UNIVERSITAT ROVIRA I VIRGILI

CLAIMS PROBLEMS: AN IMPLEMENTATION APPROACH

Maria José Solís Baltodano

UNIVERSITAT ROVIRA i VIRGILI

Claims Problems: An Implementation Approach

MARIA JOSÉ SOLÍS BALTODANO

DOCTORAL THESIS
2018

Claims Problems: An Implementation Approach

PhD dissertation by

Maria José Solís Baltodano

Submitted to the Department of Economics
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy
at the
Universitat Rovira i Virgili

Supervised by
José-Manuel Giménez-Gómez
Cori Vilella Bach

UNIVERSITAT ROVIRA i VIRGILI

Reus, 2018

"We must dare, dare again, always dare..."

Georges Jacques Danton

Acknowledgments

Gracias a todas las personas que aportaron y formaron parte de este largo proceso, sin su ayuda esta tesis no hubiese sido posible. Ahora sé que la decisión de realizarla ha sido una de las mejores de mi vida. Todo la experiencia y conocimiento adquirido en este tiempo, tanto en el aspecto personal como profesional es cuantioso.

En primer lugar, quiero agradecer a mis supervisores:

A mi director Dr. José Manuel Giménez Gómez no solo por su paciencia, y su valiosa dirección en este proyecto, sino también por su enorme calidad humana. Tengo presente las conversaciones motivadoras recibidas por su parte desde el máster, las cuales fueron el detonante para emprender este camino. Por la confianza depositada en mí para la elaboración de este trabajo y algo muy importante, por saber lidiar con mi carácter.

A mi directora Dra. Cori Vilella Bach, por el ánimo y la guía brindada durante cada paso de la realización de esta tesis. Por el cariño y por compartir sus conocimientos, su tiempo y experiencia, que han hecho que pueda seguir hasta finalizar la tesis.

A ambos, gracias por la oportunidad de trabajar a su lado, ha sido una experiencia enriquecedora.

Gracias a los profesores del departamento de Economía que me han ayudado con comentarios y recomendaciones para la elaboración de los artículos. Al Dr. Bern Theilen, Dr. Antonio Quesada, Dra. Misericordia Carles, Dr. Sebastián Cano, Dra. Maria Llop. A nivel administrativo agradecer a Eulàlia, Lourdes, Loli y Verònica, por siempre tener disposición y la habilidad para resolver cualquier trámite y problema administrativo.

A mis coautores quienes son parte fundamental de esta tesis. Al Dr. Josep Enric Peris Ferrando, por toda la ayuda, sugerencias y aportaciones a este trabajo. Un agradecimiento especial a la Dra. Arantza Estévez Fernández. Primero por haberme recibido en la Universidad Vrije de Amsterdam y trabajar conmigo. Segundo por las ideas y comentarios brindados, y en particular gracias por los consejos y el ánimo los cuales tendré siempre presente.

A mis compañeros de doctorado y a los que hoy ya son doctores. A Eva, Patricia, Elisenda, David, Françeska, Carlos, Yitez, y Carles.

Gracias por el cariño y los buenos momentos compartidos. A esos amigos que han estado ahí en los momentos difíciles de este proceso, por escucharme y darme siempre ánimos Shaidy S., Jennyfer M. y Samuel M.

Agradecer también los comentarios de los asistentes a mis seminarios y conferencias, Francesc Llerena, Elena Iñarra, Juan Vidal, Juan Antonio Duro, junto con altos cargos de la Generalitat de Catalunya como el ex-conseller Andreu Mas-Colell, Salvador Maluquer y María José Àlvarez.

Gracias a mi madre Dora por ser la promotora de mis sueños y ser un ejemplo de superación. A mi hermana Mayela, por confiar en mí y estar presente durante todo el proceso de esta tesis, gracias por tu apoyo incondicional. A mi hermano Juan, quien, aunque no esté presente físicamente, su energía y felicidad me acompañarán toda la vida. A mi padre, Juan, a mis hermanas Marisa, Yamileth y Kathy, a mis sobrinos, Josué, David y Sofía. Este también es su logro.

Agradecer el apoyo económico por parte de Universitat Rovira i Virgili, GRODE, CREIP, y Generalitat de Catalunya (bajo los proyectos 2014SGR325 y 2014SGR631), y al Ministerio de Economía y Competitividad (bajo los proyectos ECO2011-22765 y ECO2016-75410-P).

Finally, my very special thanks to professor William Thomson for agreeing to be part of the evaluation committee of my PhD dissertation. My further thanks for his interest on my work. I am sure that his comments and suggestions will contribute in a positive way to this thesis. I would also like to thank professor Antonio Quesada for accepting the invitation to create this tribunal. I am thankful to him for being always available to answer any inquiry, for all his help and recommendations. Last, but not least, I would like to express my gratitude to professor María del Carmen Marco for her attendance, and I am sure that her participation will give many benefits to this thesis.

A todos ustedes, mi mayor agradecimiento por hacer posible este trabajo.

Muchas gracias.

Contents

Introduction

Game theory is the discipline that studies how agents make strategic decisions. It was initially developed in economics to understand a large collection of economic behaviors, including firms, markets and consumers. Specifically, a game is the mathematical formalization of such conflicts, originated by Antoine Augustine Cournot (1801-1877) in 1838 with his solution of the Cournot duopoly.

Later on, and since the classical book by von Neumann and Morgenstern (1944) and the remarkable paper by Nash (1951), game theory has become an interdisciplinary approach to the study of human behaviour. Indeed, according to Maschler (1992), after this initial period, game theory was developed extensively in the 1950s by numerous authors, not only in economics, but also in many other disciplines (mathematics, politics, social network formation, behavioral science, biology and in general all social sciences).

Therefore, a game describes a situation with several agents (usually called players) where the agents, who are assumed to have independent interests, can make several decisions and the result depends on the interactions between all the chosen strategies. As mentioned above, agents aware about their own benefits, but it does not imply that they will cooperate. Thus, game theory is divided into two branches, called the non-cooperative and cooperative branches. Actually, in Aumann's words (Aumann, 1989, pp. 8-9):

> "Cooperative theory starts with a formalization of games that abstracts away altogether from procedures and [...] concentrates, instead, on the possibilities for agreement [...] There are several reasons that explain why cooperative games came to be treated separately. One is that when one does build negotiation and enforcement procedures explicitly into the model, then the results of a non-cooperative analysis depend very strongly on the precise form of the procedures, on the order of making offers and counter-offers and so on. This may be appropriate in voting situations in which precise rules of parliamentary order prevail, where a good strategist can indeed carry the day. But problems of negotiation are usually more amorphous; it is difficult to pin down just what the procedures are. More fundamentally, there is a feeling that procedures are not really all that relevant; that it is the possibilities for coalition forming, promising and threatening that are de-

> cisive, rather than whose turn it is to speak [...] Detail distracts attention from essentials. Some things are seen better from a distance; the Roman camps around Metzada are indiscernible when one is in them, but easily visible from the top of the mountain."

These two branches of game theory differ in how they formalize interdependence between the players. On the one hand, in cooperative game theory it is possible to make enforceable binding agreements. Moreover, in many real situations modelled by cooperative games, also side payments are also allowed. On the other hand, in non-cooperative game theory, agents are considered as utility-maximizer individuals. As already mentioned, the cornerstone of this theory is the notion of Nash equilibrium and a game is any interactive situation in which a player's payoff depends not only on his own choice of actions, but also on the actions of the others. In this case, pre-play communication is allowed, but enforceable binding agreements are not. Thus, strategic analysis and individual incentives play an important role.

The present work stands upon both the non-cooperative and the cooperative branches of the game theory literature. In particular, we deal with claims problems. A claims problem appears when a group of agents have claims on a resource and the available quantity of the resource is not enough to satisfy the demands of all agents. The most characteristic example is when a firm goes bankrupt and the liquidation value of the firm is not enough to satisfy all the creditors' demands. However, there are many real-life situations like this and the question is always, how should the resource be divided? A rule specifies a non-negative division of the amount available for each claims problem, which exhausts the endowment and it is bounded by the claims (O'Neill, 1982). The main rules in the literature are the proportional rule, for which awards are proportional to claims, the constrained equal awards, for which awards are as equal as possible subject to no one receiving more than her claim, and the constrained equal losses, for which losses are as equal as possible but no one can receive a negative amount (for surveys, see Moulin, 2000; Thomson, 2003, 2006, 2015). Some of these problems and rules already appear in the ancient literature, in the Talmud (a collection of writings that constitute the basis of the Jewish law).

There are different approaches that can be used to study claims problems: the direct one starting from rules, the axiomatic one starting from properties of rules and, the game theoretic one where situations are modelled as games. In our work we deal with the axiomatic approach. In this approach the solutions are based on a selection of some properties of the rules, fixed on any situation, whose mathematical expression is denoted by an axiom. We focus on the ideas of equity and stability criteria, considering that when a resource is distributed, each agent should have a fair award. Therefore, we decided to analyze different areas in this field from both its implementation to real cases

(Chapters 1 and 2), and the theoretical point of view (Chapters 3 and 4).

The remainder of the current document is as follows (note that each chapter is independent of the others, so each chapter has an introduction and a conclusion).[1]

Chapter 1 analyzes the adjustments applied on the public health budget in Catalonia. It is noteworthy that, due to the economic crisis, many economic sectors were affected by the austerity measures applied in Spain, such as health, education, transport, housing, etc. We focus on the health sector, because it is a sector that generates great social impact and dissatisfaction in the population. As far as we know, this theory has not been applied to this sector before. Therefore, we provide an alternative proposal for the distribution of the health budget to achieve a more accurate allocation with the purpose of maintaining the stability and quality of life of the community.

Chapter 2 studies the European structural and investment funds (ESIF). Our main objective is to find a solution that can reduce inequality and promote convergence among member countries. In particular, we focus on the European Regional Development Fund (ERDF) in the European Union and Spain. In both cases, we propose an alternative way of distributing the budget funds, through solutions based on the claims problems theory, and the imposition of limits (guaran-

[1]Due to the fact that each chapter corresponds to a complete independent article, some repetitions are generated mainly in the definitions.

tees) on each of the regions. These limits guarantee a certain amount to each agent (region), and this can be interpreted as an equitable distribution. This is known in the literature as a lower bound (or guarantee). Specifically, we use the lower bounds that fit better in our context: The fair lower bound (Moulin, 2002) and the min lower bound (Dominguez, 2006).

Chapter 3 provides new characterizations for the constrained equal awards rule and for the Ibn Ezra's proposal. Following the line of the lower bounds, we analyze four lower bounds already defined in the literature: The minimal right (Curiel et al., 1987), the fair lower bound (Moulin, 2002), securement (Moreno-Ternero and Villar, 2004a) and the min lower bound (Dominguez, 2006). We analyze the effect of requiring the aforementioned minimums in a mechanism of allocation or distribution of the endowment. Furthermore, we compare the allocation mechanisms along with some additional properties. As a result, we obtain the verification of the connection between the lower bounds (minimum) and the rules and, consequently, we find a particular allocation rule.

Finally, Chapter 4 shows a new proposal of claims problem, which we denote as claims sequential claims problems. In this perspective we redefine the constrained equal awards rule, and characterize it through the use of axioms studied in this field.

Chapter 1

The Catalan health budget: a claims problems approach[1]

Overview. The financial and economic crisis in Spain during recent years has induced public budget adjustments. The crisis has caused a great social impact due to the way the austerity measures have been implemented, affecting mainly key economic sectors such as the civil service, justice, education and health. Among all of these sectors, the current Chapter focuses on the health budget distribution, since the changes in the provision of the health services induce faster and clearer impacts in the social welfare. Spain is divided into 17 regions, and each region manages its own health system. Specifically, we analyze the Catalan health budget assignment since Catalonia is one of the

[1] The results of this chapter have been published at Hacienda Pública Española

most populated regions and one where the restrictions have been more evident. We study the health budget distribution for the period 1998-2014, from the point of view of the claims problem (O'Neill,1982). Accordingly, alternative allocations of the health budget are proposed by using some of the most used solutions in the body of literature. Finally, in order to choose the most appropriate solution, we require the fulfillment of (i) some equity and stability criteria, and (ii) some commonly accepted social constraints.

Keywords: Distribution problems; health; axiomatic analysis; public budget.

1.1 Introduction

Due to the crisis started in 2007, the USA and Europe experienced several consequences, such as economies in deep recession, millions of lost jobs, decreasing gross domestic product, and a fall in the stock market. The reaction of the countries against the so-called "greatest financial crisis worldwide" was heterogeneous. In the USA and Japan, the central banks decided to apply expansionary policies that led to injecting trillions of dollars in order to rescue the bankrupt financial entities. On the contrary, in Europe, following the recommendation of the European central bank, countries such as Greece, Ireland, Portugal and Spain applied austerity measures (Hemerijck, 2012).

In particular, Spain has applied economic policies that are designed to reduce public expenditure. For instance, during 2013, the education sector suffered a budget reduction of € 326.17 million more than the previous year, that is, a decrease of 14.4%; in the culture sector the budget assigned in 2013 was € 175.81 million less than in 2012, representing a reduction of 19.6%. All these spending adjustments provoked, almost immediately, negative consequences in the provision of public services. Specifically, the Spanish health sector suffered a reduction of € 8,778 million in the period 2009-2013, that is 12.5% of reduction, which induce that, according to the reports of the Sociedad Española de Salud Pública y Administración Sanitaria, many primary attention centers closed, and the numbers of beds, operating rooms, and sanitary staff, among others, were drastically reduced, inducing an increase in numbers on the waiting lists (43% from 2009 to 2012). All these adjustments have clearly affected welfare of the country either economically (Ayala and Triguero, 2017) or socially Cerno et al. (2017), for instance, with respect to the quality of the Public services.

Among all the aforementioned public services, the present work focuses on health, which was defined by the World Health Organization (WHO, 1946) as "a state of complete physical, mental and social well-being and not merely the absence of disease or infirmity".[2] Further-

[2]Preamble to the Constitution of the World Health Organization as adopted by the International Health Conference, New York, 19-22 June, 1946; signed on 22 July 1946 by the representatives of 61 States (Official Records of the World Health

more, from an economic point of view, it is also important to ensure the protection and promotion of health, because the population's vitality increases the labour force and the productive capacity (Arrow, 1963).

Therefore, we might assert that (i) the health sector generates great social impact, (ii) health is essential for social welfare, and (iii) the quality of the Spanish National Health System (SNHS) has suffered a substantial decrease, due to the way in which the budget readjustment has been applied.

In this sense, it is noteworthy that the SNHS in Spain, which is known as one of the best in the world (Stuckler et al., 2011), is managed independently by its 17 regions.

Due to the availability of data and the significance of the public budget adjustments, we focus on Catalonia.[3] Specifically, the health services in Catalonia are managed by the Health Department, which also coordinates the central organisms: the Servei Català de la Salut, and the Institut Català de la Salut.[4]

On one hand, during the period 2010-2013 the Catalan health budget has been reduced in € 1,355.85 million (14% decrease), which provokes some negative implications, such that, the number of patients

Organization, no. 2, p. 100) and entered into force on 7 April 1948.

[3]Amigot, B. (2013) "Catalonia and Castilla-La Mancha lead the health adjustments". Expansion, 22 July 2013 [online].

[4]Catalan Health Service (SCS) and Catalan Institute of Health (ICS), respectively.

in the waiting lists increased by 30,000, the waiting time increased by up to 4.57 months (that corresponds to 43%), the number of public health employees was reduced to 28,700 (that is 5,6%) following data from finance ministry, not all the primary care centers have access to 24 h emergency attention[5], several hospitals beds and operating rooms have been closed, and pharmacy spending decreased.[6]

On the other hand, the claims problem approach (O'Neill, 1982) models those situations where the available resources are not enough to totally honour the aggregate claim. Usually, this model has been used to explain how to distribute the money of a failed bank among its creditors, or an inheritance among heirs. Nonetheless, it can be applied to many different situations, such as medical assistance, budget distribution in universities (for instance, Pulido et al., 2002, propose that the funds should be allocated proportionality to the number of teachers, students, etc., of each department), and milk quota distribution among EU member states. This theory is also applied in environmental issues such as the reduction of fishing quotas (Iñarra and Prellezo, 2008; Iñarra and Skonhoft, 2008; Kampas, 2015), and in the case of global carbon budget where the allocation of CO2 emissions among countries

[5]Ferran Balsells, (2012) "Waiting lists raise 43% due to the Mas' health budget adjustments." EL PAIS, 21 March 2012 [online].
Sevillano G. Elena. (2014) "The public health staff suffer a record fall: 28.500 less personnel in two years". EL PAIS, 01 July 2014 [online].

[6]Health department will draw 456 commonly used drugs from public funding. El PAIS, 2012.

is studied (Giménez-Gómez et al., 2016). Therefore, clearly, the Catalan health budget distribution fits the claims problem approach since the available resources cannot satisfy the aggregate needs. A situation which, to the best of our knowledge, has not been studied from this perspective.

By doing the implementation of the claims approach, firstly, we analyze, during the period 2011-2014, how the budget is distributed among the different economic areas of the public health expenditure (consolidated health budgets): salary, current expenditures of goods and services, current transfer, transfer of capital, real investment, and variation of financial assets. Secondly, we apply some of the solutions that have been proposed in the literature to mediate conflicts: the proportional, the constrained equal awards, the constrained equal losses, the Talmud, the adjusted proportional and the α-min. Thirdly, since, our aim is to find the most appealing and fairest solution, we introduce the power index, which is a criterion of stability and fairness that ensures a reasonable assignment of the budget. Fourthly, in order to analyze the evenly distribution of the budget, we apply the Gini inequality coefficient. Finally, we introduce several commonly accepted social constraints in the health context; and we choose the solution that satisfies the fair criterion, the equity indexes and the social constraints.

Hence, dealing with the health budget problem in this way may be potentially more effective than the current distribution, since we provide new different allocations in terms of appealing principles of fairness and equity in terms of the actual and current needs.

The remainder of the Chapter is organized as follows. Section 1.2 provides an overview of the health sector in Catalonia and the budget problem in this sector after the crisis. Section 1.3 describes the health budget as a claims problem. Section 1.4 presents some theoretical solutions to the claims problem. Sections 1.5 and 1.6 introduce equity and stability criteria, and some commonly accepted social constraints, respectively. Finally, Section 1.7 concludes.

1.2 The health department of Catalonia

The Spanish National Health System (SNHS) is the organization responsible for the coordination, cooperation and administration of health services. It is organized in two levels: primary and specialist health care. The population can receive basic services in the primary health care centres, and if they need a specialized treatment, they can be attended to specialized centres and hospitals.

As aforementioned, Spain is divided into 17 regions, and each region administers its health system independently. Specifically, each region is responsible for the management of the centres and the health services within the region.

Among all regions, we focus on Catalonia, mainly for the availability of data, but also because it is (i) the second region with the greatest population density, (ii) the one that allocates more budget to the health sector; and, (iii) it has been the Spanish region where the most budget adjustments have been applied.

Following a report of the State Association of Directors and Managers of Social Services, Catalonia was the region that experienced the highest adjustments in the health-care system during the period 2009-2015, representing the 15% of the total SNHS budget adjustment. Focusing on the readjustment of health sector, specifically the staff's health salaries and the expenditures of goods and services suffered an adjustment of € 409.56 (19%) and € 400.39 (7%) million, respectively.

Nonetheless, the population's health needs became greater, since the total number of inhabitants during the same period increased in 41,269. Hence, meanwhile the Catalan health resources were reduced by 14%, the total population grew by 1%. Thus, the consequences of these adjustments were reflected in many aspects of Catalan health system. The waiting lists time to access to medical tests or surgical interventions increased. For instance, orthopedics went from 8 to 10 months and gynecology increased up to 7 months. Operating rooms were closed during some periods of time. The hospital staff, the number of beds and the hospital stay time were also reduced. Clearly, all of these adjustments have had a great social impact, since it induced a lower quality of the public service.[7]

The Health Department in Catalonia is the highest authority and

[7]Gallardo, A. (2016) "The public health service face a difficult situation due to the health adjustments," El periódico, 16 September 2016 [online].

manages its regional health policies. The Servei Català de la Salut **(CatSalut)** is the responsible for the funding and purchase of health services, and for supplying these services to health centres and hospitals. Regarding the provision of these health services, there exists a set of entities that supply them to the population.[8] These entities can be either public, concerted (50% public, 50% private) or fee-paying private. Figure 1.1 shows the organisation chart of the Catalan Health System.

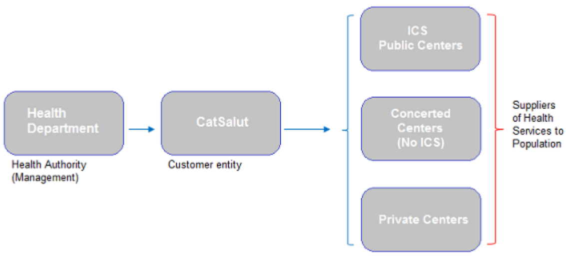

Figure 1.1: Organisation chart of the Catalan Health System.

In this work, we study the **CatSalut** because it is the supplier of health services to all centres and hospitals, and the **ICS** because it is the most important public entity that provides these health services to all users. The main objective of the **CatSalut** and the **ICS** is to ensure the equity, quality and efficiency of the health system in order to improve the population's quality of life.

[8]Catalan Health Services (CatSalut) and Catalan Institute of Health, respectively.

In order to analyze the health budget distribution as a claims problem, we formally introduce this approach in the following section.

1.3 The Catalan health budget as a claims problem

As mentioned, notice that the claims problem approach, which originates formally with O'Neill (1982), has been used by many authors to analyze conflicts of interests in actual situations.

Formally, consider a set of agents $N = \{1, 2, ..., n\}$, such that each agent has a claim $c_i \in \mathbb{R}_+$ on an infinitely divisible resource, the endowment $\mathrm{E} \in \mathbb{R}_+$. Let $c \equiv (c_i)_{i \in N}$ be the claims vector. Then, a **claims problem** is a pair (E, c) with $C = \sum_{i=1}^{n} c_i > E$, that is, the endowment is not enough to honour all the claims. Without loss of generality, we order the agents increasingly according to their claims, $c_1 \leqslant c_2 \leqslant \ldots \leqslant c_n$. We denote by $\mathcal{C}$ the set of all claims problems.

In this work, the endowment is the health budget assigned to the health sector in each one of the evaluated years (from 2011 to 2014). Besides this, we use an inflation rate by using the consumer price index **(CPI)** in order to compare the real and the nominal values of the changes in the yearly budget.

