Chapter 4

Signal and System Model with Non-Linear Distortion

In the previous chapter a complete base-band signal model was presented for the exact discrete representation of the OFDM system considering a completely linear transference. Along with this, and independently, an analog base-band model was established to characterize the non-linear HPA complex gain based on a series of q-order autocorrelations of the base-band input signal. Now, the main objective is to insert the non-linear effect into the discrete model of the transmission chain, reviewing at each step the spectral evolution of the transmitted OFDM symbol and focusing on the conditions for its final recovery at the receiver end.

4.1 Equivalent Framing Structures for Reception

In the chapter 3 the auxiliary signal frame $\widetilde{\overline{b}}_x(t)$ defined in (3.69) and represented in figure 3.17(d) was found to be helpful in formulating the discrete model that we finally expressed through the elements of the transference matrix **S**. The resulting expressions, obtained using such auxiliary (virtual) signal structure, are particularly well suited to evaluate the analog HPA output characteristics (distorted spectrum) rather than the end-to-end transmission of the M-QAM data. Therefore, the model previously developed must be assessed in terms of the exact discrete description provided for the analog stages of the transmission process. The exactitude of this linear discrete model was further verified when the diagonality of the matrix **S** was demonstrated in the section 3.4, appendix 3.C. Although the final expression to calculate the elements of **S** proved to be surprisingly intuitive –the elements of the main diagonal correspond to the channel profile in frequency–, the demonstration procedure resulted quite long. Now, in this chapter, we will be interested in establishing a ‘reception model’ to include the effect of the HPA nonlinearity. Since in this case the analysis focuses on the final recovery of the M-QAM base-band information instead of the analog channel implications, an alternative and simpler reasoning can be employed to explain the absence of ICI and, consequently, the

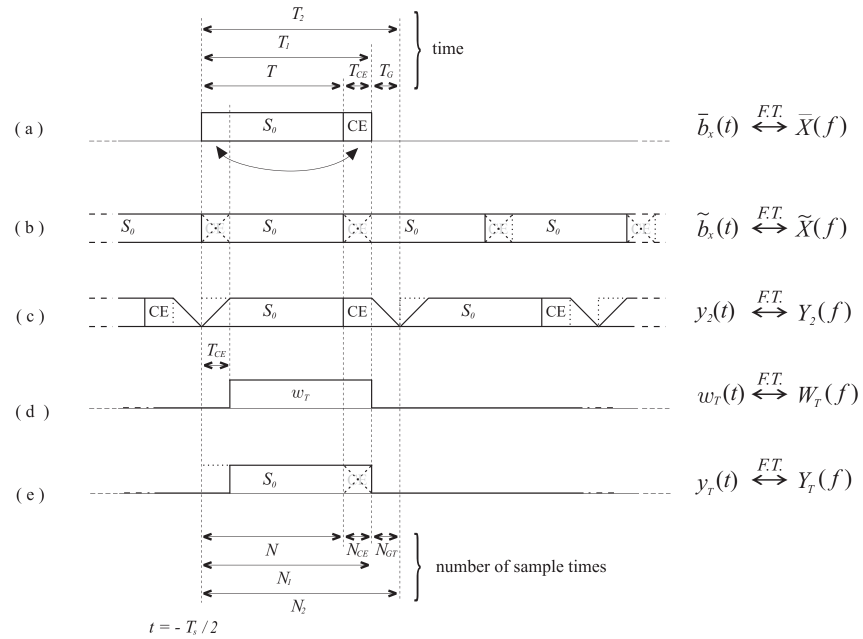

Figure 4.1: Equivalence in the definition of different frames for transmission when the CE is discarded in reception by applying the window w_T over the periodic extension of the OFDM symbol $\widetilde{b}_x(t)$ and over the signal $y_2(t)$ considering cyclic extension and guard time as defined in (3.76).

