{ij} + \mathcal{H}_0(x_i)G^0_{ij}\Big)\hat{h}_j \qquad i = 1 \div N-1$$

Note that, because of the boundary conditions ($u_0 = u_N = 0$), the index for the collocation points is running from 1 to $N-1$. Same boundary conditions verifies h and they are included in basis $g_j(x)$ of the appendix B.2. Solving this system of $(N-1)\times(N-1)$ equations a matrix $\mathbf{U}_{ij}$ which defines the cross-shore velocity component u in the collocation points x_i as a linear combination of $\hat{h}_j$, ie. the bottom perturbation, is found:

$$u_i = \sum_{j=2}^{N} \mathbf{U}_{ij}\hat{h}_j \qquad i = 0 \div N-1\,.$$

The boundary condition for u at $x = 0$ gives $\mathbf{U}_{0j} = 0$.

Back substitution into equations (B.2), yields

$$\mathbf{V}_{ij} = \alpha_1(x_i)G^0_{ij} + (\alpha_2(x_i)D_{ik} + \alpha_3(x_i)\delta_{ik})\,\mathbf{U}_{kj} \qquad i,j = 0 \div N-1$$
$$\mathbf{E}_{ij} = \beta_1(x_i)G^0_{ij} + (\beta_2(x_i)D_{ik} + \beta_3(x_i)\delta_{ik})\,\mathbf{U}_{kj} \qquad i,j = 0 \div N-1\,,$$

and from these operators, the values of v and η at the collocation points are

$$v_i = \sum_{j=2}^{N} \mathbf{V}_{ij}\hat{h}_j \qquad i = 0 \div N-1$$
$$\eta_i = \sum_{j=2}^{N} \mathbf{E}_{ij}\hat{h}_j \qquad i = 0 \div N-1\,.$$

B.3.2 Bottom evolution equation

Application of the collocation method to the equation for h and using the solutions of the FOT-problem, the equation (B.3) become

$$\omega \sum_{j=2}^{N} G^0_{ij}\hat{h}_j = \sum_{j=2}^{N} \Bigg\{ \sum_{k=0}^{N-1} \left(\left(m\frac{D_{0x}}{D_0} - (m-1)\frac{V_x}{V} \right) |V|^{m-1}\delta_{ik} + (m-1)|V|^{m-1}D^1_{ik} \right)_i \mathbf{U}_{kj} - \left(ikm\frac{|V|^{m-1}V}{D_0} \right)_i G^0_{ij} + \hat{\gamma}\left(|V|^m G^2_{ij} + m\frac{V_x}{V}|V|^m G^1_{ij} - k^2|V|^m G^0_{ij} \right)_i \Bigg\} \hat{h}_j\,.$$

The result of write this equations in the collocation points ($i = 1 \div N-1$) is a generalized eigenvalue problem

$$\omega\,\mathbf{A}\hat{h} = \mathbf{B}\hat{h}$$

The generalized eigenvectors, $\hat{h}_i$, are the components in $g_j(x)$ of the bottom perturbation.

Appendix C

Analytical approximation for $m = 1$

A simple analytical approximation of the eigenvalue problem discussed in section 2.2 can be obtained for the parameter values $\hat{\gamma} = \hat{f} = r = F = 0, m = a = 1$ and $\beta \ll 1$. In essence this reduces the model to the system studied by Trowbridge (1995), but he did not persue this method. By assuming normal mode solutions $h = \mathrm{Re}\left\{\hat{h}(x)\exp(iky + \omega t)\right\}$ it follows that the bed evolution equation (2.17) reduces for $0 \le x \le 1$ to

$$\frac{\beta}{H}\mathbf{U}h = \lambda h$$

where hats have been dropped for simplicity, $\lambda = \omega - ik$ and operator $\mathbf{U}$ is defined in appendix B. The boundary conditions are $h = 0$ at both $x = 0$ and $x = 1$: due to the absence of slope effects in the sediment transport $h = 0$ at the outer shelf. Next we assume the expansions

$$\begin{aligned} h(x) =& h_0(x) + \beta h_1(x) + \ldots \\ \lambda =& \beta\lambda_1 + \beta^2\lambda_2 + \ldots \\ \mathbf{U} =& \mathbf{U}_0 + \beta\mathbf{U}_1 + \ldots \end{aligned}$$

so that at the lowest order we will have the eigenproblem for $\mathbf{U}_0$:

$$\mathbf{U}_0 h_0 = \lambda_1 h_0 \tag{C.1}$$

Operator $\mathbf{U}_0$ involves solving equation (B.1) for *horizontal flat bottom and uniform basic current*, which, owing to $r = 0, \hat{f} = 0$, reduces to

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - k^2 u = -ik\frac{\mathrm{d}h_0}{\mathrm{d}x}$$

where $u = \mathbf{U}_0 h_0$. Therefore, solving (C.1) is equivalent to solving

$$\frac{\mathrm{d}^2 h_0}{\mathrm{d}x^2} + 2s\frac{\mathrm{d}h_0}{\mathrm{d}x} - k^2 h_0 = 0 \quad , \quad h_0(0) = h_0(1) = 0 \tag{C.2}$$

with

$$s = i\frac{k}{2\lambda_1} \tag{C.3}$$

The solutions of (C.2) are easily found to be

$$h_0(x) = e^{i\alpha_1 x} - e^{i\alpha_2 x} \tag{C.4}$$

where

$$\alpha_1 = i(s - \sqrt{s^2 + k^2}) \quad , \quad \alpha_2 = i(s + \sqrt{s^2 + k^2}) \tag{C.5}$$

and where the boundary condition at $x = 1$ requires

$$\alpha_2 - \alpha_1 = 2n\pi \quad , \quad n = \pm 1, 2, 3, \ldots \tag{C.6}$$

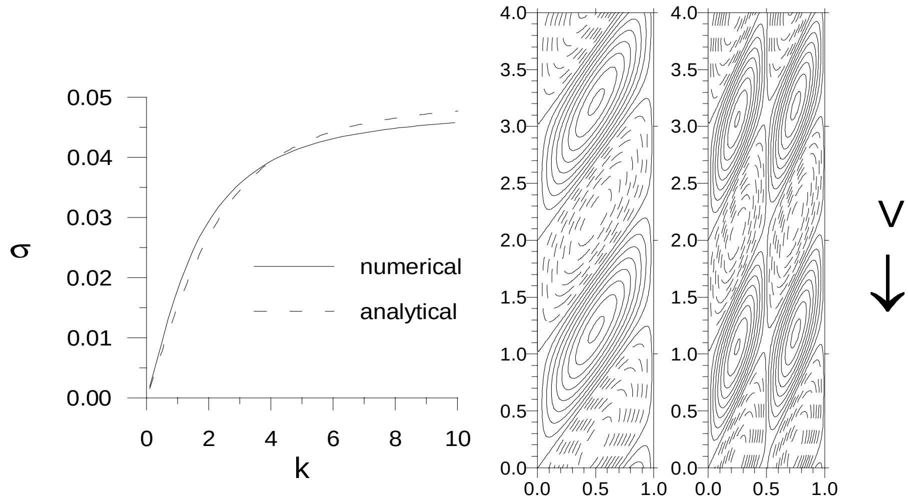

Figure C.1: Left: Comparison of the instability curves obtained either by the numerical model or by the analytical approximation, for $\hat{\gamma} = \hat{f} = r = 0, m = a = 1$ and $\beta = 0.1$. Right: Perturbation in the topography given by the analytical model, modes 1 and 2.

