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UNIVERSITAT AUTÒNOMA DE BARCELONA

Departament de Matemàtiques

Tesi Doctoral

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Contributions to the study of periodic orbits and  
invariant manifolds in dynamical systems

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Encara no m'ho crec  
i ja torno a ser un satèl·lit  
que fa voltes en línia recta.

Antònia Font



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# Preface

This thesis concerns the study of invariant manifolds and periodic orbits of discrete and continuous dynamical systems.

A *dynamical system*,  $(\mathcal{M}, \mathbb{T}, \Phi)$ , is defined as the action  $\Phi$  of a group,  $\mathbb{T}$ , which represents the time, on a set  $\mathcal{M}$ , called the *phase space*, of the form

$$\begin{aligned}\Phi : \mathbb{T} \times \mathcal{M} &\longrightarrow \mathcal{M} \\ (t, x) &\longmapsto \Phi(t, x),\end{aligned}$$

such that for all  $x \in \mathcal{M}$  and all  $t, s \in \mathbb{T}$ , it holds that  $\Phi(0, x) = x$  and  $\Phi(t + s, x) = \Phi(s, \Phi(t, x))$ .

In practice, a *discrete* dynamical system is given by a map,  $F : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ , with  $\mathbb{T} = \mathbb{Z}$ , and by the action  $\Phi(k, x) = F^k(x)$ , where  $F^k$  denotes the  $k$ -fold composition of  $F$  with itself. A *continuous* dynamical system is given by a complete vector field,  $X : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ , with  $\mathbb{T} = \mathbb{R}$ , and by the action  $\Phi(t, x) = \varphi_t(x)$ , where  $\frac{d}{dt}\varphi_t(x) = X(\varphi_t(x))$ . In the case that  $X$  is not complete, there exists an equivalent vector field  $\tilde{X}$  such that its solutions are defined for all  $t \in \mathbb{R}$ .

The memoir is divided into two parts that can be read independently. The first part is dedicated to the study of invariant manifolds associated with parabolic points and parabolic invariant tori, both for maps and vector fields. The second part concerns the study of periodic orbits of dynamical systems on manifolds. Each part starts with an introductory chapter. Then, every chapter contains an introduction section where we motivate the main problem and we describe the structure of the chapter and locate its main results.

Part I consists of six chapters. After the introduction, Chapters 2 and 3 concern the study of invariant curves of planar maps with parabolic points. The work of these two chapters is collected in the paper [22]. In Chapter 4 we study an analogous problem for planar vector fields. This work, even if it is original on its form, can be also deduced from the results of [25]. Finally, Chapters 5 and 6 are dedicated to invariant manifolds of higher dimensional maps and vector fields with parabolic invariant tori. The content of these chapters is collected in a paper in preparation, [21].

Part 2 consists on three chapters, which are also independent between them. After a common introduction in Chapter 7, Chapter 8 is dedicated to the study of the set of periods of a class of diffeomorphisms on manifolds. The work of this chapter is reflected in the paper [23]. Finally, the results presented in Chapter 9, concerning the existence of limit cycles of linear vector fields, appear in the preprint [24].



## Part I

# Invariant manifolds of parabolic points and tori with nilpotent part

# Chapter 1

## Introduction

This first part of the thesis is dedicated to the study of invariant manifolds of some families of discrete and continuous dynamical systems. Concretely, the main objective is to provide sufficient conditions for the existence and regularity of local invariant manifolds asymptotic to parabolic points and parabolic invariant tori.

Given  $U \subset \mathbb{R}^n$  and a map,  $F : U \rightarrow \mathbb{R}^n$ , or a vector field,  $X : U \rightarrow \mathbb{R}^n$ , and *invariant manifold* is a submanifold  $V$  of  $U$  such that for all  $x \in V$ ,  $F(x)$  remains in  $V$ , for the case of maps, or such that for all  $x \in V$ ,  $X(x) \in T_x V$ , for the case of vector fields, where  $T_x V$  is the space tangent to  $V$  at the point  $x$ . A *fixed point* of  $F$  is a point  $x \in U$  such that  $F(x) = x$ , and a fixed or *critical point* of  $X$  is a point  $x \in U$  such that  $X(x) = 0$ .

A fixed point  $p$  of a map or a vector field is said to be *parabolic* if the linearization of the map at  $p$  has all the eigenvalues equal to 1 or  $-1$ , or, respectively, if the linearization of the vector field at  $p$  has all the eigenvalues equal to zero. One can also generalize this concept to higher dimensional invariant sets such as invariant tori.

Invariant manifolds play a central role in the study of dynamical systems. There is a huge amount of literature devoted to studying them in many different settings. In this thesis we deal with the invariant manifolds of a type of parabolic fixed points in dimension two, and with a related generalized problem with parabolic invariant tori. The fixed points (resp. invariant tori) that we study have the particularity that the differential of the map or the vector field at the fixed point (resp. invariant torus) does not diagonalize. For this reason we call them parabolic points (resp. parabolic tori) with *nilpotent part*.

The study of parabolic invariant manifolds is relevant, apart from the interest that presents itself as a mathematical problem, because that kind of manifolds appears naturally in many problems motivated by physics, chemistry and other sciences.

Parabolic points appear generically in two-parameter families of planar maps or in one-parameter families in the case of area-preserving maps. In particular they appear when a family of maps undergoes a Bogdanov-Takens bifurcation [10, 71].

In some problems in Celestial Mechanics it is useful to consider parabolic points or parabolic orbits at infinity in order to use their invariant manifolds (provided they exist) to study features of the dynamics in the finite phase space. The local study in a neighborhood of such points is done by means of a change of variables which sends the infinity to a finite part of the

space [54]. Also, the periodic orbits become fixed points of appropriate families of Poincaré maps. In such cases the fixed points are parabolic for all values of the parameters of the family and may have invariant manifolds.

Parabolic manifolds have been also used to prove the existence of oscillatory motions in some well-known problems of Celestial Mechanics as the Sitnikov problem [67, 56] and the circular planar restricted three-body problem [50, 36, 37] using the transversal intersection of invariant manifolds of parabolic points and symbolic dynamics. The Sitnikov problem deals with a configuration of the restricted three-body problem where the two primary bodies (those with non-zero mass) describe ellipses, while the third body moves in the line through their center of mass and orthogonal to the plane where the motion of the primaries takes place. The circular planar restricted three-body problem, instead, considers the motion of a body of negligible mass moving under the gravitational action of the two primary bodies, which perform a circular motion, while all three bodies remain in the same plane.

The existence of oscillatory motions in all these instances is strongly related to some invariant objects at infinity that are either fixed points or periodic orbits and also with their stable and unstable manifolds. Although if these invariant objects are parabolic in the sense that the linearization of the vector field on them has all the eigenvalues equal to zero, they do have stable and unstable invariant manifolds in the classical sense of hyperbolic invariant objects, that is, invariant manifolds that locally govern the dynamics close to the invariant object.

Parabolic manifolds also appear in the Manev problem [20], and they play a significant role in the study of certain physical systems [48, 30].

In Chapters 2 and 3 we study the existence and regularity of invariant manifolds of planar maps having a parabolic fixed point with nilpotent part. The study is done for analytic maps (Chapter 2) and for finitely differentiable maps (Chapter 3). We distinguish three different cases depending on the nonlinear terms of the series expansion of the maps, where the generic maps are contained in case 1.

In the analytic case, we prove the existence of an analytic one-dimensional invariant manifold (away from the fixed point) under suitable conditions on some of the coefficients of the nonlinear terms of the map. The existence of an analytic invariant curve in such case is already proved in [25] using a variation of McGehee's method. However, here we use the parameterization method (see Section 2.2.2), which provides approximations of the manifolds up to any order, and we also present an *a posteriori* result.

In the  $C^r$  case, first we use our results for analytic maps applied to the Taylor polynomial of degree  $r$  of the map. In this way we obtain an analytic invariant manifold which is used as an approximation to apply the parameterization method to the original map. Moreover, we use the fiber contraction theorem to obtain the differentiability result. Concretely, we prove that if the regularity of the map is bigger than some (easily computable) value, then there exists an invariant manifold of the same regularity, away from the fixed point.

For both the analytic and the  $C^r$  cases, we provide approximations of the parameterizations of the invariant manifolds up to an order that depends on the regularity of the map. Those approximations are used to prove later on the existence of the invariant manifolds. Moreover, one can implement our algorithms in a computer program to calculate the coefficients of an approximation of the invariant manifolds.

For the same class of maps that we consider, but using different tools, some regularity results



are obtained in [75]. In that paper, the authors deal with what we denote by case 1 for  $C^\infty$  maps and obtain the existence of a stable manifold  $W_\rho^{s+}$  as the graph of some function  $\varphi$  by solving a fixed point equation equivalent to the invariance of the graph of  $\varphi$ . This equation is considered for functions  $\varphi$  in a suitable subset of the space of functions of class  $C^{\lfloor (k+1)/2 \rfloor}$ , where  $\lfloor \cdot \rfloor$  denotes integer part, and it is solved by applying the Schauder fixed point theorem. Hence, they obtain invariant manifolds of class  $C^{\lfloor (k+1)/2 \rfloor}$ . Instead, in our approach, we use the parameterization method and we obtain, away from the fixed point, analytic invariant manifolds for analytic maps and  $C^r$  invariant manifolds for  $C^r$  maps, provided  $r$  is larger than some quantity that depends on the nonlinear terms of the map.

One-dimensional manifolds of fixed points with linear part equal to the identity are studied in [3] using the parameterization method. Higher-dimensional manifolds in the same setting are considered in [2] using a generalized version of the method of McGehee, and in [5, 6] using the parameterization method, where applications to Celestial Mechanics are given. The Gevrey character of one-dimensional manifolds is studied in [4].

In Chapter 4 we consider an analogous problem as in Chapters 2 and 3, but concerning planar vector fields. We present the results of existence of invariant curves of such vector fields using the results from the previous chapters and the fact that, under suitable conditions, the invariant manifolds of a vector field are the same ones as the invariant manifolds of its time- $t$  flow.

In Chapters 5 and 6 we deal with invariant manifolds of parabolic invariant tori with nilpotent part. We consider maps (Chapter 5) and vector fields (Chapter 6) defined in  $\mathbb{R}^2 \times \mathbb{T}^d$ , having a  $d$ -dimensional invariant torus,  $\mathcal{T} = \{x = y = 0\}$ . The map (resp. vector field) restricted to  $\mathcal{T}$  defines a rotation of frequency  $\omega$ , and its differential restricted to the directions normal to  $\mathcal{T}$  does not diagonalize. In this context, we give conditions on the coefficients of the nonlinear terms of the map (resp. vector field) under which  $\mathcal{T}$  possesses stable and unstable invariant manifolds, also called *whiskers*. We also consider the same problem for non-autonomous vector fields that depend quasiperiodically on time, and we present some applications of our results.

All the results of existence of invariant manifolds, both for maps and vector fields, and for fixed points and invariant tori, are stated in two steps. In the first step we present an algorithm that allows to compute an approximation of a parameterization of the invariant manifolds. In the planar case this approximation is a polynomial. In the case of maps and vector fields with invariant tori, it is a function that depends in a polynomial way of one of the variables and where the coefficients of the polynomial are functions of the angle variables.

In the second step, we present an *a posteriori* result, which is a kind of statement that assumes that one can find a «close to invariant» manifold satisfying certain hypotheses, and then ensures that there exist a true invariant manifold closeby.

The algorithm provided in the first step of the procedure satisfies the hypotheses required in the second step, and hence, combining the two results, we obtain the existence of an invariant manifold which is well approximated by the parameterization provided in the first step.

Contrary to the case of hyperbolic fixed points or hyperbolic invariant tori, the dynamics inside the parabolic invariant manifolds can not be linearized. As a consequence of our techniques we obtain a normal form of the dynamics of the maps and vector fields restricted to the invariant manifolds, extending some of the results of Takens [70] and Voronin [74] to parabolic tori. In the planar case we recover the normal form of a one-dimensional system

around a parabolic point described in [18] and [70].

It is well accepted that some versions of invariant manifold theory, at least for the analytic case, were already known to Darboux, Poincaré and Lyapunov by the end of the 19th century. The motivation for Poincaré in [60] was the theory of special functions, where he considered equations of the form

$$F \circ K(t) = K(\lambda t), \tag{1.0.1}$$

where  $F$  is a polynomial, and which can be interpreted as saying that the system of functions given by the components of  $K$  admits a multiplication rule. The paper [60] shows that given a map  $F$  and provided that  $\lambda$ , with  $|\lambda| > 1$ , is a simple eigenvalue of  $DF(0)$  and that there are no eigenvalues of  $F$  which are powers of  $\lambda$ , one can compute a formal series expansion for  $K$ . Moreover, using the so-called majorant method one can show that the formal series of  $K$  converges.

In our setting we deal with equations similar to (1.0.1) to find an invariant manifold,  $K$ , of  $F$  (or a modified version of it for the case of vector fields). We also find a series expansion for  $K$  that provides an approximation of a parameterization of an invariant manifold. However, due to the nature of the parabolic invariant manifolds, the series of  $K$  does not converge in any neighborhood of the fixed point or the invariant torus. This is a consequence of the fact that for analytic systems one cannot expect to find analyticity of the invariant manifolds around a parabolic point or a parabolic invariant torus. We can however obtain the existence of a solution  $K$  of an equation of the form (1.0.1) by looking for a correction of the approximation of the series expansion of  $K$ .

Throughout this first part of the memoir,  $M$  and  $\rho_0$  will denote positive constants, and they do not take necessarily the same value at different places.

## Chapter 2

# Invariant manifolds of analytic maps with nilpotent parabolic points

### 2.1 Introduction

The objective of this chapter is to study the existence and regularity of invariant curves asymptotic to a parabolic nilpotent fixed point and to provide an algorithm to compute an approximation of a parameterization of such invariant curves.

We consider two-dimensional maps having a parabolic fixed point whose linearization does not diagonalize, concretely we assume it has a double eigenvalue equal to 1. By simple changes such maps can be brought to the form

$$F(x, y) = \begin{pmatrix} x + cy + f_1(x, y) \\ y + f_2(x, y) \end{pmatrix}, \quad (2.1.1)$$

with  $c > 0$ ,  $f_1(0, 0) = f_2(0, 0) = 0$  and  $Df_1(0, 0) = Df_2(0, 0) = 0$ . The origin has a center manifold of dimension two, however, inside this manifold there may exist curves that behave topologically as stable or unstable curves.

This class of maps was considered in [25] and the existence of analytic curves was proved. Concretely the (local) sets considered there and the ones we deal with are

$$W_\rho^{s+} = \{(x, y) \mid F^n(x, y) \in (0, \rho) \times (-\rho, \rho), \forall n \geq 0, \lim_{n \rightarrow \infty} F^n(x, y) = 0\}$$

and

$$W_\rho^{u+} = \{(x, y) \mid F^{-n}(x, y) \in (0, \rho) \times (-\rho, \rho), \forall n \geq 0, \lim_{n \rightarrow \infty} F^{-n}(x, y) = 0\}.$$

The main result of [25] concerns analytic stable invariant curves in the domain  $\{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \leq 0\}$  under some appropriate conditions on the higher order terms. Then, the existence of both stable and unstable curves in neighborhoods of the origin are deduced from the main result by using the symmetries  $(x, y) \mapsto (-x, y)$ ,  $(x, y) \mapsto (x, -y)$  and  $(x, y) \mapsto (-x, -y)$  and the inverse map  $F^{-1}$ . Moreover, a detailed study of the local dynamics provide

the uniqueness of such curves in the category of  $C^k$  maps where  $k$  is the minimum regularity for having a Taylor expansion providing the relevant nonlinear terms [25].

In this chapter we study the existence and regularity of stable curves in the domain  $\{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \leq 0\}$  using the parameterization method. In the analytic case we recover the existence results of [25] but we also provide approximations of the curves up to an arbitrarily high order. We consider three cases of maps of the form (2.1.1), already introduced in [25], which depend in some sense on the dominant part of the nonlinear terms. The study depends on each case. Contrary to other works we do not use the Poincaré normal form for the map, but a simple and easy-to-compute reduced form.

This class of maps, assuming the fixed point is not isolated, was studied in [17] motivated by the study of collisions in two-body problems with central force potential satisfying certain asymptotic properties at the origin. A special case of this family not previously covered is studied in [45]. These papers use an adapted form of the method of McGehee for parabolic points without nilpotent part [54]. McGehee's method consists of looking for a sector-like domain  $S$ , with the fixed point in the vertex, such that the points whose positive iterates remain on  $S$  form a graph of some function  $\varphi$ . To prove analyticity, it considers the complexified map and uses Rouché's theorem to obtain the uniqueness of  $\varphi(x)$  in terms of  $x$ , for  $x$  in a complex extension  $\overline{S}$  of  $S$ , so that then one can apply the implicit function theorem to obtain the analyticity of  $\varphi(x)$  for  $x \in \overline{S}$ .

This chapter is dedicated to the study of invariant curves of maps of the form (2.1.1) which are analytic. In Chapter 3 we consider the analogous problem but for differentiable maps. Since the proofs of the results for analytic maps are, in general, more simple than the results for differentiable maps, we will introduce the maps we will deal with for the differentiable case, which is more general, the analytic maps being a particular case of them. However, in this chapter the existence results will only concern analytic maps.

The main results of this chapter are Theorems 2.2.1 and 2.2.3, concerning the existence of analytic invariant curves of a map  $F$  of the form (2.1.1). In Section 2.2 we present them after introducing the parameterization method. The results are stated for the stable curves. In Section 2.2.4 we show that completely analogous results hold true for the unstable ones. In Section 2.3 we provide an algorithm to obtain parameterizations of approximations of the invariant curves of  $F$ . We provide the proof of existence of such curves in Section 2.4. Finally, in Section 2.5 we illustrate numerically that these curves in general are not analytic in any neighborhood of the fixed point.

## 2.2 Statement of the main results

### 2.2.1 Reduction of the maps to a simple form

We consider  $C^r$ ,  $r \geq 3$ , or analytic maps  $F : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , where  $U$  is a neighborhood of  $(0, 0)$ , of the form

$$F(x, y) = \begin{pmatrix} x + cy + f_1(x, y) \\ y + f_2(x, y) \end{pmatrix}, \quad (2.2.1)$$

with  $c > 0$  and with  $f_1(x, y), f_2(x, y) = O(\|(x, y)\|^2)$ . Via the  $C^r$  change of variables given by  $\tilde{x} = x, \tilde{y} = y + \frac{1}{c} f_1(x, y)$ ,  $F$  can be written in the form

$$F(x, y) = \begin{pmatrix} x + cy \\ y + f(x, y) \end{pmatrix},$$

with  $f(x, y) = O(\|(x, y)\|^2)$  having the same regularity as  $F$ . In the  $C^r$  case we denote by  $P(x, y)$  the Taylor polynomial of degree  $r$  of  $f(x, y)$ . We write  $P(x, y)$  in the form

$$P(x, y) = p(x) + yq(x) + u(x, y),$$

where we have collected all the terms independent of  $y$  in  $p(x)$ , the terms that are linear in  $y$  in  $yq(x)$  and all remaining terms in  $u(x, y)$ . Note that all terms in  $u(x, y)$  have the factor  $y^2$ . More precisely, we write  $p(x) = x^k(a_k + \dots + a_r x^{r-k})$  and  $q(x) = x^{l-1}(b_l + \dots + b_r x^{r-l})$ , with  $2 \leq k, l \leq r$ . Therefore we have  $f(x, y) = P(x, y) + g(x, y)$  with  $g(x, y) = o(\|(x, y)\|^r)$ .

Also, note that one can always assume that  $c > 0$ . If this is not the case, then it can be attained via the linear transformation given by  $L(x, y) = (x, -y)$ , taking the conjugate map  $\bar{F} = L^{-1} \circ F \circ L$ . Notice however that  $L$  sends the lower semi plane to the upper one. Hence, any map  $F$  of the form (2.2.1) can be written in the form

$$\bar{F}(x, y) = \begin{pmatrix} x + cy \\ y + p(x) + yq(x) + u(x, y) + g(x, y) \end{pmatrix}, \quad (2.2.2)$$

with  $c > 0$ . In the analytic case we have the same form with  $g(x, y)$  analytic. In general we will not write the dependence of  $p, q, u$  and  $g$  on  $r$ . Throughout the chapter we will refer to (2.2.2) as the *reduced form* of  $F$  and we will use the same notation  $F$ .

We will deal with maps of the form (2.2.2). We remark that in contrast with other references [25, 75] in which they work with normal forms of  $F$  à la Poincaré, we work with the reduced form obtained with a simple change of variables. This is an important advantage when one has to perform effective computations.

Following [25], we shall consider three cases depending on the indices  $k$  and  $l$ :

- Case 1:  $k < 2l - 1$  and  $a_k \neq 0$ ,
- Case 2:  $k = 2l - 1$  and  $a_k \neq 0, b_l \neq 0$ ,
- Case 3:  $k > 2l - 1$  and  $b_l \neq 0$ .

In order to deal, whenever possible, with several cases at the same time we associate to  $F$  the integers  $N$  and  $s$ :  $N = k$  in case 1 and  $N = l$  in cases 2 and 3;  $s = 2r$  in case 1 and  $s = r$  in cases 2, 3. Notice that the generic case is case 1 with  $k = 2$ .

Next we make a comment concerning notation. The superindices  $x$  and  $y$  on the symbol of a function or an operator that takes values in  $\mathbb{R}^2$  will denote the first and second components of its image, respectively. In  $\mathbb{R}^2$  and  $\mathbb{C}^2$  we will use the norm given by  $\|(x, y)\| = \max\{|x|, |y|\}$ .

## 2.2.2 The parameterization method

To study the stable curves of  $F$  we will use the parameterization method (see [14], [15], [16], [42]). It consists in looking for the curves as images of parameterizations,  $K$ , together with a representation of the dynamics of the map restricted to them,  $R$ , satisfying the invariance equation,

$$F \circ K = K \circ R. \quad (2.2.3)$$

This is a functional equation that has to be adapted to the setting of the problem at hand. Clearly, we need the range of  $R$  to be contained in the domain of  $K$ . It follows immediately from (2.2.3) that the range of  $K$  is invariant. Essentially,  $K$  is a (semi)conjugation of the map restricted to the range of  $K$  to  $R$ . Equation (2.2.3) has to be solved in a suitable space of functions. Usually it is convenient to have good approximations of  $K$  and  $R$  and look for a (small) correction of  $K$ , in some sense, while maintaining  $R$  fixed. Assuming differentiability and taking derivatives in (2.2.3) we get  $DF \circ K \cdot DK = DK \circ R \cdot DR$  which says that the range of  $DK$  has to be invariant by  $DF$ .

In our setting we look for  $K = (K^x, K^y) : [0, \rho] \rightarrow \mathbb{R}^2$  such that  $K(0) = (0, 0)$  and  $DK(t)$  satisfies  $DK^y(t)/DK^x(t) \rightarrow 0$  as  $t \rightarrow 0$ . We already know that in the parabolic case, in general, there is a loss of regularity of the invariant curves at the origin with respect to the regularity of the map [25], [5], [6]. Then we can not assume *a priori* a Taylor expansion of high degree of the curve at  $t = 0$ . However, we can obtain formal polynomial approximations,  $\mathcal{K}_n$  and  $\mathcal{R}_n$ , of  $K$  and  $R$ , satisfying (2.2.3) up to a certain order that depends on the degree of differentiability of  $F$ . Our results will then provide that these expressions are indeed approximations of true invariant curves, whose existence is rigorously established.

On the other hand we can suppose that we have approximations, obtained in some way, that satisfy some conditions and obtain that there are true invariant curves closeby.

## 2.2.3 Main results

We state the main results concerning the existence of analytic stable invariant manifolds of analytic maps of the form (2.2.2). Since an analytic map of the form (2.2.1) is analytically conjugated to a map of the form (2.2.2), the results of the next theorems provide invariant manifolds for (2.2.1).

**Theorem 2.2.1.** *Let  $F : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be an analytic map in a neighborhood  $U$  of  $(0, 0)$  of the form (2.2.2). Assume the following hypotheses according to the different cases:*

$$(case\ 1) \quad a_k > 0, \quad (case\ 2) \quad a_k > 0, b_l \neq 0, \quad (case\ 3) \quad b_l < 0.$$

*Then, there exists a  $C^1$  map  $K : [0, \rho] \rightarrow \mathbb{R}^2$ , analytic in  $(0, \rho)$ , such that*

$$K(t) = \begin{cases} (t^2, K_{k+1}^y t^{k+1}) + (O(t^3), O(t^{k+2})) & case\ 1, \\ (t, K_l^y t^l) + (O(t^2), O(t^{l+1})) & cases\ 2, 3, \end{cases} \quad (2.2.4)$$

*with  $K_{k+1}^y = -\sqrt{\frac{2a_k}{c(k+1)}}$  for case 1,  $K_l^y = \frac{b_l - \sqrt{b_l^2 + 4ca_k l}}{2cl}$  for case 2 and  $K_l^y = \frac{b_l}{c}$  for case 3, and a polynomial  $R$  of the form  $R(t) = t + R_N t^N + R_{2N-1} t^{2N-1}$ , with  $R_k = \frac{c}{2} K_{k+1}^y$  for case 1 and  $R_l = c K_l^y$  for cases 2, 3, such that*

$$F(K(t)) = K(R(t)), \quad t \in [0, \rho].$$

**Remark 2.2.2.** This theorem provides a local stable manifold parameterized by  $K : [0, \rho) \rightarrow \mathbb{R}^2$  with  $\rho$  small. The proof does not give an explicit estimate for the value of  $\rho$ . However, we can extend the domain of  $K$  by using the formula

$$K(t) = F^{-j}K(R^j(t)), \quad j \geq 1,$$

while the iterates of the inverse map  $F^{-1}$  exist (note that  $R$  is a weak contraction). In particular, if the map  $F^{-1}$  is globally defined, as it happens for example for the Hénon map, one can extend the domain of  $K$  to  $[0, \infty)$ . This observation also applies for the next theorem 2.2.3. Also, the domain of  $K$  can be extended to an open domain of  $\mathbb{C}$  that contains  $(0, \rho)$ .

Next theorem is an *a posteriori* version of Theorem 2.2.1 which, given an analytic approximation, in a certain sense, of the solutions  $K$  and  $R$  of the conjugation equation  $F \circ K = K \circ R$ , provides exact solutions of the equation, close to the approximations.

**Theorem 2.2.3.** Let  $F : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be as in Theorem 2.2.1 and let  $\hat{K} : (-\rho, \rho) \rightarrow \mathbb{R}^2$  and  $\hat{R} = (-\rho, \rho) \rightarrow \mathbb{R}$  be analytic maps satisfying

$$\hat{K}(t) = \begin{cases} (t^2, \hat{K}_{k+1}^y t^{k+1}) + (O(t^3), O(t^{k+2})) & \text{case 1,} \\ (t, \hat{K}_l^y t^l) + (O(t^2), O(t^{l+1})) & \text{cases 2, 3,} \end{cases}$$

and  $\hat{R}(t) = t + \hat{R}_N t^N + O(t^{N+1})$ ,  $\hat{R}_N < 0$ , such that

$$F(\hat{K}(t)) - \hat{K}(\hat{R}(t)) = (O(t^{n+N}), O(t^{n+2N-1})), \quad (2.2.5)$$

for some  $n \geq 2$  in case 1 or  $n \geq 1$  in cases 2, 3.

Then, there exists a  $C^1$  map  $K : [0, \rho) \rightarrow \mathbb{R}^2$ , analytic in  $(0, \rho)$ , and an analytic map  $R : (-\rho, \rho) \rightarrow \mathbb{R}$  such that

$$F(K(t)) = K(R(t)), \quad t \in [0, \rho)$$

and

$$K(t) - \hat{K}(t) = (O(t^{n+1}), O(t^{n+N})),$$

$$R(t) - \hat{R}(t) = \begin{cases} O(t^{2k-1}) & \text{if } n \leq k \\ 0 & \text{if } n > k \end{cases} \quad \text{case 1,}$$

$$R(t) - \hat{R}(t) = \begin{cases} O(t^{2l-1}) & \text{if } n \leq l-1 \\ 0 & \text{if } n > l-1 \end{cases} \quad \text{cases 2, 3.}$$

**Remark 2.2.4.** In case 1, condition (2.2.5) with  $n \geq 2$  implies the following relations

$$\hat{K}_{k+1}^y = \pm \sqrt{\frac{2a_k}{c(k+1)}}, \quad \hat{R}_k = \frac{c}{2} \hat{K}_{k+1}^y.$$

In cases 2 and 3 condition (2.2.5) with  $n \geq 1$  implies

$$\hat{R}_l = c \hat{K}_{k+1}^y, \quad \begin{cases} a_k + b_l \hat{K}_l^y = l \hat{R}_l \hat{K}_l^y & \text{case 2,} \\ b_l = l \hat{R}_l & \text{case 3.} \end{cases}$$

**Remark 2.2.5.** Theorem 2.2.3 provides the existence of a stable manifold assuming we have previously computed an approximation of it, but the theorem is independent of the way such an approximation has been obtained. Propositions 2.3.1, 2.3.4 and 2.3.5 (in Section 2.3) provide an algorithm to obtain polynomial maps  $\mathcal{K}_n$  and  $\mathcal{R}_n$  that satisfy condition (2.2.5) of Theorem 2.2.3 for any  $n$ .

**Remark 2.2.6.** The form of the map  $R$  given in the statement of Theorem 2.2.1 is the normal form of the dynamics of a one-dimensional system in a neighborhood of a parabolic point (see [18, 70, 74]).

As mentioned, using the conjugations  $(x, y) \mapsto (\pm x, \pm y)$  and  $F^{-1}$  we can obtain the local phase portraits and the location of the local invariant manifolds of  $F$  depending on the studied cases (see [25]).

**Remark 2.2.7.** The invariant manifolds obtained in Theorems 2.2.1 and 2.2.3 are unique. For that we refer to Theorem 4.1 of [25], where it is proved that if the map  $F$  is  $C^k$ , in all the considered cases the local stable set  $W_\rho^{s+}$  is a graph and therefore is unique. This is proved by checking that both the iterates of the points that are above and the ones that are below the invariant curve cannot converge to the fixed point by a detailed study of the behaviour of the iterates. However, the parameterizations are not unique because if  $K$  and  $R$  satisfy  $F \circ K = K \circ R$ , then for any invertible map  $\beta : [0, \rho] \rightarrow \mathbb{R}$ , the maps  $\tilde{K} = K \circ \beta$  and  $\tilde{R} = \beta^{-1} \circ R \circ \beta$  satisfy  $F \circ \tilde{K} = \tilde{K} \circ \tilde{R}$ .

## 2.2.4 Unstable manifolds

Assuming  $F$  satisfies the hypotheses of Theorem 2.2.1, in cases 1 and 2, the result for the stable manifold is obtained from the stated theorem without having to compute the inverse map  $F^{-1}$ . For case 3, if one assumes  $b_l > 0$  instead, then an analogous result is obtained for the existence of an unstable manifold of  $F$ .

Next, we show that the expansions of the parameterizations of the unstable curves obtained in Section 2.3 are approximations of true invariant curves, as it happens for the stable ones.

Assume we have a map of the form (2.2.2). Then, by Propositions 2.3.1, 2.3.4 or 2.3.5 (see next section) we have approximations  $\mathcal{K}_n$  and  $\mathcal{R}_n$  such that

$$\mathcal{G}_n(t) = F(\mathcal{K}_n(t)) - \mathcal{K}_n(\mathcal{R}_n(t)) = (O(t^{n+N}), O(t^{n+2N-1})), \quad (2.2.6)$$

with  $\mathcal{R}_n(t) = t + R_N t^N + O(t^{N+1})$  and  $R_N > 0$ , which means that 0 is a repeller for  $\mathcal{R}_n$ . Also,  $\mathcal{R}_n$  is locally invertible and we have

$$\mathcal{R}_n^{-1}(t) = t - R_N t^N + O(t^{N+1}),$$

and

$$F^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - cy + ca_k(x - cy)^k + cb_l y(x - cy)^{l-1} + O(x^{k+1}) + O(yx^l) \\ y - a_k(x - cy)^k - b_l y(x - cy)^{l-1} + O(x^{k+1}) + O(yx^l) \end{pmatrix}.$$

Then, composing by  $F^{-1}$  and  $\mathcal{R}_n^{-1}$  in (2.2.6) we obtain

$$F^{-1}(\mathcal{K}_n(t)) - \mathcal{K}_n(\mathcal{R}_n^{-1}(t)) = (O(t^{n+N}), O(t^{n+2N-1})).$$



Moreover, there exists a change of variables of the form  $C(x, y) = (x, -y) + O(\|(x, y)\|^N)$  that transforms  $F^{-1}$  into its reduced form  $G := C^{-1} \circ F^{-1} \circ C$ , and then  $G$  reads

$$G \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + cy \\ y + a_k x^k - b_l y x^{l-1} + O(x^{k+1}) + O(yx^l) \end{pmatrix}.$$

We also have

$$G(C^{-1}(\mathcal{K}_n(t))) - C^{-1}(\mathcal{K}_n(\mathcal{R}_n^{-1}(t))) = (O(t^{n+N}), O(t^{n+2N-1})).$$

Thus, if  $F$  is in case 1 with  $a_k > 0$  then  $G$  is also in case 1 with the same coefficient  $a_k$  positive. Also, if  $F$  is in case 2 with  $a_k > 0$  and  $b_l \neq 0$  then  $G$  is also in case 2 with the corresponding coefficients  $a_k$  positive and  $b_l$  different from 0. If  $F$  is in case 3 with  $b_l > 0$  then  $G$  is also in case 3 and the coefficient of  $yx^{l-1}$  is given by  $-b_l$ . Therefore, by Theorem 2.2.3 there exist a map  $K : [0, \rho) \rightarrow \mathbb{R}^2$ , analytic in  $(0, \rho)$  and an analytic map  $R : (-\rho, \rho) \rightarrow \mathbb{R}$  such that  $G \circ K = K \circ R$ , with

$$K(t) - C^{-1}\mathcal{K}_n(t) = (O(t^{n+1}), O(t^{n+N})), \quad (2.2.7)$$

$$R(t) - \mathcal{R}_n^{-1}(t) = \begin{cases} O(t^{2k-1}) & \text{if } n \leq k \\ 0 & \text{if } n > k \end{cases} \quad \text{case 1,}$$

$$R(t) - \mathcal{R}_n^{-1}(t) = \begin{cases} O(t^{2l-1}) & \text{if } n \leq l-1 \\ 0 & \text{if } n > l-1 \end{cases} \quad \text{cases 2, 3.}$$

Hence, we have  $F^{-1} \circ C \circ K = C \circ K \circ R$ , which means that  $C \circ K$  is a parameterization of an unstable manifold of  $F$ . Moreover, from (2.2.7) and the form of  $C$ , we have

$$C(K(t)) - \mathcal{K}_n(t) = (O(t^{n+1}), O(t^{n+N})),$$

and therefore  $\mathcal{K}_n$  is an approximation of a parameterization of such unstable manifold.

## 2.3 Formal polynomial approximation of a parameterization of the invariant curves

In this section we consider  $C^r$  maps  $F$  of the form (2.2.2) and we provide algorithms, depending on the case, to obtain two polynomial maps,  $\mathcal{K}_n$  and  $\mathcal{R}_n$ , that are approximations of solutions  $K$  and  $R$  of the invariance equation

$$F \circ K = K \circ R. \quad (2.3.1)$$

Because of the nature of the problem, the two components of  $\mathcal{K}_n$  will have different orders and different degrees. The index  $n$  has to be seen as an induction index. Higher values of  $n$  mean better approximation.

The obtained approximations correspond to formal invariant curves. They correspond to stable curves when the coefficient  $R_k$  (case 1) or  $R_l$  (cases 2, 3) of  $\mathcal{R}_n$  are negative (see the results below). When those coefficients are positive they correspond to unstable curves.

**Proposition 2.3.1** (Case 1). *Let  $F$  be a  $C^r$  map of the form (2.2.2) with  $2 \leq k \leq r$ . Assume that  $k < 2l - 1$  and  $a_k > 0$ . Then, for all  $2 \leq n \leq 2(r - k + 1)$ , there exist two pairs of polynomial maps,  $\mathcal{K}_n$  and  $\mathcal{R}_n$ , of the form*

$$\mathcal{K}_n(t) = \begin{pmatrix} t^2 + \dots + K_n^x t^n \\ K_{k+1}^y t^{k+1} + \dots + K_{n+k-1}^y t^{n+k-1} \end{pmatrix}$$

and

$$\mathcal{R}_n(t) = \begin{cases} t + R_k t^k & \text{if } 2 \leq n \leq k, \\ t + R_k t^k + R_{2k-1} t^{2k-1} & \text{if } n \geq k + 1, \end{cases}$$

such that

$$\mathcal{G}_n(t) := F(\mathcal{K}_n(t)) - \mathcal{K}_n(\mathcal{R}_n(t)) = (O(t^{n+k}), O(t^{n+2k-1})). \quad (2.3.2)$$

For the first pair we have

$$K_{k+1}^y = -\sqrt{\frac{2a_k}{c(k+1)}}, \quad R_k = -\sqrt{\frac{ca_k}{2(k+1)}} = \frac{c}{2} K_{k+1}^y,$$

and for the second one

$$K_{k+1}^y = \sqrt{\frac{2a_k}{c(k+1)}}, \quad R_k = \sqrt{\frac{ca_k}{2(k+1)}} = \frac{c}{2} K_{k+1}^y.$$

If  $F$  is  $C^\infty$  or analytic, one can compute the polynomial approximation  $\mathcal{K}_n$  up to any order.

**Remark 2.3.2.** The algorithm described in the proof of this (and the next) propositions can be implemented in a computer program to calculate  $\mathcal{R}_n$  and the expansion of  $\mathcal{K}_n$ .

**Notation 2.3.3.** Along the proof, given a  $C^r$  one-variable map  $f$ , we will denote by  $[f]_n$ ,  $0 \leq n \leq r$ , the coefficient of the term of order  $n$  of the jet of  $f$  at 0.

*Proof.* We will see that we can determine  $\mathcal{K}_n$  and  $\mathcal{R}_n$  iteratively.

For  $n = 2$ , we claim that there exist polynomial maps  $\mathcal{K}_2(t) = (t^2, K_{k+1}^y t^{k+1})$  and  $\mathcal{R}_2(t) = t + R_k t^k$ , such that  $\mathcal{G}_2(t) = F(\mathcal{K}_2(t)) - \mathcal{K}_2(\mathcal{R}_2(t)) = (O(t^{k+2}), O(t^{2k+1}))$ .

Indeed, from the expansion of  $\mathcal{G}_2$  we have

$$\mathcal{G}_2(t) = \begin{pmatrix} t^2 + c K_{k+1}^y t^{k+1} - t^2 - 2R_k t^{k+1} + O(t^{2k}) \\ K_{k+1}^y t^{k+1} + a_k t^{2k} - K_{k+1}^y t^{k+1} - (k+1) K_{k+1}^y R_k t^{2k} + O(t^{2k+1}) \end{pmatrix},$$

so, if the conditions

$$c K_{k+1}^y - 2R_k = 0, \quad a_k - (k+1) K_{k+1}^y R_k = 0,$$

are satisfied, then we clearly have  $\mathcal{G}_2(t) = (O(t^{2+k}), O(t^{2k+1}))$ , and we obtain the values of  $K_{k+1}^y$  and  $R_k$  given in the statement.

Now we assume that we have already obtained maps  $\mathcal{K}_n$  and  $\mathcal{R}_n$ ,  $2 \leq n < 2(r - k + 1)$ , such that (2.3.2) holds true, and we look for

$$\mathcal{K}_{n+1}(t) = \mathcal{K}_n(t) + \begin{pmatrix} K_{n+1}^x t^{n+1} \\ K_{n+k}^y t^{n+k} \end{pmatrix}, \quad \mathcal{R}_{n+1}(t) = \mathcal{R}_n(t) + R_{n+k-1} t^{n+k-1},$$

such that  $\mathcal{G}_{n+1}(t) = (O(t^{n+k+1}), O(t^{n+2k}))$ .

Using Taylor's theorem, we write

$$\begin{aligned}
\mathcal{G}_{n+1}(t) &= F(\mathcal{K}_n(t) + (K_{n+1}^x t^{n+1}, K_{n+k}^y t^{n+k})) \\
&\quad - (\mathcal{K}_n(t) + (K_{n+1}^x t^{n+1}, K_{n+k}^y t^{n+k})) \circ (\mathcal{R}_n(t) + R_{n+k-1} t^{n+k-1}) \\
&= \mathcal{G}_n(t) + DF(K_n(t)) \cdot (K_{n+1}^x t^{n+1}, K_{n+k}^y t^{n+k}) \\
&\quad - (K_{n+1}^x t^{n+1}, K_{n+k}^y t^{n+k}) \circ (\mathcal{R}_n(t) + R_{n+k-1} t^{n+k-1}) \\
&\quad + \int_0^1 (1-s) D^2 F(\mathcal{K}_n(t) + s(K_{n+1}^x t^{n+1}, K_{n+k}^y t^{n+k})) ds (K_{n+1}^x t^{n+1}, K_{n+k}^y t^{n+k})^{\otimes 2} \\
&\quad - D\mathcal{K}_n(\mathcal{R}_n(t)) R_{n+k-1} t^{n+k-1} \\
&\quad - \int_0^1 (1-s) D^2 \mathcal{K}_n(\mathcal{R}_n(t) + s R_{n+k-1} t^{n+k-1}) ds (R_{n+k-1} t^{n+k-1})^2.
\end{aligned}$$

Performing the computations in the previous expression we have

$$\begin{aligned}
\mathcal{G}_{n+1}(t) &= \mathcal{G}_n(t) \\
&\quad + \left( \begin{array}{c} [c K_{n+k}^y - (n+1) R_k K_{n+1}^x - 2 R_{n+k-1}] t^{n+k} + O(t^{n+k+1}) \\ [k a_k K_{n+1}^x - (n+k) R_k K_{n+k}^y - (k+1) K_{k+1}^y R_{n+k-1}] t^{n+2k-1} + O(t^{n+2k}) \end{array} \right). \quad (2.3.3)
\end{aligned}$$

Since, by the induction hypothesis,  $\mathcal{G}_n(t) = (O(t^{n+k}), O(t^{n+2k-1}))$ , to complete the induction step we need to make  $[\mathcal{G}_{n+1}^x]_{n+k}$  and  $[\mathcal{G}_{n+1}^y]_{n+2k-1}$  vanish.

From (2.3.3) we have

$$\begin{aligned}
[\mathcal{G}_{n+1}^x]_{n+k} &= [\mathcal{G}_n^x]_{n+k} + c K_{n+k}^y - (n+1) R_k K_{n+1}^x - 2 R_{n+k-1}, \\
[\mathcal{G}_{n+1}^y]_{n+2k-1} &= [\mathcal{G}_n^y]_{n+2k-1} + k a_k K_{n+1}^x - (n+k) R_k K_{n+k}^y - (k+1) K_{k+1}^y R_{n+k-1}.
\end{aligned}$$

Thus, the conditions  $[\mathcal{G}_{n+1}^x]_{n+k} = [\mathcal{G}_{n+1}^y]_{n+2k-1} = 0$  are equivalent to

$$\begin{pmatrix} -(n+1)R_k & c \\ k a_k & -(n+k) R_k \end{pmatrix} \begin{pmatrix} K_{n+1}^x \\ K_{n+k}^y \end{pmatrix} = \begin{pmatrix} -[\mathcal{G}_n^x]_{n+k} + 2 R_{n+k-1} \\ -[\mathcal{G}_n^y]_{n+2k-1} + (k+1) K_{k+1}^y R_{n+k-1} \end{pmatrix}. \quad (2.3.4)$$

If  $n \neq k$  the matrix in the left hand side of (2.3.4) is invertible, so we can take  $R_{n+k-1} = 0$  and then obtain  $K_{n+1}^x$  and  $K_{n+k}^y$  in a unique way. When  $n = k$ , the determinant of the matrix is zero. Then, choosing

$$R_{2k-1} = \frac{2k R_k [\mathcal{G}_n^x]_{2k} + c [\mathcal{G}_n^y]_{3k-2}}{2(3k+1) R_k},$$

system (2.3.4) has solutions. In this case, however,  $K_{k+1}^x$  and  $K_{2k}^y$  are not uniquely determined.  $\square$

**Proposition 2.3.4** (Case 2). *Let  $F$  be a  $C^r$  map of the form (2.2.2), with  $r \geq k \geq 2$ . We assume  $k = 2l - 1$ ,  $a_k \neq 0$ ,  $b_l \neq 0$  and  $a_k > -\frac{b_l^2}{4cl}$ . If  $a_k < 0$  we assume also  $a_k \neq \frac{1-2l}{(3l-1)^2} \frac{b_l^2}{c}$ . Then, for all  $1 \leq n \leq r - 2l + 2 = r - k + 1$ , there exist two pairs of polynomial maps,  $\mathcal{K}_n$  and  $\mathcal{R}_n$ , of the form*

$$\mathcal{K}_n(t) = \begin{pmatrix} t + \dots + K_n^x t^n \\ K_l^y t^l + \dots + K_{n+l-1}^y t^{n+l-1} \end{pmatrix} \quad (2.3.5)$$

and

$$\mathcal{R}_n(t) = \begin{cases} t + R_l t^l & \text{if } 1 \leq n \leq l-1, \\ t + R_l t^l + R_{2l-1} t^{2l-1} & \text{if } n \geq l, \end{cases} \quad (2.3.6)$$

such that

$$\mathcal{G}_n(t) := F(\mathcal{K}_n(t)) - \mathcal{K}_n(\mathcal{R}_n(t)) = (O(t^{n+l}), O(t^{n+2l-1})).$$

For the first pair we have

$$K_l^y = \frac{b_l - \sqrt{b_l^2 + 4ca_k l}}{2cl}, \quad R_l = \frac{b_l - \sqrt{b_l^2 + 4ca_k l}}{2l} = cK_l^y,$$

and for the second one

$$K_l^y = \frac{b_l + \sqrt{b_l^2 + 4ca_k l}}{2cl}, \quad R_l = \frac{b_l + \sqrt{b_l^2 + 4ca_k l}}{2l} = cK_l^y.$$

If  $a_k = \frac{1-2l}{(3l-1)^2} \frac{b_l^2}{c}$  one can compute the coefficients of  $\mathcal{K}_n$  and  $\mathcal{R}_n$  up to  $n = l-1$ .

If  $F$  is  $C^\infty$  or analytic, one can compute the polynomial approximations  $\mathcal{K}_n$  up to any order, except when  $a_k = \frac{1-2l}{(3l-1)^2} \frac{b_l^2}{c}$ .

*Proof.* The proof is analogous to the one of Proposition 2.3.1. We will see that we can determine iteratively the coefficients of  $\mathcal{K}_n$  and  $\mathcal{R}_n$ , for what we proceed by induction.

For  $n = 1$ , we see that there exist polynomial maps  $\mathcal{K}_1(t) = (t, K_l^y t^l)$  and  $\mathcal{R}_1(t) = t + R_l t^l$ , with  $R_l < 0$ , such that  $\mathcal{G}_1(t) = F(\mathcal{K}_1(t)) - \mathcal{K}_1(\mathcal{R}_1(t)) = O(t^{l+1}, t^{2l})$ .

From the series expansion of  $\mathcal{G}_1$  we have

$$\mathcal{G}_1(t) = \begin{pmatrix} t + cK_l^y t^l - t - R_l t^l + O(t^{l+1}) \\ K_l^y t^l + (a_k + K_l^y b_l) t^{2l-1} - K_l^y t^l - K_l^y l R_l t^{2l-1} + O(t^{2l}) \end{pmatrix},$$

so, if the conditions

$$K_l^y - R_l = 0, \quad a_k + K_l^y (b_l - l R_l) = 0,$$

are satisfied, then we clearly have  $\mathcal{G}_1(t) = (O(t^{l+1}), O(t^{2l}))$ , and we obtain the values  $K_l^y$  and  $R_l$  given in the statement.

Next we assume that we have already obtained maps  $\mathcal{K}_n$  and  $\mathcal{R}_n$  such that (4.3.3) holds true, and we look for

$$\mathcal{K}_{n+1}(t) = \mathcal{K}_n(t) + \begin{pmatrix} K_{n+1}^x t^{n+1} \\ K_{n+1}^y t^{n+l} \end{pmatrix}, \quad \mathcal{R}_{n+1}(t) = \mathcal{R}_n(t) + R_{n+l-1} t^{n+l-1}.$$

By the induction hypothesis, we have  $\mathcal{G}_n(t) = (O(t^{n+l}), O(t^{n+2l-1}))$ , and we want to obtain  $\mathcal{G}_{n+1}(t) = (O(t^{n+l+1}), O(t^{n+2l}))$ .

Using Taylor's theorem similarly as in the proof of Proposition 2.3.4 we see that  $\mathcal{G}_{n+1}(t) - \mathcal{G}_n(t) = (O(t^{n+l}), O(t^{n+2l-1}))$ , and hence, assuming  $\mathcal{G}_n(t) = (O(t^{n+l}), O(t^{n+2l-1}))$ , to complete the induction process we need to make  $[G_{n+1}^x]_{n+l}$  and  $[G_{n+1}^y]_{n+2l-1}$  vanish.

Concretely, performing the calculations we find

$$\begin{aligned} [G_{n+1}^x]_{n+l} &= [G_n^x]_{n+l} - (n+1)R_l K_{n+1}^x + cK_{n+l}^y - R_{n+l}, \\ [G_{n+1}^y]_{n+2l-1} &= [G_n^y]_{n+2l-1} + (k a_k + (l-1)b_l K_l^y) K_{n+1}^x + (b_l - (n+l)R_l)K_{n+l}^y - l K_l^y R_{n+l}, \end{aligned}$$

and therefore the condition  $[G_{n+1}^x]_{n+l} = [G_{n+1}^y]_{n+2l-1} = 0$  is equivalent to

$$\begin{pmatrix} -(n+1)R_l & c \\ k a_k + (l-1)b_l K_l^y & b_l - (n+l)R_l \end{pmatrix} \begin{pmatrix} K_{n+1}^x \\ K_{n+l}^y \end{pmatrix} = \begin{pmatrix} -[G_n^x]_{n+l} + R_{n+l} \\ -[G_n^y]_{n+2l-1} + l K_l^y R_{n+l} \end{pmatrix}. \quad (2.3.7)$$

If  $n \neq l-1$  the matrix in the left hand side of (2.3.7) is invertible, so we can take  $R_{n+l} = 0$  and then we obtain  $K_{n+1}^x$  and  $K_{n+l}^y$  in a unique way. When  $n = l-1$ , the determinant of the matrix is zero. Then, choosing

$$R_{2l-1} = \frac{c[G_n^y]_{3l-2} + (cK_l^y(2l-1) - b_l)[G_n^x]_{2l-1}}{cK_l^y(3l-1) - b_l}$$

system (2.3.7) has solutions. In this case, however,  $K_l^x$  and  $K_{2l-1}^y$  are not uniquely determined.

Note that the deominator of the expression above vanishes for  $a_k = \frac{1-2l}{(3l-1)^2} \frac{b_l^2}{c} < 0$ .  $\square$

**Proposition 2.3.5** (Case 3). *Let  $F$  be a  $C^r$  map of the form (2.2.2), with  $r \geq l \geq 2$ . Assume  $k > 2l-1$ ,  $b_l \neq 0$ . Then, for all  $1 \leq n \leq r-2l+2$ , there exist a pair of polynomial maps,  $\mathcal{K}_n$  and  $\mathcal{R}_n$ , of the form (2.3.5) and (2.3.6) respectively, such that*

$$\mathcal{G}_n(t) := F(\mathcal{K}_n(t)) - \mathcal{K}_n(\mathcal{R}_n(t)) = (O(t^{n+l}), O(t^{n+2l-1})).$$

We have

$$K_l^y = \frac{b_l}{cl}, \quad R_l = \frac{b_l}{l} = cK_l^y.$$

If we further assume that  $k \leq r$  and  $a_k \neq 0$ , then for  $1 \leq n \leq r - (k-l)l - 2l + 1$  there exists another pair  $\mathcal{K}_n$  and  $\mathcal{R}_n$  with

$$\mathcal{K}_n(t) = \begin{pmatrix} t + \dots + K_n^x t^n \\ K_{k-l+1}^y t^{k-l+1} + \dots + K_{n+k-l}^y t^{n+k-l} \end{pmatrix}$$

and

$$\mathcal{R}_n(t) = \begin{cases} t + R_{k-l+1} t^{k-l+1} & \text{if } 2 \leq n \leq k-l, \\ t + R_{k-l+1} t^{k-l+1} + R_{2(k-l)+1} t^{2(k-l)+1} & \text{if } n \geq k-l+1, \end{cases}$$

such that

$$\mathcal{G}_n(t) := F(\mathcal{K}_n(t)) - \mathcal{K}_n(\mathcal{R}_n(t)) = (O(t^{n+k-l+1}), O(t^{n+k})).$$

We have

$$K_{k-l+1}^y = -\frac{a_k}{b_l}, \quad R_{k-l+1} = cK_{k-l+1}^y.$$

If  $F$  is  $C^\infty$  or analytic, one can compute the polynomial approximations  $\mathcal{K}_n$  up to any order.

The proof of Proposition 2.3.5 is analogous to the one of Propositions 2.3.1 and 2.3.4, and so it will be omitted.

## 2.4 Existence of analytic invariant curves

This section is devoted to prove Theorems 2.2.1 and 2.2.3. Following the parameterization method, given a map  $F$  of the form (2.2.2), first we consider polynomial approximations  $\mathcal{K}_n : \mathbb{R} \rightarrow \mathbb{R}^2$  and  $\mathcal{R}_n : \mathbb{R} \rightarrow \mathbb{R}$  of solutions of equation (2.2.3) obtained in Section 2.3 up to a high enough order, to be determined in the proof. Then, keeping  $R = \mathcal{R}_n$  fixed, we look for a correction  $\Delta : [0, \rho) \rightarrow \mathbb{R}^2$ , for some  $\rho > 0$ , of  $\mathcal{K}_n$ , analytic on  $(0, \rho)$ , such that the pair  $K = \mathcal{K}_n + \Delta$ ,  $R = \mathcal{R}_n$  satisfies the invariance condition

$$F \circ (\mathcal{K}_n + \Delta) - (\mathcal{K}_n + \Delta) \circ R = 0. \quad (2.4.1)$$

The proof of Theorem 2.2.1 is organized as follows. First, taking into account the structure of  $F$  we rewrite equation (2.4.1) to separate the dominant linear part with respect to  $\Delta$  and the remaining terms. This motivates the introduction of two families of operators,  $\mathcal{S}_{n,R}$  and  $\mathcal{N}_{n,F}$ , and the spaces where these operators will act on. We provide the properties of these operators in Lemmas 2.4.6 and 2.4.7.

Finally, we rewrite the equation for  $\Delta$  as the fixed point equation

$$\Delta = \mathcal{T}_{n,F}(\Delta), \quad \text{where} \quad \mathcal{T}_{n,F} = \mathcal{S}_{n,R}^{-1} \circ \mathcal{N}_{n,F}$$

and we apply the Banach fixed point theorem to get the solution. The properties of the operators  $\mathcal{T}_{n,F}$  are deduced in Lemma 2.4.10. At the end of the section we also prove Theorem 2.2.3 using the preliminary results presented along the section.

### 2.4.1 The functional equation

Let  $F : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be an analytic map in a neighborhood  $U$  of  $(0,0)$ , satisfying the hypotheses of Theorem 2.2.1,

$$F(x, y) = \begin{pmatrix} x + cy \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ p(x) + yq(x) + u(x, y) + g(x, y) \end{pmatrix},$$

where  $c > 0$ ,  $p$ ,  $q$  and  $u$  are the polynomials introduced in Section 2.2.1 and  $g(x, y)$  is an analytic function. We take  $p$ ,  $q$  and  $u$  of degree at least  $k$  in case 1 and degree at least  $2l - 1$  in cases 2 and 3. Then we have  $g(x, y) = O(\|(x, y)\|^{k+1})$  for case 1 and  $g(x, y) = O(\|(x, y)\|^{2l})$  for cases 2 and 3. We denote  $v(x, y) = u(x, y) + g(x, y)$ .

From Propositions 2.3.1, 2.3.4 and 2.3.5 we take  $n$ , with  $n \geq k + 1$  in case 1 and  $n \geq l$  in cases 2 and 3, and we have that there exist polynomials  $\mathcal{K}_n$  and  $R = \mathcal{R}_n$  such that

$$\mathcal{E}_n(t) = (O(t^{n+N}), O(t^{n+2N-1})), \quad (2.4.2)$$

where  $\mathcal{E}_n = F \circ \mathcal{K}_n - \mathcal{K}_n \circ R$ . Since we are looking for the stable manifold we will take the approximations corresponding to  $R = \mathcal{R}_n$  with the coefficient  $R_N < 0$ .

Hence, we look for  $\rho > 0$  and a map  $K = \mathcal{K}_n + \Delta : [0, \rho) \rightarrow \mathbb{R}^2$ , analytic on  $(0, \rho)$  satisfying (2.4.1), where  $\mathcal{K}_n$  and  $R$  are the mentioned maps that satisfy (2.4.2). Moreover, we will ask  $\Delta$  to satisfy  $\Delta(t) = (\Delta^x(t), \Delta^y(t)) = (O(t^n), O(t^{n+N-1}))$ .

Using (2.4.2) we can rewrite (2.4.1) as

$$\begin{aligned}\Delta^x \circ R - \Delta^x &= c \Delta^y + \mathcal{E}_n^x, \\ \Delta^y \circ R - \Delta^y &= p \circ (\mathcal{K}_n^x + \Delta^x) - p \circ \mathcal{K}_n^x + \mathcal{K}_n^y \cdot (q \circ (\mathcal{K}_n^x + \Delta^x) - q \circ \mathcal{K}_n^x) \\ &\quad + \Delta^y \cdot q \circ (\mathcal{K}_n^x + \Delta^x) + v \circ (\mathcal{K}_n + \Delta) - v \circ \mathcal{K}_n + \mathcal{E}_n^y.\end{aligned}\tag{2.4.3}$$

## 2.4.2 Function spaces, operators and their properties

Next we introduce notation, suitable function spaces, and some operators.

**Definition 2.4.1.** *Given  $\beta, \rho > 0$  such that  $\rho < 1$  and  $\beta < \pi$ , let  $S$  be the sector*

$$S = S(\beta, \rho) = \{z \in \mathbb{C} \mid |\arg(z)| < \frac{\beta}{2}, 0 < |z| < \rho\}.$$

*Given a sector  $S = S(\beta, \rho)$  let  $\mathcal{X}_n$ , for  $n \in \mathbb{N}$ , be the Banach space given by*

$$\mathcal{X}_n = \{f : S \rightarrow \mathbb{C} \mid f \in \text{Hol}(S), f((0, \rho)) \subset \mathbb{R}, \|f\|_n := \sup_{z \in S} \frac{|f(z)|}{|z|^n} < \infty\},$$

*where  $\text{Hol}(S)$  denotes the space of holomorphic functions on  $S$ .*

Note that when  $n \geq 1$  the functions  $f$  in  $\mathcal{X}_n$  can be continuously extended to  $z = 0$  with  $f(0) = 0$  and, if moreover,  $n \geq 2$ , the derivative of  $f$  can be continuously extended to  $z = 0$  with  $f'(0) = 0$ .

Note also that  $\mathcal{X}_{n+1} \subset \mathcal{X}_n$ , for all  $n \in \mathbb{N}$ , and that if  $f \in \mathcal{X}_{n+1}$ , then  $\|f\|_n \leq \|f\|_{n+1}$ . Moreover if  $f \in \mathcal{X}_m$ ,  $g \in \mathcal{X}_n$ , then  $fg \in \mathcal{X}_{m+n}$  and  $\|fg\|_{m+n} \leq \|f\|_m \|g\|_n$ .

Given  $n, m \in \mathbb{N}$  we denote  $\mathcal{X}_{m,n} := \mathcal{X}_m \times \mathcal{X}_n$  the product spaces, endowed with the product norm

$$\|f\|_{m,n} = \max \{\|f^x\|_m, \|f^y\|_n\}, \quad f = (f^x, f^y) \in \mathcal{X}_{m,n}.$$

Given  $n \geq 1$ ,  $N \geq 2$ , we define the space

$$\Sigma_{n,N} = \mathcal{X}_{n,n+N-1},$$

endowed with the product norm. Also, given  $\alpha > 0$ , we define the closed ball

$$\Sigma_{n,N}^\alpha = \{f \in \Sigma_{n,N} \mid \|f\|_{\Sigma_{n,N}} \leq \alpha\}.$$

For the sake of simplicity, we will omit the parameters  $\rho$  and  $\beta$  in the notation of the spaces  $\Sigma_{n,N}$  and the balls  $\Sigma_{n,N}^\alpha$ .

Now let  $F$  be as in Theorem 2.2.1, and  $\mathcal{K}_n$  and  $R = \mathcal{R}_n$  be the polynomials provided in Section 2.3 satisfying (2.4.2) with  $n \geq k+1$  in case 1 and  $n \geq l$  in cases 2, 3.

Since  $F$  is analytic in  $U$ , it has a holomorphic extension to some neighborhood  $W$  of  $(0,0)$  in  $\mathbb{C}^2$ . Let  $d > 0$  be the radius of a ball in  $\mathbb{C}^2$  contained in the domain where  $F$  is holomorphic. Also,  $\mathcal{K}_n$  and  $R$  are defined on any complex sector  $S(\beta, \rho)$ . Then it is possible to set equation (2.4.3) in a space of holomorphic functions defined in a sector  $S(\beta, \rho)$ , and look for  $\Delta$  being an analytic function of a complex variable that takes real values when restricted to the real line.

To solve equation (2.4.3), we will consider  $n$  big enough and we will look for a solution,  $\Delta$ , in a closed ball of the space  $\Sigma_{n,N}$ . In order for the compositions in (2.4.3) to make sense we need to ensure the range of  $\mathcal{K}_n + \Delta$  to be contained in the domain where  $F$  is analytic. We take

$$\alpha = \min \left\{ \frac{1}{2}, \frac{d}{2} \right\}.$$

In this way, since  $\mathcal{K}_n(0) = (0,0)$ , taking  $\rho_K \in (0, 1)$  such that  $\sup_{z \in S(\beta, \rho_K)} \|\mathcal{K}_n(z)\| < d/2$  and  $\rho \leq \rho_K$ , if  $\Delta : S(\beta, \rho) \rightarrow \mathbb{C}^2$  belongs to the ball of radius  $\alpha$  of  $\mathcal{X}_{n,m}$ , with  $n, m \geq 0$ , we have

$$\sup_{z \in S(\beta, \rho)} \|\Delta(z)\| = \sup_{z \in S(\beta, \rho)} \max\{|\Delta^x(z)|, |\Delta^y(z)|\} \leq \max\left\{\frac{d}{2}\rho^n, \frac{d}{2}\rho^m\right\} < \frac{d}{2}.$$

Therefore, under the previous conditions, if  $\rho \leq \rho_K$  and  $\Delta \in \Sigma_{n,N}^\alpha$  then  $\|\mathcal{K}_n(z) + \Delta(z)\| < d$  and the composition  $F \circ (\mathcal{K}_n + \Delta)$  is well defined.

Next we introduce two families of operators that will be used to deal with (2.4.3). The definition of such operators is motivated by the equation itself.

First, we state the following auxiliary result (see [4]),

**Lemma 2.4.2.** *Let  $R : S(\beta, \rho) \rightarrow \mathbb{C}$  be a holomorphic function of the form  $R(z) = z + R_N z^N + O(|z|^{N+1})$ , with  $R_N < 0$ . Assume that  $0 < \beta < \frac{\pi}{N-1}$ . Then, for any  $\nu \in (0, (N-1)|R_N| \cos \lambda)$ , with  $\lambda = \beta \frac{N-1}{2}$ , there exists  $\rho > 0$  small enough such that*

$$|R^j(z)| \leq \frac{|z|}{(1 + j\nu |z|^{N-1})^{1/N-1}}, \quad \forall j \in \mathbb{N}, \quad \forall z \in S(\beta, \rho),$$

where  $R^j$  refers to the  $j$ -th iterate of the map  $R$ . In addition,  $R$  maps  $S(\beta, \rho)$  into itself.

Then, if  $f$  is defined in  $S(\beta, \rho)$ , with suitable values of the parameters  $\beta, \rho$ , and  $R$  satisfies the conditions of the lemma, the composition  $f \circ R$  is well defined.

**Definition 2.4.3.** *Given  $n \geq 1, N \geq 2$  and a polynomial  $R(z) = z + R_N z^N + O(|z|^{N+1})$  satisfying the hypotheses of Lemma 2.4.2, let  $\mathcal{S}_{n,R} : \Sigma_{n,N} \rightarrow \Sigma_{n,N}$  be the linear operator defined component-wise as  $\mathcal{S}_{n,R} = (\mathcal{S}_{n,R}^x, \mathcal{S}_{n,R}^y)$ , with*

$$\mathcal{S}_{n,R}^x f = \mathcal{S}_{n,R}^y f = f \circ R - f.$$

**Remark 2.4.4.** Notice that although both components of  $\mathcal{S}_{n,R}$  are formally identical they act on spaces of holomorphic functions of different orders.

**Definition 2.4.5.** *Let  $F$  be the holomorphic extension of an analytic map of the form (2.2.2) satisfying the hypotheses of Theorem 2.2.1. For  $n \in \mathbb{N}$ , we introduce  $\mathcal{N}_{n,F} = (\mathcal{N}_{n,F}^x, \mathcal{N}_{n,F}^y) : \Sigma_{n,N}^\alpha \rightarrow \mathcal{X}_{n+N-1, n+2N-2}$ , by*

$$\begin{aligned} \mathcal{N}_{n,F}^x(f) &= c f^y + \mathcal{E}_n^x, \\ \mathcal{N}_{n,F}^y(f) &= p \circ (\mathcal{K}_n^x + f^x) - p \circ \mathcal{K}_n^x + \mathcal{K}_n^y \cdot (q \circ (\mathcal{K}_n^x + f^x) - q \circ \mathcal{K}_n^x) \\ &\quad + f^y \cdot q \circ (\mathcal{K}_n^x + f^x) + v \circ (\mathcal{K}_n + f) - v \circ \mathcal{K}_n + \mathcal{E}_n^y. \end{aligned}$$



By the properties of  $R$  and the choice of  $\alpha$ , the operators  $\mathcal{S}_{n,R}$  and  $\mathcal{N}_{n,F}$  are well defined and  $\mathcal{S}_{n,R}$  is linear and bounded.

Using these operators, equations (2.4.3) can be written as

$$\mathcal{S}_{n,R} \Delta = \mathcal{N}_{n,F}(\Delta).$$

The following lemma states that the operators  $\mathcal{S}_{n,R}$  have a bounded right inverse and provide a bound for the norm  $\|\mathcal{S}_{n,R}^{-1}\|$ .

**Lemma 2.4.6.** *Given  $N \geq 2$  and  $n \geq 1$ , the operator  $\mathcal{S}_{n,R} : \Sigma_{n,N} \rightarrow \Sigma_{n,N}$ , has a bounded right inverse*

$$\mathcal{S}_{n,R}^{-1} : \mathcal{X}_{n+N-1, n+2N-2} \rightarrow \Sigma_{n,N} = \mathcal{X}_{n, n+N-1},$$

given by

$$\mathcal{S}_{n,R}^{-1} \eta = - \sum_{j=0}^{\infty} \eta \circ R^j, \quad \eta \in \mathcal{X}_{n+N-1, n+2N-2}. \quad (2.4.4)$$

Moreover, for any fixed  $\nu \in (0, (N-1)|R_N| \cos \lambda)$ , with  $\lambda = \beta \frac{N-1}{2}$ , there exists  $\rho > 0$  such that, taking  $S(\beta, \rho)$  with  $\beta < \frac{\pi}{N-1}$  as the domain of the functions of  $\mathcal{X}_{n+N-1, n+2N-2}$ , we have the operator norm bounds

$$\|(\mathcal{S}_{n,R}^x)^{-1}\| \leq \rho^{N-1} + \frac{1}{\nu} \frac{N-1}{n}, \quad \|(\mathcal{S}_{n,R}^y)^{-1}\| \leq \rho^{N-1} + \frac{1}{\nu} \frac{N-1}{n+N-1}.$$

*Proof.* Note that  $\mathcal{S}_{n,R}$  is defined component-wise, where each component acts in the same way, with the only difference that the spaces of definition of each component contain functions of different orders. We show the proof for  $\mathcal{S}_{n,R}^x : \mathcal{X}_{n+N-1} \rightarrow \mathcal{X}_n$ , but it is attained in the same way for the component  $\mathcal{S}_{n,R}^y$ .

A simple computation shows that (2.4.4) gives a formal right inverse of  $\mathcal{S}_{n,R}^x$ , namely

$$\mathcal{S}_{n,R}^x \circ (\mathcal{S}_{n,R}^x)^{-1} \eta = -\mathcal{S}_{n,R}^x \sum_{j=0}^{\infty} \eta \circ R^j = -\left( \sum_{j=0}^{\infty} \eta \circ R^j \right) \circ R + \sum_{j=0}^{\infty} \eta \circ R^j = \sum_{j=0}^{\infty} \eta \circ R^j - \sum_{j=1}^{\infty} \eta \circ R^j = \eta.$$

Let us see that the series given by (2.4.4) converges uniformly on  $S$ . Since  $R$  maps  $S$  into itself, one has

$$|\eta(R^j(z))| \leq \|\eta\|_{n+N-1} |R^j(z)|^{n+N-1}, \quad \forall z \in S,$$

and applying Lemma 2.4.2,

$$\begin{aligned} |\eta(R^j(z))| &\leq \|\eta\|_{n+N-1} |R^j(z)|^{n+N-1} \leq \|\eta\|_{n+N-1} \left( \frac{|z|}{(1+j\nu|z|^{N-1})^{1/N-1}} \right)^{n+N-1} \\ &\leq C \|\eta\|_{n+N-1} \frac{1}{j^{\frac{n}{N-1}+1}}, \quad \forall z \in S, \end{aligned}$$

so (2.4.4) converges uniformly on  $S$  by the Weierstrass  $M$ -test if  $n \geq 1$ , and thus  $\sum_{j=0}^{\infty} \eta \circ R^j$  is holomorphic in  $S$ .

We prove now that  $\mathcal{S}_{n,R}^{-1}$  is bounded on  $\mathcal{X}_{n+N-1}$ . From the expression obtained in (2.4.4) and by Lemma 2.4.2, one has, if  $\beta < \frac{\pi}{N-1}$  and if  $\rho$  is small enough,

$$\begin{aligned} \|(\mathcal{S}_{n,R}^x)^{-1} \eta\|_n &= \sup_{z \in S} \frac{|(\mathcal{S}_{n,R}^x)^{-1} \eta(z)|}{|z|^n} \\ &\leq \sup_{z \in S} \frac{1}{|z|^n} \sum_{j=0}^{\infty} |\eta(R^j(z))| \\ &\leq \|\eta\|_{n+N-1} \sup_{z \in S} \frac{1}{|z|^n} \sum_{j=0}^{\infty} |R^j(z)|^{n+N-1} \\ &\leq \|\eta\|_{n+N-1} \sup_{z \in S} \frac{1}{|z|^n} \sum_{j=0}^{\infty} \left( \frac{|z|}{(1+j\nu|z|^{N-1})^{1/N-1}} \right)^{n+N-1}, \end{aligned}$$

and bounding the sum by an appropriate integral we obtain, provided that  $n \geq 1$ ,

$$\begin{aligned} \frac{1}{|z|^n} \sum_{j=0}^{\infty} \left( \frac{|z|}{(1+j\nu|z|^{N-1})^{1/N-1}} \right)^{n+N-1} &= |z|^{N-1} \sum_{j=0}^{\infty} \frac{1}{(1+j\nu|z|^{N-1})^{\frac{n+N-1}{N-1}}} \\ &\leq |z|^{N-1} \left( 1 + \int_0^{\infty} \frac{1}{(1+x\nu|z|^{N-1})^{\frac{n+N-1}{N-1}}} dx \right) \\ &= |z|^{N-1} \left( 1 + \frac{1}{\nu|z|^{N-1}} \int_0^{\infty} \frac{1}{(1+y)^{\frac{n+N-1}{N-1}}} dy \right) \\ &= |z|^{N-1} \left( 1 + \frac{1}{\nu|z|^{N-1}} \frac{N-1}{n} \right) \\ &= |z|^{N-1} + \frac{1}{\nu} \frac{N-1}{n}. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \|\mathcal{S}_{n,R}^{-1}(\eta)\|_n &\leq \|\eta\|_{n+N-1} \sup_{z \in S} \left( |z|^{N-1} + \frac{1}{\nu} \frac{N-1}{n} \right) \\ &= \|\eta\|_{n+N-1} \left( \rho^{N-1} + \frac{1}{\nu} \frac{N-1}{n} \right), \quad \eta \in \mathcal{X}_{n+N-1}, \end{aligned}$$

which shows that  $(\mathcal{S}_{n,R}^x)^{-1} : \mathcal{X}_{n+N-1} \rightarrow \mathcal{X}_n$  is bounded with  $\|(\mathcal{S}_{n,R}^x)^{-1}\| \leq \rho^{N-1} + \frac{1}{\nu} \frac{N-1}{n}$ .

In the same way,  $(\mathcal{S}_{n,R}^y)^{-1} : \mathcal{X}_{n+2N-1} \rightarrow \mathcal{X}_{n+N-1}$  is bounded with  $\|(\mathcal{S}_{n,R}^y)^{-1}\| \leq \rho^{N-1} + \frac{1}{\nu} \frac{N-1}{n+N-1}$ .  $\square$

Next, we show that the operators  $\mathcal{N}_{n,F}$  are Lipschitz and we provide bounds for their Lipschitz constants.

**Lemma 2.4.7.** *For each  $n \geq 3$ , there exists a constant,  $M_n > 0$ , for which the operator  $\mathcal{N}_{n,F}$  satisfies*

$$\text{Lip } \mathcal{N}_{n,F}^x = c,$$

and

$$\begin{aligned} \text{Lip } \mathcal{N}_{n,F}^y &\leq k|a_k| + M_n\rho, \quad (\text{case 1}), \\ \text{Lip } \mathcal{N}_{n,F}^y &\leq \max\{(l-1)|K_l^y b_l| + k|a_k| + M_n\rho, |b_l| + M_n\rho\}, \quad (\text{case 2}), \\ \text{Lip } \mathcal{N}_{n,F}^y &\leq \max\{(l-1)|K_l^y b_l| + M_n\rho, |b_l| + M_n\rho\}, \quad (\text{case 3}), \end{aligned}$$

where  $\rho$  is the radius of the sector  $S(\beta, \rho)$  where the functions of  $\Sigma_{n,N}^\alpha$  are defined.