diagonality of the matrix $\mathbf{S}$ that describes such transference. This alternative reasoning is based on the fact that, when the CE is discarded at the reception end, the signal fed into the demodulator is the same whether we use the auxiliary signal $\widetilde{\widetilde{b}}_x(t)$ or the basic periodic extension of the original symbol $\widetilde{b}_x(t)$ for transmission. This is illustrated in figure 4.1[1] where the crossed regions representing the time dispersion effect of the filters are marked to show that this equivalence is not valid (due to ISI appearance) when different symbols $[S_0, S_1, \ldots, S_m]$ without guard time are considered instead of the virtual periodic extension of a single symbol S_0. In the case of a real scenario, a stream with the changing symbols will be transmitted. Nevertheless, the OFDM framing structure is defined so that the receiver observes only a window of duration T for each symbol, where no interference is present from contiguous symbols due to the adequate use of the guard time and cyclic extension. Then, if the transmission process is modeled using the periodic discrete sequence $\widetilde{b}_x[n]$ defined in (3.52), the frequency domain discrete representation of the transmission process will consider only the evolution of linear spectrums as it was shown in figure 3.15 (left branch), where we have periodic signals at each point of the analog chain up to the reception end at which the one-symbol length window $w_T[n]$ is applied. In such case, previous to the reception windowing, the effect of the filters and the D/A conversion can be modeled as an exact scaling by $H(f)$ of the spectral components of the discrete and periodic FT of the input signal which is given by (3.55) and (3.56).

[1]Recall that continuous-time windows and signal frames are suitably defined starting at $t = T_s/2$ to avoid discontinuities to coincide with discrete sampling times.

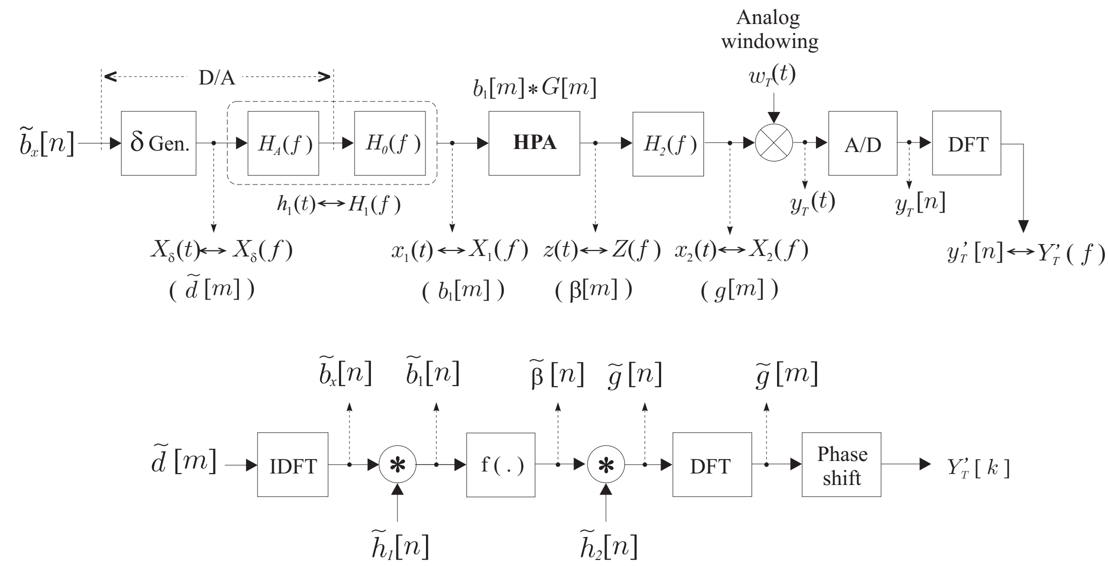

Figure 4.2: Upper: Transmission chain introducing nonlinearity. Lower: Equivalent processing chain when $h_2[n] = \delta[n]$.

Under these conditions, the analysis of the end-to-end transference does not include intermediate windowing which is equivalent to removing $\widetilde{W}_1(\cdot)$ from equation (3.100). This reduces the dependence of $\mathbf{S}$ in one dimension and leads again to the diagonality of the transference matrix.

The foregoing remarks show that for developing a general discrete reception model describing the recovery of an OFDM symbol S_0, it is possible to consider the transmission of a periodic stream[2] consisting of repetitions of S_0. Therefore, the signal model that will apply in the following derivations is the periodic stream shown in figure 4.1(b) whose characteristics, as will become clear during the model formulation, yield simpler mathematical developments.