From (C.5) , (C.6) and (C.3) the eigenvalue spectrum is found to be

$$\lambda_1 = \pm\frac{1}{2}\frac{k}{\sqrt{k^2+n^2\pi^2}} \qquad , \qquad n = 1, 2, 3, \ldots$$

From this it follows that the growth rates to first order in the slope, β, are given by

$$\omega = \mathrm{i}k \pm \frac{\beta}{2}\frac{k}{\sqrt{k^2+n^2\pi^2}} \qquad , \qquad n = 1, 2, 3, \ldots \tag{C.7}$$

Figure C.1 shows ω as a function of k for the dominant mode ($n = 1$) and for $\beta = 0.1$, computed both by the MORFO20 numerical model and by equation (C.7). Note the striking resemblance between the analytical and the numerical model. This even applies in case $\beta = 1$, see figure 4 in Trowbridge (1995) for a comparison. From (C.4) and taking into account the dependence on the alongshore coordinate, the sea bed perturbation at any time can be written as proportional to

$$h(x, y) = \cos(ky + \alpha_1 x) - \cos(ky + \alpha_2 x) \tag{C.8}$$

Each term is wave-like with straight crests parallel to $ky + \alpha x = 0$. Therefore, recalling that the basic current is $V < 0$, the topographic features are downcurrent oriented if $\alpha > 0$ and upcurrent oriented if $\alpha < 0$. In the case of growing bedforms, that is, $\lambda_1 > 0$, (C.5) and (C.6) imply

$$\alpha_1 = -(\sqrt{k^2+n^2\pi^2} + n\pi) \qquad , \qquad \alpha_2 = -(\sqrt{k^2+n^2\pi^2} - n\pi)$$

which both are negative. So, growing bedforms are upcurrent oriented. In contrast, decaying bedforms yield positive values for α_1 and α_2 and thus they are downcurrent oriented. This is in agreement with the numerical computations in section 2.3 and with the field observations described in the introduction. Finally, the topographic contours computed by means of (C.8) for $\beta = 0.1$, $k = \pi$ are shown in figure C.1 for the first and second growing modes ($n = 1$, $n = 2$). Also, a remarkable similarity with the bedforms computed with the numerical model is found (see, for instance, figure 2.2 A and B).

Appendix D

Derivation of the sediment transport parametrization

Sediment transport is a very complex process so that for practical applications it is necessary to model it by means of gross parametrizations. The latter are based on a combination of basic physics, dimension analysis and observations in the field and in the laboratory, see Dyer (1986) and Fredsoe & Deigaard (1993) for further details.

Now consider noncohesive sediment with a uniform grainsize which is transported as bedload. Then, in case of a flat bed, a frequently used parametrization for the dimensional volumetric flux per unit width is

$$\boldsymbol{q} = \nu\, \boldsymbol{u}^3,$$

i.e., the transport is proportional to the cube of the *total* velocity $\boldsymbol{u}$ near the bed. Here it is assumed that the critical velocity for erosion is much smaller than the actual velocity. The foundation of this dates back to Bagnold (1956) and has been confirmed by many experimental data. A characteristic value for the parameter ν is 10^{-5} $\mathrm{s^2 m^{-1}}$.

In order to apply this parametrization to the depth-averaged model used in this paper it is important to realize that

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{v}'.$$

Here $\boldsymbol{v}$ is the depth-averaged velocity which consists of both a steady and tidal components. Furthermore, $\boldsymbol{v}'$ is the part of the velocity field that is not explicitly accounted for by the depth-averaged model; in particular waves and small-scale turbulent motions determine the behaviour of $\boldsymbol{v}'$.

During storms we assume that the currents induced by waves and turbulence are much larger than the steady and tidal currents, in other words $|\boldsymbol{v}'| \gg |\boldsymbol{v}|$. Moreover it is assumed that $|\boldsymbol{v}'|$ is independent of the location and that waves donot induce any net sediment transport over a tidal cycle. On the inner shelf these conditions seem reasonable because water depths are generally too large to cause significant wave steepening and wave breaking. Then it follows by straightforward means that the tidally averaged volumetric flux per unit width during storms can be approximated by

$$< \boldsymbol{q} >_{\text{storms}} = \nu \langle |\boldsymbol{v}'|^2 \rangle < \boldsymbol{v} >,$$

where the proportionality factor is the so-called wave stirring factor. This result can be easily generalized to a sloping bed and motivates the use of the *dimensionless* flux $\boldsymbol{q}_1$ in equation (3.7), where $\nu_1 = \langle |\boldsymbol{v}'|^2 \rangle / U^2$ if we choose $[q] = \nu U^3$. Note that by definition $\nu_1 \gg 1$.

Likewise the situation during mild weather conditions can be analyzed, during which we assume that $|\boldsymbol{v}'| \ll |\boldsymbol{v}|$. In that case it follows for the dimensional flux

$$< \boldsymbol{q} >_{\text{quiet}} = \nu \langle \boldsymbol{v}^3 \rangle.$$

Its dimensionless analogon is the flux $\boldsymbol{q}_3$ in equation (3.7) with $\nu_3 = 1$.

Appendix E

Solution procedure: Tidal model

In chaper 3 equations for u, v and h were found, in this appendix the numerical procedure to solve those equations is gived.

E.1 Equations

The equation for the cross-shore velocity component $u(x,t)$ reads

$$\mathcal{U}_{12}u_{txx} + \mathcal{U}_{11}u_{tx} + \mathcal{U}_{10}u_t + \mathcal{U}_{02}u_{xx} + \mathcal{U}_{01}u_x + \mathcal{U}_{00}u = \mathcal{H}_1 h_x + \mathcal{H}_0 h$$

where

$$\begin{aligned}
\mathcal{U}_{12} =& 1 \\
\mathcal{U}_{11} =& \frac{H_x}{H} \\
\mathcal{U}_{10} =& -k^2 + \frac{H_{xx}}{H} - \frac{H_x^2}{H^2} \\
\mathcal{U}_{02} =& \frac{r}{H} + ik\lambda V \\
\mathcal{U}_{01} =& ik\lambda V \frac{H_x}{H} \\
\mathcal{U}_{00} =& r\left(\frac{-k^2}{H} + \frac{H_{xx}}{H^2} - \frac{2H_x^2}{H^3}\right) \\
& + ik\lambda\left(-k^2 V - V_{xx} + \frac{V_x H_x + V H_{xx} + (\hat{f}/\lambda)H_x}{H} - \frac{V H_x^2}{H^2}\right) \\
\mathcal{H}_1 =& -\lambda\frac{k^2V^2}{H} + ik\left(\frac{V_t}{H} + \frac{r(2V - \alpha U_0)}{H^2}\right) - \frac{ik\delta}{H} \\
\mathcal{H}_0 =& \lambda k^2\left(\frac{-2VV_x - (\hat{f}/\lambda)V}{H} + \frac{V^2 H_x}{H^2}\right) + ik\left(\frac{V_{tx}}{H} - \frac{V_t H_x}{H^2}\right) \\
& + ik\left(r\left(\frac{2V_x - \alpha U_{0x}}{H^2} - \frac{2(2V - \alpha U_0)H_x}{H^3}\right) + \left(\frac{\delta}{H^2}H_x + ik\frac{\tau_{sx}}{H^2}\right)\right)
\end{aligned}$$