*Proof.* The statement concerning the component  $\mathcal{N}_{n,F}^x$  is clear by the definition of the operator, since for every  $f, \tilde{f} \in \Sigma_{n,N}^\alpha$ , one has

$$\|\mathcal{N}_{n,F}^x(f) - \mathcal{N}_{n,F}^x(\tilde{f})\|_{n+N-1} = c \|f - \tilde{f}\|_{n+N-1},$$

which means that  $\text{Lip } \mathcal{N}_{n,F}^x = c$ .

We prove next the result for the component  $\mathcal{N}_{n,N}^y$ . Let  $n \geq 3$  be fixed.

Using the integral form of the mean value theorem, one can write from the definition of the operator  $\mathcal{N}_{n,F}^y$ , for every  $f, \tilde{f} \in \Sigma_{n,N}^\alpha$ ,

$$\begin{aligned} & \mathcal{N}_{n,F}^y(f) - \mathcal{N}_{n,F}^y(\tilde{f}) \\ &= \left[ \int_0^1 p' \circ (\mathcal{K}_n^x + \tilde{f}^x + s(f^x - \tilde{f}^x)) ds + (\mathcal{K}_n^y + \tilde{f}^y) \int_0^1 q' \circ (\mathcal{K}_n^x + \tilde{f}^x + s(f^x - \tilde{f}^x)) ds \right. \\ & \quad \left. + \int_0^1 D_1 v \circ (\mathcal{K}_n + \tilde{f} + s(f - \tilde{f})) ds \right] (f^x - \tilde{f}^x) \\ & \quad + \left[ q \circ (\mathcal{K}_n^x + f^x) + \int_0^1 D_2 v \circ (\mathcal{K}_n + \tilde{f} + s(f - \tilde{f})) ds \right] (f^y - \tilde{f}^y). \end{aligned} \tag{2.4.5}$$

Let us denote

$$\begin{aligned} \xi_s &= \xi_s(f, \tilde{f}) = \mathcal{K}_n + \tilde{f} + s(f - \tilde{f}), \quad s \in [0, 1], \\ \varphi &= \varphi(f, \tilde{f}) = \int_0^1 p' \circ \xi_s^x ds + (\mathcal{K}_n^y + \tilde{f}^y) \int_0^1 q' \circ \xi_s^x ds + \int_0^1 D_1 v \circ \xi_s ds, \\ \psi &= \psi(f, \tilde{f}) = q \circ (\mathcal{K}_n^x + f^x) + \int_0^1 D_2 v \circ \xi_s ds, \end{aligned}$$

so that we have

$$\|\mathcal{N}_{n,F}^y(f) - \mathcal{N}_{n,F}^y(\tilde{f})\|_{n+2N-2} \leq \|\varphi(f, \tilde{f}) (f^x - \tilde{f}^x)\|_{n+2N-2} + \|\psi(f, \tilde{f}) (f^y - \tilde{f}^y)\|_{n+2N-2}.$$

For case 1 we have  $\xi_s \in \mathcal{X}_{2,k+1}$ , for every  $f, \tilde{f} \in \Sigma_{n,k}^\alpha$ , since we have  $\mathcal{K}_n(t) = (O(t^2), O(t^{k+1}))$ , and as we chose  $n \geq 3$ , any function  $f \in \Sigma_{n,k}^\alpha$  satisfies  $f(t) = (O(t^3), O(t^{k+2}))$ . Also, the first pair of coefficients of the series expansion of  $\xi_s$  is the one of  $\mathcal{K}_n$ .

Thus we can bound the norm

$$\|\xi_s^x\|_2 = \sup_{z \in S} \frac{1}{|z|^2} |\mathcal{K}_n^x(z) + \tilde{f}^x(z) + s(f^x(z) - \tilde{f}^x(z))| \leq 1 + M \rho,$$

for all  $s \in [0, 1]$ .

Moreover, checking the growth orders of  $\varphi$  and  $\psi$  taking into account the properties of  $p, q$  and  $v$ , we have

$$\varphi \in \mathcal{X}_{2k-2}, \quad \psi \in \mathcal{X}_k \subset \mathcal{X}_{k-1}, \quad \forall f, \tilde{f} \in \Sigma_{n,k}^\alpha.$$

Then, applying the mean value theorem to (2.4.5), we have

$$\begin{aligned} \|\mathcal{N}_{n,F}^y(f) - \mathcal{N}_{n,F}^y(\tilde{f})\|_{n+2k-2} &\leq \|\varphi\|_{2k-2} \|f^x - \tilde{f}^x\|_n + \|\psi\|_{k-1} \|f^y - \tilde{f}^y\|_{n+k-1} \\ &\leq \|\varphi\|_{2k-2} \|f^x - \tilde{f}^x\|_n + \rho M_n \|f^y - \tilde{f}^y\|_{n+k-1}. \end{aligned}$$

Also, we can bound

$$\begin{aligned} \|\varphi\|_{2k-2} &\leq \sup_{s \in [0,1]} (\|p' \circ \xi_s^x\|_{2k-2} + \|(\mathcal{K}_n^y + \tilde{f}^y) q' \circ \xi_s^x + D_1 v \circ \xi_s\|_{2k-2}) \\ &\leq \sup_{s \in [0,1]} \sup_{z \in S} \frac{1}{|z|^{2k-2}} (k |a_k| |\xi_s^x(z)|^{k-1} + M_n |z|^{2k-1}) \\ &\leq \sup_{s \in [0,1]} (k |a_k| \|\xi_s^x\|_2^{k-1} + M_n |z|^{2k-1}) \\ &\leq k |a_k| + M_n \rho, \end{aligned}$$

for all  $f, \tilde{f} \in \Sigma_{n,k}^\alpha$ .

By joining the previous estimates we get

$$\begin{aligned} \|\mathcal{N}_{n,F}^y(f) - \mathcal{N}_{n,F}^y(\tilde{f})\|_{n+2k-2} \\ \leq (k |a_k| + M_n \rho) \max \{\|f^x - \tilde{f}^x\|_n, \|f^y - \tilde{f}^y\|_{n+k-1}\}, \end{aligned}$$

for all  $f, \tilde{f} \in \Sigma_{n,k}^\alpha$ , that is,

$$\text{Lip } \mathcal{N}_{n,F}^y \leq k |a_k| + M_n \rho,$$

for case 1.

For cases 2 and 3 of the reduced form of  $F$ , where  $N = l$ ,  $\mathcal{K}_n$  is of the form  $\mathcal{K}_n(t) = (t, K_l^y t^l) + (O(z^2), O(z^{l+1}))$  and one obtains, for each given  $n \geq 2$  (and thus, also for  $n \geq 3$ ), that  $\xi_s \in \mathcal{X}_{1,l}$ , and that the first pair of coefficients of  $\xi_s$  coincides with the first one of  $\mathcal{K}_n$ . Then, for every  $f, \tilde{f} \in \Sigma_{n,l}^\alpha$  we can bound

$$\|\xi_s^x\|_1 \leq 1 + M_n \rho.$$

Moreover, now checking the growth orders of  $\varphi$  and  $\psi$ , we have

$$\varphi \in \mathcal{X}_{2l-2}, \quad \psi \in \mathcal{X}_{l-1}, \quad \forall s \in [0, 1], \quad \forall f, \tilde{f} \in \Sigma_{n,l}^\alpha.$$

For case 2 we obtain the bounds

$$\begin{aligned} \|\varphi\|_{2l-2} &\leq \sup_{s \in [0,1]} (\|p' \circ \xi_s^x\|_{2l-2} + \|(\mathcal{K}_n^y + \tilde{f}^y) q' \circ \xi_s^x\|_{2l-2} + \|D_1 v \circ \xi_s\|_{2l-2}) \\ &\leq \sup_{s \in [0,1]} \sup_{z \in S} \frac{1}{|z|^{2l-2}} (k |a_k| |\xi_s^x(z)|^{k-1} + |\mathcal{K}_n^y(z) + \tilde{f}^y(z)| (l-1) |b_l| |\xi_s^x(z)|^{l-2} + M_n |z|^{2l-1}) \\ &\leq k |a_k| + (l-1) |K_l^y b_l| + M_n \rho, \end{aligned}$$

and

$$\begin{aligned} \|\psi\|_{l-1} &\leq \sup_{s \in [0,1]} (\|q \circ (\mathcal{K}_n^x + \tilde{f}^x)\|_{l-1} + \|D_2 v \circ \xi_s\|_{l-1}) \\ &\leq \sup_{z \in S} \frac{1}{|z|^{l-1}} (|b_l| |\mathcal{K}_n^x + \tilde{f}^x|^{l-1} + M_n |z|^l) \\ &\leq |b_l| + M_n \rho, \end{aligned}$$

for all  $f, \tilde{f} \in \Sigma_{n,l}^\alpha$ .

For case 3 the previous estimations can be done in the same way as for case 2. We shall only remove the terms depending on the index  $k$ , since now we have  $k > 2l - 1$ , and hence the corresponding terms are of higher order. In this case we have then

$$\begin{aligned} \|\varphi\|_{2l-2} &\leq \sup_{s \in [0,1]} (\|p' \circ \xi_s^x\|_{2l-2} + \|(\mathcal{K}_n^y + \tilde{f}^y) q' \circ \xi_s^x\|_{2l-2} + \|D_1 v \circ \xi_s\|_{2l-2}) \\ &\leq \sup_{s \in [0,1]} \sup_{z \in S} \frac{1}{|z|^{2l-2}} (|\mathcal{K}_n^y(z) + \tilde{f}^y(z)|(l-1)|b_l| |\xi_s^x(z)|^{l-2} + M_n |z|^{2l-1}) \\ &\leq (l-1)|K_l^y b_l| + M_n \rho, \end{aligned}$$

and

$$\begin{aligned} \|\psi\|_{l-1} &\leq \sup_{s \in [0,1]} (\|q \circ (\mathcal{K}_n^x + \tilde{f}^x)\|_{l-1} + \|D_2 v \circ \xi_s\|_{l-1}) \\ &\leq \sup_{s \in [0,1]} \sup_{z \in S} \frac{1}{|z|^{l-1}} (|b_l| |\xi_s^x|^{l-1} + M_n |z|^l) \\ &\leq |b_l| + M_n \rho, \end{aligned}$$

for all  $f, \tilde{f} \in \Sigma_{n,l}^\alpha$ .

Then, from (2.4.5) we have

$$\|\mathcal{N}_{n,F}^y(f) - \mathcal{N}_{n,F}^y(\tilde{f})\|_{n+2l-2} \leq \|\varphi\|_{2l-2} \|f^x - \tilde{f}^x\|_n + \|\psi\|_{l-1} \|f^y - \tilde{f}^y\|_{n+l-1},$$

from what we obtain the bounds

$$\begin{aligned} \text{Lip } \mathcal{N}_{0,F}^y &\leq \max\{(l-1)|K_l^y b_l| + k|a_k| + M_n \rho, |b_l| + M_n \rho\} \quad (\text{case 2}), \\ \text{Lip } \mathcal{N}_{0,F}^y &\leq \max\{(l-1)|K_l^y b_l| + M_n \rho, |b_l| + M_n \rho\} \quad (\text{case 3}). \end{aligned}$$

□

Now, we define the third family of operators,  $\mathcal{T}_{n,F}$ .

**Definition 2.4.8.** *Let  $F$  be the holomorphic extension of an analytic map of the form (2.2.2) satisfying the hypotheses of Theorem 2.2.1. Given  $n \geq 3$  we define  $\mathcal{T}_{n,F} : \Sigma_{n,N}^\alpha \rightarrow \Sigma_{n,N}$  by*

$$\mathcal{T}_{n,F} = \mathcal{S}_{n,R}^{-1} \circ \mathcal{N}_{n,F}.$$

**Remark 2.4.9.** Note that given a map  $F$ , to define the previous operators we always take together the associated triple  $(F, \mathcal{K}_n, R)$  satisfying  $F \circ \mathcal{K}_n - \mathcal{K}_n \circ R = \mathcal{E}_n$ . Then, the operators  $\mathcal{S}_{n,R}$ ,  $\mathcal{N}_{n,F}$  and  $\mathcal{T}_{n,F}$  are associated not only with the map  $F$  itself but to the approximation of a particular invariant manifold of  $F$ .

**Lemma 2.4.10.** *Given an analytic map  $F$  satisfying the hypotheses of Theorem 2.2.1, there exist  $n_0 > 0$  and  $\rho_0 > 0$  such that if  $\rho < \rho_0$ , then, for every  $n \geq n_0$ , we have  $\mathcal{T}_{n,F}(\Sigma_{n,N}^\alpha) \subseteq \Sigma_{n,N}^\alpha$ , and  $\mathcal{T}_{n,F}$  is a contraction operator in  $\Sigma_{n,N}^\alpha$ .*

*Proof.* Given a map  $F$  satisfying the hypotheses of Theorem 3.2.1 and the associated polynomial maps  $R$  and  $\mathcal{K}_n$ , the operator  $\mathcal{T}_{n,F}$  satisfies, by its definition, for each  $n$ ,

$$\text{Lip } \mathcal{T}_{n,F} \leq \max\{\|(\mathcal{S}_{n,R}^x)^{-1}\| \text{Lip } \mathcal{N}_{n,F}^x, \|(\mathcal{S}_{n,R}^y)^{-1}\| \text{Lip } \mathcal{N}_{n,F}^y\}. \quad (2.4.6)$$

Recall that the functions in  $\Sigma_{n,N}$  are defined in a sector  $S(\beta, \rho)$  with  $\beta < \frac{\pi}{N-1}$ .

For case 1, from (2.4.6) and the estimates obtained in Lemmas 2.4.6 and 2.4.7, given  $\nu \in (0, |R_k|(k-1)\lambda)$ , with  $\lambda = \cos(\beta \frac{k-1}{2})$ , there is  $\rho_0$  such that for  $\rho < \rho_0$  we have the bound

$$\text{Lip } \mathcal{T}_{n,F} \leq \max\left\{\left(\rho^{k-1} + \frac{1}{\nu} \frac{k-1}{n}\right) c, \left(\rho^{k-1} + \frac{1}{\nu} \frac{k-1}{n+k-1}\right) (k a_k + M_n \rho)\right\}, \quad (2.4.7)$$

for each  $n \geq 3$ .

Then, fixed  $\nu$ , we can take  $n_0$  as

$$n_0 = \min\{n \in \mathbb{N} \mid \max\left\{\left(\frac{1}{\nu} \frac{k-1}{n}\right) c, \left(\frac{1}{\nu} \frac{k-1}{n+k-1}\right) (k a_k)\right\} < 1, \},$$

and therefore, from (2.4.7), we can take also  $\rho_n < \rho_0$  such that we have  $\text{Lip } \mathcal{T}_{n,F} < 1$  provided that  $n \geq n_0$ .

For cases 2 and 3 of the normal form of  $F$  the result follows in a similar way, since, from (2.4.6) and the estimates obtained in Lemmas 2.4.6 and 2.4.7, given  $\nu \in (0, |R_l|(l-1)\lambda)$ , with  $\lambda = \cos(\beta \frac{l-1}{2})$ , we have, for  $\rho$  small enough,

$$\begin{aligned} \text{Lip } \mathcal{T}_{n,F} \leq & \max\left\{\left(\rho^{l-1} + \frac{1}{\nu} \frac{l-1}{n}\right) c, \right. \\ & \left. \left(\rho^{l-1} + \frac{1}{\nu} \frac{l-1}{n+l-1}\right) (\max\{((l-1)|K_l^y b_l| + k a_k) + M_n \rho, |b_l| + M_n \rho\})\right\}, \end{aligned}$$

for case 2, and

$$\begin{aligned} \text{Lip } \mathcal{T}_{n,F} \leq & \max\left\{\left(\rho^{l-1} + \frac{1}{\nu} \frac{l-1}{n}\right) c, \right. \\ & \left. \left(\rho^{l-1} + \frac{1}{\nu} \frac{l-1}{n+l-1}\right) (\max\{(l-1)|K_l^y b_l| + M_n \rho, |b_l| + M_n \rho\})\right\}, \end{aligned}$$

for case 3.

Let us see next that for a given  $n \in \mathbb{N}$  such that  $\mathcal{T}_{n,F}$  satisfies  $\text{Lip } \mathcal{T}_{L,F} < 1$  for  $\rho < \rho_n$ , one can find a new value for  $\rho_n$ , maybe smaller than the previous one, such that, if  $\rho < \rho_n$ , then  $\mathcal{T}_{n,F}$  maps  $\Sigma_{n,N}^\alpha$  into itself.

For all  $f \in \Sigma_{n,N}^\alpha$  we can write

$$\begin{aligned} \|\mathcal{T}_{n,F}(f)\|_{\Sigma_{n,N}} & \leq \|\mathcal{T}_{n,F}(f) - \mathcal{T}_{n,F}(0)\|_{\Sigma_{n,N}} + \|\mathcal{T}_{n,F}(0)\|_{\Sigma_{n,N}} \\ & \leq \alpha \text{Lip } \mathcal{T}_{n,F} + \|\mathcal{T}_{n,F}(0)\|_{\Sigma_{n,N}}. \end{aligned}$$

From the definition of  $\mathcal{T}_{n,F}$  and  $\mathcal{N}_{n,F}$  we have, for each  $n \in N$ ,

$$\mathcal{T}_{n,F}(0) = \mathcal{S}_{n,R}^{-1} \circ \mathcal{N}_{n,F}(0) = \mathcal{S}_{n,R}^{-1} \mathcal{E}_n.$$

Also, from the construction of  $\mathcal{E}_n$  we have  $\mathcal{E}_n = (\mathcal{E}_n^x, \mathcal{E}_n^y) \in \mathcal{X}_{n+N, n+2N-1}$ , and thus, for every  $\varepsilon > 0$ , there is  $\rho_n > 0$  such that for  $\rho < \rho_n$  we have

$$\|\mathcal{T}_{n,F}(0)\|_{\Sigma_{n,N}} \leq \|\mathcal{S}_{n,R}^{-1}\| \|\mathcal{E}_n\|_{n+N-1, n+2N-2} \leq \|\mathcal{S}_{n,R}^{-1}\| M_n \rho < \varepsilon.$$

Therefore, since we have  $\text{Lip } \mathcal{T}_{n,F} < 1$ , we can take  $\rho_n$  as

$$\rho_n = \sup \{ \rho > 0 \mid \alpha \text{Lip } \mathcal{T}_{n,F} + \|\mathcal{T}_{n,F}(0)\|_{\Sigma_{n,N}} \leq \alpha \},$$

and then for every  $\rho < \rho_n$ , we have  $\mathcal{T}_{n,F}(\Sigma_{n,N}^\alpha) \subseteq \Sigma_{n,N}^\alpha$ .  $\square$

### 2.4.3 Proofs of Theorems 2.2.1 and 2.2.3

Now we are ready to give the proofs of Theorems 2.2.1 and 2.2.3.

*Proof of Theorem 2.2.1.* First we consider the holomorphic extension of  $F$  to a neighborhood of the origin which contains a ball of radius  $d > 0$  in  $\mathbb{C}^2$  and let  $\alpha = \min \{1/2, d/2\}$ . Let  $\mathcal{K}_n$  and  $R(t) = \mathcal{R}_n(t) = t + R_N t^N + R_{2N-1} t^{2N-1}$  be the polynomials given by Propositions 2.3.1, 2.3.4 or 2.3.5, with  $n \geq k+1$  or  $n \geq l$  respectively, satisfying

$$\mathcal{E}_n(t) = F \circ \mathcal{K}_n(t) - \mathcal{K}_n \circ R(t) = (O(t^{n+N}), O(t^{n+2N-1})).$$

We also assume that  $n > n_0$ , where  $n_0$  is the integer provided by Lemma 2.4.10. We rewrite

$$F \circ (\mathcal{K}_n + \Delta) - (\mathcal{K}_n + \Delta) \circ R = 0$$

in the form (2.4.3), or using the previously defined operators,

$$\mathcal{S}_{n,R} \Delta = \mathcal{N}_{n,F}(\Delta).$$

By Lemma 2.4.6, if  $\rho$  is small,  $\mathcal{S}_{n,R}$  has a right inverse and we can rewrite the equation as

$$\Delta = \mathcal{T}_{n,F}(\Delta).$$

By Lemma 2.4.10 we have that  $\mathcal{T}_{n,F}$  maps  $\Sigma_{n,N}^\alpha$  into itself and is a contraction. Then it has a unique fixed point,  $\Delta^\infty \in \Sigma_{n,N}^\alpha$ . Note that this solution is unique once  $\mathcal{K}_n$  is fixed. Finally  $K = \mathcal{K}_n + \Delta^\infty$  satisfies the conditions in the statement.

The  $C^1$  character of  $K$  at the origin follows from the order condition of  $K$  at 0.  $\square$

*Proof of Theorem 2.2.3.* We write the proof for case 1, the other cases being almost identical except for some adjustments in the indices of the coefficients of  $\mathcal{R}_n$ . Let  $n_0$  be the integer provided by Lemma 2.4.10. If the value of  $n$  given in the statement is such that  $n < n_0$ , first we look for a better approximation  $\mathcal{K}_{n_0}$  of the form  $\mathcal{K}_{n_0}(t) = \hat{K}(t) + \sum_{j=n+1}^{n_0} \hat{K}^j(t)$  with  $\hat{K}^j(t) = (\hat{K}_j^x t^j, \hat{K}_{j+k-1}^y t^{j+k-1})$  and

$$\mathcal{R}_{n_0}(t) = \begin{cases} \hat{R}(t) & \text{if } n \geq k+1, \\ \hat{R}(t) + \hat{R}_{2k-1} t^{2k-1} & \text{if } n \leq k. \end{cases}$$

The coefficients  $\hat{K}_j^x, \hat{K}_{j+k-1}^y$  and  $\hat{R}_{2k-1}$  are obtained imposing the condition

$$F \circ \mathcal{K}_{n_0}(t) - \mathcal{K}_{n_0} \circ R(t) = (O(t^{n_0+k}), O(t^{n_0+2k-1})).$$

Proceeding as in Proposition 2.3.1, we obtain  $\hat{K}^j$  iteratively. We denote  $\mathcal{K}_j(t) = \hat{K}(t) + \sum_{m=n+1}^j \hat{K}^m(t)$  and  $\mathcal{R}_j(t) = \hat{R}(t) + \tilde{R}_j(t)$ , where  $\tilde{R}_j(t) = \delta_{j,k+1} \hat{R}_{2k-1} t^{2k-1}$ . In the iterative step we have

$$F \circ \mathcal{K}_j(t) - \mathcal{K}_j \circ \mathcal{R}_j(t) = (O(t^{j+k}), O(t^{j+2k-1})).$$

Then,

$$\begin{aligned} & F(\mathcal{K}_j(t) + \hat{K}^{j+1}(t)) - (\mathcal{K}_j + \hat{K}^{j+1}) \circ (\hat{R}(t) + \tilde{R}_j(t)) \\ &= F(\mathcal{K}_j(t)) - \mathcal{K}_j(\hat{R}(t)) \\ & \quad + DF(\mathcal{K}_j(t))\hat{K}^{j+1}(t) - \hat{K}^{j+1}(\hat{R}(t) + \tilde{R}_j(t)) \\ & \quad + \int_0^1 (1-s) D^2 F(\mathcal{K}_j(t) + s\hat{K}^{j+1}(t)) (\hat{K}^{j+1}(t))^{\otimes 2} ds \\ & \quad - D\mathcal{K}_j(\hat{R}(t))\tilde{R}_j(t) \\ & \quad - \int_0^1 (1-s) D^2 \mathcal{K}_j(\hat{R}(t) + s\tilde{R}_j(t)) (\tilde{R}_j(t))^2 ds. \end{aligned}$$

The condition

$$F \circ \mathcal{K}_{j+1}(t) - \mathcal{K}_{j+1} \circ \mathcal{R}_{j+1}(t) = (O(t^{j+k+1}), O(t^{j+2k}))$$

leads to the same equation (2.3.4) as in Proposition 2.3.1 which we solve in the same way. From this point we can proceed as in the proof of Theorem 2.2.1 and look for  $\Delta \in \mathcal{X}_{n_0, n_0+k-1}$  such that the pair  $K = \mathcal{K}_{n_0} + \Delta$ ,  $R = \mathcal{R}_{n_0}$  satisfies  $F \circ K = K \circ R$ . We have that

$$K(t) - \hat{K}(t) = \mathcal{K}_{n_0}(t) - \hat{K}(t) + \Delta(t) = (O(t^{n+1}), O(t^{n+k})) + (O(t^{n_0}), O(t^{n_0+k-1})),$$

with  $n < n_0$ .

If  $n \geq n_0$  we look for  $\mathcal{K}^*(t) = \hat{K}(t) + \hat{K}^{n+1}(t)$  with

$$\hat{K}^{n+1}(t) = (\hat{K}_{n+1}^x t^{n+1}, \hat{K}_{n+k}^y t^{n+k})$$

and

$$\mathcal{R}^*(t) = \begin{cases} \hat{R}(t) & \text{if } n \geq k+1, \\ \hat{R}(t) + \hat{R}_{2k-1} t^{2k-1} & \text{if } n \leq k. \end{cases}$$

We determine  $\hat{K}_{n+1}^x$ ,  $\hat{K}_{n+k}^y$  so that  $F \circ \mathcal{K}^*(t) - \mathcal{K}^* \circ \mathcal{R}^*(t) = (O(t^{n+k+1}), O(t^{n+2k}))$  as in the previous case and we look for  $\Delta \in \Sigma_{n,N}^\alpha \subset \mathcal{X}_{n+1, n+k}$  such that the pair  $K = \mathcal{K}^* + \Delta$ ,  $R = \mathcal{R}^*$  satisfies  $F \circ K = K \circ R$ . As before we obtain  $K(t) - \hat{K}(t) = (O(t^{n+1}), O(t^{n+k}))$ . Again, the  $C^1$  character of  $K$  at 0 follows from the order condition of  $K$ .  $\square$

## 2.5 Numerical estimates for the Gevrey constant of the invariant curves

In this short section, we illustrate the fact that an analytic map  $F : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with  $F(0) = 0$  and such that the origin is a nilpotent parabolic fixed point, in general cannot have a stable invariant curve that is analytic in any neighborhood of the fixed point.



We have chosen two polynomial maps,  $F_1$  and  $F_2$ , such that

$$DF_1(0,0) = DF_2(0,0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

namely

$$F_1(x,y) = \begin{pmatrix} x + y + x^2 + xy + y^2 \\ y + x^2 + xy + y^2 \end{pmatrix}$$

and

$$F_2(x,y) = \begin{pmatrix} x + y + \sum_{n=0}^2 x^{2-n}y^n + \dots + \sum_{n=0}^8 x^{8-n}y^n \\ y + 3x^2 \end{pmatrix},$$

and we have written a code in C language that computes an approximation of a parameterization,  $\mathcal{K}_n$ , of the stable invariant curve of  $F_1$  and  $F_2$  and the *Gevrey constant* of such parameterizations.

**Definition 2.5.1.** *Given  $\gamma > 0$ , we say that a formal series of the form  $\sum_{n=0}^{\infty} a_n t^n$  is  $\gamma$ -Gevrey if there exist positive constants  $C, D$  such that*

$$|a_n| \leq C D^n (n!)^\gamma, \quad \forall n \in \mathbb{N}.$$

This class of series was first introduced and studied in [31], and in [4] the Gevrey properties of parabolic invariant curves of analytic maps are studied.

Our scope is to estimate numerically if the formal approximation  $\mathcal{K}_n$  is a series of Gevrey type and to compute its Gevrey constant,  $\gamma$ .

From Definition 2.5.1, one has that if a series  $\sum_{n=0}^{\infty} a_n t^n$  is  $\gamma$ -Gevrey, then

$$\log |a_n| \leq \log C + n \log D + \gamma \log(n!),$$

and so

$$\frac{\log |a_n| - \log C}{\log(n!)} \leq \frac{n \log D}{\log(n!)} + \gamma,$$

which shows that  $\gamma$  can be bounded as

$$\gamma \geq \lim_{n \rightarrow \infty} \frac{\log |a_n|}{\log(n!)},$$

in the case that such a limit exists. Hence, the quantity  $\gamma^* = \lim_{n \rightarrow \infty} \frac{\log |a_n|}{\log(n!)}$  is a lower bound of the Gevrey constant of the formal series. If  $\gamma^* > 0$ , then the coefficients  $\{a_n\}_n$  grow in a factorial way, and in this case we say that the series is *strictly* of Gevrey type.

Note that after performing the change of variables given by  $\tilde{x} = x, \tilde{y} = y + \frac{1}{c} f_1(x, y)$ , presented in Section 2.2.1, the maps  $F_1$  and  $F_2$  correspond to case 1 of the reduced form (2.2.2), with  $k = 2$ . Then, our code implements the algorithm provided in the proof of Proposition 2.3.1 to obtain the coefficients of  $\mathcal{K}_n$ .

Moreover, the program computes the quantities

$$\alpha_n = \frac{\log |K_n^x|}{\log(n!)}, \quad \beta_n = \frac{\log |K_n^y|}{\log(n!)}, \quad n \leq 300, \quad (2.5.1)$$

where  $\{K_n^x, K_n^y\}_n$  are the coefficients of each component of the parameterization  $\mathcal{K}_n$ . Since the value of the coefficient  $K_3^x$  is free, one has to choose it before running the program. We have performed the simulation with  $K_3^x = 0$ ,  $K_3^x = 1$  and  $K_3^x = -1$  for both  $F_1$  and  $F_2$ .

The program obtains the approximation  $\mathcal{K}_n$  up to degree 300 for any given value of  $K_3^x$  and computes the sequence of values  $\{\alpha_n\}_n$  and  $\{\beta_n\}_n$  defined in (2.5.1). In order to estimate the Gevrey behavior of the invariant curve  $K$ , we are interested in the values of  $\alpha_n$  and  $\beta_n$  for  $n$  large.

In Figures 2.3 and 2.4 we have represented the values of  $\alpha_n$  and  $\beta_n$ , respectively, versus  $n$ , being  $F_1$  the input function, and in Figures 2.3 and 2.4 we have represented the values of  $\alpha_n$  and  $\beta_n$ , respectively, versus  $n$ , being  $F_2$  the input function.

From the results plotted in the figures it appears that the values of  $\alpha_n$  and  $\beta_n$  may tend respectively to some constants  $\alpha$  and  $\beta$  as  $n$  tends to infinity. Hence, we suggest that the invariant curves associated to the origin for the given maps may be functions of Gevrey type. For the case of  $F_1$  we have  $\alpha, \beta \in (0.4, 0.5)$  and for the case of  $F_2$  we have  $\alpha, \beta \in (0.5, 0.6)$ .

Observe that in both cases the values of  $\alpha_n$  and  $\beta_n$ , for  $n$  big enough, do not seem to depend on the initial value chosen for  $K_3^x$ . That is, different parameterizations of the same stable curve have the same Gevrey constant. Also, it holds that for both  $F_1$  and  $F_2$  the limits of  $\{\alpha_n\}_n$  and  $\{\beta_n\}_n$  appear to be the same, that is, the Gevrey constants are the same for both components of  $K$ .

The fact that the polynomial approximations obtained for the invariant curves associated to  $F_1$  and  $F_2$  are series of strictly Gevrey type shows that the series associated to these curves cannot converge in any neighborhood of the origin, due to the factorial growth of the coefficients. Therefore, the invariant curves associated with a nilpotent parabolic point given in Theorem 2.2.1 can not be analytic functions in any neighborhood of the origin. This is indeed the reason for which in Section 2.4.2 we consider spaces of functions defined on a sector  $S(\beta, \rho)$ , otherwise, it would not be possible to obtain an analytic function  $\Delta$  satisfying the functional equation established in Section 2.4.1.

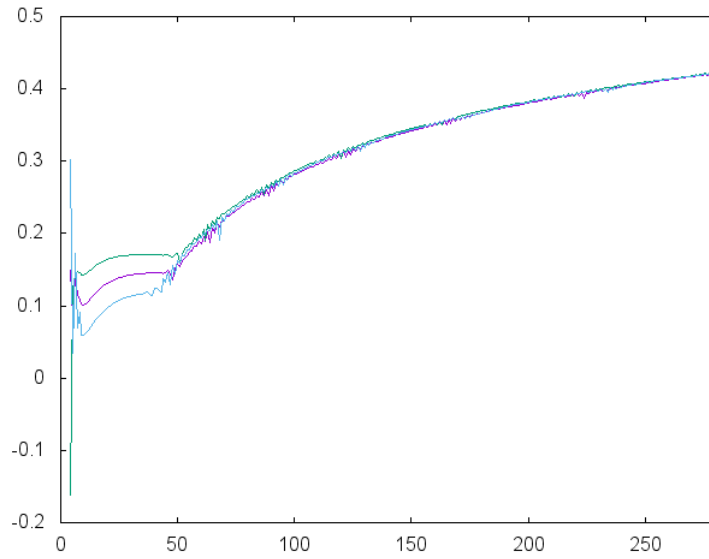


Figure 2.1: Representation of the constants  $\alpha_n$  versus  $n$  for the map  $F_1$ . The three different plots correspond to the simulation starting with  $K_3^x = 1$ ,  $K_3^x = 0$  and  $K_3^x = -1$ .

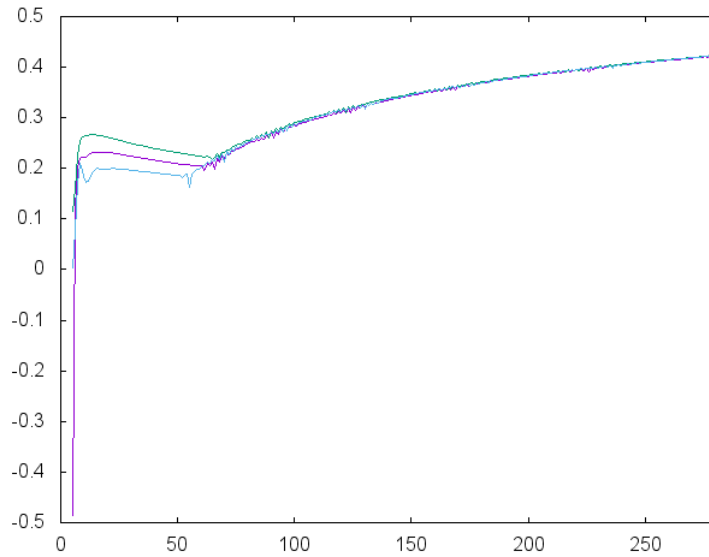


Figure 2.2: Representation of the constants  $\beta_n$  versus  $n$  for the map  $F_1$ . The three different plots correspond to the simulation starting with  $K_3^x = 1$ ,  $K_3^x = 0$  and  $K_3^x = -1$ .

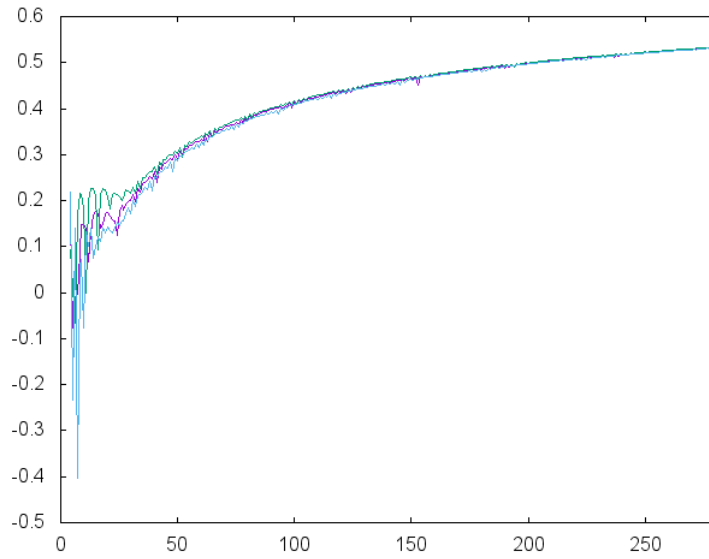


Figure 2.3: Representation of the constants  $\alpha_n$  versus  $n$  for the map  $F_2$ . The three different plots correspond to the simulation starting with  $K_3^x = 1$ ,  $K_3^x = 0$  and  $K_3^x = -1$ .

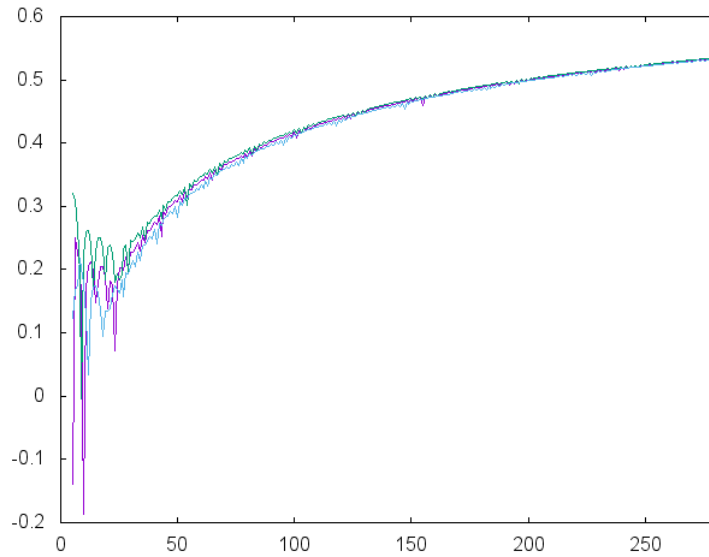


Figure 2.4: Representation of the constants  $\beta_n$  versus  $n$  for the map  $F_2$ . The three different plots correspond to the simulation starting with  $K_3^x = 1$ ,  $K_3^x = 0$  and  $K_3^x = -1$ .

## Chapter 3

# Invariant manifolds of differentiable maps with nilpotent parabolic points

### 3.1 Introduction

This chapter is the continuation of Chapter 2, and is devoted to state and prove the existence of invariant curves for a differentiable planar map,  $F$ , having a parabolic nilpotent fixed point. Concretely, the main results of the chapter show that, when they exist, the stable and unstable invariant curves of  $F$  have the same degree of differentiability that the map  $F$ , away from the fixed point.

First, for the convenience of the reader, we recall the setting of the problem, which was already presented with more detail in Chapter 2.

We consider  $C^r$ ,  $r \geq 3$ , maps  $F : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , where  $U$  is a neighborhood of  $(0, 0)$ , of the form

$$F(x, y) = \begin{pmatrix} x + cy + f_1(x, y) \\ y + f_2(x, y) \end{pmatrix}, \quad (3.1.1)$$

with  $c > 0$  and with  $f_1(x, y), f_2(x, y) = O(\|(x, y)\|^2)$ . Via the  $C^r$  change of variables given by  $\tilde{x} = x, \tilde{y} = y + \frac{1}{c} f_1(x, y)$ ,  $F$  can be written in the form

$$F(x, y) = \begin{pmatrix} x + cy \\ y + f(x, y) \end{pmatrix},$$

with  $f(x, y) = O(\|(x, y)\|^2)$  having the same regularity as  $F$ . Therefore, along this chapter we will always deal with maps of the form

$$\bar{F}(x, y) = \begin{pmatrix} x + cy \\ y + p(x) + yq(x) + u(x, y) + g(x, y) \end{pmatrix}, \quad (3.1.2)$$

with

$$p(x) = x^k(a_k + \cdots + a_r x^{r-k}), \quad q(x) = x^{l-1}(b_l + \cdots + b_r x^{r-l}),$$

with  $2 \leq k, l \leq r$ , where  $u(x, y)$  is a polynomial of degree  $r$  that contains the factor  $y^2$ , and where  $g(x, y) = o(\|(x, y)\|^r)$ . We also assume  $c > 0$ .

Throughout the chapter, as in the previous one, we will refer to (3.1.2) as the *reduced form* of  $F$  and we will use the same notation  $F$ .

Again, we consider the following three cases for the reduced fom of  $F$  depending on the indices  $k$  and  $l$ , namely,

- Case 1:  $k < 2l - 1$  and  $a_k \neq 0$ ,
- Case 2:  $k = 2l - 1$  and  $a_k \neq 0, b_l \neq 0$ ,
- Case 3:  $k > 2l - 1$  and  $b_l \neq 0$ .

Also recall that in order to deal with several cases at the same time we associate to a map  $F$  of the form (3.1.2) the integers  $N$  and  $s$  as  $N = k$  in case 1 and  $N = l$  in cases 2 and 3;  $s = 2r$  in case 1 and  $s = r$  in cases 2, 3.

In Section 3.2 we present the main results of the chapter, concerning the existence of  $C^r$  invariant curves for maps of the form (3.1.2). The rest of the chapter is devoted to prove these results. In Sections 3.3 – 3.5 we present the setting and some preliminary lemmas. We end the chapter proving the main theorems in Section 3.6.

## 3.2 Main results

The following are the main results concerning the existence and regularity of stable invariant manifolds of  $C^r$  maps of the form (3.1.2). As in the analytic case in Chapter 2, the results provide also the existence of invariant manifolds for maps of the form (3.1.1).

**Theorem 3.2.1.** *Let  $F : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a  $C^r$  map in a neighborhood  $U$  of  $(0, 0)$  of the form (3.1.2) with  $r \geq 3$ .*

*Assume the following hypotheses according to the different cases:*

- (case 1)  $a_k > 0$  and  $r \geq \frac{3}{2}k$ ,
- (case 2)  $a_k > 0, b_l \neq 0, r > k$  and

$$\max \left\{ \frac{\beta}{(r-2l+2)(r-l+1)} (2l(l-1) + \frac{ck a_k}{b_l^2} \beta), \frac{2l\beta}{r-l+1} \right\} < 1,$$

$$\text{where } \beta = \frac{2l|b_l|}{|b_l - \sqrt{b_l^2 + 4c a_k l}|}.$$

- (case 3)  $b_l < 0, r > 2l - 1$  and  $\frac{l(l-1)}{(r-2l+2)(r-l+1)} < 1$ .

*Then, there exists a  $C^1$  map  $H : [0, \rho) \rightarrow \mathbb{R}^2, H \in C^r(0, \rho)$ , of the form*

$$H(t) = \begin{cases} (t^2, H_{k+1}^y t^{k+1}) + (O(t^3), O(t^{k+2})) & \text{case 1,} \\ (t, H_l^y t^l) + (O(t^2), O(t^{l+1})) & \text{cases 2, 3,} \end{cases} \quad (3.2.1)$$

with  $H_{k+1}^y = -\sqrt{\frac{2a_k}{c(k+1)}}$  for case 1,  $H_l^y = \frac{b_l - \sqrt{b_l^2 + 4ca_k l}}{2cl}$  for case 2 and  $H_l^y = \frac{b_l}{cl}$  for case 3, and a polynomial  $R$  of the form  $R(t) = t + R_N t^N + R_{2N-1} t^{2N-1}$ , with  $R_k = \frac{c}{2} H_{k+1}^y$  for case 1 and  $R_l = cH_l^y$  for cases 2, 3, such that

$$F(H(t)) = H(R(t)), \quad t \in [0, \rho).$$

If the map  $F$  is  $C^\infty$  then the parameterization  $H$  is  $C^\infty$  in  $(0, \rho)$ .

**Remark 3.2.2.** The assumptions  $a_k > 0$  and  $k \leq r$  for cases 1 and 2 and  $b_l < 0$  and  $l \leq r$  for case 3 are necessary conditions for the existence of a formal, locally unique stable invariant curve of  $F$  asymptotic to  $(0, 0)$ . The other hypotheses of the theorem are nondegeneracy conditions on the reduced form of  $F$ , sufficient to ensure the existence of a stable invariant curve of class  $C^r$  asymptotic to  $(0, 0)$ . We do not claim that these conditions on  $r$  are sharp.

**Remark 3.2.3.** For case 2, the condition on the coefficients of  $F$  is always satisfied provided that  $r$  is sufficiently larger than  $l$ . Another sufficient condition for it to be satisfied is that  $\beta$  is small enough. The smallness of the coefficient  $\beta$  is a measure of how fast the dynamics on the associated invariant manifold is. For case 3, a sufficient nondegeneracy condition for the stable manifold to exist is given by  $r \geq \frac{4}{3}(2l - 1)$ . Notice that the assumption  $r \geq 2l - 1$  is necessary for the constructions we will do.

We also provide an *a posteriori* version of Theorem 3.2.1.

**Theorem 3.2.4.** Let  $F : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a map satisfying the hypotheses of Theorem 3.2.1 and let  $\hat{K} : (-\rho, \rho) \rightarrow \mathbb{R}^2$  and  $\hat{R} : (-\rho, \rho) \rightarrow \mathbb{R}$  be analytic maps satisfying

$$\hat{K}(t) = \begin{cases} (t^2, \hat{K}_{k+1}^y t^{k+1}) + (O(t^3), O(t^{k+2})) & \text{case 1,} \\ (t, \hat{K}_l^y t^l) + (O(t^2), O(t^{l+1})) & \text{cases 2, 3,} \end{cases}$$

and  $\hat{R}(t) = t + \hat{R}_N t^N + O(t^{N+1})$ ,  $\hat{R}_N < 0$ , such that

$$F(\hat{K}(t)) - \hat{K}(\hat{R}(t)) = (O(t^{n+N}), O(t^{n+2N-1})), \quad (3.2.2)$$

for some  $2 \leq n \leq 2r - 2k + 1$  in case 1 or  $1 \leq n \leq r - 2l + 1$  in cases 2, 3.

Then, there exists a  $C^1$  map  $H : [0, \rho) \rightarrow \mathbb{R}^2$ ,  $H \in C^r(0, \rho)$ , and an analytic map  $R : (-\rho, \rho) \rightarrow \mathbb{R}$  such that

$$F(H(t)) = H(R(t)), \quad t \in [0, \rho)$$

and

$$H(t) - \hat{K}(t) = (O(t^{n+1}), O(t^{n+N})),$$

$$R(t) - \hat{R}(t) = \begin{cases} O(t^{2k-1}) & \text{if } n \leq k \\ 0 & \text{if } n > k \end{cases} \quad \text{case 1,}$$

$$R(t) - \hat{R}(t) = \begin{cases} O(t^{2l-1}) & \text{if } n \leq l - 1 \\ 0 & \text{if } n > l - 1 \end{cases} \quad \text{cases 2, 3.}$$

As mentioned, using the conjugations  $(x, y) \mapsto (\pm x, \pm y)$  and  $F^{-1}$  we can obtain the local phase portraits and the location of the local invariant manifolds of  $F$  depending on the studied cases (see [25]).

**Remark 3.2.5.** As well as in the previous chapter, the invariant manifolds obtained in Theorems 3.2.1 and 3.2.4 are unique (see Remark 2.2.7).

**Remark 3.2.6** (Unstable manifolds for  $C^r$  maps). As in Chapter 2, we can obtain analogous results of existence of the unstable manifolds for a given map  $F$  without having to compute explicitly the inverse map  $F^{-1}$ , as explained in Section 2.2.4.

As in the analytic case, the expansions of the parameterizations of the unstable curves obtained in Section 2.3 are approximations of true invariant curves, as it happens for the stable ones.

Following the notation of Section 2.2.4, we have that if  $F$  satisfies the conditions of case 1,  $G$  also does. The same happens for case 3 if we assume  $b_l > 0$  instead of  $b_l < 0$ . If  $F$  satisfies the conditions of case 2, since the coefficient  $b_l$  of  $F$  becomes  $-b_l$  for  $G$ , one has to check the condition involving the maximum taking now  $\beta$  as  $\beta = \frac{2l|b_l|}{|-b_l - \sqrt{b_l^2 + 4ca_k l}|}$ . Then, for cases 1 and 3 or for case 2 when that condition holds, we conclude analogous results as the ones explained for the analytic case.

### 3.3 The functional equation

The rest of the chapter is dedicated to the proofs of Theorems 3.2.1 and 3.2.4. To this end, we will use some of the results presented in Chapter 2, and we will often refer to them.

Concretely, along this chapter we will focus on the proof of Theorem 3.2.1. The structure of the proof of Theorem 3.2.4 is presented at the end of the chapter as a corollary of the previous results. It is indeed analogous to the one of Theorem 2.2.3 of Chapter 2 and uses the constructions of the approximations in the proofs of Theorems 2.2.1 and 3.2.1.

As in the analytic case, we use the parameterization method (see Section 2.2.2). To get an initial approximation of a parameterization of the invariant manifolds of  $F$ , we first consider the Taylor polynomial of  $F$  of degree  $r$  which we denote by  $F^{\leq}$  and reads

$$F^{\leq}(x, y) = \begin{pmatrix} x + cy \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ p(x) + yq(x) + u(x, y) \end{pmatrix}.$$

Since  $F^{\leq}$  is analytic, Theorem 2.2.1 provides a  $C^1$  map  $K : [0, \rho) \rightarrow \mathbb{R}$ , analytic on  $(0, \rho)$  and a polynomial,  $R$ , such that

$$F^{\leq} \circ K - K \circ R = 0 \quad \text{on} \quad [0, \rho). \quad (3.3.1)$$

Then, we look for  $\rho > 0$  and a  $C^r$  function,  $H = K + \Delta : (0, \rho) \rightarrow \mathbb{R}^2$ , such that

$$F \circ (K + \Delta) - (K + \Delta) \circ R = 0. \quad (3.3.2)$$

Moreover, we ask  $\Delta$  to satisfy  $\Delta(t) = (\Delta^x(t), \Delta^y(t)) = (O(t^{2r-2k+2}), O(t^{2r-k+1}))$  for case 1 and  $\Delta(t) = (O(t^{r-2l+2}), O(t^{r-l+1}))$  for cases 2 and 3.

Next, we establish a functional equation for  $\Delta$  obtained from (3.3.2) which will be the object of our study. Later, in Section 3.4 we describe the function spaces where we will set such



an equation and the operators  $\mathcal{S}_{L,R}$  and  $\mathcal{N}_{L,F}$  together with their properties (Lemmas 3.4.6 and 3.4.7). Notice that although the notation of the operators is similar to the one of the operators in Chapter 2, both pair of families of operators are different.

In Section 3.5 we recall the fiber contraction theorem and we also introduce the family of operators  $\mathcal{T}_{L,F}$  given by  $\mathcal{T}_{L,F} = \mathcal{S}_{L,R}^{-1} \circ \mathcal{N}_{L,F}$  and we describe its properties in Lemmas 3.5.2 and 3.5.3. Finally, in Section 3.6 we prove the existence of a solution of the functional equation and we conclude the proof of Theorems 3.2.1 and 3.2.4.

Let  $F : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a  $C^r$  map of the form (3.1.2) satisfying the hypotheses of Theorem 3.2.1. Along the section, once having taken a  $C^r$  map  $F$  of the form (3.1.2), the maps  $K$  and  $R$  will always refer to the analytic solutions of  $F^{\leq} \circ K - K \circ R = 0$ , on some interval  $[0, \rho]$  given by Theorem 2.2.1.

Using (3.3.1) and the previous notation, condition (3.3.2) can be rewritten as

$$\begin{aligned} \Delta^x \circ R - \Delta^x &= c \Delta^y, \\ \Delta^y \circ R - \Delta^y &= p \circ (K^x + \Delta^x) - p \circ K^x + K^y \cdot (q \circ (K^x + \Delta^x) - q \circ K^x) \\ &\quad + \Delta^y \cdot q \circ (K^x + \Delta^x) + u \circ (K + \Delta) - u \circ K + g \circ (K + \Delta). \end{aligned} \quad (3.3.3)$$

Clearly, a continuous function  $\Delta$  satisfies (3.3.2) if and only if it satisfies (3.3.3). Since we want to prove differentiability of  $\Delta$ , next we derive  $r$  equations for the derivatives of  $\Delta$  by formally differentiating equation (3.3.3). In our approach we will look for continuous solutions of these equations.

After having differentiated (3.3.3)  $L$  times,  $1 \leq L \leq r$ , we obtain

$$\begin{aligned} D^L \Delta^x \circ R (DR)^L - D^L \Delta^x &= c D^L \Delta^y + \mathcal{J}_{L,N}^x(\Delta, \dots, D^{L-1} \Delta), \\ D^L \Delta^y \circ R (DR)^L - D^L \Delta^y &= p' \circ (K^x + \Delta^x) D^L \Delta^x + (K^y + \Delta^y) q' \circ (K^x + \Delta^x) D^L \Delta^x \\ &\quad + q \circ (K^x + \Delta^x) D^L \Delta^y + (Du + Dg) \circ (K + \Delta) \cdot D^L \Delta \\ &\quad + \mathcal{J}_{L,N}^y(\Delta, \dots, D^{L-1} \Delta), \end{aligned} \quad (3.3.4)$$

where  $\mathcal{J}_{L,F}^x$  and  $\mathcal{J}_{L,F}^y$  are given by

$$\begin{aligned} \mathcal{J}_{L,F}^x(f_0, \dots, f_{L-1}) &= \Lambda_{L,R}^x(f_0^x, \dots, f_{L-1}^x), \\ \mathcal{J}_{L,F}^y(f_0, \dots, f_{L-1}) &= \Lambda_{L,R}^y(f_0^y, \dots, f_{L-1}^y) + \Omega_{L,F}(f_0, \dots, f_{L-1}), \end{aligned} \quad (3.3.5)$$

and  $\Lambda_{L,R}^i$ ,  $i = x, y$ , by

$$\begin{aligned} \Lambda_{1,R}^i(f_0^i) &= 0, \\ \Lambda_{2,R}^i(f_0^i, f_1^i) &= -f_1^i \circ R D^2 R, \\ \Lambda_{L,R}^i(f_0^i, \dots, f_{L-1}^i) &= D[\Lambda_{L-1,R}^i(f_0^i, \dots, f_{L-2}^i)] \\ &\quad - (L-1) f_{L-1}^i \circ R (DR)^{L-2} D^2 R, \quad L \in \{3, \dots, r\}, \end{aligned} \quad (3.3.6)$$

where in the expansion of the derivative  $D[\Lambda_{L-1,R}^i(f_0^i, \dots, f_{L-2}^i)]$  we substitute  $Df_i$  by  $f_{i+1}$ .

Note that  $\Lambda_{L,R}^i$  does not depend on  $f_0$ . Moreover,  $\Omega_{L,F}$  is given by

$$\begin{aligned}
\Omega_{1,F}(f_0) &= DK^x(p' \circ (K^x + f_0^x) - p' \circ K^x) + DK^y \cdot (q \circ (K^x + f_0^x) - q \circ K^x) \\
&\quad + K^y \cdot DK^x(q' \circ (K^x + f_0^x) - q' \circ K^x) + f_0^y \cdot DK^x q' \circ (K^x + f_0^x) \\
&\quad + (Du \circ (K + f_0) - Du \circ K)DK + Dg \circ (K + f_0)DK, \\
\Omega_{L,F}(f_0, \dots, f_{L-1}) &= D[\Omega_{L-1,F}(f_0, \dots, f_{L-2})] + D[p' \circ (K^x + f_0^x)]f_{L-1}^x \\
&\quad + D[(K^y + f_0^y)q' \circ (K^x + f_0^x)]f_L^x + D[q \circ (K^x + f_0^x)]f_{L-1}^y \\
&\quad + D[(Du + Dg) \circ (K + f_0)] \cdot f_{L-1}, \quad L \in \{2, \dots, r\}.
\end{aligned} \tag{3.3.7}$$

Note that  $\Lambda_{L,R}(f_0, \dots, f_{L-1})$  comes from the differentiation on the left hand side of (3.3.3) and  $\Omega_{L,F}(f_0, \dots, f_{L-1})$  comes from the differentiation on the right hand side of the second equation of (3.3.3). Expanding the derivatives in (3.3.6) and (3.3.7) and changing  $Df_i$  by  $f_{i+1}$  we obtain expressions that have to be understood as operators acting on  $(f_0, \dots, f_{L-1})$ , considering the  $f_j$ 's as independent variables.

It is important to note that  $\Lambda_{L,R}^i$  and  $\Omega_{L,F}^i$ ,  $i = x, y$ , depend in a polynomial way on  $f_j$  for  $j \geq 1$ , but not on  $f_0$ .

### 3.4 Function spaces, the operators $\mathcal{S}_{L,R}$ and $\mathcal{N}_{L,F}$ and their properties

We introduce next the notation and the function spaces that we will use to study the functional equations (3.3.3) and (3.3.4).

**Definition 3.4.1.** *Given  $0 < \rho < 1$ , let  $\mathcal{Y}_n$ , for  $n \in \mathbb{Z}$ , be the Banach space given by*

$$\mathcal{Y}_n = \{f : (0, \rho) \rightarrow \mathbb{R} \mid f \in C^0(0, \rho), \|f\|_n := \sup_{(0, \rho)} \frac{|f(t)|}{|t|^n} < \infty\},$$

where  $C^0(0, \rho)$  denotes the space of continuous functions on  $(0, \rho)$ .

Note that when  $n \geq 1$  the functions  $f$  in  $\mathcal{Y}_n$  can be continuously extended to  $t = 0$  with  $f(0) = 0$  and, if moreover,  $n \geq 2$ , the derivative of  $f$  can be continuously extended to  $t = 0$  with  $f'(0) = 0$ . For  $n < 0$  the functions contained in  $\mathcal{Y}_n$  may be unbounded in a neighborhood of 0.

Note also that  $\mathcal{Y}_{n+1} \subset \mathcal{Y}_n$ , for all  $n \in \mathbb{Z}$ . If  $f \in \mathcal{Y}_m$ ,  $g \in \mathcal{Y}_n$ , then  $fg \in \mathcal{Y}_{m+n}$  and  $\|fg\|_{m+n} \leq \|f\|_m \|g\|_n$ . If  $f \in \mathcal{Y}_{n+1}$ , then  $\|f\|_n \leq \|f\|_{n+1}$ .

Given  $n, m \in \mathbb{Z}$  we denote  $\mathcal{Y}_{m,n} := \mathcal{Y}_m \times \mathcal{Y}_n$  the product space, endowed with the product norm

$$\|f\|_{m,n} = \max \{\|f^x\|_m, \|f^y\|_n\}, \quad f = (f^x, f^y) \in \mathcal{Y}_m \times \mathcal{Y}_n.$$

Given  $s, r, N$  positive integer numbers and  $L \in \{0, \dots, r\}$ , we define the spaces

$$\Sigma_{L,N} = \prod_{j=0}^L (\mathcal{Y}_{s-2N+2-j, s-N+1-j}), \quad 0 \leq L \leq r$$

and

$$D\Sigma_{L-1,N} = \mathcal{Y}_{s-2N+2-L, s-N+1-L}, \quad 1 \leq L \leq r$$

both endowed with the product norm. Clearly, we have  $\Sigma_{L,N} = \Sigma_{L-1,N} \times D\Sigma_{L-1,N}$ , and  $\Sigma_{L,N} = \Sigma_{0,N} \times \prod_{i=1}^L D\Sigma_{i-1,N}$ , for  $1 \leq L \leq r$ .

For notational convenience we also write  $D\Sigma_{-1,N} = \Sigma_{0,N}$ .

Also, let  $\alpha_i > 0$ ,  $1 \leq i \leq r$ . Given  $L$  we write  $\alpha = (\alpha_0, \dots, \alpha_L)$ . We define the closed balls

$$\begin{aligned} \Sigma_{0,N}^{\alpha_0} &= \{f \in \Sigma_{0,N} \mid \|f\|_{\Sigma_{0,N}} \leq \alpha_0\}, \\ D\Sigma_{i-1,N}^{\alpha_i} &= \{f \in D\Sigma_{i-1,N} \mid \|f\|_{D\Sigma_{i-1,N}} \leq \alpha_i\}, \quad i \in \{1, \dots, r\}, \end{aligned}$$

and the products of balls

$$\Sigma_{L,N}^\alpha = \Sigma_{0,N}^{\alpha_0} \times \prod_{i=1}^L D\Sigma_{i-1,N}^{\alpha_i}, \quad L \in \{1, \dots, r\},$$

For notational convenience we will write  $\Sigma_{0,N}^\alpha = \Sigma_{0,N}^{\alpha_0}$ .

An element of  $\Sigma_{L,N}$  will be denoted by  $(f_0, \dots, f_L)$ , with  $f_0 = (f_0^x, f_0^y) \in \Sigma_{0,N}$ , and  $f_i = (f_i^x, f_i^y) \in D\Sigma_{i-1,N}$ , for  $i = 1, \dots, L$ .

For the sake of simplicity we do not write the dependence with respect to  $r$ ,  $s$  and  $\rho$  in the notation of the previous objects.

To solve the functional equation (3.3.2), we look for a solution,  $f_0$ , of (3.3.3) contained in a closed ball  $\Sigma_{0,N}^{\alpha_0}$ , and for a solution,  $(f_1, \dots, f_L)$ , of (3.3.4) in a product  $\Sigma_{L,N}^\alpha$ , for each  $L \in \{1, \dots, r\}$ . In order for the compositions in (3.3.4) to be meaningful we have to deal with  $f_0$  in a ball of sufficiently small radius. Arguing as in the analytic case we take  $\alpha_0 = \min\{\frac{1}{2}, \frac{d}{2}\}$ , where  $d$  is the radius of a ball contained in the domain where  $F$  is  $C^r$ . The values of the radii  $\alpha_i$ ,  $1 \leq i \leq r$ , will be determined later (see proof of Lemma 3.5.3).

In the differentiable case we consider analogous operators as in the analytical case but now we need a family of them, depending on  $L$ , to deal with the equations (3.3.4) for the derivatives of  $\Delta$ . Their definitions are determined by the structure of such equations.

First, we state two auxiliary results about the iterates of  $R$  and their derivatives.

**Lemma 3.4.2.** *Let  $R : [0, \rho) \rightarrow \mathbb{R}$  be a differentiable map of the form  $R(t) = t + R_N t^N + O(|t|^{N+1})$ , with  $R_N < 0$ . Then, for any  $\nu, \mu$  such that  $0 < \nu < (N-1)|R_N| < \mu$ , there exists  $\rho > 0$  such that*

$$\frac{t}{(1 + j \mu t^{N-1})^{1/N-1}} < R^j(t) < \frac{t}{(1 + j \nu t^{N-1})^{1/N-1}}, \quad \forall j \geq 1, \quad \forall t \in (0, \rho). \quad (3.4.1)$$

As a consequence,  $R$  maps  $(0, \rho)$  into itself.

If  $R$  were a polynomial the upper bound in Lemma 3.4.2 would be an immediate corollary of Lemma 2.4.2.

*Proof.* Let  $\lambda > 0$  and  $\varphi_\lambda(t) = \frac{t}{(1 + \lambda t^{N-1})^{1/N-1}}$  for  $t \geq 0$ . A computation shows that  $\frac{d}{dt} \varphi_\lambda(t) = \frac{1}{(1 + \lambda t^{N-1})^{N/N-1}} > 0$  and hence  $\varphi_\lambda$  is increasing. We prove (3.4.1) by induction. When  $j = 1$ ,

it is easy to see that there exists  $\rho > 0$  such that

$$\varphi_\mu(t) = \frac{t}{(1 + \mu t^{N-1})^{1/N-1}} < R(t) < \frac{t}{(1 + \nu t^{N-1})^{1/N-1}} = \varphi_\nu(t), \quad \forall t \in (0, \rho).$$

Assuming (3.4.1) for  $j \geq 1$ ,

$$\begin{aligned} R^{j+1}(t) &= R(R^j(t)) < \varphi_\nu(R^j(t)) < \varphi_\nu\left(\frac{t}{(1 + j \nu t^{N-1})^{1/N-1}}\right) \\ &= \frac{t}{(1 + (j+1) \nu t^{N-1})^{1/N-1}} \end{aligned}$$

in the same interval  $(0, \rho)$ . The lower bound is obtained in a completely analogous way using  $\varphi_\mu$ .  $\square$

**Lemma 3.4.3.** *Let  $R : [0, \rho) \rightarrow \mathbb{R}$  be a differentiable map of the form  $R(t) = t + R_N t^N + O(|t|^{N+1})$ , with  $R_N < 0$ , such that  $DR(t) = 1 + N R_N t^{N-1} + O(|t|^N)$ . For any  $\nu, \mu$  such that  $0 < \nu < (N-1)|R_N| < \mu$ , let  $\kappa = \nu/\mu$ . Then, there exists  $\rho > 0$  such that*

$$DR^j(t) \leq \frac{1}{(1 + j \mu t^{N-1})^{\kappa N/N-1}}, \quad \forall j \in \mathbb{N}, \quad \forall t \in (0, \rho). \quad (3.4.2)$$

*Proof.* Since  $N|R_N| > \nu \frac{N}{N-1}$ , by the form of the derivative  $DR$ , there exists  $\rho > 0$  such that

$$0 < DR(t) < 1 - \frac{\nu N}{N-1} t^{N-1}, \quad \forall t \in (0, \rho).$$

Using the chain rule  $DR^j(t) = \Pi_{m=0}^{j-1} DR(R^m(t))$  and the lower bound in (3.4.1) we can write

$$\begin{aligned} DR^j(t) &= \exp \sum_{m=0}^{j-1} \log DR(R^m(t)) \leq \exp \sum_{m=0}^{j-1} \log \left(1 - \frac{\nu N}{N-1} (R^m(t))^{N-1}\right) \\ &\leq \exp \left( \frac{-\nu N}{N-1} \sum_{m=0}^{j-1} (R^m(t))^{N-1} \right) \leq \exp \left( \frac{-\nu N}{N-1} \sum_{m=0}^{j-1} \frac{t^{N-1}}{(1 + m \mu t^{N-1})} \right) \\ &\leq \exp \left( \frac{-\nu N}{N-1} \int_0^j \frac{t^{N-1}}{(1 + s \mu t^{N-1})} ds \right) = \exp \left( \frac{-\nu N}{\mu(N-1)} \int_0^{j \mu t^{N-1}} \frac{1}{1 + \xi} d\xi \right) \\ &= \exp \left( \frac{-\kappa N}{N-1} \log(1 + j \mu t^{N-1}) \right) = \frac{1}{(1 + j \mu t^{N-1})^{\kappa N/N-1}}. \end{aligned}$$

$\square$

From now on we assume  $R$  is as in the previous lemmas and  $\rho$  satisfies the conclusions of them, in particular,  $R(0, \rho) \subset (0, \rho)$ .

**Definition 3.4.4.** *Given  $L \in \{0, \dots, r\}$ , let  $\mathcal{S}_{L,R} : D\Sigma_{L-1,N} \rightarrow D\Sigma_{L-1,N}$  be the linear operator defined component-wise as  $\mathcal{S}_{L,R} = (\mathcal{S}_{L,R}^x, \mathcal{S}_{L,R}^y)$ , with*

$$\mathcal{S}_{L,R}^x f = \mathcal{S}_{L,R}^y f = f \circ R(DR)^L - f.$$

Notice that although both components are formally identical, they act on different domains.

**Definition 3.4.5.** Given a map  $F$  of class  $C^r$  satisfying the hypotheses of Theorem 3.2.1, let  $\mathcal{N}_{0,F} : \Sigma_{0,N}^\alpha \rightarrow \mathcal{Y}_{s-N+1,s}$  be the operator given by

$$\begin{aligned}\mathcal{N}_{0,F}^x(f_0) &= c f_0^y, \\ \mathcal{N}_{0,F}^y(f_0) &= p \circ (K^x + f_0^x) - p \circ K^x + K^y \cdot [q \circ (K^x + f_0^x) - q \circ K^x] \\ &\quad + f_0^y \cdot q \circ (K^x + f_0^x) + u \circ (K + f_0) - u \circ K + g \circ (K + f_0),\end{aligned}$$

and let  $\mathcal{N}_{L,F} : \Sigma_{L,N}^\alpha \rightarrow \mathcal{Y}_{s-N+1-L,s-L}$ ,  $L \in \{1, \dots, r\}$ , be the operator given by

$$\begin{aligned}\mathcal{N}_{L,F}^x(f_0, \dots, f_L) &= c f_L^y + \mathcal{J}_{L,N}^x(f_0, \dots, f_{L-1}), \\ \mathcal{N}_{L,F}^y(f_0, \dots, f_L) &= p' \circ (K^x + f_0^x) \cdot f_L^x + (K^y + f_0^y) \cdot q' \circ (K^x + f_0^x) f_L^x \\ &\quad + q \circ (K^x + f_0^x) \cdot f_L^y + (Du + Dg) \circ (K + f_0) \cdot f_L \\ &\quad + \mathcal{J}_{L,N}^y(f_0, \dots, f_{L-1}),\end{aligned}$$

where  $\mathcal{J}_{L,N}$  are already introduced in (3.3.5), (3.3.6) and (3.3.7).

From the definition of the operators  $\mathcal{S}_{L,R}$  and  $\mathcal{N}_{L,F}$ , the recursive expressions of  $\Lambda_{L,R}$  and  $\Omega_{L,F}$  obtained in (3.3.6) and (3.3.7) and the choice of  $\alpha_0$  it is clear that the operators  $\mathcal{S}_{L,R}$  and  $\mathcal{N}_{L,F}$  are well defined and that  $\mathcal{S}_{L,R}$  is linear and bounded.

Note that with the operators introduced above, equations (3.3.3) and (3.3.4) can be written now as

$$\mathcal{S}_{L,R} D^L \Delta = \mathcal{N}_{L,F}(\Delta, \dots, D^L \Delta), \quad (\Delta, \dots, D^L \Delta) \in \Sigma_{L,N}^\alpha,$$

for each  $L \in \{0, \dots, r\}$  and  $\alpha_0$  as fixed previously and some  $\alpha_i > 0$ ,  $1 \leq i \leq L$ .

In the following lemmas we prove that each of the operators  $\mathcal{S}_{L,R}$  has a bounded right inverse and we provide a bound for the norm  $\|\mathcal{S}_{L,R}^{-1}\|$ . We also show that each of the operators  $\mathcal{N}_{L,F}$  is Lipschitz with respect to the last variable and we provide a uniform bound for the Lipschitz constant for the family  $\mathcal{N}_{L,F}$ ,  $L \in \{0, \dots, r\}$ .

**Lemma 3.4.6.** Let  $0 \leq L \leq r$ . Assume  $r > k$  in case 1 and  $r > 2l - 1$  in cases 2 and 3. Then, given  $0 < \nu < (N - 1)|R_N| < \mu$  such that  $\kappa = \nu/\mu$  satisfies  $\kappa > 1/N$ , there exists  $\rho > 0$  small enough such that, taking  $(0, \rho)$  as the domain of the functions of  $\mathcal{Y}_{s-N+1-L,s-L}$ , the operator  $\mathcal{S}_{L,R} : D\Sigma_{L-1,N} \rightarrow D\Sigma_{L-1,N}$  has a bounded right inverse,

$$\mathcal{S}_{L,R}^{-1} : \mathcal{Y}_{s-N+1-L,s-L} \rightarrow D\Sigma_{L-1,N} = \mathcal{Y}_{s-2N+2-L,s-N+1-L},$$

given by

$$\mathcal{S}_{L,R}^{-1} \eta = - \sum_{j=0}^{\infty} \eta \circ R^j (DR^j)^L, \quad \eta \in \mathcal{Y}_{s-N+1-L,s-L}, \quad (3.4.3)$$

and we have the operator norm bound

$$\begin{aligned}\|(\mathcal{S}_{L,R}^x)^{-1}\| &\leq \rho^{N-1} + \frac{1}{\nu} \frac{N-1}{s-2N+2+L(\kappa N-1)}, \\ \|(\mathcal{S}_{L,R}^y)^{-1}\| &\leq \rho^{N-1} + \frac{1}{\nu} \frac{N-1}{s-N+1+L(\kappa N-1)}.\end{aligned}$$

*Proof.* A simple computation shows that the expression (3.4.3) of  $\mathcal{S}_{L,R}$  formally satisfies  $\mathcal{S}_{L,R} \circ (\mathcal{S}_{L,R})^{-1} \eta = \eta$ , for  $\eta \in \mathcal{Y}_{s-N+1-L,s-L}$ .

We give the details of the proof for the second component  $\mathcal{S}_{L,R}^y : \mathcal{Y}_{s-N+1-L} \rightarrow \mathcal{Y}_{s-N+1-L}$  of the operator  $\mathcal{S}_{L,R}$ , the details for  $\mathcal{S}_{L,R}^x : \mathcal{Y}_{s-2N+2-L} \rightarrow \mathcal{Y}_{s-2N+2-L}$  being completely analogous. The results for  $\mathcal{S}_{L,R}$  follow immediately because the components of the operator are uncoupled.

We take  $\kappa > 1/N$  and  $\mu, \nu$  such that  $0 < \nu < (N-1)|R_N| < \mu$  and  $\nu/\mu = \kappa$ . By Lemmas 3.4.2 and 3.4.3 there exists  $\rho > 0$  such that  $R$  maps the interval  $(0, \rho)$  into itself and the bounds (3.4.1) and (3.4.2) hold. Then, given  $\eta \in \mathcal{Y}_{s-L}$

$$\begin{aligned} |(\eta \circ R^j(DR^j)^L)(t)| &\leq \|\eta\|_{s-L} |R^j(t)|^{s-L} |DR^j(t)|^L \\ &\leq \|\eta\|_{s-L} \frac{t^{s-L}}{(1+j\nu t^{N-1})^{\frac{s-L}{N-1}}} \frac{1}{(1+j\mu t^{N-1})^{\frac{\kappa NL}{N-1}}} \\ &\leq M \|\eta\|_{s-L} \frac{1}{j^{\frac{s+L(\kappa N-1)}{N-1}}}, \quad \forall t \in (0, \rho), \end{aligned}$$

hence, since  $s \geq r \geq N > N-1$ , (3.4.3) converges uniformly on  $(0, \rho)$  by the Weierstrass  $M$ -test. Thus,  $(\mathcal{S}_{L,R}^y)^{-1} \eta = -\sum_{j=0}^{\infty} \eta \circ R^j(DR^j)^L$  is continuous on  $(0, \rho)$ .