4.2 Signal Model for the Non-linear Processing Chain

4.2.1 Non Linear System Structure and Definitions

Throughout this chapter we will consider the new system structure shown in figure 4.2. The processing chain therein illustrated contains all the linear processing blocks described up to this point but now incorporating the non-linear block that introduces

[2]Periodic sequences in time or frequency domain as well as periodic extensions of time limited sequences will be herein denoted with the symbol $\sim$ on top.

the effect of the HPA. The aim is to include all the effects associated with D/A and A/D conversions, filtering, amplifier operation (spectral regrowth) and sampling rate (aliasing) into a base-band exact equivalent discrete model that incorporates the theoretically infinite analog bandwidth of the process.

Among other differences, unlike the linear case described in section 3.4, the presence of the HPA in the non linear chain does not allow all the analog filtering stages and channel responses to be merged into a single $H(f)$. Therefore, the notation along the processing chain shall be redefined[3] whenever necessary regarding the new resulting structure. Let us first consider the discrete input sequence $\widetilde{b}_x[n]$, generated by the discrete OFDM modulator and previously defined in (3.52), which is here redefined as

$$\widetilde{b}_x[n] = \sum_{m=0}^{N-1} \widetilde{d}_N[m] e^{j2\pi \frac{m}{N} n} \tag{4.1}$$

where $\widetilde{d}_N[m]$ is the periodic extension (with period N) of the input data vector $d[m]$ defined by

$$\widetilde{d}_N[m] = \sum_{k=-\infty}^{+\infty} d[m-kN] = d[m] \ast \sum_{k=-\infty}^{+\infty} \delta[m-kN]. \tag{4.2}$$

Thus, the FT of the discrete and periodic sequence $\widetilde{b}_x[n]$ can be expressed as

$$\begin{aligned} X(e^{j2\pi f_n}) &= \sum_{n=-\infty}^{+\infty} \widetilde{b}_x[n] e^{-j2\pi f_n n} \\ &= \sum_{m=-\infty}^{+\infty} \widetilde{d}_N[m] \delta\left(f_n - \frac{m}{N}\right) \end{aligned} \tag{4.3}$$

where the M-QAM symbols $d[m]$ modulating the N subcarriers are contained in their N-periodic extension $\widetilde{d}_N[m]$. In this case, as represented in figure 4.2, the D/A conversion is modeled as a delta generator (DG) plus a rectangular (time-limited) impulse response filter, or aperture filter $H_A(f)$. With $f_s = 1/T_s$, the sampling frequency for the D/A converter, the FT at the output of the DG is denoted as

$$X_\delta(f) = \mathcal{F}\big(\widetilde{x}_\delta(t)\big) = \frac{1}{T_s} \sum_{m=-\infty}^{+\infty} \widetilde{d}_N[m] \delta\left(f - \frac{m}{T}\right). \tag{4.4}$$

Then, since the inherent filter $H_A(f)$ of the D/A converter and the global response $H_0(f)$ associated to the up-conversion stages are both previous to the HPA, they can be combined in a single response $H_1(f) = H_A(f)H_0(f)$. Hence, the spectrum at the input of the base-band HPA model is

[3] Samples in time domain are indexed with [n] while samples in frequency domain use the index [m].

$$\begin{aligned} X_1(f) &= H_1(f)X_\delta(f) \\ &= \frac{1}{T_s}\sum_{m=-\infty}^{+\infty} \widetilde{d}_N[m]H_1\left(\frac{m}{T}\right)\delta\left(f-\frac{m}{T}\right) && (4.5) \\ &= \frac{1}{T_s}\sum_{m=-\infty}^{+\infty} b_1[m]\delta\left(f-\frac{m}{T}\right) && (4.6) \end{aligned}$$

where, for notation convenience, the spectral coefficients $b_1[m]$ observed at the HPA input were defined as,

$$b_1[m] = \widetilde{d}_N[m]H_1\left(\frac{m}{T}\right). \tag{4.7}$$

Note that $b_1[m]$ is not a periodic set according to the spectrum representation used in chapter 3 in the left branch of figure 3.15.