The equation for the perturbation in the longshore flow $v(x,t)$ reads

$$v = \frac{V}{H}h + \frac{i}{k}\left(u_x + \frac{H_x}{H}u\right)$$

And the equation for the bottom perturbation $h(x,t)$ is

$$\frac{\partial h}{\partial \tau} = -\epsilon_q \beta \langle \vec{\nabla} \cdot \vec{q}_{11} \rangle_{\alpha=\alpha_1} - (1-\beta)\langle \vec{\nabla} \cdot \vec{q}_{31} \rangle_{\alpha=\alpha_2}$$

where $\langle\cdot\rangle$ denotes the average over a tidal period

$$\vec{\nabla}\cdot\vec{q}_{m1} = -\nu_m |V|^{m-1}\Big\{ \left(m\frac{H_x}{H} - (m-1)\frac{V_x}{V}\right) u + (m-1)u_x \\ -ikm\frac{V}{H}h + |V|\hat{\gamma}\left(h_{xx} + m\frac{V_x}{V}h_x - k^2 h\right)\Big\}$$

E.2 Discretization

To solve the linear stability problem the Chebyschev collocation method of the appendix B in x and a Fourier Galerkin method in t has been used. The approximate solution are expanded as

$$u(x_n,t,\tau) = e^{\omega\tau}\sum_{p=-M}^{M} u_n^p e^{ipt} \qquad v(x_n,t,\tau) = e^{\omega\tau}\sum_{p=-M}^{M} v_n^p e^{ipt}$$
$$h(x_n,\tau) = e^{\omega\tau} h_n$$

The expansions for the derivatives of u and h at the collocation points are

$$\left.\frac{\partial^3 u}{\partial t x^2}\right|_{x_n} = e^{\omega\tau}\sum_{\substack{-M\le p\le M\\ 1\le j\le N-1}} ipD_{nj}^2 u_j^p e^{ipt} \qquad \left.\frac{\partial^2 u}{\partial x^2}\right|_{x_n} = e^{\omega\tau}\sum_{\substack{-M\le p\le M\\ 1\le j\le N-1}} D_{nj}^2 u_j^p e^{ipt}$$
$$\left.\frac{\partial^2 u}{\partial t x}\right|_{x_n} = e^{\omega\tau}\sum_{\substack{-M\le p\le M\\ 1\le j\le N-1}} ipD_{nj}^1 u_j^p e^{ipt} \qquad \left.\frac{\partial u}{\partial x}\right|_{x_n} = e^{\omega\tau}\sum_{\substack{-M\le p\le M\\ 1\le j\le N-1}} D_{nj}^1 u_j^p e^{ipt}$$
$$\left.\frac{\partial u}{\partial t}\right|_{x_n} = e^{\omega\tau}\sum_{\substack{-M\le p\le M\\ 1\le j\le N-1}} ip\delta_{nj} u_j^p e^{ipt} \qquad u|_{x_n} = e^{\omega\tau}\sum_{\substack{-M\le p\le M\\ 1\le j\le N-1}} \delta_{nj} u_j^p e^{ipt}$$

$$\left.\frac{\partial^2 h}{\partial x^2}\right|_{x_n} = e^{\omega\tau}\sum_{j=2}^{N} G_{nj}^2\hat{h}_j \qquad \left.\frac{\partial h}{\partial x}\right|_{x_n} = e^{\omega\tau}\sum_{j=2}^{N} G_{nj}^1\hat{h}_j \qquad h|_{x_n} = e^{\omega\tau}\sum_{j=2}^{N} G_{nj}^0\hat{h}_j$$

E.2.1 FOT-problem

Application of the collocation method to the equation for u yields

$$\sum_{\substack{-M\le p\le M\\ 1\le j\le N-1}} \Big(ip\mathcal{U}_{12}(x_n,t)D_{nj}^2 + ip\mathcal{U}_{11}(x_n,t)D_{nj}^1 + ip\mathcal{U}_{10}(x_n,t)\delta_{nj} \\ + \mathcal{U}_{02}(x_n,t)D_{nj}^2 + \mathcal{U}_{01}(x_n,t)D_{nj}^1 + \mathcal{U}_{00}(x_n,t)\delta_{nj}\Big) u_j^p e^{ipt} = \\ \sum_{j=2}^{N}\Big(\mathcal{H}_1(x_n,t)G_{nj}^1 + \mathcal{H}_0(x_n,t)G_{nj}^0\Big)\hat{h}_j \qquad n = 1 \div N-1$$

Galerkin projection of the various terms results in

$$\Big(\sum_{\substack{-M\le p\le M\\ 1\le j\le N-1}} ip\mathcal{U}_{12}(x_n,t)D_{nj}^2 u_j^p e^{ipt}, e^{-iqt}\Big) = \sum_{\substack{-M\le p\le M\\ 1\le j\le N-1}} ipI_{p-q}^0 D_{nj}^2 u_j^p$$

$$\Big(\sum_{\substack{-M\le p\le M\\ 1\le j\le N-1}} ip\mathcal{U}_{11}(x_n,t)D_{nj}^1 u_j^p e^{ipt}, e^{-iqt}\Big) = \sum_{\substack{-M\le p\le M\\ 1\le j\le N-1}} ip\left(\frac{H_x}{H}\right)_n I_{p-q}^0 D_{nj}^1 u_j^p$$

$$\Big(\sum_{\substack{-M\le p\le M\\1\le j\le N-1}} ip\mathcal{U}_{10}(x_n,t)\delta_{nj}u_j^p e^{ipt}, e^{-iqt}\Big) = \sum_{p=-M}^{M} ip\Big(-k^2+\frac{H_{xx}}{H}-\frac{H_x^2}{H^2}\Big)_n I^0_{p-q}u_n^p$$

$$\Big(\sum_{\substack{-M\le p\le M\\1\le j\le N-1}} \mathcal{U}_{02}(x_n,t)D^2_{nj}u_j^p e^{ipt}, e^{-iqt}\Big) = \sum_{\substack{-M\le p\le M\\1\le j\le N-1}} \Big(\frac{rI^0_{p-q}}{H}+ik\lambda I^1_{n,p-q}\Big)_n D^2_{nj}u_j^p$$

$$\Big(\sum_{\substack{-M\le p\le M\\1\le j\le N-1}} \mathcal{U}_{01}(x_n,t)D^1_{nj}u_j^p e^{ipt}, e^{-iqt}\Big) = \sum_{\substack{-M\le p\le M\\1\le j\le N-1}} \Big(ik\lambda I^1_{n,p-q}\frac{H_x}{H}\Big)_n D^1_{nj}u_j^p$$

$$\Big(\sum_{\substack{-M\le p\le M\\1\le j\le N-1}} \mathcal{U}_{00}(x_n,t)\delta_{nj}u_j^p e^{ipt}, e^{-iqt}\Big) = \sum_{p=-M}^{M}\Big(r\Big\{\frac{-k^2}{H}+\frac{H_{xx}}{H^2}-\frac{2H_x^2}{H^3}\Big\}_n I^0_{p-q}$$
$$+ik\lambda\Big\{-k^2 I^1_{n,p-q}-I''^1_{p-q}$$
$$+\frac{I'^1_{p-q}H_x+I^1_{n,p-q}H_{xx}+(\hat f/\lambda)I^0_{p-q}H_x}{H}-\frac{I^1_{n,p-q}H_x^2}{H^2}\Big\}\Big)_n u_n^p$$