Now, we prove that  $(\mathcal{S}_{L,R}^y)^{-1}$  is a bounded operator from  $\mathcal{Y}_{s-L}$  to  $\mathcal{Y}_{s-N+1-L}$  and we obtain a bound for its norm. Again, having chosen  $\kappa = \nu/\mu$ , from Lemmas 3.4.2 and 3.4.3 one has,

$$\begin{aligned} \|(\mathcal{S}_{L,R}^y)^{-1} \eta\|_{s-N+1-L} &\leq \sup_{t \in (0, \rho)} \frac{1}{t^{s-N+1-L}} \sum_{j=0}^{\infty} |\eta(R^j(t))(DR^j(t))^L| \\ &\leq \|\eta\|_{s-L} \sup_{t \in (0, \rho)} \frac{1}{t^{s-N+1-L}} \sum_{j=0}^{\infty} \frac{t^{s-L}}{(1+j\nu t^{N-1})^{\frac{s-L}{N-1}}} \frac{1}{(1+j\mu t^{N-1})^{\frac{\kappa NL}{N-1}}}, \end{aligned}$$

and, bounding the sum by an appropriate integral, we obtain the bound

$$\begin{aligned} &\frac{1}{t^{s-N+1-L}} \sum_{j=0}^{\infty} \frac{t^{s-L}}{(1+j\nu t^{N-1})^{\frac{s-L}{N-1}}} \frac{1}{(1+j\mu t^{N-1})^{\frac{\kappa NL}{N-1}}} \\ &\leq t^{N-1} \left( 1 + \int_0^{\infty} \frac{1}{(1+x\nu t^{N-1})^{\frac{s-L+\kappa NL}{N-1}}} dx \right) \\ &= t^{N-1} + \frac{1}{\nu} \frac{N-1}{s-N+1+L(\kappa N-1)}. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \|(\mathcal{S}_{L,R}^y)^{-1} \eta\|_{s-N+1-L} &\leq \|\eta\|_{s-L} \sup_{t \in (0, \rho)} \left( t^{N-1} + \frac{1}{\nu} \frac{N-1}{s-N+1+L(\kappa N-1)} \right), \quad \eta \in \mathcal{X}_{s-L}, \end{aligned}$$

which shows that  $(\mathcal{S}_{L,R}^y)^{-1} : \mathcal{Y}_{s-L} \rightarrow \mathcal{Y}_{s-N+1-L}$  is bounded and

$$\|(\mathcal{S}_{L,R}^y)^{-1}\| \leq \rho^{N-1} + \frac{1}{\nu} \frac{N-1}{s-N+1+L(\kappa N-1)}.$$

In the same way,  $(\mathcal{S}_{L,R}^x)^{-1} : \mathcal{Y}_{s-N+1-L} \rightarrow \mathcal{Y}_{s-2N+2-L}$  is bounded and

$$\|(\mathcal{S}_{L,R}^x)^{-1}\| \leq \rho^{N-1} + \frac{1}{\nu} \frac{N-1}{s-2N+2+L(\kappa N-1)}.$$

□

**Lemma 3.4.7.** *Let  $0 \leq L \leq r$ . Assume  $r > k$  in case 1 and  $r > 2l - 1$  in cases 2 and 3. There exists a constant,  $M > 0$ , for which the family of operators  $\mathcal{N}_{L,F}$  satisfy, for each  $L \in \{0, \dots, r\}$ ,*

$$\text{Lip } \mathcal{N}_{L,F}^x(f_0, \dots, f_{L-1}, \cdot) = c,$$

and

$$\text{Lip } \mathcal{N}_{L,F}^y(f_0, \dots, f_{L-1}, \cdot) \leq k|a_k| + M\rho, \quad (\text{case 1}),$$

$$\begin{aligned} \text{Lip } \mathcal{N}_{L,F}^y(f_0, \dots, f_{L-1}, \cdot) \\ \leq \max\{((l-1)|K_l^y b_l| + k|a_k|) + M\rho, |b_l| + M\rho\}, \quad (\text{case 2}), \end{aligned}$$

$$\text{Lip } \mathcal{N}_{L,F}^y(f_0, \dots, f_{L-1}, \cdot) \leq \max\{(l-1)|K_l^y b_l| + M\rho, |b_l| + M\rho\}, \quad (\text{case 3}),$$

where  $(0, \rho)$  is the domain of the functions of  $\Sigma_{L,N}^\alpha$ .

*Proof.* To distinguish the roles of the variables  $(f_0, \dots, f_{L-1})$  and  $f_L$  we will denote the latter by  $h_L$ . The statement concerning the component  $\mathcal{N}_{L,F}^x$  is clear by the definition of  $\mathcal{N}_{L,F}$ .

For  $\mathcal{N}_{L,N}^y$  we first deal with the case  $L = 0$ .

Since  $g(x, y) = o(\|(x, y)\|^r)$  and  $g \in C^r$  we have  $D_i g(x, y) = o(\|(x, y)\|^{r-1})$ ,  $i = 1, 2$ .

For every  $h_0, \tilde{h}_0 \in \Sigma_{0,N}^\alpha$ , from the definition of the operator  $\mathcal{N}_{0,F}^y$ , one can write

$$\begin{aligned} & \mathcal{N}_{0,F}^y(h_0) - \mathcal{N}_{0,F}^y(\tilde{h}_0) \\ &= \left( \int_0^1 p' \circ (K^x + \tilde{h}_0^x + s(h_0^x - \tilde{h}_0^x)) ds \right. \\ & \quad \left. + (K^y + h_0^y) \int_0^1 q' \circ (K^x + \tilde{h}_0^x + s(h_0^x - \tilde{h}_0^x)) ds \right. \\ & \quad \left. + \int_0^1 (D_1 u + D_1 g) \circ (K + \tilde{h}_0 + s(h_0 - \tilde{h}_0)) ds \right) (h_0^x - \tilde{h}_0^x) \\ & \quad + \left( q \circ (K^x + \tilde{h}_0^x) + \int_0^1 (D_2 u + D_2 g) \circ (K + \tilde{h}_0 + s(h_0 - \tilde{h}_0)) ds \right) (h_0^y - \tilde{h}_0^y). \end{aligned}$$

Let us denote, for  $s \in [0, 1]$

$$\begin{aligned} \xi_s &= \xi_s(h_0, \tilde{h}_0) = K + \tilde{h}_0 + s(h_0 - \tilde{h}_0), \\ \varphi &= \varphi(h_0, \tilde{h}_0) = \int_0^1 p' \circ \xi_s^x ds + (K^y + h_0^y) \int_0^1 q' \circ \xi_s^x ds + \int_0^1 (D_1 u + D_1 g) \circ \xi_s ds, \\ \psi &= \psi(h_0, \tilde{h}_0) = q \circ (K^x + \tilde{h}_0^x) + \int_0^1 (D_2 u + D_2 g) \circ \xi_s ds, \end{aligned}$$

so that we have

$$\|\mathcal{N}_{0,F}^y(h_0) - \mathcal{N}_{0,F}^y(\tilde{h}_0)\|_s \leq \|\varphi(h_0, \tilde{h}_0)(h_0^x - \tilde{h}_0^x)\|_s + \|\psi(h_0, \tilde{h}_0)(h_0^y - \tilde{h}_0^y)\|_s. \quad (3.4.4)$$

For case 1 we have  $K \in \mathcal{Y}_{2,k+1}$  and, since  $s = 2r$  and  $r > k$ , then for every  $h_0, \tilde{h}_0 \in \Sigma_{0,k}^\alpha$  we have  $(h_0, \tilde{h}_0) \in \mathcal{Y}_{4,k+2}$ . Thus we can bound the norm

$$\|\xi_s^x\|_2 = \sup_{t \in (0, \rho)} \frac{1}{t^2} |K^x(t) + \tilde{h}_0^x(t) + s(h_0^x(t) - \tilde{h}_0^x(t))| \leq 1 + M\rho,$$

for all  $s \in [0, 1]$ .

Moreover, checking the orders of  $\varphi$  and  $\psi$ , taking into account the properties of  $p$ ,  $q$ ,  $u$  and  $g$ , we have

$$\varphi \in \mathcal{Y}_{2k-2}, \quad \psi \in \mathcal{Y}_k \subset \mathcal{Y}_{k-1}, \quad \forall h_0, \tilde{h}_0 \in \Sigma_{0,k}^\alpha.$$

More precisely, we can bound

$$\begin{aligned} \|\varphi\|_{2k-2} &\leq \sup_{s \in [0,1]} (\|p' \circ \xi_s^x\|_{2k-2} + \|(K^y + h_0^y) q' \circ \xi_s^x + D_1 g \circ \xi_s + D_1 u \circ \xi_s\|_{2k-2}) \\ &\leq \sup_{s \in [0,1]} \sup_{t \in (0,\rho)} \frac{1}{t^{2k-2}} (k |a_k| |\xi_s^x(t)|^{k-1} + M t^{2k-1}) \\ &\leq k |a_k| + M\rho, \end{aligned} \tag{3.4.5}$$

$$\|\psi\|_{k-1} \leq M\rho, \tag{3.4.6}$$

for all  $h_0, \tilde{h}_0 \in \Sigma_{0,k}^{\alpha_0}$ .

Then, from (3.4.4) we have

$$\begin{aligned} \|\mathcal{N}_{0,F}^y(h_0) - \mathcal{N}_{0,F}^y(\tilde{h}_0)\|_s &\leq \|\varphi\|_{2k-2} \|h_0^x - \tilde{h}_0^x\|_{s-2k+2} + \|\psi\|_{k-1} \|h_0^y - \tilde{h}_0^y\|_{s-k+1} \\ &\leq (k |a_k| + M\rho) \|h_0^x - \tilde{h}_0^x\|_{s-2k+2} + \rho M \|h_0^y - \tilde{h}_0^y\|_{s-k+1}, \end{aligned}$$

which proves that  $\text{Lip } \mathcal{N}_{0,F}^y \leq k |a_k| + M\rho$ , for case 1.

For cases 2 and 3 the bounds for  $\text{Lip } \mathcal{N}_{0,F}^y$  are obtained in an analogous way. In these cases we have  $K \in \mathcal{Y}_{1,l}$  and we obtain  $\xi_s \in \mathcal{Y}_{2,l+1}$ . Take  $h_0, \tilde{h}_0 \in \Sigma_{0,l}^\alpha$ . Since  $r > 2l - 1$ ,

$$\varphi \in \mathcal{Y}_{2l-2}, \quad \psi \in \mathcal{Y}_{l-1},$$

with the following bounds for their norms,

$$\|\varphi\|_{2l-2} \leq k |a_k| + (l-1) |K_l^y b_l| + M\rho, \quad \|\psi\|_{l-1} \leq |b_l| + M\rho, \tag{3.4.7}$$

in case 2 and

$$\|\varphi\|_{2l-2} \leq (l-1) |K_l^y b_l| + M\rho, \quad \|\psi\|_{l-1} \leq |b_l| + M\rho, \tag{3.4.8}$$

in case 3.

The proof for  $L \geq 1$  is similar. Given  $f_0, \dots, f_{L-1}$  and  $h_L, \tilde{h}_L \in D\Sigma_{L-1,N}$ , from the definition of  $\mathcal{N}_{L,N}^y$ , we have

$$\begin{aligned} \mathcal{N}_{L,F}^y(f_0, \dots, f_{L-1}, h_L) - \mathcal{N}_{L,F}^y(f_0, \dots, f_{L-1}, \tilde{h}_L) &= (p' \circ (K^x + f_0^x) + (K^y + f_0^y) q' \circ (K^x + f_0^x) \\ &\quad + (D_1 u + D_1 g) \circ (K + f_0))(h_L^x - \tilde{h}_L^x) \\ &\quad + (q \circ (K^x + f_0^x) + (D_2 u + D_2 g) \circ (K + f_0))(h_L^y - \tilde{h}_L^y). \end{aligned}$$

Given  $f_0 \in \Sigma_{0,N}^\alpha$ , we denote

$$\begin{aligned} \tilde{\varphi} &= \tilde{\varphi}(f_0) = p' \circ (K^x + f_0^x) + (K^y + f_0^y) q' \circ (K^x + f_0^x) + (D_1 u + D_1 g) \circ (K + f_0), \\ \tilde{\psi} &= \tilde{\psi}(f_0) = q \circ (K^x + f_0^x) + (D_2 u + D_2 g) \circ (K + f_0), \end{aligned}$$



so that we can write

$$\begin{aligned} & \|\mathcal{N}_{L,F}^y(f_0, \dots, f_{L-1}, h_L) - \mathcal{N}_{L,F}^y(f_0, \dots, f_{L-1}, \tilde{h}_L)\|_s \\ & \leq \|\tilde{\varphi}(f_0)(h_L^x - \tilde{h}_L^x)\|_s + \|\tilde{\psi}(f_0)(h_L^y - \tilde{h}_L^y)\|_s. \end{aligned}$$

The orders of  $\tilde{\varphi}$  and  $\tilde{\psi}$  are the same as the ones of the corresponding  $\varphi$  and  $\psi$  when  $L = 0$ , respectively, for each of the cases 1, 2 and 3. That is,

$$\tilde{\varphi} \in \mathcal{Y}_{2k-2}, \quad \tilde{\psi} \in \mathcal{Y}_k \subset \mathcal{Y}_{k-1},$$

for case 1 and

$$\tilde{\varphi} \in \mathcal{Y}_{2l-2}, \quad \tilde{\psi} \in \mathcal{Y}_{l-1},$$

for cases 2 and 3. As in the case  $L = 0$ , for each  $f_0 \in \Sigma_{0,N}^{\alpha_0}$ , the order of  $K + f_0$  is the same as the one of  $K$ . Therefore we get the same bounds for the norms of  $\tilde{\varphi}$  and  $\tilde{\psi}$ , namely those obtained in (3.4.5) - (3.4.8), and finally the bounds in the statement.  $\square$

Note that the bound we have found for  $\text{Lip } \mathcal{N}_{L,F}(f_0, \dots, f_{L-1}, \cdot)$  does not depend on  $L$ , and the obtained bounds for  $\|(\mathcal{S}_{0,R}^x)^{-1}\|$  and  $\|(\mathcal{S}_{0,R}^y)^{-1}\|$  do not depend on  $\kappa$ .

### 3.5 Main lemmas and the fiber contraction theorem

From  $\mathcal{S}_{L,R}$  and  $\mathcal{N}_{L,F}$  introduced in Section 3.4, we can define the operators  $\mathcal{T}_{L,F}$  and  $\mathcal{T}_{L,F}^\times$ .

**Definition 3.5.1.** *Given a map  $F$  of class  $C^r$  satisfying the hypotheses of Theorem 3.2.1, let  $\mathcal{T}_{L,F} : \Sigma_{L,N}^\alpha \rightarrow D\Sigma_{L-1,N}$  be the operator given by*

$$\mathcal{T}_{L,F} = \mathcal{S}_{L,R}^{-1} \circ \mathcal{N}_{L,F}, \quad L \in \{0, \dots, r\},$$

and let  $\mathcal{T}_{L,F}^\times : \Sigma_{L,N}^\alpha \rightarrow \Sigma_{L,N}$  be the operator given by

$$\mathcal{T}_{L,F}^\times = (\mathcal{T}_{0,F}, \dots, \mathcal{T}_{L,F}), \quad L \in \{1, \dots, r\}.$$

In the following results we show that, under appropriate conditions, the operators  $\mathcal{T}_{L,F}$  have some properties strongly related to the hypotheses of the fiber contraction theorem.

**Lemma 3.5.2.** *Let  $F$  be a  $C^r$  map satisfying the hypotheses of Theorem 3.2.1,  $\alpha_i > 0$ ,  $1 \leq i \leq r$ , and  $\alpha = (\alpha_0, \dots, \alpha_L)$ ,  $0 \leq L \leq r$ . Then, for every  $L \in \{0, \dots, r-1\}$ , the operator  $\mathcal{T}_{L,F} : \Sigma_{L,N}^\alpha \rightarrow D\Sigma_{L-1,N}$  is Lipschitz on  $\Sigma_{L,N}^\alpha$  with respect to  $(f_0, \dots, f_{L-1})$ , with Lipschitz constant independent of  $f_L$ .*

Moreover, the operator  $\mathcal{T}_{r,F} : \Sigma_{r,N}^\alpha \rightarrow D\Sigma_{r-1,N}$  can be decomposed as  $\mathcal{T}_{r,F}^{(1)} + \mathcal{T}_{r,F}^{(2)}$ , where  $\mathcal{T}_{r,F}^{(1)}$  is Lipschitz on  $\Sigma_{r,N}^\alpha$  with respect to  $(f_0, \dots, f_{r-1})$ , with Lipschitz constant independent of  $f_r$  and

$$\mathcal{T}_{r,F}^{(2)} = \left(0, (\mathcal{S}_{r,R}^y)^{-1} \circ (D^r g \circ (K + f_0)(DK + f_1)^r)\right),$$

which is continuous with respect to  $(f_0, f_1)$ .

*Proof.* As before, to distinguish the roles of the variables  $f_L$  and  $(f_0, \dots, f_{L-1})$  we will denote the former by  $h_L$ . Since  $\mathcal{T}_{L,F} = \mathcal{S}_{L,R}^{-1} \circ \mathcal{N}_{L,F}$  and  $\mathcal{S}_{L,R}^{-1}$  is linear and bounded, along the proof we will deal only with  $\mathcal{N}_{L,F}$ .

Given a function  $h_L \in D\Sigma_{L-1,N}^{\alpha_L}$  we decompose

$$\mathcal{N}_{L,F}(f_0, \dots, f_{L-1}, h_L) = \mathcal{A}_{h_L,F}(f_0) + \mathcal{J}_{L,F}(f_0, \dots, f_{L-1}),$$

where  $\mathcal{A}_{h_L,F} := (\mathcal{A}_{h_L,F}^x, \mathcal{A}_{h_L,F}^y) : \Sigma_{0,N}^\alpha \rightarrow \mathcal{Y}_{s-N+1-L, s-L}$  is the auxiliary operator

$$\begin{aligned} \mathcal{A}_{h_L,F}^x(f_0) &= c h_L^y, \\ \mathcal{A}_{h_L,F}^x(f_0) &= p' \circ (K^x + f_0^x) \cdot h_L^x + (K^y + f_0^y) \cdot q' \circ (K^x + f_0^x) h_L^x \\ &\quad + q \circ (K^x + f_0^x) \cdot h_L^y + (Du + Dg) \circ (K + f_0) \cdot h_L, \end{aligned}$$

and we will work on  $\mathcal{A}_{h_L,F}$  and  $\mathcal{J}_{L,F}$  separately.

Clearly  $\mathcal{A}_{h_L,F}^x$  is uniformly Lipschitz on  $\Sigma_{0,N}^\alpha$ . To deal with  $\mathcal{A}_{h_L,F}^y$ , let  $f_0, \tilde{f}_0 \in \Sigma_{0,N}^\alpha$ . Then

$$\begin{aligned} \mathcal{A}_{h_L,F}^y(f_0) - \mathcal{A}_{h_L,F}^y(\tilde{f}_0) &= \varphi_{h_L}(f_0, \tilde{f}_0)(f_0^x - \tilde{f}_0^x) + \psi_{h_L}(f_0, \tilde{f}_0)(f_0^y - \tilde{f}_0^y) \\ &\quad + \theta(f_0, \tilde{f}_0)(f_0 - \tilde{f}_0) \cdot h_L, \end{aligned}$$

with

$$\begin{aligned} \varphi_{h_L} &= \varphi_{h_L}(f_0, \tilde{f}_0) = h_L^x \int_0^1 p'' \circ (K^x + \tilde{f}_0^x + s(f_0^x - \tilde{f}_0^x)) ds \\ &\quad + h_L^y \int_0^1 q' \circ (K^x + \tilde{f}_0^x + s(f_0^x - \tilde{f}_0^x)) ds \\ &\quad + h_L^x (K^y + f_0^y) \int_0^1 q'' \circ (K^x + \tilde{f}_0^x + s(f_0^x - \tilde{f}_0^x)) ds, \\ \psi_{h_L} &= \psi_{h_L}(f_0, \tilde{f}_0) = h_L^x q' \circ (K^x + \tilde{f}_0^x), \\ \theta &= \theta(f_0, \tilde{f}_0) = \int_0^1 (D^2u + D^2g) \circ (K + \tilde{f}_0 + s(f_0 - \tilde{f}_0)) ds. \end{aligned}$$

First we deal with case 1. By similar arguments as in Lemma 3.4.7 we have  $\varphi_{h_L}(f_0, \tilde{f}_0) \in \mathcal{Y}_{2r-2-L} \subseteq \mathcal{Y}_{2k-2-L}$ ,  $\psi_{h_L}(f_0, \tilde{f}_0) \in \mathcal{Y}_{r-1-L} \subseteq \mathcal{Y}_{k-1-L}$ . All the entries of the matrix  $\theta(f_0, \tilde{f}_0)$  belong to  $\mathcal{Y}_0$ . Also, it is clear that the quantities  $\|\varphi_{h_L}(f_0, \tilde{f}_0)\|_{2k-2-L}$ ,  $\|\psi_{h_L}(f_0, \tilde{f}_0)\|_{k-1-L}$  and the  $\|\cdot\|_{\mathcal{Y}_0}$ -norm of the entries of  $\theta(f_0, \tilde{f}_0)$  are uniformly bounded for  $f_0, \tilde{f}_0 \in \Sigma_{0,N}^\alpha$ , the norm depending on  $\alpha_0$  in the form  $\rho^m \alpha_0$  for some  $m > 0$  and depending linearly on  $\alpha_L$ .

Then, since  $h_L$  is fixed, we get

$$\begin{aligned} \|\mathcal{A}_{h_L,F}^y(f_0) - \mathcal{A}_{h_L,F}^y(\tilde{f}_0)\|_{2r-L} &\leq \|\varphi_{h_L}(f_0, \tilde{f}_0)\|_{2k-2-L} \|f_0^x - \tilde{f}_0^x\|_{2r-2k+2} \\ &\quad + \|\psi_{h_L}(f_0, \tilde{f}_0)\|_{k-1-L} \|f_0^y - \tilde{f}_0^y\|_{2r-k+1} \\ &\quad + M \|h_L\|_{D\Sigma_{L-1,k}} \|f_0 - \tilde{f}_0\|_{\Sigma_{0,k}} \\ &\leq M \alpha_L \|f_0 - \tilde{f}_0\|_{\Sigma_{0,k}}. \end{aligned}$$

Similarly we also obtain  $\|\mathcal{A}_{h_L,F}^y(f_0) - \mathcal{A}_{h_L,F}^y(\tilde{f}_0)\|_{2r-L} \leq M \alpha_L \|f_0 - \tilde{f}_0\|_{\Sigma_{0,k}}$  for cases 2 and 3, where in these cases we have  $\varphi_{h_L} \in \mathcal{Y}_{r-L-1}$ ,  $\psi_{h_L} \in \mathcal{Y}_{r-l-L}$  and the entries of  $\theta$  belong to  $\mathcal{Y}_0$ . This proves that  $\mathcal{A}_{h_L,F}$  is uniformly Lipschitz on  $\Sigma_{0,N}^\alpha$ .

Next we deal with  $\mathcal{J}_{L,F}$ . Recall that we have, for every  $L \in \{1, \dots, r\}$ ,

$$\begin{aligned}\mathcal{J}_{L,F}^x(f_0, \dots, f_{L-1}) &= \Lambda_{L,R}^x(f_0^x, \dots, f_{L-1}^x), \\ \mathcal{J}_{L,F}^y(f_0, \dots, f_{L-1}) &= \Lambda_{L,R}^y(f_0^y, \dots, f_{L-1}^y) + \Omega_{L,F}(f_0, \dots, f_{L-1}),\end{aligned}$$

where  $\Lambda_{L,R}^x$  and  $\Lambda_{L,R}^y$  are given recursively in (3.3.6) and  $\Omega_{L,F}$  is given recursively in (3.3.7).

From (3.3.6),  $\Lambda_{1,R}^i = 0$  and, for  $L \geq 2$ , one can check by induction that

$$\Lambda_{L,R}^i(f_1^i, \dots, f_{L-1}^i) = \sum_{j=1}^{L-1} P_{L,j} f_j^i \circ R, \quad i = x, y, \quad (3.5.1)$$

where each function  $P_{L,j}$  is a polynomial on the variable  $t$ .

Indeed,  $P_{2,1}(t) = -D^2R(t) \in \mathcal{Y}_{N-2}$ . Assuming (3.5.1) and applying the recurrence (3.3.6) we have

$$\begin{aligned}\Lambda_{L+1,R}^i &= \sum_{j=1}^{L-1} P'_{L,j} f_j^i \circ R + \sum_{j=1}^{L-1} P_{L,j} f_{j+1}^i \circ R DR - L f_L^i \circ R (DR)^{L-1} D^2R \\ &= P'_{L,1} f_1^i \circ R + \sum_{j=2}^{L-1} (P'_{L,j} + P_{L,j-1} DR) f_j^i \circ R \\ &\quad + (P_{L,L-1} DR - L (DR)^{L-1} D^2R) f_L^i \circ R.\end{aligned}$$

We also have the recurrences

$$\begin{aligned}P_{L+1,1}(t) &= P'_{L,1}(t), \\ P_{L+1,j}(t) &= P'_{L,j} + P_{L,j-1} DR, \quad 2 \leq j \leq L-1, \\ P_{L+1,L}(t) &= P_{L,L-1} DR - L (DR)^{L-1} D^2R,\end{aligned}$$

and then we also deduce by induction that  $P_{L,j} = \mathcal{Y}_{N+j-1-L}$ .

From this, it is clear that  $\Lambda_{L,R} = (\Lambda_{L,R}^x, \Lambda_{L,R}^y) : \Sigma_{L-1,N}^\alpha \rightarrow \mathcal{Y}_{s-N+1-L,s-L}$  is linear and bounded, so it is uniformly Lipschitz in  $\Sigma_{L-1,N}^\alpha$ .

Also, from (3.3.7), one can see that  $\Omega_{L,F}$  is a polynomial operator on the variables  $f_1, \dots, f_{L-1}$  having coefficients depending on  $f_0$ .

When  $L = 1$ ,

$$\begin{aligned}\Omega_{1,F}(f_0) - \Omega_{1,F}(\tilde{f}_0) &= DK^x \int_0^1 (p'' \circ (K^x + \tilde{f}_0^x + s(f_0^x - \tilde{f}_0^x))) ds (f_0^x - \tilde{f}_0^x) \\ &\quad + DK^y \int_0^1 (q' \circ (K^x + \tilde{f}_0^x + s(f_0^x - \tilde{f}_0^x))) ds (f_0^x - \tilde{f}_0^x) \\ &\quad + (K^y + \tilde{f}_0^y) DK^x \int_0^1 q'' \circ (K^x + \tilde{f}_0^x + s(f_0^x - \tilde{f}_0^x)) ds (f_0^x - \tilde{f}_0^x) \\ &\quad + (f_0^y - \tilde{f}_0^y) DK^x q' \circ (K^x + f_0^x) \\ &\quad + DK^x \int_0^1 (D^2u + D^2g) \circ (K + \tilde{f}_0 + s(f_0 - \tilde{f}_0)) ds (f_0 - \tilde{f}_0)\end{aligned}$$

and hence there exists  $M > 0$  depending on  $F$  and  $\alpha_0$  such that

$$\|\Omega_{1,F}(f_0) - \Omega_{1,F}(\tilde{f}_0)\|_{s-1} \leq M \|f_0 - \tilde{f}_0\|_{\Sigma_{0,N}}.$$

For  $L > 1$ , we decompose  $\Omega_{L,F} = \Omega_{L,F}^{(1)} + \Omega_{L,F}^{(2)}$ , where

$$\Omega_{L,F}^{(2)} = \Omega_{L,F}^{(2)}(f_0, f_1) = D^L g \circ (K + f_0)(DK + f_1)^L,$$

and  $\Omega_{L,F}^{(1)} = \Omega_{L,F} - \Omega_{L,F}^{(2)}$ . The difference  $\Omega_{L,F}^{(1)}(f_0, \dots, f_{L-1}) - \Omega_{L,F}^{(1)}(\tilde{f}_0, \dots, \tilde{f}_{L-1})$  is a sum of terms of the form  $c_L(f_0, \tilde{f}_0)\Pi$ , where  $\Pi$  is a product of factors among  $f_j^{x,y}$ ,  $\tilde{f}_j^{x,y}$  and  $f_j^{x,y} - \tilde{f}_j^{x,y}$  and such that  $c_L(f_0, \tilde{f}_0)\Pi \in \mathcal{Y}_{s-L}$ . From (3.3.7) we estimate  $\Omega_{L,F}^{(1)}(f_0, \dots, f_{L-1}) - \Omega_{L,F}^{(1)}(\tilde{f}_0, \dots, \tilde{f}_{L-1})$  iteratively, where a part of it comes from

$$D[\Omega_{L-1,F}^{(1)}(f_0, \dots, f_{L-2}) - \Omega_{L-1,F}^{(1)}(\tilde{f}_0, \dots, \tilde{f}_{L-2})].$$

When one differentiates formally the terms  $c_{L-1}(f_0)\Pi$ , the new terms  $c_{L-1}(f_0, \tilde{f}_0)'\Pi$  and  $c_{L-1}(f_0, \tilde{f}_0)\Pi'$  appear.

The factors of each function  $c_L(f_0, \tilde{f}_0)$  are derivatives of  $K^i$ ,  $f_j^i$ ,  $\tilde{f}_j^i$ ,  $\int_0^1 (Q_1(K^i + \tilde{f}_0^i + s(f_0^i - \tilde{f}_0^i)) ds (f_0^i - \tilde{f}_0^i)$  and  $(Q_2(K^i + \tilde{f}_0^i))$ , where  $Q_1, Q_2$  are polynomials (derivatives of  $p, q$  or  $u$ ), and the derivative of

$$\int_0^1 D^m g \circ (K + \tilde{f}_0 + s(f_0 - \tilde{f}_0)) ds (f_0 - \tilde{f}_0), \quad m \leq L - 1. \quad (3.5.2)$$

When taking a derivative, each term generates several terms, each one having bigger order, the same order or the same order minus one unit. The term  $\Omega_{L,F}^{(2)}$  is Lipschitz when  $L < r$ . When  $L = r$ , it is continuous (in the given topology) since  $D^r g$  is uniformly continuous in closed balls.

On the other hand, when taking a derivative to  $\Pi$  we obtain terms which have the same factors except one which is transformed to its derivative, that is,  $f_j^{x,y}$  is transformed to  $f_{j+1}^{x,y}$  or  $f_j^{x,y} - \tilde{f}_j^{x,y}$  is transformed to  $f_{j+1}^{x,y} - \tilde{f}_{j+1}^{x,y}$ . In any case the order decreases by one unit so we have that their  $\|\cdot\|_{s-L}$ -norm is bounded by  $M_L \|(f_0, \dots, f_{L-1}) - (\tilde{f}_0, \dots, \tilde{f}_{L-1})\|_{\Sigma_{L,N}}$ , where the constant  $M_L$  depends on  $\alpha_0, \dots, \alpha_L$  and  $F$  but not on the  $(f_j^i)'$ s.  $\square$

Next we introduce a convenient rescaling. Given  $\gamma > 0$ , let

$$T_\gamma(x, y) = (x, \gamma y). \quad (3.5.3)$$

We define  $\tilde{F} = T_\gamma^{-1} \circ F \circ T_\gamma$ . If  $K$  and  $R$  are analytic maps associated to  $F$ , then the corresponding analytic maps associated to  $\tilde{F}$  will be given by  $\tilde{K} = T_\gamma^{-1} \circ K$  and  $\tilde{R} = R$ . Concretely, the parameterizations of  $\tilde{F}$  and  $\tilde{K}$  with respect to the coefficients of  $F$  and  $K$  will be given by

$$\tilde{F}(x, y) = \begin{pmatrix} x + \gamma cy \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \gamma^{-1} a_k x^k + b_l y x^{l-1} + \dots \end{pmatrix},$$

and

$$\begin{aligned}\tilde{K}(t) &= \begin{pmatrix} t^2 + \dots \\ \gamma^{-1} K_{k+1}^y t^{k+1} + \dots \end{pmatrix}, & \text{for case 1,} \\ \tilde{K}(t) &= \begin{pmatrix} t + \dots \\ \gamma^{-1} K_l^y t^l + \dots \end{pmatrix}, & \text{for cases 2 and 3.}\end{aligned}$$

**Lemma 3.5.3.** *Given a  $C^r$  map  $F$  satisfying the hypotheses of Theorem 3.2.1, there exist  $\rho_0 > 0$  and a linear transformation  $T_\gamma$  as in (3.5.3) such that if  $\rho < \rho_0$ , then the operator  $\mathcal{T}_{L, \tilde{F}} : \Sigma_{L, N}^\alpha \rightarrow D\Sigma_{L-1, N}$  associated to  $\tilde{F} = T_\gamma^{-1} \circ F \circ T_\gamma$ , for  $L \in \{0, \dots, r\}$ , is contractive with respect to the variable  $f_L \in D\Sigma_{L-1, N}^\alpha$ . Moreover, for a proper choice of  $\alpha = (\alpha_0, \dots, \alpha_L)$ ,  $\mathcal{T}_{L, \tilde{F}}$  maps  $\Sigma_{L, N}^\alpha$  into  $D\Sigma_{L-1, N}^{\alpha_L}$ , for each  $L \in \{0, \dots, r\}$ .*

*Proof.* By its definition, the operator  $\mathcal{T}_{L, F}$  satisfies

$$\begin{aligned}\text{Lip } \mathcal{T}_{L, F}(f_0, \dots, f_{L-1}, \cdot) &\leq \max\{\|(\mathcal{S}_{L, R}^x)^{-1}\| \text{Lip } \mathcal{N}_{L, F}^x(f_0, \dots, f_{L-1}, \cdot), \\ &\quad \|(\mathcal{S}_{L, R}^y)^{-1}\| \text{Lip } \mathcal{N}_{L, F}^y(f_0, \dots, f_{L-1}, \cdot)\}.\end{aligned}\tag{3.5.4}$$

From the estimates obtained in Lemmas 3.4.6 and 3.4.7 we have that the bounds of  $\text{Lip } \mathcal{N}_{L, F}(f_0, \dots, f_{L-1}, \cdot)$  do not depend on  $L$ , and taking  $\kappa < 1$  close to 1 the obtained bounds for  $\|\mathcal{S}_{L, R}^{-1}\|$  decrease as  $L$  increases, so that it holds

$$\text{Lip } \mathcal{T}_{L, F}(f_0, \dots, f_{L-1}, \cdot) \leq \text{Lip } \mathcal{T}_{0, F}, \quad \forall L \in \{0, \dots, r\}.$$

Actually, this inequality is for the obtained bounds for the Lipschitz constants of the family  $\{\mathcal{T}_{L, F}\}_L$ . Note also that  $\text{Lip } \mathcal{T}_{0, F}$  does not depend on  $\kappa$ .

To prove the first part of the lemma we will find an appropriate map  $T_\gamma$  given in (3.5.3) (that is, an appropriate value for  $\gamma$ ) such that if the coefficients of  $F$  satisfy the hypotheses of Theorem 3.2.1, then the corresponding operator  $\mathcal{T}_{L, \tilde{F}}$  associated to  $\tilde{F} = T_\gamma^{-1} \circ F \circ T_\gamma$  satisfies  $\text{Lip } \mathcal{T}_{L, \tilde{F}}(f_0, \dots, f_{L-1}, \cdot) < 1$ .

We start by considering case 1. From (5.7.9) and the estimates obtained in Lemmas 3.4.6 and 3.4.7, given  $\nu \in (0, (k-1)|R_k|)$  there is  $\tilde{\rho}_0$  such that for  $\rho < \tilde{\rho}_0$  we have the bound

$$\begin{aligned}\text{Lip } \mathcal{T}_{L, \tilde{F}}(f_0, \dots, f_{L-1}, \cdot) &\leq \max\left\{ \left(\rho^{k-1} + \frac{1}{\nu} \frac{k-1}{2r-2k+2}\right) \gamma |c|, \right. \\ &\quad \left. \left(\rho^{k-1} + \frac{1}{\nu} \frac{k-1}{2r-k+1}\right) (\gamma^{-1} k a_k + M \rho) \right\}.\end{aligned}$$

Clearly, the condition

$$\max\left\{ \gamma \frac{|c|}{|R_k|} \frac{1}{2r-2k+2}, \gamma^{-1} \frac{k a_k}{|R_k|} \frac{1}{2r-k+1} \right\} < 1, \tag{3.5.5}$$

is sufficient to ensure that there exists  $0 < \rho_0 < \tilde{\rho}_0$  such that  $\text{Lip } \mathcal{T}_{L, \tilde{F}}(f_0, \dots, f_{L-1}, \cdot) < 1$  for  $\rho < \rho_0$ , since keeping  $\kappa$  fixed one can choose a value for  $\nu$  close enough to  $(k-1)|R_k|$ .

Then, taking  $\gamma = \sqrt{\frac{k a_k}{c} \frac{2r-2k+2}{2r-k+1}}$ , condition (3.5.5) is given by

$$\frac{2k(k+1)}{(2r-2k+2)(2r-k+1)} < 1,$$

which holds for any  $k \geq 2$  and  $r \geq \frac{3}{2}k$ . Hence, if  $r \geq \frac{3}{2}k$ , the operator  $\mathcal{T}_{L, \tilde{F}}$  associated to  $\tilde{F} = T_\gamma^{-1} \circ F \circ T_\gamma$  for the chosen value of  $\gamma$  satisfies  $\text{Lip } \mathcal{T}_{L, \tilde{F}}(f_0, \dots, f_{L-1}, \cdot) < 1$ , for every  $L \in \{0, \dots, r\}$ , provided that  $\rho < \rho_0$ .

For cases 2 and 3 of the reduced form of  $F$  the result follows in a similar way choosing an appropriate value for the parameter  $\gamma$ .

For case 2 we have, from (5.7.9) and the estimates obtained in Lemmas 3.4.6 and 3.4.7, that the condition

$$\max \left\{ \gamma \frac{|c|}{|R_l|} \frac{1}{r-2l+2}, \gamma^{-1} \frac{(l-1)|K_l^y b_l| + k a_k}{|R_l|} \frac{1}{r-l+1}, \frac{|b_l|}{|R_l|} \frac{1}{r-l+1} \right\} < 1, \quad (3.5.6)$$

is sufficient to ensure that there exists  $\rho_0 > 0$  such that  $\text{Lip } \mathcal{T}_{L,F}(f_0, \dots, f_{l-1}, \cdot) < 1$  for  $\rho < \rho_0$ .

Then, taking  $\gamma = \sqrt{\frac{(l-1)|K_l^y b_l| + k a_k}{c} \frac{r-2l+2}{r-l+1}}$ , condition (3.5.6) is given by

$$\max \left\{ \frac{\beta}{(r-2l+2)(r-l+1)} \left( (l-1) + \frac{c k a_k}{b_l^2} \beta \right), \frac{\beta}{r-l+1} \right\} < 1,$$

where  $\beta = \frac{2l|b_l|}{|b_l - \sqrt{b_l^2 + 4c a_k l}|}$ , which is the condition for  $F$  assumed for case 2.

For case 3 we have, again from (5.7.9) and the estimates obtained in Lemmas 3.4.6 and 3.4.7, that the condition

$$\max \left\{ \gamma \frac{|c|}{|R_l|} \frac{1}{r-2l+2}, \gamma^{-1} \frac{(l-1)|K_l^y b_l|}{|R_l|} \frac{1}{r-l+1}, \frac{|b_l|}{|R_l|} \frac{1}{r-l+1} \right\} < 1, \quad (3.5.7)$$

is sufficient to ensure that there exists  $\rho_0 > 0$  such that  $\text{Lip } \mathcal{T}_{L,F}(f_0, \dots, f_{l-1}, \cdot) < 1$  for  $\rho < \rho_0$ .

Taking  $\gamma = \frac{|b_l|}{|c|} \sqrt{\frac{(l-1)(r-2l+2)}{l(r-l+1)}}$ , condition (3.5.7) is given by

$$\max \left\{ \frac{l(l-1)}{(r-2l+2)(r-l+1)}, \frac{l}{r-l+1} \right\} < 1,$$

that is,

$$\frac{l(l-1)}{(r-2l+2)(r-l+1)} < 1,$$

which is the condition for  $F$  assumed for case 3.

Finally we prove that given a map  $F$  satisfying the hypotheses of Theorem 3.2.1 such that the associated operators  $\mathcal{T}_{L,F}$  satisfy  $\text{Lip } \mathcal{T}_{L,F}(f_0, \dots, f_{L-1}, \cdot) < 1$  for  $\rho < \rho_0$ , one can find a new  $\rho_0$ , maybe smaller than the previous one, and a choice for the values  $\alpha_1, \dots, \alpha_r$  such that, if  $\rho < \rho_0$ , then  $\mathcal{T}_{L,F}$  maps  $\Sigma_{L,N}^\alpha$  into  $D\Sigma_{L-1,N}^{\alpha_L}$ , for every  $L \in \{0, \dots, r\}$ .

For later use, we estimate  $\|\mathcal{T}_{L,F}(0, \dots, 0)\|_{D\Sigma_{L-1,N}}$ . From Definition 3.4.5 of  $\mathcal{N}_{L,F}$  and the definition of  $\mathcal{J}_{L,F}$  in (3.3.5) we have

$$\mathcal{N}_{L,F}(0, \dots, 0) = \mathcal{J}_{L,F}(0, \dots, 0) = (0, D^L(g \circ K)).$$

Moreover  $D^L(g \circ K)(t) = o(|t|^{s-L})$ . Therefore, for every  $\varepsilon > 0$ , there is  $\rho_0 > 0$  such that if  $\rho < \rho_0$ , then

$$\begin{aligned} \|\mathcal{T}_{L,F}(0, \dots, 0)\|_{D\Sigma_{L-1,N}} &\leq \|(\mathcal{S}_{L,R}^y)^{-1}\| \|\mathcal{N}_{L,F}^y(0, \dots, 0)\|_{s-N+1-L, s-L} \\ &\leq \|(\mathcal{S}_{L,R}^y)^{-1}\| \sup_{t \in (0, \rho)} \frac{|D^L(g \circ K)(t)|}{t^{s-L}} \leq \|(\mathcal{S}_{L,R}^y)^{-1}\| \varepsilon. \end{aligned} \quad (3.5.8)$$

Next we proceed by induction. For  $L = 0$ , we have, for all  $f_0 \in \Sigma_{0,N}^{\alpha_0}$ ,

$$\begin{aligned} \|\mathcal{T}_{0,F}(f_0)\|_{\Sigma_{0,N}} &\leq \|\mathcal{T}_{0,F}(f_0) - \mathcal{T}_{0,F}(0)\|_{\Sigma_{0,N}} + \|\mathcal{T}_{0,F}(0)\|_{\Sigma_{0,N}} \\ &\leq \alpha_0 \text{Lip } \mathcal{T}_{0,F} + \|\mathcal{T}_{0,F}(0)\|_{\Sigma_{0,N}}. \end{aligned}$$

We need to see then that there exists  $\rho_0 > 0$  such that  $\|\mathcal{T}_{0,F}(f_0)\|_{\Sigma_{0,N}} \leq \alpha_0$  provided that  $\rho < \rho_0$ . Clearly this holds from the estimate obtained in (3.5.8) since we have  $\text{Lip } \mathcal{T}_{0,F} < 1$ , and then one can take  $\rho_0$  such that  $\alpha_0 \text{Lip } \mathcal{T}_{0,F} + \|\mathcal{T}_{0,F}(0)\|_{\Sigma_{0,N}} \leq \alpha_0$  for  $\rho < \rho_0$ . Hence we have  $\mathcal{T}_{0,F}(\Sigma_{0,N}^{\alpha_0}) \subseteq \Sigma_{0,N}^{\alpha_0}$ .

Now, we take  $\rho_1 < \rho_0$  and we denote by  $\varepsilon_L$  the quantity

$$\varepsilon_L = \|\mathcal{T}_{L,F}(0, \dots, 0)\|_{D\Sigma_{L-1,N}}, \quad L \in \{1, \dots, r\},$$

taking as the domain of the functions of  $\Sigma_{L,N}^\alpha$  the interval  $(0, \rho_1)$ .

Continuing with the induction procedure, for each  $L \in \{1, \dots, r\}$ , we decompose

$$\begin{aligned} \|\mathcal{T}_{L,F}(f_0, \dots, f_L)\|_{D\Sigma_{L-1,N}} &\leq \|\mathcal{T}_{L,F}(f_0, \dots, f_L) - \mathcal{T}_{L,F}(f_0, \dots, f_{L-1}, 0)\|_{D\Sigma_{L-1,N}} \\ &\quad + \|\mathcal{T}_{L,F}(f_0, \dots, f_{L-1}, 0) - \mathcal{T}_{L,F}(0, \dots, 0)\|_{D\Sigma_{L-1,N}} \\ &\quad + \|\mathcal{T}_{L,F}(0, \dots, 0)\|_{D\Sigma_{L-1,N}}. \end{aligned} \quad (3.5.9)$$

Also, from the definitions of  $\mathcal{T}_{L,F}$  and  $\mathcal{N}_{L,F}$  we have

$$\mathcal{T}_{L,F}(f_0, \dots, f_{L-1}, 0) = \mathcal{S}_{L,R}^{-1} \circ \mathcal{N}_{L,F}(f_0, \dots, f_{L-1}, 0) = \mathcal{S}_{L,R}^{-1} \circ \mathcal{J}_{L,F}(f_0, \dots, f_{L-1}).$$

Now we have to consider separately the cases  $L < r$  and  $L = r$ . For  $L < r$  we have, from Lemma 3.5.2, that  $\mathcal{T}_{L,F}(f_0, \dots, f_L)$  is uniformly Lipschitz with respect to  $(f_0, \dots, f_{L-1})$  in  $\Sigma_{L,N}^\alpha$ , and in particular,

$$\text{Lip } \mathcal{T}_{L,F}(\cdot, 0) = \text{Lip } (\mathcal{S}_{L,R}^{-1} \circ \mathcal{J}_{L,F}).$$

Therefore, from (3.5.9) we have

$$\begin{aligned} \|\mathcal{T}_{L,F}(f_0, \dots, f_L)\|_{D\Sigma_{L-1,N}} &\leq \text{Lip } \mathcal{T}_{L,F}(f_0, \dots, f_{L-1}, \cdot) \|f_L\|_{D\Sigma_{L-1,N}} \\ &\quad + \text{Lip } (\mathcal{S}_{L,R}^{-1} \circ \mathcal{J}_{L,F}) \|(f_0, \dots, f_{L-1})\|_{\Sigma_{L-1,N}} + \|\mathcal{T}_{L,F}(0, \dots, 0)\|_{D\Sigma_{L-1,N}} \\ &\leq \alpha_L \text{Lip } \mathcal{T}_{L,F}(f_0, \dots, f_{L-1}, \cdot) + \max\{\alpha_0, \dots, \alpha_{L-1}\} \text{Lip } (\mathcal{S}_{L,R}^{-1} \circ \mathcal{J}_{L,F}) + \varepsilon_L. \end{aligned} \quad (3.5.10)$$

Then we can choose a value for the radius  $\alpha_L$  of  $D\Sigma_{L-1,N}^{\alpha_L}$  to ensure that  $\mathcal{T}_{L,F}$  maps  $\Sigma_{L,N}^\alpha$  into  $D\Sigma_{L-1,N}^{\alpha_L}$ . Since we have  $\text{Lip } \mathcal{T}_{L,F}(f_0, \dots, f_{L-1}, \cdot) < 1$ , then taking

$$\alpha_L = \frac{\varepsilon_L + \text{Lip } (\mathcal{S}_{L,R}^{-1} \circ \mathcal{J}_{L,F}) \max \{\alpha_0, \dots, \alpha_{L-1}\}}{1 - \text{Lip } \mathcal{T}_{L,F}(f_0, \dots, f_{L-1}, \cdot)},$$

we have, applying (3.5.10),

$$\|\mathcal{T}_{L,F}(f_0, \dots, f_L)\|_{D\Sigma_{L-1,N}} \leq \alpha_L,$$

for each  $(f_0, \dots, f_L) \in \Sigma_{L,N}^\alpha$ , as we wanted to see.

For  $L = r$  we proceed in an analogous way, except for the fact that we use the decomposition  $\mathcal{T}_{r,F}^{(1)} + \mathcal{T}_{r,F}^{(2)}$  given in Lemma 3.5.2. Since  $\mathcal{T}_{r,F}^{(1)}$  is Lipschitz with respect to  $(f_0, \dots, f_r)$ , its contribution is as in the cases  $L < r$ . As we also have  $\mathcal{T}_{r,F}(f_0, \dots, f_{r-1}, 0) = \mathcal{S}_{r,R}^{-1} \circ \mathcal{J}_{r,F}(f_0, \dots, f_{r-1})$  and  $\mathcal{S}_{r,R}^{-1}$  is linear, we can denote  $\mathcal{T}_{r,F}^{(i)}(f_0, \dots, f_{r-1}, 0) = \mathcal{S}_{r,R}^{-1} \circ \mathcal{J}_{r,F}^{(i)}(f_0, \dots, f_{r-1})$ , for  $i = 1, 2$ , with  $\mathcal{J}_{r,F}^{(2)}(f_0, \dots, f_{r-1}) = (0, D^r g \circ (K + f_0)(DK + f_1)^r)$ .

We proceed as in (3.5.9), but now for the second term of the sum we have, applying Lemma 3.5.2,

$$\begin{aligned} & \|\mathcal{T}_{r,F}(f_0, \dots, f_{r-1}, 0) - \mathcal{T}_{r,F}(0, \dots, 0)\|_{D\Sigma_{r-1,N}} \\ & \leq \text{Lip } (\mathcal{S}_{r,R}^{-1} \circ \mathcal{J}_{r,F}^{(1)}) \|(f_0, \dots, f_{r-1})\|_{\Sigma_{r-1,N}} \\ & \quad + \|(\mathcal{S}_{r,R}^y)^{-1}\| \|D^r g \circ (K + f_0)(DK + f_1)^r\|_{s-r}. \end{aligned}$$

To bound the quantity  $\|D^r g \circ (K + f_0)(DK + f_1)^r\|_{s-r}$ , note that we have  $D^r g(x, y) = o(\|(x, y)\|^0)$ .

For case 1 of the reduced form of  $F$  we have  $(DK + f_1)^r \in \mathcal{Y}_r$  and thus, for every  $\varepsilon > 0$  there is  $\rho_0$  such that if  $\rho < \rho_0$ , then

$$\|D^r g \circ (K + f_0)(DK + f_1)^r\|_r = \sup_{t \in (0, \rho)} \frac{1}{t^r} |D^r g \circ (K + f_0)(t)(DK + f_1)^r(t)| < \varepsilon.$$

Similarly, for cases 2 and 3 we have  $(DK + f_1)^r \in \mathcal{Y}_0$  and

$$\|D^r g \circ (K + f_0)(DK + f_1)^r\|_0 = \sup_{t \in (0, \rho)} |D^r g \circ (K + f_0)(t)(DK + f_1)^r(t)| < \varepsilon.$$

Then, for the chosen radius  $\rho_1$  we denote  $\hat{\varepsilon} = \|D^r g \circ (K + f_0)(DK + f_1)^r\|_{s-r}$  and similarly as in (3.5.10) we have

$$\begin{aligned} & \|\mathcal{T}_{r,F}(f_0, \dots, f_r)\|_{D\Sigma_{r-1,N}} \\ & \leq \alpha_r \text{Lip } \mathcal{T}_{r,F}(f_0, \dots, f_{r-1}, \cdot) + \max \{\alpha_0, \dots, \alpha_{r-1}\} \text{Lip } (\mathcal{S}_{r,R} \circ \mathcal{J}_{r,F}^{(1)}) + \varepsilon_r + \hat{\varepsilon}, \end{aligned}$$

and therefore the statement of the lemma follows choosing

$$\alpha_r = \frac{\hat{\varepsilon} + \varepsilon_r + \text{Lip } (\mathcal{S}_{r,R}^{-1} \circ \mathcal{J}_{r,F}^{(1)}) \max \{\alpha_0, \dots, \alpha_{r-1}\}}{1 - \text{Lip } \mathcal{T}_{r,F}(f_0, \dots, f_{r-1}, \cdot)}.$$

□



**Remark 3.5.4.** The value  $\alpha_0$  denoting the radius of the ball  $\Sigma_{0,N}^{\alpha_0}$ , obtained previously, is forced by the definition of  $\mathcal{N}_{0,F}$  (and thus, of  $\mathcal{T}_{0,N}$ ). Indeed, since we will look for the invariant curves of  $F$  as parameterizations of  $\Sigma_{0,N}^{\alpha_0}$ , their image must be contained in the domain where  $F$  is  $C^r$ . This is not the case for the derivatives of the invariant curves, for which we do not need to put a bound on them to have the operators well defined. Also, the definition of  $\mathcal{T}_{L,F}$ , for  $L \in \{1, \dots, r\}$  does not force any restriction to the size of the arguments  $f_1, \dots, f_L$  since the dependence with respect to these variables is polynomial. The values  $\alpha_1, \dots, \alpha_r$  obtained in Lemma 3.5.3 provide then upper bounds for the norms of the derivatives of the invariant curves of  $F$ .

Finally, for the convenience of the reader, we recall the fiber contraction theorem [58] which will be used in the proof of Theorem 3.2.1. We use a version of it stated in [26].

**Theorem 3.5.5** (Fiber contraction theorem). *Let  $\Sigma$  and  $D\Sigma$  be metric spaces,  $D\Sigma$  complete, and  $\Gamma : \Sigma \times D\Sigma \rightarrow \Sigma \times D\Sigma$  a map of the form  $\Gamma(\gamma, \varphi) = (G(\gamma), H(\gamma, \varphi))$ . Assume that*

- (a)  $G$  has an attracting fixed point,  $\gamma_\infty \in \Sigma$ ,
- (b)  $H$  is contractive with respect to the second variable, i.e., for all  $\gamma \in \Sigma$ ,  $\text{Lip } H(\gamma, \cdot) < 1$ .

Let  $\varphi_\infty \in D\Sigma$  be the fixed point of  $H(\gamma_\infty, \cdot)$ .

- (c)  $H$  is continuous with respect to  $\gamma$  at  $(\gamma_\infty, \varphi_\infty)$ .

Then,  $(\gamma_\infty, \varphi_\infty)$  is an attracting fixed point of  $\Gamma$ .

### 3.6 Proofs of Theorems 3.2.1 and 3.2.4

We give next the proof of Theorems 3.2.1 and 3.2.4, where we use the setting and the results obtained along the previous sections.

*Proof of Theorem 3.2.1.* Let  $F$  be as in the statement and  $T_\gamma$ ,  $\gamma > 0$ , be defined by (3.5.3). It is clear that given maps  $H$  and  $R$ , the triple  $(F, H, R)$  satisfies  $F \circ H = H \circ R$  if and only if  $(\tilde{F}, \tilde{H}, \tilde{R})$  satisfies  $\tilde{F} \circ \tilde{H} = \tilde{H} \circ \tilde{R}$ , where  $\tilde{F} = T_\gamma^{-1} \circ F \circ T_\gamma$ ,  $\tilde{H} = T_\gamma^{-1} \circ H$  and  $\tilde{R} = R$ . Clearly  $F$  and  $\tilde{F}$  belong to the same case 1, 2 or 3 of the reduced form (3.1.2).

To prove the theorem, we shall look for  $\rho > 0$  and a function  $H : (0, \rho) \rightarrow \mathbb{R}^2$ , with  $H(0) = 0$  and  $H \in C^r(0, \rho)$ , and a map of the form  $R(t) = t + R_N t^N + R_{2N-1} t^{2N-1}$ , with  $R_N < 0$ , such that

$$F \circ H = H \circ R, \tag{3.6.1}$$

with  $N = k$  for case 1 of (3.1.2) and  $N = l$  for cases 2 and 3.

We take the value  $\gamma > 0$  associated with  $F$  provided in Lemma 3.5.3, and we set  $\tilde{F} = T_\gamma^{-1} \circ F \circ T_\gamma$ . Let  $\tilde{F}^{\leq r}$  be the Taylor polynomial of  $\tilde{F}$  of degree  $r$  at the origin. Then it is a polynomial of the form

$$\tilde{F}^{\leq r}(x, y) = \begin{pmatrix} x + \gamma c y \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \gamma^{-1} a_k x^k + b_l y x^{l-1} + h.o.t. \end{pmatrix}.$$

Since we assumed  $a_k > 0$  for cases 1 and 2 and  $b_l < 0$  for case 3, then by Theorem 2.2.1, there exists, for each case, an analytic map  $\tilde{K}$  and a polynomial  $R$  of the form  $R(t) = t + R_N t^N + R_{2N-1} t^{2N-1}$ , with  $R_N < 0$ , satisfying  $\tilde{F} \circ \tilde{K} - \tilde{K} \circ R = 0$ .

Given such maps  $\tilde{K}$  and  $R$ , we look for  $\rho > 0$  and a function  $\Delta : (0, \rho) \rightarrow \mathbb{R}^2$ ,  $\Delta \in C^r(0, \rho)$ , such that

$$\tilde{F} \circ (\tilde{K} + \Delta) - (\tilde{K} + \Delta) \circ R = 0. \quad (3.6.2)$$

To do so, we consider the set of  $r$  equations described in (3.3.3) and (3.3.4). We take  $\alpha = (\alpha_0, \dots, \alpha_r)$  with  $\alpha_0 = \min\{\frac{1}{2}, \frac{d}{2}\}$ , where  $d$  is the radius of a centered ball in  $\mathbb{R}^2$  contained in the domain where  $\tilde{F}$  is of class  $C^r$ , and  $\alpha_1, \dots, \alpha_r$  given in Lemma 3.5.3. We also take the value  $\rho > 0$  associated to  $\tilde{F}$  provided in Lemma 3.5.3.

Given such values of  $\rho$  and  $\alpha$ , we take the function spaces  $\Sigma_{L,N}^\alpha$ , for  $L \in \{0, \dots, r\}$ , with domain  $(0, \rho) \subset \mathbb{R}$ . Concretely, we look for a solution  $\Delta$  of (3.6.2) with  $\Delta \in \mathcal{Y}_{2r-2k+2} \times \mathcal{Y}_{2r-k+1}$  for case 1 and  $\Delta \in \mathcal{Y}_{r-2l+2} \times \mathcal{Y}_{r-l+1}$  for cases 2 and 3.

With the operators introduced in Definition 3.5.1, equation (3.3.3) can be written as

$$f_0 = \mathcal{T}_{0,\tilde{F}}(f_0), \quad f_0 \in \Sigma_{0,N}^\alpha, \quad (3.6.3)$$

and each of the equations (3.3.4) can be written as

$$f_L = \mathcal{T}_{L,\tilde{F}}(f_0, \dots, f_L), \quad (f_0, \dots, f_L) \in \Sigma_{L,N}^\alpha,$$

for  $L \in \{1, \dots, r\}$ , or equivalently, all of them together as a unique equation,

$$(f_0, \dots, f_r) = \mathcal{T}_{r,\tilde{F}}^\times(f_0, \dots, f_r), \quad (f_0, \dots, f_r) \in \Sigma_{r,N}^\alpha. \quad (3.6.4)$$

By Lemma 3.5.3 and the Banach fixed point theorem,  $\mathcal{T}_{0,\tilde{F}}$  has an unique attracting fixed point,  $f_0^\infty \in \Sigma_{0,N}^\alpha$ , which is a solution of equation (3.6.3) and which ensures that there exists a continuous solution,  $\Delta^\infty$ , of (3.6.2). We will see next that in fact the solution  $f_0^\infty$  of (3.6.3) is a function of class  $C^r$ .

We will proceed by induction. First we prove that  $f_0^\infty$  is  $C^1$ .

Let us pick a  $C^1$  function  $f_0^0 \in \Sigma_{0,N}^{\alpha_0}$  such that  $f_1^0 := Df_0^0$  belongs to  $D\Sigma_{0,N}^{\alpha_1}$ . For simplicity we take  $f_0^0 = 0$ . Then we take the sequence  $(f_0^j, f_1^j) = (\mathcal{T}_{1,\tilde{F}}^\times)^j(f_0^0, f_1^0)$ . From the definition of the operator  $\mathcal{T}_{1,\tilde{F}}$ , we have

$$D(\mathcal{T}_{0,\tilde{F}}(f_0^0)) = \mathcal{T}_{1,\tilde{F}}(f_0^0, f_1^0). \quad (3.6.5)$$

Applying (3.6.5) inductively we have that  $f_1^j = Df_0^j$ , for all  $j$ . Also, since  $f_0^0$  is  $C^1$  and  $f_1^0 = Df_0^0$ , all the iterates  $f_0^j = (\mathcal{T}_{0,\tilde{F}})^j(f_0^0)$  are  $C^1$ , and as we have said the sequence converges in  $\Sigma_{0,N}^{\alpha_0}$  to  $f_0^\infty$ .

Again, by Lemma 3.5.3, the operator  $\mathcal{T}_{1,\tilde{F}} : \Sigma_{0,N}^\alpha \times D\Sigma_{0,N}^{\alpha_1} \rightarrow D\Sigma_{0,N}^{\alpha_1}$  is contractive with respect to the variable  $f_1 \in D\Sigma_{0,N}^\alpha$ . Thus,  $\mathcal{T}_{1,\tilde{F}}(f_0^\infty, \cdot)$  has a unique attracting fixed point,  $f_1^\infty \in D\Sigma_{0,N}^\alpha$ .

Moreover, by Lemma 3.5.2,  $\mathcal{T}_{1,\tilde{F}}$  is continuous with respect to  $f_0$  at any point  $(f_0, f_1) \in \Sigma_{1,N}^\alpha$ . Hence, by the fiber contraction theorem,  $(f_0^\infty, f_1^\infty) \in \Sigma_{1,N}^\alpha$  is an attracting fixed point of  $\mathcal{T}_{1,\tilde{F}}^\times$ ,

which means that the sequence  $f_1^i = Df_0^j$  converges in  $D\Sigma_{0,N}$ . That is,  $f_1^i$  converges uniformly in  $C^0(0, \rho)$  and therefore we have  $f_1^\infty = Df_0^\infty$  and thus,  $f_0^\infty \in C^1(0, \rho)$ .

Now, for every  $L \in \{2, \dots, r\}$ , we assume that there exists a unique attracting fixed point of  $\mathcal{T}_{L-1, \tilde{F}}$ , given by  $(f_0^\infty, \dots, f_{L-1}^\infty) \in \Sigma_{L-1, N}^\alpha$ , such that  $f_0^\infty \in C^{L-1}(0, \rho)$  and

$$f_1^\infty = Df_0^\infty, \dots, f_{L-1}^\infty = D^{L-1}f_0^\infty.$$

We will see next that in fact  $f_0^\infty$  is of class  $C^L$ .

Let us pick again the function  $f_0^0 = 0 \in C^L(0, \rho)$ , and let us take also  $f_1^0 := Df_0^0, \dots, f_L^0 := D^L f_0^0$ . Then we have  $(f_0^0, \dots, f_{L-1}^0) \in \Sigma_{L-1, N}^\alpha$  and  $f_L^0 \in D\Sigma_{L-1, N}^{\alpha L}$ .

From the definition of the operator  $\mathcal{T}_{L, \tilde{F}}$ , we have

$$D(\mathcal{T}_{L-1, \tilde{F}}(f_0^0, \dots, f_{L-1}^0)) = \mathcal{T}_{L, \tilde{F}}(f_0^0, \dots, f_L^0). \quad (3.6.6)$$

Then let  $(f_0^j, \dots, f_L^j) = (\mathcal{T}_{L, \tilde{F}}^\times)^j(f_0^0, \dots, f_L^0)$ . Applying (3.6.6) inductively we have  $f_1^j = Df_0^j, \dots, f_L^j = D^L f_0^j$ , for all  $j$ , and then the iterates  $(f_0^j, \dots, f_{L-1}^j) = (\mathcal{T}_{L-1, N}^\times)^j(f_0^0, \dots, f_{L-1}^0)$  are such that  $f_m^j \in C^{L-m}$ , for  $m \in \{0, \dots, L-1\}$ . By the induction hypothesis, the sequence  $(f_0^j, \dots, f_{L-1}^j)$  converges in  $\Sigma_{L-1, N}$  to the solution  $(f_0^\infty, \dots, f_{L-1}^\infty)$  and

$$f_1^\infty = Df_0^\infty, \dots, f_{L-1}^\infty = D^{L-1}f_0^\infty.$$

Also, applying Lemmas 3.5.2 and 3.5.3 and the fiber contraction theorem, the sequence  $f_L^j = D^L f_0^j$  converges in  $D\Sigma_{L-1, N}$ . That is,  $f_L^j$  converges uniformly in  $C^0(0, \rho)$  and therefore we have  $f_L^\infty = D^L f_0^\infty$  and thus,  $f_0^\infty \in C^L(0, \rho)$ . In conclusion  $f_0^\infty \in C^r(0, \rho)$ .

Finally, the  $C^r$  map  $\tilde{H} = \tilde{K} + \Delta$  with  $\Delta = f_0^\infty$  parameterizes the stable manifold of  $\tilde{F}$  and therefore it is  $C^r$ .

When  $F$  is  $C^\infty$ , to see that the stable manifold is  $C^\infty$  we take  $r_1$  satisfying the hypotheses of the theorem and  $r_2 > r_1$ . The previous proof provides  $H_1 = K_{r_1} + \Delta_1$  and  $H_2 = K_{r_2} + \Delta_2$  defined in  $(0, \rho_1)$  and  $(0, \rho_2)$  and of class  $C^{r_1}$  and  $C^{r_2}$  respectively that parameterize stable manifolds  $W_1$  and  $W_2$ . Theorem 4.1 of [25], which is proved by geometric methods, provides the uniqueness of the stable manifold in this setting. If  $\rho_2 < \rho_1$ , since we deal with stable manifolds we can extend  $W_2$  iterating by  $F^{-1}$  to recover  $W_1$ . Then  $W_1$  is  $C^{r_2}$  for all  $r_2 > r_1$ .  $\square$

Finally, as a corollary of the previous results, we give a short proof of Theorem 3.2.4.

*Proof of Theorem 3.2.4.* We write the proof for case 1, the other cases being almost identical except for some adjustments in the index  $n$ . We write  $F = F^\leq + (0, g)$  where  $F^\leq$  denotes the Taylor polynomial of degree  $r$  of  $F$ . Then, from (3.2.2) we have

$$F(\hat{K}(t)) - \hat{K}(\hat{R}(t)) = F^\leq(\hat{K}(t)) + (0, g(\hat{K}(t))) - \hat{K}(\hat{R}(t)) = (O(t^{n+k}), O(t^{n+2k-1})),$$

and thus, since  $g(\hat{K}(t)) = o(t^{2r})$  and  $n \leq 2r - 2k + 1$ , we have

$$F^\leq(\hat{K}(t)) - \hat{K}(\hat{R}(t)) = (O(t^{n+k}), O(t^{n+2k-1})) + (0, O(t^{2r})) = (O(t^{n+k}), O(t^{n+2k-1})).$$

Clearly,  $F^{\leq}$  is analytic and satisfies the hypotheses of Theorem 2.2.3. Then, there exists a  $C^1$  map  $K : [0, \rho) \rightarrow \mathbb{R}^2$ , analytic in  $(0, \rho)$ , and an analytic map  $R : (-\rho, \rho) \rightarrow \mathbb{R}$  such that

$$F^{\leq}(K(t)) = K(R(t)), \quad t \in [0, \rho),$$

and

$$\begin{aligned} K(t) - \hat{K}(t) &= (O(t^{n+1}), O(t^{n+k})), \\ R(t) - \hat{R}(t) &= \begin{cases} O(t^{2k-1}) & \text{if } n \leq k, \\ 0 & \text{if } n > k. \end{cases} \end{aligned}$$

Also, following the proof of Theorem 3.2.1, there exists a  $C^1$  map  $H : [0, \rho) \rightarrow \mathbb{R}^2$ ,  $H \in C^r(0, \rho)$ , given by  $H = K + \Delta$ , with  $\Delta = (O(t^{2r-2k+2}), O(t^{2r-k+1}))$ ,  $\Delta \in C^r(0, \rho)$ , such that

$$F(H(t)) = H(R(t)), \quad t \in [0, \rho).$$

To complete the proof of the theorem, note that we have

$$\begin{aligned} H(t) - \hat{K}(t) &= K(t) - \hat{K}(t) + \Delta(t) \\ &= (O(t^{n+1}), O(t^{n+k})) + (O(t^{2r-2k+2}), O(t^{2r-k+1})) = (O(t^{n+1}), O(t^{n+k})). \end{aligned}$$

□

## Chapter 4

# Integral curves of planar vector fields asymptotic to a parabolic point

### 4.1 Introduction

The objective of this chapter is to use the results of existence of invariant curves for maps obtained in Chapters 2 and 3 to present analogous results concerning the existence of invariant curves of planar vector fields.

Let  $X : U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a vector field of class  $C^r$  of the form

$$X(x, y) = \begin{pmatrix} cy + f_1(x, y) \\ f_2(x, y) \end{pmatrix} \quad (4.1.1)$$

with  $f_1(x, y), f_2(x, y) = O(\|(x, y)\|^2)$ . The origin is a critical point of parabolic type, concretely with

$$DX(0, 0) = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix},$$

and therefore the only eigenspace of  $DX(0, 0)$  is  $\langle(1, 0)\rangle$ .

We study the existence of solutions for the differential system  $(\dot{x}, \dot{y}) = X(x, y)$ , where  $X$  is of the form (4.1.1), that are asymptotic to the critical point at the origin. A neighbourhood of the origin is contained in the center manifold, but in some cases there may exist solutions of the differential equation that have the origin as  $\alpha$ -limit or  $\omega$ -limit. We also obtain a representation of the dynamics inside the stable and unstable curves.

In the particular case of planar vector fields, an invariant curve is the image of a solution of the vector field. Thus, the invariant curves of a planar vector field can be parametrized by the time variable as solutions of the system. We will however consider the invariant manifolds as planar curves parametrized by a real variable. We call this variable  $u$  to distinguish it from the space variables  $(x, y)$  and from the time variable,  $t$ .

As in the previous chapters, the superindices  $x$  and  $y$  on the symbol of a function or an operator that takes values in  $\mathbb{R}^2$  will denote the first and second components of its image, respectively.

For our study we use the parameterization method for invariant manifolds, presented in Section 2.2.2, but now slightly modified and adapted to the vector field setting. Concretely, we look for a parameterization,  $K$ , and a one-dimensional vector field,  $Y$ , such that

$$X \circ K = DK \cdot Y, \quad (4.1.2)$$

with  $DK(0,0) = (1,0)$ . Equation (4.1.2) expresses that at the range of  $K$ , the vector field  $X$  is tangent to the range of  $K$ , and therefore, the image of  $K$  is invariant under the flow of  $X$ . Moreover, the vector field  $Y$  is a representation of  $X$  restricted to the invariant manifold  $K$ .

Here we will not look directly for solutions  $K$  and  $Y$  that satisfy (4.1.2) by studying the properties of (4.1.2) as a functional equation, as we did for the map case. We will instead obtain the results for vector fields from the corresponding results for maps presented in Chapters 2 and 3. The main tool we will use is the fact that the invariant curves of a vector field are the same invariant curves of the map given by its time $-t$  flow.

Similarly as with the maps described in Section 2.2.1, performing the change of variables given by  $\tilde{x} = x$ ,  $\tilde{y} = y + \frac{1}{c} f_1(x, y)$ , (4.1.1) can be brought to the form

$$X(x, y) = \begin{pmatrix} cy \\ p(x) + yq(x) + u(x, y) + g(x, y) \end{pmatrix} \quad (4.1.3)$$

with  $c > 0$ ,  $2 \leq k, l \leq r$ , where  $p(x) = x^k(a_k + \dots + a_r x^{r-k})$ ,  $q(x) = x^{l-1}(b_l + \dots + b_r x^{r-l})$ , where  $u(x, y)$  is a polynomial containing the factor  $y^2$  and where  $g(x, y) = o(\|(x, y)^r\|)$ . We denote (4.1.3) as the *reduced form* of  $X$ . We also consider the following three cases for this reduced form depending on the indices  $k$  and  $l$ , analogous to the cases presented for maps:

- Case 1:  $k < 2l - 1$  and  $a_k \neq 0$ ,
- Case 2:  $k = 2l - 1$  and  $a_k \neq 0$ ,  $b_l \neq 0$ ,
- Case 3:  $k > 2l - 1$  and  $b_l \neq 0$ .

In order to deal with several cases at the same time we associate to  $X$  the integer  $N$  as  $N = k$  in case 1 and  $N = l$  in cases 2 and 3.

The main results of this chapter are Theorems 4.5.1 and 4.5.2, concerning the existence of analytic invariant curves of an analytic vector field of the form (4.1.3), and Theorems 4.5.3 and 4.5.4, concerning the existence of differentiable invariant curves. Since any vector field of the form (4.1.1) is  $C^r$ -conjugate to a vector field of the form (4.1.3), the existence results given in the main theorems also provide invariant manifolds for (4.1.1). As for the map case, we cannot expect the invariant curves of  $X$  to have sharp regularity around a parabolic critical point.

In Section 4.3 we provide an algorithm to obtain parameterizations of approximations of the invariant curves of  $X$ .

In Section 4.4 we present some results about the connections between a vector field  $X$  of the form (4.1.3) and its time $-t$  flow,  $\varphi_t$ .

Finally we present the main results of the chapter in Section 4.5. The results are stated for the stable curves. The existence of the unstable ones is obtained from the corresponding study of the stable curves of  $-X$ . Moreover, using the conjugations  $(x, y) \mapsto (\pm x, \pm y)$  one can obtain the local phase portraits and the location of the local invariant manifolds of  $X$  depending on the studied cases.

The existence results 4.5.2 and 4.5.3 presented in Section 4.5 could also be obtained from the results for maps of [25] using the tools presented in Section 4.4. However, our setting based on the parameterization method and the results for maps from Chapters 2 and 3 allows to state Theorems 4.5.1 and 4.5.4 as *a posteriori* results and to provide an effective algorithm to compute an approximation of a parameterization of the invariant manifolds of  $X$ .

## 4.2 Preliminary results

In this short section we introduce some preliminary results about vector fields and flows that will be used later on.