Although the formulation of the reception model will be based on the use of periodic signals throughout the processing chain, whenever a time-limited signal is to be recovered at any point in the chain, a suitable analog or discrete rectangular window can be applied using either the original or extended OFDM symbol duration (see Fig. 3.17). These rectangular windows must be carefully defined, specially regarding the time-alignment necessary to recover the symbol without interferences at the output of those blocks introducing a memory effect. For instance, since the DG is memoryless, the time-limited extended symbol at its output can be obtained as $w_1(t)x_\delta(t)$, using [4]$w_1(t) = \Pi\left(\frac{t-T_1/2+T_s/2}{T_1}\right)$. Then, at the output of $H_1(f)$ the receiver would observe $w_T(t)x_1(t)$, where the one-symbol length window $w_T(t) = \Pi\left(\frac{t-T_{CE}-T/2+T_s/2}{T}\right)$ takes into account the limited duration D_{H_1} of the time response of $H_1(f)$ with a shifting factor $T_{CE} > D_{H_1}$.

4.2.2 Spectral Modeling for the Non-linear Transference (discrete equivalent)

In section 3.2.2 we developed a special power series model to represent the HPA gain. Let us recall that in equations (3.25) and (3.26) the HPA's base-band nonlinearity was suitably modeled as an amplitude-dependent multiplicative gain with a functional dependence restricted to consider only even powers of the input signal (squared modulus). Given the modulus of the base-band input $u_x(t)$, the non-linear complex gain of the HPA was expressed in (3.25) as

$$G(u_x^2(t)) = \sum_{q=0}^{+\infty} g_q u_x^{2q}(t).$$

[4]$\Pi\left(\frac{t}{T_p}\right)$ is the unit rectangular pulse on the support $[-T_p/2, +T_p/2]$.

Here we particularize the proposed model for periodic signals according to the assumptions on the signal model previously expressed in section 4.1. In general, for periodic signals $a_i(t)$ defined by

$$a_i(t) \overset{FT}{\longleftrightarrow} A_i(f) = \sum_{k=-\infty}^{+\infty} c_i[k]\delta\left(f - \frac{k}{T}\right), \tag{4.8}$$

the product

$$a_i(t)a_{i'}(t) \overset{FT}{\longleftrightarrow} A_i(f) * A_{i'}(f) = \sum_{k=-\infty}^{+\infty} \beta[k]\delta\left(f - \frac{k}{T}\right) \tag{4.9}$$

with

$$\beta[k] = c_i[k] * c_{i'}[k]$$

also results in a T-periodic signal corresponding to the squared modulus $|a_i(t)|^2$ when $a_i(t)a_{i'}^*(t)$ is evaluated for $i = i'$. Thence, the squared modulus of the base-band signal at the output of $H_1(f)$ is obtained as

$$|x_1(t)|^2 \;=\; x_1(t)x_1^*(t) = u_1^2(t) \overset{FT}{\longleftrightarrow} U_1(f) \tag{4.10}$$

where

$$\begin{aligned}
U_1(f) &= X_1(f) * X_1^*(-f) \\
&= \sum_{k,k'=-\infty}^{+\infty} b_1[k]b_1^*[k']\delta\left(f - \frac{k}{T}\right) * \delta^*\left(-f + \frac{k'}{T}\right) \\
&= \sum_{k,k'=-\infty}^{+\infty} b_1[k]b_1^*[k']\delta\left(f - \frac{k}{T}\right) * \delta\left(f - \frac{k'}{T}\right) \\
&= \sum_{k,k'=-\infty}^{+\infty} b_1[k]b_1^*[k']\delta\left(f - \frac{(k+k')}{T}\right) \\
&= \sum_{m=-\infty}^{+\infty} \delta\left(f - \frac{m}{T}\right) \sum_{k+k'=m} b_1[k]b_1^*[k'] \\
&= \sum_{m=-\infty}^{+\infty} r_b[m]\delta\left(f - \frac{m}{T}\right).
\end{aligned} \tag{4.11}$$

Using $k' = m - k$, the autocorrelation term $r_b[m]$ can also be expressed as the following discrete convolution:

$$r_b[m] = \sum_{k=-\infty}^{+\infty} b[k]b^*[m-k] = b_1[m] * b_1^*[m]. \tag{4.12}$$