Furthermore, since we focus our analysis on the financial adjustment that the health sector suffered from the crisis to the present day, we use the economic classification of the public health expenditure to define who the claimants are. Specifically, there are six claimants: salaries **(S)**, current expenditures of goods and services **(EGS)**, current transfers **(CT)**, transfers of capital **(TC)**, real investment **(RI)**, and variation of financial assets **(VFA)**.

Finally, in order to define the amount of resources that the six economic areas will claim from the year 2010 on, it is noteworthy that the number of inhabitants has increased. Additionally, as Table 1.1 shows, the health budget has been diminishing in all economic areas from the year 2011 on. Therefore, it seems natural to assume that each economic area would claim at least, the same resources it has before the crisis. Likewise, we define the claims with the health budget assigned to each claimant (economic area) for the year 2010.

	2009	2010	2011	2012	2013	2014
TC	24.0	44.1	43.0	39.6	36.9	37.4
VFA	66.4	82.1	69.4	70.4	65.6	74.2
RI	192.2	207.8	147.5	131.8	123.1	114.8
CT	1,872.3	1,497.7	1,353.7	1,028.5	959.5	950.4
S	1,946.0	2,080.6	1,922.6	1,861.5	1,736.8	1,735.5
EGS	5,183.1	5,391.1	5,416.7	5,272.0	4,918.7	4,929.9
Total	9,194.0	9,302.8	8,952.8	8,403.8	7,840.6	7,841.8

Table 1.1: Current health expenditure budget for the period 2009-2014 disaggregated by economic chapters (in € million). As a reference point we include 2009, the year before the adjustments were made.

Summing up, our set-up corresponds with (E, c) = (CPI revised annual health budget, $(TC; VFA; RI; CT; S; EGS)$), so that,

- There are four different endowments, corresponding with each health annual budget (in € million), considering the inflation rate (see Table 1.2): 8,952.8; 8,403.8; 7,840.6; and, 7,841.8 for 2011, 2012, 2013 and 2014, respectively. Hence, there are four claims problems, one per each year during the period 2011-2014.

- There are six claimants, corresponding to the economic classification of expenditures: TC, VFA, RI, CT, S, and EGS (increasingly ordered with respect to the claims). In this sense, and due to the increase of population, the claims are the largest amount the claimants received before the adjustments (2010), considering the inflation rate, i.e., $c = (44.1; 82.1; 207.8; 1,497.7; 2,080.6; 5,391.1)$.

Since, we propose an alternative way to allocate the Catalan health budget, in the next section we introduce some different proposals (rules) considered in the literature of claims problems.

1.4 How to distribute the health budget

Once the claims problem is properly defined, some methods are provided by the literature to allocate the endowment. These methods, called rules, propose a distribution of the endowment among the agents taking into account their claims.

Formally, a **rule** is a function $\varphi : \mathcal{C} \to \mathbb{R}^n_+$ that associates with each claims problem an awards vector for it, such that $\varphi_i(E, c) \geqslant 0$, for all $i \in N$ (**non-negativity**), $\varphi_i(E, c) \leqslant c_i$, for all $i \in N$ (**claim-boundedness**), and $\sum_{i=1}^{n} \varphi_i(E, c) = E$ (**efficiency**).

According to our framework, a rule distributes the total health budget among all the economic areas with respect to their claims. In other words, the application of a rule implies that no economic area can receive a negative amount (i.e., no area is lending money), no area will receive an award higher than its claim, and the total health budget is distributed.

Among all the rules that have been proposed in the claims problems literature, we introduce those that have been used actually in similar situations: The proportional, the constrained equal awards, the constrained equal losses, the Talmud, the adjusted proportional and the α^{min} rules. For the sake of comprehension, we define the rules applying them to our framework.

The **proportional (P)** rule divides the health budget proportionally with respect to each economic area's claim.
For each $(E, c) \in \mathcal{C}$ and each $i \in N$, $P_i(E, c) \equiv \lambda c_i$, where $\lambda = \frac{E}{\sum_{i \in N} c_i}$.

The **constrained equal awards (CEA)** rule (Maimonides, 1135, 1204), proposes an equal distribution of the health budget subject to

no one can receive more than her claim.
For each $(E, c) \in \mathcal{C}$ and each $i \in N$, $CEA_i(E, c) \equiv \min\{c_i, \mu\}$, where μ is such that $\sum\limits_{i \in N} \min\{c_i, \mu\} = E$.

The **constrained equal losses (CEL)** rule (Maimonides, 1135, 1204; Aumann and Maschler, 1985) focuses on distributing losses, that is, all the economic areas must lose equally, but none of them must receive a negative amount.
For each $(E, c) \in \mathcal{C}$ and each $i \in N$, $CEL_i(E, c) \equiv \max\{0, c_i - \lambda\}$, where λ is such that $\sum\limits_{i \in N} \max\{0, c_i - \lambda\} = E$.

The **Talmud (T)** rule (Aumann and Maschler, 1985) contains the CEA and the CEL. It takes the middle of the claims as a reference point. If the half of the aggregate claim is lower than the health budget, then the CEA is applied over the half-claims. Otherwise, each economic area receives the half of its claim and the CEL is applied in order to distribute the remaining budget.
For each $(E, c) \in \mathcal{C}$, and each $i \in N$, $T_i(E, c) \equiv CEA_i(E, (\frac{c_i}{2})_{i \in N})$ if $E \leqslant \frac{\sum_{i \in N} C_i}{2}$; or $T_i(E, c) \equiv \frac{c_i}{2} + CEL_i(E - \frac{\sum_{i \in N} C_i}{2}, (\frac{c_i}{2})_{i \in N})$, otherwise.

The **Adjusted Proportional (AP)** rule (Curiel et al., 1987) ensures that each economic area receives its minimal right m (O'Neill, 1982), which, for each $(E, c) \in \mathcal{C}$ and each $i \in N$, guarantees to each agent the not unclaimed part of the endowment, i.e., $m_i(E, c) = \max\{E - \sum\limits_{j \neq i \in N} c_j, 0\}$. Afterwards, it divides the remaining health budget in proportion to the revised claims, given that if a claim is greater than the available budget, it is truncated accordingly.

For each $(E, c) \in \mathcal{C}$ and each $i \in N$, $AP_i(E, c) = m_i(E, c) + P_i(E - \sum_{j \in N} m_j(E, c), (\min\left\{c_i - m_i(E, c), E - \sum_{j \in N} m_j(E, c)\right\})_{i \in N})$.

The α**-min** (α^{min}) rule (Giménez-Gómez and Peris, 2014) ensures, for each $(E, c) \in \mathcal{C}$, an equal division of the health budget among the economic areas as far as the smallest claim is totally honoured; then, the remaining budget is distributed proportionally.

For each $(E, c) \in \mathcal{C}$ and each $i \in N$, if $c_1 > \frac{E}{n}$ then, $\alpha_i^{min}(E, c) = \frac{E}{n}$, or, $\alpha_i^{min}(E, c) = c_1 + P(E - nc_1, (c_i - c_1)_{i \in N})$, otherwise.

Next, in Table 1.2 we summarize the comparison among the introduced rules for each of the claims problems defined in Section 1.3. Recall that, we consider six economic areas (TC; VFA; RI; CT; S; EGS), whose claims are $c = (44.1; 82.1; 207.8; 1,497.7; 2,080.6;\ 5.391.1)$, and the CPI revised consolidated health budget is 8,952.8; 8,403.8; 7,840.6; and, 7,841.8 for the years 2011, 2012, 2013 and 2014, respectively.

Claims: $TC = 44.1; VFA = 82.1; RI = 207.8; CT = 1,497.7; S = 2,080.6; EGS = 5,391.1$								
		Actual	P	CEA	CEL	T	AP	α^{min}
Health Budget 2011: $8,952.8$	TC	43.0	42.4	44.1	00.0	22.1	32.9	44.1
	VFA	69.4	79.0	82.1	20.8	41.1	61.3	80.6
	RI	147.5	200.0	207.8	146.5	135.9	155,2	201.5
	CT	1,353.7	1,441.3	1,497.7	1,436.4	1,425.8	1,409.0	1,441.3
	S	1,922.6	2,002.2	2,080.6	2,019.3	2,008.7	1,991.9	2,001.6
	EGS	5,416.7	5,187.9	5,040.5	5,329.8	5,319.2	5,302.4	5,183.7
Health Budget 2012: $8,403.8$	TC	39.6	39.8	44.1	00.0	22.1	31.0	44.1
	VFA	70.4	74.2	82.1	00.0	41.1	57.8	78.3
	RI	131.8	187.7	207.8	14.45	103.9	146.16	191.5
	CT	1,028.5	1,352.9	1,497.7	1,304.4	1,253.5	1,230.9	1,353.0
	S	1,861.5	1,879.41	2,080.6	1,887.3	1,836.4	1,813.8	1,877.9
	EGS	5,272.0	4,869.8	4,491.5	5,197.7	5,146.9	5,124.3	4,858.9
Health Budget 2013: $7,840.6$	TC	36.9	37.2	44.1	00.0	22.1	30.4	44.1
	VFA	65.6	69.2	82.1	00.0	41.1	56.7	76.0
	RI	123.1	175.1	207.8	00.0	103.9	143.4	181.3
	CT	959.5	1262.2	1497.7	1121.4	1,065.8	1,044.6	1,262.5
	S	1,736.8	1,753.5	2,080.6	1,704.3	1,648.7	1,627.5	1,751.0
	EGS	4,918.7	4,345.4	3,928.3	5,014.8	4,959.2	4,938.0	4,525.8
Health Budget 2014: $7,841.8$	TC	37.4	37.2	44.1	00.0	22.1	30.4	44.1
	VFA	74.2	69.2	82.1	00.0	41.1	56.7	76.0
	RI	114.8	175.2	207.8	00.0	103.9	143.4	181.3
	CT	950.4	1,261.4	1,497.7	1,121.8	1,066.2	1,045.0	1,262.6
	S	1,735.5	1,753.7	2,080.6	1,704.7	1,649.1	1,627.9	1,751.3
	EGS	4,929.9	4,544.1	3,929.5	5,015.2	4,959.6	4,938.4	4,526.5

Table 1.2: Allocation of each health budget according to each considered rule between the period 2011-2014. Within each year, rows provide the allocations recommended to each of the six considered economic areas.

Among all possible allocations, the natural question that arises is which is the most appealing way to distribute the available public health budget among all the economic areas? As a response, we propose to use an equity criteria that induces to the most suitable rule in our framework.

1.5 Equity and stability criteria

Following Robert (1974), "the complete principle of distributive justice would say simply that a distribution is just if everyone is entitled to the holdings they possess under the distribution." Hence, in order to find out the rule that induces a larger commitment among the different economic agents involved in the health budget distribution, we are introducing some equity criteria.

Firstly, it is noteworthy that there are different inequality indexes widely used: the Atkinson index (Atkinson, 1970), the generalized entropy index (Theil, 1967), and the Gini index (Gini, 1921). Among them, the latter is the most popular one, vastly used in both official and scientific reports, and considered in the literature as the best single measure of inequality (see, for instance, Atkinson, 1970, and Aaberge and Brandolini, 2015).

The **Gini index (Gi)** (Gini, 1921), is formally defined as:

$$Gi = \frac{1}{2N^2\mu}\sum_{i=1}^{k}\sum_{j<i}|r_i - r_{j<i}|,$$

where N is the total number of agents $n_1, n_2 ... n_k$, r_i is the the ith claimant's allocation of the health budget proposed by a particular rule, and μ is the average of $r_1, r_2, ..., r_k$. Note that this index considers the average distribution μ and the differences between an economic area and the next one, following an increasingly ordering. Hence, it takes values in the interval $[0, 1]$, where $Gi = 0$ means perfect equality, and $Gi = 1$ means complete inequality, so the lower the index the more equality the allocation.

Table 1.3 shows the computation of this coefficient for each studied year and for each proposed rule. By comparing the obtained results with our baseline (the actual way in which the health budget was distributed in 2010), it might be plausible to choose only those rules that induce no more inequality in the way of allocating the available budget: the P, CEA and α^{min} rules.

		P*	CEA*	CEL	T	AP	α^{min}*	Baseline
Gini index	2011	0.609	0.601	0.632	0.627	0.622	0.609	0.609
	2012	0.609	0.585	0.653	0.638	0.631	0.608	
	2013	0.609	0.568	0.666	0.648	0.641	0.606	
	2014	0.609	0.570	0.670	0.650	0.641	0.604	

Table 1.3: Computation of Gini coefficient. Each row shows the Gini index for each of the considered rules in each studied year. The "*" denotes the rules that propose a lower inequality distribution than the baseline.

Secondly, notice that the economic areas with a larger budget relevance might be damaged by using only one equity criterion, since it is not considering any priority measure. Nonetheless, and after considering the information provided by the economic resources department of the Catsalut, there are no previously established priority parameters to make the allocation of the health budget. Hence, the Moulin (2000)'s method implementation becomes not feasible. For the sake of facing this issue, as a measure of stability, we introduce the coefficient of variation, which has been applied to select stable solutions for cooperative problems (Dinar and Howitt, 1997; Read et al., 2014).

In doing so, we consider that each economic area $i \in N$ should be treated differently, depending on its long-run average health budget share W_i. This long-run average health budget share of the i's agent is the average of the resources that agent i receives form 1997 to 2014.

To compute the CV, we consider r_i^{max}, the best distribution (i.e. the rule that assigns a greater amount) for the i economic area across all the rules, and r_{ik}, the actual amount proposed by each of the rules in comparison to the others. Furthermore, for each economic area $i \in N$, we compute also its Power Index, $PI_i = \dfrac{W_i(r_i^{max} - r_{ik})}{\sum\limits_j W_j(r_j^{max} - r_{jk})}$.

Therefore the **coefficient of variation** (CV), is formally defined as:

$$CV = \frac{\sigma}{\mu(PI)}$$

where σ and $\mu(PI)$ are the standard deviation and the mean of the

Power Index PI, respectively. Note that the higher the value of CV is, the greater the instability (Dinar and Howitt, 1997; Kampas, 2015).

Next, Table 1.4 presents the CV index for each rule and for each year analyzed. Note that, the rules that have a lower index in comparison to the baseline (that is, the real-life way of applying the distribution in the year 2010) are P, CEL, T, AP and α^{min} rules.

		P*	CEA	CEL*	T*	AP*	α^{min}*	Baseline
	2011	1.819	2.449	1.228	1.235	1.254	1.828	1.917
CV	2012	1.819	2.449	1.237	1.260	1.268	1.828	
	2013	1.528	2.449	1.325	1.351	1.224	1.546	
	2014	1.528	2.449	1.504	1.271	1.224	1.546	

Table 1.4: Computation of the coefficient of variation. Each row shows the CV for each of the considered rules and each studied year. The "$*$" denotes those rules that propose a lower CV than the baseline.

It is noteworthy that this CV measure depends on the PI, which means the satisfaction degree of the parts involved in the distribution problem with the final allocation, so none of them has incentives to deviate from the proposed allocation. In this regard, Dinar and Howitt (1997) point out that Shapley and Shubik (1954) suggest this index as a method of measuring power in voting games: "...the power of an individual member depends on the chance he has of being critical to the success of a winning coalition".

Thirdly, we study which rules satisfy both the equity and priority criteria. By doing so, the intersection of Tables 1.3 and 1.4 show that there are only two rules having a lower Gini index and satisfying the CV criterion: the P and the α^{min} rules.[9]

	P	α^{min}
Transfer current	0	1
Variation of financial assets	0	1
Real investment	0	1
Current transfer	0	1
Salaries	1	0
Current expenditures of goods and services	1	0
Total	2	4

Table 1.5: Borda count for the P and α^{min} rules. Each economic area assigns 1 point for its preferred way of distributing the budget (rule).

[9] See Thomson, 2007, Bosmans and Lauwers, 2011a, and Giménez-Gómez and Peris, 2014 for a Lorenz (Gini) comparison among the proposed rules. In this sense, note that the CEA and CEL rules are the most and the less equitable ways of distributing the resources, respectively. There is no a fixed relationship among the rest of the rules in this terms. Consequently, the results with respect the Gini and CV analysis observed in the current work remains true for the CEA and CEL rules, but not in general for the other rules.

Finally, we apply an election method to select one of the remaining rules. The idea is to select (among the proportional and α^{min}) the rule preferred by most economic areas. By doing so, we introduce the Borda count election method: each economic area assigns 1 point to its preferred rule, and zero, otherwise. Consequently, the rule that gets more votes will be chosen. Formally,

The **Borda count (B)** (Black, 1976) is given by, $B = \underset{m}{max}(B_m)$, where m is each one of the feasible rules, R_{im} denotes the points assigned by each economic area $i \in N$ to each of the proposals, and $B_m = \sum_{i \in N} R_{im}$.

As shown in Table 1.5 the rule with more votes is the α^{min} rule. Therefore, we may conclude that the economic areas prefer the allocation of the health budget proposed by the α^{min} rule for each one of the analyzed years.

For the sake of comparison, through Table 1.6 we observe a remarkable difference in the allocation of the health budget. Note that the agents with a lower claim get a larger share of the resources than the actually assigned amount. Specifically, the salaries area receives more resources, which, as aforementioned, could affect positively to the social impact about the quality of the public health service.

		Actually	α^{min}
Health Budget 2011: 8,952.8	TC	43.0	44.1
	VFA	69.4	80.6
	RI	147.5	201.5
	CT	1353.7	1,441.3
	S	1,922.6	2,001.6
	EGS	5,416.7	5,183.7
Health Budget 2012: 8,403.8	TC	39.6	44.1
	VFA	70.4	78.3
	RI	131.8	191.5
	CT	1,028.5	1,353.0
	S	1,861.5	1,877.9
	EGS	5,272.0	4,858.9
Health Budget 2013: 7,840.6	TC	36.9	44.1
	VFA	65.6	76.0
	RI	123.1	181.3
	CT	959.5	1,262.6
	S	1,736.8	1,751.0
	EGS	4,918	4,525.8
Health Budget 2014: 7,841.8	TC	37.4	44.1
	VFA	74.2	76.0
	RI	114.8	181.3
	CT	950.4	1,262.6
	S	1,735.5	1,751.3
	EGS	4,929.9	4,526.5

Table 1.6: Comparison between the α^{min} rule and the real distribution of the health budget between the period 2011-2014.

The following section provides some commonly accepted social constraints in order to enrich the comparison among the proposed rules.

1.6 Commonly accepted social constraints

In this section, we provide an axiomatic justification of the proposed allocations through some commonly accepted social constraints that should determine the way of distributing the Catalan health budget. Notice that in our context, this approach is totally suitable since there is a regulatory entity (the Health Department) that manages the assignments of the budget among the different economic areas, in accordance with some principles or constraints.

Next, we introduce some properties that adapt to our context. By doing so, we propose those commonly accepted social constraints (see, for instance, Moulin, 2000, and Thomson, 2015) that gather the idea of ensuring a fair distribution and treatment among all economic areas, not only taking into account an equity criterion, but also the relative relevance of each economic area on the total health budget distribution.

Equal treatment of equals says that economic areas with similar claims should be rewarded with the same health budget allocation: for each $(E, c) \in \mathcal{C}$, and each $\{i, j\} \subseteq N$, if $c_i = c_j$, then $\varphi_i(E, c) = \varphi_j(E, c)$.

Note that this property gathers the simply idea of fairness that equal economic areas should be treated equally, i.e., they should receive the same award.

Order preservation (Aumann and Maschler, 1985) requires respect-ing the ordering of the economic areas: if $i's$ claim is at least as large as $j's$ claim, agent i should receive and lose at least as much as j does, respectively: for each $(E,c) \in \mathcal{C}$, and each $i,j \in N$, such that $c_i \geqslant c_j$, then $\varphi_i(E,c) \geqslant \varphi_j(E,c)$, and $c_i - \varphi_i(E,c) \geqslant c_j - \varphi_j(E,c)$.

This property is relevant because it maintains the order of the areas when distributing the health budget. That is, the larger the historical relevance of an economic area with respect the health budget is, the larger the allocation received.

Resource monotonicity (Curiel et al., 1987), Young (1987) says that if the health budget increases, then all economic areas should get at least the awards they received initially: for each $(E,c) \in \mathcal{C}$ and each $E' \in \mathbb{R}_+$ such that $C > E' > E$, then $\varphi_i(E',c) \geqslant \varphi_i(E,c)$, for each $i \in N$.

Resource monotonicity implies that the larger the health budget is, the larger the financial support received by each economic area.

Super-modularity (Dagan et al., 1997) requires that if the health budget increases, the economic areas with the greater claim experience a larger gain than the others: for each $(E,c) \in \mathcal{C}$, all $E' \in \mathbb{R}_+$ and each $i,j \in N$ such that $C > E' > E$ and $c_i \geqslant c_j$, then $\varphi_i(E',c) - \varphi_i(E,c) \geqslant \varphi_j(E',c) - \varphi_j(E,c)$.

Note that this property gives, somehow, priority to those economic areas with a larger historical relevance in the health budget, since they receive a greater share of the budget increasing.

	P	CEA	CEL	T	AP	α^{min}
Equal treatment of equals	Yes	Yes	Yes	Yes	Yes	Yes
Order preservation	Yes	Yes	Yes	Yes	Yes	Yes
Resource monotonicity	Yes	Yes	Yes	Yes	Yes	Yes
Super-modularity	Yes	Yes	Yes	Yes	Yes	Yes
Reasonable lower bounds on awards	No	Yes	No	Yes	No	Yes

Table 1.7: The considered rules and the commonly accepted social constraints. Each row shows the introduced properties and each column the considered rules. For further discussion about the fulfillment of properties, see Thomson (2003, 2015), and Giménez-Gómez and Peris (2014).

Reasonable lower bounds on awards (Moreno-Ternero and Villar, 2004b; Dominguez and Thomson, 2006a) ensures that each economic area receives at least the minimum of (i) its claim divided by the number of areas, and (ii) the health budget divided by the number of areas: for each $(E, c) \in \mathcal{C}$ and each $i \in N$, $\varphi_i(E, c) \geqslant \frac{\min\{c_i, E\}}{n}$.

This is an important property since it ensures a minimum amount for each economic area, so no one can be completely punished. In other words, by doing so, it induces a fair distribution.

Table 1.7 summarizes the axiomatic comparative among the considered rules. Note that, the α^{min} rule, not only is the unique rule satisfying the equity and stability criteria (introduced in Section 1.5), but also it fulfills all the commonly accepted social constraints that may be considered as the basic criteria to guarantee a fair allocation of the health budget.

