Universitat Autònoma de Barcelona

Limit cycles and critical periods for some polynomial differential equations

Iván Sánchez Sánchez

Universitat Autònoma de Barcelona

Limit cycles and critical periods for some polynomial differential equations

Iván Sánchez Sánchez

A thesis submitted in fulfillment of the requirements for the degree of Doctor in Mathematics

in the

Dynamical Systems Group
Faculty of Sciences

May 2021

I hereby certify that this memory has been elaborated by Iván Sánchez Sánchez under my supervision, and that it represents his thesis to aspire to the degree of Doctor in Mathematics issued by Universitat Autònoma de Barcelona.

Joan Torregrosa Arús
Bellaterra, May 2021

Dedicado a la memoria de mi abuela, que nos dejó durante el tiempo que estuve trabajando en esta tesis.

Acknowledgements

Voldria començar agraïnt a tot el Departament de Matemàtiques de la Universitat Autònoma de Barcelona, i en particular als membres del Grup de Sistemes Dinàmics, haver-me tractat des del primer moment com un company més. És curiós tenir unes professores i uns professors que en poc més d'un any passin a ser els teus companys de feina, però amb vosaltres ha estat tot molt fàcil. Igualment, vull valorar l'enorme feina del personal de secretaria del Departament, així com del servei d'informàtica, sempre disposat a col·laborar per qualsevol tema que un necessités i amb una eficàcia a la qual estem poc acostumats en el que a tràmits burocràtics es refereix. Una menció especial també pels meus companys i companyes doctorands, tant dins de la mateixa UAB com tots aquells membres del Seminari Caòtic, el seminari interuniversitari UAB–UB–UPC organitzat pels doctorands de Sistemes Dinàmics de les diferents universitats. Guardo amb molt afecte el record de totes les nostres quedades i els congressos que hem viscut plegats. A tots ells una forta abraçada i els desitjo el millor en les seves prometedores carreres professionals.

Não me esqueço das pessoas que encontrei em São Carlos, cidade que foi minha casa durante os três meses que estive no Brasil. Nunca estive tanto tempo longe de Barcelona e não posso estar mais grato pelo acolhimento que recebi. Uma obrigado muito especial à Professora Regilene, que foi minha orientadora acadêmica durante minha estadia. Obrigado por todas as horas que me dedicou e por estar sempre disponível para o que fosse preciso. Também quero agradecer todos os membros do ICMC e da UFSCar que me ajudaram em algum momento. Agradeço ao Leonardo, que sempre se preocupou comigo e me incluía em seus planos para tornar meu tempo fora de casa o mais agradável possível. Também quero citar o grupo de amigos que fiz nessa época: Sorenza, Lucy, Josnei, Dalia, Alessandra e todos os demais. Obrigado pela companhia durante todo esse tempo. Por supuesto no puedo dejarme a Angelica, así como a toda su familia. Desde el primer momento me abriste las puertas de tu casa como a uno más, me lo ofreciste todo. No esperaba cogerte tanto cariño en tan poco tiempo, fuisteis mi familia durante tres meses, y a pesar de la distancia sé que nuestra amistad perdurará. Un fuerte abrazo para todos.

A continuación quiero agradecer a mi familia y amigos. Gracias a mis padres por haberme enseñado lo que es el trabajo y la constancia, y por haberme dado todo lo que tenían para ofrecerme la oportunidad de ser quien yo quisiera ser; si he llegado hasta aquí es gracias a vosotros. También a mis hermanas Irene e Inés, por enseñarme que una dosis de locura también viene bien de tanto en tanto. Quisiera agradecer a mis amigas de siempre, Rocío, Alicia, Andrea y Sara, el haberme hecho sentir siempre que ahí estaban para cualquier cosa que necesitara.

Y por último, quiero dar las gracias a mi prometido Víctor, por apoyarme en todas las decisiones, por tener que soportar mis cambios de humor repentinos cuando era víctima del estrés, y sobretodo por ser un pilar fundamental para mí durante estos años.

Finalment, però per descomptat no menys important, vull fer un agraïment molt especial al meu director de tesi Joan Torregrosa, la persona que en definitiva ha guiat el meu treball durant aquests gairebé quatre anys (de fet sis, si comptem els treballs de final de grau i de màster). Poques vegades es té el plaer de conèixer algú amb tanta passió pel que fa, i en aquest sentit em puc considerar afortunat d'haver treballat amb tu tot aquest temps. Gràcies per haver-me acollit des del primer moment, per tot el temps que m'has dedicat, per la paciència que has tingut (que no ha estat poca!), i per haver-me fet les coses molt més fàcils en aquest camí no sempre fàcil d'escriure una tesi doctoral.

Badalona, mayo 2021.

Contents

Introduction

It is often said that mathematics is the language of nature and science. Since humankind became conscious of the world around them, there has been an imperative need to understand the underlaying rules behind natural phenomena, and no one can deny the central role that mathematics plays in this game. They allow to explain what we see and to predict what will happen in a rigorous and universal language. In particular, differential equations have proved to be one of the most efficient tools for modelling the relations between objects or events from the reality in which we live, not only in terms of describing the laws of nature but also for example to explain the behaviour of some social processes. Although seen as a simple modelling tools at first, ordinary differential equations have given rise to a whole and elaborated theory themselves.

The birth of differential equations is intrinsically linked to the development of infinitesimal calculus in the 17th century by I. Newton (1642–1727) and G. W. Leibnitz (1646–1716). These two brilliant minds built the foundations for the theory of ordinary differential equations which would be developed during the following 350 years. During the 18th century, the study of differential equations was fostered by L. Euler (1707–1783), who tackled the resolution of some problems in mechanics, as well as by the two French mathematicians J. L. Lagrange (1736–1813) and P. S. Laplace (1749–1827), who also introduced the notion of partial differential equations.

In the last years of the 19th century, the work of H. Poincaré (1854–1912) implied a new point of view in the study of ordinary differential equations, which led to the beginning of what today is known as *qualitative theory of differential equations*. Poincaré gave a geometrical sense to differential equations in his series of works *Mémoire sur les courbes définies par une équation différentielle*, published between 1881 and 1886. This new approach consisted on the study of the topological structure of the solutions of a differential equation, which allowed to deduce properties about such solutions without explicitly finding them.

The idea of limit cycle as a periodic orbit for which at least one trajectory of the vector field approaches in positive or negative time was introduced by Poincaré. Usually, an alternative definition is given: a limit cycle is periodic orbit which is isolated in the set of periodic orbits of a differential equation. He also defined other fundamental objects such as *phase portrait*, a name for the compilation of the minimal information which enables to determine the topological structure of the orbits of a differential system, or the notion of *return map*, which is also known as

Poincaré map. The work of Poincaré together with the contribution of I. Bendixson (1861–1935) during the first years of the 20th century resulted in the well-known Poincaré–Bendixson's Theorem, which states that under compacity conditions every solution tends to a singular solution which can be either a critical point, a periodic orbit or a connected set.

In 1900, D. Hilbert published a series of problems which would be very influential for mathematics during the 20th century. Ten of these problems were presented at the International Congress of Mathematicians in Paris. Among them there is the 16th Hilbert Problem, whose second part can be outlined, according to [Rou98], as

> *"Proving that for any $n \geq 2$ there exists a finite number $\mathcal{H}(n)$ such that any polynomial differential equation whose degree is lower or equal than n has less than $\mathcal{H}(n)$ limit cycles."*

This problem remains unsolved, but a number of researchers have made a lot of advances. More details about it can be found in the review of Y. Ilyashenko in [Ily02] and J. Li did a nice review of the state of the problem in [Li03]. About global lower bounds, the work of C. Christopher and N. Lloyd in [CL95] is remarkable, improved some years ago by M. Han and J. Li in [HL12] and recently in [Álv+20]. Regarding summaries of known lower bounds for $\mathcal{H}(n)$ for low values of the degree, the best ones can be found in [PT19].

The interest in the study of limit cycles arises from the large number of phenomena in nature or in social sciences where periodic behaviour can be observed. A classical example is the periodicity of the oscillations in RLC circuits and the existence of some isolated periodic orbits. In this line, the works of the engineer A. Liénard (1869–1958) and the physicist B. van der Pol (1889–1959) are highly remarkable. Years later, A. A. Andronov (1901–1952) and his partners tackled the problem of mechanical and electronical oscillators focusing on the limit cycles analysis. Actually, the van der Pol and Liénard equations have been frequently generalized, and up to the present days there is still a strong interest and research on the study of limit cycles in such generalized models.

Let us consider a system of differential equations in the plane

$$\begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{cases} \tag{1}$$

being P, Q analytic functions. We say that (x_0, y_0) is an equilibrium point of system (1) if $P(x_0, y_0) = Q(x_0, y_0) = 0$, and γ is a periodic orbit if it is a solution of the system such that $\gamma(0) = \gamma(T)$ for some $T > 0$. From now on, we will consider that P, Q in (1) are polynomials.

A classical result in qualitative theory is the Hartman's Theorem, which classifies equilibrium points which are hyperbolic, but does not conclude for other types of critical points. Let us consider system (1) taking the form

$$\begin{cases} \dot{x} = -y + X(x,y), \\ \dot{y} = x + Y(x,y), \end{cases} \tag{2}$$

being X, Y polynomials without constant nor linear terms. The linear part of such system has eigenvalues $\pm\mathrm{i}$, so the origin is a nonhyperbolic equilibrium point and Hartman's Theorem does not apply. This case is known as *monodromic nondegenerate*, and the orbits near the origin spin around it. This situation leads to the so-called *center problem* or *center-focus problem*, which aims to determine whether orbits in a neighbourhood of the origin are closed –in which case the origin is a *center*– or they spiral towards the origin when times goes forward or backward –in which case the origin is a *weak focus*. In the latter case, we are also interested in determining the stability of the focus, this is whether it is attracting or repelling. In many contexts, we will have that (2) consists of a parametric family of equations and we aim to find conditions on the parameters which define the centers of such family.

Another problem we will consider is a local version of the Hilbert problem that consists on finding the maximum number of limit cycles of small amplitude that bifurcate from an equilibrium point for a planar polynomial vector field (1) of degree n. This number is usually called the cyclicity of the equilibrium, which gives name to the *cyclicity problem*. The most standard way to get lower bounds for this number is to analyze the local *return map* or *Poincaré map* defined in a neighborhood of a monodromic equilibrium point, which maps a radial initial condition ρ to the radial component $\Pi(\rho)$ after a 2π loop on the angular component. This study is usually done by studying the maximum codimension of a degenerated Hopf bifurcation and the most recent progress in the aforementioned problem for small degrees can be found in [GGT21], analyzing such bifurcation near very special centers that have high codimension.

A. M. Lyapunov (1957–1918) defined the functions which give the stability in the resolution of the center problem, also known as *Lyapunov constants*; some authors also refer to them as Lyapunov quantities or focal values. Lyapunov, in his work [Lia66], proves that in a neighbourhood of the origin of system (2) Poincaré map takes the form

$$\Pi(\rho) = r(2\pi, \rho) = \rho + \sum_{j=2}^{\infty} V_j \rho^j,$$

being $r(\varphi, \rho)$ the solution such that $r(0, \rho) = \rho$. The first nonzero coefficient in the expression of $\Pi(\rho)$ gives the stability of the origin. As we will see, the first nonzero coefficient V_k provided that $V_l = 0$ for $l < k$ has odd subscript, and we define it as the $\frac{k-1}{2}$th Lyapunov constant of the system. These Lyapunov constants are the main tool to address the center and cyclicity problems, but finding these quantities turns out to be a highly demanding challenge in computational terms, impossible to achieve in most of the situations, as we will see throughout this memory.

Let us assume now that system (1) has a center at the origin having the form (2). One can then consider the *period function* on the period annulus of the center, which maps each periodic orbit ρ to the time $T(\rho)$ needed by such orbit to perform a complete loop and return to the original point. It can be shown that this period function takes the form

$$T(\rho) = 2\pi + \sum_{j=1}^{\infty} T_j \rho^j,$$

and the first nonzero coefficient T_k provided that $T_l = 0$ for $l < k$ is known as the $\frac{k}{2}$th *period constant* of the system, a consistent definition since such first nonzero coefficient T_k has even subscript k, as we will justify.

From the described situation one can propose the *isochronicity problem*, which aims to determine whether all the periodic orbits in the period annulus of the center have the same period or not, in which case we say that the center is *isochronous*. This question has been classically considered in relevant physical models, such as the pendulum equation or certain conservative systems. In the last decades, there has been an increasing interest on studying the monotonicity of the period function and seeing the existence of oscillations or critical points of it. These oscillations are known as *critical periods*, and we refer to this question as bifurcation of critical periods or *criticality problem*. The isochronicity and criticality problems have a strong analogy to the center and cyclicity problems, respectively, and the period constants fulfill the same role as the Lyapunov constants. Actually, the computational difficulties of dealing with Lyapunov constants can also be extrapolated to the case of period constants, as in some sense they are part of the same mathematical object as we will see in the last chapter of this work.

The present doctoral thesis is framed in the study of the described problems, in the context of the qualitative theory of differential equations. This memory is organized in three chapters, which are described with more detail followingly together with their main results.

The first chapter deals with the center and cyclicity problems. We start by giving a deeper and more exhaustive description of the center and cyclicity problems, together with a brief introduction to the main tools about polynomial ideals that will be needed to cover the topics. Our attention is focused on centers of the form (2). Some classical techniques for classifying centers are also presented, such as symmetries or Darboux Integrability Theory. We proceed then to a more detailed analysis on Lyapunov constants, by showing methods to compute them and how these calculations can be computationally implemented, stressing the importance of parallelization in the used techniques. Later, we deal with the center and cyclicity problems for some families of differential equations in the plane. Many of these first introductory sections and examples are based on the work done during my master's thesis, which consisted on a first approach to the center and cyclicity problems.

Section 1.4 collects a series of advances in the computation of Lyapunov constants and the determination of their structure. We explain how we have been able to reach the 14th Lyapunov constant of a complete cubic system in the plane, with the indispensable support of parallelization. Also, we present a reconstruction technique which allows to see whether a specific Lyapunov constant belongs to the ideal generated by the previous ones, and this has been applied to three different families. The results in this section have led to the publication of the work *New advances on the Lyapunov constants of some families of planar differential systems* in the book *Extended Abstracts Spring 2018* ([ST19]).

In the last section of Chapter 1 we study the Hopf-bifurcation in 3-dimensional polynomial vector fields, with the objective to find the highest possible number of limit cycles for different degrees. We explain how the classical computation algorithm of Lyapunov constants can be extrapolated to $\mathbb{R}^3$, and we highlight again the importance of parallelization. The Poincaré–Miranda's Theorem is also introduced here, as it represents an essential tool to prove our results. The used techniques have enabled to find 11 limit cycles for quadratic systems, 31 for cubic systems, 54 for quartic systems, and 92 for quintic systems, which to the best of our knowledge are the highest numbers found so far. These findings have resulted in the paper *Hopf-bifurcation in 3-dimensional polynomial vector fields*, which is submitted for its publication ([ST21b]). Even though this work was initially planned to be developed during a research stay in the Shanghai Jiao-Tong University (China) in 2020, this had to be cancelled due to the COVID-19 pandemic.

The second chapter is devoted to the study of isochronicity and criticality. We start by defining both concepts in more detail, working on the notions of period function and critical periods. Then, the equivalence between isochronicity and linearizability is introduced, together with other tools to study isochronicity which are the Lie bracket and commuting transversal systems. A section which deals with the computation of period constants is presented, where two methods are described: the classical one and a new approach which uses the Lie bracket. A result on linear parts of period constants is also given at this stage.

The next section in Chapter 2 aims to find the maximum number of critical periods which unfold from low degree n planar polynomial centers when perturbing reversible holomorphic isochronous centers inside the reversible class. This is done by studying the local bifurcation of zeros of the first derivative of the period function. We prove a result on the criticality of isochronous centers which is based on the Implicit Function Theorem, and by using the isochronicity parameters of the system extra critical periods can be obtained. The result is that 6 critical periods are seen for the cubic case, 10 for the quartic case, $(n^2+n-2)/2$ for $5 \leq n \leq 9$, and $(n^2+n-4)/2$ for $10 \leq n \leq 16$. This work has been published this year under the name *New lower bounds of the number of critical periods in reversible centers* in the *Journal of Differential Equations* ([ST21c]).

The final section of this chapter introduces the idea of using an equivalent of Melnikov functions for the bifurcation of critical periods instead of limit cycles. This will allow to study criticality by using period constants only up to their first-order Taylor development, which will imply a significant reduction of computation time and improved results. In particular, we obtain 10, 22, 37, 57, 80, 106, and 136 critical periods for $n = 4, 6, 8, 10, 12, 14$, and 16, respectively. We also classify some isochronous centers throughout this section. This work has been published also this year on the paper *Criticality via first order development of the period constants* in *Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal* ([ST21a]).

The third and last chapter of this memory presents a new problem that, to the best of our knowledge, has never been considered before. This problem consists on simultaneously studying the bifurcation of limit cycles and critical periods for a system of differential equations in the plane, obtaining a value (k,l) which means that k limit cycles and l critical periods can simulteneously unfold. In this line, we will define the term *bi-weakness* $[k,l]$ as a concept for the degree of the first nonzero coefficients in the return map k and the period function l at the same time, being both the center and isochronicity properties not kept. During Chapter 3 we study the bi-weakness for different classical families: we obtain a $[5,4]$ case for cubic Liénard systems, a $[7,6]$ for quartic Liénard, a $[5,4]$ for the complete quadratic family, and a $[7,6]$ for the cubic homogeneous nonlinearities family. In addition, we give a complete classification of the simultaneous cyclicity and criticality for the cubic Liénard system, proving that the center case can only have one critical period, that for the noncenter case we can have the configurations $(1,3)$ and $(2,3)$, and that isochronous foci do not exist for this family. We also show the isochronicity for some Liénard families in this part.

The work to elaborate Chapter 3 was mainly developed during a research stay in the *Instituto de Ciências Matemáticas e de Computação* from the São Paulo University (Brazil) in 2019, together with professor Dr Regilene D. S. Oliveira. The results here have led to the preprint named *Simultaneous bifurcation of limit cycles and critical periods* ([OST21]), which has been recently submitted for its publication.

Chapter 1

Center problem and limit cycles

The second part of the 16th Hilbert problem aims to determine the maximum number of isolated periodic solutions which a system of polynomial differential equations in the plane has. Two related problems are the center and cyclicity problems, which consist on identifying whether the origin of a system having a specific form is a center or a focus and determining the maximum number of limit cycles, respectively. Here, we aim to work on the necessary tools for the study of these problems for polynomial differential equations, and this means to analyze the stability in a neighborhood of a monodromic nondegenerate point. An essential mathematical object to deal with these problems are the Lyapunov constants, which determine whether the origin is a center or a focus and its stability. In this chapter, we introduce two procedures to find these quantities: the Lyapunov method and an interpolation technique. Both algorithms are computationally implemented, and the differences between both methods are discussed. The idea of implementing parallelization on these methods is also introduced, and we show the necessity of considering a parallel approach for the computation of the Lyapunov constants in terms of efficiency and computational speed.

The developed codes are tested on some examples of planar systems, for which we show how to solve the center and cyclicity problems. It is worth remarking that the idea of developing a parallelization approach here is to consolidate the proposed techniques rather than to apply it to the resolution of the center and cyclicity problems of many new planar families. In this sense, such techniques will be extrapolated to polynomial three-dimensional systems in Section 1.5 and to period constants in the following chapter to obtain a number of new results. Also, the implemented algorithms are used in Section 1.4 to obtain new findings on the Lyapunov constants of some systems in $\mathbb{R}^2$. In particular, we use parallelization to obtain 14 Lyapunov constants for the complete cubic family, and we present a reconstruction technique that enables to find new Lyapunov constants of a few planar systems. Finally, we present some studies on the cyclicity of polynomial vector fields in $\mathbb{R}^3$, and prove the unfolding of limit cycles for several degrees. To this end, we consider a 3-dimensional Hopf bifurcation to provide lower bounds for the number of limit cycles.

1.1 The center and cyclicity problems

As we already stated, the idea of this chapter is to analyze the center and cyclicity problems for several types of polynomial differential equations systems. This consists on studying the stability in a neighborhood of the origin when it is a monodromic nondegenerate point –this is the name given to points which do not have any arrival direction, particularly those whose linear part has zero trace and positive determinant. If the readers are familiar with these topics, they could skip directly to Section 1.2.

The *center problem*, also known as the Poincaré center problem or center-focus problem, consists on identifying whether the origin of a system of differential equations in the plane whose origin is a monodromic nondegenerate point is a center or a focus. This means distinguishing whether the solutions near the origin are all periodic orbits or not. Answering this question is a first step towards the *cyclicity problem*, which aims to determine the maximum number of limit cycles[1] which can appear when perturbing a system. All this is related to the second part of the 16th Hilbert problem, a question related to finding the maximum number of limit cycles $\mathcal{H}(n)$ that a planar polynomial system can have as a function of its degree n. This problem remains unsolved for most of the polynomial families of differential equations. Actually, the center and cyclicity problems have been solved only for a few systems, and during this chapter some families for which the problem can be completed will be introduced.

Let us consider a system of differential equations in the plane with an equilibrium point at the origin such that its differential matrix has eigenvalues $\pm\beta\,\mathrm{i}$ with $\beta \neq 0$ and zero trace –in later sections, when we deal with cyclicity, we will consider the same system with nonvanishing trace. With an appropriate time change, we can assume $\beta = 1$ and write the system in its normal form

$$\begin{cases} \dot{x} = -y + X(x,y,\lambda), \\ \dot{y} = x + Y(x,y,\lambda), \end{cases} \tag{1.1}$$

where $X(x,y,\lambda)$ and $Y(x,y,\lambda)$ are polynomials without constant nor linear terms in x and y and parameters $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$ in the coefficients. For the sake of simplicity, from now on these polynomials will be denoted as $X(x,y)$ and $Y(x,y)$ in order to simplify notation, but the dependence on parameters λ must be taken into account. $X(x,y)$ and $Y(x,y)$ are such that

$$X(x,y) = X_2(x,y) + X_3(x,y) + X_4(x,y) + \cdots,$$
$$Y(x,y) = Y_2(x,y) + Y_3(x,y) + Y_4(x,y) + \cdots,$$

[1]Recall that a limit cycle is a periodic orbit which is isolated in the set of periodic orbits of a system.

where $X_k(x,y)$ and $Y_k(x,y)$ are homogeneous kth degree polynomials with parameters $\lambda = (\lambda_1, \ldots, \lambda_d)$ in the coefficients, which means that the coefficients polynomially depend on λ.

It will be useful to have system (1.1) in complex coordinates $z = x + \mathrm{i}\, y$. Let us derive z with respect to time,

$$\dot{z} = \dot{x} + \mathrm{i}\,\dot{y} = (-y + X(x,y)) + \mathrm{i}\,(x + Y(x,y)) = \mathrm{i}(x + \mathrm{i}\,y) + (X(x,y) + \mathrm{i}\,Y(x,y)).$$

Let $w = x - \mathrm{i}\,y$ be the conjugate of z. We define

$$Z(z,w) := X(x,y) + \mathrm{i}\,Y(x,y) = X\left(\frac{z+w}{2}, \frac{z-w}{2\,\mathrm{i}}\right) + \mathrm{i}\,Y\left(\frac{z+w}{2}, \frac{z-w}{2\,\mathrm{i}}\right).$$

Then the system of differential equations (1.1) can be rewritten in complex variables simply as

$$\dot{z} = \mathrm{i}\,z + Z(z,w). \tag{1.2}$$

Notice that the equation in $\dot{w}$ is redundant because it is the complex conjugate of $\dot{z}$ due to the fact that this complex system is associated to a real vector field. In (1.2), $Z(z,w)$ is such that

$$Z(z,w) = Z_2(z,w) + Z_3(z,w) + Z_4(z,w) + \cdots,$$

being $Z_k(z,w)$ homogeneous kth degree polynomials. Equation (1.2) will also have a set of parameters λ, which could be the same as in (1.1) or could be transformed into new complex parameters. In the latter case we will also denote them as $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$ with a slight abuse of notation.

As the linear part of system (1.1) has eigenvalues $\pm\,\mathrm{i}$ at the origin, the Hartman Theorem cannot be applied for studying the stability of the origin, so other techniques are required. A transformation into polar coordinates shows that if the origin is monodromic nondegenerate then the orbits near the origin spin around it, so the origin will be either a center or a focus. We aim to detect whether the origin consists of a center or a focus and to determine the maximum number of limit cycles which can appear when a perturbation occurs in a given degree family of polynomials.

To deal with this problem, we will start by introducing the notion of Poincaré map (see for example [Wig03]). Let Σ be a transversal section to an orbit γ which is in a neighborhood of the origin. The Poincaré map is a map $\Pi : \Sigma \to \Sigma$ such that, for an initial value $r(0) = \rho \in \gamma \cap \Sigma$ (where $r(\varphi)$ is the radial coordinate of system (1.1) in polar coordinates), then $\Pi(\rho)$ is the first intersection point of orbit γ with Σ in positive time. This is schematized in Figure 1.1. A classical result is that the Poincaré map can be analytically extended to $\rho = 0$ (see [And+73; Rou98]), so we can consider its Taylor expansion

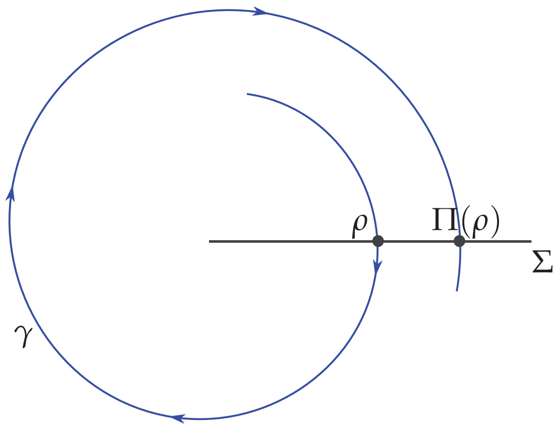

FIGURE 1.1: The Poincaré Map $\Pi(\rho)$.

$$\Pi(\rho) = \rho + V_2\rho^2 + V_3\rho^3 + V_4\rho^4 + \cdots = \rho + \sum_{j=2}^{\infty} V_j\rho^j, \tag{1.3}$$

for certain values V_j which depend on the parameters λ of (1.1). Observe that the center-focus problem is equivalent to determine whether all V_j are zero or not, since periodic orbits are fixed points of the Poincaré map. In the case for which not all the coefficients in (1.3) vanish, the origin of the system is a focus and its stability is determined by the first nonzero coefficient. As a consequence, the center-focus problem reduces to the problem of finding the coefficients V_j of the Poincaré map.

1.1.1 Polynomial ideals and radicality

In later sections we will see that the coefficients of the return map (1.3) are polynomials in the parameters λ of the corresponding system (1.1). The ideals generated by these polynomials will be crucial to address the center and cyclicity problems. Therefore, we will need some concepts and results from ring theory, which will be reviewed in this section. For the definitions and results outlined here the reader is referred for example to [CLO07].

Definition 1.1. *Let I be an ideal of a ring R. The radical of I is the set*

$$\operatorname{Rad} I := \{r \in R \mid \exists\, n \in \mathbb{N},\, r^n \in I\}.$$

Definition 1.2. *An ideal I of a ring R is said to be radical if $I = \operatorname{Rad} I$.*

For our study we will need to use the well-known Hilbert Basis Theorem, outlined as follows.

Theorem 1.3 ([CLO07]). ***(Hilbert Basis Theorem)*** *Let K be a field and we denote by $K[x_1, \ldots, x_n]$ the ring of polynomials with coefficients in K. Then every ideal I of*

$K[x_1, \dots, x_n]$ is finitely generated, i.e. there exist $F_1, \dots, F_s \in K[x_1, \dots, x_n]$ such that $I = \langle F_1, \dots, F_s \rangle$.

Another important result which will be useful for us is the Hilbert Zeros Theorem (in German, *Hilberts Nullstellensatz*). This theorem requires some previous definitions.

Definition 1.4. *Given a set of points $U \in \mathbb{A}^n$, for a set $\mathbb{A}$, we define the ideal of U as*

$$\mathcal{I}(U) = \{F \in K[x_1, \dots, x_n] \mid F(u) = 0 \text{ for all } u \in U\}.$$

It can be proved that $\mathcal{I}(U)$ is actually an ideal (see [CLO07]).

Definition 1.5. *Given a field K and a set $\mathbb{A}$, for every ideal in $K[x_1, \dots, x_n]$ the set of zeros or variety of I is*

$$V(I) = \{x \in \mathbb{A}^n(K) \mid P(x) = 0 \text{ for all } P \in I\}.$$

Now we can finally outline the theorem.

Theorem 1.6 ([CLO07]). ***(Hilbert Zeros Theorem)*** *Let K be an algebraically closed field and let us denote by $K[x_1, \dots, x_n]$ the ring of polynomials with coefficients in K. If I is an ideal of $K[x_1, \dots, x_n]$, then*

$$\mathcal{I}(V(I)) = \operatorname{Rad} I.$$

As a consequence, using this theorem and the definition of radical ideal we have the next result.

Corollary 1.7. *With the same notation as in Theorem 1.6, if I is a radical ideal then:*

$$\mathcal{I}(V(I)) = I.$$

1.1.2 The center problem and center characterization

Let us express system (1.1) in polar coordinates as follows,

$$\frac{dr}{dt} = r^2 P_2(\varphi) + r^3 P_3(\varphi) + \cdots, \tag{1.4}$$

$$\frac{d\varphi}{dt} = 1 + r\, Q_2(\varphi) + r^2 Q_3(\varphi) + \cdots, \tag{1.5}$$

where

$$P_i(\varphi) = \cos\varphi\, X_i(\cos\varphi, \sin\varphi) + \sin\varphi\, Y_i(\cos\varphi, \sin\varphi),$$
$$Q_i(\varphi) = \cos\varphi\, Y_i(\cos\varphi, \sin\varphi) - \sin\varphi\, X_i(\cos\varphi, \sin\varphi).$$

Now we divide equation (1.4) by (1.5),

$$\frac{dr}{d\varphi} = \frac{r^2\, P_2(\varphi) + r^3\, P_3(\varphi) + \cdots}{1 + r\, Q_2(\varphi) + r^2\, Q_3(\varphi) + \cdots}.$$

We can observe that this function is analytic in a neighborhood of the origin because the denominator does not vanish in $r = 0$. Let us then expand this function as a power series in r, which leads to an analytic differential equation,

$$\frac{dr}{d\varphi} = R_2(\varphi)\, r^2 + R_3(\varphi)\, r^3 + \cdots, \tag{1.6}$$

for certain $R_i(\varphi)$. This is an analytic differential equation with particular solution $r = 0$. Let $r(\varphi, \rho)$ be the solution of equation (1.6) which $r(0, \rho) = \rho$. This solution is analytic in ρ, which is the initial value, so it can be expanded in the following way:

$$r(\varphi, \rho) = \rho + \sum_{j=2}^{\infty} u_j(\varphi)\, \rho^j,$$

As $r(0, \rho) = \rho$, it immediately follows that $u_j(0) = 0$ for every j. Let us study the stability near the origin, $r = 0$, by using

$$r(2\pi, \rho) = \rho + V_k\, \rho^k + \sum_{j=k+1}^{\infty} V_j \rho^j, \tag{1.7}$$

where $V_j := u_j(2\pi)$ for $j \geq k$ and V_k is the first coefficient which does not vanish. The following classic result about these coefficients will be highly useful.

Lemma 1.8 ([And+73]). *With the used notation, the first nonidentically zero coefficient V_k has odd k. Furthermore, quantities V_k are polynomials in the coefficients of the original equation.*

Applying this Lemma, expression (1.7) can be rewritten as

$$r(2\pi, \rho) = \rho + V_{2n+1}\, \rho^{2n+1} + \sum_{j=2n+2}^{\infty} V_j \rho^j.$$

This first nonzero coefficient $L_n := V_{2n+1}$ with odd subscript is known as the nth *Lyapunov constant* of the system, also known as the nth Lyapunov quantity or focal value. Notice that the Lyapunov constant V_{2n+1} or L_n is technically defined under the conditions on the parameters λ of the coefficients in (1.1) such that $V_j = 0$ for every $j < 2n + 1$. It is worth remarking that the expressions of the Lyapunov constants can differ in a positive multiplicative factor when being found by different methods due to the nature of the used techniques in each case.

Despite this, the stability and center conditions do not vary when using different computation algorithms.

Observe that $r(2\pi, \rho)$ indicates the radius after a whole loop starting in the initial value ρ, so it turns out to be the Poincaré map defined in Section 1.1,

$$\Pi(\rho) := r(2\pi, \rho) = \rho + \sum_{j=3}^{\infty} V_j \rho^j.$$

Alternatively, this can be written as

$$\Delta(\rho) := \Pi(\rho) - \rho = \sum_{j=3}^{\infty} V_j \rho^j,$$

and this function is known as the displacement map. With this last expression of the Poincaré map, the next properties follow immediately.

Proposition 1.9. *The Poincaré map $\Pi(\rho)$ satisfies the following properties:*

(a) A certain initial condition ρ_0 defines a periodic orbit of system (1.1) if and only if $\Delta(\rho_0) = \Pi(\rho_0) - \rho_0 = 0$.

(b) Furthermore, this periodic orbit is a limit cycle if and only if ρ_0 is an isolated zero in the set of zeros of function $\Delta(\rho) = \Pi(\rho) - \rho$.

(c) The origin of the system (1.1) is a center if and only if $\Delta(\rho) = \Pi(\rho) - \rho \equiv 0$ in a neighborhood of the origin.

According to Proposition 1.9c, solving the center problem and characterizing a center is equivalent to determine under which conditions the Lyapunov constants are $L_n = 0$ for all $n \geq 1$. There are two possible cases: either $L_n = 0$ for all n or there exists an ℓ such that $L_\ell \neq 0$.

(i) If all L_n vanish, then $\Pi(\rho) - \rho \equiv 0$ so the origin is a center.

(ii) Otherwise, let L_ℓ be the first nonzero Lyapunov constant. In this case the origin is a focus, namely a weak focus of order ℓ, and furthermore

- If $L_\ell < 0$ then it is an attracting focus (asymptotically stable).
- If $L_\ell > 0$ then it is a repelling focus.

These conclusions follow immediately from the Lyapunov Stability Theorem, an elementary theorem in dynamical systems theory which can be found for example in [Chi06].

Even though we will computationally implement some codes to compute the first N Lyapunov constants, this is not enough to determine centers. The reason is that finding that $L_n = 0$ for $n \leq N$ for some $N \in \mathbb{N}$ does not guarantee that all the Lyapunov constants will vanish, and we would need to compute an infinite number of L_n.

Let us consider the ideal generated by all the Lyapunov constants of a system $\langle L_1, L_2, L_3, \ldots \rangle$, which according to Lemma 1.8 are polynomials –the ideals generated by Lyapunov constants are also known as *Bautin ideals*. This is an ideal of the ring of polynomials $\mathbb{C}[\lambda]$, where $\lambda \in \mathbb{C}^d$ denotes the array of parameters in the coefficients of the differential system (1.2). Then, according to the Hilbert Basis Theorem, this ideal is finitely generated, so there must exist $m \in \mathbb{N}$ such that

$$\langle L_i \rangle_{i=1,\ldots,\infty} = \langle L_1, \ldots, L_m \rangle. \tag{1.8}$$

Knowing the value of m would significantly simplify the problem, because by computing the first m Lyapunov constants of a system we would obtain its center conditions. Nevertheless, as [CR06] states, there are no general methods find this m, so in each particular case in which a point is a candidate to be a center we have to manage to see whether actually $L_n = 0$ for all $n \geq 1$. This means that there are not general methods to find the center conditions of a system, and this is the reason why the center problem has been solved only for certain polynomial families. In this line, in the following subsections we will introduce a few techniques and results which may be useful for center characterization.

Liénard systems

We will start by briefly introducing Liénard systems and a center characterization theorem for them, which will be useful because many other systems can be written in Liénard form.

Definition 1.10. *A Liénard system is a ordinary system of differential equations in the plane which has the form*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -g(x) - y f(x), \end{cases} \tag{1.9}$$

where f(x) and g(x) are real polynomials such that

$$g(0) = 0, \qquad g'(0) > 0.$$

Observe that a Liénard system with $g(x) = x + \tilde{g}(x)$, where $\tilde{g}(x)$ does not have constant nor linear terms, is a particular case of the system of equations (1.1) under a time change $t \to -t$.

Let $F(x)$ and $G(x)$ be the primitive functions of $f(x)$ and $g(x)$, respectively,

$$F(x) = \int_0^x f(s)\, ds, \qquad G(x) = \int_0^x g(s)\, ds.$$

It is easy to see that, under the so-called Liénard transformation $(x, y) \to (x, y + F(x))$, system (1.9) can be rewritten as

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -g(x), \end{cases} \tag{1.10}$$

and this way of expressing Liénard systems will be helpful afterwards.

For Liénard systems, [Chr99] outlines and proves the following necessary and sufficient condition for the so-called composition centers; this center classification can also be found in [GT98].

Theorem 1.11 ([Chr99]). *The origin of system* (1.9) *is a center if and only if* $F(x) = \Phi(G(x))$ *for any analytic function* Φ *such that* $\Phi(0) = 0$.

Darboux Integrability Theory

In this subsection we will see how the existence of first integrals is related to invariant curves through Darboux Integrability Theory. Let us define $P := -y + X(x, y, \lambda)$ and $Q := x + Y(x, y, \lambda)$ in system (1.1). We denote by $\mathbb{K}$ either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$, and by $\mathbb{K}[x, y]$ the ring of polynomials in the variables x and y and coefficients in $\mathbb{K}$. For a reference on the ideas and the proofs of the results presented here the reader is referred to [DLA06].

We start by introducing the basic concepts of first integral and invariant algebraic curve.

Definition 1.12. ***(Integrability and first integral)*** *The polynomial system* (1.1) *is integrable on an open subset* $U \subset \mathbb{K}^2$ *if there exists a nonconstant analytic function* $H : U \to \mathbb{K}$, *called a first integral of the system on* U, *which is constant on all solution curves of system* (1.1) *contained in* U.

Definition 1.13. ***(Invariant algebraic curve)*** *Consider* $f \in \mathbb{C}[x, y]$, f *nonidentically zero. The algebraic curve* $f(x, y) = 0$ *is an invariant algebraic curve of system* (1.1) *if*

$$(P, Q) \cdot \vec{\nabla} f = K f,$$

for some polynomial $K \in \mathbb{C}[x, y]$. *The polynomial* K *is called the cofactor of the invariant algebraic curve* $f = 0$. *Finally, if* f *is an irreducible polynomial in* $\mathbb{C}[x, y]$ *we say that* $f = 0$ *is an irreducible invariant algebraic curve.*

Invariant algebraic curves are also known as *algebraic partial integrals*. Observe that in the definition of invariant algebraic curve we always allow this curve to be complex, even in the case of a real polynomial system. This is due to the fact that for real polynomial systems the existence of a real first integral can be forced by the existence of complex invariant algebraic curves. For more details see [DLA06].

To study Darboux Integrability, we also need the notion of integrating factor.

Definition 1.14. ***(Integrating factor)*** *Let U be an open subset of $\mathbb{K}^2$ and let $R : U \to \mathbb{K}$ be an analytic function which is not identically zero on U. The function R is an integrating factor of system (1.1) on U if one of the following three equivalent conditions holds on U:*

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y}, \quad \mathrm{div}(RQ, RP) = 0, \quad P\frac{\partial R}{\partial x} + Q\frac{\partial R}{\partial y} = -R\ \mathrm{div}(Q, P).$$

Definition 1.15. *The first integral H associated to the integrating factor R is given by*

$$H(x,y) = \int R(x,y)P(x,y)\,dy + \tilde{h}(x),$$

where $\tilde{h}$ is chosen such that $\frac{\partial H}{\partial x} = -RQ$; then

$$\begin{cases} \dot{x} = RP = \frac{\partial H}{\partial y}, \\ \dot{y} = RQ = -\frac{\partial H}{\partial x}. \end{cases}$$

There is another mathematical concept, the so-called exponential factor, which plays the same role that invariant algebraic curves in obtaining a first integral of a polynomial system.

Definition 1.16. ***(Exponential factor)*** *Let $h,g \in \mathbb{C}[x,y]$ and assume that h and g are relatively prime in the ring $\mathbb{C}[x,y]$ or that $h \equiv 1$. Then the function $\exp(g/h)$ is called an exponential factor of the system (1.1) if*

$$(P,Q) \cdot \vec{\nabla} \exp\left(\frac{g}{h}\right) = L \exp\left(\frac{g}{h}\right),$$

for some polynomial $L \in \mathbb{C}[x,y]$ of degree at most $\max\{\deg(P), \deg(Q)\} - 1$. We say that the polynomial L is the cofactor of the exponential factor $\exp(g/h)$.

For the same reason that for invariant algebraic curves, in the definition of exponential factor we always allow the function to be complex even in the case of a real polynomial system.

Now we can outline the following Darboux Integrability Theorem.

Theorem 1.17 ([DLA06])**.** ***(Darboux Integrability)*** *Suppose that system (1.1) admits p irreducible invariant algebraic curves $f_i = 0$ with cofactors K_i for $i = 1, \ldots, p$ and q exponential factors $\exp(g_j/h_j)$ with cofactors L_j for $j = 1, \ldots, q$.*

(i) There exist $\lambda_i,\mu_j \in \mathbb{R}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$ if and only if the function

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} \left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1} \cdots \left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q} \tag{1.11}$$

is a first integral of system (1.1).

(ii) *There exist* $\lambda_i, \mu_j \in \mathbb{R}$ *not all zero such that* $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -\operatorname{div}(P, Q)$ *if and only if function* (1.11) *is an integrating factor of system* (1.1).

If a system has a focus on the origin then it cannot exist a first integral in a neighborhood U of the origin. The reason is that if this first integral existed we would have that H is constant in all U, which contradicts the definition of first integral. As a consequence, in the center-focus problem analysis, if we manage to find a first integral –for example by means of Darboux Integrability Theorem– we can guarantee that the origin is a center.

Symmetries: reversibility with respect to straight lines

Another tool to determine whether a system has a center at the origin is identifying if there exists any kind of symmetry which allows to detect if orbits close on themselves or not. Even though there are many other types of reversibilities, here we will only introduce the time-reversibility with respect to straight lines, which is the one that we will use throughout this memory. For more details about the relation between the center problem and reversibility the reader is referred to [GM11]; the work [BBT21] presents a generalization of the most usual symmetries in differential equations.

Let us consider a system which has the form (1.1) such that is invariant under a time and coordinates change $(x, y, t) \to (x, -y, -t)$. This means that the orbits near the origin are symmetric with respect to the $x-$axis, so they follow a specific direction for $y > 0$ and the opposite direction but symmetrically for $y < 0$, as the left-hand side image in Figure 1.2 shows. This implies that all the orbits near the origin will close on themselves defining periodic orbits, which proves that the system has a center at the origin. On the other hand, if the system remains invariant when applying a transformation $(x, y, t) \to (-x, y, -t)$ then the orbits near the origin are symmetric with respect to the $y-$axis, as the right-hand side image in Figure 1.2 shows. Therefore, by an analogous reasoning to the previous case we deduce that this system also has a center at the origin.

Two monomial differential equations

In this subsection, a center characterization technique is introduced for the family of differential equations

$$\dot{z} = \mathrm{i}\, z + A z^k w^l + B z^m w^n, \tag{1.12}$$

where $k + l \leq m + n$, $(k, l) \neq (m, n)$ and $A, B \in \mathbb{C}$. The integer values defined as

$$\alpha = k - l - 1, \quad \beta = m - n - 1,$$

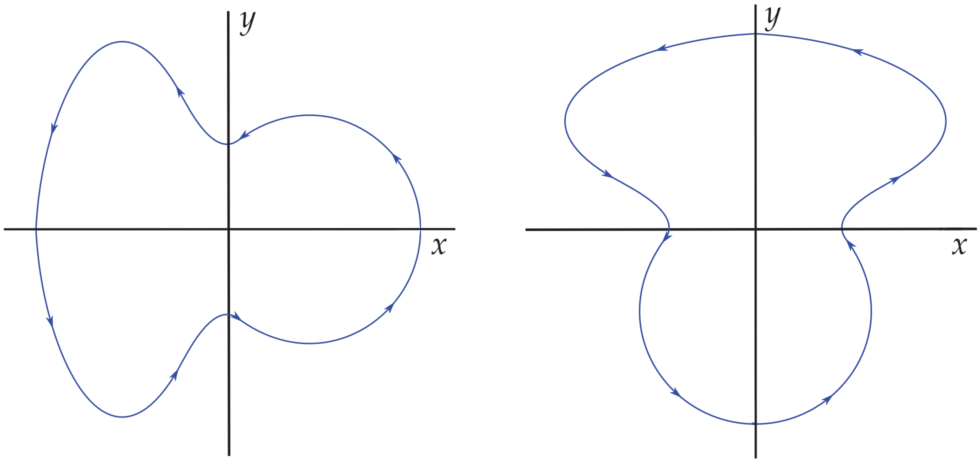

FIGURE 1.2: Periodic orbits which are symmetric with respect to the coordinate axes.

will play a key role in our study. One of the reasons for this special role of both numbers is that when $\alpha = 0$ (resp. $\beta = 0$) the monomial $z^k w^l$ (resp. $z^m w^n$) appears as a resonant monomial.

The center characterization result we present here has been extracted from [GGT16], and it is formulated as follows.

Theorem 1.18 ([GGT16]). *The origin of equation* (1.12) *is a center when one of the following (nonexclusive) conditions holds:*

(i) $k = n = 2$ *and* $l = m = 0$;

(ii) $l = n = 0$;

(iii) $k = m$ *and* $(l - n)\alpha \neq 0$.

Although these center conditions are only valid for the particular example of equations which have the form (1.12), they will be useful in some examples which we will analyze in later sections.

Inverse integrating factor

To end this center characterization part, we will recall the notion of inverse integrating factor. Let U be an open subset of $\mathbb{R}^2$. A class $\mathcal{C}^1(U)$ function $V : U \to \mathbb{R}$ is an *inverse integrating factor* of system (2.1) if V verifies the partial differential equation

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) V \tag{1.13}$$

in U. The name inverse integrating factor arises from the fact that if $V(x,y)$ satisfies (1.13) then its reciprocal $1/V(x,y)$ is an integrating factor of (2.1). This implies that the system can be transformed into an integrable system by means of being multiplied by $1/V(x,y)$. For more information the reader is referred to [DLA06].

1.1.3 The cyclicity problem

The second problem that we are dealing with in this chapter is the cyclicity problem, which aims to determine the maximum number of limit cycles that unfold in a neighborhood of the origin when slightly perturbing a system whose origin is a center or a focus. The problem can be outlined as follows: given a differential system whose origin is a monodromic nondegenerate point for a certain set of parameters λ_0, which is the maximum number of limit cycles that will bifurcate when perturbing the problem parameters? Applying Proposition 1.9, we can reformulate the problem as, given a set of parameters $\lambda_0 \in \mathbb{C}^d$ such that

$$\Pi(\rho,\lambda_0) - \rho = V_k \rho^k + \sum_{j=k+1}^{\infty} V_j \rho^j, \tag{1.14}$$

with odd k and being $V_k \neq 0$ the first nonzero Lyapunov constant if the origin is a focus, or

$$\Pi(\rho,\lambda_0) - \rho = 0,$$

if the origin is a center, which is the maximum number of positive zeros that function $\Pi(\rho,\lambda) - \rho$ will have for $\lambda \sim \lambda_0$?

In order to analyze the problem let us consider system (1.1) slightly perturbed with nonzero trace: we choose $\alpha \sim 0$ but $\alpha \neq 0$, this is, α arbitrarily close to 0, and rewrite the system as

$$\begin{cases} \dot{x} = \alpha x - y + X(x,y), \\ \dot{y} = x + \alpha y + Y(x,y), \end{cases}$$

so the linear part has trace $2\alpha \sim 0$, $\alpha \neq 0$. We proceed analogously to Subsection 1.1.2: rewrite the system in polar coordinates, and we obtain expressions which are similar to (1.4) and (1.5):

$$\begin{aligned} \frac{dr}{dt} &= \alpha r + r^2\, \widetilde{P_2}(\varphi) + r^3\, \widetilde{P_3}(\varphi) + \cdots, \\ \frac{d\varphi}{dt} &= 1 + r\, \widetilde{Q_2}(\varphi) + r^2\, \widetilde{Q_3}(\varphi) + \cdots. \end{aligned}$$

Now divide both equations and expand in power series,

$$\frac{dr}{d\varphi} = \frac{\alpha r + r^2\,\widetilde{P}_2(\varphi) + r^3\,\widetilde{P}_3(\varphi) + \cdots}{1 + r\,\widetilde{Q}_2(\varphi) + r^2\,\widetilde{Q}_3(\varphi) + \cdots} = \alpha r + \widetilde{R}_2(\varphi,\alpha)\,r^2 + \widetilde{R}_3(\varphi,\alpha)\,r^3 + \cdots, \tag{1.15}$$

for certain $\widetilde{R}_i(\varphi,\alpha)$. Let $r(\varphi,\rho,\alpha)$ be the solution of (1.15) such that $r(0,\rho,\alpha) = \rho$. This solution is analytic in ρ, which is the initial value, so it can be expanded as

$$r(\varphi,\rho,\alpha) = \mathrm{e}^{\alpha\varphi}\,\rho + \sum_{j=2}^{\infty} \widetilde{u}_j(\varphi,\alpha)\,\rho^j.$$

Let us set $W_j(\alpha) := \widetilde{u}_j(2\pi,\alpha)$, so we can rewrite the previous expression evaluated at 2π as

$$\Pi(\rho,\alpha) - \rho = \left(\mathrm{e}^{2\pi\,\alpha} - 1\right)\rho + \sum_{j=2}^{\infty} W_j(\alpha)\,\rho^j,$$

where $\Pi(\rho,\alpha) := r(2\pi,\rho,\alpha)$ is the Poincaré map. Notice that, as expected, if we set the trace to be zero we recover expression (1.7), being V_j equal to W_j for $\alpha = 0$. If our differential equation has parameters $\lambda \in \mathbb{C}^d$, the trace α will depend on them, $\alpha = \alpha(\lambda)$, so we can actually write $\Pi(\rho,\lambda)$ instead of $\Pi(\rho,\alpha)$, where λ are the parameters of the original system.

Assume that $\alpha(\lambda_0) = 0$ for certain λ_0 and that the origin is a focus ($V_k \neq 0$), so we have expression (1.14). We aim to study what occurs when the system is perturbed, taking $\lambda \sim \lambda_0$. We define a displacement map $\Delta(\rho,\lambda) := \Pi(\rho,\lambda) - \rho$ and consider $\lambda \sim \lambda_0$:

$$\Delta(\rho,\lambda) = \left(\mathrm{e}^{2\pi\,\alpha(\lambda)} - 1\right)\rho + \sum_{j=2}^{\infty} W_j(\lambda)\,\rho^j, \tag{1.16}$$

$$\Delta(\rho,\lambda_0) = V_k\rho^k + \sum_{j=k+1}^{\infty} V_j\rho^j.$$

Notice that, if $\lambda = \lambda_0$ and hence $\alpha = 0$, all $W_j(\alpha = 0) = V_j$ are polynomials in λ_0. Now we can apply on function $\Delta(\rho,\lambda)$ the so-called Weierstrass Preparation Theorem, proved for example in [Wal04] and outlined as follows.

Theorem 1.19 ([Wal04]). ***(Weierstrass Preparation Theorem)*** *Let $f(x,\lambda)$ be an analytic function with $x \in \mathbb{C}$ and $\lambda \in \mathbb{C}^d$ near the origin. Let k be the lowest integer such that*

$$f(0,0) = 0,\ \frac{\partial f}{\partial x}(0,0) = 0, \ldots, \frac{\partial^{k-1} f}{\partial x^{k-1}}(0,0) = 0,\ \frac{\partial^k f}{\partial x^k}(0,0) \neq 0.$$

Then, near the origin, function f can be written in a unique way as a product of an analytic function c which is nonidentically zero at the origin by an analytic function *which consists of a kth degree polynomial in x, i.e.*

$$f(x,\lambda) = c(x,\lambda)\left(x^k + a_{k-1}(\lambda)x^{k-1} + \cdots + a_1(\lambda)x + a_0(\lambda)\right),$$

where functions $c(x,\lambda)$ and $a_i(\lambda)$ are analytic and $c(x,\lambda)$ is nonidentically zero at the origin.

When perturbing a focus, the first derivative of the displacement map $\Delta(\rho,\lambda)$ which does not vanish in $(\rho = 0, \lambda = \lambda_0)$ is that of order k, since $V_k \neq 0$. Therefore, applying the Weierstrass Preparation Theorem, we can rewrite $\Delta(\rho,\lambda)$ as

$$\Delta(\rho,\lambda) = c(\rho,\lambda)\left(\rho^k + a_{k-1}(\lambda)\rho^{k-1} + \cdots + a_1(\lambda)\rho\right),$$

for $(\rho,\lambda) \sim (0,\lambda_0)$ and where the involved functions are analytic. Notice that, according to (1.16), using the theorem notation we have $a_0(\lambda) = 0$.

As a consequence, the problem of finding the positive zeros of $\Delta(\rho,\lambda)$ for $(\rho,\lambda) \sim (0,\lambda_0)$ reduces to finding the positive zeros of $\rho^k + a_{k-1}(\lambda)\rho^{k-1} + \cdots + a_1(\lambda)\rho$, since function $c(\rho,\lambda)$ does not vanish near λ_0. Our aim is then to solve equation

$$\rho^k + a_{k-1}(\lambda)\rho^{k-1} + \cdots + a_1(\lambda)\rho = 0. \tag{1.17}$$

The following lemma, which is proved in [And+73], will be helpful.

Lemma 1.20. *With the previous notation, the displacement map satisfies $\Delta(-\rho) = -\Delta(\rho)$.*

One of the k solutions of (1.17) is $\rho = 0$. Applying Lemma 1.20 we have that if $\Delta(\rho_0) = 0$ then $\Delta(-\rho_0) = 0$, so every positive solution is associated with a negative one and vice versa. As we also have $\Delta(0) = 0$, we can conclude that the number of positive zeros is at most $\frac{k-1}{2}$, being $k \geq 3$ is odd, and this will be the maximum number of limit cycles which can unfold.

Finally, let us see what happens when the origin is a center. According to (1.8), there will be a certain m such that, if $L_n = 0$ for every $n \leq m$, the origin is already a center. In this case, $\Delta(\rho,\lambda_0) \equiv 0$, so all the successive derivatives are also identically zero and the Weierstrass Preparation Theorem cannot be applied to complete the cyclicity study. We will see in a later section an example of cubic polynomials for which we will justify that if the ideal $\langle L_1, \ldots, L_m \rangle$ is radical then the problem can be solved in a relatively simple way. However, if the ideal is not radical the problem gets much more complicated, as happens for example with the quadratic polynomials that Bautin studied in [Bau52], which we will also analyze.

For this center case, we will end by providing a theorem originally proved by C. Christopher in [Chr05], which uses linear parts of Lyapunov constants of a center to study the cyclicity of the system, and this result will be highly useful in some of our studies.

Theorem 1.21 ([GT21])**.** *Suppose that S is a point on the center variety and that the first k Lyapunov constants, $L_1, \ldots, L_k$, have independent linear parts (with respect to the expansion of L_i about S), then S lies on a component of the center variety of codimension at least k and there are bifurcations which produce k limit cycles locally from the center corresponding to the parameter value S. If, furthermore, we know that S lies on a component of the center variety of codimension k, then S is smooth point of the variety, and the cyclicity of the center for the parameter value S is exactly k. In the latter case, k is also the cyclicity of a generic point on this component of the center variety.*

1.2 Lyapunov constants computation

At this point, the importance of Lyapunov constants in the center and cyclicity problems has been made clear, as they are the main mathematical object to tackle such problems. In this section we aim to present some methods which allow to compute these quantities, as well as their computational implementation.

The main technique we will see is the Lyapunov method, which uses Lyapunov functions[2] to find the quantities. This method will be introduced in Subsection 1.2.1 and implemented afterwards. Later, a new method for more complicated systems is presented. It is based on applying the previous Lyapunov method to some simple systems and using interpolation so as to obtain the Lyapunov constants of the original differential equation.

Apart from the two approaches we will see here, there are many other procedures to compute the Lyapunov constants. Another interesting technique is the Andronov method, which is actually the classical technique to find Lyapunov constants. This method consists on writing the system in polar coordinates and, by means of derivation, obtain equivalent expressions whose coefficients can be equalized, which gives some integrals which can be solved in order to obtain the Lyapunov constants; see [And+73] for more details. This method is not useful for us at the current stage at a practical level due to the size of the involved expressions. It follows the idea introduced in Subsection 1.1.2, and we will give an overview of it in the last chapter of this thesis because it will be interesting from the theoretical point of view for the topics addressed there. It can be proved that Andronov method is equivalent to Lyapunov method, as stability and center conditions cannot depend on the used procedure.

[2]Given a differential system in $\mathbb{R}^n$ with and an open subset $U \subset \mathbb{R}^n$, we define a Lyapunov function (resp. a strict Lyapunov function) for an equilibrium point $x_0 \in \mathbb{R}^n$ as a scalar function $F : \mathbb{R}^n \to \mathbb{R}$ such that is continuous, has continuous derivatives, $F(x_0) = 0$ but $F(x) > 0$ locally near x_0 and $\dot{F}(x(t)) \leq 0$ (resp. $\dot{F}(x(t)) < 0$) locally near x_0.

1.2.1 The Lyapunov method

This method is based on the utilization of a Lyapunov function type of system (1.1). The computation could be made using real values –see [Shi81]–, but the obtained expressions are shorter if complex coordinates are considered. For this reason, here the method will be developed using complex variables. The objective is then to find a Lyapunov function type F of system (1.2),

$$F = F_2 + F_3 + F_4 + \cdots, \tag{1.18}$$

with F_k an homogeneous kth degree polynomial. Let us start with degree 2. We aim to study the sign of $\dot{F}$ and whether it vanishes or not. We compute

$$\dot{F} = F_z\,\dot{z} + F_w\,\dot{w} = F_z\,(\mathrm{i}\,z + Z(z,w)) + F_w\,(-\mathrm{i}\,w + \overline{Z(z,w)}) = \sum_{k\geq 1} L_k\,(zw)^{k+1}. \tag{1.19}$$

The last equality is a consequence of the following theorem applied to (1.2):

Theorem 1.22 ([DLA06]). *Given the system of differential equations* (1.1) *(resp.* (1.2)*), if there exists a first integral F of the system, then in suitable coordinates F is analytic on x^2+y^2 (resp. on zw). As a consequence, $\dot{F}$ is also analytic on x^2+y^2 (resp. on zw).*

Observe that, in expression (1.19), if all L_k vanish then $\dot{F} = 0$, and therefore F is a first integral so the origin is a center. Otherwise, if any L_k is nonzero, according to Lyapunov Stability Theorem the origin will be a focus, either attracting or repelling depending on the sign of the first nonzero L_k. Therefore, these coefficients L_k are actually the Lyapunov constants, maybe differing from those presented in Section 1.1 by a multiplicative constant.

Now we will show how to find F recursively. We impose equation (1.19) and perform formal operations as follows,

$$\begin{aligned} &(F_{2z} + F_{3z} + F_{4z} + \cdots)\,(\mathrm{i}\,z + Z_2 + Z_3 + Z_4 + \cdots) + \\ &+ (F_{2w} + F_{3w} + F_{4w} + \cdots)\,(-\mathrm{i}\,w + \overline{Z_2} + \overline{Z_3} + \overline{Z_4} + \cdots) = \\ &= L_1\,(zw)^2 + L_2\,(zw)^3 + L_3\,(zw)^4 + \cdots. \end{aligned}$$

Here we equal those terms having the same degree.

- 2nd degree:

$$\begin{aligned} \mathrm{i}\,z\,F_{2z} - \mathrm{i}\,w\,F_{2w} &= 0, \\ z\,F_{2z} - w\,F_{2w} &= 0. \end{aligned}$$

  By deriving the corresponding polynomials and making their coefficients equal, it is easy to see that the solution to this equation is $F_2(z,w) = c\,zw$ for any constant $c \in \mathbb{C}$. We set for example $c = 1/2$, and we obtain

$$F_2(z,w) = \frac{zw}{2}.$$

- 3rd degree:
$$\mathrm{i}\, z\, F_{3z} - \mathrm{i}\, w\, F_{3w} + Z_2\, F_{2z} + \overline{Z_2}\, F_{2w} = 0.$$
The term $Z_2\, F_{2z} + \overline{Z_2}\, F_{2w}$ is already known, so by writing F_3 as an homogeneous 3rd degree polynomial in z and w and unknown coefficients, taking the derivative and using the previous equation F_3 can be determined, if there exists a solution.

- 4th degree:
$$\mathrm{i}\, z\, F_{4z} - \mathrm{i}\, w\, F_{4w} + Z_3\, F_{2z} + \overline{Z_3}\, F_{2w} + Z_2\, F_{3z} + \overline{Z_2}\, F_{3w} = L_1\, (zw)^2.$$
We operate analogously to the 3rd degree case.

Using this reasoning and notation $\phi_{lk} := F_{lz}\, Z_k + F_{lw}\, \overline{Z_k}$ we can write the pth degree equation as follows:

$$-\,\mathrm{i}\, z\, F_{pz} + \mathrm{i}\, w\, F_{pw} = \sum_{k=2}^{p-1} \phi_{p-k+1,k} - L_{\frac{p}{2}-1}\, (zw)^{\frac{p}{2}}, \tag{1.20}$$

with $L_{\frac{p}{2}-1} = 0$ if p is odd. As we will see, for even p the system matrix will have zero determinant. Therefore, so that the system has a solution we must force a suitable independent term so that the system is compatible. Now we will write this equation for degree p (1.20) in matrix form, and we will see how the whole problem can be reduced to solve a simple system of linear equations. Let us denote

$$F_p(z,w) := \sum_{j=0}^{p} h_{p-j,j}\, z^{p-j} w^j. \tag{1.21}$$

Now we derive F_p with respect to z and w:

$$F_{pz} = \sum_{j=0}^{p} (p-j)\, h_{p-j,j}\, z^{p-j-1} w^j, \;\; F_{pw} = \sum_{j=0}^{p} j\, h_{p-j,j}\, z^{p-j} w^{j-1}.$$

Thus,

$$
\begin{aligned}
-\mathrm{i}\, z\, F_{pz} + \mathrm{i}\, w\, F_{pw} &= -\mathrm{i}\, z \left(\sum_{j=0}^{p} (p-j)\, h_{p-j,j}\, z^{p-j-1} w^{j} \right) + \mathrm{i}\, w \left(\sum_{j=0}^{p} j\, h_{p-j,j}\, z^{p-j} w^{j-1} \right) \\
&= \mathrm{i} \sum_{j=0}^{p} \left(-(p-j)\, h_{p-j,j}\, z^{p-j} w^{j} + j\, h_{p-j,j}\, z^{p-j} w^{j} \right) \\
&= \mathrm{i} \sum_{j=0}^{p} (2j-p)\, h_{p-j,j}\, z^{p-j} w^{j}.
\end{aligned}
$$

Then, substituting in equation (1.20) we obtain

$$
\mathrm{i} \sum_{j=0}^{p} (2j-p)\, h_{p-j,j}\, z^{p-j} w^{j} = \sum_{k=2}^{p-1} \phi_{p-k+1,k} - L_{\frac{p}{2}-1}\, (zw)^{\frac{p}{2}},
$$

and multiplying both members by $-\mathrm{i}$ we finally obtain that equation (1.20) can be rewritten as

$$
\sum_{j=0}^{p} (2j-p)\, h_{p-j,j}\, z^{p-j} w^{j} = -\mathrm{i} \sum_{k=2}^{p-1} \phi_{p-k+1,k} + \mathrm{i}\, L_{\frac{p}{2}-1}\, (zw)^{\frac{p}{2}}. \tag{1.22}
$$

Our aim is to determine the coefficients $h_{p-j,j}$ of the pth degree term F_p in the Lyapunov function F. By equating coefficients in equation (1.22), we can outline a simple diagonal system of linear equations as follows.