Then, as expressed in the analog non-linear model in section 3.2.2, equations (3.26) and (3.28), we have the HPA output given by

$$\begin{aligned} z(t) &= x_1(t)\sum_{k=0}^{+\infty} g_k u_1^{2k}(t) &(4.13)\\ Z(f) &= X_1(f) * \sum_{k=0}^{+\infty} g_k U_1^{(k)}(f) &(4.14)\end{aligned}$$

where, in the last expression, the k-order term in frequency domain can be calculated as

$$U_1^{(k)}(f) = \underbrace{U_1(f) * \cdots * U_1(f)}_{k-\text{times}}$$

and from (4.8),(4.9) and (4.11),

$$\begin{aligned} U_1^{(k)}(f) &= \sum_{m=-\infty}^{+\infty} (r_b[m] * \cdots * r_b[m])\delta\left(f - \frac{m}{T}\right)\\ &= \sum_{m=-\infty}^{+\infty} r_b^{(k)}[m]\delta\left(f - \frac{m}{T}\right). &(4.15)\end{aligned}$$

Hence, we replace (4.6) and (4.15) in (4.14) obtaining

$$\begin{aligned} Z(f) &= X_1(f) * \sum_{k=0}^{+\infty} g_k \sum_{m=-\infty}^{+\infty} r_b^{(k)}[m]\delta\left(f - \frac{m}{T}\right)\\ &= \frac{1}{T_s}\sum_{m'=-\infty}^{+\infty} b_1[m']\delta\left(f - \frac{m'}{T}\right) * \sum_{k=0}^{+\infty}\sum_{m=-\infty}^{+\infty} g_k r_b^{(k)}[m]\delta\left(f - \frac{m}{T}\right)\\ &= \frac{1}{T_s}\sum_{m=-\infty}^{+\infty} \beta[m]\delta\left(f - \frac{m}{T}\right) &(4.16)\end{aligned}$$

where

$$\beta[m] = b_1[m] * \sum_{k=0}^{+\infty} g_k r_b^{(k)}[m] = b_1[m] * G[m]. \tag{4.17}$$

Since the HPA distortion is a memoryless operation, the signal eventually observed at this point by the receiver and fed to a demodulator would be $w_T(t)z(t)$.

4.2.3 Signal model from the HPA output up to DFT demodulator

After passing the HPA output through the down-conversion filters, whose global response is expressed by $H_2(f)$, the corresponding input signal $x_2(t) = z(t) * h_2(t)$ to the analog-to-digital converter (ADC) is expressed in frequency domain as

$$\begin{aligned} X_2(f) &= Z(f)H_2(f) \\ &= \frac{1}{T_s}\sum_{m=-\infty}^{+\infty}\beta[m]H_2\left(\frac{m}{T}\right)\delta\left(f-\frac{m}{T}\right) \quad (4.18) \\ &= \frac{1}{T_s}\sum_{m=-\infty}^{+\infty}g[m]\delta\left(f-\frac{m}{T}\right) \quad (4.19) \end{aligned}$$

where we have defined the spectral coefficients observed by the receiver as

$$g[m] = \beta[m]H_2\left(\frac{m}{T}\right). \quad (4.20)$$

Now, as shown in figure 4.2, the receiver observation window $w_T(t)$ is applied and the time-limited signal $y_T(t) = w_T(t)x_2(t)$ is then sampled as $y_T[n] = y_T(nT_s)$ to construct the received vector $\mathbf{y}_T$,

$$\mathbf{y}_T = \left[y_T[N_{CE}], \cdots, y_T[N+N_{CE}-1]\right]^T.$$

Note that the correct time-alignment of the window $w_T(t) = \Pi\left(\frac{t-T_{CE}-T/2+T_s/2}{T}\right)$ requires the total impulse response of the filters to be considered so that $(D_{H1}+D_{H2}) \leq T_{CE}$. Finally, the received OFDM symbol is demodulated when the receiver computes the DFT of $\mathbf{y}_T$ as

$$\begin{aligned} Y_T'[k] &= \sum_{n=0}^{N-1} y_T[n+N_{CE}]e^{-j2\pi\frac{k}{N}n} \\ &= e^{-j2\pi N_{CE}\frac{k}{N}}X_2[k] \quad (4.21) \end{aligned}$$

where

$$X_2[k] = \sum_{n=0}^{N-1} x_2\left(nT_s\right)e^{-j2\pi\frac{k}{N}n}. \quad (4.22)$$