1.7 Final Remarks

Spain applied the economic policy of austerity in order to address the crisis. As a consequence, some areas that affect the social welfare, as health, education and culture, have been affected significantly. In this work we focus on the Catalan health system. Specifically, by implementing the classical claims problem approach (O'Neill, 1982), we propose an alternative way of allocating the health budget among the different economic areas.

We consider some rules together with some equity and stability criteria to evaluate the different allocations. Accordingly, by using the Gini index and the coefficient of variation, we look for the most appropriate way to distribute the available health budget. Furthermore, we analyze this problem from an axiomatic point of view, that is, we study the fulfillment of some commonly accepted social constraints, widely used in the related literature. Among all of the considered rules, we find out that the α^{min} rule is the only one satisfying all of the aforementioned criteria.

Chapter 2

Distributing the European structural and investment funds from a claims problem approach

Overview. In order to support economic development across all EU regions, € 351.8 billion –almost a third of the total EU budget– has been set aside for the Cohesion Policy during the period 2014-2020. The distribution of this budget is made throughout five main structural and investment funds, after long and though negotiations among the EU member states. The current Chapter analyzes the problem of allocating the limited resources of the European Regional Development

Fund (ERDF) as a claims problem (O'Neill[17]). Specifically, we show how this approach fits this actual problem, and we propose an alternative way of distributing the budget via (i) claims solutions or (ii) the imposition of bounds (guarantees) to each of the regions. We apply this approach to European Union and Spanish evidences. In both cases we obtain that the constrained equal losses rule reduces inequality and promotes convergence more properly.

Keywords: Claims problems; public budget distribution; European Regional Development Fund; EU convergence.

2.1 Introduction

The main objective of the European Union (EU) is to strengthen the social and economic cohesion of the EU regions, as well as to reduce the inequalities among them. In doing so, and in accordance with the objectives of the Europe 2020 strategy, the European Structural and Investment Funds (ESIF) are implemented throughout five main funds: the European Regional Development Fund (ERDF), the European Social Fund (ESF), the Cohesion Fund (CF), the European Agricultural Fund for Rural Development (EAFRD) and the European Maritime and Fisheries Fund (EMFF).[1]

[1] https://cohesiondata.ec.europa.eu/funds

In order to support job creation, business competitiveness, economic growth, sustainable development, and improve citizens' quality of life, the Regional Policy has set € 351.8 billion -almost a third of the total EU budget- to the Cohesion Policy funds for the period 2014-2020. Following the magazine Panorama Inforegio, the support of the EU's cohesion policy has achieved member states to experience a 5% growth in *per capita* gross domestic product.[2] The bulk of Cohesion Policy funding, above the 50%, is allocated to less developed European regions in order to help them to catch up and to reduce the economic, social and territorial disparities that still exist in the EU.
It is noteworthy that the available budget does not honor all the claims of the EU regions which are involved. Accordingly, the current work aims to implement the claims problems approach (originated with O'Neill (1982), and which fits situations such as inheritance problems, divorces, the failure of the company or bank, for instance) in order to achieve the aforementioned goals in a proper way. In doing so, once we define the claims problem associated to the distribution of EU funds, we apply well known solution concepts, so-called rules. By comparison, our results provide a rule that clearly performs better than the others, and also better than the current allocation.

Among all the aforementioned funds, the present Chapter focuses

[2]http://ec.europa.eu/regional_policy/es/information/publications/panorama-magazine/2017/panorama-61-cohesion-policy-looks-to-the-future

on the European Regional Development Fund (ERDF), which represents almost the 44% of the total budget. These funds are allocated at the NUTS 2 level, which is a regional classification providing a harmonized hierarchy of regions: the NUTS classification subdivides each member state into regions at three different levels, from larger to smaller areas. For practical reasons the NUTS classification generally mirrors the territorial administrative division of the member states, which supports the availability of data and the policy implementation capacity. Specifically, the NUTS regulation defines minimum and maximum population thresholds for the size of the NUTS regions: NUTS 2 level corresponds to regions whose population is between 800000 and 3000000 inhabitants. Taking into account this division, the regional eligibility for the ERDF is calculated on the basis of regional GDP per inhabitant (*per capita*), and NUTS 2 regions were ranked and split into three groups:

1. Less developed regions (where GDP *per capita* was less than 75 % of the EU-27 average).

2. Transition regions (where GDP *per capita* was between 75 % and 90 % of the EU-27 average).

3. More developed regions (where GDP *per capita* was more than 90 % of the EU-27 average).

Related literature

There are many papers analyzing the importance of ESIF funds in order to achieve greater social cohesion and economic growth among the European Union countries, most of them looking for the results obtained through the policies applied. For instance, Rodríguez-Pose and Fratesi (2004) apply cross-sectional and panel data analyses to observe the impact of European Structural Funds in Objective 1 regions; also Puigcerver-Peñalver (2007) studies the impact of the ESIF funds in the economic growth of the regions; Mohl and Hagen (2010) analyze the economic growth of the European Union countries, using the financial aspect for the NUTS 1 and NUTS 2 regions; Bouayad-Agha et al. (2013) consider an econometric model to analyze the effect of the cohesion policies on the European economies; and Dall'Erba and Fang (2017) apply a meta-analysis with the objective of studying the impact generated for the ESIF funds on the development of the recipient regions.

Our approach complements the aforementioned studies by providing a new point of view of this problem: the implementation of the theoretical claims approach to the distribution of the ERDF funds. Other economic and social sectors have been analyzed through this approach: in the education sector Pulido et al. (2002) to obtain an efficient allocation of the university funds; in the fishing sector to search possible solutions to face fish shortages, where it is proposed to distribute fishing quotas among a number of agents within an established perimeter (Iñarra and Prellezo, 2008; Iñarra and Skonhoft, 2008; Kampas, 2015); or, in the negotiations of the CO_2 emissions, a relevant issue nowadays, in which Giménez-Gómez et al. (2016) propose an appealing distribution by using the commonly accepted principles.

We propose the use of rules to distribute the EU funds in order to achieve social cohesion, convergence and equality among state members, properly. In doing so, we define some of the usual rules and compare them from a convergence perspective by the application of the Lorenz dominance (comparing the inequality of the proposals), the Gini index (comparing the inequality across regions after a proposal is implemented) and a convergence ratio.

Our results show that the allocations proposed by all of the rules reduce (i) the divergence among regions, and (ii) the inequality Gini index. Nevertheless, only the constrained equal losses rule performs better than the current allocation.

The remainder of the Chapeter is organized as follows. Next, Section 2.2 presents the ERDF claims problem. Section 2.3 proposes different solutions to the EU evidence. Section 2.4 compares the different rules from the convergence point of view. Section 2.5 analyzes and compares the proposed allocations from the point of view of equity, and Section 2.6 studies the problem by ensuring some guarantees (in awards and in losses) to all regions. Section 2.7 implements our approach to the detailed Spanish evidence. Finally, Section 2.8 concludes.

2.2 The ERDF claims problem

A **claims problem** is defined by a set of agents (regions), $R_1, R_2, \ldots, R_n$. Each region R_i is identified by its **claim** c_i on the total available **budget** E. Let $c = (c_i)_{i \in N}$ be the claims vector. The **aggregate claim** C is given by $C = \sum_{i=1}^{n} c_i$. Therefore, the claims problem appears whenever the claims cannot be simultaneously honored by the available budget: $C > E$. The pair (E, c) represents the claims problem. We denote by $\mathcal{C}$ the set of all claims problems.

As aforementioned, we implement our approach to the ERDF European Union evidence. In this situation, two facts have to be considered. Firstly, each region has a proposal with the amount that they plan to spend on the projects: this is the claim each region demands. Secondly, the actual amount that is decided to be assigned to each of the regions, that is the actual expenses that each region has for projects throughout the ERDF funds, which is always lower than the claims, so in a natural manner a claims problem appears.

Therefore, in our scenario the proposal for the endowment E is the ERDF budget currently allocated to all regions in EU (in absolute terms). The claims c_i correspond to the sum of the total budget demanded by the regions in each category (less developed, in transition and more developed regions) for the period 2014-2020.

In order to compare the claims of these three categories of regions, and the allocations they receive, it is necessary to analyze the problems in terms of *per capita* resources, since the populations are very different. Then we obtain the claims, current allocations and GDP/head. Table 2.1 reflects these data.

The endowment $\mathbf{E} = 188,007,299,928$			
Absolute	Claim	Current	Population
More developed	61,901,153,827	32,300,565,888	280,056,802
Transition	36,181,081,146	25,396,981,020	51,298,111
Less developed	166,509,560,350	130,309,753,020	118,577,982
Per capita	Claim	Current	GDP
More developed	221.03	115.14	29,713.20
Transition	705.31	495.09	21,332.85
Less developed	1,404.22	1,098.94	10,587.31

Table 2.1: Current allocation of ERDF budget according to each category of region (€). In the first row we have the estate, in absolute terms. The first column presents the three different regions. The second column provides the claim of each of the regions (first in absolute terms and then in *per capita* terms). The third column shows the actual distribution of the ERDF budget. Finally, last column reflects population of each category of region (inhabitants) and the GDP/head.

2.3 A way to distribute the ERDF budget

There are many well known solution concepts defined for solving claims problems, called rules. A **rule** is a single valued function φ such that for each claims problem (E, c) assigns an amount $\varphi_i(E, c)$ to each region R_i, fulfilling: $0 \leqslant \varphi_i(E, c) \leqslant c_i$ (**non-negativity** and **claim-boundedness**); and $\sum_{i=1}^{n} \varphi_i(E, c) = E$ (**efficiency**).

That is, the total budget is distributed among the regions and any region receives neither a negative amount, nor an amount exceeding its claim.

We now briefly introduce and analyze the behavior of some commonly used rules: the proportional, the constrained equal awards, the constrained equal losses, the Talmud and the $\alpha^{\min}$ rules.

The **proportional (P)** rule is the most popular one since it divides the available budget proportionally to the claim of the regions.
For each (E, c) and each region R_i, $\quad P_i(E, c) \equiv \lambda c_i$, where $\lambda = \dfrac{E}{C}$.

The **constrained equal awards (CEA)** rule (Maimonides, 1135, 1204) equalizes the amount each region receives, such that no region receives more than its demand.
For each (E, c) and each region R_i, $CEA_i(E, c) \equiv \min\{c_i, \lambda\}$, where λ is chosen so that $\sum_{i=1}^{n} \min\{c_i, \lambda\} = E$.

The **constrained equal losses (CEL)** (Maimonides, 1135, 1204; Aumann and Maschler, 1985) rule tries to analyze the problem from the point of view of losses (what the regions do not receive with respect to their claims), hence it proposes equalizing losses, such that no region receives a negative amount.
For each (E, c) and each region R_i, $CEL_i(E, c) \equiv \max\{0, c_i - \lambda\}$, where λ is chosen so that $\sum_{i=1}^{n} \max\{0, c_i - \lambda\} = E$.

The **Talmud (T)** rule (Aumann and Maschler, 1985), is a combination of the CEA and the CEL rules, which takes in account the half of the aggregate claim C as a reference. If C is lower than the available resource, then the CEA rule is applied over the half-claims. Otherwise, each region receives the half of its claim and the CEL rule is applied in order to distribute the remaining budget with respect to the remaining claims (the other half).
For each (E, c), $T(E, c)$ $CEA(E, \frac{1}{2}c)$ if $E \leqslant \frac{1}{2}C$ or $\frac{1}{2}c + CEL(E - \frac{1}{2}C, \frac{1}{2}c)$ if $E \geqslant \frac{1}{2}C$.

The $\alpha^{\min}$ rule (Giménez-Gómez and Peris, 2014) guarantees a minimum amount to each region: if possible, all regions first receive an amount that coincides with the lowest claim and then, the remaining budget is distributed proportionally to the reduced claims (the initial claims minus the amount already received). If the budget does not allow each region to receive at least the lowest claim, then all regions receive the same amount. That is:

For each (E, c), $\alpha^{\min}(E, c) \equiv \frac{1}{n}E$ if $E \leqslant nk$, or $k+P(E-nk, (c_i-k)_{i\in N})$ if $E \geqslant nk$, where $k = \min\{c_i\}_{i\in N}$ and n is the number of regions.

Per capita rules

Due to the fact that the considered regions have different population, the determination to which category they belong (less developed, transition, or more developed) is made in GDP/head terms. So, in order to compare the treatment each one receives with respect to its claim, we might use the claims *per capita* and adapt the rules, accordingly. It is noteworthy that this adaptation, with differences, is somewhat related to the **weighted constrained rules** (Casas-Méndez et al., 2011).

Specifically, consider n categories of regions $R_1, R_2, \ldots R_n$, with respective populations $p_1, p_2, \ldots, p_n$ that claim $c_1, c_2, \ldots, c_n$ of a budget E. Then, the *per capita* claim is

$$c_i^H = \frac{c_i}{p_i} \qquad i = 1, 2, 3$$

Therefore, the rules are accordingly defined, such as, the P rule equalizes the portion of the claim that is satisfied, i.e., $P_i^H = \frac{c_i^H}{\sum\limits_{j=1}^{n} c_j^H}\lambda$, λ such that $\sum\limits_{i=1}^{n} p_i P_i^H = E$; the CEA rule tries to equalize the awards, $CEA_i^H = \min\left\{c_i^H, \lambda\right\}$, λ such that $\sum\limits_{i=1}^{n} p_i CEA_i^H = E$; or the CEL rule tries to equalize the losses, $CEL_i^H = \max\left\{0, c_i^H - \lambda\right\}$, λ such that $\sum\limits_{i=1}^{n} p_i CEL_i^H = E$.

Straightforwardly, the same adaptation is applied to the remaining rules, and the results are shown in Table 2.2.

Per capita	Claim	Current	P	CEA	CEL	T	$\alpha^{\min}$
More developed	221.03	115.34	157.05	221.03	50.82	50.82	221.03
Transition	705.31	495.09	501.16	705.31	535.10	535.10	722.41
Less developed	1,404.22	1,098.94	997.78	758.36	1,234.01	1,234.01	750.96

Table 2.2: Allocation of ERDF budget according to each considered rule (€). The first column presents the three different regions. Within each region, rows provide the *per capita* allocations recommended to each of the three considered regions. The second column provides the *per capita* claim of each of the regions.

Absolute	Claim	Current	P	CEA
More developed	61,901,153,827	32,300,565,888	43,984,239,115	61,901,153,827
Transition	36,181,081,146	25,396,981,020	25,708,685,964	36,181,070,669
Less developed	166,509,560,350	130,309,753,020	118,314,374,848	89,925,075,432

	CEL	T	$\alpha^{\min}$
More developed	14,231,803,350	14,231,803,350	61,901,153,827
Transition	27,449,468,078	27,449,468,078	24,660,387,099
Less developed	146,326,028,500	146,326,028,500	101,445,759,002

Table 2.3: Allocation of ERDF budget according to each considered rule in absolute terms (€). The first column presents the three different regions. Within each region, rows provide the absolute term allocations recommended to each of the three considered regions. The second column provides the absolute term claim of each of the regions.

Once the problem of distributing the ERDF funds among the EU regions has been translated into a claims problem, and the allocations are calculated in terms of the *per capita* claims, Table 2.3 shows the distribution of the budget proposed by the rules in absolute terms, i.e., the final distribution of the total ERDF budget.

Furthermore, and for the sake of facilitating the analysis, Table 2.4 provides data about the percentage of the claims that rules allocates to each of the regions.

Absolute	Claim (€)	Current	P	CEA	CEL	T	$\alpha^{\min}$
More developed	61,901,153,827	52.2%	71.1%	100%	23%	23%	100%
Transition	36,181,081,146	70.2%	71.1%	100%	75.9%	75.9%	68.2%
Less developed	166,509,560,350	78.3%	71.1%	54.0%	87.9%	87.9%	60.9%

Table 2.4: Percentages of claims satisfied by current allocation and rules proposals. The first column shows the three different regions. Each row presents the percentages of claim satisfied by each allocation rule for each of the three regions.

In order to choose one proposal among all the considered allocations, the following two sections compare the different rules in terms of convergence and equity.

2.4 Convergence among regions

As aforementioned, one of the main objectives of the EU through the ERDF funds is to promote convergence among regions of different categories. So, how the introduced rules affects this concerns is our natural next step. Specifically, consider two regions R_i and R_j with the following features:

- R_i belongs to the less developed regions, has a GDP/head r_i and a claim *per capita* c_i.
- R_j belongs to the more developed regions, has a GDP/head r_j and a claim *per capita* c_j.
- $r_j > r_i$.
- $c_j < c_i$ (the claim *per capita* is greater for the less developed region, in order to obtain convergence).
- Hence, some funds E should be allocated to these regions taking into account their claims.

Firstly, on the one hand, we measure the initial divergence d^0 between these regions by,

$$d^0 = 1 - \frac{r_i}{r_j}$$

It is noteworthy that each of the proposed rule satisfies the so-called **order preservation** property, that is, the larger the claim, the larger the resources allocated to the region. Formally, if we denote by x_i, x_j the *per capita* allocation to regions R_i and R_j, respectively, made by a rule φ, then $x_i \geqslant x_j$.

Secondly, after the rule φ is applied to allocate the funds, the new divergence ratio $d^1(\varphi)$ is obtained by,

$$1 - d^1(\varphi) = \frac{r_i + x_i}{r_j + x_j} \geqslant \frac{r_i + x_j}{r_j + x_j} > \frac{r_i}{r_j} \quad \Rightarrow \quad d^1(\varphi) < d^0$$

Therefore, the proposed rules always reduce the divergence ratio.

On the other hand, it is easy to observe that $c_i > c_j$ implies that the application of the CEL rule always provide to the less developed region an allocation greater or equal that the one provided by other rules:

$$CEL_i > \varphi_i \qquad \text{for} \quad \varphi = P, CEA, T, \alpha^{\min}$$

so,

$$d^1(CEL) < d^1(\varphi) \qquad \text{for} \quad \varphi = P, CEA, T, \alpha^{\min}$$

that is, the rule better promoting convergence is CEL.

If we compute the divergence ratio (in percentages) from Table 2.2 we observe these facts. Indeed, Table 2.5 highlights that the more reducing proposal is given by CEL rule (that, in this case coincides with the T rule). Note that it is the only rule that reduces all divergence ratios with respect to the current allocation.

Divergence	Initial d^0	Current d^1	$d^1(P)$	$d^1(CEA)$	$d^1(CEL)$	$d^1(T)$	$d^1(\alpha^{\min})$
$R_2VS.R_1$	28%	27%	27%	26%	27%	27%	27%
$R_3VS.R_1$	64%	61%	61%	62%	60%	60%	62%
$R_3VS.R_2$	50%	46%	47%	49%	46%	46%	48%

Table 2.5: This table provides the divergence ratio after applying current allocation and rules proposals. In the first column, R_1 corresponds to the more developed regions, R_2 for transition regions and R_3 for less developed regions. The rows show the percentage value of the divergence ratio corresponding to each of the rules applied.

2.5 Reducing the inequality: fair criteria

Following Robert (1974), "the complete principle of distributive justice would say simply that a distribution is just if everyone is entitled to the holdings they possess under the distribution." Hence, in order to find out the rule that induces a larger commitment among the different regions involved in the ERDF budget distribution, we introduce some equity criteria.

Lorenz dominance is a criterion used to check whether a solution is more favourable to smaller claimants relative to larger claimants.[3] So, a Lorenz dominant solution is intended to equalize the allocations

[3] The Lorenz criterion is a key concept in the literature on income distribution. See, e.g., Sen (1973).

among claimants, regardless of their claims. Let $\mathbb{R}^n_\leqslant$ be the set of positive n-dimensional vectors $x = (x_1, x_2, \ldots, x_n)$ ordered from small to large; i.e., $0 < x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$. Let x and y be in $\mathbb{R}^n_\leqslant$. We say that x **Lorenz dominates** y, denoted by $x \succ_L y$, if for each $k = 1, 2, \ldots, n-1$, $x_1+x_2+\cdots+x_k \geqslant y_1+y_2+\ldots+y_k$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. If $x \succ_L y$ and $x \neq y$, then at least one of these $n-1$ inequalities is a strict inequality. Given two rules, φ and ψ, it is said that φ **Lorenz dominates** ψ, $\varphi \succ_L \psi$, if $\varphi(E, c) \succ_L \psi(E, c)$, for each claims problem (E, c).

Hence, a Lorenz dominated rule, in some sense, respects the claims. Bosmans and Lauwers (2011b) obtain a Lorenz dominance comparison among several rules:[4] $CEA \succ_L \alpha^{\min} \succ_L P \succ_L T \succ_L CEL$.

So, the CEA rule distributes the budget as egalitarian as possible, maintaining the existent differences before the budget was allocated. On the contrary, the CEL rule provides the less egalitarian distribution of the funds. Then, if one of the objectives is reducing previous inequalities, the CEL solution may be more appropriate.
Next, Figure 2.1 depicts the graphical expression of this dominance, the so-called Lorenz curve.[5] Note that the CEA rule is the closest

[4]The following result is true whenever $C \leqslant 2E$, which is the case in our applied problem. In the general case, the proportional and Talmud rules are not related, but the other relationships are also true.

[5]It it noteworthy that we represent the allocation provided by different rules and we do not represent the final situation of each region.

to the line of perfect equality, whereas the CEL rule is the farthest one, thus the two extreme allocations are proposed by the CEA and CEL rules, the most and the least equitable distributions, respectively. Furthermore, the Lorenz dominance suggest to select the dominated solution (that is, the more unequal proposal in order to favour the less developed regions). Since we depart from an unequal situation (unequal GDP/head regions) thus the most unequal Lorenz solution (the CEL rule) provides greater convergence.

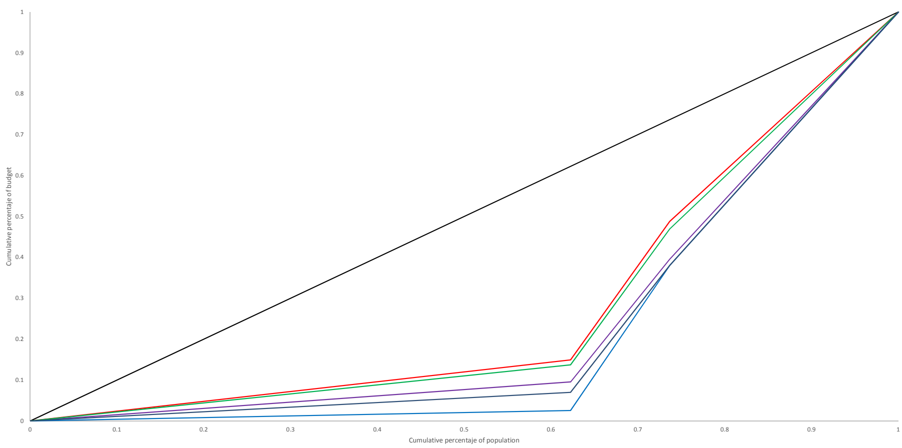

Figure 2.1: Lorenz representation of the allocations proposed by the considered rules. The black line means the perfect equality and the dark blue is the current allocation. The purple line corresponds to the P rule; the red line is the CEA rule; the blue line is equivalent to the CEL and T rule. Finally the green line corresponds to $\alpha^{\min}$ rule.