- If p is odd, there is no $L_{\frac{p}{2}-1}$ and then the system can be written as

$$
\begin{pmatrix}
-p & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
0 & -p+2 & 0 & & & & & 0 \\
\vdots & \ddots & \ddots & \ddots & & & & \vdots \\
0 & & 0 & -1 & 0 & & & 0 \\
0 & & & 0 & 1 & 0 & & 0 \\
\vdots & & & & \ddots & \ddots & \ddots & \vdots \\
0 & & & & & 0 & p-2 & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & p
\end{pmatrix}
\begin{pmatrix}
h_{p,0} \\ h_{p-1,1} \\ \vdots \\ h_{\frac{p+1}{2},\frac{p-1}{2}} \\ h_{\frac{p-1}{2},\frac{p+1}{2}} \\ \vdots \\ h_{1,p-1} \\ h_{0,p}
\end{pmatrix}
=
\begin{pmatrix}
\widetilde{\phi_0} \\ \widetilde{\phi_1} \\ \vdots \\ \widetilde{\phi_{\frac{p-1}{2}}} \\ \widetilde{\phi_{\frac{p+1}{2}}} \\ \vdots \\ \widetilde{\phi_{p-1}} \\ \widetilde{\phi_p}
\end{pmatrix},
$$

where $\widetilde{\phi_j}$ are the coefficients corresponding to $-\mathrm{i}\sum_{k=2}^{p-1} \phi_{p-k+1,k}$ in equation

(1.22) and they are known values. Let us observe that, in this case, the system has a unique solution and values $h_{p-j,j}$ can be trivially computed as

$$h_{p-j,j} = \frac{\widetilde{\phi}_j}{2j-p}. \tag{1.23}$$

- If p is even, then the system can be written as

$$\begin{pmatrix} -p & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & -p+2 & 0 & & & & & & 0 \\ \vdots & \ddots & \ddots & \ddots & & & & & \vdots \\ 0 & & 0 & -2 & 0 & & & & 0 \\ 0 & & & 0 & 0 & 0 & & & 0 \\ 0 & & & & 0 & 2 & 0 & & 0 \\ \vdots & & & & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & & & & 0 & p-2 & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & p \end{pmatrix} \begin{pmatrix} h_{p,0} \\ h_{p-1,1} \\ \vdots \\ h_{\frac{p}{2}+1,\frac{p}{2}-1} \\ h_{\frac{p}{2},\frac{p}{2}} \\ h_{\frac{p}{2}-1,\frac{p}{2}+1} \\ \vdots \\ h_{1,p-1} \\ h_{0,p} \end{pmatrix} = \begin{pmatrix} \widetilde{\phi}_0 \\ \widetilde{\phi}_1 \\ \vdots \\ \widetilde{\phi_{\frac{p}{2}-1}} \\ \widetilde{\phi_{\frac{p}{2}}} + \mathrm{i}\, L_{\frac{p}{2}-1} \\ \widetilde{\phi_{\frac{p}{2}+1}} \\ \vdots \\ \widetilde{\phi_{p-1}} \\ \widetilde{\phi}_p \end{pmatrix},$$

where $\widetilde{\phi}_j$ are the coefficients corresponding to $-\mathrm{i}\sum_{k=2}^{p-1} \phi_{p-k+1,k}$ in equation (1.22) and, as before, they are known. In the same way as in the odd case, coefficients $h_{p-j,j}$ for $j \neq \frac{p}{2}$ can be trivially determined using expression (1.23). Observe that for $j = \frac{p}{2}$ the equation

$$0\, h_{\frac{p}{2},\frac{p}{2}} = \widetilde{\phi_{\frac{p}{2}}} + \mathrm{i}\, L_{\frac{p}{2}-1},$$

is obtained, so $h_{\frac{p}{2},\frac{p}{2}}$ remains as a free parameter, and for the sake of simplicity we set $h_{\frac{p}{2},\frac{p}{2}} = 0$. This equation also allows to find the Lyapunov constant $L_{\frac{p}{2}-1}$ as

$$L_{\frac{p}{2}-1} = \mathrm{i}\, \widetilde{\phi_{\frac{p}{2}}}.$$

Example

Let us provide an example to illustrate the presented Lyapunov method. Consider the system of differential equations

$$\begin{cases} \dot{x} = -y + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5, \\ \dot{y} = x, \end{cases} \tag{1.24}$$

for $a_2, a_3, a_4, a_5 \in \mathbb{R}$. We will show how to use the explained Lyapunov method to find some coefficients of a Lyapunov function F of this system and its first Lyapunov constant. Using the notation of equation (1.1), we have that $X(x,y) =$

$a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$ and $Y(x,y) = 0$. Writing in complex coordinates we obtain the system

$$\dot{z} = \mathrm{i}\, z + a_2 \left(\frac{z+w}{2}\right)^2 + a_3 \left(\frac{z+w}{2}\right)^3 + a_4 \left(\frac{z+w}{2}\right)^4 + a_5 \left(\frac{z+w}{2}\right)^5,$$

and we have that $Z_k(z,w) = \overline{Z_k(z,w)} = a_k \left(\frac{z+w}{2}\right)^k$, for $k = 2,3,4,5$.

- 2nd degree. We have seen that the 2nd degree term of the Lyapunov function F (expression (1.18)) is

$$F_2 = \frac{zw}{2}.$$

- 3rd degree. We write the 3rd degree term of the Lyapunov function F as

$$F_3 = h_{30}z^3 + h_{21}z^2w + h_{12}zw^2 + h_{03}w^3,$$

where $h_{kj} = a_{kj} + \mathrm{i}\, b_{kj}$. We can find ϕ_{22} by doing

$$\phi_{22} = F_{2z}Z_2 + F_{2w}\overline{Z_2} = \frac{w}{2}a_2 \left(\frac{z+w}{2}\right)^2 + \frac{z}{2}a_2 \left(\frac{z+w}{2}\right)^2 = a_2 \left(\frac{z+w}{2}\right)^3.$$

We can write the system matrix introduced in the method explanation, and we can then compute the coefficients h_{kj} by means of (1.23),

$$\begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} h_{30} \\ h_{21} \\ h_{12} \\ h_{03} \end{pmatrix} = \begin{pmatrix} -\mathrm{i}\,\frac{1}{8}a_2 \\ -\mathrm{i}\,\frac{3}{8}a_2 \\ -\mathrm{i}\,\frac{3}{8}a_2 \\ -\mathrm{i}\,\frac{1}{8}a_2 \end{pmatrix},$$

$$h_{30} = \mathrm{i}\,\frac{1}{24}a_2, \qquad h_{21} = \mathrm{i}\,\frac{3}{8}a_2, \qquad h_{12} = -\mathrm{i}\,\frac{3}{8}a_2, \qquad h_{03} = -\mathrm{i}\,\frac{1}{24}a_2. \tag{1.25}$$

Then,

$$F_3 = \mathrm{i}\,\frac{a_2}{24}\left(z^3 + 9z^2w - 9zw^2 - w^3\right).$$

- 4th degree. The 4th degree term of F is

$$F_4 = h_{40}z^4 + h_{31}z^3w + h_{22}z^2w^2 + h_{13}zw^3 + h_{04}w^4.$$

Let us find ϕ_{23} and ϕ_{32}:

$$\begin{aligned}\phi_{23} =& F_{2z}Z_3 + F_{2w}\overline{Z_3} = \frac{w}{2}a_3\left(\frac{z+w}{2}\right)^3 + \frac{z}{2}a_3\left(\frac{z+w}{2}\right)^3 = a_3\left(\frac{z+w}{2}\right)^4,\\ \phi_{32} =& F_{3z}Z_2 + F_{3w}\overline{Z_2} =\\ &= \mathrm{i}\,\frac{a_2}{24}(3z^2+18zw-9w^2)a_2\left(\frac{z+w}{2}\right)^2 +\\ &+ \mathrm{i}\,\frac{a_2}{24}(9z^2-18zw-3w^2)a_2\left(\frac{z+w}{2}\right)^2 = \mathrm{i}\,{a_2}^2(z-w)\left(\frac{z+w}{2}\right)^3.\end{aligned}$$

Therefore, the degree 4 system can be written as follows and the coefficients can be found,

$$\begin{pmatrix} -4 & 0 & 0 & 0 & 0\\ 0 & -2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 0 & 4\end{pmatrix}\begin{pmatrix} h_{40}\\ h_{31}\\ h_{22}\\ h_{13}\\ h_{04}\end{pmatrix} = \begin{pmatrix} \frac{1}{8}{a_2}^2 - \frac{1}{16}\,\mathrm{i}\,a_3\\ \frac{1}{4}{a_2}^2 - \frac{1}{4}\,\mathrm{i}\,a_3\\ \mathrm{i}\,L_1 - \frac{3}{8}\,\mathrm{i}\,a_3\\ -\frac{1}{4}{a_2}^2 - \frac{1}{4}\,\mathrm{i}\,a_3\\ -\frac{1}{8}{a_2}^2 - \frac{1}{16}\,\mathrm{i}\,a_3\end{pmatrix},$$

$$h_{40} = -\frac{1}{32}{a_2}^2 + \frac{1}{64}\,\mathrm{i}\,a_3, \qquad h_{31} = -\frac{1}{8}{a_2}^2 + \frac{1}{8}\,\mathrm{i}\,a_3,$$
$$h_{22} = 0 \text{ (we impose)},$$
$$h_{13} = -\frac{1}{8}{a_2}^2 - \frac{1}{8}\,\mathrm{i}\,a_3, \qquad h_{04} = -\frac{1}{32}{a_2}^2 - \frac{1}{64}\,\mathrm{i}\,a_3,$$
$$L_1 = \frac{3}{8}a_3. \tag{1.26}$$

Then, the coefficients of the 4th degree terms of the Lyapunov function have been obtained, as well as the first Lyapunov constant. In the same recurrent way one can continue to find the expressions for the following Lyapunov constants up to the desired order; we have applied this to obtain the expressions of L_2 and L_3:

$$\begin{aligned} L_2 &= -\frac{53}{32}a_3a_2^2 + \frac{5}{16}a_5,\\ L_3 &= \frac{31393}{3072}a_3a_2^4 - \frac{215}{96}a_2^2a_5 - \frac{1673}{384}a_2a_3a_4 - \frac{663}{2048}a_3^3.\end{aligned}$$

1.2.2 Interpolation technique

Let us consider the system of differential equations in complex coordinates (1.2),

$$\dot{z} = \mathrm{i}\, z + Z_2(z,w) + Z_3(z,w) + Z_4(z,w) + \cdots ,$$

being every $Z_k(z,w)$ an homogeneous kth degree polynomial having the form $Z_k(z,w) = \sum_{j=0}^{k} r_{k-j,j} z^{k-j} w^j$, where $r_{k-j,j} \in \mathbb{C}$. Recall that w denotes the conjugate of z, this is $w = \overline{z}$.

Before presenting our results and our method, we need some introductory definitions.

Definition 1.23. *M is a monomial of (1.2) when $M = \prod_{k,l} r_{k,l}^{m_{k,l}} \overline{r}_{k,l}^{n_{k,l}}$, with $m_{k,l}, n_{k,l} \in \mathbb{N}$, where the product is finite and $r_{k,l}$ is any coefficient of $Z_{k+l}(z,w)$.*

Definition 1.24. *A monomial M of (1.2) is a monomial of the nth Lyapunov constant L_n if the expression of L_n has a term with either $\mathrm{Re}(M)$ or $\mathrm{Im}(M)$.*

Definition 1.25. *Let M be a monomial as defined above. We define the degree, $\deg(M)$, the quasi degree, $\mathrm{qdeg}(M)$, and the weight of M, $\mathrm{w}(M)$, respectively, as*

$$\begin{aligned} \deg(M) &= \sum_{k,l} (m_{k,l} + n_{k,l}), \\ \mathrm{qdeg}(M) &= \sum_{k,l} (k + l - 1)(m_{k,l} + n_{k,l}), \\ \mathrm{w}(M) &= \sum_{k,l} (1 - k + l)(m_{k,l} - n_{k,l}). \end{aligned}$$

Definition 1.26. *A monomial M of weight zero is basic if $M'|M$ and $\mathrm{w}(M') = 0$ imply that $M' = \pm M$. In other words, the basic monomials are the prime factors of the monomials of weight zero.*

With the above notation, the following result is well known; see [Cim+97; LL90; LL91; Sib76; Zol94].

Theorem 1.27 ([GGM99])**.** *Let M be a monomial of the Lyapunov constant L_n. Then $\mathrm{qdeg}(M) = 2n$ and $\mathrm{w}(M) = 0$.*

This theorem gives some information about the monomials that appear in the Lyapunov constants. [GGM99] proves a result which improves Theorem 1.27 by describing how these monomials are distributed according to their degree. This improvement is due to the fact that the new theorem restricts even more the monomials which can appear in the Lyapunov constant. The result is as follows.

Theorem 1.28 ([GGM99]). *Let $M_1, \ldots, M_k$ be monomials of L_n with even degree, and $M_{k+1}, \ldots, M_{k+l}$ monomials of L_n with odd degree. Then*

$$L_n = \sum_{i=1}^{k} \alpha_i \operatorname{Im}(M_i) + \sum_{i=k+1}^{k+l} \beta_i \operatorname{Re}(M_i)$$

for some $\alpha_i, \beta_i \in \mathbb{R}$.

Observe that, for any differential equation, the Lyapunov constants L_n are real numbers. Therefore, if M is a monomial of (1.2), then L_n must have the form $L_n = \alpha M + \overline{\alpha}\overline{M} + N$, where N denotes the sum of the other monomials appearing in the expression, and hence $L_n = 2\operatorname{Re}(\alpha)\operatorname{Re}(M) - 2\operatorname{Im}(\alpha)\operatorname{Im}(M) + N$. As a consequence, Theorem 1.28 reduces by half the estimation of the length of the Lyapunov constants obtained using only the monomials predicted by Theorem 1.27.

We present now a method to compute the general formula of the Lyapunov constants via interpolation. Let us suppose that we want to find the expression of the nth Lyapunov quantity L_n for a differential equation of the form (1.2). We proceed as follows:

1. By using Theorems 1.27 and 1.28, we list all the monomials involved in L_n, that is, we write L_n as a linear function of products of basic monomials and their unknown coefficients.

2. Once the monomials are listed, we search all the undetermined coefficients by computing the Lyapunov constants for some particular systems. This can be done by applying the Lyapunov mehtod introduced in Section 1.2.1. Then, by interpolation, we obtain the general expression of the constant L_n.

Finally, let us briefly justify the use of interpolation. One can wonder why to use interpolation if the Lyapunov method from the previous section already computes the Lyapunov constants. The answer is that, for some cumbersome differential equations, or polynomial equations whose coefficients contain many parameters, using the Lyapunov method can be slow and inefficient, and even some problems could not be solved. In this case, the interpolation technique described here reduces the problem to applying the Lyapunov method to many simple differential equations adequately chosen, and then find the Lyapunov constant for the original equation. Therefore, a complicated and slow problem can be split into several simple and faster problems which can even be parallelized; then, when these simple problems have been solved, the solution of the initial problem can be found by means of interpolation.

1.2.3 Computational implementation and parallelization

The aim of this subsection is to perform the computational implementation of the previously presented methods, as well as to introduce parallelization tools in order to increase computational efficiency and highly reduce execution times. Before starting we would like to point out the following. The parallelization approaches of these methods are developed for the computation of Lyapunov constants of systems in the plane, which we have seen that are the principal tool to deal with the center and cyclicity problems. Such parallelization will enable us to get some new results about Lyapunov constants in Section 1.4. Nevertheless, the center and cyclicity problems in $\mathbb{R}^2$ have been exhaustively studied for decades, and we barely present new results in this sense apart from using some known systems to check our implementations and studying a quartic and a quintic system in Subsections 1.3.3 and 1.3.4, respectively. The idea is rather to consolidate the computing and parallelization mechanisms so that they can be extrapolated to limit cycles in $\mathbb{R}^3$ in Section 1.5 and to the calculation of period constants to deal with isochronicity and criticality in Chapter 2, where we do provide more powerful results.

Programming language choice

The two methods to calculate Lyapunov constants presented in the previous section have been computationally implemented. The programming languages selected for this purpose have been Maple and PARI/GP, or simply PARI. Maple ([Map]) is a symbolic and numeric computing environment as well as a multi-paradigm programming language. PARI ([Par]) is a specialized computer algebra system which, according to its creators, is designed to users whose primary need is speed, since its main advantage is execution velocity. This software is suitable when working with rational numbers with a lot of digits, and works properly when not much polynomial algebra is needed. However, although quite an amount of symbolic manipulation is possible, PARI does badly compared to systems like Maple. This is the reason why some of the outputs of our PARI codes will be afterwards treated with Maple also. The Lyapunov method has been implemented in both languages, while the interpolation technique only in PARI.

Verification of the codes with some examples

After implementing the codes, we will check that they work correctly by applying them to some examples. The codes whose verification is shown here are those written in PARI, as it is the language in which we have implemented both codes. The Lyapunov method implementation in Maple has also been tested and works properly, but it is not presented here for the sake of brevity.

The first system we will consider for this verification is (1.24), for which in Subsection 1.2.1 we already found the first Lyapunov constant and the coefficients of 3rd and 4th degree terms of the Lyapunov function. Now we want to perform the same computation using the elaborated code in PARI/GP that implements the Lyapunov method, and check that the same result is obtained to verify that the program works properly. The execution in the PARI calculator is as follows:

```
? X=a2*x^2+a3*x^3+a4*x^4+a5*x^5;
? Y=0;
? Z=complex(X,Y);
? lyapunov_method(Z,1);
? L
%21 = [3/8*a3]
? h
%22 =
[                    0                    0         0           -1/24*I*a2 -1/32*a2^2 - 1/64*I*a3]

[                    0                 1/2 -3/8*I*a2 -1/8*a2^2 - 1/8*I*a3                      0]

[                    0            3/8*I*a2         0                    0                      0]

[         1/24*I*a2 -1/8*a2^2 + 1/8*I*a3         0                    0                      0]

[-1/32*a2^2 + 1/64*I*a3                    0         0                    0                      0]
```

Coefficient h_{ij} in the Lyapunov function corresponds to the component $(i+1, j+1)$ from matrix `h`, so we see that the results obtained by executing the code for these coefficients and for L_1 are the same that those we obtained by manually applying the method in (1.25) and (1.26).

Let us study a second example, extracted from [LT17]. The system is

$$\dot{z} = \mathrm{i}\, z - \frac{n}{n-2} z^n + z w^{n-1} + \mathrm{i}\, \tau_n w^n. \tag{1.27}$$

[LT17] states that for $n = 6, 8, 10, 12, 14, 16, 18$ there exist values τ_n such that equation (1.27) has a weak focus of order $n^2 + n - 2$ at the origin. Let us check this for $n = 6$:

```
? n=6;
? lyapunov_method(I*z-(n/(n-2))*z^n+z*w^(n-1)+I*tn*w^n,n^2+n-2);
? L
%25 = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
, 0, 0, 0, 0, 0, 0, 0, 0, 0, -41683536624647/2142402696*tn^7 + 1573547024015/805
15134*tn^5, 0, 0, 0, 0, 1794626188667717/16246553778*tn^9 - 12386004853480749489
787/22812598487378700*tn^7 + 30386878785608837/80095055040*tn^5]
```

Observe that

$$L_{35} = -\frac{41683536624647}{2142402696}\tau_n^7 + \frac{1573547024015}{80515134}\tau_n^5.$$

If we want this weak focus to be of order $n^2 + n - 2 = 6^2 + 6 - 2 = 40$ then L_{35} must vanish. The values of τ_n for which $L_{35} = 0$ are

$$\tau_{n_1} = 0,$$
$$\tau_{n_2} = \frac{6}{958721342366881}\sqrt{256460829573982160754100026 55},$$
$$\tau_{n_3} = -\frac{6}{958721342366881}\sqrt{25646082957398216075410002655}.$$

We have also obtained

$$L_{40} = \frac{1794626188667717}{16246553778}\tau_n^9 - \frac{1238600485348074948978 7}{22812598487378700}\tau_n^7 + \\ + \frac{30386878785608837}{80095055040}\tau_n^5.$$

Now we can check

$$L_{40}(\tau_n = \tau_{n_1}) = 0,\ L_{40}(\tau_n = \tau_{n_2}) \approx -55149 \neq 0,\ L_{40}(\tau_n = \tau_{n_3}) \approx 55149 \neq 0.$$

These results have been rounded because fractions with a lot of irrelevant digits have been obtained, and our only purpose is to see whether they vanish or not. Thus, we have checked what the article states, since we have found $\tau_n = \tau_{n_2}$ and $\tau_n = \tau_{n_3}$ such that $L_{n^2+n-2} = L_{40} \neq 0$ and $L_j = 0$ for $j < 40$, so there is a weak focus of order 40 at the origin.

We will show a last verification example to check that the interpolation technique implementation is also working as expected. Let us consider the following differential equation, also extracted from [LT17],

$$\dot{z} = \mathrm{i}\, z + w^{n-1} + z^n.$$

According to Theorem 1.3 from [LT17], the origin of this equation is an stable (resp. unstable) weak focus of order $(n-1)^2$ when n is even (resp. odd), for $3 \leq n \leq 100$. Let us check this for $n = 3, 4, 5, 6$ with the implemented PARI code. To apply the interpolation technique we can write

```
? Ln=vector((3-1)^2);
? for(q=1,(3-1)^2,Ln[q]=substvec(interpolation([[3,0],[0,3-1]],q,0),[r30,Cr30,r02,Cr02],[1,1,1,1]));
? print(Ln);
[0, 0, 0, 1]
? Ln=vector((4-1)^2);
? for(q=1,(4-1)^2,Ln[q]=substvec(interpolation([[4,0],[0,4-1]],q,0),[r40,Cr40,r03,Cr03],[1,1,1,1]));
? print(Ln);
[0, 0, 0, 0, 0, 0, 0, 0, -91/40]
? Ln=vector((5-1)^2);
? for(q=1,(5-1)^2,Ln[q]=substvec(interpolation([[5,0],[0,5-1]],q,0),[r50,Cr50,r04,Cr04],[1,1,1,1]));
? print(Ln);
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 167501/17325]
? Ln=vector((6-1)^2);
? for(q=1,(6-1)^2,Ln[q]=substvec(interpolation([[6,0],[0,6-1]],q,0),[r60,Cr60,r05,Cr05],[1,1,1,1]));
? print(Ln);
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3991590097/62741952]
```

Now in order to compare with the Lyapunov method, we create the following loop and obtain the same results:

```
? for(n=3,6,lyapunov_method(I*z+w^(n-1)+z^n,(n-1)^2);print(L);)
[0, 0, 0, 1]
[0, 0, 0, 0, 0, 0, 0, 0, -91/40]
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 167501/17325]
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3991590097/62741952]
```

We have stated that for a certain n there is a focus of order $(n-1)^2$. This means that the first Lyapunov constant different from 0 is the $(n-1)^2$th, and this is why for each n the first $(n-1)^2$ first constants have been computed. Observe that we obtain what we expected, as we see that for each n we have $L_k = 0$ for $k < (n-1)^2$ and $L_{(n-1)^2} \neq 0$. Furthermore, as [LT17] claims, we observe in our results that for odd n ($n = 3, 5$) then $L_{(n-1)^2} > 0$, so it is an unstable weak focus, and for even n ($n = 4, 6$) then $L_{(n-1)^2} < 0$, so it is a stable weak focus.

A first approach to parallelization

The high computational cost of the methods developed in this thesis justifies the necessity of implementing a parallelization technique on them. In this subsection, we take a first approach to the technique of parallelization by applying it to the interpolation method for a particular example, and in later sections we will develop this technique by using it in other algorithms such as finding linear parts of Lyapunov and period constants.

As previously explained, the interpolation technique intoduced above allows to split a complicated and slow problem into several simple and fast problems. Particularly, instead of applying the Lyapunov method to the original equation it can be applied to several simpler equations. Furthermore, this approach has the advantage that the many calls to this function can be parallelized, which reduces execution times and increases efficiency. Therefore, the part of the interpolation technique which can be potentially parallelized are the many executions of the Lyapunov method.

For the parallelization of codes during this thesis we will connect to Antz, the computing servers in the Mathematics Department of UAB (Universitat Autònoma de Barcelona). The software used to perform the parallelization will be PBala ([Sal]), a distributed execution software for Antz developed by Oscar Saleta, a former research support specialist in the Mathematics Department of UAB. PBala is a parallellization interface for single threaded scripts, which allows to distribute executions in Parallel Virtual Machine enabled clusters using single program multiple data paradigm. This interface lets the user execute the same script or program over multiple input data in several CPUs located at the Antz computing servers. It supports memory management so nodes do not run out of RAM due to too many processes being started in the same node. It also reports resource usage data after execution. PBala allows the parallelization of codes in many different languages: Maple, C, Python, PARI/GP, Sage, and Octave. For the current case we will use the PARI/GP option, as the interpolation code we want to parallelize is written in this language, but in later sections we will also parallelize Maple codes.

Let us see now with a bit more of detail how to perform this parallelization. First, we need to have a data file which contains the data to be passed as arguments in the parallel code. Each row is a single execution and has each value separated by a comma, line format being "tasknumber,arg1,arg2,. . . ,argN". The first value in each row must be a number and it is the task identification number, which will be stored in `taskId`. The rest of each row are the arguments to be passed to the code, which are stored in `taskArgs`. For row `i`, the arguments are contained in `taskArgs[i]`. A node file also needs to be created, which contains the number of processes to be assigned to each node. In Antz there are nine nodes, called a01, a02, . . . , a09, and the node file line format is "nodename number_of_processes".

With this, the instruction to execute the parallel code in our case will be

```
time ./PBala -eh 3 solve_lyapunov_system.gp datafile.txt nodefile.txt
results
```

The options `-eh` tell PBala that we want to generate error files –in case something goes wrong– and a slave file –that tells us which node has performed each execution. Number `3` in this line tells PBala that the language to use is Pari/GP, and each programming language is represented by a different number. Then the file containing the code to be executed, the data file and the node file must also be indicated. Finally, `results` is the output directory for storing the results and the information about each process. The option `time` at the beginning makes the execution show the runtime.

As we have already stated, we will not perform a general parallelization of the problem, but we will present a particular example of this and show how execution times are reduced with parallelization. To carry out this analysis we will consider an example from [GGM99] for computing some Lyapunov constants of the differential equation

$$\dot{z} = \mathrm{i}\,z + A\,z^2 + B\,zw + C\,w^2 + D\,z^3 + E\,z^2w + F\,zw^2 + G\,w^3 + H\,z^4 + I\,z^3w \\ + J\,z^2w^2 + K\,zw^3 + L\,w^4 + M\,z^5 + N\,z^4w + O\,z^3w^2 + P\,z^2w^3 + Q\,zw^4 + R\,w^5.$$

To simplify the problem, we will use the interpolation technique and consider a set of simpler nonlinearities to apply the Lyapunov method on them. According to [GGM99], a suitable set of 28 simple polynomial nonlinearities $Z(z, w)$ which allows to solve this problem is the following:

$$\begin{array}{cccc}
z^3w^2 & w^2 + \mathrm{i}\,zw^3 & z^2 + w^2 + zw^2 & z^2 + \mathrm{i}\,zw + (1+\mathrm{i})z^2w \\
z^3 - \mathrm{i}\,zw^2 & z^2 + zw^2 & zw + w^2 + zw^2 & z^2 + (1+\mathrm{i})zw + (1+\mathrm{i})z^2w \\
z^2 - \mathrm{i}\,z^3w & zw - z^3 & zw + w^2 + z^3 & (1+\mathrm{i})z^2 + zw + (1+\mathrm{i})z^2w \\
z^2 + \mathrm{i}\,z^2w^2 & zw + zw^2 & z^2 + w^2 + w^3 & z^2 + (1+\mathrm{i})zw + z^2w \\
zw - \mathrm{i}\,z^3w & z^2 + zw + zw^2 & zw + w^2 + w^3 & z^2 - \mathrm{i}\,zw + w^2 - z^2w \\
zw + \mathrm{i}\,z^2w^2 & z^2 + zw - z^3 & \mathrm{i}\,z^2 + w^2 & \mathrm{i}\,z^2 + zw + w^2 + z^2w \\
w^2 + \mathrm{i}\,z^4 & z^2 + w^2 - z^3 & zw + \mathrm{i}\,w^2 & \mathrm{i}\,z^2 + \mathrm{i}\,zw + w^2
\end{array}$$

These 28 polynomials are then included in the data file to be taken as the arguments of the Lyapunov method. Our aim is to compute the first N Lyapunov constants for the differential equations (1.2), taking as $Z(z, w)$ the previous 28 polynomials. This will be done with different levels of parallelization[3] and their execution times will be compared. In particular, we have taken as numbers of threads $1, 2, 4, 7, 14, 21$, and 28, and this information is given to PBala by means of the node file. Then the Lyapunov constants computation has been performed taking $N = 30$ and $N = 50$. The execution time results are shown in Table 1.1, and graphically in Figure 1.3.

TABLE 1.1: Execution times for different levels of parallelization

Number of threads	Execution time for $N = 30$ (seconds)	Execution time for $N = 50$ (seconds)
1	29.741	218.823
2	15.798	157.654
4	9.096	81.911
7	6.146	54.347
14	5.005	37.431
21	4.001	27.603
28	3.409	24.633

[3]The level of parallelization refers to the number of threads, this is, the number of parallelized tasks that the system will perform at the same time.

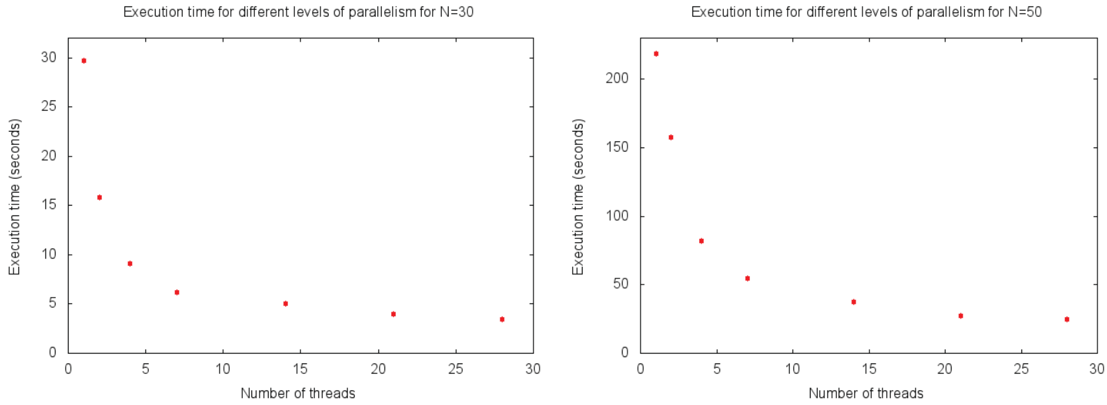

FIGURE 1.3: Execution time as a function of the level of paralellism for the computation of the first 30 Lyapunov constants (first graph) and the first 50 Lyapunov constants (second graph).

We can make some observations on the pictures in Figure 1.3. The most obvious and expectable observation is the fact that, as the level of paralellization increases, the execution time is reduced, because the execution is distributed among a greater number of threads and this makes it faster. Furthermore, as both graphs have the same shape, we can deduce that the relative improvement by parallelization is always the same independently on the problem size N.

The difference between both N is that, as can be seen in Table 1.1, when passing from $N = 30$ to $N = 50$, the execution time increases approximately at least 7 times. This shows that the Lyapunov constants computation complexity does not increse linearly with the problem size. Taking into account the Lyapunov method algorithm this is what we expect, since for computing the first N constants it is necessary to solve linear equations systems whose matrices have size $N \times N$, which quickly increases the computational cost (at least with order N^2). For this reason, as the problem size is increased parallelization becomes more useful, and even indispensable.

Moreover, we can see in Figure 1.3 that both plots become flatter as the number of threads increases, which indicates that the improvement by parallelization is lower for high levels of parallelism. This fact corroborates the so-called Amdahl's Law, a principle which states that there is a speedup limitation that makes that, for a certain number of threads, increasing even more the number of threads is not worth –see [CA12; YMG14] for more information on this topic.

This example justifies the need of parallelization and the great benefit it provides in the computation of the Lyapunov constants and, more generally, in the resolution of the center and cyclicity problems. The reason is that the systems to solve quickly become enormous and they require high computational efficiency.

This is just what we have seen with the significant improvement in the computation time of Lyapunov constants for big problem sizes: comparing the execution time between 1 thread (no parallelization) and 28 threads (maximum parallelization) in Table 1.1, we can see a reduction of approximately 90% in computation time.

1.3 Resolution for some families in $\mathbb{R}^2$

In this section we address the center and cyclicity problems for a few polynomial systems of differential equations. Firstly, we will solve such problems for a cubic polynomial family and the so-called Bautin's quadratic polynomials. The results for these families are classical and well-known, but they are outlined here for completeness and in order to illustrate how the tools introduced in previous sections can be applied to specific examples. We also show how the radicality of Bautin ideals helps to the resolution of the cyclicity problem, a method that has enabled us to solve it for the considered cubic family in a new way. Nevertheless, if the readers are familiar with these topics they can skip directly to Subsections 1.3.3 and 1.3.4, where we bound the cyclicity for a quartic rigid system and solve the center problem for a particular polynomial family with quintic homogeneous nonlinearities, respectively.

1.3.1 A cubic polynomial family

Let us consider the cubic polynomial family

$$\begin{cases} \dot{x} = -y + a_2x^2 + a_3x^3, \\ \dot{y} = x + b_2x^2 + b_3x^3. \end{cases} \tag{1.28}$$

Notice that by applying the time change $t \to -t$ these equations have the form of a Liénard system (1.10). Let us start by solving the center problem for this family. The result is as follows.

Theorem 1.29. *The system of differential equations* (1.28) *has a center at the origin if and only if at least one of the following conditions is satisfied:*

(i) $a_2 = a_3 = 0$;

(ii) $b_2 = a_3 = 0$; *or*

(iii) $b_3 = 0$ *and* $a_3 = \frac{2}{3}a_2b_2$.

Proof. We must check for which parameters the Lyapunov constants of the system satisfy $L_k = 0$ for every $k \geq 1$. Using the software developed during the previous section, we can compute the two first Lyapunov constants of (1.28),

$$L_1 = \frac{3}{8}a_3 - \frac{1}{4}a_2b_2, \; L_2 = \frac{5}{24}a_2b_2b_3.$$

We will solve $L_1 = 0$ and $L_2 = 0$ and use methods to prove that in this case the origin is a center, so $L_k = 0$ for every $k \geq 3$. This will be the only case in which the origin can be a center, since if $L_1 \neq 0$ or $L_2 \neq 0$ we know that the origin must be a focus.

Let us solve the system $\{L_1 = L_2 = 0\}$, which has three possible solutions: $S_1 = \{a_2 = a_3 = 0\}, S_2 = \{b_2 = a_3 = 0\}$, and $S_3 = \{b_3 = 0, a_3 = \frac{2}{3}a_2b_2\}$. For solution S_1, the system becomes invariant under the change $(x, y, t) \to (x, -y, -t)$, so according to what we saw in Subsection 1.1.2 orbits close on themselves and form periodic orbits, hence the origin of the system is a center. For the second solution S_2, the system is invariant under the coordinate change $(x, y, t) \to (-x, y, -t)$, so it also has a center at the origin. For the latter case S_3, by applying a time change $t \to -t$ the system takes the form of a Liénard system (1.10) with $F(x) = a_2x^2 + \frac{2}{3}a_2b_2x^3$ and $g(x) = x + b_2x^2$. A primitive $G(x)$ of $g(x)$ is $G(x) = \frac{1}{2}x^2 + \frac{1}{3}b_2x^3$. Comparing $F(x)$ and $G(x)$ we see that

$$F(x) = a_2x^2 + \frac{2}{3}a_2b_2x^3 = 2a_2\left(\frac{1}{2}x^2 + \frac{1}{3}b_2x^3\right) = 2a_2G(x) =: \Phi(G(x)),$$

where we define function $\Phi(x) = 2a_2x$. This function is clearly analytic and satisfies $\Phi(0) = 0$, so we can apply Theorem 1.11 to conclude that the origin of this system is a center.

These are the three only possibilities for the origin of being a center. Thus, the conditions we have found here which make L_1 and L_2 vanish are the center conditions of system (1.28), and the result follows. □

A complex coordinates version of this theorem can be found in [Zol94]. Thus, we have seen that for system (1.28), if $L_1 = L_2 = 0$ then the system has a center at the origin and $L_k = 0$ for every $k \geq 3$. A question that naturally arises is whether this is due to the fact that every L_k with $k \geq 3$ belongs to the ideal generated by L_1 and L_2. This is actually true, and we aim to prove the following result which was already presented in [Tor98], but is included here in order to illustrate how the radicality of Bautin ideals can be used to bound the local cyclicity.

Theorem 1.30. *The Bautin ideal of system* (1.28) *satisfies that*

$$\langle L_i \rangle_{i=1,\ldots,\infty} = \langle L_1, L_2 \rangle, \tag{1.29}$$

and this implies that at most two limit cycles bifurcate from the origin when the trace parameter is added.

Proof. Let us start by considering the case $L_1 \neq 0$, for which there exists a set of parameters $\lambda_1 := (a_2^{(1)}, a_3^{(1)}, b_2^{(1)}, b_3^{(1)})$ for which

$$\Pi(\rho, \lambda_1) - \rho = L_1\rho^3 + \cdots .$$

Now let us slightly perturb the system by taking $\lambda := (a_2, a_3, b_2, b_3)$ such that $\lambda \sim \lambda_1$. We want to compute the zeros of the displacement map $\Delta(\rho, \lambda \sim \lambda_1)$. As we saw in Subsection 1.1.3, by applying the Weierstrass Preparation Theorem this problem reduces to finding the zeros of polynomial $\rho^3 + s_2(\lambda)\rho^2 + s_1(\lambda)\rho$. We also justified that, as in this case $k = 3$, then the system has at most $(k-1)/2 = (3-1)/2 = 1$ limit cycle.

Assume that $L_1 = 0$ and $L_2 \neq 0$, then there exists $\lambda_2 := (a_2^{(2)}, a_3^{(2)}, b_2^{(2)}, b_3^{(2)})$ such that

$$\Pi(\rho, \lambda_2) - \rho = L_2\rho^5 + \cdots .$$

Analogously, by applying the Weierstrass Preparation Theorem, we know that finding the zeros of $\Delta(\rho, \lambda \sim \lambda_2)$ is equivalent to finding the zeros of polynomial $\rho^5 + t_4(\lambda)\rho^4 + t_3(\lambda)\rho^3 + t_2(\lambda)\rho^2 + t_1(\lambda)\rho$. In this case, as $k = 5$ we can conclude that the system has at most $(k-1)/2 = (5-1)/2 = 2$ limit cycles.

The problem gets a bit more complicated when $L_1 = L_2 = 0$. We know that in this center case

$$\Pi(\rho, \lambda_0) - \rho \equiv 0,$$

for $\lambda_0 := (a_2^{(0)}, a_3^{(0)}, b_2^{(0)}, b_3^{(0)})$ satisfying one of the conditions of Theorem 1.29. In this situation, the Weierstrass Preparation Theorem cannot be applied to study what occurs when perturbing λ_0 because the Poincaré map is identically zero and then the conditions of the theorem are not satisfied. Therefore, in this case other techniques must be used to study function $\Delta(\rho, \lambda) = \Pi(\rho, \lambda) - \rho$ for $\lambda \sim \lambda_0$.

Let J be the ideal generated by L_1 and L_2, this is $J = \langle L_1, L_2 \rangle$. Using instruction `IsRadical` from the package `PolynomialIdeals` in Maple we can straightforwardly check that this ideal J is radical. Now let $V(J)$ be the set of zeros of ideal J –recall that this is the set of parameters for which the elements of $J = \langle L_1, L_2 \rangle$ vanish. Consider now the ideal of set $V(J)$, denoted as $\mathcal{I}(V(J))$.

As we have seen that if $L_1 = L_2 = 0$ then the system has a center and $L_k = 0$ for every $k \geq 3$, we have that the set of parameters which make L_1 and L_2 vanish will automatically make $L_k = 0$ vanish for every $k \geq 3$. This implies that $L_k \in \mathcal{I}(V(J))$ for every $k \geq 3$. Therefore, due to Corollary 1.7 of the Hilbert Zeros Theorem, we also have that $L_k \in J = \langle L_1, L_2 \rangle$ for every $k \geq 3$, and this finally proves that $\langle L_i \rangle_{i=1,\ldots,\infty} = \langle L_1, L_2 \rangle$. As a consequence, if $k \geq 3$ then

$$L_k = r_k(x)L_1 + s_k(x)L_2,$$

for certain polynomials $r_k(x)$ and $s_k(x)$. We have then that the displacement map for a perturbation $\lambda \sim \lambda_0$ is

$$\begin{aligned}\Delta(\rho,\lambda) =& L_1\rho^3 + V_4\rho^4 + V_5\rho^5 + V_6\rho^6 + V_7\rho^7 + \cdots = \\ =& L_1\rho^3 \left(1 + A_1(\rho,\lambda)\rho + A_2(\rho,\lambda)\rho^2 + \cdots\right) + \\ &+ L_2\rho^5 \left(1 + B_1(\rho,\lambda)\rho + B_2(\rho,\lambda)\rho^2 + \cdots\right),\end{aligned}$$

for certain polynomials $A_i(\rho,\lambda), B_i(\rho,\lambda)$. The Weierstrass Preparation Theorem can be applied on each of the terms in the previous expression, and we obtain that there exist analytic functions $A(\rho,\lambda), B(\rho,\lambda)$ such that

$$\Delta(\rho,\lambda) = A(\rho,\lambda)\rho^3 + B(\rho,\lambda)\rho^5.$$

As it is a degree $k = 5$ polynomial, function $\Delta(\rho,\lambda)$ for $\lambda \sim \lambda_0$ has at most $(5-1)/2 = 2$ positive zeros, which means that at most 2 limit cycles can bifurcate in a neighborhood of the origin when perturbing the system also considering the trace parameter. □

Notice that the fact that the polynomials introduced here have two positive zeros is due to the fact that L_1 and L_2 in (1.29) take an arbitrary value. Thus, the procedure we have seen is general for any system where $\langle L_1, L_2\rangle$ is a radical ideal which satisfies (1.29).

1.3.2 Bautin's quadratic polynomials

In his work [Bau52] from 1952, Bautin solved the problem of finding the maximum number of limit cycles for quadratic differential systems, these are systems of the form

$$\begin{cases}\dot{x} = a_{10}\,x + a_{01}\,y + a_{20}\,x^2 + a_{11}\,xy + a_{02}\,y^2, \\ \dot{y} = b_{10}\,x + b_{01}\,y + b_{20}\,x^2 + b_{11}\,xy + b_{02}\,y^2.\end{cases} \tag{1.30}$$

In this subsection we will explicitly show how to solve the center problem for system (1.30) and we will give an overview on how the maximum number of limit cycles can be found. First, we will illustrate the center characterization for this family in the following result.

Theorem 1.31 ([DLA06]). ***(Kapteyn-Bautin Theorem)*** *A quadratic system* (1.30) *that has a center at the origin can be written in the form*

$$\begin{cases}\dot{x} = -y - bx^2 - Cxy - dy^2, \\ \dot{y} = x + ax^2 + Axy - ay^2.\end{cases} \tag{1.31}$$

This system has a center at the origin if and only if at least one of the following conditions is satisfied:

(i) $A = 2b$ *and* $C = -2a$*;*

(ii) $C = a = 0$*;*

(iii) $d = -b$*; or*

(iv) $C = -2a$*,* $A + 3b + 5d = 0$ *and* $a^2 + bd + 2d^2 = 0$*.*

Proof. A quadratic system can have a center only if it can be rewritten in the form (1.31), by applying a linear transformation and a time rescaling –see [DLA06] for more details on this.

Using the implemented algorithm, the first Lyapunov constants of (1.31) can be computed,

$$\begin{aligned}
L_1 &= \frac{1}{8}(b+d)(2a+C),\\
L_2 &= -\frac{1}{96}C(b+d)(A-2b)(A+3b+5d),\\
L_3 &= -\frac{5}{512}C(b+d)^2(C^2+4bd+8d^2)(A-2b).
\end{aligned}$$

We will see that these 3 first Lyapunov constants are enough to tackle the problem, so solve system $\{L_1 = L_2 = L_3 = 0\}$ and obtain four solutions: $S_1 = \{A = 2b, C = -2a\}, S_2 = \{C = a = 0\}, S_3 = \{d = -b\}$, and $S_4 = \{C = -2a, A + 3b + 5d = 0, a^2 + bd + 2d^2 = 0\}$. These are the four cases to analyze whether the origin is a center. For S_1, the system is Hamiltonian[4], and it is straightforward to check that its Hamiltonian function is $H = \frac{1}{2}(x^2 + y^2) + \frac{a}{3}x^3 + bx^2y - axy^2 + \frac{d}{3}y^3$, which is a first integral defined near the origin. Thus, as there exists a first integral the origin must be a center. In the case S_2, the system is invariant under the change $(x, y, t) \to (-x, y, -t)$, and this symmetry proves that the origin is a center.

For the third solution S_3, if $a \neq 0$ then there exists a certain rotation such that the new $a' = 0$, so we can assume $a = 0$ and $d = -b$ –see [DLA06] for more details. This system has the invariant curves $f_1 = 1 + Ay = 0$ (if $A \neq 0$) with cofactor $K_1 = Ax$ and $f_2 = (1 - by)^2 + C(1 - by)x - b(A + b)x^2 = 0$ with cofactor $K_2 = -2bx - Cy$. As $\operatorname{div}(P, Q) = -2bx - Cy + Ax = K_1 + K_2$, we have that there exist $\lambda_1 = -1, \lambda_2 = -1$ such that $\lambda_1 K_1 + \lambda_2 K_2 = -\operatorname{div}(P, Q)$, and by applying Darboux Theorem 1.17ii we have that $f_1^{-1} f_2^{-1}$ is an integrating factor. As the first integral associated to the integrating factor is well-defined near the

[4] A Hamiltonian system is a differential system for which there exists a scalar function H, called the Hamiltonian of the system, such that $\dot{x} = -\partial H/\partial y$ and $\dot{y} = \partial H/\partial x$. Furthermore, it can be trivially checked that the Hamiltonian H is a first integral of the system.

origin, we can conclude that the origin is a center. If $A = 0$ then f_1 is not an invariant curve, but in this case the system divergence is K_2 and analogously we see that the integrating factor is f_2^{-1} and the origin is a center.

For the last case S_4, let us assume that $d \neq 0$ –if $d = 0$ we can reduce the problem to case S_2. The resulting system has an invariant curve $f_1 = (a^2 + d^2)\left[(dy - ax)^2 + 2dy\right] + d^2 = 0$ with cofactor $K_1 = 2(a^2 + d^2)x/d$. If we compute the divergence of the system in this case we obtain $\frac{5}{2}K_1$, and by applying Darboux Theorem 1.17ii we deduce that $f_1^{-5/2}$ is an integrating factor. As $d \neq 0$, the associated first integral is defined in a neighborhood of the origin, and therefore the origin is a center.

We have seen the conditions under which system (1.31) has a center. Observe that these are the only possibilities for the system to have a center at the origin, because if they are not satisfied then either $L_1 \neq 0, L_2 \neq 0$, or $L_3 \neq 0$, and the origin would be a focus, so the theorem follows. □

We have just proved that, for the considered quadratic family, if $L_1 = L_2 = L_3 = 0$ then the system has a center at the origin and therefore $L_k = 0$ for every $k \geq 4$. Using what we saw about cyclicity in Subsection 1.1.3, we have that if $L_1 \neq 0$ then at most 1 limit cycle will appear when perturbing the system; if $L_1 = 0$ and $L_2 \neq 0$, at most 2 limit cycles can bifurcate; if $L_1 = L_2 = 0$ and $L_3 \neq 0$ at most 3 limit cycles can unfold. We can deduce this because the Weierstrass Preparation Theorem holds. Nevertheless, in the center case $L_1 = L_2 = L_3 = 0$,

$$\Pi(\rho, \lambda_0) - \rho \equiv 0,$$

so the Weierstrass Preparation Theorem cannot be applied.

As we saw in the example of the cubic polynomials in Subsection 1.3.1, the fact that the ideal generated by the two first Lyapunov constants was radical already solved the problem. In the current case of the quadratic polynomials, again by using for example Maple we see that the ideal $\langle L_1, L_2, L_3 \rangle$ is not radical, so we cannot apply the same techniques we used for the cubic system (1.28) and the problem becomes much more complicated. Despite this, Bautin proved in his article [Bau52] that the maximum number of limit cycles which can unfold near the origin when perturbing the system is 3. In this article, Bautin explicitly deduced how the Lyapunov constants L_k for $k \geq 4$ are a linear combination of constants L_1, L_2, and L_3, so proceeding as in the cubic example one can conclude that the maximum number of limit cycles which can unfold when perturbing the system is 3. We will not go deeper in the problem of the cyclicity for this quadratic family; it has only been introduced here as an example of how the problem becomes much harder when the ideal generated by the Lyapunov constants is not radical, as in this case the resolution of the cyclicity problem is not straightforward.

1.3.3 A quartic rigid system

Rigid systems are those having constant angular velocity, or equivalently, $\dot{\varphi} = 1$ being φ the angular component in polar coordinates. In this subsection we will show the resolution of the center and cyclicity problems for the quartic rigid system

$$\begin{cases} \dot{x} = -y + x\,f(x,y), \\ \dot{y} = x + y\,f(x,y), \end{cases} \tag{1.32}$$

being $f(x,y) = a_{10}x + a_{01}y + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3$ for $a_{ij} \in \mathbb{R}$. The following result characterizes the centers of (1.32), and uses this to bound its cyclicity by means of the radicality of Bautin ideals, an approach which to the best of our knowledge has not been considered formerly for this system.

Theorem 1.32. *System (1.32) has a center at the origin if and only if, modulo a rotation, one of the following set of conditions holds:*

(i) $a_{10} = a_{01} = 0$; *or*

(ii) $a_{10} = a_{30} = a_{12} = 0$.

In this case, at most 5 *limit cycles bifurcate from the origin when the trace parameter is added.*

Proof. Let us start by computing the Lyapunov constants of system (1.32), which modulo multiplicative constants take the form

$$\begin{aligned}
L_1 =& 0, \\
L_2 =& -a_{10}a_{21} - 3a_{03}a_{10} + 3a_{30}a_{01} + a_{12}a_{01}, \\
L_3 =& a_{03}a_{10}^3 - 3a_{03}a_{10}a_{01}^2 + 2a_{30}a_{01}^3 - a_{12}a_{10}^2a_{01} + a_{12}a_{01}^3, \\
L_4 =& 3a_{01}^2a_{03}a_{30} - a_{01}^2a_{12}a_{21} - 2a_{01}^2a_{21}a_{30} + a_{01}a_{10}a_{12}^2 + 3a_{01}a_{10}a_{12}a_{30} - a_{03}a_{10}^2a_{12} \\
& - 3a_{03}a_{10}^2a_{30}, \\
L_5 =& 9a_{01}a_{03}^2a_{30} - 3a_{01}a_{03}a_{12}a_{21} - 3a_{01}a_{03}a_{21}a_{30} + a_{01}a_{12}^3 + 6a_{01}a_{12}^2a_{30} - a_{01}a_{12}a_{21}^2 \\
& + 9a_{01}a_{12}a_{30}^2 - 2a_{01}a_{21}^2a_{30} - a_{03}a_{10}a_{12}^2 - 6a_{03}a_{10}a_{12}a_{30}9a_{03}a_{10}a_{30}^2.
\end{aligned}$$

If we solve the system $\{L_1 = L_2 = L_3 = L_4 = L_5 = 0\}$, we find that modulo rotations there are two solutions $S_1 = \{a_{10} = a_{01} = 0\}$ and $S_2 = \{a_{10} = a_{30} = a_{12} = 0\}$. For S_1, the system has $f(x,y) = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3$, which is cubic homogeneous. As a system being linear plus nth degree homogeneous takes the form $\dot{r} = r^n\,P(\varphi)$ in polar coordinates for some trigonometric polynomial $P(\varphi)$, it can be trivially integrated by separation of variables, so in our case the origin is a center. For S_2, we have $y(a_{01} + a_{21}x^2 + a_{03}y^2)$ and the origin is a center because there exists a reversibility with respect to the horizontal axis, and therefore the system is symmetric –this nonhomogeneous case was actually solved in [CGG01].

Now we check by using Maple that the Bautin ideal generated by the first 5 Lyapunov constants of the system is radical, and therefore the resolution of the center problem also bounds the cyclicity of the system. Indeed, we have that equality $\langle L_i \rangle_{i=1,\dots,\infty} = \langle L_1, L_2, L_3, L_4, L_5 \rangle$ holds, and as we have seen in previous sections this implies that at most 5 limit cycles can bifurcate from the origin, so the result follows. □

1.3.4 A system with quintic homogeneous nonlinearities

Let us consider the following example of polynomial family with fifth degree homogeneous nonlinearities,

$$\dot{z} = \mathrm{i}\, z + a_{50} z^5 + a_{32} z^3 w^2 + a_{14} z w^4. \tag{1.33}$$

The following result shows the center characterization for this family.

Theorem 1.33. *Equation (1.33) has a center at the origin if and only if one of the following set of conditions holds:*

(i) $\overline{a_{32}} = -a_{32}$, $a_{50} = 0$; *or*

(ii) $\overline{a_{32}} = -a_{32}$, $\overline{a_{14} a_{50}} = a_{14} a_{50}$.

Proof. The first 20 Lyapunov constants of (1.33) have been computed, but they are not shown here for the sake of brevity. What we can see is that those constants L_k with odd $k \leq 20$ are identically zero, while those with even $k \leq 20$ are not. With the help of Maple we check that $L_{20} = 0$ provided that $L_2 = L_4 = L_6 = L_8 = L_{10} = L_{12} = L_{14} = L_{16} = L_{18} = 0$. Therefore, it is expected that the first 9 nonidentically zero constants are enough to solve the center problem.

Let us first find the conditions for which $L_2 = L_4 = \cdots = L_{18} = 0$. This is done by using Maple, and two possible solutions are obtained: $S_1 = \{\overline{a_{32}} = -a_{32}, a_{50} = 0\}$ and $S_2 = \{\overline{a_{32}} = -a_{32}, \overline{a_{14} a_{50}} = a_{14} a_{50}\}$. Now we will prove that under these conditions the system has a center at the origin, which will complete the resolution of the center problem for this family. To simplify notation, we will define $a_{kl} := b_{kl} + \mathrm{i}\, c_{kl}$ being $b_{kl}, c_{kl} \in \mathbb{R}$ for any pair of subscripts $k, l \in \mathbb{N} \cup \{0\}$.

Let us start by studying the first solution S_1. Observe that the condition $\overline{a_{32}} = -a_{32}$ implies that a_{32} is purely imaginary, so $a_{32} = \mathrm{i}\, c_{32}$ for $c_{32} \in \mathbb{R}$. Then, as $a_{50} = 0$, in this case equation (1.33) can be written as

$$\dot{z} = \mathrm{i}\, z + \mathrm{i}\, c_{32} z^3 w^2 + a_{14} z w^4, \tag{1.34}$$

As (1.34) has the form (1.12), we can apply Theorem 1.18 to show that the origin is a center. Using the notation of the theorem, $k = 3$, $l = 2$, $m = 1$, $n = 4$

$(k+l = 3+2 \leq 1+4 = m+n)$, $\alpha = 3-2-1 = 0$, and $\beta = 1-4-1 = -4$. Then, by taking $A = \mathrm{i}\, c_{32} = -\overline{\mathrm{i}\, c_{32}}\, \mathrm{e}^{\mathrm{i}\, 0\varphi}$ and choosing an appropriate φ so that $B = a_{14} = -\overline{a_{14}}\, \mathrm{e}^{-4\,\mathrm{i}\,\varphi}$, condition (iii) of the theorem is satisfied and therefore the origin is a center.

For the second solution S_2, we have again that the first condition $\overline{a_{32}} = -a_{32}$ implies that $a_{32} = \mathrm{i}\, c_{32}$ for $c_{32} \in \mathbb{R}$, so the system has the form

$$\begin{cases} \dot{z} = \mathrm{i}\, z + a_{50} z^5 + \mathrm{i}\; c_{32} z^3 w^2 + a_{14} z w^4, \\ \dot{w} = -\,\mathrm{i}\, w + \overline{a_{50}} w^5 - \mathrm{i}\; c_{32} z^2 w^3 + \overline{a_{14}} z^4 w, \end{cases} \tag{1.35}$$

where we have also written the complex conjugate equation. In this case, it will be useful to express the system in polar coordinates (r, φ) instead of (z, w), so we consider $z = r\, \mathrm{e}^{\mathrm{i}\,\varphi}$ and $w = r\, \mathrm{e}^{-\,\mathrm{i}\,\varphi}$. By taking the corresponding derivatives,

$$\dot{z} = \frac{dz}{dr}\dot{r} + \frac{dz}{d\varphi}\dot{\varphi} = \mathrm{e}^{\mathrm{i}\,\varphi}\,\dot{r} + \mathrm{i}\, z\dot{\varphi}, \quad \dot{w} = \frac{dw}{dr}\dot{r} + \frac{dw}{d\varphi}\dot{\varphi} = \mathrm{e}^{-\,\mathrm{i}\,\varphi}\,\dot{r} - \mathrm{i}\, w\dot{\varphi}. \tag{1.36}$$

We are interested in finding an expression for $\dot{r}$. This can be done by substituting (1.36) in $\dot{z}w + \dot{w}z$ as follows,

$$\begin{aligned} \dot{z}w + \dot{w}z &= \left(\mathrm{e}^{\mathrm{i}\,\varphi}\,\dot{r} + \mathrm{i}\, z\dot{\varphi}\right) w + \left(\mathrm{e}^{-\,\mathrm{i}\,\varphi}\,\dot{r} - \mathrm{i}\, w\dot{\varphi}\right) z, \\ \dot{z}r\, \mathrm{e}^{-\,\mathrm{i}\,\varphi} + \dot{w}r\, \mathrm{e}^{\mathrm{i}\,\varphi} &= \mathrm{e}^{\mathrm{i}\,\varphi}\,\dot{r}r\, \mathrm{e}^{-\,\mathrm{i}\,\varphi} + \mathrm{e}^{-\,\mathrm{i}\,\varphi}\,\dot{r}r\, \mathrm{e}^{\mathrm{i}\,\varphi}, \\ \dot{r} &= \frac{\dot{z}\, \mathrm{e}^{-\,\mathrm{i}\,\varphi} + \dot{w}\, \mathrm{e}^{\mathrm{i}\,\varphi}}{2}. \end{aligned}$$

Substituting (1.35) in this expression and using polar coordinates we obtain

$$\begin{aligned} \dot{r} &= r^5 \frac{\left(a_{50}\, \mathrm{e}^{4\,\mathrm{i}\,\varphi} + \overline{a_{50}}\, \mathrm{e}^{-4\,\mathrm{i}\,\varphi}\right) + \left(a_{14}\, \mathrm{e}^{-4\,\mathrm{i}\,\varphi} + \overline{a_{14}}\, \mathrm{e}^{4\,\mathrm{i}\,\varphi}\right)}{2} \\ &= r^5\, \mathrm{Re}\left(a_{50}\, \mathrm{e}^{4\,\mathrm{i}\,\varphi} + a_{14}\, \mathrm{e}^{-4\,\mathrm{i}\,\varphi}\right). \end{aligned}$$

Using notation $a_{kl} = b_{kl} + \mathrm{i}\, c_{kl}$ and Euler's fomula[5] this can be rewritten as

$$\dot{r} = r^5 \left(b_{50}\cos(4\varphi) - c_{50}\sin(4\varphi) + b_{14}\cos(4\varphi) + c_{14}\sin(4\varphi)\right).$$

Now we aim to follow an analogous procedure to find an expression for the derivative of the angular component $\dot{\varphi}$. To this end we will subtitute (1.36) in

[5] Euler's formula states that, for any real number ϕ, $\mathrm{e}^{\mathrm{i}\,\phi} = \cos\phi + \mathrm{i}\sin\phi$.

expression $\dot{z}\,e^{-i\varphi} - \dot{w}\,e^{i\varphi}$ as follows,

$$\begin{aligned} \dot{z}\,e^{-i\varphi} - \dot{w}\,e^{i\varphi} &= \left(e^{i\varphi}\dot{r} + i z\dot{\varphi}\right) e^{-i\varphi} - \left(e^{-i\varphi}\dot{r} - i w\dot{\varphi}\right) e^{i\varphi}, \\ \dot{z}\,e^{-i\varphi} - \dot{w}\,e^{i\varphi} &= i r\,e^{i\varphi}\,\dot{\varphi}\,e^{-i\varphi} + i r\,e^{-i\varphi}\,\dot{\varphi}\,e^{i\varphi}, \\ \dot{\varphi} &= \frac{\dot{z}\,e^{-i\varphi} - \dot{w}\,e^{i\varphi}}{2 i r}. \end{aligned}$$

Again we can substitute (1.35) in this expression and use polar coordinates to obtain

$$\begin{aligned} \dot{\varphi} =& 1 + r^4\left(c_{32} + \frac{\left(a_{50}\,e^{4i\varphi} - \overline{a_{50}}\,e^{-4i\varphi}\right) + \left(a_{14}\,e^{-4i\varphi} - \overline{a_{14}}\,e^{4i\varphi}\right)}{2i}\right) \\ =& 1 + r^4\left(c_{32} + \operatorname{Im}\left(a_{50}\,e^{4i\varphi} + a_{14}\,e^{-4i\varphi}\right)\right). \end{aligned}$$

With notation $a_{kl} = b_{kl} + i\,c_{kl}$ and Euler's fomula, this can be expressed as

$$\dot{\varphi} = 1 + r^4\left(c_{32} + b_{50}\sin(4\varphi) + c_{50}\cos(4\varphi) - b_{14}\sin(4\varphi) + c_{14}\cos(4\varphi)\right),$$

Therefore, we have seen that system (1.35) in polar coordinates takes the form

$$\begin{cases} \dot{r} = r^5\left(b_{50}\cos(4\varphi) - c_{50}\sin(4\varphi) + b_{14}\cos(4\varphi) + c_{14}\sin(4\varphi)\right), \\ \dot{\varphi} = 1 + r^4\left(c_{32} + b_{50}\sin(4\varphi) + c_{50}\cos(4\varphi) - b_{14}\sin(4\varphi) + c_{14}\cos(4\varphi)\right). \end{cases} \tag{1.37}$$

Recall that the second condition in S_2 is $\overline{a_{14}a_{50}} = a_{14}a_{50}$. Let us write in polar coordinates $a_{14} = r_{14}\,e^{i\varphi_{14}}$ and $a_{50} = r_{50}\,e^{i\varphi_{50}}$ for $r_{14}, \varphi_{14}, r_{50}, \varphi_{50} \in \mathbb{R}$. Using this notation, the aforementioned condition is rewritten as $e^{-i(\varphi_{14}+\varphi_{50})} = e^{i(\varphi_{14}+\varphi_{50})}$, and after Euler's formula this yields to $-\sin(\varphi_{14}+\varphi_{50}) = \sin(\varphi_{14}+\varphi_{50})$, which implies $\sin(\varphi_{14}+\varphi_{50}) = 0$. The solutions for this equation are $\varphi_{14}+\varphi_{50} = 0$ and $\varphi_{14}+\varphi_{50} = \pi$. Let us study both cases separately.