Given a vector field  $X : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ , we denote by  $\varphi(t, x)$  its flow. First, we recall the properties of  $\varphi$  in the two following well known results (see for example [69]),

**Theorem 4.2.1.** *Let  $X$  be a vector field of class  $C^r$ ,  $r \geq 1$ , defined in an open set,  $U \subset \mathbb{R}^n$ .*

(a) *For each point  $x \in U$ , there exists an interval,  $I_x$ , where a unique maximal integral curve of  $X$  passing through  $x$ ,  $\varphi_x : I_x \rightarrow U$ , is defined. That is,  $\varphi_x$  satisfies in  $I_x$  the differential equation  $\frac{dy}{dt} = X(y)$ ,  $y(0) = x$ .*

(b) *If  $y = \varphi_x(t)$ , for some  $t \in I_x$ , then*

$$I_y = I_x - t = \{\tau - t \mid \tau \in I_x\},$$

*and  $\varphi_y(s) = \varphi_x(t + s)$ , for all  $s \in I_y$ .*

(c) *The set  $D = \{(t, x) \mid x \in U, t \in I_x\} \subseteq \mathbb{R}^{n+1}$  is open, and the mapping  $\varphi : D \rightarrow \mathbb{R}^n$  defined as  $\varphi(t, x) = \varphi_x(t)$  is of class  $C^r$  in  $D$ .*

**Theorem 4.2.2.** *Let  $X : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a vector field,  $X \in C^1(U)$ . If  $x \in U$  and  $I_x = (\omega_x^-, \omega_x^+)$  is such that  $\omega_x^+ < \infty$  (resp.  $\omega_x^- > -\infty$ ), then  $\varphi_x(t)$  tends to  $\partial U$  when  $t \rightarrow \omega_x^+$  (resp. when  $t \rightarrow \omega_x^-$ ). That is, for each compact  $K \subset U$  there exists  $\varepsilon = \varepsilon(K) > 0$  such that if  $t \in [\omega_x^+ - \varepsilon, \omega_x^+)$  (resp.  $t \in (\omega_x^-, \omega_x^- + \varepsilon]$ ), then  $\varphi_x(t) \notin K$ .*

As a consequence of Theorem 4.2.1, the flow  $\varphi(t, x)$  of a vector field  $X : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$  defines a map,  $\varphi_t : V \rightarrow \mathbb{R}^n$ , where  $V$  is an open subset of  $U$  that depends on  $t$ , such that  $\varphi_t \in C^r(U)$ , and where  $\varphi_t$  is given by  $\varphi_t(x) = \varphi(t, x)$ . We call  $\varphi_t$  the *time- $t$  flow* of  $X$ .

We will refer to the flow of a given vector field  $X$  as  $\varphi(t, x)$ ,  $\varphi_x(t)$  or  $\varphi_t(x)$  to emphasize in each case the dependence on the variable we are interested in.

In the next result we show that under suitable conditions, a vector field and its time- $t$  flow have the same invariant manifolds.

**Proposition 4.2.3.** *Let  $X_1 : U_1 \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $X_2 : U_2 \subset \mathbb{R}^m \rightarrow \mathbb{R}^m$  be vector fields of class  $C^1$ , where  $n \leq m$ , and let  $\varphi_1 : D_1 \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ ,  $\varphi_2 : D_2 \subset \mathbb{R}^{m+1} \rightarrow \mathbb{R}^m$  be their flows, respectively. Let  $K : U_1 \rightarrow U_2$  be a map of class  $C^1$ . Then,*

$$X_2(K(x)) = DK(x)X_1(x), \quad \forall x \in U_1,$$

*if and only if*

$$\varphi_2(t, K(x)) = K(\varphi_1(t, x)), \quad \forall (t, x) \in D_1.$$

*Proof.* We define, for each  $x \in U_1$ , the functions

$$\alpha(t) = K(\varphi_1(t, x)), \quad \beta(t) = \varphi_2(t, K(x)).$$

Observe that

$$\beta'(t) = \frac{\partial}{\partial t} \varphi_2(t, K(x)) = X_2(\varphi_2(t, K(x))) = X_2(\beta(t)),$$

and we also have, using (4.2.3),

$$\alpha'(t) = \frac{\partial}{\partial t} K(\varphi_1(t, x)) = DK(\varphi_1(t, x))X_1(\varphi_1(t, x)) = X_2(K(\varphi_1(t, x))) = X_2(\alpha(t)).$$

Moreover,  $\alpha(0) = \beta(0) = K(x)$ . We have then that for each  $x \in U_1$ ,  $\alpha(t)$  and  $\beta(t)$  are functions satisfying the same differential equation and with the same initial condition, and thus  $\alpha(t) \equiv \beta(t)$ .

To see (4.2.3), since we assume  $\alpha(t) \equiv \beta(t)$ , we have  $\alpha'(t) \equiv \beta'(t)$ , and then

$$DK(\varphi_1(t, x))X_1(\varphi_1(t, x)) = X_2(\varphi_2(t, K(x))), \quad \forall (t, x) \in D_1.$$

In particular for  $t = 0$  we have  $DK(x)X_1(x) = X_2(K(x))$ , for all  $x \in U_1$ . □

### 4.3 Formal polynomial approximation of a parameterization of the invariant curves

In this section we present analogous results to the ones in Section 2.3, but in this case concerning planar vector fields.

Concretely, we consider  $C^r$  vector fields  $X$  of the form (4.1.3) and we provide algorithms, depending on the cases 1, 2 or 3, to obtain a polynomial map  $\mathcal{K}_n$  and a one-dimensional vector field,  $\mathcal{Y}_n$ , that are approximations of solutions  $K$  and  $Y$  of the invariance equation

$$X \circ K = DK \cdot Y. \tag{4.3.1}$$

As in Section 2.3, the two components of  $\mathcal{K}_n$  will have different order and different degree, and the index  $n$  has to be seen as an induction index. Therefore, higher values of  $n$  mean better approximation.

The obtained approximations correspond to formal invariant curves. They correspond to stable curves when the coefficient  $Y_k$  (case 1) or  $Y_l$  (cases 2, 3) of  $\mathcal{Y}_n$  in the statements below are negative. When those coefficients are positive they would correspond to unstable curves.



The parameterizations obtained in the propositions below have a completely analogous form to the parameterizations obtained for maps in Section 2.3. Also, the obtained form of the one-dimensional vector field  $\mathcal{Y}_n$ , namely  $\mathcal{Y}_n(u) = Y_N u^N + Y_{2N-1} u^{2N-1}$ , is the normal form of a one-dimensional vector field around a parabolic singularity [70].

**Proposition 4.3.1** (Case 1). *Let  $X$  be a  $C^r$  vector field of the form (4.1.3) with  $2 \leq k \leq r$ . Assume that  $k < 2l - 1$  and  $a_k > 0$ . Then, for all  $2 \leq n \leq 2(r - k + 1)$ , there exist two polynomials,  $\mathcal{K}_n$ , and two vector fields  $\mathcal{Y}_n$ , of the form*

$$\mathcal{K}_n(u) = \begin{pmatrix} u^2 + \dots + K_n^x u^n \\ K_{k+1}^y u^{k+1} + \dots + K_{n+k-1}^y u^{n+k-1} \end{pmatrix}$$

and

$$\mathcal{Y}_n(u) = \begin{cases} Y_k u^k & \text{if } 2 \leq n \leq k, \\ Y_k u^k + Y_{2k-1} u^{2k-1} & \text{if } n \geq k + 1, \end{cases}$$

such that

$$\mathcal{G}_n(t) := X(\mathcal{K}_n(u)) - D\mathcal{K}_n(u) \cdot \mathcal{Y}_n(u) = (O(u^{n+k}), O(u^{n+2k-1})). \quad (4.3.2)$$

For the first pair we have

$$K_{k+1}^y = -\sqrt{\frac{2a_k}{c(k+1)}}, \quad Y_k = -\sqrt{\frac{ca_k}{2(k+1)}} = \frac{c}{2} K_{k+1}^y,$$

and for the second one

$$K_{k+1}^y = \sqrt{\frac{2a_k}{c(k+1)}}, \quad Y_k = \sqrt{\frac{ca_k}{2(k+1)}} = \frac{c}{2} K_{k+1}^y.$$

If  $X$  is  $C^\infty$  or analytic, one can compute the polynomial approximation  $\mathcal{K}_n$  up to any order.

**Remark 4.3.2.** The algorithm described in the proof of this and the next propositions can be implemented in a computer program to calculate  $\mathcal{R}$  and the expansion of  $\mathcal{K}_n$ .

**Notation.** Along the proof, given a  $C^r$  one-variable map  $f$ , we will denote  $[f]_n$ ,  $0 \leq n \leq r$ , the coefficient of the term of order  $n$  of the jet of  $f$  at 0.

*Proof.* We will see that we can determine  $\mathcal{K}_n$  and  $\mathcal{Y}_n$  iteratively.

For  $n = 2$ , we claim that there exist polynomials  $\mathcal{K}_2(u) = (u^2, K_{k+1}^y u^{k+1})$  and  $\mathcal{Y}_2(u) = Y_k u^k$ , such that  $\mathcal{G}_2(u) = F(\mathcal{K}_2(u)) - D\mathcal{K}_2 \cdot \mathcal{Y}_2(u) = (O(u^{k+2}), O(u^{2k+1}))$ .

Indeed, from the expansion of  $\mathcal{G}_2$  we have

$$\mathcal{G}_2(u) = \begin{pmatrix} c K_{k+1}^y u^{k+1} - 2Y_k u^{k+1} \\ a_k u^{2k} - (k+1) K_{k+1}^y Y_k u^{2k} + O(u^{2k+1}) \end{pmatrix},$$

so, if the conditions

$$c K_{k+1}^y - 2Y_k = 0, \quad a_k - (k+1) K_{k+1}^y Y_k = 0,$$

are satisfied, then we clearly have  $\mathcal{G}_2(u) = (O(u^{2+k}), O(u^{2k+1}))$ , and we obtain the values of  $K_{k+1}^y$  and  $Y_k$  given in the statement.

Now we assume that we have already obtained polynomials  $\mathcal{K}_n$  and  $\mathcal{Y}_n$ ,  $2 \leq n < 2(r-k+1)$  such that (4.3.2) holds true, and we look for

$$\mathcal{K}_{n+1}(u) = \mathcal{K}_n(u) + \begin{pmatrix} K_{n+1}^x u^{n+1} \\ K_{n+k}^y u^{n+k} \end{pmatrix}, \quad \mathcal{Y}_{n+1}(u) = \mathcal{Y}_n(u) + Y_{n+k-1} u^{n+k-1},$$

such that  $\mathcal{G}_{n+1}(u) = (O(u^{n+k+1}), O(u^{n+2k}))$ .

Using Taylor's theorem, we write

$$\begin{aligned} \mathcal{G}_{n+1}(u) &= X(\mathcal{K}_n(u) + (K_{n+1}^x u^{n+1}, K_{n+k}^y u^{n+k})) \\ &\quad - (D\mathcal{K}_n(u) + D(K_{n+1}^x u^{n+1}, K_{n+k}^y u^{n+k})) \cdot (\mathcal{Y}_n(u) + Y_{n+k-1} u^{n+k-1}) \\ &= \mathcal{G}_n(u) + DX(\mathcal{K}_n(u)) \cdot (K_{n+1}^x u^{n+1}, K_{n+k}^y u^{n+k}) \\ &\quad + \int_0^1 (1-s) D^2 X(\mathcal{K}_n(u) + s(K_{n+1}^x u^{n+1}, K_{n+k}^y u^{n+k})) ds (K_{n+1}^x u^{n+1}, K_{n+k}^y u^{n+k})^{\otimes 2} \\ &\quad - D(K_{n+1}^x u^{n+1}, K_{n+k}^y u^{n+k}) \cdot \mathcal{Y}_n(u) - D\mathcal{K}_{n+1}(u) \cdot Y_{n+k-1} u^{n+k-1}. \end{aligned}$$

Performing the computations in the previous expression we have

$$\mathcal{G}_{n+1}(u) = \mathcal{G}_n(u) + \begin{pmatrix} [c K_{n+k}^y - (n+1) Y_k K_{n+1}^x - 2 Y_{n+k-1}] u^{n+k} + O(u^{n+k+1}) \\ [k a_k K_{n+1}^x - (n+k) Y_k K_{n+k}^y - (k+1) K_{k+1}^y Y_{n+k-1}] u^{n+2k-1} + O(u^{n+2k}) \end{pmatrix}. \quad (4.3.3)$$

Since, by the induction hypothesis,  $\mathcal{G}_n(u) = (O(u^{n+k}), O(u^{n+2k-1}))$ , to complete the induction step we need to make  $[\mathcal{G}_{n+1}^x]_{n+k}$  and  $[\mathcal{G}_{n+1}^y]_{n+2k-1}$  vanish.

From (4.3.3) we have

$$\begin{aligned} [\mathcal{G}_{n+1}^x]_{n+k} &= [\mathcal{G}_n^x]_{n+k} + c K_{n+k}^y - (n+1) Y_k K_{n+1}^x - 2 Y_{n+k-1}, \\ [\mathcal{G}_{n+1}^y]_{n+2k-1} &= [\mathcal{G}_n^y]_{n+2k-1} + k a_k K_{n+1}^x - (n+k) Y_k K_{n+k}^y - (k+1) K_{k+1}^y Y_{n+k-1}. \end{aligned}$$

Thus, the conditions  $[\mathcal{G}_{n+1}^x]_{n+k} = [\mathcal{G}_{n+1}^y]_{n+2k-1} = 0$  are equivalent to

$$\begin{pmatrix} -(n+1) Y_k & c \\ k a_k & -(n+k) Y_k \end{pmatrix} \begin{pmatrix} K_{n+1}^x \\ K_{n+k}^y \end{pmatrix} = \begin{pmatrix} -[\mathcal{G}_n^x]_{n+k} + 2 Y_{n+k-1} \\ -[\mathcal{G}_n^y]_{n+2k-1} + (k+1) K_{k+1}^y Y_{n+k-1} \end{pmatrix}. \quad (4.3.4)$$

If  $n \neq k$  the matrix in the left hand side of (4.3.4) is invertible, so we can take  $Y_{n+k-1} = 0$  and then obtain  $K_{n+1}^x$  and  $K_{n+k}^y$  in a unique way. When  $n = k$ , the determinant of the matrix is zero. Then, choosing

$$Y_{2k-1} = \frac{2k Y_k [\mathcal{G}_n^x]_{2k} + c [\mathcal{G}_n^y]_{3k-2}}{2(3k+1) Y_k},$$

system (4.3.4) has solutions. In this case, however,  $K_{k+1}^x$  and  $K_{2k}^y$  are not uniquely determined.  $\square$

**Proposition 4.3.3** (Case 2). *Let  $X$  be a  $C^r$  vector field of the form (4.1.3), with  $r \geq k \geq 2$ . We assume  $k = 2l - 1$ ,  $a_k \neq 0$ ,  $b_l \neq 0$  and  $a_k > -\frac{b_l^2}{4cl}$ . If  $a_k < 0$  we assume also  $a_k \neq \frac{1-2l}{(3l-1)^2} \frac{b_l^2}{c}$ . Then, for all  $1 \leq n \leq r - 2l + 2 = r - k + 1$ , there exist two polynomials,  $\mathcal{K}_n$ , and two vector fields,  $\mathcal{Y}_n$ , of the form*

$$\mathcal{K}_n(u) = \begin{pmatrix} u + \cdots + K_n^x u^n \\ K_l^y u^l + \cdots + K_{n+l-1}^y u^{n+l-1} \end{pmatrix} \quad (4.3.5)$$

and

$$\mathcal{Y}_n(u) = \begin{cases} Y_l u^l & \text{if } 1 \leq n \leq l - 1, \\ Y_l u^l + Y_{2l-1} u^{2l-1} & \text{if } n \geq l, \end{cases} \quad (4.3.6)$$

such that

$$\mathcal{G}_n(u) := X(\mathcal{K}_n(u)) - D\mathcal{K}_n(u) \cdot \mathcal{Y}_n(u) = (O(u^{n+l}), O(u^{n+2l-1})).$$

For the first pair we have

$$K_l^y = \frac{b_l - \sqrt{b_l^2 + 4ca_k l}}{2cl}, \quad Y_l = \frac{b_l - \sqrt{b_l^2 + 4ca_k l}}{2l} = cK_l^y,$$

and for the second one

$$K_l^y = \frac{b_l + \sqrt{b_l^2 + 4ca_k l}}{2cl}, \quad Y_l = \frac{b_l + \sqrt{b_l^2 + 4ca_k l}}{2l} = cK_l^y.$$

If  $a_k = \frac{1-2l}{(3l-1)^2} \frac{b_l^2}{c}$  one can compute the coefficients of  $\mathcal{K}_n$  and  $\mathcal{Y}_n$  up to  $n = l - 1$ . If  $X$  is  $C^\infty$  or analytic, one can compute those coefficients up to any order, except when  $a_k = \frac{1-2l}{(3l-1)^2} \frac{b_l^2}{c}$ .

**Proposition 4.3.4** (Case 3). *Let  $X$  be a  $C^r$  vector field of the form (4.1.3), with  $r \geq l \geq 2$ . Assume  $k > 2l - 1$ ,  $b_l \neq 0$ . Then, for all  $1 \leq n \leq r - 2l + 2$ , there exist two polynomials,  $\mathcal{K}_n$ , and two vector fields,  $\mathcal{Y}_n$ , of the form (4.3.5) and (4.3.6) respectively, such that*

$$\mathcal{G}_n(u) := X(\mathcal{K}_n(u)) - D\mathcal{K}_n(u) \cdot \mathcal{Y}_n(u) = (O(u^{n+l}), O(u^{n+2l-1})).$$

We have

$$K_l^y = \frac{b_l}{cl}, \quad Y_l = \frac{b_l}{l} = cK_l^y.$$

If we further assume that  $k \leq r$  and  $a_k \neq 0$ , then for  $1 \leq n \leq r - (k-l)l - 2l + 1$  there exists another pair,  $\mathcal{K}_n$  and  $\mathcal{Y}_n$ , with

$$\mathcal{K}_n(u) = \begin{pmatrix} u + \cdots + K_n^x u^n \\ K_{k-l+1}^y u^{k-l+1} + \cdots + K_{n+k-l}^y u^{n+k-l} \end{pmatrix}$$

and

$$\mathcal{Y}_n(u) = \begin{cases} Y_{k-l+1} u^{k-l+1} & \text{if } 2 \leq n \leq k-l, \\ Y_{k-l+1} u^{k-l+1} + Y_{2(k-l)+1} u^{2(k-l)+1} & \text{if } n \geq k-l+1, \end{cases}$$

such that

$$\mathcal{G}_n(u) := X(\mathcal{K}_n(u)) - D\mathcal{K}_n(u) \cdot \mathcal{Y}_n(u) = (O(u^{n+k-l+1}), O(u^{n+k})).$$

We have

$$K_{k-l+1}^y = -\frac{a_k}{b_l}, \quad Y_{k-l+1} = cK_{k-l+1}^y.$$

If  $F$  is  $C^\infty$  or analytic, one can compute the polynomial approximations  $\mathcal{K}_n$  up to any order.

The proofs of Propositions 4.3.3 and 4.3.4 are analogous to the one of Proposition 4.3.1.

## 4.4 From vector fields to flows

In this section we present some features of the relationship between a vector field  $X$  of the form (4.1.3) and its time- $t$  flow,  $\varphi_t$ .

We recall the notation for maps introduced in Section 2.2.1 of Chapter 2. We consider maps  $F : U \rightarrow \mathbb{R}^2$ ,  $F \in C^r(U)$ , of the form

$$F(x, y) = \begin{pmatrix} x + cy \\ y + p(x) + yq(x) + u(x, y) + g(x, y) \end{pmatrix}, \quad (4.4.1)$$

with  $p(x) = x^k(a_k + \dots + a_r x^{r-k})$ ,  $q(x) = x^{l-1}(b_l + \dots + b_r x^{r-l})$ ,  $2 \leq k, l \leq r$ , and where  $u(x, y)$  is a polynomial of degree  $r$  with the factor  $y^2$ , and  $g(x, y) = o(\|(x, y)\|^r)$ . For such maps we consider the three cases 1, 2 and 3 already defined in Section 2.2.1. Of course the three cases distinguished for (4.4.1) and the ones of (4.1.3) are completely analogous.

Along this section we will see, roughly speaking, that if  $X$  is a vector field of the form (4.1.3), then its time- $t$  flow  $\varphi_t$  has analogous properties to the corresponding map of the form (4.4.1). Actually, we have that

$$D\varphi_t(0, 0) = \begin{pmatrix} 1 & ct \\ 0 & 1 \end{pmatrix},$$

and thus  $\varphi_t$  has a nilpotent parabolic fixed point at  $(0, 0)$ . The properties concerning the map  $\varphi_t$  (or  $\varphi_1$ ) that we are interested in are collected in Corollary 4.4.5. Such corollary is proved as a consequence of the following Lemmas 4.4.1 and 4.4.4, which contain many tedious but straightforward computations.

We will also show that a given approximation of a parameterization of an invariant curve of  $X$  is also an approximation of a parameterization of an invariant curve of the time- $t$  flow  $\varphi_t$  (Lema 4.4.7).

**Lemma 4.4.1.** *Let  $X : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a vector field in a neighborhood  $U$  of  $(0, 0)$  of the form (4.1.3). Then, its time- $t$  flow,  $\varphi : D \rightarrow \mathbb{R}^2$ , where  $D = \{(t, x) \mid x \in U, t \in I_x\}$ , has the form*

$$\varphi(t, x, y) = \begin{pmatrix} x + cty + p_1(t, x) + yq_1(t, x) + u_1(t, x, y) + g_1(t, x, y) \\ y + p_2(t, x) + yq_2(t, x) + u_2(t, x, y) + g_2(t, x, y) \end{pmatrix}$$

where, for  $i = 1, 2$ ,  $p_i$  is a polynomial of degree  $r$  on the variable  $x$ ,  $q_i$  is a polynomial of degree  $r - 1$  on the variable  $x$ ,  $u_i$  is a polynomial of degree  $r$  on the variables  $(x, y)$  containing the factor  $y^2$  and  $g_i(t, x, y) = o(\|(x, y)\|^r)$ .

Moreover, if  $k > l$ , we have

$$\begin{aligned} p_1(t, x) &= \frac{1}{2}ca_k t^2 x^k + O(x^{k+1}), & q_1(t, x) &= \frac{1}{2}cb_l t^2 x^{l-1} + O(x^l), \\ p_2(t, x) &= a_k t x^k + O(x^{k+1}), & q_2(t, x) &= b_l t x^{l-1} + O(x^l), \end{aligned}$$

and if  $k \leq l$ , we have

$$\begin{aligned} p_1(t, x) &= \frac{1}{2}ca_k t^2 x^k + O(x^{k+1}), & q_1(t, x) &= O(x^{k-1}), \\ p_2(t, x) &= a_k t x^k + O(x^{k+1}), & q_2(t, x) &= O(x^{k-1}). \end{aligned}$$

*Proof.* By Theorem 4.2.1, the flow  $\varphi_t$  is of class  $C^r$  in  $U$ . Thus in a neighborhood of 0,  $\varphi_t$  can be written as

$$\varphi_t(x, y) = \begin{pmatrix} \varphi_t^x(x, y) \\ \varphi_t^y(x, y) \end{pmatrix} = \begin{pmatrix} \sum_{i+j \leq r} a_{ij}^x(t) x^i y^j + o(\|(x, y)\|^r) \\ \sum_{i+j \leq r} a_{ij}^y(t) x^i y^j + o(\|(x, y)\|^r) \end{pmatrix}.$$

By its definition,  $\varphi_t$  satisfies

$$\varphi_t(x, y) = (x, y) + \int_0^t X(\varphi_s(x, y)) ds, \quad (t, x, y) \in D. \quad (4.4.2)$$

Since  $(0, 0)$  is a critical point of  $X$ , then  $\varphi_t(0, 0) = (0, 0)$ , and hence  $a_{00}^x(t) \equiv a_{00}^y(t) \equiv 0$ .

From (4.4.2), for the second component of  $\varphi_t$  we have

$$\begin{aligned} \varphi_t^y(x, y) &= y + \int_0^t p\left(\sum_{i+j=1}^r a_{ij}^x(s) x^i y^j\right) ds + \int_0^t \left(\sum_{i+j=1}^r a_{ij}^y(s) x^i y^j\right) q\left(\sum_{i+j=1}^r a_{ij}^x(s) x^i y^j\right) ds \\ &\quad + \int_0^t u\left(\sum_{i+j=1}^r a_{ij}^x(s) x^i y^j, \sum_{i+j=1}^r a_{ij}^y(s) x^i y^j\right) ds + o(\|(x, y)\|^r). \end{aligned} \quad (4.4.3)$$

In order to analyze each term of (4.4.3) we define

$$\begin{aligned} \xi_{s,1}(x, y) &:= p\left(\sum_{i+j=1}^r a_{ij}^x(s) x^i y^j\right), \\ \xi_{t,2}(x, y) &:= \left(\sum_{i+j=1}^r a_{ij}^y(s) x^i y^j\right) q\left(\sum_{i+j \leq r} a_{ij}^x(s) x^i y^j\right), \\ \xi_{t,3}(x, y) &:= u\left(\sum_{i+j=1}^r a_{ij}^x(s) x^i y^j, \sum_{i+j=1}^r a_{ij}^y(s) x^i y^j\right), \end{aligned}$$

and then from the properties of  $p$ ,  $q$  and  $u$ , we have

$$\xi_{s,1}(x, y) = a_k x^k (a_{10}^x(s))^k + a_k y x^{k-1} (a_{10}^x(s))^{k-1} a_{01}^x(s) + O(x^{k+1}) + yO(x^k) + O(y^2), \quad (4.4.4)$$

$$\begin{aligned} \xi_{s,2}(x, y) &= \left( \sum_{i+j=1}^r a_{ij}^y(s) x^i y^j \right) [b_l x^{l-1} (a_{10}^x(s))^{l-1} + O(x^l) + b_l y^{l-1} (a_{01}^x(s))^{l-1} + O(y^l)] \\ &\quad + b_l y x^{l-2} (a_{10}^x(s))^{l-2} a_{01}^x(s) + yO(x^{l-1})] \\ &= \left( \sum_{i=1}^r a_{i0}^y(s) x^i \right) [b_l x^{l-1} (a_{10}^x(s))^{l-1} + O(x^l) + b_l y^{l-1} (a_{01}^x(s))^{l-1} + O(y^l)] \\ &\quad + b_l y x^{l-2} (a_{10}^x(s))^{l-2} a_{01}^x(s) + yO(x^{l-1})] + b_l y x^{l-1} a_{01}^y(s) (a_{10}^x(s))^{l-1} + yO(x^l), \end{aligned} \quad (4.4.5)$$

$$\begin{aligned} \xi_{s,3}^y(x, y) &= \left[ \left( \sum_{i=1}^r a_{i0}^y(s) x^i \right)^2 + 2 \left( \sum_{i=1}^r a_{i0}^y(s) x^i \right) y (a_{01}^y(s) + O(\|(x, y)\|)) \right] \\ &\quad + O(y^2) [C(s) + O(\|(x, y)\|)], \end{aligned} \quad (4.4.6)$$

where  $C(s)$  is some function of  $s$ .

Note that from (4.4.3), since  $k, l \geq 2$ , we have

$$\varphi_t^y(x, y) = \sum_{i+j=1}^r a_{ij}^y(t) x^i y^j + o(\|(x, y)\|^r) = y + O(\|(x, y)\|^2),$$

and hence

$$a_{10}^y(t) \equiv 0, \quad a_{01}^y(t) \equiv 1, \quad (4.4.7)$$

and concerning  $\varphi_t^x$  we have

$$\begin{aligned} \varphi_t^x(x, y) &= x + \int_0^t c \varphi_s^y(x, y) ds = x + cx \int_0^t a_{10}^y(s) ds + cy \int_0^t a_{01}^y(s) ds + O(\|(x, y)\|^2) \\ &= x + cty + O(\|(x, y)\|^2), \end{aligned} \quad (4.4.8)$$

and

$$a_{10}^x(t) \equiv 1, \quad a_{01}^x(t) = ct. \quad (4.4.9)$$

Now, let  $a_t(x)$  denote the nonlinear polynomial terms of  $\varphi_t^y(x, y)$  containing only powers of  $x$ , and let  $yb_t(x)$  denote the nonlinear polynomial terms of  $\varphi_t^y$  of the form  $yO(x)$ . We can write then  $\varphi_t^y(x, y) = y + a_t(x) + yb_t(x) + O(y^2) + o(\|(x, y)\|^r)$ . Let us define also the polynomial

$$A_s(x) = \sum_{i=1}^r a_{i0}^y(s) x^i.$$

From the expressions of  $\xi_{s,1}^y$ ,  $\xi_{s,2}^y$ , and  $\xi_{s,3}^y$ , and the values obtained in (4.4.7) and (4.4.9),  $a_t$  and  $b_t$  can be written as

$$a_t(x) = a_k t x^k + b_l t x^{l-1} \int_0^t A_s(x) ds + \int_0^t (A_s(x))^2 ds + h.o.t., \quad (4.4.10)$$

$$b_t(x) = \begin{cases} b_l t x^{l-1} + a_k c \frac{t^2}{2} x^{k-1} + b_l c x^{l-2} \int_0^t A_s(x) s ds + 2 \int_0^t A_s(x) ds + h.o.t. & (\text{if } l \geq 3), \\ b_l t x^{l-1} + a_k c \frac{t^2}{2} x^{k-1} + b_l c x^{l-2} \int_0^t A_s(x) s ds + 2 \int_0^t A_s(x) ds \\ \quad + b_l c^{l-1} \int_0^t A_s(x) s^{l-1} ds + h.o.t. & (\text{if } l = 2). \end{cases} \quad (4.4.11)$$

We prove next that  $a_t(x) = a_k t x^k + O(x^{k+1})$ . From (4.4.7) we have  $A_s(x) = O(x^2)$  and thus

$$a_t(x) = a_k t x^k + O(x^m), \quad m = \min\{l+1, 4\}.$$

If  $k = l = 2$ , then we are done. If  $k \geq 3$  we have then  $a_t(x) = O(x^3)$  which implies that  $a_{20}^y(t) \equiv 0$ . But with this assumption we have then  $A_s(x) = O(x^3)$  and

$$a_t(x) = a_k t x^k + O(x^m), \quad m = \min\{l+2, 6\}$$

If  $k < \min\{l+2, 6\}$ , then we are done. If not, we have then,  $\min\{l+2, 6\} \geq 4$  and  $a_t(x) = O(x^4)$ , which implies that  $a_{30}^y(t) \equiv 0$ . We repeat this process successively and we obtain  $a_{i0}^y(t) \equiv 0$  for  $i = 0, \dots, k-1$ ,  $a_{k0}^y(t) = a_k t$ . Note that in particular this implies that we have

$$A_s(x) = a_k t x^k + O(x^{k+1}). \quad (4.4.12)$$

Next we deal with  $b_t(x)$ . With the expression given in (4.4.11) and the properties of  $A_s(x)$  in (4.4.12) we have, both for  $l = 2$  and  $l \geq 3$ ,

$$b_t(x) = b_l t x^{l-1} + O(x^{k-1}). \quad (4.4.13)$$

Recall that we have  $\varphi_t^y(x, y) = y + a_t(x) + y b_t(x) + O(y^2) + o(\|(x, y)\|^r)$ . Then, if  $k > l$ , from (4.4.13) we have  $b_t(x) = b_l t x^{l-1} + O(x^l)$  and the statement of the lemma concerning the component  $\varphi_t^y$  is proved.

If  $k \leq l$ , then we have  $\varphi_t^y(x, y) = y + a_k t x^k + O(x^{k+1}) + y O(x^{k-1}) + O(y^2)$ .

Note also that for the case  $k > l$  we have obtained  $a_{l-1,1}^y(t) = b_l t$ .

Finally we go back to the first component of the flow,  $\varphi_t^x$ . From (4.4.8) we obtain directly

$$\begin{aligned} \varphi_t^x(x, y) &= x + cty + \frac{1}{2} c a_k t^2 x^k + O(x^{k+1}) + \frac{1}{2} c b_l t^2 y x^{l-1} + y O(x^l) + O(y^2), & \text{if } k > l, \\ \varphi_t^x(x, y) &= x + cty + \frac{1}{2} c a_k t^2 x^k + O(x^{k+1}) + y O(x^{k-1}) + O(y^2), & \text{if } k \leq l, \end{aligned}$$

as we wanted to see. □

**Remark 4.4.2.** Note that if  $k \leq l$ , then  $k < 2l - 1$  and we are in case 1.

A very simplified version of the previous lemma gives the following property in dimension one,

**Remark 4.4.3.** Let  $Y$  be a  $C^r$  one-dimensional vector field of the form  $Y(u) = a_n u^n + O(u^{n+1})$ , with  $n \leq r$ . Then its time- $t$  flow has the form  $\varphi_t(u) = u + a_n(t) u^n + O(u^{n+1})$ , as a direct consequence of the fact that  $\varphi_t(u) = u + \int_0^t Y(\varphi_s(u)) ds$ .

In the next lemma we show that the *reduced form* of the flow of a vector field  $X$  of the form (4.1.3) is actually of the form (4.4.1).

**Lemma 4.4.4.** Let  $F : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a map of class  $C^r$  in a neighborhood  $U$  of 0, of the form

$$F(x, y) = \begin{pmatrix} x + cy + f_1(x, y) \\ y + f_2(x, y) \end{pmatrix}, \quad (4.4.14)$$

with  $c \neq 0$ ,  $f_1(x, y), f_2(x, y) = O(\|(x, y)\|^2)$ .

Consider the  $C^r$  change of variables  $T : U \rightarrow \mathbb{R}^2$  given by explicitly by its inverse by

$$T^{-1}(x, y) = (x, y + \frac{1}{c}f_1(x, y)) \quad (4.4.15)$$

Then, the map  $\tilde{F} := T^{-1} \circ F \circ T$  has the form

$$\tilde{F}(x, y) = \begin{pmatrix} x + cy \\ y + f_3(x, y) \end{pmatrix}, \quad (4.4.16)$$

where  $f_3(x, y) = O(\|(x, y)\|^2)$ . Moreover,

(a) If  $f_1$  and  $f_2$  are of the form

$$\begin{aligned} f_1(x, y) &= a_k^x x^k + O(x^{k+1}) + b_l^x y x^{l-1} + yO(x^l) + O(y^2), \\ f_2(x, y) &= a_k^y x^k + O(x^{k+1}) + b_l^y y x^{l-1} + yO(x^l) + O(y^2), \end{aligned} \quad (4.4.17)$$

for some  $l \geq 2$ ,  $k \geq l$ , then  $f_3$  is of the form

$$f_3(x, y) = a_k^y x^k + O(x^{k+1}) + b_l^y y x^{l-1} + yO(x^l) + O(y^2). \quad (4.4.18)$$

(b) If  $f_1$  and  $f_2$  are of the form

$$\begin{aligned} f_1(x, y) &= a_k^y x^k + O(x^{k+1}) + yO(x) + O(y^2), \\ f_2(x, y) &= a_k^y x^k + O(x^{k+1}) + yO(x) + O(y^2), \end{aligned} \quad (4.4.19)$$

for some  $k \geq 2$ , then  $f_3$  is of the form

$$f_3(x, y) = a_k^y x^k + O(x^{k+1}) + yO(x) + O(y^2). \quad (4.4.20)$$

*Proof.* From the relation  $\tilde{F} \circ T^{-1} = T^{-1} \circ F$ , a simple computation shows that  $\tilde{F}$  is of the form (4.4.16). Actually, if  $\tilde{F}$  is as in (4.4.16), then we have

$$\tilde{F}(T^{-1}(x, y)) = \begin{pmatrix} x + cy + f_1(x, y) \\ y + \frac{1}{c}f_1(x, y) + f_3(x, y + \frac{1}{c}f_1(x, y)) \end{pmatrix},$$

and

$$T^{-1}(F(x, y)) = \begin{pmatrix} x + cy + f_1(x, y) \\ y + \frac{1}{c}f_1(x, y) + f_2(x, y) \end{pmatrix}.$$

To prove (a), we assume that  $f_3$  is of the form (4.4.18) and we see that  $f_3(x, y + \frac{1}{c}f_1(x, y))$  is of the form (4.4.17) given for  $f_2$ .



We have

$$\begin{aligned}
f_3(x, y + \frac{1}{c}f_1(x, y)) &= a_k^y x^k + O(x^{k+1}) + b_l^y (y + \frac{1}{c}f_1(x, y))x^{l-1} + (y + \frac{1}{c}f_1(x, y))O(x^l) \\
&\quad + O((y + \frac{1}{c}f_1(x, y))^2) \\
&= a_k^y x^k + O(x^{k+1}) + b_l^y yx^{l-1} + \frac{b_l^y}{c}(O(x^k) + yO(x^{l-1}) + O(y^2))x^{l-1} \\
&\quad + yO(x^l) + \frac{1}{c}(O(x^k) + yO(x^{l-1}) + O(y^2))O(x^l) + O((y + \frac{1}{c}f_1(x, y))^2) \\
&= a_k^y x^k + O(x^{k+1}) + b_l^y yx^{l-1} + yO(x^l) + O((y + \frac{1}{c}f_1(x, y))^2).
\end{aligned} \tag{4.4.21}$$

Also, we have

$$\begin{aligned}
(y + \frac{1}{c}f_1(x, y))^2 &= y^2 + \frac{2}{c}y(O(x^k) + yO(x^{l-1}) + O(y^2)) + \frac{1}{c^2}(O(x^k) + yO(x^{l-1}) + O(y^2))^2 \\
&= O(x^{2k}) + yO(x^k) + O(y^2).
\end{aligned} \tag{4.4.22}$$

Then, if  $k \geq l$ , from (4.4.21) and (4.4.22), we have

$$f_3(x, y + \frac{1}{2}f_1(x, y)) = a_k^y x^k + O(x^{k+1}) + b_l^y yx^{l-1} + yO(x^l) + O(y^2),$$

as the form given for  $f_2$  in (4.4.17).

To prove statement (b), note that it is a generalization of statement (a). Analogously as in (4.4.21) we have

$$f_3(x, y + \frac{1}{c}f_1(x, y)) = a_k^y x^k + O(x^{k+1}) + yO(x) + O((y + \frac{1}{c}f_1(x, y))^2), \tag{4.4.23}$$

with  $(y + \frac{1}{c}f_1(x, y))^2$  as in (4.4.22). Hence, from (4.4.23) and (4.4.22), we get

$$f_3(x, y + \frac{1}{2}f_1(x, y)) = a_k^y x^k + O(x^{k+1}) + yO(x) + O(y^2),$$

as the form given for  $f_2$  in (4.4.19). □

As a consequence of the previous lemmas we obtain the following corollary, which shows how one can relate the coefficients of a vector field  $X$  of the form (4.1.3) with the coefficients of the reduced form of its time-1 flow.

**Corollary 4.4.5.** *Let  $X : U \rightarrow \mathbb{R}^2$  be a vector field of the form (4.1.3) and let  $\varphi_1$  be its time-1 flow. Consider the map  $T$  given in (4.4.15) and define  $\tilde{\varphi}_1 = T^{-1} \circ \varphi_1 \circ T$ . Then,  $\tilde{\varphi}_1$  is of the form (4.4.1). Moreover, if  $X$  is in case 1, then the coefficient  $a_k$  and the index  $k$  are the same for  $\tilde{\varphi}_1$ . If  $X$  is in case 2 or 3, the coefficients  $a_k$  and  $b_l$  and the indices  $k$  and  $l$  are the same for  $\tilde{\varphi}_1$ . In particular, the cases are preserved from the expression of  $X$  to the expression of  $\tilde{\varphi}_1$ .*

*Proof.* It is a direct consequence of Lemmas 4.4.1 and 4.4.4. Putting  $t = 1$  in Lemma 4.4.1, we have that the second component of the map  $\varphi_1$  has the same form as the second component of the map  $F$  given in (4.4.1). Then, applying Lemma 4.4.4 one has that  $\tilde{\varphi}_1 = T \circ \varphi_1 \circ T$  has the whole form (4.4.1) with the stated properties. □

Even if it is understood by the context, we remark that we have called  $\tilde{\varphi}_1 = T^{-1} \circ \varphi_1 \circ T$  the *reduced form* of  $\varphi_1$ .

**Remark 4.4.6.** Note that if  $X$  is in case 1, the values  $l$  and  $b_l$  may be different from the respective ones in  $\tilde{\varphi}_1$ , but in that case those coefficients are not relevant concerning the existence of invariant curves of  $X$  asymptotic to  $(0, 0)$ .

Finally, in the following lemma we show that a formal approximation of a parameterization of an invariant curve of a vector field  $X$  of the form (4.1.3) is also a formal approximation of the same order of a parameterization of the corresponding invariant curve of the time- $t$  map,  $\varphi_t$ , of  $X$ .

**Lemma 4.4.7.** *Let  $X : U \in \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a vector field of class  $C^r$  in a neighborhood  $U$  of 0 of the form (4.1.3), and let  $\hat{K} : (-\rho, \rho) \rightarrow \mathbb{R}^2$  and  $\hat{Y} : (-\rho, \rho) \rightarrow \mathbb{R}^2$  be an analytic map and an analytic vector field, respectively, satisfying*

$$\hat{K}(u) = \begin{cases} (u^2, \hat{K}_{k+1}^y u^{k+1}) + (O(u^3), O(u^{k+2})) & \text{case 1,} \\ (u, \hat{K}_l^y u^l) + (O(u^2), O(u^{l+1})) & \text{cases 2, 3,} \end{cases}$$

and  $\hat{Y}(u) = \hat{Y}_N u^N + O(u^{N+1})$ , and such that

$$F(\hat{K}(u)) - D\hat{K}(u)\hat{Y}(u) = (O(u^n), O(u^{n+N-1})), \quad (4.4.24)$$

for some  $n \geq 2$  in case 1 or  $n \geq 1$  in cases 2, 3.

Let  $\phi_t$  and  $\varphi_t$  be the time- $t$  flows of  $X$  and  $\hat{Y}$ , respectively, and assume that they are defined for all  $t \in [-1, 1]$ .

Then,

$$\phi_t(\hat{K}(u)) - \hat{K}(\varphi_t(u)) = (O(u^n), O(u^{n+N-1})), \quad u \in (0, \rho),$$

uniformly for all  $t \in [-1, 1]$ .

*Proof.* We define  $E(u) = X(\hat{K}(u)) - D\hat{K}(u) \cdot \hat{Y}(u)$ ,  $\varepsilon(t, u) = \phi(t, \hat{K}(u)) - \hat{K}(\varphi(t, u))$  and

$$g(t, u) = \frac{|\varepsilon^x(t, u)|}{u^n} + \frac{|\varepsilon^y(t, u)|}{u^{n+N-1}},$$

for  $t \in [-1, 1]$  and  $u \in (0, \rho)$ , and we see that there exists a constant  $M > 0$  such that  $\sup_{t \in [-1, 1]} \sup_{u \in (0, \rho)} g(t, u) \leq M$ .

By the properties of the flows  $\phi$  and  $\varphi$  we have

$$\begin{aligned} \varepsilon(t, u) &= \int_0^t \frac{\partial}{\partial s} [\phi(s, \hat{K}(u)) - \hat{K}(\varphi(s, u))] ds \\ &= \int_0^t [X(\phi(s, \hat{K}(u))) - D\hat{K}(\varphi(s, u)) \cdot \hat{Y}(\varphi(s, u))] ds \\ &= \int_0^t [X(\phi(s, \hat{K}(u))) - X(\hat{K}(\varphi(s, u)))] ds + \int_0^t [X(\hat{K}(\varphi(s, u))) - D\hat{K}(\varphi(s, u)) \cdot \hat{Y}(\varphi(s, u))] ds \\ &= \int_0^t [X(\phi(s, \hat{K}(u))) - X(\hat{K}(\varphi(s, u)))] ds + \int_0^t E(\varphi(s, u)) ds. \end{aligned} \quad (4.4.25)$$

By the mean value theorem we also have

$$\begin{aligned} X(\phi(s, \hat{K}(u))) - X(\hat{K}(\varphi(s, u))) &= \int_0^1 DX(\gamma(s, u, \nu)) d\nu [\phi(s, \hat{K}(u))] - \hat{K}(\varphi(s, u)) \\ &= \int_0^1 DX(\gamma(s, u, \nu)) d\nu \varepsilon(s, u), \end{aligned} \quad (4.4.26)$$

where

$$\gamma(s, u, \nu) = \nu \phi(s, K(u)) + (1 - \nu) K(\varphi(s, u)).$$

Summarising, from (4.4.25) and (4.4.26), we have

$$|\varepsilon(t, u)| \leq \left( \sup_{\nu \in [0,1]} \sup_{s \in [-1,1]} |DX(\gamma(s, u, \nu))| \right) \int_0^t |\varepsilon(s, u)| ds + \int_0^t |E(\varphi(s, u))| ds, \quad u \in (0, \rho), \quad (4.4.27)$$

where the absolute value is taken component by component, and the absolute value of a matrix denotes the matrix of its absolute values.

By the properties of the flows  $\phi_t$  and  $\varphi_t$  around the critical point at the origin (see Lemma 4.2.3 and Remark 4.4.3), we have

$$\phi_t(x, y) = O(\|(x, y)\|), \quad \varphi_t(u) = O(u),$$

and then by the form of  $\hat{K}$  we have  $\gamma(s, u, \nu) = (O(u^2), O(u^{k+1}))$  (case 1) and  $\gamma(s, u, \nu) = (O(u), O(u^l))$  (cases 2, 3). Finally, by the form of  $X$  we have, for some positive constants  $c_1, c_2, c_3$ ,

$$\begin{aligned} \sup_{\nu \in [0,1]} \sup_{s \in [-1,1]} |DX(\gamma(s, u, \nu))| &\leq \begin{pmatrix} 0 & c \\ c_1|u|^{2k-2} & c_3|u|^{2l-2} \end{pmatrix}, \quad u \in (0, \rho), \quad (\text{case 1}), \\ \sup_{\nu \in [0,1]} \sup_{s \in [-1,1]} |DX(\gamma(s, u, \nu))| &\leq \begin{pmatrix} 0 & c \\ c_1|u|^{k-1} + c_2|u|^{2l-2} & c_3|u|^{l-1} \end{pmatrix}, \quad u \in (0, \rho), \quad (\text{cases 2, 3}). \end{aligned}$$

Then, using (4.4.27) and the estimates above we can bound each of the components of  $\varepsilon(t, u)$  as follows. For case 1 we have

$$\begin{aligned} |\varepsilon^x(t, u)| &\leq c \int_0^t |\varepsilon^y(s, u)| ds + \int_0^t |E^x(\varphi(s, u))| ds, \\ |\varepsilon^y(t, u)| &\leq c_1|u|^{2k-2} \int_0^t |\varepsilon^x(s, u)| ds + c_3|u|^{2l-2} \int_0^t |\varepsilon^y(s, u)| ds + \int_0^t |E^y(\varphi(s, u))| ds. \end{aligned}$$

By hypothesis we have  $E(u) = (O(u^n), O(u^{n+N-1}))$  and thus also

$$E(\varphi(t, u)) = (O(u^n), O(u^{n+N-1})).$$

Then, taking into account, by the definition of  $g$ , that  $|\varepsilon^x(t, u)| \leq |u|^n g(t, u)$  and  $|\varepsilon^y(t, u)| \leq |u|^{n+k-1} g(t, u)$ , we can write

$$\begin{aligned} g(t, u) &\leq \frac{1}{|u|^n} \int_0^t |E^x(\varphi(s, u))| + \frac{1}{|u|^{n+k-1}} \int_0^t |E^y(\varphi(s, u))| ds \\ &\quad + (u^{k-1} + c_1 u^{k-1} + c_3 u^{2l-2}) \int_0^t g(s, u) ds \\ &\leq \tilde{M} + (u^{k-1} + c_1 u^{k-1} + c_3 u^{2l-2}) \int_0^t g(s, u) ds, \quad t \in [-1, 1], \quad u \in (0, \rho). \end{aligned}$$

Finally, applying Gronwall's lemma, we have

$$g(t, u) \leq \tilde{M} \exp\{(u^{k-1} + c_1 u^{k-1} + c_3 u^{2l-2}) t\}, \quad t \in [-1, 1], u \in (0, \rho),$$

and hence,

$$\sup_{t \in [-1, 1]} \sup_{u \in (0, \rho)} g(t, u) \leq \tilde{M} \exp((1 + c_1) \rho^{k-1} + c_3 \rho^{2l-2}) < M,$$

as we wanted to see.

Similarly, for cases 2 and 3 we get

$$|\varepsilon^x(t, u)| \leq c \int_0^t |\varepsilon^y(s, u)| ds + \int_0^t |E^x(\varphi(s, u))| ds,$$

$$|\varepsilon^y(t, u)| \leq (c_1 |u|^{k-1} + c_2 |u|^{2l-1}) \int_0^t |\varepsilon^x(s, u)| ds + c_3 |u|^{l-1} \int_0^t |\varepsilon^y(s, u)| ds + \int_0^t |E^y(\varphi(s, u))| ds,$$

and  $E(\varphi(t, u)) = (O(u^n), O(u^{n+N-1}))$ .

In particular, for case 3, since we have  $k > 2l - 1$  we can write

$$|\varepsilon^y(t, u)| \leq c_2 |u|^{2l-1} \int_0^t |\varepsilon^x(s, u)| ds + c_3 |u|^{l-1} \int_0^t |\varepsilon^y(s, u)| ds + \int_0^t |E^y(\varphi(s, u))| ds.$$

In these cases we have  $|\varepsilon^x(t, u)| \leq |u|^n g(t, u)$  and  $|\varepsilon^y(t, u)| \leq |u|^{n+l-1} g(t, u)$  and then we write

$$\begin{aligned} g(t, u) &\leq \frac{1}{|u|^n} \int_0^t |E^x(\varphi(s, u))| + \frac{1}{|u|^{n+l-1}} \int_0^t |E^y(\varphi(s, u))| ds \\ &\quad + (u^{l-1} + c_1 u^{k-1} + c_2 u^{2l-2} + c_3 u^{l-1}) \int_0^t g(s, u) ds \\ &\leq \tilde{M} + (u^{l-1} + c_1 u^{k-1} + c_2 u^{2l-2} + c_3 u^{2l-2}) \int_0^t g(s, u) ds, \quad t \in [-1, 1], u \in (0, \rho). \end{aligned}$$

Applying Gronwall's lemma, we have

$$g(t, u) \leq \tilde{M} \exp\{(u^{l-1} + c_1 u^{k-1} + c_2 u^{2l-2} + c_3 u^{2l-2}) t\}, \quad t \in [-1, 1], u \in (0, \rho),$$

and hence,

$$\sup_{t \in [-1, 1]} \sup_{u \in (0, \rho)} g(t, u) \leq \tilde{M} \exp(\rho^{l-1} + a_1 \rho^{k-1} + c_2 \rho^{2l-2} + c_3 \rho^{2l-2}) < M, \quad (\text{case 1}),$$

$$\sup_{t \in [-1, 1]} \sup_{u \in (0, \rho)} g(t, u) \leq \tilde{M} \exp(\rho^{l-1} + c_2 \rho^{2l-2} + c_3 \rho^{2l-2}) < M, \quad (\text{cases 2, 3}).$$

□

## 4.5 Existence of stable curves

In this section we introduce the main results of the chapter, which are Theorems 4.5.1 and 4.5.2, concerning analytic vector fields, and Theorems 4.5.3 and 4.5.4, concerning  $C^r$  vector fields. In the proofs of these theorems we will refer recurrently to the Theorems 2.2.1, 2.2.3, 3.2.1 and 3.2.4 stated in Chapter 2 and Chapter 3, respectively.

### 4.5.1 The analytic case

In the following results we provide the existence and regularity of stable curves asymptotic to the critical point at the origin for analytic vector fields of the form (4.1.3).

Theorem 4.5.1 is an *a posteriori* result which shows that, given an approximation of a parameterization of a stable curve of  $X$ , there exists a true stable curve close to this approximation. In Propositions 4.3.1, 4.3.3 and 4.3.4 we showed that one can compute explicitly such approximation.

In Theorem 4.5.2 we obtain the existence of an analytic stable curve of  $X$  without having an approximation of it and we give an expression of the restricted dynamics on the invariant curve, where we recover the normal form of a one-dimensional vector field around a parabolic singularity given by Takens in [70].

**Theorem 4.5.1.** *Let  $X : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be an analytic vector field in a neighborhood  $U$  of  $(0, 0)$  of the form (4.1.3), and let  $\hat{K} : (-\rho, \rho) \rightarrow \mathbb{R}^2$  and  $\hat{Y} : (-\rho, \rho) \rightarrow \mathbb{R}$  be an analytic map and an analytic vector field, respectively, satisfying*

$$\hat{K}(u) = \begin{cases} (u^2, \hat{K}_{k+1}^y u^{k+1}) + (O(u^3), O(u^{k+2})) & \text{case 1,} \\ (u, \hat{K}_l^y u^l) + (O(u^2), O(u^{l+1})) & \text{cases 2, 3,} \end{cases}$$

and  $\hat{Y}(u) = \hat{Y}_N u^N + O(u^{N+1})$ , with  $\hat{Y}_N < 0$ , and such that

$$X(\hat{K}(u)) - D\hat{K}(u) \cdot \hat{Y}(u) = (O(u^{n+N}), O(u^{n+2N-1})), \quad (4.5.1)$$

for some  $n \geq 2$  in case 1 or  $n \geq 1$  in cases 2, 3.

Then, there exists a  $C^1$  map  $K : [0, \rho) \rightarrow \mathbb{R}^2$ , analytic in  $(0, \rho)$ , and an analytic vector field  $Y : (-\rho, \rho) \rightarrow \mathbb{R}$  such that

$$X(K(u)) = DK(u) \cdot Y(u), \quad u \in [0, \rho) \quad (4.5.2)$$

and

$$\begin{aligned} K(u) - \hat{K}(u) &= (O(u^{n+1}), O(u^{n+N})), \\ Y(u) - \hat{Y}(u) &= \begin{cases} O(u^{2k-1}) & \text{if } n \leq k \\ 0 & \text{if } n > k \end{cases} \quad \text{case 1,} \\ Y(u) - \hat{Y}(u) &= \begin{cases} O(u^{2l-1}) & \text{if } n \leq l-1 \\ 0 & \text{if } n > l-1 \end{cases} \quad \text{cases 2, 3.} \end{aligned}$$

*Proof.* We write the proof for case 1, the other cases being almost identical except for some adjustments on the indices of the coefficients of  $\hat{K}$  and  $\hat{Y}$ .

Let us first define the maps  $\check{K}$  and  $\check{Y}$  as follows. If  $n \leq k$  we define  $\check{K}(u) = \hat{K}(u) + \sum_{j=n+1}^{k+1} \hat{K}_j(u)$ , with  $\hat{K}_j(u) = (\hat{K}_j^x u^j, \hat{K}_{j+k-1}^y u^{j+k-1})$ , and  $\check{Y}(u) = \hat{Y}(u) + \hat{Y}_{2k-1} u^{2k-1}$ , where the coefficients  $\hat{K}_j^x$ ,  $\hat{K}_{j+k-1}^y$  and  $\hat{Y}_{2k-1}$  are obtained imposing the condition

$$X(\check{K}(u)) - D\check{K}(u) \cdot \check{Y}(u) = (O(u^{2k+1}), O(u^{3k})).$$

Proceeding as in Proposition 4.3.1 one can obtain  $\hat{K}_j$  iteratively. We denote  $\mathcal{K}_j(u) = \hat{K}(u) + \sum_{m=n+1}^j \hat{K}_m(u)$  and  $\mathcal{Y}_j(u) = \hat{Y}(u) + \hat{Y}_j(u)$ , where  $\hat{Y}_j(u) = \delta_{j,k+1} \hat{Y}_{2k-1} u^{2k-1}$ . In the iterative step we have

$$X(\mathcal{K}_j(u)) - D\mathcal{K}_j(u) \cdot \mathcal{Y}_j(u) = (O(u^{j+k}), O(u^{j+2k-1})).$$

Then, similarly as in Proposition 4.3.1, applying Taylor's Theorem, we have

$$\begin{aligned} X(\mathcal{K}_j(u) + \hat{K}_{j+1}(u)) - D(\mathcal{K}_j(u) + \hat{K}_{j+1}(u)) \cdot (\hat{Y}(u) + \hat{Y}_j(u)) \\ = X(\mathcal{K}_j(u)) - D\mathcal{K}_j(u) \cdot \hat{Y}(u) + DX(\mathcal{K}_j(u)) \cdot \hat{K}_{j+1}(u) \\ + \int_0^1 (1-s) D^2 X(\mathcal{K}_j(u) + s\hat{K}_{j+1}(u)) ds (\hat{K}_{j+1}(u))^{\otimes 2} \\ - D\hat{K}_{j+1}(u) \cdot \mathcal{Y}_n(u) - D(\mathcal{K}_j(u) + \hat{K}_{j+1}(u)) \cdot \hat{Y}_j(u), \end{aligned}$$

and the condition  $X(\mathcal{K}_{j+1}(u)) - D\mathcal{K}_{j+1}(u) \cdot \mathcal{Y}_{j+1}(u) = (O(u^{j+k+1}), O(u^{j+2k}))$  leads to the same equation (4.3.4), which we solve in the same way. Otherwise, if  $n > k$  we take  $\check{K} = \hat{K}$  and  $\check{Y} = \hat{Y}$ . We also denote  $n_0 = k + 1$  if  $n \leq k$  and  $n_0 = n$  if  $n > k$ .

We have then,

$$X(\check{K}(u)) - \check{K}(\check{Y}(u)) = (O(u^{n_0+k}), O(u^{n_0+2k-1})).$$

Let  $\phi_t$  and  $\check{\varphi}_t$  be the flows of  $X$  and  $\check{Y}$ , respectively. Without loss of generality we assume that  $\phi_t$  and  $\check{\varphi}_t$  are defined for  $t \in [-1, 1]$ , as we already remarked. By Lemma 4.4.7, we have

$$\phi_t(\check{K}(u)) - \check{K}(\check{\varphi}_t(u)) = (O(u^{n_0+k}), O(u^{n_0+2k-1})), \quad u \in [0, \rho),$$

uniformly for all  $t \in [-1, 1]$ . In particular, for  $t = 1$  we have

$$\phi_1(\check{K}(u)) - \check{K}(\check{\varphi}_1(u)) = (O(u^{n_0+k}), O(u^{n_0+2k-1})), \quad u \in [0, \rho).$$

Next we consider the map  $T^{-1}$  given in (4.4.15) and we take  $\tilde{\phi}_1 = T^{-1} \circ \phi_1 \circ T$ . By the form of  $T^{-1}$ , and using Taylor's theorem, we have

$$\begin{aligned} \tilde{\phi}_1(T^{-1}(\check{K}(u))) - T^{-1}(\check{K}(\check{\varphi}_1(u))) &= T^{-1}(\phi_1(\check{K}(u)) - T^{-1}(\check{K}(\check{\varphi}_1(u)))) \\ &= DT^{-1}(\check{K}(\check{\varphi}_1(u)))(\phi(\check{K}(u)) - \check{K}(\check{\varphi}_1(u))) + h.o.t. \\ &= (O(u^{n_0+k}), O(u^{n_0+2k-1})). \end{aligned}$$

Moreover, by Corollary 4.4.5,  $\tilde{\phi}_1$  satisfies the hypotheses of Theorem 2.2.3, and also, by Remark 4.4.3 we have that  $\check{\varphi}_1$  is of the form  $\check{\varphi}_1(u) = u + \hat{Y}_k u^k + O(u^{k+1})$ . Then, by Theorem 2.2.3, there exist a map  $G : [0, \rho) \rightarrow \mathbb{R}^2$ , analytic in  $(0, \rho)$ , and an analytic map  $R : (-\rho, \rho) \rightarrow \mathbb{R}$ , such that  $\tilde{\phi}_1 \circ G = G \circ R$ , with

$$G(u) - T^{-1}\check{K}(u) = (O(u^{n_0+1}), O(u^{n_0+k})), \quad (4.5.3)$$

and  $R(u) \equiv \check{\varphi}_1(u)$ .

Finally, taking  $K = T \circ G$  we have

$$\phi_1(K(u)) = K(\check{\varphi}_1(u)), \quad u \in [0, \rho), \quad (4.5.4)$$

which means that  $K$  is a parameterization of the stable manifold of  $\phi_1$ , where  $\phi_1$  is the time-1 flow of  $X$  and  $\check{\varphi}_1$  is the time-1 flow of  $\check{Y}$ . Moreover, such  $K$  is unique once we fixed the approximation  $\hat{K}$ .

Next we define  $A_t(u) = \phi(-t, K(\check{\varphi}(t, u)))$ , for  $t \in [0, 1]$ , and using (4.5.4) we have

$$\begin{aligned} A_t(\check{\varphi}(1, u)) &= \phi_1(-t, K(\check{\varphi}(t, \check{\varphi}(1, u)))) = \phi(-t, K(\check{\varphi}(1, \check{\varphi}_1(t, u)))) \\ &= \phi(-t, \phi_1(1, K(\check{\varphi}(t, u)))) = \phi(1, \phi(-t, K(\check{\varphi}(t, u)))) = \phi(1, A_t(u)), \end{aligned}$$

that is,  $A_t(u)$  also satisfies (4.5.4), concretely with  $\phi_1(A_t(u)) = A_t(\check{\varphi}_1(u))$ . As a consequence, by the uniqueness of  $K$ , we have that  $K = A_t$  for all  $t \in [0, 1]$ , and thus,

$$\phi_t(K(u)) = K(\check{\varphi}_t(u)), \quad u \in [0, \rho), \quad t \in [0, 1].$$

Then, by Proposition 4.2.3 it holds that

$$X(K(u)) = DK(u) \cdot \check{Y}(u), \quad u \in [0, \rho),$$

and the statement (4.5.2) is proved taking  $Y = \check{Y}$ .

Finally, from (4.5.3) and the form of  $T$ , we have, using Taylor's theorem,

$$\begin{aligned} K(u) - \check{K}(u) &= T(G(u)) - \check{K}(u) = DT(G(u))(T^{-1}(\check{K}(u)) - G(u)) + h.o.t. \\ &= (O(u^{n_0+1}), O(u^{n_0+k})), \end{aligned}$$

and therefore, in the case when  $n > k$ , we obtain

$$K(u) - \hat{K}(u) = K(u) - \check{K}(u) = (O(u^{n_0+1}), O(u^{n_0+k})) = (O(u^{n+1}), O(u^{n+k})).$$

Otherwise, if  $n \leq k$  we have then

$$\begin{aligned} K(u) - \hat{K}(u) &= K(u) - \check{K}(u) + \check{K}(u) - \hat{K}(u) = K(u) - \check{K}(u) + \sum_{j=n+1}^{k+1} \hat{K}_j(u) \\ &= (O(u^{n_0+1}), O(u^{n_0+k})) + (O(u^{n+1}), O(u^{n+k})) \\ &= (O(u^{k+2}), O(u^{2k+1})) + (O(u^{n+1}), O(u^{n+k})) = (O(u^{n+1}), O(u^{n+k})). \end{aligned}$$

Similarly, we obtain the estimates for  $Y - \hat{Y}$ , where  $Y = \check{Y}$ . Indeed, if  $n > k$  we have  $Y - \hat{Y} = \check{Y} - \check{Y} = 0$ . If  $n \leq k$  we have  $Y - \hat{Y} = \check{Y} - \hat{Y} = \check{Y} - \check{Y} + \hat{Y}_{2k-1} u^{2k-1} = O(u^{2k-1})$ .

The  $C^1$  character of  $K$  at the origin follows from the order condition of  $K$  at 0.  $\square$

**Theorem 4.5.2.** *Let  $X : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be an analytic vector field in a neighborhood  $U$  of  $(0, 0)$  of the form (4.1.3). Assume the following hypotheses according to the different cases:*

$$(case\ 1) \quad a_k > 0, \quad (case\ 2) \quad a_k > 0, b_l \neq 0, \quad (case\ 3) \quad b_l < 0.$$

*Then, there exists a  $C^1$  map  $K : [0, \rho) \rightarrow \mathbb{R}^2$ , analytic in  $(0, \rho)$ , such that*

$$K(u) = \begin{cases} (u^2, K_{k+1}^y u^{k+1}) + (O(u^3), O(u^{k+2})), & case\ 1, \\ (u, K_l^y u^l) + (O(u^2), O(u^{l+1})), & cases\ 2, 3, \end{cases}$$

with  $K_{k+1}^y = -\sqrt{\frac{2a_k}{c(k+1)}}$  for case 1,  $K_l^y = \frac{b_l - \sqrt{b_l^2 + 4ca_k l}}{2cl}$  for case 2 and  $K_l^y = \frac{b_l}{cl}$  for case 3, and a polynomial one-dimensional vector field  $Y$  of the form  $Y(u) = Y_N u^N + Y_{2N-1} u^{2N-1}$ , with  $Y_k = \frac{c}{2} K_{k+1}^y$  for case 1 and  $Y_l = cK_l^y$  for cases 2, 3, such that

$$X(K(u)) = DK(u) \cdot Y(u), \quad u \in [0, \rho].$$

*Proof.* For case 1, we consider the maps  $\mathcal{K}_n$  and  $\mathcal{Y}_n$  provided in Proposition 4.3.1, for some  $n \geq k+1$ . Concretely we take  $\mathcal{Y}_n(u) = Y_k u^k + Y_{2k-1} u^{2k-1}$ . The coefficients  $K_{k+1}^y$  and  $Y_k$  of these maps are the ones provided in the statement, and moreover,  $\mathcal{K}_n$  and  $\mathcal{Y}_n$  clearly satisfy the hypotheses of Theorem 4.5.1 required for  $\hat{K}$  and  $\hat{Y}$ , respectively. Thus, by Theorem 4.5.1, there exists a  $C^1$  map  $K : [0, \rho] \rightarrow \mathbb{R}^2$ , analytic in  $(0, \rho)$ , and an analytic vector field  $Y : (-\rho, \rho) \rightarrow \mathbb{R}$  such that

$$X(K(u)) = DK(u) \cdot Y(u), \quad u \in [0, \rho],$$

and also, since  $n \geq k+1$ , we have  $Y(u) \equiv \mathcal{Y}_n(u) = Y_k u^k + Y_{2k-1} u^{2k-1}$ . Also, it is clear that  $K$  has the form (4.5.2) since we have  $K(u) - \mathcal{K}_n(u) = (O(u^{n+1}), O(u^{n+N}))$ .

Again, the  $C^1$  character of  $K$  at the origin follows from the order condition of  $K$  at 0.

The proof for cases 2 and 3 follows in an analogous way using Propositions 4.3.3 and 4.3.4, respectively.  $\square$

## 4.5.2 The differentiable case

Next Theorem 4.5.3 concerns the existence of a stable curve of a vector field  $X$  of the form (4.1.3), asymptotic to 0, in the case when  $X$  is of class  $C^r$ . It is well known that the integral curves of a  $C^r$  vector field are also  $C^r$ . Hence, to prove the existence of a  $C^r$  stable curve of a vector field  $X$  of the form (4.1.3), it would be sufficient to show that there exists an integral curve of  $X$  asymptotic to 0. For this reason, to use our method based on finding an integral curve of  $X$  by knowing the existence of an invariant curve of the time-1 flow  $\varphi_1$  of  $X$ , it is sufficient to have a continuous stable curve,  $K$ , of  $\varphi_1$ . The existence of such a stable curve for the corresponding class of maps is already proved in [25], and therefore the results of this section can be deduced from the ones of that paper. However, here we have used the parameterization method and we present a proof of Theorem 4.5.3 based on the results presented in Section 3. As in the analytic case, we also provide an *a posteriori* result (Theorem 4.5.4).

**Theorem 4.5.3.** *Let  $X : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a  $C^r$  vector field in a neighborhood  $U$  of  $(0, 0)$  of the form (4.1.3) with  $r \geq 3$ .*

*Assume the following hypotheses according to the different cases:*

- (case 1)  $a_k > 0$  and  $r \geq \frac{3}{2}k$ ,
- (case 2)  $a_k > 0$ ,  $b_l \neq 0$ ,  $r > k$  and

$$\max \left\{ \frac{\beta}{(r-2l+2)(r-l+1)} (2l(l-1) + \frac{ck a_k}{b_l^2} \beta), \frac{2l\beta}{r-l+1} \right\} < 1,$$

$$\text{where } \beta = \frac{2l|b_l|}{|b_l - \sqrt{b_l^2 + 4ca_k l}|}.$$



- (case 3)  $b_l < 0$ ,  $r > 2l - 1$  and  $\frac{l(l-1)}{(r-2l+2)(r-l+1)} < 1$ .

Then, there exists a  $C^1$  map  $H : [0, \rho) \rightarrow \mathbb{R}^2$ ,  $H \in C^r(0, \rho)$ , of the form

$$H(u) = \begin{cases} (u^2, H_{k+1}^y u^{k+1}) + (O(u^3), O(u^{k+2})), & \text{case 1,} \\ (u, H_l^y u^l) + (O(u^2), O(u^{l+1})), & \text{cases 2, 3,} \end{cases}$$

with  $H_{k+1}^y = -\sqrt{\frac{2a_k}{c(k+1)}}$  for case 1,  $H_l^y = \frac{b_l - \sqrt{b_l^2 + 4ca_k l}}{2cl}$  for case 2 and  $H_l^y = \frac{b_l}{cl}$  for case 3, and a polynomial one-dimensional vector field  $Y$  of the form  $Y(u) = Y_N u^N + Y_{2N-1} u^{2N-1}$ , with  $Y_k = \frac{c}{2} H_{k+1}^y$  for case 1 and  $Y_l = c H_l^y$  for cases 2, 3, such that

$$X(H(u)) = DH(u) \cdot Y(u), \quad u \in [0, \rho).$$

If the vector field  $X$  is  $C^\infty$  then the parameterization  $H$  is  $C^\infty$  in  $(0, \rho)$ .

*Proof.* We write the proof for case 1, the other cases being almost identical except for some adjustments in the indices of the coefficients of  $H$  and  $Y$ . We write  $X = X^\leq + (0, g)$  where  $X^\leq$  denotes the Taylor polynomial of degree  $r$  of  $X$ . Then, by Theorem 4.5.2, there exist a  $C^1$  map  $K : [0, \rho) \rightarrow \mathbb{R}^2$ , analytic in  $(0, \rho)$ , such that

$$K(u) = (u^2, K_{k+1}^y u^{k+1}) + (O(u^3), O(u^{k+2})),$$

with  $K_{k+1}^y = -\sqrt{\frac{2a_k}{c(k+1)}}$ , and a polynomial one-dimensional vector field  $Y$  of the form  $Y(u) = Y_k u^k + Y_{2k-1} u^{2k-1}$ , with  $Y_k = \frac{c}{2} K_{k+1}^y$  such that

$$X^\leq(K(u)) = DK(u) \cdot Y(u), \quad u \in [0, \rho).$$

Next let us define  $\phi_t$  as the time- $t$  flow of  $X$ ,  $\phi_t^\leq$  as the time- $t$  flow of  $X^\leq$ , and  $\varphi_t$  as the time- $t$  flow of  $Y$ . Without loss of generality we assume that those flows are defined for all  $t \in [-1, 1]$ . By Proposition 4.2.3 we have

$$\phi_t^\leq(K(u)) = K(\varphi_t(u)), \quad u \in [0, \rho), \quad t \in [-1, 1],$$

and therefore, putting  $t = 1$ ,

$$\phi_1^\leq(K(u)) = K(\varphi_1(u)), \quad u \in [0, \rho). \quad (4.5.5)$$

Also, by the definition and the properties of the flow  $\phi_t$  we have

$$\begin{aligned} \phi_t(x, y) &= (x, y) + \int_0^t X(\phi_s(x, y)) ds = (x, y) + \int_0^t X^\leq(\phi_s(x, y)) ds + \int_0^t (0, g(\phi_s(x, y))) ds \\ &= \phi_t^\leq(x, y) + (0, \bar{g}_t(x, y)), \end{aligned}$$

where  $\phi_t^\leq$  is analytic in  $U$  and  $\bar{g}_t$  is a function of class  $C^r$  in  $U$  with  $\bar{g}_t(x, y) = o(\|(x, y)\|^r)$ , for each  $t \in [-1, 1]$ .

From (4.5.5) we have that  $K$  is an analytic invariant curve of the map  $\phi_1^\leq$ . Next we look for a  $C^r$  invariant curve,  $H$ , of the map  $\phi_1$ , given by  $H = K + \Delta$ , with  $\Delta \in C^r(0, \rho)$  and  $\Delta(u) = (O(u^{2r-2k+2}), O(u^{2r-k+1}))$ . To do so we proceed as in the proof of Theorem 3.2.1.

We consider the map  $T$  given in (4.4.15) and we take  $\tilde{\phi}_1 = T^{-1} \circ \phi_1 \circ T$  and  $\tilde{\phi}_1^{\leq} = T^{-1} \circ \phi_1^{\leq} \circ T$ . By Corollary 4.4.5 and taking into account the hypotheses on the coefficients stated for  $X$ ,  $\tilde{\phi}_1$  satisfies the hypotheses of Theorem 3.2.1, and  $\tilde{\phi}_1^{\leq}$  satisfies the hypotheses of Theorem 2.2.1. Also, composing by  $T^{-1}$  in (4.5.5), we have that  $\tilde{K} := T^{-1} \circ K$  is an analytic map in  $(0, \rho)$  that satisfies

$$\tilde{\phi}_1^{\leq}(\tilde{K}(u)) = \tilde{K}(\varphi_1(u)), \quad u \in [0, \rho).$$

Then, following the proof of Theorem 3.2.1, there exist  $\tilde{\Delta} \in C^r(0, \rho)$ , with  $\tilde{\Delta}(u) = (O(u^{2r-2k+2}), O(u^{2r-k+1}))$ , such that  $\tilde{H} = \tilde{K} + \tilde{\Delta}$  satisfies

$$\tilde{\phi}_1(\tilde{H}(u)) = \tilde{H}(\varphi_1(u)), \quad u \in [0, \rho).$$

Therefore, taking  $\Delta = T \circ \tilde{\Delta}$  and  $H = T \circ \tilde{H} = K + \Delta$  we have

$$\phi_1(H(u)) = H(\varphi_1(u)), \quad u \in [0, \rho),$$

which means that  $H$  is a parameterization of an invariant curve of  $\phi_1$ . Also, such  $H$  is unique once we fixed  $K$ , and by the same argument as in the proof of Theorem 4.5.1 we have

$$\phi_t(H(u)) = H(\varphi_t(u)), \quad u \in [0, \rho), \quad t \in [0, 1].$$

Finally, applying again Proposition 4.2.3 we have

$$X(H(u)) = DH(u) \cdot Y(u), \quad u \in [0, \rho).$$

Note also that by the form of  $T$  we have  $\Delta(u) = T(\tilde{\Delta}(u)) = (O(u^{2r-2k+2}), O(u^{2r-k+1}))$ , and the coefficients of  $H$  and  $Y$  are the ones in the statement since we have  $H(u) - K(u) = (O(u^{2r-2k+2}), O(u^{2r-k+1}))$ , and thus,  $H_{k+1}^y = K_{k+1}^y$  and  $Y_k = \frac{c}{2}K_{k+1}^y$ . Also, the  $C^1$  character of  $H$  at the origin follows from the order condition of  $K$  at 0.  $\square$

**Theorem 4.5.4.** *Let  $X : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a  $C^r$  vector field satisfying the hypotheses of Theorem 4.5.3 and let  $\hat{K} : (-\rho, \rho) \rightarrow \mathbb{R}^2$  and  $\hat{Y} : (-\rho, \rho) \rightarrow \mathbb{R}$  be analytic maps satisfying*

$$\hat{K}(u) = \begin{cases} (u^2, \hat{K}_{k+1}^y u^{k+1}) + (O(u^3), O(u^{k+2})) & \text{case 1,} \\ (u, \hat{K}_l^y u^l) + (O(u^2), O(u^{l+1})) & \text{cases 2, 3,} \end{cases}$$

and  $\hat{Y}(u) = \hat{Y}_N u^N + O(u^{N+1})$ ,  $\hat{Y}_N < 0$ , such that

$$F(\hat{K}(u)) - \hat{K}(u) \cdot \hat{Y}(u) = (O(u^{n+N}), O(u^{n+2N-1})), \quad (4.5.6)$$

for some  $2 \leq n \leq 2r - 2k + 1$  in case 1 or  $1 \leq n \leq r - 2l + 1$  in cases 2, 3.