Apart from the above mentioned divergence ratio, it is noteworthy that there are different indexes widely used to measure the inequality among regions: the Atkinson index (Atkinson, 1970), the generalized entropy index (Theil, 1967), and the Gini index (Gini, 1921). Among them, the latter is the most popular one, vastly used in both official and scientific reports, and considered in the literature as the best single measure of inequality (see, for instance, Atkinson, 1970, and Aaberge and Brandolini, 2015). We use the weighted version of it. Formally, given k regions with population n_1, n_2, ..., n_k, and (analyzed) variables r_1, r_2, ..., r_k, the **Gini/head index (*HGi*)** (Gini, 1921) of these variables in that regions is defined by

$$Gi = \frac{1}{N^2\mu}\sum_{i=1}^{k}\sum_{j<i}|r_i - r_j|n_i n_j.$$

where $N = n_1 + n_2 + \ldots + n_k$ and μ is the average of r_1, r_2, ..., r_k.[6]

The Gini/head index takes values in the interval $[0, 1]$, where $HGi = 0$ means perfect equality, and $HGi = 1$ means complete inequality, so the lower the index the more equality the allocation. We analyze how the allocations provided by all the introduced rules modify the Gini/head index in our applied problem.

[6]This definition is a variation of the original version of Gini index adapting it according to the claims *per capita*.

It is clear that favouring regions with lower GDP/head (that is, favouring the agents with higher claims) reduces the Gini/head index and so the inequality among regions. As happened with the divergence ratio, The CEL rule is the one that more reduces the Gini/head index.

We now compute this index for the initial situation (considering the 2013 GDP/head of the three categories of regions) and the result after the application of the current proposal and the allocations provided by rules. Table 2.6 shows the Gini/head index for each of the considered rules. If we compare these indices, we observe that all distributions of the ERDF funds reduce the inequality (in terms of the Gini/head index), but only the one provided by CEL and Talmud rules (which coincide) reduce the Gini/head index of the current allocation. So, this index also supports the implementation of the CEL rule.

	Initial	Current	P	CEA	CEL	T	$\alpha^{\min}$
Gini/head index	19.74%	18.23%	18.38%	18.62%	17.99%	17.99%	18.62%

Table 2.6: Gini/head inequality index (in percentage) of the initial and the current allocations, as well as each of the allocations proposed by the considered rules.

2.6 Establishing guarantees

An alternative approach that appears in the claims problems literature consists on ensuring a certain amount to each agent (region), which depends on the total budget and the quantity that each region claims (indeed, the definition of a rule imposes a lower bound by the non-negative constraint). This amount is known as lower bound (or guarantee). Some commonly used lower bounds that perfectly fit in our context are the fair lower bound (Moulin, 2002) and the min lower bound (Dominguez, 2013).

The **fair lower bound,** f**,** (Moulin, 2002) establishes that all regions should receive at least the amount assigned to each of them in an equal division, or their full claim. Formally,
For each $(E,c) \in \mathcal{C}$ and each region R_i, $f_i(E,c) = \min\left\{c_i, \dfrac{E}{n}\right\}$.

The **min lower bound,** m**,** (Dominguez, 2013) proposes that all regions receive an equal amount that consists (if possible) in the n-th part of the smallest claim (in other case, it guarantees an equal division of the endowment). Formally,
For each $(E,c) \in \mathcal{C}$ and each region R_i, $\mathrm{m}_i(E,c) = \frac{1}{n}\min\left\{\min_{j \in N} c_j, E\right\}$.

If we analyze the problem from the point of view of losses (the unsatisfied part of the claim), then ensuring a lower bound in losses is equivalent to establish an upper bound in awards. In this sense we define the following upper bound.

The **up upper bound,** *up***,** establishes that all regions should incur in the same loss, restricted to the fact that no region may end with a negative allocation. We denote by L the aggregate losses, that is $L = C - E$. Formally,

For each $(E, c) \in \mathcal{C}$ and each region R_i, $up_i(E, c) = \max\{0, c_i - L\}$.

Table 2.7 provides these lower and upper bounds to each of the regions. It is noteworthy that the f and m bounds guarantee a more egalitarian distribution of the budget, whereas the upper bound benefits to the less developed region (since it has the larger *per capita* claim).

If we try to apply jointly one of the lower bounds and the upper bound, we observe that it is not possible for the more developed regions category (since the lower bound is greater that the upper bound). With respect to the other regions, we obtain an interval that should contain the final allocation.

Per capita	Claim	Current	*f*	*m*	*up*
More developed	221.03	115.34	56.74	56.74	50.82
Transition	705.31	495.09	56.74	56.74	535.10
Less developed	1,404.22	1,098.94	56.74	56.74	1,234.01
Absolute	Claim	Current	*f*	*m*	*up*
More developed	61,901,153,827	32,300,565,888	15,889,783,492	15,889,783,492	14,231,803,350
Transition	36,181,081,146	25,396,981,020	2,910,537,689	2,910,537,689	27,449,468,078
Less developed	166,509,560,350	130,309,753,020	6,727,843,950	6,727,843,950	146,326,028,500

Table 2.7: Guarantees assigned to each region by lower bounds (€). The first column presents the three different economic regions. Within each region, rows provide the guarantees recommended to each of the three considered economic regions. The second column provides the claim of each of the regions in *per capita* terms. The third column shows the actual distribution of the health budget in per capita terms, meanwhile the rest of the columns show the allocations recommended by each of the bounds for each economic region. Finally, note that rows 2-4 show the values are in *per capita* terms, and rows 6-8 the values are in absolute terms.

In order to distribute the remaining budget, if any, Giménez-Gómez et al. (2017) propose some axioms that depend on the lower bound being used. They show that by asking for some natural properties, we recover the usual rules.[7] An alternative approach to distribute the non-

[7]In particular, they show that the fair and min lower bound provide the CEA

allocated budget is by recursively applying the obtained guarantees. This process is defined in the following way: once the first guarantee is allocated to the regions, we compute new guarantees in the problem defined by the non distributed budget and the unsatisfied claims (the initial claim minus the received guarantee). Once these new guarantees are allocated to the regions, we repeat the process until the budget is completely distributed.

As Table 2.8 shows, by recursively applying the previously introduced bounds to our problem we recover either the CEA rules (by using f and m) or the CEL rule (through up).

Therefore, we obtain, as in the previous section, that those bounds that favor the largest claimant end-up a more equitable distribution of the budget in terms of convergence, since they favor the less developed region (which is the largest claimant in per capita terms).

rule, whereas the up upper bound recovers the CEL rule. See Giménez-Gómez et al. (2017) for further details.

Per capita	Claim	Current	f	m	up
More developed	221.03	115.34	221.03	221.03	50.82
Transition	705.31	495.09	705.31	705.31	535.10
Less developed	1,404.22	1,098.94	758.36	758.36	1,234.01
Absolute	Claim	Current	f	m	up
More developed	61,901,153,827	32,300,565,888	61,901,153,827	61,901,153,827	14,231,803,350
Transition	36,181,081,146	25,396,981,020	36,181,070,669	36,181,070,669	27,449,468,078
Less developed	166,509,560,350	130,309,753,020	89,925,075,432	89,925,075,432	146,326,028,500

Table 2.8: Recursive application of guarantees (€). The first column presents the three different economic regions. Within each region, rows provide the allocation recommended to each of the three considered economic regions. The second column provides the claim of each of the regions in *per capita* terms. The third column shows the actual distribution of the health budget in *per capita* terms, meanwhile the rest of the columns show the allocations recommended by each of the recursive application of the bounds for each economic region. Finally, note that rows 2-4 show the values are in *per capita* terms, and in rows 6-8 the values are in absolute terms.

2.7 The ERDF Spanish evidence

For the sake of going deeply in the analysis by NUTS 2, and due to the impossibility of exposing the analysis of the total number of the EU NUTS 2 regions, we implement the aforementioned approach to the Spanish evidence that help to introduce insights in the detailed problem. That is, as Figure 2.2 depicts, Spain is formed by 19 regions, divided into three different groups, but analyzed in a individual way. Therefore, in our scenario the proposal for the endowment E is the ERDF budget currently allocated to all regions in Spain (in absolute terms) and its claims c_i correspond to the sum of the total budget they demanded for the period 2014-2020.

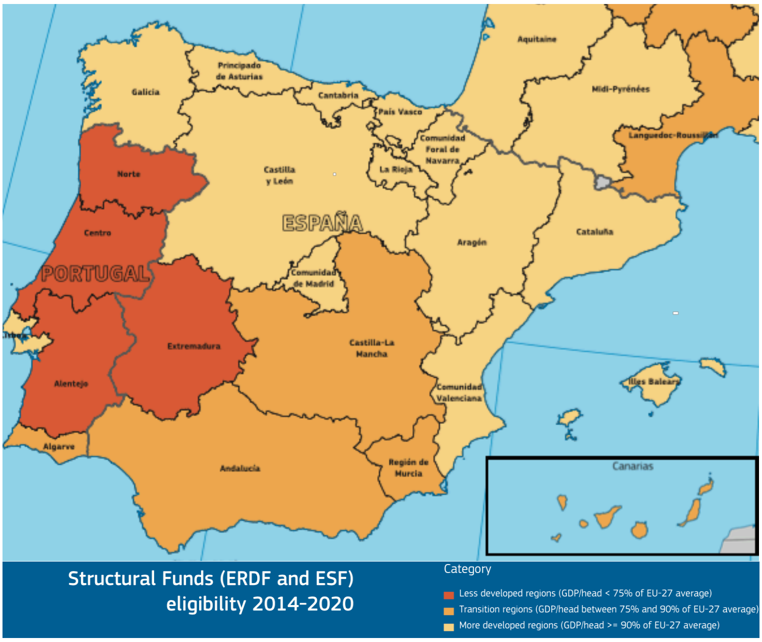

Figure 2.2: NUTS 2 regions in Spain: different development levels. The orange area denotes the less developed region, meanwhile the yellow and the light yellow areas highlight the transition and more development regions, respectively. Source: www.ec.europa.eu/regional_policy, www.ec.europa.eu/esf.

In order to compare the claims of these 19 regions, and the allocations they receive, it is necessary to analyze the problems in terms of *per capita* resources, since the populations are very different. Then we obtain the claims, current allocations and population. Note that the regions are ordered with respect to their claim *per capita* (from lowest to highest). Tables 2.9 and 2.10 reflects this data.

The endowment $\mathbf{E} = 9,760,853,165.00$			
Absolute	Claim	Current	Population
Comunidad de Madrid	474,688,914	249,844,457	6,476,838
Navarra	89,018,434	44,509,217	640,353
País Vasco	352,899,958	176,449,979	2,167,323
Aragón	239,894,676	119,947,338	1,316,072
Cantabria	112,598,206	56,299,103	581,490
La Rioja	67,613,030	33,806,515	312,624
Cataluña	1,671,234,350	835,617,175	7,441,284
Islas Baleares	267,392,822	133,696,411	1,150,962
Comunidad Valenciana	1,180,510,000	590,255,000	4,935,182
Castilla y León	669,877,226	334,938,613	2,435,951
Región de Murcia	416,855,908	333,484,725	1,472,991
Asturias	329,723,791	263,779,031	1,034,302
Castilla-La Mancha	747,447,717	597,958,172	2,040,977
Galicia	1,142,109,802	913,687,840	2,710,216
Andalucía	3,990,192,722	3,200,907,333	8,408,976
Islas Canarias	1,220,044,945	1,037,038,201	2,154,978
Ceuta	56,721,428	45,377,141	85,034
Melilla	65,830,519	52,664,377	84,946
Extremadura	925,740,673	740,592,537	1,077,525

Table 2.9: Claim and current allocation of ERDF budget according to each Spanish region (€).

The endowment $\mathbf{E} = 9,760,853,165.00$			
per capita	Claim	Current	GDP
Comunidad de Madrid	73.29	38.58	30,188
Navarra	139.01	69.51	247,442
País Vasco	162.83	81.41	28,858
Aragón	182.28	91.14	24,417
Cantabria	193.64	96.82	19,965
La Rioja	216.28	108.14	23,726
Cataluña	224.59	112.29	25,945
Islas Baleares	232.32	116.16	22,924
Comunidad Valenciana	239.20	119.60	19,176
Castilla y León	275.00	137.50	20,688
Región de Murcia	283.00	226.40	18,122
Asturias	318.79	255.03	19,445
Castilla-La Mancha	366.22	292.98	17,557
Galicia	421.41	337.13	19,508
Andalucía	474.52	380.65	16,379
Islas Canarias	566.15	481.23	18,761
Ceuta	667.04	533.64	18,434
Melilla	774.97	619.97	16,670
Extremadura	859.14	687.31	15,280

Table 2.10: Claim and current allocation of ERDF budget according to each Spanish region (€).

Taking into account the data in Table 2.10, the introduced rules recommend the allocations shown by Tables 2.11 and 2.12 in *per capita* and absolute terms, respectively. Furthermore, Table 2.13 shows the percentage of the claims satisfied by each of the rules.

Per capita	Claim	Current	P	CEA	CEL	T	$\alpha^{\min}$
Comunidad de Madrid	73.29	38.58	51.02	73.29	0.00	36.65	73.29
Navarra	139.01	69.51	96.78	139.01	44.51	69.51	112.63
País Vasco	162.83	81.41	113.36	162.83	68.33	81.41	126.88
Aragón	182.28	91.14	126.90	182.28	87.78	91.14	138.53
Cantabria	193.64	96.82	134.81	193.64	99.14	96.82	145.32
La Rioja	216.28	108.14	150.57	216.28	121.78	113.72	158.87
Cataluña	224.59	112.29	156.36	224.59	130.09	122.03	163.85
Islas Baleares	232.32	116.16	161.74	232.32	137.82	129.76	168.48
Comunidad Valenciana	239.20	119.60	166.53	239.20	144.70	136.65	172.60
Castilla y León	275.00	137.50	191.45	246.68	180.50	172.44	194.02
Región de Murcia	283.00	226.40	197.02	246.68	188.50	180.44	198.81
Asturias	318.79	255.03	221.94	246.68	224.29	216.23	220.23
Castilla-La Mancha	366.22	292.98	254.96	246.68	271.72	263.66	248.62
Galicia	421.41	337.13	293.38	246.68	326.91	318.85	281.66
Andalucía	474.52	380.65	330.35	246.68	380.02	371.96	313.44
Islas Canarias	566.15	481.23	394.15	246.68	471.65	463.59	368.29
Ceuta	667.04	533.64	464.39	246.68	572.54	564.49	428.68
Melilla	774.97	619.97	539.53	246.68	680.47	672.41	493.28
Extremadura	859.14	687.31	598.12	246.68	764.64	756.58	543.66

Table 2.11: Allocation of ERDF Spanish budget according to each considered rule (€).

Absolute	P	CEA	CEL	T	$\alpha^{\min}$
Comunidad de Madrid	330,473,481.57	474,688,914.00	0.00	237,344,457.00	474,688,914.00
Navarra	61,973,707.29	89,018,434.00	28,504,820.25	44,509,217.00	72,122,606.73
País Vasco	245,685,277.93	352,899,958.00	148,087,070.57	176,449,979.00	274,995,688.16
Aragón	167,012,176.71	239,894,676.00	115,525,347.39	119,947,338.00	182,310,651.50
Cantabria	78,389,699.14	112,598,206.00	57,647,169.21	56,299,103.00	84,504,341.82
La Rioja	47,071,487.80	67,613,030.00	38,069,937.38	35,551,095.89	49,667,836.68
Cataluña	1,163,495,960.98	1,671,234,350.00	968,030,045.79	908,074,908.28	1,219,255,484.55
Islas Baleares	186,156,099.77	267,392,822.00	158,626,454.21	149,353,043.20	193,911,671.21
Comunidad Valenciana	821,858,775.76	1,180,510,000.00	714,133,333.76	674,370,099.96	851,797,527.15
Castilla y León	466,361,552.95	600,892,276.93	439,678,885.49	420,052,195.10	472,625,579.46
Región de Murcia	290,210,744.69	363,352,512.38	277,657,671.34	265,789,641.85	292,847,433.60
Asturias	229,550,271.67	255,138,171.42	231,981,839.71	223,648,369.49	227,787,335.54
Castilla-La Mancha	520,365,321.46	668,547,053.37	554,574,576.94	538,130,229.87	507,434,142.14
Galicia	795,124,957.56	503,461,406.52	885,993,309.67	864,156,839.92	763,349,117.13
Andalucía	2,777,930,644.85	2,074,298,183.85	3,195,541,138.06	3,127,789,213.46	2,635,735,851.05
Islas Canarias	849,382,593.01	531,582,793.39	1,016,398,664.99	999,035,800.96	793,660,575.75
Ceuta	39,488,867.84	20,975,903.82	48,685,681.10	48,000,554.03	36,452,423.04
Melilla	45,830,522.19	20,954,196.27	57,803,088.14	57,118,670.09	41,902,064.27
Extremadura	644,491,021.83	265,800,277.06	823,914,130.98	815,232,408.91	585,803,921.22

Table 2.12: Allocation of ERDF budget according to each considered rule in absolute terms (€).

Absolute	Claim (€)	Current	P	CEA	CEL	T	$\alpha^{\min}$
Comunidad de Madrid	474,688,914.00	53%	70%	100%	0%	50%	100%
Navarra	89,018,434.00	50%	70%	100%	32%	50%	81%
País Vasco	352,899,958.00	50%	70%	100%	42%	50%	78%
Aragón	239,894,676.00	50%	70%	100%	48%	50%	76%
Cantabria	112,598,206.00	50%	70%	100%	51%	50%	75%
La Rioja	67,613,030.00	50%	70%	100%	56%	53%	73%
Cataluña	1,671,234,350.00	50%	70%	100%	58%	54%	73%
Islas Baleares	267,392,822.00	50%	70%	100%	59%	56%	73%
Comunidad Valenciana	1,180,510,000.00	50%	70%	100%	60%	57%	72%
Castilla y León	669,877,226.00	50%	70%	90%	66%	63%	71%
Región de Murcia	416,855,908.00	80%	70%	87%	67%	64%	70%
Asturias	329,723,791.00	80%	70%	77%	70%	68%	69%
Castilla-La Mancha	747,447,717.00	80%	70%	89%	75%	72%	68%
Galicia	1,142,109,802.00	80%	70%	44%	78%	76%	67%
Andalucía	3,990,192,722.00	80%	70%	52%	80%	78%	66%
Islas Canarias	1,220,044,945.00	85%	70%	44%	83%	82%	65%
Ceuta	56,721,428.00	80%	70%	37%	86%	85%	64%
Melilla	65,830,519.00	80%	70%	32%	88%	87%	64%
Extremadura	925,740,673.00	80%	70%	29%	89%	88%	63%

Table 2.13: Percentages of claims satisfied by current allocation and rules proposals for the ERDF Spanish evidence.

Next, Figure 2.3 and Table 2.14 provide insights about the equity behavior of the rules and the final allocation.

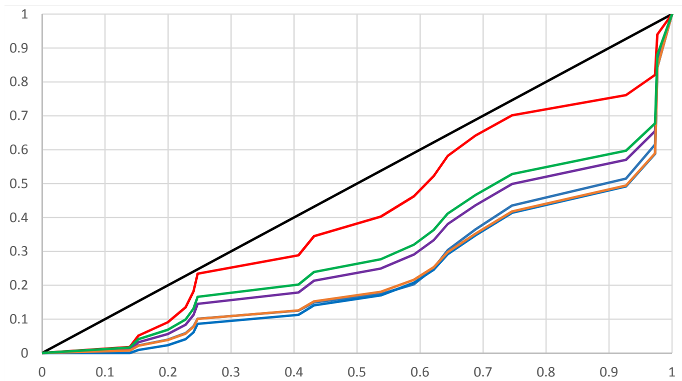

Figure 2.3: Lorenz representation of the allocations proposed by the considered rules for the ERDF Spanish evidence. The black line means the perfect equality and the dark blue is the current allocation. The purple line corresponds to the P rule; the red line is the CEA rule; the blue line is equivalent to the CEL and T rules. The green line corresponds to the $\alpha^{\min}$ rule.

	Initial	Current	P	CEA	CEL	T	$\alpha^{\min}$
Gini/head index	13.11%	12.91%	12.95%	13.04%	12.90%	12.91%	12.97%

Table 2.14: Gini/head inequality index (in percentage) of the initial and the current allocations, as well as each of the allocations proposed by the considered rules.

Madrid VS.	AND	ARA	AST	CAN	C-L	C-M	CAT	CEU	CV
GDP/H	184%	124%	155%	151%	146%	172%	116%	164%	157%
GDP+current/H	180%	123%	153%	151%	146%	169%	116%	159%	157%
GDP+CEL/H	180%	123%	153%	150%	145%	169%	116%	159%	156%
	EXT	GAL	I-B	I-C	RIO	MEL	NAV	P-V	MUR
GDP/H	198%	155%	132%	161%	127%	181%	110%	105%	167%
GDP+current/H	189%	152%	131%	157%	127%	175%	110%	104%	165%
GDP+CEL/H	188%	152%	131%	157%	127%	174%	110%	104%	165%

Table 2.15: Divergence ratio after applying current allocation and rules proposals for Madrid (the Spanish richest region).

As shown by Tables 2.15 and 2.16, the CEL rule is the one that further reduce the divergence among regions. For the sake of clarity, we only provide what happens to the richest and the poorest regions in Spain (Madrid and Extremadura, respectively). The rest of the data may be provided by the authors under request.

Finally, by applying the guarantees introduce in Section 2.6, the results remain valid. That is, the f and m lower bounds retrieve the CEA rule, and the *up* upper bound, the CEL rule.