If $\varphi_{14}+\varphi_{50} = 0$ then $\varphi_{14} = -\varphi_{50}$, so $a_{14} = r_{14}\,e^{-i\varphi_{50}} = \frac{r_{14}}{r_{50}}\overline{a_{50}}$. Using notation $a_{kl} = b_{kl} + i\,c_{kl}$, we have then that $b_{14} = \frac{r_{14}}{r_{50}}b_{50}$ and $c_{14} = -\frac{r_{14}}{r_{50}}c_{50}$. Substituting this in (1.37) we obtain

$$\begin{cases} \dot{r} = r^5\left(1 + \frac{r_{14}}{r_{50}}\right)\left(b_{50}\cos(4\varphi) - c_{50}\sin(4\varphi)\right), \\ \dot{\varphi} = 1 + r^4\left(c_{32} + \left(1 - \frac{r_{14}}{r_{50}}\right)\left(b_{50}\sin(4\varphi) + c_{50}\cos(4\varphi)\right)\right). \end{cases}$$

Let us define $G_1(\varphi) := \left(1 + \frac{r_{14}}{r_{50}}\right)\left(b_{50}\cos(4\varphi) - c_{50}\sin(4\varphi)\right)$ as the part of the $\dot{r}$

equation which only depends on the angular variable. Then we integrate such equation as follows,

$$\int_{r_0}^{r} \frac{dr}{r^5} = \int_0^t G_1(\varphi)\, dt \Rightarrow -\frac{1}{4r^4} + \frac{1}{4r_0^4} = G_1(\varphi)\, t.$$

By denoting $\widetilde{r_0} := \dfrac{1}{4r_0^4}$ and isolating r, we get the expression

$$r(t,\varphi) = \frac{1}{\sqrt[4]{4\left(\widetilde{r_0} - G_1(\varphi)\, t\right)}}. \tag{1.38}$$

Now we can use Maple to solve the differential equation on $\dot{\varphi}$ to find $\varphi(t,r)$, and then isolate t on this expression to find how t depends on φ. The result is

$$t(\varphi) = -\frac{1}{2}\frac{2K_1\sqrt{f_1} - r_{50}\arctan\left(\dfrac{f_2\tan(2\varphi) + f_3}{\sqrt{f_4}}\right)}{\sqrt{f_5}}, \tag{1.39}$$

where K_1 is an integration constant and f_i for $i = 1,2,3,4,5$ are expressions of sums and products such that $f_i = f_i(r, b_{14}, c_{14}, r_{14}, b_{50}, c_{50}, r_{50}, c_{32})$, so they not depend on the angular variable φ. Observe that, for any angle φ_0, $\tan(2(\varphi_0 + \pi)) = \tan(2(\varphi_0 - \pi))$. Therefore, using expression (1.39) we see that $t(\varphi_0+\pi) = t(\varphi_0 - \pi)$ for any angle φ_0, which implies that for any initial angle φ_0 the time $t(\varphi_0+\pi)$ needed to go forward π is the same as the time $t(\varphi_0-\pi)$ needed to go backward π. Furthermore, as $\cos(4(\varphi_0+\pi)) = \cos(4(\varphi_0-\pi))$ and $\sin(4(\varphi_0+\pi)) = \sin(4(\varphi_0-\pi))$ are also true for any angle φ_0, by using the definition of G_1 we trivially see that $G_1(\varphi_0+\pi) = G_1(\varphi_0-\pi)$. Using this and the fact that $t(\varphi_0+\pi) = t(\varphi_0-\pi)$, according to expression (1.38) we obtain that $r(t(\varphi+\pi), \varphi+\pi) = r(t(\varphi-\pi), \varphi-\pi)$. This implies that, as the radius of the orbit when moving forward a π angle is the same as moving backward a π angle for any initial angle φ_0 we can conclude that under these conditions the orbits close on themselves so the origin of the system is a center.

The case $\varphi_{14} + \varphi_{50} = \pi$ can be solved analogously to the previous one. If $\varphi_{14} + \varphi_{50} = \pi$ then $\varphi_{14} = \pi - \varphi_{50}$, so $a_{14} = r_{14}\, \mathrm{e}^{\mathrm{i}(\pi - \varphi_{50})} = -\dfrac{r_{14}}{r_{50}}\overline{a_{50}}$. This implies $b_{14} = -\dfrac{r_{14}}{r_{50}} b_{50}$ and $c_{14} = \dfrac{r_{14}}{r_{50}} c_{50}$, and substituting in system (1.37) we obtain

$$\begin{cases} \dot{r} = r^5\left(1 - \dfrac{r_{14}}{r_{50}}\right)\left(b_{50}\cos(4\varphi) - c_{50}\sin(4\varphi)\right), \\ \dot{\varphi} = 1 + r^4\left(c_{32} + \left(1 + \dfrac{r_{14}}{r_{50}}\right)\left(b_{50}\sin(4\varphi) + c_{50}\cos(4\varphi)\right)\right). \end{cases}$$

Now we define $G_2(\varphi) := \left(1 - \frac{r_{14}}{r_{50}}\right)(b_{50}\cos(4\varphi) - c_{50}\sin(4\varphi))$, and by solving the $\dot{r}$ equation we obtain

$$r(t,\varphi) = \frac{1}{\sqrt[4]{4\left(\widetilde{r_0} - G_2(\varphi)\, t\right)}}, \tag{1.40}$$

for a certain $\widetilde{r_0}$. The differential equation in $\dot{\varphi}$ can be solved using Maple and the temporal variable isolated to obtain

$$t(\varphi) = -\frac{1}{2}\frac{2K_2\sqrt{h_1} - r_{50}\arctan\left(\dfrac{h_2\tan(2\varphi) + h_3}{\sqrt{h_4}}\right)}{\sqrt{h_5}},$$

where K_2 is an integration constant and h_i for $i = 1,2,3,4,5$ are expressions of sums and products such that $h_i = h_i(r, b_{14}, c_{14}, r_{14}, b_{50}, c_{50}, r_{50}, c_{32})$, so they do not depend on the angular variable φ. Again, for any initial angle φ_0 we have that $G_2(\varphi_0 + \pi) = G_2(\varphi_0 - \pi)$ and $t(\varphi_0 + \pi) = t(\varphi_0 - \pi)$ so, according to (1.40), $r(t(\varphi + \pi), \varphi + \pi) = r(t(\varphi - \pi), \varphi - \pi)$. As a consequence, for any initial angle φ_0 the radius when moving forward or backward a π angle is the same, which implies that the origin of the system is a center. □

1.4 New advances on Lyapunov constants

This section presents some advances regarding the Lyapunov constants of some families of planar polynomial differential systems, as a first step towards the resolution of the center and cyclicity problems. Firstly, a parallelization approach is computationally implemented to achieve the 14th Lyapunov constant of the complete cubic family. It is worth remarking that, to the best of our knowledge, this is the first time that 14 Lyapunov constants for the complete cubic family are found, and this achievement would not have been possible without the indispensable support of the developed parallelization techniques. We notice that despite having found these Lyapunov constants, solving the center problem for this family has not been feasible due to the high computational cost. Secondly, a technique based on interpolating some specific quantities so as to reconstruct the structure of the Lyapunov constants is used to study a Kukles system, some fifth-degree homogeneous systems, and a quartic system with two invariant lines.

1.4.1 The complete cubic family

In Subsection 1.2.1 we described the Lyapunov method algorithm to find Lyapunov constants of a system, which is based on the utilization of a first integral of system (1.2). Here we will use the PARI implementation of this method. As the computation of Lyapunov constants is a highly computationally expensive procedure, this algorithm has been optimized and improved by means of parallelization, which allows to significantly increase computation velocity. The idea is to find each of the Lyapunov constants and the coefficients $h_{p-j,j}$ $(j = 0, \ldots, p)$ from F_p of degree p from (1.21) in terms of the coefficients of lower degree, this is, as a function of $h_{k-j,j}$ being $k < p$ and $j = 0, \ldots, k$. This part is relatively fast computationally speaking, since the manipulated expressions are not too large. Then we parallelize the substitution of those coefficients with their actual value, and here parallelization is essential because this process deals with very large expressions.

The results of this parallelization technique are amazing, and its efficiency has allowed our method to find Lyapunov constants in a relatively short time for cases which had not been solved before due to the huge amount of time and computational complexity required. In particular, we have applied this method to the complete cubic system

$$\begin{cases} \dot{z} = \mathrm{i}\, z + \hat{r}_{20}z^2 + \hat{r}_{11}zw + \hat{r}_{02}w^2 + \hat{r}_{30}z^3 + \hat{r}_{21}z^2w + \hat{r}_{12}zw^2 + \hat{r}_{03}w^3, \\ \dot{w} = -\,\mathrm{i}\, w + \hat{s}_{20}w^2 + \hat{s}_{11}wz + \hat{s}_{02}z^2 + \hat{s}_{30}w^3 + \hat{s}_{21}w^2z + \hat{s}_{12}wz^2 + \hat{s}_{03}z^3. \end{cases} \tag{1.41}$$

We have observed that if time is rescaled by dividing by the imaginary unit i, computations are much more efficient and the calculation time decreases. Actually, the computations we describe here cannot be performed if this time rescaling is not done, and it has turned out to be as important as parallelization. This seems to be because of how computer algebra systems manipulate the expressions with the imaginary unit i, and eliminating it from the problem to consider only rational coefficients has proved to be an indispensable part to achieve the presented results. If we denote $r_{jk} = \frac{\hat{r}_{jk}}{\mathrm{i}}$ and $s_{jk} = \frac{\hat{s}_{jk}}{\mathrm{i}}$, system (1.41) in the new time variable can be written as

$$\begin{cases} z' = z + r_{20}z^2 + r_{11}zw + r_{02}w^2 + r_{30}z^3 + r_{21}z^2w + r_{12}zw^2 + r_{03}w^3, \\ w' = -w + s_{20}w^2 + s_{11}wz + s_{02}z^2 + s_{30}w^3 + s_{21}w^2z + s_{12}wz^2 + s_{03}z^3. \end{cases} \tag{1.42}$$

Up to our knowledge, the highest known Lyapunov constant for the above system is the 10th. However, with our parallelization technique we have been able to reach the 14th. To perform this computation we have used the computer network of our department Antz, and the parallelization has been done with the software PBala which we already introduced in previous sections. The nodes of this server work with Intel Xeon 2.60GHz processors, the total used memory is 640GB, and the maximum number of threads run at the same time has been 96. The found Lyapunov constants are not shown here due to their enormous length, but their sizes are shown in Table 1.2. The total computing time has been around

22 hours.

TABLE 1.2: Size of the computed Lyapunov constants of (1.42)

Lyapunov constant	Size (MB)
11	111
12	261
13	588
14	1282

The last step to obtain a complete characterization of cubic centers would be how can we solve the nonlinear system $\{L_1 = L_2 = \cdots = 0\}$, and not how to construct it because we think that we have computed enough Lyapunov constants to achieve it. This is a very demanding problem computationally speaking and we have not been able to tackle it yet, but by finding 14 Lyapunov constants we have taken a step towards its resolution.

Let us finish by making the following observations regarding the number of necessary Lyapunov constants to address the problem for the complete cubic family. At the moment when the calculations in this section were done, it was a well-known fact –see for example [Chr05]– that the solution of the center problem for the general cubic differential system needs at least 11 Lyapunov constants, and we thought that 14 constants would actually be needed. Later, a real 12th order weak focus was found in [GGT21], which made clear that 12 Lyapunov constants were needed in the real case by correcting some missing points in the arguments of the original proof from [YT14]. Very recently, in 2020, Sadovskii proved the existence of a complex cubic system with a 14th order weak focus at the origin ([Sad20]), which strengthens the idea that 14 Lyapunov constants are necessary to solve the center problem for the cubic family. It is worth noticing that in such work the author finds the first 14 Lyapunov constants for the particular cubic system he studies, while we have reached the first 14 constants for a general complete cubic.

The fact of having a complex weak focus with 14th order shows the great difficulty of the problem. This is because more elements will be needed to find the generators of the Bautin ideal, but maybe not for the center problem as these complex objects may not help to solve the problem in real coordinates. What could happen here is that the conditions which vanish the first 14 Lyapunov constants in complex coordinates have no real associated solutions, so fewer constants may be necessary to solve the problem in real coordinates, maybe 12 as in the 12th order focus from [GGT21]. However, classically extra Lyapunov constants have been used in order to help to solve the problem, and to deal with all these mathematical object is an extremely demanding problem in computational terms.

1.4.2 The reconstruction technique

Let $\mathcal{B}_k := \langle L_1, \ldots, L_k \rangle$ be the Bautin ideal generated by the first k Lyapunov constants. The method suggested in this subsection aims to check whether a certain Lyapunov constant L_n belongs to $\mathcal{B}_{n-1}$, and therefore it vanishes when the previous are equal to zero. It is important to remark that to apply this techique at this step we assume that we have been able to compute the first n Lyapunov constants.

Let us start by writing

$$L_n = \sum_{j=1}^{n-1} A_j L_j, \tag{1.43}$$

where A_j are polynomials whose variables are the parameters of the original system (1.2). Our method consists on trying to see whether we can determine these polynomials A_j, since this will tell if expression (1.43) is possible or not. Using the notation of (1.42) for the parameters, let us consider a monomial $M = \prod_{k,l} r_{k,l}^{p_{k,l}} \bar{r}_{k,l}^{q_{k,l}}$, where $r_{k,l}$ denotes the coefficients of $z^k w^l$ in $Z(z,w)$ from (1.2), and $\bar{r}_{k,l}$ denotes their complex conjugates. Recalling Theorem 1.27, the monomials of a Lyapunov constant L_j have quasi-degree $2j$ and weight 0. Now using these properties together with the degree of L_j, we can select which monomials are candidates to be part of each A_j, but with undetermined coefficients. Thus, we have that A_j are polynomials whose monomials have been selected and have undetermined coefficients, and these coefficients of A_j are what we try to compute.

Once known the structure of A_j, we would substitute it in (1.43), expand the products and the sum and finally equate the coefficients of monomials with the same literal part. This gives a set of linear equations consisting of the coefficients of equality (1.43). If this system of linear equations is compatible, then the polynomials A_j do exist and L_n vanishes when $L_1, \ldots, L_{n-1}$ are zero; otherwise, if the system is incompatible then the polynomials A_j do not exist and L_n does not belong to $\mathcal{B}_{n-1}$. Therefore, instead of explicitly solving the system of equations, it is enough to compare the ranks of the system matrices to see if they are equal or not, and this is how we have proceeded.

With this method we have studied three different polynomial families and we have obtained the following results. Notice that we have not explicitly shown their proofs, because they consist simply on applying the described method to the considered systems and check the belonging of L_n to $\mathcal{B}_{n-1}$ in each case.

Proposition 1.34. *Consider the Kukles differential system*

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3, \end{cases}$$

which in complex coordinates is written as

$$\begin{cases} \dot{z} = \mathrm{i}\, z + r_{20}z^2 + r_{11}zw + r_{02}w^2 + r_{30}z^3 + r_{21}z^2w + r_{12}zw^2 + r_{03}w^3, \\ \dot{w} = -\mathrm{i}\, w - r_{20}z^2 - r_{11}zw - r_{02}w^2 - r_{30}z^3 - r_{21}z^2w - r_{12}zw^2 - r_{03}w^3. \end{cases}$$

Then L_9 does not belong to $\mathcal{B}_8$. In the case $r_{12} = 0$, L_9 again does not belong to $\mathcal{B}_8$, but L_{10} does belong to $\mathcal{B}_9$ since there exist A_j such that $L_{10} = \sum_{j=1}^{9} A_j L_j$.

From the above result, we can guess that only the first 9 Lyapunov constants are enough to solve the center problem for this family when $r_{12} = 0$. This interpolation method works better than the standard approach using Groebner basis for the simplifications (see [RS09] for more details on Groebner basis).

The center problem for 5th degree homogeneous perturbations of the linear oscillator is an open problem, and even the value m in (1.8) is unknown. The mechanism proposed here fails for the general family due to the size of the computations. The next result presents some particular cases.

Proposition 1.35. *Consider the linear plus homogeneous 5th degree polynomial differential system*

$$\begin{cases} \dot{z} = \mathrm{i}\, z + r_{41}z^4w + r_{32}z^3w^2 + r_{23}z^2w^3 + r_{14}zw^4 + r_{05}w^5, \\ \dot{w} = -\mathrm{i}\, w + s_{41}w^4z + s_{32}w^3z^2 + s_{23}w^2z^3 + s_{14}wz^4 + s_{05}z^5. \end{cases}$$

Then the next properties hold:

- *If $r_{41} = s_{41} = 0$, then L_{10} does not belong to $\mathcal{B}_9$ but L_{11} does belong to $\mathcal{B}_{10}$.*
- *If $r_{32} = s_{32} = 0$, then L_{11} and L_{12} do not belong to $\mathcal{B}_{10}$ and $\mathcal{B}_{11}$, respectively.*

The last considered family is a special quartic differential system with four invariant straight lines, which has been considered here because it has turned out to work well with the proposed reconstruction technique.

Proposition 1.36. *Consider the system with two parallel invariant straight lines*

$$\begin{cases} \dot{x} = (1 - x^2)(-y + a_{20}x^2 + a_{11}xy + a_{02}y^2), \\ \dot{y} = (1 - y^2)(x + b_{20}x^2 + b_{11}xy + b_{02}y^2), \end{cases}$$

which can be written in complex coordinates as

$$\begin{cases} \dot{z} = \mathrm{i}\, z + r_{40}z^4 + r_{31}z^3w + r_{22}z^2w^2 + r_{14}zw^3 + r_{04}w^4 + r_{21}z^2w + r_{03}w^3 + \\ \qquad r_{20}z^2 + r_{11}zw + r_{02}w^2, \\ \dot{w} = -\mathrm{i}\, w + s_{40}w^4 + s_{31}w^3z + s_{22}w^2z^2 + s_{14}wz^3 + s_{04}z^4 + s_{21}w^2z + s_{03}z^3 + \\ \qquad s_{20}w^2 + s_{11}wz + s_{02}z^2. \end{cases}$$

For this system both L_7, L_8, and L_9 do not belong to $\mathcal{B}_6, \mathcal{B}_7$, and $\mathcal{B}_8$, respectively. However, when $r_{11} = s_{11} = 0$, L_7 does belong to $\mathcal{B}_6$.

1.5 Hopf bifurcation for polynomial vector fields in $\mathbb{R}^3$

In this section we consider the Hopf bifurcation in families of polynomial differential systems of equations in $\mathbb{R}^3$, and we aim to find as many limit cycles as possible for systems of several degrees n. It is widely known that, unlike for planar systems, systems in $\mathbb{R}^3$ can exhibit infinitely many limit cycles, as it is the case, for example, in any vector field with a Shilnikov homoclinic orbit, see [GH90; Shi65]. Other interesting bifurcations also exhibiting infinitely many can be found in [BZ07], where a counterexample to a multidimensional version of the weakened Hilbert's 16th problem is presented, or the ones appearing on an infinite family of algebraic invariant surfaces, even for the quadratic case ([BG96; YH15]). There is also the example of the bifurcation of infinitely many limit cycles near a Hopf-zero equilibrium point. Indeed, geometrical arguments to show the existence of Shilnikov homoclinic orbits around a Hopf-zero point were already provided by Guckenheimer and Holmes ([GH90]). Formal statements were proved in [BV84] and later in [Dum+13], but the first rigorous proof was done very recently in [BIS20].

In our case, we will restrict the problem of finding lower bounds for the maximum number of limit cycles of small amplitude to the center manifold. In fact, we will study bifurcations from systems having an equilibrium point such that the corresponding Jacobian matrix has eigenvalues $\{\pm\,\mathrm{i}, 1\}$, this is having a center in the center manifold and a hyperbolic eigenvalue in the third direction. The fact that the linear part of the third equation $\dot{z}$ is different from 0 and thus hyperbolic makes that the z direction is tightening the solutions towards the center manifold, and in this sense the considered problem is far from the Hopf-zero situation. In this line, the problem is more similar to finding the cyclicity in a two-dimensional case, and therefore makes sense to consider lower bounds for the maximum number of local limit cycles despite the global problem being unbounded. This problem was also considered in [BZ03; BZ05a; BZ05b; SMB06] and more recently in [GMS18; GMS19]. We will also take the advantage that, as explained in some of these works, from the computational point of view it is not necessary to do the changes of variables to transform the problem to a planar one. The scenario being considered is then a local cyclicity problem of a certain object inside a particular class of vector fields which is far from the 0 eigenvalue degeneration in the third component. We notice that in the nonpolynomial case this problem makes no sense because also an infinite number of small limit cycles can bifurcate from the origin, see [YS19].

Let us consider then a three-dimensional system

$$\begin{cases} \dot{x} = \alpha x - y + X(x,y,z), \\ \dot{y} = x + \alpha y + Y(x,y,z), \\ \dot{z} = z + Z(x,y,z), \end{cases} \tag{1.44}$$

where X, Y, Z are polynomials of degree $n \geq 2$ having no constant nor linear terms in x, y, z. The main result of this section is the following.

Theorem 1.37. *There exist systems of the form* (1.44) *(with $\alpha \approx 0$) such that at least* 11, 31, 54, 92 *limit cycles of small amplitude bifurcate from the origin for $n = 2, 3, 4, 5$, respectively.*

To the best of our knowledge, the highest number of limit cycles found so far for the degenerate Hopf bifurcation in the quadratic case is 10 (see [YH15]), and we are not aware of any studies neither on cubic, quartic, nor quintic polynomial vector fields in $\mathbb{R}^3$, probably due to the computational difficulties.

This section is devoted to prove the above main theorem and is structured as follows. First, we present a subsection which introduces the main tools necessary for the proof of our result: the Lyapunov constants method computation in $\mathbb{R}^3$ and a couple of results that will be useful for the proofs. Subsections 1.5.2 and 1.5.3 use 2-parametric families to study the Hopf bifurcation and prove Theorem 1.37 for $n = 2$ and $n = 3$, respectively. In the last subsection, we extend to $\mathbb{R}^3$ the parallelization approach for $\mathbb{R}^2$ presented in [LT15] for the Lyapunov constants computation and apply it to achieve our above main result for the fourth and fifth degree cases.

1.5.1 Preliminary tools

The usual way to study the Hopf bifurcation in $\mathbb{R}^3$ is the restriction to the central manifold where a center-focus type problem can be considered, but in many cases the necessary normal form changes to go further in the computations make the problem impossible to be solved. Then, we have opted for the approach of working directly in $\mathbb{R}^3$ as in [SMB06] but with the algorithm described in [BGM12]. Additionally, as we will explain later, we have avoided the situation when the center manifold is the invariant plane $z = 0$, because the corresponding obtained results are worse than when the center manifold is not a plane. In fact we will prove that the number of limit cycles in the Hopf bifurcation depends on the center manifold when the considered vector field is a family depending on parameters. We will restrict our computations to families with (at most) two parameters because of the computational difficulties.

Lyapunov constants in $\mathbb{R}^3$

The main tool to study the local cyclicity of a Hopf equilibrium point are the Lyapunov constants, and we present here two methods to find such quantities in $\mathbb{R}^3$. The first method simply uses a first integral and finds center conditions, whereas the second one consists on performing the corresponding transformation to the center manifold in order to consider it as a planar problem. This reduction to the center manifold presents some computational difficulties, so the first technique is the one we will use throughout the section to find Lyapunov constants. However, we will see an example and check that both methods lead to the same result.

Even though the center notion can be only considered in even dimensional spaces, some authors introduce the notion of center in $\mathbb{R}^3$ to simplify the reading. With this aim we can say that the origin is a center for an analytic system in $\mathbb{R}^3$ when the eigenvalues of the Jacobian matrix are $\{\pm \mathrm{i}, \lambda\}$, with $\lambda \neq 0$, and the system has a center on the 2-dimensional center manifold. The next result is a classical one in the study of the existence of 2-dimensional center varieties having centers in a three-dimensional space. It can be found in [BGM12] and is proved in [Bib79].

Theorem 1.38 ([BGM12]). *The origin is a center for the analytic system* (1.44) *if and only if $\alpha = 0$ and it admits a real analytic local first integral of the form $H(x, y, z) = x^2 + y^2 + O_3(x, y, z)$ in a neighborhood of the origin in $\mathbb{R}^3$, being $O_3(x, y, z)$ a sum of terms of degree at least 3. Moreover, when there is a center, the local center manifold is unique and analytic.*

The method we propose to find Lyapunov constants in $\mathbb{R}^3$ consists on using Theorem 1.38 to construct a first integral $H(x, y, z) = x^2 + y^2 + \cdots$ with unknown coefficients up to a certain degree. It is a well-known fact that condition

$$\dot{H} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} + \frac{\partial H}{\partial z}\dot{z} \equiv 0 \tag{1.45}$$

is equivalent to the system having a center at the origin. In contrast, if the system does not have a center at the origin then (1.45) is not identically zero, and it can be proved ([DLA06]) that actually it is an analytic function in $x^2 + y^2$, this is

$$\dot{H} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} + \frac{\partial H}{\partial z}\dot{z} = \sum_{\ell \geq 1} L_\ell (x^2 + y^2)^{\ell+1}, \tag{1.46}$$

for some coefficients $L_\ell \in \mathbb{R}$. In this context, the coefficients L_ℓ are the so-called Lyapunov constants, and they have the property that they all vanish if and only if the system has a center.

The algorithm that we have implemented to find the coefficients in (1.46) works as follows. First, we define a first integral up to a certain degree N having the form

$$H = x^2 + y^2 + \sum_{i+j+k=3}^{N} h_{ijk} x^i y^j z^k, \tag{1.47}$$

being h_{ijk} the unknown coefficients for degrees between 3 and N. We aim to find these coefficients together with Lyapunov constants. First, we calculate $\dot{H} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} + \frac{\partial H}{\partial z}\dot{z}$ for (1.47). Then, by imposing equality (1.46), the coefficients of both sides are equated degree by degree starting at 3, and this allows to recursively determine coefficients h_{ijk} together with the Lyapunov constants when the degree is even. Actually, the extra term $L_\ell(x^2+y^2)^{\ell+1}$ is added so that the resulting systems for even degrees are not underdetermined, but in our case and for simplicity we have equivalently considered $L_\ell x^{2\ell+2}$ as the adding term (see [Chr05] for more details on this change).

This is essentially the usual Lyapunov method used to find Lyapunov constants in the plane but adapted to the three-dimensional case. As we have commented above, this approach decreases the computational time because the restriction of being on the center manifold given by Theorem 1.38 is not necessary. We can also observe that we have considered the method in real coordinates, since the complex coordinates approach which in previous sections simplified the computations cannot be applied in a three-dimensional space.

Remark 1.39. *If the linear part of (1.44) was not written in its real Jordan normal form, we could use the same approach but finding which coefficients would make function $H(x,y,z) = a_{20}x^2 + a_{11}xy + a_{02}y^2 + O_3(x,y,z)$ be a first integral.*

The algorithm explained here has been computationally implemented with Maple ([Map]), and used to calculate the Lyapunov constants throughout the rest of this section. Actually, we will see that for the results we want to prove we will only need those Lyapunov constants up to first-order in the perturbative parameters, which highly simplifies the calculations and reduces the computation time.

The second method to find Lyapunov constants in 3-dimensional systems consists on an algorithm to compute Lyapunov constants on the center manifold (see for instance [WHW11]). According to the center manifold theorem (see [Car81]), system (1.44) has an approximation to the center manifold by means of the change

$$z \to Z = \chi(x,y) = \chi_2(x,y) + O_3(x,y), \tag{1.48}$$

where χ_2 is a quadratic homogeneous polynomial in x and y, and O_3 denotes the terms with orders greater than or equal to 3. This process can be iterated to up to an arbitrary degree M to remove all the possible terms of the third equation in (1.44), by following the idea of normal form theory. The obtained change of variable can be performed into the equations of system (1.44), and we obtain a generic two-differential system with center-focus type linear part: these are the equations on the center manifold. This way, the center manifold can be translated to the plane $Z = 0$, which highly simplifies the problem as it is reduced to finding Lyapunov constants of a system in the plane and the usual method can be applied.

As an example to check that the two presented methods are equivalent, we will see that the Lyapunov constants obtained by both methods are the same when computing the $N = 2$ first Lyapunov constants of system

$$\begin{cases} \dot{x} = -y + x^2 + \sum_{i+j+k=2}^{2} a_{ijk} x^i y^j z^k, \\ \dot{y} = x + \sum_{i+j+k=2}^{2} b_{ijk} x^i y^j z^k, \\ \dot{z} = z + \sum_{i+j+k=2}^{2} c_{ijk} x^i y^j z^k, \end{cases}$$

for $a_{ijk}, b_{ijk}, c_{ijk} \in \mathbb{R}$. Its first integral would have the form $H(x,y,z) = x^2 + y^2 + \cdots$, and to find $N = 2$ Lyapunov constants we need degree $2N + 2$, so we define

$$H(x,y,z) = x^2 + y^2 + \sum_{i+j+k=3}^{2N+2} h_{ijk} x^i y^j z^k.$$

Now by imposing condition (1.45) and applying the usual algorithm, we determine coefficients h_{ijk} and Lyapunov constants up to second-order in the perturbative parameters, which in our case are

$$\begin{aligned} L_1 = & \frac{1}{4}a_{110} - \frac{1}{2}b_{200} - \frac{1}{10}c_{020}a_{011} - \frac{1}{20}c_{110}a_{011} + \frac{1}{10}c_{200}a_{011} + \frac{1}{4}a_{020}a_{110} \\ & + \frac{1}{2}a_{020}b_{020} - \frac{9}{20}c_{020}a_{101} - \frac{1}{10}c_{110}a_{101} - \frac{11}{20}c_{200}a_{101} + \frac{1}{4}a_{110}a_{200} \\ & - \frac{1}{2}b_{200}a_{200} - \frac{11}{20}c_{020}b_{011} + \frac{1}{10}c_{110}b_{011} - \frac{9}{20}c_{200}b_{011} - \frac{1}{4}b_{020}b_{110} \\ & - \frac{1}{10}b_{101}c_{020} - \frac{1}{20}b_{101}c_{110} + \frac{1}{10}b_{101}c_{200} - \frac{1}{4}b_{200}b_{110}, \end{aligned} \tag{1.49}$$

$$
\begin{aligned}
L_2 =& \frac{5}{24}a_{110} - \frac{1}{4}b_{020} - \frac{5}{12}b_{200} - \frac{121}{600}c_{020}a_{011} - \frac{121}{1200}c_{110}a_{011} + \frac{2}{75}c_{200}a_{011} \\
& - \frac{2}{3}a_{020}a_{110} + 2a_{020}b_{200} - \frac{221}{300}c_{020}a_{101} - \frac{121}{600}c_{110}a_{101} - \frac{343}{600}c_{200}a_{101} \\
& - 2a_{110}a_{200} - \frac{31}{48}a_{110}b_{110} - \frac{3}{4}a_{200}b_{020} + 4b_{200}a_{200} - \frac{263}{600}c_{020}b_{011} \\
& - \frac{13}{1200}c_{110}b_{011} - \frac{97}{600}c_{200}b_{011} + \frac{67}{300}b_{101}c_{020} + \frac{67}{600}b_{101}c_{110} \\
& + \frac{53}{300}b_{101}c_{200} + \frac{4}{3}b_{200}b_{110}.
\end{aligned} \tag{1.50}
$$

These are the constants obtained by the first method presented above.

Let us now find the same Lyapunov constants but applying the algorithm of reducing the problem to the center manifold. As we want to find $N = 2$ Lyapunov constants, we need to reach the normal form up to degree $M = 2N + 1 = 5$. To this end, we consider the Mth order truncation of (1.48), and we determine the coefficients of this change which remove as many terms as possible in the $\dot{z}$ equation in (1.44) to obtain the corresponding normal form up to 5th degree. We do not show here neither the explicit change of variables nor the explicit form of the equations of $\dot{x}$ and $\dot{y}$ after the change due to the length of the expressions, but the expression of $\dot{z}$ in the new variable becomes

$$
\begin{aligned}
\dot{Z} =& Z + \Big(a_{200}c_{011} + \frac{6}{5}a_{101}c_{200} + \frac{2}{5}a_{101}c_{110} + \frac{4}{5}a_{101}c_{020} + c_{011}a_{020} - \frac{2}{5}a_{011}c_{200} \\
& + \frac{1}{5}a_{011}c_{110} + \frac{2}{5}a_{011}c_{020} - c_{101}b_{200} - \frac{2}{5}b_{101}c_{200} + \frac{1}{5}b_{101}c_{110} + \frac{2}{5}b_{101}c_{020} \\
& - c_{101}b_{020} + \frac{4}{5}b_{011}c_{200} - \frac{2}{5}b_{011}c_{110} + \frac{6}{5}b_{011}c_{020} - 2c_{002}c_{200} - 2c_{002}c_{020} \\
& + c_{011} \Big) x^2 Z + \cdots .
\end{aligned}
$$

Finally, we can set $Z = 0$ to move the invariant center manifold to the plane $Z = 0$, and therefore the problem becomes a system of differential equations in the plane for which the usual $\mathbb{R}^2$ algorithm can be applied to find the Lyapunov constants. Using the expressions of $\dot{x}$ and $\dot{y}$ after the change with $Z = 0$, we have checked that the obtained Lyapunov constants are the same as those in (1.49) and (1.50).

The Poincaré–Miranda's Theorem

Here we formulate Poincaré–Miranda's Theorem, a result which could be described as a generalization of Bolzano's Theorem to higher dimensions.

Theorem 1.40 ([Maw19]). *(Poincaré–Miranda's Theorem) Let* $\mathcal{B} = \{x \in \mathbb{R}^m : |x_j| \leq h, \text{ for } 1 \leq j \leq m\}$ *and suppose that the mapping* $F = (f_1, f_2, \ldots, f_m) : \mathcal{B} \to \mathbb{R}^m$ *is continuous on* $\mathcal{B}$ *and such that* $F(x) \neq (0, 0, \ldots, 0)$ *for* x *on the boundary* $\partial\mathcal{B}$ *of* $\mathcal{B}$*, and*

(i) $f_j(x_1, x_2, \ldots, x_{j-1}, -h, x_{j+1}, \ldots, x_m) \geq 0$ *for* $1 \leq j \leq m$*, and*

(ii) $f_j(x_1, x_2, \ldots, x_{j-1}, +h, x_{j+1}, \ldots, x_m) \leq 0$ *for* $1 \leq j \leq m$*.*

Then, $F(x) = (0, 0, \ldots, 0)$ *has a solution in* $\mathcal{B}$*.*

We observe that in all our examples of using the above result, the inequalities on $\partial\mathcal{B}$ are always strict. For the proof of this theorem the reader is referred to [Maw19] or [Vra89].

A result on the number of limit cycles of parametric systems

The idea behind the proof of Theorem 1.37 is studying the structure of the Lyapunov constants near centers up to first-order Taylor development and analyzing the rank of such linear parts. The last preliminary tool we present is a theorem which, given a family of centers which depends on some parameters and under certain conditions, enables to obtain extra limit cycles to those seen only with the ranks of linear parts. The following result is based on the fact that we study the local cyclicity on a center component, which takes a generical value that can be increased on some special curves. For example, in Proposition 1.42 we have a 2-dimensional space of center parameters (a, b) such that, generically, 9 limit cycles of small amplitude bifurcate from the origin under quadratic perturbations. But there exists a curve of special centers where 10 limit cycles bifurcate and, on this curve, there exist special points (we prove the existence of at least one) for which, from the corresponding center, 11 limit cycles bifurcate. In fact, this is like describing a bifurcation diagram on the center component because the local cyclicity depends on the parameters of the center family. In all our results we are only providing lower bounds for the local cyclicity value.

Theorem 1.41 ([GGT21]). *We denote by* $L_j^{(1)}(\lambda, b)$ *the first-order development, with respect to* $\lambda \in \mathbb{R}^k$*, of the j-Lyapunov constant of system*

$$\begin{cases} \dot{x} = \alpha y + P_c(x, y, \mu) + P(x, y, \lambda), \\ \dot{y} = \alpha x + Q_c(x, y, \mu) + Q(x, y, \lambda), \end{cases} \tag{1.51}$$

being $(\dot{x}, \dot{y}) = (P_c(x, y, \mu), Q_c(x, y, \mu))$ *a family of polynomial centers of degree* n *depending on a parameter* $\mu \in \mathbb{R}^\ell$ *and having a non-degenerate center equilibrium point at the origin, and being* $P(x, y, \lambda), Q(x, y, \lambda)$ *polynomials of degree* n *having no constant* *nor linear terms with perturbative parameters $\lambda \in \mathbb{R}^{n^2+3n-4}$. We assume that, after a change of variables in the parameter space if necessary, we can write*

$$L_j = \begin{cases} \lambda_j + O_2(\lambda), \text{ for } j = 1, \dots, k-1, \\ \sum_{l=1}^{k-1} g_{j,l}(\mu)\lambda_l + f_{j-k}(\mu)\lambda_k + O_2(\lambda), \text{ for } j = k, \dots, k+\ell, \end{cases}$$

where with $O_2(\lambda)$ we denote all the monomials of degree higher or equal than 2 in λ with coefficients analytic functions in μ. If there exists a point μ^* such that $f_0(\mu^*) = \cdots = f_{\ell-1}(\mu^*) = 0$, $f_\ell(\mu^*) \neq 0$, and the Jacobian matrix of $(f_0, \dots, f_{\ell-1})$ with respect to μ has rank ℓ at μ^*, then system (1.51) has $k+\ell$ hyperbolic limit cycles of small amplitude bifurcating from the origin.

This result is proved in [GGT21], where it is used to study the cyclicity of some planar families of vector fields. Even though we aim to study cyclicity in $\mathbb{R}^3$, the same technique from Theorem 1.41 can be automatically extrapolated to vector fields in the space as the whole problem is analogous.

1.5.2 11 limit cycles for a quadratic system

In this subsection we present a quadratic system in $\mathbb{R}^3$ and show that it unfolds 11 limit cycles by using the techniques from Subsection 1.5.1. This proves the case $n = 2$ in Theorem 1.37.

The presented result improves by one the previous best lower bound found in [YH15]. In such work, the authors use a family of planar centers inspired by a family of Lotka-Volterra systems in $\mathbb{R}^2$, because this family defines a center component (depending on parameters) denoted by Q_3^{LV} according to the classification of quadratic planar centers provided by [Zol94]. However, this family does not have the maximum generic local cyclicity unlike family Q_4. Family Q_4 has a first integral of the form F^3G^{-2}, as detailed in [BZ05b], but no free parameters. Here, $F = 0$ and $G = 0$ are two special invariant algebraic curves of Q_4, such as in (1.53). Our goal is to take advantage of Theorem 1.41 in a family of centers extending Q_4 to $\mathbb{R}^3$. Hence, we add parameters to the third component instead of the first two, and from Theorem 1.38 we know the existence of a local center manifold. We remark that the next result is saying that the local cyclicity can increase when the parameters change. We recall that these families have already been found in Theorem 1.31.

From the computational point of view, it is important to remark that the best centers are the ones whose linear part is in the usual Jordan normal form, namely $(x', y', z') = (-y, x, z)$. But sometimes, as in our case, the changes of variables to achieve this form add square roots in the coefficients, and this increases the computational complexity, so we use Remark 1.39 to avoid these difficulties.

The following result provides, up to our knowledge, the best lower bound for the local cyclicity near a Hopf point in quadratic vector fields in $\mathbb{R}^3$.

Proposition 1.42. *The quadratic system*

$$\begin{cases} \dot{x} = -\frac{1}{3}y - 5x^2 - 2xy + \frac{1}{3}y^2, \\ \dot{y} = x - 3x^2 - 10xy + y^2, \\ \dot{z} = z + z^2 + ax^2 + by^2, \end{cases} \tag{1.52}$$

has a center at the origin, and there exist a and b such that 11 *limit cycles of small amplitude bifurcate from the origin under a complete quadratic perturbation.*

Proof. The two first equations in (1.52) define a center in the plane, as it can be trivially checked that the corresponding system has a rational first integral of the form

$$H_2(x,y) = \frac{(36x^2 - 24xy + 4y^2 - 8y + 1)^3}{(108x^3 - 108x^2y + 36xy^2 - 4y^3 - 36xy + 12y^2 - 12y + 1)^2}. \tag{1.53}$$

This center in $\mathbb{R}^2$ can be embedded in $\mathbb{R}^3$ by adding a third component that, from Theorem 1.38, adds a 2-dimensional center manifold where system (1.52) has a center and the origin is a equilibrium point of Hopf type. Let us add a quadratic perturbation and the trace parameter to (1.52) in the following way

$$\begin{cases} \dot{x} = \alpha x - \frac{1}{3}y - 5x^2 - 2xy + \frac{1}{3}y^2 + \sum_{i+j+k=2}^{2} a_{ijk}x^i y^j z^k, \\ \dot{y} = x + \alpha y - 3x^2 - 10xy + y^2 + \sum_{i+j+k=2}^{2} b_{ijk}x^i y^j z^k, \\ \dot{z} = z + z^2 + ax^2 + by^2 + \sum_{i+j+k=2}^{2} c_{ijk}x^i y^j z^k, \end{cases} \tag{1.54}$$

for $\alpha, a_{ijk}, b_{ijk}, c_{ijk} \in \mathbb{R}$ perturbative parameters.

The next step is to find, for $\alpha = 0$, the first 11 Lyapunov constants of (1.54) up to first-order in the perturbative parameters. Notice that, due to Remark 1.39, in this case we will consider a first integral with the form $H(x,y,z) = \frac{1}{2}x^2 + \frac{1}{6}y^2 + O_3(x,y,z)$. Once we have the linear parts of the 11 first Lyapunov constants of the system, we check that their rank is generically 9. Hence, by the Implicit Function Theorem and adding α, we generically have 9 limit cycles of small amplitude bifurcating from the origin.

After a linear change of coordinates in the parameters space, we obtain that the Lyapunov constants have the following form:

$$\begin{aligned} L_k &= u_k + O_2, \text{ for } k = 1, \dots, 8, \\ L_9 &= \frac{f_1(a,b)}{d(a,b)} u_9 + O_2, \\ L_{10} &= \frac{f_2(a,b)}{d(a,b)} u_9 + O_2, \\ L_{11} &= \frac{f_3(a,b)}{d(a,b)} u_9 + O_2, \end{aligned}$$

being $f_1(a,b), f_2(a,b), f_3(a,b), d(a,b)$ certain polynomials with integer coefficients in the variables a and b. We do not show the complete polynomials here due to their large size. They have respectively total degree $38, 39, 40$, and 29, their number of monomials are respectively $744, 784, 825$, and 444, and the coefficients are integers having between 72 and 158 figures. Then, by Theorem 1.41, to prove the bifurcation of 11 limit cycles we just have to check that there exists a point $(\hat{a}, \hat{b})$ in the parameters space such that $f_1(\hat{a}, \hat{b}) = f_2(\hat{a}, \hat{b}) = 0$, $f_3(\hat{a}, \hat{b}) \neq 0, d(\hat{a}, \hat{b}) \neq 0$, and the Jacobian determinant $\det \mathrm{Jac}_{(f_1,f_2)}(\hat{a}, \hat{b}) \neq 0$.

The situation is represented on the graph in Figure 1.4, where the zero level curves of the considered polynomials are represented. In it, we can see how the curves $f_1(a,b) = 0$ and $f_2(a,b) = 0$ intersect at a point $(\hat{a}, \hat{b})$ which does not belong to the curves $f_3(a,b) = 0$ and $d(a,b) = 0$. We aim to analytically prove the existence of such point by means of Poincaré–Miranda's Theorem (Theorem 1.40). To do this, we will provide a computer-assisted proof by using rational interval analysis.

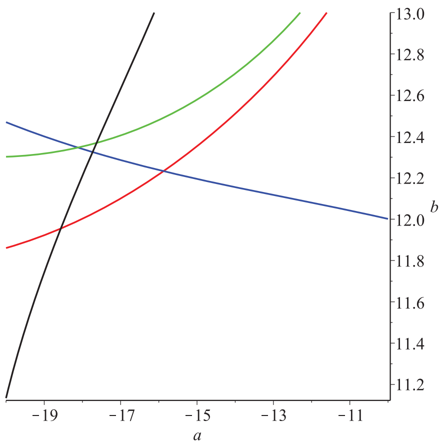

FIGURE 1.4: Plot of the zero level curves of $f_1(a,b), f_2(a,b), f_3(a,b)$, and $d(a,b)$ in color red, blue, green, and black, respectively.

Let us start by finding a numerical approximation for an intersection point

$$(\hat{a}, \hat{b}) \approx (-15.87687966375324925, 12.23255254136248609) \quad (1.55)$$

of $f_1(a,b) = 0$ and $f_2(a,b) = 0$, which can be seen in Figure 1.4. To simplify the application of Poincaré–Miranda's Theorem, we will perform a linear change of variables $(a,b) \to (u,v)$ with rational coefficients. To this end, we will consider the approximation $(\hat{a}, \hat{b}) \approx (-\frac{159}{10}, \frac{61}{5}) = (-15.9, 12.2)$ and define u and v as the numerical tangent lines at this point. Then, one can isolate (a,b) as a function of (u,v), consider the solution with 3 significant digits and convert it to rational values, obtaining

$$(a,b) = \left(-\frac{159}{10} - \frac{577}{100}u + \frac{577}{100}v, \frac{61}{5} + \frac{129}{500}u + \frac{371}{500} \right). \quad (1.56)$$

If we substitute (1.56) in $f_1(a,b), f_2(a,b), f_3(a,b)$, and $d(a,b)$ we obtain the polynomials in the new variables, which we will denote by $F_1(u,v), F_2(u,v), F_3(u,v)$, and $D(u,v)$. These new polynomials are represented in Figure 1.5, where we can see that now the intersection point $(\hat{u}, \hat{v})$ of $F_1(u,v) = 0$ and $F_2(u,v) = 0$ has shifted near $(0,0)$ and the application of Poincaré–Miranda's Theorem will be easier.

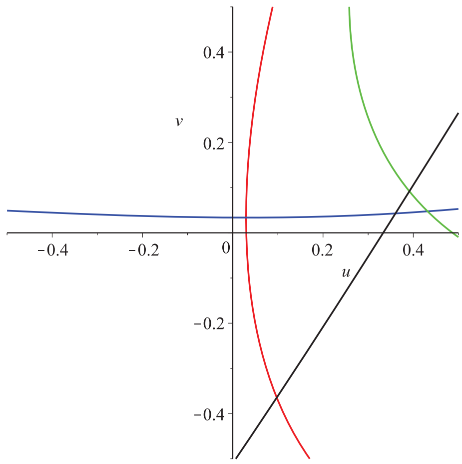

FIGURE 1.5: Plot of the zero level curves of $F_1(u,v), F_2(u,v), F_3(u,v)$, and $D(u,v)$ in color red, blue, green, and black, respectively.

Taking $h = 1/10$ in Theorem 1.40, we will show that in the square $[-h,h]^2$ there must be a zero of $F_1(u,v)$ and $F_2(u,v)$. The proof follows by checking also that $F_3(u,v)$, $D(u,v)$, and the Jacobian determinant $J(u,v) := \det \text{Jac}_{(F_1,F_2)}(u,v)$ do not vanish in the whole square. Observe that $F_1(u,v)$ and $F_2(u,v)$ are continuous because they are polynomials. Then, there will be a point $(\hat{u}, \hat{v}) \in (-h,h)^2$

such that $F_1(\hat{u},\hat{v}) = 0$ and $F_2(\hat{u},\hat{v}) = 0$ by applying the Poincaré–Miranda's Theorem because the following conditions hold.

(a) $F_1(h,v) > 0$ and $F_1(-h,v) < 0$ for $v \in [-h,h]$.
To prove this, we will show that $F_1(h,v)$ is inferiorly bounded by a positive number in $v \in [-h,h]$ and $F_1(-h,v)$ is superiorly bounded by a negative number in $v \in [-h,h]$. Indeed,

$$4 \cdot 10^{145} < F_1(h,0) - \sum_{i=1}^{38} |A_i| h^i \leq F_1(h,0) + \sum_{i=1}^{38} A_i v^i = F_1(h,v),$$
$$F_1(-h,v) = F_1(-h,0) + \sum_{i=1}^{38} B_i v^i \leq F_1(-h,0) + \sum_{i=1}^{38} |B_i| h^i < -8 \cdot 10^{145},$$

where $A_i, B_i \in \mathbb{Q}$ are the coefficients of the corresponding polynomials. Notice that, due to how Theorem 1.40 is formulated, it should be applied to $-F_1(u,v)$ rather than $F_1(u,v)$, but the conclusion is exactly the same.

(b) $F_2(u,-h) > 0$ and $F_2(u,h) < 0$ for $u \in [-h,h]$.
Analogously, we will show that $F_2(u,-h)$ is inferiorly bounded by a positive number in $u \in [-h,h]$ and $F_2(u,h)$ is superiorly bounded by a negative number in $u \in [-h,h]$. Indeed,

$$1 \cdot 10^{158} < F_2(0,-h) - \sum_{i=1}^{39} |C_i| h^i \leq F_2(0,-h) + \sum_{i=1}^{39} C_i u^i = F_2(u,-h),$$
$$F_2(u,h) = F_2(0,h) + \sum_{i=1}^{39} D_i u^i \leq F_2(0,h) + \sum_{i=1}^{39} |D_i| h^i < -8 \cdot 10^{157},$$

where $C_i, D_i \in \mathbb{Q}$ are the coefficients of the corresponding polynomials.

The last step of the proof is to ensure that $F_3(u,v), D(u,v)$, and $J(u,v)$ do not vanish in $[-h,h]^2$. More concretely, we will check that the three functions are negative in the whole square by seeing that they are superiorly bounded by a negative number,

$$F_3(u,v) = F_3(0,0) + \sum_{i+j=1}^{40} G_{ij} u^i v^j \leq F_3(0,0) + \sum_{i+j=1}^{40} |G_{ij}| h^{i+j} < -6 \cdot 10^{171},$$
$$D(u,v) = D(0,0) + \sum_{i+j=1}^{29} H_{ij} u^i v^j \leq D(0,0) + \sum_{i+j=1}^{29} |H_{ij}| h^{i+j} < -7 \cdot 10^{128},$$
$$J(u,v) = J(0,0) + \sum_{i+j=1}^{75} K_{ij} u^i v^j \leq J(0,0) + \sum_{i+j=1}^{75} |K_{ij}| h^{i+j} < -3 \cdot 10^{305},$$

where $G_{ij}, H_{ij}, K_{ij} \in \mathbb{Q}$ are the coefficients of the corresponding polynomials. Hence, all the above bounds are obtained by adding rational numbers with no error in the computations.

All the computations have been done with $\alpha = 0$, and 10 limit cycles are obtained. The proof follows by adding an extra limit cycle which bifurcates from the origin when using the usual Hopf bifurcation moving the trace parameter α adequately. □

From the proof of Proposition 1.42, we can extract from Figure 1.4 the bifurcation diagram as we explained at the beginning of Subsection 1.5.1, obtaining Figure 1.6.

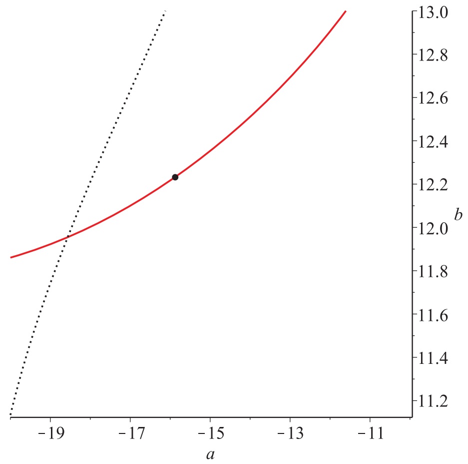

FIGURE 1.6: Avoiding the dots line, we have (generically) 9 small limit cycles bifurcating from (1.52), 10 on the red line, and 11 on the black point.

In fact, there are other intersection points for the zero level curves of f_1 and f_2, but they are more difficult to find and to prove their transversal intersection. See Figure 1.7 for a better understanding of the difficulty of finding intersection points like (1.55).

Before finishing this subsection, we would like to make two final comments about quadratic systems in $\mathbb{R}^3$.

In Proposition 1.42, we have added a third component in (1.52) to extend the problem from $\mathbb{R}^2$ to $\mathbb{R}^3$. This procedure can be done in many different ways and for each third component we obtain different center manifolds. We have tested different possibilities, and we have observed that when the center manifold is $z = 0$, the number of limit cycles is the same as in the planar problem. For this reason, we have added some terms including x or y for increasing such number. Furthermore, we have observed by adding different third components which include x and y that the result does not improve.

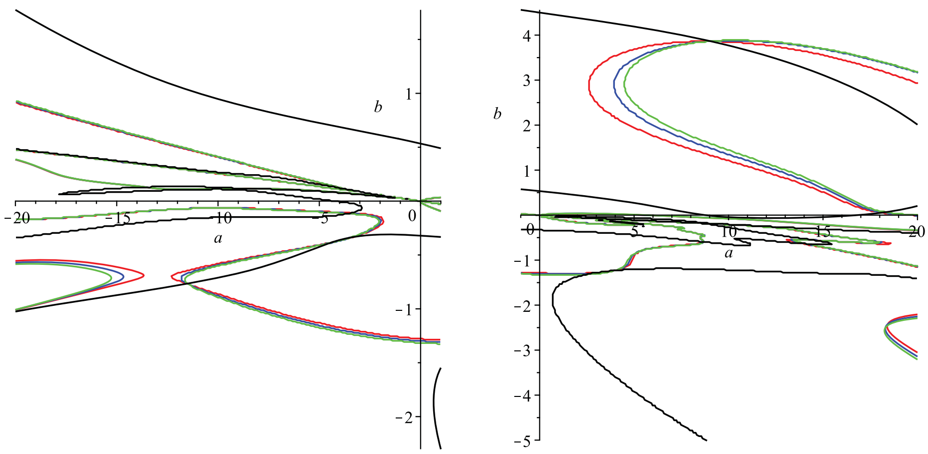

FIGURE 1.7: Plot of the zero level curves of f_1, f_2, f_3, and d in color red, blue, green, and black, respectively, in different zones of the plane (a, b).

We believe that the result obtained in Proposition 1.42 is the maximum cyclicity that can be obtained by using the presented technique. In (1.54) we have a total of 18 perturbative parameters, and we have observed in our calculations that only 12 of them actually play a role when finding the ranks of linear parts of the Lyapunov constants. Actually, the ones which play a role are a_{ijk} and b_{ijk}, these are the ones in $\dot{x}$ and $\dot{y}$; parameters c_{ijk} in $\dot{z}$ do not appear in the computation of the linear parts of the Lyapunov constants. Perhaps more limit cycles could be obtained by studying higher-order developments, but this is a very difficult computational problem due to the size of the corresponding polynomials, assuming that such polynomials could be found.

1.5.3 31 limit cycles for a cubic system

Here we prove the case $n = 3$ in Theorem 1.37, by presenting a cubic system in $\mathbb{R}^3$ having a Hopf point at the origin from which 31 limit cycles bifurcate.

Proposition 1.43. *The cubic system*

$$\begin{cases} \dot{x} = -y(1 - 68x + 1183x^2), \\ \dot{y} = x - 58x^2 - 44xy + 30y^2 + 672x^3 + 1484x^2y - 945xy^2 - 84y^3, \\ \dot{z} = z + x^2 + ax^3 + by^3, \end{cases} \tag{1.57}$$

has a center at the origin, and there exist a and b such that 31 limit cycles of small amplitude bifurcate from the origin under a complete cubic perturbation.

Proof. The two first equations in (1.57) define a system in the plane with a center at the origin because it has a rational first integral (see [BS08]). Then, we extend this system to (1.57) by adding a third equation, and there exists a center manifold tangent to $z = 0$ guaranteed by Theorem 1.38 on which (1.57) has a center. Let us add a perturbation with cubic and quadratic terms to (1.57) and the trace parameter as follows,

$$\begin{cases} \dot{x} = & \alpha x - y(1 - 68x + 1183x^2) + \sum_{i+j+k=2}^{3} a_{ijk} x^i y^j z^k, \\ \dot{y} = & x + \alpha y - 58x^2 - 44xy + 30y^2 + 672x^3 + 1484x^2y - 945xy^2 - 84y^3 \\ & + \sum_{i+j+k=2}^{3} b_{ijk} x^i y^j z^k, \\ \dot{z} = & z + x^2 + ax^3 + by^3 + \sum_{i+j+k=2}^{3} c_{ijk} x^i y^j z^k, \end{cases} \tag{1.58}$$

for $\alpha, a_{ijk}, b_{ijk}, c_{ijk} \in \mathbb{R}$ perturbative parameters.

The proof for the unfolding of 31 limit cycles is analogous to that of Proposition 1.42. We first take $\alpha = 0$ and find the linear parts of the first 31 Lyapunov constants of (1.58) with respect to the perturbative parameters. We see then that generically their rank is 29. After a linear change of coordinates in the parameters space we obtain that the first 28 Lyapunov constants have the following form:

$$L_k = u_k + O_2, \text{ for } k = 1, \ldots, 28.$$

Next, we consider an analytic change of coordinates, by using the Implicit Function Theorem, such that the Lyapunov constants write as $L_k = v_k$, for $k = 1, \ldots, 28$. Assuming that $v_k = 0$, for $k = 1, \ldots, 28$, and vanishing the nonessential perturbative parameters, we get

$$\begin{aligned} L_{29} &= \frac{f_1(a,b)}{d(a,b)} u_{29} + O_2(u_{29}), \\ L_{30} &= \frac{f_2(a,b)}{d(a,b)} u_{29} + O_2(u_{29}), \\ L_{31} &= \frac{f_3(a,b)}{d(a,b)} u_{29} + O_2(u_{29}), \end{aligned}$$

being $f_1(a,b), f_2(a,b), f_3(a,b), d(a,b)$ certain polynomials with integer coefficients in the variables a and b. We do not show the complete polynomials here due to their large size, but we see that both $f_1(a,b), f_2(a,b)$, and $f_3(a,b)$ have degree 28 and 434 monomials, and $d(a,b)$ has degree 25 and 350 monomials. To see the bifurcation of 31 limit cycles we can use Theorem 1.41 and check the conditions on it, this is to find a point $(\hat{a}, \hat{b})$ in the parameters space such that $f_1(\hat{a}, \hat{b}) = f_2(\hat{a}, \hat{b}) = 0$, $f_3(\hat{a}, \hat{b}) \neq 0, d(\hat{a}, \hat{b}) \neq 0$, and the Jacobian determinant $\det \mathrm{Jac}_{(f_1, f_2)}(\hat{a}, \hat{b}) \neq 0$.

We have drawn $f_1(a,b) = 0, f_2(a,b) = 0, f_3(a,b) = 0$, and $d(a,b) = 0$ in Figure 1.8. Although it is not discernible in this graph, what is happening is that curves $f_1(a,b) = 0, f_2(a,b) = 0$, and $f_3(a,b) = 0$ are practically overlapping, so the intersection point that we are searching cannot be appreciated; $d(a,b) = 0$ is not shown because it remains out of the plotted region. To sight the intersection point of $f_1(a,b) = 0$ and $f_2(a,b) = 0$ we will perform a change of variables such that the intersection is transversal, and we will apply Poincaré–Miranda's Theorem (Theorem 1.40) to analytically show its existence. This will be done by means of a computer-assisted proof and using rational interval analysis.

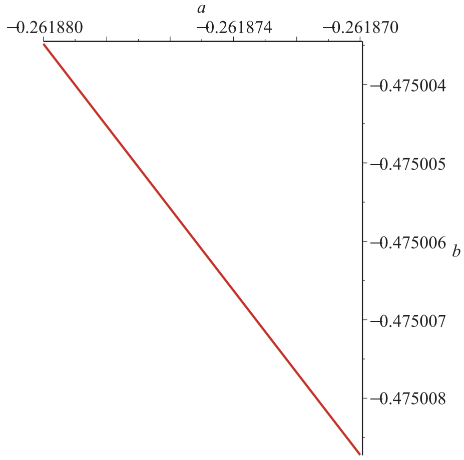

FIGURE 1.8: Plot of the zero level curves of $f_1(a,b), f_2(a,b)$, and $f_3(a,b)$ in color red, blue and green, respectively.

First we find a numerical approximation

$$(\hat{a}, \hat{b}) = (-0.26187466968713249428115457453967889567 98,\\ -0.47500627278383963054665090584849081940 11)$$

for an intersection point of $f_1(a,b) = 0$ and $f_2(a,b) = 0$. We need a good rational approximation so that the zero level curves drawn in Figure 1.8 are separated, and this way the transversality can be appreciated. Let us perform a change of variables $(a,b) \to (u,v)$ such that u and v are the first-order Taylor expansion at $(\hat{a}, \hat{b})$ of $f_1(a,b)$ and $f_2(a,b)$, respectively. Now we can find the inverse of such change and convert the coefficients to rational numbers, which gives

$$
\begin{aligned}
(a,b) = \Big(&-\frac{3125780069700516145310827}{1193616806633094860249265 5} \\
&+\frac{1427237216612}{940600247814793427315253720871652057}u \\
&-\frac{6400609904497}{4224724520267912944601259158052139159}v, \\
&-\frac{2950612633153916740411853}{6211734038503208283147559} \\
&-\frac{3951139552423}{4971433349089293973487138537373615874}u \\
&+\frac{2249793630741}{2835231811366846850770950557241968074}v\Big).
\end{aligned}
$$

These expressions can be substituted in $f_1(a,b)$, $f_2(a,b)$, $f_3(a,b)$, and $d(a,b)$ to obtain the polynomials in the new variables, which we will be denoted respectively by $F_1(u,v)$, $F_2(u,v)$, $F_3(u,v)$, and $D(u,v)$. The new polynomials are represented in Figure 1.9, where we can see that now the intersection point $(\hat{u},\hat{v})$ of $F_1(u,v) = 0$ and $F_2(u,v) = 0$ has shifted near $(0,0)$ and its transversality can be clearly seen. $F_3(u,v)$ and $D(u,v)$ are not in the graph because they stay out of the plotted region, so they will be nonvanishing at the intersection point.

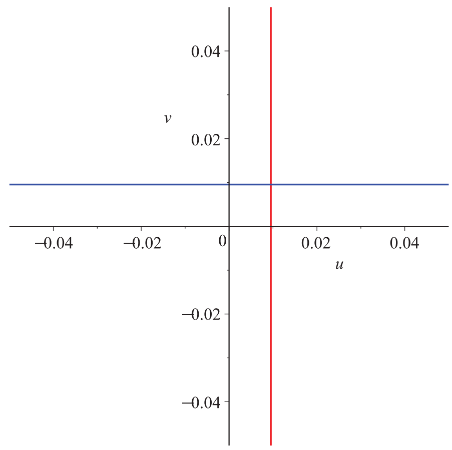

FIGURE 1.9: Plot of the zero level curves of $F_1(u,v)$ and $F_2(u,v)$ in color red and blue, respectively.