$$\Big(\sum_{j=2}^{N}\mathcal{H}_1(x_n)G^1_{nj}\hat h_j, e^{-iqt}\Big) = \sum_{j=2}^{N}\Big(-\frac{\lambda k^2 I^2_{n,-q}}{H}+$$
$$ik\Big\{\frac{K^0_{n,-q}}{H}+r\left(\frac{(2I^1_{n,-q}-\alpha U_0 I^0_{-q})}{H^2}\right)-\frac{\delta}{H}I^0_{-q}\Big\}\Big)_n G^1_{nj}\hat h_j$$

$$\Big(\sum_{j=2}^{N}\mathcal{H}_0(x_n)G^0_{nj}\hat h_j, e^{-iqt}\Big) = \sum_{j=2}^{N}\Big(\lambda k^2\Big\{\frac{-2I'^2_{n,-q}-(\hat f/\lambda)I^1_{n,-q}}{H}+\frac{I^2_{n,-q}H_x}{H^2}\Big\}$$
$$+ik\Big\{\frac{K^1_{n,-q}}{H}-\frac{K^0_{n,-q}H_x}{H^2}$$
$$+r\left(\frac{2I'^1_{n,-q}-\alpha U_{0x}I^0_{-q}}{H^2}-\frac{2(2I^1_{n,-q}-\alpha U_0 I^0_{n,-q})H_x}{H^3}\right)$$
$$+\left(\delta\frac{H_x}{H^2}+ik\frac{\tau_{sx}}{H^2}\right)I^0_{-q}\Big\}\Big)_n G^0_{nj}\hat h_j$$

Thus the values of the indices are $-M\le q\le M$ and $0\le i\le N-1$. The 'I,J,K'–coefficients will be specified later.

This defines a system of $(2M+1)\times(N-1)$ equations which can be solved by standard methods. The results are summarized in a matrix $\mathbf{U}^p_{nj}$, which defines the cross-shore velocity component u in the collocation points and for the different Fourier modes. It is defined as

$$u_n = e^{\omega\tau}\sum_{\substack{-M\le p\le M\\2\le j\le N}}\mathbf{U}^p_{nj}e^{ipt}\hat h_j \qquad n = 0\div N-1$$

The boundary conditions for u at $x=0$ gives $\mathbf{U}^p_{0j}=0$.

Substitution in the equation for $v(x,t)$ yields

$$\mathbf{V}^p_{nj} = \left(\frac{I^1_{n,-p}}{2\pi H}\right)_n G^0_{nj} + \frac{i}{k}\left(\sum_{k=0}^{N-1} D^0_{nk} + \frac{H_x}{H}\delta_{nk}\right)_n \mathbf{U}^p_{kj}$$

Then the values of v in the collocation points are

$$v_n = e^{\omega\tau} \sum_{\substack{-M\le p\le M\\ 2\le j\le N}} \mathbf{V}^p_{nj} e^{ipt}\hat{h}_j \qquad n = 0 \div N-1$$

In particular the tidally averaged velocity components read

$$\langle u_n\rangle = e^{\omega\tau}\sum_{j=2}^{N} \mathbf{U}^0_{nj}\hat{h}_j \qquad \langle v_n\rangle = e^{\omega\tau}\sum_{j=2}^{N} \mathbf{V}^0_{nj}\hat{h}_j$$

E.2.2 Bottom evolution equation

Application of the collocation method to the equation for h results in

$$\omega\sum_{j=2}^{N} G^0_{nj}\hat{h}_j = \sum_{j=2}^{N}\{-\epsilon_q\beta\langle\vec{\nabla}\cdot\vec{q}_{11}\rangle_{\alpha=\alpha_1,j} - (1-\beta)\langle\vec{\nabla}\cdot\vec{q}_{31}\rangle_{\alpha=\alpha_2,j}\}\hat{h}_j$$

at the $n = 1 \div N-1$ collocation points.

Using the solutions of the FOT-problem the terms $\langle\vec{\nabla}\cdot\vec{q}_{m1}\rangle_{\alpha=\alpha_{1/2},j}$ become

$$\left\langle \sum_{p=-M}^{M}\left(-m\nu_m|V|^{m-1}\frac{D_x}{D}\right)_n \mathbf{U}^p_{nj}e^{ipt}\right\rangle = \frac{1}{2\pi}\left(-m\nu_m\frac{D_x}{D}\right)_n \sum_{p=-M}^{M} I^{m-1}_{n,p}\mathbf{U}^p_{nij}$$

$$\left\langle \sum_{p=-M}^{M}\left((m-1)\nu_m|V|^{m-1}\frac{V_x}{V}\right)_n \mathbf{U}^p_{nj}e^{ipt}\right\rangle = \frac{1}{2\pi}\left((m-1)\nu_m\right)\sum_{p=-M}^{M} {I'}^{m-1}_{n,p}\mathbf{U}^p_{nj}$$

$$\left\langle \sum_{\substack{-M\le p\le M\\ 1\le k\le N-1}}\left(-(m-1)\nu_m|V|^{m-1}\right)_n D^1_{nk}\mathbf{U}^p_{kj}e^{ipt}\right\rangle = \frac{1}{2\pi}\left(-(m-1)\nu_m\right)\sum_{\substack{-M\le p\le M\\ 1\le k\le N-1}} I^{m-1}_{n,p}D^1_{nk}\mathbf{U}^p_{kj}$$

$$\left\langle\left(ikm\nu_m|V|^{m-1}\frac{V}{D}\right)_n G^0_{nj}\right\rangle = \frac{1}{2\pi}\left(ikm\frac{\nu_m}{D}\right)_n I^m_{n,0}G^1_{nj}$$

$$\left\langle\left(-\hat{\gamma}\nu_m|V|^m\right)_n G^2_{nj}\right\rangle = \frac{1}{2\pi}\left(-\hat{\gamma}\nu_m\right)J^m_{n,0}G^2_{nj}$$

$$\left\langle\left(-\hat{\gamma}m\nu_m|V|^m\frac{V_x}{V}\right)_n G^1_{nj}\right\rangle = \frac{1}{2\pi}\left(-\hat{\gamma}m\nu_m\right){J'}^m_{n,0}G^1_{nj}$$

$$\left\langle\left(-\hat{\gamma}\nu_m|V|^m k^2\right)_n G^0_{nj}\right\rangle = \frac{1}{2\pi}\left(-\hat{\gamma}\nu_m k^2\right)J^m_{n,0}G^0_{nj}$$

Having calculated the operator $\mathbf{U}^p_{nj}$ for $\alpha=\alpha_1$ and for $\alpha=\alpha_2$, determines $\langle\vec{\nabla}\cdot\vec{q}_{m1}\rangle_{\alpha=\alpha_1,j}$ and the term $\langle\vec{\nabla}\cdot\vec{q}_{m1}\rangle_{\alpha=\alpha_2,j}$, respectively.

E.3 Galerkin integrals

The Galerkin method and the tidal averages give rise the 'I, J, K'–coefficients. These coefficients arise from integrals which involve e^{ipt} and the basic velocity $V(x,t)$. In this section these coefficients are written explicily. First, in order to compute the integrals with absolute values of the velocity are involved, sign functions, $\Sigma(x,t)$ and $\sigma(x,t)$, are difined. Afterwards the 'I, J, K'–coefficients are written by means of some auxiliar coefficients, which are given at the end of this section.