2.8 Final Remarks

The European Union tries to promote the social and economic cohesion of the countries members, as well as to reduce the inequalities among

Extremadura VS.	AND	ARA	AST	CAN	C-L	C-M	CAT	CEU	MAD
GDP/H	7%	37%	21%	23%	26%	13%	41%	17%	49%
GDP+current/H	5%	35%	19%	20%	23%	11%	39%	16%	47%
GDP+CEL/H	4%	35%	18%	20%	23%	10%	38%	16%	47%
	CV	GAL	I-B	I-C	RIO	MEL	NAV	P-V	MUR
GDP/H	20%	22%	33%	19%	36%	8%	44%	47%	16%
GDP+current/H	17%	20%	31%	17%	33%	8%	42%	45%	13%
GDP+CEL/H	17%	19%	30%	17%	33%	8%	42%	45%	12%

Table 2.16: Divergence ratio after applying current allocation and rules proposals for Extremadura (the Spanish poorest region).

them. By doing so, it uses some financial instruments, being one of them the the European Regional Development Fund (ERDF).

In this Chapter we focus on these funds, due to their important social impact. By implementing the classical claims problem approach (O'Neill, 1982), we propose an alternative way of allocating the budget among the different regions in EU, and, in a detailed way, to the Spanish regions case.

We analyze the most usual rules in order to obtain alternative allocations of the budget. In order to compare different proposals, we observe, throughout different equity criteria, that the CEL rule performs better when looking for convergence and reducing inequalities across regions. By using the Lorenz dominance, a divergence ratio or the well known Gini index, always the CEL rule is the better proposal:

it is the most unequal (then reducing initial inequalities), it is the one that reduces divergence the most and provides the lowest inequality Gini index. So, this way of allocating resources may be proved to be a strong candidate for future policy changes concerning the allocation of the EU funds.

Chapter 3

Resource allocations with guaranteed awards in claims problems

Overview. The establishment of guarantees that ensure a minimum award to each agent when rationing a resource, or in the adjudication of conflicting claims, has been widely analyzed in the body of literature by introducing the notion of lower bound on awards. Indeed, this concept has a key role in most of the approaches related to the problem of fair allocation (Thomson, 2015) and a range of such lower bounds have been proposed: The minimal right (Curiel et al., 1987), the fair lower bound (Moulin, 2002), securement (Moreno-Ternero and Villar, 2004a) and the min lower bound (Dominguez, 2006). The aim of this Chapter

is to show that there is a correspondence between lower bounds and rules; i.e., associated to each particular lower bound, we will find a specific way of distributing the resources. In doing so, we provide new characterizations for some well known rules: The constrained equal awards, as well as the Ibn Ezra's rule. A dual analysis, by using lower bounds on losses (or, equivalently, upper bounds on awards) will provide characterizations of the dual of the previously mentioned rules: The constrained equal losses rule and the dual of the Ibn Ezra's rule.

Keywords: Claims problem; guarantees; lower bounds; constrained equal awards rule; Ibn Ezra's rule.

3.1 Introduction

The so-called claims problem reflects a situation where the agents' claims cannot be totally honored when a resource must be distributed among them. The way of rationing this endowment among the agents, taking into account their claims, is prescribed by a rule: A method with desirable properties that prescribes how the resource is allocated. In this context, we analyze how to distribute any increment of the endowment in terms of two general concepts: First, establishing that each agent should be guaranteed a minimum award, which is determined by a particular lower bound (respect of the lower bound); and then requiring that agents with equal guarantees, should be treated equally (equal treatment of equals).

It is noteworthy that the concern of ensuring some minimum individual rights has figured in a large number of contexts. Specifically, the Universal Basic Income is a classical issue that has attracted much attention in the social policy literature and the political agenda during the last two decades (Noguera, 2010).[1] The establishment of a minimum wage in the labor market, the debate about ensuring a universal minimum health coverage in the U.S. Senate, the European Structural and Investment Funds (ESIF), ensuring minimum quantities in heritage laws, fishing quotas (Iñarra and Prellezo, 2008; Iñarra and Skonhoft, 2008; Kampas, 2015); or, the negotiations of CO^2 emissions, a relevant issue nowadays (Giménez-Gómez et al., 2016), are further real-life examples.

From a theoretical point of view, the idea of establishing minimum guarantees in awards underlies the analysis of claims problems from its beginning (O'Neill, 1982) up to the present day (Giménez-Gómez and Marco-Gil, 2014). Indeed, the formal definition of a rule already includes the requirement that, for each problem, awards be non-negative, which represents a lower bound on awards. The impact of requiring that a claims rule fulfills a lower bound was first analyzed by Dominguez and Thomson (2006b) and Yeh (2008). Afterwards, the recursive application of a lower bound has been analyzed in the litera-

[1] See, for instance, Sonia Sodha (2017) "Is Finland's basic universal income a solution to automation, fewer jobs and lower wages?". The Guardian.

ture, showing that (under some mild conditions) this process provides a unique rule. In particular, Dominguez (2013) and Giménez-Gómez and Marco-Gil (2014), among others, find out that some well known rules are retrieved by recursively applying lower bounds and, consequently, they provide new axiomatic characterizations of classical rules.

Our present approach elaborates on these previous works but, instead of applying a lower bound recursively, we combine the requirement that rules should fulfill the lower bound with some additional requirements on the distribution of the resources that depend on the lower bound being used. Specifically, we require that a rule (i) guarantees to each individual at least the amount determined by the particular lower bound being used (respect of the lower bound); and, (ii) fulfills properties related to equal treatment of equals (conditional equal treatment), or related to some monotonicity behaviour (conditional resource monotonicity, conditional equal bound monotonicity, or priority). The idea behind these properties is to compare the guaranteed awards among the agents and, on this basis, to determine the way of distributing the endowment whenever it increases.

A key point in our study is the selection of a specific lower bound on which the aforementioned axioms are based. Hence, we need to choose a meaningful lower bound in the sense that it should be different from zero, whenever the claim is different from zero (quoting Dominguez (2013) words, "these lower bounds satisfy positivity"). In doing so, by focusing on three lower bounds (the fair lower bound (Moulin, 2002), securement (Moreno-Ternero and Villar, 2004a), and the min lower bound (Dominguez, 2006)), our main results show how these axioms provide new characterizations of the constrained equal awards and the Ibn Ezra's rule.

Finally, note that when facing a claims problem, each individual has a claim on the endowment that represents the maximum amount she can receive and, at the same time, the maximum amount she can lose. The agent's loss is equal to the difference between her claim and her award. By focusing on losses (the so-called dual approach), a lower bound on awards provides the maximum amount that individual can lose; that is, we are considering upper bounds on losses. Analogously, a lower bound on losses provides an upper bound on awards. By analyzing the implications of the existence of lower bounds on losses, we straightforwardly obtain from the previous results characterizations of their dual rules: The constrained equal losses and the dual Ibn Ezra's rule.

The remainder of the Chapter is organized as follows. The next section presents the model and introduces the lower bounds. Section 3.3 introduces the axioms and Section 3.4 provides our main results. Finally, Section 3.5 comments on the dual approach and mentions some possible future research. The proofs are relegated to the Appendix.

3.2 Preliminaries

3.2.1 Claims problems and rules

Throughout this work we consider a set of agents $N = \{1, 2, ..., n\}$, such that each agent has a claim $c_i \in \mathbb{R}_+$ on an infinitely divisible resource, the endowment, $E \in \mathbb{R}_+$. Let $c \equiv (c_i)_{i \in N}$ be the claims vector.

A **claims problem** appears whenever the endowment is not enough to satisfy the aggregate claim. Without loss of generality, we assume that the agents are indexed according to their claims, $c_1 \leqslant c_2 \leqslant \ldots \leqslant c_n$. The pair $(E, c) \in \mathbb{R}_+ \times \mathbb{R}^n_+$ represents the claims problem, and $\mathcal{C}$ denotes the set of all claims problems.

A **rule** is a single-valued function $\varphi : \mathcal{C} \to \mathbb{R}^n_+$ such that for each problem $(E, c) \in \mathcal{C}$, and each $i \in N$, $0 \leqslant \varphi_i(E, c) \leqslant c_i$ (**non-negativity** and **claim-boundedness**), and $\sum_{i=1}^{n} \varphi_i(E, c) = E$ (**efficiency**).

Two of the most important rules in the literature are the *constrained equal awards* and the *constrained equal losses* (Maimonides, 12th century).[2] These rules propose an egalitarian distribution of the awards and losses, respectively, among the claimants, given some constraints. Specifically,

The **constrained equal awards (CEA)** rule (Maimonides, 1135, 1204), proposes an equal distribution of the health budget subject to

[2]Other important rules are the Proportional or the Concede-and-Divide rules. See Thomson (2003, 2015) for complete and updated surveys on claims problems.

no one can receive more than Head Gini claim.

For each $(E,c) \in \mathcal{C}$ and each $i \in N$, the CEA, recommends: $CEA_i(E,c) \equiv \min\{c_i, \lambda\}$, where λ is chosen so that $\sum_{i=1}^{n} \min\{c_i, \lambda\} = E$.

The **constrained equal losses (CEL)** (Maimonides, 1135, 1204; Aumann and Maschler, 1985) rule tries to analyze the problem from the point of view of losses (what the regions do not receive with respect to their claims), hence it proposes equalizing losses, such that no region receives a negative amount.

For each $(E,c) \in \mathcal{C}$ and each $i \in N$, the CEL, proposes: $CEL_i(E,c) \equiv \max\{0, c_i - \mu\}$, where μ is chosen so that $\sum_{i=1}^{n} \max\{0, c_i - \mu\} = E$.

The **Ibn Ezra**'s rule is another classical proposal for solving claims problems.[3] This rule is only defined whenever the endowment is lower than the greatest claim; so it requires a restriction on the domain of claims problems: $\mathcal{C}_{IE} = \{(E,c) \in \mathcal{C} : E \leqslant \max_i\{c_i\}\}$. Within this context,

For each $(E,c) \in \mathcal{C}_{IE}$ and each $i \in N$, the Ibn Ezra's rule, IE, assigns the awards: $IE_i(E,c) \equiv \sum_{k=1}^{i} \frac{\min\{c_k,E\}-\min\{c_{k-1},E\}}{n-k+1}$, where, for notational convenience, we set $c_0 = 0$.

[3] Attributed to Rabbi Abraham Ibn Ezra (Spain,12th century). See O'Neill (1982) and Alcalde et al. (2005) for additional details on this rule.

3.2.2 Lower bounds on awards

A lower bound on awards is a function such that, for each claims problem (E, c) and each agent $i \in N$, $b_i(E, c)$ represents the guaranteed minimum amount that agent i should receive in this situation, according to such a bound. According to the formal definition of a rule, a lower bound should fulfill two compulsory conditions:

1. **Rationality**: The guaranteed minimum award is non-negative and lower than the agent's claim.

2. **Feasibility**: The endowment allows the allocation of these guaranteed awards to the agents.

A **lower bound** is a function $b : \mathcal{C} \rightarrow \mathbb{R}^n_+$, which maps each claims problem $(E, c) \in \mathcal{C}$ to a vector $b(E, c)$ such that for each $i \in N$, $0 \leqslant b_i(E, c) \leqslant c_i$, and $\sum_{i=1}^{n} b_i(E, c) \leqslant E$.

Remark 1. *There are other conditions that should be included in the above definition of a lower bound. Indeed, the following conditions are satisfied by all lower bounds defined in the literature. For each claims problem $(E, c) \in \mathcal{C}$ and each $i \in N$*

- ***Resource monotonicity**: $b_i(E, c)$ increases with E.*

- ***Order preserving**: If $c_i \leqslant c_j$ then $b_i(E, c) \leqslant b_j(E, c)$.*

- ***Positivity**: If $c \neq 0$, and $E \neq 0$ then, $b(E, c) \neq 0$.*

- ***Continuity***: *$b_i(E,c)$ is a continuous function on its arguments.*

Remark 2. *A clear example of a (non-trivial) lower bound is obtained by considering a constant guarantee across agents $k \leqslant c_1$, i.e., the same guarantee for all the agents. This is the idea of rationing with a minimum (survival) allocation.*

3.2.3 An inventory of lower bounds

The first formally defined lower bound, the so-called minimal right (Curiel et al., 1987), requires that each agent receives what is available whenever the other agents have already received their claim in full, or zero if this is not possible.

Minimal right, mr: For each $(E,c) \in \mathcal{C}$ and each $i \in N$, $mr_i(E,c) = \max\left\{0, E - \sum_{j \in N \setminus \{i\}} c_j\right\}$.

Remark 3. *As mentioned in Thomson (2015), it follows directly from the definition that any rule proposes an allocation above mr. So, guaranteeing to each agent the award provided by mr does not discriminate among rules. In order to compare mr among agents, it is noteworthy that it always benefits individuals with relatively large claims, hurting those agents with lower claims. In this sense, note that $mr_i(E,c) > 0$ implies $c_i > \frac{E}{n}$, although the converse is not true in general. So, for each agent i such that $c_i \leqslant \frac{E}{n}$, her minimal right equals to zero, $mr_i(E,c) = 0$.*

Moulin (2002) introduces the fair lower bound, which establishes that all agents should receive at least the amount assigned to each of them in an equal division, or their full claim.

Fair lower bound, f: For each $(E, c) \in \mathcal{C}$ and each $i \in N$, $f_i(E, c) = \min\left\{c_i, \frac{E}{n}\right\}$.

Moreno-Ternero and Villar (2004a) propose the securement lower bound, that guarantees (if possible) the *n-th* part of each agent's claim (otherwise, this bound guarantees an equal division of the endowment).

Securement, s: For each $(E, c) \in \mathcal{C}$ and each $i \in N$, $s_i(E, c) = \frac{1}{n}\min\{c_i, E\}$.

Finally, Dominguez (2006) introduces the min lower bound, that proposes that each agent receives (if possible) the *n-th* part of the smallest claim (otherwise, this bound guarantees an equal division of the endowment).

Min lower bound, m: For each $(E, c) \in \mathcal{C}$ and each $i \in N$, $m_i(E, c) = \frac{1}{n}\min\left\{\min_{j \in N} c_j, E\right\}$.

Remark 4. *If we consider the agents ordered according to their claims, then the minimum claim corresponds to the first agent,* $c_1 = \min\{c_i, i \in N\}$*, and the* min *lower bound can be defined as* $m_i(E, c) = \frac{1}{n}\min\{c_1, E\}$*. Then, it comes straightforwardly from the definition that for each* $(E, c) \in \mathcal{C}$ *and each individual* $i \in N$*,* $0 \leqslant m_i(E, c) \leqslant$

$s_i(E,c) \leqslant f_i(E,c) \leqslant c_i$. *That is, the fair lower bound guarantees the largest awards to all involved agents (with respect to the min and securement lower bounds).*

3.3 Axiomatic analysis

We introduce some properties on rules, which refer to a fixed lower bound b, and are based on axioms considered as a minimum requirement of fairness in claims problems (Thomson, 2003). The first property is our basic assumption: The required lower bound is satisfied by the claims rule.

Respect of the lower bound, RB^b: For each $(E,c) \in \mathcal{C}$, and each $i \in N$, $\varphi_i(E,c) \geqslant b_i(E,c)$.

RB^b requires that each agent receives at least her lower bound; i.e., agents have a guaranteed minimum level on awards. Note that this condition is meaningless when applied with the minimal rights lower bound, since all rules satisfy it.

Figure 3.1 shows the fulfillment of this axiom in the two-agent case (with $c_1 \leqslant c_2$). A rule fulfills RB^b if it provides efficient allocations ($x_1 + x_2 = E$) that lie between the dashed lines. When these lines coincide they appear as a solid black line, so a unique allocation is determined. Note that in the two-agent case, the RB^f condition provides the allocation determined by the CEA rule. In any case, if the endowment is below c_1, $E \leqslant c_1$, the rules should divide this endowment equally between the two agents. If $E > c_1$, then there is some room for different rules (the region between the dashed lines) that varies from one lower bound to another. As we know from Remark 4 the min lower bound is the less restrictive one, so more rules will fulfill RB^m.

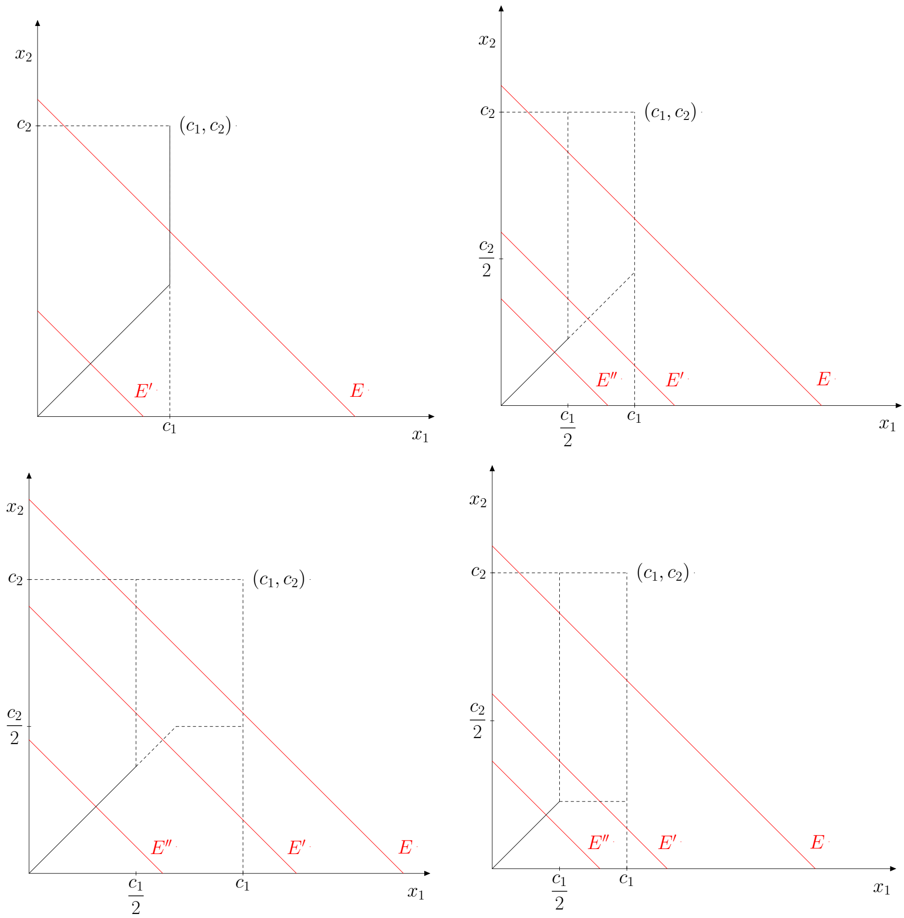

Figure 3.1: RB^b for $b = f$, $b = s$, with $c_1 < \frac{c_2}{2}$, $b = s$, with $c_1 \geqslant \frac{c_2}{2}$ and $b = m$, from left to right, top to bottom. We consider a two-agent claims problem $(E, (c_1, c_2))$, with $c_1 \leqslant c_2$. A rule φ fulfills RB^b if the allocations it provides lie between the dashed lines. When these lines coincide they appear depicted as a solid black line.

Why should agents with equal bounds receive different awards? The next axiom is based on the idea that equal claimants should be treated equally.

Constrained equal treatment of equals, $ETEB^b$: For each $(E, c) \in \mathcal{C}$, and each $i, j \in N$ such that $c_i \leqslant c_j$, then $b_i(E, c) = b_j(E, c)$ implies $\varphi_i(E, c) = \varphi_j(E, c)$, or $\varphi_i(E, c) = c_i \leqslant \varphi_j(E, c)$.

$ETEB^b$ demands equal treatment for equal agents (regarding their lower bounds), unless one of them has her demand met in full. Note that, if the lower bound is order preserving (all considered lower bounds satisfy this condition), the above property implies that agents with the same claims receive the same award.

The following properties analyze the effects of an increase in the endowment. We propose that changes in the final allocation depend on the changes in the guarantees of the agents (lower bounds). Specifically, we consider that the final allocation should change, at least, as much as the lower bounds increase, and, furthermore, that equal changes in the lower bounds, should induce equal changes in the final allocation. Finally, we also require that only those agents who experiment an increase in their guarantees, might benefit from the increase in the endowment.

Constrained resource monotonicity, CRM^b: If $(E, c), (E', c') \in \mathcal{C}$ are two claims problems such that $c = c'$ and $E > E'$, then for each $i \in N$, $\varphi_i(E, c) - \varphi_i(E', c) \geqslant b_i(E, c) - b_i(E', c)$, or $\varphi_i(E, c) = c_i$.

CRM^b requires that any change in the awards received by an agent due to a change in the endowment E should be at least equal to the change in her bound. As before, we need to restrict this idea so that no one receives more than her claim.

Constrained equal bound monotonicity, $CEBM^b$: If (E, c), $(E', c') \in \mathcal{C}$ are two claims problems such that $c = c'$ and $E > E'$, then for each $i, j \in N$ with $c_i \leqslant c_j$, $b_i(E, c) - b_i(E', c) = b_j(E, c) - b_j(E', c)$ implies $\varphi_i(E, c) - \varphi_i(E', c) = \varphi_j(E, c) - \varphi_j(E', c)$, or $\varphi_i(E, c) = c_i \leqslant \varphi_j(E, c)$.

$CEBM^b$ demands that the increment in the endowment might be shared equally among agents who experience an equal change in their lower bound. As before, this increment needs to be limited to the claim.

Priority in allocation, PRI^b: If $(E, c), (E', c') \in \mathcal{C}$ are two claims problems such that $c = c'$ and $E > E'$, then for each $i \in N$ $\varphi_i(E, c) - \varphi_i(E', c) > 0$ if and only if $b_i(E, c) - b_i(E', c) > 0$.

PRI^b states that only those agents whose lower bound increases might benefit from an increment in the endowment.

3.4 Main results

In this section we analyze, in terms of the selected lower bound, how some combinations of the aforementioned axioms uniquely determine a claims rule satisfying them. In particular, we provide some characterizations of the constrained equal awards rule and Ibn Ezra's rule. All proofs are relegated to the Appendix.

Giménez-Gómez and Peris (2015) analyze, in the framework of redistribution problems, the effect of ensuring that each individual obtains the guarantee defined by the minimal right lower bound. This guarantee, under some premises, defines a new rule which is somewhat related to the constrained equal losses. The results obtained can be easily adapted to claims problems. Henceforth, the current Chapter focuses on the remaining lower bounds: The fair lower bound, the securement and the min lower bound.

Hereinafter, let $\mathcal{L}$ denote the family of these lower bounds: $\mathcal{L} = \{f,\ s,\ m\}$.
Table 3.1 summarizes the fulfillment of the axioms (with respect to each of the lower bounds in $\mathcal{L}$) by some relevant rules. Besides the introduced rules, we also consider the Proportional (Pr) and the Talmud (TAL) rules.