Then, substituting the corresponding time-domain expression (Fourier Series) of (4.19) in (4.22), it is possible to find the relation between $X_2[k]$ and $g[m]$ as follows:

$$\begin{aligned} X_2[k] &= \sum_{n=0}^{N-1}\left(\sum_{m=-\infty}^{+\infty}g[m]e^{j2\pi\frac{m}{N}n}\right)e^{-j2\pi\frac{k}{N}n} \\ &= \sum_{m=-\infty}^{+\infty}g[m]\sum_{n=0}^{N-1}e^{-j2\pi\frac{k-m}{N}n} \\ &= \sum_{m=-\infty}^{+\infty}g[m]\widetilde{U}_N\left(\frac{k-m}{N}\right) \quad (4.23) \end{aligned}$$

where $\widetilde{U}_N(\cdot)$ corresponds to the evaluation of the FT of the N-length discrete unitary window described previously in appendix 3.B, equation (3.113). Hence, from the properties shown later in appendix 4.A, the expression in (4.23) can be equivalently written as

$$X_2[k] = g[k] * \widetilde{U}_N\left(\frac{k}{N}\right) = \widetilde{g}_N[k] \circledast \widetilde{U}_N\left(\frac{k}{N}\right) \tag{4.24}$$

where we use the periodic extension

$$\widetilde{g}_N[k] = \sum_{m'=-\infty}^{+\infty} g[k + Nm']. \tag{4.25}$$

Now, let us recall the expression (3.113) for convenience

$$\widetilde{U}_N\left(\frac{k}{N}\right) = \sum_{n=0}^{N-1} e^{-j2\pi\frac{k}{N}n} = \frac{\sin(\pi k)}{\sin(\pi k/N)} e^{-j\pi k\frac{N-1}{N}}$$

and observe that $\widetilde{U}_N\left(\frac{k}{N}\right) = N\delta[k]$, for $0 \le k \le N-1$. Hence, we obtain from (4.24) the final relationship

$$X_2[k] = N\widetilde{g}_N[k] \tag{4.26}$$

and thence (4.21) is expressed as

$$Y_T'[k] = e^{-j2\pi N_{CE}\frac{k}{N}} N\widetilde{g}_N[k]. \tag{4.27}$$

The spectral coefficients $g[m]$ at the receiver input can now be expressed using (4.17) and (4.20) in the form

$$\begin{aligned} g[m] &= H_2[m] \cdot (G[m] * b[m]) \\ &= H_2[m] \cdot \Big(G[m] * (H_1[m]\widetilde{d}_N[m])\Big). \end{aligned} \tag{4.28}$$

When the corresponding periodic extension $\widetilde{g}_N[m]$ is considered, we can verify the following equivalences[5]:

[5]For the sake of brevity, whenever the expressions appear too long in $[\cdot]$, we will use the shorthand notation $x_1[m] \star x_2[m] \equiv (x_1 \star x_2)[m]$, where $\star$ stands for the product or convolution between operators.

$$
\begin{aligned}
\widetilde{g}_N[m] &= \sum_{k=-\infty}^{+\infty} H_2[m+kN] \cdot (G * b)[m+kN] && (4.29)\\
&= \sum_{k=-\infty}^{+\infty} H_2[m+kN] \sum_{\ell=-\infty}^{+\infty} G[\ell](H_1 \cdot \widetilde{d}_N)[m+kN-\ell] \\
&= \sum_{\ell=-\infty}^{+\infty} G[\ell] \sum_{k=-\infty}^{+\infty} H_2[m+kN](H_1 \cdot \widetilde{d}_N)[m+kN-\ell] \\
&= \sum_{\ell=-\infty}^{+\infty} G[\ell]\widetilde{d}_N[m-\ell] \sum_{k=-\infty}^{+\infty} H_2[m+kN]H_1[m+kN-\ell] && (4.30)\\
&= \sum_{\ell=-\infty}^{+\infty} G[\ell]\widetilde{d}_N[m-\ell]\Gamma[m,\ell]. && (4.31)
\end{aligned}
$$

The expression in (4.30) is obtained using $\widetilde{d}_N[m+kN-\ell] = \widetilde{d}_N[m-\ell]$ and then, in (4.31), the inner summation is defined as a two-variable dependent term

$$
\Gamma[m,\ell] = \sum_{k=-\infty}^{+\infty} H_2[m+kN]H_1[m+kN-\ell] \tag{4.32}
$$

which is clearly N-periodic in the variable m but not in ℓ.