Then, there exists a  $C^1$  map  $H : [0, \rho) \rightarrow \mathbb{R}^2$ ,  $H \in C^r(0, \rho)$ , and an analytic vector field  $Y : (-\rho, \rho) \rightarrow \mathbb{R}$  such that

$$F(H(u)) = H(u) \cdot Y(u), \quad u \in [0, \rho)$$

and

$$\begin{aligned} H(t) - \hat{K}(t) &= (O(t^{n+1}), O(t^{n+N})), \\ Y(u) - \hat{Y}(u) &= \begin{cases} O(u^{2k-1}) & \text{if } n \leq k \\ 0 & \text{if } n > k \end{cases} \quad \text{case 1,} \\ Y(u) - \hat{Y}(u) &= \begin{cases} O(u^{2l-1}) & \text{if } n \leq l - 1 \\ 0 & \text{if } n > l - 1 \end{cases} \quad \text{cases 2, 3.} \end{aligned}$$

*Proof.* We write the proof for case 1, the other cases being almost identical except for some adjustments on the indices of the coefficients of  $\hat{K}$  and  $\hat{Y}$ . We write  $X = X^{\leq} + (0, g)$  where  $X^{\leq}$  denotes the Taylor polynomial of degree  $r$  of  $X$ . Then, from (4.5.6) we have

$$X(\hat{K}(u)) - D\hat{K}(u) \cdot \hat{Y}(u) = X^{\leq}(\hat{K}(u)) + (0, g(\hat{K}(u))) - D\hat{K}(u) \cdot \hat{Y}(u) = (O(u^{n+k}), O(u^{n+2k-1})),$$

and thus, since  $g(\hat{K}(u)) = o(u^{2r})$  and  $n \leq 2r - 2k + 1$ , we have

$$X^{\leq}(\hat{K}(u)) - \hat{K}(u) \cdot \hat{Y}(u) = (O(u^{n+k}), O(u^{n+2k-1})) + (0, O(u^{2r})) = (O(u^{n+k}), O(u^{n+2k-1})).$$

Clearly,  $X^{\leq}$  is analytic and satisfies the hypotheses of Theorem 4.5.1. Then, there exists a  $C^1$  map  $K : [0, \rho) \rightarrow \mathbb{R}^2$ , analytic in  $(0, \rho)$ , and an analytic vector field  $Y : (-\rho, \rho) \rightarrow \mathbb{R}^2$  such that

$$X^{\leq}(K(u)) = DK(u) \cdot Y(u), \quad u \in [0, \rho),$$

and

$$\begin{aligned} K(u) - \hat{K}(u) &= (O(u^{n+1}), O(u^{n+k})), \\ Y(u) - \hat{Y}(u) &= \begin{cases} O(u^{2k-1}) & \text{if } n \leq k, \\ 0 & \text{if } n > k. \end{cases} \end{aligned}$$

Also, following the proof of Theorem 4.5.3, there exists a  $C^1$  map  $H : [0, \rho) \rightarrow \mathbb{R}^2$ ,  $H \in C^r(0, \rho)$ , given by  $H = K + \Delta$ , with  $\Delta = (O(u^{2r-2k+2}), O(u^{2r-k+1}))$ , such that

$$X(H(u)) = DH(u) \cdot Y(u), \quad u \in [0, \rho).$$

To complete the proof of the theorem, note that we have

$$\begin{aligned} H(u) - \hat{K}(u) &= K(u) - \hat{K}(u) + \Delta(u) \\ &= (O(u^{n+1}), O(u^{n+k})) + (O(u^{2r-2k+2}), O(u^{2r-k+1})) = (O(u^{n+1}), O(u^{n+k})). \end{aligned}$$

Again, the  $C^1$  character of  $H$  at the origin follows from the order condition of  $K$  at 0.  $\square$

## Chapter 5

# Whiskered parabolic tori with nilpotent part. Map case.

### 5.1 Introduction

The objective of this chapter is to study the existence and regularity of invariant manifolds of analytic maps asymptotic to an invariant parabolic torus where its complementary dimension is two.

We consider analytic maps  $G : U \times \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$  of the form

$$G(x, y, \theta, \lambda) = \begin{pmatrix} x + c(\theta, \lambda)y + f_1(x, y, \theta, \lambda) \\ y + f_2(x, y, \theta, \lambda) \\ \theta + \omega + f_3(x, y, \theta, \lambda) \end{pmatrix}, \quad (5.1.1)$$

with  $(x, y) \in U \subset \mathbb{R}^2$ ,  $\theta \in \mathbb{T}^d$ ,  $\omega \in \mathbb{R}^d$ ,  $\lambda \in \Lambda \subset \mathbb{R}^m$ , and with  $f_1(x, y, \theta, \lambda), f_2(x, y, \theta, \lambda) = O(\|(x, y)\|^2)$ , and  $f_3(x, y, \theta, \lambda) = O(\|(x, y)\|)$ .

By simple changes of variables, similarly as in Section 2.2.1 of Chapter 2, such maps can be brought to the form

$$F(x, y, \theta, \lambda) = \begin{pmatrix} x + c(\theta, \lambda)y \\ y + a_k(\theta, \lambda)x^k + A(x, y, \theta, \lambda) \\ \theta + \omega + d_p(\theta, \lambda)x^p + B(x, y, \theta, \lambda) \end{pmatrix}, \quad (5.1.2)$$

with  $k \geq 2$ ,  $p \geq 1$ , where  $A(x, y, \theta, \lambda) = yO(\|(x, y)\|^{k-1}) + O(\|(x, y)\|^{k+1})$ ,  $B(x, y, \theta, \lambda) = yO(\|(x, y)\|^{p-1}) + O(\|(x, y)\|^{p+1})$ , and where  $c(\theta, \lambda)$  has positive mean, namely  $\bar{c} > 0$ .

The set

$$\mathcal{T} = \{(0, 0, \theta) \in U \times \mathbb{T}^d\}$$

is an invariant torus of  $F$ , that is, for all  $\lambda \in \Lambda$ ,  $F(\mathcal{T}, \lambda) \subset \mathcal{T}$ . We say that  $\mathcal{T}$  is a *parabolic torus with nilpotent part* because the top-left  $2 \times 2$  box of the matrix  $DF(0, 0, \theta)$  is

$$\begin{pmatrix} 1 & c(\theta, \lambda) \\ 0 & 1 \end{pmatrix}.$$

In this chapter we study the existence and regularity of  $(d+1)$ -dimensional invariant manifolds of analytic maps of the form (5.1.2). Such maps are, in some sense, a generalization of the maps studied in Chapters 2 and 3 because they present an analogous form when restricted to the first two components, that here represent the dynamics in the directions normal to  $\mathcal{T}$ .

We use similar methods to the ones used in [7] for the study of the existence of invariant manifolds of analytic maps defined on  $\mathbb{R}^n \times \mathbb{T}^d$ , and where the first  $n \times n$  box of the linear part is equal to the identity. There, applications to the study of the planar  $(n+1)$ -body problem are provided.

Contrary to the planar case studied in Chapters 2 and 3, here we will not consider different cases for maps of the form (5.1.2) depending on the indices  $k$  and  $p$  related to the first order terms of the expansion of  $F$ . For simplicity on the notation we will only consider those maps with  $a_k(\theta, \lambda), d_p(\theta, \lambda) \neq 0$ , which include the generic case.

Invariant manifolds of dynamical systems asymptotic to invariant tori are often called *whiskers* in the literature. We will sometimes refer to them also by this name.

As for the planar case, we will provide an algorithm to compute an approximation of a parameterization of the invariant manifolds and two results of existence, one of them given as an *a posteriori* result. Moreover we will give results concerning the analytic dependence on parameters. However, to avoid cumbersome notation we will sometimes omit the dependence of the functions we work with on the parameter  $\lambda$  when there is no danger of confusion. Concretely, we present the statements, the setting and the function spaces with full detail but we skip the dependence on the parameters in the lemmas and proofs.

Also as in the planar case, we cannot expect the invariant manifolds obtained for maps of the form (5.1.2) to be analytic in a neighborhood of the invariant torus,  $\mathcal{T}$ .

For our study we will use again the parameterization method (see Section 2.2.2) adapted to the current setting. Here we will look for maps  $K(t, \theta, \lambda) : [0, \rho) \times \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$  and  $R(t, \theta, \lambda) : [0, \rho) \times \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R} \times \mathbb{T}^d$  satisfying the invariance equation

$$F(K(t, \theta, \lambda), \lambda) = K(R(t, \theta, \lambda), \lambda),$$

and such that  $K(0, \theta, \lambda) = (0, 0, \theta)$ ,  $R(0, \theta, \lambda) = (0, \theta + \omega)$ , and  $\partial_t K^x / \partial_t K^x \rightarrow 0$  as  $t \rightarrow 0$ .

Following the notation of Chapter 2, here  $t$  denotes a real variable that parameterizes the invariant manifolds of  $F$ . In Chapters 4 and 6, dedicated to vector fields, we use  $u$  to denote that real parameter and  $t$  to denote the time variable.

The main results of this chapter are Theorems 5.3.1 and 5.3.2, concerning the existence of analytic invariant manifolds of a map  $F$  of the form (5.1.2). The results are stated for the stable manifolds. In Section 5.5 we show that completely analogous results hold true for the unstable ones. In Section 5.4 we provide an algorithm to obtain parameterizations of approximations of the invariant manifolds of  $F$ . The rest of the chapter is dedicated to introduce the techniques used for the proofs of the main theorems. Finally we provide the proofs of the main results in Section 5.8.

## 5.2 Preliminaries and notation

In this section we present some notation and preliminary results that will be used along the chapter. We start with some notation and definitions,

- Real and complex  $d$ -torus: the *real torus* is  $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ . Given  $\sigma > 0$ , the *complex torus* is

$$\mathbb{T}_\sigma^d = \{\theta = (\theta_1, \dots, \theta_d) \in (\mathbb{C}/\mathbb{Z})^d \mid |\operatorname{Im} z| < \sigma\}.$$

- Given  $\beta, \rho > 0$  such that  $\rho < 1$  and  $\beta < \pi$ , let  $S$  be the complex sector

$$S = S(\beta, \rho) = \{z \in \mathbb{C} \mid |\arg(z)| < \frac{\beta}{2}, 0 < |z| < \rho\}.$$

- We will often consider functions depending on a parameter,  $\lambda \in \Lambda$ , with  $\Lambda \subset \mathbb{R}^m$ . We denote by  $\Lambda_{\mathbb{C}} \subset \mathbb{C}^m$  a complex neighborhood of  $\Lambda$ .
- Let  $U \subset \mathbb{R}^k \times \mathbb{T}^d$  and  $V \subset \mathbb{R}^{k'} \times \mathbb{T}^{d'}$  be open sets. If  $\lambda \in \Lambda$  is a parameter, given functions  $g : U \times \Lambda \rightarrow V$  and  $h : V \times \Lambda \rightarrow \mathbb{R}^{k''} \times \mathbb{T}^{d''}$ , the composition  $f = h \circ g$  is defined as

$$f(x, \lambda) = h(g(x, \lambda), \lambda).$$

- Given a map  $h : \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R}^n$ , we define the *average* of  $h$  with respect to  $\theta \in \mathbb{T}^d$  as

$$\bar{h}(\lambda) = \frac{1}{\operatorname{vol}(\mathbb{T}^d)} \int_{\mathbb{T}^d} h(\theta, \lambda) d\theta,$$

and the *oscillatory* part of  $h$  as

$$\tilde{h}(\theta, \lambda) = h(\theta, \lambda) - \bar{h}(\lambda).$$

- The superindices  $x, y$  and  $\theta$  on the symbol of a function or an operator with values in  $\mathbb{R}^2 \times \mathbb{T}^d$  will denote its respective components.

Next we introduce some preliminary theory concerning Diophantine vectors and the small divisors equation.

We say that  $\omega \in \mathbb{R}^d$  is *Diophantine (in the map setting)* if there exists  $c > 0$  and  $\tau \geq d$  such that

$$|\omega \cdot k - l| \geq c|k|^{-\tau}, \quad \text{for all } k \in \mathbb{Z}^d \setminus \{0\}, l \in \mathbb{Z},$$

where  $|k| = |k_1| + \dots + |k_d|$  and  $\omega \cdot k$  denotes the scalar product.

Along the proofs of some of the results, when solving cohomological equations to compute approximations of parameterizations of invariant manifolds, we will encounter the so-called *small divisors equation*. In the map setting such equation has the following form,

$$\varphi(\theta + \omega, \lambda) - \varphi(\theta, \lambda) = h(\theta, \lambda), \tag{5.2.1}$$

with  $h : \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R}^n$  and  $\omega \in \mathbb{R}^d$ .

In order to find a solution  $\varphi(\theta, \lambda)$  of (5.2.1) we develop  $h$  as a Fourier series with respect to  $\theta$ ,

$$h(\theta, \lambda) = \sum_{k \in \mathbb{Z}^d} h_k(\lambda) e^{2\pi i k \cdot \theta},$$

with

$$h_k(\lambda) = \int_0^1 h(\theta, \lambda) e^{-2\pi i k \cdot \theta} d\theta, \quad k \cdot \theta = k_1 \theta_1 + \cdots + k_d \theta_d.$$

If  $h$  has zero average and  $k \cdot \omega \notin \mathbb{Z}$  for all  $k \neq 0$ , then equation (5.2.1) has the formal solution

$$\varphi(\theta, \lambda) = \sum_{k \in \mathbb{Z}^d} \varphi_k(\lambda) e^{2\pi i k \cdot \theta}, \quad \varphi_k(\lambda) = \frac{h_k(\lambda)}{1 - e^{2\pi i k \cdot \omega}}, \quad k \neq 0.$$

Note that all coefficients  $\varphi_k$  are uniquely determined except for  $\varphi_0$  (the average of  $\varphi$ ), which is free.

The following well-known result establishes the existence of a solution to equation (5.2.1) when  $h$  is analytic.

**Theorem 5.2.1** (Small divisors lemma for maps). *Let  $h : \mathbb{T}_\sigma^d \times \Lambda_{\mathbb{C}} \rightarrow \mathbb{C}^n$  be analytic with zero average and let  $\omega \in \mathbb{R}^d$  be Diophantine with  $\tau \geq d$ . Then, there exists a unique analytic solution  $\varphi : \mathbb{T}_\sigma^d \times \Lambda_{\mathbb{C}} \rightarrow \mathbb{C}^n$  of (5.2.1) with zero average. Moreover,*

$$\sup_{(\theta, \lambda) \in \mathbb{T}_{\sigma-\delta}^d \times \Lambda_{\mathbb{C}}} \|\varphi(\theta, \lambda)\| \leq C \delta^{-\tau} \sup_{(\theta, \lambda) \in \mathbb{T}_\sigma^d \times \Lambda_{\mathbb{C}}} \|h(\theta, \lambda)\|, \quad 0 < \delta < \sigma,$$

where  $C$  depends on  $\tau$  and  $d$  but not on  $\delta$ .

The proof with close to optimal estimates is due to Russmann [63].

We will denote by  $\mathcal{SD}(h)$  the unique solution of (5.2.1) with zero average.

### 5.3 Main results

In this section we state two theorems of existence of analytic stable invariant manifolds of a map  $F$  of the form (5.1.2) asymptotic to its invariant torus  $\mathcal{T}$ . The second Theorem is an *a posteriori* result, which provides the existence of a stable manifold assuming it has been previously approximated but the statement is independent of the way such an approximation has been obtained.

**Theorem 5.3.1.** *Let  $F : U \times \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$  be an analytic map of the form (5.1.2). Assume that  $2p > k - 1$ ,  $\bar{a}_k(\lambda) > 0$  for  $\lambda \in \Lambda$ , and that  $\omega$  is Diophantine. Then, there exists  $\rho > 0$  and a  $C^1$  map  $K : [0, \rho) \times \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$ , analytic in  $(0, \rho) \times \mathbb{T}^d \times \Lambda$ , of the form*

$$K(t, \theta, \lambda) = (t^2, \bar{K}_{k+1}^y(\lambda) t^{k+1}, \theta + \bar{K}_{2p-k+1}^\theta(\lambda) t^{2p-k+1}) + (O(t^3), O(t^{k+2}), O(t^{2p-k+2})),$$

and a polynomial  $R$  of the form

$$R(t, \theta, \lambda) = \begin{pmatrix} t + \bar{R}_k^x(\lambda) t^k + \bar{R}_{2k-1}^x(\lambda) t^{2k-1} \\ \theta + \omega \end{pmatrix},$$

with  $\bar{R}_k^x(\lambda) < 0$ , such that

$$F(K(t, \theta, \lambda), \lambda) = K(R(t, \theta, \lambda), \lambda), \quad (t, \theta, \lambda) \in [0, \rho] \times \mathbb{T}^d \times \Lambda.$$

Moreover, we have

$$\bar{K}_{k+1}^y(\lambda) = -\sqrt{\frac{2\bar{a}_k(\lambda)}{\bar{c}(\lambda)(k+1)}}, \quad \bar{K}_{2p-k+1}^\theta(\lambda) = -\frac{\bar{d}_p(\lambda)}{2p-k+1} \sqrt{\frac{2(k+1)}{\bar{c}(\lambda)\bar{a}_k}}, \quad \bar{R}_k^x(\lambda) = -\sqrt{\frac{\bar{c}(\lambda)\bar{a}_k(\lambda)}{2(k+1)}}.$$

The statement of Theorem 5.3.1 provides a local stable manifold parameterized by  $K : [0, \rho] \times \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$  with  $\rho$  small and the proof does not give an explicit estimate for the value of  $\rho$ . However, as in the case of planar maps (see Chapters 2 and 3), we can extend the domain of  $K$  by using the formula

$$K(t) = F^{-j}K(R^j(t)), \quad j \geq 1,$$

while the iterates of the inverse map  $F^{-1}$  exist.

We also note that the first component of the map  $R$  (corresponding to the directions normal to the invariant torus) given in the statement of Theorem 5.3.1 is the normal form of the dynamics of a one-dimensional system in a neighborhood of a parabolic point ([18, 70]), already seen in Chapters 2 and 3. In the second component,  $R$  defines a rigid rotation of frequency  $\omega$ .

**Theorem 5.3.2.** *Let  $F : U \times \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$  be an analytic map of the form (5.1.2), and let  $\hat{K} : (-\rho, \rho) \times \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$  and  $\hat{R} : (-\rho, \rho) \times \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R} \times \mathbb{T}^d$  be analytic maps of the form*

$$\hat{K}(t, \theta, \lambda) = (t^2, \bar{K}_{k+1}^y(\lambda)t^{k+1}, \theta + \bar{K}_{2p-k+1}^\theta(\lambda)t^{2p-k+1}) + (O(t^3), O(t^{k+2}), O(t^{2p-k+2})),$$

and

$$\hat{R}(t, \theta, \lambda) = \begin{pmatrix} t + \bar{R}_k^x(\lambda)t^k + O(t^{k+1}) \\ \theta + \omega \end{pmatrix},$$

with  $\bar{R}_k^x(\lambda) < 0$ , satisfying

$$F(\hat{K}(t, \theta, \lambda), \lambda) - \hat{K}(\hat{R}(t, \theta, \lambda), \lambda) = (O(t^{n+k}), O(t^{n+2k-1}), O(t^{n+2p-1})),$$

for some  $n \geq 2$ .

Then, there exists a  $C^1$  map  $K : [0, \rho] \times \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$ , analytic in  $(0, \rho) \times \mathbb{T}^d \times \Lambda$ , and an analytic map  $R : (-\rho, \rho) \times \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R} \times \mathbb{T}^d$  such that

$$F(K(t, \theta, \lambda), \lambda) = K(R(t, \theta, \lambda), \lambda), \quad (t, \theta) \in [0, \rho] \times \mathbb{T}^d,$$

and

$$K(t, \theta, \lambda) - \hat{K}(t, \theta, \lambda) = (O(t^{n+1}), O(t^{n+k}), O(t^{n+2p-k})),$$

$$R(t, \theta, \lambda) - \hat{R}(t, \theta, \lambda) = \begin{cases} (O(t^{2k-1}), 0) & \text{if } n \leq k, \\ (0, 0) & \text{if } n > k. \end{cases}$$

As in the case of planar maps, the invariant manifolds obtained in Theorems 5.3.1 and 5.3.2 are unique (see Remark 2.2.7).



## 5.4 Formal approximation of a parameterization of the whiskers

In this section we show how to compute formal approximations of a parameterization of the invariant manifolds (*whiskers*) of a map  $F$  of the form 5.1.2. Similarly as for planar analytic maps (see Chapter 2), we provide an algorithm to obtain two maps,  $\mathcal{K}_n$  and  $\mathcal{R}_n$ , that are approximations of solutions  $K$  and  $R$  of the invariance equation

$$F \circ K = K \circ R.$$

However, in this case the map  $\mathcal{K}_n$  will be  $d + 2$ -dimensional, and the map  $\mathcal{R}_n$  will be  $d + 1$ -dimensional. The first component of the map  $\mathcal{R}_n$  represents the dynamics in the directions normal to the invariant torus,  $\mathcal{T}$ .

As in the planar case, the obtained approximation correspond to the stable manifold when the coefficient  $\bar{R}_k^x(\lambda)$  of  $\mathcal{R}_n$  is negative. When this coefficient is positive it corresponds to the unstable manifold.

The obtained approximations  $\mathcal{K}_n$  and  $\mathcal{R}_n$  will satisfy the hypotheses of Theorem 5.3.2, and therefore  $\mathcal{K}_n$  provides an approximation of a true invariant manifold of  $F$ .

The algorithm presented here is somehow analogous to the one used in Propositions 2.3.1 and 2.3.4 of Chapter 2. In this case, however, when solving the cohomological equations to obtain the coefficients of  $\mathcal{K}_n$  and  $\mathcal{R}_n$  we will encounter the small divisors equation, which can be solved when the frequency  $\omega$  is Diophantine.

Even if in the statement we ask for  $F$  to be analytic, the result holds if  $F$  is only  $C^\infty$ , since the proof requires only formal computations.

**Proposition 5.4.1.** *Let  $F$  be an analytic map of the form (5.1.2). Assume that  $2p > k - 1$ ,  $\bar{a}_k(\lambda) > 0$  for  $\lambda \in \Lambda$ , and that  $\omega$  is Diophantine. Then, for all  $n \geq 2$ , there exist two pairs of maps,  $\mathcal{K}_n : \mathbb{R} \times \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$  and  $\mathcal{R}_n : \mathbb{R} \times \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R} \times \mathbb{T}^d$ , of the form*

$$\mathcal{K}_n(t, \theta, \lambda) = \begin{pmatrix} t^2 + \sum_{i=3}^n \bar{K}_i^x(\lambda)t^i + \sum_{i=k+1}^{n+k-1} \tilde{K}_i^x(\theta, \lambda)t^i \\ \sum_{i=k+1}^{n+k-1} \bar{K}_i^y(\lambda)t^i + \sum_{i=2k}^{n+2k-2} \tilde{K}_i^y(\theta, \lambda)t^i \\ \theta + \sum_{i=2p-k+1}^{n+2p-k-1} \bar{K}_i^\theta(\lambda)t^i + \sum_{i=2p}^{n+2p-2} \tilde{K}_i^\theta(\theta, \lambda)t^i \end{pmatrix}$$

and

$$\mathcal{R}_n(t, \theta, \lambda) = \begin{cases} \begin{pmatrix} t + \bar{R}_k^x(\lambda)t^k \\ \theta + \omega \end{pmatrix} & \text{if } 2 \leq n \leq k, \\ \begin{pmatrix} t + \bar{R}_k^x(\lambda)t^k + \bar{R}_{2k-1}^x(\lambda)t^{2k-1} \\ \theta + \omega \end{pmatrix} & \text{if } n \geq k + 1, \end{cases}$$

such that

$$\mathcal{G}_n(t, \theta, \lambda) := F(\mathcal{K}_n(t, \theta, \lambda), \lambda) - \mathcal{K}_n(\mathcal{R}_n(t, \theta, \lambda), \lambda) = (O(t^{n+k}), O(t^{n+2k-1}), O(t^{n+2p-1})). \quad (5.4.1)$$

Moreover, for the first coefficients we obtain

$$\begin{aligned} \bar{K}_{k+1}^y(\lambda) &= \pm \sqrt{\frac{2\bar{a}_k(\lambda)}{\bar{c}(\lambda)(k+1)}}, \quad \bar{K}_{2p-k+1}^\theta(\lambda) = \pm \frac{\bar{d}_p(\lambda)}{2p-k+1} \sqrt{\frac{2(k+1)}{\bar{c}(\lambda)\bar{a}_k}}, \quad \bar{R}_k^x(\lambda) = \pm \sqrt{\frac{\bar{c}(\lambda)\bar{a}_k(\lambda)}{2(k+1)}}, \\ \tilde{K}_{k+1}^x(\theta, \lambda) &= \mathcal{SD}(\tilde{c}(\theta, \lambda)\bar{K}_{k+1}^y(\lambda)), \quad \tilde{K}_{2k}^y(\theta, \lambda) = \mathcal{SD}(\tilde{a}_k(\theta, \lambda)), \quad \tilde{K}_{2p}^\theta(\theta, \lambda) = \mathcal{SD}(\tilde{d}_p(\theta, \lambda)). \end{aligned}$$

**Notation 5.4.2.** Along the proof, given a map  $f(t, \theta)$ , we will denote by  $[f]_n$  the coefficient of the term of order  $n$  of the jet of  $f$  with respect to  $t$  at 0 .

*Proof.* We prove the result by induction and show that we can determine  $\mathcal{K}_n$  and  $\mathcal{R}_n$  iteratively.

For the first induction step,  $n = 2$ , we claim that there exist maps of the form

$$\mathcal{K}_2(t, \theta) = \begin{pmatrix} t^2 + \tilde{K}_{k+1}^x(\theta)t^{k+1} \\ \bar{K}_{k+1}^y t^{k+1} + \tilde{K}_{2k}^y(\theta)t^{2k} \\ \theta + \bar{K}_{2p-k+1}^\theta t^{2p-k+1} + \tilde{K}_{2p}^\theta(\theta)t^{2p} \end{pmatrix}, \quad \mathcal{R}_2(t, \theta) = \begin{pmatrix} t + \bar{R}_k^x t^k \\ \theta + \omega \end{pmatrix},$$

such that  $\mathcal{G}_2(t, \theta) = F(\mathcal{K}_2(t, \theta)) - \mathcal{K}_2(\mathcal{R}_2(t, \theta)) = (O(t^{k+2}), O(t^{2k+1}), O(t^{2p+1}))$ .

Indeed, from the expansion of  $\mathcal{G}_2$  we have

$$\begin{aligned} \mathcal{G}_2^x(t, \theta) &= t^{k+1}[\tilde{K}_{k+1}^x(\theta) - \tilde{K}_{k+1}^x(\theta + \omega) + c(\theta)\bar{K}_{k+1}^y - 2\bar{R}_k^x] + O(t^{k+2}), \\ \mathcal{G}_2^y(t, \theta) &= t^{2k}[\tilde{K}_{2k}^y(\theta) - \tilde{K}_{2k}^y(\theta + \omega) + a_k(\theta) - (k+1)\bar{K}_{k+1}^y\bar{R}_k^x] + O(t^{2k+1}), \\ \mathcal{G}_2^\theta(t, \theta) &= t^{2p}[\tilde{K}_{2p}^\theta(\theta) - \tilde{K}_{2p}^\theta(\theta + \omega) + d_p(\theta) - (2p-k+1)\bar{K}_{2p-k+1}^\theta\bar{R}_k^x] + O(t^{2p+1}). \end{aligned}$$

To obtain  $\mathcal{G}_2^x(t, \theta) = O(t^{k+2})$  we solve the equation

$$\tilde{K}_{k+1}^x(\theta) - \tilde{K}_{k+1}^x(\theta + \omega) + c(\theta)\bar{K}_{k+1}^y - 2\bar{R}_k^x = 0$$

as follows. First, we separate the average and the oscillatory part of the functions that depend on  $\theta$ , so that we obtain

$$\tilde{K}_{k+1}^x(\theta) - \tilde{K}_{k+1}^x(\theta + \omega) + \bar{c}\bar{K}_{k+1}^y + \tilde{c}(\theta)\bar{K}_{k+1}^y - 2\bar{R}_k^x = 0.$$

Then we split the equation into two parts, one containing the terms that are independent of  $\theta$ , namely  $\bar{c}\bar{K}_{k+1}^y = 2\bar{R}_k^x$ , and the other being a small divisors equation of functions with zero average,  $\tilde{K}_{k+1}^x(\theta + \omega) - \tilde{K}_{k+1}^x(\theta) = \tilde{c}(\theta)\bar{K}_{k+1}^y$ .

We proceed in the same way to get  $\mathcal{G}_2^y(t, \theta) = O(t^{2k+1})$  and  $\mathcal{G}_2^\theta(t, \theta) = O(t^{2p+1})$ . If  $\bar{a}_k > 0$  and  $\omega$  is Diophantine, the obtained equations have formal solutions given by

$$\begin{aligned} \bar{K}_{k+1}^y &= \pm \sqrt{\frac{2\bar{a}_k}{\bar{c}(k+1)}}, \quad \bar{K}_{2p-k+1}^\theta = \pm \frac{\bar{d}_p}{2p-k+1} \sqrt{\frac{2(k+1)}{\bar{c}\bar{a}_k}}, \quad \bar{R}_k^x = \pm \sqrt{\frac{\bar{c}\bar{a}_k}{2(k+1)}}, \\ \tilde{K}_{k+1}^x(\theta) &= \mathcal{SD}(\tilde{c}(\theta)\bar{K}_{k+1}^y), \quad \tilde{K}_{2k}^y(\theta) = \mathcal{SD}(\tilde{a}_k(\theta)), \quad \tilde{K}_{2p}^\theta(\theta) = \mathcal{SD}(\tilde{d}_p(\theta)). \end{aligned}$$

Next we perform the induction procedure. We assume that we have already obtained maps  $\mathcal{K}_n$  and  $\mathcal{R}_n$ ,  $n \geq 2$ , such that (5.4.1) holds true, and we look for

$$\begin{aligned} \mathcal{K}_{n+1}(t, \theta) &= \mathcal{K}_n(t, \theta) + \begin{pmatrix} \bar{K}_{n+1}^x t^{n+1} + \tilde{K}_{n+k}^x(\theta)t^{n+k} \\ \bar{K}_{n+k}^y t^{n+k} + \tilde{K}_{n+2k-1}^y(\theta)t^{n+2k-1} \\ \bar{K}_{n+2p-k}^\theta t^{n+2p-k} + \tilde{K}_{n+2p-1}^\theta(\theta)t^{n+2p-1} \end{pmatrix}, \\ \mathcal{R}_{n+1}(t, \theta) &= \mathcal{R}_n(t, \theta) + \begin{pmatrix} \bar{R}_{n+k-1}^x t^{n+k-1} \\ 0 \end{pmatrix}, \end{aligned}$$

such that  $\mathcal{G}_{n+1}(t, \theta) = (O(t^{n+k+1}), O(t^{n+2k}), O(t^{n+2p}))$ . To simplify the notation, we denote  $\mathcal{K}_{n+1}^+ = \mathcal{K}_{n+1} - \mathcal{K}_n$  and  $\mathcal{R}_{n+1}^+ = \mathcal{R}_{n+1} - \mathcal{R}_n$ .

Using Taylor's theorem, we write

$$\begin{aligned}\mathcal{G}_{n+1}(t, \theta) &= F(\mathcal{K}_n(t, \theta) + \mathcal{K}_{n+1}^+(t, \theta)) - (\mathcal{K}_n(t, \theta) + \mathcal{K}_{n+1}^+(t, \theta)) \circ (\mathcal{R}_n(t, \theta) + \mathcal{R}_{n+1}^+(t, \theta)) \\ &= \mathcal{G}_n(t, \theta) + DF(\mathcal{K}_n(t, \theta)) \cdot \mathcal{K}_{n+1}^+(t, \theta) - \mathcal{K}_{n+1}^+(t, \theta) \circ (\mathcal{R}_n(t, \theta) + \mathcal{R}_{n+1}^+(t, \theta)) \\ &\quad + \int_0^1 (1-s) D^2 F(\mathcal{K}_n(t, \theta) + s \mathcal{K}_{n+1}^+(t, \theta)) ds \mathcal{K}_{n+1}^+(t, \theta)^{\otimes 2} \\ &\quad - D\mathcal{K}_n \circ \mathcal{R}_n(t, \theta) \cdot \mathcal{R}_{n+1}^+(t, \theta) \\ &\quad - \int_0^1 (1-s) D^2 \mathcal{K}_n(\mathcal{R}_n(t, \theta) + s \mathcal{R}_{n+1}^+(t, \theta)) ds \mathcal{R}_{n+1}^+(t, \theta)^{\otimes 2}.\end{aligned}$$

Performing the computations in the previous expression we have

$$\begin{aligned}\mathcal{G}_{n+1}^x(t, \theta) &= \mathcal{G}_n^x(t, \theta) \\ &\quad + t^{n+k} [\tilde{K}_{n+k}^x(\theta) - \tilde{K}_{n+k}^x(\theta + \omega) + c(\theta) \bar{K}_{n+k}^y - (n+1) \bar{K}_{n+1}^x \bar{R}_k^x - 2 \bar{R}_{n+k-1}^x] + O(t^{n+k+1}), \\ \mathcal{G}_{n+1}^y(t, \theta) &= \mathcal{G}_n^y(t, \theta) \\ &\quad + t^{n+2k-1} [\tilde{K}_{n+2k-1}^y(\theta) - \tilde{K}_{n+2-1k}^y(\theta + \omega) + k a_k(\theta) \bar{K}_{n+1}^x - (n+k) \bar{K}_{n+k}^y \bar{R}_k^x \\ &\quad - (k+1) \bar{K}_{k+1}^y \bar{R}_{n+k-1}^x] + O(t^{n+2k}), \\ \mathcal{G}_{n+1}^\theta(t, \theta) &= \mathcal{G}_n^\theta(t, \theta) \\ &\quad + t^{n+2p-1} [\tilde{K}_{n+2p-1}^\theta(\theta) - \tilde{K}_{n+2p-1}^\theta(\theta + \omega) + p d_p(\theta) \bar{K}_{n+1}^x \\ &\quad - (n+2p-k) \bar{K}_{n+2p-k}^\theta \bar{R}_k^x - (2p-k+1) \bar{K}_{2p-k+1}^\theta \bar{R}_{n+k-1}^x] + O(t^{n+2p}).\end{aligned}\tag{5.4.2}$$

Since, by the induction hypothesis,  $\mathcal{G}_n(t, \theta) = (O(t^{n+k}), O(t^{n+2k-1}), O(t^{n+2p-1}))$ , to complete the induction step we need to make  $[\mathcal{G}_{n+1}^x]_{n+k}$ ,  $[\mathcal{G}_{n+1}^y]_{n+2k-1}$  and  $[\mathcal{G}_{n+1}^\theta]_{n+2p-1}$  vanish. From the expansions obtained in (5.4.2), such condition leads to the following cohomological equations,

$$\begin{aligned}\tilde{K}_{n+k}^x(\theta) - \tilde{K}_{n+k}^x(\theta + \omega) + c(\theta) \bar{K}_{n+k}^y - (n+1) \bar{K}_{n+1}^x \bar{R}_k^x - 2 \bar{R}_{n+k-1}^x + [\mathcal{G}_n^x(\theta)]_{n+k} &= 0, \\ \tilde{K}_{n+2k-1}^y(\theta) - \tilde{K}_{n+2-1k}^y(\theta + \omega) + k a_k(\theta) \bar{K}_{n+1}^x - (n+k) \bar{K}_{n+k}^y \bar{R}_k^x \\ &\quad - (k+1) \bar{K}_{k+1}^y \bar{R}_{n+k-1}^x + [\mathcal{G}_n^y(\theta)]_{n+2k-1} = 0, \\ \tilde{K}_{n+2p-1}^\theta(\theta) - \tilde{K}_{n+2p-1}^\theta(\theta + \omega) + p d_p(\theta) \bar{K}_{n+1}^x \\ &\quad - (n+2p-k) \bar{K}_{n+2p-k}^\theta \bar{R}_k^x - (2p-k+1) \bar{K}_{2p-k+1}^\theta \bar{R}_{n+k-1}^x + [\mathcal{G}_n^\theta(\theta)]_{n+2p-1} = 0.\end{aligned}\tag{5.4.3}$$

Taking averages with respect to  $\theta$  in the previous equations and separating the terms that depend on  $\theta$  from the constant ones, we split (5.4.3) into three small divisors equations of functions with zero average, namely,

$$\begin{aligned}\tilde{K}_{n+k}^x(\theta + \omega) - \tilde{K}_{n+k}^x(\theta) &= \tilde{c}(\theta) \bar{K}_{n+k}^y + [\tilde{\mathcal{G}}_n^x(\theta)]_{n+k}, \\ \tilde{K}_{n+2-1k}^y(\theta + \omega) - \tilde{K}_{n+2k-1}^y(\theta) &= k \tilde{a}_k(\theta) \bar{K}_{n+1}^x + [\tilde{\mathcal{G}}_n^y(\theta)]_{n+2k-1}, \\ \tilde{K}_{n+2p-1}^\theta(\theta + \omega) - \tilde{K}_{n+2p-1}^\theta(\theta) &= p \tilde{d}_p(\theta) \bar{K}_{n+1}^x + [\tilde{\mathcal{G}}_n^\theta(\theta)]_{n+2p-1},\end{aligned}\tag{5.4.4}$$

and the following linear system of equations with constant coefficients,

$$\begin{aligned} & \begin{pmatrix} -(n+1)\bar{R}_k^x & \bar{c} & 0 \\ k\bar{a}_k & -(n+k)\bar{R}_k^x & 0 \\ p\bar{d}_p & 0 & -(n+2p-k)\bar{R}_k^x \end{pmatrix} \begin{pmatrix} \bar{K}_{n+1}^x \\ \bar{K}_{n+k}^y \\ \bar{K}_{n+2p-k}^\theta \end{pmatrix} \\ &= \begin{pmatrix} -[\bar{\mathcal{G}}_n^x]_{n+k} + 2\bar{R}_{n+k-1}^x \\ -[\bar{\mathcal{G}}_n^y]_{n+2k-1} + (k+1)\bar{K}_{k+1}^y \bar{R}_{n+k-1}^x \\ -[\bar{\mathcal{G}}_n^\theta]_{n+2p-1} + (2p-k+1)\bar{K}_{2p-k+1}^\theta \bar{R}_{n+k-1}^x \end{pmatrix}. \end{aligned} \quad (5.4.5)$$

Note that the determinant of the matrix in the left hand side of (5.4.5) is  $(n+2p-k)\bar{R}_k^x k\bar{c}\bar{a}_k - (n+2p-k)(n+1)(n+k)(\bar{R}_k^x)^3$ , which vanishes when  $k\bar{c}\bar{a}_k - (n+1)(n+k)(\bar{R}_k^x)^2 = 0$ . Then, if  $n \neq k$  the matrix is invertible, so we can take  $\bar{R}_{n+k-1}^x = 0$  and then obtain  $\bar{K}_{n+1}^x$ ,  $\bar{K}_{n+k}^y$  and  $\bar{K}_{n+2p-k}^\theta$  in a unique way. When  $n = k$ , the determinant of the matrix is zero. Then, choosing

$$\bar{R}_{2k-1}^x = \frac{2k\bar{R}_k^x [\bar{\mathcal{G}}_n^x]_{2k} + \bar{c} [\bar{\mathcal{G}}_n^y]_{3k-2}}{2(3k+1)\bar{R}_k^x},$$

system (5.4.5) has solutions. In this case, however,  $\bar{K}_{k+1}^x$ ,  $\bar{K}_{2k}^y$  and  $\bar{K}_{2p}^\theta$  are not uniquely determined.

Once we have chosen solutions  $\bar{K}_{k+1}^x$ ,  $\bar{K}_{2k}^y$  and  $\bar{K}_{k+2p-1}^\theta$  for system (5.4.5), we solve the small divisors equations (5.4.4) taking

$$\begin{aligned} \tilde{K}_{n+k}^x(\theta) &= \mathcal{SD}(\tilde{c}(\theta)\bar{K}_{n+k}^y + [\tilde{\mathcal{G}}_n^x(\theta)]_{n+k}), \\ \tilde{K}_{n+2k-1}^y(\theta) &= \mathcal{SD}(k\tilde{a}_k(\theta)\bar{K}_{n+1}^x + [\tilde{\mathcal{G}}_n^y(\theta)]_{n+2k-1}), \\ \tilde{K}_{n+2p-1}^\theta(\theta) &= \mathcal{SD}(p\tilde{d}_p(\theta)\bar{K}_{n+1}^x + [\tilde{\mathcal{G}}_n^\theta(\theta)]_{n+2p-1}). \end{aligned}$$

In this way all equations in (5.4.3) are solved and one can proceed to the next induction step.  $\square$

## 5.5 Unstable manifolds

In Theorem 5.3.1 we showed that a map  $F$  of the form (5.1.2) possesses a stable manifold asymptotic to  $\mathcal{T}$  provided that the coefficients of  $F$  satisfy that  $\bar{a}_k > 0$  and that  $\omega$  is Diophantine, and we also showed that such stable manifold,  $K$ , can be approximated by a parameterization,  $\mathcal{K}_n$ , provided in Proposition 5.4.1. Moreover, in that proposition we also obtained approximations,  $\mathcal{K}_n$  and  $\mathcal{R}_n$ , such that  $F \circ \mathcal{K}_n - \mathcal{K}_n \circ \mathcal{R}_n = (O(t^{n+k}), O(t^{n+2k-1}), O(t^{n+2p-1}))$ , with  $\bar{R}_k^x > 0$ , that is,  $\mathcal{K}_n$  being an approximation of a parameterization of an unstable manifold of  $F$ .

From now on, to clarify the notation, we will refer to  $\mathcal{K}_n^-$  and  $\mathcal{R}_n^-$  as the parameterizations obtained in Proposition 5.4.1 corresponding to the stable manifold and the restricted dynamics on it, respectively, and to  $\mathcal{K}_n^+$  and  $\mathcal{R}_n^+$  as the parameterizations of the unstable manifold and the restricted dynamics inside it.

In this section we show that Theorem 5.3.2 also holds for unstable manifolds without having to compute explicitly the map  $F^{-1}$ . Concretely, we show that the approximation  $\mathcal{K}_n^+$  provided

in Proposition 5.4.1 is an approximation of a parameterization of a true unstable manifold,  $\hat{K}^+$ , of  $F$ , asymptotic to  $\mathcal{T}^d$ . Moreover, the dynamics on  $\hat{K}^+$  can be parameterized by a map  $\hat{R}^+$  that is also approximated by  $\mathcal{R}_n^+$ . As in the stable case, such pairs of maps also satisfy

$$\hat{K}^+(t, \theta) - \mathcal{K}_n^+(t, \theta) = (O(t^{n+1}), O(t^{n+k}), O(t^{n+2p-k})),$$

and

$$\hat{R}^+(t, \theta) - \mathcal{R}_n^+(t, \theta) = \begin{cases} (O(t^{2k-1}), 0) & \text{if } n \leq k, \\ (0, 0) & \text{if } n > k. \end{cases}$$

To avoid cumbersome computations, in this section we will only consider maps  $F$  satisfying the hypotheses of Theorem 5.3.2 with the additional hypothesis  $p \geq k - 2$ , which is a bit more restrictive.

Assume we have a map of the form (5.1.2). By Proposition 5.4.1, there exist approximations  $\mathcal{K}_n^+$  and  $\mathcal{R}_n^+$  such that

$$\mathcal{G}_n = F \circ \mathcal{K}_n^+ - \mathcal{K}_n^+ \circ \mathcal{R}_n^+ = (O(t^{n+k}), O(t^{n+2k-1}), O(t^{n+2p-1})), \quad (5.5.1)$$

with

$$\mathcal{R}_n^+(t, \theta) = \begin{pmatrix} t + \bar{R}_k^x t^k + O(t^{k+1}) \\ \theta + \omega \end{pmatrix}$$

and  $\bar{R}_k^x > 0$ , which means that  $\mathcal{R}_n^+$  is a repeller in the normal directions of  $\mathcal{T}$ . Also,  $\mathcal{R}_n^+$  is invertible and we have

$$(\mathcal{R}_n^+)^{-1}(t, \theta) = \begin{pmatrix} t - \bar{R}_k^x t^k + O(t^{k+1}) \\ \theta - \omega \end{pmatrix},$$

and

$$F^{-1} \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} = \begin{pmatrix} x - c(\theta - \omega)y + c(\theta - \omega)a_k(\theta - \omega)(x - c(\theta - \omega)y)^k + \tilde{A}(x, y, \theta) \\ y - a_k(\theta - \omega)(x - c(\theta - \omega)y)^k + \tilde{B}(x, y, \theta) \\ \theta - \omega - d_p(\theta - \omega)(x - c(\theta - \omega)y)^p + \tilde{C}(x, y, \theta) \end{pmatrix},$$

with  $\tilde{A}(x, y, \theta), \tilde{B}(x, y, \theta) = O(\|(x, y)\|^{k+1}) + yO(\|(x, y)\|^{k-1})$ , and  $\tilde{C}(x, y, \theta) = O(\|(x, y)\|^{p+1}) + yO(\|(x, y)\|^{p-1})$ .

Composing by  $F^{-1}$  by the left in (5.5.1) and using Taylor's Theorem, we get

$$\begin{aligned} \mathcal{K}_n^+ &= F^{-1} \circ (\mathcal{K}_n^+ \circ \mathcal{R}_n^+ + \mathcal{G}_n) \\ &= F^{-1} \circ (\mathcal{K}_n^+ \circ \mathcal{R}_n^+) + DF^{-1} \circ (\mathcal{K}_n^+ \circ \mathcal{R}_n^+) \cdot \mathcal{G}_n + O(\mathcal{G}_n^2), \end{aligned} \quad (5.5.2)$$

that is,

$$\mathcal{K}_n^+ - F^{-1} \circ (\mathcal{K}_n^+ \circ \mathcal{R}_n^+) = (O(t^{n+k}), O(t^{n+2k-1}), O(t^{n+2p-1})),$$

and then composing by  $(\mathcal{R}_n^+)^{-1}$  by the right we obtain

$$F^{-1} \circ \mathcal{K}_n^+ - \mathcal{K}_n^+ \circ (\mathcal{R}_n^+)^{-1} = (O(t^{n+k}), O(t^{n+2k-1}), O(t^{n+2p-1})). \quad (5.5.3)$$

Next we consider the following analytic changes of variables,  $\phi_1, \phi_2$ , given by

$$\begin{aligned}\phi_1(x, y, \theta) &= (x, -y, \theta), \\ \phi_2(x, y, \theta) &= (x, y, \theta + \omega),\end{aligned}$$

and  $\phi_3$  given explicitly by its inverse,

$$\phi_3^{-1}(x, y, \theta) = \left(x, y + \frac{1}{c(\theta)}[c(\theta)a_k(\theta)(x - c(\theta)y)^k + \tilde{A}(x, y, \theta)], \theta\right).$$

Then, taking  $\phi = \phi_3 \circ \phi_2 \circ \phi_1$ , we have that  $H := \phi^{-1} \circ F^{-1} \circ \phi$  is of the form

$$H \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} = \begin{pmatrix} x + c(\theta)y \\ y + a_k(\theta)x^k + \tilde{D}(x, y, \theta) \\ \theta - \omega - d_p(\theta)x^p + \tilde{G}(x, y, \theta) \end{pmatrix},$$

with

$$\tilde{D}(x, y, \theta) = O(\|(x, y)\|^{k+1}) + yO(\|(x, y)\|^{k-1})$$

and

$$\tilde{G}(x, y, \theta) = O(\|(x, y)\|^{p+1}) + yO(\|(x, y)\|^{p-1}),$$

where we have assumed  $p \geq k - 2$ . Namely, the map  $H$  is of the form (5.1.2) (with  $\omega$  of opposite sign) and the functions  $\tilde{D}$  and  $\tilde{G}$  have the same properties as the functions  $A$  and  $B$  in (5.1.2). Moreover, composing by  $\phi^{-1}$  by the left in (5.5.3) and using Taylor's Theorem as in (5.5.2), we get

$$\phi^{-1} \circ F^{-1} \circ \mathcal{K}_n^+ - \phi^{-1} \circ \mathcal{K}_n^+ \circ (\mathcal{R}_n^+)^{-1} = (O(t^{n+k}), O(t^{n+2k-1}), O(t^{n+2p-1})),$$

which is equivalent to

$$H \circ \phi^{-1} \circ \mathcal{K}_n^+ - \phi^{-1} \circ \mathcal{K}_n^+ \circ (\mathcal{R}_n^+)^{-1} = (O(t^{n+k}), O(t^{n+2k-1}), O(t^{n+2p-1})).$$

Hence,  $H, \phi^{-1} \circ \mathcal{K}_n^+$  and  $(\mathcal{R}_n^+)^{-1}$  are analytic maps that satisfy the hypotheses of Theorem 5.3.2, where here the frequency of rotation is  $-\omega$ . Therefore, by Theorem 5.3.2, there exist a map  $K^+ : [0, \rho) \times \mathbb{T}^d \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$ , analytic in  $(0, \rho) \times \mathbb{T}^d$ , and an analytic map  $R^+ : (-\rho, \rho) \times \mathbb{T}^d \rightarrow \mathbb{R} \times \mathbb{T}^d$  such that

$$H \circ K^+ = K^+ \circ R^+, \tag{5.5.4}$$

and moreover it holds that

$$K^+(t, \theta) - \phi^{-1} \mathcal{K}_n^+(t, \theta) = (O(t^{n+1}), O(t^{n+k}), O(t^{n+2p-k})), \tag{5.5.5}$$

$$R^+(t, \theta) - (\mathcal{R}_n^+)^{-1}(t, \theta) = \begin{cases} (O(t^{2k-1}), 0) & \text{if } n \leq k, \\ (0, 0) & \text{if } n > k. \end{cases} \tag{5.5.6}$$

Also, composing by  $\phi$  by the left in (5.5.4) we have

$$F^{-1} \circ \phi \circ K^+ = \phi \circ K^+ \circ R^+,$$

which means that  $\phi \circ K^+$  is a parameterization of a stable manifold of  $F^{-1}$ , and the restricted dynamics on this stable manifold is given by the map  $R^+$ , which, using (5.5.6), is of the form

$$R^+(t, \theta) = \begin{pmatrix} t - \bar{R}_k^x t^k + O(t^{k+1}) \\ \theta - \omega \end{pmatrix} \quad (5.5.7)$$

with  $\bar{R}_k^x > 0$ .

As a consequence,  $\phi \circ K^+$  is a parameterization of an unstable manifold of  $F$ , analytic in  $[0, \rho) \times \mathbb{T}^d$ , for some  $\rho > 0$ . Moreover, composing by  $\phi$  in (5.5.5) and using Taylor's Theorem, we have

$$\phi(K^+(t, \theta)) - \mathcal{K}_n^+(t, \theta) = (O(t^{n+1}), O(t^{n+k}), O(t^{n+2p-k})),$$

that is,  $\phi \circ K^+$  is approximated by the parameterization  $\mathcal{K}_n^+$  obtained in Proposition 5.4.1. Denoting  $\hat{K}^+ := \phi \circ K^+$  we recover the notation used at the beginning of the section.

Finally, note that since  $R^+$  represents the restricted dynamics of  $F^{-1}$  in the stable manifold  $\phi \circ K^+$ , then  $(R^+)^{-1}$  represents the restricted dynamics of  $F$  in the unstable manifold  $\phi \circ K^+$ . By the form of (5.5.7) we have

$$(R^+)^{-1}(t, \theta) = \begin{pmatrix} t + \bar{R}_k^x t^k + O(t^{k+1}) \\ \theta + \omega \end{pmatrix},$$

with  $\bar{R}_k^x > 0$ , and hence, finally,

$$(R^+)^{-1}(t, \theta) - \mathcal{R}_n^+(t, \theta) = \begin{cases} (O(t^{2k-1}), 0) & \text{if } n \leq k, \\ (0, 0) & \text{if } n > k, \end{cases}$$

as we claimed at the beginning of the section. Concretely, we recover the notation given at the beginning of the section denoting  $\hat{R}^+ := (R^+)^{-1}$ .

## 5.6 The functional equation

To study the existence of invariant manifolds of a map of the form (5.1.2) following the parameterization method we proceed similarly as in the case of planar maps presented in Chapter 2. First we consider approximations  $\mathcal{K}_n : \mathbb{R} \times \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$  and  $\mathcal{R}_n : \mathbb{R} \times \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R} \times \mathbb{T}^d$  of solutions of the equation

$$F \circ K = K \circ R, \quad (5.6.1)$$

obtained in Section 5.4 up to a high enough order, to be determined later. Then, keeping  $R = \mathcal{R}_n$  fixed, we look for a correction  $\Delta : [0, \rho) \times \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$ , for some  $\rho > 0$ , of  $\mathcal{K}_n$ , analytic on  $(0, \rho) \times \mathbb{T}^d \times \Lambda$ , such that the pair  $K = \mathcal{K}_n + \Delta$ ,  $R = \mathcal{R}_n$  satisfies the invariance condition

$$F \circ (\mathcal{K}_n + \Delta) - (\mathcal{K}_n + \Delta) \circ R = 0. \quad (5.6.2)$$

Note that, contrary to the planar case, here the approximations  $\mathcal{K}_n$  are no more polynomial maps, but they depend in a polynomial way only on the variable  $u$ . Also, in this case equation

(5.6.1) has  $d + 2$  components. Along the section we will often write the last  $d$  components of such equation in a same expression.

The proof of Theorems 5.3.1 and 5.3.2 is organized in an analogous way as in the planar case. First, taking into account the structure of  $F$  we rewrite equation (5.6.2) to separate the dominant linear part with respect to  $\Delta$  and the remaining terms. This motivates the introduction of two families of operators,  $\mathcal{S}_{n,R}^\times$  and  $\mathcal{N}_{n,F}$ , and the spaces where these operators will act on. We provide the properties of these operators in Lemmas 5.7.8 and 5.7.9, in particular the invertibility of  $\mathcal{S}_{n,R}^\times$ .

Finally, we rewrite the equation for  $\Delta$  as the fixed point equation

$$\Delta = \mathcal{T}_{n,F}(\Delta), \quad \text{where} \quad \mathcal{T}_{n,F} = (\mathcal{S}_{n,R}^\times)^{-1} \circ \mathcal{N}_{n,F},$$

and we apply the Banach fixed point theorem to get the solution. The properties of the operators  $\mathcal{T}_{n,F}$  are deduced in Lemma 5.7.14. Note that the symbols used for the operators  $\mathcal{S}_{n,R}^\times$ ,  $\mathcal{N}_{n,F}$  and  $\mathcal{T}_{n,F}$  are very similar to the ones used for the planar case in Chapter 2. However, they are different operators and should not be confused.

Let  $F : U \times \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$  be an analytic map of the form (5.1.2). Along the rest of the Section, to simplify the notation we rewrite  $F$  in the following form,

$$F(x, y, \theta, \lambda) = \begin{pmatrix} x + c(\theta, \lambda)y \\ y + P(x, y, \theta, \lambda) \\ \theta + \omega + Q(x, y, \theta, \lambda) \end{pmatrix},$$

where  $P(a, y, \theta, \lambda) = a_k(\theta, \lambda)x^k + A(x, y, \theta, \lambda)$  and  $Q(x, y, \theta, \lambda) = d_p(\theta, \lambda)x^p + B(x, y, \theta, \lambda)$  have the properties described back in (5.1.2).

From Proposition 5.4.1, given  $n \geq 2$  there exist polynomials  $\mathcal{K}_n$  and  $R = \mathcal{R}_n$  such that

$$F \circ \mathcal{K}_n - \mathcal{K}_n \circ R = \mathcal{E}_n, \tag{5.6.3}$$

where  $\mathcal{E}_n(t, \theta) = (O(t^{n+k}), O(t^{n+2k-1}), O(t^{n+2p-1}))$ . Since we are looking for a stable manifold of  $F$  we will take the approximations corresponding to  $R = \mathcal{R}_n$  with the coefficient  $\bar{R}_k^x(\lambda) < 0$ .

Hence, we look for  $\rho > 0$  and a map  $K = \mathcal{K}_n + \Delta : [0, \rho) \times \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$ , analytic in  $(0, \rho) \times \mathbb{T}^d \times \Lambda$ , satisfying (5.6.2), where  $\mathcal{K}_n$  and  $R$  are the mentioned maps that satisfy (5.6.3). Moreover, we ask  $\Delta$  to satisfy  $\Delta = (\Delta^x, \Delta^y, \Delta^\theta) = (O(t^n), O(t^{n+k-1}), O(t^{n+2p-k-1}))$ .

Using (5.6.3) we can rewrite (5.6.2) as

$$\begin{aligned} \Delta^x \circ R - \Delta^x &= \mathcal{K}_n^y [c \circ (\mathcal{K}_n^\theta + \Delta^\theta) - c \circ \mathcal{K}_n^\theta] + \Delta^y c \circ (\mathcal{K}_n^\theta + \Delta^\theta) + \mathcal{E}_n^x, \\ \Delta^y \circ R - \Delta^y &= P \circ (\mathcal{K}_n + \Delta) - P \circ \mathcal{K}_n + \mathcal{E}_n^y, \\ \Delta^\theta \circ R - \Delta^\theta &= Q \circ (\mathcal{K}_n + \Delta) - Q \circ \mathcal{K}_n + \mathcal{E}_n^\theta. \end{aligned} \tag{5.6.4}$$

As we mentioned before, note that (5.6.2) is indeed a set of  $d + 2$  equations, that is, the last expression of (5.6.2) has  $d$  components.



## 5.7 Function spaces, operators and their properties

Next we introduce notation, function spaces, and some operators.

For the rest of the chapter, we fix  $0 < \beta < \frac{\pi}{k-1}$  and we consider the sector  $S(\beta, \rho)$  for some  $0 < \rho < 1$ .

**Definition 5.7.1.** *Given a sector  $S = S(\beta, \rho)$ , and  $\sigma > 0$ , let  $\mathcal{W}_n$ , for  $n \in \mathbb{N}$ , be the Banach space defined as*

$$\mathcal{W}_n = \left\{ f : S \times \mathbb{T}_\sigma^d \times \Lambda_{\mathbb{C}} \rightarrow \mathbb{C} \mid f \text{ real analytic, } \|f\|_n := \sup_{(z, \theta, \lambda) \in S \times \mathbb{T}_\sigma^d \times \Lambda_{\mathbb{C}}} \frac{|f(z, \theta, \lambda)|}{|z|^n} < \infty \right\}.$$

Note that when  $n \geq 1$  the functions  $f$  in  $\mathcal{W}_n$  can be continuously extended to  $z = 0$  with  $f(0, \theta, \lambda) = 0$  and, if moreover we have  $n \geq 2$ , the derivative of  $f$  with respect to  $z$  can be continuously extended to  $z = 0$  with  $\frac{\partial f}{\partial z}(0, \theta, \lambda) = 0$ .

Note also that  $\mathcal{W}_{n+1} \subset \mathcal{W}_n$ , for all  $n \in \mathbb{N}$ , and that if  $f \in \mathcal{W}_{n+1}$ , then  $\|f\|_n \leq \|f\|_{n+1}$ . More concretely it holds that  $\|f\|_n \leq \rho \|f\|_{n+1}$ . Moreover if  $f \in \mathcal{W}_m$ ,  $g \in \mathcal{W}_n$ , then  $fg \in \mathcal{W}_{m+n}$  and  $\|fg\|_{m+n} \leq \|f\|_m \|g\|_n$ .

Given a product of spaces,  $\prod_i \mathcal{W}_i$ , we endow it with the product norm

$$\|f\|_{\prod_i \mathcal{W}_i} = \max_i \|f_i\|_{\mathcal{W}_i},$$

where  $f_i = \pi_i \circ f$ , and  $\pi_i$  is the canonical projection from  $\prod_j \mathcal{W}_j$  to  $\mathcal{W}_i$ .

Next we define the spaces

$$\Gamma_n = \mathcal{W}_n \times \mathcal{W}_{n+k-1} \times \mathcal{W}_{n+2p-k-1}^d,$$

endowed with the product norm defined above, where the functions in  $\mathcal{W}_{n+2p-k-1}$  are mapped into  $\mathbb{C}/\mathbb{Z}$ . Also, given  $\alpha > 0$  we define

$$\Gamma_n^\alpha = \{f = (f^x, f^y, f^\theta) \in \Gamma_n \mid \|f\|_{\Gamma_n} = \max\{\|f^x\|_n, \|f^y\|_{n+k-1}, \|f^\theta\|_{n+2p-k-1}\} \leq \alpha\}.$$

For the sake of simplicity, we will omit the parameters  $\rho$ ,  $\beta$  and  $\sigma$  in the notation of the spaces  $\mathcal{W}_n$ .

Let  $F$  be an analytic map of the form (5.1.2), and  $\mathcal{K}_n$  and  $R = \mathcal{R}_n$  be the polynomials provided in Section 5.4 satisfying (5.6.3) with  $n \geq k + 1$ .

Since  $F$  is analytic in  $U \times \mathbb{T}^d \times \Lambda$ , it has a holomorphic extension to some complex neighborhood that contains  $U \times \mathbb{T}^d \times \Lambda$ , of the form  $V \times \mathbb{T}_\sigma^d \times \Lambda_{\mathbb{C}}$ , where  $V$  is a neighborhood of  $(0, 0)$  in  $\mathbb{C}^2$ ,  $\mathbb{T}_\sigma^d$  is a complex  $d$ -dimensional torus and  $\Lambda_{\mathbb{C}}$  is a complex extension of  $\Lambda$ .

On the other hand, since  $\mathcal{K}_n$  and  $R$  are analytic maps, they are defined on a complex domain of the form  $S(\beta, \rho) \times \mathbb{T}_\sigma^d \times \Lambda_{\mathbb{C}}$ .

Then it is possible to set equation (5.6.4) in a space of holomorphic functions defined on  $S(\beta, \rho) \times \mathbb{T}_\sigma^d \times \Lambda_{\mathbb{C}}$ , and to look for  $\Delta$  being an analytic function of complex variables that takes values in  $\mathbb{R}^2 \times \mathbb{T}^d$  when restricted to  $(0, \rho) \times \mathbb{T}^d \times \Lambda$ .

To solve equation (5.6.4), we will consider  $n$  big enough and we will look for a solution,  $\Delta \in \Gamma_n^\alpha$ , for some  $\alpha > 0$ . In what follows, we describe what value of  $\alpha$  must be considered. In

order for the compositions in (5.6.4) to make sense we need to ensure the range of  $\mathcal{K}_n + \Delta$  to be contained in the domain where  $F$  is analytic. Also, we look for an invariant manifold parameterized as  $\mathcal{K}_n + \Delta$  where  $\Delta$  has to be considered as a small correction of  $\mathcal{K}_n$ .

Let  $b > 0$  be the radius of a closed ball in  $\mathbb{C}^2$  contained in  $V$ , and let  $\tilde{\sigma} < \sigma$ . We need to consider  $K$  and  $\Delta$  such that  $((\mathcal{K}_n + \Delta)^x, (\mathcal{K}_n + \Delta)^y) \in V$ ,  $(\mathcal{K}_n + \Delta)^\theta \in \mathbb{T}_\sigma^d$ . To this end we will ensure that

$$|((\mathcal{K}_n + \Delta)^x, (\mathcal{K}_n + \Delta)^y)| \leq b \quad \text{and} \quad |\text{Im}((\mathcal{K}_n + \Delta)^\theta)| \leq \tilde{\sigma}. \quad (5.7.1)$$

We choose  $\rho$  and  $\sigma'$  small enough such that  $\sup_{S(\beta, \rho) \times \mathbb{T}_{\sigma'}^d \times \Lambda_{\mathbb{C}}} |(\mathcal{K}_n^x(z, \theta, \lambda), \mathcal{K}_n^y(z, \theta, \lambda))| \leq \frac{b}{2}$  and such that  $\sup_{S(\beta, \rho) \times \mathbb{T}_{\sigma'}^d \times \Lambda_{\mathbb{C}}} |\text{Im}(\mathcal{K}_n^\theta(z, \theta, \lambda))| \leq \frac{\tilde{\sigma}}{2}$ . Later on we will modify the size of  $\rho$  to a smaller value.

Then, we take

$$\alpha = \min \left\{ \frac{1}{2}, \frac{b}{2}, \frac{\tilde{\sigma}}{2} \right\},$$

and we set  $\Delta \in \Gamma_n^\alpha$ . This way, we get

$$\sup_{S \times \mathbb{T}_{\sigma'}^d \times \Lambda_{\mathbb{C}}} |\Delta^x(z, \theta, \lambda)| \leq \sup_S \|\Delta^x\|_n |z|^n \leq \alpha \rho^n \leq \frac{b}{2} \rho^n,$$

and similarly,  $\sup_{S \times \mathbb{T}_{\sigma'}^d \times \Lambda_{\mathbb{C}}} |\Delta^y(z, \theta, \lambda)| \leq \frac{b}{2} \rho^{n+k-1}$ , and

$$\sup_{S \times \mathbb{T}_{\sigma'}^d \times \Lambda_{\mathbb{C}}} |\Delta^\theta(z, \theta, \lambda)| \leq \sup_S \|\Delta^\theta\|_{n+2p-k+1} |z|^{n+2p-k+1} \leq \alpha \rho^{n+2p-k+1} \leq \frac{\tilde{\sigma}}{2} \rho^{n+2p-k+1},$$

and in particular,  $|\text{Im}(\Delta^\theta)| \leq \frac{\tilde{\sigma}}{2}$ . Hence, with these considerations one obtains the bounds required in (5.7.1).

**Remark 5.7.2.** Along the section, the value  $\alpha$  denoting the radius of the ball  $\Gamma_n^\alpha$  is always fixed and given by  $\alpha = \min\{1/2, b/2, \tilde{\sigma}/2\}$ . We may modify instead the value  $\rho$  denoting the radius of the sector  $S(\beta, \rho)$  where the functions of the spaces  $\mathcal{W}_n$  are defined.

**Notation 5.7.3.** By the considerations described above, along this chapter  $\mathbb{T}_\sigma^d$  is a complex torus of thickness  $2\sigma$  where the complex extension of  $F$  is defined, and  $\mathbb{T}_{\sigma'}^d$  is a complex torus of thickness  $2\sigma'$  where the functions of  $\mathcal{W}_n$  are defined. The value  $\tilde{\sigma}$  that appeared in the paragraph above is an auxiliary parameter.

Next we introduce two families of operators that will be used to deal with (5.6.4). The definition of such operators is motivated by the equation itself.

We will need again the auxiliary Lemma 2.4.2 from [4] stated in Chapter 2 (extracted form [4]). We state it here again for the convenience of the reader, with a slightly modified notation adapted to the setting of this chapter.

**Lemma 5.7.4.** *Let  $R^x : S(\beta, \rho) \rightarrow \mathbb{C}$  be a holomorphic function of the form  $R^x(z) = z + R_k z^k + O(|z|^{k+1})$ , with  $R_k < 0$  and  $k \geq 2$ . Assume that  $0 < \beta < \frac{\pi}{k-1}$ . Then, for any  $\mu \in (0, (k-1)|R_k| \cos \lambda)$ , with  $\lambda = \beta \frac{k-1}{2}$ , there exists  $\rho > 0$  small enough such that*

$$|(R^x)^j(z)| \leq \frac{|z|}{(1 + j \mu |z|^{k-1})^{1/k-1}}, \quad \forall j \in \mathbb{N}, \quad \forall z \in S(\beta, \rho),$$

where  $(R^x)^j$  refers to the  $j$ -th iterate of the map  $R^x$ . In addition,  $R^x$  maps  $S(\beta, \rho)$  into itself.

**Definition 5.7.5.** Let  $n \geq 0$ ,  $\beta < \frac{\pi}{k-1}$ , and let  $R : S(\beta, \rho) \times \mathbb{T}_{\sigma'}^d \rightarrow S(\beta, \rho) \times \mathbb{T}_{\sigma'}^d$  be an analytic map of the form

$$R(z, \theta) = \begin{pmatrix} z + R_k z^k + O(z^{k+1}) \\ \theta + \omega \end{pmatrix}, \quad (5.7.2)$$

where the terms  $O(z^{k+1})$  do not depend on  $\theta$ , and with  $R_k < 0$ .

We define  $\mathcal{S}_{n,R} : \mathcal{W}_n \rightarrow \mathcal{W}_n$ , as the linear operator given by

$$\mathcal{S}_{n,R} f = f \circ R - f.$$

**Remark 5.7.6.** By Lemma 5.7.4, for a map  $R$  as in Definition (5.7.5), we have that  $R^x(z, \theta) = R^x(z)$  maps  $S(\beta, \rho)$  into itself, and also, since we have  $\omega \in \mathbb{R}^d$ , then  $R^\theta(z, \theta) = R^\theta(\theta)$  maps  $\mathbb{T}_{\sigma'}^d$  into itself. Moreover, the functions  $f \in \mathcal{W}_n$  are defined on  $S(\beta, \rho) \times \mathbb{T}_{\sigma'}^d$ , and thus the compositions in the definition of  $\mathcal{S}_{n,R}$  are well defined.

**Definition 5.7.7.** Let  $F$  be the holomorphic extension of an analytic map of the form (5.1.2) satisfying the hypotheses of Theorem 5.3.1. Given  $n \geq 3$ , we introduce  $\mathcal{N}_{n,F} = (\mathcal{N}_{n,F}^x, \mathcal{N}_{n,F}^y, \mathcal{N}_{n,F}^\theta) : \Gamma_n^\alpha \rightarrow \mathcal{W}_{n+k-1} \times \mathcal{W}_{n+2k-2} \times (\mathcal{W}_{n+2p-2})^d$ , given by

$$\begin{aligned} \mathcal{N}_{n,F}^x(f) &= \mathcal{K}_n^y [c \circ (\mathcal{K}_n^\theta + f^\theta) - c \circ \mathcal{K}_n^\theta] + f^y c \circ (\mathcal{K}_n^\theta + f^\theta) + \mathcal{E}_n^x, \\ \mathcal{N}_{n,F}^y(f) &= P \circ (\mathcal{K}_n + f) - P \circ \mathcal{K}_n + \mathcal{E}_n^y, \\ \mathcal{N}_{n,F}^\theta(f) &= Q \circ (\mathcal{K}_n + f) - Q \circ \mathcal{K}_n + \mathcal{E}_n^\theta. \end{aligned}$$

The following lemma states that the operators  $\mathcal{S}_{n,R}$  have a bounded right inverse and provides a bound for the norm  $\|\mathcal{S}_{n,R}^{-1}\|$ .

**Lemma 5.7.8.** Given  $k \geq 2$ , for all  $n \geq 1$ , the operator  $\mathcal{S}_{n,R} : \mathcal{W}_n \rightarrow \mathcal{W}_n$  (see Definition 5.7.5), has a bounded right inverse,

$$\mathcal{S}_{n,R}^{-1} : \mathcal{W}_{n+k-1} \rightarrow \mathcal{W}_n$$

given by

$$\mathcal{S}_{n,R}^{-1} \eta = - \sum_{j=0}^{\infty} \eta \circ R^j, \quad \eta \in \mathcal{W}_{n+k-1}. \quad (5.7.3)$$

Moreover, for any fixed  $\mu \in (0, (k-1)|R_k^x| \cos \lambda)$ , with  $\lambda = \beta \frac{k-1}{2}$ , there exists  $\rho > 0$  such that, taking  $S(\beta, \rho) \times \mathbb{T}_{\sigma'}^d$  as the domain of the functions of  $\mathcal{W}_{n+k-1}$ , we have the operator norm bound

$$\|(\mathcal{S}_{n,R})^{-1}\| \leq \rho^{k-1} + \frac{1}{\mu} \frac{k-1}{n}.$$

*Proof.* A simple computation shows that (5.7.3) gives a formal right inverse of  $\mathcal{S}_{n,R}$ , as shown in the proof of Lemma 2.4.6.

Denoting  $R(z, \theta) = (R^x(z), R^\theta(z, \theta))$ , note that one has

$$(R^j(z, \theta))^x = (R^x)^j(z), \quad (R^j(z, \theta))^\theta = \theta + j\omega,$$

and therefore the composition  $\eta \circ R^j$  is well defined for every  $j \in \mathbb{N}$ .

By Lemma 5.7.4,  $R^x$  maps  $S(\beta, \rho)$  into itself, and hence it is clear that  $(R^j(z, \theta))^x \in S(\beta, \rho)$ , for any  $j$ , and also  $(R^j(z, \theta))^\theta \in \mathbb{T}_{\sigma'}^d$ .

Then we see that the series given by (5.7.3) converges uniformly on  $S \times \mathbb{T}_{\sigma'}^d$ , using the Weierstrass M-test. Indeed, using again Lemma 5.7.4 we have

$$\begin{aligned} |\eta(R^j(z, \theta))| &\leq \|\eta\|_{n+k-1} |(R^x)^j(z)|^{n+k-1} \leq \|\eta\|_{n+k-1} \left( \frac{|z|}{(1 + j\mu|z|^{k-1})^{1/k-1}} \right)^{n+k-1} \\ &\leq C \|\eta\|_{n+k-1} \frac{1}{j^{\frac{n}{k-1}+1}}, \quad \forall (z, \theta) \in S \times \mathbb{T}_{\sigma'}^d, \end{aligned}$$

so (5.7.3) converges uniformly on  $S \times \mathbb{T}_{\sigma'}^d$ , and  $\sum_{j=0}^{\infty} \eta \circ R^j$  is holomorphic. Finally, we obtain the claimed bound for  $\|(\mathcal{S}_{n,R})^{-1}\|$  in a completely analogous way as in the proof of Lemma 2.4.6.  $\square$

In the following Lemma we show that the operators  $\mathcal{N}_{n,F}$  are Lipschitz and we provide bounds for their Lipschitz constants. Even if the statement of this lemma looks analogous to the one of Lemma 2.4.7 of Chapter 2, here the proof uses more delicate estimates that have to be taken into account because of the dependence on  $\theta$  of the functions of  $\mathcal{W}_n$ .