	Pr			TAL			CEL			CEA			IE		
Bounds	f	s	m	f	s	m	f	s	m	f	s	m	f	s	m
RB	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\surd$	$\surd$	$\otimes$	$\otimes$	$\otimes$	$\surd$	$\surd$	$\surd$	$\otimes$	$\surd$	$\surd$
$ETEB$	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\surd$	$\surd$	$\otimes$	$\otimes$	$\otimes$	$\surd$	$\surd$	$\surd$	$\otimes$	$\surd$	$\otimes$
CRM	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\surd$	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\surd$	$\surd$	$\surd$	$\otimes$	$\surd$	$\otimes$
$CEBM$	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\surd$	$\surd$	$\surd$	$\otimes$	$\surd$	$\otimes$
PRI	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\surd$	$\otimes$	$\otimes$	$\otimes$	$\surd$	$\otimes$

Table 3.1: Axiom fulfillment. For each considered lower bound, we analyze if the rule satisfies the required axiom.

3.4.1 Constrained Equal Awards rule

We analyze the effect of the introduced properties on a rule, when applied to the fair and min lower bounds. We will obtain that, with these bounds, the CEA rule is characterized. Our first result shows that CEA fulfills RB^b and $ETEB^b$ for each lower bound in $\mathcal{L}$. It is noteworthy that the first part in Proposition 1 comes directly from Remark 4 and Moulin (2002), who characterizes CEA by means of RB^f, composition up, and null claims consistency.[4]

[4]Moulin (2002) calls lower bound what we denote by RB^f, respect of the fair lower bound.

Proposition 1. *For each $b \in \mathcal{L}$,*

1. *CEA fulfills RB^b.*

2. *CEA fulfills $ETEB^b$.*

Now, if we fix the fair lower bound, f, Theorem 1 shows that RB^f and $ETEB^f$ retrieve the constrained equal awards rule.

Theorem 1. *CEA is the only rule satisfying RB^f and $ETEB^f$.*

Remark 5. *The above result shows that the respect of the fair lower bound almost provides the CEA rule. A mild axiom ensuring equal treatment of agents with the same claim, or agents with claims larger than the equal allocation of the endowment, $c_i > \frac{E}{n}$, leads to the constrained equal awards rule.*

If, instead of requiring RB^f, we ask for the monotonicity condition CRM^f, again the CEA rule is obtained. Note that, when requiring this property, the increase in the endowment is allocated in terms of the increase in the fair bound. So, only agents with claims larger than the equal allocation of the endowment, are guaranteed an increase in their awards.

Theorem 2. *CEA is the only rule satisfying CRM^f and $ETEB^f$.*

Lemma 2 in the Appendix shows that the axioms used in the characterizations obtained in Theorem 1 and Theorem 2 are independent.

In Lemma 1 (see Appendix) we analyze some relationships among the axioms used in the above theorems, that allow us to combine RB^f and $CEBM^f$, and CRM^f and $CEBM^f$, so that we obtain alternative characterizations in which the equal treatment condition is substituted by the equal bound monotonicity.

Corollary 1. *CEA is the only rule satisfying RB^f and $CEBM^f$.*

Corollary 2. *CEA is the only rule satisfying CRM^f and $CEBM^f$.*

If, instead of the fair lower bound, we use the min lower bound, the following results show that the CEA rule is characterized by a single property. It is noteworthy that the min lower bound only depends on the endowment, E, and on the minimum claim, c_1. So, it guarantees the same amount to all agents. In this case, as shown in Lemma 3 (see Appendix), $ETEB^m$ implies RB^m, whereas $CEBM^m$ implies CRM^m and the following characterization result is obtained.

Theorem 3. *CEA is the only rule satisfying either $ETEB^m$ or $CEBM^m$.*

3.4.2 Ibn Ezra's rule

From the previous results, it seems that the proposed axioms will characterize the CEA rule under any lower bound being considered. We prove that this is not true when the securement lower bound is used to fix the guarantees of the agents. In this regard, Theorem 4 shows that requiring RB^s, $CEBM^s$ and PRI^s we characterize the Ibn Ezra's rule.

Theorem 4. *IE is the only rule in the class of claims problems $\mathcal{C}_{IE}$ satisfying RB^s, $CEBM^s$ and PRI^s.*

Lemma 4 in the Appendix shows the independence of the axioms used in Theorem 4.

Remark 6. *As far as we know, the only existing characterization of the IE rule is the one provided by Alcalde et al. (2005). In that Chapter, the Ibn Ezra's rule is characterized in terms of anonymity, transitional dummy and worth-generators composition. These properties are based on the cooperative game associated to a claims problem (O'Neill, 1982). Alcalde et al. (2005) define a new cooperative game that compares the behavior of a rule whenever the endowment increases (transitional game). Our result in Theorem 4 characterizes this rule by means of some properties that, in this context, may be easily interpreted in terms of the primitives of the claims problem: The agents' claims and the endowment:*

a) RB^s, that guarantees to any agent the $n-th$ part of her claim (if $c_i < E$, small creditors), or $\frac{E}{n}$ otherwise (large creditors).

b) $CEBM^s$ requires an equal treatment among the same kind of agents (small creditors, large creditors).

c) PRI^s implies that only large creditors increase their awards when the endowment increases.

Since CRM^s implies RB^s (Lemma 1, see Appendix), and the Ibn Ezra's rule fulfills this stronger axiom, Corollary 3 provides a new characterization result for this rule.

Corollary 3. *IE is the only rule in the class of claims problems $\mathcal{C}_{IE}$ satisfying $CEBM^s$, PRI^s and CRM^s.*

3.5 Final Remarks

Throughout this Chapter, we have shown how lower bounds can be associated with a particular rule: The fair and min lower bounds are linked to the constrained equal awards rule, and the securement lower bound is associated to the Ibn Ezra's rule.[5]

In the analysis of the constrained equal losses, agents are concerned about the losses they incur (what they do not receive with respect to their claims). In this regard, an important tool is the notion of duality. The dual rule allocates losses in the same way that the primitive rule allocates awards (for instance, Herrero (2003) proves that the constrained equal awards and the constrained equal losses rules are dual

[5]The minimal right lower bound is linked with a new rule, somewhat related to the constrained equal losses rule, that we name minimal right based egalitarian rule (Giménez-Gómez and Peris, 2015).

rules). The dual axiom is defined so that whenever a rule satisfies the axiom, its dual rule also satisfies it. In an analogous way, given a lower bound (on awards) b the dual lower bound on losses (see Dominguez (2006)) is defined, for each $(E, c) \in \mathcal{C}$, as $b^d(E, c) = c - b(E, c)$.

Taking into account this point of view, and as a consequence of the results in Section 3.4, characterizations of the constrained equal losses or the dual of the Ibn Ezra's rules can be obtained in a straightforward way. For instance, the dual of Theorem 1 can be stated as follows: CEL is the only rule satisfying RB^{fd} and $ETEB^{fd}$, where the dual fair lower bound on losses, f^d, is defined as $f_i^d(E, c) = c_i - \min\left\{c_i, \frac{E}{n}\right\} = \max\left\{0, \frac{E}{n} - c_i\right\}$. Note that a lower bound on losses indicates the maximum award an agent can obtain. So, the f^d bound implies that agents with claims below $\frac{E}{n}$ will obtain zero awards (then, it is not surprising that CEL be the resultant rule).

Finally, it is noteworthy that the analyzed correspondence between lower bounds and rules makes us wonder about bounds that are linked to other important rules such as the the proportional, the Talmud, etc., a question that remains open. Furthermore, we put forward for discussion the converse question: If we propose reasonable guarantees for all agents (a lower bound on awards) or a maximum award they can receive (a lower bound on losses), is it possible to define a unique rule satisfying the required axioms? Although the positivity condition allows us to associate a unique rule satisfying the recursive extension of a lower bound (Dominguez, 2006), we speculate whether it might be possible to combine RB^b with some other condition to define a unique rule.

3.6 Appendix

Proof of Proposition 1: For each $b \in \mathcal{L}$, CEA fulfills RB^b and $ETEB^b$.

Proof. The first part comes from Remark 4 and Moulin (2002). To prove the second part, let $b \in \mathcal{L}$, $(E, c) \in \mathcal{C}$ and $i, j \in N$ such that $c_i \leqslant c_j$ and $b_i(E, c) = b_j(E, c)$. Then, $CEA_i(E, c) = \min\{c_i, \lambda\} \leqslant \min\{c_j, \lambda\} = CEA_j(E, c)$, which implies $CEA_i(E, c) = CEA_j(E, c)$, if the minimum is λ in both cases, or $CEA_i(E, c) = c_i \leqslant CEA_j(E, c)$, whenever the first minimum is c_i. □

In order to prove the main results of the Chapter, we introduce the following lemmas, which analyze some relationships among the introduced axioms.

Lemma 1. *For each lower bound $b \in \mathcal{L}$,*

1. *CRM^b implies RB^b.*
2. *$CEBM^b$ implies $ETEB^b$.*

Proof. Consider $(E, c) \in \mathcal{C}$ and $(E', c) \in \mathcal{C}$ with $E' = 0$. For each $b \in \mathcal{L}$, $b_i(E', c) = 0$ and $\varphi_i(E', c) = 0$, for each $i \in N$.

1. If φ is a rule that satisfies CRM^b, then either $\varphi_i(E,c) \geqslant b_i(E,C)$ or $\varphi_i(E,c) = c_i$. Since $b_i(E,c) \leqslant c_i$, RB^b is fulfilled.

2. Immediate, since $CEBM^b$, applied to problems $(E,c), (0,c)$, coincides with $ETEB^b$. ■

Remark 7. *The results in Proposition 1 and Lemma 1 are also true for the minimal rights lower bound.*

Lemma 2. *For each lower bound* $b \in \{f, s\}$

1. RB^b *and* $ETEB^b$ *are independent.*

2. RB^b *and* $CEBM^b$ *are independent.*

3. CRM^b *and* $ETEB^b$ *are independent.*

4. CRM^b *and* $CEBM^b$ *are independent.*

Proof. The independence of these axioms is shown throughout Examples 1 and 2. Note that these examples can be easily extended to the $n-$agent case by considering additional individuals with null claims. □

Example 1. *Let* $n = 3$ *and* φ^a *be defined by:*

$$\varphi_i^a(E,(c_1,c_2,c_3)) = \begin{cases} \min\left\{c_i, \frac{E}{3}\right\} & i = 1,2 \\ \\ E - \min\left\{c_1, \frac{E}{3}\right\} - \min\left\{c_2, \frac{E}{3}\right\} & i = 3 \end{cases}$$

It is clear that φ^a satisfies CRM^b and RB^b for $b = f$ or $b = s$. Consider now the claims problem $(E, c) = (9, (1, 9, 10))$. Then, $\varphi^a(E, c) = (1, 3, 5)$, whereas $b_2(E, c) = b_3(E, c)$ for $b = f$ or $b = s$. Therefore, φ^a does not satisfy $ETEB^b$, hence neither does $CEBM^b$.

Example 2. *Let $n = 3$ and φ^* be defined by:*

$$\varphi_i^*(E,c) = \begin{cases} CEA(E,c) & \text{if } f_1(E,c) = f_2(E,c) = f_3(E,c) \\ \\ CEA(E,c) + (-x,-x,2x) & \text{if } f_1(E,c) = f_2(E,c) < f_3(E,c) \\ \\ CEA(E,c) + (-2x,x,x) & \text{if } f_1(E,c) < f_2(E,c) \end{cases}$$

It is clear that φ^* fulfills $ETEB^b$ and $CEBM^b$ for $b = f$ or $b = s$. Nevertheless, if we consider the problem $(E, c) = (12, (1, 9, 10))$, then $\varphi^*(E, c) = (0, 6, 6)$, hence RB^b and CRM^b are not satisfied.

Lemma 3. *If we consider the* min *lower bound, m,*

1. *$ETEB^m$ implies RB^m.*

2. *$CEBM^m$ implies CRM^m.*

Proof. Note that the guaranteed award provided by the min lower bound coincides for all agents:

$$(1)\ m_i(E,c) = \tfrac{c_1}{n} \leqslant \tfrac{E}{n}, \quad \text{or} \quad (2)\ m_i(E,c) = \tfrac{E}{n} \leqslant \tfrac{c_1}{n}.$$

Consider a claims rule φ satisfying $ETEB^m$. As the lower bound coincides for all agents, this axiom implies that agents receive the same award, or receive their claim in full. In both cases, $\varphi_i(E,c) \geqslant m_i(E,c)$ and then RB^m holds. Analogously, it is straightforward to prove that $CEBM^m$ implies CRM^m. □

Lemma 4. *If we consider the securement lower bound, s,*

1. *RB^s and PRI^s are independent.*
2. *$ETEB^s$ and PRI^s are independent.*

Proof.

1. Consider φ^a and the problem (E,c) introduced in Example 1, and the claims problem $(E',c) = (12,(1,9,10))$. Then, if we compare the securement lower bound of both problems, $s(E,c) = \left(\frac{1}{3},3,3\right)$ and $s(E',c) = \left(\frac{1}{3},3,\frac{10}{3}\right)$; so, $s_2(E,c) = s_2(E',c)$. Nevertheless, $\varphi^a(E,c) = (1,3,5)$ and $\varphi^a(E',c) = (1,4,7)$, contradicting PRI^s.

   On the other hand, CEL fulfills PRI^s and does not satisfy RB^s nor CRM^s.

2. Let $n = 3$ and consider CEA. It is clear that $ETEB^s$ and $CEBM^s$ are fulfilled. Now consider the problems $(E,c) = (3,(3,6,9))$ and $(E',c) = (6,(3,6,9))$, then $CEA(E,c) = (1,1,1)$, and $CEA(E',c) = (2,2,2)$. Note that $s(E,c) = (1,1,1)$, and $s(E',c) = (1,2,2)$, hence PRI^s is not satisfied.

   On the other hand, CEL fulfills PRI^s and does not satisfy $ETEB^s$ nor $CEBM^s$. ■

Proof of Theorem 1: CEA is the only rule satisfying RB^f and $ETEB^f$.

Proof. From Proposition 1 we know that CEA satisfies RB^f and $ETEB^f$.

Let φ satisfy RB^f and $ETEB^f$. For each $(E, c) \in \mathcal{C}$, as $E < \sum_{i=1}^{n} c_i \leqslant nc_n$, there is some $k \in N$ such that $E < nc_k$. Note that, by definition, the fair lower bound is the same for each agent whenever $E \leqslant nc_1$. Furthermore, this lower bound changes as $\frac{E}{n}$ increases. Henceforth, we use this fact to divide all the possible cases that cause variation in the agents' fair lower bound.

If $E < nc_1$, then $f_i(E, c) = \frac{E}{n} \leqslant c_i$, for each $i \in N$. By RB^f and *efficiency*, $\varphi_i(E, c) = \frac{E}{n} = CEA_i(E, c)$ for each $i \in N$.

Otherwise, there is some $k \in N$ such that $nc_{k-1} \leqslant E < nc_k$. For each $i \leqslant k-1$, $f_i(E, c) = c_i$, and for each $i \geqslant k$, $f_i(E, c) = \frac{E}{n}$. By RB^f and *claim-boundedness*, for each $i \leqslant k-1$, $\varphi_i(E, c) = c_i$. $ETEB^f$ and *efficiency* imply an equal sharing of $E' = E - (c_1 + c_2 + \ldots + c_{k-1})$, among agents $i = k, \ldots, n$, unless some of those agents get more than her claim.

If $\frac{E'}{n-(k-1)} > c_k$, then $ETEB^f$ and *claim-boundedness* imply $\varphi_k(E, c) = c_k$. Now, by $ETEB^f$, $\varphi_i(E, c) = \varphi_j(E, c)$, for each $i, j > k$, and *efficiency* imply $\varphi_i(E, c) = \frac{E - \sum_{i=1}^{k} c_i}{n-k}$ for each $i > k$, unless this amount is greater than some claims.

If $\frac{E''}{n-k} > c_{k+1}$, $E'' = E - (c_1 + c_2 + \ldots + c_k)$, $ETEB^f$ and *claim-boundedness* imply $\varphi_{k+1}(E, c) = c_{k+1}$ and the remainder must be distributed equally by $ETEB^f$ and *efficiency*, unless this amount is greater than some claims. This argument is repeated until no one gets more than their claim, and we observe that the result is $\varphi(E, c) = CEA(E, c)$. □

Proof of Theorem 2: CEA is the only rule satisfying CRM^f and $ETEB^f$.

Proof. By Proposition 1, CEA satisfies $ETEB^f$. In order to prove that it also fulfills CRM^f, let (E, c) and $(E', c) \in \mathcal{C}$ be such that $E' < E$. If for some $i \in N$ $CEA_i(E, c) < c_i$, then $\min\{c_i, \lambda\} = \lambda < c_i$, so $CEA_i(E', c) = \min\{c_i, \lambda'\} = \lambda' < c_i$, since $E' < E$. Therefore,

$$CEA_i(E,c) - CEA_i(E',c) = \lambda - \lambda', \qquad f_i(E,c) = \frac{E}{n}, \quad f_i(E',c) = \frac{E'}{n}.$$

From the definition of CEA,

$$\lambda = \frac{E - (c_1 + c_2 + \ldots + c_r)}{n - r} \qquad r = \max_k \{CEA_k(E,c) = c_k\},$$

$$\lambda' = \frac{E' - (c_1 + c_2 + \ldots + c_s)}{n - s} \qquad s = \max_k \{CEA_k(E',c) = c_k\}.$$

As $E' < E$, $s \leqslant r$ and

$$\lambda' \leqslant \frac{E' - (c_1 + c_2 + \ldots + c_r)}{n - r} \qquad \Rightarrow \qquad \lambda - \lambda' \geqslant \frac{E - E'}{n - r} \geqslant \frac{E - E'}{n}$$

Hence, CRM^f is fulfilled in this case. On the other hand, if $CEA_i(E,c) = c_i$, the axiom is obviously fulfilled.

Consider now a rule φ satisfying axioms $ETEB^f$ and CBM^f. From Lemma 1, φ fulfills RB^f, and Theorem 1 implies $\varphi = CEA$. □

Proof of Corollary 1: CEA is the only rule satisfying RB^f and $CEBM^f$.

Proof. By Proposition 1, CEA satisfies RB^f. In order to prove that it also fulfills $CEBM^f$, let (E,c) and $(E',c) \in \mathcal{C}$ be such that $E' < E$, and two agents $i, j \in N$ with $c_i \leqslant c_j$. We suppose that $f_i(E,c) - f_i(E',c) = f_j^l(E,c) - f_j^l(E',c)$. We distinguish several possible cases:

a) If $f_i(E,c) = \frac{E}{n}$, then $f_i(E',c) = \frac{E'}{n}$. In this case, either

   (i) $CEA_i(E,c) = CEA_j(E,c) = \lambda < c_i$, in which case $CEA_i(E',c) = CEA_j(E',c) = \lambda' < c_i$, since $E' < E$, and $CEA_i(E,c) - CEA_i(E',c) = CEA_j(E,c) - CEA_j(E',c) = \lambda - \lambda'$; or

   (ii) $CEA_i(E,c) = c_i \leqslant CEA_j(E,c)$.

(b) If $f_i(E,c) = c_i$, then $CEA_i(E,c) = c_i \leqslant CEA_j(E,c)$.

Hence CEA satisfies $CEBM^f$.

Consider now a rule φ satisfying axioms RB^f and $CEBM^f$. From Lemma 1, φ fulfills $ETEB^f$, so that Theorem 1 implies $\varphi = CEA$. □

Proof of Corollary 2: CEA is the only rule satisfying CRM^f and $CEBM^f$.

Proof. From Lemma 1, CBM^f implies RB^f, and $CEBM^f$ implies $ETEB^f$. Moreover, by Theorem 1 and Theorem 2 and Corollary 1, CEA fulfills the four axioms. So, this results comes straightforwardly.

□

Proof of Theorem 3: CEA is the only rule satisfying either $ETEB^m$ or $CEBM^m$.

Proof. 1. From Proposition 1 we know that CEA fulfills $ETEB^m$.

Now, consider a rule φ satisfying $ETEB^m$ and a claims problem $(E, c) \in \mathcal{C}$. As agents are ordered according to their claims, $m_i(E, c) = \min\left\{\frac{c_1}{n}, \frac{E}{n}\right\}$, the same for each $i \in N$. There are two possibilities:

1.1) If $E \leqslant c_1$, then $m_i(E, c) = \frac{E}{n}$. By $ETEB^m$ and *efficiency*, $\varphi_i(E, c) = \frac{E}{n} = CEA_i(E, c)$.

1.2) If $E > c_1$, then $m_i(E, c) = \frac{c_1}{n}$. By $ETEB^m$, all individuals receive the same amount λ unless they receive $c_i \leqslant \lambda$, and this coincides with $CEA(E, c)$.

Hence, φ coincides with the constrained awards rule.

2. First, we prove that CEA fulfills $CEBM^m$. Consider two claims problems $(E, c), (E', c) \in \mathcal{C}$, $E' < E$ and two agents $i, j \in N$, with $c_i \leqslant c_j$. As $CEA_i(E, c) = \min\{\lambda, c_i\}$, we have the following two possibilities:

   2.1) If $CEA_i(E, c) = c_i$, the condition is fulfilled.

   2.2) If $CEA_i(E, c) = \lambda$, then $CEA_j(E, c) = \lambda$ and, as $E' < E$, $CEA_i(E', c) = CEA_j(E', c) = \lambda' < \lambda$. Then,

$$CEA_i(E, c) - CEA_i(E', c) = CEA_j(E, c) - CEA_j(E', c)$$
$$= \lambda - \lambda'.$$

   Hence, CEA fulfills $CEBM^m$.

   Now, let φ satisfy $CEBM^m$. From Lemma 1 we know that $ETEB^m$ is fulfilled and then, as we have just proved, $\varphi = CEA$.

□

Proof of Theorem 4. *IE is the only rule in $\mathcal{C}_{IE}$ satisfying RB^s, $CEBM^s$ and PRI^s.*

Proof. To prove that IE satisfies the required axioms, let $(E,c),(E',c) \in \mathcal{C}_{IE}$ such that $E' < E$.

(RB^s) If $c_1 \geqslant E$, then $c_i \geqslant E$ and $IE_i(E,c) = \frac{E}{n} = s_i(E,c)$, for each $i \in N$.