Now, by applying a polyphase decomposition of the index ℓ in the form

$$
\ell = \ell_1 + \ell_2 N
$$

it is possible to find an alternative expression for (4.31) obtaining

$$
\begin{aligned}
\widetilde{g}_N[m] &= \sum_{\ell_1=0}^{N-1} \sum_{\ell_2=-\infty}^{+\infty} G[\ell_1+\ell_2 N]\widetilde{d}_N[m-\ell_1-\ell_2 N]\Gamma[m,\ell_1+\ell_2 N] \\
&= \sum_{\ell_1=0}^{N-1} \widetilde{d}_N[m-\ell_1] \sum_{\ell_2=-\infty}^{+\infty} G[\ell_1+\ell_2 N]\Gamma[m,\ell_1+\ell_2 N] \\
&= \sum_{\ell_1=0}^{N-1} \widetilde{d}_N[m-\ell_1]\Upsilon[m,\ell_1]. && (4.33)
\end{aligned}
$$

Here we have redefined the two-variable dependent term as

$$
\Upsilon[m,\ell_1] = \sum_{\ell_2=-\infty}^{+\infty} G[\ell_1+\ell_2 N]\Gamma[m,\ell_1+\ell_2 N]
$$

which is now N-periodic in m and in ℓ_1. Then, we can see in (4.33) that the elements $\Upsilon[m, \ell_1]$ can be interpreted as the coefficients of a transfer (square) matrix between the transmitted $\widetilde{d}_N[\ell_1]$ and the received $\widetilde{g}_N[m]$ symbols at subcarriers ℓ_1 and m respectively. Given the dimension of this discrete transference model in frequency domain, the corresponding exact representation in time domain will involve multi-dimensional convolutions to include the memory effects of $h_1[n]$ and $h_2[n]$. Such complexity is reduced to a minimum in the case when $h_2[n] = \delta[n]$. The processing chain under this restriction is represented in the lower half of figure 4.2 and its analysis is now discussed.

Using the property (4.42), the N-periodic extension of $\beta[m]$ from (4.17) fulfils

$$\widetilde{\beta}_N[m] = \widetilde{b}_{1N}[m] \circledast \widetilde{G}_N[m] = \widetilde{b}_{1N}[m] \circledast \sum_{k=0}^{+\infty} g_k \widetilde{\left(r_b^{(k)}\right)}_N[m] \tag{4.34}$$

where the periodic extension of the k-order autocorrelation term can be calculated as

$$\widetilde{\left(r_b^{(k)}\right)}_N[m] = \widetilde{\left(r_b^{(k-1)}\right)}_N[m] \circledast \widetilde{\left(r_b\right)}_N[m]. \tag{4.35}$$

To define the discrete time-domain equivalent of (4.34) we let the N-point inverse DFT of the circular convolution operators be denoted as

$$\mathrm{DFT}_N^{-1}\left(\widetilde{b}_{1N}[m]\right) = \widetilde{b}_1[n]$$
$$\mathrm{DFT}_N^{-1}\left(\widetilde{\left(r_b\right)}_N[m]\right) = \widetilde{r}_b[n].$$

Hence, we obtain for $\mathrm{DFT}_N^{-1}\left(\widetilde{\beta}_N[m]\right)$ the expression

$$\widetilde{\beta}_N[n] = \widetilde{b}_{1N}[n] \cdot \widetilde{G}_N[n] = \widetilde{b}_{1N}[n] \cdot \sum_{k=0}^{+\infty} g_k \widetilde{r}_b^{(k)}[n] = \mathrm{f}\left(\widetilde{b}_1[n]\right) \tag{4.36}$$

where $\mathrm{f}(\cdot)$ is the HPA function. The periodicity of (4.36) suggests that the non-linearity can be completely simulated in discrete time using the same series expansion based model. Furthermore, evaluating this expression for only N values of $\widetilde{b}_{1N}[n]$ will be sufficient to completely characterize the HPA output.