The proof follows as in the case of Proposition 1.42, also taking $h = 1/10$ and applying the Poincaré–Miranda's Theorem, since the following conditions hold:

(a) $F_1(h,v) > 0$ and $F_1(-h,v) < 0$ for $v \in [-h,h]$.
We will see that $F_1(h,v)$ is inferiorly bounded by a positive number in $v \in$

$[-h,h]$ and $F_1(-h,v)$ is superiorly bounded by a negative number in $v \in [-h,h]$. Indeed,

$$\frac{9}{100} < F_1(h,0) - \sum_{i=1}^{28} |A_i| h^i \le F_1(h,0) + \sum_{i=1}^{28} A_i v^i = F_1(h,v),$$

$$F_1(-h,v) = F_1(-h,0) + \sum_{i=1}^{28} B_i v^i \le F_1(-h,0) + \sum_{i=1}^{28} |B_i| h^i < -\frac{1}{10},$$

where $A_i, B_i \in \mathbb{Q}$ are the coefficients of the corresponding polynomials.

(b) $F_2(u,h) > 0$ and $F_2(u,-h) < 0$ for $u \in [-h,h]$.
Analogously, we will show that $F_2(u,h)$ is inferiorly bounded by a positive number in $u \in [-h,h]$ and $F_2(u,-h)$ is superiorly bounded by a negative number in $u \in [-h,h]$:

$$\frac{9}{100} < F_2(0,h) - \sum_{i=1}^{28} |C_i| h^i \le F_2(0,h) + \sum_{i=1}^{28} C_i u^i = F_2(u,h),$$

$$F_2(u,-h) = F_2(0,-h) + \sum_{i=1}^{28} D_i u^i \le F_2(0,-h) + \sum_{i=1}^{28} |D_i| h^i < -\frac{1}{10},$$

where $C_i, D_i \in \mathbb{Q}$ are the coefficients of the corresponding polynomials.

Notice that, due to how Theorem 1.40 is formulated, it should be applied to $-F_1(u,v)$ and $-F_2(u,v)$ rather than $F_1(u,v)$ and $F_2(u,v)$, but the conclusion is exactly the same.

Finally, we will check that $F_3(u,v)$ is negative in $[-h,h]^2$, and $D(u,v)$ and $J(u,v)$ are positive, so none of those functions vanish in the whole square.

$$F_3(u,v) = F_3(0,0) + \sum_{i+j=1}^{28} G_{ij} u^i v^j \le F_3(0,0) + \sum_{i+j=1}^{28} |G_{ij}| h^{i+j} < -2 \cdot 10^{20},$$

$$2 \cdot 10^{1032} < D(0,0) - \sum_{i+j=1}^{25} |H_{ij}| h^{i+j} \le D(0,0) + \sum_{i+j=1}^{25} H_{ij} u^i v^j = D(u,v),$$

$$\frac{99}{100} < J(0,0) - \sum_{i+j=1}^{54} |K_{ij}| h^{i+j} \le J(0,0) + \sum_{i+j=1}^{54} K_{ij} u^i v^j = J(u,v)$$

where $G_{ij}, H_{ij}, K_{ij} \in \mathbb{Q}$ are the coefficients of the corresponding polynomials.

The proof finishes adding, as in Proposition 1.42, the trace parameter. □

As we already noticed for quadratic systems, we think that we have almost obtained the maximum cyclicity for the cubic case in $\mathbb{R}^3$ by using this approach.

Here we have 48 perturbative parameters and, by considering linear Taylor developments, only 32 do appear.

1.5.4 A parallelization approach for quartic and quintic systems

In this subsection we will show how the cases $n = 4$ and $n = 5$ from Theorem 1.37 are achieved. The idea, as we have done in the previous subsections, is to consider centers in the plane such that generically unfold a high number of limit cycles, and then extend them to $\mathbb{R}^3$ by adding the third equation. For this case, we will not use the technique described in Theorem 1.41 as we did for quadratic and cubic systems of taking the best known quartic and quintic systems due to the difficulty to deal with the obtained constants because of their huge size. We have considered the cubic system (1.57) adding one or two straight lines of equilibria. Furthermore, for the Lyapunov constants computation during this subsection we will consider a parallelization approach in order to reduce the executing times of the processes.

The technique we have used to parallelize the computation of linear parts of Lyapunov constants is inspired by [LT15], and is described as follows. Let us consider a system with perturbative parameters $\lambda_1, \ldots, \lambda_d$. We select some $k \in \{1, \ldots, d\}$ and consider the same system with $\lambda_l = 0$ for $l \in \{1, \ldots, d\} \setminus \{k\}$, this is a system with only one perturbative parameter λ_k. Then, we find its Lyapunov constants up to first-order $L_{j,k}^{(1)}$, and repeat this process for every $k = 1, \ldots, d$. This step can be easily parallelized by assigning to each thread of the execution the computation of the Lyapunov constants of the system with a different nonzero perturbative parameter λ_k, so we would have a parallelization paradigm with d threads, as many as perturbative parameters. Once this has been done, the linear part of the jth Lyapunov constant L_j of the original system with all the perturbative parameters $\lambda_1, \ldots, \lambda_d$ would be

$$L_j^{(1)} = \sum_{k=1}^{d} L_{j,k}^{(1)}.$$

These calculations have been performed using a cluster of servers. The parallelization has been implemented by using the previously introduced PBala ([Sal]), a parallellization interface for single threaded scripts which allows to distribute executions in Parallel Virtual Machine enabled clusters using single program multiple data paradigm. This interface lets the user execute a same script/program over multiple input data in several CPUs located at the cluster. It supports memory management, so nodes do not run out of RAM due to too many processes being started at the same node.

The parallelization approach presented here has proved to be highly efficient. For instance, to find the necessary Lyapunov constants to prove Proposition 1.44, the computing time has been reduced from 30 hours to 4 hours when parallelizing. However, the computational requirements of the studied problem have caused that we cannot go further than 5th degree, which is the case solved in Proposition 1.45 and for which about 10 days of computing were needed even with parallelization in a cluster with 5 servers

The first result, related to a quartic system, is as follows.

Proposition 1.44. *The quartic system*

$$\begin{cases} \dot{x} = -y(1-68x+1183x^2)(1-x-y), \\ \dot{y} = (x-58x^2-44xy+30y^2+672x^3+1484x^2y-945xy^2-84y^3)(1-x-y), \\ \dot{z} = z+x^2+x^3+x^4, \end{cases} \tag{1.59}$$

has a center at the origin and unfolds 54 *limit cycles of small amplitude under a complete quartic perturbation.*

Proof. The two first equations in (1.59) define a system in the plane with a center at the origin because they are the same center defined by the two first equations of (1.57) multiplied by a fixed points straight line. Then, by adding the third equation with a center manifold tangent to $z = 0$ we have a center in $\mathbb{R}^3$. The proof follows as the first part of the proofs of Propositions 1.42 and 1.43. First, we consider a perturbation having the trace parameter terms and a quartic perturbation starting with degree 2 terms. Second, we take $\alpha = 0$ and compute the first 54 Lyapunov constants of the perturbed system up to first-order with respect to the perturbative parameters by using the parallelization algorithm described above. Finally, we see that generically they have rank 54, which by adding the trace parameter proves the unfolding of 54 limit cycles of small amplitude. We notice that the generical rank does not increase when we compute 6 more Lyapunov constants. □

In all our computations we have observed that, for none of the systems studied so far, the perturbative parameters c_{ijk} from the third equation $\dot{z}$ appear in the expressions of any first-order Taylor series of the Lyapunov constants. Even though this fact has not been proved, we believe that this is a general behavior. Therefore, in this sense, when considering the perturbed system for the following quintic system in Proposition 1.45 we will ignore the perturbation in the third equation to simplify the computations and reduce the execution times, as we think that this will make no difference. Actually, for this quintic system (1.60) we have checked that the linear parts of the first 70 Lyapunov constants do not include the perturbative parameters in $\dot{z}$, which confirms what we expected. This fact also reduces by $2/3$ the maximum cyclicity that can be obtained by only looking at Lyapunov constants up to first-order regarding the number of perturbative parameters, as $1/3$ of such parameters will not increase the cyclicity of the system.

For the quintic case we have the following result.

Proposition 1.45. *The quintic system*

$$\begin{cases} \dot{x} = -y(1 - 68x + 1183x^2)f(x,y), \\ \dot{y} = (x - 58x^2 - 44xy + 30y^2 + 672x^3 + 1484x^2y - 945xy^2 - 84y^3)f(x,y), \\ \dot{z} = z + x^2 + x^3 + x^4 + x^5, \end{cases} \tag{1.60}$$

with $f(x,y) = (1 - x - y)(1 + 2x - y)$, has a center at the origin and unfolds 92 *limit cycles of small amplitude under a complete quintic perturbation.*

Proof. System (1.60) has a center at the origin for the same reason that system (1.59), in this case with two straight lines filled with equilibrium points and also having a center manifold tangent to $z = 0$. To simplify the calculations we have considered perturbations only on the first two equations. The proof finishes as the previous one. Here we have computed the first 92 Lyapunov constants, and we have checked that when computing three more the rank does not increase. □

We have also made an attempt to find the linear parts of Lyapunov constants for a degree $n = 6$ system, but as we already commented the problem soon becomes highly demanding computationally speaking. In particular, for the tested sextic case we have reached the memory limit and the process is using 16GB of RAM memory. We have reached the 124th Lyapunov constant, and to find only this constant for only one perturbative parameter the required time has been approximately 3 days. For this reason, we have stopped the problem at 5th degree, as we believe that going higher in the degree is impossible at this stage in computational terms.

It is worth making a final comment about the expected local cyclicity from the used approach. The total number of perturbative parameters –also considering the trace parameter– is $(n^3 + 6n^2 + 11n - 16)/2$ but, as we explained above, the parameters from the third equation do not seem to appear in the linear part of the Lyapunov constants. Hence, the maximum number of essential parameters is $(n^3 + 6n^2 + 11n - 15)/3$ and, consequently, the best lower bound for the number of limit cycles of small amplitude in Hopf point in $\mathbb{R}^3$ will be one less, that is $\mathcal{C}(n) \geq (n^3 + 6n^2 + 11n - 18)/3$. This function takes the values 12 and 32 for degrees $n = 2$ and $n = 3$, respectively, which are very close to the ones obtained in our main result, but the values corresponding to $n = 4$ and $n = 5$ are a bit quite far from the ones we obtained. The reason is that the first two systems are built from optimal planar systems. Our achievement is that we have been able to work with such degrees –4th and 5th– because of the designed parallelized algorithm. We notice that, up to our knowledge, all the obtained values are the highest ones found so far.

Chapter 2

Isochronicity and critical periods

With his work on the cycloidal pendulum in the 17th century, the mathematician and physicist C. Huygens (1629–1695) was the forerunner of isochronicity studies and aroused the interest of this line of research, see [CF05]. In the last 30 years many authors have studied the existence of differential equations with equilibrium points of center type that satisfy this isochronicity property, see for example [CMV99; MMJR97], the interesting survey of Chavarriga and Sabatini [CS99], and the approach to the problem via normal form in [AFG00; AR08] or via Lie brackets in [Sab97]. Closely connected are the problems of the isochronicity when having a focus instead of a center ([AR08; Gin03; GG05]) and the problems associated to the flight return function of a focus, see [BGG19]. There is another intimately related question, the bifurcation of critical periods or criticality problem. In analogy to cyclicity, the criticality problem aims to determine the number of critical periods[1] that can unfold from a system. In this chapter we deal with these isochronicity and criticality problems. We will introduce the mathematical object known as period constants and we will describe a few efficient methods to compute them, as well as some techniques to optimize the obtained results, which will allow us address the isochronicity and criticality problems for some systems.

2.1 The isochronicity and criticality problems

Let us consider a real polynomial system of differential equations in the plane with a nondegenerate center at the origin, this is the linear part at the equilibrium point having zero trace and positive determinant. It is a well known fact that, by a suitable change of coordinates and time rescaling, it can be written in the form

$$\begin{cases} \dot{x} = -y + X(x,y) =: P(x,y), \\ \dot{y} = x + Y(x,y) =: Q(x,y), \end{cases} \tag{2.1}$$

[1] As we will see later with more detail, critical periods are the oscillations of the period function of a system.

where X and Y are polynomials of degree $n \geq 2$ which start at least with quadratic monomials. This system in complex coordinates $(z, w) = (z, \bar{z}) = (x + \mathrm{i}\, y, x - \mathrm{i}\, y)$ can be written as

$$\begin{cases} \dot{z} = \mathrm{i}\, z + Z(z, w) =: \mathcal{Z}(z, w), \\ \dot{w} = -\mathrm{i}\, w + \overline{Z}(z, w) =: \overline{\mathcal{Z}}(z, w), \end{cases} \tag{2.2}$$

where Z is a polynomial starting without linear nor constant terms and $\overline{Z}$ is its complex conjugate.

We define the *period annulus* of a center as the largest neighborhood Ω of the origin with the property that the orbit of every point in $\Omega \setminus \{(0,0)\}$ is a simple closed curve that encloses the origin, so these trajectories are periodic. Suppose the origin is a center for system (2.1) and that the number $\rho^* > 0$ is so small that the segment $\Sigma = \{(x, y) : 0 < x < \rho^*, y = 0\}$ of the x-axis lies wholly within the period annulus. For ρ satisfying $0 < \rho < \rho^*$, let $T(\rho)$ denote the lowest period of the trajectory through $(x, y) = (\rho, 0) \in \Sigma$. The function $T(\rho)$ is the *period function* of the center, which by the Implicit Function Theorem is real analytic. Moreover, we say that the center of system (2.1) is *isochronous* if its period function $T(\rho)$ is constant, which means that every periodic orbit in a neighborhood of the origin has the same period.

By performing a change to polar coordinates, one can deduce that the period function takes the form

$$T(\rho) = 2\pi + \sum_{k=1}^{\infty} \widehat{T}_k \rho^k, \tag{2.3}$$

where the $\widehat{T}_k$ are known as the *period constants* of the center, see for example [RS09]. In the next section we will see how to compute these period constants. In the case that (2.1) depends on some parameters, the period constants are polynomials on them ([Cim+97]). A direct consequence of (2.3) is that, in the considered situation, system (2.1) has an isochronous center at the origin if and only if $\widehat{T}_k = 0$ for all $k \in \mathbb{N}$. This result is also justified by Shafer and Romanovski in [RS09]. This shows that the period constants play the same role when studying isochronicity as Lyapunov constants when characterizing centers. Every value $\rho > 0$ for which $T'(\rho) = 0$ is called a *critical period*. In addition, if it is a simple zero of T', i.e. $T''(\rho) \neq 0$, we call it a *simple* or *hyperbolic critical period*. The number of simple critical periods provides the number of oscillations of the period function. For a family of vector fields having an equilibrium point of center type, we can say that it has criticality c if the maximum number of oscillations of the period function is not higher than c. For some examples of works about critical periods of some families the reader is referred to [LH14; PLF15; PLF17]. In analogy to the local cyclicity finiteness conjecture in the 16th Hilbert problem ([Rou88]) we think that, in any class of planar polynomial vector fields of degree n having a center of type (2.1), the number of oscillations of the period function will be uniformly bounded by a function depending only on the degree n.

About the problem of the monotonicity of the period function (2.3), it is usually studied in polynomial center families ([Chi87; Sab12; VZ20]). The uniqueness of critical periods is studied for example in [GGJ06] for a class of polynomial complex centers. Recently, this uniqueness problem has also been considered for some Hamiltonian and quadratic Loud families in [RV18; VZ20]. For the quadratic family we recommend the nice work done by Chicone and Jacobs in [CJ89]. The study of critical periods for the classical quadratic Loud family was extended to some generalized Loud's centers, see [MV13]. For cubics, in particular for homogenenous cubics nonlinearities, we refer the reader to [GV10; RT93]. For more information on the period function and the criticality problem we suggest the reading of [MRV16] and [RS09].

In our work we will consider the class of *time-reversible*, or simply *reversible*, planar polynomial vector fields of degree n. The most common symmetry is reversibility with respect to straight lines. As the linear part of system (2.1) is invariant with respect to any rotation, without loss of generality we can consider only differential systems which are invariant under the change $(x, y, t) \mapsto (x, -y, -t)$. This classic reversibility makes the system have a symmetry with respect to the straight line $x = 0$ and have a center at the origin, as we saw in the previous chapter. The finiteness property described above should also be true if we restrict our attention to this time-reversible polynomial vector fields class.

The problem of bifurcations of critical periods or *criticality problem* is addressed to find the maximum number of zeros of T' which can bifurcate. Let us denote by $\mathcal{C}(n)$ the criticality restricted to the degree n class; as the general criticality problem is very difficult, we will focus on the bifurcation of local critical periods in the period annulus of the origin in the reversible class. We will denote by $\mathcal{C}_\ell(n)$ the maximum number of local critical periods that can bifurcate from the origin of an nth degree reversible planar system. Our aim is to find the highest possible lower bound of this number for different values of the degree n. This question is considered in analogy to the local cyclicity problem, whose purpose is to find the maximum number of limit cycles –these are zeros of the Poincaré map– that bifurcate from an equilibrium. Observe that the concept of hyperbolic critical period is also defined in analogy to a hyperbolic limit cycle, following the idea of having multiplicity one.

2.1.1 Isochronicity characterization

In this subsection we present three different methods which may help to check whether a center is isochronous or not. Actually, all three methods are equivalent in terms of characterizing the isochronicity of a system ([AFG00; CS99]). We will start by justifying that the isochronicity property is equivalent to linearizability, and we will provide the linearization tools known as Darboux linearization. The observations and results introduced here are based on [RS09].

Let us consider the canonical linear center

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x, \end{cases} \tag{2.4}$$

which is trivially isochronous. Since isochronicity does not depend on the coordinates in use, without changing time, any system with a center which can be brought to (2.4) by means of an analytic change of coordinates must be isochronous. Such a change of coordinates is called a *linearization*, and in this case we say that the system is *linearizable*. From this observation the next result follows.

Theorem 2.1 ([RS09]). *The origin of system* (2.1) *is an isochronous center if and only if there is an analytic change of coordinates* $(x, y) \mapsto (x + \zeta(x, y), y + \eta(x, y))$ *that reduces* (2.1) *to the canonical linear center* (2.4).

We notice that, in the above result, $\zeta(0, 0) = \eta(0, 0) = 0$ and $\nabla\zeta(0, 0) = \nabla\eta(0, 0) = 0$. This theorem tells us that the isochronicity of a planar analytic system is equivalent to its linearizability, so the linarizability of a system can be studied to prove its isochronicity. In this line, we present now one of the most efficient tools for checking linearizability, which is Darboux linearization.

Definition 2.2. *For* $z, w \in \mathbb{C}$*, a Darboux linearization of a polynomial system* (2.2) *is an analytic change of variables* $(z, w) \mapsto (\chi(z, w), \xi(z, w))$ *whose inverse linearizes* (2.2) *and is such that* $\chi(z, w)$ *and* $\xi(z, w)$ *are of the form*

$$\begin{cases} \chi(z, w) = \prod_{j=0}^{m_1} f_j^{\alpha_j}(z, w) = z + \widetilde{\chi}(z, w), \\ \xi(z, w) = \prod_{j=0}^{m_2} g_j^{\beta_j}(z, w) = w + \widetilde{\xi}(z, w), \end{cases} \tag{2.5}$$

for some m_1, m_2*, where* $f_j, g_j \in \mathbb{C}[z, w]$*,* $\alpha_j, \beta_j \in \mathbb{C}$*, and* $\widetilde{\chi}(z, w)$ *and* $\widetilde{\xi}(z, w)$ *begin with terms of order at least two. A system is Darboux linearizable if it admits a Darboux linearization.*

In our case, as the considered vector fields come from real systems the conjugacy relationship $\xi(z, w) = \overline{\chi}(z, w)$ holds, so throughout this section we will only give the first component $\chi(z, w)$ of the provided linearizations.

The next concept we will see is the notion of Lie bracket, which will be highly useful for isochronicity studies and period constants computation.

Definition 2.3. *We define the Lie bracket of two complex planar vector fields* $\mathcal{Z}, \mathcal{U}$*, corresponding to two real vector fields, as*

$$[\mathcal{Z}, \mathcal{U}] = \frac{\partial \mathcal{Z}}{\partial z}\mathcal{U} + \frac{\partial \mathcal{Z}}{\partial w}\overline{\mathcal{U}} - \frac{\partial \mathcal{U}}{\partial z}\mathcal{Z} - \frac{\partial \mathcal{U}}{\partial w}\overline{\mathcal{Z}}. \tag{2.6}$$

This definition appears also in [GGJ06]. We notice that, as we have mentioned above, both vector fields $\mathcal{Z}$ and $\mathcal{U}$ are described only from their first components, because the second ones are obtained by complex conjugation. The first proof of the next geometrical equivalence was done by Sabatini in [Sab97].

Theorem 2.4 ([AFG00]). *Equation* (2.2) *has an isochronous center at the origin if and only if there exists* $\dot{z} = \mathcal{U}(z, w) = z + O(|z, w|^2)$ *such that* $[\mathcal{Z}, \mathcal{U}] = 0$.

It is a well-known fact that holomorphic systems are isochronous, see for example [AFG00]. The paper [GGJ10] also deals with this problem, and gives $\dot{z} = \mathrm{i}\, f(z)$ as a linearizing system. As a first example of application of Theorem 2.4, we can straightforwardly prove this same result and see that actually $\mathcal{U} : \dot{z} = k\, f(z)$ for any $k \in \mathbb{C}$ is a linearizing system. Indeed,

$$\begin{aligned}[\mathcal{Z}, \mathcal{U}] =& [f(z), k\, f(z)] = \frac{\partial f(z)}{\partial z} k\, f(z) + \frac{\partial f(z)}{\partial w} \overline{k\, f(z)} - \frac{\partial\, (k\, f(z))}{\partial z} f(z) \\ &= - \frac{\partial\, (k\, f(z))}{\partial w} \overline{f(z)} \frac{\partial f(z)}{\partial z} k\, f(z) + 0 - k \frac{\partial f(z)}{\partial z} f(z) - 0 = 0.\end{aligned}$$

Finally, we will deal with the utility of commuting systems (see [CS99]). We will start by defining the notion of two systems which commute.

Definition 2.5. *Let us consider two systems of the form* (2.1) *and denote by* $\phi(t, (x_0, y_0))$ *and* $\psi(s, (x_0, y_0))$ *their respective solutions such that* $\phi(0, (x_0, y_0)) = (x_0, y_0)$ *and* $\psi(0, (x_0, y_0)) = (x_0, y_0)$. *Let* τ_1, τ_2 *be positive real numbers, and* $S = [0, \tau_1] \times [0, \tau_2]$ *be a rectangle, which will be called a parametric rectangle. We say that the local flows* $\phi(t, (x_0, y_0))$ *and* $\psi(s, (x_0, y_0))$ *commute if, for every parametric rectangle* S *such that both* $\phi(t, \psi(s, (x_0, y_0)))$ *and* $\psi(s, \phi(t, (x_0, y_0)))$ *exist whenever* $(t, s) \in S$, *one has*

$$\phi(t, \psi(s, (x_0, y_0))) = \psi(s, \phi(t, (x_0, y_0))).$$

By a classical result, two local flows commute if and only if the Lie bracket (2.6) of their corresponding vector fields vanishes identically (see [Arn89; Olv86]). In this case we say that the vector fields commute. It is then natural to think that commutativity can actually be used to characterize isochronous centers, a fact proved in [Sab97] and stated in next theorem.

Theorem 2.6 ([CS99]). *The center at the origin of system* (2.1) *is isochronous if and only if there exists a second vector field defined in a neighbourhood of the origin which is transversal to* (2.1) *at nonsingular points and both commute.*

2.2 Period constants computation

2.2.1 The classical method

We start this subsection by presenting the classical method to find period constants (see [RS09]). By performing the usual change to polar coordinates $(x, y) = (r\cos\varphi, r\sin\varphi)$, one can rewrite system (2.1) as

$$\begin{cases} \dot{r} = \sum_{k=1}^{n-1} U_k(\varphi) r^{k+1}, \\ \dot{\varphi} = 1 + \sum_{k=1}^{n-1} W_k(\varphi) r^k, \end{cases} \tag{2.7}$$

where $U_k(\varphi)$ and $W_k(\varphi)$ are homogeneous polynomials in $\sin\varphi$ and $\cos\varphi$ of degree $k+2$. Eliminating time and doing the Taylor series expansion in r we obtain

$$\frac{dr}{d\varphi} = \sum_{k=2}^{\infty} R_k(\varphi) r^k, \tag{2.8}$$

where $R_k(\varphi)$ are 2π-periodic functions of φ and the series is convergent for all φ and for all sufficiently small r. The initial value problem for (2.8) with the initial condition $(r, \varphi) = (\rho, 0)$ has a unique truncated solution

$$r(\varphi) = \rho + \sum_{j=2}^{M} A_j(\varphi)\rho^j, \tag{2.9}$$

up to some finite order $M \in \mathbb{N}$. Let us see how to find the coefficients $A_j(\varphi)$. By the chain rule, we have

$$\frac{dr}{d\varphi}\frac{d\varphi}{dt} - \frac{dr}{dt} = 0. \tag{2.10}$$

If we substitute (2.7) and (2.9) in (2.10), we obtain

$$\left(\sum_{j=2}^{M} A_j'(\varphi)\rho^j\right)\left(1 + \sum_{k=1}^{n-1} W_k(\varphi)\left(\rho + \sum_{j=2}^{M} A_j(\varphi)\rho^j\right)^k\right) - \sum_{k=1}^{n-1} U_k(\varphi)\left(\rho + \sum_{j=2}^{M} A_j(\varphi)\rho^j\right)^{k+1} = 0. \tag{2.11}$$

Now for j from 2 to M, we can extract the coefficient of ρ^j from the left hand side of (2.11) and equate it to zero, this is

$$A_j'(\varphi) - C_j(\varphi) = 0,$$

where $-C_j$ denotes the remaining part after A_j'. Observe that due to the structure of (2.11), for a certain j we have that $C_j(\varphi)$ can only contain terms $A_i(\varphi)$ for $i < j$. With a slight abuse of notation, this allows to constructively obtain the expressions for A_j as

$$A_j(\varphi) = \int_0^{\varphi} C_j(\theta)\, d\theta. \tag{2.12}$$

Let us now substitute the solution (2.9) into the second equation of (2.7), which yields a differential equation of the form

$$\frac{d\varphi}{dt} = 1 + \sum_{k=1}^{M+n-1} F_k(\varphi)\rho^k.$$

Rewriting this equation as

$$dt = \frac{d\varphi}{1 + \sum\limits_{k=1}^{M+n-1} F_k(\varphi)\rho^k} = \left(1 + \sum_{k=1}^{\infty} \Psi_k(\varphi)\rho^k\right) d\varphi$$

and integrating from 0 to 2π yields

$$T(\rho) = \int_0^{T(\rho)} dt = \int_0^{2\pi} \left(1 + \sum_{k=1}^{\infty} \Psi_k(\varphi)\rho^k\right) d\varphi = 2\pi + \sum_{k=1}^{\infty} \left(\int_0^{2\pi} \Psi_k(\varphi)\, d\varphi\right) \rho^k, \tag{2.13}$$

where the series converges for $0 \le \varphi \le 2\pi$ and sufficiently small $\rho \ge 0$. From (2.13) it follows that the lowest period of the trajectory of (2.1) passing through $(x, y) = (\rho, 0)$ for $\rho \neq 0$ is given by (2.3), which is the period function. Then we can directly see that the period constants $\widehat{T}_k$ are given by the expression

$$\widehat{T}_k = \int_0^{2\pi} \Psi_k(\varphi)\, d\varphi. \tag{2.14}$$

Assume now that system (2.7) has an isochronous center at the origin and we add a perturbation which depends on some parameters $\lambda \in \mathbb{R}^d$ and such that the center property is kept. We can follow exactly the same procedure as before, and now we have that the period constants

$$\widehat{T}_k(\lambda) = \int_0^{2\pi} \Psi_k(\varphi, \lambda)\, d\varphi \tag{2.15}$$

are polynomials in the parameters λ (see [Cim+97]).

Even though this is the classical method of finding period constants, the integrals in (2.14) and (2.15) easily become too difficult to be explicitly solved, so this technique is not useful in many cases for high degree polynomial vector fields. In the next subsection we will describe a second method which turns out to be more efficient in many cases.

2.2.2 The Lie bracket method

The second algorithm we will introduce for the computation of period constants is equivalent to the previous one, but has the advantage that it avoids integrals and reduces the problem to solving linear systems of equations. Our method is based on the ideas given in [AFG00] and uses the Lie bracket and normal form theory.

We will consider system (2.2) in complex coordinates. In this case, Z and $\overline{Z}$ do not actually need to be polynomials, they can be convergent series which start at least with quadratic terms. For the sake of simplicity, we will deal with

$$\dot{z} = \mathrm{i}\, z + Z(z, w) = \mathcal{Z}(z, w) \tag{2.16}$$

instead of (2.2) and using $w = \bar{z}$, taking into account that the second component is the complex conjugate of the first one. By applying near the identity changes of variables, as the spirit of normal form transformations, system (2.16) can be simplified to

$$\dot{z} = \mathrm{i}\, z + \sum_{j=1}^{N} (\alpha_{2j+1} + \mathrm{i}\, \beta_{2j+1}) z (zw)^j + O_{2N+3}(z, w),$$

where $N \in \mathbb{N}$ is arbitrary and $\alpha_{2j+1}, \beta_{2j+1} \in \mathbb{R}$. The above normal form can be expressed in polar coordinates as follows,

$$\begin{cases} \dot{r} = \sum\limits_{j=1}^{N} \alpha_{2j+1} r^{2j+1} + O_{2N+3}(r), \\ \dot{\varphi} = 1 + \sum\limits_{j=1}^{N} \beta_{2j+1} r^{2j} + O_{2N+2}(r). \end{cases} \tag{2.17}$$

As we are considering system (2.1), which has a center at the origin, the normal form of system (2.17) becomes

$$\begin{cases} \dot{r} = r^{2N+3}\mathcal{R}(r,\varphi), \\ \dot{\varphi} = 1 + \beta_3 r^2 + \beta_5 r^4 + \cdots + \beta_{2N+1} r^{2N} + r^{2N+2}\Theta(r,\varphi), \end{cases} \tag{2.18}$$

where $\beta_3, \beta_5, \ldots, \beta_{2N+1} \in \mathbb{R}$, and the functions $\mathcal{R}(r,\varphi)$ and $\Theta(r,\varphi)$ are analytical in r and 2π periodic in φ.

The following theorem establishes a relationship between these coefficients β_{2j+1} and the period constants defined in (2.3). From this result it becomes clear that coefficients β_{2j+1} play the same role as the period constants, in the sense that a center is isochronous if and only if $\beta_{2j+1} = 0$ for all $j \geq 1$.

Theorem 2.7 ([AFG00]). *For all $m \geq 1$, the period constants defined in (2.3) satisfy*

(i) $\widehat{T}_{2m-1} = 0$,

(ii) $\widehat{T}_{2m} = 2\pi \displaystyle\sum_{\substack{n_1+\cdots+n_l=2m \\ n_j \text{ even},\, l\geq 1}} (-1)^l \beta_{n_1+1} \cdots \beta_{n_l+1}$.

From this theorem, one deduces that only period constants with even subscript actually play a role, in the sense that if for a certain m we vanish $\widehat{T}_1, \ldots, \widehat{T}_{2m}$ then $\widehat{T}_{2m+1} = 0$. Therefore, it is convenient to define $T_m := \widehat{T}_{2m}$, and we will use this notation from now on during this chapter.

Now we can bring this result together with Theorem 2.4 to propose a constructive method to find the first N period constants of a system. We define

$$\mathcal{U} = z + \sum_{m=2}^{2N+1}\sum_{l=0}^{m} u_{l,m-l} z^l w^{m-l}, \quad \overline{\mathcal{U}} = w + \sum_{m=2}^{2N+1}\sum_{l=0}^{m} \overline{u}_{l,m-l} w^l z^{m-l},$$

and use it together with $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ in (2.16) to compute the Lie bracket $[\mathcal{Z},\mathcal{U}]$ from (2.6). Observing the structure of the normal form of a center (2.18) and considering Theorems 2.4 and 2.7, it is straightforward to see that we can also write the Lie bracket as

$$[\mathcal{Z},\mathcal{U}] = \widetilde{T}_1\, z(zw) + \widetilde{T}_2\, z(zw)^2 + \cdots + \widetilde{T}_N\, z(zw)^N + O_{2N+3}(z,w).$$

We have now two expressions for the Lie bracket of $\mathcal{Z}$ and $\mathcal{U}$, and equating the coefficients with the same degree from both expressions, we can constructively determine the coefficients $u_{l,m-l}$, $\overline{u}_{l,m-l}$, and $\widetilde{T}_m$ for $m = 1, \ldots, N$, simply by solving linear systems of equations. Then we have that the first nonvanishing period constants obtained above and the one provided by (2.14) may differ only in a nonzero multiplicative constant. As both methods are equivalent for our purposes and as in this chapter we will use the later, for the sake of simplicity we will denote T_m instead of $\widetilde{T}_m$.

The benefit of this approach is that it reduces the problem of finding period constants to the resolution of linear systems of equations, instead of dealing with integrals which can become cumbersome or even unsolvable. We have checked that this new approach allows us to go further in the computation of period constants than the classical previously explained method. This algorithm has been computationally implemented with Maple ([Map]) and used to calculate all the necessary period constants throughout this chapter.

2.2.3 Linear parts of period constants

To end this section, we will prove the following result inspired by [LT15] which provides a useful method to compute the linear parts of the period constants by means of parallelization. We already used this idea for the linear parts of Lyapunov constants in Subsection 1.5.4.

Proposition 2.8. *Consider a system, as in* (2.16)*, with a center at the origin*

$$\dot{z} = \mathrm{i}\, z + Z(z, w, \lambda), \tag{2.19}$$

where $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$ *are parameters such that for* $\lambda = 0$ *the origin is an isochronous center and* $Z \in \mathcal{C}^1(\lambda)$*. Assume that for every* $j = 1, \ldots, d$*, the kth period constant of system* (2.19) *with* $\lambda_r = 0$ *for every* $r = 1, \ldots, d$ *such that* $r \neq j$ *takes the form*

$$T_k^{(j)} = \tau_k^{(j)} \lambda_j + O_2(\lambda_j),$$

for some coefficient $\tau_k^{(j)} \in \mathbb{R}$*, where* $O_2(\lambda_j)$ *denotes a sum of monomials of degree at least* 2 *in* λ_j*. Then the kth period constant of system* (2.19) *takes the form*

$$T_k = \sum_{j=1}^{d} \tau_k^{(j)} \lambda_j + O_2(\lambda_1, \ldots, \lambda_d),$$

where $O_2(\lambda)$ *denotes a sum of monomials of degree at least* 2 *in the parameters.*

Proof. The proof is straightforward by using the linearity property in the first-order terms of the period constants. The kth period constant of system (2.19) must have the form

$$T_k = \sum_{j=1}^{d} \eta_k^{(j)} \lambda_j + O_2(\lambda_1, \ldots, \lambda_d),$$

for some coefficients $\eta_k^{(1)}, \ldots, \eta_k^{(d)} \in \mathbb{R}$ and $O_2(\lambda_1, \ldots, \lambda_d)$ being a sum of monomials of degree at least 2 on the parameters. Now if for some $j = 1, \ldots, d$ we impose $\lambda_r = 0$ for every $r = 1, \ldots, d$ such that $r \neq j$, we obtain that the kth period constant of the corresponding system has the form

$$T_k^{(j)} = \eta_k^{(j)} \lambda_j + O_2(\lambda_j),$$

which shows that $\eta_k^{(j)} = \tau_k^{(j)}$, where $\tau_k^{(j)}$ is as defined in the statement of the proposition. Repeating this process for every $j = 1, \ldots, d$, the statement is proved. □

Remark 2.9. *The structure outlined in Proposition 2.8 can be used together with parallelization to find the linear part of the period constants of a given center in a way which is much more efficient, in computational terms, than directly applying the Lie bracket method. The idea is to consider each perturbative monomial instead of all of them together. One can separately use this method up to first-order Taylor development to obtain the linear part of the corresponding kth period constant, and then add all of them to find the linear part of T_k. It is relevant to observe that the computed linear parts are not obtained by calculating each complete period constant and then finding its power series expansion up to first-order, but by directly computing its first-order terms at each step.*

The advantage of this approach is that it is much easier in computational terms to find the first-order part of the period constants of a number of systems with only one parameter than computing them for only one system with many parameters. As a matter of fact, what we are doing is to apply the same Lie bracket method to these simpler systems instead of directly to the initial one. Furthermore, this technique allows to parallelize the computation for each family, which allows to highly decrease the total execution time.

2.3 Lower bounds in criticality for reversible centers

The main objective of this section is to find the highest possible lower bound for $\mathcal{C}_\ell(n)$, by fixing our attention to lower bounds of local criticality for low degree planar polynomial centers. The main technique is the study of perturbations of reversible holomorphic isochronous centers, inside the reversible centers class. More concretely, we study the Taylor developments of the period constants with respect to the perturbation parameters.

The problem of finding the maximum number of local critical periods which can bifurcate from a plane vector field is completely solved only for the quadratic case $n = 2$. This is done by Chicone and Jacobs in [CJ89]: their result states that $\mathcal{C}_\ell(2) = 2$. To the best of our knowledge, for cubic reversible systems the highest number of critical periods achieved so far is 6, a result given in [YH09] by Yu and Han. In the case of Hamiltonian systems, [YHZ10] shows that such bound increases to 7. There are also a few works dealing with lower bounds for general families of degree n. One is given by Cima, Gasull, and da Silva in [CGS08] proving that $\mathcal{C}_\ell(n) \geq 2\left[(n-2)/2\right]$, where $[\cdot]$ denotes the integer part. Another one is the bound that Gasull, Liu, and Yang propose in [GLY10], which grows as $n^2/4$. Very recently, in 2020, Cen proves in [Cen21] a lower bound of $(n^2-4)/2$ for even n and $(n^2+2n-5)/2$ for odd n. In our work we have improved some of these bounds up to $n = 16$. Our main result from this section is as follows.

Theorem 2.10. *The number of local critical periods in the family of polynomial time-reversible centers of degree n is*

$$\mathcal{C}_\ell(n) \geq \begin{cases} 6, & \text{for } n = 3, \\ 10, & \text{for } n = 4, \\ (n^2+n-2)/2, & \text{for } 5 \leq n \leq 9, \\ (n^2+n-4)/2, & \text{for } 10 \leq n \leq 16. \end{cases}$$

The essential tool for proving the above result is the local bifurcation of zeros of the first derivative of the period function (2.3). That is, for each degree n, finding the highest value for the multiplicity of a zero of T' and its unfolding in the corresponding reversible polynomial centers family, more concretely by perturbing some special isochronous centers. This is as the usual mechanism for limit cycles of small amplitude in polynomial vector fields known as *degenerate Hopf bifurcation*, see [RS09]. The number of critical periods bifurcating from a center clearly depends on the family to which it belongs and it is closely related to the number of free parameters. The most studied families in this problem are the reversible and the Hamiltonian ones, having $(n^2+3n-4)/2$ and $(n^2+5n-6)/2$ parameters, respectively. As usual in this kind of bifurcation mechanisms, at least one of the monomials in the perturbation terms can be rescaled to be one, so regarding the number of parameters and using this bifurcation technique, the maximum number of critical periods we expect to find in the class of nth degree time-reversible systems is

$$\mathcal{C}_\ell(n) = \frac{n^2+3n-6}{2}.$$

This is the value obtained in Theorem 2.10 for $n = 3$, and it is only one more than our lower bound for $n = 4$. In later subsections, we will discuss more about this explicit value and why we expect that it will be the value for the maximum number of local critical periods. Observe also that for $n = 2$ this value $\mathcal{C}_\ell(2) = 2$ coincides with the one provided by [CJ89] that we already mentioned. The reason why we choose reversible families instead of Hamiltonian is that they can be more easily generated. For a study on the criticality of third-order Hamiltonian systems the reader is referred to [YHZ10], where the unfolding of 7 critical periods is proved, one more than in the reversible class.

This whole section is devoted to prove Theorem 2.10 and has the following structure. In Subsection 2.3.1 we present a technique that can be used to increase the number of critical periods with respect to the bounds obtained by linear developments. Subsection 2.3.2 explains the choice of the family of isochronous centers that will be perturbed to obtain as many local critical periods as possible. All this is used in Subsection 2.3.3 to increase the number of local critical periods to 5 in the cubic case. Nevertheless, the complete proof of Theorem 2.10 for $n = 3$ is done in Subsection 2.3.6, where it is shown that actually 6 critical periods can unfold in cubic reversible centers family, but perturbing from an isochronous center only having linear terms. Despite being a previous result (given by [YH09]), we present an alternative proof for the existence of 6 critical periods in cubic reversible systems. With the same technique we also increase the number of local critical periods up to 10 for $n = 4$ and the ones stated in Theorem 2.10 for $5 \leq n \leq 9$, respectively in Subsections 2.3.4 and 2.3.5. The last bounds of $\mathcal{C}_\ell(n)$ for $10 \leq n \leq 16$ are also obtained in Subsection 2.3.5, studying only first-order developments. We finish with a last short discussion in Subsection 2.3.7 about these increment values. We notice that all the computations have been done using the computer algebra system Maple ([Map]).

Finally, we would like to say a few words about the computational difficulties and what about going further in the degree n to improve Theorem 2.10. As we will see during the section, some of the results have been obtained thanks to developing particular algorithms using parallelized computations. The main difficulty is related to the fact that there are no general classifications of reversible isochronous centers, so we have used holomorphic centers because they provide isochronous reversible centers for every degree n. But as they have many free parameters, the necessary computations to improve our main result would involve the explicit resolution of nonlinear systems of equations with several variables, concretely $n - 1$ for families of degree n. This is actually the hardest point to go further in the degree.

2.3.1 A result on the criticality of isochronous centers

Let us consider a family of isochronous centers with some parameters, and add a perturbation which keeps the center property. In this subsection we will prove a theorem which outlines how the criticality of such a family can increase under some conditions on the isochronicity parameters. The idea behind this result is inspired by [HY12] but better developed in [GGT21], a recent work about cyclicity in families of centers that proves Theorem 1.41. We already used such theorem for cyclicity in Section 1.5, and the aim now is to extrapolate it to period constants and the study of criticality.

First we present a technical result which shows the structure of the first-order terms of the period constants for a perturbed family of isochronous centers. This is essentially an extension of Proposition 2.8 adapted to the case where the unperturbed system is a parametric family of isochronous centers instead of a fixed one.

Proposition 2.11. *Let us consider a polynomial family of isochronous centers being parametrized by $A \in \mathbb{R}^P$ for some $P \in \mathbb{N}$ and add a polynomial perturbation with coefficients $\lambda \in \mathbb{R}^N$ for some $N \in \mathbb{N}$ which does not break the center property.*

(i) The kth period constant T_k of the perturbed system is a polynomial on the perturbative parameters λ whose coefficients are polynomials in A and takes the form

$$T_k = \sum_{j=1}^{N} g_k^{(j)}(A)\lambda_j + O_2(\lambda), \tag{2.20}$$

for some polynomials $g_k^{(j)}(A)$ in A which are the coefficients of the linear part of T_k with respect to λ.

(ii) Let us consider the $m \times m$ matrix $G_m(A)$ whose element in position (i,j) is $g_i^{(j)}(A)$ from expression (2.20). This is the matrix of coefficients of linear parts of the first m period constants. Then if $\det G_N(A) = 0$ and $\det G_{N-1}(A) \neq 0$ there exists a linear change of variables such that the first $N-1$ first period constants take the form

$$T_k = u_k + O_2(u_1, \ldots, u_N) \tag{2.21}$$

for $k = 1, \ldots, N-1$, where the linear part of T_k is u_k, $u_N := \lambda_N$, and the higher-order terms are denoted by $O_2(u_1, \ldots, u_N)$.

(iii) Under the same assumptions of (ii), the first $N+M$ period constants for some $M \in \mathbb{N}$ can be written as

$$T_k = \begin{cases} v_k, & \text{if } k = 1, \ldots, N-1, \\ \sum_{j=1}^{N-1} \tilde{g}_k^{(j)}(A)v_j + f_{k-N}(A)u_N + O_2(v, u_N), & \text{if } k = N, \ldots, N+M, \end{cases} \tag{2.22}$$

where $v = (v_1, \ldots, v_{N-1})$ are new variables, $f_{k-N}(A)$ and $\tilde{g}_k^{(j)}(A)$ are the corresponding coefficients of $v_1, \ldots, v_{N-1}, u_N$ which are rational functions in $A \in \mathbb{R}^P$, and $O_2(v, u_N)$ are analytical functions of order two in $v_1, \ldots, v_{N-1}, u_N$.

Proof. Recall that the period constants are polynomials in the parameters of the system. As parameters A do not break the isochronicity of the system they cannot appear isolated, so when considering the power series expansion of the period constant T_k, it is straightforward to see that its linear part must be a linear combination of the perturbative parameters λ with the coefficients being polynomials in A, and (i) follows.

To see (ii), as $\det G_{N-1}(A) \neq 0$, we can apply Cramer's rule to the system of $N-1$ equations $\sum_{j=1}^{N} g_k^{(j)}(A)\lambda_j = u_k$ or equivalently $\sum_{j=1}^{N-1} g_k^{(j)}(A)\lambda_j = u_k - g_k^{(N)}(A)\lambda_N =: u_k - g_k^{(N)}(A)u_N$ for $k = 1, \ldots, N-1$, with unknowns $\lambda_1, \ldots, \lambda_{N-1}$. Then we can explicitly find the linear change of variables that proves (2.21). By using this method it is clear that the coefficients which define the change of variables are rational functions in A.

Let us consider new variables $v_1, \ldots, v_{N-1}$ to perform the following change, using (2.21), in $\mathbb{R}^N$:

$$v_k = T_k = u_k + O_2(u_1, \ldots, u_N), \qquad \text{for } k = 1, \ldots, N-1.$$

As $u_1, \ldots, u_{N-1}$ are independent and have rank $N-1$, the Implicit Function Theorem can be applied to write $u_1, \ldots, u_{N-1}$ as functions of $v_1, \ldots, v_{N-1}, u_N$. This is

$$u_k = F_k(v_1, \ldots, v_{N-1}, u_N), \qquad \text{for } k = 1, \ldots, N-1, \tag{2.23}$$

for some real functions F_k. Then by applying (2.20) from part (i) of the statement together with the change (2.23), the period constants take the form (2.22) where $\tilde{g}_{N+d}^{(j)}(A)$ and $f_d(A)$ for $d = 0, \ldots, M$ and $j = 1, \ldots, N-1$ are the corresponding coefficients of $v_1, \ldots, v_{N-1}, u_N$ respectively, and are functions in $A \in \mathbb{R}^M$, and each $O_2(v, u_N)$ is an analytical function of order two in $v_1, \ldots, v_{N-1}, u_N$ due to the application of the Implicit Function Theorem. Then the statement follows. □

Now we can present the aforementioned theorem.

Theorem 2.12. *Let us consider a polynomial family of isochronous centers parametrized by $A \in \mathbb{R}^P$ for some $P \in \mathbb{N}$ and a polynomial perturbation with coefficients $\lambda \in \mathbb{R}^N$ for some $N \in \mathbb{N}$ which does not break the center property. Let us denote by $G_m(A)$ the $m \times m$ matrix as defined in Proposition 2.11.*

(i) If there exists $A^* \in \mathbb{R}^P$ such that $\det G_N(A^*) \neq 0$, then the linear parts of the first period constants have rank N and at least $N-1$ simple critical periods can bifurcate.

(ii) If there exists $A^* \in \mathbb{R}^P$ such that $\det G_N(A^*) = 0$, $\det G_{N-1}(A^*) \neq 0$, $f_i(A^*) = 0$ for $i = 0, \ldots, M-1$, $f_M(A^*) \neq 0$ (where $f_0, \ldots, f_M$ are those defined in (2.22)) and the Jacobian determinant satisfies $J(A^*) := \det \mathrm{Jac}_{(f_0, \ldots, f_{M-1})}(A^*) \neq 0$, then M extra critical periods can bifurcate, which leads to a total of $N+M-1$ critical periods.

Proof. If there exists $A^* \in \mathbb{R}^P$ such that $\det G_N(A^*) \neq 0$, we can apply the same technique as in Proposition 2.11ii to obtain a change of variables to N new independent variables $u_1, \ldots, u_N$. By applying Weierstrass Preparation Theorem, this implies that $N-1$ critical periods can bifurcate and the first statement follows.

Now we move to prove statement (ii). Firstly, as we are under the assumption $\det G_{N-1}(A^*) \neq 0$ for some $A^* \in \mathbb{R}^P$, we can apply Proposition 2.11 and write the first $N+M$ period constants as (2.22). If we set the problem in the manifold $\{v_1 = \cdots = v_{N-1} = 0\}$ –this means vanishing the first $N-1$ period constants–, the structure becomes

$$T_k = \begin{cases} 0, & \text{for } k = 1, \ldots, N-1, \\ u_N \left(f_{k-N}(A) + \sum_{l=1}^{\infty} f_{k-N}^{(l)}(A) u_N^l \right), & \text{for } k = N, \ldots, N+M, \end{cases}$$

for some functions $f_d^{(l)}(A)$ with $d = 0, \ldots, M$. As by assumption there exists $A^* \in \mathbb{R}^P$ such that the Jacobian determinant $J(A^*) \neq 0$, the Implicit Function Theorem guarantees that in a neighbourhood of $A = A^*$ and $u_N = 0$ the following change of variables can be performed in $T_N, \ldots, T_{N+M-1}$:

$$v_{N+k} = f_k(A) + \sum_{l=1}^{\infty} f_k^{(l)}(A) u_N^l, \text{ for } k = 0, \ldots, M-1.$$

As we suppose that $f_i(A^*) = 0$ for $i = 0, \ldots, M-1$ but $f_M(A^*) \neq 0$, we can rewrite

$$T_{N+k} = \begin{cases} u_N v_{N+k}, & \text{for } k = 0, \ldots, M-1, \\ u_N \left(f_M(A^*) + \sum_{l=1}^{\infty} f_M^{(l)}(A^*) u_N^l \right) =: u_N v_{N+M}, & \text{for } k = M. \end{cases}$$

Finally, by again the Implicit Function Theorem, as we have obtained M new independent variables we get the existence of M extra critical periods. □

A natural consequence of the last result is the following corollary.

Corollary 2.13. *With the notation from Theorem 2.12, if* $\det G_N(A)$ *is not identically zero then generically at least* $N-1$ *simple critical periods bifurcate from the origin. The same conclusion is valid also when the number of parameters is greater than or equal to* N*. Clearly, in this second case the corresponding matrix* G_m *would be a nonsquare matrix having rank* N*.*

Proof. The proof is straightforward by following the ideas in the proof of the previous theorem. If $\det G_N(A)$ is not identically zero, then as it is a polynomial we have that $\det G_N(A) \neq 0$ except for a set of zero Lebesgue measure, which implies that the rank of $G_N(A)$ is N and therefore $N-1$ critical periods unfold. □

This last property is equivalent to the one for bifurcation of limit cycles from [Chr05]. The idea of using just linear parts appeared previously in [CJ89]. It is important to notice that in some cases the above determinant is identically zero, then the generic condition is never satisfied. This is the case for the analogous case of limit cycles bifurcation from holomorphic polynomial centers of degree 3, see [LT15].

2.3.2 The main reversible families

As we have previously mentioned, to get the bounds outlined in Theorem 2.10 we have considered nth degree polynomial differential systems which are time-reversible with respect to straight lines. We can assume without loss of generality that the equilibrium is at the origin and that the symmetry line with respect to which the reversibility is considered is the horizontal axis. These differential systems take the form

$$\begin{cases} \dot{x} = -y + yf(x,y^2), \\ \dot{y} = x + g(x,y^2), \end{cases} \tag{2.24}$$

where $f(x,y^2)$ and $g(x,y^2)$ are polynomials in x and y of degrees $n-1$ and n, respectively. Clearly, system (2.24) is invariant under the classical reversibility change of coordinates $(x,y,t) \mapsto (x,-y,-t)$.

The next proposition shows that the condition of a system being reversible with respect to the horizontal axis in complex coordinates $z = x + \mathrm{i}\, y$ and $w = \bar{z} = x - \mathrm{i}\, y$ is that its coefficients are purely imaginary.

Proposition 2.14. *A system (2.24), which is reversible with respect to the horizontal axis, takes in complex coordinates the form*

$$\dot{z} = \mathrm{i}\, z + \mathrm{i} \sum_{l+m\geq 2}^{n} c_{lm} z^l w^m, \tag{2.25}$$

where $c_{lm} \in \mathbb{R}$.

Proof. A change to complex coordinates shows that system (2.24) is written as

$$\begin{cases} \dot{z} = \mathrm{i}\, z + \displaystyle\sum_{l+m\geq 2}^{n} b_{lm} z^l w^m, \\ \dot{w} = -\,\mathrm{i}\, w + \displaystyle\sum_{l+m\geq 2}^{n} \bar{b}_{lm} w^l z^m, \end{cases} \tag{2.26}$$

for certain parameters $b_{lm} \in \mathbb{C}$ and their conjugate values $\bar{b}_{lm} \in \mathbb{C}$. Observe that the reversibility change $(x,y,t) \mapsto (x,-y,-t)$ takes the form $(z,w,t) = (x + \mathrm{i}\, y, x - \mathrm{i}\, y, t) \mapsto (x - \mathrm{i}\, y, x + \mathrm{i}\, y, -t) = (w,z,-t)$ in complex coordinates. Thus, when applied to (2.26), one obtains

$$\begin{cases} -\dot{w} = \mathrm{i}\, w + \sum\limits_{l+m\geq 2}^{n} b_{lm} w^l z^m, \\ -\dot{z} = -\mathrm{i}\, z + \sum\limits_{l+m\geq 2}^{n} \overline{b}_{lm} z^l w^m. \end{cases} \tag{2.27}$$

Now imposing that the system must remain invariant under this change, we have that systems (2.26) and (2.27) must be equal, so we see that $\overline{b}_{lm} = -b_{lm}$. The proof follows from this condition. The reversibility property in complex coordinates is given by the parameters being purely imaginary, this is $b_{lm} = \mathrm{i}\, c_{lm}$ with $c_{lm} \in \mathbb{R}$. Notice that in (2.25) there is no need to write the equation in $\dot{w}$ because it is the complex conjugate of the equation in $\dot{z}$. □

Let us consider an nth degree polynomial system of the form

$$\begin{cases} \dot{z} = \mathrm{i}\, F(z,w), \\ \dot{w} = -\mathrm{i}\, \overline{F}(z,w), \end{cases} \tag{2.28}$$

having an isochronous center at the origin, where $F(z,w) = z + \widetilde{F}(z,w)$ being $\widetilde{F}(z,w)$ a sum of monomials of degree at least 2. As the $\dot{w}$ equation in (2.28) is the complex conjugate of the $\dot{z}$ equation, from now on we will simply write the equation in $\dot{z}$ to describe the system. We will also consider adding a reversible nth degree polynomial perturbation as follows

$$\dot{z} = \mathrm{i}\, F(z,w) + \mathrm{i} \sum_{l+m\geq 2}^{n} r_{lm} z^l w^m, \tag{2.29}$$

where r_{lm} are real perturbative parameters so that the perturbation is reversible and thus the center property is kept.

A well-known fact is that holomorphic systems are isochronous (see [GGJ04]). We are interested in perturbing holomorphic isochronous centers by adding non-holomorphic perturbations, in which case equation (2.29) can be rewritten as

$$\dot{z} = \mathrm{i}\, z + \mathrm{i} \sum_{j=2}^{n} A_j z^j + \mathrm{i} \sum_{\substack{l+m\geq 2 \\ m\geq 1}}^{n} r_{lm} z^l w^m, \tag{2.30}$$

for certain holomorphy parameters $A_j \in \mathbb{R}$, and $r_{lm} \in \mathbb{R}$ are perturbative parameters of the isochronous center

$$\dot{z} = \mathrm{i}\, z + \mathrm{i} \sum_{j=2}^{n} A_j z^j, \tag{2.31}$$

which keep the center property due to being real.

In our work we have considered perturbations of the family of isochronous centers

$$\dot{z} = \mathrm{i}\, z \prod_{j=1}^{n-1} \left(1 - a_j z\right), \tag{2.32}$$

where $n > 1$ and $a_j \in \mathbb{R} \setminus \{0\}$ are real parameters such that $a_j \neq a_i$ for every $i, j \in \{1, \ldots, n-1\}$, $i \neq j$. Observe that this family takes the form (2.31), so it is isochronous due to the holomorphy property. These systems are also Darboux linearizable, as we will see at the end of this subsection.

Our study will focus on reversible families of the form (2.32) being perturbed also inside the reversible polynomial class. The choice of these holomorphic systems is due to the fact that it is the easiest family that can be considered for any degree n. Moreover, as we will see, these particular systems are the most suitable for our study, in the sense that they provide quite a high number of oscillations of the period function without being too demanding computationally. Additionally, in the following subsection we also perturb some other cubic isochronous centers obtained from [CR10], where a complete classification of all reversible cubic isochronous centers is done.

The next result is a direct consequence of applying Theorem 2.12 to (2.30).

Theorem 2.15. *Consider the polynomial differential system of degree n defined in* (2.30) *with* $n \geq 3$ *and* $A_2 = 1$. *Let us denote by* $G_m(A)$ *the* $m \times m$ *matrix as defined in Proposition* 2.11 *and* $N := (n^2 + n - 2)/2$ *the number of perturbative parameters.*

(i) If there exists $A^* = (A_3^*, \ldots, A_n^*) \in \mathbb{R}^{n-2}$ *such that* $\det G_N(A^*) \neq 0$, *then the linear parts of the first period constants have rank* N *and at least* $N - 1$ *simple critical periods bifurcate from the origin.*

(ii) If there exists $A^* = (A_3^*, \ldots, A_n^*) \in \mathbb{R}^{n-2}$ *such that* $\det G_N(A^*) = 0$, $\det G_{N-1}(A^*) \neq 0$, $f_i(A^*) = 0$ *for* $i = 0, \ldots, M-1$, $f_M(A^*) \neq 0$ *(where* $f_0, \ldots, f_M$ *are those defined in* (2.22)*) and the Jacobian determinant satisfies* $J(A^*) := \det \mathrm{Jac}_{(f_0, \ldots, f_{M-1})}(A^*) \neq 0$, *then* M *extra critical periods bifurcate from the origin, which leads to a total of* $N + M - 1$ *critical periods.*

Experimentally, we have observed that we get more criticality when all the parameters A_j are nonvanishing. Then, after a variables rescaling and without loss of generality, we can fix $A_2 = 1$, when $A_2 \neq 0$. Subsection 2.3.5 uses the first statement fixing specific values for A. The second statement, choosing $M = n - 2$, is used in Subsections 2.3.3 and 2.3.4 for perturbations of holomorphic polynomial vector fields of degree 3 and 4, respectively. In these last cases we have achieved the maximum value for the corresponding criticality when $A_2 = 1$. This statement is also used in Subsection 2.3.5 but only with $M = 1$ for some small values of the degree n. Finally, in the above result we have not considered quadratic vector fields because this case was completely solved in [CJ89].

Darboux linearization for (2.32)

Even though we already know that (2.32) is an isochronous center due to being holomorphic, we will prove that it is also Darboux linearizable by explicitly finding its linearization, as an alternative proof for the isochronicity of the system. This will be done to illustrate an example of how Darboux linearizations can be found.

Let us consider equation (2.32) together with its complex conjugate, and perform a change of time to eliminate the imaginary unit i,

$$\begin{cases} z' = z \prod_{j=1}^{n-1} (1 - a_j z) =: Z(z), \\ w' = -w \prod_{j=1}^{n-1} (1 - a_j w) =: W(w). \end{cases} \tag{2.33}$$

The idea to find the linearization is to apply the following result, which is Theorem 4.4.2 from [RS09] adapted to our case.

Theorem 2.16 ([RS09])**.** *The polynomial system (2.33) is Darboux linearizable if and only if there exist $s+1 \geq 1$ algebraic partial integrals $f_0, \ldots, f_s$ with corresponding cofactors $K_0, \ldots, K_s$ and $t+1 \geq 1$ algebraic partial integrals $g_0, \ldots, g_t$ with corresponding cofactors $L_0, \ldots, L_t$ with the following properties:*

(i) $f_0(z,w) = z + \cdots$ *but* $f_j(0,0) = 1$ *for* $j \geq 1$;

(ii) $g_0(z,w) = w + \cdots$ *but* $g_j(0,0) = 1$ *for* $j \geq 1$; *and*

(iii) there are $s+t$ constants $\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_t \in \mathbb{C}$ such that

$$K_0 + \alpha_1 K_1 + \cdots + \alpha_s K_s = 1 \tag{2.34}$$

and

$$L_0 + \beta_1 L_1 + \cdots + \beta_t L_t = -1. \tag{2.35}$$

The Darboux linearization is then given by $Y_1(z,w) = f_0 f_1^{\alpha_1} \cdots f_s^{\alpha_s}$ and $Y_2(z,w) = g_0 g_1^{\beta_1} \cdots g_t^{\beta_t}$.

Now the following proposition gives some algebraic partial integrals of system (2.33) and their corresponding cofactors.

Proposition 2.17. *Polynomials $f_0 = z$ and $f_m = 1 - a_m z$ with $m \in \{1, \ldots, n-1\}$ are algebraic partial integrals of (2.33) with cofactors $K_0 = \prod_{j=1}^{n-1} (1 - a_j z)$ and $K_m = -a_m z \prod_{j=1, j \neq m}^{n-1} (1 - a_j z)$ with $m \in \{1, \ldots, n-1\}$, respectively. Also, polynomials $g_0 =$*

w and $g_m = 1 - a_m w$ with $m \in \{1, \ldots, n-1\}$ are algebraic partial integrals of (2.33) with cofactors $L_0 = -\prod_{j=1}^{n-1}(1 - a_j w)$ and $L_m = a_m w \prod_{j=1, j\neq m}^{n-1}(1 - a_j w)$ with $m \in \{1, \ldots, n-1\}$, respectively.

Proof. Observe that

$$\frac{\partial f_0}{\partial z} Z + \frac{\partial f_0}{\partial w} W = Z = z \prod_{j=1}^{n-1}(1 - a_j z),$$

so $f_0 = z$ is an algebraic partial integral with cofactor $K_0 = \prod_{j=1}^{n-1}(1 - a_j z)$.
Now take $f_m = 1 - a_m z$ for $m \in \{1, \ldots, n-1\}$, and see that

$$\begin{aligned}\frac{\partial f_m}{\partial z} Z + \frac{\partial f_m}{\partial w} W &= -a_m Z = -a_m z \prod_{j=1}^{n-1}(1 - a_j z) \\ &= \left(-a_m z \prod_{j=1, j\neq m}^{n-1}(1 - a_j z) \right)(1 - a_m z),\end{aligned}$$

which shows that $f_m = 1 - a_m z$ is an algebraic partial integral with cofactor $K_m = -a_m z \prod_{j=1, j\neq m}^{n-1}(1 - a_j z)$.
Analogously,

$$\frac{\partial g_0}{\partial z} Z + \frac{\partial g_0}{\partial w} W = W = -w \prod_{j=1}^{n-1}(1 - a_j w),$$

so $g_0 = w$ is an algebraic partial integral with cofactor $L_0 = -\prod_{j=1}^{n-1}(1 - a_j w)$.
Finally,

$$\begin{aligned}\frac{\partial g_m}{\partial z} Z + \frac{\partial g_m}{\partial w} W &= -a_m W = a_m w \prod_{j=1}^{n-1}(1 - a_j w) \\ &= \left(a_m w \prod_{j=1, j\neq m}^{n-1}(1 - a_j w) \right)(1 - a_m w),\end{aligned}$$

so $g_m = 1 - a_m w$ for $m \in \{1, \ldots, n-1\}$ is an algebraic partial integral with cofactor $L_m = a_m w \prod_{j=1, j\neq m}^{n-1}(1 - a_j w)$. □

From now on, for the sake of simplicity we will denote $N := n-1$. Observe that $n > 1$ is the degree of the polynomial system (2.33), so $N > 0$ represents the number of factors in the product $\prod_{j=1}^{n-1}(1-a_j z)$ of (2.33). The next result is fundamental to reach our purpose.