The basic state velocity field reads

$$V(x,t) = \alpha U_0(x) + \sigma(1-|\alpha|)U_1(x)\sin\left(t+\varphi(x)\right)$$

where

$$U_1(x) = \frac{H}{\sqrt{H^2+r^2}} \qquad \varphi(x) = \arctan\left(\frac{r}{H}\right)$$

Now define the following sign-functions:

$$\Sigma(x,t) = \frac{|V(x,t)|}{V(x,t)} \qquad \sigma(x,t) = \frac{|\alpha|}{\alpha}$$

It appears than they are related according to

$$\Sigma = \sigma\left(1 - 2(\theta(t-t_1) - \theta(t-t_2))\right)$$

where θ is the Heavyside function and

$$t_1 = \pi - \varphi + \arcsin\left(\frac{-|\alpha|U_0}{(1-|\alpha|)U_1}\right)$$
$$t_2 = 2\pi - \varphi - \arcsin\left(\frac{-|\alpha|U_0}{(1-|\alpha|)U_1}\right)$$

The subsindex n refers to the numerical value of the functions in a collocation point x_n.

E.3.1 'I, J, K'-coefficients

'I, J, K'-coefficients are computed by means of the following auxiliar coefficients

$$X^{kl}_{n,-q} = \int_0^{2\pi} \Sigma(x_n,t)\sin^k(t+\varphi_n)\cos^l(t+\varphi_n)e^{-iqt}dt = \sigma(Y^{kl}_{n,-q} - 2Z^{kl}_{n,-q})$$
$$Y^{kl}_{n,-q} = \int_0^{2\pi} \sin^k(t+\varphi_n)\cos^l(t+\varphi_n)e^{-iqt}dt$$
$$Z^{kl}_{n,-q} = \int_{\pi+\theta_n-\varphi_n}^{2\pi-\theta_n-\varphi_n} \sin^k(t+\varphi_n)\cos^l(t+\varphi_n)e^{-iqt}dt$$

They will be computed in the next subsection.

'I'-coefficients

$$
\begin{aligned}
I^0_{-q} =& \int_0^{2\pi} e^{-iqt} dt \\
=& Y^{00}_{-q} \\
I^1_{n,-q} =& \int_0^{2\pi} V(x_n,t) e^{-iqt} dt \\
=& \alpha U_{0n} Y^{00}_{-q} + \sigma(1-|\alpha|) U_{1n} Y^{10}_{n,-q} \\
I^2_{n,-q} =& \int_0^{2\pi} V^2(x_n,t) e^{-iqt} dt \\
=& \alpha^2 U^2_{0n} Y^{00}_{-q} + 2|\alpha|(1-|\alpha|) U_{0n} U_{1n} Y^{10}_{n,-q} + (1-|\alpha|)^2 U^2_{1n} Y^{20}_{n,-q} \\
I^3_{n,-q} =& \int_0^{2\pi} V^3(x_n,t) e^{-iqt} dt \\
=& \alpha^3 U^3_{0n} Y^{00}_{n,-q} + 3\alpha^2 \sigma(1-|\alpha|) U^2_{0n} U_{1n} Y^{10}_{n,-q} \\
& + 3\alpha(1-|\alpha|)^2 U_{0n} U^2_{1n} Y^{20}_{n,-q} + \sigma(1-|\alpha|)^3 U^3_{1n} Y^{30}_{n,-q}
\end{aligned}
$$

$$
\begin{aligned}
{I'}^1_{n,-q} =& \int_0^{2\pi} V_x(x_n,t) e^{-iqt} dt \\
=& \alpha U'_{0n} Y^{00}_{-q} + \sigma(1-|\alpha|)\{U'_{1n} Y^{10}_{n,-q} + U_{1n} \varphi'_n Y^{01}_{n,-q}\} \\
{I'}^2_{n,-q} =& \int_0^{2\pi} V(x_n,t) V_x(x_n,t) e^{-iqt} dt \\
=& \alpha^2 U_{0n} U'_{0n} Y^{00}_{-q} \\
& + |\alpha|(1-|\alpha|)\{(U'_{0n} U_{1n} + U_{0n} U'_{1n}) Y^{10}_{n,-q} + U_{0n} U_{1n} \varphi'_n Y^{01}_{n,-q}\} \\
& + (1-|\alpha|)^2 \{U_{1n} U'_{1n} Y^{20}_{n,-q} + U^2_{1n} \varphi'_n Y^{11}_{n,-q}\} \\
{I''}^1_{n,-q} =& \int_0^{2\pi} V_{xx}(x_n,t) e^{-iqt} dt \\
=& \alpha U''_{0n} Y^{00}_{-q} \\
& + \sigma(1-|\alpha|)\{(U''_{1n} - U_{1n}(\varphi'_n)^2) Y^{10}_{n,-q} + (2U'_{1n}\varphi'_n + U_{1n}\varphi''_n) Y^{01}_{n,-q}\}
\end{aligned}
$$

'J'-coefficients

$$
\begin{aligned}
J^0_{n,-q} =& \int_0^{2\pi} \frac{|V(x_n,t)|}{V(x_n,t)} e^{-iqt} dt \\
=& X^{00}_{n,-q} \\
J^1_{n,-q} =& \int_0^{2\pi} |V(x_n,t)| e^{-iqt} dt \\
=& \alpha U_{0n} X^{00}_{n,-q} + \sigma(1-|\alpha|) U_{1n} X^{10}_{n,-q} \\
J^2_{n,-q} =& \int_0^{2\pi} V(x_n,t)|V(x_n,t)| e^{-iqt} dt \\
=& \alpha^2 U^2_{0n} X^{00}_{n,-q} + 2|\alpha|(1-|\alpha|) U_{0n} U_{1n} X^{10}_{n,-q} + (1-|\alpha|)^2 U^2_{1n} X^{20}_{n,-q} \\
J^3_{n,-q} =& \int_0^{2\pi} V^2(x_n,t)|V(x_n,t)| e^{-iqt} dt \\
=& \alpha^3 U^3_{0n} X^{00}_{n,-q} + 3\alpha^2 \sigma(1-|\alpha|) U^2_{0n} U_{1n} X^{10}_{n,-q} \\
& + 3\alpha(1-|\alpha|)^2 U_{0n} U^2_{1n} X^{20}_{n,-q} + \sigma(1-|\alpha|)^3 U^3_{1n} X^{30}_{n,-q}
\end{aligned}
$$

$$\begin{aligned}
{J'}^1_{n,-q} =& \int_0^{2\pi} \frac{|V(x_n,t)|}{V(x_n,t)} V_x(x_n,t) e^{-iqt} dt \\
=& \alpha U'_{0n} X^{00}_{-q} + \sigma(1-|\alpha|)\{U'_{1n} X^{10}_{n,-q} + U_{1n}\varphi'_n X^{01}_{n,-q}\} \\
{J'}^2_{n,-q} =& \int_0^{2\pi} V_x(x_n,t)|V(x_n,t)| e^{-iqt} dt \\
=& \alpha^2 U_{0n} U'_{0n} X^{00}_{-q} \\
&+ |\alpha|(1-|\alpha|)\{(U'_{0n}U_{1n} + U_{0n}U'_{1n})X^{10}_{n,-q} + U_{0n}U_{1n}\varphi'_n X^{01}_{n,-q}\} \\
&+ (1-|\alpha|)^2\{U_{1n}U'_{1n}X^{20}_{n,-q} + U^2_{1n}\varphi'_n X^{11}_{n,-q}\} \\
{J'}^3_{n,-q} =& \int_0^{2\pi} V(x_n,t) V_x(x_n,t)|V(x_n,t)| e^{-iqt} dt \\
=& \alpha^3 U^2_{0n} U'_{0n} X^{00}_{-q} \\
&+ \alpha^2\sigma(1-|\alpha|)\{(2U_{0n}U'_{0n}U_{1n} + U^2_{0n}U'_{1n})X^{10}_{n,-q} + U^2_{0n}U_{1n}\varphi'_n X^{01}_{n,-q}\} \\
&+ \alpha(1-|\alpha|)^2\{(U'_{0n}U^2_{1n} + 2U_{0n}U_{1n}U'_{1n})X^{20}_{n,-q} + 2U_{0n}U^2_{1n}\varphi'_n X^{11}_{n,-q}\} \\
&+ \sigma(1-|\alpha|)^3\{U^2_{1n}U'_{1n}X^{30}_{n,-q} + U^3_{1n}\varphi'_n X^{21}_{n,-q}\}
\end{aligned}$$