The coefficients of the spectral components at the HPA input have been defined as $b_1[m] = \widetilde{d}_N[m] H_1\left(\frac{m}{T}\right)$, where only $\widetilde{d}_N[m]$ is N-periodic. Then, the periodic extension of $b_1[m]$ is

$$\widetilde{b}_{1N}[m] = \widetilde{d}_N[m] \widetilde{\left(H_1\right)}_N \left(\frac{m}{T}\right) \tag{4.37}$$

and from (4.1) we have

$$\mathrm{DFT}_N^{-1}\left(\widetilde{d}_N[m]\right) = \frac{1}{N}\widetilde{b}_x[n].$$

Respectively, for the filter $H_1(\cdot)$ we have

$$\mathrm{DFT}_N^{-1}\left(\widetilde{(H_1)}_N\left(\frac{m}{T}\right)\right) = \widetilde{h}_1[n].$$

Then, the inverse DFT for (4.37) gives

$$\widetilde{b}[n] = \frac{1}{N}\widetilde{b}_x[n] \circledast \widetilde{h}_1[n]. \tag{4.38}$$

Hence, the final time-domain relationship for simulating the complete processing chain can be established as

$$\boxed{\widetilde{g}[n] = \widetilde{h}_2[n] \circledast \mathrm{f}\left(\frac{1}{N}\widetilde{b}_x[n] \circledast \widetilde{h}_1[n]\right)} \tag{4.39}$$

only when $h_2[n] = \delta[n]$.

4.A Appendix: General Properties for Periodically Extended Sequences

In this appendix some useful properties for circular convolutions and periodic extensions of infinite-lenght finite-energy sequences are briefly examined. Let us denote the N-periodic extension of the sequence $x[n]$ as

$$\widetilde{x}_N[n] = \sum_{m=-\infty}^{+\infty} x[n + Nm].$$

Given the definition of discrete convolution,

$$y[n] = h[n] * x[n] = \sum_{m=-\infty}^{+\infty} h[m]x[n - m], \tag{4.40}$$

the following properties are verifiable:

$$\begin{aligned} \widetilde{y}_N[n] &= \widetilde{h}_N[n] * x[n] && (4.41) \\ \widetilde{y}_N[n] &= \widetilde{h}_N[n] \circledast \widetilde{x}_N[n]. && (4.42) \end{aligned}$$

The first property is proved by

$$\begin{aligned} \widetilde{h}_N[n] * x[n] &= \sum_{k=-\infty}^{+\infty} h[n + kN] * x[n] \\ &= \sum_{k=-\infty}^{+\infty} y[n + kN] = \widetilde{y}_N[n]. && (4.43) \end{aligned}$$

The second property is proved by

$$\begin{aligned} \widetilde{h}_N[n] \circledast \widetilde{x}_N[n] &= \sum_{m=0}^{N-1} \widetilde{h}_N[m]\widetilde{x}_N[n - m] \\ &= \sum_{m=0}^{N-1} \sum_{k=-\infty}^{+\infty} h[m + kN]\widetilde{x}_N[n - m] \\ &= \sum_{\ell=-\infty}^{+\infty} h[\ell]\widetilde{x}_N[n - \ell] \\ &= h[n] * \widetilde{x}_N[n] = \widetilde{y}_N[n]. && (4.44) \end{aligned}$$

A third property is

$$\widetilde{\left(\widetilde{x}_N[n]h[n]\right)}_N[n] = \widetilde{x}_N[n]\widetilde{h}_N[n] \tag{4.45}$$

and can be proved by

$$\begin{aligned}\widetilde{\left(\widetilde{x}_N[n]h[n]\right)}_N[n] &= \sum_{k=-\infty}^{+\infty} \widetilde{x}_N[n+kN]h[n+kN] \\ &= \sum_{k=-\infty}^{+\infty} \widetilde{x}_N[n]h[n+kN] = \widetilde{x}_N[n]\widetilde{h}_N[n]. \end{aligned} \tag{4.46}$$