Lemma 2.18. *If we denote*

$$\alpha_m = -\frac{a_m{}^{N-1}}{\prod_{j=1, j\neq m}^{N}(a_m - a_j)}$$

for $m \in \{1, \dots, N\}$, then equality

$$K_0 + \alpha_1 K_1 + \dots + \alpha_N K_N = 1$$

holds, where $K_0, \dots, K_N$ are as defined in Proposition 2.17.

Proof. The Lemma will be proved by induction on N. The case $N = 1$ is trivial,

$$K_0 + \alpha_1 K_1 = (1 - a_1 z) + (-1)(-a_1 z) = 1.$$

Now we will assume that the statement is true for N, this is

$$K_0 + \sum_{m=1}^{N} \alpha_m K_m = 1,$$

$$\prod_{j=1}^{N}(1-a_j z) + \sum_{m=1}^{N}\left(-\frac{a_m{}^{N-1}}{\prod_{j=1, j\neq m}^{N}(a_m - a_j)}\right)\left(-a_m z \prod_{j=1, j\neq m}^{N}(1-a_j z)\right) = 1,$$

$$\prod_{j=1}^{N}(1-a_j z) + z \sum_{m=1}^{N}\left(\frac{a_m{}^{N}}{\prod_{j=1, j\neq m}^{N}(a_m - a_j)} \prod_{j=1, j\neq m}^{N}(1-a_j z)\right) = 1. \tag{2.36}$$

Assuming that equality (2.36) holds, we will deduce that then it must also hold for $N+1$, which means that

$$\prod_{j=1}^{N+1}(1-a_j z) + z \sum_{m=1}^{N+1}\left(\frac{a_m{}^{N+1}}{\prod_{j=1, j\neq m}^{N+1}(a_m - a_j)} \prod_{j=1, j\neq m}^{N+1}(1-a_j z)\right) = 1. \tag{2.37}$$

First, let us multiply both sides of equality (2.36) by $(1 - a_{N+1}z)$, which gives

$$\prod_{j=1}^{N+1}(1-a_jz) + z\sum_{m=1}^{N}\left(\frac{{a_m}^N}{\prod\limits_{j=1,j\neq m}^{N}(a_m-a_j)}\prod_{j=1,j\neq m}^{N+1}(1-a_jz)\right) = 1 - a_{N+1}z. \quad (2.38)$$

Let us now compute the following sum, which will be useful later:

$$\frac{{a_m}^N}{\prod\limits_{j=1,j\neq m}^{N}(a_m-a_j)} + \frac{{a_m}^N a_{N+1}}{\prod\limits_{j=1,j\neq m}^{N+1}(a_m-a_j)} = \frac{{a_m}^{N+1}}{\prod\limits_{j=1,j\neq m}^{N+1}(a_m-a_j)}. \quad (2.39)$$

Now add the term

$$z\sum_{m=1}^{N}\left(\frac{{a_m}^N a_{N+1}}{\prod\limits_{j=1,j\neq m}^{N+1}(a_m-a_j)}\prod_{j=1,j\neq m}^{N+1}(1-a_jz)\right) + \frac{{a_{N+1}}^{N+1}}{\prod\limits_{j=1}^{N}(a_{N+1}-a_j)}z\prod_{j=1}^{N}(1-a_jz) \quad (2.40)$$

on both sides of (2.38). By applying (2.39), the left-hand side of equation (2.38) when adding (2.40) results in

$$\begin{aligned}
&\prod_{j=1}^{N+1}(1-a_jz) + z\sum_{m=1}^{N}\left(\frac{{a_m}^{N+1}}{\prod\limits_{j=1,j\neq m}^{N+1}(a_m-a_j)}\prod_{j=1,j\neq m}^{N+1}(1-a_jz)\right) \\
&\quad + \frac{{a_{N+1}}^{N+1}}{\prod\limits_{j=1}^{N}(a_{N+1}-a_j)}z\prod_{j=1}^{N}(1-a_jz) \\
&= \prod_{j=1}^{N+1}(1-a_jz) + z\sum_{m=1}^{N+1}\left(\frac{{a_m}^{N+1}}{\prod\limits_{j=1,j\neq m}^{N+1}(a_m-a_j)}\prod_{j=1,j\neq m}^{N+1}(1-a_jz)\right),
\end{aligned}$$

and this is equal to the left-hand side of equation (2.37) which we want to prove. Now to finish the proof, we must see that the right-hand side of (2.38) when adding (2.40) equals 1 as in (2.37). This means that we must check

$$1 - a_{N+1}z + z\sum_{m=1}^{N}\left(\frac{a_m{}^N a_{N+1}}{\prod\limits_{j=1,j\neq m}^{N+1}(a_m - a_j)}\prod_{j=1,j\neq m}^{N+1}(1 - a_j z)\right)$$
$$+\frac{a_{N+1}{}^{N+1}}{\prod\limits_{j=1}^{N}(a_{N+1} - a_j)}z\prod_{j=1}^{N}(1 - a_j z) = 1,$$

which is equivalent to

$$\sum_{m=1}^{N+1}\left(\frac{a_m{}^N}{\prod\limits_{j=1,j\neq m}^{N+1}(a_m - a_j)}\prod_{j=1,j\neq m}^{N+1}(1 - a_j z)\right) = 1.$$

This equality can be rewritten as

$$\sum_{m=1}^{N+1}\prod_{j=1,j\neq m}^{N+1}\frac{a_m - a_m a_j z}{a_m - a_j} = 1.$$

To prove that this equality holds we will proceed as follows. Observe that the left-hand side of the equality is a degree N polynomial in z. If it is evaluated in $z = \frac{1}{a_m}$ the result is 1 for all $m \in \{1, \ldots, N+1\}$. Therefore, as we have a degree N polynomial which takes the same value 1 for $N+1$ points, we conclude that this must be a constant polynomial and equals 1. □

From this lemma we can deduce the following analogous one.

Lemma 2.19. *If we denote $\beta_m := \alpha_m$ for $m \in \{1, \ldots, N\}$ where the α_m are defined as in Lemma 2.18, then equality*

$$L_0 + \beta_1 L_1 + \cdots + \beta_N L_N = -1$$

holds, where $L_0, \ldots, L_N$ are as defined in Proposition 2.17.

Proof. The proof is straightforward by applying Lemma 2.18. Let us compute

$$\begin{aligned} L_0 + \sum_{m=1}^{N} \beta_m L_m &= -\prod_{j=1}^{N}(1-a_j w) + \sum_{m=1}^{N} \alpha_m a_m w \prod_{j=1, j\neq m}^{N} (1-a_j w) \\ &= -\left(\prod_{j=1}^{N}(1-a_j w) + \sum_{m=1}^{N} \alpha_m \left(-a_m w \prod_{j=1, j\neq m}^{N} (1-a_j w)\right)\right) \\ &= -\left(K_0(w) + \sum_{m=1}^{N} \alpha_m K_m(w)\right) = -1, \end{aligned}$$

where we have used Lemma 2.18 in the last equality. Notice that $K_m(w)$ for $m \in \{0,\dots,N\}$ are simply the polynomials K_m switching the variable z by w. □

Finally, we can outline our result.

Proposition 2.20. *System (2.33) is Darboux linearizable and its linearization is given by* $Y_1(z,w) = f_0 f_1^{\alpha_1} \cdots f_N^{\alpha_N}$ *and* $Y_2(z,w) = g_0 g_1^{\beta_1} \cdots g_N^{\beta_N}$*, where*

- $f_0 = z$ *and* $f_m = 1 - a_m z$ *for* $m \in \{1,\dots,N\}$*;*
- $g_0 = w$ *and* $g_m = 1 - a_m w$ *with* $m \in \{1,\dots,N\}$*; and*
- *the exponents* α_m *and* β_m *for* $m \in \{1,\dots,N\}$ *are*

$$\alpha_m = \beta_m = -\frac{{a_m}^{N-1}}{\prod_{j=1, j\neq m}^{N} (a_m - a_j)}.$$

Proof. We will check that all conditions in Theorem 2.16 are fulfilled. First, by Proposition 2.17 we know that $f_0,\dots,f_N$ and $g_0,\dots,g_N$ as defined in the statement of the proposition are algebraic partial integrals of the system, and have resepctively cofactors $K_0,\dots,K_N$ and $L_0,\dots,L_N$ that have been explicitly found. Observe also that $f_0(z,w) = z$, $f_m(0,0) = 1$ for $m \in \{1,\dots,N\}$, $g_0(z,w) = w$ and $g_m(0,0) = 1$ for $m \in \{1,\dots,N\}$, so conditions (i) and (ii) of the theorem are satisfied. Finally, Lemmas 2.18 and 2.19 show that equalities (2.34) and (2.35) of condition (iii) in the theorem hold, which finishes the proof. □

2.3.3 Perturbing cubic isochronous systems

The first part of this subsection is focused on the cubic systems of the form (2.32), this is for $n = 3$. In the second part we study lower bounds for the criticality of some reversible isochronous centers appearing in [CR10]. We will see that at least 5 critical periods can unfold in the reversible cubic polynomial class. Actually, in

Subsection 2.3.6 we will show that 6 critical periods can unfold in cubic systems, but not bifurcating from centers having nonlinear terms.

The first 4 critical periods appear by studying specific isochronous centers such that, after reversible perturbation, the rank of the linear parts of their period constants is 5. In all the studied cases this is the maximum found rank. Then, by Proposition 2.11, we can write the first 5 period constants in the form

$$T_k = u_k + O_2, \text{ for } k = 1, \ldots, 5,$$

where O_2 denotes the terms of degree at least 2, or directly $T_k = u_k$ for $k = 1, \ldots, 5$ if we use the Implicit Function Theorem. We have checked that the next three linear parts are a linear combination of these 5 variables. Consequently, in all the studied cases, no more critical periods can be found using only first-order developments. We need to use higher-order developments or pay attention to the nongeneric cases in some parameter families of isochronous centers.

Perturbing holomorphic centers

In the next result we will study the critical periods bifurcation diagram of a 1-parameter cubic holomorphic system. We show how, by applying Theorem 2.15, we can obtain 5 critical periods when choosing the values for which the rank is not maximal. In the following subsection, these 5 critical periods will appear from higher-order developments.

Proposition 2.21. *Let $a \in \mathbb{R} \setminus \{0\}$. Consider the 1-parameter family of cubic (holomorphic) reversible systems*

$$\dot{z} = \mathrm{i}\, z\,(1-z)\,(1-az)\,. \tag{2.41}$$

The number of critical periods bifurcating from the origin when perturbing in the class of reversible cubic systems is at least 5 for $a \in \{-3/2, -1, -2/3, 1/2, 2\}$ and 4 otherwise.

Proof. As we have explained in Subsection 2.3.2, system (2.41) is time-reversible holomorphic and therefore it has an isochronous center at the origin.

We can consider system (2.41) without losing generality with respect to the general cubic case (2.32), which is $\dot{z} = \mathrm{i}\, z\,(1-a_1 z)\,(1-a_2 z)\,$, with $|a_1| > |a_2|$. Both systems are equivalent after the rescaling $(z,w) \mapsto (a_1^{-1} z, a_1^{-1} w)$ and we get $a := a_1^{-1} a_2$. Thus, we can reduce our study to $a \in [-1,1) \setminus \{0\}$. Notice that the case $a = 1$ is not included in (2.32) because $a_1 \neq a_2$.

As in (2.30), we consider the time-reversible cubic perturbation without the holomorphic monomials,

$$\begin{cases} \dot{z} = \mathrm{i}\, z\,(1-z)\,(1-az) + \mathrm{i}\,(r_{11} z w + r_{02} w^2 + r_{21} z^2 w + r_{12} z w^2 + r_{03} w^3), \\ \dot{w} = -\,\mathrm{i}\, w\,(1-w)\,(1-aw) - \mathrm{i}\,(r_{11} w z + r_{02} z^2 + r_{21} w^2 z + r_{12} w z^2 + r_{03} z^3). \end{cases}$$

When $a \in \mathbb{R} \setminus \{-1, 0, 1/2, 2\}$, the rank of the linear developments of first four period constants of this system with respect to $(r_{11}, r_{02}, r_{21}, r_{12})$ is 4. The explicit expressions of those linear developments are not shown here due to the fact that they are quite long. Then, after using the Implicit Function Theorem, the period constants take the form

$$T_k = u_k, \text{ for } k = 1, \ldots, 4.$$

Taking $u_1 = u_2 = u_3 = u_4 = 0$ and $r_{03} = u_5$, the fifth and sixth period constants take the form

$$\begin{aligned} T_5 &= \frac{5}{24} \frac{P(a)}{3a^2 + 2a + 3} u_5 + u_5^2 \sum_{j=0}^{\infty} f_j(a)\, u_5^j, \\ T_6 &= -\frac{1}{42} \frac{Q(a)}{3a^2 + 2a + 3} u_5 + u_5^2 \sum_{j=0}^{\infty} g_j(a)\, u_5^j, \end{aligned} \tag{2.42}$$

where $P(a) = a^3(a-2)(3a+2)(2a+3)(2a-1)$, $Q(a) = a^3(a-2)(2a-1)(834a^2 + 1735a + 834)(a+1)^2$, and f_j and g_j are rational functions. Applying Theorem 2.15 we have 4 critical periods when $P(a) \neq 0$ and 5 when $P(a) = 0$, $P'(a) \neq 0$, and $Q(a) \neq 0$. Then, as $a \neq 0$, the statement follows except for the remaining cases $a \in \{-1, 1/2, 2\}$.

For the cases $a \in \{-1, 1/2, 2\}$ we need to add the holomorphic monomials, then the time-reversible cubic perturbation is now

$$\begin{cases} \dot{z} = \mathrm{i}\, z\,(1 - z)\,(1 - az) + \mathrm{i} \displaystyle\sum_{k+l=2}^{3} r_{kl} z^k w^l, \\ \dot{w} = -\,\mathrm{i}\, w\,(1 - w)\,(1 - aw) - \mathrm{i} \displaystyle\sum_{k+l=2}^{3} r_{kl} w^k z^l. \end{cases} \tag{2.43}$$

When computing the linear parts of the period constants we observe that they have rank 3 with respect to three of the parameters in $\{r_{20}, r_{11}, r_{02}, r_{30}, r_{21}, r_{12}, r_{03}\}$. Then, similarly to what we did above, we have $T_k = u_k$, for $k = 1, 2, 3$ and we should study the second-order developments of T_4, T_5, T_6 under the condition $u_1 = u_2 = u_3 = 0$ with respect to the remaining parameters.

For $a = 2$ (and similarly for its equivalent case $a = 1/2$) we write the remaining parameters, as in a blow-up procedure, as $r_{03} = u_4 v_1$, $r_{12} = u_4 v_2$, $r_{20} = u_4$, $r_{30} = 0$. Then,

$$T_k = u_4^2 F_{k-3}(v_1, v_2) + u_4^3 \sum_{j=0}^{\infty} f_{kj}(v_1, v_2)\, u_4^j, \text{ for } k = 4, 5, 6, \tag{2.44}$$

with
$$F_1(v_1,v_2) = -96\,v_1 - \frac{304}{5}\,v_2 - 24\,v_1^2 - \frac{1016}{5}\,v_1v_2 - \frac{1178}{15}\,v_2^2,$$
$$F_2(v_1,v_2) = \frac{112}{3}v_2 - 350\,v_1^2 - \frac{922}{3}\,v_1v_2 - \frac{1297}{126}\,v_2^2,$$
$$F_3(v_1,v_2) = -\frac{1080}{7}\,v_1^2 + \frac{6264}{49}\,v_1v_2 + \frac{212634}{1715}\,v_2^2.$$
Next we show that the zero level curves of F_1 and F_2 have a transversal intersection point

$$(v_1^*,v_2^*) = \left(-\frac{6972965}{1901}\alpha^2 - \frac{807195}{7604}\alpha - \frac{1743}{3802}, \frac{105}{2}\alpha\right),$$

being α the unique simple real zero of $p(\alpha) = 5578372\alpha^3 + 183328\alpha^2 + 1789\alpha + 7$, where $F_3(v_1^*,v_2^*)$ is nonvanishing. This follows because $F_1(v_1^*,v_2^*) = F_2(v_1^*,v_2^*) = 0$,

$$F_3(v_1^*,v_2^*) = p_1(\alpha) = \frac{1051652160\alpha^2 + 17223840\alpha + 120960}{1901} \neq 0,$$
$$\det \mathrm{Jac}_{(F_1,F_2)}(v_1^*,v_2^*) = p_2(\alpha) = \frac{-103534584320\alpha^2 - 571544320\alpha + 7499520}{1901} \neq 0,$$

and the resultants $\mathrm{Res}(p,p',\alpha)$, $\mathrm{Res}(p,p_1,\alpha)$, and $\mathrm{Res}(p,p_2,\alpha)$ are all nonvanishing.

Then, after dividing (2.44) by u_4^2 and using again the Implicit Function Theorem at $(v_1,v_2,u_4) = (v_1^*,v_2^*,0)$, we obtain that 5 critical periods unfold for this value of the parameter a.

The proof for the case $a=-1$, also considering the perturbation (2.43), follows similarly taking in $r_{02}=u_4$, $r_{11}=u_4v_1$, $r_{20}=u_4v_2$, $r_{30}=0$. Now we have

$$F_1(v_1,v_2) = -8 + \frac{192}{5}v_1 - \frac{16}{5}v_2,$$
$$F_2(v_1,v_2) = \frac{1277}{56} + \frac{145}{24}v_1 - \frac{85}{8}v_1^2 + \frac{5}{8}v_1v_2,$$
$$F_3(v_1,v_2) = \frac{12}{35} - \frac{144}{7}v_1.$$

Here, the zero level curves of F_1 and F_2 have two transversal intersection points, both of them written as
$$(v_1^*,v_2^*) = \frac{1}{5}(\alpha, 12\alpha - 5),$$

being α each simple real zero of $p(\alpha) = 42\alpha^2 - 301\alpha - 7662$. Additionally, we see that $F_3(v_1^*,v_2^*) = p_1(\alpha) = 12(-12\alpha+1)/35$ and $\det \mathrm{Jac}_{(F_1,F_2)}(v_1^*,v_2^*) = p_2(\alpha) =$

$(-12\alpha + 43)/3$.

Finally, we would like to consider an alternative proof for the special case $a = -3/2$ (similarly for its equivalent case $a = -2/3$), which is a simple zero of P that does not vanish Q in (2.42). We will consider (2.43) and second-order developments, as in the previous cases for which the generic result for every a does not apply.

Here, the linear parts of the first four period constants have rank 4. Then, by using the Implicit Function Theorem, $T_k = u_k$ for $k = 1, \ldots, 4$, and vanishing these first four we get the next two period constants which depend on the remaining parameters (u_5, u_6, u_7),

$$\begin{aligned} T_5 &= u_5 u_6 + O_3(u_5, u_6, u_7), \\ T_6 &= u_5\Big(\frac{9}{2} - \frac{2552689}{12348}u_5 - \frac{1439}{245}u_6 - 15u_7\Big) + O_3(u_5, u_6, u_7), \end{aligned} \tag{2.45}$$

where $r_{03} = u_5, r_{20} = (16000u_5 - 4536u_6 - 6615u_7)/39690$, and $r_{30} = u_7$. To solve $T_5 = 0$ we need to know the different branches of the variety $T_5 = 0$ near the origin. The blow-up mechanism can help to discover them. This is the procedure proposed by Loud in [Lou61], where he considered it as a *singular use* of the Implicit Function Theorem. As we would like to find a branch where T_5 vanishes but T_6 does not, we will not use the tangent variety to $u_5 = 0$ because it is not clear from (2.45) whether T_6 vanishes on it or not. Then, assuming u_5 small but not zero and using the blow-up $u_6 = u_5 v_1$ and $u_7 = u_5 v_2$, the expressions (2.45) write as

$$\begin{aligned} T_5 &= u_5^2\Big(v_1 + u_5 \sum_{j=0}^{\infty} f_j(v_1, v_2)\, u_5^j\Big), \\ T_6 &= u_5\Big(\frac{9}{2} + u_5 \sum_{j=0}^{\infty} g_j(v_1, v_2)\, u_5^j, \Big). \end{aligned}$$

Clearly, we can use the usual Implicit Function Theorem to write $T_5 = u_5^2 w_1$. Then, on the variety $w_1 = 0$ we have $T_5 = 0$ but $T_6 \neq 0$, and the unfolding of 5 critical periods is proved. □

We notice that we have not considered $a = 0$ because in this case the unperturbed system is only quadratic, and up to first and second-orders only one and two critical periods appear, respectively.

Perturbing other isochronous

This section is devoted to see the existence of other cubic reversible isochronous systems from which, after perturbation inside the cubic reversible class, also 5 critical periods bifurcate from the origin. All these systems appear in the full classification of cubic reversible isochronous systems of Chen and Romanovski, see

[CR10]. We have not checked all of them and neither the ones in [CS99] because, as we have commented previously, we believe that there will be no more critical periods bifurcating from the centers different from the harmonic oscillator.

In the following results the cubic reversible perturbations are considered as in (2.43), because first we switch them to complex coordinates and then we apply the mechanism explained in Subsection 2.2.2. Recall that the bifurcation mechanisms are the direct application of the limit cycles bifurcation mechanisms described in [Chr05; GGT21].

Proposition 2.22. *Consider the cubic reversible isochronous systems*

$$\begin{cases} \dot{x} = -y + \dfrac{16}{3}xy, \\ \dot{y} = x - \dfrac{16}{3}x^2 + 4y^2 + \dfrac{256}{27}x^3, \end{cases} \qquad \begin{cases} \dot{x} = -y - 3x^2y, \\ \dot{y} = x + 2x^3 - 9xy^2. \end{cases}$$

The number of critical periods bifurcating from the origin when perturbing in the class of reversible cubic systems is at least 5.

Proof. The existence of the respective unfoldings of 5 critical periods follows as in Proposition 2.21, so we only describe the main differences.

For the first system, after using the Implicit Function Theorem we get $T_k = u_k$ for $k = 1, \ldots, 5$. Then, after vanishing them, the sixth writes as

$$T_6 = -\frac{2928640}{81}u_6^3 + O_4(u_6, u_7).$$

For the second system we need again the Implicit Function Theorem but a little more work is required. First, we get $T_k = u_k$ for $k = 1, \ldots, 3$. Then, after vanishing them and from the order two developments of the next three period constants, we have that there exists a curve in the parameters space such that, along it, the zero level curves of T_4 and T_5 intersect transversally and T_6 does not vanish at this point. The curve is defined by

$$\Lambda := (r_{02}(\lambda), r_{11}(\lambda), r_{20}(\lambda)) = \left(\frac{3\alpha}{2}, 1, \frac{1288836\alpha^2 - 33437\alpha + 8492}{2(182687\alpha - 14408)}\right)\lambda + O_2(\lambda),$$

where α is the unique simple zero of the polynomial $p(\alpha) = 14865206\alpha^3 - 9450402\alpha^2 + 5998353\alpha - 494789$. On such curve, T_4 and T_5 vanish and

$$T_6(\Lambda) = \frac{4428675\, p_1(\alpha)}{98996508541251328(182687\alpha - 14408)^2}\lambda^2 + O_3(\lambda),$$

$$\det \operatorname{Jac}_{(T_4,T_5)}(r_{02}, r_{20})(\Lambda) = \frac{12695535\, p_2(\alpha)}{118921648(182687\alpha - 14408)^2}\lambda^2 + O_3(\lambda),$$

with

$$\begin{aligned} p_1(\alpha) &= 860144811862228359035 9\alpha^2 - 9039597241380812188234\alpha \\ &\quad + 7675022628311829018 77, \\ p_2(\alpha) &= 62303007298924\alpha^2 + 70835816547508\alpha - 7694925309941. \end{aligned}$$

Moreover, the resultants with respect to α of (p, p'), (p, p_1), and (p, p_2) are nonzero rational numbers. □

Proposition 2.23. *Consider the cubic reversible isochronous systems*

$$\begin{cases} \dot{x} = -y + \dfrac{4}{3}xy, \\ \dot{y} = x - \dfrac{4}{3}x^2 + 4y^2 + \dfrac{16}{27}x^3, \end{cases} \qquad \begin{cases} \dot{x} = -y - \dfrac{14}{15}xy + \dfrac{16}{175}x^2y, \\ \dot{y} = x + \dfrac{16}{15}x^2 - \dfrac{46}{15}y^2 + \dfrac{64}{175}x^3 + \dfrac{48}{175}xy^2. \end{cases}$$

Up to a sixth-order study, the number of critical periods bifurcating from the origin when perturbing in the class of reversible cubic systems is only 4.

Proof. The proof follows just by checking that the linear parts of the first five period constants have rank 5. Straightforward computations show that, after using the Implicit Function Theorem and vanishing them, the next two period constants vanish up to a sixth-order study. □

Proposition 2.24. *Let $a \in \mathbb{R} \setminus \{0, \pm\sqrt{3}, \pm\sqrt{5}\}$. Consider the* 1*-parameter family of cubic isochronous reversible systems*

$$\begin{cases} \dot{x} = -y + 2(1-a^2)a^{-1}xy + 2x^2y - 2y^3, \\ \dot{y} = x + ax^2 + (2-a^2)a^{-1}y^2 + 4xy^2. \end{cases}$$

The number of critical periods bifurcating from the origin when perturbing in the class of reversible cubic systems is at least 5 *for $a \in \{\pm\sqrt{7/3}, \pm 2, \pm 3\}$ and* 4 *otherwise.*

Proof. The proof follows using Theorem 2.12 as the proof of Proposition 2.21. Here, the linear part of the first four period constants have rank 4, then there exists a change of variables such that $T_k = u_k$ for $k = 1, \ldots, 4$. The differences are only the expressions of T_5 and T_6 which are, after vanishing the first period constants,

$$\begin{aligned} T_5 &= -\frac{70(a-2)(a-3)(a+3)(a+2)(3a^2-7)a^4}{44a^8 + 90a^6 + 129a^4 + 167a^2 + 30} u_5, \\ T_6 &= \frac{4(834a^{10} - 16310a^8 + 115767a^6 - 387870a^4 + 629063a^2 - 401940)a^2}{44a^8 + 90a^6 + 129a^4 + 167a^2 + 30} u_5. \end{aligned}$$

□

In the above result we have not considered $a \in \{\pm\sqrt{3}, \pm\sqrt{5}\}$ because for these values more computations and higher-order developments should be studied, and we suspect that no more than 5 oscillations of the period function will appear. Let us explain the main difficulties. Let $\mathcal{R}_\ell(a) = (R_1, \ldots, R_\ell)$ be the sequence of ranks of the linear developments of the ordered period constants for a fixed value of the parameter a, being $R_k = \text{Rank}(T_1^{(1)}, \ldots, T_k^{(1)})$. Then, we have that $\mathcal{R}_{10}(\pm\sqrt{3}) = (1,2,3,3,4,4,4,4,4,4)$ and $\mathcal{R}_{10}(\pm\sqrt{5}) = (1,2,2,3,4,4,5,5,5,5)$ while for the other values, that is for $a \in \mathbb{R} \setminus \{0, \pm\sqrt{3}, \pm\sqrt{7/3}, \pm 2, \pm\sqrt{5}, \pm 3\}$, we have $\mathcal{R}_7(a) = (1,2,3,4,5,5,5)$.

2.3.4 Perturbing quartic isochronous systems

In this section we will prove that there exist quartic reversible centers for which at least 10 critical periods bifurcate by using Theorem 2.15. This proves the statement of Theorem 2.10 corresponding to $n = 4$, that is $\mathcal{C}_\ell(4) \geq 10$. Basically we will follow the same scheme as in the previous section for the holomorphic case. Assuming that the linear parts of the period constants of a quartic system have rank 9, we rewrite the 9 first period constants as

$$T_k = u_k + O_2, \text{ for } k = 1, \ldots, 9,$$

where the u_k are new variables which depend on the original perturbative parameters and O_2 denotes a sum of monomials of degree at least 2. Linear parts of higher period constants would be a linear combination of these u_k. For convenience we can also write directly, by using the Implicit Function Theorem, $T_k = u_k$ for $k = 1, \ldots, 9$.

By using Poincaré–Miranda's Theorem (Theorem 1.40) together with Theorem 2.15, in the following result we present a family of quartic isochronous reversible centers from which at least 10 critical periods can bifurcate.

Proposition 2.25. *Let $a, b \in \mathbb{R}$. Consider the 2-parameter family of quartic (holomorphic) reversible systems*

$$\dot{z} = \mathrm{i}\, z\,(1-z)\,(1-az)\,(1-bz)\,. \tag{2.46}$$

Generically, at least 8 critical periods bifurcate from the origin when perturbing in the class of reversible quartic centers. Moreover, in this perturbation class there exists a point (a,b) such that at least 10 critical periods bifurcate from the origin.

Proof. System (2.46) is time-reversible holomorphic and therefore it has an isochronous center at the origin. Let us add a time-reversible quartic perturbation with no holomorphic terms as in (2.30), this is, being $r_{lm} \in \mathbb{R}$,

$$\dot{z} = \mathrm{i}\, z\,(1-z)\,(1-az)\,(1-bz) + \mathrm{i} \sum_{\substack{l+m\geq 2\\ m\geq 1}}^{4} r_{lm} z^l w^m.$$

Straightforward computations show that the coefficients of the linear parts of the first 9 period constants, with respect to the only 9 perturbation parameters in the above equation, form a square matrix. Its determinant is a polynomial of degree 64 in the parameters of the family (a, b). We do not show it here because of its size. Then, the first statement follows from Theorem 2.15i.

The proof of the second statement needs more computations. After a linear change of coordinates in the parameters space we obtain that, generically, the period constants have the following form:

$$\begin{aligned}
T_k &= u_k + O_2, \text{ for } k = 1, \ldots, 8,\\
T_9 &= \frac{G(a,b)P(a,b)}{D(a,b)} u_9 + O_2,\\
T_{10} &= \frac{G(a,b)Q(a,b)}{D(a,b)} u_9 + O_2,\\
T_{11} &= \frac{G(a,b)R(a,b)}{D(a,b)} u_9 + O_2,
\end{aligned}$$

with $G(a,b) = (ab-a-b+2)(ab-2b^2-a+b)(2a^2-ab-a+b)a^3b^3$ and $P(a,b)$, $Q(a,b)$, $R(a,b)$, and $D(a,b)$ certain polynomials with rational coefficients in the variables a and b. We do not show the complete polynomials here because they are too large. They have respectively total degree $37, 39, 41$, and 37. Their number of monomials are respectively $657, 736, 819$, and 606. Then, the second statement follows directly from Theorem 2.15ii just checking that there exists a point (a_0, b_0) in the parameters space such that $P(a_0,b_0) = Q(a_0,b_0) = 0$, $R(a_0,b_0) \neq 0$, $\det \mathrm{Jac}_{(P,Q)}(a_0,b_0) \neq 0$, and $D(a_0,b_0) \neq 0$. To show the difficulty to find this special point, the zero level curves of the polynomials P, Q, R, and D in the square $[-1,1]^2$ are depicted in Figure 2.1. The point (a_0, b_0) should be in the intersection of the red and blue curves but not in the green and black ones, although the curves are very close to see the point.

Before proving analytically the existence of at least one intersection point (a_0, b_0), we will start by doing some numerical simulations in order to apply later the Poincaré–Miranda's Theorem.

After some tedious work zooming some zones of the figure together with some tricks, we have found a numerical approximation of this special point. Increasing the number of digits in the computations up to see the stabilization of

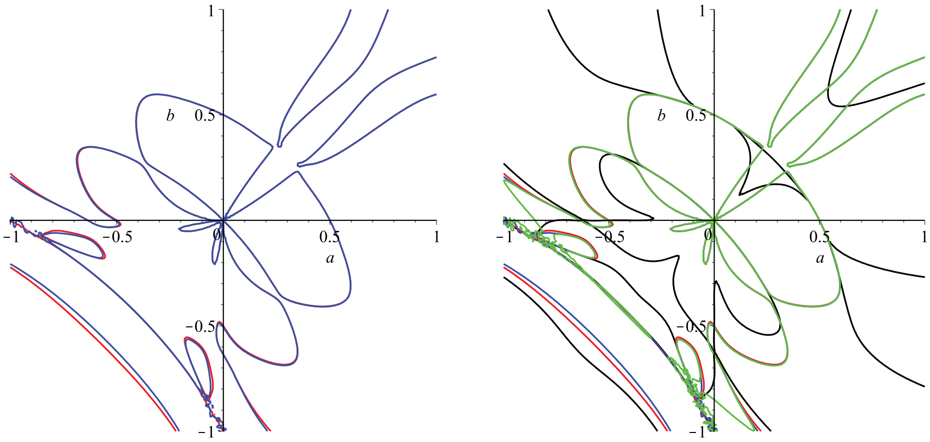

FIGURE 2.1: Plot of the zero level curves of $P(a,b)$, $Q(a,b)$, $R(a,b)$, and $D(a,b)$ for $(a,b) \in [-1,1]^2$, in color red, blue, green, and black, respectively.

the results, we obtain

$$\begin{aligned} a_0 &\approx 0.62577035826746384070691323127, \\ b_0 &\approx 0.71179266608573393310773491596, \\ R(a_0,b_0) &\approx -1.44391455520361722121698980760 \cdot 10^{13}, \\ \det \mathrm{Jac}_{(P,Q)}(a_0,b_0) &\approx -7.71411995359481041501433585645 \cdot 10^{29}, \\ D(a_0,b_0) &\approx -9.87896448642393578498609236141 \cdot 10^{13}. \end{aligned} \tag{2.47}$$

For the sake of simplicity of the expressions, we will divide each of the polynomials P, Q, R, and D by the coefficient of its highest power in a and, with a slight abuse of notation, we call them P, Q, R, and D again. Now we perform a linear change of variables which allows to separate the curves. The (numerical) Taylor expansion of $P(a,b)$ and $Q(a,b)$ at the above numerical approximation (a_0,b_0) is

$$\begin{aligned} P(a,b) \approx\ & 14476.3555282622425927110069492 \\ & - 1516162.34376751076199474015954\,a \\ & + 1312591.63242100192712169534384\,b + O_2(a,b), \\ Q(a,b) \approx\ & 78319.0710623740477702760304 2 \\ & - 8048108.27358418867430264665612\,a \\ & + 6965439.18320811849214303073248\,b + O_2(a,b), \end{aligned}$$

where O_2 are sums of monomials of degree at least 2. Consider now a change of variables from (a,b) to new parameters (u,v) such that u and v are respectively the above linear parts. By solving these two equations with respect to a and b, we obtain that

$$\begin{aligned} a &= 0.62577035826746384070691324177 3\\ &\quad + 0.002216189934882979960135882844 94\, u\\ &\quad - 0.000417626554172747676923930511920\, v,\\ b &= 0.71179266608573393310773492810 3\\ &\quad + 0.002560662160939406513270900787 41\, u\\ &\quad - 0.000482396534881316802873914874492\, v. \end{aligned} \tag{2.48}$$

We notice that at $(u, v) = (0, 0)$ we approximately recover the values for (a_0, b_0) at (2.47). Figure 2.2 shows the zero level curves of the polynomials P, Q, R, and D near $(0, 0)$ after this change of variables. Now it is clear that the four zero level curves do not intersect simultaneously at such point. Moreover, the ones corresponding to P and Q are transversal. Observe that $D(u, v)$ is not seen in the graph because it stays out of the plotted region. This intersection point has shifted to near $(0, 0)$ in the new variables, and is not exactly at $(0, 0)$ due to the rounding errors.

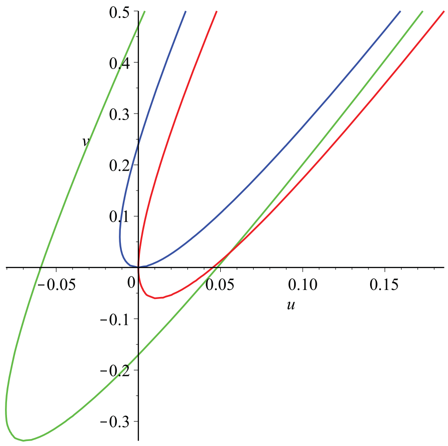

FIGURE 2.2: Plot of the zero level curves of $P(u, v)$, $Q(u, v)$, and $R(u, v)$ in color red, blue, green, respectively; the curve corresponding to $D(u, v)$ is out of the plotted region.

The last step is the analytical proof of the existence of the point (a_0, b_0), which we have seen above that exists numerically. We will do a computer-assisted proof checking the properties in Theorem 1.40 by using rational interval analysis, because all the involved polynomials have rational coefficients. We start by writing

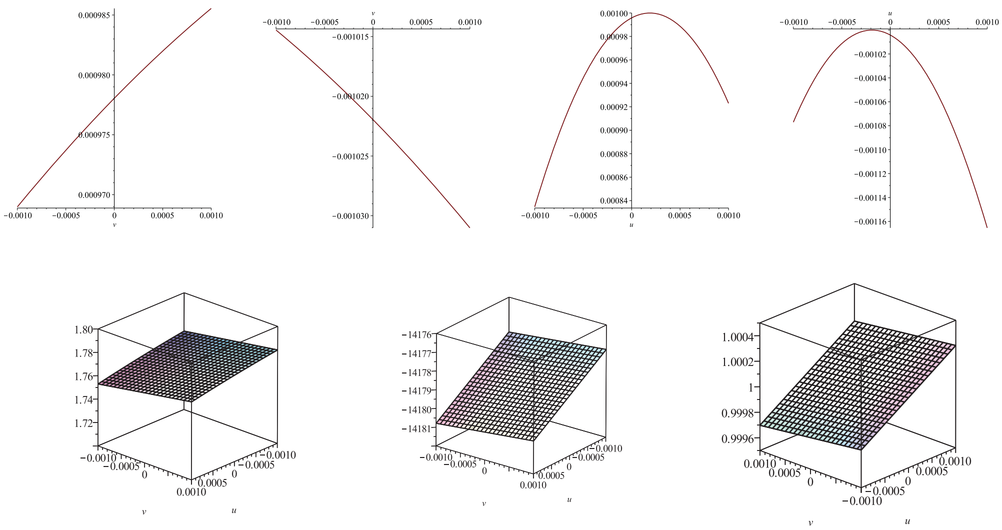

FIGURE 2.3: Plot of rescaled polynomials P and Q at the boundaries of $[-h,h]^2$ and the polynomials $R(u,v)$, $D(u,v)$, and $J(u,v)$ in the full square $[-h,h]^2$.

the relation (2.48) as rational numbers with a 30 digits precision,

$$a = \frac{803010141443820}{1283234545763833} + \frac{599808603999959}{27064855523371976}u - \frac{5287648183641}{12661187682653458}v,$$
$$b = \frac{480154601557585}{674570874968458} + \frac{4931930765653}{1926037273048026}u - \frac{4470981572020}{9268270496843407}v.$$

We will set $h = 10^{-3}$ in Theorem 1.40, and we will show that in the square $\mathcal{B} = [-h,h]^2$ there must be a zero of $P(u,v)$ and $Q(u,v)$. The proof follows checking also that $R(u,v)$, $D(u,v)$, and the Jacobian determinant $J(u,v) := \det \mathrm{Jac}_{(P,Q)}(u,v)$ do not vanish in the whole square. The draws in Figure 2.3 show that these conditions hold. Observe that $P(u,v)$ and $Q(u,v)$ are continuous because they are polynomials. Then there will be a point $(u_0,v_0) \in (-h,h)^2$ such that $P(u_0,v_0) = 0$ and $Q(u_0,v_0) = 0$ by applying the Poincaré–Miranda's Theorem because the following conditions hold.

(a) $P(h,v) > 0$ and $P(-h,v) < 0$ for $v \in [-h,h]$.
First we find the first derivatives of $P(h,v)$ and $P(-h,v)$ with respect to v. Then we compute all its real roots and see that none of them belongs to the interval $(-h,h)$, which implies that there are no local maxima nor minima in this interval. Now we check that $P(h,-h) > 0$, $P(h,h) > 0$, $P(-h,-h) < 0$, and $P(-h,h) < 0$, which together with the fact that there are not any local extrema means that the function $P(h,v)$ is strictly positive in the whole interval while $P(-h,v)$ is strictly negative.

(b) $Q(u,h) > 0$ and $Q(u,-h) < 0$ for $u \in [-h,h]$.
The proof follows checking that the first derivatives of $Q(u,h)$ and $Q(u,-h)$ with respect to u have only one real root in the interval $(-h,h)$, which means only one extremum. We also see that the second derivatives of $Q(u,h)$ and $Q(u,-h)$ again with respect to u at those points take a negative value, so these only local extrema are local maxima. Also, the value of $Q(u,h)$ and $Q(u,-h)$ evaluated at the u which gives the maxima are positive and negative, respectively. Additionally, $Q(-h,h) > 0$, $Q(h,h) > 0$, $Q(-h,-h) < 0$, and $Q(h,-h) < 0$. Then, the functions $Q(u,h)$ and $Q(u,-h)$ are respectively strictly positive and negative in the whole interval.

Strictly speaking, we observe that due to how Theorem 1.40 is formulated we should apply it to $-P(u,v)$ and $-Q(u,v)$ rather than $P(u,v)$ and $Q(u,v)$, but the conclusion is exactly the same.

The last step of the proof is to ensure that $R(u,v)$, $D(u,v)$, and $J(u,v)$ do not vanish in the whole square.

First we will prove that there exists $\widetilde{R} \in \mathbb{Q}^+$ such that $R(u,v) \geq \widetilde{R} > 0$ for $(u,v) \in [-h,h]^2$. It is clear that $R(u,v)$ can be written as

$$R(u,v) = R(0,0) + \sum_{i=0}^{\hat{k}} \sum_{\substack{j=0 \\ (i,j)\neq(0,0)}}^{\hat{l}} a_{ij} u^i v^j \tag{2.49}$$

for certain rational coefficients a_{ij}, where $\hat{k}$ and $\hat{l}$ denote the degree of $R(u,v)$ with respect to u and v, respectively. Observe that

$$R(u,v) = R(0,0) + \sum_{i=0}^{\hat{k}} \sum_{\substack{j=0 \\ (i,j)\neq(0,0)}}^{\hat{l}} a_{ij} u^i v^j \geq R(0,0) - \sum_{i=0}^{\hat{k}} \sum_{\substack{j=0 \\ (i,j)\neq(0,0)}}^{\hat{l}} |a_{ij}| h^{i+j} =: \widetilde{R},$$

where we have used that $|u| \leq h$ and $|v| \leq h$. The right part of the inequality can be easily computed and we obtain a positive rational number $\widetilde{R} \approx 1.7529595059$.

The proof that there exists $\widetilde{J} \in \mathbb{Q}^+$ such that $J(u,v) \geq \widetilde{J} > 0$ for $(u,v) \in [-h,h]^2$ follows analogously to the one for $R(u,v)$, just by writing the equivalent expression (2.49) for function J and adequately changing the values for the degrees $\hat{k}, \hat{l}$, and the rational coefficients a_{ij}. The positive rational lower bound is $\widetilde{J} \approx 0.9996974188$. Similarly, we can prove that there exists $\widetilde{d} \in \mathbb{Q}^-$ such that $D(u,v) \leq \widetilde{d} < 0$ for $(u,v) \in [-h,h]^2$. In this case, as well as changing the values for the degrees $\hat{k}, \hat{l}$ and the rational coefficients a_{ij} we have to invert all inequalities. The upper bound is the negative rational number $\widetilde{d} \approx -14177.3096985157$.

We notice that these values for the lower and upper bounds obtained above are far from the values in (2.47) because we have rescaled all the involved functions. □

2.3.5 Perturbing higher degree systems

In this subsection we will use period constants only up to first-order in the perturbative parameters to obtain as many critical periods as possible by bifurcating, in the class of reversible systems, from some reversible holomorphic systems. The idea is to consider an isochronous center of the form (2.32) perturbed as in (2.30), being $r_{lm} \in \mathbb{R}$. Using linear terms of the period constants one can deduce that at least $(n^2+n-4)/2$ critical periods bifurcate from the origin. In Proposition 2.26 this is proved for $3 \le n \le 16$. This provides the lower bound for $\mathcal{C}_\ell(n)$ given in Theorem 2.10 for $10 \le n \le 16$. In fact, we notice that for $n=3$ and $n=4$ we have already found better bounds in the previous sections, but we also include them for the sake of completeness. According to Theorem 2.15, under certain conditions the system could unfold up to $n-2$ extra critical periods with respect to those $(n^2+n-4)/2$ obtained by using only linear parts, as the system has $n-2$ holomorphy parameters a_j. Nevertheless, we will see that this is unfeasible even for degree 5 due to the large size of the obtained polynomials, but we will add at least one extra critical period in Proposition 2.27 for $5 \le n \le 9$. This gives the lower bound for $\mathcal{C}_\ell(n)$ given in Theorem 2.10 for $5 \le n \le 9$.

Proposition 2.26. *For $3 \le n \le 16$, consider the system*

$$\dot{z} = \mathrm{i}\, z \prod_{k=2}^{n} \left(1 - \Phi\left(\left[\frac{k}{2} \right] \right)^{(-1)^k} z \right), \tag{2.50}$$

where $\Phi(j)$ is the jth prime number and $[\cdot]$ denotes the integer part function. Then, when perturbing in the class of reversible centers at least $(n^2+n-4)/2$ critical periods bifurcate from the origin, which is of isochronous reversible center type.

Proof. The nth degree system (2.50) can alternatively be written as

$$\begin{cases} \dot{z} = \mathrm{i}\, z\,(1-2z)\,(1-2^{-1}z)\,(1-3z)\,(1-3^{-1}z)\,(1-5z)\,(1-5^{-1}z)\cdots, \\ \dot{w} = -\mathrm{i}\, w\,(1-2w)\,(1-2^{-1}w)\,(1-3w)\,(1-3^{-1}w)\,(1-5w)\,(1-5^{-1}w)\cdots. \end{cases}$$

This system is reversible and holomorphic, so it has an isochronous center at the origin. Now add an nth degree perturbation with real parameters r_{lm} as in (2.30).

The next step is to compute the first $N=(n^2+n-2)/2$ period constants of the perturbed system up to first-order. To this end, we apply the method presented in Subsection 2.2.3 which uses Proposition 2.8. We have performed these calculations for degree $3 \le n \le 16$ by using Maple plus the parallelization with

PBala (see [Sal]), and we have found that the rank of the linear part of the first N period constants is precisely N, thus we obtain maximal rank. Therefore, by applying Theorem 2.15i this implies that $N-1$ critical periods bifurcate from the origin, which is the lower bound given in the statement. □

It is worth saying that we would not have been able to reach degree $n = 16$ in the above result without using the technique presented in Proposition 2.8. The reason why for a certain degree n we can obtain rank $N = (n^2+n-2)/2$ in the linear parts of the corresponding period constants is as follows. By basic combinatorics one can see that the number of perturbative terms in a reversible degree $n \geq 3$ system is

$$\sum_{j=3}^{n+1} j = \sum_{j=1}^{n+1} j - 2 - 1 = \frac{(n+2)(n+1)}{2} - 2 - 1 = \frac{n^2+3n-4}{2}. \tag{2.51}$$

However, observe that the terms of the form $c_{j0}z^j = A_j z^j$ belong to the holomorphic part of the system and are not considered perturbative parameters, so they cannot appear in the linear part of the period constants. As a consequence, for degree n the terms $A_j z^j$ for $2 \leq j \leq n$ do not count when computing ranks of linear parts of period constants, so the number of perturbative parameters which can actually play a part results from subtracting $n-1$ to the total number (2.51), which results in N. This means that with Proposition 2.26 we have reached the maximum number of critical periods that can bifurcate by studying the rank when perturbing a fixed holomorphic system using linear parts only.

As we can theoretically get rank N for degree n, then $N-1$ critical periods could bifurcate from the origin. In Proposition 2.26 we proved that this number of critical periods can actually appear for $3 \leq n \leq 16$, and for higher degrees the problem gets too demanding in computational terms. Nevertheless, it is natural to think that this lower bound will hold for any degree $n \geq 3$. For the computations we have used the cluster of servers Antz described in Subsection 1.2.3, with more than 100 cores and more than 300 GB of RAM in total.

In the next result we provide one more critical period than the obtained in the previous proposition, considering the holomorphic reversible system of degree n

$$\dot{z} = \mathrm{i}\, z(1-z) \prod_{j=1}^{n-2} (1 - a_j z), \tag{2.52}$$

with $5 \leq n \leq 9$ and $a_j \in \mathbb{R}$, but with only one free parameter instead of $n-2$, $(a_1, \ldots, a_{n-2})$, because of the difficulties in the analytical computations.

Proposition 2.27. *Let $5 \le n \le 9$ be a natural number and $a \in \mathbb{R}$. For the (holomorphic) reversible 1-parameter family*

$$\dot{z} = \mathrm{i}\, z(1 - az) \prod_{k=1}^{n-2} (1 - kz)\,, \tag{2.53}$$

there exists a real value a such that at least $(n^2 + n - 2)/2$ critical periods bifurcate from the origin when perturbing in the class of polynomial reversible centers of degree n.

Proof. System (2.53) is time-reversible holomorphic, so it has an isochronous center at the origin. We consider the time-reversible polynomial perturbation of degree n with no holomorphic terms as in (2.30) and we compute the first-order developments of its $(n^2 + n)/2$ first period constants as a function of a. Notice that this system has $N := (n^2 + n - 2)/2$ perturbative parameters, which is the maximal rank that the linear parts can have. In the case that we have rank $N - 1$ instead, as in Theorem 2.15ii, a perturbative parameter is still not used. We have checked that, after a linear change of parameters, for each degree $5 \le n \le 9$, the period constants have the form

$$\begin{aligned} T_k &= u_k + O_2 \text{ for } k = 1, \ldots, N-1, \\ T_N &= a^2 C_n(a) \frac{P_n(a)}{D_n(a)} u_N + O_2, \\ T_{N+1} &= a^2 C_n(a) \frac{Q_n(a)}{D_n(a)} u_N + O_2, \end{aligned}$$

for certain polynomials $P_n(a)$, $Q_n(a)$, $D_n(a)$, and $C_n(a)$ in the variable a with rational coefficients. These polynomials are not shown here because of their large size: $P_n(a)$ has degree 100, 206, 374, 626, and 986 for $n = 5, 6, 7, 8$, and 9, respectively; $Q_n(a)$ has degree 102, 208, 376, 628, and 988 for $n = 5, 6, 7, 8$, and 9, respectively; $D_n(a)$ has degree 89, 188, 349, 593, and 944 for $n = 5, 6, 7, 8$, and 9, respectively. The polynomials $C_n(a)$ are $C_5(a) = 2a - 3$ and $C_n(a) = 1$ for $n = 6, 7, 8, 9$.

To prove the unfolding of an extra critical period by following the ideas in Theorem 2.15, we should see that there exists some value a_n such that $P_n(a_n) = 0$, $P_n'(a_n) \ne 0$, $Q_n(a_n) \ne 0$, and $D_n(a_n) \ne 0$ for $5 \le n \le 9$. Straightforward computations show that $P_5(a)$ has a root a_5 in the interval $[0.75, 0.76]$, $P_6(a)$ has a root a_6 in the interval $[1.27, 1.28]$, $P_7(a)$ has a root a_7 in the interval $[0.11, 0.12]$, $P_8(a)$ has a root a_8 in the interval $[0.58, 0.59]$ and $P_9(a)$ has a root a_9 in the interval $[0.12, 0.13]$. Thus, we know that for each $n = 5, 6, 7, 8$, and 9, $P_n(a)$ has a real root a_n.

Finally, we find that the resultant of $P_n(a)$ with $P_n'(a)$, the resultant of $P_n(a)$ with $Q_n(a)$, and the resultant of $P_n(a)$ with $D_n(a)$ are nonzero rational numbers for each $n = 5, 6, 7, 8$, and 9, which means that $P_n(a)$ has no common zeros with

$P_n'(a)$, $Q_n(a)$, and $D_n(a)$. Therefore, by applying Theorem 2.15ii, we can conclude that for degrees from 5 to 9 one extra critical period unfolds. □

As we already commented, due to the fact that system (2.52) has $n-2$ holomorphic free parameters, according to Theorem 2.15ii one could expect to see $n-2$ extra critical periods. However, when computing the period constants of (2.52) even for $n=5$, we observe that we cannot deal with them: after appropriately handling the following three period constants T_{14}, T_{15}, and T_{16}, the obtained polynomials that we need to apply the theorem have approximately half a million of monomials, with degrees $154, 156$, and 158. Moreover, their coefficients are integer numbers between 40 and 80 digits long. Because of this, we have not been able to see numerically the existence of a transversal intersection of them. Nevertheless, by working with two parameters we have numerical evidence that for $n=5$ actually 2 additional critical periods unfold, as we will see followingly. Even for this case, the size of the expressions is too large to achieve an analytical proof.

Let us consider the system

$$\begin{cases} \dot{z} = \mathrm{i}\, z(1-z)^2(1-az)(1-bz), \\ \dot{w} = -\mathrm{i}\, w(1-w)^2(1-aw)(1-bw). \end{cases} \tag{2.54}$$

Firstly, we compute the first 16 period constants of system (2.54), consider their linear parts and denote by $P(a,b)$, $Q(a,b)$, and $R(a,b)$ the numerator of the coefficient of the corresponding linear part and $D(a,b)$ the common denominator, following the notation in the proof of Proposition 2.27. Working with enough precision up to see the stabilization of the values of the intersection of the zero level curves of P and Q, together with the value of R, D, and the Jacobian determinant of (P,Q), we can find a transversal intersection at

$$(a_0, b_0) \approx (0.63824202454687891, -1.75185147414301379).$$

In Figure 2.4 we represent graphically the intersection of the zero level curves of the polynomials $P(a,b)$ and $Q(a,b)$.

2.3.6 Six critical periods on cubic systems

This subsection is devoted to prove the part of the statement of Theorem 2.10 corresponding to $n=3$, this is $\mathcal{C}_\ell(3) \geq 6$. This result will not arise from a perturbation of isochronous centers as we did in Subsection 2.3.3, in the sense that the perturbative parameters are not 'small'.

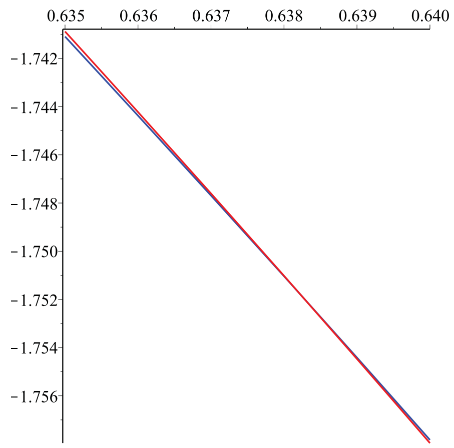

FIGURE 2.4: Plot of the zero level curves of polynomials $P(a, b)$ and $Q(a, b)$ in red and blue color, respectively; the zero level curves of the polynomials R, D, and the Jacobian determinant of P and Q do not appear because they stay out of figure. The intersection of $P(a, b)$ and $Q(a, b)$ can be clearly seen.

Proposition 2.28. *There exist values of $r_{20}, r_{11}, r_{02}, r_{30}, r_{21}, r_{12}, r_{03} \in \mathbb{R}$ for which the origin of the cubic reversible system*

$$\dot{z} = \mathrm{i}\Big(z - z^3 + \sum_{l+m=2}^{3} r_{lm} z^l w^m\Big), \tag{2.55}$$

unfolds 6 *local critical periods.*

Proof. The proof will consist on the following steps. First we compute the first 7 period constants of system (2.55). Then we show the existence of a point in the parameters space, with $r_{20} = 1$, for which $T_1 = \cdots = T_6 = 0$ but $T_7 \neq 0$. The complete unfolding is proved checking that the determinant of the Jacobian matrix of $(T_1, \ldots, T_6)$ with respect to the remaining 6 parameters is not zero.

The first 7 period constants of system (2.55) have been obtained by using the method described in Subsection 2.2.2. Because of their size, here we only show the first two,

$$T_1 = -2r_{11}r_{20} + 2r_{21} - \frac{4}{3}r_{02}^2 - 2r_{11}^2,$$

$$
\begin{aligned}
T_2 =& 4r_{12} - 8r_{11}^2 + 4r_{11}r_{20} - 4r_{12}^2 - 3r_{03}^2 - 4r_{12}r_{30} + 8r_{11}^2r_{30} + 8r_{11}^2r_{21} + \frac{8}{3}r_{02}^2r_{21} \\
&+ 16r_{11}^2r_{12} - \frac{8}{3}r_{20}r_{02} - \frac{40}{3}r_{02}r_{11}^3 - \frac{44}{3}r_{11}r_{02} - \frac{4}{3}r_{02}^2r_{20}^2 - 15r_{02}^2r_{11}^2 + 20r_{12}r_{11}r_{02} \\
&+ \frac{8}{3}r_{02}r_{12}r_{20} + 4r_{11}r_{12}r_{20} - 4r_{11}r_{20}r_{21} + \frac{4}{3}r_{03}r_{20}r_{02} + \frac{44}{3}r_{30}r_{11}r_{02} + \frac{58}{3}r_{02}r_{03}r_{11} \\
&+ \frac{8}{3}r_{30}r_{20}r_{02} - 4r_{11}r_{20}r_{30} - \frac{28}{3}r_{02}^2r_{11}r_{20} - \frac{8}{3}r_{02}r_{20}^2r_{11} - 12r_{02}r_{20}r_{11}^2.
\end{aligned}
$$

The number of monomials of the following constants, T_3, T_4, T_5, T_6, T_7, are respectively $164, 576, 1645, 3861, 8303$, and their degrees are $6, 8, 10, 12, 14$.

Now the second step is to check that there exists some point in the parameters space such that the first 6 period constants vanish but T_7 does not. Let us start by imposing $r_{20} = 1$ and solving $T_1 = T_2 = 0$ provided that $D := 3r_{12} + 3r_{11} - 11r_{02}r_{11} - 2r_{02} - 6r_{11}^2 \neq 0$. Then

$$
\begin{aligned}
r_{21} =& r_{11} + \frac{2}{3}r_{02}^2 + r_{11}^2, \\
r_{30} =& \frac{1}{12\left(3r_{12} + 3r_{11} - 11r_{02}r_{11} - 2r_{02} - 6r_{11}^2\right)} \Big(16r_{02}^4 - 63r_{02}^2r_{11}^2 - 84r_{02}^2r_{11} \\
&- 12r_{02}^2 - 120r_{02}r_{11}^3 - 108r_{02}r_{11}^2 - 24r_{02}r_{11}^2 + 72r_{11}^4 + 36r_{11}^3 - 36r_{11}^2 \\
&+ 174r_{02}r_{03}r_{11} + 12r_{02}r_{03} + 180r_{02}r_{11}r_{12} + 24r_{02}r_{12} + 144r_{11}^2r_{12} \\
&+ 36r_{11}r_{12} - 132r_{02}r_{11} - 24r_{02} - 27r_{03}^2 - 72r_{11}^2 + 36r_{11} - 36r_{12}^2 + 36r_{12}\Big).
\end{aligned}
$$

Under the above condition $D \neq 0$, the Jacobian determinant of T_1 and T_2 with respect to r_{21} and r_{30} is nonzero. This implies that the study of the complete versal unfolding of the 6 critical periods can be restricted to the study of the remaining period constants with respect to the four free parameters $r_{11}, r_{02}, r_{12}, r_{03}$.

To simplify the manipulation of $T_3, \ldots, T_7$, we take their numerators and divide them by their highest coefficient in absolute value; with a slight abuse of notation, we call them again $T_3, \ldots, T_7$, respectively.

Before the analytical proof, we will provide numerical evidence that there exists a solution for $\{T_3 = 0, T_4 = 0, T_5 = 0, T_6 = 0\}$ such that T_7, the denominator D, and the Jacobian determinant J of (T_3, T_4, T_5, T_6) do not vanish. We have increased the precision up to see the stabilization of the results. A 30-digits approximation to this intersection point is

$$
\begin{aligned}
S := \{r_{11} &= 0.332239671964981276819848124224, \\
r_{02} &= -1.14623564863006725151534814297, \\
r_{12} &= 0.707146879073682873590033571024, \\
r_{03} &= -0.857479316438844353902485565632\}
\end{aligned}
$$

and, at this point,

$$\begin{aligned} T_7 &= -1.8462057344648559009728611 8 \cdot 10^{-9}, \\ D &= -4.9242326181310472013221146319 1 \cdot 10^{-14}, \\ J &= -8.9374062674613686846226017250 3. \end{aligned}$$

Even though T_7 and D might seem too close to zero, the numerical values of T_3, T_4, T_5, T_6 at S are about 20 orders of magnitude lower, so we can actually consider that T_7 and D are nonzero.

Having this numerical evidence, we will proceed with the analytical proof by following a computer-assisted proof as we have done in the proof of Proposition 2.25.

Let us consider the rational approximation of the first-order Taylor expansion of the period constants T_3, T_4, T_5, T_6 at the point S,

$$\begin{aligned} T_3^{(1)} &= -\frac{73352896192857}{1157958866091236} + \frac{66262571735671}{670216015479518} r_{11} - \frac{119234362424303}{776335803127460} r_{02} \\ &\quad + \frac{7903848963503}{675876388619388} r_{12} + \frac{55731331328881}{310685226195660} r_{03}, \\ T_4^{(1)} &= -\frac{25841873263308}{2144739207215017} + \frac{40160593855699}{1426912747264762} r_{11} - \frac{117691544210802}{4702223212288759} r_{02} \\ &\quad + \frac{7773205101075}{2079218586073918} r_{12} + \frac{110363479645312}{3304887976984249} r_{03}, \\ T_5^{(1)} &= -\frac{16219703349568}{10414082088666585} + \frac{42608385876433}{9737281715798994} r_{11} - \frac{29366717293918}{9762325804542787} r_{02} \\ &\quad + \frac{6748894626740}{9711312046719413} r_{12} + \frac{6430413960561}{1437481484699156} r_{03}, \\ T_6^{(1)} &= -\frac{1959207228925}{14543712037193487} + \frac{11134499629945}{27672511586934129} r_{11} - \frac{5248411486748}{20480381071923191} r_{02} \\ &\quad + \frac{3134463695044}{45208235327076605} r_{12} + \frac{8433554097025}{21161093903316966} r_{03}, \end{aligned}$$

and the change of variables

$$\{T_3^{(1)} = u_1, T_4^{(1)} = u_2, T_5^{(1)} = u_3, T_6^{(1)} = u_4\}. \tag{2.56}$$

Now one can solve this system to obtain the inverse change. To deal with shorter rational numbers, we convert the coefficients of the resulting expressions to a

30-digit approximation and then reconvert it to rational, obtaining

$$
\begin{aligned}
r_{11} = {} & \frac{114216314885635}{343776871106692} + \frac{3690270297600200}{19535367373429}u_1 - \frac{24306493749268230}{4889036398111}u_2 \\
& + \frac{3565552496655516}{44921198443}u_3 - \frac{74580049035068047}{133328816009}u_4, \\
r_{02} = {} & -\frac{2830661790614852}{2469528664544625} - \frac{8579444837165377}{6135160885376}u_1 + \frac{62633434081451044}{1884512347855}u_2 \\
& - \frac{218159188810346297}{437320083677}u_3 + \frac{433555893724556147}{125890356506}u_4, \\
r_{12} = {} & \frac{308981620175863}{436941220161516} + \frac{14794405087051724}{7814232065819}u_1 - \frac{52171010172694907}{1180772594709}u_2 \\
& + \frac{207814747586335205}{323119349554}u_3 - \frac{91326167194000251}{20903373436}u_4, \\
r_{03} = {} & -\frac{301994834117308}{352189059639955} - \frac{28146991231557103}{19831964964997}u_1 + \frac{65898129221474685}{1933801767914}u_2 \\
& - \frac{292164620132414823}{569752028783}u_3 + \frac{570903623821683593}{161190849884}u_4.
\end{aligned}
$$

Using these expressions we can rewrite the whole $T_3, \ldots, T_7$ in these new variables. For simplicity we denote them by $U_j(u_1, u_2, u_3, u_4) := T_{j+2}(r_{11}, r_{02}, r_{12}, r_{03})$ for $j = 1, \ldots, 5$. Observe that the first-order Taylor expansion of $U = (U_1, U_2, U_3, U_4)$ with respect to the variables $u = (u_1, u_2, u_3, u_4)$ is near the identity. Consequently, the problem reduces to proving the existence of some point $u^* = (u_1^*, u_2^*, u_3^*, u_4^*)$ near the origin for which $U_1(u^*) = U_2(u^*) = U_3(u^*) = U_4(u^*) = 0$, and $U_5(u^*)$, the denominator $D(u^*)$, and the Jacobian determinant $J(u^*) := \det \mathrm{Jac}_U(u^*)$ do not vanish. The existence of such point will be shown applying again Poincaré–Miranda's Theorem (Theorem 1.40).