**Lemma 5.7.9.** *For each  $n \geq 3$ , there exists a constant,  $M_n > 0$ , for which the operator  $\mathcal{N}_{n,F}$  satisfies*

$$\begin{aligned} \text{Lip } \mathcal{N}_{n,F}^x &\leq \sup_{\theta \in \mathbb{T}_{\sigma'}^d} |c(\theta)| + M_n \rho, \\ \text{Lip } \mathcal{N}_{n,F}^y &\leq k \sup_{\theta \in \mathbb{T}_{\sigma'}^d} |a_k(\theta)| + M_n \rho, \\ \text{Lip } \mathcal{N}_{n,F}^\theta &\leq p \sup_{\theta \in \mathbb{T}_{\sigma'}^d} |d_p(\theta)| + M_n \rho, \end{aligned}$$

where  $\rho$  is the radius of the sector  $S(\beta, \rho)$ .

*Proof.* We deal with the three components of  $\mathcal{N}_{n,F}$  separately, obtaining bounds for their Lipschitz constants. Recall that from the definition of  $\mathcal{N}_{n,F}$  we have  $2p > k - 1$ .

Along the proof we use recurrently the following type of argument: if a given analytic function  $g$  defined in  $S(\beta, \rho) \times \mathbb{T}_{\sigma'}^d$  satisfies  $g(z, \theta) = O(|z|^n)$ , for some integer  $n$ , then  $g$  belongs to the space  $\mathcal{W}_n$ , and consequently, there exists a constant,  $M_n > 0$ , such that  $\|g\|_n < M_n$ .

First we prove the lemma for  $\mathcal{N}_{n,F}^x$ . For each  $f, \tilde{f} \in \Gamma_n^\alpha$  we have,

$$\begin{aligned} \mathcal{N}_{n,F}^x(f) - \mathcal{N}_{n,F}^x(\tilde{f}) &= \mathcal{K}_n^y(c \circ (\mathcal{K}_n^\theta + f^\theta) - c \circ (\mathcal{K}_n^\theta + \tilde{f}^\theta)) + f^y c \circ (\mathcal{K}_n^\theta + f^\theta) - \tilde{f}^y c \circ (\mathcal{K}_n^\theta + \tilde{f}^\theta) \\ &= (\mathcal{K}_n^y + f^y) \int_0^1 Dc \circ (\mathcal{K}_n^\theta + \tilde{f}^\theta + s(f^\theta - \tilde{f}^\theta)) ds (f^\theta - \tilde{f}^\theta) \\ &\quad + c \circ (\mathcal{K}_n^\theta + \tilde{f}^\theta) (f^y - \tilde{f}^y). \end{aligned}$$

We can then bound, for some  $M_n > 0$ ,

$$\begin{aligned}
& \|(\mathcal{K}_n^y + f^y) \int_0^1 Dc \circ (\mathcal{K}_n^\theta + \tilde{f}^\theta + s(f^\theta - \tilde{f}^\theta)) ds (f^\theta - \tilde{f}^\theta)\|_{n+k-1} \\
& \leq \sup_{\mathbb{T}_\sigma^d} |Dc(\theta)| \sup_{S \times \mathbb{T}_{\sigma'}^d} \frac{1}{|z|^{n+k-1}} |\mathcal{K}_n^y(z, \theta) + f^y(z, \theta)| |f^\theta(z, \theta) - \tilde{f}^\theta(z, \theta)| \\
& \leq \|f^\theta - \tilde{f}^\theta\|_{n+2p-k-1} \|\mathcal{K}_n^y + f^y\|_{k+1} \sup_{\mathbb{T}_\sigma^d} |Dc(\theta)| \sup_{S \times \mathbb{T}_{\sigma'}^d} |z|^{2p-k+1} \\
& \leq M_n \rho^{2p-k+1} \|f^\theta - \tilde{f}^\theta\|_{n+2p-k-1},
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
\|c \circ (\mathcal{K}_n^\theta + \tilde{f}^\theta)(f^y - \tilde{f}^y)\|_{n+k-1} &= \sup_{S \times \mathbb{T}_{\sigma'}^d} |c \circ (\mathcal{K}_n^\theta + \tilde{f}^\theta)(z, \theta)| \frac{|f^y(z, \theta) - \tilde{f}^y(z, \theta)|}{|z|^{n+k-1}} \\
&\leq \sup_{\mathbb{T}_\sigma^d} |c(\theta)| \|f^y - \tilde{f}^y\|_{n+k-1},
\end{aligned}$$

and thus, we obtain

$$\|\mathcal{N}_{n,F}^x(f) - \mathcal{N}_{n,F}^x(\tilde{f})\|_{n+k-1} \leq (\sup_{\mathbb{T}_\sigma^d} |c(\theta)| + M_n \rho) \max\{\|f^y - \tilde{f}^y\|_{n+k-1}, \|f^\theta - \tilde{f}^\theta\|_{n+2p-k-1}\},$$

that is,

$$\text{Lip } \mathcal{N}_{n,F}^x \leq \sup_{\mathbb{T}_\sigma^d} |c(\theta)| + M_n \rho.$$

Next we consider  $\mathcal{N}_{n,F}^y$ . By Taylor's Theorem we have, for each  $f, \tilde{f} \in \Gamma_n^\alpha$ ,

$$\mathcal{N}_{n,F}^y(f) - \mathcal{N}_{n,F}^y(\tilde{f}) = \int_0^1 DP \circ (\mathcal{K}_n + \tilde{f} + s(f - \tilde{f})) ds (f - \tilde{f}) \in \mathcal{W}_{n+2k-2}. \quad (5.7.4)$$

Indeed, by the form of  $P$ , it is clear that the leading terms of (5.7.4) are contained in

$$a_k \circ (\mathcal{K}_n^\theta + f^\theta)(\mathcal{K}_n^x + f^x)^k - a_k \circ (\mathcal{K}_n^\theta + \tilde{f}^\theta)(\mathcal{K}_n^x + \tilde{f}^x)^k \in \mathcal{W}_{n+2k-2}. \quad (5.7.5)$$

We denote the rest of the terms in (5.7.4) by  $\Phi(f, \tilde{f})(f - \tilde{f})$ . Since  $\Phi(f, \tilde{f})(f - \tilde{f})$  contains only higher order terms than (5.7.5), it belongs to  $\mathcal{W}_{n+2k-1}$ , and thus, we have

$$\|\Phi(f, \tilde{f})(f - \tilde{f})\|_{n+2k-2} \leq M_n \rho \|f - \tilde{f}\|_{\Gamma_n}.$$

We will therefore focus on bounding (5.7.5) in  $\mathcal{W}_{n+2k-2}$ . To simplify the notation, we define, for  $s \in [0, 1]$ ,

$$\xi_s = \xi_s(f, \tilde{f}) = \mathcal{K}_n + \tilde{f} + s(f - \tilde{f}) \in \mathcal{W}_2 \times \mathcal{W}_{k+1} \times (\mathcal{W}_0)^d.$$

Note that indeed we have

$$\xi_s^x(z, \theta) = z^2 + O(|z|^3), \quad \xi_s^y(z, \theta) = \bar{K}_{k+1}^y z^{k+1} + O(|z|^{k+2}), \quad \xi_s^\theta(z, \theta) = \theta + O(|z|),$$

since the presence of  $f$  does not affect the lower order terms of  $\xi_s$ , and since the coefficients depending on  $\theta$  of  $\mathcal{K}_n(z, \theta)$  are bounded for  $\theta \in \mathbb{T}_\sigma^d$ , as a consequence of the small divisors lemma (Theorem 5.2.1).

Next we proceed to bounding (5.7.5). One can write

$$\begin{aligned} & a_k \circ (\mathcal{K}_n^\theta + f^\theta)(\mathcal{K}_n^x + f^x)^k - a_k \circ (\mathcal{K}_n^\theta + \tilde{f}^\theta)(\mathcal{K}_n^x + \tilde{f}^x)^k \\ &= a_k \circ (\mathcal{K}_n^\theta + f^\theta) k \int_0^1 (\xi_s^x)^{k-1} ds (f^x - \tilde{f}^x) + (\mathcal{K}_n^x + \tilde{f}^x)^k \int_0^1 Da_k \circ (\xi_s^\theta) ds (f^\theta - \tilde{f}^\theta), \end{aligned} \quad (5.7.6)$$

and we have, for the first term of the sum,

$$\begin{aligned} & \|a_k \circ (\mathcal{K}_n^\theta + f^\theta) k \int_0^1 (\xi_s^x)^{k-1} ds (f^x - \tilde{f}^x)\|_{n+2k-2} \\ & \leq \|a_k \circ (\mathcal{K}_n^\theta + f^\theta) k \int_0^1 (\xi_s^x)^{k-1} ds\|_{2k-2} \|f^x - \tilde{f}^x\|_n \\ & \leq \sup_{s \in [0,1]} \sup_{S \times \mathbb{T}_\sigma^d} \frac{1}{|z|^{2k-2}} (k |a_k \circ (\mathcal{K}_n^\theta + f^\theta)(z, \theta)| |\xi_s^x(z, \theta)|^{k-1}) \|f^x - \tilde{f}^x\|_n \\ & \leq k \sup_{\mathbb{T}_\sigma^d} |a_k(\theta)| \sup_{s \in [0,1]} \sup_{S \times \mathbb{T}_\sigma^d} \frac{1}{|z|^{2k-2}} |\xi_s^x(z, \theta)|^{k-1} \|f^x - \tilde{f}^x\|_n \\ & \leq (k \sup_{\mathbb{T}_\sigma^d} |a_k(\theta)| + M_n \rho) \|f^x - \tilde{f}^x\|_n. \end{aligned}$$

For the second term of the sum in (5.7.6) we have

$$\begin{aligned} & \|(\mathcal{K}_n^x + \tilde{f}^x)^k \int_0^1 Da_k \circ (\xi_s^\theta) ds (f^\theta - \tilde{f}^\theta)\|_{n+2k-2} \\ & \leq \sup_{\mathbb{T}_\sigma^d} |Da_k(\theta)| \sup_{S \times \mathbb{T}_\sigma^d} \frac{1}{|z|^{n+2k-2}} |(\mathcal{K}_n^x(z, \theta) + \tilde{f}^x(z, \theta))^k| \|f^\theta(z, \theta) - \tilde{f}^\theta(z, \theta)\| \\ & \leq \|f^\theta - \tilde{f}^\theta\|_{n+2p-k-1} \|(\mathcal{K}_n^x + \tilde{f}^x)^k\|_{2k} \sup_{\mathbb{T}_\sigma^d} |Da_k(\theta)| \sup_{S \times \mathbb{T}_\sigma^d} |z|^{2p-k+1} \\ & \leq M_n \rho^{2p-k+1} \|f^\theta - \tilde{f}^\theta\|_{n+2p-k-1}, \end{aligned}$$

where  $2p - k + 1 \geq 1$ .

Finally, putting together the obtained bounds, we have

$$\begin{aligned} \|\mathcal{N}_{n,F}^y(f) - \mathcal{N}_{n,F}^y(\tilde{f})\|_{n+2k-2} & \leq \|a_k(\mathcal{K}_n^\theta + f^\theta)(\mathcal{K}_n^x + f^x)^k - a_k(\mathcal{K}_n^\theta + \tilde{f}^\theta)(\mathcal{K}_n^x + \tilde{f}^x)^k\|_{n+2k-2} \\ & \quad + \|\Gamma(f, \tilde{f})(f - \tilde{f})\|_{n+2k-2} \\ & \leq (k \sup_{\mathbb{T}_\sigma^d} |a_k(\theta)| + M_n \rho) \|f^x - \tilde{f}^x\|_n + M_n \rho \|f - \tilde{f}\|_{\Gamma_n} \\ & \leq (k \sup_{\mathbb{T}_\sigma^d} |a_k(\theta)| + M_n \rho) \|f - \tilde{f}\|_{\Gamma_n}, \end{aligned}$$

as we wanted to see.

Finally we prove the result for  $\mathcal{N}_{n,F}^\theta$  in an analogous way as with  $\mathcal{N}_{n,F}^y$ .

Here we have, for each  $f, \tilde{f} \in \Gamma_n^\alpha$ ,

$$\mathcal{N}_{n,F}^\theta(f) - \mathcal{N}_{n,F}^\theta(\tilde{f}) = \int_0^1 DQ \circ (\xi_s) ds (f - \tilde{f}) \in \mathcal{W}_{n+2p-2}, \quad (5.7.7)$$

and by the form of  $Q$ , the leading terms of (5.7.7) are contained in

$$d_p \circ (\mathcal{K}_n^\theta + f^\theta)(\mathcal{K}_n^x + f^x)^p - d_p \circ (\mathcal{K}_n^\theta + \tilde{f}^\theta)(\mathcal{K}_n^x + \tilde{f}^x)^p \in \mathcal{W}_{n+2p-2}. \quad (5.7.8)$$

As before, the rest of the terms are not relevant in the norm  $\|\mathcal{N}_{n,F}^\theta(f) - \mathcal{N}_{n,F}^\theta(\tilde{f})\|_{n+2p-2}$ , so we focus the attention on bounding (5.7.8) in  $\mathcal{W}_{n+2p-2}$ .

We have

$$\begin{aligned} & d_p \circ (\mathcal{K}_n^\theta + f^\theta)(\mathcal{K}_n^x + f^x)^p - d_p \circ (\mathcal{K}_n^\theta + \tilde{f}^\theta)(\mathcal{K}_n^x + \tilde{f}^x)^p \\ &= d_p \circ (\mathcal{K}_n^\theta + f^\theta) p \int_0^1 (\xi_s^x)^{p-1} ds (f^x - \tilde{f}^x) + (\mathcal{K}_n^x + \tilde{f}^x)^p \int_0^1 Dd_p \circ (\xi_s^\theta) ds (f^\theta - \tilde{f}^\theta). \end{aligned}$$

We can bound the two terms of the sum above as

$$\begin{aligned} & \|d_p \circ (\mathcal{K}_n^\theta + f^\theta) p \int_0^1 (\xi_s^x)^{p-1} ds (f^x - \tilde{f}^x)\|_{n+2p-2} \\ & \leq \|d_p \circ (\mathcal{K}_n^\theta + f^\theta) p \int_0^1 (\xi_s^x)^{p-1} ds\|_{2p-2} \|f^x - \tilde{f}^x\|_n \\ & \leq \sup_{s \in [0,1]} \sup_{S \times \mathbb{T}_\sigma^d} \frac{1}{|z|^{2p-2}} (p |d_p \circ (\mathcal{K}_n^\theta + f^\theta)(z, \theta)| |\xi_s^x(z, \theta)|^{p-1}) \|f^x - \tilde{f}^x\|_n \\ & \leq \sup_{s \in [0,1]} \sup_{S \times \mathbb{T}_\sigma^d} \frac{1}{|z|^{2p-2}} (p |d_p(\theta)| |\xi_s^x(z, \theta)|^{p-1}) \|f^x - \tilde{f}^x\|_n \\ & \leq (p \sup_{\mathbb{T}_\sigma^d} |d_p(\theta)| + M_n \rho) \|f^x - \tilde{f}^x\|_n, \end{aligned}$$

and

$$\begin{aligned} & \|(\mathcal{K}_n^x + \tilde{f}^x)^p \int_0^1 Dd_p \circ (\xi_s^\theta) ds (f^\theta - \tilde{f}^\theta)\|_{n+2p-2} \\ & \leq \sup_{\mathbb{T}_\sigma^d} |Dd_p(\theta)| \sup_{S \times \mathbb{T}_\sigma^d} \frac{1}{|z|^{n+2p-2}} |(\mathcal{K}_n^x(z, \theta) + \tilde{f}^x(z, \theta))^p| \|f^\theta(z, \theta) - \tilde{f}^\theta(z, \theta)\| \\ & \leq \|f^\theta - \tilde{f}^\theta\|_{n+2p-k-1} \|(\mathcal{K}_n^x + \tilde{f}^x)^p\|_{2p} \sup_{\mathbb{T}_\sigma^d} |Dd_p(\theta)| \sup_S |z|^{2p-k+1} \\ & \leq M_n \rho^{2p-k+1} \|f^\theta - \tilde{f}^\theta\|_{n+2p-k-1}, \end{aligned}$$

and therefore, taking into account the obtained bounds we get the estimate claimed in the statement,

$$\|\mathcal{N}_{n,F}^\theta(f) - \mathcal{N}_{n,F}^\theta(\tilde{f})\|_{n+2p-2} \leq (p \sup_{\mathbb{T}_\sigma^d} |d_p(\theta)| + M_n \rho) \|f - \tilde{f}\|_{\Gamma_n}.$$

□

Next, we define some more operators and we introduce the family  $\mathcal{T}_{n,F}$ .

**Definition 5.7.10.** For  $n > 2p - k - 1$ , we denote by  $\mathcal{S}_{n,R}^\times : \Gamma_n \rightarrow \Gamma_n$  the linear operator defined component-wise as  $\mathcal{S}_{n,R}^\times = (\mathcal{S}_{n,R}, \mathcal{S}_{n+k-1,R}, (\mathcal{S}_{n+2p-k-1,R})^d)$ .

**Remark 5.7.11.** Since  $\mathcal{S}_{n,R}^\times$  is defined component-wise, its inverse,

$$(\mathcal{S}_{n,R}^\times)^{-1} : \mathcal{W}_{n+k-1} \times \mathcal{W}_{n+2k-2} \times (\mathcal{W}_{n+2p-2})^d \rightarrow \Gamma_n,$$

is given by

$$(\mathcal{S}_{n,R}^\times)^{-1} = (\mathcal{S}_{n,R}^{-1}, \mathcal{S}_{n+k-1,R}^{-1}, (\mathcal{S}_{n+2p-k-1,R}^{-1})^d).$$

**Definition 5.7.12.** Let  $F$  be the holomorphic extension of an analytic map of the form (5.1.2) satisfying the hypotheses of Theorem 5.3.1. Given  $n \geq 3$ , we define  $\mathcal{T}_{n,F} : \Gamma_n^\alpha \rightarrow \Gamma_n$  by

$$\mathcal{T}_{n,F} = (\mathcal{S}_{n,R}^\times)^{-1} \circ \mathcal{N}_{n,F}.$$

**Remark 5.7.13.** Note that given a map  $F$ , to define the previous operators we always take together the associated triple  $(F, \mathcal{K}_n, R = \mathcal{R}_n)$  satisfying  $F \circ \mathcal{K}_n - \mathcal{K}_n \circ R = \mathcal{E}_n$ . Then, the operators  $\mathcal{S}_{n,R}$ ,  $\mathcal{N}_{n,F}$  and  $\mathcal{T}_{n,F}$  are associated not only with the map  $F$  itself but to the approximation of a particular invariant manifold of  $F$ .

Using the introduced operators, equations (5.6.4) can be written as

$$\mathcal{S}_{n,R}^\times \Delta = \mathcal{N}_{n,F}(\Delta).$$

**Lemma 5.7.14.** There exist  $m_0 > 0$  and  $\rho_0 > 0$  such that if  $\rho < \rho_0$ , then, for every  $n \geq m_0$ , we have  $\mathcal{T}_{n,F}(\Gamma_n^\alpha) \subseteq \Gamma_n^\alpha$  and  $\mathcal{T}_{n,F}$  is a contraction operator in  $\Gamma_n^\alpha$ .

*Proof.* By its definition, the operator  $\mathcal{T}_{n,F}$  satisfies

$$\begin{aligned} \text{Lip } \mathcal{T}_{n,F} \leq & \max\{\|(\mathcal{S}_{n,R})^{-1}\| \text{Lip } \mathcal{N}_{n,F}^x, \|(\mathcal{S}_{n+k-1,R})^{-1}\| \text{Lip } \mathcal{N}_{n,F}^y, \\ & \|(\mathcal{S}_{n+2p-k-1,R})^{-1}\| \text{Lip } \mathcal{N}_{n,F}^\theta\}. \end{aligned} \quad (5.7.9)$$

From (5.7.9) and the estimates obtained in Lemmas 5.7.8 and 5.7.9, given  $\mu \in (0, (k-1)|\tilde{R}_k^x| \cos \lambda)$ , with  $\lambda = \beta \frac{k-1}{2}$ , there is  $\rho_0 > 0$  such that for  $\rho < \rho_0$  we have the bound

$$\begin{aligned} \text{Lip } \mathcal{T}_{n,F} \leq & \max \left\{ (\rho^{k+1} + \frac{1}{\mu} \frac{k-1}{n}) (\sup_{T_\sigma^d} |c(\theta)| + M_n \rho), \right. \\ & \left. (\rho^{k+1} + \frac{1}{\mu} \frac{k-1}{n+k-1}) (\sup_{T_\sigma^d} |a_k(\theta)| + M_n \rho), (\rho^{k+1} + \frac{1}{\mu} \frac{k-1}{n+2p-k-1}) (\sup_{T_\sigma^d} |d_p(\theta)| + M_n \rho) \right\}, \end{aligned}$$

taking  $S(\beta, \rho) \times \mathbb{T}_{\sigma'}^d$  as the domain of the functions of  $\Gamma_n^\alpha$ .

Then, choosing  $\rho < \rho_0$  small enough, it is clear that one can choose  $m_0$  such that, for  $n \geq m_0$ , one has  $\text{Lip } \mathcal{T}_{n,F} < 1$ .

Hence, for the chosen  $\rho < \rho_0$  and  $n \geq m_0$ ,  $\mathcal{T}_{n,F}$  is a contraction in  $\Gamma_n^\alpha$ .

Next we prove that one can find  $\tilde{\rho}_0 > 0$ , maybe smaller than  $\rho_0$ , such that taking  $S(\beta, \rho) \times \mathbb{T}_{\sigma'}^d$ , with  $\rho < \tilde{\rho}_0$  as the domain of the functions of  $\Gamma_n^\alpha$ , then  $\mathcal{T}_{n,F}$  maps  $\Gamma_n^\alpha$  into itself.

For each  $f \in \Gamma_n^\alpha$  we can write

$$\begin{aligned} \|\mathcal{T}_{n,F}(f)\|_{\Gamma_n} & \leq \|\mathcal{T}_{n,F}(f) - \mathcal{T}_{n,F}(0)\|_{\Gamma_n} + \|\mathcal{T}_{n,F}(0)\|_{\Gamma_n} \\ & \leq \alpha \text{Lip } \mathcal{T}_{n,F} + \|\mathcal{T}_{n,F}(0)\|_{\Gamma_n}. \end{aligned}$$



From the definition of  $\mathcal{T}_{n,F}$  and  $\mathcal{N}_{n,F}$  we have, for each  $n \in \mathbb{N}$ ,

$$\mathcal{T}_{n,F}(0) = (\mathcal{S}_{n,R}^\times)^{-1} \circ \mathcal{N}_{n,F}(0) = (\mathcal{S}_{n,R}^\times)^{-1} \mathcal{E}_n.$$

Also, we have  $\mathcal{E}_n = (\mathcal{E}_n^x, \mathcal{E}_n^y, \mathcal{E}_n^\theta) \in \mathcal{W}_{n+k} \times \mathcal{W}_{n+2k-1} \times (\mathcal{W}_{n+2p-1})^p$ , and thus, for every  $\varepsilon > 0$ , there is  $\rho_n > 0$  such that for  $\rho < \rho_n$  one has

$$\|\mathcal{T}_{n,F}(0)\|_{\Gamma_n} \leq \|(\mathcal{S}_{n,R}^\times)^{-1}\| \max\{\|\mathcal{E}_n^x\|_{n+k-1}, \|\mathcal{E}_n^y\|_{n+2k-2}, \|\mathcal{E}_n^\theta\|_{n+2p-2}\} \leq \|(\mathcal{S}_{n,R}^\times)^{-1}\| M_n \rho < \varepsilon.$$

Moreover, since we have  $\text{Lip } \mathcal{T}_{n,F} < 1$ , we can take  $\rho_n$  as

$$\rho_n = \sup \{\rho > 0 \mid \alpha \text{Lip } \mathcal{T}_{n,F} + \|\mathcal{T}_{n,F}(0)\|_{\Gamma_n} \leq \alpha\},$$

and then for every  $\rho < \rho_n$  it holds that  $\mathcal{T}_{n,F}(\Gamma_n^\alpha) \subseteq \Gamma_n^\alpha$ .  $\square$

## 5.8 Proofs of the main results

Next we give the proofs of Theorems 5.3.1 and 5.3.2 using the results presented along the chapter.

*Proof of Theorem 5.3.1.* Let  $m_0$  be the integer provided by Lemma 5.7.14, and let  $n_0 = \max\{m_0, k+1\}$ . We take the maps  $\mathcal{K}_{n_0}$  and  $R = \mathcal{R}_{n_0}$  given by Proposition 5.4.1, which satisfy

$$\mathcal{E}_{n_0}(t, \theta) = F(\mathcal{K}_{n_0}(t, \theta)) - \mathcal{K}_{n_0}(R(t, \theta)) = (O(t^{n_0+k}), O(t^{n_0+2k-1}), O(t^{n_0+2p-1})).$$

We will look for  $\rho > 0$  and for a function  $\Delta : [0, \rho) \times \mathbb{T}^d$ ,  $\Delta$  analytic in  $(0, \rho) \times \mathbb{T}^d$ , satisfying

$$F \circ (\mathcal{K}_{n_0} + \Delta) - (\mathcal{K}_{n_0} + \Delta) \circ R = 0. \quad (5.8.1)$$

Next, consider the holomorphic extension of  $F$  to a neighborhood  $V \times \mathbb{T}_\sigma^d$  of  $(0, 0) \times \mathbb{T}^d$ , where  $V \subset \mathbb{C}^2$  contains the centered closed ball of radius  $b > 0$  and take  $\alpha = \min\{\frac{1}{2}, \frac{b}{2}, \frac{\tilde{\sigma}}{2}\}$ , with  $0 < \tilde{\sigma} < \sigma$ . With this setting we rewrite (5.8.1) as

$$\begin{aligned} \Delta^x \circ R - \Delta^x &= \mathcal{K}_n^y [c \circ (\mathcal{K}_n^\theta + \Delta^\theta) - c \circ \mathcal{K}_n^\theta] + \Delta^y c \circ (\mathcal{K}_n^\theta + \Delta^\theta) + \mathcal{E}_n^x, \\ \Delta^y \circ R - \Delta^y &= P \circ (\mathcal{K}_n + \Delta) - P \circ \mathcal{K}_n + \mathcal{E}_n^y, \\ \Delta^\theta \circ R - \Delta^\theta &= Q \circ (\mathcal{K}_n + \Delta) - Q \circ \mathcal{K}_n + \mathcal{E}_n^\theta, \end{aligned}$$

with  $\Delta \in \Gamma_{n_0}^\alpha$ , or using the previously defined operators and function spaces,

$$\mathcal{S}_{n_0,R}^\times \Delta = \mathcal{N}_{n_0,F}(\Delta), \quad \Delta \in \Gamma_{n_0}^\alpha.$$

By Lemma 5.7.8, if  $\rho$  is small enough,  $\mathcal{S}_{n_0,R}^\times$  has a bounded right inverse and we can rewrite the equation as

$$\Delta = \mathcal{T}_{n_0,F}(\Delta), \quad \Delta \in \Gamma_{n_0}^\alpha.$$

By Lemma 5.7.14, since  $n_0 \geq m_0$ , we have that  $\mathcal{T}_{n_0,F}$  maps  $\Gamma_{n_0}^\alpha$  into itself and is a contraction. Then it has a unique fixed point,  $\Delta^\infty \in \Gamma_{n_0}^\alpha$ . Note that this solution is unique once  $\mathcal{K}_{n_0}$  is fixed. Finally  $K = \mathcal{K}_{n_0} + \Delta^\infty$  satisfies the conditions in the statement.

The  $C^1$  character of  $K$  at the origin follows from the order condition of  $K$  at 0.  $\square$

*Proof of Theorem 5.3.2.* Let  $m_0$  be the integer provided by Lemma 5.7.14, and let  $n_0 = \max\{m_0, k + 1\}$ . If the value of  $n$  given in the statement is such that  $n < n_0$ , first we look for a better approximation  $\mathcal{K}_{n_0}$  of the form

$$\mathcal{K}_{n_0}(t, \theta) = \hat{K}(t, \theta) + \sum_{j=n+1}^{n_0} \hat{K}_j(t, \theta),$$

with

$$\hat{K}_j(t, \theta) = \begin{pmatrix} \bar{K}_j^x t^j + \tilde{K}_{j+k-1}^x(\theta) t^{j+k-1} \\ \bar{K}_{j+k-1}^y t^{j+k-1} + \tilde{K}_{j+2k-2}^y(\theta) t^{j+2k-2} \\ \bar{K}_{j+2p-k-1}^\theta t^{j+2p-k-1} + \tilde{K}_{j+2p-2}^\theta(\theta) t^{j+2p-2} \end{pmatrix},$$

and

$$\mathcal{R}_{n_0}(t, \theta) = \hat{R}(t, \theta) + \sum_{j=n+1}^{n_0} \hat{R}_j(t),$$

with

$$\hat{R}_j^x(t) = \begin{cases} \delta_{j,k+1} \bar{R}_{2k-1}^x t^{2k-1} & \text{if } n \leq k, \\ 0 & \text{if } n > k, \end{cases} \quad \hat{R}_j^\theta(t) = 0.$$

The coefficients of  $\mathcal{K}_{n_0}(t, \theta)$  and  $\mathcal{R}_{n_0}(t, \theta)$  are obtained imposing the condition

$$F(\mathcal{K}_{n_0}(t, \theta)) - \mathcal{K}_{n_0}(\mathcal{R}_{n_0}(t, \theta)) = (O(t^{n_0+k}), O(t^{n_0+2k-1}), O(t^{n_0+2p-1})).$$

Proceeding as in Proposition 5.4.1, we obtain such coefficients iteratively. We denote  $\mathcal{K}_j(t, \theta) = \hat{K}(t, \theta) + \sum_{m=n+1}^j \hat{K}_m(t, \theta)$  and  $\mathcal{R}_j(t, \theta) = \hat{R}(t, \theta) + \sum_{m=n+1}^j \hat{R}_m(t)$ . In the iterative step we have

$$F(\mathcal{K}_j(t, \theta)) - \mathcal{K}_j(\mathcal{R}_j(t, \theta)) = (O(t^{j+k}), O(t^{j+2k-1}), O(t^{j+2p-1})).$$

Then,

$$\begin{aligned} & F(\mathcal{K}_j(t, \theta) + \hat{K}_{j+1}(t, \theta)) - (\mathcal{K}_j + \hat{K}_{j+1}) \circ (\mathcal{R}_j(t, \theta) + \hat{R}_{j+1}(t)) \\ &= F(\mathcal{K}_j(t, \theta)) - \mathcal{K}_j(\mathcal{R}_j(t, \theta)) \\ & \quad + DF(\mathcal{K}_j(t, \theta)) \hat{K}_{j+1}(t, \theta) - \hat{K}_{j+1}(\mathcal{R}_j(t, \theta) + \hat{R}_{j+1}(t)) \\ & \quad + \int_0^1 (1-s) D^2 F(\mathcal{K}_j(t, \theta) + s \hat{K}_{j+1}(t, \theta)) ds (\hat{K}_{j+1}(t, \theta))^{\otimes 2} \\ & \quad - D\mathcal{K}_j(\mathcal{R}_j(t, \theta)) \hat{R}_{j+1}(t) \\ & \quad - \int_0^1 (1-s) D^2 \mathcal{K}_j(\mathcal{R}_j(t, \theta) + s \hat{R}_{j+1}(t)) ds (\hat{R}_{j+1}(t))^{\otimes 2}. \end{aligned}$$

The condition

$$F(\mathcal{K}_{j+1}(t, \theta)) - \mathcal{K}_{j+1}(\mathcal{R}_{j+1}(t, \theta)) = (O(t^{j+k+1}), O(t^{j+2k}), O(t^{j+2p}))$$

leads to the same equations (5.4.4) and (5.4.5) as in Proposition 5.4.1, which we solve in the same way.

From this point we can proceed as in the proof of Theorem 5.3.1 and look for  $\Delta \in \Gamma_n^\alpha \subset \mathcal{W}_{n_0} \times \mathcal{W}_{n_0+k-1} \times (\mathcal{W}_{n_0+2p-k-1})^d$  such that the pair  $K = \mathcal{K}_{n_0} + \Delta$ ,  $R = \mathcal{R}_{n_0}$  satisfies  $F \circ K = K \circ R$ .

Finally, for the map  $K$ , we also have

$$\begin{aligned} K(t, \theta) - \hat{K}(t, \theta) &= \mathcal{K}_{n_0}(t, \theta) - \hat{K}(t, \theta) + \Delta(t, \theta) \\ &= \sum_{j=n+1}^{n_0} \hat{K}_j(t, \theta) + \Delta(t, \theta) \\ &= (O(t^{n+1}), O(t^{n+k}), O(t^{n+2p-k})) + (O(t^{n_0}), O(t^{n_0+k-1}), O(t^{n_0+2p-k-1})), \end{aligned}$$

with  $n < n_0$ . We also have  $n + 2p - k \leq n_0 + 2p - k - 1$ , and therefore,

$$K(t, \theta) - \hat{K}(t, \theta) = (O(t^{n+1}), O(t^{n+k}), O(t^{n+2p-k})).$$

For the map  $R$  we have

$$R(t, \theta) - \hat{R}(t, \theta) = \mathcal{R}_{n_0}(t, \theta) - \hat{R}(t, \theta) = \sum_{j=n+1}^{n_0} \hat{R}_j(t) = \begin{cases} (O(t^{2k-1}), 0) & \text{if } n \leq k, \\ (0, 0) & \text{if } n > k. \end{cases}$$

If  $n \geq n_0$  we look for  $\mathcal{K}^*(t, \theta) = \hat{K}(t, \theta) + \hat{K}_{n+1}(t, \theta)$  with

$$\hat{K}_{n+1}(t, \theta) = \begin{pmatrix} \bar{K}_{n+1}^x t^{n+1} + \tilde{K}_{n+k}^x(\theta) t^{n+k} \\ \bar{K}_{n+k}^y t^{n+k} + \tilde{K}_{n+2k-1}^y(\theta) t^{n+2k-1} \\ \bar{K}_{n+2p-k}^\theta t^{n+2p-k} + \tilde{K}_{n+2p-1}^\theta(\theta) t^{n+2p-1} \end{pmatrix},$$

and  $\mathcal{R}_n^*(t, \theta) = \hat{R}(t, \theta) + \hat{R}_{n+1}(t)$  with

$$\hat{R}_{n+1}^x(t) = \begin{cases} \bar{R}_{2k-1}^x t^{2k-1} & \text{if } n \leq k, \\ 0 & \text{if } n > k, \end{cases} \quad \hat{R}_{n+1}^\theta(t) = 0.$$

We determine the coefficients of  $\hat{K}_{n+1}(t, \theta)$  so that

$$F \circ \mathcal{K}^*(t) - \mathcal{K}^* \circ \mathcal{R}^*(t) = (O(t^{n+k+1}), O(t^{n+2k}), O(t^{n+2p}))$$

as in the previous case and we look for  $\Delta \in \Gamma_{n+1}^\alpha \subset \mathcal{W}_{n+1} \times \mathcal{W}_{n+k} \times (\mathcal{W}_{n+2p-k})^d$  such that the pair  $K = \mathcal{K}^* + \Delta$ ,  $R = \mathcal{R}^*$  satisfies  $F \circ K = K \circ R$ .

Similarly as before we obtain

$$\begin{aligned} K(t, \theta) - \hat{K}(t, \theta) &= \mathcal{K}^*(t, \theta) - \hat{K}(t, \theta) + \Delta(t, \theta) \\ &= \hat{K}_{n+1}(t, \theta) + \Delta(t, \theta) \\ &= (O(t^{n+1}), O(t^{n+k}), O(t^{n+2p-1})) + (O(t^{n+1}), O(t^{n+k}), O(t^{n+2p-k})) \\ &= (O(t^{n+1}), O(t^{n+k}), O(t^{n+2p-k})), \end{aligned}$$

and

$$R(t, \theta) - \hat{R}(t, \theta) = \mathcal{R}^*(t, \theta) - \hat{R}(t, \theta) = \hat{R}_{n+1}(t) = \begin{cases} (O(t^{2k-1}), 0) & \text{if } n \leq k, \\ (0, 0) & \text{if } n > k. \end{cases}$$

Again, the  $C^1$  character of  $K$  at  $\mathbb{T}^d$  follows from the order condition of  $K$  at  $t = 0$ .  $\square$

## Chapter 6

# Whiskered parabolic tori with nilpotent part. Vector field case.

### 6.1 Introduction

In this chapter we study the existence and regularity of invariant manifolds of analytic vector fields having an analogous form as the maps studied in Chapter 5. Moreover, we consider non-autonomous vector fields, concretely vector fields that depend quasiperiodically on time.

The class of vector fields we consider is the following one. Let  $U \subset \mathbb{R}^2$  be a neighborhood of 0 and let  $X : U \times \mathbb{T}^d \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$  be an analytic vector field of the form

$$X(x, y, \theta, t, \lambda) = \begin{pmatrix} c(\theta, t, \lambda)y \\ a_k(\theta, t, \lambda)x^k + A(x, y, \theta, t, \lambda) \\ \omega + d_p(\theta, t, \lambda)x^p + B(x, y, \theta, t, \lambda) \end{pmatrix}, \quad (6.1.1)$$

depending quasiperiodically on  $t$  (see the definition below) and being  $\nu \in \mathbb{R}^{d'}$  the time frequencies of  $X$ , and with  $(x, y) \in \mathbb{R}^2$ ,  $\theta \in \mathbb{T}^d$ ,  $\omega \in \mathbb{R}^d$ . We assume that  $\bar{c} > 0$ ,  $k \geq 2$ ,  $p \geq 1$ , and that  $A, B$  are of the form

$$\begin{aligned} A(x, y, \theta, t, \lambda) &= y O(\|(x, y)\|^{k-1}) + O(\|(x, y)\|^{k+1}), \\ B(x, y, \theta, t, \lambda) &= y O(\|(x, y)\|^{p-1}) + O(\|(x, y)\|^{p+1}). \end{aligned}$$

The set

$$\mathcal{T} = \{(0, 0, \theta) \in U \times \mathbb{T}^d\}$$

is an invariant torus of  $X$ , that is, for any point  $x \in \mathcal{T}$ ,  $X(x, \lambda)$  is tangent to  $\mathcal{T}$  at  $x$ , and the first  $2 \times 2$  box of  $DX(0, 0, \theta, t, \lambda)$  is

$$\begin{pmatrix} 0 & c(\theta, t, \lambda) \\ 0 & 0 \end{pmatrix}.$$

We say then that  $\mathcal{T}$  is a parabolic torus with nilpotent part.

As for the map case in Chapter 5, we use the parameterization method, here adapted to the vector field setting. We look for a map  $K(u, \theta, t, \lambda)$  and a vector field  $Y(u, \theta, t, \lambda)$  that satisfy

$$X(K(u, \theta, t, \lambda), t, \lambda) - \partial_{(u, \theta)} K(u, \theta, t, \lambda) \cdot Y(u, \theta, t, \lambda) - \partial_t K(u, \theta, t, \lambda) = 0 \quad (6.1.2)$$

with  $K(0, \theta, t, \lambda) = (0, 0, \theta) \in \mathbb{R}^2 \times \mathbb{T}^d$  and  $Y(0, \theta, t, \lambda) = (0, \omega) \in \mathbb{R} \times \mathbb{T}^d$ .

Equation (6.1.2) is the time-dependent version of (4.1.2). It is obtained by adding the equation  $\dot{t} = 1$  to the system  $\dot{x} = X$ ,  $x \in U \times \mathbb{T}^d$ , and applying (4.1.2) to the new vector field, where here  $K$  and  $Y$  also depend on  $t$ . Then, equation (6.1.2) expresses that at the range of  $K$ , the vector field  $(X, 1)$  is tangent to the range of  $K$ , and thus the image of  $K$  is invariant under the flow of  $(X, 1)$ .

Along this chapter, as in Chapter 4,  $t$  denotes the time variable and we use  $u \in \mathbb{R}$  to denote the first variable that parameterizes the invariant manifolds.

In Chapter 4, the methodology to study invariant manifolds of vector fields was based in using the results obtained for maps and showing how they could be translated into the vector field setting. Here instead we will use a self-contained method consisting in studying the existence of solutions  $K$  and  $Y$  of equation (6.1.2) using appropriate function spaces and operators.

As in Chapter 5 we will give results concerning the analytic dependence on parameters, but we will skip it inside the lemmas and proofs.

The structure of this chapter is the following. We introduce preliminaries and notation in Section 6.2 and we state the main results of existence of invariant manifolds in Section 6.3. The results are stated for the stable manifolds. The existence of the unstable ones is obtained from the corresponding study of the stable manifolds of  $-X$ . As in the previous chapters we also provide an algorithm to compute an approximation of a parameterization of the invariant manifolds and of the restricted dynamics in it (Section 6.4.) Then we present the functional equation and the function spaces and operators we deal with in Sections 6.5 and 6.6. Here, by the nature of the problem, the operators that correspond to composition operators in Chapter 5 will be differential operators. We give the proofs on the main results in Section 6.7. Finally, in Section 6.8 we present in more detail a particular case inside the class of vector fields of the form (6.3.3), namely a family of planar non-autonomous vector fields. We finish the chapter presenting some applications of our results to physical and chemical problems.

## 6.2 Preliminaries and notation

In this section we present some notation and preliminary results that will be used along the chapter.

We start with some notation and definitions that complement the ones presented in Section 5.2. Concretely, we only state here the notation that is new with respect to Chapter 5 or that has to be adapted to the current setting.

- Given  $\sigma > 0$ , we define the *complex strip*  $\mathbb{H}_\sigma$  as

$$\mathbb{H}_\sigma = \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < \sigma\}.$$

- If  $t$  denotes the time variable, then given two functions  $g(x, t)$  and  $h(x, t)$  the composition  $f = h \circ g$  will mean

$$f(x, t) = h(g(x, t), t).$$

- We say that a function  $h : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $h = h(x, t)$ , depends *quasiperiodically* on  $t$  if there exists a vector  $\nu = (\nu_1, \dots, \nu_{d'}) \in \mathbb{R}^{d'}$  and a function  $\check{h} : \mathbb{R}^n \times \mathbb{T}^{d'} \rightarrow \mathbb{R}$  such that

$$h(x, t) = \check{h}(x, \nu t).$$

We call  $\nu$  the vector of *time frequencies* of  $h$ . If  $d' = 1$  then  $h$  is a periodic function of  $t$ .

- The superindices  $x, y$  and  $\theta$  on the symbol of a function or an operator with values in  $\mathbb{R}^2 \times \mathbb{T}^d$  denotes the respective components of the function or the operator. We also use the superindices  $u, \theta$  and  $t$ , respectively, for functions or operators that take values in  $S(\beta, \rho) \times \mathbb{T}_\sigma^d \times \mathbb{T}_\sigma^{d'}$ .

Next we introduce some basic theory concerning Diophantine vectors and the small divisors equation for vector fields.

We say that  $\omega \in \mathbb{R}^d$  is *Diophantine (in the vector field setting)* if there exist  $c > 0$  and  $\tau \geq d - 1$  such that

$$|\omega \cdot k| \geq c|k|^{-\tau} \quad \text{for all } k \in \mathbb{Z}^d \setminus \{0\},$$

where  $|k| = |k_1| + \dots + |k_d|$  and  $\omega \cdot k$  denotes the scalar product.

The small divisors equation in the vector field setting is

$$\partial_\theta \varphi(\theta, \lambda) \cdot \omega = h(\theta, \lambda), \tag{6.2.1}$$

with  $h : \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R}^n$  and  $\omega \in \mathbb{R}^d$ .

To look for a solution of (6.2.1) we develop  $h$  in Fourier series,

$$h(\theta, \lambda) = \sum_{k \in \mathbb{Z}^d} h_k(\lambda) e^{2\pi i k \cdot \theta},$$

with

$$h_k(\lambda) = \int_0^1 h(\theta, \lambda) e^{-2\pi i k \cdot \theta} d\theta, \quad k \cdot \theta = k_1 \theta_1 + \dots + k_d \theta_d.$$

If  $h$  has zero average and  $k \cdot \omega \notin \mathbb{Z}$  for all  $k \neq 0$ , then equation (6.2.1) has the formal solution

$$\varphi(\theta, \lambda) = \sum_{k \in \mathbb{Z}^d} \varphi_k(\lambda) e^{2\pi i k \cdot \theta}, \quad \varphi_k(\lambda) = \frac{h_k(\lambda)}{2\pi i k \cdot \omega}, \quad k \neq 0,$$

where all the coefficients  $\varphi_k$  are uniquely determined except for  $\varphi_0$  which is free.

We state the small divisors lemma for the vector field setting.

**Theorem 6.2.1** (Small divisors lemma for vector fields). *Let  $h : \mathbb{T}_\sigma^d \times \Lambda_{\mathbb{C}} \rightarrow \mathbb{C}^n$  be analytic with zero average and  $\omega$  Diophantine with  $\tau \geq d - 1$ . Then, there exists a unique analytic solution  $\varphi : \mathbb{T}_\sigma^d \times \Lambda \rightarrow \mathbb{C}^n$  of (6.2.1) with zero average. Moreover,*

$$\sup_{(\theta, \lambda) \in \mathbb{T}_{\sigma-\delta}^d \times \Lambda_{\mathbb{C}}} \|\varphi(\theta, \lambda)\| \leq C\delta^{-\tau} \sup_{(\theta, \lambda) \in \mathbb{T}_\sigma^d \times \Lambda_{\mathbb{C}}} \|h(\theta, \lambda)\|, \quad 0 < \delta < \sigma,$$

where  $C$  depends on  $\tau$  and  $d$  but not on  $\delta$ .

We denote by  $\mathcal{SD}(h)$  the unique solution of (6.2.1) with zero average.

As a consequence, if  $h : \mathbb{T}_\sigma^d \times \mathbb{H}_\sigma \times \Lambda_{\mathbb{C}} \rightarrow \mathbb{C}^n$  is quasiperiodic with respect to  $t \in \mathbb{H}$  with frequencies  $\nu \in \mathbb{R}^{d'}$ ,  $(\omega, \nu) \in \mathbb{R}^{d+d'}$  is Diophantine and  $h$  has zero average, then the equation

$$(\partial_\theta \varphi(\theta, t, \lambda), \partial_t \varphi(\theta, t, \lambda)) \cdot (\omega, 1) = h(\theta, t, \lambda), \quad (\theta, t, \lambda) \in \mathbb{T}_\sigma^d \times \mathbb{H}_\sigma \times \Lambda_{\mathbb{C}}, \quad (6.2.2)$$

has a unique solution with zero average, defined in  $\mathbb{T}_\sigma^d \times \mathbb{H}_\sigma \times \Lambda_{\mathbb{C}}$  and bounded in  $\mathbb{T}_{\sigma'}^d \times \mathbb{H}_{\sigma'} \times \Lambda_{\mathbb{C}}$  for any  $0 < \sigma' < \sigma$ . Indeed, since  $h$  is quasiperiodic in  $t$ , equation (6.2.2) is equivalent to

$$(\partial_\theta \check{\varphi}(\theta, \tau, \lambda), \partial_\tau \check{\varphi}(\theta, \tau, \lambda)) \cdot (\omega, \nu) = \check{h}(\theta, \tau, \lambda), \quad (\theta, \tau, \lambda) \in \mathbb{T}_{\sigma'}^{d+d'} \times \Lambda_{\mathbb{C}}, \quad (6.2.3)$$

where  $\tau = \nu t$  and  $h(\theta, t, \lambda) = \check{h}(\theta, \tau, \lambda)$ . Then, applying Theorem 6.2.1 to equation (6.2.3) taking  $(\omega, \nu)$  as the frequency vector, we obtain a unique solution  $\check{\varphi} : \mathbb{T}_{\sigma'}^{d+d'} \times \Lambda_{\mathbb{C}} \rightarrow \mathbb{C}^n$  with zero average, and thus  $\varphi(\theta, t, \lambda) = \check{\varphi}(\theta, \tau, \lambda)$  is the unique solution of equation (6.2.2) with zero average. We also denote it by  $\mathcal{SD}(h)$ . We use the same notation to denote the solution of a small divisors equation that is either time dependent or independent, as such dependence will be understood by the context.

### 6.3 Main results

Next we state the main results concerning the existence of analytic stable invariant manifolds of a vector field  $X$  of the form (6.3.3) depending quasiperiodically on time. These results are analogous to Theorems 5.3.1 and 5.3.2, which concern invariant manifolds for maps. The second Theorem is an *a posteriori* result.

**Theorem 6.3.1.** *Let  $X$  be an analytic vector field of the form (6.3.3) and let  $\nu \in \mathbb{R}^{d'}$  be the time frequencies of  $X$ . Assume that  $2p > k - 1$ . Assume also that  $(\omega, \nu)$  is Diophantine and that  $\bar{a}_k(\lambda) > 0$  for  $\lambda \in \Lambda$ .*

*Then, there exists  $\rho > 0$  and a  $C^1$  map  $K : [0, \rho) \times \mathbb{T}^d \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$ , analytic in  $(0, \rho) \times \mathbb{T}^d \times \mathbb{R} \times \Lambda$ , of the form*

$$K(u, \theta, t, \lambda) = (u^2, \bar{K}_{k+1}^y(\lambda)u^{k+1}, \theta + \bar{K}_{2p-k+1}^\theta(\lambda)u^{2p-k+1}) + (O(u^3), O(u^{k+2}), O(u^{2p-k+2})),$$

*depending quasiperiodically on  $t$  with the same frequencies as  $X$ , and a polynomial vector field  $Y$  of the form*

$$Y(u, \theta, t, \lambda) = Y(u, \lambda) = \begin{pmatrix} \bar{Y}_k^x(\lambda)u^k + \bar{Y}_{2k-1}^x(\lambda)u^{2k-1} \\ \omega \end{pmatrix},$$

*with  $\bar{Y}_k^x(\lambda) < 0$ , such that*

$$X(K(u, \theta, t, \lambda), t, \lambda) - \partial_{(u, \theta)} K(u, \theta, t, \lambda) \cdot Y(u, \theta, t, \lambda) - \partial_t K(u, \theta, t, \lambda) = 0, \\ (u, \theta, t, \lambda) \in [0, \rho) \times \mathbb{T}^d \times \mathbb{R} \times \Lambda.$$

**Theorem 6.3.2.** *Let  $X$  be an analytic vector field of the form (6.3.3) and let  $\nu \in \mathbb{R}^{d'}$  be the time frequencies of  $X$ . Let  $\hat{K} : (-\rho, \rho) \times \mathbb{T}^d \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$  and  $\hat{Y} = (-\rho, \rho) \times \mathbb{T}^d \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R} \times \mathbb{T}^d$  be an analytic map and an analytic vector field, respectively, of the form*

$$K(u, \theta, t, \lambda) = (u^2, \bar{K}_{k+1}^y(\lambda)u^{k+1}, \theta + \bar{K}_{2p-k+1}^\theta(\lambda)u^{2p-k+1}) + (O(u^3), O(u^{k+2}), O(u^{2p-k+2})),$$

and

$$Y(u, \theta, t, \lambda) = \begin{pmatrix} \bar{Y}_k^x(\lambda)u^k + O(u^{k+1}) \\ \omega \end{pmatrix},$$

with  $\bar{Y}_k^x(\lambda) < 0$ , depending quasiperiodically on  $t$  with the same frequencies as  $X$ , satisfying

$$\begin{aligned} X(\hat{K}(u, \theta, t, \lambda), t, \lambda) - \partial_{(u, \theta)} \hat{K}(u, \theta, t, \lambda) \cdot \hat{Y}(u, \theta, t, \lambda) - \partial_t \hat{K}(u, \theta, t, \lambda) \\ = (O(u^{n+k}), O(u^{n+2k-1}), O(u^{n+2p-1})), \end{aligned}$$

for some  $n \geq 2$ .

Then, there exists a  $C^1$  map  $K : [0, \rho) \times \mathbb{T}^d \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$ , analytic in  $(0, \rho) \times \mathbb{T}^d \times \mathbb{R} \times \Lambda$ , and an analytic vector field  $Y : (-\rho, \rho) \times \mathbb{T}^d \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R} \times \mathbb{T}^d$  such that

$$\begin{aligned} X(K(u, \theta, t, \lambda), t, \lambda) - \partial_{(u, \theta)} K(u, \theta, t, \lambda) \cdot Y(u, \theta, t, \lambda) - \partial_t K(u, \theta, t, \lambda) = 0, \\ (u, \theta, t, \lambda) \in [0, \rho) \times \mathbb{T}^d \times \mathbb{R} \times \Lambda, \end{aligned}$$

and

$$\begin{aligned} K(u, \theta, t, \lambda) - \hat{K}(u, \theta, t, \lambda) &= (O(u^{n+1}), O(u^{n+k}), O(u^{n+2p-k})), \\ Y(u, \theta, t, \lambda) - \hat{Y}(u, \theta, t, \lambda) &= \begin{cases} (O(u^{2k-1}), 0) & \text{if } n \leq k, \\ (0, 0) & \text{if } n > k. \end{cases} \end{aligned}$$

Finally, the following result is a particular case of a slightly modified version of Theorem 6.3.1. It will be used later on in Section 6.9 applied to the study of the scattering of helium atoms off copper surfaces. The proof, which is completely analogous to the one of Theorem 6.3.1, will be omitted.

**Theorem 6.3.3.** *Let  $X$  be an analytic vector field of the form*

$$X(x, y, \theta, t, \lambda) = \begin{pmatrix} c(\theta, t, \lambda)y \\ b(\theta, t, \lambda)xy + O(y^2) \\ \omega + d(\theta, t, \lambda)y + O(\|(x, y)\|^2) \end{pmatrix},$$

with  $(x, y) \in \mathbb{R}^2$ ,  $\theta \in \mathbb{T}^d$ ,  $\omega \in \mathbb{R}^d$ , and depending quasiperiodically on  $t$  with time frequencies  $\nu \in \mathbb{R}^{d'}$ . Assume that  $\bar{c}(\lambda) > 0$ ,  $\bar{b}(\lambda) \neq 0$  and  $\bar{d}(\lambda) \neq 0$ . Assume also that  $(\omega, \nu)$  is Diophantine.

Then, there exists  $\rho > 0$  and a  $C^1$  map  $K : [0, \rho) \times \mathbb{T}^d \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$ , analytic in  $(0, \rho) \times \mathbb{T}^d \times \mathbb{R} \times \Lambda$ , of the form

$$K(u, \theta, t, \lambda) = (u, \bar{K}_2^y(\lambda)u^2, \theta + \bar{K}_1^\theta(\lambda)u) + (O(u^2), O(u^3), O(u^2)),$$

depending quasiperiodically on  $t$  with the same frequencies as  $X$ , and a polynomial vector field  $Y$  of the form

$$Y(u, \theta, t, \lambda) = Y(u, \lambda) = \begin{pmatrix} \bar{Y}_2^x(\lambda)u^2 + \bar{Y}_3^x(\lambda)u^3 \\ \omega \end{pmatrix},$$

with  $\bar{Y}_2^x(\lambda) < 0$ , such that

$$\begin{aligned} X(K(u, \theta, t, \lambda), t, \lambda) - \partial_{(u, \theta)} K(u, \theta, t, \lambda) \cdot Y(u, \theta, t, \lambda) - \partial_t K(u, \theta, t, \lambda) = 0, \\ (u, \theta, t, \lambda) \in [0, \rho) \times \mathbb{T}^d \times \mathbb{R} \times \Lambda. \end{aligned}$$



## 6.4 Formal approximation of a parameterization of the whiskers

In this section we consider analytic vector fields  $X$  of the form (6.3.3), depending quasiperiodically on time, and we provide an algorithm to obtain a polynomial map,  $\mathcal{K}_n(u, \theta, t, \lambda)$  and a vector field,  $\mathcal{Y}_n(u, \theta, t, \lambda)$ , that are approximations of solutions  $K$  and  $Y$  of the invariance equation

$$X \circ (K, t) - \partial_{(u, \theta)} K \cdot Y - \partial_t K = 0. \quad (6.4.1)$$

The first component of the vector field  $\mathcal{Y}_n$  represents the dynamics in the directions normal to the invariant torus,  $\mathcal{T}$ . Similarly to the planar case presented in Chapter 4, the obtained approximations correspond to stable manifolds when the coefficient  $\bar{Y}_k^x(\lambda)$  of  $\mathcal{Y}_n$  is negative. When this coefficient is positive they correspond to unstable manifolds.

The obtained approximations  $\mathcal{K}_n$  and  $\mathcal{Y}_n$  satisfy the hypotheses of Theorem 6.3.2 and therefore  $\mathcal{K}_n$  provides an approximation of a true invariant manifold of  $X$ .

In this case, when solving the cohomological equations to obtain the coefficients of  $\mathcal{K}_n$  and  $\mathcal{Y}_n$  we will encounter the small divisors equation for vector fields, which can be solved when the vector  $(\omega, \nu)$  is Diophantine, where  $\nu \in \mathbb{R}^{d'}$  are the time frequencies of  $X$ .

As for the map case, even if in the statement we ask for  $X$  to be analytic, the result holds if  $X$  is only  $C^\infty$  since the proof requires only formal computations.

Note that the obtained map  $\mathcal{Y}_n$  neither depends on  $\theta$  nor on  $t$ . Moreover, in the first component of  $\mathcal{Y}_n$  we recover the expression of the normal form of a one-dimensional vector field around a parabolic point ([70]).

**Proposition 6.4.1.** *Let  $X$  be an analytic vector field of the form (6.3.3). Assume that  $2p > k - 1$ . Assume also that  $(\omega, \nu)$  is Diophantine and  $\bar{a}_k(\lambda) > 0$  for  $\lambda \in \Lambda$ . Then, for all  $n \geq 2$ , there exist two maps,  $\mathcal{K}_n : \mathbb{R} \times \mathbb{T}^d \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$ , of the form*

$$\mathcal{K}_n(u, \theta, t, \lambda) = \begin{pmatrix} u^2 + \sum_{i=3}^n \bar{K}_i^x(\lambda) u^i + \sum_{i=k+1}^{n+k-1} \tilde{K}_i^x(\theta, t, \lambda) u^i \\ \sum_{i=k+1}^{n+k-1} \bar{K}_i^y(\lambda) u^i + \sum_{i=2k}^{n+2k-2} \tilde{K}_i^y(\theta, t, \lambda) u^i \\ \theta + \sum_{i=2p-k+1}^{n+2p-k-1} \bar{K}_i^\theta(\lambda) u^i + \sum_{i=2p}^{n+2p-2} \tilde{K}_i^\theta(\theta, t, \lambda) u^i \end{pmatrix},$$

depending quasiperiodically on time with the same frequencies as  $X$ , and two vector fields,  $\mathcal{Y}_n : \mathbb{R} \times \mathbb{T}^d \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R} \times \mathbb{T}^d$ , of the form

$$\mathcal{Y}_n(u, \theta, t, \lambda) = \mathcal{Y}_n(u, \lambda) = \begin{cases} \begin{pmatrix} \bar{Y}_k^x(\lambda) u^k \\ \omega \end{pmatrix} & \text{if } 2 \leq n \leq k, \\ \begin{pmatrix} \bar{Y}_k^x(\lambda) u^k + \bar{Y}_{2k-1}^x(\lambda) u^{2k-1} \\ \omega \end{pmatrix} & \text{if } n \geq k+1, \end{cases}$$

such that

$$\begin{aligned} \mathcal{G}_n(u, \theta, t, \lambda) &:= X(\mathcal{K}_n(u, \theta, t, \lambda), t, \lambda) - \partial_{(u, \theta)} \mathcal{K}_n(u, \theta, t, \lambda) \cdot \mathcal{Y}_n(u, \theta, t, \lambda) - \partial_t \mathcal{K}_n(u, \theta, t, \lambda) \\ &= (O(u^{n+k}), O(u^{n+2k-1}), O(u^{n+2p-1})). \end{aligned} \quad (6.4.2)$$

Moreover, for the first coefficients we obtain

$$\begin{aligned}\bar{K}_{k+1}^y(\lambda) &= \pm \sqrt{\frac{2\bar{a}_k(\lambda)}{\bar{c}(\lambda)(k+1)}}, & \bar{K}_{2p-k+1}^\theta(\lambda) &= \pm \frac{\bar{d}_p(\lambda)}{2p-k+1} \sqrt{\frac{2(k+1)}{\bar{c}(\lambda)\bar{a}_k(\lambda)}}, & \bar{Y}_k^x(\lambda) &= \pm \sqrt{\frac{\bar{c}(\lambda)\bar{a}_k(\lambda)}{2(k+1)}}, \\ \tilde{K}_{k+1}^x(\theta, t, \lambda) &= \mathcal{SD}(\tilde{c}(\theta, t, \lambda))\bar{K}_{k+1}^y(\lambda), & \tilde{K}_{2k}^y(\theta, t, \lambda) &= \mathcal{SD}(\tilde{a}_k(\theta, t, \lambda)), & \tilde{K}_{2p}^\theta(\theta, t, \lambda) &= \mathcal{SD}(\tilde{d}_p(\theta, t, \lambda)).\end{aligned}$$

**Notation 6.4.2.** Along the proof, given a differentiable map  $f(u, \theta, t)$ , we will denote by  $[f]_n$  the coefficient of the term of order  $n$  of the jet of  $f$  with respect to  $u$  at 0.

*Proof.* We prove it by induction and show that we can determine  $\mathcal{K}_n$  and  $\mathcal{Y}_n$  iteratively.

For the first induction step,  $n = 2$ , we claim that there exist a map and a vector field,

$$\mathcal{K}_2(u, \theta, t) = \begin{pmatrix} u^2 + \tilde{K}_{k+1}^x(\theta, t)u^{k+1} \\ \bar{K}_{k+1}^y u^{k+1} + \tilde{K}_{2k}^y(\theta, t)u^{2k} \\ \theta + \bar{K}_{2p-k+1}^\theta u^{2p-k+1} + \tilde{K}_{2p}^\theta(\theta, t)u^{2p} \end{pmatrix}, \quad \mathcal{Y}_2(u, \theta, t) = \begin{pmatrix} \bar{Y}_k^x u^k \\ \omega \end{pmatrix},$$

such that

$$\begin{aligned}\mathcal{G}_2(u, \theta, t) &= X(\mathcal{K}_2(u, \theta, t), t) - \partial_{(u, \theta)}\mathcal{K}_2(u, \theta, t) \cdot \mathcal{Y}_2(u, \theta, t) - \partial_t\mathcal{K}_2(u, \theta, t) \\ &= (O(u^{k+2}), O(u^{2k+1}), O(u^{2p+1})).\end{aligned}$$

Indeed, from the expansion of  $\mathcal{G}_2$  we have

$$\begin{aligned}\mathcal{G}_2^x(u, \theta, t) &= u^{k+1}[c(\theta, t)\bar{K}_{k+1}^y - 2\bar{Y}_k^x - \partial_\theta\tilde{K}_{k+1}^x(\theta, t) \cdot \omega - \partial_t\tilde{K}_{k+1}^x(\theta, t)] + O(u^{k+2}), \\ \mathcal{G}_2^y(u, \theta, t) &= u^{2k}[a_k(\theta, t) - (k+1)\bar{K}_{k+1}^y\bar{Y}_k^x - \partial_\theta\tilde{K}_{2k}^y(\theta, t) \cdot \omega - \partial_t\tilde{K}_{2k}^y(\theta, t)] + O(u^{2k+1}), \\ \mathcal{G}_2^\theta(u, \theta, t) &= u^{2p}[d_p(\theta, t) - (2p-k+1)\bar{K}_{2p-k+1}^\theta - \partial_\theta\tilde{K}_{2p}^\theta(\theta, t) - \partial_t\tilde{K}_{2p}^\theta(\theta, t)] + O(u^{2p+1}).\end{aligned}$$

To obtain  $\mathcal{G}_2^x(u, \theta, t) = O(u^{2+k})$  we solve the equation

$$c(\theta, t)\bar{K}_{k+1}^y - 2\bar{Y}_k^x - \partial_\theta\tilde{K}_{k+1}^x(\theta, t) \cdot \omega - \partial_t\tilde{K}_{k+1}^x(\theta, t) = 0$$

as follows. First, we separate the average and the oscillatory part of the functions that depend on  $\theta$  and  $t$ , so that we obtain

$$\bar{c}\bar{K}_{k+1}^y + \tilde{c}(\theta, t)\bar{K}_{k+1}^y - 2\bar{Y}_k^x - \partial_\theta\tilde{K}_{k+1}^x(\theta, t) \cdot \omega - \partial_t\tilde{K}_{k+1}^x(\theta, t) = 0.$$

Then we split the equation into two parts, one containing the terms that are independent of  $(\theta, t)$ , namely  $\bar{c}\bar{K}_{k+1}^y = 2\bar{Y}_k^x$ , and the other being a small divisors equation of functions with zero average,  $\partial_\theta\tilde{K}_{k+1}^x(\theta, t) \cdot \omega + \partial_t\tilde{K}_{k+1}^x(\theta, t) = \tilde{c}(\theta, t)\bar{K}_{k+1}^y$ .

We proceed in the same way to get  $\mathcal{G}_2^y(u, \theta, t) = O(u^{2k+1})$  and  $\mathcal{G}_2^\theta(u, \theta, t) = O(u^{2p+1})$ . If  $\bar{a}_k > 0$  and  $(\omega, \nu)$  is Diophantine, the obtained equations have formal solutions given by

$$\begin{aligned}\bar{K}_{k+1}^y &= \pm \sqrt{\frac{2\bar{a}_k}{\bar{c}(k+1)}}, & \bar{K}_{2p-k+1}^\theta &= \pm \frac{\bar{d}_p}{2p-k+1} \sqrt{\frac{2(k+1)}{\bar{c}\bar{a}_k}}, & \bar{Y}_k^x &= \pm \sqrt{\frac{\bar{c}\bar{a}_k}{2(k+1)}}, \\ \tilde{K}_{k+1}^x(\theta, t) &= \mathcal{SD}(\tilde{c}(\theta, t))\bar{K}_{k+1}^y, & \tilde{K}_{2k}^y(\theta, t) &= \mathcal{SD}(\tilde{a}_k(\theta, t)), & \tilde{K}_{2p}^\theta(\theta, t) &= \mathcal{SD}(\tilde{d}_p(\theta, t)).\end{aligned}$$

Next we perform the induction procedure. We assume that we have already obtained a map  $\mathcal{K}_n$  and a vector field  $\mathcal{Y}_n$ ,  $n \geq 2$ , such that (6.4.2) holds true, and we look for

$$\begin{aligned}\mathcal{K}_{n+1}(u, \theta, t) &= \mathcal{K}_n(u, \theta, t) + \begin{pmatrix} \bar{K}_{n+1}^x u^{n+1} + \tilde{K}_{n+k}^x(\theta, t) u^{n+k} \\ \bar{K}_{n+k}^y u^{n+k} + \tilde{K}_{n+2k-1}^y(\theta, t) u^{n+2k-1} \\ \bar{K}_{n+2p-k}^\theta u^{n+2p-k} + \tilde{K}_{n+2p-1}^\theta(\theta, t) u^{n+2p-1} \end{pmatrix}, \\ \mathcal{Y}_{n+1}(u, \theta, t) &= \mathcal{Y}_n(u, \theta, t) + \begin{pmatrix} \bar{Y}_{n+k-1}^x u^{n+k-1} \\ 0 \end{pmatrix},\end{aligned}$$

such that  $\mathcal{G}_{n+1}(u, \theta, t) = (O(u^{n+k+1}), O(u^{n+2k}), O(u^{n+2p}))$ . To simplify the notation, we denote  $\mathcal{K}_{n+1}^+ = \mathcal{K}_{n+1} - \mathcal{K}_n$  and  $\mathcal{Y}_{n+1}^+ = \mathcal{Y}_{n+1} - \mathcal{Y}_n$ .

Using Taylor's theorem, one can write

$$\begin{aligned}\mathcal{G}_{n+1} &= X(\mathcal{K}_n, t) + \partial_{(x,y,\theta)} X(\mathcal{K}_n, t) \cdot \mathcal{K}_{n+1}^+ \\ &\quad + \int_0^1 (1-s) \partial_{(x,y,\theta)}^2 X((\mathcal{K}_n, t) + s(\mathcal{K}_{n+1}^+, t)) ds (\mathcal{K}_{n+1}^+)^{\otimes 2} \\ &\quad - \partial_{(u,\theta)} \mathcal{K}_n \cdot \mathcal{Y}_n - \partial_{(u,\theta)} \mathcal{K}_{n+1}^+ \cdot \mathcal{Y}_n - \partial_{(u,\theta)} \mathcal{K}_{n+1} \cdot \mathcal{Y}_{n+1}^+ - \partial_t \mathcal{K}_n - \partial_t \mathcal{K}_{n+1}^+ \\ &= \mathcal{G}_n + \partial_{(x,y,\theta)} X(\mathcal{K}_n, t) \cdot \mathcal{K}_{n+1}^+ - \partial_{(u,\theta)} \mathcal{K}_{n+1}^+ \cdot \mathcal{Y}_n - \partial_{(u,\theta)} \mathcal{K}_{n+1} \cdot \mathcal{Y}_{n+1}^+ - \partial_t \mathcal{K}_{n+1}^+ \\ &\quad + \int_0^1 (1-s) \partial_{(x,y,\theta)}^2 X((\mathcal{K}_n, t) + s(\mathcal{K}_{n+1}^+, t)) ds (\mathcal{K}_{n+1}^+)^{\otimes 2}.\end{aligned}$$

Performing the computations in the previous expression we have

$$\begin{aligned}\mathcal{G}_{n+1}^x(u, \theta, t) &= \mathcal{G}_n^x(u, \theta, t) \\ &\quad + u^{n+k} [c(\theta, t) \bar{K}_{n+k}^y - (n+1) \bar{K}_{n+1}^x \bar{Y}_k^x - 2 \bar{Y}_{n+k-1}^x - \partial_\theta \tilde{K}_{n+k}^x(\theta, t) \cdot \omega - \partial_t \tilde{K}_{n+k}^x(\theta, t)] \\ &\quad + O(u^{n+k+1}), \\ \mathcal{G}_{n+1}^y(u, \theta, t) &= \mathcal{G}_n^y(t, \theta) \\ &\quad + u^{n+2k-1} [k a_k(\theta, t) \bar{K}_{n+1}^x - (n+k) \bar{K}_{n+k}^y \bar{Y}_k^x - (k+1) \bar{K}_{k+1}^y \bar{Y}_{n+k-1}^x - \partial_\theta \tilde{K}_{n+2k-1}^y(\theta, t) \cdot \omega \\ &\quad - \partial_t \tilde{K}_{n+2k-1}^y(\theta, t)] + O(u^{n+2k}), \\ \mathcal{G}_{n+1}^\theta(u, \theta, t) &= \mathcal{G}_n^\theta(t, \theta) \\ &\quad + u^{n+2p-1} [p d_p(\theta, t) \bar{K}_{n+1}^x - (n+2p-k) \bar{K}_{n+2p-k}^\theta \bar{Y}_k^x - (2p-k+1) \bar{K}_{2p-k+1}^\theta \bar{Y}_{n+k-1}^x \\ &\quad - \partial_\theta \tilde{K}_{n+2p-1}^\theta(\theta, t) \cdot \omega - \partial_t \tilde{K}_{n+2p-1}^\theta(\theta, t)] + O(u^{n+2p}).\end{aligned}\tag{6.4.3}$$

Since, by the induction hypothesis,  $\mathcal{G}_n(u, \theta, t) = (O(u^{n+k}), O(u^{n+2k-1}), O(u^{n+2p-1}))$ , to complete the induction step we need to make  $[\mathcal{G}_{n+1}^x]_{n+k}$ ,  $[\mathcal{G}_{n+1}^y]_{n+2k-1}$  and  $[\mathcal{G}_{n+1}^\theta]_{n+2p-1}$  vanish. From the expansions obtained in (6.4.3), such condition leads to the following cohomological

equations,

$$\begin{aligned}
c(\theta, t)\bar{K}_{n+k}^y - (n+1)\bar{K}_{n+1}^x\bar{Y}_k^x - 2\bar{Y}_{n+k-1}^x - \partial_\theta\tilde{K}_{n+k}^x(\theta, t) \cdot \omega - \partial_t\tilde{K}_{n+k}^x(\theta, t) \\
+ [\mathcal{G}_n^x(\theta, t)]_{n+k} = 0, \\
k a_k(\theta, t)\bar{K}_{n+1}^x - (n+k)\bar{K}_{n+k}^y\bar{Y}_k^x - (k+1)\bar{K}_{k+1}^y\bar{Y}_{n+k-1}^x - \partial_\theta\tilde{K}_{n+2k-1}^y(\theta, t) \cdot \omega \\
- \partial_t\tilde{K}_{n+2k-1}^y(\theta, t) + [\mathcal{G}_n^y(\theta, t)]_{n+2k-1} = 0, \\
p d_p(\theta, t)\bar{K}_{n+1}^x - (n+2p-k)\bar{K}_{n+2p-k}^\theta\bar{Y}_k^x - (2p-k+1)\bar{K}_{2p-k+1}^\theta\bar{Y}_{n+k-1} \\
- \partial_\theta\tilde{K}_{n+2p-1}^\theta(\theta, t) \cdot \omega - \partial_t\tilde{K}_{n+2p-1}^\theta(\theta, t) + [\mathcal{G}_n^\theta(\theta, t)]_{n+2p-1} = 0.
\end{aligned} \tag{6.4.4}$$

Taking averages with respect to  $(\theta, t)$  in the previous equations and separating the terms that depend on  $(\theta, t)$  from the constant ones, we split (6.4.4) into three small divisors equations of functions with zero average, namely,

$$\begin{aligned}
\partial_\theta\tilde{K}_{n+k}^x(\theta, t) \cdot \omega + \partial_t\tilde{K}_{n+k}^x(\theta, t) &= \tilde{c}(\theta, t)\bar{K}_{n+k}^y + [\tilde{\mathcal{G}}_n^x(\theta, t)]_{n+k}, \\
\partial_\theta\tilde{K}_{n+2k-1}^y(\theta, t) \cdot \omega + \partial_t\tilde{K}_{n+2k-1}^y(\theta, t) &= k\tilde{a}_k(\theta, t)\bar{K}_{n+1}^x + [\tilde{\mathcal{G}}_n^y(\theta, t)]_{n+2k-1}, \\
\partial_\theta\tilde{K}_{n+2p-1}^\theta(\theta, t) \cdot \omega + \partial_t\tilde{K}_{n+2p-1}^\theta(\theta, t) &= p\tilde{d}_p(\theta, t)\bar{K}_{n+1}^x + [\tilde{\mathcal{G}}_n^\theta(\theta, t)]_{n+2p-1},
\end{aligned} \tag{6.4.5}$$

and the following linear system of equations with constant coefficients,

$$\begin{aligned}
&\begin{pmatrix} -(n+1)\bar{Y}_k^x & \bar{c} & 0 \\ k\bar{a}_k & -(n+k)\bar{Y}_k^x & 0 \\ p\bar{d}_p & 0 & -(n+2p-k)\bar{Y}_k^x \end{pmatrix} \begin{pmatrix} \bar{K}_{n+1}^x \\ \bar{K}_{n+k}^y \\ \bar{K}_{n+2p-k}^\theta \end{pmatrix} \\
&= \begin{pmatrix} -[\bar{\mathcal{G}}_n^x]_{n+k} + 2\bar{Y}_{n+k-1}^x \\ -[\bar{\mathcal{G}}_n^y]_{n+2k-1} + (k+1)\bar{K}_{k+1}^y\bar{Y}_{n+k-1}^x \\ -[\bar{\mathcal{G}}_n^\theta]_{n+2p-1} + (2p-k+1)\bar{K}_{2p-k+1}^\theta\bar{Y}_{n+k-1}^x \end{pmatrix}.
\end{aligned} \tag{6.4.6}$$

Note that the determinant of the matrix in the left hand side of (6.4.6) vanishes when  $k\bar{c}\bar{a}_k - (n+1)(n+k)(\bar{Y}_k^x)^2 = 0$ . Then, if  $n \neq k$  that matrix is invertible, so we can take  $\bar{Y}_{n+k-1}^x = 0$  and then obtain  $\bar{K}_{n+1}^x$ ,  $\bar{K}_{n+k}^y$  and  $\bar{K}_{n+2p-k}^\theta$  in a unique way. When  $n = k$ , the determinant of the matrix is zero. Then, choosing

$$\bar{Y}_{2k-1}^x = \frac{2k\bar{Y}_k^x [\bar{\mathcal{G}}_n^x]_{2k} + \bar{c} [\bar{\mathcal{G}}_n^y]_{3k-2}}{2(3k+1)\bar{Y}_k^x},$$

system (6.4.6) has solutions. In this case, however,  $\bar{K}_{k+1}^x$ ,  $\bar{K}_{2k}^y$  and  $\bar{K}_{2p}^\theta$  are not uniquely determined.