Otherwise, if $c_1 < E$, $IE_1(E,c) = \frac{c_1}{n} = s_1(E,c)$. Moreover, for $i \geqslant 2$,

$$IE_i(E,c) = IE_{i-1}(E,c) + \frac{\min\{c_i, E\} - \min\{c_{i-1}, E\}}{n-(i-1)}. \quad (3.1)$$

If we assume $IE_i(E,c) \geqslant \frac{\min\{c_i,E\}}{n}$, from Equation (3.1) we obtain $IE_{i+1}(E,c) = IE_i(E,c) + \frac{\min\{c_{i+1},E\}-\min\{c_i,E\}}{n-i} \geqslant \frac{\min\{c_{i+1},E\}}{n} = s_{i+1}(E,c)$ and, by induction, RB^s is fulfilled.

($CEBM^s$) Let $i,j \in N$ be such that $c_i \leqslant c_j$ and $s_i(E,c) - s_i(E',c) = s_j(E,c) - s_j(E',c)$. It is easy to observe that only the two following possibilities for the values of the securement lower bound are compatible with the above conditions:

a) $s_i(E,c) = s_j(E,c) = \frac{E}{n}$, $s_i(E',c) = s_j(E',c) = \frac{E'}{n}$.

This case corresponds with $E' < E \leqslant c_i \leqslant c_j$, which implies that $IE_i(E,c) = IE_j(E,c)$ and $IE_i(E',c) = IE_j(E',c)$. Then, $CEBM^s$ is satisfied.

b) $s_i(E,c) = s_i(E',c) = \dfrac{c_i}{n}$, $s_j(E,c) = s_j(E',c) = \dfrac{c_j}{n}$.
This case corresponds with $c_i \leqslant c_j \leqslant E' < E$, which implies that $IE_i(E,c) = IE_j(E,c) = \frac{E}{n}$ and $IE_i(E',c) = IE_j(E',c) = \frac{E'}{n}$. Then, $CEBM^s$ is also satisfied.

(PRI^s) Let $i \in N$ such that $IE_i(E,c) > IE_i(E',c)$. We distinguish two cases:

a) If $E' < c_i$, then $s_i(E,c) = \min\left\{\frac{E}{n}, \frac{c_i}{n}\right\} > \frac{E'}{n} = s_i(E',c)$, and PRI^s is fulfilled.

b) If $c_i \leqslant E' < E$, then the definition of the *Ibn Ezra*'s rule implies $IE_i(E,c) = IE_i(E',c)$, a contradiction.

To prove the uniqueness let us consider $(E,c) \in \mathcal{C}_{IE}$. We distinguish several cases:

a) If $E \leqslant c_1$, then $s_i(E,c) = \frac{E}{n}$ for each $i \in N$. By RB^s and *efficiency*, $\varphi_i(E,c) = \frac{E}{n} = IE_i(E,c)$.

b) If $c_1 < E \leqslant c_2$, $s_1(E,c) = \frac{c_1}{n}$ and, for each $j \geqslant 2$, $s_j(E,c) = \frac{E}{n}$. By RB^s, $\varphi_1(E,c) \geqslant \frac{c_1}{n}$, and $\varphi_j(E,c) \geqslant \frac{E}{n}$. Now, we consider the claims problem (E',c), with $E' = c_1$. Then, $s_j(E',c) = \frac{c_1}{n}$, for each $j \in N$, and this problem is in case a), so $\varphi_i(E',c) = \frac{c_1}{n} = IE_i(E',c)$. By $CEBM^s$ and PRI^s, only agents j, who have increased their lower bound, should receive an equal increase of their allocation, i.e., $\varphi_1(E, c) = \frac{c_1}{n}$, and $\varphi_j(E, c) = \frac{c_1}{n} + \frac{E'-c_1}{n-1}$, that coincides with $IE(E, c)$.

c) If $c_i < E \leqslant c_{i+1}$, we repeat the previous argument, by considering the claims problem (E', c), with $E' = c_i$.

Hence, $\varphi(E, c) = IE(E, c)$. □

Proof of Corollary 3. IE is the only rule in $\mathcal{C}_{IE}$ satisfying $CEBM^s$, PRI^s and CRM^s.

Proof. Note that we only need to prove that IE fulfills CRM^s. Let $(E, c), (E', c) \in \mathcal{C}_{IE}$ such that $E' < E$. We need to prove that

$$IE_i(E, c) - IE_i(E'c) \geqslant \frac{1}{n}(\min\{c_i, E\} - \min\{c_i, E'\}) \tag{3.2}$$

For $i \in N$, the following cases are possible:

1. $c_i \leqslant E' < E$
2. $c_{i-1} \leqslant E' < c_i \leqslant E$
3. $E' < c_{i-1} \leqslant c_i \leqslant E$
4. $c_{i-1} \leqslant E' < E < c_i$
5. $E' < c_{i-1} \leqslant E < c_i$
6. $E' < E < c_{i-1} \leqslant c_i$

In any case, it is easy to check that Equation (3.2) holds. Hence, CRM^s is fulfilled. □

Chapter 4

Sequential claims problems

Overview. A claims problem is a situation where a group of agents has to distribute an insufficient resource to satisfy all their requests. The current paper analyzes this kind of situations from a sequential point of view, i.e., it considers that agents are linearly ordered. Two applications of sequential claims problems are sharing the water of an international river (Ansink and Weikard, 2012) and sharing rewards due to expedition in projects (Estévez-Fernández, 2012). Within this context, we propose three mechanisms to generalize well-known rules to our setting: the upward, the downward, and the two-step mechanisms. Besides, we analyze the constrained equal awards rule through some of the main well-known axioms used to characterize it.

Keywords: Claims problems; sequential claims problems; constrained equal awards

4.1 Introduction

It is noteworthy that this chapter aims to present the work-in-progress ideas of this PhD thesis. Specifically, this chapter focuses on analyzing the sequential claims problems, and proposes alternative solutions in cases where there are conflicts of interests.

The sequential claims problem can be studied from a claims problem perspective (O'Neill, 1982; Young, 1987; Aumann, 1989; Moulin, 2002), considering that, generally, the agents' demands are greater than the divisible resource (endowment), which has to be distributed. In this framework, several solutions are proposed, usually called rules, in order to get a suitable way to distribute the available resource.

In the literature, many authors have analyzed problems corresponding to sequential claims problems in different situations. For instance, in the river sharing problem (Parrachino, 2006; Carraro et al., 2007; Ambec and Ehlers, 2008; Ansink and Weikard, 2012, among others), there is a number of agents located along a river, who have a demand on a specific part of the the river's water. Hence, some agents may have the right to demand on the same part of the river. Note that the water of the river is divided into parts by territory, so that there are several sub-problems within the general problem. In this sense, allocating a resource over which property rights are not well defined is notoriously problematic, since efficiency often requires that upstream agents limit their own consumption so as to increase that of downstream agents whose marginal benefits are higher.

Another example is the sharing of penalties and rewards in projects (Bergantiños and Sánchez, 2002; Brânzei et al., 2002; Estévez-Fernández, 2012, among others). In this case, there exists a general project which depends on several intermediate steps to be completed. These steps are connected to each other, since for a step to be completed, it is necessary to end the previous steps. In a project, there are groups of interconnected activities that need to be carried out. These activities can be divided into paths. A path specifies a sequence of activities that have to be performed one after another.

The duration of a path is the sum of the duration of its activities. A path is critical if its duration is the highest of all the path duration, it is second critical if its duration is the second highest duration, and so on. In order to expedite a project, activities in all critical paths need to be expedited. If we want to further expedite the project, activities in second critical paths also need to be expedited, and so on. When sharing the rewards obtained from the expedition of a project, activities in critical paths can claim over the total reward, while activities in second critical paths can claim over a smaller part of the total reward.

This chapter studies this kind of problems. In doing so, we propose a new definition of sequential claims problems and their associated rules. Furthermore, we provide three different methods to solve sequential problems and we start analyzing the four main well-known rules in claims problems in our set-up.

The remainder of the paper is organized as follows. Section 4.2 provides the definition of basic concepts in claims problems. Section 4.3 defines a sequential claims problems model. Section 4.4 introduces the upward mechanism and focuses on the upward constrained equal awards rule. Finally, Section 4.5 presents open questions and future research.

4.2 Preliminaries

This section gives a brief survey of existing concepts in the literature of claims problems.

Throughout the chapter we consider a set of agents $N = \{1, 2, ..., n\}$, such that each agent has a claim $c_i \in \mathbb{R}_+$ on an infinitely divisible resource, the endowment, $E \in \mathbb{R}_+$. Let $c \equiv (c_i)_{i \in N}$ be the claim vector. A **claims problem** appears whenever the endowment is not enough to satisfy the aggregate claim. Without loss of generality, we assume that the agents are indexed according to their claims, $c_1 \leqslant c_2 \leqslant \ldots \leqslant c_n$. The pair $(E, c) \in \mathbb{R}_+ \times \mathbb{R}^n_+$ represents the claims problem, and $\mathcal{C}$ denotes the set of all claims problems.

A **rule** is a single-valued function $\varphi : \mathcal{C} \to \mathbb{R}^n_+$ such that for each problem $(E, c) \in \mathcal{C}$, and each $i \in N$, $0 \leqslant \varphi_i(E, c) \leqslant c_i$ (**non-negativity and claim-boundedness**), and $\sum\limits_{i=1}^{n} \varphi_i(E, c) = E$ (**efficiency**).

We now briefly introduce and analyze the behavior of some commonly used rules: the *proportional rule*, the *constrained equal awards rule*, the *constrained equal losses rule* and the *Talmud rule*.

The **proportional (P)** rule is the most popular one since it divides the available budget proportionally to the claim of the agents.
For each (E, c) and each agent i, $P_i(E, c) \equiv \lambda c_i$, where $\lambda = \dfrac{E}{\sum_{i=1}^{n} c_i}$.

The **constrained equal awards (CEA)** rule (Maimonides, 1135, 1204) equalizes the amount each agent receives, such that no agent receives more than her demand.
For each (E, c) and each agent i, $CEA_i(E, c) \equiv \min\{c_i, \lambda\}$, where λ is chosen so that $\sum_{i=1}^{n} \min\{c_i, \lambda\} = E$.

The **constrained equal losses (CEL)** rule (Maimonides, 1135, 1204; Aumann and Maschler, 1985) tries to analyze the problem from the point of view of losses (what the regions do not receive with respect to their claims), hence it proposes equalizing losses, such that no agent receives a negative amount.
For each (E, c) and each agent i, $CEL_i(E, c) \equiv \max\{0, c_i - \lambda\}$, where λ is chosen so that $\sum_{i=1}^{n} \max\{0, c_i - \lambda\} = E$.

The **Talmud (T)** rule (Aumann and Maschler, 1985), is a combination of the CEA and the CEL rules, which takes into account half of the aggregate claim C as a reference. If C is lower than the

available resource, then the CEA rule is applied over the half-claims. Otherwise, each agent receives half of her claim and the CEL rule is applied in order to distribute the remaining budget with respect to the remaining claims (the other half).
For each (E, c), $T(E, c) \equiv CEA(E, \frac{1}{2}c)$ if $E \leqslant \frac{1}{2}C$, or $\frac{1}{2}c + CEL(E - \frac{1}{2}C, \frac{1}{2}c)$ if $E \geqslant \frac{1}{2}C$.

4.3 The sequential claims problem

Next, we introduce the sequential problems and the definition of a rule associated to them.
We consider the set of agents N, $c \in R_+^N$ is the vector of claims, for each $i \in N$, and $E \in R_+$ is the endowment that has to be shared among the claimants. Furthermore, there is an exogenous partition of the set of agents $N_1, \ldots, N_m$.

For each $i \in N$, we denote by $l(i) \in \{1, \ldots, m\}$ the index with $i \in N_{l(i)}$.

Definition 1. *A sequential claims problem is a tuple* $(N_1, \ldots, N_m, E_1, \ldots, E_m, c)$ *as described above satisfying* $\sum_{k=1}^{l} \sum_{i \in N_k} c_i \geqslant \sum_{k=1}^{l} E_k$ *for each* $l = \{1, \ldots, m\}$.

Let $\mathcal{SC}$ denote the set of sequential claims problems. For the sake of exposition, since throughout the chapter we consider that the set of claimants is fixed, we denote $(E_1, \ldots, E_m, c) \in \mathcal{SC}$ instead of

$(N_1, \ldots, N_m, E_1, \ldots, E_1, c) \in \mathcal{SC}$.

In this context, a rule is defined as follows.

Definition 2. *A sequential rule is a function that associates with each* $(E_1, \ldots, E_1, c) \in \mathcal{SC}$ *a vector* $x \in R^N$ *satisfying*

$$
\begin{aligned}
&0 \leqslant x \leqslant c, \\
&\sum_{i \in N} x_i = \sum_{k=1}^{m} E_k, \qquad \textit{and} \\
&\sum_{k=1}^{l} \sum_{i \in N_k} x_i \geqslant E_l \quad \textit{for each } l = 1, \ldots, m.
\end{aligned}
$$

4.4 Upward mechanism

Given a rule φ, the upward mechanism generalizes φ as follows.

Definition 3. *For each* $(E_1, \ldots, E_m, c) \in \mathcal{SC}$,

$$
\varphi^{up}(E_1, \ldots, E_m, c) = \sum_{l=1}^{m} \varphi \left(N_1 \cup \ldots \cup N_m, E_l, c^l - \sum_{\bar{l}=0}^{l-1} x^{\bar{l}} \right)
$$

where $x^0, \ldots, x^{m-1}$ *are recursively defined by*

$$
\begin{aligned}
x^0 &= (0, \ldots, 0) \\
x^l &= \varphi \left(N_1 \cup \ldots \cup N_m, E_l, c^l - \sum_{\bar{l}=0}^{l-1} x^{\bar{l}} \right) \quad \textit{for } l = 1, \ldots, m-1.
\end{aligned}
$$

The upward mechanism first allocates E_1 among the members of N_1 using φ. Then, the allocation of E_2 among N_1 and N_2 is carried out after updating the claims of N_1, and so on.

Example 3. *For instance, we have three groups* $N_1 = \{1, 2\}$, $N_2 = \{3\}$, $N_3 = \{4, 5\}$, *where agent 1 and 2 are in group 1, 3 is in the group 2, and 4 and 5 are in group 3;* $c = (3, 5, 5, 1, 2)$; $E_1 = 3, E_2 = 5, E_3 = 4$. *Let* $x^0 = (0, 0, 0, 0, 0)$.

a) Let us apply the CEA *rule to distribute the endowment.*

- $x^1 = CEA(E_1, c^1 - x^0) =$
  $CEA(3, (3, 5, 0, 0, 0)) = (1.50, 1.50, 0, 0, 0)$.
- $x^2 = CEA(E_2, c^2 - x^1) =$
  $CEA(5, (1.50, 3.50, 5, 0, 0)) = (1.50, 1.75, 1.75, 0, 0)$.
- $x^3 = CEA(E_3, c^3 - x^1 - x^2) =$
  $CEA(4, (0, 1.75, 3.25, 1, 2)) = (0, 1, 1, 1, 1)$.

Then, $CEA^{up}(E_1, E_2, E_3, c) = (x^1 + x^2 + x^3) =$

$$(3, 4.25, 2.75, 1, 1).$$

b) Let us apply the P *rule to distribute the endowment.*

- $x^1 = P(E_1, c^1 - x^0) =$
  $P(3, (3, 5, 0, 0, 0)) = \frac{3}{8}(3, 5, 0, 0, 0) = (1.13, 1.87, 0, 0, 0)$.
- $x^2 = P(E_2, c^2 - x^1) =$
  $P(5, (1.87, 3.12, 5, 0, 0)) = \frac{5}{10}(1.87, 3.13, 5, 0, 0) =$
  $(0.94, 1.56, 2.5, 0, 0)$.
- $x^3 = P(E_3, c^3 - x^1 - x^2) =$
  $P(4, (0.93, 1.56, 2.5, 1, 2)) = \frac{4}{8}(0.93, 1.56, 2.5, 1, 2) =$

$$(0.47, 0.78, 1.25, 0.50, 1).$$

Then, $P^{up}(E_1, E_2, E_3, c) = (x^1 + x^2 + x^3) =$

$$(2.54, 4.21, 3.75, 0.5, 1).$$

c) Let us apply the CEL *rule to distribute the endowment.*

- $x^1 = CEL(E_1, c^1 - x^0) =$
  $CEL(3, (3, 5, 0, 0, 0)) = (0.50, 2.50, 0, 0, 0).$
- $x^2 = CEL(E_2, c^2 - x^1) =$
  $CEL(5, (2.50, 2.50, 5, 0, 0)) = (0.83, 0.83, 3.34, 0, 0).$
- $x^3 = CEL(E_3, c^3 - x^1 - x^2)$
  $CEL(4, (1.67, 1.67, 1.66, 1, 2)) =$
  $(0.87, 0.87, 0.86, 0.20, 1.20).$

Then, $CEL^{up}(E_1, E_2, E_3, c) = (x^1 + x^2 + x^3) =$

$$(2.20, 4.20, 4.20, 0.20, 1.20).$$

d) Let us apply the T *rule to distribute the endowment.*

- $x^1 = T(E_1, c^1 - x^0) =$
  $T(3, (3, 5, 0, 0, 0)) = (1.50, 1.50, 0, 0, 0).$
- $x^2 = T(E_2, c^2 - x^1) =$
  $T(5, (1.50, 3.50, 5, 0, 0)) = (0.75, 1.75, 2.50, 0, 0).$
- $x^3 = T(E_3, c^3 - x^1 - x^2) =$
  $T(4, (0.75, 1.75, 2.5, 1, 2)) = (0.375, 0.875, 1.25, 0.50, 1).$

Then, $T^{up}(E_1, E_2, E_3, c) = (x^1 + x^2 + x^3) =$

$$(2.625, 4.125, 3.75, 0.50, 1).$$

4.4.1 The upward constrained equal awards rule

Two of the most important rules in the literature are the CEA and the CEL rules (Maimonides, 12th century). These rules propose an egalitarian distribution of the awards and losses, respectively, among the claimants, given that no agent should neither get more, nor lose more than her claim. Specifically from among all the aforementioned rules, we focus on the CEA rule since we consider it a relevant rule in terms of equality.

Given the formal definition of this rule in claims problems, we extend it to the sequential claims problems.

For the sake of studying the behavior of the rule presented, we aim to characterize it by generalizing some of the well-known characterizations of the CEA rule to our new setting. One of them is based on three commonly used axioms in the field of claims problem.

Equal treatment of equals indicates that if two agents have the same claim, they should receive the same award.
For each $(E, c) \in \mathcal{C}$, and each $\{i, j\} \subseteq N$, if $c_i = c_j$, then $\varphi_i(E, c) = \varphi_j(E, c)$.

Invariance under claims truncation (Curiel et al., 1987; Dagan and Volij, 1993) proposes that if the claims of some of the agents are truncated at the endowment, the obtained vector of awards should not be affected.
For each $(E, c) \in \mathcal{C}$, if $c_i = c_j$, then $\varphi(E, c) = \varphi(E, t(E, c))$ where t is the vector of truncated claims: $t_i = \min\{c_i, E\}$, for each $i \in N$.

Composition up (Young, 1988) states that if there is an increment in the endowment, the vector of awards obtained should be the same when (i) applying the rule directly to the new endowment, or when (ii) by dividing the initial endowment, and afterward adjusting the claims by subtracting this first assignment and dividing the remaining endowment.
For each $(E, c) \in \mathcal{C}$, and each $E' > E$, such that $\sum c_i \geqslant E'$, we have $\varphi(E', c) = \varphi(E, c) + \varphi(E' - E, c - \varphi(E, c))$.

Theorem 5 (Dagan, 1996)**.** *The constrained equal awards rule is the only rule that satisfies equal treatment of equals, invariance under claims truncation and composition up.*

Next, we redefine the axioms introduced above for the sequential claims problems.
Equal treatment of equals: For each $(E_1, \ldots, E_m, c) \in \mathcal{SC}$, if $i, j \in N_r$, $r \in \{1, \ldots, m\}$, with $c_i = c_j$, then $\varphi_i(E, c) = \varphi_j(E, c)$.

Invariance under claims truncation: For each $(E_1, \ldots, E_m, c) \in \mathcal{SC}$, $\varphi(E_1, \ldots, E_m, c) = \varphi(E_1, \ldots, E_m, \bar{c})$ with $\bar{c} \in \mathbb{R}^N$ defined by $\bar{c}_i = \min\left\{c_i, \sum_{l=r}^{m} E_l\right\}$ for each $i \in N_r, r \in \{1, \ldots, m\}$.

Composition up: For each $(E_1, \ldots, E_r, \ldots, E_m, c)$, and $(E_1, \ldots, \bar{E}_r, \ldots, E_m, c) \in \mathcal{SC}$, with $\bar{E}_r \geqslant E_r$, $\varphi(E_1, \ldots, \bar{E}_r, \ldots, E_m, c) = \varphi(E_1, \ldots, E_{r-1}, E_r, 0, \ldots, 0, c) + \varphi(0, \ldots, 0, \bar{E}_r - E_r, E_{r+1}, \ldots, E_m, c - \varphi(E_1, \ldots, E_r, 0, \ldots, 0, c))$.

Henceforth, the study of the CEA^{up} through the axioms used previously, in the context of sequential claims problems is still on going. Our conjecture is: "CEA^{up} is the only sequential rule that satisfies equal treatment of equals, invariance under claims truncation and composition up".

4.5 Final Remarks

It is noteworthy that this chapter studies other relevant rules used in the literature, such as the proportional rule, the constrained equal losses rule, and the Talmud rule. Furthermore, the study of additional axioms and their possible characterizations arises in a natural way.

Finally, we also consider alternative ways to distribute the endowment: The two-step and downward mechanisms. Formally,

Definition 4. *Let φ be a rule. We define the **two-step generalization** of φ, φ^{ts}, as follows:*

For $(E_1, \ldots, E_m, c) \in \mathcal{SC}$*,*

(1) Define the claims problem on the groups: (M, E, d)*, where*

$$M = \{1, \ldots, m\}$$
$$E = E_1 + \ldots + E_m$$
$$d \in \mathbb{R}^M_+, \quad d_l = \min\left\{\sum_{i \in N_l} c_i, \sum_{\bar{l}=l}^{m} E_{\bar{l}}\right\} \text{ for each } l \in M.$$

Then, $\varphi(M, E, d)$ *allocates the estates* $E_1, \ldots, E_m$ *among the groups.*

(2) We use φ *to allocate* $\varphi_l(M, E, d) = E_l$ *among the members of* N_l*:*

$$\varphi^{ts}(E_1, \ldots, E_m, c) = (\varphi(N_l, E_l, c_l))_{l \in M}.$$

The two-step mechanism first recalculate the groups' demand. Then, with the new demand vector, the general E is allocate among groups. Afterwards, each E_m is allocate by each group.