Let us set $h = 10^{-12}$. We have implemented an algorithm which provides rational upper and lower bounds to a given function with m variables in $\mathcal{B} = [-h, h]^m$, for $m = 3, 4$. Using it as a computer-assisted proof, we have been able to find the following bounds.

- For U_1, we have $0 < \hat{u}_1 < U_1(h, u_2, u_3, u_4)$ and $U_1(-h, u_2, u_3, u_4) < -\hat{u}_1 < 0$ for all $u_2, u_3, u_4 \in [-h, h]$, where $\hat{u}_1 \approx 2.67 \cdot 10^{-13}$.
- For U_2, we have $0 < \hat{u}_2 < U_2(u_1, h, u_3, u_4)$ and $U_2(u_1, -h, u_3, u_4) < -\hat{u}_2 < 0$ for all $u_1, u_3, u_4 \in [-h, h]$, where $\hat{u}_2 \approx 8.78 \cdot 10^{-13}$.
- For U_3, we have $0 < \hat{u}_3 < U_3(u_1, u_2, h, u_4)$ and $U_3(u_1, u_2, -h, u_4) < -\hat{u}_3 < 0$ for all $u_1, u_2, u_4 \in [-h, h]$, where $\hat{u}_3 \approx 9.85 \cdot 10^{-13}$.
- For U_4, we have $0 < \hat{u}_4 < U_4(u_1, u_2, u_3, h)$ and $U_4(u_1, u_2, u_3, -h) < -\hat{u}_4 < 0$ for all $u_1, u_2, u_3 \in [-h, h]$, where $\hat{u}_4 \approx 9.98 \cdot 10^{-13}$.

This means that U_j is positive in $u_j = h$ and negative in $u_j = -h$ for $j = 1, 2, 3, 4$. Therefore, by applying Poincaré–Miranda's Theorem we can conclude that there exists some point in $[-h, h]^4$ which vanishes U_1, U_2, U_3, U_4.

By following an analogous computer-assisted proof, one can see that functions U_5 and D satisfy $U_5(u_1, u_2, u_3, u_4) < -\hat{u}_5 < 0$ and $D(u_1, u_2, u_3, u_4) < -\hat{d} < 0$ for all $u_1, u_2, u_3, u_4 \in [-h, h]$, where $\hat{u}_5 \approx 1.84 \cdot 10^{-9}$ and $\hat{d} \approx 8.93$, so both functions are always negative in $[-h, h]^4$ and do not vanish in the box.

The last part of the proof will be to check that the Jacobian determinant $J(u)$ is also nonzero in $[-h, h]^4$. From the change (2.56), it is clear that the Jacobian matrix Jac_U is close to the identity matrix I and we can write $\mathrm{Jac}_U = I + M$ for some matrix M. By adapting and using the previously implemented algorithm, we find upper and lower bounds for each one of the 16 entries (k, l) of M, proving that for every entry M_{kl} of the matrix there exists a positive rational number $\hat{m}_{kl}$ such that $-\hat{m}_{kl} < M_{kl} < \hat{m}_{kl}$.

It is straightforward to check that the Jacobian determinant $J(u)$ has the following structure,

$$J(u) = 1 + \sum_{s=1}^{64} \mathcal{M}_s,$$

where every $\mathcal{M}_s$ is a product of entries of matrix M which may be either positive or negative. Let us denote by $\hat{\mathcal{M}}_s$ the rational number resulting of the substitution of every factor M_{kl} by $\hat{m}_{kl}$ in $\mathcal{M}_s$. We have then a rational lower bound $\hat{J}$ for which $J(u)$ satisfies

$$J(u) = 1 + \sum_{s=1}^{64} \mathcal{M}_s \geq 1 - \sum_{s=1}^{64} |\hat{\mathcal{M}}_s| = \hat{J} \approx 0.9918518555136.$$

This justifies that the determinant is positive for every $u_1, u_2, u_3, u_4 \in [-h, h]$, so we can guarantee that it does not vanish in $[-h, h]^4$ and the result follows. □

2.3.7 Some remarks for arbitrary degree

The method used in Subsection 2.3.3 for cubics and Subsection 2.3.4 for quartics can be theoretically extended to systems of any degree n. We have seen that for the cubic case we can obtain families with an extra parameter which gives one extra oscillation, and for the quartic case we have families with two extra parameters which give two extra oscillations. Indeed, holomorphic reversible systems of degree n of the form (2.32) can be rescaled as $z \mapsto a_1^{-1} z$ to obtain

$$\dot{z} = \mathrm{i}\, z\,(1 - z)\,(1 - b_1 z) \cdots (1 - b_{n-2} z)\,,$$

where we have defined the $n-2$ new parameters $b_j := a_{j+1}a_1^{-1}$ for $j = 1, \ldots, n-2$. By adding a time-reversible perturbation, with the same technique from Subsections 2.3.3 and 2.3.4 we should be able to obtain $n-2$ extra critical periods. Even though this method seems pretty clear from a theoretical point of view, when trying to make the calculations one realises that it soon becomes too demanding in computational terms, and this is the reason why we have not gone further than $n = 4$. However, we think that these $n-2$ extra critical periods must appear near the holomorphic reversible centers, by bifurcation in the class of polynomial reversible systems of degree n. Then the *local criticality of polynomial holomorphic reversible* systems of degree n in the class of polynomial reversible vector fields also of degree n would be $\mathcal{C}_\ell^h(n) \geq (n^2+3n-8)/2$. We notice that we have not considered here the harmonic oscillator, $\dot{z} = \mathrm{i}\,z$, because it is not strictly a degree n system.

As we have seen in Subsection 2.3.6, if we consider the complete polynomial reversible center family of 3rd degree, an extra oscillation can be found when using the total number of parameters except the scaled one. This rescaling is like stating that the harmonic oscillator will be the reversible center with the highest criticality. We think that what is happening for degree 3 is a bifurcation phenomenon that will occur for every degree, being $\mathcal{C}_\ell(n) \geq (n^2+3n-6)/2$.

As a summary, being $N = (n^2+3n-4)/2$ the total number of parameters in reversible nondegenerate centers, we think that $\mathcal{C}_\ell(n) \geq N-1$ while $\mathcal{C}_\ell^h(n) \geq N-2$.

2.4 Criticality via first-order development of period constants

Melnikov functions are widely used on the well-known problem of limit cycles bifurcation in planar systems of differential equations, an issue related to the second part of the 16th Hilbert Problem ([Gin07; Li03]). In analogy to this question, some authors have proposed an equivalent approach for studying the number of oscillations of the period function of a center. Works such as [CRZ11; GLY10; ZLH13] propose this technique to deal with the lower bounds on the number of critical periods by using the equivalent to the first-order Melnikov function for the period.

The main objective of this section is to present a method which allows to obtain high numbers of (local) critical periods with less computational effort, and to apply it to some low degrees n systems having a center at the origin and considering only one period annulus. The bifurcation technique uses the development (2.3) and usually each local oscillation is obtained from a perturbative parameter. To perform this criticality study, we present an approach which is equivalent to the use of the first nonidentically zero Melnikov function in the problem of limit cycles bifurcation, but adapted to the period function. We prove that the Taylor development of this first order function can be found from the linear terms of the corresponding period constants. Later, we consider families which are isochronous centers being perturbed inside the reversible centers class, and we prove our criticality results by finding the first-order Taylor developments of the period constants with respect to the perturbation parameters.

Using this technique we have been able to improve the lower bound of $\mathcal{C}_\ell(n)$ known so far for some even values of the degree n, as the following theorem states.

Theorem 2.29. *The number of local critical periods in the family of polynomial time-reversible centers of degree n is at least $\kappa(n)$, this is $\mathcal{C}_\ell(n) \geq \kappa(n)$, where*

n	4	6	8	10	12	14	16
$\kappa(n)$	10	22	37	57	80	106	136

To the best of our knowledge, the highest lower bound for $\mathcal{C}_\ell(4)$ is what we achieved in Section 2.3 and is also 10. Observe that we do not improve this number, but we will see that here we obtain the same lower bound for the local criticality with a much simpler method both in conceptual and computational terms. As in the previous section, the way to prove this result is the local bifurcation of zeros of the first derivative of the period function (2.3), by finding the highest value for the multiplicity of a zero of T' for each degree n. More concretely, this is done by perturbing inside the time-reversible class some isochronous centers with homogeneous polynomial nonlinearities.

Let us first introduce how we write the families that we will use throughout the section. Assume that system (2.2) has an isochronous center, and add a perturbation starting with quadratic terms such that in (z, w) coordinates is written as

$$\begin{cases} \dot{z} = \mathcal{Z}(z,w) + \sum\limits_{l+m\geq 2}^{\nu} b_{lm} z^l w^m, \\ \dot{w} = \overline{\mathcal{Z}}(z,w) + \sum\limits_{l+m\geq 2}^{\nu} \bar{b}_{lm} z^m w^l, \end{cases} \tag{2.57}$$

where ν is the perturbation degree and $b_{lm} \in \mathbb{C}$ are perturbative parameters. In general, we will have perturbations such that $\nu = n$, this meaning that the perturbation degree is actually the system degree. However, in Subsection 2.4.3 we will consider some cases in which $\nu = n + 1$, as we will justify later.

We are interested in reversible perturbations so that the center property is kept. As Proposition 2.14 states, a perturbation of the form (2.57) is reversible if it satisfies $\bar{b}_{lm} = -b_{lm}$, or equivalently, it is purely imaginary and $b_{lm} = \mathrm{i}\, c_{lm}$ for some $c_{lm} \in \mathbb{R}$. Therefore, throughout this section we will deal with perturbed systems of the form

$$\begin{cases} \dot{z} = \mathcal{Z}(z,w) + \mathrm{i} \sum\limits_{l+m\geq 2}^{\nu} c_{lm} z^l w^m \\ \dot{w} = \overline{\mathcal{Z}}(z,w) - \mathrm{i} \sum\limits_{l+m\geq 2}^{\nu} c_{lm} z^m w^l, \end{cases} \tag{2.58}$$

with $c_{lm} \in \mathbb{R}$, which still have a center at the origin despite the perturbation and being $\dot{z} = \mathcal{Z}(z,w)$ a planar polynomial system of degree n having an nondegenerate isochronous center at the origin.

This whole section is devoted to prove Theorem 2.29, and has the following structure. Subsection 2.4.1 proves the main result that will be used to obtain the criticality results from Theorem 2.29, and which is equivalent to Melnikov functions when dealing with cyclicity. Later, Subsection 2.4.2 characterizes some isochronous centers of 6th degree and general even degree n. Finally, these isochronous centers are used in Subsection 2.4.3 to show the bifurcation of critical periods which proves Theorem 2.29. We remark that all the computations have been done using Maple.

2.4.1 Melnikov technique for the period function

The method we propose to obtain lower bounds on the number of critical periods is based on the equivalence of a first Melnikov type function for the period of the perturbation of an isochronous system and the linear developments with respect to the perturbation parameters of the period constants also near the same isochronous system. This is our main technique and is presented in the following result.

Theorem 2.30. *Let $\lambda = (a_{20}, a_{11}, \ldots, b_{20}, b_{11}, \ldots) \in \mathbb{R}^{(n^2+3n-4)/2}$ be perturbative parameters such that the next polynomial perturbations of a system of differential equations in the plane of the form (2.1),*

$$\begin{cases} \dot{x} = -y + X_c(x,y) + \sum\limits_{k+l=2}^{n} a_{kl}\, x^k y^l, \\ \dot{y} = x + Y_c(x,y) + \sum\limits_{k+l=2}^{n} b_{kl}\, x^k y^l, \end{cases} \tag{2.59}$$

and

$$\begin{cases} \dot{x} = -y + X_c(x,y) + \varepsilon \sum\limits_{k+l=2}^{n} a_{kl}\, x^k y^l, \\ \dot{y} = x + Y_c(x,y) + \varepsilon \sum\limits_{k+l=2}^{n} b_{kl}\, x^k y^l, \end{cases} \tag{2.60}$$

have a center at the origin which is isochronous respectively when $\lambda = 0$ and $\varepsilon = 0$. Let us denote by $T_k^{(1)}(\lambda)$ the first-order truncation of the Taylor series, with respect to λ, of the period constants $T_k(\lambda)$ of (2.59). If we write the Taylor series in ε of the period function of system (2.60) as

$$T(\rho, \lambda, \varepsilon) = 2\pi + \sum_{k=1}^{\infty} \mathcal{T}_k(\rho, \lambda)\varepsilon^k, \tag{2.61}$$

then, for ρ small enough, the first averaging function $\mathcal{T}_1(\rho, \lambda)$ writes as

$$\mathcal{T}_1(\rho, \lambda) = \sum_{k=1}^{N} T_k^{(1)}(\lambda) \left(1 + \sum_{j=1}^{\infty} \alpha_{kj0}\rho^j\right) \rho^{2k}, \tag{2.62}$$

with the Bautin ideal[2] satisfying $\langle T_1, \ldots, T_N, \ldots \rangle = \langle T_1, \ldots, T_N \rangle$.

We notice that, by the isochronicity property of the unperturbed system, $T_k^{(1)}(0) = 0$. Let us briefly interpret what this theorem is expressing. If we consider a privileged perturbative parameter ε such that the perturbed system is written as (2.60), by taking its period function (2.61) we can express the power series of $\mathcal{T}_1(\rho, \lambda)$ with respect to ρ and rewrite (2.61) as

$$T(\rho, \lambda, \varepsilon) = 2\pi + \left(\sum_{j=1}^{\infty} \theta_j(\lambda)\rho^j\right) \varepsilon + \sum_{k=2}^{\infty} \mathcal{T}_k(\rho, \lambda)\varepsilon^k, \tag{2.63}$$

for some functions $\theta_j(\lambda)$. This idea is equivalent to the Melnikov method when studying limit cycles. Theorem 2.30 states that the first-order coefficients in $\mathcal{T}_1(\rho, \lambda)$ from (2.63), these are functions $\theta_j(\lambda)$, are exactly the first-order truncation of the Taylor series of the period constants in (2.15) with respect to λ. This is inspired by [GT20], where the authors prove the equivalence between the first-order truncation of the Lyapunov constants and the first Melnikov function for limit cycles.

The utility of the above result in terms of finding a high number of critical periods lies in its following corollary.

Corollary 2.31. *Let us consider the $m \times l$ matrix G_m whose element in position (i, j) is the coefficient of the jth perturbative parameter in the first-order expression of the ith period constant of a perturbed system (2.59), so G_m is the matrix of coefficients of the first-order truncation of the Taylor series of the first m period constants. If the rank of G_m is N then at least $N - 1$ critical periods bifurcate from the origin of the center (2.59) or (2.60).*

[2] The notion of Bautin ideal for period constants is defined in analogy to the Bautin ideal for Lyapunov constants

Observe that the size of matrix G_m is determined by the number of considered period constants m and the number of perturbative parameters l.

Before the proof of Theorem 2.30 and its Corollary 2.31, we will start by illustrating the equivalence between both methods with a particular example. Consider the next polynomial system with homogeneous nonlinearities of degree 6 written in polar coordinates as

$$\begin{cases} \dfrac{dr}{dt} = r^6 \left(\sin\varphi + 2\sin(3\varphi)\right) =: r^6 U(\varphi), \\ \dfrac{d\varphi}{dt} = 1 - \dfrac{5}{3} r^5 \left(3\cos\varphi + 2\cos(3\varphi)\right) =: 1 + r^5 V(\varphi), \end{cases} \tag{2.64}$$

which has the form (2.7). It can be shown that this system has a reversible isochronous center at the origin by using that it has a rational first integral, written in Cartesian coordinates as

$$H(x,y) = \frac{(x^2+y^2)^5}{1 - \frac{50}{3}x^5 - \frac{20}{3}x^3y^2 + 10xy^4},$$

and applying Proposition 2.32. Here, the time-reversibility condition is moved to the invariance with respect to the change $(r,\varphi,t) \mapsto (r,-\varphi,-t)$. First we consider a change of variables $\hat{r} := r^5$ to simplify notation, then $d\hat{r}/dt = (d\hat{r}/dr)\cdot(dr/dt) = 5r^4 dr/dt$. Therefore, system (2.64) becomes

$$\begin{cases} \dfrac{d\hat{r}}{dt} = 5\hat{r}^2\, U(\varphi), \\ \dfrac{d\varphi}{dt} = 1 + \hat{r}\, V(\varphi). \end{cases}$$

Now we add a time-reversible polynomial perturbation with parameters $\lambda = (\lambda_1,\ldots,\lambda_7) \in \mathbb{R}^7$ also corresponding to homogeneous nonlinearities of degree 6, and having the form

$$\begin{cases} \dfrac{d\hat{r}}{dt} = 5\hat{r}^2 \left(U(\varphi) + \widetilde{U}(\varphi,\lambda)\right), \\ \dfrac{d\varphi}{dt} = 1 + \hat{r}\left(V(\varphi) + \widetilde{V}(\varphi,\lambda)\right), \end{cases} \tag{2.65}$$

where

$$\begin{aligned} \widetilde{U}(\varphi,\lambda) :=& \lambda_1 \sin\varphi + \lambda_2 \sin(3\varphi) + \lambda_3 \sin(5\varphi) + \lambda_4 \sin(7\varphi), \\ \widetilde{V}(\varphi,\lambda) :=& -(5\lambda_1 - \lambda_5)\cos\varphi - \frac{1}{3}(5\lambda_2 - 3\lambda_6)\cos(3\varphi) + \lambda_7\cos(5\varphi) + \lambda_1\cos(7\varphi). \end{aligned}$$

Let us propose a truncated solution up to fourth-order as in (2.9), this is

$$\hat{r} = \rho + A_2(\varphi)\rho^2 + A_3(\varphi)\rho^3 + A_4(\varphi)\rho^4.$$

By using (2.11) and (2.12), we obtain that $A_2(\varphi) = A_3(\varphi) = A_4(\varphi) = 0$. Now applying formula (2.14) to first-order terms, we finally write the linear parts with respect to λ of the first and second period constants as

$$\begin{aligned} T_1^{(1)} &= -\frac{5}{2}\lambda_5 - \frac{5}{3}\lambda_6, \\ T_2^{(1)} &= \frac{625}{27}\lambda_3 - \frac{1000}{63}\lambda_4 - \frac{3125}{6}\lambda_5 - \frac{3250}{9}\lambda_6 + \frac{625}{27}\lambda_7. \end{aligned} \tag{2.66}$$

To exemplify the second method we will consider system (2.65) with a privileged perturbative parameter ε, this is

$$\begin{cases} \dfrac{d\hat{r}}{dt} = 5\hat{r}^2\left(U(\varphi) + \varepsilon\widetilde{U}(\varphi,\lambda)\right), \\ \dfrac{d\varphi}{dt} = 1 + \hat{r}\left(V(\varphi) + \varepsilon\widetilde{V}(\varphi,\lambda)\right). \end{cases}$$

In this case we can express the period function as a power series in ε (see equation (2.63)), so

$$T(\rho,\lambda,\varepsilon) = 2\pi + \mathcal{T}_1(\rho,\lambda)\varepsilon + \sum_{k=2}^{\infty}\mathcal{T}_k(\rho,\lambda)\varepsilon^k,$$

and then $\mathcal{T}_1(\rho,\lambda) = \sum_{j=1}^{\infty}\theta_j(\lambda)\rho^j$. Finally, after performing the calculations we check that the two first nonzero coefficients $\theta_j(\lambda)$ are the linear parts of period constants obtained in (2.66).

Followingly we present the proofs of Theorem 2.30 and Corollary 2.31.

Proof of Theorem 2.30. Consider the series expansions of the perturbative parameters λ in terms of a privileged parameter ε,

$$\lambda_l(\varepsilon) = \sum_{j=0}^{\infty}\lambda_{jl}\varepsilon^j, \tag{2.67}$$

we have that the period function writes

$$T(\rho,\lambda) = \sum_{k=1}^{N} T_k(\lambda)\rho^{2k}\left(1 + \sum_{j=1}^{\infty}\alpha_{kj}(\lambda)\rho^j\right),$$

with α_{kj} vanishing at zero in the variables λ. We can now consider the power series expansion in ε of the period function

$$T(\rho,\varepsilon) = \sum_{k=1}^{\infty} \tau_k(\rho)\varepsilon^k = \sum_{k=1}^{\infty} \frac{1}{k!}\left(\left.\frac{\partial^k T(\rho,\varepsilon)}{\partial \varepsilon^k}\right|_{\varepsilon=0} \right) \varepsilon^k.$$

Notice that the series representation of the period function is only local, but the Global Bifurcation Lemma, see [CJ89], implies that the coefficients

$$\tau_k(\rho) = \frac{1}{k!} \left.\frac{\partial^k T(\rho,\varepsilon)}{\partial \varepsilon^k}\right|_{\varepsilon=0}$$

are defined and analytic in the period annulus of the center.

Considering the power series expansions (2.67), we have that for each k

$$T_k(\lambda(\varepsilon)) = \sum_{m=1}^{\infty} T_k^{(m)}(\lambda(\varepsilon))\varepsilon^m,$$

and

$$\alpha_{kj}(\lambda(\varepsilon)) = \sum_{i=0}^{\infty} \alpha_{kji}\varepsilon^i.$$

Rearranging the series for ε and ρ small enough it follows that

$$T(\rho,\varepsilon) = \sum_{k=1}^{N} \sum_{m=1}^{\infty} T_k^{(m)} \varepsilon^m \left(1 + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \alpha_{kji}\rho^j \varepsilon^i \right) \rho^{2k}.$$

Hence, choosing the coefficient of ε in the equation above –this is $m = 1$ and $i = 0$–, for ρ small we have the expression

$$\mathcal{T}_1(\rho) = \tau_1(\rho) = \sum_{k=1}^{N} T_k^{(1)} \left(1 + \sum_{j=1}^{\infty} \alpha_{kj0}\rho^j \right) \rho^{2k},$$

where all $T_k^{(m)}$ depend on λ and, consequently, the first-order truncation of $T_k^{(1)}$ are linear combinations of the original parameters λ in the statement. □

Proof of Corollary 2.31. If the rank of G_m is N, one can rearrange the terms of the linear parts $T_k^{(1)}$ from expression (2.62) in Theorem 2.30 according to the linear relationship between the parameters, and by applying Weierstrass Preparation Theorem (Theorem 1.19), this implies that $N-1$ critical periods can bifurcate from the origin and the statement follows. □

2.4.2 Isochronicity of some even degree systems

In this section we will present some results about the isochronicity of some even degree systems. As we have already mentioned, the studied polynomial systems have homogeneous nonlinearities of degree n. We will consider systems with even n, and the reason is as follows. It is a well-known fact that, as we saw for Lyapunov constants in Theorem 1.27, given a parametric family of systems, its period constants are polynomials whose variables are the parameters of the system and having a particular structure based on their weight and quasi-degree –for more details see for instance [Cim+97; GGM99]. It can be checked that this structure implies that, when the nonlinearities are homogeneous of degree n, some of the corresponding period constants are identically zero. When n is even and $k = i(n-1)$, for $i \in \mathbb{N}$, we obtain $T_k \not\equiv 0$, while when n is odd this property holds for $k = i(n-1)/2$. Therefore, the computational effort is lower using only homogeneous nonlinearities when the objective is to get systems having at the origin a point with the highest multiplicity value for the period function. Clearly, for even degrees we can go further with less computations and this allows us to obtain higher criticality. This fact was already observed in the analogous problem of studying cyclicity using Lyapunov constants –for example, Giné took advantage of it in [Gin12a; Gin12b].

We will start with the following proposition that characterizes a class of systems of even degree n, whose proof is a generalization of a reasoning inspired by reading [CS99].

Proposition 2.32. *Let $n > 1$ be a natural number and $p(x,y)$ a homogeneous polynomial of degree $n-1$ such that $p(x,-y) \equiv p(x,y)$. The system*

$$\begin{cases} \dot{x} = -y + X_n(x,y), \\ \dot{y} = x + Y_n(x,y), \end{cases} \tag{2.68}$$

with $X_n(x,y)$ and $Y_n(x,y)$ homogeneous polynomials of degree n, associated to the first integral

$$H(x,y) = \frac{(x^2+y^2)^{n-1}}{1+p(x,y)}, \tag{2.69}$$

has a time-reversible (with respect to the x-axis) isochronous center at the origin.

Proof. System (2.68) has a center at the origin because the first integral (2.69) is well defined and, moreover, it is time-reversible since also the first integral is so. To see the isochronicity let us first write the first integral (2.69) in polar coordinates,

$$H(r,\varphi) = \frac{r^{2(n-1)}}{1+r^{n-1}\Phi(\varphi)}, \tag{2.70}$$

where $\Phi(\varphi)$ is a trigonometric polynomial in φ.

Due to the reversible linear plus homogeneous structure and the parity of the polynomials being n even, $\Phi(\varphi) = \sum_{k=1}^{n/2} a_k \cos((2k-1)\varphi)$. Here we have used the well-known fact that $\cos(m\varphi) = f_m(\cos\varphi)$ and $\sin((m+1)\varphi) = g_m(\cos\varphi)\sin\varphi$, where f_m and g_m are the mth degree Chebyshev polynomials of the first and second kind, respectively (see [Riv90] for more information on this topic).

Let us see that this function $\Phi(\varphi)$ is actually directly related to the expression of system (2.68) in polar coordinates. As (2.70) is a first integral, it satisfies $\frac{\partial H}{\partial r}\dot{r} + \frac{\partial H}{\partial \varphi}\dot{\varphi} = 0$, so

$$\frac{\dot{r}}{\dot{\varphi}} = -\frac{\frac{\partial H}{\partial \varphi}}{\frac{\partial H}{\partial r}} = \frac{r^n \Phi'(\varphi)}{(n-1)\left(2 + r^{n-1}\Phi(\varphi)\right)}.$$

Therefore, system (2.68) is written in polar coordinates as

$$\begin{cases} \dot{r} = r^n \dfrac{\Phi'(\varphi)}{2(n-1)}, \\ \dot{\varphi} = 1 + r^{n-1}\dfrac{\Phi(\varphi)}{2}. \end{cases} \tag{2.71}$$

From the level curve $H(r,\varphi) = 1/h$, where h is an arbitrary nonzero real number, we obtain $hr^{2(n-1)} = 1 + r^{n-1}\Phi(\varphi)$, and solving this second degree equation in r^{n-1} we get

$$r^{n-1} = \frac{\Phi(\varphi) \pm \sqrt{\Phi^2(\varphi) + 4h}}{2h}. \tag{2.72}$$

From the second differential equation in (2.71) and using (2.72), we obtain that the period function of the system is

$$\begin{aligned} T(r) &= \int_0^{2\pi} \frac{d\varphi}{1 + r^{n-1}\frac{\Phi(\varphi)}{2}} = \int_0^{2\pi} \left(1 \pm \frac{\Phi(\varphi)}{\sqrt{\Phi^2(\varphi) + 4h}}\right) d\varphi \\ &= 2\pi \pm \int_0^{2\pi} \frac{\Phi(\varphi)}{\sqrt{\Phi^2(\varphi) + 4h}} d\varphi. \end{aligned}$$

Finally, as $\Phi(\varphi)$ is a sum of terms of the form $\cos((2k-1)\varphi)$, it is easy to see that the last integral is zero by making the change $\theta = \varphi + \pi$ and using the periodicity of $\Phi(\varphi)$. Therefore, the period function is constant and the statement follows. □

The next results prove the isochronicity of some 6th degree polynomial systems, mainly by finding linearizations of them.

Proposition 2.33. *The time-reversible system (with respect to the x-axis) with polynomial homogeneous nonlinearities of 6th degree*

$$\begin{cases} \dot{x} = -y + \dfrac{32}{3}x^5y + \dfrac{80}{9}x^3y^3 - \dfrac{2}{3}xy^5, \\ \dot{y} = x - \dfrac{80}{9}x^6 - \dfrac{8}{3}x^4y^2 + \dfrac{55}{9}x^2y^4 + y^6, \end{cases} \tag{2.73}$$

has an isochronous center at the origin.

Proof. The system has a center due to the fact that it is time-reversible with respect to the x-axis, since it remains invariant under the change $(x, y, t) \mapsto (x, -y, -t)$. The statement follows just checking that the system has a Darboux linearization (in complex coordinates) of the form (2.5),

$$\chi(z, w) = z\, \chi_1^{-1/5}\, \chi_2^{4/5}\, \chi_3^{1/10}\, \chi_4^{-3/10},$$

with

$$\begin{aligned}
\chi_1(z,w) &= 1 - \frac{5}{144}z^5 - \frac{35}{36}z^4w - \frac{55}{8}z^3w^2 - \frac{35}{36}z^2w^3 - \frac{5}{144}zw^4, \\
\chi_2(z,w) &= 1 - \frac{5}{144}z^4w - \frac{35}{36}z^3w^2 - \frac{55}{8}z^2w^3 - \frac{35}{36}zw^4 - \frac{5}{144}w^5, \\
\chi_3(z,w) &= 1 - \frac{40}{27}z^4w - \frac{40}{9}z^3w^2 - \frac{40}{9}z^2w^3 - \frac{40}{27}zw^4, \\
\chi_4(z,w) &= 1 + \frac{125}{7776}z^{12}w^3 + \frac{2375}{2592}z^{11}w^4 + \frac{12875}{648}z^{10}w^5 + \frac{128375}{648}z^9w^6 \\
&\quad + \frac{1081625}{1296}z^8w^7 + \frac{1081625}{1296}z^7w^8 + \frac{128375}{648}z^6w^9 + \frac{12875}{648}z^5w^{10} \\
&\quad + \frac{2375}{2592}z^4w^{11} + \frac{125}{7776}z^3w^{12} + \frac{25}{72}z^8w^2 + \frac{325}{36}z^7w^3 + \frac{3575}{72}z^6w^4 \\
&\quad - \frac{2125}{18}z^5w^5 + \frac{3575}{72}z^4w^6 + \frac{325}{36}z^3w^7 + \frac{25}{72}z^2w^8 - \frac{5}{3}z^4w - \frac{35}{3}z^3w^2 \\
&\quad - \frac{35}{3}z^2w^3 - \frac{5}{3}zw^4.
\end{aligned}$$

□

Proposition 2.34. *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3 \in \mathbb{R}[x, y]$ be nonidentically zero homogeneous polynomials with degrees $5, 5$, and 10, respectively, such that $\mathcal{H}_i(x, -y) \equiv \mathcal{H}_i(x, y)$, for $i = 1, 2, 3$. A time-reversible polynomial system (with respect to the x-axis) of degree $n = 6$ of the form (2.1) having an isochronous center at the origin with an inverse integrating factor of the form $V(x, y) = (x^2 + y^2)\, U_1(x, y)\, U_2(x, y)$, being*

$U_1(x,y) = 1 + \mathcal{H}_1(x,y)$ and $U_2(x,y) = 1 + \mathcal{H}_2(x,y) + \mathcal{H}_3(x,y)$, writes as

$$\begin{cases} \dot{x} = -y + \frac{6}{5}x^5y - \frac{4}{5}x^3y^3, \\ \dot{y} = x - x^6 + \frac{21}{5}x^4y^2 + \frac{16}{5}x^2y^4, \end{cases} \tag{2.74}$$

or

$$\begin{cases} \dot{x} = -y + \frac{6}{5}x^5y - \frac{6}{5}xy^5, \\ \dot{y} = x - x^6 + \frac{6}{5}x^4y^2 + 3x^2y^4 + \frac{4}{5}y^6. \end{cases} \tag{2.75}$$

Proof. We notice that $U_1(x,y) = 0$ and $U_2(x,y) = 0$ are two algebraic invariant curves which, as well as the inverse integrating factor, are invariant with respect to the change $(x,y) \mapsto (x,-y)$. Due to the reversibility, the considered systems take the form

$$\begin{cases} \dot{x} = -y + p_1x^5y + p_2x^3y^3 + p_3xy^5 =: P(x,y), \\ \dot{y} = x + q_1x^6 + q_2x^4y^2 + q_3x^2y^4 + q_4y^6 =: Q(x,y), \end{cases} \tag{2.76}$$

and the invariant curves write as

$$\begin{aligned} U_1(x,y) =& 1 + a_1x^5 + a_2x^3y^2 + a_3xy^4, \\ U_2(x,y) =& 1 + b_1x^5 + b_2x^3y^2 + b_3xy^4 \\ &+ c_1x^{10} + c_2x^8y^2 + c_3x^6y^4 + c_4x^4y^6 + c_5x^2y^8 + c_6y^{10}. \end{aligned}$$

From the statement it is clear that $P, Q, U_1, U_2 \in \mathbb{R}[x,y]$.

As V is actually an inverse integrating factor of system (2.76), the relation (1.13) must be satisfied. Now equating the corresponding coefficients we obtain a system of polynomial equations, which can be solved by means of a computer algebra system. Among the obtained solutions are only interested in those which satisfy that $U_1(x,y) \neq 0$, $U_2(x,y) \neq 0$, and $r' \neq 0$, were r is the radial component in the usual polar coordinates. The latter condition is imposed in order to avoid trivial cases, as the fact that $r' = 0$ implies that the system can be rescaled to the canonical linear center (2.4).

The next step is to test those solutions and check if they could correspond to isochronous centers by computing some period constants. We must reject those which give period constants that cannot be vanished at the same time, since this means that they are not isochronous. Finally, we have only two solutions which are candidates to be isochronous, and correspond to systems (2.74) and (2.75). To prove the isochronicity of such systems we will propose a linearization in complex coordinates and a transversal commuting system for each of them, and then apply Theorems 2.1 and 2.6.

The functions U_1, U_2 for systems (2.74) and (2.75) that we have obtained are respectively

$$U_1^A(x,y) = 1 - \frac{4}{3}x^5 - \frac{4}{3}x^3y^2,$$
$$U_2^A(x,y) = 1 - 2x^5 + x^{10} + x^8y^2,$$

and

$$U_1^B(x,y) = 1 - 2x^5 - 4x^3y^2 - 2xy^4,$$
$$U_2^B(x,y) = 1 - 2x^5 - 2x^3y^2 + x^{10} + 3x^8y^2 + 3x^6y^4 + x^4y^6.$$

The corresponding (complex) linearizations are $\chi^A(z,w) = z\,\chi_1^A\,\chi_2^A$ with

$$\chi_1^A(z,w) = 1 - \frac{1}{6}zw^4 - \frac{1}{2}z^2w^3 - \frac{1}{2}z^3w^2 - \frac{1}{6}z^4w,$$
$$\chi_2^A(z,w) = 1 + \frac{1}{4}w^5 + \frac{7}{16}zw^4 - \frac{1}{4}z^2w^3 - \frac{7}{8}z^3w^2 - \frac{1}{2}z^4w - \frac{1}{16}z^5,$$

and $\chi^B(z,w) = z\,\chi_1^B\,\chi_2^B$ with

$$\chi_1^B(z,w) = 1 - z^3w^2 - z^2w^3,$$
$$\chi_2^B(z,w) = 1 - \frac{1}{2}z^4w - \frac{5}{4}z^3w^2 + \frac{3}{4}zw^4.$$

For the sake of completeness in the isochronicity characterization we have also found the (real) transversal commuting systems

$$\begin{cases} \dot{x} = x\left(1 - x^5 + x^3y^2\right) U_1^A(x,y), \\ \dot{y} = y\left(1 - 6x^5 - 4x^3y^2\right) U_1^A(x,y), \end{cases}$$

and

$$\begin{cases} \dot{x} = x\left(1 - x^5 + 2x^3y^2 + 3xy^4\right) U_1^B(x,y), \\ \dot{y} = y\left(1 - 6x^5 - 8x^3y^2 - 2xy^4\right) U_1^B(x,y), \end{cases}$$

associated to (2.74) and (2.75), respectively. We notice that the second functions U_2^A and U_2^B do not appear in the above transversal systems. □

2.4.3 Critical periods unfolding

In this subsection we will apply Corollary 2.31 to obtain lower bounds on the number of critical periods for some polynomial systems to prove Theorem 2.29.

Before that, we will introduce a notation that will be useful throughout the subsection.

Consider a system (2.59) and let $\mathbf{r}_\ell = (r_1, \ldots, r_\ell)$ be the sequence of ranks of the matrices obtained from the first-order truncated Taylor series of the first ℓ ordered period constants with respect to the parameters λ, being $r_k = \operatorname{Rank} G_k$ and the matrix G_k as defined in Corollary 2.31 from the coefficients of the linear homogeneous polynomials $T_1^{(1)}(\lambda), \ldots, T_k^{(1)}(\lambda)$. In the case that a consecutive subsequence of length m of ranks takes a constant value $\widetilde{r}$ ($r_k = r_{k+1} = \cdots = r_{k+m-1} = \widetilde{r}$ for some $k, m \in \mathbb{N}$) we will substitute the whole subsequence $r_k, r_{k+1}, \ldots, r_{k+m-1}$ by $\widetilde{r}_m$.

4th degree systems

Let us consider the following systems with quartic homogeneous nonlinearities,

$$(\dot{x}, \dot{y}) = \left(-y + (a+4b)x^3y + axy^3, x + (a+4b)x^2y^2 + ay^4\right), \tag{2.77}$$

$$(\dot{x}, \dot{y}) = \left(-y - 7x^3y + 5xy^3, x + 3x^4 - 10x^2y^2 - y^4\right), \tag{2.78}$$

$$(\dot{x}, \dot{y}) = \left(-y - 4x^3y + 2xy^3, x + 3x^4 - 7x^2y^2 - 4y^4\right), \tag{2.79}$$

$$(\dot{x}, \dot{y}) = \left(-y + 4x^3y + 10xy^3, x - 5x^2y^2 + y^4\right), \tag{2.80}$$

$$\begin{aligned}(\dot{x}, \dot{y}) = \Big(&-y - (4a+2b)x^3y - (4a-4b)xy^3, \\ &x + ax^4 + (2a-5b)x^2y^2 - (a-b)y^4\Big),\end{aligned} \tag{2.81}$$

$$(\dot{x}, \dot{y}) = \left(-y + x^3y + xy^3, x\right), \tag{2.82}$$

$$\begin{aligned}(\dot{x}, \dot{y}) = \Big(&-y + 100(a+3)^2x^3y + 4(5a-81)(5a-9)xy^3, \\ &x - 75(a+3)^2x^4 - 10(a+3)(5a-201)x^2y^2 + (5a-9)^2y^4\Big),\end{aligned} \tag{2.83}$$

for $a, b \in \mathbb{R}$. All these systems are reversible with respect to the x-axis, as they are invariant under the change $(x, y, t) \mapsto (x, -y, -t)$. The isochronicity of these systems is studied in [CS99], where the authors make an attempt to characterize all the isochronous centers of a linear center perturbed with homogeneous polynomials of degree 4. They conclude that the first 6 systems are all the possibilities, but they do not manage to prove the isochronicity of the latter. The above ordered list of systems corresponds to the ones in [CS99] labeled as H4$_i$, for $i = 1, \ldots, 7$. Notice that we have rescaled the systems for the sake of simplicity and switched their symmetry so that they are reversible with respect to the x-axis as in the rest of this section.

Let us observe that we are presenting a technique for the perturbation of isochronous centers and the isochronicity of (2.83) has not been proved. Despite this, if it was not isochronous the method would be valid anyway, since we could ask for the vanishing only of the first k period constants for a certain k and the approach would work anyway if we are not dealing with higher period constants.

In the following proposition we give lower bounds for the criticality of systems (2.77)–(2.83).

Proposition 2.35. *For each system* (2.77)–(2.83)*, let us consider a quartic perturbation inside the reversible class which starts with quadratic terms as in* (2.58)*. Then the perturbation of systems* (2.78)*,* (2.79)*, and* (2.80) *unfold at least* 10 *critical periods, while the perturbation of systems* (2.77)*,* (2.82)*,* (2.81)*, and* (2.83) *unfold at least* $7, 8, 9$*, and* 9 *local critical periods, respectively.*

Proof. Firstly, for each system we will find the linear part with respect to the perturbative parameters of the first 20 period constants perturbed in the reversible polynomial class detailed in the statement. Secondly, we will evaluate the corresponding sequence of ranks $\mathbf{r}_{20}$. Finally, the statement will follow applying Corollary 2.31. We notice that each lower bound will be the maximum achieved rank minus 1.

Straightforward computations show that for systems (2.77), (2.82), (2.81) we have

$$\mathbf{r}_{20} = (1,2,3,4,5,6,7_2,8_{12}),$$
$$\mathbf{r}_{20} = (1,2,3,4,5,6_2,7,8_3,9_9),$$
$$\mathbf{r}_{20} = (1,2,3,4,5,6,7,8,9,10_{11}),$$

respectively, so at least $7, 8$, and 9 local critical periods bifurcate from the origin, respectively. For system (2.83) we obtain the same sequence as for (2.81) and, consequently, the same number of local critical periods. For all three systems (2.78), (2.79), and (2.80) we have obtained the best result for these families with homogeneous nonlinearities because the sequence of ranks is

$$\mathbf{r}_{20} = (1,2,3,4,5,6,7,8,9,10_2,11_9).$$

Hence, at least 10 local critical periods bifurcate from the origin and the proof is finished. □

Notice that we have computed a few extra period constants to check that, in some sense, the sequence of ranks stabilizes and that no extra oscillation of the period function will easily appear by applying this first-order bifurcation mechanism. We remark that the 10 local critical periods obtained above prove the part of Theorem 2.29 corresponding to degree 4.

6th degree systems

In this subsection we will study lower bounds for the local criticality of the 6th degree isochronous centers from Subsection 2.4.2 using the tools provided by Theorem 2.30 and Corollary 2.31, in a similar way to the previous quartic case. The result is as follows.

Proposition 2.36. *For each system* (2.73)–(2.75)*, let us consider a sextic perturbation inside the reversible class which starts with quadratic terms as in* (2.58)*. Then the perturbation of system* (2.73) *unfolds at least* 22 *critical periods, while the perturbation of systems* (2.74) *and* (2.75) *unfolds at least* 20 *local critical periods.*

Proof. The proof follows analogously as we have done in Proposition 2.35, the only difference being the corresponding sequences of ranks. The described perturbation provides

$$\mathbf{r}_{35} = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17_2, 18, 19_2, 20_2, 21, 22_2, 23_9),$$

for (2.73) and

$$\mathbf{r}_{35} = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17_2, 18, 19_2, 20_3, 21_{11}),$$

for both systems (2.74) and (2.75). Consequently, the respective lower bounds are the ones detailed in the statement. □

According to our previous result in Section 2.3, the highest achieved lower bound for $\mathcal{C}_\ell(6)$ is 20. Notice that in Proposition 2.36 we have obtained the same lower bound with systems (2.74) and (2.75) but with a more efficient technique and, moreover, we have improved it with system (2.73). Actually, the fact that we obtain at least 22 local critical periods for system (2.73) proves $\mathcal{C}_\ell(6) \geq 22$ in Theorem 2.29.

nth degree systems

Here we will study the bifurcation of local critical periods for nth degree isochronous systems, provided by Proposition 2.32, for several values of n. As we have already mentioned, systems with homogeneous nonlinearities and even degree will usually have higher criticality than those with odd degree, so we will take advantage of this fact to also study odd degree systems by perturbing systems of even degree $n-1$ with an odd nth degree perturbation.

Let us start with the following genericity criticality result for 4th and 6th degrees.

Proposition 2.37. *Isochronous systems* (2.68) *of degrees $n = 4$ and $n = 6$ with a first integral of the form* (2.69)*, when they are perturbed in the class of reversible polynomials of degree n, generically unfold 9 and 21 local critical periods, respectively.*

Proof. For the case $n = 4$ we have a first integral

$$H_4(x,y) = \frac{(x^2+y^2)^3}{1+ax^3+bxy^2},$$

with $a, b \in \mathbb{R}$, and the corresponding reversible isochronous system is

$$\begin{cases} \dot{x} = -y - \left(a - \frac{1}{3}b\right) x^3 y - \frac{2}{3}bxy^3, \\ \dot{y} = x + \frac{1}{2}ax^4 - \left(\frac{1}{2}a - \frac{5}{6}b\right) x^2y^2 - \frac{1}{6}by^4. \end{cases}$$

Now if we change to complex coordinates and add a quartic reversible perturbation as in (2.58), we can find the first-order developments of the first 10 period constants and compute their determinant with respect to the perturbative parameters $c_{02}, c_{03}, c_{04}, c_{11}, c_{12}, c_{13}, c_{20}, c_{21}, c_{22}, c_{30}$, which after being rescaled via a multiplicative constant is

$$\begin{aligned}(-3045a^5 - 17535a^4b - 19362a^3b^2 - 5166a^2b^3 + 1975ab^4 + 125b^5)(-42735a^5 \\ -126049a^4b - 6974a^3b^2 + 35766a^2b^3 + 6909ab^4 + 475b^5)(-295507521a^7 \\ -165909573a^6b + 517786803a^5b^2 + 19400559a^4b^3 - 132219763a^3b^4 \\ -14086623a^2b^5 + 4613697ab^6 + 320885b^7)(a-b)^6(3a+b)^7.\end{aligned}$$

This determinant is nonzero except for a set of null measure. Therefore, generically we obtain rank 10 which means 9 local critical periods by using Corollary 2.31.

For the case $n = 6$, the first integral is

$$H_6(x,y) = \frac{(x^2+y^2)^5}{1+ax^5+bx^3y^2+cxy^4},$$

with $a, b, c \in \mathbb{R}$, and the corresponding system is

$$\begin{cases} \dot{x} = -y - \left(a - \frac{1}{5}b\right) x^5 y - \left(\frac{4}{5}b - \frac{2}{5}c\right) x^3y^3 - \frac{3}{5}cxy^5, \\ \dot{y} = x + \frac{1}{2}ax^6 - \left(\frac{1}{2}a - \frac{7}{10}b\right) x^4y^2 - \left(\frac{3}{10}b - \frac{9}{10}c\right) x^2y^4 - \frac{1}{10}cy^6. \end{cases}$$

Analogously to the quartic case, we find the first-order developments of the first 22 period constants of this system after being perturbed and compute their determinant with respect to 22 perturbative parameters. The resulting determinant, which is a polynomial of degree 92 in (a, b, c), has such a long expression to be written here. We conclude that the rank is generically 22 and the finishes using again Corollary 2.31. □

We have also dealt with systems of higher even degrees $n = 8, 10, 12, 14$, and 16, as the following proposition states.

Proposition 2.38. *There exist isochronous reversible systems of degrees* $n = 8, 10, 12, 14$, *and* 16 *having a first integral of the form* (2.69) *which unfold at least* $37, 57, 80, 106$, *and* 136 *local critical periods under a polynomial reversible perturbation of degree* n, *respectively.*

Proof. Here we will consider perturbations of the form (2.58) being $\nu = n$, this is, both the isochronous system and the perturbation having the same degree n.

Due to Proposition 2.32, all the chosen systems have an isochronous reversible center at the origin, so we can follow the same idea and notation as in the proofs of Propositions 2.35 and 2.36. Hence, by evaluating the sequence of ranks $\mathbf{r}_\ell$ for a high enough number of period constants and applying Corollary 2.31, we deduce the lower bound for the criticality values detailed in the statement. We will only list the first integrals, the systems and the sequences of ranks.

For the case $n = 8$, we propose a first integral

$$H_8(x,y) = \frac{(x^2+y^2)^7}{1 + x^7 + 2x^5y^2 + 3x^3y^4 + 4xy^6},$$

and the corresponding system

$$\begin{cases} \dot{x} = -y - \dfrac{5}{7}x^7y - \dfrac{6}{7}x^5y^3 - \dfrac{3}{7}x^3y^5 - \dfrac{16}{7}xy^7, \\ \dot{y} = x + \dfrac{1}{2}x^8 + \dfrac{11}{14}x^6y^2 + \dfrac{23}{14}x^4y^4 + \dfrac{43}{14}x^2y^6 - \dfrac{2}{7}y^8. \end{cases}$$

In this case we have

$$\mathbf{r}_{64} = (1, 2, 3, \ldots, 36_3, 37_4, 38_{13}).$$

In the case $n = 10$ the first integral and system are, respectively,

$$H_{10}(x,y) = \frac{(x^2+y^2)^9}{1 + 8x^9 + 90x^7y^2 + \frac{6}{7}x^5y^4 + 5x^3y^6 - 54xy^8}$$

and

$$\begin{cases} \dot{x} = -y + 2x^9y - \dfrac{1676}{21}x^7y^3 + x^5y^5 - \dfrac{82}{3}x^3y^7 + 30xy^9, \\ \dot{y} = x + 4x^{10} + 51x^8y^2 - \dfrac{722}{21}x^6y^4 + \dfrac{55}{14}x^4y^6 - \dfrac{311}{6}x^2y^8 + 3y^{10}. \end{cases} \tag{2.84}$$

The first 100 period constants of this system provide the following sequence of ranks

$$\mathbf{r}_{100} = (1, 2, 3, \ldots, 56_4, 57_5, 58_{16}).$$

For $n = 12$ we take the first integral

$$H_{12}(x,y) = \frac{(x^2+y^2)^{11}}{1 + 4x^{11} + 99x^9y^2 + \frac{1023}{2}x^7y^4 + \frac{3047}{24}x^5y^6 + \frac{770}{3}x^3y^8 + 44xy^{10}}$$

corresponding to system

$$\begin{cases} \dot{x} = -y + 5x^{11}y + 3x^9y^3 - \dfrac{3071}{8}x^7y^5 + x^5y^7 - \dfrac{430}{3}x^3y^9 - 24xy^{11}, \\ \dot{y} = x + 2x^{12} + \dfrac{113}{2}x^{10}y^2 + \dfrac{1233}{4}x^8y^4 - \dfrac{3103}{48}x^6y^6 + \dfrac{3085}{16}x^4y^8 + 7x^2y^{10} - 2y^{12}, \end{cases} \tag{2.85}$$

which has

$$\mathbf{r}_{140} = (1, 2, 3, \ldots, 79_6, 80_5, 81_{21}).$$

For $n = 14$ the first integral and the corresponding system are, respectively,

$$H_{14}(x,y) = \frac{(x^2+y^2)^{13}}{1 + 10x^{13} + 221x^{11}y^2 + \frac{2691}{2}x^9y^4 - 3x^7y^6 - x^5y^8 - \frac{13}{8}x^3y^{10}}$$

and

$$\begin{cases} \dot{x} = -y + 7x^{13}y + 3x^{11}y^3 - \dfrac{29619}{26}x^9y^5 + 2x^7y^7 + \dfrac{7}{104}x^5y^9 + x^3y^{11}, \\ \dot{y} = x + 5x^{14} + \dfrac{245}{2}x^{12}y^2 + \dfrac{3145}{4}x^{10}y^4 - \dfrac{24333}{52}x^8y^6 - \dfrac{259}{208}x^4y^{10} + \dfrac{3}{16}x^2y^{12}. \end{cases} \tag{2.86}$$

The linear parts of the period constants of the above system provide the following sequence of ranks:

$$\mathbf{r}_{200} = (1, 2, 3, \ldots, 105_6, 106_7, 107_{39}).$$

Finally, for degree $n = 16$ we propose the first integral

$$H_{16}(x,y) = \frac{(x^2+y^2)^{15}}{1 - 2x^{15} + 45x^{13}y^2 + \frac{735}{2}x^{11}y^4 + \frac{3215}{2}x^9y^6 - \frac{40}{11}x^7y^8 - 2x^5y^{10} - \frac{5}{3}x^3y^{12}}$$

corresponding to system

$$\begin{cases} \dot{x} = & -y + 5x^{15}y + 7x^{13}y^3 + 3x^{11}y^5 - \dfrac{42470}{33}x^9y^7 + 2x^7y^9 + \dfrac{2}{3}x^5y^{11} + x^3y^{13}, \\ \dot{y} = & x - x^{16} + \dfrac{53}{2}x^{14}y^2 + \dfrac{853}{4}x^{12}y^4 + \dfrac{1981}{2}x^{10}y^6 - \dfrac{64025}{132}x^8y^8 - \dfrac{9}{11}x^6y^{10} \\ & -\dfrac{7}{6}x^4y^{12} + \dfrac{1}{6}x^2y^{14}. \end{cases} \tag{2.87}$$

The corresponding sequence of ranks for the linear parts of its period constants is

$$\mathbf{r}_{260} = (1, 2, 3, \ldots, 135_7, 136_8, 137_{44}).$$

□

The above result provides the proof of all the cases for even $n \geq 8$ from Theorem 2.29. We have not gone further in the degree because we have reached the computational limit of our computing machines. Inside the considered family having a first integral of the form (2.69), with the values found in this subsection for $n = 4, 6, 8, 10$ we have provided a good lower bound $\mathcal{C}_\ell(n) \geq (n^2 + 2n - 6)/2$, but the ones for $n = 12, 14, 16$ are lower than expected. Therefore, this general family is not good enough to get the previously conjectured value for $\mathcal{C}_\ell(n)$, although they are the best values obtained so far.

Finally, we will present a last result concerning systems with odd degrees.

Proposition 2.39. *There exist reversible isochronous systems of degrees* $n = 10, 12, 14,$ *and* 16 *having a first integral of the form* (2.69) *which unfold at least* 66, 91, 119, *and* 151 *critical periods, respectively, under a reversible perturbation of odd degree* $\nu = n + 1$.

Proof. Here we consider the even degree n reversible isochronous systems (2.84), (2.85), (2.86), and (2.87) in (2.58) but perturbed with reversible odd degree $\nu = n + 1$. The proof follows similarly to the previous results, so we only indicate the respective sequences of ranks for a high enough number of period constants in order to get the lower bounds written in the statement:

$$\begin{aligned} \mathbf{r}_{120} &= (1, 2, 3, \ldots, 65_9, 66_9, 67_{17}), \\ \mathbf{r}_{170} &= (1, 2, 3, \ldots, 90_{11}, 91_{11}, 92_{22}), \\ \mathbf{r}_{230} &= (1, 2, 3, \ldots, 118_{13}, 119_{13}, 120_{29}), \\ \mathbf{r}_{300} &= (1, 2, 3, \ldots, 150_{15}, 151_{15}, 152_{38}). \end{aligned}$$

□

This technique of using an even degree system with an odd degree perturbation to obtain higher criticality was already introduced in [GLY10], and has resulted in a higher criticality than directly perturbing all our best candidates with homogeneous nonlinearities of odd degree. It is worth noticing that we have also tested this approach with odd degrees $5, 7$, and 9, but we have not presented them here because they do not improve the local criticality we already obtained in Section 2.3. The bounds we obtain for degrees $11, 13, 15$, and 17 in Proposition 2.39 are better than those from Section 2.3 but do not improve the ones from [Cen21]. However, we have explained them anyway because it is interesting to illustrate how this method works and its efficiency.

Chapter 3

Simultaneous cyclicity and criticality

As we have seen in the previous chapters, a classical problem in the study of qualitative theory of planar differential equations in the plane is the second part of the 16th Hilbert Problem, related to the bifurcation of limit cycles or isolated periodic orbits in a polynomial class of fixed degree. A large number of works in this line of research have been published so far for several polynomial families of differential equations. A different problem that has aroused interest during the last decades is the study of the isochronicity of a system, as well as its bifurcation of critical periods. These problems consist on analyzing the flatness and the oscillations of the period function of the system, respectively. As a matter of fact, the bifurcation of limit cycles and critical periods are analogous in terms of the techniques that can be used to be approached. For this reason, in this chapter we suggest the study of the bifurcation of limit cycles and critical periods simultaneously, a problem that to the best of our knowledge has not been formulated yet.

3.1 Introduction

Let us consider a real polynomial system of differential equations in the plane whose origin is a nondegenerate monodromic equilibrium point, so the matrix associated to the differential system evaluated at the origin has zero trace and positive determinant. It is a well-known fact that, by a suitable change of coordinates and time rescaling, it can be written in the form

$$\begin{cases} \dot{x} = \alpha x - y + X(x,y) =: P(x,y), \\ \dot{y} = x + \alpha y + Y(x,y) =: Q(x,y), \end{cases} \tag{3.1}$$

being $\alpha = 0$, where X and Y are polynomials of degree $n \geq 2$ which start at least with quadratic monomials. We can consider system (3.1) in complex coordinates $(z,w) = (z,\bar{z}) = (x + \mathrm{i}\, y, x - \mathrm{i}\, y)$, which will be represented by only one equation as

$$\dot{z} = (\alpha + \mathrm{i})z + Z(z,w) = \mathcal{Z}(z,w), \tag{3.2}$$

where Z is a polynomial starting with monomials of at least second degree.

Let us consider system (3.1), and perform a (near the identity) change of variables to a normal form type with action-angle coordinates (ϱ, θ). We will denote by $O_d(\varrho)$ a sum of monomials in ϱ of at least degree d, and the system will become one of the following normal form type structures:

L The transformed system linearizes, that is, it takes the form $(\dot{\varrho}, \dot{\theta}) = (0, 1)$. We can also say that the origin is an *isochronous center*.

C The transformed system takes the form $(\dot{\varrho}, \dot{\theta}) = (0, 1 + T_l \varrho^l + O_{l+1}(\varrho))$. The origin is a *center with weakness of order l on the period*.

W The transformed system takes the form $(\dot{\varrho}, \dot{\theta}) = (V_k \varrho^k + O_{k+1}(\varrho), 1)$. The origin is an *isochronous weak focus of order k*.

B The remaining case is when both the center and isochronicity properties are not kept at the same time. Hence, the transformed system takes the general form

$$\begin{cases} \dot{\varrho} = V_k \varrho^k + O_{k+1}(\varrho), \\ \dot{\theta} = 1 + T_l \varrho^l + O_{l+1}(\varrho). \end{cases} \tag{3.3}$$

For more details on normal forms theory for planar vector fields we refer the reader to [CLW94; HY12].

It is usual to restrict the study of the period function to the class of centers, i.e. systems that remain in type **C** in the aforementioned normal form changes of variables classification. The study of the global monotonicity or the number of total oscillations of such function are difficult problems, see for example [Chi87; CMV99; MMJR97; Zha02] and the references therein. There are not so many global studies of the period function for general classes of centers. Gavrilov ([Gav93]) proved the existence of at most one critical period for the Hamiltonian potentials $x'' + x + ax^2 + bx^3 = 0$, a problem started by Chow and Sanders ([CS86]) in 1986. In 2006, Mañosas and Villadelprat ([MV06]) proved that the derivative of the period function for Hamiltonian potentials $x'' + x + ax^3 + bx^5 = 0$ has only one zero. Some years later, Grau and Villadelprat ([GV10]) proved that only two critical periods appear in some cubic homogeneous nonlinearity classes. In those cases, we say that the systems have one and two critical periods, respectively. For centers in the quadratic class, the most relevant study was done by Chicone and Jacobs in 1989 ([CJ89]). Among others, they studied the local problem for the quadratic family, proving that only two critical periods bifurcate from the center equilibrium point. The answer for the global problem remains open. The greatest difficulty to deal with is the fact that the outer boundary of the period annulus changes together with the parameters inside a fixed family, see for example [GMM02; Swi99]. Hence, the usual perturbation techniques are not useful and new tools need to be developed ([MV21]). As we have described, the maximal number of zeros of the derivative of the period function under perturbation (in some fixed class) is known as the criticality of the center. This problem has been studied for low degree polynomial vector fields in the class of reversible centers in Sections 2.3 and 2.4 (see also [ST21c; ST21a]).

Similarly to the above problem, we can restrict our analysis to the class of vector fields that remain in type **W**. This is the case associated to the problem of studying isochronous foci, a problem that was addressed for example by Giné in [Gin03; GG05; GL05]. In this special class, the cyclicity problem is also an interesting problem to be approached, which up to our knowledge is not completely solved even for low degree vector fields. A special family of systems in this class are the so-called rigid (or uniformly isochronous) systems. They satisfy that $\dot{\varphi} = 1$ in the usual polar coordinates. Inside this class, quadratics have no limit cycles and there are cubics with at least two ([GPT05]), but there is no answer for the global question about the total number of limit cycles in rigid cubic systems.

With this chapter we aim to initiate a sort of mixed or simultaneous bifurcation problem. From the above explanation, we will start by introducing the notions of simultaneous cyclicity and criticality and bi-weak monodromic equilibrium point.

Definition 3.1. *We say that the simultaneous cyclicity and criticality of the origin of system (3.1) is $(k, l) \in (\mathbb{N} \cup \{\infty\})^2$, or that the system has configuration (k, l), if k limit cycles and l critical periods bifurcate from the origin. If $\alpha = 0$, we denote $k = \infty$ when the origin is a center and $l = \infty$ when it is isochronous.*

Definition 3.2. *We say that the origin of system (3.1) is a bi-weak monodromic equilibrium point with normal form of type $[k, l]$, or for brevity bi-weak $[k, l]$ type of bi-weak of type $[k, l]$, if there exists a (near the identity) change to normal form such that the system becomes (3.3) in a neighborhood of the origin. We denote $k = \infty$ if the origin is a center and $l = \infty$ if the origin is isochronous. In the case $k \neq \infty$, we will also refer to a bi-weak $[k, l]$ type as a bi-weak $[k, l]$ focus.*

When $\alpha = 0$, the origin of system (3.1) can be a either a center or a weak focus of a certain order. This classical notion of order will be recalled in the next section, where we will generalize it to the bi-weak type by adding also the study of the first nonzero term in the second component of (3.3). This gives birth to the idea of duality in weakness of a nonisochronous focus. In this chapter we will restrict our analysis to the case in which the transversal section is the horizontal axis.

As we will see in Section 3.2, we start by observing that for a bi-weak $[k, l]$ focus k will always be odd, since it is a well-known fact that the first nonzero Lyapunov constant of a system has odd subscript (Lemma 1.8). We also know that the first nonzero period constant of a center has even subscript (Theorem 2.7), so if $k = \infty$ then l is even. However, this is not the case for systems which do not have a center at the origin. For instance, let us consider the quadratic system

$$\begin{cases} \dot{x} = -y + x^2 - \dfrac{10}{9}xy, \\ \dot{y} = x + x^2 + 4xy - \dfrac{25}{9}y^2. \end{cases} \tag{3.4}$$

For this system, the origin is bi-weak of type $[3,3]$, because we can easily find that the first nonzero Lyapunov constant is $V_3 = \pi$ and the first nonzero coefficient of the period function is $T_3 = \pi$, so the origin is not a center because $V_3 \neq 0$ and we have $T_3 \neq 0$. Therefore, if the center property is not kept then the property which states that the first nonzero period constant has even subscript does not hold.