Once we have chosen solutions  $\bar{K}_{k+1}^x$ ,  $\bar{K}_{2k}^y$  and  $\bar{K}_{n+2p-k}^\theta$  for system (6.4.6), we solve the small divisors equations (6.4.5) taking

$$\begin{aligned}
\tilde{K}_{n+k}^x(\theta, t) &= \mathcal{SD}(\tilde{c}(\theta, t)\bar{K}_{n+k}^y + [\tilde{\mathcal{G}}_n^x(\theta, t)]_{n+k}), \\
\tilde{K}_{n+2k-1}^y(\theta, t) &= \mathcal{SD}(k\tilde{a}_k(\theta, t)\bar{K}_{n+1}^x + [\tilde{\mathcal{G}}_n^y(\theta, t)]_{n+2k-1}), \\
\tilde{K}_{n+2p-1}^\theta(\theta, t) &= \mathcal{SD}(p\tilde{d}_p(\theta, t)\bar{K}_{n+1}^x + [\tilde{\mathcal{G}}_n^\theta(\theta, t)]_{n+2p-1}).
\end{aligned}$$

In this way all equations in (6.4.4) are solved and one can proceed to the next induction step.  $\square$

## 6.5 The functional equation

To study the existence of invariant manifolds of a time-dependent vector field of the form (6.3.3) following the parameterization method we proceed in a similar way as we did in Chapter 5.

First we consider approximations  $\mathcal{K}_n : \mathbb{R} \times \mathbb{T}^d \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$  and  $\mathcal{Y}_n : \mathbb{R} \times \mathbb{T}^d \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R} \times \mathbb{T}^d$  of solutions of equation (6.4.1) obtained in Section 6.4 up to a high enough order. Then, keeping  $Y = \mathcal{Y}_n$  fixed, we look for a correction  $\Delta : [0, \rho) \times \mathbb{T}^d \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$ , for some  $\rho > 0$ , of  $\mathcal{K}_n$ , analytic on  $(0, \rho) \times \mathbb{T}^d \times \mathbb{R} \times \Lambda$ , such that the pair  $K = \mathcal{K}_n + \Delta$ ,  $Y = \mathcal{Y}_n$  satisfies the invariance condition

$$X \circ (\mathcal{K}_n + \Delta, t) - \partial_{(u, \theta)}(\mathcal{K}_n + \Delta) \cdot Y - \partial_t(\mathcal{K}_n + \Delta) = 0. \quad (6.5.1)$$

To be able to deal with equation (6.5.1) in a suitable space of analytic functions, we rewrite the vector field (6.3.3) as  $\check{X}(x, y, \theta, \tau, \lambda) = X(x, y, \theta, t, \lambda)$ , with  $\tau = \nu t$  and  $\nu \in \mathbb{T}^{d'}$ , so that the corresponding differential system reads

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \check{c}(\theta, \tau, \lambda)y \\ \check{a}_k(\theta, \tau, \lambda)x^k + \check{A}(x, y, \theta, \tau, \lambda) \\ \omega + \check{d}_p(\theta, \tau, \lambda)x^p + \check{B}(x, y, \theta, \tau, \lambda) \end{pmatrix} \quad (6.5.2)$$

where  $\check{c} : \mathbb{T}^d \times \mathbb{T}^{d'} \times \Lambda \rightarrow \mathbb{R}$ ,  $\check{c}(\theta, \tau, \lambda) = c(\theta, t, \lambda)$ , and similarly for the other quantities with an inverted hat.

Note that now the vector field  $\check{X}$  is defined in a domain of the form  $U \times \mathbb{T}^n$ , with  $n = d + d'$ , and thus the new variables  $(\theta, \tau)$  can be thought as angles.

We also introduce

$$\check{\mathcal{K}}_n(u, \theta, \tau, \lambda) = \mathcal{K}_n(u, \theta, t, \lambda), \quad \check{Y}(u, \theta, \tau, \lambda) = Y(u, \theta, t, \lambda),$$

and

$$J(u, \theta, \tau, \lambda) = \begin{pmatrix} \check{Y}(u, \theta, \tau, \lambda) \\ \nu \end{pmatrix}.$$

Therefore, equation (6.5.1) can be written as

$$\check{X} \circ (\check{\mathcal{K}}_n + \Delta, \tau) - D(\check{\mathcal{K}}_n + \Delta) \cdot J = 0, \quad (6.5.3)$$

and then we look for a solution  $\Delta = \Delta(u, \theta, \tau, \lambda)$ ,  $\Delta : [0, \rho) \times \mathbb{T}^d \times \mathbb{T}^{d'} \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$ .

The proofs of Theorems 6.3.1 and 6.3.2 are organized in a similar way as the ones of Theorems 5.3.1 and 5.3.2 from Chapter 5. Taking into account the structure of  $\check{X}$  we rewrite equation (6.5.1) to separate the dominant linear part with respect to  $\Delta$  and the remaining terms. The obtained equation motivates the introduction of the families of operators  $\mathcal{S}_{n,J}^\times$  and  $\mathcal{N}_{n,X}$  and the spaces where these operators will act on. Note that the symbols used to name the operators  $\mathcal{S}_{n,J}^\times$  and  $\mathcal{N}_{n,X}$  are the same that in Chapter 5, but they correspond to different operators that act on different function spaces. As for the map case, we rewrite the equation for  $\Delta$  as the fixed point equation

$$\Delta = \mathcal{T}_{n,X}(\Delta), \quad \text{where} \quad \mathcal{T}_{n,X} = (\mathcal{S}_{n,J}^\times)^{-1} \circ \mathcal{N}_{n,X},$$

and we apply the Banach fixed point theorem to get the solution. The properties of the operators  $\mathcal{T}_{n,X}$  are deduced in Lemma 6.6.12.

From Proposition 6.4.1, given  $n$  there exist a map  $\mathcal{K}_n$  and a vector field  $Y = \mathcal{Y}_n$  such that

$$X \circ (\mathcal{K}_n, t) - \partial_{(u,\theta)} \mathcal{K}_n \cdot Y - \partial_t \mathcal{K}_n = \mathcal{E}_n,$$

or equivalently,

$$\check{X} \circ (\check{\mathcal{K}}_n, \tau) - D\check{\mathcal{K}}_n \cdot J = \check{\mathcal{E}}_n, \quad (6.5.4)$$

where  $\check{\mathcal{E}}_n(u, \tau, \lambda) = (O(u^{n+k}), O(u^{n+2k-1}), O(u^{n+2p-1}))$ . Since we are looking for a stable manifold we will take the approximations corresponding to  $\check{Y} = \check{\mathcal{Y}}_n$  with the coefficient  $\check{Y}_k^x(\lambda) < 0$ .

We look for  $\rho > 0$  and a map  $\check{K} = \check{\mathcal{K}}_n + \Delta : [0, \rho) \times \mathbb{T}^{d+d'} \times \Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^d$ , analytic on  $(0, \rho) \times \mathbb{T}^{d+d'} \times \Lambda$  satisfying (6.5.3), where  $\check{\mathcal{K}}_n$  and  $J$  satisfy (6.5.4). Moreover, we ask  $\Delta$  to be of the form  $\Delta = (\Delta^x, \Delta^y, \Delta^\theta) = (O(u^n), O(u^{n+k-1}), O(u^{n+2p-k-1}))$ .

To simplify the notation, similarly as in Chapter 5, we write

$$P(x, y, \theta, \tau, \lambda) = \check{a}_k(\theta, \tau, \lambda)x^k + \check{A}(x, y, \theta, \tau, \lambda)$$

and

$$Q(x, y, \theta, \tau, \lambda) = \check{d}_p(\theta, \tau, \lambda)x^p + \check{B}(x, y, \theta, \tau, \lambda).$$

Then, using (6.5.4) we can rewrite (6.5.3) as

$$\begin{aligned} D\Delta^x \cdot J &= \check{\mathcal{K}}_n^y [\check{c} \circ (\check{\mathcal{K}}_n^\theta + \Delta^\theta, \tau) - \check{c} \circ (\check{\mathcal{K}}_n^\theta, \tau)] + \Delta^y c \circ (\check{\mathcal{K}}_n^\theta + \Delta^\theta, \tau) + \check{\mathcal{E}}_n^x, \\ D\Delta^y \cdot J &= P \circ (\check{\mathcal{K}}_n + \Delta, \tau) - P \circ (\check{\mathcal{K}}_n, \tau) + \check{\mathcal{E}}_n^y, \\ D\Delta^\theta \cdot J &= Q \circ (\check{\mathcal{K}}_n + \Delta, \tau) - Q \circ (\check{\mathcal{K}}_n, \tau) + \check{\mathcal{E}}_n^\theta, \end{aligned} \quad (6.5.5)$$

or in a compact form,

$$D\Delta \cdot J = X \circ (\check{K}_n + \Delta, \tau) - X \circ (\check{K}_n, \tau) + \check{\mathcal{E}}_n.$$

Note that this functional equation has  $d + 2$  components.

## 6.6 Function spaces, operators and their properties

To deal with equation (6.5.4) we need to define suitable function spaces and operators as we did for the map case in Chapter 5. The spaces and operators we will use here are somehow analogous to the ones used in Chapter 5, but here we will have to take into account also the time dependence. Also, the operator corresponding to  $\mathcal{S}_{n,J}$  in Chapter 5, namely  $\mathcal{S}_{n,R}$ , which was a composition operator, will be now an integral operator due to the nature of the invariance equation for vector fields, (6.5.1).

We fix  $0 < \beta < \frac{\pi}{k-1}$  and we take the sector  $S(\beta, \rho)$  for some  $0 < \rho < 1$ .

**Definition 6.6.1.** *Given a sector  $S = S(\beta, \rho)$ , and  $\sigma > 0$ , let  $\mathcal{Z}_n$ , for  $n \in \mathbb{N}$ , be the Banach space*

$$\mathcal{Z}_n = \left\{ f : S \times \mathbb{T}_\sigma^{d+d'} \times \Lambda_{\mathbb{C}} \rightarrow \mathbb{C} \mid f \text{ real analytic}, \right. \\ \left. \|f\|_n := \sup_{(u,\theta,\tau,\lambda) \in S \times \mathbb{T}_\sigma^{d+d'} \times \Lambda_{\mathbb{C}}} \frac{|f(u, \theta, \tau, \lambda)|}{|u|^n} < \infty \right\}.$$

Given a product of spaces,  $\prod_i \mathcal{Z}_i$ , we endow it with the product norm

$$\|f\|_{\prod_i \mathcal{Z}_i} = \max_i \|\pi_i \circ f\|_{\mathcal{Z}_i},$$

where  $\pi_i$  is the canonical projection from  $\prod_j \mathcal{Z}_j$  to  $\mathcal{Z}_i$ .

We also define the space

$$\Omega_n = \mathcal{Z}_n \times \mathcal{Z}_{n+k-1} \times \mathcal{Z}_{n+2p-k-1}^d,$$

endowed with the product norm defined above, where the functions in  $\mathcal{Z}_{n+2p-k-1}$  are mapped into  $\mathbb{C}/\mathbb{Z}$ . Also, given  $\alpha > 0$  we define

$$\Omega_n^\alpha = \{f = (f^x, f^y, f^\theta) \in \Omega_n \mid \|f\|_{\Omega_n} \leq \alpha\}.$$

For the sake of simplicity, we will omit the parameters  $\rho$ ,  $\beta$  and  $\sigma$  in the notation of the spaces  $\mathcal{Z}_n$ .

Since  $\check{X}$  is analytic in  $U \times \mathbb{T}^{d+d'} \times \Lambda$ , which is relatively compact, it has a holomorphic extension to some neighborhood of the form  $V \times \mathbb{T}_\sigma^{d+d'} \times \Lambda_{\mathbb{C}}$ ,  $V \subset \mathbb{C}^2$ . Also since  $\check{K}_n$  and  $J$  are analytic in  $\mathbb{R} \times \mathbb{T}^{d+d'} \times \Lambda$  they can be defined on a complex domain of the form  $S(\beta, \rho) \times \mathbb{T}_{\sigma'}^{d+d'} \times \Lambda_{\mathbb{C}}$ . Then it is possible to set equation (6.5.5) in a space of holomorphic functions defined in  $S(\beta, \rho) \times \mathbb{T}_{\sigma'}^{d+d'} \times \Lambda_{\mathbb{C}}$ , and to look for  $\Delta$  being a real analytic function of complex variables that takes values in  $\mathbb{R}^2 \times \mathbb{T}^d$  when restricted to  $(0, \rho) \times \mathbb{T}^{d+d'} \times \Lambda$ .

To solve equation (6.5.5), we will consider  $n$  big enough and we will look for a solution,  $\Delta \in \Omega_n^\alpha$ , for some  $\alpha > 0$ . To determine which value of  $\alpha$  must be considered we proceed as for the map case. In order for the compositions in (6.5.5) to make sense we need to ensure the range of  $\check{K}_n + \Delta$  to be contained in the domain where  $\check{X}$  is analytic. Also, we look for an invariant manifold parameterized as  $\check{K}_n + \Delta$  where  $\Delta$  has to be considered as a small correction of  $\check{K}_n$ .

Let  $b > 0$  be the radius of a closed ball in  $\mathbb{C}^2$  contained in  $V$ , and let  $\tilde{\sigma} < \sigma$ . We need to consider  $\check{K}_n$  and  $\Delta$  such that  $((\check{K}_n + \Delta)^x, (\check{K}_n + \Delta)^y) \in V$ ,  $(\check{K}_n + \Delta)^\theta \in \mathbb{T}_\sigma^d$ . To this end we want to ensure that

$$|((\check{K}_n + \Delta)^x, (\check{K}_n + \Delta)^y)| \leq b \quad \text{and} \quad |\text{Im}((\check{K}_n + \Delta)^\theta)| \leq \tilde{\sigma}. \quad (6.6.1)$$

We choose  $\rho$  and  $\sigma'$  small enough such that

$$\sup_{S(\beta, \rho) \times \mathbb{T}_{\sigma'}^{d+d'} \times \Lambda_{\mathbb{C}}} |(\check{K}_n^x(u, \theta, \tau, \lambda), \check{K}_n^y(u, \theta, \tau, \lambda))| \leq \frac{b}{2}$$

and such that

$$\sup_{S(\beta, \rho) \times \mathbb{T}_{\sigma'}^{d+d'} \times \Lambda_{\mathbb{C}}} |\text{Im}(\check{K}_n^\theta(u, \theta, \tau, \lambda))| \leq \frac{\tilde{\sigma}}{2}.$$

Later on we may modify the size of  $\rho$  to a smaller value.

Then, we take

$$\alpha = \min \left\{ \frac{1}{2}, \frac{b}{2}, \frac{\tilde{\sigma}}{2} \right\},$$

and we set  $\Delta \in \Omega_n^\alpha$ . In this way, we get

$$\sup_{S \times \mathbb{T}_{\sigma'}^{d+d'} \times \Lambda_{\mathbb{C}}} |\Delta^x(u, \theta, \tau, \lambda)| \leq \sup_S \|\Delta^x\|_n |u|^n \leq \alpha \rho^n \leq \frac{b}{2} \rho^n,$$

and similarly,  $\sup_{S \times \mathbb{T}_{\sigma'}^{d+d'} \times \Lambda_{\mathbb{C}}} |\Delta^y(u, \theta, \tau, \lambda)| \leq \frac{b}{2} \rho^{n+k-1}$ , and

$$\sup_{S \times \mathbb{T}_{\sigma'}^{d+d'} \times \Lambda_{\mathbb{C}}} |\Delta^\theta(u, \theta, \tau, \lambda)| \leq \sup_S \|\Delta^\theta\|_{n+2p-k+1} |u|^{n+2p-k-1} \leq \alpha \rho^{n+2p-k-1} \leq \frac{\tilde{\sigma}}{2} \rho^{n+2p-k-1},$$

and in particular,  $|\operatorname{Im}(\Delta^\theta)| \leq \frac{\tilde{\sigma}}{2}$ . Hence, with these considerations one obtains the bounds required in (6.6.1).

**Remark 6.6.2.** As for the map case, along the section, the value of the radius of the ball  $\Omega_n^\alpha$  is always fixed and given by  $\alpha = \min\{1/2, b/2, \tilde{\sigma}/2\}$ . We may modify the value  $\rho$  denoting the radius of the sector  $S(\beta, \rho)$  where the functions of the spaces  $\mathcal{Z}_n$  are defined.

Next we introduce two families of operators that will be used to deal with (6.5.5).

**Definition 6.6.3.** Let  $n \geq 0$ ,  $\beta < \frac{\pi}{k-1}$ , and let  $J : S(\beta, \rho) \times \mathbb{T}_{\sigma'}^{d+d'} \rightarrow \mathbb{C} \times \mathbb{T}_{\sigma'}^{d+d'}$  be an analytic vector field of the form

$$J(u, \theta, \tau) = (Y_k u^k + O(u^{k+1}), \omega, \nu), \quad (6.6.2)$$

with  $Y_k < 0$ , and where the terms  $O(u^{k+1})$  do not depend on  $(\theta, \tau)$ .

We define  $\mathcal{S}_{n,J} : \mathcal{Z}_n \rightarrow \mathcal{Z}_n$ , as the linear operator given by

$$\mathcal{S}_{n,J} f = Df \cdot J = \partial_u f \cdot J^x + \partial_\theta f \cdot \omega + \partial_\tau f \cdot \nu.$$

Next we introduce an auxiliary lemma concerning the properties of the solutions of vector fields of the form (6.6.2) that will be used later on.

**Lemma 6.6.4.** Let  $J(u, \theta, \tau)$  be an analytic vector field as in Definition 6.6.3 and let  $\varphi_s$  be its time- $s$  flow. That is,  $\varphi_s = (\varphi_s^u, \varphi_s^\theta, \varphi_s^\tau)$  is the flow of

$$\begin{cases} \dot{u} = Y_k u^k + O(u^{k+1}), \\ \dot{\theta} = \omega, \\ \dot{\tau} = \nu. \end{cases}$$

Then,  $\varphi_s$  is of the form

$$\varphi_s(u, \theta, \tau) = (\varphi_s^u(u), \theta + \omega s, \tau + \nu s),$$

and for any fixed  $\mu \in (0, (k-1)|Y_k^x| \cos \lambda)$ , with  $\lambda = \beta \frac{k-1}{2}$ , there exists  $\rho > 0$  small enough such that

$$|\varphi_s^u(u)| \leq \frac{|u|}{(1 + s\mu|u|^{k-1})^{\frac{1}{k-1}}}, \quad \forall u \in S(\beta, \rho), \quad \forall s \in [0, \infty). \quad (6.6.3)$$

In addition, we have  $\varphi_s(u) \in S(\beta, \rho)$  for all  $s \in [0, \infty)$  and  $u \in S(\beta, \rho)$ .

*Proof.* By its definition the time- $s$  flow of  $J$  is given by

$$\varphi_s(u, \theta, \tau) = (u, \theta, \tau) + \int_0^s J \circ \varphi_s ds. \quad (6.6.4)$$

and thus, integrating the system we obtain

$$\varphi_s^\theta(u, \theta, t) = \theta + \omega s, \quad \varphi_s^\tau(u, \theta, t) = \tau + \nu s,$$



and that  $\varphi_s^u$  is independent of  $\theta$  and  $\tau$ .

To show that (6.6.3) holds we will use Lemma 5.7.4 stated in Chapter 5. We start by writing  $\varphi_s(u)$  developed as a Taylor series with respect to  $u$  around  $u = 0$ ,

$$\varphi_s^u(u) = \alpha_0(s) + \alpha_1(s)u + \alpha_2(s)u^2 + \cdots, \quad (6.6.5)$$

and then one gets  $\alpha_0(s) \equiv 0$ , since  $u = 0$  is a critical point of  $J^x$ . By equating (6.6.4) to (6.6.5) we obtain

$$\begin{aligned} \alpha_1(s)u + \alpha_2(s)u^2 + \cdots &= u + \int_0^s [Y_k(\alpha_1(s)u + \alpha_2(s)u^2 + \cdots)^k + O(\varphi_s(u))^{k+1}] ds \\ &= u + Y_k u^k \int_0^s \alpha_1(s) ds + O(u^{k+1}). \end{aligned} \quad (6.6.6)$$

From the previous expression we infer that  $\alpha_1(s) \equiv 1$ , since for  $s = 0$  the flow is  $\varphi_0^u(u) = u$ , and hence, by replacing this in the second line of (6.6.6) we obtain that  $\varphi_s^u(u)$  is given by

$$\varphi_s^u(u) = u + sY_k u^k + O(u^{k+1}). \quad (6.6.7)$$

Then, from (6.6.7) and by the definition of  $J^x$  it is clear that for every fixed  $s$ , the map  $\varphi_s : S(\beta, \rho) \rightarrow \mathbb{C}$  satisfies the hypotheses of Lemma 5.7.4. Namely, we have that given  $s > 0$  and given  $\mu \in (0, (k-1)|Y_k^x| \cos \lambda)$ , with  $\lambda = \beta \frac{k-1}{2}$ , there exists  $\rho_s = \rho_s(\mu) > 0$  small enough such that

$$|(\varphi_s^u)^j(u)| = |\varphi_{js}^u(u)| \leq \frac{|u|}{(1 + j\mu|u|^{k-1})^{\frac{1}{k-1}}}, \quad u \in S(\beta, \rho_s), \quad j \in \mathbb{N},$$

and therefore, there exists  $\rho = \rho(\mu)$  such that

$$\sup_{s \in [0,1]} |\varphi_{js}^u(u)| \leq \frac{|u|}{(1 + j\mu|u|^{k-1})^{\frac{1}{k-1}}}, \quad u \in S(\beta, \rho), \quad j \in \mathbb{N}.$$

Moreover, clearly it holds that

$$\frac{|u|}{(1 + j\mu|u|^{k-1})^{\frac{1}{k-1}}} \leq \frac{|u|}{(1 + sj\mu|u|^{k-1})^{\frac{1}{k-1}}} \quad u \in S(\beta, \rho), \quad s \in [0, 1], \quad j \in \mathbb{N}.$$

By joining the two previous estimates we get

$$|\varphi_{js}^u(u)| \leq \frac{|u|}{(1 + sj\mu|u|^{k-1})^{\frac{1}{k-1}}} \quad u \in S(\beta, \rho), \quad s \in [0, 1], \quad j \in \mathbb{N},$$

which is equivalent to

$$|\varphi_s^u(u)| \leq \frac{|u|}{(1 + s\mu|u|^{k-1})^{\frac{1}{k-1}}} \quad u \in S(\beta, \rho), \quad s \in [0, \infty).$$

Finally, also as a direct consequence of Lemma 5.7.4, we have that  $\varphi_s^u$  maps  $S(\beta, \rho)$  into itself for all  $s \in [0, \infty)$ .  $\square$

**Definition 6.6.5.** Let  $X$  be a vector field satisfying the hypotheses of Theorem 6.3.1, and let  $\check{X}(x, y, \theta, \tau) = X(x, y, \theta, t)$ , defined in  $V \times \mathbb{T}_\sigma^d \times \mathbb{T}_\sigma^{d'}$ ,  $V \in \mathbb{C}^2$ . Given  $n \geq 3$ , we introduce  $\mathcal{N}_{n,X} = (\mathcal{N}_{n,X}^x, \mathcal{N}_{n,X}^y, \mathcal{N}_{n,X}^\theta) : \Omega_n^\alpha \rightarrow \mathcal{Z}_{n+k-1} \times \mathcal{Z}_{n+2k-2} \times (\mathcal{Z}_{n+2p-2})^d$ , given by

$$\begin{aligned}\mathcal{N}_{n,X}^x(f) &= \check{\mathcal{K}}_n^y[\check{c} \circ (\check{\mathcal{K}}_n^\theta + f^\theta, \tau) - \check{c} \circ (\check{\mathcal{K}}_n^\theta, \tau)] + f^y \check{c} \circ (\check{\mathcal{K}}_n^\theta + f^\theta, \tau) + \check{\mathcal{E}}_n^x, \\ \mathcal{N}_{n,X}^y(f) &= P \circ (\check{\mathcal{K}}_n + f, \tau) - P \circ (\check{\mathcal{K}}_n, \tau) + \check{\mathcal{E}}_n^y, \\ \mathcal{N}_{n,X}^\theta(f) &= Q \circ (\check{\mathcal{K}}_n + f, \tau) - Q \circ (\check{\mathcal{K}}_n, \tau) + \check{\mathcal{E}}_n^\theta.\end{aligned}$$

The following lemma states that the integral operators  $\mathcal{S}_{n,J}$  have a bounded right inverse and provide a bound for the norm  $\|\mathcal{S}_{n,J}^{-1}\|$ .

**Lemma 6.6.6.** Given  $k \geq 2$ , for all  $n \geq 1$ , the operator  $\mathcal{S}_{n,J} : \mathcal{Z}_n \rightarrow \mathcal{Z}_n$  associated with the vector field  $J$  (see Definition 6.6.3), has a bounded right inverse,

$$\mathcal{S}_{n,J}^{-1} : \mathcal{Z}_{n+k-1} \rightarrow \mathcal{Z}_n,$$

given by

$$\mathcal{S}_{n,J}^{-1} \eta = - \int_0^\infty \eta \circ \varphi_s ds, \quad \eta \in \mathcal{Z}_{n+k-1}, \quad (6.6.8)$$

where  $\varphi_s$  denotes the time- $s$  flow of  $J$ .

Moreover, for any fixed  $\mu \in (0, (k-1)|Y_k^x| \cos \lambda)$ , with  $\lambda = \beta \frac{k-1}{2}$ , there exists  $\rho > 0$  such that, taking  $S(\beta, \rho) \times \mathbb{T}_{\sigma'}^{d+d'}$  as the domain of the functions of  $\mathcal{Z}_{n+k-1}$ , we have the operator norm bound

$$\|(\mathcal{S}_{n,J})^{-1}\| \leq \frac{1}{\mu} \frac{k-1}{n}.$$

*Proof.* To show that (6.6.8) is a formal expression for a right inverse of  $\mathcal{S}_{n,J}$ , note that  $\varphi_0(u, \theta, \tau) = (u, \theta, \tau)$  and that  $\lim_{s \rightarrow \infty} \varphi_s^u(u, \theta, \tau) = 0$ , and recall that  $\varphi_s(u, \theta, \tau) = (\varphi_s^u(u), \theta + \omega s, \tau + \nu s)$  is the time- $s$  flow of  $J$ . By differentiating under the integral sign one has

$$\mathcal{S}_{n,J} \circ (\mathcal{S}_{n,J})^{-1} \eta = - \int_0^\infty \partial_u(\eta \circ \varphi_s) ds J^x - \int_0^\infty \partial_\theta(\eta \circ \varphi_s) ds \cdot \omega - \int_0^\infty \partial_\tau(\eta \circ \varphi_s) ds \cdot \nu.$$

Moreover, the following relations hold true,

$$\begin{aligned}\int_0^\infty \partial_\theta(\eta \circ \varphi_s) ds \cdot \omega &= \int_0^\infty \partial_\theta \eta \circ \varphi_s \partial_s \varphi_s^\theta ds, \\ \int_0^\infty \partial_\tau(\eta \circ \varphi_s) ds \cdot \nu &= \int_0^\infty \partial_\tau \eta \circ \varphi_s \partial_s \varphi_s^\tau ds, \\ \int_0^\infty \partial_u(\eta \circ \varphi_s) ds J^x &= \int_0^\infty \partial_u \eta \circ \varphi_s \partial_s \varphi_s^u ds.\end{aligned} \quad (6.6.9)$$

Indeed, the first two equalities above are trivial. To prove the third one, observe that we have

$$\int_0^\infty \partial_u(\eta \circ \varphi_s) J^x ds = \int_0^\infty \partial_u \eta \circ \varphi_s \partial_u \varphi_s^u J^x ds = \int_0^\infty \partial_u \eta \circ \varphi_s \partial_u \varphi_s^u J^x \frac{J^x \circ \varphi_s}{J^x \circ \varphi_s} ds,$$

and then, denoting  $g(s, u) = \partial_u \eta \circ \varphi_s J^x \circ \varphi_s$  and  $v(s, u) = \partial_u \varphi_s^u \frac{J^x}{J^x \circ \varphi_s}$ , we can write

$$\int_0^\infty \partial_u(\eta \circ \varphi_s) J^x ds = \int_0^\infty g(s, u) v(s, u) ds. \quad (6.6.10)$$

Note that we also have  $v(s, u) \equiv 1$ . Indeed,

$$\partial_s v(s, u) = J^x \partial_s \frac{\partial_u \varphi_s^u}{J^x \circ \varphi_s} = J^x \left( \frac{\partial_u J^x \circ \varphi_s \partial_u \varphi_s^u}{J^x \circ \varphi_s} - \frac{\partial_u \varphi_s^u \partial_u J^x \circ \varphi_s J^x \circ \varphi_s}{(J^x \circ \varphi_s)^2} \right) \equiv 0,$$

and then,

$$v(s, u) = v(0, u) = \partial_u \varphi_0^u \frac{J^x}{J^x \circ \varphi_0} \equiv 1.$$

Therefore, from (6.6.10) we have

$$\int_0^\infty \partial_u (\eta \circ \varphi_s) J^x ds = \int_0^\infty g(s, u) ds = \int_0^\infty \partial_u \eta \circ \varphi_s J^x \circ \varphi_s ds = \int_0^\infty \partial_u \eta \circ \varphi_s \partial_s \varphi_s^u ds,$$

and the third equality of (6.6.9) is proved. Finally, using (6.6.9) we obtain

$$\begin{aligned} \mathcal{S}_{n,J} \circ (\mathcal{S}_{n,J})^{-1} \eta &= - \int_0^\infty \partial_u \eta \circ \varphi_s \partial_s \varphi_s^u ds - \int_0^\infty \partial_\theta \eta \circ \varphi_s \partial_s \varphi_s^\theta ds - \int_0^\infty \partial_\tau \eta \circ \varphi_s \partial_s \varphi_s^\tau ds \\ &= - \int_0^\infty \partial_s (\eta \circ \varphi_s) ds = \eta \circ \varphi_0 - \lim_{s \rightarrow \infty} \eta \circ \varphi_s = \eta. \end{aligned}$$

Also, by Lemma 6.6.4 we have  $\varphi_s(u, \theta, \tau) = (\varphi_s^u(u), \theta + \omega s, \tau + \nu s)$ , and that  $\varphi_s^u(u)$  belongs to  $S(\beta, \rho)$  for all  $s \in [0, \infty)$ . Then clearly one has  $\varphi_s \in S(\beta, \rho) \times \mathbb{T}_{\sigma'}^{d+d'}$  and the composition  $\eta \circ \varphi_s$  is well defined for all  $s$ .

We check next that the integral (6.6.8) converges uniformly on  $S \times \mathbb{T}_{\sigma'}^{d+d'}$ . Using again Lemma 6.6.4 we have, for  $\rho$  small enough,

$$\begin{aligned} |\eta \circ \varphi_s(u, \theta, \tau)| &\leq \|\eta\|_{n+k-1} |\varphi_s^u(u)|^{n+k-1} \leq \|\eta\|_{n+k-1} \left( \frac{|u|}{(1 + s\mu|u|^{k-1})^{1/k-1}} \right)^{n+k-1} \\ &\leq C \|\eta\|_{n+k-1} \frac{1}{s^{(1+\frac{n}{k-1})}}, \quad \forall (u, \theta, \tau) \in S \times \mathbb{T}_{\sigma'}^{d+d'}, \quad \forall s \in [0, \infty), \end{aligned}$$

and therefore the integral  $\int_0^\infty \eta \circ \varphi_s ds$  defines a holomorphic function in  $S \times \mathbb{T}_{\sigma'}^{d+d'}$ .

We prove next that  $\mathcal{S}_n^{-1}$  is bounded on  $\mathcal{Z}_{n+k-1}$ . From the expression obtained in (6.6.8) and by Lemma 6.6.4, one has, if  $\beta < \frac{\pi}{k-1}$  and for  $\rho$  small enough,

$$\begin{aligned} \|(\mathcal{S}_{n,J})^{-1} \eta\|_n &= \sup_{S \times \mathbb{T}_{\sigma'}^{d+d'}} \frac{|(\mathcal{S}_{n,J})^{-1} \eta(u, \theta, \tau)|}{|u|^n} \\ &\leq \sup_{S \times \mathbb{T}_{\sigma'}^{d+d'}} \frac{1}{|u|^n} \int_0^\infty |(\eta \circ \varphi_s)(u, \theta, \tau)| ds \\ &\leq \|\eta\|_{n+k-1} \sup_S \frac{1}{|u|^n} \int_0^\infty |\varphi_s^u(u)|^{n+k-1} \\ &\leq \|\eta\|_{n+k-1} \sup_S \frac{1}{|u|^n} \int_0^\infty \left( \frac{|u|}{(1 + s\mu|u|^{k-1})^{1/k-1}} \right)^{n+k-1} ds, \end{aligned}$$

and we also have

$$\begin{aligned}
\frac{1}{|u|^n} \int_0^\infty \left( \frac{|u|}{(1+s\mu|u|^{k-1})^{1/k-1}} \right)^{n+k-1} ds &= |u|^{k-1} \int_0^\infty \frac{1}{(1+s\mu|u|^{k-1})^{\frac{n+k-1}{k-1}}} ds \\
&= \frac{|u|^{k-1}}{\mu|u|^{k-1}} \int_0^\infty \frac{1}{(1+y)^{\frac{n+k-1}{k-1}}} dy \\
&= \frac{1}{\mu} \frac{k-1}{n}, \quad \forall u \in S(\beta, \rho).
\end{aligned}$$

Therefore, we get

$$\|(\mathcal{S}_{n,J})^{-1}\eta\|_n \leq \|\eta\|_{n+k-1} \left(\frac{1}{\mu} \frac{k-1}{n}\right), \quad \eta \in \mathcal{Z}_{n+k-1},$$

which shows that  $(\mathcal{S}_{n,J})^{-1} : \mathcal{Z}_{n+k-1} \rightarrow \mathcal{Z}_n$  is bounded with  $\|(\mathcal{S}_{n,J})^{-1}\| \leq \frac{1}{\mu} \frac{k-1}{n}$ .  $\square$

The operators  $\mathcal{N}_{n,X}$  are Lipschitz and we provide bounds for their Lipschitz constants.

**Lemma 6.6.7.** *For each  $n \geq 3$ , there exists a constant,  $M_n > 0$ , for which the operator  $\mathcal{N}_{n,X}$  satisfies*

$$\begin{aligned}
\text{Lip } \mathcal{N}_{n,X}^x &\leq \sup_{(\theta,\tau) \in \mathbb{T}_\sigma^{d+d'}} |\check{c}(\theta, \tau)| + M_n \rho, \\
\text{Lip } \mathcal{N}_{n,X}^y &\leq k \sup_{(\theta,\tau) \in \mathbb{T}_\sigma^{d+d'}} |\check{a}_k(\theta, \tau)| + M_n \rho, \\
\text{Lip } \mathcal{N}_{n,X}^\theta &\leq p \sup_{(\theta,\tau) \in \mathbb{T}_\sigma^{d+d'}} |\check{d}_p(\theta, \tau)| + M_n \rho,
\end{aligned}$$

where  $\rho$  is the radius of the sector  $S(\beta, \rho)$ .

*Proof.* The proof is completely analogous to the one of Lemma 5.7.9, with the only difference that here the vector field  $X$  and the functions of  $\Omega_n^\alpha$  also depend on  $\tau$ . To avoid redundancy, we write the proof only for the component  $\mathcal{N}_{n,X}^x$ .

As in Lemma 5.7.9, along the proof we use that if a given analytic function  $g$  defined in  $S(\beta, \rho) \times \mathbb{T}_\sigma^{d+d'}$  satisfies  $g(u, \theta, t) = O(|u|^n)$ , for some integer  $n$ , then there exists  $M_n > 0$  such that  $\|g\|_n < M_n$ , and also, the coefficients of  $\check{\mathcal{K}}_n$  that depend on  $(\theta, \tau)$  are bounded for  $(\theta, \tau) \in \mathbb{T}_\sigma^{d+d'}$  as a consequence of the small divisors lemma.

For the component  $\mathcal{N}_{n,X}^x$ , we have, for each  $f, \tilde{f} \in \Omega_n^\alpha$ ,

$$\begin{aligned}
\mathcal{N}_{n,X}^x(f) - \mathcal{N}_{n,X}^x(\tilde{f}) &= \check{\mathcal{K}}_n^y [\check{c} \circ (\check{\mathcal{K}}_n^\theta + f^\theta, \tau) - \check{c} \circ (\check{\mathcal{K}}_n^\theta + \tilde{f}^\theta, \tau)] + f^y \check{c} \circ (\check{\mathcal{K}}_n^\theta + f^\theta, \tau) \\
&\quad - \tilde{f}^y \check{c} \circ (\check{\mathcal{K}}_n^\theta + \tilde{f}^\theta, \tau) \\
&= (\check{\mathcal{K}}_n^y + f^y) \int_0^1 D\check{c} \circ [\check{\mathcal{K}}_n^\theta + \tilde{f}^\theta + s(f^\theta - \tilde{f}^\theta), \tau] ds (f^\theta - \tilde{f}^\theta) \\
&\quad + \check{c} \circ (\check{\mathcal{K}}_n^\theta + \tilde{f}^\theta, \tau) (f^y - \tilde{f}^y).
\end{aligned}$$

We can then bound, for some  $M_n > 0$ ,

$$\begin{aligned}
& \|(\check{\mathcal{K}}_n^y + f^y) \int_0^1 D\check{c} \circ (\check{\mathcal{K}}_n^\theta + \tilde{f}^\theta + s(f^\theta - \tilde{f}^\theta), \tau) ds (f^\theta - \tilde{f}^\theta)\|_{n+k-1} \\
& \leq \sup_{\mathbb{T}_\sigma^{d+d'}} |D\check{c}(\theta, \tau)| \sup_{S \times \mathbb{T}_{\sigma'}^{d+d'}} \frac{1}{|u|^{n+k-1}} |\check{\mathcal{K}}_n^y(u, \theta, \tau) + f^y(u, \theta, \tau)| |f^\theta(u, \theta, \tau) - \tilde{f}^\theta(u, \theta, \tau)| \\
& \leq \|f^\theta - \tilde{f}^\theta\|_{n+2p-k-1} \|\check{\mathcal{K}}_n^y + f^y\|_{k+1} \sup_{\mathbb{T}_\sigma^{d+d'}} |D\check{c}(\theta, \tau)| \sup_{S \times \mathbb{T}_{\sigma'}^{d+d'}} |u|^{2p-k+1} \\
& \leq M_n \rho^{2p-k+1} \|f^\theta - \tilde{f}^\theta\|_{n+2p-k-1},
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
\|\check{c} \circ (\check{\mathcal{K}}_n^\theta + \tilde{f}^\theta, \tau) (f^y - \tilde{f}^y)\|_{n+k-1} &= \sup_{S \times \mathbb{T}_{\sigma'}^{d+d'}} |\check{c} \circ (\check{\mathcal{K}}_n^\theta + \tilde{f}^\theta, \tau)(u, \theta, \tau)| \frac{|f^y(u, \theta, \tau) - \tilde{f}^y(u, \theta, \tau)|}{|u|^{n+k-1}} \\
&\leq \sup_{\mathbb{T}_\sigma^{d+d'}} |\check{c}(\theta, \tau)| \|f^y - \tilde{f}^y\|_{n+k-1},
\end{aligned}$$

and thus, we obtain

$$\|\mathcal{N}_n^x(f) - \mathcal{N}_n^x(\tilde{f})\|_{n+k-1} \leq \left( \sup_{\mathbb{T}_\sigma^{d+d'}} |\check{c}(\theta, \tau)| + M_n \rho \right) \max\{\|f^y - \tilde{f}^y\|_{n+k-1}, \|f^\theta - \tilde{f}^\theta\|_{n+2p-k-1}\},$$

that is,

$$\text{Lip } \mathcal{N}_n^x \leq \sup_{\mathbb{T}_\sigma^{d+d'}} |\check{c}(\theta, \tau)| + M_n \rho.$$

□

We define some more operators and we introduce the family  $\mathcal{T}_{n,X}$  similarly as we did in Chapter 5 for the map case.

**Definition 6.6.8.** For  $n > 2p - k - 1$ , we denote by  $\mathcal{S}_{n,J}^\times : \Omega_n \rightarrow \Omega_n$  the linear operator defined component-wise as  $\mathcal{S}_{n,J}^\times = (\mathcal{S}_{n,J}, \mathcal{S}_{n+k-1,J}, (\mathcal{S}_{n+2p-k-1,J})^d)$ .

**Remark 6.6.9.** Since  $\mathcal{S}_{n,J}^\times$  is defined component-wise, its inverse,

$$(\mathcal{S}_{n,J}^\times)^{-1} : \mathcal{Z}_{n+k-1} \times \mathcal{Z}_{n+2k-2} \times (\mathcal{Z}_{n+2p-2})^d \rightarrow \Omega_n,$$

is given by

$$(\mathcal{S}_{n,J}^\times)^{-1} = (\mathcal{S}_{n,J}^{-1}, \mathcal{S}_{n+k-1,J}^{-1}, (\mathcal{S}_{n+2p-k-1,J}^{-1})^d).$$

**Definition 6.6.10.** Let  $X$  be a vector field satisfying the hypotheses of Theorem 6.3.1, and let  $\check{X}(x, y, \theta, \tau) = X(x, y, \theta, t)$ , defined in  $V \times \mathbb{T}_\sigma^d \times \mathbb{T}_\sigma^{d'}$ ,  $V \in \mathbb{C}^2$ . Given  $n \geq 3$ , we define  $\mathcal{T}_{n,X} : \Omega_n^\alpha \rightarrow \Omega_n$  by

$$\mathcal{T}_{n,X} = (\mathcal{S}_{n,J}^\times)^{-1} \circ \mathcal{N}_{n,X}.$$

**Remark 6.6.11.** Note that given the vector field  $X$ , to define the previous operators we always take together the associated triple  $(\check{X}, \check{\mathcal{K}}_n, J)$  satisfying  $\check{X} \circ \check{\mathcal{K}}_n - D\check{\mathcal{K}}_n \cdot J = \check{\mathcal{E}}_n$ .

Using the introduced operators, equations (6.5.5) can be written as

$$\mathcal{S}_{n,J}^\times \Delta = \mathcal{N}_{n,X}(\Delta).$$

**Lemma 6.6.12.** *There exist  $m_0 > 0$  and  $\rho_0 > 0$  such that if  $\rho < \rho_0$ , then, for every  $n \geq m_0$ , we have  $\mathcal{T}_{n,X}(\Omega_n^\alpha) \subseteq \Omega_n^\alpha$  and  $\mathcal{T}_{n,X}$  is a contraction operator in  $\Omega_n^\alpha$ .*

*Proof.* By its definition, the operator  $\mathcal{T}_{n,X}$  satisfies

$$\text{Lip } \mathcal{T}_{n,X} \leq \max\{\|\mathcal{S}_{n,J}^{-1}\| \text{Lip } \mathcal{N}_{n,X}^x, \|\mathcal{S}_{n+k-1,J}^{-1}\| \text{Lip } \mathcal{N}_{n,X}^y, \|\mathcal{S}_{n+2p-k-1,J}^{-1}\| \text{Lip } \mathcal{N}_{n,X}^\theta\}. \quad (6.6.11)$$

From (6.6.11) and the estimates obtained in Lemmas 6.6.6 and 6.6.7, given  $\mu \in (0, (k-1)|\bar{Y}_k^x| \cos \lambda)$ , with  $\lambda = \beta \frac{k-1}{2}$ , there is  $\rho_0 > 0$  such that for  $\rho < \rho_0$  we have the bound

$$\begin{aligned} \text{Lip } \mathcal{T}_{n,X} \leq & \max\left\{\frac{1}{\mu} \frac{k-1}{n} \left(\sup_{\mathbb{T}_\sigma^{d+d'}} |\check{c}(\theta, \tau)| + M_n \rho\right), \right. \\ & \left. \frac{1}{\mu} \frac{k-1}{n+k-1} \left(\sup_{\mathbb{T}_\sigma^{d+d'}} |\check{a}_k(\theta, \tau)| + M_n \rho\right), \frac{1}{\mu} \frac{k-1}{n+2p-k-1} \left(\sup_{\mathbb{T}_\sigma^{d+d'}} |\check{d}_p(\theta, \tau)| + M_n \rho\right)\right\}, \end{aligned}$$

taking  $S(\beta, \rho) \times \mathbb{T}_{\sigma'}^{d+d'}$  as the domain of the functions of  $\Omega_n^\alpha$ .

Then, choosing  $\rho < \rho_0$  small enough, it is clear that one can choose  $m_0$  such that, for  $n \geq m_0$ , one has  $\text{Lip } \mathcal{T}_{n,X} < 1$ . For the chosen  $\rho < \rho_0$  and  $n \geq m_0$ ,  $\mathcal{T}_{n,X}$  is a contraction in  $\Omega_n^\alpha$ .

Next we prove that one can find  $\tilde{\rho}_0 > 0$ , maybe smaller than  $\rho_0$ , such that taking  $S(\beta, \rho) \times \mathbb{T}_{\sigma'}^{d+d'}$ , with  $\rho < \tilde{\rho}_0$  as the domain of the functions of  $\Omega_n^\alpha$ , then  $\mathcal{T}_{n,X}$  maps  $\Omega_n^\alpha$  into itself.

For each  $f \in \Omega_n^\alpha$  we can write

$$\begin{aligned} \|\mathcal{T}_{n,X}(f)\|_{\Omega_n} & \leq \|\mathcal{T}_{n,X}(f) - \mathcal{T}_{n,X}(0)\|_{\Omega_n} + \|\mathcal{T}_{n,X}(0)\|_{\Omega_n} \\ & \leq \alpha \text{Lip } \mathcal{T}_{n,X} + \|\mathcal{T}_{n,X}(0)\|_{\Omega_n}. \end{aligned}$$

From the definition of  $\mathcal{T}_{n,X}$  and  $\mathcal{N}_{n,X}$  we have, for each  $n \in \mathbb{N}$ ,

$$\mathcal{T}_{n,X}(0) = (\mathcal{S}_{n,J}^\times)^{-1} \circ \mathcal{N}_{n,X}(0) = (\mathcal{S}_{n,J}^\times)^{-1} \check{\mathcal{E}}_n.$$

Also, we have  $\check{\mathcal{E}}_n = (\check{\mathcal{E}}_n^x, \check{\mathcal{E}}_n^y, \check{\mathcal{E}}_n^\theta) \in \mathcal{Z}_{n+k} \times \mathcal{Z}_{n+2k-1} \times (\mathcal{Z}_{n+2p-1})^p$ , and thus, for every  $\varepsilon > 0$ , there is  $\rho_n > 0$  such that for  $\rho < \rho_n$  one has

$$\|\mathcal{T}_{n,X}(0)\|_{\Omega_n} \leq \|(\mathcal{S}_{n,J}^\times)^{-1}\| \max\{\|\check{\mathcal{E}}_n^x\|_{n+k-1}, \|\check{\mathcal{E}}_n^y\|_{n+2k-2}, \|\check{\mathcal{E}}_n^\theta\|_{n+2p-2}\} \leq \|(\mathcal{S}_{n,J}^\times)^{-1}\| M_n \rho < \varepsilon.$$

Therefore, since we have  $\text{Lip } \mathcal{T}_{n,X} < 1$ , we can take  $\rho_n$  as

$$\rho_n = \sup\{\rho > 0 \mid \alpha \text{Lip } \mathcal{T}_{n,X} + \|\mathcal{T}_{n,X}(0)\|_{\Omega_n} \leq \alpha\},$$

and then for every  $\rho < \rho_n$  it holds that  $\mathcal{T}_{n,X}(\Omega_n^\alpha) \subseteq \Omega_n^\alpha$ .  $\square$

## 6.7 Proofs of the main results

We give next the proofs of Theorems 6.3.1 and 6.3.2, where we show that there exists a solution  $\Delta$  of equation (6.5.5). The invariant manifold of  $X$  we are looking for will be given by  $\mathcal{K}_n(u, \theta, t) + \hat{\Delta}(u, \theta, t)$ , with  $\hat{\Delta}(u, \theta, t) = \Delta(u, \theta, \tau)$  and  $\tau = \nu t$ ,  $t \in \mathbb{R}$ .

*Proof of Theorem 6.3.1.* Let  $m_0$  be the integer provided by Lemma 6.6.12, and let  $n_0 = \max\{m_0, k + 1\}$ . We take the approximations  $\mathcal{K}_{n_0}$  and  $Y = \mathcal{Y}_{n_0}$  given by Proposition 6.4.1, which satisfy

$$\begin{aligned}\mathcal{E}_{n_0}(u, \theta, t) &= X(\mathcal{K}_{n_0}(u, \theta, t), t) - \partial_{(u, \theta)} \mathcal{K}_{n_0}(u, \theta, t) \cdot Y(u, \theta, t) - \partial_t \mathcal{K}_{n_0}(u, \theta, t) \\ &= (O(u^{n_0+k}), O(u^{n_0+2k-1}), O(u^{n_0+2p-1})).\end{aligned}$$

We will look for  $\rho > 0$  and a function  $\Delta : [0, \rho] \times \mathbb{T}^d \times \mathbb{R}$ ,  $\Delta$  analytic in  $(0, \rho) \times \mathbb{T}^d \times \mathbb{R}$ , satisfying

$$X \circ (\mathcal{K}_{n_0} + \Delta, t) - \partial_{(u, \theta)} (\mathcal{K}_{n_0} + \Delta) \cdot Y - \partial_t (\mathcal{K}_{n_0} + \Delta) = 0. \quad (6.7.1)$$

We consider  $\check{X}(x, y, \theta, \tau) = X(x, y, \theta, t)$ , and we take the holomorphic extension of  $\check{X}$  to a neighborhood  $V \times \mathbb{T}_\sigma^{d+d'}$  of  $(0, 0) \times \mathbb{T}^{d+d'}$ , where  $V \subset \mathbb{C}^2$  contains the centered closed ball of radius  $b > 0$ , and we take also  $\alpha = \min\{\frac{1}{2}, \frac{b}{2}, \frac{\tilde{\sigma}}{2}\}$  with  $0 < \tilde{\sigma} < \sigma$ . This setting allows to rewrite (6.7.1) as

$$\begin{aligned}D\Delta^x \cdot J &= \check{K}_n^y [\check{c} \circ (\check{K}_n^\theta + \Delta^\theta, \tau) - \check{c} \circ (\check{K}_n^\theta, \tau)] + \Delta^y c \circ (\check{K}_n^\theta + \Delta^\theta, \tau) + \check{\mathcal{E}}_n^x, \\ D\Delta^y \cdot J &= P \circ (\check{K}_n + \Delta, \tau) - P \circ (\check{K}_n, \tau) + \check{\mathcal{E}}_n^y, \\ D\Delta^\theta \cdot J &= Q \circ (\check{K}_n + \Delta, \tau) - Q \circ (\check{K}_n, \tau) + \check{\mathcal{E}}_n^\theta,\end{aligned}$$

with  $\Omega_{n_0}^\alpha$ , or using the operators defined previously,

$$\mathcal{S}_{n_0, J}^\times \Delta = \mathcal{N}_{n_0, X}(\Delta), \quad \Delta \in \Omega_{n_0}^\alpha.$$

By Lemma 6.6.6, if  $\rho$  is small,  $\mathcal{S}_{n_0}^\times$  has a bounded right inverse and we can rewrite this equation as

$$\Delta = \mathcal{T}_{n_0, X}(\Delta), \quad \Delta \in \Omega_{n_0}^\alpha.$$

By Lemma 6.6.12, since  $n_0 \geq m_0$ , we have that  $\mathcal{T}_{n_0, X}$  maps  $\Omega_{n_0}^\alpha$  into itself and is a contraction. Then it has a unique fixed point,  $\Delta^\infty \in \Omega_{n_0}^\alpha$ . Note that this solution is unique once  $\check{K}_{n_0}$  is fixed. Finally we take  $\tilde{\Delta}^\infty(u, \theta, t) = \Delta^\infty(u, \theta, \tau)$ , and then  $K = \mathcal{K}_{n_0} + \tilde{\Delta}^\infty$  satisfies the conditions in the statement.

The  $C^1$  character of  $K$  at the origin follows from the order condition of  $K$  at  $u = 0$ .  $\square$

*Proof of Theorem 6.3.2.* Let  $m_0$  be the integer provided by Lemma 6.6.12, and let  $n_0 = \max\{m_0, k + 1\}$ . If the value of  $n$  given in the statement of the theorem is such that  $n < n_0$ , first we look for a better approximation  $\mathcal{K}_{n_0}$  of the form

$$\mathcal{K}_{n_0}(u, \theta, t) = \hat{K}(u, \theta, t) + \sum_{j=n+1}^{n_0} \hat{K}_j(u, \theta, t),$$

with

$$\hat{K}_j(u, \theta, t) = \begin{pmatrix} \bar{K}_j^x u^j + \tilde{K}_{j+k-1}^x(\theta, t) u^{j+k-1} \\ \bar{K}_{j+k-1}^y u^{j+k-1} + \tilde{K}_{j+2k-2}^y(\theta, t) u^{j+2k-2} \\ \bar{K}_{j+2p-k-1}^\theta u^{j+2p-k-1} + \tilde{K}_{j+2p-2}^\theta(\theta, t) u^{j+2p-2} \end{pmatrix},$$

and

$$\mathcal{Y}_{n_0}(u, \theta, t) = \hat{Y}(u, \theta, t) + \sum_{j=n+1}^{n_0} \hat{Y}_j(u),$$

with

$$\hat{Y}_j^x(u) = \begin{cases} \delta_{j,k+1} \bar{Y}_{2k-1}^x u^{2k-1} & \text{if } n \leq k, \\ 0 & \text{if } n > k, \end{cases} \quad \hat{Y}_j^\theta(u) = 0.$$

The coefficients of  $\mathcal{K}_{n_0}(u, \theta, t)$  and  $\mathcal{Y}_{n_0}(u, \theta, t)$  are obtained imposing the condition

$$\begin{aligned} X(\mathcal{K}_{n_0}(u, \theta, t), t) - \partial_{(u,\theta)} \mathcal{K}_{n_0}(u, \theta, t) \cdot Y(u, \theta, t) - \partial_t \mathcal{K}_{n_0}(u, \theta, t) \\ = (O(u^{n_0+k}), O(u^{n_0+2k-1}), O(u^{n_0+2p-1})). \end{aligned}$$

Proceeding as in Proposition 6.4.1, we obtain such coefficients iteratively. We denote  $\mathcal{K}_j(u, \theta, t) = \hat{K}(u, \theta, t) + \sum_{m=n+1}^j \hat{K}_m(u, \theta, t)$  and  $\mathcal{Y}_j(u, \theta, t) = \hat{Y}(u, \theta, t) + \sum_{m=n+1}^j \hat{Y}_m(u)$ . In the iterative step we have

$$X(\mathcal{K}_j(u, \theta, t), t) - \partial_{(u,\theta)} \mathcal{K}_j(u, \theta, t) \cdot \mathcal{Y}_j(u, \theta, t) - \partial_t \mathcal{K}_j(u, \theta, t) = (O(u^{j+k}), O(u^{j+2k-1}), O(u^{j+2p-1})).$$

Then,

$$\begin{aligned} X(\mathcal{K}_j + \hat{K}_{j+1}, t) - \partial_{(u,\theta)} (\mathcal{K}_j + \hat{K}_{j+1}) \cdot (\mathcal{Y}_j + \hat{Y}_{j+1}) - \partial_t (\mathcal{K}_j + \hat{K}_{j+1}) \\ = X(\mathcal{K}_j, t) - \partial_{(u,\theta)} \mathcal{K}_j \cdot \mathcal{Y}_j - \partial_t \mathcal{K}_j \\ + \partial_{(x,y,\theta)} X(\mathcal{K}_j, t) \cdot \hat{K}_{j+1} - \partial_{(u,\theta)} \hat{K}_{j+1} \cdot \mathcal{Y}_j - \partial_{(u,\theta)} \mathcal{K}_{j+1} \cdot \hat{Y}_{j+1} - \partial_t \hat{K}_{j+1} \\ + \int_0^1 (1-s) \partial_{(x,y,\theta)}^2 X((\mathcal{K}_j, t) + s(\hat{K}_{j+1}, t)) ds (\hat{K}_{j+1})^{\otimes 2}. \end{aligned}$$

The condition

$$\begin{aligned} X(\mathcal{K}_{j+1}(u, \theta, t), t) - \partial_{(u,\theta)} \mathcal{K}_{j+1}(u, \theta, t) \cdot \mathcal{Y}_{j+1}(u, \theta, t) - \partial_t \mathcal{K}_{j+1}(u, \theta, t) \\ = (O(u^{j+k+1}), O(u^{j+2k}), O(u^{j+2p})), \end{aligned}$$

leads to the same equations (6.4.5) and (6.4.6) as in Proposition 6.4.1, which we solve in the same way.

From this point we can proceed as in the proof of Theorem 6.3.1 and look for  $\Delta \in \Omega_n^\alpha \subset \mathcal{Z}_{n_0} \times \mathcal{Z}_{n_0+k-1} \times (\mathcal{Z}_{n_0+2p-k-1})^d$  such that the pair  $K = \mathcal{K}_{n_0} + \Delta$ ,  $Y = \mathcal{Y}_{n_0}$  satisfies  $X(K, t) - \partial_{(u,\theta)} K \cdot Y - \partial_t K = 0$ .

Finally, for the map  $K$ , we also have

$$\begin{aligned} K(u, \theta, t) - \hat{K}(u, \theta, t) &= \mathcal{K}_{n_0}(u, \theta, t) - \hat{K}(u, \theta, t) + \Delta(u, \theta, t) \\ &= \sum_{j=n+1}^{n_0} \hat{K}_j(u, \theta, t) + \Delta(u, \theta, t) \\ &= (O(u^{n+1}), O(u^{n+k}), O(u^{n+2p-k})) + (O(u^{n_0}), O(u^{n_0+k-1}), O(u^{n_0+2p-k-1})), \end{aligned}$$

with  $n < n_0$ . Then we have  $n + 2p - k \leq n_0 + 2p - k - 1$  and thus,

$$K(u, \theta, t) - \hat{K}(u, \theta, t) = (O(u^{n+1}), O(u^{n+k}), O(u^{n+2p-k})).$$

For the vector field  $Y$  we have

$$Y(u, \theta, t) - \hat{Y}(u, \theta, t) = \mathcal{Y}_{n_0}(u, \theta, t) - \hat{Y}(u, \theta, t) = \sum_{j=n+1}^{n_0} \hat{Y}_j(u) = \begin{cases} (O(u^{2k-1}), 0) & \text{if } n \leq k, \\ (0, 0) & \text{if } n > k. \end{cases}$$



If  $n \geq n_0$  we look for  $\mathcal{K}^*(u, \theta, t) = \hat{K}(u, \theta, t) + \hat{K}_{n+1}(u, \theta, t)$  with

$$\hat{K}_{n+1}(u, \theta, t) = \begin{pmatrix} \bar{K}_{n+1}^x u^{n+1} + \tilde{K}_{n+k}^x(\theta, t) u^{n+k} \\ \bar{K}_{n+k}^y u^{n+k} + \tilde{K}_{n+2k-1}^y(\theta, t) u^{n+2k-1} \\ \bar{K}_{n+2p-k}^\theta u^{n+2p-k} + \tilde{K}_{n+2p-1}^\theta(\theta, t) u^{n+2p-1} \end{pmatrix}$$

and  $\mathcal{Y}_n^*(u, \theta, t) = \hat{Y}(u, \theta, t) + \hat{Y}_{n+1}(u)$  with

$$\hat{Y}_{n+1}^x(u) = \begin{cases} \bar{Y}_{2k-1}^x u^{2k-1} & \text{if } n \leq k, \\ 0 & \text{if } n > k, \end{cases} \quad \hat{Y}_{n+1}^\theta(u) = 0.$$

We determine the coefficients of  $\hat{K}_{n+1}(t, \theta)$  so that

$$X(\mathcal{K}^*(u, \theta, t), t) - \partial_{(u, \theta)} \mathcal{K}^*(u, \theta, t) \cdot \mathcal{Y}^*(u, \theta, t) - \partial_t \mathcal{K}^*(u, \theta, t) = (O(u^{n+k+1}), O(u^{n+2k}), O(u^{n+2p})),$$

as in the previous case, and we look for  $\Delta \in \Lambda_{n+1}^\alpha \subset \mathcal{Z}_{n+1} \times \mathcal{Z}_{n+k} \times (\mathcal{Z}_{n+2p-k})^d$  such that the pair  $K = \mathcal{K}^* + \Delta$ ,  $Y = \mathcal{Y}^*$  satisfies  $X \circ (K, t) - \partial_{(u, \theta)} K \cdot Y - \partial_t K = 0$ .

Similarly as before we obtain

$$\begin{aligned} K(u, \theta, t) - \hat{K}(u, \theta, t) &= \mathcal{K}^*(u, \theta, t) - \hat{K}(u, \theta, t) + \Delta(u, \theta, t) \\ &= \hat{K}_{n+1}(u, \theta, t) + \Delta(u, \theta, t) \\ &= (O(u^{n+1}), O(u^{n+k}), O(u^{n+2p-1})) + (O(u^{n+1}), O(u^{n+k}), O(u^{n+2p-k})) \\ &= (O(u^{n+1}), O(u^{n+k}), O(u^{n+2p-k})), \end{aligned}$$

and

$$Y(u, \theta, t) - \hat{Y}(u, \theta, t) = \mathcal{Y}^*(u, \theta, t) - \hat{Y}(u, \theta, t) = \hat{Y}_{n+1}(u) = \begin{cases} (O(u^{2k-1}), 0) & \text{if } n \leq k, \\ (0, 0) & \text{if } n > k. \end{cases}$$

□

## 6.8 Planar vector fields depending quasiperiodically on time

A particular case in the family of vector fields of the form (6.3.3) is the class obtained taking the vector fields of such family that do not depend on the variable  $\theta$ . Those are then planar vector fields depending quasiperiodically on time with a critical point at the origin with

$$DX(0, 0) = \begin{pmatrix} 0 & c(t, \lambda) \\ 0 & 0 \end{pmatrix}.$$

This class of vector fields are much simpler to study than the ones of the form (6.3.3). In this section we present a bigger class of planar vector fields containing those ones and we recover the three cases of the reduced form presented in Chapter 4.

Let  $U \subset \mathbb{R}^2$  be a neighborhood of 0, and let  $X : U \times \Lambda \rightarrow \mathbb{R}^2$  be an analytic, non autonomous vector field of the form

$$X(x, y, t, \lambda) = \begin{pmatrix} c(t, \lambda)y \\ p(x, t, \lambda) + yq(x, t, \lambda) + u(x, y, t, \lambda) + g(x, y, t, \lambda) \end{pmatrix} \quad (6.8.1)$$

with  $\bar{c}(\lambda) > 0$ ,

$$\begin{aligned} p(x, t, \lambda) &= x^k(a_k(t, \lambda) + \cdots + a_r(t, \lambda)x^{r-k}), \\ q(x, t, \lambda) &= x^{l-1}(b_l(t, \lambda) + \cdots + b_r(t, \lambda)x^{r-l}), \end{aligned}$$

for some  $k, l \geq 2$  and  $r \geq k + 1$ ,  $r \geq l + 1$ , and where  $u(x, y, t, \lambda)$  is a polynomial on the variables  $(x, y)$  containing the factor  $y^2$  and  $g(x, y, t, \lambda) = O(\|(x, y)^r\|)$ . Moreover,  $X$  depends quasiperiodically on  $t$  with time frequencies  $\nu \in \mathbb{R}^d$ .

We consider the following three cases for the class of vector fields of the form (6.8.1), depending on the indices  $k$  and  $l$ , analogous to the ones used in Chapter 4,

- Case 1:  $k < 2l - 1$  and  $\bar{a}_k(\lambda) \neq 0$ ,
- Case 2:  $k = 2l - 1$  and  $\bar{a}_k(\lambda), \bar{b}_l(\lambda) \neq 0$ ,
- Case 3:  $k > 2l - 1$  and  $\bar{b}_l(\lambda) \neq 0$ .

We note that the vector fields of the form (6.3.3) would correspond to case 1 once one removes the dependence on the variable  $\theta$ .

As in Chapter 4, to provide a uniform notation, we define the integer  $N$  as  $N = k$  for case 1 and  $N = l$  for cases 2 and 3.

### 6.8.1 Approximation of a parameterization of the invariant curves

Similarly as in Section 6.4, we provide a parameterization of an approximation of an invariant curve of a vector field of the form (6.8.1) and a representation of the dynamics inside this invariant curve.

Concretely, here we look for  $\mathcal{K}_n(u, t, \lambda)$  and  $\mathcal{Y}_n(u, t, \lambda)$  being approximations of solutions of the invariance equation

$$X \circ (K, t) - DK \cdot Y - \partial_t K = 0.$$

To do so we proceed similarly as in Proposition 6.4.1 removing the dependence on  $\theta$ , that is, we look for  $\mathcal{K}_n(u, t, \lambda)$  and  $\mathcal{Y}_n(u, t, \lambda)$  satisfying

$$X(\mathcal{K}_n(u, t, \lambda), t, \lambda) - \partial_u \mathcal{K}_n(u, t, \lambda) \cdot \mathcal{Y}_n(u, t, \lambda) - \partial_t \mathcal{K}_n(u, t, \lambda) = (O(u^{n+k}), O(u^{n+2k-1})).$$

We state three results corresponding to the three cases defined above. The proofs combine the techniques from Proposition 6.4.1 and from the proofs of Propositions 4.3.1, 4.3.3 and 4.3.4 of Chapter 4, and thus we will omit them.

**Proposition 6.8.1** (Case 1). *Let  $X$  be a  $C^\infty$  vector field of the form (6.8.1), with  $k < 2l - 1$ . Assume that  $\nu$  is Diophantine and that  $\bar{a}_k(\lambda) > 0$ , for  $\lambda \in \Lambda$ . Then, for all  $n \geq 2$ , there exist two maps,  $\mathcal{K}_n : \mathbb{R} \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}^2$  and two corresponding vector fields,  $\mathcal{Y}_n : \mathbb{R} \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}$ , of the form*

$$\mathcal{K}_n(u, t, \lambda) = \begin{pmatrix} u^2 + \sum_{i=3}^n \bar{K}_i^x(\lambda) u^i + \sum_{i=k+1}^{n+k-1} \tilde{K}_i^x(t, \lambda) u^i \\ \sum_{i=k+1}^{n+k-1} \bar{K}_i^y(\lambda) u^i + \sum_{i=2k}^{n+2k-2} \tilde{K}_i^y(t, \lambda) u^i \end{pmatrix}$$

and

$$\mathcal{Y}_n(u, t, \lambda) = \begin{cases} \bar{Y}_k(\lambda) u^k & \text{if } 2 \leq n \leq k, \\ \bar{Y}_k(\lambda) u^k + \bar{Y}_{2k-1}(\lambda) u^{2k-1} & \text{if } n \geq k+1, \end{cases}$$

such that

$$\begin{aligned} \mathcal{G}_n(u, t, \lambda) &:= X(\mathcal{K}_n(u, t, \lambda), t, \lambda) - \partial_u \mathcal{K}_n(u, t, \lambda) \cdot \mathcal{Y}_n(u, t, \lambda) - \partial_t \mathcal{K}_n(u, t, \lambda) \\ &= (O(u^{n+k}), O(u^{n+2k-1})). \end{aligned} \quad (6.8.2)$$

For the first coefficients of  $\mathcal{K}_n$  and  $\mathcal{Y}_n$  we have

$$\begin{aligned} \bar{K}_{k+1}^y(\lambda) &= \pm \sqrt{\frac{2\bar{a}_k(\lambda)}{\bar{c}(\lambda)(k+1)}}, & \bar{Y}_k(\lambda) &= \pm \sqrt{\frac{\bar{c}(\lambda)\bar{a}_k(\lambda)}{2(k+1)}}, \\ \tilde{K}_{k+1}^x(t, \lambda) &= \mathcal{SD}(\tilde{c}(t, \lambda) \bar{K}_{k+1}^y(\lambda)), & \tilde{K}_{2k}^y(t, \lambda) &= \mathcal{SD}(\tilde{a}_k(t, \lambda)). \end{aligned}$$

Note that the expression for the restricted dynamics inside the invariant curves is independent of the time  $t$ .

**Proposition 6.8.2** (Case 2). *Let  $X$  be a  $C^\infty$  vector field of the form (6.8.1) with  $k = 2l - 1$ . Assume that  $\bar{a}_k(\lambda) \neq 0$ ,  $\bar{b}_l(\lambda) \neq 0$  and  $\bar{a}_k(\lambda) > -\frac{(\bar{b}_l(\lambda))^2}{4\bar{c}(\lambda)l}$ , for  $\lambda \in \Lambda$ . If  $\bar{a}_k(\lambda) < 0$  assume also that  $\bar{a}_k(\lambda) \neq \frac{1-2l}{(3l-1)^2} \frac{(\bar{b}_l(\lambda))^2}{\bar{c}(\lambda)}$ . Assume that  $\nu$  is Diophantine. Then, for all  $n \geq 1$ , there exist two maps,  $\mathcal{K}_n : \mathbb{R} \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}^2$  and two vector fields,  $\mathcal{Y}_n : \mathbb{R} \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}$ , of the form*

$$\mathcal{K}_n(u, t, \lambda) = \begin{pmatrix} u + \sum_{i=2}^n \bar{K}_i^x(\lambda) u^i + \sum_{i=l}^{n+l-1} \tilde{K}_i^x(t, \lambda) u^i \\ \sum_{i=l}^{n+l-1} \bar{K}_i^y(\lambda) u^i + \sum_{i=2l-1}^{n+2l-2} \tilde{K}_i^y(t, \lambda) u^i \end{pmatrix} \quad (6.8.3)$$

and

$$\mathcal{Y}_n(u, t, \lambda) = \begin{cases} \bar{Y}_l(\lambda) u^l & \text{if } 2 \leq n \leq l-1, \\ \bar{Y}_l(\lambda) u^l + \bar{Y}_{2l-1}(\lambda) u^{2l-1} & \text{if } n \geq l, \end{cases} \quad (6.8.4)$$

such that

$$\begin{aligned} \mathcal{G}_n(u, t, \lambda) &:= X(\mathcal{K}_n(u, t, \lambda), t, \lambda) - \partial_u \mathcal{K}_n(u, t, \lambda) \cdot \mathcal{Y}_n(u, t, \lambda) - \partial_t \mathcal{K}_n(u, t, \lambda) \\ &= (O(u^{n+l}), O(u^{n+2l-1})). \end{aligned} \quad (6.8.5)$$

For the first pair we have

$$\begin{aligned} K_l^y(\lambda) &= \frac{\bar{b}_l(\lambda) - \sqrt{(\bar{b}_l(\lambda))^2 + 4\bar{c}(\lambda)\bar{a}_k(\lambda)l}}{2\bar{c}(\lambda)l}, \\ Y_l(\lambda) &= \frac{\bar{b}_l(\lambda) - \sqrt{(\bar{b}_l(\lambda))^2 + 4\bar{c}(\lambda)\bar{a}_k(\lambda)l}}{2l} = \bar{c}(\lambda) \bar{K}_l^y(\lambda), \\ \tilde{K}_l^x(t, \lambda) &= \mathcal{SD}(\tilde{c}(t, \lambda) \bar{K}_l^y(\lambda)), & \tilde{K}_{2l-1}^y(t, \lambda) &= \mathcal{SD}(\tilde{a}_k(t, \lambda) + \tilde{b}_l(t, \lambda) \bar{K}_l^y(\lambda)), \end{aligned}$$

and for the second one we have

$$\begin{aligned} K_l^y(\lambda) &= \frac{\bar{b}_l(\lambda) + \sqrt{(\bar{b}_l(\lambda))^2 + 4\bar{c}(\lambda)\bar{a}_k(\lambda)l}}{2\bar{c}(\lambda)l}, \\ Y_l(\lambda) &= \frac{\bar{b}_l(\lambda) + \sqrt{(\bar{b}_l(\lambda))^2 + 4\bar{c}(\lambda)\bar{a}_k(\lambda)l}}{2l} = \bar{c}(\lambda)\bar{K}_l^y(\lambda), \\ \tilde{K}_l^x(t, \lambda) &= \mathcal{SD}(\tilde{c}(t, \lambda)\bar{K}_l^y(\lambda)), \quad \tilde{K}_{2l-1}^y(t, \lambda) = \mathcal{SD}(\tilde{a}_k(t, \lambda) + \tilde{b}_l(t, \lambda)\bar{K}_l^y(\lambda)). \end{aligned}$$

If  $\bar{a}_k(\lambda) = \frac{1-2l}{(3l-1)^2} \frac{(\bar{b}_l(\lambda))^2}{\bar{c}(\lambda)}$  one can compute the coefficients of  $\mathcal{K}_n$  and  $\mathcal{Y}_n$  up to  $n = l - 1$ .