Example 4. *For instance, we have three groups* $N_1 = \{1, 2\}$*,* $N_2 = \{3\}$*,* $N_3 = \{4, 5\}$*, where agent 1 and 2 are in group 1, 3 is in group 2, and 4 and 5 are in group 3.* $c = (3, 5, 5, 1, 2)$*;* $E_1 = 3, E_2 = 5, E_3 = 4$*. Let* $x^0 = (0, 0, 0, 0, 0)$ *First step: Applying the CEA rule by group*

1. *Let us calculate the new demand vector d.*

   - $d_1 = \min\{c_1 + c_2, E_1 + E_2 + E_3\} = \min\{8, 12\} = 8.$

- $d_2 = \min\{c_3, E_2 + E_3\} = \min\{5, 9\} = 5.$

- $d_3 = \min\{c_4 + c_5, E_3\} = \min\{3, 4\} = 3.$

2. Next, we apply the rules.

a) Two-step constrained equal awards rule.

a.1) First step: Applying the CEA rule by group: $E = 12$; $d(8, 5, 3)$.

- $CEA(E, d_1, d_2, d_3) = CEA(12(8, 5, 3)) = (4.5, 4.5, 3).$

a.2) Second step: Applying the CEA rule within each group:

- $CEA(N_1, CEA_1(E, (d_1, d_2, d_3), c_{N_1}) = CEA(\{1, 2\}, 4.5, (3, 5)) = (2.25, 2.25).$

- $CEA(N_2, CEA_2(E, (d_1, d_2, d_3), c_{N_2}) = CEA(\{3\}, 4.5, (5)) = (4.50).$

- $CEA(N_3, CEA_3(E, (d_1, d_2, d_3), c_{N_3}) = CEA(\{4, 5\}, 3, (1, 2)) = (1, 2).$

Then, $CEA^{ts}(E_1, E_2, E_3, c) = (x^1 + x^2 + x^3) = (2.25, 2.25, 4.50, 1, 2).$

b) The two-step proportional rule.

b.1) First step: Applying the P rule by group: $E = 12;\ d(8, 5, 3)$

- $P(E, d_1, d_2, d_3) = P(12, (8, 5, 3) = (6, 3.75, 2.25).$

b.2) Second step: Applying the P rule within each group:

- $P(N_1, P_1(E, (d_1, d_2, d_3), c_{N_1}) =$
  $P(\{1, 2\}, 6, (3, 5)) = (2.25, 3.75).$
- $P(N_2, P_2(E, (d_1, d_2, d_3), c_{N_2}) =$
  $P(\{3\}, 3.75, (5)) = (3.75).$
- $P(N_3, P_3(E, (d_1, d_2, d_3), c_{N_3}) =$
  $P(\{4, 5\}, 2.25, (1, 2)) = (0.75, 1.50).$

Then, $P^{ts}(E_1, E_2, E_3, c) = (x^1 + x^2 + x^3) =$

$$(2.25, 3.75, 3.75, 0.75, 1.50).$$

c) The two-step constrained equal losses rule.

c.1) First step: applying the CEL *rule by group:* $E = 12$; $d(8, 5, 3)$.

- $CEL(E, d_1, d_2, d_3) = CEL(12(8, 5, 3)) =$
  $(6.67, 3.67, 1.67).$

c.2) Second step: Applying the CEL *rule within each group:*

- $CEL(N_1, CEL_1(E, (d_1, d_2, d_3), c_{N_1}) =$
  $CEL(\{1, 2\}, 6.67, (3, 5)) = (2.34, 4.34).$
- $CEL(N_2, CEL_2(E, (d_1, d_2, d_3), c_{N_2}) =$
  $CEL(\{3\}, 3.67, (5)) = (3.67).$
- $CEL(N_3, CEL_3(E, (d_1, d_2, d_3), c_{N_3}) =$
  $CEL(\{4, 5\}, 1.67, (1, 2)) = (0.34, 1.34).$

Then, $CEL^{ts}(E_1, E_2, E_3, c) = (x^1 + x^2 + x^3) =$

$$(2.34, 4.34, 3.67, 0.34, 1.34).$$

d) The two-step Talmud rule.

c.1) First step: Applying the T *rule by group:* $E = 12;\ d(8, 5, 3)$.

- $T(E, d_1, d_2, d_3) = T(12(8, 5, 3)) = (6.67, 3.67, 1.67)$.

c.2) Second step: Applying the T *rule within each group:*

- $T(N_1, T_1(E, (d_1, d_2, d_3), c_{N_1}) =$
  $T(\{1, 2\}, 6.67, (3, 5)) = (2.34, 4.34)$.
- $T(N_2, T_2(E, (d_1, d_2, d_3), c_{N_2}) =$
  $T(\{3\}, 3.67, (5)) = (3.67)$.
- $T(N_3, T_3(E, (d_1, d_2, d_3), c_{N_3}) =$
  $T(\{4, 5\}, 1.67, (1, 2)) = (0.50, 1.17)$.

Then, $T^{ts}(E_1, E_2, E_3, c) = (x^1 + x^2 + x^3) =$

$$(2.34, 4.34, 3.67, 0.50, 1.17).$$

Definition 5. *Let φ be a rule. We define the* ***downward generalization*** *of φ, φ^{dw}, recursively as follows: For $(E_1, \ldots, E_m, c) \in \mathcal{SC}$, let $y_0^{m+1} = 0$ and for $l = 1, \ldots, m-1$,*

$$d^{m-l+1} \in \mathbb{R}_+^{N_{m-l+1} \cup \{0\}} \quad with \quad d^{m-l+1}_{N_{m-l+1}} = c_{N_{m-l+1}},$$

with $d_0^{m-l+1} = \sum_{i \in N_1 \cup \ldots \cup N_{m-l}} c_i - \sum_{\bar{l}=1}^{m-l} E_{\bar{l}}\,;$

$$y^{m-l+1} = \varphi(N_{m-l+1} \cup \{0\}, E_{m-l+1} + y_0^{m-l+2}, d^{m-l+1})$$

and for $l = m$,

$$d^1 \in \mathbb{R}_+^{N_1} \quad with \quad d^1 = c_{N_1}, \quad y^1 = \varphi(N_1, E_1 + y_0^2, d^1).$$

Then, $\varphi^{dw} = (E_1, \ldots, E_m, c) = (y^1_{N_1}, \ldots, y^m_{N_m})$.

The downward mechanism first allocates the last endowment E_m using a specific rule taking into account that the claims are organized as follows: those agents that claim only E_m are considered individually (N_m), and the rest of the agents $(N_1, N_2, \ldots, N_{m-1})$ are considered jointly. In doing so, it aggregates the claims of the agents in $N_1, N_2, \ldots, N_{m-1}$ minus the endowments they ask for, except E_m, i.e, E_1, E_2, E_{m-1}. Afterwards, E_{m-1} plus the resources allocated to N_{m-1} (y_0^1) are assigned among the agents individually N_{m-1} and the rest of the agents $(N_1, N_2, \ldots, N_{m-2})$ are considered jointly. Thus, it aggregates the claims of the agents in $N_1, N_2, \ldots, N_{m-2}$ minus the endowments they ask for, except E_{m-1}, i.e, E_1, E_2, E_{m-2}. Then, this method is applied recursively.

Example 5. *For instance, we have three groups* $N_1 = \{1, 2\}$, $N_2 = \{3\}$, $N_3 = \{4, 5\}$, *where agent 1 and 2 are in group 1, 3 is in group 2, and 4 and 5 are in group 3.* $c = (3, 5, 5, 1, 2)$; $E_1 = 3, E_2 = 5, E_3 = 4$. *Let* $y_0^{m+1} = (0, 0, 0, 0, 0)$.

a) Let us apply the CEA *rule to distribute the endowment.*

- $y^1 = CEA(\{0, 4, 5\}, E_3, (c_1 + c_2 + c_3 - E_1 - E_2, c_4, c_5))$
  $CEA(\{0, 4, 5\}, 4, (3 + 5 + 5 - 3 - 5, 1, 2)) =$
  $CEA(\{0, 4, 5\}, 4, (5, 1, 2)) = (1.5, 1, 1.5)$.

- $y^2 = CEA(\{0, 3\}, E_2 + y_0^1, (c_1 + c_2 - E_1, c_3))$
  $CEA(\{0, 3\}, 5 + 1.5, (3 + 5 - 3, 5)) =$
  $CEA(\{0, 3\}, 6.5, (5, 5)) = (3.25, 3.25)$.

- $y^3 = CEA(\{1, 2\}, E_1 + y_0^2, (c_1, c_2))$
  $CEA(\{1, 2\}, 3 + 3.25, (3, 5)) =$
  $CEA(\{1, 2\}, 6.25, (3, 5)) = (3, 3.25)$.

Then, $CEA^{dw}(E_1, E_2, E_3, c) = (y^1 + y^2 + y^3) =$

$$(3, 3.25, 3.25, 1, 1.50).$$

b) Let us apply the P *rule to distribute the endowment.*

- $y^1 = P(\{0, 4, 5\}, E_3, (c_1 + c_2 + c_3 - E_1 - E_2, c_4, c_5))$
  $P(\{0, 4, 5\}, 4, (3 + 5 + 5 - 3 - 5, 1, 2)) =$
  $P(\{0, 4, 5\}, 4, (5, 1, 2)) = (2.50, 0.50, 1)$.

- $y^2 = P(\{0,3\}, E_2 + y_0^1, (c_1 + c_2 - E_1, c_3))$
  $P(\{0,3\}, 5 + 2.50, (3 + 5 - 3, 5)) =$
  $P(\{0,3\}, 7.50, (5,5)) = (3.75, 3.75).$

- $y^3 = P(\{1,2\}, E_1 + y_0^2, (c_1, c_2))$
  $P(\{1,2\}, 3 + 3.75, (3,5)) =$
  $P(\{1,2\}, 6.75, (3,5)) = (2.34, 4.22).$

Then, $P^{dw}(E_1, E_2, E_3, c) = (y^1 + y^2 + y^3) =$

$$(2.53, 4.22, 3.75, 0.50, 1).$$

c) Let us apply the CEL *rule to distribute the endowment.*

- $y^1 = CEL(\{0,4,5\}, E_3, (c_1 + c_2 + c_3 - E_1 - E_2, c_4, c_5))$
  $CEL(\{0,4,5\}, 4, (3 + 5 + 5 - 3 - 5, 1, 2)) =$
  $CEL(\{0,4,5\}, 4, (5,1,2)) = (3.50, 0, 0.50).$

- $y^2 = CEL(\{0,3\}, E_2 + y_0^1, (c_1 + c_2 - E_1, c_3))$
  $CEL(\{0,3\}, 5 + 3.50, (3 + 5 - 3, 5)) =$
  $CEL(\{0,3\}, 8.50, (5,5)) = (4.25, 4.25).$

- $y^3 = CEL(\{1,2\}, E_1 + y_0^2, (c_1, c_2))$
  $CEL(\{1,2\}, 3 + 4.25, (3,5)) =$
  $CEL(\{1,2\}, 7.25, (3,5)) = (2.625, 4.625).$

Then, $CEL^{dw}(E_1, E_2, E_3, c) = (y^1 + y^2 + y^3) =$

$$(2.625, 4.625, 4.25, 0, 0.50).$$

c) Let us apply the T rule to distribute the endowment.

- $y^1 = T(\{0,4,5\}, E_3, (c_1 + c_2 + c_3 - E_1 - E_2, c_4, c_5))$
  $T(\{0,4,5\}, 4, (3+5+5-3-5, 1, 2)) =$
  $T(\{0,4,5\}, 4, (5,1,2)) = (2.50, 0.50, 1).$
- $y^2 = T(\{0,3\}, E_2 + y^1_0, (c_1 + c_2 - E_1, c_3))$
  $T(\{0,3\}, 5 + 2.50, (3+5-3, 5)) =$
  $T(\{0,3\}, 7.50, (5,5)) = (3.75, 3.75).$
- $y^3 = T(\{1,2\}, E_1 + y^2_0, (c_1, c_2))$
  $T(\{1,2\}, 3 + 3.75, (3,5)) =$
  $T(\{1,2\}, 6.75, (3,5)) = (2.375, 4.375).$

Then, $T^{dw}(E_1, E_2, E_3, c) = (y^1 + y^2 + y^3) =$

$$(2.375, 4.375, 3.75, 0.50, 1).$$

Therefore, the comparison among the three aforementioned mechanisms is an ongoing issue.

Bibliography

Aaberge, R., Brandolini, A., 2015. Multidimensional poverty and inequality. In: Handbook of income distribution. Vol. 2. Elsevier, pp. 141–216.

Alcalde, J., Marco, M. d. C., Silva, J. A., 2005. Bankruptcy games and the Ibn Ezra's proposal. Economic Theory 26 (1), 103–114.

Ambec, S., Ehlers, L., 2008. Sharing a river among satiable agents. Games and Economic Behavior 64 (1), 35–50.

Ansink, E., Weikard, H.-P., 2012. Sequential sharing rules for river sharing problems. Social Choice and Welfare 38 (2), 187–210.

Arrow, K. J., 1963. Uncertainty and the welfare economics of medical care. The American economic review 53 (5), 941–973.

Atkinson, A. B., 1970. On the measurement of inequality. Journal of economic theory 2 (3), 244–263.

Aumann, R. J., 1989. Lectures on Game Theory. Underground Classics in Economics, Westview Press, Boulder, CO.

Aumann, R. J., Maschler, M., 1985. Game theoretic analysis of a bankruptcy from the Talmud. Journal of Economic Theory 36, 195–213.

Ayala, L., Triguero, Á., 2017. Economic downturns, endogenous government policy and welfare caseloads. Hacienda Pública Española (220), 107–136.

Bergantiños, G., Sánchez, E., 2002. How to distribute costs associated with a delayed project. Annals of Operations Research 109 (1-4), 159–174.

Black, D., 1976. Partial justification of the borda count. Public Choice 28 (1), 1–15.

Bosmans, K., Lauwers, L., 2011a. Lorenz comparisons of nine rules for the adjudication of conflicting claims. International Journal of Game Theory 40 (4), 791–807.

Bosmans, K., Lauwers, L., 2011b. Lorenz comparisons of nine rules for the adjudication of conflicting claims. International Journal of Game Theory Forthcoming.

Bouayad-Agha, S., Turpin, N., Védrine, L., 2013. Fostering the development of european regions: A spatial dynamic panel data analysis of the impact of cohesion policy. Regional Studies 47 (9), 1573–1593.

Brânzei, R., Ferrari, G., Fragnelli, V., Tijs, S., 2002. Two approaches to the problem of sharing delay costs in joint projects. Annals of Operations Research 109 (1-4), 359–374.

Carraro, C., Marchiori, C., Sgobbi, A., 2007. Negotiating on water: insights from non-cooperative bargaining theory. Environment and Development Economics 12 (2), 329–349.

Casas-Méndez, B., Fragnelli, V., García-Jurado, I., 2011. Weighted bankruptcy rules and the museum pass problem. European Journal of Operational Research 215 (1), 161–168.

Cerno, L., López, C. P., Arcega, E. S., 2017. Determinantes de la satisfacción de los españoles con las prestaciones y servicios públicos: un enfoque de sociología tributaria con microdatos. Hacienda Pública Española (220), 57–87.

Curiel, J., Maschler, M., Tijs, S., 1987. Bankruptcy games. Zeitschrift für Operations Research 31, A143–A159.

Dagan, N., 1996. New characterizations of old bankruptcy rules. Social Choice and Welfare 13 (1), 51–59.

Dagan, N., Serrano, R., Volij, O., 1997. A non-cooperative view of consistent bankruptcy rules. Games and Economic Behavior 18, 55 – 72.

Dagan, N., Volij, O., 1993. The bankruptcy problem: a cooperative bargaining approach. Mathematical Social Sciences 26, 287 – 287.

Dall'Erba, S., Fang, F., 2017. Meta-analysis of the impact of european union structural funds on regional growth. Regional Studies 51 (6), 822–832.

Dinar, A., Howitt, R. E., 1997. Mechanisms for allocation of environmental control cost: empirical tests of acceptability and stability. Journal of Environmental Management 49 (2), 183–203.

Dominguez, D., 2006. Lower bounds and recursive methods for the problem of adjudicating conflicting claims. *mimeo*.

Dominguez, D., 2013. Lower bounds and recursive methods for the problem of adjudicating conflicting claims. Social Choice and Welfare 40 (3), 663–678.

Dominguez, D., Thomson, W., 2006a. A new solution to the problem of adjudicating conflicting claims. Economic Theory 28 (2), 283–307.

Dominguez, D., Thomson, W., 2006b. A new solution to the problem of adjudicating conflicting claims. Economic Theory 28 (2), 283–307.

Estévez-Fernández, A., 2012. A game theoretical approach to sharing penalties and rewards in projects. European Journal of Operational Research 216 (3), 647–657.

Giménez-Gómez, J.-M., Marco-Gil, M. C., 2014. A new approach for bounding awards in bankruptcy problems. Social Choice and Welfare 43 (2), 447–469.

Giménez-Gómez, J.-M., Peris, J. E., 2014. A proportional approach to claims problems with a guaranteed minimum. European Journal of Operational Research 232 (1), 109–116.

Giménez-Gómez, J.-M., Peris, J. E., 2015. Participation and solidarity in redistribution mechanisms. Czech Economic Review (1), 36–48.

Giménez-Gómez, J. M., Peris, J. E., Solís-Baltodano, M. J., Dec. 2017. Resource Allocation with Warranties in Claims Problems. QM&ET working papers, Universitat d'Alacant, Quantitative Methods and Economic Theory.

Giménez-Gómez, J.-M., Teixidó-Figueras, J., Vilella, C., 2016. The global carbon budget: a conflicting claims problem. Climatic Change, 1–11.

Gini, C., 1921. Measurement of inequality of incomes. The Economic Journal 31 (121), 124–126.

Hemerijck, A., 2012. Changing welfare states. OUP Oxford.

Herrero, C., 2003. Equal awards vs. equal losses: duality in bankruptcy. In: Advances in economic design. Springer, pp. 413–426.

Iñarra, E., Prellezo, R., 2008. Bankruptcy of fishing resources: The northern european anglerfish fishery. Marine Resource Economics 17, 291–307.

Iñarra, E., Skonhoft, A., 2008. Restoring a fish stock: a dynamic bankruptcy problem. Land Economics 84 (2), 327–339.

Kampas, A., 2015. Combining fairness and stability concerns for global commons: The case of east atlantic and mediterranean tuna. Ocean & Coastal Management 116, 414–422.

Maimonides, M., 1135, 1204. Book of Judgements, (translated by Rabbi Elihahu Touger, 2000). Moznaim Publishing Corporation, New York; Jerusalem.

Maschler, M., 1992. The bargaining set, kernel, and nucleolus. In Aumann R and Hart S (Eds.) Handbook of game theory with economic applications 1, 591–667.

Mohl, P., Hagen, T., 2010. Do eu structural funds promote regional growth? new evidence from various panel data approaches. Regional Science and Urban Economics 40 (5), 353–365.

Moreno-Ternero, J. D., Villar, A., 2004a. The Talmud rule and the securement of agents' awards. Mathematical Social Sciences 47 (2), 245–257.

Moreno-Ternero, J. D., Villar, A., 2004b. The Talmud rule and the securement of agents' awards. Mathematical Social Sciences 47 (2), 245–257.

Moulin, H., 2000. Priority Rules and Other Asymmetric Rationing Methods. Econometrica 68 (3), 643–684.

Moulin, H., 2002. Axiomatic cost and surplus sharing. Handbook of social choice and welfare 1, 289–357.

Nash, J., 1951. Non-cooperative games. Annals of mathematics, 286–295.

Noguera, J., 2010. The universal basic income: reasons and strategies. Policy Papers 5, 541–559.

O'Neill, B., 1982. A problem of rights arbitration from the Talmud. Mathematical Social Sciences 2 (4), 345–371.

Parrachino, I., 2006. Cooperative game theory and its application to natural, environmental and water resource issues. Vol. 4072. World Bank Publications.

Puigcerver-Peñalver, M.-C., 2007. The impact of structural funds policy on european regions' growth. a theoretical and empirical approach. The European Journal of Comparative Economics 4 (2), 179.

Pulido, M., Sánchez-Soriano, J., Llorca, N., 2002. Game Theory Techniques for University Management. Annals of Operations Research, 129–142.

Read, L., Madani, K., Inanloo, B., 2014. Optimality versus stability in water resource allocation. Journal of environmental management 133, 343–354.

Robert, N., 1974. Anarchy, state and utopia. New York basic book.

Rodríguez-Pose, A., Fratesi, U., 2004. Between development and social policies: the impact of european structural funds in objective 1 regions. Regional Studies 38 (1), 97–113.

Sen, A., 1973. On economic inequality. Clarendon Press Oxford.

Shapley, L. S., Shubik, M., 1954. A method for evaluating the distribution of power in a committee system. American political science review 48 (3), 787–792.

Stuckler, D., Basu, S., Suhrcke, M., Coutts, A., McKee, M., 2011. Effects of the 2008 recession on health: a first look at european data. The Lancet 378 (9786), 124–125.

Theil, H., 1967. Economics and information theory (No. 04; HB74. M3, T4).

Thomson, W., 2003. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. Mathematical Social Sciences 45 (3), 249–297.

Thomson, W., 2006. A characterization of a family of rules for the adjudication of conflicting claims. RCER Working Papers (530).

Thomson, W., 2007. Lorenz rankings of rules for the adjudication of conflicting claims. Economic Theory, 1–23.

Thomson, W., 2015. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: an update. Mathematical Social Sciences 74, 41–59.

von Neumann, J., Morgenstern, O., 1944. Theory of games and economic behavior. Princeton, NJ: Princeton University Press.

Yeh, C. H., 2008. Secured lower bound, composition up, and minimal rights first for bankruptcy problems. Journal of Mathematical Economics 44 (9), 925 – 932.

Young, P., 1987. On dividing an amount according to individual claims or liabilities. Mathematics of Operations Research 12, 198–414.

Young, P., 1988. Distributive justice in taxation. Journal of Economic Theory 43, 321–335.

UNIVERSITAT
ROVIRA i VIRGILI