As we have established, $k = \infty$ means that the system has a center at the origin, so all Lyapunov constants would vanish and the Poincaré return map Π is the identity, i.e. $\Pi(x) \equiv x$. Analogously, $l = \infty$ means isochronicity and all period constants would vanish in this case, so the period function T is constant. Consequently, an isochronous center would be $[\infty, \infty]$. Setting these notations and fixing a class of systems, we have two different problems, one being to find the maximal simultaneous cyclicity and criticality (k,l), and another being to find the highest finite bi-weak $[k,l]$ type –more concretely, when both components of the pair are finite. Observe that this last problem is different from finding the highest finite values k and l of bi-weak $[k,\infty]$ or $[\infty,l]$ types. It is also different from finding the classical highest weak focus order of a monodromic equilibrium point.

As we will see in the following, the values for k, l will be not the same in both problems. This is due to the fact that in the classical Hopf bifurcation, when only one limit cycle bifurcate from the origin of (3.1), the stability of the equilibrium point (with $\alpha = 0$) is given by the sign of V_3.

The main purpose of this chapter is to present the problems of simultaneous cyclicity and criticality and bi-weak systems with $[k,l]$ normal form type, as well as some useful tools to deal with them. The first result we present is related to bi-weak $[k,l]$ foci in Liénard, quadratic, and linear plus cubic homogeneous systems.

Theorem 3.3. *(i) There exist cubic Liénard systems of the form*

$$\begin{cases} \dot{x} = -y + a_2x^2 + a_3x^3, \\ \dot{y} = x + b_2x^2 + b_3x^3, \end{cases} \tag{3.5}$$

being $a_2, a_3, b_2, b_3 \in \mathbb{R}$, which have a bi-weak $[5,4]$ focus at the origin.

(ii) *There exist quartic Liénard systems of the form*

$$\begin{cases} \dot{x} = -y + a_2x^2 + a_3x^3 + a_4x^4, \\ \dot{y} = x + b_2x^2 + b_3x^3 + b_4x^4, \end{cases} \tag{3.6}$$

being $a_2, a_3, a_4, b_2, b_3, b_4 \in \mathbb{R}$, *which have a bi-weak* $[7,6]$ *focus at the origin.*

(iii) *There exist quadratic systems of the form*

$$\begin{cases} \dot{x} = -y + a_{20}x^2 + a_{11}xy + a_{02}y^2, \\ \dot{y} = x + b_{20}x^2 + b_{11}xy + b_{02}y^2, \end{cases} \tag{3.7}$$

being $a_{20}, a_{11}, a_{02}, b_{20}, b_{11}, b_{02} \in \mathbb{R}$, *which have a bi-weak* $[5,4]$ *focus at the origin.*

(iv) *There exist linear plus cubic homogeneous systems of the form*

$$\begin{cases} \dot{x} = -y + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ \dot{y} = x + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3, \end{cases} \tag{3.8}$$

being $a_{30}, a_{21}, a_{12}, a_{03}, b_{30}, b_{21}, b_{12}, b_{03} \in \mathbb{R}$, *which have a bi-weak* $[7,6]$ *focus at the origin.*

The second result solves the complete classification of the simultaneous cyclicity and criticality for cubic Liénard systems.

Theorem 3.4. *For the cubic Liénard family* (3.5), *adding the trace parameter* α *as in* (3.1), *we have the following possible configurations of simultaneous cyclicity and criticality:*

(i) *for the center case we can obtain either* $(\infty, 1)$, *i.e. at most* 1 *critical period, or an isochronous center* (∞, ∞);

(ii) *if the center property is not kept then the maximal simultaneous cyclicity and criticality has configurations* $(1,3)$ *and* $(2,3)$, *and if* $k \geq 3$ *then* $(k,l) = (\infty, l)$ *so it is a center;*

(iii) *the system does not have isochronous foci at the origin, i.e. if* (k, ∞) *then* $k = \infty$.

This chapter is structured as follows. In Section 3.2 we introduce the classical methods to find Lyapunov and period constants as well as a method which uses the Lie bracket. We also discuss the pros and cons of each of the approaches. The presented Lie bracket method is used in Section 3.3 to study the bi-weakness of Liénard, quadratic, and linear plus cubic homogeneous systems in order to prove Theorem 3.3. Finally, in Section 3.4 we show some properties of several Liénard systems and give the isochronicity conditions for some cases, and finish by proving Theorem 3.4. We remark however that the aim of this chapter is not to provide any isochronicity characterization, not even for an a priori simple class of systems as the Liénard family, a characterization of which has been obtained very recently in [Ame21].

3.2 Lyapunov and period constants computation

Throughout this section we will introduce two methods to find Lyapunov and period constants. First, we present a generalization of the mathematical object known as Lie bracket introduced in Chapter 2. Then, we describe the classical method to find Lyapunov and period constants by means of integrating the system. Notice that we have already seen this method in the previous chapters, but we include them also here for completeness to show how one actually can find Lyapunov and period constants analogously. Then we describe a technique based on the Lie bracket which also enables to find Lyapunov and period constants under certain conditions, and we finish the section with a discussion of its advantages and drawbacks.

3.2.1 Generalization of the Lie bracket

In this subsection we present the Lie bracket, a powerful tool to study the isochronicity of a system and to find its Lyapunov and period constants under certain conditions that we will see. Even though we gave a first overview on the Lie bracket in Subsection 2.1.1 from Chapter 2, we present here a more general point of view. In particular, here we generalize the notion of Lie bracket to general complex vector fields, instead of considering it only for complex vector fields associated to real vector fields.

Definition 3.5. *We define the Lie bracket of two planar vector fields* $F_1 = (F_1^1, F_1^2)$ *and* $F_2 = (F_2^1, F_2^2)$ *in variables* $(x, y) \in \mathbb{K}^2$*, where* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*, as a new vector field which has the form*

$$[F_1, F_2] = \left(\frac{\partial F_1^1}{\partial x} F_2^1 + \frac{\partial F_1^1}{\partial y} F_2^2 - \frac{\partial F_2^1}{\partial x} F_1^1 - \frac{\partial F_2^1}{\partial y} F_1^2, \right. \\ \left. \frac{\partial F_1^2}{\partial x} F_2^1 + \frac{\partial F_1^2}{\partial y} F_2^2 - \frac{\partial F_2^2}{\partial x} F_1^1 - \frac{\partial F_2^2}{\partial y} F_1^2 \right). \tag{3.9}$$

Let us consider now the Lie bracket in the case of having two complex planar vector fields $\mathcal{Z}, \mathcal{U}$, corresponding to two real vector fields. Observe that, in such situation, second components are obtained by complex conjugation of first components, thus both vector fields $\mathcal{Z}$ and $\mathcal{U}$ and their Lie bracket can be described only from their first components, so in some cases we will simply write the Lie bracket as in (2.6).

3.2.2 The classical method

We start by presenting the classical method of finding Lyapunov and period constants. Let us write system (3.1), with $\alpha = 0$, in polar coordinates by performing the usual change $(x, y) = (r\cos\varphi, r\sin\varphi)$, and one obtains

$$\begin{cases} \dot{r} = \sum_{i=1}^{n-1} U_i(\varphi) r^{i+1}, \\ \dot{\varphi} = 1 + \sum_{i=1}^{n-1} W_i(\varphi) r^i, \end{cases} \tag{3.10}$$

where $U_i(\varphi)$ and $W_i(\varphi)$ are homogeneous polynomials in $\sin\varphi$ and $\cos\varphi$ of degree $i+2$. Eliminating time and doing the Taylor series expansion in r we obtain

$$\frac{dr}{d\varphi} = \sum_{j=2}^{\infty} R_j(\varphi) r^j. \tag{3.11}$$

The initial value problem for (3.11) with the initial condition $(r, \varphi) = (\rho, 0)$ has a unique analytic solution which can be expanded as

$$r(\varphi, \rho) = \rho + \sum_{j=2}^{\infty} u_j(\varphi)\rho^j. \tag{3.12}$$

As $r(0, \rho) = \rho$, it immediately follows that $u_j(0) = 0$ for every j. Let us study the stability near the origin, $r = 0$, by using

$$r(2\pi, \rho) = \rho + V_k \rho^k + O_{k+1}(\rho), \tag{3.13}$$

where $V_k := u_k(2\pi)$ is the first coefficient which does not vanish. It is a well-known fact that the first nonidentically zero V_k has odd k ([And+73; RS09]). As a consequence, expression (3.13) can be rewritten as

$$r(2\pi, \rho) = \rho + V_{2m+1}\rho^{2m+1} + O_{2m+2}(\rho).$$

We will denote the V_{2m+1} with odd subscript as $L_m := V_{2m+1}$, and these values are the Lyapunov constants. These objects are the key tool to study the center and cyclicity problems of a system of the form (3.1). Observe that $r(2\pi, \rho)$ indicates the radius after a whole loop starting in the initial value ρ, and then we define the

Poincaré map

$$\Pi(\rho) := \rho + \sum_{j=3}^{\infty} V_j \rho^j.$$

Alternatively, this can be written as

$$\Delta(\rho) := \Pi(\rho) - \rho = \sum_{j=3}^{\infty} V_j \rho^j, \tag{3.14}$$

which is the so-called displacement map. We recall that we are taking a nondegenerate equilibrium point with zero trace, i.e. $\alpha = 0$.

Now let us illustrate how the period constants can be found, for which the reader is referred to [Mn95; RS09; ST21a], or to Chapter 2. First, we substitute the power series (3.12) into the second equation of (3.10), which yields a differential equation of the form

$$\frac{d\varphi}{dt} = 1 + \sum_{i=1}^{\infty} F_i(\varphi)\rho^i,$$

for some trigonometric polynomials $F_i(\varphi)$. Rewriting this equation as

$$dt = \frac{d\varphi}{1 + \sum_{i=1}^{\infty} F_i(\varphi)\rho^i} = \left(1 + \sum_{i=1}^{\infty} \Psi_i(\varphi)\rho^i\right) d\varphi$$

and integrating φ from 0 to 2π yields the period function

$$T(\rho) = \int_0^{T(\rho)} dt = \int_0^{2\pi} \left(1 + \sum_{i=1}^{\infty} \Psi_i(\varphi)\rho^i\right) d\varphi = 2\pi + \sum_{i=1}^{\infty} \left(\int_0^{2\pi} \Psi_i(\varphi)\, d\varphi\right) \rho^i, \tag{3.15}$$

where the series converges for $0 \leq \varphi \leq 2\pi$ and sufficiently small $\rho \geq 0$. In the center case, (3.15) can be seen as the lowest period of the trajectory of (3.1) passing through $(x, y) = (\rho, 0)$ for $\rho \neq 0$, and this is known as the period function. The coefficients T_i of this function are then given by the expression

$$T_i = \int_0^{2\pi} \Psi_i(\varphi)\, d\varphi, \tag{3.16}$$

and the first nonzero T_l is the lth period constant of the system.

If we assume now that system (3.10) depends on some parameters $\lambda \in \mathbb{R}^d$, we can follow exactly the same procedure as before, and now we have that both the

Lyapunov constants $V_k = u_k(2\pi, \lambda)$ and the period constants

$$T_l(\lambda) = \int_0^{2\pi} \Psi_l(\varphi, \lambda)\, d\varphi \tag{3.17}$$

are polynomials in the parameters λ (see [Cim+97; RS09]).

Finally, we consider the case with nonzero trace ($\alpha \neq 0$). The following result studies how the return map and the period function, together with cyclicity and criticality, change when the considered system has nonzero trace.

Lemma 3.6. *Let us consider a system of the form* (3.1) *with* $V_3 \neq 0$, $T_2 \neq 0$, *and nonzero trace, written as*

$$\begin{cases} \dot{x} = \alpha x - y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \widetilde{X}(x,y), \\ \dot{y} = x + \alpha y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + \widetilde{Y}(x,y), \end{cases} \tag{3.18}$$

where $\widetilde{X}$ *and* $\widetilde{Y}$ *are polynomials of degree* $n \geq 3$ *which start at least with cubic monomials, and* $\alpha \neq 0$.

(i) The displacement map (3.14) *in this case has the form*

$$\Delta(\rho) = (\mathrm{e}^{2\pi\alpha} - 1)\rho + V_3\rho^3 + O_4(\rho). \tag{3.19}$$

If $\alpha V_3 < 0$, *then a unique limit cycle bifurcates from the origin due to a Hopf bifurcation.*

(ii) The period function (3.15) *in this case has the form*

$$T(\rho) = 2\pi + (\mathrm{e}^{2\pi\alpha} - 1)\widetilde{T}_1(\alpha)\,\rho + T_2\rho^2 + O_3(\rho), \tag{3.20}$$

where

$$\widetilde{T}_1(\alpha) = \frac{-\alpha^3 b_{20} - \alpha^2 a_{20} + \alpha^2 b_{11} + 2\alpha a_{11} - 2\alpha b_{02} - 7\alpha b_{20} - 6a_{02} - 3a_{20} + 3b_{11}}{\alpha^4 + 10\alpha^2 + 9}.$$

If $(\mathrm{e}^{2\pi\alpha} - 1)\,\widetilde{T}_1(\alpha)\,T_2 < 0$, *then a unique critical period bifurcates from the origin.*

Proof. It is a well-known fact and a straightforward computation that the displacement map of system (3.18) has the form (3.19). Thus, as α and $\mathrm{e}^{2\pi\alpha} - 1$ have the same sign, it is immediate to see that if $\alpha V_3 < 0$ then $\Delta(\rho)$ has an extra change of sign in the coefficients of ρ, so a new positive zero can appear and this implies the unfolding of an extra limit cycle of small amplitude.

For the case of the period, we can change system (3.18) to polar coordinates and integrate the angular equation of $d\varphi/dt$ as we did above. After doing the computations, we obtain that the period function in the case of nonzero trace becomes (3.20). Notice that in (3.18) we have only considered the coefficients of the quadratic part because by construction they are the only ones which can actually play a part in the coefficient of ρ in the period function, in the sense that higher degree coefficients would appear in higher degree terms of the period function. Now if $(e^{2\pi\alpha} - 1)\widetilde{T}_1(\alpha)\, T_2 < 0$, there is an extra change of sign in the coefficients of the period function, which implies the bifurcation of an extra critical period. □

3.2.3 Lyapunov and period constants via the Lie bracket

In the previous subsection we presented the classical method to compute Lyapunov and period constants. Such method involves some integrals (equations (3.16), (3.17)) which easily become too difficult to be explicitly obtained, so this technique is not useful in many cases. To deal with this inconvenience, a method to find period constants which is based on the Lie bracket tool was introduced in [Mn95] and recently used in [ST21c; ST21a] (corresponding to Sections 2.3,2.4 from Chapter 2). However, this technique was only valid when the origin is a center, which is not our case in this chapter, since we aim to study cyclicity and criticality simultaneously. Here we present a new approach based on the Lie bracket which will allows to find both Lyapunov and period constants at the same time. This method will provide some valuable advantages, even though it also has its limitations.

Let us consider system (3.2) with $\alpha = 0$. By applying near the identity changes of variables, as the spirit of normal form transformations, such system can be simplified to

$$\dot{z} = \mathrm{i}\, z + \sum_{j=1}^{N} (\alpha_{2j+1} + \mathrm{i}\, \beta_{2j+1}) z (zw)^j + O_{2N+3}(z, w), \tag{3.21}$$

where $N \in \mathbb{N}$ is arbitrary, $O_{2N+3}(z,w)$ denotes a sum of monomials of degree at least $2N+3$ in z and w, and $\alpha_{2j+1}, \beta_{2j+1} \in \mathbb{R}$ (see [AFG00] for more details). Let us consider the truncation of differential system (3.21)

$$\dot{z} = \mathrm{i}\, z + \sum_{j=1}^{N} (\alpha_{2j+1} + \mathrm{i}\, \beta_{2j+1}) z (zw)^j. \tag{3.22}$$

Proposition 3.7. *System (3.22) in polar coordinates takes the form*

$$\begin{cases} \dot{r} = \sum\limits_{j=1}^{N} \alpha_{2j+1} r^{2j+1}, \\ \dot{\varphi} = 1 + \sum\limits_{j=1}^{N} \beta_{2j+1} r^{2j}. \end{cases} \tag{3.23}$$

Proof. The proof is straightforward by applying the change of variables $(r, \varphi) \mapsto (z, w) = \left(r\,e^{i\varphi}, r\,e^{-i\varphi}\right)$ to differential equation (3.23). Indeed,

$$\begin{aligned}\dot{z} =& \dot{r}\,e^{i\varphi} + i\,r\dot{\varphi}\,e^{i\varphi} = \left(\sum_{j=1}^{N} \alpha_{2j+1} r^{2j+1}\right) e^{i\varphi} + i\,r\,e^{i\varphi}\left(1 + \sum_{j=1}^{N} \beta_{2j+1} r^{2j}\right) \\ =& z\left(\sum_{j=1}^{N} \alpha_{2j+1} r^{2j}\right) + i\,z\left(1 + \sum_{j=1}^{N} \beta_{2j+1} r^{2j}\right) = i\,z + \sum_{j=1}^{N} (\alpha_{2j+1} + i\,\beta_{2j+1}) z (zw)^{j},\end{aligned}$$

where we have used the trivial relation $r^2 = zw$. □

The next result shows how we can compute the Lyapunov and period constants of (3.22) by using the Lie bracket. This result was previously obtained in [Mn95], but its proof is included here for completeness.

Theorem 3.8. *Let us denote by* $\mathcal{Z}$ *the vector field* (3.22)*, and consider the vector field* $\mathcal{U}$ *defined by the differential equation* $\dot{z} = z + \sum_{k=2N+2}^{\infty} \sum_{l=0}^{k} A_{k-l,l} z^{k-l} w^{l}$*. Then the Lie bracket between* $\mathcal{Z}$ *and* $\mathcal{U}$ *has the form*

$$[\mathcal{Z}, \mathcal{U}] = \left(\sum_{j=1}^{N} p_{2j+1} z (zw)^{j} + O_{2N+2}(z, w), \sum_{j=1}^{N} \overline{p}_{2j+1} w (zw)^{j} + O_{2N+2}(z, w)\right),$$

where $O_{2N+2}(z, w)$ *denotes a sum of monomials of degree at least* $2N + 2$ *in* z *and* w*, and*

$$V_{2j+1} = \frac{1}{2j} \operatorname{Re}(p_{2j+1}) = \frac{p_{2j+1} + \overline{p}_{2j+1}}{4j} = \alpha_{2j+1}, \tag{3.24}$$

$$T_{2j} = \frac{1}{2j} \operatorname{Im}(p_{2j+1}) = \frac{p_{2j+1} - \overline{p}_{2j+1}}{4j\,i} = \beta_{2j+1}, \tag{3.25}$$

are the coefficients of the return map and the period function of system (3.22)*, respectively.*

Proof. Observe that, by Proposition 3.7, the quantities α_{2j+1} and β_{2j+1} from (3.22) are in fact the Lyapunov and period constants, respectively, of the system in this normal form.

Let us compute the Lie bracket between $\mathcal{Z}$ and $\mathcal{U}$. Actually, we only need to find the first component $[\mathcal{Z}, \mathcal{U}]^1$ of such operation, since its second component is its complex conjugate.

$$
\begin{aligned}
[\mathcal{Z},\mathcal{U}]^1 =& \left(\mathrm{i} + \sum_{j=1}^{N} (\alpha_{2j+1} + \mathrm{i}\,\beta_{2j+1})(j+1)(zw)^j \right)\left(z + \sum_{k=2N+2}^{\infty} \sum_{l=0}^{k} A_{k-l,l} z^{k-l} w^l \right) \\
&+ \left(\sum_{j=1}^{N} (\alpha_{2j+1} + \mathrm{i}\,\beta_{2j+1}) j z^{j+1} w^{j-1} \right)\left(w + \sum_{k=2N+2}^{\infty} \sum_{l=0}^{k} \overline{A}_{k-l,l} z^l w^{k-l} \right) \\
&- \left(1 + \sum_{k=2N+2}^{\infty} \sum_{l=0}^{k-1} A_{k-l,l} (k-l) z^{k-l-1} w^l \right)\left(\mathrm{i}\, z + \sum_{j=1}^{N} (\alpha_{2j+1} + \mathrm{i}\,\beta_{2j+1}) z (zw)^j \right) \\
&- \left(\sum_{k=2N+2}^{\infty} \sum_{l=1}^{k} A_{k-l,l}\, l z^{k-l} w^{l-1} \right)\left(-\mathrm{i}\, w + \sum_{j=1}^{N} (\alpha_{2j+1} - \mathrm{i}\,\beta_{2j+1}) w (zw)^j \right) \\
=& \sum_{j=1}^{N} 2j(\alpha_{2j+1} + \mathrm{i}\,\beta_{2j+1}) z (zw)^j + O_{2N+2}(z,w) \\
=:& \sum_{j=1}^{N} p_{2j+1} z (zw)^j + O_{2N+2}(z,w),
\end{aligned}
$$

and the second component is, by conjugation,

$$
\begin{aligned}
[\mathcal{Z},\mathcal{U}]^2 &= \sum_{j=1}^{N} 2j(\alpha_{2j+1} - \mathrm{i}\,\beta_{2j+1}) w (zw)^j + O_{2N+2}(z,w) \\
&= \sum_{j=1}^{N} \overline{p}_{2j+1} w (zw)^j + O_{2N+2}(z,w).
\end{aligned}
$$

Notice that we have denoted by p_{2j+1} the coefficient of the Lie bracket with degree $2j+1$, and it has the expression $p_{2j+1} = 2j(\alpha_{2j+1} + \mathrm{i}\,\beta_{2j+1})$. Finally, as we observed that α_{2j+1} and β_{2j+1} are the Lyapunov and period constants of system (3.22), formulas (3.24) and (3.25) follow. □

Remark 3.9. *It is worth remarking that this Lie bracket method does not allow to obtain the general expression of the Lyapunov and period constants V_{2j+1} and T_{2j}, but only under the conditions $V_{2i+1} = T_{2i} = 0$ for every $i < j$. In this sense, and with a slight abuse of notation, when using the Lie bracket method throughout this chapter we will denote by V_{2j+1} the jth Lyapunov constant assuming that $V_{2i+1} = T_{2i} = 0$ for every $i < j$, and by T_{2j} the jth period constant assuming that $V_{2i+1} = T_{2i} = 0$ for every $i < j$.*

We have seen that, when having a system in normal form (3.22), one can use the Lie bracket to find the corresponding Lyapunov and period constants. The last step is to prove that, in fact, the system does not have to be necessarily in such normal form. In other words, we will show that the method is equally valid if a change of variables is performed on the system, and this will allow to apply the Lie bracket method to general systems of the form (3.2) and not only to those which are in normal form.

Before the main result we present the following lemma.

Lemma 3.10. *Let $\phi = (\phi_1, \phi_2) : \mathbb{K}^2 \to \mathbb{K}^2$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, be a $\mathcal{C}^2$-diffeomorphism which maps $(u,v) \in \mathbb{K}^2$ to new variables $(x,y) = \phi(u,v) = (\phi_1(u,v), \phi_2(u,v)) \in \mathbb{K}^2$, and whose inverse is $(u,v) = \phi^{-1}(x,y) = (\phi_1^{-1}(x,y), \phi_2^{-1}(x,y))$. Then the following equivalences hold:*

$$\frac{\partial \phi_1^{-1}}{\partial x}(x,y) \cdot \frac{\partial \phi_1}{\partial u}(\phi^{-1}(x,y)) + \frac{\partial \phi_1^{-1}}{\partial y}(x,y) \cdot \frac{\partial \phi_2}{\partial u}(\phi^{-1}(x,y)) = 1, \tag{3.26}$$

$$\frac{\partial \phi_2^{-1}}{\partial x}(x,y) \cdot \frac{\partial \phi_1}{\partial v}(\phi^{-1}(x,y)) + \frac{\partial \phi_2^{-1}}{\partial y}(x,y) \cdot \frac{\partial \phi_2}{\partial v}(\phi^{-1}(x,y)) = 1, \tag{3.27}$$

$$\frac{\partial \phi_1^{-1}}{\partial x}(x,y) \cdot \frac{\partial \phi_1}{\partial v}(\phi^{-1}(x,y)) + \frac{\partial \phi_1^{-1}}{\partial y}(x,y) \cdot \frac{\partial \phi_2}{\partial v}(\phi^{-1}(x,y)) = 0, \tag{3.28}$$

$$\frac{\partial \phi_2^{-1}}{\partial x}(x,y) \cdot \frac{\partial \phi_1}{\partial u}(\phi^{-1}(x,y)) + \frac{\partial \phi_2^{-1}}{\partial y}(x,y) \cdot \frac{\partial \phi_2}{\partial u}(\phi^{-1}(x,y)) = 0. \tag{3.29}$$

Proof. The proof is straightforward by applying the chain rule for two variable functions. For (3.26), we use that $\phi_1(u,v) = x$ and $\phi_2(u,v) = y$ and the chain rule to rewrite it as

$$\frac{\partial \phi_1^{-1}}{\partial x}(x,y) \cdot \frac{\partial x}{\partial u}(\phi^{-1}(x,y)) + \frac{\partial \phi_1^{-1}}{\partial y}(x,y) \cdot \frac{\partial y}{\partial u}(\phi^{-1}(x,y)) = \frac{\partial \phi_1^{-1}(x,y)}{\partial u} = \frac{\partial u}{\partial u} = 1.$$

For (3.27), (3.28), and (3.29), we follow an analogous procedure and we obtain that they are equivalent to $\partial v/\partial v = 1$, $\partial u/\partial v = 0$, and $\partial v/\partial u = 0$, respectively. □

Theorem 3.11. *Let us consider two vector fields $F_1 = (F_1^1, F_1^2)$ and $F_2 = (F_2^1, F_2^2)$ in variables $(u,v) \in \mathbb{K}^2$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\phi = (\phi_1, \phi_2) : \mathbb{K}^2 \to \mathbb{K}^2$ be a $\mathcal{C}^2$-diffeomorphism which maps $(u,v) \in \mathbb{K}^2$ to new variables $(x,y) = \phi(u,v) = (\phi_1(u,v), \phi_2(u,v)) \in \mathbb{K}^2$, and whose inverse is $(u,v) = \phi^{-1}(x,y) = (\phi_1^{-1}(x,y), \phi_2^{-1}(x,y))$. Let us denote by $G_1 = (G_1^1, G_1^2)$ and $G_2 = (G_2^1, G_2^2)$ the vector fields F_1 and F_2, respectively, in variables (x,y) after applying the change of variables $(x,y) = \phi(u,v)$. Then the following equivalence between the Lie brackets of F_1, F_2 and G_1, G_2 holds:*

$$[G_1, G_2]^T = J_\phi(\phi^{-1}(x,y)) \cdot [F_1, F_2]^T_{\phi^{-1}(x,y)}, \tag{3.30}$$

where $J_\phi(\phi^{-1}(x,y))$ is the Jacobian matrix of $\phi(u,v)$ evaluated at $(u,v) = \phi^{-1}(x,y)$, $[F_1, F_2]_{\phi^{-1}(x,y)}$ denotes the Lie bracket between F_1 and F_2 also evaluated at $(u,v) = \phi^{-1}(x,y)$, and the superscript ^{T} denotes the transpose vector.

Proof. Let us start by observing that equivalence (3.30) can be rewritten in matrix form as

$$\begin{pmatrix} [G_1, G_2]^1 \\ [G_1, G_2]^2 \end{pmatrix} = \begin{pmatrix} \frac{\partial \phi_1}{\partial u}(\phi^{-1}(x,y)) & \frac{\partial \phi_1}{\partial v}(\phi^{-1}(x,y)) \\ \frac{\partial \phi_2}{\partial u}(\phi^{-1}(x,y)) & \frac{\partial \phi_2}{\partial v}(\phi^{-1}(x,y)) \end{pmatrix} \cdot \begin{pmatrix} [F_1, F_2]^1_{\phi^{-1}(x,y)} \\ [F_1, F_2]^2_{\phi^{-1}(x,y)} \end{pmatrix}. \tag{3.31}$$

We will only prove the first component of equivalence (3.31), since the second one can be analogously seen. Then, our aim is to show that

$$[G_1, G_2]^1 = \frac{\partial \phi_1}{\partial u}(\phi^{-1}(x,y)) \cdot [F_1, F_2]^1_{\phi^{-1}(x,y)} + \frac{\partial \phi_1}{\partial v}(\phi^{-1}(x,y)) \cdot [F_1, F_2]^2_{\phi^{-1}(x,y)}. \tag{3.32}$$

The idea of the proof is to develop the expression of $[G_1, G_2]^1$ when performing the change of variables ϕ and to check that it satisfies (3.32). First we will see how G_1 and G_2 can be expressed in terms of F_1 and F_2. Let us denote by $\widetilde{G}_i^j(u,v)$ the vector field such that $G_i^j(x,y) = (\widetilde{G}_i^j \circ \phi^{-1})(x,y)$ for $i,j = 1,2$. As vector fields G_i^1 and G_i^2 correspond respectively to $\dot{x}$ and $\dot{y}$, we can take the derivative of $(x,y) = \phi(u,v)$ with respect to time and apply the chain rule to see that

$$\widetilde{G}_i^j = \frac{\partial \phi_j}{\partial u} F_i^1 + \frac{\partial \phi_j}{\partial v} F_i^2, \tag{3.33}$$

since vector fields F_i^1 and F_i^2 correspond respectively to $\dot{u}$ and $\dot{v}$.

The first partial derivative we need to find for $[G_1, G_2]^1$ is $\partial G_1^1/\partial x$ and, by applying the chain rule, it is

$$\begin{aligned} \frac{\partial G_1^1}{\partial x}(x,y) &= \frac{\partial(\widetilde{G}_1^1 \circ \phi^{-1})}{\partial x}(x,y) \\ &= \frac{\partial \widetilde{G}_1^1}{\partial u}(\phi^{-1}(x,y)) \cdot \frac{\partial u}{\partial x} + \frac{\partial \widetilde{G}_1^1}{\partial v}(\phi^{-1}(x,y)) \cdot \frac{\partial v}{\partial x}. \end{aligned} \tag{3.34}$$

We can find the expression for $\partial \widetilde{G}_1^1/\partial u$ by taking the derivative of (3.33) with respect to u, and we get

$$\frac{\partial \widetilde{G}_1^1}{\partial u} = \frac{\partial}{\partial u}\left(\frac{\partial \phi_1}{\partial u} F_1^1 + \frac{\partial \phi_1}{\partial v} F_1^2 \right) = \frac{\partial^2 \phi_1}{\partial u^2} F_1^1 + \frac{\partial \phi_1}{\partial u} \frac{\partial F_1^1}{\partial u} + \frac{\partial^2 \phi_1}{\partial u \partial v} F_1^2 + \frac{\partial \phi_1}{\partial v} \frac{\partial F_1^2}{\partial u}.$$

To find $\partial \widetilde{G}_1^1/\partial v$ we proceed analogously. Now, by substituting these two expressions in (3.34), we get $\partial G_1^1/\partial x$ in terms of the derivatives of F_1 and F_2 as we wanted. The same procedure can be applied to $\partial G_1^1/\partial y$, $\partial G_2^1/\partial x$, and $\partial G_2^1/\partial y$, and then we have all the terms of $[G_1, G_2]^1$ expressed with F_1 and F_2.

For the sake of compactness, from now on we will use the following notation to write this expression of $[G_1, G_2]^1$:

$$\phi_{ju} = \frac{\partial \phi_j}{\partial u}(\phi^{-1}(x,y)), \quad \phi_{jv} = \frac{\partial \phi_j}{\partial v}(\phi^{-1}(x,y)), \quad \phi_{juu} = \frac{\partial^2 \phi_j}{\partial u^2}(\phi^{-1}(x,y)),$$
$$\phi_{juv} = \frac{\partial^2 \phi_j}{\partial u \partial v}(\phi^{-1}(x,y)) = \frac{\partial^2 \phi_j}{\partial v \partial u}(\phi^{-1}(x,y)), \quad \phi_{jvv} = \frac{\partial^2 \phi_j}{\partial v^2}(\phi^{-1}(x,y)).$$

Also, with a slight abuse of notation, we will denote the derivatives $(\partial F_i^j/\partial u)(\phi^{-1}(x,y))$ and $(\partial F_i^j/\partial v)(\phi^{-1}(x,y))$ simply as $\partial F_i^j/\partial u$ and $\partial F_i^j/\partial v$, respectively.

The expression of $[G_1, G_2]^1$ can then be written as

$$\begin{aligned}
[G_1, G_2]^1 =& \frac{\partial G_1^1}{\partial x} G_2^1 + \frac{\partial G_1^1}{\partial y} G_2^2 - \frac{\partial G_2^1}{\partial x} G_1^1 - \frac{\partial G_2^1}{\partial y} G_1^2 \\
=& \left(\left(\phi_{1uu} F_1^1 + \phi_{1u} \frac{\partial F_1^1}{\partial u} + \phi_{1uv} F_1^2 + \phi_{1v} \frac{\partial F_1^2}{\partial u} \right) \frac{\partial u}{\partial x} \right. \\
& \left. + \left(\phi_{1uv} F_1^1 + \phi_{1u} \frac{\partial F_1^1}{\partial v} + \phi_{1vv} F_1^2 + \phi_{1v} \frac{\partial F_1^2}{\partial v} \right) \frac{\partial v}{\partial x} \right) \left(\phi_{1u} F_2^1 + \phi_{1v} F_2^2 \right) \\
& + \left(\left(\phi_{1uu} F_1^1 + \phi_{1u} \frac{\partial F_1^1}{\partial u} + \phi_{1uv} F_1^2 + \phi_{1v} \frac{\partial F_1^2}{\partial u} \right) \frac{\partial u}{\partial y} \right. \\
& \left. + \left(\phi_{1uv} F_1^1 + \phi_{1u} \frac{\partial F_1^1}{\partial v} + \phi_{1vv} F_1^2 + \phi_{1v} \frac{\partial F_1^2}{\partial v} \right) \frac{\partial v}{\partial y} \right) \left(\phi_{2u} F_2^1 + \phi_{2v} F_2^2 \right) \\
& - \left(\left(\phi_{1uu} F_2^1 + \phi_{1u} \frac{\partial F_2^1}{\partial u} + \phi_{1uv} F_2^2 + \phi_{1v} \frac{\partial F_2^2}{\partial u} \right) \frac{\partial u}{\partial x} \right. \\
& \left. + \left(\phi_{1uv} F_2^1 + \phi_{1u} \frac{\partial F_2^1}{\partial v} + \phi_{1vv} F_2^2 + \phi_{1v} \frac{\partial F_2^2}{\partial v} \right) \frac{\partial v}{\partial x} \right) \left(\phi_{1u} F_1^1 + \phi_{1v} F_1^2 \right) \\
& - \left(\left(\phi_{1uu} F_2^1 + \phi_{1u} \frac{\partial F_2^1}{\partial u} + \phi_{1uv} F_2^2 + \phi_{1v} \frac{\partial F_2^2}{\partial u} \right) \frac{\partial u}{\partial y} \right. \\
& \left. + \left(\phi_{1uv} F_2^1 + \phi_{1u} \frac{\partial F_2^1}{\partial v} + \phi_{1vv} F_2^2 + \phi_{1v} \frac{\partial F_2^2}{\partial v} \right) \frac{\partial v}{\partial y} \right) \left(\phi_{2u} F_1^1 + \phi_{2v} F_1^2 \right).
\end{aligned} \tag{3.35}$$

This expression can be expanded, and a long sum of terms in products of $F_1^1, F_1^2, F_2^1, F_2^2$ and their derivatives is obtained. Even though we will not show the complete expansion here due to length reasons, we will split it into several parts and see what happens in each case.

First, it is straightforward to see that the terms with $F_1^1 F_2^1$ cancel with each other, as well as those terms with $F_1^2 F_2^2$. Now let us focus on the terms with $F_1^1 F_2^2$ and $F_1^2 F_2^1$. The coefficient of $F_1^1 F_2^2$ in (3.35) is

$$\left(\left(\frac{\partial v}{\partial x}\phi_{1v} + \frac{\partial v}{\partial y}\phi_{2v}\right) - \left(\frac{\partial u}{\partial x}\phi_{1u} + \frac{\partial u}{\partial y}\phi_{2u}\right)\right)\phi_{1uv} +$$
$$\left(\frac{\partial u}{\partial x}\phi_{1v} + \frac{\partial u}{\partial y}\phi_{2v}\right)\phi_{1uu} - \left(\frac{\partial v}{\partial x}\phi_{1u} + \frac{\partial v}{\partial y}\phi_{2u}\right)\phi_{1vv}.$$

We can show that this expression is zero by applying Lemma 3.10. In particular, the coefficient of ϕ_{1uv} equals zero due to (3.26) and (3.27), and the coefficients of ϕ_{1uu} and ϕ_{1vv} also equal zero due to (3.28) and (3.29), respectively. Therefore, the term with $F_1^1 F_2^2$ vanishes in (3.35). Analogously, by applying Lemma 3.10, we can see that the coefficient of $F_1^2 F_2^1$ also vanishes.

Now let us see how the Lie bracket of F_1 and F_2 appears in (3.35). One can see that the coefficient of $\frac{\partial F_1^1}{\partial u} F_2^1$ in this expression is

$$\phi_{1u}\left(\frac{\partial u}{\partial x}\phi_{1u} + \frac{\partial u}{\partial y}\phi_{2u}\right) = \phi_{1u},$$

where we have applied (3.26) from Lemma 3.10. Analogously, applying both (3.26) and (3.27) from such lemma, we see that the coefficients of $\frac{\partial F_1^1}{\partial v} F_2^2$, $\frac{\partial F_2^1}{\partial u} F_1^1$, and $\frac{\partial F_2^1}{\partial v} F_1^2$ in (3.35) are respectively $\phi_{1u}, -\phi_{1u}$, and $-\phi_{1u}$. Therefore, as these are the terms of the first component of the Lie bracket according to (3.9), we obtain that those terms altogether equal $\phi_{1u}[F_1, F_2]^1$. The same procedure can be followed with terms $\frac{\partial F_1^2}{\partial u} F_2^1$, $\frac{\partial F_1^2}{\partial v} F_2^2$, $\frac{\partial F_2^2}{\partial u} F_1^1$, and $\frac{\partial F_2^2}{\partial v} F_1^2$ from (3.35), and we see that these terms equal $\phi_{1v}[F_1, F_2]^2$.

After all this, expression (3.35) can be rewritten as

$$\begin{aligned}
[G_1, G_2]^1 = &\phi_{1u}[F_1, F_2]^1 + \phi_{1v}[F_1, F_2]^2 \\
&+ \left(\frac{\partial u}{\partial x}\phi_{1v} + \frac{\partial u}{\partial y}\phi_{2v}\right)\left(\phi_{1u}\frac{\partial F_1^1}{\partial u}F_2^2 + \phi_{1v}\frac{\partial F_1^2}{\partial u}F_2^2 - \phi_{1u}\frac{\partial F_2^1}{\partial u}F_1^2 - \phi_{1v}\frac{\partial F_2^2}{\partial u}F_1^2\right) \\
&+ \left(\frac{\partial v}{\partial x}\phi_{1u} + \frac{\partial v}{\partial y}\phi_{2u}\right)\left(\phi_{1u}\frac{\partial F_1^1}{\partial v}F_2^1 + \phi_{1v}\frac{\partial F_1^2}{\partial v}F_2^1 - \phi_{1u}\frac{\partial F_2^1}{\partial v}F_1^1 - \phi_{1v}\frac{\partial F_2^2}{\partial v}F_1^1\right).
\end{aligned} \tag{3.36}$$

Finally, by applying (3.28) and (3.29) on the first factor in the second and third lines of (3.36) and seeing that they are zero, we prove relation (3.32) for the first component of $[G_1, G_2]$.

To prove the relation for the second component, this is

$$[G_1, G_2]^2 = \frac{\partial \phi_2}{\partial u}(\phi^{-1}(x,y)) \cdot [F_1, F_2]^1_{\phi^{-1}(x,y)} + \frac{\partial \phi_2}{\partial v}(\phi^{-1}(x,y)) \cdot [F_1, F_2]^2_{\phi^{-1}(x,y)},$$

we follow an analogous procedure and see that

$$[G_1, G_2]^2 = \phi_{2u}[F_1, F_2]^1 + \phi_{2v}[F_1, F_2]^2,$$

and the proof follows. □

Now let us consider the Lie bracket of two planar real vector fields written (abusing notation) in complex coordinates as $(F_1, \overline{F}_1)$ and $(F_2, \overline{F}_2)$, and the Lie bracket of new vector fields $(G_1, \overline{G}_1)$ and $(G_2, \overline{G}_2)$ after a change of variables $(\phi, \overline{\phi})$ on F_1 and F_2, respectively. As we already stated, these Lie brackets can be described only from their first components, as the second ones are the complex conjugates, so in such case we will denote the first components of the Lie bracket simply by $[F_1, F_2]$ and $[G_1, G_2]$, with a slight abuse of notation as in (2.6). Therefore, relation (3.30) from the theorem can be written, in this case, as

$$[G_1, G_2] = \phi_u[F_1, F_2] + \overline{\phi}_u\overline{[F_1, F_2]}.$$

With this relation, the following corollary from Theorem 3.11 is straightforward.

Corollary 3.12. *Let $(F_1, \overline{F}_1)$ and $(F_2, \overline{F}_2)$ be two planar complex vector fields which correspond to two real vector fiels in the plane, and let $(G_1, \overline{G}_1)$ and $(G_2, \overline{G}_2)$, respectively, be the same vector fields after a change of variables $(\phi, \overline{\phi})$. If $[F_1, F_2] = 0$ then $[G_1, G_2] = 0$.*

3.2.4 Some remarks on the Lie bracket method

The Lie bracket method introduced in Theorem 3.8 allows to find Lyapunov and period constants equivalently to the classical approach from Subsection 3.2.2, but with the difference that the constants V_{2j+1}, T_{2j} obtained by the Lie bracket method are found under conditions $V_{2i+1} = T_{2i} = 0$ for $i < j$ (see Remark 3.9). Even though the exact expressions of the Lyapunov and period constants may differ from a multiplicative constant when using one method or the other –in particular, we have checked that the classical method gives an extra π–, the cyclicity and criticality or the center and isochronicity conditions deduced in both ways are identical.

As we have already stated, the Lie bracket method has the advantage that it does not involve cumbersome integrals while the classical approach does, and also that in contrast to the Lie bracket method introduced in Chapter 2, the current technique does not require that the center conditions are fulfilled. On the other hand, this method also has a few drawbacks. The nature of the presented Lie bracket method requires that Lyapunov and period constants of the same order are found at the same time. This is in fact the same mechanism that finding the linearization condition, see for example [RS09]. In this sense, if we are interested in finding some period constant we are forced to find simultaneously the Lyapunov constant corresponding to the same order, and vice versa. For this reason, if both the classical and the Lie bracket methods work for a system then the classical may be better, but theoretically the classical approach will get stuck quite fast due to the complexity of the integrals to solve, in which case only the Lie bracket method would be able to go further in the order. This fact allows us to split case (3.3) into the two following subcases:

(3.3.1) Finding Lyapunov and period constants of the same order simultaneously,

$$\begin{cases} \dot{\varrho} = V_{2m+1}\varrho^{2m+1} + O_{2m+2}(\varrho), \\ \dot{\theta} = 1 + T_{2m}\varrho^{2m} + O_{2m+1}(\varrho), \end{cases}$$

in which case both the classical approach and the Lie bracket method are valid; using the Lie bracket method here may be slower but able to arrive further in the order.

(3.3.2) Finding Lyapunov and period constants of the same order separately,

$$\begin{cases} \dot{\varrho} = V_k\varrho^k + O_{k+1}(\varrho), \\ \dot{\theta} = 1 + T_l\varrho^l + O_{l+1}(\varrho), \end{cases}$$

in which case the classical method works but not the Lie bracket method; this approach should be faster but would get stuck sooner in comparison to the previous one.

The presented Lie bracket technique has been computationally implemented with Maple, and used to calculate the Lyapunov and period constants in some parts of this chapter.

3.3 Bi-weakness for certain families

This section is devoted prove Theorem 3.3. The way to do this is to apply the Lie bracket method introduced in Section 3.2, so all the found bi-weak types will be of the form $[2m+1, 2m]$. Actually, the proved results are maximal in the sense that if we vanish V_{2k+1} and T_{2k} for $k \leq m$ then $V_{2m+3} = 0$ and $T_{2m+2} = 0$. However, this does not mean that such bi-weak type is the maximal for the system, but the maximal which can be found by means of the Lie bracket method, since such method only detects bi-weak types of the form $[2m+1, 2m]$.

Proof of Theorem 3.3. (i) Let us start by studying system (3.5). We find V_3 and T_2 by using the Lie bracket method,

$$V_3 = \frac{1}{4}(2b_2a_2 - 3a_3), \; T_2 = \frac{1}{12}(4a_2^2 + 10b_2^2 - 9b_3). \tag{3.37}$$

System $\{V_3 = 0, T_2 = 0\}$ has solution $S_1^{(1)} = \{3a_3 - 2a_2b_2 = 0, 9b_3 - 4a_2^2 - 10b_2^2 = 0\}$. The next step is to find V_5 and T_4 assuming that conditions in $S_1^{(1)}$ hold. This gives

$$V_5 = -\frac{5}{27}(2b_2a_2^3 + 5b_2^3a_2), \; T_4 = -\frac{5}{27}(11a_2^2b_2^2 - 14b_2^4),$$

and $\{V_5 = 0, T_4 = 0\}$ has the solution $S_1^{(2)} = \{b_2 = 0\}$. Finally, if we compute V_7 and T_6 under conditions $S_1^{(1)} \cup S_1^{(2)}$, we find $V_7 = 0$ and $T_6 = 0$, and this case corresponds to system

$$\begin{cases} \dot{x} = -y + a_2x^2, \\ \dot{y} = x + \frac{4}{9}a_2^2x^3, \end{cases} \tag{3.38}$$

which is an isochronous center as we will see in Proposition 3.14. System (3.5) under condition $S_1^{(1)}$ has then a bi-weak $[5,4]$ type, which is the maximal of the form $[2m+1, 2m]$ as we have seen that $[7,6]$ does not appear.

(ii) For the quartic Liénard (3.6), we proceed analogously. By using the Lie bracket technique we find that V_3 and T_2 are the same that in the cubic case, (3.37), so they vanish again for $S_2^{(1)} = \{3a_3 - 2a_2b_2 = 0, 9b_3 - 4a_2^2 - 10b_2^2 = 0\}$. Now we find the next constants, which are

$$\begin{aligned} V_5 =& \frac{1}{54}(-20a_2^3b_2 - 50a_2b_2^3 + 27a_2b_4 + 90a_4b_2), \\ T_4 =& \frac{1}{54}(-110a_2^2b_2^2 - 140b_2^4 + 72a_2a_4 + 189b_2b_4). \end{aligned}$$

We consider then system $\{V_5 = 0, T_4 = 0\}$ which has the two following solutions:

$$\begin{aligned} S_2^{(2a)} &= \left\{ a_4 = -\frac{5}{3}\frac{b_2^2a_2(a_2^2 + 7b_2^2)}{4a_2^2 - 35b_2^2}, b_4 = \frac{10}{27}\frac{b_2(8a_2^4 - 35a_2^2b_2^2 - 70b_2^4)}{4a_2^2 - 35b_2^2} \right\}, \\ S_2^{(2b)} &= \{a_2 = 0, b_2 = 0\}. \end{aligned}$$

The next step is to find V_7 and T_6, which under conditions $S_2^{(1)} \cup S_2^{(2a)}$ are

$$V_7^{(a)} = -\frac{35}{216}\frac{b_2^3 a_2(8a_2^6 + 525a_2^4 b_2^2 - 2520a_2^2 b_2^4 - 4900b_2^6)}{(4a_2^2 - 35b_2^2)^2},$$
$$T_6^{(a)} = \frac{35}{648}\frac{b_2^4(430a_2^6 - 3900a_2^4 b_2^2 + 46431a_2^2 b_2^4 + 56350b_2^6)}{(4a_2^2 - 35b_2^2)^2},$$

and under $S_2^{(1)} \cup S_2^{(2b)}$,

$$V_7^{(b)} = \frac{21}{16}a_4 b_4, \quad T_6^{(b)} = \frac{1}{80}(84a_4^2 + 189b_4^2).$$

System $\{V_7^{(a)} = 0, T_6^{(a)} = 0\}$ has the solution $S_2^{(3a)} = \{b_2 = 0\}$, and $\{V_7^{(b)} = 0, T_6^{(b)} = 0\}$ has the solution $S_2^{(3b)} = \{a_4 = 0, b_4 = 0\}$. Finally, if we substitute $S_2^{(1)} \cup S_2^{(2a)} \cup S_2^{(3a)}$ and $S_2^{(1)} \cup S_2^{(2b)} \cup S_2^{(3b)}$ in V_9 and T_8 we obtain $V_9 = 0$ and $T_8 = 0$ in both cases, so the maximal bi-weak type is of the form $[2m+1, 2m]$ is $[7, 6]$ and this proves Theorem 3.3ii. We notice that case $S_2^{(1)} \cup S_2^{(2a)} \cup S_2^{(3a)}$ corresponds to system (3.38) and case $S_2^{(1)} \cup S_2^{(2b)} \cup S_2^{(3b)}$ is the linear center, so both systems are isochronous centers.

(iii) Let us study the quadratic case. First, for the sake of simplicity in the obtained expressions, we will consider system (3.7) in complex coordinates as

$$\begin{cases} \dot{z} = \mathrm{i}\, z + r_{20}z^2 + r_{11}zw + r_{02}w^2, \\ \dot{w} = -\mathrm{i}\, w + s_{02}z^2 + s_{11}zw + s_{20}w^2, \end{cases}$$

being $s_{ij} = \bar{r}_{ij}$. The first Lyapunov and period constants are

$$V_3 = \mathrm{i}(-r_{11}r_{20} + s_{11}s_{20}), \quad T_2 = \frac{4}{3}r_{02}s_{02} + 2r_{11}s_{11} - r_{11}r_{20} - s_{11}s_{20}.$$

In this case, solving the systems for Lyapunov and period constants equal to zero as we did in Liénard families is cumbersome, as many more solutions are obtained. For this reason, we will study Lyapunov and period constants V_{2m+1} and T_{2m} in the Bautin ideal $\mathcal{B}_{m-1} := \langle V_3, T_2, \ldots, V_{2m-1}, T_{2m-2} \rangle$. First we want to check whether V_5 and T_4 belong to $\mathcal{B}_1 = \langle V_3, T_2 \rangle$. The expressions of V_5 and T_4 in $\mathcal{B}_1$ are

$$V_5 = \frac{1}{3}\mathrm{i}(-4r_{02}r_{20}^2 s_{11} + 6r_{02}r_{20}s_{11}^2 + 4r_{02}s_{11}^3 - 4r_{11}^3 s_{02} - 6r_{11}^2 s_{02}s_{20} + 4r_{11}s_{02}s_{20}^2),$$

$$T_4 = \frac{1}{6}\Big(-8r_{02}r_{20}^2s_{11} + 36r_{02}r_{20}s_{11}^2 - 40r_{02}s_{11}^3 - 40r_{11}^3s_{02} - 96r_{11}^2r_{20}^2 + 219r_{11}^2r_{20}s_{11} + 36r_{11}^2s_{02}s_{20} - 135r_{11}^2s_{11}^2 + 12r_{11}r_{20}^2s_{20} - 8r_{11}s_{02}s_{20}^2\Big).$$

As we can easily find explicit values of the parameters such that these constants are not identically zero, we have a bi-weak $[5,4]$ type.

Let us check now that $[7,6]$ does not appear, so we find the expressions of V_7 and T_6 in $\mathcal{B}_2 = \langle V_3, T_2, V_5, T_4\rangle$,

$$V_7 = \frac{1}{64}\,\mathrm{i}\Big(-80r_{02}r_{11}r_{20}^2s_{11}^2 + 360r_{02}r_{11}r_{20}s_{11}^3 - 100r_{02}r_{11}s_{11}^4 - 300r_{11}^4s_{02}s_{11} + 60r_{11}^3r_{20}^3 - 480r_{11}^3r_{20}^2s_{11} + 1095r_{11}^3r_{20}s_{11}^2 - 675r_{11}^3s_{11}^3\Big),$$

$$T_6 = \frac{1}{480}\Big(3032r_{02}r_{11}r_{20}^2s_{11}^2 - 4548r_{02}r_{11}r_{20}s_{11}^3 - 2282r_{02}r_{11}s_{11}^4 + 750r_{11}^4s_{02}s_{11} + 94178r_{11}^3r_{20}^3 - 249834r_{11}^3r_{20}^2s_{11} + 287559r_{11}^3r_{20}s_{11}^2 - 117863r_{11}^3s_{11}^3 - 14832r_{11}^2r_{20}^3s_{20} + 792r_{11}r_{20}^3s_{20}^2\Big).$$

However, we can see that actually $V_7^2 = 0$ in $\mathcal{B}_2$, i.e. $V_7 = 0$ in the variety of zeros defined by the elements in $\mathcal{B}_2$, so the bi-weak $[7,6]$ type cannot appear as we have a center and in this case $[5,4]$ is maximal. We notice that there are quadratic centers with $T_6 \neq 0$, which makes sense as it is a classical result that locally quadratic centers can have at most 2 oscillations in the period function (see [CJ89]), because in the center case only T_2, T_4, and T_6 are necessary to characterize the isochronicity property. In fact, $T_8^2 = 0$ in $\mathcal{B}_3 = \langle V_3, T_2, V_5, T_4, T_6\rangle$.

Let us make the following observation regarding the quadratic case. It is a well-known fact that the general quadratic family (3.7) classically unfolds 3 limit cycles, hence the case $V_3 = V_5 = 0$ with nonvanishing V_7 exists. This is not the case for the simultaneous study of Lyapunov and period constants we have seen here, but as we are adding the conditions for vanishing the corresponding period constants, the fact that the obtained Lyapunov constants vanish at lower orders is not inconsistent.

(iv) Finally, let us study the bi-weakness of the cubic homogeneous nonlinearity family (3.8). To simplify the expressions we will perform a change to complex coordinates, obtaining

$$\begin{cases} \dot{z} = \mathrm{i}\,z + r_{30}z^3 + r_{21}z^2w + r_{12}zw^2 + r_{03}w^3, \\ \dot{w} = -\mathrm{i}\,w + s_{03}z^3 + s_{12}z^2w + s_{21}zw^2 + s_{30}w^3, \end{cases}$$

being $s_{ij} = \bar{r}_{ij}$. We will proceed analogously to the quadratic case, by studying V_{2m+1} and T_{2m} in the Bautin ideal $\mathcal{B}_{m-1} := \langle V_{2k+1}, T_{2k}\rangle_{k\le m-1}$.

The first Lyapunov and period constants are

$$V_3 = -r_{21} - s_{21}, \; T_2 = \mathrm{i}(r_{21} - s_{21}).$$

The next step is to find the expressions of V_5 and T_4 simplified with respect to $\mathcal{B}_1 = \langle V_3, T_2 \rangle$,

$$V_5 = -2\,\mathrm{i}(r_{12}r_{30} - s_{12}s_{30}), \; T_4 = 3r_{03}s_{03} - 2r_{12}r_{30} + 4r_{12}s_{12} - 2s_{12}s_{30}.$$

Being in $\mathcal{B}_2 = \langle V_3, T_2, V_5, T_4 \rangle$ we have

$$\begin{aligned} V_7 =& \frac{3}{8}(3r_{03}r_{30}^2 - 8r_{03}r_{30}s_{12} - 3r_{03}s_{12}^2 - 3r_{12}^2 s_{03} - 8r_{12}s_{03}s_{30} + 3s_{03}s_{30}^2), \\ T_6 =& -\frac{3}{8}\,\mathrm{i}(3r_{03}r_{30}^2 - 16r_{03}r_{30}s_{12} + 21r_{03}s_{12}^2 - 21r_{12}^2 s_{03} + 16r_{12}s_{03}s_{30} - 3s_{03}s_{30}^2). \end{aligned}$$

As they are not identically zero, this proves the existence of a bi-weak [7, 6] type. The proof finishes due to the fact that $V_9 = 0$ in $\mathcal{B}_3 = \langle V_3, T_2, V_5, T_4, V_7, T_6 \rangle$. □

3.4 Some results on Liénard families

This section is devoted to the study of Liénard systems and it is divided into three parts. First, we use the Lie bracket method to deduce the structure of Lyapunov and period constants of a Liénard system starting with an odd and an even degree monomials on its first differential equation. Second, we classify the centers and the isochronicity of a Liénard family. Finally, we use the previous results to provide a complete study of the simultaneous bifurcation of limit cycles and critical periods of the cubic Liénard family, which proves Theorem 3.4.

3.4.1 A Liénard family starting with an odd and an even degree monomials

In this subsection we will consider the Liénard family

$$\begin{cases} \dot{x} = -y + a_m x^m + a_n x^n + x^d P(x), \\ \dot{y} = x, \end{cases} \tag{3.39}$$

where m and n are even and odd natural numbers, respectively, $d = \max(m, n) + 1$, and $P(x)$ is a polynomial in x.

In order to motivate the problem, let us start by considering (3.39) being $P_d(x) = 0$. It is a well-known fact that in this classical Liénard family the coefficients corresponding to odd powers control the center property, so if $a_n = 0$ we have a center at the origin. In this case the even power controls the isochronicity property, so if $a_m = 0$ the system has an isochronous center. A study in this line is presented for example in [Cim+97], but in our case we will present our result by using the Lie bracket method introduced in Subsection 3.2.3 to find the Lyapunov and period constants. It is worth recalling that different methods can lead to Lyapunov and period constants that differ in a multiplicative constant, but the dependence on parameters a_m and a_n is the same and the center and linearizability conditions are also kept. The result is as follows.

Theorem 3.13. *For system (3.39),*

(i) if $n < 2m-1$, then the first nonidentically zero Lyapunov constant is

$$V_n = \frac{a_n}{2^n}(n-1)\binom{n}{\frac{n-1}{2}}, \tag{3.40}$$

and the system vanishes all its period constants up to order n;

(ii) if $n = 2m-1$, then the first nonidentically zero Lyapunov and period constants are (3.40) and

$$T_{2m-2} = -\frac{a_m^2}{2^{2m-1}}\frac{m^2(m-1)}{(m+1)^2}\binom{2m}{m}, \tag{3.41}$$

respectively;

(iii) if $n > 2m-1$, then the first nonidentically zero period constant is (3.41), and the system vanishes all its Lyapunov constants constants up to order $2m-1$.

Proof. Let us start by writing system (3.39) in complex coordinates. By using the Binomial Theorem,

$$\begin{cases} \dot z = \mathrm{i}\, z + \dfrac{a_m}{2^m}\displaystyle\sum_{j=0}^{m}\binom{m}{j} z^{m-j}w^j + \dfrac{a_n}{2^n}\sum_{k=0}^{n}\binom{n}{k} z^{n-k}w^k + \left(\dfrac{z+w}{2}\right)^d P(z,w), \\ \dot w = -\mathrm{i}\, w + \dfrac{a_m}{2^m}\displaystyle\sum_{j=0}^{m}\binom{m}{j} z^{j}w^{m-j} + \dfrac{a_n}{2^n}\sum_{k=0}^{n}\binom{n}{k} z^{k}w^{n-k} + \left(\dfrac{z+w}{2}\right)^d \overline{P}(z,w), \end{cases} \tag{3.42}$$

where we denote the first equation by $\dot z = \mathcal{Z}(z,w)$ and the second one by $\dot w = \overline{\mathcal{Z}}(z,w)$. We will consider the system

$$\begin{cases} \dot z = z + \displaystyle\sum_{l=0}^{m}(A_{m-l,l} + \mathrm{i}\, B_{m-l,l}) z^{m-l}w^l + \sum_{p=0}^{n}(A_{n-p,p} + \mathrm{i}\, B_{n-p,p}) z^{n-p}w^p, \\ \dot w = w + \displaystyle\sum_{l=0}^{m}(A_{m-l,l} - \mathrm{i}\, B_{m-l,l}) z^{l}w^{m-l} + \sum_{p=0}^{n}(A_{n-p,p} - \mathrm{i}\, B_{n-p,p}) z^{p}w^{n-p}, \end{cases} \tag{3.43}$$

where we denote the first equation by $\dot{z} = \mathcal{U}(z,w)$ and the second by $\dot{w} = \overline{\mathcal{U}}(z,w)$. To calculate the Lyapunov and period constants we will find the structure of the Lie bracket of (3.42) and (3.43). Observe that, due to the fact that they are associated to a real vector field, by using (2.6) their Lie bracket can be described only from its first component:

$$\begin{aligned}[\mathcal{Z},\mathcal{U}]^1 &= \frac{\partial \mathcal{Z}}{\partial z}\mathcal{U} + \frac{\partial \mathcal{Z}}{\partial w}\overline{\mathcal{U}} - \frac{\partial \mathcal{U}}{\partial z}\mathcal{Z} - \frac{\partial \mathcal{U}}{\partial w}\overline{\mathcal{Z}} \\ &= \mathcal{H}_m + \mathcal{H}_n + \mathcal{H}_{2m-1} + \mathcal{H}_{2n-1} + O_{m+n-1}(z,w),\end{aligned} \tag{3.44}$$

where each $\mathcal{H}_q$ is an homogeneous qth degree polynomial in z, w. These polynomials have been found, but they are not written here for the sake of brevity.

We have that

$$\mathcal{H}_m = \sum_{l=0}^{m} \left[(-B_{m-l,l} + \mathrm{i}\, A_{m-l,l})(2l-m+1) + \frac{a_m}{2^m}(m-1)\binom{m}{l} \right] z^{m-l} w^l. \tag{3.45}$$

This homogeneous mth degree part vanishes taking

$$A_{m-l,l} = 0,\; B_{m-l,l} = \frac{a_m}{2^m}\frac{m-1}{2l-m+1}\binom{m}{l} \text{ for } l \in \{0,\dots,m\}. \tag{3.46}$$

The homogeneous nth degree polynomial $\mathcal{H}_n$ is analogous to (3.45), only changing m by n. We can take then

$$\begin{aligned} A_{n-p,p} &= 0 \text{ for } p \in \{0,\dots,n\}, \\ B_{n-p,p} &= \frac{a_n}{2^n}\frac{n-1}{2p-n+1}\binom{n}{p} \text{ for } p \in \{0,\dots,n\} \backslash \left\{\frac{n-1}{2}\right\}, \end{aligned} \tag{3.47}$$

as for the term corresponding to $p = \frac{n-1}{2}$ we have that $2p - n + 1 = 0$, and therefore the coefficient of $z^{\frac{n+1}{2}} w^{\frac{n-1}{2}} = z(zw)^{\frac{n-1}{2}}$ cannot be vanished; in this case we take $B_{\frac{n+1}{2},\frac{n-1}{2}} = 0$. Notice that this fact occurs with n because it is an odd number, but not with m which is even.