'K'-coefficients

$$\begin{aligned}
K^0_{n,-q} =& \int_0^{2\pi} V_t(x_n,t) e^{-iqt} dt = \sigma(1-|\alpha|)U_{1n} Y^{01}_{n,-q} \\
K^1_{n,-q} =& \int_0^{2\pi} V_{tx}(x_n,t) e^{-iqt} dt = \sigma(1-|\alpha|)\{U'_{1n} Y^{01}_{n,-q} - U_{1n}\varphi'_n Y^{10}_{n,-q}\}
\end{aligned}$$

E.3.2 Auxiliary integrals

Now defining

$$Y^j_{n,-q} = \int_0^{2\pi} e^{ij(t+\varphi_n)} e^{-iqt} dt = e^{ij\varphi_n} 2\pi \delta_{0,j-q}$$

Then the Y-integrals can be written as

$$\begin{aligned}
Y^{00}_{n,-q} =& Y^0_{n,-q} \\
Y^{10}_{n,-q} =& \frac{i}{2}\left\{Y^{-1}_{n,-q} - Y^{+1}_{n,-q}\right\} \\
Y^{01}_{n,-q} =& \frac{1}{2}\left\{Y^{-1}_{n,-q} + Y^{+1}_{n,-q}\right\} \\
Y^{20}_{n,-q} =& \frac{1}{2} Y^0_{n,-q} - \frac{1}{4}\left\{Y^{-2}_{n,-q} + Y^{+2}_{n,-q}\right\} \\
Y^{11}_{n,-q} =& \frac{i}{4}\left\{Y^{-2}_{n,-q} - Y^{+2}_{n,-q}\right\} \\
Y^{02}_{n,-q} =& \frac{1}{2} Y^0_{n,-q} + \frac{1}{4}\left\{Y^{-2}_{n,-q} + Y^{+2}_{n,-q}\right\} \\
Y^{30}_{n,-q} =& \frac{3i}{8}\left\{Y^{-1}_{n,-q} - Y^{+1}_{n,-q}\right\} - \frac{i}{8}\left\{Y^{-3}_{n,-q} - Y^{+3}_{n,-q}\right\} \\
Y^{21}_{n,-q} =& \frac{1}{8}\left\{Y^{-1}_{n,-q} + Y^{+1}_{n,-q}\right\} - \frac{1}{8}\left\{Y^{-3}_{n,-q} + Y^{+3}_{n,-q}\right\}
\end{aligned}$$

Finally, introducing

$$\begin{aligned}
Z^j_{n,-q} =& \int_{\pi+\theta_n-\varphi_n}^{2\pi-\theta_n-\varphi_n} e^{ij(t+\varphi_n)} e^{-iqt} dt \\
=& e^{ij\varphi_n}(\pi - 2\theta_n)\delta_{0,j-q} - i\frac{e^{iq\varphi_n}}{j-q}\left\{e^{-i(j-q)\theta_n} - (-1)^{j-q} e^{i(j-q)\theta_n}\right\}
\end{aligned}$$

Then the Z-integrals can be written as

$$
\begin{aligned}
Z^{00}_{n,-q} &= Z^{0}_{n,-q} \\
Z^{10}_{n,-q} &= \frac{i}{2}\left\{Z^{-1}_{n,-q} - Z^{+1}_{n,-q}\right\} \\
Z^{01}_{n,-q} &= \frac{1}{2}\left\{Z^{-1}_{n,-q} + Z^{+1}_{n,-q}\right\} \\
Z^{20}_{n,-q} &= \frac{1}{2}Z^{0}_{n,-q} - \frac{1}{4}\left\{Z^{-2}_{n,-q} + Z^{+2}_{n,-q}\right\} \\
Z^{11}_{n,-q} &= \frac{i}{4}\left\{Z^{-2}_{n,-q} - Z^{+2}_{n,-q}\right\} \\
Z^{02}_{n,-q} &= \frac{1}{2}Z^{0}_{n,-q} + \frac{1}{4}\left\{Z^{-2}_{n,-q} + Z^{+2}_{n,-q}\right\} \\
Z^{30}_{n,-q} &= \frac{3i}{8}\left\{Z^{-1}_{n,-q} - Z^{+1}_{n,-q}\right\} - \frac{i}{8}\left\{Z^{-3}_{n,-q} - Z^{+3}_{n,-q}\right\} \\
Z^{21}_{n,-q} &= \frac{1}{8}\left\{Z^{-1}_{n,-q} + Z^{+1}_{n,-q}\right\} - \frac{1}{8}\left\{Z^{-3}_{n,-q} + Z^{+3}_{n,-q}\right\}
\end{aligned}
$$

Since $\theta_n = \arcsin(\frac{|\alpha|U_{0n}}{(1-|\alpha|)U_{1n}})$, the phases of the solution are in the interval $[0, \frac{\pi}{2}]$. In the points where $\frac{|\alpha|U_{0i}}{(1-|\alpha|)} > U_{1i}$ it follows that $\theta_n = \frac{\pi}{2}$, hence in these points $Z^{0}_{n,-q}$,$Z^{1}_{n,-q}$,$Z^{2}_{n,-q}$ y $Z^{3}_{n,-q}$ are zero.

Appendix F

F.1 Adjoint operator

Defining an inner product $\langle\cdot|\cdot\rangle$ as:

$$\langle f|g\rangle = \sum_{i=1}^{4}(f_i, g_i)$$

and

$$(f_i, g_i) = \int_0^\infty \int_0^{\lambda_0} \overline{f_i(x,y)} g_i(x,y)\mathrm{d}x\mathrm{d}y\ ,$$

from the definition of adjoint operator $\mathcal{L}^+$, ie. $\langle\Psi|\mathcal{L}\Phi\rangle = \langle\mathcal{L}^+\Psi|\Phi\rangle$, it follows that

$$(\omega_{kn_k} - \overline{\omega^+_{k'n_{k'}}})\langle\Psi^+_{k'n_{k'}}|\mathcal{S}\Psi_{kn_k}\rangle = 0\ ,$$

where Ψ_{kn_k} (resp. $\Psi^+_{k'n_{k'}}$) are the eigenvectors of $\mathcal{L}$ (resp. $\mathcal{L}^+$) with eigenvalue ω_{kn_k}(resp. $\omega^+_{k'n_{k'}}$), or

$$(\omega_{kn_k} - \overline{\omega^+_{k'n_{k'}}})(h^+_{k'n_{k'}}e^{\mathrm{i}k'y}, h_{kn_k}e^{\mathrm{i}ky}) = 0$$

Under the hypothesis that the domain of $\mathcal{L}$ is the same than the domain of $\mathcal{L}^+$, the relations written above mean that the set of eigenvectors Ψ_{kn_k} and $\Psi^+_{k'n_{k'}}$ are a bi–orthogonal set under the inner product $\langle\mathcal{S}\cdot|\cdot\rangle$, ie.

$$\langle\Psi^+_{k'n_{k'}}|\mathcal{S}\Psi_{kn_k}\rangle = \delta_{kk'}\delta_{n_kn_{k'}}$$

or that

$$(h^+_{k'n_{k'}}e^{\mathrm{i}k'y}, h_{kn_k}e^{\mathrm{i}ky}) \begin{cases} = 0 & \text{if } \omega_{kn_k} \neq \overline{\omega^+_{k'n_{k'}}} \\ \neq 0 & \text{if } \omega_{kn_k} = \overline{\omega^+_{k'n_{k'}}} \end{cases}\ .$$

In particular, given $h_{k\,n_k}$, we always can assume that there exists a $h^+_{k'\,n'_k}$ which is non-orthogonal to it.

The adjoint operator of $\mathcal{L}_k$ reads

$$\mathcal{L}^+_k = \begin{pmatrix} -\mathrm{i}kV + \dfrac{r}{H} & V_x + \hat{f} & -H\dfrac{\mathrm{d}}{\mathrm{d}x} & \dfrac{\mathrm{d}}{\mathrm{d}x} \\ -\hat{f} & -\mathrm{i}kV + \dfrac{r}{H} & \mathrm{i}kH & \mathrm{i}k \\ -\dfrac{\mathrm{d}}{\mathrm{d}x} & -\mathrm{i}k & 0 & 0 \\ 0 & -\dfrac{\delta}{H} & \mathrm{i}kV & \hat{\gamma}|V|(\dfrac{V_x}{V}\dfrac{\mathrm{d}}{\mathrm{d}x} + \dfrac{\mathrm{d}^2}{\mathrm{d}^2x} - k^2) \end{pmatrix}$$

The pertinent boundary conditions are that u^+ and h^+ should vanish at $x = 0$ and $x \to \infty$ for each solution kn_k.

F.2 Nonlinear system

In this appendix equations (4.10) are explicitly written.

u–kn_k equation

Equation (4.10a) gives

$$0 = \sum_{n'_k} (\mathcal{L}_{11}(k;n_k,n'_k)\hat{u}_{kn'_k} + \mathcal{L}_{12}(k;n_k,n'_k)\hat{v}_{kn'_k} + \mathcal{L}_{13}(k;n_k,n'_k)\hat{\eta}_{kn'_k}) + \mathcal{N}_1(k;n_k)\,,$$

and these coefficients read

$$\mathcal{L}_{11}(k;n_k,n'_k) = \int_0^\infty \overline{u^+_{kn_k}} u_{kn'_k}(\mathrm{i}kV + \frac{r}{H})\mathrm{d}x$$
$$\mathcal{L}_{12}(k;n_k,n'_k) = \int_0^\infty \overline{u^+_{kn_k}} v_{kn'_k}(-\hat{f})\mathrm{d}x$$
$$\mathcal{L}_{13}(k;n_k,n'_k) = \int_0^\infty \overline{u^+_{kn_k}} (\eta_{kn'_k})_x\mathrm{d}x$$
$$\mathcal{N}_1(k;n_k) = \int_0^\infty \overline{u^+_{kn_k}} \mathcal{F}_k\{u\partial_x u + v\partial_y u + \frac{ruh}{H(H-h)}\}\mathrm{d}x$$

where $\mathcal{F}_k$ is the k–component of the Fourier expansion.

v–kn_k equation

Equation (4.10b) gives

$$0 = \sum_{n'_k} (\mathcal{L}_{21}(k;n_k,n'_k)\hat{u}_{kn'_k} + \mathcal{L}_{22}(k;n_k,n'_k)\hat{v}_{kn'_k} + \mathcal{L}_{23}(k;n_k,n'_k)\hat{\eta}_{kn'_k} + \mathcal{L}_{24}(k;n_k,n'_k)\hat{h}_{kn'_k}) + \mathcal{N}_2(k;n_k)\,,$$

and these coefficients read

$$\mathcal{L}_{21}(k;n_k,n'_k) = \int_0^\infty \overline{v^+_{kn_k}} u_{kn'_k}(V_x + \hat{f})\mathrm{d}x$$
$$\mathcal{L}_{22}(k;n_k,n'_k) = \int_0^\infty \overline{v^+_{kn_k}} v_{kn'_k}(\mathrm{i}kV + \frac{r}{H})\mathrm{d}x$$
$$\mathcal{L}_{23}(k;n_k,n'_k) = \int_0^\infty \overline{v^+_{kn_k}} \eta_{kn'_k}(\mathrm{i}k)\mathrm{d}x$$
$$\mathcal{L}_{24}(k;n_k,n'_k) = \int_0^\infty \overline{v^+_{kn_k}} h_{kn'_k}(-\frac{\delta}{H})\mathrm{d}x$$
$$\mathcal{N}_2(k;n_k) = \int_0^\infty \overline{v^+_{kn_k}} \mathcal{F}_k\{u\partial_x v + v\partial_y v + \frac{(rv-\delta h)h}{H(H-h)}\}\mathrm{d}x\,.$$

η–kn_k equation

Equation (4.10c) gives

$$0 = \sum_{n'_k} (\mathcal{L}_{31}(k;n_k,n'_k)\hat{u}_{kn'_k} + \mathcal{L}_{32}(k;n_k,n'_k)\hat{v}_{kn'_k} + \mathcal{L}_{34}(k;n_k,n'_k)\hat{h}_{kn'_k}) + \mathcal{N}_3(k;n_k)\,,$$

and these coefficients read

$$\mathcal{L}_{31}(k;n_k,n'_k) = \int_0^\infty \overline{\eta^+_{kn_k}} (H_x u_{kn'_k} + H(u_{kn'_k})_x)\mathrm{d}x$$
$$\mathcal{L}_{32}(k;n_k,n'_k) = \int_0^\infty \overline{\eta^+_{kn_k}} v_{kn'_k}(\mathrm{i}kH)\mathrm{d}x$$
$$\mathcal{L}_{34}(k;n_k,n'_k) = \int_0^\infty \overline{\eta^+_{kn_k}} h_{kn'_k}(-\mathrm{i}kV)\mathrm{d}x$$
$$\mathcal{N}_3(k;n_k) = \int_0^\infty \overline{\eta^+_{kn_k}} \mathcal{F}_k\{-\partial_x(uh) - \partial_y(vh)\}\mathrm{d}x\,.$$

h–kn_k equation

Equation (4.10d) gives

$$\sum_{n'_k} \mathcal{S}_{44}(k;n_k,n'_k)\frac{\mathrm{d}\hat{h}_{kn'_k}}{\mathrm{d}t} = \sum_{n'_k}(\mathcal{L}_{41}(k;n_k,n'_k)\hat{u}_{kn'_k} + \mathcal{L}_{42}(k;n_k,n'_k)\hat{v}_{kn'_k} + \mathcal{L}_{44}(k;n_k,n'_k)\hat{h}_{kn'_k}) + \mathcal{N}_4(k;n_k)\,,$$