Let us see the structure of $\mathcal{H}_{2m-1}$. After substituting (3.46), we obtain

$$\begin{aligned}\mathcal{H}_{2m-1} = \mathrm{i}\,\frac{a_m^2}{2^{2m}} \sum_{j,l=0}^{m} \frac{m-1}{2l-m+1}\binom{m}{l}\binom{m}{j} \Big[&(l-j) z^{2m-l-j-1} w^{j+l} \\ &- j z^{m+l-j} w^{m+j-l-1} - l z^{m+j-l} w^{m+l-j-1} \Big].\end{aligned} \tag{3.48}$$

We know by Theorem 3.8 that the period constant associated to (3.48) is the coefficient of $z(zw)^{m-1}$. Then, after some basic combinatorial computations, this coefficient can be written as

$$p_{2m-1} = \mathrm{i}\,\frac{a_m^2}{2^{2m}}(m-1)\left(\sum_{j=0}^{m-1}\binom{m}{j}\binom{m}{j+1} - \sum_{j=0}^{m}\frac{2j}{2j-m+1}\binom{m}{j}^2\right). \tag{3.49}$$

Let us write this expression in a more compact way. We can use the generating functions $(1+x)^m$ and $(1+x)^{2m}$ to rewrite the first summation in (3.49). In the relation

$$\sum_{k=0}^{2m}\binom{2m}{k}x^k = (1+x)^{2m} = ((1+x)^m)^2 = \left(\sum_{k=0}^{m}\binom{m}{k}x^k\right)^2 = \sum_{k,j=0}^{m}\binom{m}{k}\binom{m}{j}x^{k+j},$$

we equate the coefficients of x^{m-1}, and we obtain

$$\sum_{j=0}^{m-1}\binom{m}{j}\binom{m}{j+1} = \binom{2m}{m-1} = \frac{m}{m+1}\binom{2m}{m}. \tag{3.50}$$

For the second summation in (3.49), the equality

$$\sum_{j=0}^{m}\frac{2j}{2j-m+1}\binom{m}{j}^2 = \frac{m(3m+1)}{(m+1)^2}\binom{2m}{m} \tag{3.51}$$

holds. This equality has been obtained with a computer algebra system by means of the Zeilberger's algorithm. For more details on how to use such method, the reader is referred to [Zei90; Zei91], as well as [Koe14] and the references therein. Notice that this Zeilberger's algorithm can also be used to justify (3.50).

Adding (3.50) and (3.51), one can rewrite (3.49) as

$$p_{2m-1} = -\,\mathrm{i}\,\frac{a_m^2}{2^{2m-1}}\frac{m^2(m-1)}{(m+1)^2}\binom{2m}{m}. \tag{3.52}$$

After these calculations we can finally prove our result. To this end, we will consider (3.44) assuming (3.46) and (3.47), so $\mathcal{H}_m = 0$ and $\mathcal{H}_n$ only has the $z(zw)^{\frac{n-1}{2}}$ term. If $n < 2m-1$, then the lowest degree homogeneous polynomial is the nth degree one,

$$\mathcal{H}_n = \frac{a_n}{2^n}(n-1)\binom{n}{\frac{n-1}{2}}z(zw)^{\frac{n-1}{2}},$$

and due to Theorem 3.8 this coefficient is the first nonidentically zero Lyapunov constant (3.40), while all period constants vanish up to this order.

If $n = 2m-1$, then the lowest degree homogeneous polynomial in (3.44) is $\mathcal{H}_n + \mathcal{H}_{2m-1}$, so the coefficient of $z(zw)^{\frac{n-1}{2}} = z(zw)^{m-1}$ is (3.40) plus (3.52). Therefore, by Theorem 3.8 the real part is the Lyapunov constant (3.40) and the imaginary part is the first nonidentically zero period constant (3.41).

Finally, for $n > 2m-1$, an analogous procedure shows that the lowest degree homogeneous polynomial in (3.44) is $\mathcal{H}_{2m-1}$, so the period constant is the imaginary part of (3.52) and all Lyapunov constants vanish up to this order. □

3.4.2 Center and isochronicity classification of a Liénard family

In this subsection we present, as an application of our approach, a characterization of isochronous centers in a Liénard family. For a general proof we refer the reader to the very recent work of Amel'kin ([Ame21]). We start with the following result.

Proposition 3.14. *Let us consider the Liénard system*

$$\begin{cases} \dot{x} = -y + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6, \\ \dot{y} = x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6. \end{cases} \tag{3.53}$$

The only isochronous center having the form (3.53) is (3.38).

Proof. We start by finding the first five Lyapunov and period constants of (3.53) by using the Lie bracket method from Subsection 3.2.3. Then we solve the resulting system $\{V_3 = V_5 = V_7 = V_9 = V_{11} = T_2 = T_4 = T_6 = T_8 = T_{10} = 0\}$, and we compute that the only nontrivial solution is $\{a_3 = 0, a_4 = 0, a_5 = 0, b_2 = 0, 9b_3 - 4a_2^2 = 0, b_4 = 0, b_5 = 0\}$, which corresponds to system (3.38) and has a center at the origin due to Theorem 1.11.

To show the isochronicity of (3.38) we will start by rescaling the system via the change $(x,y) \to \left(\frac{3}{2a_2}x, \frac{3}{2a_2}y\right)$ for $a_2 \neq 0$, which results in

$$\begin{cases} \dot{x} = -y + \dfrac{3}{2}x^2, \\ \dot{y} = x + x^3. \end{cases} \tag{3.54}$$

Observe that for the case $a_2 = 0$, the system is isochronous because it corresponds to the linear center. System (3.54) is isochronous because it commutes with

$$\begin{cases} \dot{x} = xA(x,y), \\ \dot{y} = \left(y + \dfrac{1}{2}x^2\right) A(x,y), \end{cases}$$

where $A(x,y) = -1 - y + \frac{1}{2}x^2$, as the Lie bracket between both systems is 0.

We will present an alternative proof for the isochronicity of (3.54), because we consider that it is interesting as it simply uses a first integral and the system itself. A first integral of (3.54) is

$$H(x,y) = \frac{x^4 - 4x^2y + 4x^2 + 4y^2}{(x^2 - 2y - 2)^2}, \tag{3.55}$$

which satisfies that $H(0,0) = 0$. The idea is to prove the isochronicity of (3.54) by using (3.55) to find an expression for the level curves γ_h and check that its integral through a whole loop is 2π.

Let us consider $H(x,y) = h^2$ with $h > 0$ and solve it for x. As it has degree 4 in x we obtain four solutions, two of which are imaginary for values of h close to 0. The other two solutions are

$$X_\pm(y,h) = \pm\frac{\sqrt{-2(h^2-1)(-h^2y - h^2 + \sqrt{2h^2y + 3h^2 - 2y + 1} + y - 1)}}{h^2 - 1},$$

which are real for $0 < h < 1$, and they correspond to the level curve on the right and left side of the vertical axis. By solving $H(0,y) = h^2$ with respect to y, we can find that the intersections of the level curves with the vertical axis for level h^2 are $y_\pm = -\frac{h}{h\mp 1}$, being $0 < h < 1$. Considering the second equation in (3.54), we aim to calculate the integral for the time

$$T(h) = \int_0^{T(h)} dt = \int_{\gamma_h} \frac{dy}{\dot{y}} = 2\int_{y_-}^{y_+} \frac{dy}{X_+(y,h) + X_+^3(y,h)} =: 2I(h),$$

and see that we obtain 2π. Here we have used that, due to the symmetry of the problem, we can simply integrate from y_- to y_+ and check that we obtain $I(h) = \pi$, which represents half a loop.

To simplify the expression of $I(h)$ we will consider the change of variables $z^2 = 2h^2y + 3h^2 - 2y + 1$, or equivalently $y = -\frac{1}{2}\frac{3h^2 - z^2 + 1}{h^2 - 1}$. After applying this change both to $X_+(y,h)$ and the integration limits $y_\pm$, the integral becomes

$$I(h) = \int_{1+h}^{1-h} -\frac{1 - h^2}{(z-2)\sqrt{-h^4 + h^2z^2 - 2h^2z + 2h^2 - z^2 + 2z - 1}} dz.$$

This integral can be simplified a little more so that the integration limits become ± 1. Let us apply the change $z = 1 - hw$ to a new variable w, and we have

$$I(h) = \int_{-1}^{1} \frac{\sqrt{1-h^2}}{\sqrt{1-w^2}(1+hw)} dw = \lim_{k \to 1} \int_{-k}^{k} \frac{\sqrt{1-h^2}}{\sqrt{1-w^2}(1+hw)} dw$$
$$= \lim_{k \to 1} \left[\arctan \frac{h+w}{\sqrt{1-h^2}\sqrt{1-w^2}} \right]_{w=-k}^{w=k} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi,$$

for $0 < h < 1$, so the result follows. □

By studying the Liénard system from the previous result we have come across the Liénard family

$$\begin{cases} \dot{x} = -y + ax^n, \\ \dot{y} = x + bx^{2n-1}, \end{cases} \tag{3.56}$$

for $a, b \in \mathbb{R}, n \in \mathbb{N}$, being $n \geq 2$. Indeed, (3.38) is a particular case of (3.56). The centers and isochronicity of this new family are classified on Theorem 3.16. For the proof of this classification result, we first need the following proposition.

Proposition 3.15. *System* (3.56) *with even* n, $b = \frac{n^2}{(n+1)^2}a^2$, *and* $a \neq 0$ *has a first integral of the form*

$$H(x,y) = \frac{A(x,y)}{(x^2 + y^2 - 2x^n y + x^{2n})^{\frac{n-1}{2}}}, \tag{3.57}$$

being $A(x,y)$ *a polynomial in* x, y.

Proof. For $b = \frac{n^2}{(n+1)^2}a^2$ and $a \neq 0$, system (3.56) can be rewritten as

$$\begin{cases} \dot{x} = -y + (n+1)x^n =: P(x,y), \\ \dot{y} = x + n^2 x^{2n-1} =: Q(x,y), \end{cases} \tag{3.58}$$

after the rescaling $(x,y) \to \left(\left(\frac{n+1}{a}\right)^{\frac{1}{n-1}} x, \left(\frac{n+1}{a}\right)^{\frac{1}{n-1}} y \right)$. It can be checked that function $F(x,y) = x^2 + y^2 - 2x^n y + x^{2n}$ is an invariant curve of system (3.58) with cofactor $K = 2nx^{n-1}$, and the divergence of the vector field is $\operatorname{div}(P,Q) = n(n+1)x^{n-1}$. Now according to Darboux integrability theory (see Theorem 1.17ii), for λ satisfying $\lambda K = -\operatorname{div}(P,Q)$ we have that $R(x,y) = F(x,y)^{\lambda}$ is an integrating factor of the system. In our case, $\lambda K = -\operatorname{div}(P,Q)$ is $\lambda 2nx^{n-1} = -n(n+1)x^{n-1}$, so $\lambda = -\frac{n+1}{2}$ and $R(x,y) = \left(x^2 + y^2 - 2x^n y + x^{2n}\right)^{-\frac{n+1}{2}}$ is an integrating factor.

Having an integrating factor $R(x,y)$ of the system, we know that a first integral $H(x,y)$ satisfies $\partial H/\partial x = Q(x,y)R(x,y)$ and $\partial H/\partial y = -P(x,y)R(x,y)$, so we can integrate this second equation to find the form of such first integral,

$$\begin{aligned} H(x,y) &= \int -P(x,y)R(x,y)dy = \int \frac{y - nx^n - x^n}{(x^2+y^2-2x^ny+x^{2n})^{\frac{n+1}{2}}}dy \\ &= \int \frac{y - x^n}{(x^2+y^2-2x^ny+x^{2n})^{\frac{n+1}{2}}}dy - n\int \frac{x^n}{(x^2+y^2-2x^ny+x^{2n})^{\frac{n+1}{2}}}dy. \end{aligned} \tag{3.59}$$

The first term in (3.59) is an immediate integral and its result is $\frac{-1}{(n-1)(x^2+y^2-2x^ny+x^{2n})^{\frac{n-1}{2}}}$, taking into account that due to being a first integral we can consider that the integration constant is 0. For the second integral we will perform the change of variables $\omega = \frac{x}{\sqrt{x^2+y^2-2x^ny+x^{2n}}}$, so $dy = -\frac{x}{\omega^2\sqrt{1-\omega^2}}d\omega$ and the integral can be written as

$$\int \frac{x^n}{(x^2+y^2-2x^ny+x^{2n})^{\frac{n+1}{2}}}dy = -\int \frac{\omega^{n-1}}{\sqrt{1-\omega^2}}d\omega.$$

Now we can apply a trigonometric change of variables $\omega = \sin\phi$, and

$$\begin{aligned} -\int \frac{\omega^{n-1}}{\sqrt{1-\omega^2}}d\omega &= -\int \frac{\sin^{n-1}\phi}{\sqrt{1-\sin^2\phi}}\cos\phi d\phi = -\int \sin^{n-1}\phi d\phi \\ &= -\int \sin\phi \sin^{n-2}\phi d\phi. \end{aligned}$$

As n is even, so is $n-2$ and the integral becomes

$$\begin{aligned} -\int \sin\phi \left(\sin^2\phi\right)^{\frac{n-2}{2}} d\phi &= -\int \sin\phi \left(1-\cos^2\phi\right)^{\frac{n-2}{2}} d\phi \\ &= -\int \sin\phi \left(\sum_{j=0}^{\frac{n-2}{2}} a_j \cos^{2j}\phi\right) d\phi, \end{aligned}$$

for certain coefficients a_j, where in the last equality we have used the Binomial Theorem. Then, after swapping the sum and the integral we get

$$-\sum_{j=0}^{\frac{n-2}{2}} a_j \int \sin\phi\cos^{2j}\phi d\phi = \sum_{j=0}^{\frac{n-2}{2}} \frac{a_j}{2j+1}\cos^{2j+1}\phi = \cos\phi \sum_{j=0}^{\frac{n-2}{2}} \frac{a_j}{2j+1}\cos^{2j}\phi.$$

By using $\cos^{2j}\phi = (1-\sin^2\phi)^j$ and applying the Binomial Theorem again, we have that for new coefficients b_j,

$$\cos\phi \sum_{j=0}^{\frac{n-2}{2}} \frac{a_j}{2j+1}\cos^{2j}\phi = \cos\phi \sum_{j=0}^{\frac{n-2}{2}} b_j \sin^{2j}\phi = \sqrt{1-\omega^2}\sum_{j=0}^{\frac{n-2}{2}} b_j\omega^{2j},$$

after undoing the change $\omega = \sin\phi$. To finish the proof, we can substitute $\omega = \frac{x}{\sqrt{x^2+y^2-2x^ny+x^{2n}}}$, and after some trivial calculations we get

$$\sqrt{1-\omega^2}\sum_{j=0}^{\frac{n-2}{2}} b_j\omega^{2j} = \frac{(y-x^n)\sum_{j=0}^{\frac{n-2}{2}} b_j x^{2j}(x^2+y^2-2x^ny+x^{2n})^{\frac{n-2}{2}-j}}{(x^2+y^2-2x^ny+x^{2n})^{\frac{n-1}{2}}},$$

so joining both terms from (3.59) the result follows. □

Theorem 3.16. *The Liénard family (3.56) satisfies that,*

(i) for odd n, the system is a center if and only if $a=0$ –in which case it is a reversible center–, and it is an isochronous center if and only if $a=b=0$;

(ii) for even n, the system is a reversible center for any a and b, and it is an isochronous center if and only if $b = \frac{n^2}{(n+1)^2}a^2$.

Proof. Let us start with the case of n being odd. If $a=0$, it is immediate to see that (3.56) is a time-reversible center with respect to the vertical axis. Consider the function

$$H(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{b}{2n}x^{2n}, \tag{3.60}$$

and compute

$$\nabla H\cdot(\dot{x},\dot{y}) = \left(x+bx^{2n-1},y\right)\cdot\left(-y+ax^n, x+bx^{2n-1}\right) = ax^{n+1}\left(1+bx^{2n-2}\right).$$

We can see then that for $a=0$ function H is a first integral of the system, and for $a\neq 0$ it is a Lyapunov function of the system. In the latter case, the origin is a focus and the sign of a determines its stability near the origin.

We will prove now the isochronicity condition provided that we have a center, i.e. $a=0$. If $b=0$, (3.56) becomes the linear center so it is isochronous. Now let us show that if $b\neq 0$ then the system is not isochronous, for which we will integrate the second equation in (3.56) along the level curves γ_h determined by the first integral (3.60). The integral to solve is

$$T(h) = \int_0^{T(h)} dt = \int_{\gamma_h}\frac{dy}{\dot{y}} = \int_{\gamma_h}\frac{dy}{x+bx^{2n-1}}. \tag{3.61}$$

By considering (3.60) on a level curve such that $H(x,y) = \frac{h^2}{2}$ being $h > 0$, we obtain that $y^2 = h^2 - x^2 - \frac{b}{n}x^{2n}$, and we can use this relation to perform a change of variables from y to x in (3.61) as follows:

$$\begin{aligned} T(h) &= \int_{\gamma_h} \frac{dy}{x + bx^{2n-1}} = 2\int_{x_h^-}^{x_h^+} \frac{dx}{\sqrt{h^2 - x^2 - \frac{b}{n}x^{2n}}} \\ &= 4\int_0^{x_h} \frac{dx}{\sqrt{h^2 - x^2 - \frac{b}{n}x^{2n}}}, \end{aligned} \tag{3.62}$$

where we have used the symmetry of the integral and $x_h^-, x_h^+ = x_h$, are the intersections of the level curve with the horizontal axis, i.e. the real solutions of $H(x,0) = \frac{h^2}{2}$ or equivalently $x^2 + \frac{b}{n}x^{2n} - h^2 = 0$. As x_h depends on h, we will perform a second change of variables $x = x_h z$ so that the integration limits are constant, so we rewrite (3.62) as

$$T(h) = 4\int_0^1 \frac{x_h}{\sqrt{h^2 - {x_h}^2 z^2 - \frac{b}{n}{x_h}^{2n} z^{2n}}} dz. \tag{3.63}$$

As we are not able to find the explicit solution of $L(h) := x^2 + \frac{b}{n}x^{2n} - h^2 = 0$ to have an expression of x_h, we consider a power series expansion of x_h with respect to h. One can check that such series expansion starts

$$x_h = h - \frac{b}{2n}h^{2n-1} + O_{4n-3}(h), \tag{3.64}$$

as for this x_h we have

$$\begin{aligned} L(h) = &\left(h - \frac{b}{2n}h^{2n-1} + O_{4n-3}(h)\right)^2 + \frac{b}{n}\left(h - \frac{b}{2n}h^{2n-1} + O_{4n-3}(h)\right)^{2n} - h^2 \\ =&h^2 - \frac{b}{n}h^{2n} + O_{4n-2}(h) + \frac{b}{n}\left(h^{2n} + O_{4n-2}(h)\right) - h^2 = O_{4n-2}(h). \end{aligned}$$

Now by substituting (3.64) in (3.63) and developing the expression we have

$$T(h) = 4\int_0^1 \frac{1 - \frac{b}{2n}h^{2n-2} + O_{4n-4}(h)}{\sqrt{1 - z^2 + \frac{b}{n}\left(z^2 - z^{2n}\right)h^{2n-2} + O_{4n-4}(h)}} dz,$$

where we can expand the denominator as a power series and obtain

$$\begin{aligned} T(h) =& 4\int_0^1 \left(1-\frac{b}{2n}h^{2n-2}+O_{4n-4}(h)\right)\\ &\left(\frac{1}{\sqrt{1-z^2}}-\frac{b}{2n}\frac{z^2-z^{2n}}{(1-z^2)^{3/2}}h^{2n-2}+O_{4n-4}(h)\right)dz,\\ =& \ 4\int_0^1\left[\frac{1}{\sqrt{1-z^2}}-\frac{b}{2n}\left(\frac{1}{\sqrt{1-z^2}}+\frac{z^2-z^{2n}}{(1-z^2)^{3/2}}\right)h^{2n-2}+O_{4n-4}(h)\right]dz,\\ =& 4\int_0^1\left[\frac{1}{\sqrt{1-z^2}}-\frac{b}{2n}\frac{1+z^2+z^4+\cdots+z^{2n-2}}{\sqrt{1-z^2}}h^{2n-2}+O_{4n-4}(h)\right]dz. \end{aligned}$$

Notice that on the last equality we have used the formula for the sum of terms in a geometric progression. Finally, this integral becomes

$$\begin{aligned} T(h) &= 4\int_0^1\frac{dz}{\sqrt{1-z^2}}-\frac{2b}{n}h^{2n-2}\int_0^1\frac{1+z^2+z^4+\cdots+z^{2n-2}}{\sqrt{1-z^2}}dz+O_{4n-4}(h)\\ &= 2\pi-\frac{2b}{n}h^{2n-2}\int_0^1\frac{1+z^2+z^4+\cdots+z^{2n-2}}{\sqrt{1-z^2}}dz+O_{4n-4}(h). \end{aligned}$$

As the remaining integral has a strictly positive integrand in the considered interval, the integral is strictly positive and nonzero, so if $b\neq 0$ the term of order $2n-2$ in h is nonzero and the center is not isochronous, hence the isochronicity condition is proved.

For the case of n being even, it is straightforward to see that (3.56) is a time-reversible center for any value of the parameters a and b. To prove the isochronicity condition, we will see that if $b=\frac{n^2}{(n+1)^2}a^2$ then there exists a transversal system such that its Lie bracket with the original system vanishes, and for the reciprocal we will check that if $b\neq\frac{n^2}{(n+1)^2}a^2$ then the period function is not constant.

Let us assume that $b=\frac{n^2}{(n+1)^2}a^2$, and consider the rescaled system (3.58). According to Proposition 3.15, this system has a first integral of the form (3.57). Observe that, due to the fact of being a first integral, the condition $\frac{\partial H}{\partial x}P+\frac{\partial H}{\partial y}Q=0$ must be satisfied. In our case, when finding the partial derivatives of the first integral (3.57) we obtain

$$\begin{aligned} \frac{\partial H}{\partial x}P+\frac{\partial H}{\partial y}Q=-&\left(n(n-1)x^nA(x,y)+\left(xy-(n+1)x^{n+1}\right)\frac{\partial A}{\partial x}\right.\\ &\left.-\left(x^2+n^2x^{2n}\right)\frac{\partial A}{\partial y}\right)F(x,y), \end{aligned}$$

where $F(x,y) = x^2 + y^2 - 2x^n y + x^{2n}$ is the invariant curve of the system found in the proof of Proposition 3.15. Therefore, the condition that must be fulfilled is

$$n(n-1)x^n A(x,y) + \left(xy - (n+1)x^{n+1}\right)\frac{\partial A}{\partial x} - \left(x^2 + n^2 x^{2n}\right)\frac{\partial A}{\partial y} = 0. \tag{3.65}$$

Let us consider system

$$\begin{cases} \dot{x} = xA(x,y), \\ \dot{y} = (y + (n-1)x^n)\, A(x,y), \end{cases} \tag{3.66}$$

where $A(x,y)$ is the numerator of the first integral (3.57) of the system. A straightforward computation shows that the Lie bracket between systems (3.58) and (3.66) is exactly the left hand side of equality (3.65), so by construction it equals 0 and the system is isochronous due to Theorem 2.4.

Finally, we have to check that if $b \neq \frac{n^2}{(n+1)^2}a^2$, then the center is not isochronous. To this end, we will use the Lie bracket method to find the first period constant and check that it only vanishes for $b = \frac{n^2}{(n+1)^2}a^2$. System (3.56) can be rewritten in complex coordinates as

$$\dot{z} = \mathrm{i}\, z + a\left(\frac{z+w}{2}\right)^n + \mathrm{i}\, b\left(\frac{z+w}{2}\right)^{2n-1}, \tag{3.67}$$

which is actually system (3.42) choosing $a_m = a$, $a_n = \mathrm{i}\, b$, and $P(z,w) = 0$, and taking into account that (m,n) in (3.42) corresponds to $(n, 2n-1)$ in (3.67). By applying the Lie bracket method analogously to the proof of Theorem 3.13, one can see that the first nonzero term is that of degree $2n-1$, whose coefficient is

$$p_{2n-1} = \mathrm{i}\, \frac{b}{2^{2n-1}}(2n-2)\binom{2n-1}{n-1} - \mathrm{i}\, \frac{a^2}{2^{2n-1}} \frac{n^2(n-1)}{(n+1)^2}\binom{2n}{n}. \tag{3.68}$$

By Theorem 3.8, the imaginary part of (3.68) is the period constant

$$\begin{aligned} T_{2n-2} =& \frac{b}{2^{2n-1}}(n-1)\binom{2n}{n} - \frac{a^2}{2^{2n-1}}\frac{n^2(n-1)}{(n+1)^2}\binom{2n}{n} \\ =& \left(b - \frac{n^2}{(n+1)^2}a^2\right)\frac{n-1}{2^{2n-1}}\binom{2n}{n}, \end{aligned}$$

which only vanishes for $b = \frac{n^2}{(n+1)^2}a^2$. □

After presenting the new isochronous family (3.56) for even n and $b = \frac{n^2}{(n+1)^2}a^2$, a natural question is to wonder if these isochronous centers will provide a high criticality under perturbation –notice that we do not consider odd n because in this case the only isochronous center is the linear one. To inquire into this question, we find the number of critical periods which can bifurcate from the system by considering only linear parts in the perturbative parameters of the period constants, following the ideas presented in Chapter 2. In particular, we have performed this study for $n = 2, 4, 6, 8$, which correspond to systems of degrees $2n - 1 = 3, 7, 11, 15$ respectively, and by adding either a time-reversible perturbation with respect to the horizontal axis $(x, y, t) \to (x, -y, -t)$ or with respect to the vertical axis $(x, y, t) \to (-x, y, -t)$, none of which breaks the center property. After computing period constants up to first-order and calculating the corresponding ranks, we find the number of critical periods presented in the next chart. We also show the number of critical periods $(n^2 + n - 4)/2$ obtained in Section 2.3 for a class of holomorphic systems also with linear parts, which were found with a reversible perturbation with respect to the horizontal axis.

n	Degree	Critical periods (reversible respect horizontal axis)	Critical periods (reversible respect vertical axis)	$\frac{n^2+n-4}{2}$
2	3	4	2	4
4	7	20	13	26
6	11	44	30	64
8	15	76	53	118

As we can check, the obtained criticality of the isochronous family (3.56) for even n found up to linear parts in the period constants is much worse than the one obtained for the holomorphic family in Section 2.3 also with linear parts. This leads us to believe that family (3.56) will not provide a high number of oscillations of the period function, so we will not go further in the study of its criticality.

3.4.3 The cubic Liénard family (k, l) classification

The aim of this subsection is to study the simultaneous bifurcation of limit cycles and critical periods of the cubic Liénard system (3.5) adding the trace parameter α as in (3.1). First, we will start by finding some Lyapunov and period constants of this system being $\alpha = 0$, obtaining the following results:

$$\begin{aligned} V_3 &= -\frac{1}{4}\pi\left(2a_2b_2 - 3a_3\right), \; V_5 = \frac{5}{12}\pi a_2 b_2 b_3, \\ T_2 = \frac{1}{12}\pi\left(4a_2^2 + 10b_2^2 - 9b_3\right), &\; T_3 = \frac{1}{12}a_2\pi\left(2a_2b_2 - 3a_3\right), \\ T_4 &= -\frac{1}{3456}\pi\left(3484a_2^2b_2^2 + 4480b_2^4 + 81a_3^2\right), \\ T_5 &= -\frac{1}{162}b_2\pi\left(10a_2^4 + 3193a_2^2b_2^2 + 4032b_2^4\right). \end{aligned} \tag{3.69}$$

Notice that these expressions are the corresponding constants given that the previous constants equal zero. These constants have been found by using the classical method instead of the Lie bracket method.

Let us make the following observation. As we already saw for the quadratic system (3.4), in a system not having a center at the origin it is not generally true that its first period constant must have even subscript. The presented cubic Liénard system is another example of this fact. If we compare V_3 and T_3 we can observe this: for a center we would have $V_3 = 0$ which means $2a_2b_2 - 3a_3 = 0$, and this implies $T_3 = 0$, but T_3 can be nonzero if $V_3 \neq 0$. We can also see by comparing V_5 and T_5 that this equality in the factors of V_3 and T_3 is not a general fact for any pair V_{2k+1} and T_{2k+1}.

Our aim is to prove Theorem 3.4, which provides a complete classsification of the simultaneous cyclicity and criticality for the cubic Liénard system (3.5).

Proof of Theorem 3.4. (i) Let us start by finding the centers in family (3.5). This was already done in Theorem 1.29, but we recall it here for completeness. Solving the system formed by the two first Lyapunov constants of (3.5) being equal to 0, this is $\{V_3 = 0, V_5 = 0\}$, we obtain three solutions:

$$(\dot{x}, \dot{y}) = (-y, x + b_2x^2 + b_3x^3), \tag{3.70}$$

$$(\dot{x}, \dot{y}) = (-y + a_2x^2, x + b_3x^3), \tag{3.71}$$

$$(\dot{x}, \dot{y}) = \left(-y + a_2x^2 + \frac{2}{3}a_2x^3, x + b_2x^2\right). \tag{3.72}$$

Equations (3.70) and (3.71) are time-reversible families with respect to the $x-$axis and $y-$axis, respectively, and (3.72) corresponds to a center due to Theorem 1.11. Observe that such theorem is also an alternative proof for (3.71) having a center at the origin. Therefore, we have seen that for (3.5) only two Lyapunov constants are necessary to solve the center problem, this is $V_3 = V_5 = 0$ implies $V_{2k+1} = 0$ for any $k \geq 3$, which we already saw in Subsection 1.3.1.

Now we will check that (3.70) unfolds one critical period. To see this, we find from (3.69) that for this family $T_2 = \frac{1}{12}\pi\left(10b_2^2 - 9b_3\right)$, $T_3 = 0$ –as it is a center–, and $T_4 = -\frac{35}{27}\pi b_2^4$. Then, if $b_3 = \frac{10}{9}b_2^2$ we see that $T_2 = 0$ but T_4 can be different from 0, so 1 critical period unfolds and this proves the case $(\infty, 1)$. Now if we vanish T_2 and T_4, automatically $b_2 = b_3 = 0$ and (3.70) becomes the linear center, so it is isochronous and the case (∞, ∞) is also proved.

We will also see that for (3.71) and (3.72), when vanishing T_2 an isochronous system is obtained so no critical periods appear, thus the critical period obtained from (3.70) is maximal. For (3.71) we have that $T_2 = \frac{1}{12}\pi\left(4a_2^2 - 9b_3\right)$, and it vanishes for $b_3 = \frac{4}{9}a_2^2$, which corresponds to system (3.38) and it is isochronous due to Proposition 3.14. For (3.71), $T_2 = \frac{1}{12}\pi\left(4a_2^2 + 10b_2^2\right)$, which only vanishes for $a_2 = b_2 = 0$ and in this case the system becomes the linear center, hence is isochronous. This finishes the proof of the statement.

(ii) To prove this statement, let us recall that two Lyapunov constants characterize the center for the Liénard system given in (3.5), so for studying the simultaneous cyclicity and criticality we aim to find the highest possible $l \neq \infty$ for $(1, l)$ and $(2, l)$, as for $k \geq 3$ we have a center and $(k, l) = (\infty, l)$.

First we will prove the case $(1, l)$, so we start by assuming $\alpha = 0$ and $V_3 \neq 0$. If we vanish T_2, T_3, and T_4, we obtain system (3.38) and it is an isochronous center due to Proposition 3.14, so $l = \infty$ and this case is dismissed. When solving the system of equations $\{T_2 = 0, T_3 = 0\}$ we obtain two solutions: $S_1 = \{a_2 = \frac{3}{2}\frac{a_3}{b_2}, b_3 = \frac{1}{9}\frac{10b_2^4+9a_3^2}{b_2^2}\}$, $S_2 = \{a_2 = 0, b_3 = \frac{10}{9}b_2^2\}$. S_1 is dismissed because it vanishes V_3, but for S_2 we see that both V_3 and T_4 are not identically zero, and T_4 has the expression

$$T_4 = \frac{1}{3456}\pi(4480b_2^4 + 81a_3^2).$$

We can see then that T_4 can vanish only for $b_2 = a_3 = 0$, in which case we would have the linear center and therefore isochronous, so at most 3 critical periods could appear, this is $l \leq 3$. We will show now that these 3 critical periods can actually bifurcate from the origin by using S_2.

Let us set the free parameters a_3 and b_2 from S_2 to 1 and -3, respectively, and consider adding a perturbation by performing the change $(a_2, a_3, b_2, b_3) \to (\varepsilon e_1, 1 + \varepsilon e_2, -3 + \varepsilon e_3, 10 + \varepsilon e_4)$ on the Lyapunov and period constants, where ε is a small perturbative parameter. After this change,

$$T_2 = \frac{1}{12}\pi\varepsilon\left(-60e_3 - 9e_4 + 4\varepsilon e_1^2 + 10\varepsilon e_3^2\right),$$
$$T_3 = \frac{1}{12}\pi\varepsilon e_1\left(-3 - 6\varepsilon e_1 - 3\varepsilon e_2 + 2\varepsilon^2 e_1 e_3\right),$$

and we define $f_2 := T_2/\varepsilon$ and $f_3 := T_3/(\varepsilon e_1)$. We isolate e_4 in the expression of f_2 and e_2 in the expression of f_3, and we substitute them in V_3 and T_4, performing a change of variables $(e_1, e_2, e_3, e_4) \to (e_1, f_3, e_3, f_2)$. If we consider a nonzero trace 2α, according to (3.19) and (3.20) we have the following expressions for the displacement map and the period function for ρ small enough:

$$
\begin{aligned}
\Delta(\rho) &\approx (\mathrm{e}^{2\pi\alpha} - 1)\rho - 3f_3\rho^3 + O_4(\rho),\\
T(\rho) &\approx 2\pi + (\mathrm{e}^{2\pi\alpha} - 1)\widetilde{T}_1(\alpha)\,\rho + \varepsilon f_2\rho^2 + \varepsilon e_1 f_3\rho^3\\
&\quad - \left(105\pi + \frac{3}{8}\frac{f_3^2}{\pi} + \varepsilon f_4\right)\rho^4 + O_5(\rho),
\end{aligned}
$$

where f_4 is a polynomial on $\varepsilon, e_1, f_3, e_3,$ and f_2, and in our case

$$
\begin{aligned}
\widetilde{T}_1(\alpha) &= \frac{-\alpha^3(-3+\varepsilon e_3) - \alpha^2\varepsilon e_1 - 7\alpha(-3+\varepsilon e_3) - 3\varepsilon e_1}{\alpha^4 + 10\alpha^2 + 9}\\
&= \frac{3\alpha(\alpha^2+7) - \varepsilon(\alpha^3 e_3 + \alpha^2 e_1 + 7\alpha e_3 + 3e_1)}{\alpha^4 + 10\alpha^2 + 9}.
\end{aligned}
$$

For simplicity, we have written the above approximations of functions Δ and T in the form

$$
\xi(\rho) \approx \sum_{i=0}^{N} \kappa_i\rho^i + O_{N+1}(\rho), \tag{3.73}
$$

which actually corresponds to

$$
\xi(\rho) = \sum_{i=0}^{N} \kappa_i\rho^i\,(1 + O_1(\rho)) + O_{N+1}(\rho).
$$

Finally, if we take $\alpha > 0$, $f_3 > 0$, $f_2 < 0$, and $e_1 > 0$, for sufficiently small ε we get 1 change of signs in the coefficients of $\Delta(\rho)$ and 3 changes of signs in the coefficients of $T(\rho)$, and applying Lemma 3.6, we conclude that 1 limit cycle and 3 critical periods bifurcate, which corresponds to a $(1,3)$ configuration.

To see the configuration $(2,3)$ we will follow an analogous procedure; we aim to analyze the case $(2,l)$, so we assume that $\alpha = V_3 = 0$ and $V_5 \neq 0$. Using the Lyapunov and period constants (3.69) we solve the system $\{V_3 = 0, T_2 = 0, T_3 = 0\}$ and obtain the solution $S = \{a_3 = \frac{2}{3}a_2b_2, b_3 = \frac{4}{9}a_2^2 + \frac{10}{9}b_2^2\}$, for which V_5 and T_4 are not identically zero and have the expressions

$$
V_5 = \frac{5}{54}\pi a_2 b_2(2a_2^2 + 5b_2^2), \quad T_4 = -\frac{5}{54}\pi b_2^2(11a_2^2 + 14b_2^2).
$$

Observe that $T_4 = 0$ would mean $b_2 = 0$, which would imply also $V_5 = 0$ and we would have a center. Therefore, the highest nonzero period constant that we can have is T_4, so at most 3 critical periods could appear, this is $l \leq 3$. We will check now that actually $l = 3$.

Let us set the two free parameters a_2 and b_2 from S to -3, and consider adding a perturbation by performing the change $(a_2, a_3, b_2, b_3) \to (-3+\varepsilon e_1, 6+\varepsilon e_2, -3+$

$\varepsilon e_3, 14+\varepsilon e_4)$ on the Lyapunov and period constants, where ε is a small perturbative parameter. After this change,

$$V_3 = -\frac{1}{4}\pi\varepsilon(-6e_1 - 3e_2 - 6e_3 + 2\varepsilon e_1 e_3),$$
$$T_2 = \frac{1}{12}\pi\varepsilon(-24e_1 - 60e_3 - 9e_4 + 4\varepsilon e_1^2 + 10\varepsilon e_3^2),$$

and we define $g_3 := \frac{V_3}{\varepsilon}$ and $f_2 := \frac{T_2}{\varepsilon}$. We isolate e_2 in the expression of g_3 and e_4 in the expression of f_2, and we substitute them in V_5 and T_4, performing a change of variables $(e_1, e_2, e_3, e_4) \to (e_1, g_3, e_3, f_2)$. If we consider a nonzero trace 2α, according to (3.19) and (3.20) we have the following approximate expressions for the displacement map and the period function for ρ small enough (following the notation in (3.73)):

$$\Delta(\rho) \approx (\mathrm{e}^{2\pi\alpha} - 1)\rho + g_3\rho^3 + \left(\frac{105}{2}\pi + \varepsilon g_5\right)\rho^5 + O_6(\rho),$$

$$T(\rho) \approx 2\pi + (\mathrm{e}^{2\pi\alpha} - 1)\widetilde{T}_1(\alpha)\,\rho + \varepsilon f_2\rho^2 + \varepsilon g_3\left(1 - \frac{\varepsilon e_1}{3}\right)\rho^3$$
$$- \left(\frac{375}{2}\pi + \varepsilon f_4\right)\rho^4 + O_5(\rho),$$

where g_5 and f_4 are polynomials on $\varepsilon, e_1, g_3, e_3$, and f_2, and in our case

$$\widetilde{T}_1(\alpha) = \frac{-\alpha^3(-3+\varepsilon e_3) - \alpha^2(-3+\varepsilon e_1) - 7\alpha(-3+\varepsilon e_3) - 3(-3+\varepsilon e_1)}{\alpha^4 + 10\alpha^2 + 9}$$
$$= \frac{3\alpha^3 + 3\alpha^2 + 21\alpha + 9 - \varepsilon(\alpha^3 e_3 + \alpha^2 e_1 + 7\alpha e_3 + 3e_1)}{\alpha^4 + 10\alpha^2 + 9}.$$

If we take $\alpha > 0$, $g_3 < 0$, and $f_2 < 0$, for sufficiently small ε we can get two changes of signs in the coefficients of $\Delta(\rho)$ and three changes of signs in the coefficients of $T(\rho)$, and applying Lemma 3.6 we conclude that 2 limit cycles and 3 critical periods bifurcate, which corresponds to a $(2,3)$ configuration.

(iii) Finally, we know that if an isochronous focus exists it must be either $(1,\infty)$ or $(2,\infty)$, because any other case would become a center. Actually, we have already shown that none of these two configurations is possible: in both cases we have seen that when vanishing T_2, T_3, and T_4 the system automatically becomes a center, so no isochronous foci exist for this system and the result follows. □

Bibliography

[AFG00] A. Algaba, E. Freire, and E. Gamero. “Isochronicity via normal form”. In: *Qual. Theory Dyn. Syst.* 1.2 (2000), pp. 133–156.

[AR08] A. Algaba and M. Reyes. “Isochronous centres and foci via commutators and normal forms”. In: *Proc. Roy. Soc. Edinburgh Sect. A* 138.1 (2008), pp. 1–13.

[Álv+20] M. J. Álvarez, B. Coll, P. De Maesschalck, and R. Prohens. “Asymptotic lower bounds on Hilbert numbers using canard cycles”. In: *J. Differential Equations* 268.7 (2020), pp. 3370–3391.

[Ame21] V. V. Amel’kin. “Positive Solution of One Conjecture in the Theory of Polynomial Isochronous Centers of Liénard Systems”. In: *Differential Equations* 57.2 (2021), pp. 133–138.

[And+73] A. A. Andronov, E. A. Leontovich, I. I. Gordon, and A. G. Maĭer. *Theory of bifurcations of dynamic systems on a plane*. Translated from the Russian. Halsted Press [A division of John Wiley & Sons], New York-Toronto; Israel Program for Scientific Translations, Jerusalem-London, 1973, pp. xiv+482.

[Arn89] V. I. Arnol’d. *Mathematical methods of classical mechanics*. Vol. 60. Graduate Texts in Mathematics. Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein, Corrected reprint of the second (1989) edition. Springer-Verlag, New York, [1989], pp. xvi+516.

[BBT21] J. L. R. Bastos, C. A. Buzzi, and J. Torregrosa. “Orbitally symmetric systems with applications to planar centers”. Preprint. 2021.

[BG96] V. I. Bulgakov and A. A. Grin′. “On a bifurcation of a non-rough focus of a third-order autonomous system”. In: *Differ. Uravn.* 32.12 (1996), pp. 1703, 1727.

[BGG19] A. Buică, J. Giné, and M. Grau. “On the critical points of the flight return time function of perturbed closed orbits”. In: *J. Differential Equations* 266.12 (2019), pp. 8344–8369.

[BGM12] A. Buică, I. A. García, and S. Maza. “Existence of inverse Jacobi multipliers around Hopf points in $\mathbb{R}^3$: emphasis on the center problem”. In: *J. Differential Equations* 252.12 (2012), pp. 6324–6336.

[BIS20] I. Baldomá, S. Ibáñez, and T. M. Seara. "Hopf-zero singularities truly unfold chaos". In: *Commun. Nonlinear Sci. Numer. Simul.* 84 (2020), pp. 105162, 19.

[BS08] Y. L. Bondar′ and A. P. Sadovskiĭ. "On a theorem of Żołądek". In: *Differ. Uravn.* 44.2 (2008), pp. 263–265, 287.

[BV84] H. W. Broer and G. Vegter. "Subordinate Šil′nikov bifurcations near some singularities of vector fields having low codimension". In: *Ergodic Theory Dynam. Systems* 4.4 (1984), pp. 509–525.

[BZ03] M. Bobieński and H. Żołądek. "Limit cycles for multidimensional vector fields. The elliptic case". In: *J. Dynam. Control Systems* 9.2 (2003), pp. 265–310.

[BZ05a] M. Bobieński and H. Żołądek. "Limit cycles of three-dimensional polynomial vector fields". In: *Nonlinearity* 18.1 (2005), pp. 175–209.

[BZ05b] M. Bobieński and H. Żołądek. "The three-dimensional generalized Lotka-Volterra systems". In: *Ergodic Theory Dynam. Systems* 25.3 (2005), pp. 759–791.

[BZ07] M. Bobieński and H. Żołądek. "A counterexample to a multidimensional version of the weakened Hilbert's 16th problem". In: *Mosc. Math. J.* 7.1 (2007), pp. 1–20, 166.

[Bau52] N. N. Bautin. "On the number of limit cycles appearing with variation of the coefficients from an equilibrium state of the type of a focus or a center". In: *Mat. Sbornik N.S.* 30(72) (1952), pp. 181–196.

[Bib79] Y. N. Bibikov. *Local theory of nonlinear analytic ordinary differential equations*. Vol. 702. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1979, p. 147.

[CA12] A. S. Cassidy and A. G. Andreou. "Beyond Amdahl's law: an objective function that links multiprocessor performance gains to delay and energy". In: *IEEE Trans. Comput.* 61.8 (2012), pp. 1110–1126.

[CF05] G. Cassiday and G. Fowles. *Analytical mechanics*. Thomson Brooks-Cole, 2005.

[CGG01] J. Chavarriga, I. A. García, and J. Giné. "On integrability of differential equations defined by the sum of homogeneous vector fields with degenerate infinity". In: *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 11.3 (2001), pp. 711–722.

[CGS08] A. Cima, A. Gasull, and P. R. da Silva. "On the number of critical periods for planar polynomial systems". In: *Nonlinear Anal.* 69.7 (2008), pp. 1889–1903.

[CJ89] C. Chicone and M. Jacobs. "Bifurcation of critical periods for plane vector fields". In: *Trans. Amer. Math. Soc.* 312.2 (1989), pp. 433–486.

[CL95] C. J. Christopher and N. G. Lloyd. "Polynomial systems: a lower bound for the Hilbert numbers". In: *Proc. Roy. Soc. London Ser. A* 450.1938 (1995), pp. 219–224.

[CLO07] D. Cox, J. Little, and D. O'Shea. *Ideals, varieties, and algorithms*. Third. Undergraduate Texts in Mathematics. An introduction to computational algebraic geometry and commutative algebra. Springer, New York, 2007, pp. xvi+551.

[CLW94] S.-N. Chow, C. Li, and D. Wang. *Normal forms and bifurcation of planar vector fields*. Cambridge University Press, Cambridge, 1994, pp. viii+472.

[CMV99] A. Cima, F. Mañosas, and J. Villadelprat. "Isochronicity for several classes of Hamiltonian systems". In: *J. Differential Equations* 157.2 (1999), pp. 373–413.

[CR06] L. A. Cherkas and V. G. Romanovski. "The center conditions for a Liénard system". In: *Comput. Math. Appl.* 52.3-4 (2006), pp. 363–374.

[CR10] X. Chen and V. G. Romanovski. "Linearizability conditions of time-reversible cubic systems". In: *J. Math. Anal. Appl.* 362.2 (2010), pp. 438–449.

[CRZ11] X. Chen, V. G. Romanovski, and W. Zhang. "Critical periods of perturbations of reversible rigidly isochronous centers". In: *J. Differential Equations* 251.6 (2011), pp. 1505–1525.

[CS86] S.-N. Chow and J. A. Sanders. "On the number of critical points of the period". In: *J. Differential Equations* 64.1 (1986), pp. 51–66.

[CS99] J. Chavarriga and M. Sabatini. "A survey of isochronous centers". In: *Qual. Theory Dyn. Syst.* 1.1 (1999), pp. 1–70.

[Car81] J. Carr. *Applications of centre manifold theory*. Vol. 35. Applied Mathematical Sciences. Springer-Verlag, New York-Berlin, 1981, pp. vi+142.

[Cen21] X. Cen. "New lower bound for the number of critical periods for planar polynomial systems". In: *J. Differential Equations* 271 (2021), 480–498.

[Chi06] C. Chicone. *Ordinary differential equations with applications*. Second. Vol. 34. Texts in Applied Mathematics. Springer, New York, 2006, pp. xx+636.

[Chi87] C. Chicone. "The monotonicity of the period function for planar Hamiltonian vector fields". In: *J. Differential Equations* 69.3 (1987), pp. 310–321.

[Chr05] C. Christopher. “Estimating limit cycle bifurcations from centers”. In: *Differential equations with symbolic computation*. Trends Math. Birkhäuser, Basel, 2005, pp. 23–35.

[Chr99] C. Christopher. “An algebraic approach to the classification of centers in polynomial Liénard systems”. In: *J. Math. Anal. Appl.* 229.1 (1999), pp. 319–329.

[Cim+97] A. Cima, A. Gasull, V. Mañosa, and F. Mañosas. “Algebraic properties of the Liapunov and period constants”. In: *Rocky Mountain J. Math.* 27.2 (1997), pp. 471–501.

[DLA06] F. Dumortier, J. Llibre, and J. C. Artés. *Qualitative theory of planar differential systems*. Universitext. Springer-Verlag, Berlin, 2006, pp. xvi+298.

[Dum+13] F. Dumortier, S. Ibáñez, H. Kokubu, and C. Simó. “About the unfolding of a Hopf-zero singularity”. In: *Discrete Contin. Dyn. Syst.* 33.10 (2013), pp. 4435–4471.

[GG05] J. Giné and M. Grau. “Characterization of isochronous foci for planar analytic differential systems”. In: *Proc. Roy. Soc. Edinburgh Sect. A* 135.5 (2005), pp. 985–998.

[GGJ04] A. Garijo, A. Gasull, and X. Jarque. “Normal forms for singularities of one dimensional holomorphic vector fields”. In: *Electron. J. Differential Equations* (2004), No. 122, 7.

[GGJ06] A. Garijo, A. Gasull, and X. Jarque. “On the period function for a family of complex differential equations”. In: *J. Differential Equations* 224.2 (2006), pp. 314–331.

[GGJ10] A. Garijo, A. Gasull, and X. Jarque. “A note on the period function for certain planar vector fields”. In: *J. Difference Equ. Appl.* 16.5-6 (2010), pp. 631–645.

[GGM99] A. Gasull, A. Guillamon, and V. Mañosa. “An analytic-numerical method for computation of the Liapunov and period constants derived from their algebraic structure”. In: *SIAM J. Numer. Anal.* 36.4 (1999), pp. 1030–1043.

[GGT16] A. Gasull, J. Giné, and J. Torregrosa. “Center problem for systems with two monomial nonlinearities”. In: *Commun. Pure Appl. Anal.* 15.2 (2016), pp. 577–598.

[GGT21] J. Giné, L. F. S. Gouveia, and J. Torregrosa. “Lower bounds for the local cyclicity for families of centers”. In: *J. Differential Equations* 275 (2021), pp. 309–331.

[GH90] J. Guckenheimer and P. Holmes. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*. Vol. 42. Applied Mathematical Sciences. Revised and corrected reprint of the 1983 original. Springer-Verlag, New York, 1990, pp. xvi+459.

[GL05] J. Giné and J. Llibre. "A family of isochronous foci with Darboux first integral". In: *Pacific J. Math.* 218.2 (2005), pp. 343–355.

[GLY10] A. Gasull, C. Liu, and J. Yang. "On the number of critical periods for planar polynomial systems of arbitrary degree". In: *J. Differential Equations* 249.3 (2010), pp. 684–692.

[GM11] J. Giné and S. Maza. "The reversibility and the center problem". In: *Nonlinear Anal.* 74.2 (2011), pp. 695–704.

[GMM02] A. Gasull, V. Mañosa, and F. Mañosas. "Stability of certain planar unbounded polycycles". In: *J. Math. Anal. Appl.* 269.1 (2002), pp. 332–351.

[GMS18] I. A. García, S. Maza, and D. S. Shafer. "Cyclicity of polynomial nondegenerate centers on center manifolds". In: *J. Differential Equations* 265.11 (2018), pp. 5767–5808.

[GMS19] I. A. García, S. Maza, and D. S. Shafer. "Center cyclicity of Lorenz, Chen and Lü systems". In: *Nonlinear Anal.* 188 (2019), pp. 362–376.

[GPT05] A. Gasull, R. Prohens, and J. Torregrosa. "Limit cycles for rigid cubic systems". In: *J. Math. Anal. Appl.* 303.2 (2005), pp. 391–404.

[GT20] L. F. S. Gouveia and J. Torregrosa. "The local cyclicity problem. Melnikov method using Lyapunov constants". Preprint. 2020.

[GT21] L. F. S. Gouveia and J. Torregrosa. "Lower bounds for the local cyclicity of centers using high order developments and parallelization". In: *J. Differential Equations* 271 (2021), pp. 447–479.

[GT98] A. Gasull and J. Torregrosa. "Center problem for several differential equations via Cherkas' method". In: *J. Math. Anal. Appl.* 228.2 (1998), pp. 322–343.

[GV10] M. Grau and J. Villadelprat. "Bifurcation of critical periods from Pleshkan's isochrones". In: *J. Lond. Math. Soc. (2)* 81.1 (2010), pp. 142–160.

[Gav93] L. Gavrilov. "Remark on the number of critical points of the period". In: *J. Differential Equations* 101.1 (1993), pp. 58–65.

[Gin03] J. Giné. "Isochronous foci for analytic differential systems". In: *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 13.6 (2003), pp. 1617–1623.

[Gin07] J. Giné. "On some open problems in planar differential systems and Hilbert's 16th problem". In: *Chaos Solitons Fractals* 31.5 (2007), pp. 1118–1134.

[Gin12a] J. Giné. "Higher order limit cycle bifurcations from non-degenerate centers". In: *Appl. Math. Comput.* 218.17 (2012), pp. 8853–8860.

[Gin12b] J. Giné. "Limit cycle bifurcations from a non-degenerate center". In: *Appl. Math. Comput.* 218.9 (2012), pp. 4703–4709.

[HL12] M. Han and J. Li. "Lower bounds for the Hilbert number of polynomial systems". In: *J. Differential Equations* 252.4 (2012), pp. 3278–3304.

[HY12] M. Han and P. Yu. *Normal forms, Melnikov functions and bifurcations of limit cycles*. Vol. 181. Applied Mathematical Sciences. Springer, London, 2012, pp. xii+401.

[Ily02] Y. Ilyashenko. "Centennial history of Hilbert's 16th problem". In: *Bull. Amer. Math. Soc. (N.S.)* 39.3 (2002), pp. 301–354.

[Koe14] W. Koepf. *Hypergeometric summation*. Second. Universitext. An algorithmic approach to summation and special function identities. Springer, London, 2014, pp. xviii+279.

[LH14] C. Liu and M. Han. "Bifurcation of critical periods from the reversible rigidly isochronous centers". In: *Nonlinear Anal.* 95 (2014), pp. 388–403.

[LL90] Y. R. Liu and J. B. Li. "Theory of values of singular point in complex autonomous differential systems". In: *Sci. China Ser. A* 33.1 (1990), pp. 10–23.

[LL91] Y. P. Lin and J. B. Li. "The normal form of a planar autonomous system and critical points of the period of closed orbits". In: *Acta Math. Sinica* 34.4 (1991), pp. 490–501.

[LT15] H. Liang and J. Torregrosa. "Parallelization of the Lyapunov constants and cyclicity for centers of planar polynomial vector fields". In: *J. Differential Equations* 259.11 (2015), pp. 6494–6509.

[LT17] H. Liang and J. Torregrosa. "Weak-foci of high order and cyclicity". In: *Qual. Theory Dyn. Syst.* 16.2 (2017), pp. 235–248.

[Li03] J. Li. "Hilbert's 16th problem and bifurcations of planar polynomial vector fields". In: *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 13.1 (2003), pp. 47–106.

[Lia66] A. M. Liapunov. *Stability of motion*. With a contribution by V. A. Pliss and an introduction by V. P. Basov. Translated from the Russian by Flavian Abramovici and Michael Shimshoni. Mathematics in Science and Engineering, Vol. 30. Academic Press, New York-London, 1966, pp. xi+203.

[Lou61] W. S. Loud. "Some singular cases of the implicit function theorem". In: *Amer. Math. Monthly* 68 (1961), pp. 965–977.

[MMJR97] P. Mardešić, L. Moser-Jauslin, and C. Rousseau. "Darboux linearization and isochronous centers with a rational first integral". In: *J. Differential Equations* 134.2 (1997), pp. 216–268.

[MRV16] F. Mañosas, D. Rojas, and J. Villadelprat. "The criticality of centers of potential systems at the outer boundary". In: *J. Differential Equations* 260.6 (2016), pp. 4918–4972.

[MV06] F. Mañosas and J. Villadelprat. "A note on the critical periods of potential systems". In: *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 16.3 (2006), pp. 765–774.

[MV13] D. Marín and J. Villadelprat. "The period function of generalized Loud's centers". In: *J. Differential Equations* 255.10 (2013), pp. 3071–3097.

[MV21] D. Marín and J. Villadelprat. "Asymptotic expansion of the Dulac map and time for unfoldings of hyperbolic saddles: general setting". In: *J. Differential Equations* 275 (2021), pp. 684–732.

[Map] Maplesoft, a division of Waterloo Maple Inc. *Maple 2018*. http://www.maplesoft.com/, Waterloo, Ontario (Canada).

[Maw19] J. Mawhin. "Simple proofs of the Hadamard and Poincaré-Miranda theorems using the Brouwer fixed point theorem". In: *Amer. Math. Monthly* 126.3 (2019), pp. 260–263.

[Mn95] V. Mañosa. *Liapunov and period constants for planar vector fields*. In Catalan. Bellaterra: Master's Thesis, Universitat Autònoma de Barcelona, 1995.

[OST21] R. Oliveira, I. Sánchez-Sánchez, and J. Torregrosa. "Simultaneous bifurcation of limit cycles and critical periods". Preprint. 2021.

[Olv86] P. J. Olver. *Applications of Lie groups to differential equations*. Vol. 107. Graduate Texts in Mathematics. Springer-Verlag, New York, 1986, pp. xxvi+497.

[PLF15] L. Peng, L. Lu, and Z. Feng. "A higher-order period function and its application". In: *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 25.10 (2015), pp. 1550140, 23.

[PLF17] L. Peng, L. Lu, and Z. Feng. "Perturbed rigidly isochronous centers and their critical periods". In: *J. Math. Anal. Appl.* 453.1 (2017), pp. 366–382.

[PT19] R. Prohens and J. Torregrosa. "New lower bounds for the Hilbert numbers using reversible centers". In: *Nonlinearity* 32.1 (2019), pp. 331–355.

[Par] The PARI Group. *PARI 2.3.3*. http://pari.math.u-bordeaux.fr/, Bordeaux, Nouvelle-Aquitaine (France).

[RS09] V. G. Romanovski and D. S. Shafer. *The center and cyclicity problems: a computational algebra approach*. Birkhäuser Boston Ltd., Boston MA, 2009, pp. xvi+330.

[RT93] C. Rousseau and B. Toni. "Local bifurcation of critical periods in vector fields with homogeneous nonlinearities of the third degree". In: *Canad. Math. Bull.* 36.4 (1993), pp. 473–484.

[RV18] D. Rojas and J. Villadelprat. "A criticality result for polycycles in a family of quadratic reversible centers". In: *J. Differential Equations* 264.11 (2018), pp. 6585–6602.

[Riv90] T. J. Rivlin. *Chebyshev polynomials*. Second. Pure and Applied Mathematics (New York). From approximation theory to algebra and number theory. John Wiley & Sons, Inc., New York, 1990, pp. xvi+249.

[Rou88] R. Roussarie. "A note on finite cyclicity property and Hilbert's 16th problem". In: *Dynamical systems, Valparaiso 1986*. Vol. 1331. Lecture Notes in Math. Springer, Berlin, 1988, pp. 161–168.

[Rou98] R. Roussarie. *Bifurcations of planar vector fields and Hilbert's sixteenth problem*. Modern Birkhäuser Classics. [2013] reprint of the 1998 edition [MR1628014]. Birkhäuser/Springer, Basel, 1998, pp. xviii+204.

[SMB06] J. Sotomayor, L. F. Mello, and D. d. C. Braga. "Stability and Hopf bifurcation in the Watt governor system". In: *Comm. Appl. Nonlinear Anal.* 13.4 (2006), pp. 1–17.

[ST19] I. Sánchez-Sánchez and J. Torregrosa. "New advances on the Lyapunov constants of some families of planar differential systems". In: *Extended abstracts Spring 2018*. Vol. 11. Trends Math. Res. Perspect. CRM Barc. Birkhäuser/Springer, Cham, [2019] ©2019, pp. 161–167.

[ST21a] I. Sánchez-Sánchez and J. Torregrosa. "Criticality via first order development of the period constants". In: *Nonlinear Anal.* 203 (2021), pp. 112–164, 21.

[ST21b] I. Sánchez-Sánchez and J. Torregrosa. "Hopf bifurcation in 3-dimensional polynomial vector fields". Preprint. 2021.

[ST21c] I. Sánchez-Sánchez and J. Torregrosa. "New lower bounds of the number of critical periods in reversible centers". In: *J. Differential Equations* 292 (2021), pp. 427–460.

[Sab12] M. Sabatini. "The period functions' higher order derivatives". In: *J. Differential Equations* 253.10 (2012), pp. 2825–2845.

[Sab97] M. Sabatini. "Characterizing isochronous centres by Lie brackets". In: *Differential Equations Dynam. Systems* 5.1 (1997), pp. 91–99.

[Sad20] A. P. Sadovskii. "Existence of complex cubic systems with a 14th-order focus". In: *Differ. Equ.* 56.1 (2020), pp. 140–142.

[Sal] O. Saleta. *PBala 6.0.2*. https://github.com/oscarsaleta/PBala, Bellaterra, Barcelona (Spain).

[Shi65] L. P. Šil′nikov. "A case of the existence of a denumerable set of periodic motions". In: *Dokl. Akad. Nauk SSSR* 160 (1965), pp. 558–561.

[Shi81] S. L. Shi. "A method of constructing cycles without contact around a weak focus". In: *J. Differential Equations* 41.3 (1981), pp. 301–312.

[Sib76] K. S. Sibirskiĭ. *Algebraic invariants of differential equations and matrices*. Izdat. "Štiinca", Kishinev, 1976, 268 pp. (errata insert).

[Swi99] G. Świrszcz. "Cyclicity of infinite contour around certain reversible quadratic center". In: *J. Differential Equations* 154.2 (1999), pp. 239–266.

[Tor98] J. Torregrosa. *Singular points and periodic orbits for vector fields*. In Catalan. Bellaterra: Doctoral Thesis, Universitat Autònoma de Barcelona, 1998.

[VZ20] J. Villadelprat and X. Zhang. "The period function of Hamiltonian systems with separable variables". In: *J. Dynam. Differential Equations* 32.2 (2020), pp. 741–767.

[Vra89] M. N. Vrahatis. "A short proof and a generalization of Miranda's existence theorem". In: *Proc. Amer. Math. Soc.* 107.3 (1989), pp. 701–703.

[WHW11] Q. Wang, W. Huang, and H. Wu. "Bifurcation of limit cycles for 3D Lotka-Volterra competitive systems". In: *Acta Appl. Math.* 114.3 (2011), pp. 207–218.

[Wal04] C. T. C. Wall. *Singular points of plane curves*. Vol. 63. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2004, pp. xii+370.

[Wig03] S. Wiggins. *Introduction to applied nonlinear dynamical systems and chaos*. Second. Vol. 2. Texts in Applied Mathematics. Springer-Verlag, New York, 2003, pp. xxx+843.

[YH09] P. Yu and M. Han. "Critical periods of planar revertible vector field with third-degree polynomial functions". In: *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 19.1 (2009), pp. 419–433.

[YH15] P. Yu and M. Han. "Ten limit cycles around a center-type singular point in a 3-d quadratic system with quadratic perturbation". In: *Appl. Math. Lett.* 44 (2015), pp. 17–20.

[YHZ10] P. Yu, M. Han, and J. Zhang. "Critical periods of third-order planar Hamiltonian systems". In: *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 20.7 (2010), pp. 2213–2224.

[YMG14] L. Yavits, A. Morad, and R. Ginosar. "The effect of communication and synchronization on Amdahl's law in multicore systems". In: *Parallel Comput.* 40.1 (2014), pp. 1–16.

[YS19] K. E. Yao and Y. Shi. "Hopf bifurcation in three-dimensional based on chaos entanglement function". In: *Chaos, Solitons & Fractals: X* 4 (2019), p. 100027.

[YT14] P. Yu and Y. Tian. "Twelve limit cycles around a singular point in a planar cubic-degree polynomial system". In: *Commun. Nonlinear Sci. Numer. Simul.* 19.8 (2014), pp. 2690–2705.

[ZLH13] J. Zhou, N. Li, and M. Han. "Critical periods of perturbations of reversible rigidly isochronous centers". In: *Abstr. Appl. Anal.* (2013), Art. ID 481501, 12.

[Zei90] D. Zeilberger. "A fast algorithm for proving terminating hypergeometric identities". In: *Discrete Math.* 80.2 (1990), pp. 207–211.

[Zei91] D. Zeilberger. "The method of creative telescoping". In: *J. Symbolic Comput.* 11.3 (1991), pp. 195–204.

[Zha02] Y. Zhao. "The monotonicity of period function for codimension four quadratic system Q_4^1". In: *J. Differential Equations* 185.1 (2002), pp. 370–387.

[Zol94] H. Żołądek. "Quadratic systems with center and their perturbations". In: *J. Differential Equations* 109.2 (1994), pp. 223–273.