and these coefficients read

$$\mathcal{S}_{44}(k;n_k,n'_k) = \int_0^\infty \overline{h^+_{kn_k}} h_{kn'_k}\mathrm{d}x$$
$$\mathcal{L}_{41}(k;n_k,n'_k) = \int_0^\infty \overline{h^+_{kn_k}}(-u_{kn'_k})_x\mathrm{d}x$$
$$\mathcal{L}_{42}(k;n_k,n'_k) = \int_0^\infty \overline{h^+_{kn_k}} v_{kn'_k}(-\mathrm{i}k)\mathrm{d}x$$
$$\mathcal{L}_{44}(k;n_k,n'_k) = \int_0^\infty \overline{h^+_{kn_k}}(\hat{\gamma}|V|(\frac{V_x}{V}(h_{kn'_k})_x + (h_{kn'_k})_{xx} - k^2 h_{kn'_k})\mathrm{d}x$$

$$\mathcal{N}_4(k;n_k) = \int_0^\infty \overline{h^+_{kn_k}}\mathcal{F}_k\{\partial_x(\hat{\gamma}(|\boldsymbol{v}|-|V|)\partial_x h) + \partial_y(\hat{\gamma}(|\boldsymbol{v}|-|V|)\partial_y h)\}\mathrm{d}x\,.$$

F.3 Time integration scheme

A stiffly–stable type scheme (see Karniadakis, Israeli & Orszag, 1991, sec 4.2) is then used to carry out time integration of the system (4.11):

$$0 = L_1 U^{n+1} + M_1 h^{n+1} + \sum_{q=0}^{J_e-1}\beta_q f(U^{n-q},h^{n-q})$$
$$\frac{\gamma_0 S h^{n+1} - \sum_{q=0}^{J_i-1}\alpha_q S h^{n-q}}{\Delta t} = L_2 U^{n+1} + M_2 h^{n+1} + \sum_{q=0}^{J_e-1}\beta_q g(U^{n-q},h^{n-q})$$

The values of the coefficients from Karniadakis *et al.* (1991) are reproduced in table F.1. Note that an Euler–forward/backward integration rule corresponds to the frist-order scheme. For higher orders, the scheme is implicit for the linear terms and explicit for the nonlinear terms. The nonlinear part is computed as an extrapolation at$n+1$ time from the previous J_e steps.

Coefficient	1st Order	2nd Order	3rd Order
γ_0	1	3/2	11/6
α_0	1	2	3
α_1	0	-1/2	-3/2
α_2	0	0	1/3
β_0	1	2	3
β_1	0	-1	-3
β_2	0	0	1

Table F.1: Stiffly–Stable Scheme Coefficients

F.4 $k = 0$ mode

The nonlinear self–interaction of any mode with wavenumber k excite a component with $k = 0$, ie. alongshore uniform, in the expansion (4.9). However, this was not taken into account in the computations where modes with $k = 0$ have been neglected. The reason for this is that this self–interaction was found to be very weak. For this propouse the dynamics of an alongshore uniform botom perturvation has been study. This can be shown by using the analytical approximation to the linear modes in case of $m = 1$, $\gamma = 0$ and $r = 0$ developped in appendixC.

As start, let us split the perturbation of the basic state, in a longshore mean and a periodic function in a longshore length $2L$:

$$u = \langle u \rangle + u' \qquad \eta = \langle \eta \rangle + \eta'$$
$$v = \langle v \rangle + v' \qquad h = \langle h \rangle + h'$$

where $\langle \cdot \rangle = (1/2L) \int_{-L}^{+L} \cdot \, \mathrm{d}y$. $\langle u' \rangle = \langle v' \rangle = \langle \eta' \rangle = \langle h' \rangle = 0$ and $\langle u \rangle$, $\langle v \rangle$, $\langle \eta \rangle$, and $\langle h \rangle$ refer to $k = 0$ modes. Using this notation, the finality of this appendix is to show that $\langle h \rangle = 0$.

From mass conservation (4.6) and sediment conservation (4.7) equations in case $\gamma = 0$,

$$\partial_x((H-h)u) + \partial_y((H-h)(V-v)) = 0$$

$$\partial_t h + \partial_x u + \partial_y v = 0$$

and making averages in the longshore direction, $\langle \cdot \rangle$, equations for $\langle h \rangle$ and $\langle u \rangle$ are found:

$$\partial_x((H - \langle h \rangle)\langle u \rangle - \langle h'u' \rangle) = 0\,,$$

hence,

$$\langle u \rangle \propto \frac{\langle h'u' \rangle}{H - \langle h \rangle}\,,$$

and

$$\partial_t \langle h \rangle = -\partial_x \langle u \rangle\,.$$

Now, we consider self-intercations of $\{k, n\}$ modes $h' = \hat{h}(x)\, e^{iky}$ and $u' = \hat{u}(x)\, e^{iky}$. From the apporximated modes of appendixC:

$$\hat{h}(x) = e^{\mathrm{i}\alpha_1 x} - e^{\mathrm{i}\alpha_2 x} \qquad \alpha_{1,2} = \mp n\pi - \sqrt{k^2 + n^2\pi^2}$$

and

$$\frac{\mathrm{d}^2\hat{u}}{\mathrm{d}x^2} - k^2\hat{u} = -\mathrm{i}k\frac{\mathrm{d}\hat{h}}{\mathrm{d}x}$$

Therefore,

$$\hat{u} = \frac{\alpha_1 k}{\alpha_1^2 + k^2}\left(e^{\mathrm{i}\alpha_1 x} - 1\right) - \frac{\alpha_2 k}{\alpha_2^2 + k^2}\left(e^{\mathrm{i}\alpha_2 x} - 1\right)$$

Then

$$\begin{aligned}\langle h'u' \rangle &= \langle [\hat{h}(x)\, e^{\mathrm{i}ky} + \hat{h^*}(x)\, e^{-\mathrm{i}ky}][\hat{u}(x)\, e^{\mathrm{i}ky} + \hat{u^*}(x)\, e^{-\mathrm{i}ky}] \rangle \\ &= \hat{h^*}(x)\hat{u}(x) + \text{c.c.} + \langle 2k\text{-modes} \rangle\end{aligned}$$

and

$$\begin{aligned}\hat{h^*}(x)\hat{u}(x) = &\frac{\alpha_1 k}{\alpha_1^2 + k^2}\left(e^{2\mathrm{i}\alpha_1 x} - e^{\mathrm{i}\alpha_1 x} - e^{\mathrm{i}(\alpha_1+\alpha_2)x} + e^{\mathrm{i}\alpha_2 x}\right) \\ &- \frac{\alpha_2 k}{\alpha_2^2 + k^2}\left(e^{\mathrm{i}(\alpha_1+\alpha_2)x} - e^{\mathrm{i}\alpha_1 x} - e^{2\mathrm{i}\alpha_2 x} + e^{\mathrm{i}\alpha_2 x}\right)\,.\end{aligned}$$

Sustituting expressions for $\alpha_{1,2}$ in $\hat{h^*}(x)\hat{u}(x)$ all terms cancel: self-interaction of modes $\{k, n\}$ only forces $2k$-mode whose mean vanishes. Hence $\langle h'u' \rangle = 0$ and then $\langle u \rangle = 0$ and $\langle h \rangle = 0$.