**Proposition 6.8.3** (Case 3). *Let  $X$  be a  $C^\infty$  vector field of the form (6.8.1) with  $k > 2l - 1$ . Assume that  $\nu$  is Diophantine and that  $\bar{b}_l(\lambda) \neq 0$  for  $\lambda \in \Lambda$ . Then, for all  $n \geq 1$ , there exist a map,  $\mathcal{K}_n : \mathbb{R} \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}^2$  and a vector field,  $\mathcal{Y}_n : \mathbb{R} \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}$ , of the form (6.8.3) and (6.8.4), respectively, such that (6.8.5) holds. For the first coefficients of  $\mathcal{K}_n$  and  $\mathcal{Y}_n$  we have*

$$\begin{aligned} \bar{K}_l^y(\lambda) &= \frac{\bar{b}_l(\lambda)}{\bar{c}(\lambda)l}, \quad \bar{Y}_l(\lambda) = \frac{\bar{b}_l(\lambda)}{l}, \\ \tilde{K}_l^x(t, \lambda) &= \mathcal{SD}(\tilde{c}(t, \lambda)\bar{K}_l^y(\lambda)), \quad \tilde{K}_{2l-1}^y(t, \lambda) = \mathcal{SD}(\tilde{b}_l(t, \lambda)\bar{K}_l^y(\lambda)). \end{aligned}$$

## 6.8.2 Existence results

We present an existence result and an *a posteriori* result for invariant curves of vector fields of the form (6.8.1). These results come as a direct consequence of the combination of the techniques used for the proofs of Theorems 6.3.1 and 6.3.2 and the setting and function spaces used in Section 2.4.2 for planar maps. Therefore we will not write the proofs in detail. We will state the results and we will sketch the proofs and expose how do they change when dealing with cases 2 and 3 of (6.8.1) instead of case 1.

**Theorem 6.8.4.** *Let  $X : U \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}^2$  be an analytic vector field of the form (6.8.1). Assume the following hypotheses according to the different cases, for  $\lambda \in \Lambda$ ,*

$$(case\ 1) \quad \bar{a}_k(\lambda) > 0, \quad (case\ 2) \quad \bar{a}_k(\lambda) > 0, \bar{b}_l(\lambda) \neq 0, \quad (case\ 3) \quad \bar{b}_l(\lambda) < 0.$$

Then, there exists a  $C^1$  map  $K : [0, \rho) \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}^2$ , analytic in  $(0, \rho) \times \mathbb{R} \times \Lambda$ , of the form

$$K(u, t, \lambda) = \begin{cases} (u^2, \bar{K}_{k+1}^y(\lambda)u^{k+1}) + (O(u^3), O(u^{k+2})), & case\ 1, \\ (u, \bar{K}_l^y(\lambda)u^l) + (O(u^2), O(u^{l+1})), & cases\ 2, 3, \end{cases}$$

with  $\bar{K}_{k+1}^y(\lambda) = -\sqrt{\frac{2\bar{a}_k(\lambda)}{\bar{c}(\lambda)(k+1)}}$  for case 1,  $\bar{K}_l^y(\lambda) = \frac{\bar{b}_l(\lambda) - \sqrt{(\bar{b}_l(\lambda))^2 + 4\bar{c}(\lambda)\bar{a}_k(\lambda)l}}{2\bar{c}(\lambda)l}$  for case 2 and  $\bar{K}_l^y(\lambda) = \frac{\bar{b}_l(\lambda)}{\bar{c}(\lambda)l}$  for case 3, and a one-dimensional vector field  $Y$  of the form  $Y(u, t, \lambda) = Y(u, \lambda) = \bar{Y}_N(\lambda)u^N + \bar{Y}_{2N-1}(\lambda)u^{2N-1}$ , with  $\bar{Y}_k(\lambda) = \frac{\bar{c}(\lambda)}{2}\bar{K}_{k+1}^y(\lambda)$  for case 1 and  $\bar{Y}_l(\lambda) = \bar{c}(\lambda)\bar{K}_l^y(\lambda)$  for cases 2, 3, such that

$$X(K(u, t, \lambda), t, \lambda) - \partial_u K(u, t, \lambda) \cdot Y(u, t, \lambda) - \partial_t K(u, t, \lambda) = 0, \quad (u, t, \lambda) \in [0, \rho) \times \mathbb{R} \times \Lambda.$$

**Theorem 6.8.5.** Let  $X : U \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}^2$  be an analytic vector field of the form (6.8.1), and let  $\hat{K} : (-\rho, \rho) \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}^2$  and  $\hat{Y} : (-\rho, \rho) \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}$  be an analytic map and an analytic vector field, respectively, of the form

$$\hat{K}(u, t, \lambda) = \begin{cases} (u^2, \hat{K}_{k+1}^y(\lambda)u^{k+1}) + (O(u^3), O(u^{k+2})) & \text{case 1,} \\ (u, \hat{K}_l^y(\lambda)u^l) + (O(u^2), O(u^{l+1})) & \text{cases 2, 3,} \end{cases}$$

and  $\hat{Y}(u, t, \lambda) = \hat{Y}_N(\lambda)u^N + O(u^{N+1})$ ,  $\hat{Y}_N(\lambda) < 0$ , such that

$$X(\hat{K}(u, t, \lambda), t, \lambda) - \partial_u \hat{K}(u, t, \lambda) \cdot \hat{Y}(u, t, \lambda) - \partial_t \hat{K}(u, t, \lambda) = (O(u^{n+N}), O(u^{n+2N-1})),$$

for some  $n \geq 2$  in case 1 or  $n \geq 1$  in cases 2, 3.

Then, there exists a  $C^1$  map  $K : [0, \rho) \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}^2$ , analytic in  $(0, \rho) \times \mathbb{R} \times \Lambda$ , and an analytic vector field  $Y : (-\rho, \rho) \times \mathbb{R} \times \Lambda \rightarrow \mathbb{R}$  such that

$$X(K(u, t, \lambda), t, \lambda) - \partial_u K(u, t, \lambda) \cdot Y(u, t, \lambda) - \partial_t K(u, t, \lambda) = 0, \quad (u, t, \lambda) \in [0, \rho) \times \mathbb{R} \times \Lambda,$$

and

$$\begin{aligned} K(u, t, \lambda) - \hat{K}(u, t, \lambda) &= (O(u^{n+1}), O(u^{n+N})), \\ Y(u, t, \lambda) - \hat{Y}(u, t, \lambda) &= \begin{cases} O(u^{2k-1}) & \text{if } n \leq k \\ 0 & \text{if } n > k \end{cases} & \text{case 1,} \\ Y(u, t, \lambda) - \hat{Y}(u, t, \lambda) &= \begin{cases} O(u^{2l-1}) & \text{if } n \leq l-1 \\ 0 & \text{if } n > l-1 \end{cases} & \text{cases 2, 3.} \end{aligned}$$

To prove Theorem 6.8.4, take  $\tau = \nu t$ , with  $\tau \in \mathbb{T}^d$ , and let

$$\begin{aligned} \check{X}(x, y, \tau, \lambda) &= X(x, y, t, \lambda), \quad \check{\mathcal{K}}_n(u, \tau, \lambda) = \mathcal{K}_n(u, t, \lambda), \quad \check{Y}(u, \tau, \lambda) = Y(u, t, \lambda), \\ \text{and } J(u, \tau, \lambda) &= \begin{pmatrix} \check{Y}(u, \tau, \lambda) \\ \nu \end{pmatrix}. \end{aligned}$$

We look for a  $C^1$  function,  $\Delta = \Delta(u, \tau, \lambda)$ ,  $\Delta : [0, \rho) \times \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R}^2$ , analytic in  $(0, \rho) \times \mathbb{T}^d \times \Lambda$  satisfying

$$\check{X} \circ (\check{\mathcal{K}}_n + \Delta, \tau) - D(\check{\mathcal{K}}_n + \Delta) \cdot J = 0. \quad (6.8.6)$$

Therefore an invariant manifold of  $X$  will be given by  $K(u, t, \lambda) = \mathcal{K}_n(u, t, \lambda) + \tilde{\Delta}^*(u, t, \lambda)$ , where  $\tilde{\Delta}^*(u, t, \lambda) = \Delta^*(u, \tau, \lambda)$  and  $\Delta^*(u, \tau, \lambda)$  is a solution of equation (6.8.6).

From Propositions 6.8.1, 6.8.2 and 6.8.3, given  $n$  there exist  $\mathcal{K}_n$  and  $Y = \mathcal{Y}_n$  such that

$$X \circ (\mathcal{K}_n, t) - \partial_{(u, \theta)} \mathcal{K}_n \cdot Y - \partial_t \mathcal{K}_n = \mathcal{E}_n,$$

or equivalently,

$$\check{X} \circ (\check{\mathcal{K}}_n, \tau) - D\check{\mathcal{K}}_n \cdot J = \check{\mathcal{E}}_n, \quad (6.8.7)$$

with  $\check{\mathcal{E}}_n(u, \tau, \lambda) = (O(u^{n+N}), O(u^{n+2N-1}))$ , where we denote  $N = k$  for case 1 and  $N = l$  for cases 2 and 3.

Hence, we look for  $\rho > 0$  and a map  $\check{K} = \check{K}_n + \Delta : [0, \rho) \times \mathbb{T}^d \times \Lambda \rightarrow \mathbb{R}^2$ , analytic on  $(0, \rho) \times \mathbb{T}^d \times \Lambda$  satisfying (6.8.6), where  $\check{K}_n$  and  $J$  satisfy (6.8.7). Moreover, we ask  $\Delta$  to be of the form  $\Delta = (\Delta^x, \Delta^y) = (O(u^n), O(u^{n+N-1}))$ .

Using (6.8.7) we can rewrite (6.8.6) as

$$\begin{aligned} D\Delta^x \cdot J &= \Delta^y \check{c}(\tau) + \check{\mathcal{E}}_n^x, \\ D\Delta^y \cdot J &= P \circ (\check{K}_n + \Delta, \tau) - P \circ (\check{K}_n, \tau) + \check{\mathcal{E}}_n^y, \end{aligned} \quad (6.8.8)$$

which is the functional equation that needs to be solved, and where  $P(x, y, \tau)$  denotes the second component of  $\check{X}$ .

We fix  $0 < \beta < \frac{\pi}{k-1}$  and we consider the sector  $S(\beta, \rho)$  for some  $0 < \rho < 1$ . Here we take the Banach spaces, for  $n \in \mathbb{N}$ , defined as

$$\mathcal{Z}_n = \left\{ f : S \times \mathbb{T}_\sigma^d \times \Lambda_{\mathbb{C}} \rightarrow \mathbb{C} \mid f \text{ real analytic, } \|f\|_n := \sup_{(u, \tau, \lambda) \in S \times \mathbb{T}_\sigma^d \times \Lambda_{\mathbb{C}}} \frac{|f(u, \tau, \lambda)|}{|u|^n} < \infty \right\}.$$

We consider the product spaces

$$\Omega_{n,k} = \mathcal{Z}_n \times \mathcal{Z}_{n+k-1} \quad (\text{case 1}),$$

and

$$\Omega_{n,l} = \mathcal{Z}_n \times \mathcal{Z}_{n+l-1} \quad (\text{cases 2, 3}),$$

endowed with the product norm. Also, given  $\alpha > 0$  we define

$$\Omega_{n,N}^\alpha = \{f = (f^x, f^y) \in \Omega_{n,N} \mid \|f\|_{\Omega_{n,N}} \leq \alpha\}.$$

We choose the values for  $\sigma, \sigma', \alpha$  and  $\rho$  as we described in Section 6.6 without taking into account the dependence on  $\theta$ .

Then we can define operators  $\mathcal{S}_{n,J}$  and  $\mathcal{N}_{n,X}$ , analogously as in in Section 6.6 as the left hand side and the right hand side of (6.8.8), respectively.

**Definition 6.8.6.** *Let  $n \geq 0$ ,  $\beta < \frac{\pi}{N-1}$ , and let  $J : S(\beta, \rho) \times \mathbb{T}_{\sigma'}^d \rightarrow \mathbb{C} \times \mathbb{T}_{\sigma'}^d$ , be an analytic vector field of the form*

$$J(u, t) = (Y_N u^N + O(u^{N+1}), \nu) = (J^x, \nu), \quad (6.8.9)$$

with  $Y_N < 0$ , and where the terms  $O(u^{N+1})$  do not depend on  $\tau$ .

Given  $J$ , we define  $\mathcal{S}_{n,J} : \mathcal{Z}_n \rightarrow \mathcal{Z}_n$ , as the linear operator given by

$$\mathcal{S}_{n,J} f = Df \cdot J = \partial_u f \cdot J^x + \partial_\tau f \cdot \nu.$$

**Definition 6.8.7.** *Let  $X$  be a vector field satisfying the hypotheses of Theorem 6.8.4, and let  $\check{X}(x, y, \tau) = X(x, y, t)$ , defined in the complex domain  $V \times \mathbb{T}_\sigma^d$ ,  $V \in \mathbb{C}^2$ . Given  $n \geq 3$ , we introduce  $\mathcal{N}_{n,X} = (\mathcal{N}_{n,X}^x, \mathcal{N}_{n,X}^y) : \Omega_{n,N}^\alpha \rightarrow \mathcal{Z}_{n+N-1} \times \mathcal{Z}_{n+2N-2}$ , given by*

$$\begin{aligned} \mathcal{N}_{n,X}^x(f) &= f^y \check{c}(\tau) + \check{\mathcal{E}}_n^x, \\ \mathcal{N}_{n,X}^y(f) &= P \circ (\check{K}_n + f, \tau) - P \circ (\check{K}_n, \tau) + \check{\mathcal{E}}_n^y. \end{aligned}$$

**Definition 6.8.8.** For each  $n \in \mathbb{N}$  we denote by  $\mathcal{S}_{n,J}^\times : \Omega_{n,N} \rightarrow \Omega_{n,N}$  the linear operator defined component-wise as  $\mathcal{S}_{n,J}^\times = (\mathcal{S}_{n,J}, \mathcal{S}_{n+N-1,J})$ .

With the introduced operators, equation (6.8.8) can be written as

$$\mathcal{S}_{n,J}^\times \Delta = \mathcal{N}_{n,X}(\Delta).$$

Moreover, the operators  $\mathcal{S}_{n,J}^\times$  have a bounded right inverse.

**Definition 6.8.9.** Given  $n \geq 3$ , we define  $\mathcal{T}_{n,X} : \Omega_{n,N}^\alpha \rightarrow \Omega_{n,N}$  by

$$\mathcal{T}_{n,X} = (\mathcal{S}_{n,J}^\times)^{-1} \circ \mathcal{N}_{n,X}.$$

**Lemma 6.8.10.** There exist  $m_0 > 0$  and  $\rho_0 > 0$  such that if  $\rho < \rho_0$ , then, for every  $n \geq m_0$ , we have  $\mathcal{T}_{n,X}(\Omega_{n,N}^\alpha) \subseteq \Omega_{n,N}^\alpha$  and  $\mathcal{T}_{n,X}$  is a contraction operator in  $\Omega_{n,N}^\alpha$ .

As a consequence of the Lemma 6.8.10, and provided that we computed an approximation  $\mathcal{K}_n$  up to a high enough order, we obtain that equation (6.8.8) has a unique solution,  $\Delta^\infty \in \Omega_n$ . Hence, we take  $\hat{\Delta}^\infty(u, \theta, t, \lambda) = \Delta(u, \theta, \tau, \lambda)$ , and then  $K = \mathcal{K}_n + \hat{\Delta}^\infty$  satisfies the conditions in the statement of Theorem 6.8.4. We use this same setting to prove Theorem 6.8.5.

## 6.9 Applications

In this section we present some applications of the results of this chapter. The first one is a toy-model example, while the second one is motivated by a chemistry problem.

### 6.9.1 A quasiperiodically forced oscillator

As an example of a family of systems with a parabolic nilpotent singularity at the origin of the form (6.8.1), we consider a generalization of a perturbed one-dimensional harmonic oscillator.

Consider a particle moving along a straight line under the action of a potential  $V(x)$ , with  $V(x) = cx^{2n}$ ,  $c > 0$ ,  $n \in \mathbb{N}$ . If  $n = 1$ , the system is a harmonic oscillator. The equation of motion for this particle is

$$\ddot{x} = -V'(x) = -2ncx^{2n-1},$$

where  $\ddot{x}$  is the acceleration. Denoting  $y = \dot{x}$  the velocity of the particle, the equations of motion are written

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -2ncx^{2n-1}. \end{aligned} \tag{6.9.1}$$

This system has the first integral  $H(x, y) = \frac{1}{2}y^2 + cx^{2n}$ , and hence, the phase space is foliated by periodic orbits around the origin, corresponding to the closed curves  $\frac{1}{2}y^2 + cx^{2n} = h$ , for each energy  $h > 0$ .

Next we will see that perturbing system (6.9.1) with an external force one can break the center behaviour of the system and introduce a parabolic stable invariant manifold. Assume

now that the particle moving under the action of the potential  $V(x)$  is also submitted to a force that depends quasiperiodically on the time  $t$  with frequency  $\nu \in \mathbb{R}^d$ . Then the equation of motion for this particle is given by

$$\ddot{x} = -V'(x) + F(x, \dot{x}, t),$$

where  $F$  is the one dimensional quasiperiodic applied force, that may also depend on the position  $x$  and the velocity  $\dot{x}$ . We assume also that  $F$  is an analytic function of  $x, y$  and  $t$  around  $(x, y) = (0, 0)$ , and that  $F(x, y, t) = O(\|(x, y)\|^2)$ .

Now the equations of motion read

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -2ncx^{2n-1} + F(x, y, t), \end{aligned} \tag{6.9.2}$$

and the origin of (6.9.2) is a parabolic critical point with linear part of the form

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

That is, system (6.9.2) is of the form (6.8.1). Since  $V$  and  $F$  are analytic around  $(0, 0)$ , with  $F(x, y, t) = O(\|(x, y)\|^2)$ , to study whether the origin of (6.9.2) has a stable invariant manifold one can apply Theorem 6.8.4. One can write

$$-V'(x) + F(x, y, t) = p(x, t) + yq(x, t) + u(x, y, t),$$

with  $u(x, y, t) = O(y^2)$  and

$$p(x, t) = x^k(a_k(t) + \dots + a_r(t)x^{r-k}), \quad q(x, t) = x^{l-1}(b_l(t) + \dots + b_r(t)x^{r-l}),$$

for some  $k, l \geq 2$  and  $r \geq k + 1, r \geq l + 1$ , and then classify system (6.9.2) to case 1, 2 or 3. Then, Theorem 6.8.4 provides the existence of a stable curve asymptotic to the origin,  $K(u, t) : [0, \rho) \times \mathbb{R} \rightarrow \mathbb{R}^2$ , analytic in  $[0, \rho) \times \mathbb{R}$ , provided that  $\nu$  is Diophantine and that  $\bar{a}_k > 0$  (cases 1 and 2) or  $\bar{b}_l < 0$  (case 3).

As a concrete example, we take  $n = 2$  and we assume that the particle is subject to the external force  $F(x, y, t) = dx^2g(t)$ , with  $d > 0$  and where  $g$  is a quasiperiodic function of  $t$  with  $\bar{g} > 0$  and frequency  $\nu \in \mathbb{R}^d$ ,  $\nu$  Diophantine. This system is modelled by the equations

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -4cx^3 + dx^2g(t). \end{aligned} \tag{6.9.3}$$

System (6.9.3) corresponds to case 1 of the reduced form (6.8.1) with  $a_k(t) = a_2(t) = dg(t)$ , and moreover we have  $\bar{a}_2 = d\bar{g} > 0$ .

Therefore by Theorem 6.8.4 there exists a solution of system (6.9.3) asymptotic to 0, and analytic away from 0, for any  $d > 0$ . Moreover, Proposition 6.8.1 gives an approximation of a parameterization of such curve, which is located in the lower right plane.

To look for an unstable invariant curve of system (6.9.3) we consider the vector field obtained after changing the sign of the time,  $t \mapsto -t$ , in (6.9.3). The stable curves of such system, namely

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= 4cx^3 - dx^2g(t), \end{aligned} \tag{6.9.4}$$



will be unstable curves of system (6.9.3). Performing the change of variables  $y \mapsto -y$  to system (6.9.4) we can apply again Theorem 6.8.4 to obtain the existence of an analytic stable curve of such system. Finally, by undoing the last change of variables, we conclude that for all  $d > 0$ , system (6.9.3) has a stable invariant curve in the lower right plane and an unstable invariant curve in the upper right plane, both of them analytic away from the origin.

As a consequence, the system (6.9.3) seen as a family of systems depending on  $d$  undergoes a bifurcation from a center to a stable and an unstable invariant manifolds. That means that for every  $d > 0$ , any external force applied of the form  $dx^2g(t)$ , with the conditions stated before, breaks the oscillatory behavior of the system and induces a solution that brings the particle to the origin.

## 6.9.2 Scattering of He atoms off Cu corrugated surfaces

In [35], the authors study the phase-space structure of a differential system modelling the scattering of helium atoms off copper corrugated surfaces. Concretely, elastic collisions of  ${}^4\text{He}$  atoms with corrugated Cu surfaces are considered, in particular those made of Cu(110) and Cu(117). The system, which can be adequately treated at the classical level, can be modeled by the following two degrees of freedom Hamiltonian describing the motion of a  ${}^4\text{He}$  atom,

$$H(x, z, p_x, p_z) = \frac{p_x^2 + p_z^2}{2m} + V(x, z), \quad (6.9.5)$$

where  $x$  is the coordinate parallel to the copper surface and  $z$  is the coordinate perpendicular to it,  $p_x$  and  $p_z$  are the respective momenta, and  $m$  is the mass of the atom. The potential energy  $V(x, z)$  is given by

$$V(x, z) = V_M(z) + V_C(x, z),$$

where  $V_M(z) = D(1 - e^{-\alpha z})^2$  is the Morse potential and  $V_C(x, z) = De^{-2\alpha z}g(x)$  is the coupling potential, with  $D = 6.35$  meV,  $\alpha = 1.05 \text{ \AA}^{-1}$ , and  $g(x)$  being a periodic function. Thus the variable  $x$  can be thought as an angle. For more information on the coefficients of the Morse and coupling potentials, see Table 1 of [35].

The equations of motion derived from the Hamiltonian function (6.9.5) are

$$\dot{x} = \frac{p_x}{m}, \quad \dot{z} = \frac{p_z}{m}, \quad \dot{p}_x = -De^{-2\alpha z}g'(x), \quad \dot{p}_z = -2D\alpha e^{-\alpha z} + 2D\alpha e^{-2\alpha z}(1 + g(x)). \quad (6.9.6)$$

We will use the results presented along this chapter to show that system (6.9.6) has a parabolic periodic orbit at infinity (concretely for  $z \rightarrow \infty$ ) and that such periodic orbit has stable and unstable invariant manifolds. This means that for certain initial conditions the helium atom leaves the copper surface and moves away spiraling asymptotically to a periodic orbit, and vice-versa, for certain initial conditions with position close to infinity, the atom leaves the periodic orbit and goes down to the surface.

Since (6.9.6) is a Hamiltonian system, the energy  $H$  is conserved, and thus each solution of the system is contained in a level set  $H(x, z, p_x, p_z) = h$ . Therefore we can reduce system (6.9.6) to a three equations system restricting it to an energy level,  $H(x, z, p_x, p_z) = h$ , and

removing the equation for  $\dot{p}_x$ . The obtained system reads

$$\begin{aligned}\dot{x} &= \frac{1}{m}(2m(h - D(1 - e^{-\alpha z})^2 - De^{-2\alpha z}g(x)) - p_z^2)^{1/2}, \\ \dot{z} &= \frac{p_z}{m}, \\ \dot{p}_z &= -2D\alpha e^{-\alpha z} + 2D\alpha e^{-2\alpha z}(1 + g(x)).\end{aligned}$$

Next, to study the motion at  $z \rightarrow \infty$  we perform the change of variables given by  $y = -e^{-\alpha z}$ . Now the set  $y = 0$  corresponds to infinity distance with respect to the copper surface. To adapt the notation to the one of rest of the chapter we write  $\theta := x$  and  $p := p_z$ . The obtained system reads

$$\begin{aligned}\dot{p} &= 2D\alpha y + 2D\alpha y^2(1 + g(\theta)), \\ \dot{y} &= -\frac{\alpha}{m}py, \\ \dot{\theta} &= \frac{1}{m}(2m(h - D(1 - y)^2 - Dy^2g(\theta)) - p^2)^{1/2},\end{aligned}\tag{6.9.7}$$

with  $y \leq 0$ ,  $p \in \mathbb{R}$  and  $\theta \in \mathbb{T} = \mathbb{S}^1$ .

Therefore system (6.9.7) models the dynamics of a  ${}^4\text{He}$  atom off a Cu corrugated surface for a fixed energy value  $h$ . The set  $p = y = 0$  is invariant, and corresponds to a periodic orbit at the infinity of system (6.9.6). For system (6.9.7) we have the following result.

**Theorem 6.9.1.** *Let  $X$  be the vector field associated to system (6.9.7), and assume that  $h > D$ . Then, the set  $\gamma = \{p = y = 0\}$  is a periodic orbit and it has stable and unstable invariant manifolds. Concretely, there exist  $\rho > 0$  and two  $C^1$  maps,  $K^-, K^+ : [0, \rho) \times \mathbb{T} \rightarrow \mathbb{R}^2 \times \mathbb{T}$ , analytic on  $(0, \rho) \times \mathbb{T}$ , and two analytic vector fields,  $Y^-, Y^+ : [0, \rho) \times \mathbb{T} \rightarrow \mathbb{R} \times \mathbb{T}$  of the form*

$$K^-(u, \theta) = \begin{pmatrix} u + O(u^2) \\ K_2^y u^2 + O(u^3) \\ K_1^\theta u + O(u^2) \end{pmatrix}, \quad Y^-(u, \theta) = \begin{pmatrix} -Y_2 u^2 + Y_3 u^3 \\ \omega \end{pmatrix},\tag{6.9.8}$$

corresponding to the stable manifold, and

$$K^+(u, \theta) = \begin{pmatrix} -u + O(u^2) \\ K_2^y u^2 + O(u^3) \\ K_1^\theta u + O(u^2) \end{pmatrix}, \quad Y^+(u, \theta) = \begin{pmatrix} Y_2 u^2 + \hat{Y}_3 u^3 \\ \omega \end{pmatrix},\tag{6.9.9}$$

corresponding to the unstable manifold, with

$$K_2^y = -\frac{1}{4mD}, \quad K_1^\theta = -\frac{1}{\alpha\sqrt{2m(h-D)}}, \quad Y_2 = \frac{\alpha}{2m}, \quad \omega = \sqrt{\frac{2(h-D)}{m}},\tag{6.9.10}$$

such that

$$X \circ K^-(u, \theta) = DK^- \cdot Y^-(u, \theta) \quad \text{and} \quad X \circ K^+(u, \theta) = DK^+ \cdot Y^+(u, \theta), \quad (u, \theta) \in [0, \rho) \times \mathbb{T}.$$

*Proof.* We perform the following analytic change of variables to system (6.9.7),

$$\tilde{p} = p, \quad \tilde{y} = y + (1 + g(\theta))y^2, \quad \tilde{\theta} = \theta, \quad (6.9.11)$$

and we develop the right hand side of the third equation Taylor series around  $(p, y) = (0, 0)$ , so that the new system reads, writing the new variables without tilde,

$$\dot{p} = 2D\alpha y, \quad \dot{y} = -\frac{\alpha}{m}py + O(y^2), \quad \dot{\theta} = \omega - d_1y + O(\|(p, y)\|^2), \quad (6.9.12)$$

whith  $d_1 = \frac{D}{\sqrt{2m(h-D)}}$ .

It is clear that system (6.9.12) has a periodic orbit,  $\gamma$ , at  $p = y = 0$  parameterized by  $\gamma(t) = (0, 0, \omega t)$ . Moreover, such system satisfies the hypotheses of Theorem 6.3.3 with  $d' = 0$ ,  $c(\theta, \lambda) = 2D\alpha$ ,  $b(\theta, \lambda) = -\frac{\alpha}{m}$ , and  $d(\theta, \lambda) = d_1$ .

Then, the stated results are a direct consequence of Theorem 6.3.3, which provides the existence of an analytic stable invariant manifold,  $\tilde{K}^-$ , of system (6.9.12). An analogous argument to the one of the proof of Proposition 6.4.1 provides an approximation of a parameterization of  $\tilde{K}^-$  and  $\tilde{Y}^-$ . In particular one obtains the expressions given in (6.9.8) and (6.9.10).

By undoing the change of variables (6.9.11) we recover the original parameterizations,  $K^-$  and  $Y^-$ , of the stable manifold of  $\gamma$  and the restricted dynamics on it, whose first coefficients coincide with the ones in (6.9.10).

The existence of the unstable manifold is obtained simply through the study of the system obtained after applying the change  $t \mapsto -t$  to (6.9.12). Indeed, performing such change of variables and  $p \mapsto -p$ , we obtain

$$\dot{p} = 2D\alpha y, \quad \dot{y} = -\frac{\alpha}{m}py + O(y^2), \quad \dot{\theta} = -\omega + d_1y + O(\|(p, y)\|^2). \quad (6.9.13)$$

Then we can apply again Theorem 6.3.3 to system (6.9.13), which provides an analytic stable invariant manifold,  $\tilde{K}^+$ , asymptotic to  $\gamma = \{p = y = 0\}$ , and an expression for the restricted dynamics,  $\tilde{Y}^+$ , parameterized by

$$\tilde{K}^+(u, \theta) = \begin{pmatrix} u + O(u^2) \\ K_2^y u^2 + O(u^3) \\ -K_1^\theta u + O(u^2) \end{pmatrix}, \quad \tilde{Y}^+(u, \theta) = \begin{pmatrix} -Y_2 u^2 + \tilde{Y}_3 u^3 \\ -\omega \end{pmatrix}.$$

Finally, by undoing the changes  $t \mapsto -t$  and  $p \mapsto -p$  we recover the parameterizations of the unstable manifold of  $\gamma$  and the restricted dynamics on it, namely  $K^+$  and  $Y^+$ , given in (6.9.9).  $\square$

## Part II

### Periodic orbits of maps and vector fields on manifolds

# Chapter 7

## Introduction

This second part of the thesis is devoted to the study of periodic orbits, both for discrete and continuous dynamical systems.

Let  $U \subset \mathbb{R}^n$  be a domain and let  $f : U \rightarrow \mathbb{R}^n$  be a continuous map. We denote by  $f^m$  the  $m$ -th iterate of  $f$ . That is,  $f^m$  is the  $m$ -fold composition of  $f$  with itself. A point  $x \in U$  such that  $f(x) = x$  is called a *fixed point*, or a *periodic point of period 1* of  $f$ . A point  $x \in U$  is called *periodic of period  $k > 1$*  if  $f^k(x) = x$  and  $f^m(x) \neq x$  for all  $m = 1, \dots, k - 1$ , and the set formed by the iterates of  $x$ , i.e.  $\{x, f(x), \dots, f^{k-1}(x)\}$ , is called the *periodic orbit* of the periodic point  $x$ .

Given a map  $f : U \rightarrow \mathbb{R}^n$ , the *set of periods of  $f$*  is the set of natural numbers such that  $f$  has a periodic orbit with its period that natural number.

Understanding the periodic orbits and the set of periods of a map is an important problem in dynamical systems. Concretely, knowing the set of periods of a map  $f$ , or even a subset of it, allows us to determine the existence of periodic orbits for  $f$  for a given period, and in particular the existence of fixed points.

The Lefschetz numbers are one of the most useful tools to study the existence of fixed points and periodic orbits of self-maps on compact manifolds, and it is based on understanding the connection between the dynamics of the map and its action induced on the homology groups of the manifold. In Chapter 8, we use this tool to obtain information on the set of periods of certain diffeomorphisms on compact manifolds using the Lefschetz zeta function, which is a generating function of the Lefschetz numbers of the iterates of a map.

Concretely, we study a class of maps called Morse–Smale diffeomorphisms. Morse–Smale diffeomorphisms have a big dynamical interest because they form a structurally stable family among the class of all diffeomorphisms. This means that given a Morse–Smale diffeomorphism  $f$ , there is a neighborhood around  $f$  where all diffeomorphisms are topologically equivalent to  $f$ .

We consider the class of Morse–Smale diffeomorphisms defined on the  $n$ -dimensional sphere  $\mathbb{S}^n$ , on products of two spheres of arbitrary dimension  $\mathbb{S}^m \times \mathbb{S}^n$  with  $m \neq n$ , on the  $n$ -dimensional complex projective space  $\mathbb{C}\mathbb{P}^n$ , and on the  $n$ -dimensional quaternion projective space  $\mathbb{H}\mathbb{P}^n$ , the latter ones taken as real manifolds of dimension  $2n$  and  $4n$ , respectively. Then, we describe the minimal sets of Lefschetz periods for such Morse–Smale diffeomorphisms,

which is a subset of the set of periods that are preserved under homotopy equivalence.

In the last quarter of the 20th century there appeared some papers dedicated to understanding the connections between the dynamics of the Morse–Smale diffeomorphisms and the topology of the manifold where they are defined. Without trying to be exhaustive, see for instance [28, 57, 65, 66, 68]. This interest continues during this first part of the 21st century, see for example [9, 11, 33, 34, 51].

The set of periods for Morse–Smale diffeomorphisms on a product of any number of spheres of the same dimension has been studied in [9]. For the particular case of  $\mathbb{S}^n$ , we give more detailed results considering the orientation of the diffeomorphisms. The set of periods of Morse–Smale diffeomorphisms on the two-dimensional sphere has been studied with more details in [8, 38].

In [32] the authors study the Lefschetz periodic point free self-maps on  $\mathbb{S}^m \times \mathbb{S}^n$ ,  $\mathbb{C}\mathbf{P}^n$  and  $\mathbb{H}\mathbf{P}^n$ . Our results give extended information in the same direction for the Morse–Smale diffeomorphisms.

Contrary to Chapter 8, in Chapter 9 we focus on continuous dynamical systems, namely, dynamical systems defined by vector fields. In that chapter, the main goal is to show that linear vector fields defined in manifolds different from  $\mathbb{R}^n$  can exhibit limit cycles.

Given a differential system  $\dot{x} = X(x)$ , with  $x \in \mathbb{R}^n$  and  $X : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ , we say that a solution of the system,  $x(t)$ , is *T-periodic* if there exists  $T > 0$  such that  $x(t + T) = x(t)$ , and  $x(t + s) \neq x(t)$  for all  $s \in (0, T)$ .

A *limit cycle* is defined as a periodic orbit of a differential system that is isolated in the set of all periodic orbits of the system.

The study of periodic orbits and limit cycles of differential systems play an important role in the qualitative theory of ordinary differential equations. There are many works concerning the study of limit cycles and their applications (see for instance [19, 41, 44] and the references quoted therein).

It is well known that linear vector fields in  $\mathbb{R}^n$  can not have limit cycles, because either they do not have periodic orbits or their periodic orbits form a continuum. But this is not the case if one considers linear vector fields defined in other manifolds different from  $\mathbb{R}^n$ .

The objective of Chapter 9 is to study the existence of limit cycles of linear vector fields on the manifolds  $(\mathbb{S}^2)^m \times \mathbb{R}^n$ . In that chapter we show that such linear differential systems can have limit cycles and we consider the question of how many limit cycles can they have at most.

The problem of studying limit cycles of linear vector fields on manifolds different from  $\mathbb{R}^n$  was already treated in [53], where the authors consider linear vector fields on  $(\mathbb{S}^1)^m \times \mathbb{R}^n$ , and they conjecture that such vector fields may have at most one limit cycle.

Linear autonomous differential systems, namely, systems of the form  $\dot{x} = Ax + b$ , where  $A$  is a  $n \times n$  real matrix and  $b$  is a vector in  $\mathbb{R}^n$ , are the easiest systems to study because their solutions can be completely determined (see [1, 69]), but still they play an important role in the theory of differential systems. Also, it is well known that when a nonlinear differential system has a hyperbolic equilibrium point, the dynamics around that point is determined by the linearization of the vector field at that point (Hartman–Grossman theorem, see [43]).

Moreover, linear differential systems of the form  $\dot{x} = Ax + Bu$ , where  $x$  are the state variables and  $u$  is the control input, are applied in control theory for the modeling of hybrid systems (see [47, 46]).

Given a family of differential systems,  $\dot{x} = X_a(x) = X(a, x)$ , where  $a$  is a real parameter, we say that the family undergoes a *bifurcation* at  $a = a_0$  if for any neighborhood  $V$  of  $a_0$ , there is a value  $a_1 \in V$  such that  $X_{a_0}$  and  $X_{a_1}$  are not topologically conjugate, that is, they exhibit a big difference in the behavior of their solutions. An example where this kind of behavior appears is to consider a differential system having a continuum of periodic orbits, namely a *center*, that is perturbed and so it bifurcates to a system with limit cycles.

In the same direction, linear vector fields having invariant subspaces of periodic orbits can be perturbed inside a concrete class of nonlinear differential systems to obtain limit cycles of these nonlinear systems bifurcating from the periodic orbits of the linear system (see for instance [49, 52]).

## Chapter 8

# Periods of Morse–Smale diffeomorphisms on manifolds

### 8.1 Introduction

In this chapter we study the set of periods of the Morse–Smale diffeomorphisms on the  $n$ -dimensional sphere  $\mathbb{S}^n$ , on products of two spheres of arbitrary dimension  $\mathbb{S}^m \times \mathbb{S}^n$ , with  $m \neq n$ , on the  $n$ -dimensional complex projective space  $\mathbb{C}\mathbf{P}^n$  and on the  $n$ -dimensional quaternion projective space  $\mathbb{H}\mathbf{P}^n$ . More precisely, our goal is to describe the minimal set of Lefschetz periods,  $\text{MPer}_L(f)$  (see Definition 8.3.1), that those diffeomorphisms can exhibit for arbitrary values of  $n$  and  $m$ .

Along this chapter, let  $M$  denote a compact,  $C^1$ , Riemannian manifold, namely, a  $C^1$  compact manifold endowed with a metric  $d : M \times M \rightarrow \mathbb{R}$ .

As usual  $\mathbb{N}$  denotes the set of all positive integers. Given a continuous map  $f : M \rightarrow M$ , we define the *set of periods of  $f$*  as the set

$$\text{Per}(f) := \{k \in \mathbb{N} : f \text{ has a periodic orbit of period } k\}.$$

In order to define what a Morse–Smale diffeomorphism is, we introduce some more definitions.

A fixed point  $x$  of a  $C^1$  map  $f : M \rightarrow M$  is called *hyperbolic* if all the eigenvalues of  $Df(x)$  have modulus different than one. A periodic point  $x$  of  $f$  of period  $k$  is called a *hyperbolic periodic point* if it is a hyperbolic fixed point of  $f^k$ .

We denote by  $\text{Diff}(M)$  the space of all  $C^1$  diffeomorphisms on  $M$ . We say that two maps  $f, g \in \text{Diff}(M)$  are *topologically equivalent* if there exists a homeomorphism  $h : M \rightarrow M$  such that  $h \circ f = g \circ h$ . A map  $f \in \text{Diff}(M)$  is called *structurally stable* if there exists a neighbourhood  $U \subset \text{Diff}(M)$  of  $f$  such that every  $g \in U$  is topologically equivalent to  $f$ .

We say that  $x \in M$  is a *nonwandering point* of  $f$  if for any neighborhood  $U$  of  $x$  there is a positive integer  $m$  such that  $f^m(U) \cap U \neq \emptyset$ . We denote by  $\Omega(f)$  the set of nonwandering points of  $f$ . Clearly if  $\gamma$  is a periodic orbit of  $f$ , then  $\gamma \subseteq \Omega(f)$ .

Let  $d$  be the metric on  $M$  and suppose that  $x \in M$  is a hyperbolic fixed point of  $f$ . We define



the *stable manifold* of  $x$  as the set

$$W^s(x) = \{y \in M : d(x, f^m(y)) \rightarrow 0 \text{ as } m \rightarrow \infty\},$$

and the *unstable manifold* of  $x$  as the set

$$W^u(x) = \{y \in M : d(x, f^{-m}(y)) \rightarrow 0 \text{ as } m \rightarrow \infty\}.$$

In the same way we define the stable and unstable manifolds of a hyperbolic periodic point  $x \in M$  of period  $k$  as the stable and unstable manifolds of the hyperbolic fixed point  $x$  under  $f^k$ , respectively. We say that the submanifolds  $W^s(x)$  and  $W^u(x)$  have a *transversal intersection* if at every point of intersection, their separate tangent spaces at that point together generate the tangent space of the ambient manifold  $M$  at that point.

**Definition 8.1.1.** *A map  $f \in \text{Diff}(M)$  is Morse–Smale if*

- (1)  $\Omega(f)$  is finite,
- (2) all the periodic points of  $f$  are hyperbolic, and
- (3) for every  $x, y \in \Omega(f)$ ,  $W^s(x)$  and  $W^u(y)$  have a transversal intersection.

Clearly, condition (1) implies that  $\Omega(f)$  is the set of all periodic points of  $f$ .

The dynamical importance of the Morse–Smale diffeomorphisms relies on the fact that they are structurally stable inside the class of all diffeomorphisms. That is, the set of Morse–Smale diffeomorphisms defined on a manifold  $M$  is structurally stable inside the set  $\text{Diff}(M)$  (see for details [29, 59, 61, 68]).

The main results of this chapter are Theorems 8.4.1, 8.5.1 and 8.6.1, where we characterize the possible sets for  $\text{MPer}_L(f)$  depending on the action of  $f$  on the homology and on the parity of the numbers  $n$  and  $m$ . Those results are stated in Sections 8.4 – 8.6. The main tools that we use are the Lefschetz zeta function and the fact that Morse–Smale diffeomorphisms are quasi-unipotent on homology. In Sections 8.2 – 8.3 we introduce the rest of definitions and basic theory that will be used to prove the main results.

Related with our results the reader can look at the set of periods for homeomorphisms (respectively continuous maps) on  $\mathbb{S}^n$  and on  $\mathbb{S}^m \times \mathbb{S}^n$  which have been studied in [39] (respectively [40]).

We also remark that the results obtained along this chapter hold in any compact manifold with the same homology as the manifolds considered here. More precisely, they hold for any manifold homotopy equivalent to  $\mathbb{S}^n$ ,  $\mathbb{S}^m \times \mathbb{S}^n$ ,  $\mathbb{C}\mathbb{P}^n$  and  $\mathbb{H}\mathbb{P}^n$ , respectively.

## 8.2 Lefschetz numbers and the Lefschetz zeta function

Let  $f : M \rightarrow M$  be a continuous map on a compact manifold of dimension  $n$ . We denote by  $H_0(M, \mathbb{Q}), \dots, H_n(M, \mathbb{Q})$  the homology groups of  $M$  with rational coefficients. A continuous map  $f : M \rightarrow M$  induces  $n + 1$  morphisms on the homology groups of  $M$ ,  $f_{*i} : H_i(M, \mathbb{Q}) \rightarrow H_i(M, \mathbb{Q})$ ,  $i \in \{0, \dots, n\}$ . For more details, see [73].

A map  $f \in \text{Diff}(M)$  is said to be *quasi unipotent* on homology if all the eigenvalues of the nontrivial induced maps on the homology groups of  $M$  with rational coefficients, are roots of unity. We denote by  $\mathcal{F}(M)$  the set of elements of  $\text{Diff}(M)$  being quasi unipotent on homology and having finitely many periodic points, all of them hyperbolic. Among the maps in  $\mathcal{F}(M)$  there are the Morse–Smale diffeomorphisms (see [29, 65]). Therefore Morse–Smale diffeomorphisms are quasi-unipotent on homology.

In fact, even if we will refer to Morse–Smale diffeomorphisms along the chapter because of their dynamical interest, all the results hold for any map in the class  $\mathcal{F}(M)$ .

**Definition 8.2.1.** *Let  $M$  be a compact manifold and let  $f : M \rightarrow M$  be a continuous map. The Lefschetz number of  $f$  is defined as*

$$L(f) = \sum_{i=0}^n (-1)^i \text{tr}(f_{*i}),$$

where  $\text{tr}(f_{*i})$  denotes the trace of  $f_{*i}$ .

A very important result which relates the Lefschetz number of a map  $f$  and the existence of fixed points of  $f$  is the Lefschetz Fixed Point Theorem, which says that if  $L(f) \neq 0$ , then  $f$  has a fixed point. For a proof, see [12].

In order to obtain information about the set of periods of a map  $f$  it is useful to have information of the sequence  $\{L(f^m)\}_{m=0}^{\infty}$  of the Lefschetz numbers of all the iterates of  $f$ .

**Definition 8.2.2.** *Let  $f : M \rightarrow M$  be a continuous map on a compact manifold. The Lefschetz zeta function of  $f$  is defined as*

$$\mathcal{Z}_f(t) = \exp \left( \sum_{m=1}^{\infty} \frac{L(f^m)}{m} t^m \right).$$

This function generates the whole sequence of Lefschetz numbers of  $f$ , and it may be independently computed through

$$\mathcal{Z}_f(t) = \prod_{k=0}^n \det (I_{n_k} - t f_{*k})^{(-1)^{k+1}}, \quad (8.2.1)$$

(see [27]) where  $n$  is the dimension of  $M$ ,  $n_k$  is the dimension of  $H_k(M, \mathbb{Q})$ ,  $I_{n_k}$  is the  $n_k \times n_k$  identity matrix, and we take  $\det (I_{n_k} - t f_{*k}) = 1$  if  $n_k = 0$ . Note that the expression given in (8.2.1) is a rational function of  $t$ .

For a  $C^1$  map  $f$  having a finite number of periodic points, all of them being hyperbolic, we give another characterization of the Lefschetz zeta function introduced by Franks in [27].

Let  $f : M \rightarrow M$  be a  $C^1$  map on a compact manifold without boundary, and let  $\gamma$  be a hyperbolic periodic orbit of  $f$  of period  $p$ . For each  $x \in \gamma$ , let  $E_x^u$  denote the linear subspace of the tangent space  $T_x M$  of  $M$  at  $x$ , generated by the eigenvectors of  $Df^p$  corresponding to the eigenvalues whose moduli are greater than 1, and let  $E_x^s$  denote the linear subspace of  $T_x M$  generated by the remaining eigenvectors. We denote by  $u$  and  $s$  the dimensions of the spaces  $E_x^u$  and  $E_x^s$ , respectively.

We define the *orientation type*  $\Delta$  of  $\gamma$  to be  $+1$  if  $Df^p(x) : E_x^u \rightarrow E_x^u$  preserves orientation, that is, if  $\det Df^p(x) > 0$  with  $x \in \gamma$ , and to be  $-1$  if it reverses orientation, that is, if

$\det Df^p(x) < 0$ . Note that the definitions of  $\Delta$  and  $u$  do not depend on the periodic point  $x$ , but only on the periodic orbit  $\gamma$ .

For a  $C^1$  map  $f : M \rightarrow M$  having only finitely many periodic orbits, all of them hyperbolic, we define the *periodic data*,  $\Phi$ , as the collection composed by all triples  $(p, u, \Delta)$  corresponding to all the hyperbolic periodic orbits of  $f$ , where a same triple can occur more than once provided that it corresponds to different periodic orbits. The following result was proved by Franks in [27].

**Theorem 8.2.3.** *Let  $f : M \rightarrow M$  be a  $C^1$  map on a compact manifold without boundary, having finitely many periodic points, all hyperbolic, and let  $\Phi$  be the periodic data of  $f$ . Then the Lefschetz zeta function of  $f$  satisfies*

$$\mathcal{Z}_f(t) = \prod_{(p, u, \Delta) \in \Phi} (1 - \Delta t^p)^{(-1)^{u+1}}.$$

Note that the Morse–Smale diffeomorphisms satisfy the hypotheses of Theorem 8.2.3. Then that theorem will be useful to obtain information on the set of periods for Morse–Smale diffeomorphisms from the comparison of the expressions of the Lefschetz zeta functions in Theorem 8.2.3 and in (8.2.1).

### 8.3 Minimal set of Lefschetz periods of Morse–Smale diffeomorphisms

**Definition 8.3.1.** *Let  $f$  be a map satisfying the hypotheses of Theorem 8.2.3. The minimal set of Lefschetz periods of  $f$ ,  $MPer_L(f)$ , is the set given by the intersection of all sets of periods forced by the different representations of  $\mathcal{Z}_f(t)$  as products of the form  $(1 \pm t^p)^{\pm 1}$ .*

As an example, consider the following Lefschetz zeta function of a Morse–Smale diffeomorphism  $f$  on the four-dimensional torus  $\mathbb{T}^4$ ,

$$\begin{aligned} \mathcal{Z}_f(t) &= \frac{(1-t^3)^2(1+t^3)}{(1-t)^6(1+t)^3} = \frac{(1-t^3)(1-t^6)}{(1-t)^6(1+t)^3} = \frac{(1-t^3)(1-t^6)}{(1-t)^3(1-t^2)^3} \\ &= \frac{(1-t^3)^2(1+t^3)}{(1-t)^3(1-t^2)^3}, \end{aligned} \tag{8.3.1}$$

which can be expressed in four ways as a product of factors of the form  $(1 \pm t^p)^{\pm 1}$  as a quotient of polynomials of degree 9.

By Theorem 8.2.3, the first expression of the Lefschetz zeta function (8.3.1) ensures the existence of periodic orbits of periods 1 and 3 for  $f$ . In the same way, the second expression of  $\mathcal{Z}_f(t)$  provides the periods  $\{1, 3, 6\}$  for  $f$ , the third expression of  $\mathcal{Z}_f(t)$  provides the periods  $\{1, 2, 3, 6\}$ , and finally the fourth expression provides the periods  $\{1, 2, 3\}$ . In this case we have that the minimal set of Lefschetz periods of  $f$  is

$$MPer_L(f) = \{1, 3\} \cap \{1, 3, 6\} \cap \{1, 2, 3, 6\} \cap \{1, 2, 3\} = \{1, 3\}.$$

Note that if  $\mathcal{Z}_f(t)$  is constant equal to 1, then  $MPer_L(f) = \emptyset$ .

**Remark 8.3.2.** Even if the minimal set of Lefschetz periods of a Morse–Smale diffeomorphism is empty, one can still obtain some information on the set of periods from Theorem 8.2.3. For example, suppose that the Lefschetz zeta function of a map  $f$  satisfying the hypotheses of Theorem 8.2.3 is  $\mathcal{Z}_f(t) = 1 + t^2$ . It can be expressed as products of terms of the form  $(1 \pm t^p)^{\pm 1}$  in infinitely many ways,

$$1 + t^2 = \frac{1 - t^4}{1 - t^2} = \frac{1 - t^4}{(1 + t)(1 - t)} = \frac{1 - t^8}{(1 + t)(1 - t)(1 + t^4)},$$

and so on. In this case we have clearly  $\text{MPer}_L(f) = \emptyset$ , but each of the infinitely many expressions of  $\mathcal{Z}_f(t)$  forces either the period 2 or the periods  $\{2, 4\}$ , or the periods  $\{1, 4\}$ . Then,  $f$  has either a periodic orbit of period 2, or periodic orbits of periods 2 and 4, or periodic orbits of periods 1 and 4.

**Remark 8.3.3.** By Theorem 8.2.3 an even period  $n$ , can never be contained in the set  $\text{MPer}_L(f)$ . Indeed, the following expressions

$$1 - t^n = (1 + t^{n/2})(1 - t^{n/2}), \quad 1 + t^n = \frac{1 - t^{2n}}{1 - t^n} = \frac{1 - t^{2n}}{(1 + t^{n/2})(1 - t^{n/2})},$$

show that if the term  $1 - t^n$  or  $1 + t^n$ , with  $n$  even, appears in one of the expressions of  $\mathcal{Z}_f(t)$ , one can always obtain a new expression of  $\mathcal{Z}_f(t)$  where the period  $n$  does not appear.

**Remark 8.3.4.** Along the chapter, for every possible Lefschetz zeta function of a given map  $f$ , in general we will write only one of the possible equivalent expressions of  $\mathcal{Z}_f(t)$ . We will provide the expression of  $\mathcal{Z}_f(t)$  that forces a smaller set of periods, and consequently it will be sufficient to describe the set  $\text{MPer}_L(f)$ . From remark 8.3.3 we note that providing an expression of  $\mathcal{Z}_f(f)$  that forces only even periods is sufficient to ensure that  $\text{MPer}_L(f) = \emptyset$ .

Finally note that  $\text{MPer}_L(f)$  is contained in the set of periods that are conserved under homotopy. Indeed, for a Morse–Smale diffeomorphism  $f : M \rightarrow M$  on a compact manifold consider the set  $\text{MPer}_{ms}(f) := \bigcap_{h \sim f} \text{Per}(h)$ , where  $h$  runs over all the Morse–Smale diffeomorphisms of  $M$  which are homotopic to  $f$ . Then it is clear that

$$\text{MPer}_L(f) \subseteq \text{MPer}_{ms}(f),$$

because two homotopic maps on a manifold  $M$  induce the same Lefschetz zeta functions.

## 8.4 Periods of Morse–Smale diffeomorphisms on $\mathbb{S}^n$

Let  $n \in \mathbb{N}$  and let  $\mathbb{S}^n$  be the  $n$ -dimensional sphere. The homology groups of  $\mathbb{S}^n$  over  $\mathbb{Q}$  are

$$H_k(\mathbb{S}^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k \in \{0, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

For a continuous map  $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$  the nontrivial induced maps on the homology can be written as the integer matrices  $f_{*0} = (1)$  and  $f_{*n} = (d)$ , where  $d$  is called the *degree* of  $f$ .

Let  $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$  be a Morse–Smale diffeomorphism. As we already mentioned before, the linear maps induced on the homology are quasi unipotent, which means that all their eigenvalues are roots of unity. Then we must have either  $d = 1$ , or  $d = -1$ . Also, the orientation of  $f$  is constant on  $\mathbb{S}^n$  and is determined by the sign of  $d$ .

**Theorem 8.4.1.** *Let  $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$  be a Morse–Smale diffeomorphism. Then,*

- (a) *If  $n$  is even and  $f$  preserves the orientation, then  $MPer_L(f) = \{1\}$ .*
- (b) *If  $n$  is even and  $f$  reverses the orientation, then  $MPer_L(f) = \emptyset$  but  $\{1, 2\} \cap Per(f) \neq \emptyset$ .*
- (c) *If  $n$  is odd and  $f$  preserves the orientation, then  $MPer_L(f) = \emptyset$ .*
- (d) *If  $n$  is odd and  $f$  reverses the orientation, then  $MPer_L(f) = \{1\}$ .*

*Proof.* Computing the Lefschetz zeta function for  $f$  using equation (8.2.1), we get

$$\mathcal{Z}_f(t) = \prod_{k=0}^n \det(I_{n_k} - t f_{*k})^{(-1)^{k+1}} = (1-t)^{-1} (1-td)^{(-1)^{n+1}},$$

and so for  $n$  even we have

$$\mathcal{Z}_f(t) = \frac{1}{(1-t)^2} \text{ if } d = 1, \quad \mathcal{Z}_f(t) = \frac{1}{(1-t)(1+t)} = \frac{1}{1-t^2} \text{ if } d = -1,$$

and for  $n$  odd we have

$$\mathcal{Z}_f(t) \equiv 1 \text{ if } d = 1, \quad \mathcal{Z}_f(t) = \frac{1+t}{1-t} \text{ if } d = -1.$$

It is clear that  $f$  satisfies the hypotheses of Theorem 8.2.3. Then the statements follow directly applying Theorem 8.2.3 to each of the expressions obtained for the Lefschetz zeta function  $\mathcal{Z}_f(t)$ .

Indeed, for  $1/(1-t)^2$  and  $(1+t)/(1-t)$  we have  $MPer_L(f) = \{1\}$ , because any other expression of the same Lefschetz zeta function as a product of terms the form  $(1 \pm t^p)^{\pm 1}$  would provide at least the period 1.

For the function  $1/(1-t^2) = 1/((1-t)(1+t))$  it is clear that  $MPer_L(f) = \emptyset$ , but by Remarks 8.3.2 and 8.3.3, we can ensure that  $\{1, 2\} \cap Per(f) \neq \emptyset$ . Finally we have  $MPer_L(f) = \emptyset$  when  $\mathcal{Z}_f(t) \equiv 1$ .  $\square$

## 8.5 Periods of Morse–Smale diffeomorphisms on $\mathbb{S}^m \times \mathbb{S}^n$

Let  $m, n \in \mathbb{N}$ ,  $m \neq n$ , and consider the product of spheres  $\mathbb{S}^m \times \mathbb{S}^n$ , with  $m \neq n$ . Applying Künneth’s formula, the homology groups over  $\mathbb{Q}$  of  $\mathbb{S}^m \times \mathbb{S}^n$  can be easily computed and are given by

$$H_k(\mathbb{S}^m \times \mathbb{S}^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k \in \{0, m, n, m+n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then for a continuous map  $f : \mathbb{S}^m \times \mathbb{S}^n \rightarrow \mathbb{S}^m \times \mathbb{S}^n$  the linear maps induced on the homology are given as the integer matrices  $f_{*0} = (1)$ ,  $f_{*m} = (a)$ ,  $f_{*n} = (b)$ ,  $f_{*m+n} = (d)$ , where  $d$  is the degree of  $f$ . The rest of the induced maps are the zero map.

Let  $f : \mathbb{S}^m \times \mathbb{S}^n \rightarrow \mathbb{S}^m \times \mathbb{S}^n$  be a Morse–Smale diffeomorphism. Since the linear maps induced by  $f$  on the homology are quasi unipotent, in this case we have  $a, b, d \in \{-1, 1\}$ . Moreover,

by the structure of the cohomology ring of  $\mathbb{S}^m \times \mathbb{S}^n$ , one has always that  $ab = d$  (see [32]) and hence the possibilities for these values are restricted to  $a = b = d = 1$ ,  $\{a, b\} = \{-1, 1\}$  and  $d = -1$ , and  $a = b = -1$ ,  $d = 1$ .

**Theorem 8.5.1.** *Let  $M = \mathbb{S}^m \times \mathbb{S}^n$  be with  $m \neq n$ , and let  $f : M \rightarrow M$  be a Morse–Smale diffeomorphism.*

(i) *If  $m$  and  $n$  are even, then*

(a) *If  $a = b = 1$ , then  $MPer_L(f) = \{1\}$ .*

(b) *Otherwise,  $MPer_L(f) = \emptyset$  but  $\{1, 2\} \cap Per(f) \neq \emptyset$ .*

(ii) *If  $m$  and  $n$  are odd, then*

(a) *If  $a = b = -1$ , then  $MPer_L(f) = \{1\}$ .*

(b) *Otherwise,  $MPer_L(f) = \emptyset$ .*

(iii) *If  $m$  is even and  $n$  is odd, then*

(a) *If  $b = -1$  and  $a = 1$ , then  $MPer_L(f) = \{1\}$ .*

(b) *Otherwise,  $MPer_L(f) = \emptyset$ .*

(iv) *If  $m$  is odd and  $n$  is even, then*

(a) *If  $a = -1$  and  $b = 1$ , then  $MPer_L(f) = \{1\}$ .*

(b) *Otherwise,  $MPer_L(f) = \emptyset$ .*

*Proof.* Computing the Lefschetz zeta function for  $f$  using equation (8.2.1) we obtain

$$\begin{aligned} \mathcal{Z}_f(t) &= \prod_{k=0}^{m+n} \det(I_{n_k} - t f_{*k})^{(-1)^{k+1}} \\ &= (1-t)^{-1} (1-at)^{(-1)^{m+1}} (1-bt)^{(-1)^{n+1}} (1-dt)^{(-1)^{m+n+1}}. \end{aligned}$$

Depending on whether  $m$  and  $n$  are even or odd, and for each allowed value of  $a, b, d \in \{-1, 1\}$  with  $ab = d$ , the expressions obtained for the Lefschetz zeta functions in each case are displayed in Tables 8.1, 8.2, 8.3 and 8.4.

The proof follows directly from the information obtained from the mentioned tables and applying Theorem 8.2.3, taking into account the considerations of Remarks 8.3.2, 8.3.3 and 8.3.4.  $\square$

Values for $a, b$	$\mathcal{Z}_f(t)$
$a = b = 1$	$\frac{1}{(1-t)^4}$
$\{a, b\} = \{-1, 1\}$	$\frac{1}{(1-t)^2} \frac{1}{(1+t)^2} = \frac{1}{(1-t^2)^2}$

Table 8.1:  $\mathcal{Z}_f(t)$  for a quasi-unipotent diffeomorphism  $f$  on  $\mathbb{S}^m \times \mathbb{S}^n$ , with  $m \neq n$  and  $m, n$  even.

Values for $a, b$	$\mathcal{Z}_f(t)$
$a = b = 1$	1
$a = b = -1$	$\frac{(1+t)^2}{(1-t)^2}$
$\{a, b\} = \{-1, 1\}$	1

Table 8.2:  $\mathcal{Z}_f(t)$  for a quasi-unipotent diffeomorphism  $f$  on  $\mathbb{S}^m \times \mathbb{S}^n$ , with  $m \neq n$  and  $m, n$  odd.

Values for $a, b, d$	$\mathcal{Z}_f(t)$
$a = b = 1$	1
$b = -1, a = 1$	$\frac{(1+t)^2}{(1-t)^2}$
$\{b, d\} = \{-1, 1\}$	1

Table 8.3:  $\mathcal{Z}_f(t)$  for a quasi-unipotent diffeomorphism  $f$  on  $\mathbb{S}^m \times \mathbb{S}^n$ , with  $m \neq n$  and  $m$  even,  $n$  odd.

Values for $a, b, d$	$\mathcal{Z}_f(t)$
$a = b = 1$	1
$a = -1, b = 1$	$\frac{(1+t)^2}{(1-t)^2}$
$\{a, d\} = \{-1, 1\}$	1

Table 8.4:  $\mathcal{Z}_f(t)$  for a quasi-unipotent diffeomorphism on  $\mathbb{S}^m \times \mathbb{S}^n$ , with  $m \neq n$  and  $m$  odd,  $n$  even.

## 8.6 Periods of Morse–Smale diffeomorphisms on $\mathbb{C}\mathbf{P}^n$ and $\mathbb{H}\mathbf{P}^n$

Let  $n \in \mathbb{N}$ ,  $n \geq 1$  and let  $\mathbb{C}\mathbf{P}^n$  be the  $n$ -dimensional complex projective space and let  $\mathbb{H}\mathbf{P}^n$  be the  $n$ -dimensional quaternion projective space. Along this section we consider these manifolds as real manifolds of dimension  $2n$  and  $4n$ , respectively.

The homology groups of  $\mathbb{C}\mathbf{P}^n$  over  $\mathbb{Q}$  can be easily computed applying Künneth's formula and are given by

$$H_k(\mathbb{C}\mathbf{P}^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k \in \{0, 2, \dots, 2n\}, \\ 0 & \text{otherwise.} \end{cases}$$

For a continuous map  $f : \mathbb{C}\mathbf{P}^n \rightarrow \mathbb{C}\mathbf{P}^n$ , the induced linear maps on the homology can be written as integer matrices as  $f_{*k} = (d^{k/2})$  for  $k \in \{0, 2, 4, \dots, 2n\}$ , with  $d \in \mathbb{Z}$ , and  $f_{*k} = (0)$  otherwise (see [73]).

Similarly the homology groups of  $\mathbb{H}\mathbf{P}^n$  over  $\mathbb{Q}$  are given by

$$H_k(\mathbb{H}\mathbf{P}^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k \in \{0, 4, \dots, 4n\}, \\ 0 & \text{otherwise,} \end{cases}$$

and for a continuous map  $f : \mathbb{H}\mathbf{P}^n \rightarrow \mathbb{H}\mathbf{P}^n$ , the induced linear maps on the homology can be written as  $f_{*k} = (d^{k/4})$  for  $k \in \{0, 4, 8, \dots, 4n\}$ , with  $d \in \mathbb{Z}$ , and  $f_{*k} = (0)$  otherwise (see [73]).

Let  $f : \mathbb{C}\mathbf{P}^n \rightarrow \mathbb{C}\mathbf{P}^n$  (respectively,  $f : \mathbb{H}\mathbf{P}^n \rightarrow \mathbb{H}\mathbf{P}^n$ ) be a Morse–Smale diffeomorphism. Since the linear maps induced on the homology are quasi unipotent, we will have either  $d = 1$  or  $d = -1$ , which determines also whether  $f$  preserves or reverses the orientation.

**Theorem 8.6.1.** *Let  $f : \mathbb{C}\mathbf{P}^n \rightarrow \mathbb{C}\mathbf{P}^n$  (respectively,  $f : \mathbb{H}\mathbf{P}^n \rightarrow \mathbb{H}\mathbf{P}^n$ ) be a Morse–Smale diffeomorphism. Then,*

- (a) *If  $n$  is odd and  $f$  reverses the orientation, then  $MPer_L(f) = \emptyset$ , but  $\{1, 2\} \cap Per(f) \neq \emptyset$ .*
- (b) *Otherwise,  $MPer_L(f) = 1$ .*

*Proof.* We develop the proof for  $\mathbb{C}\mathbf{P}^n$ , being the proof for  $\mathbb{H}\mathbf{P}^n$  completely analogous.

We start with the case  $d = 1$ . Computing the Lefschetz zeta function for  $f$  using (8.2.1), we get

$$\mathcal{Z}_f(t) = \prod_{k=0}^{2n} \det(I_{n_k} - t f_{*k})^{(-1)^{k+1}} = \frac{1}{(1-t)^{n+1}},$$

since we have  $f_{*k} = (1)$  for each  $k \in \{0, 2, \dots, 2n\}$ .

We consider next the case  $d = -1$ . As before, we have  $n + 1$  nontrivial induced maps, with  $f_{*k} = (1)$  if  $k/2$  is even and  $f_{*k} = (-1)$  if  $k/2$  is odd.

For  $n$  even, computing the Lefschetz zeta function for  $f$  using equation (8.2.1), we get

$$\mathcal{Z}_f(t) = \prod_{k=0}^{2n} \det(I_{n_k} - t f_{*k})^{(-1)^{k+1}} = \frac{1}{(1-t)^{\binom{n}{2}+1}} \frac{1}{(1+t)^{\frac{n}{2}}} = \frac{1}{(1-t^2)^{\frac{n}{2}}} \frac{1}{(1-t)},$$



and for  $n$  odd we have

$$\mathcal{Z}_f(t) = \prod_{k=0}^{2n} \det(I_{n_k} - t f_{*k})^{(-1)^{k+1}} = \frac{1}{(1-t)^{\frac{n+1}{2}}} \frac{1}{(1+t)^{\frac{n+1}{2}}} = \frac{1}{(1-t^2)^{\frac{n+1}{2}}}.$$

It is clear that  $f$  satisfies the hypotheses of Theorem 8.2.3. Then the results follow directly applying Theorem 8.2.3 to each of the expressions obtained for the Lefschetz zeta function  $\mathcal{Z}_f(t)$ . For  $\mathcal{Z}_f(t) = \frac{1}{(1-t)^{n+1}}$  and  $\mathcal{Z}_f(t) = \frac{1}{(1-t^2)^{\frac{n}{2}}} \frac{1}{(1-t)}$  we have  $\text{MPer}_L(f) = \{1\}$ , since any other expression of the same Lefschetz zeta function as a product of terms the form  $(1 \pm t^p)^{\pm 1}$  would provide at least the period 1.

For  $\mathcal{Z}_f(t) = \frac{1}{(1-t)^{\frac{n+1}{2}}} \frac{1}{(1+t)^{\frac{n+1}{2}}} = \frac{1}{(1-t^2)^{\frac{n+1}{2}}}$  it is clear that  $\text{MPer}_L(f) = \emptyset$ , but taking into account the considerations of Remarks 8.3.2 and 8.3.3, one has  $\{1, 2\} \cap \text{Per}(f) \neq \emptyset$ .  $\square$

**Remark 8.6.2.** A map  $f$  is called *Lefschetz periodic point free* if  $L(f^m) = 0$ , for all  $m \in \mathbb{N}$ . In [32] is claimed that there are no Lefschetz periodic point free maps on  $\mathbb{C}\mathbf{P}^n$  and  $\mathbb{H}\mathbf{P}^n$ , that is, all self maps of  $\mathbb{C}\mathbf{P}^n$  and  $\mathbb{H}\mathbf{P}^n$  have periodic points. Here we see that in particular Morse–Smale diffeomorphisms in  $\mathbb{C}\mathbf{P}^n$  and  $\mathbb{H}\mathbf{P}^n$  always have fixed points, unless when  $n$  is odd and  $d = -1$ , where in this case they always have either fixed points or periodic points of period 2.

## Chapter 9

# Limit cycles of linear vector fields on manifolds

### 9.1 Introduction

The objective of this chapter is to show that linear vector fields defined on manifolds different from  $\mathbb{R}^n$  can exhibit limit cycles. Concretely, we study the existence of limit cycles bifurcating from a continuum of periodic orbits of linear vector fields defined on manifolds of the form  $(\mathbb{S}^2)^m \times \mathbb{R}^n$ , when such vector fields are perturbed inside the class of all linear vector fields.

Let  $M$  be a smooth connected manifold of dimension  $n$ , and let  $TM$  be its tangent bundle. A *vector field* on  $M$  is a map  $X : M \rightarrow TM$  such that  $X(x) \in T_xM$ , where  $T_xM$  is the tangent space of  $M$  at the point  $x$ .

A *linear vector field* in  $\mathbb{R}^n$  is a vector field of the form  $X(x) = Ax + b$ , with  $x, b \in \mathbb{R}^n$  and where  $A$  is a  $n \times n$  real matrix. As it is well known linear vector fields on  $\mathbb{R}^n$  either do not have periodic orbits or their periodic orbits form a continuum, and therefore they do not have limit cycles.

We consider linear vector fields on some manifolds of the form  $(\mathbb{S}^2)^m \times \mathbb{R}^n$ , where  $\mathbb{S}^2$  denotes the unit two-dimensional sphere. Here the sphere  $\mathbb{S}^2$  is parameterized by the coordinates  $(\theta, \varphi)$ , where  $\theta \in [-\pi, \pi)$  denotes the azimuth angle and  $\varphi \in [-\pi/2, \pi/2]$  is the polar angle. Hence the curve  $\{\varphi = 0\}$  is the equator of the sphere.

Let  $(\theta_1, \varphi_1, \dots, \theta_m, \varphi_m, x_1, \dots, x_n)$  denote the coordinates of the space  $(\mathbb{S}^2)^m \times \mathbb{R}^n$ . Then we say that a vector field  $X$  is *linear on*  $M = (\mathbb{S}^2)^m \times \mathbb{R}^n$  if the expression of  $X$  in the coordinates  $z = (\theta_1, \varphi_1, \dots, \theta_m, \varphi_m, x_1, \dots, x_n) \in M$  is of the form  $X(z) = Az + b$ , with  $b \in M$  and where  $A$  is a  $(2m + n) \times (2m + n)$  real matrix.

A simple example in which a linear differential system on the manifold  $(\mathbb{S}^2)^m \times \mathbb{R}^n$  has a limit cycle is the following. Take  $m = 1$ ,  $n = 0$  and consider the linear system on the sphere  $\mathbb{S}^2$  given by

$$\dot{\theta} = 1, \quad \dot{\varphi} = \varphi,$$

for  $\theta \in [-\pi, \pi)$  and  $\varphi \in (-\pi/2, \pi/2)$ , and

$$\dot{\theta} = 0, \quad \dot{\varphi} = 0$$

for  $\varphi = \pm\pi/2$ . Then, clearly the equator of the sphere  $\{\varphi = 0\}$  is the only periodic orbit of the system, and therefore it is a limit cycle.

We will deal with systems that, as the one above, may not be continuous in some points of the phase space. However, we are interested in the behavior of those systems in some domain that contains limit cycles, and where they are continuous systems.

The main results of the chapter are Theorems 9.3.1, 9.3.2, and 9.3.3. The key tool that we use for proving those theorems is the averaging theory. For a general introduction to this theory, see the books [64, 72]. In Section 9.2 we present the basic results that we will need in order to prove the main results of the chapter.

As it will be shown through the proofs of Theorems 9.3.1 – 9.3.3, our method based on the averaging theory can produce at most one limit cycle for the studied systems. Therefore the following open question is natural:

*Let  $m$  and  $n$  be two non-negative integers. Is it true that a linear vector field on the manifold  $(\mathbb{S}^m)^m \times \mathbb{R}^n$  can have at most one limit cycle?*

A similar open question was stated in [53] concerning linear vector fields on the manifold  $(\mathbb{S}^1)^m \times \mathbb{R}^n$ .

## 9.2 Basic results on the averaging theory

In this section we state some basic results from the averaging theory that will be used later on.

Let  $M$  be a smooth connected manifold of dimension  $n$ , and let  $F_0, F_1 : \mathbb{R} \times M \rightarrow \mathbb{R}^n$  and  $F_2 : \mathbb{R} \times M \times [0, \varepsilon_0) \rightarrow \mathbb{R}^n$  be  $C^2$ ,  $T$ -periodic functions. Given the differential system

$$\dot{x}(t) = F_0(t, x), \tag{9.2.1}$$

we consider a perturbation of this system of the form

$$\dot{x}(t) = F_0(t, x) + \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x, \varepsilon). \tag{9.2.2}$$

The objective is to study the bifurcation of  $T$ -periodic solutions of system (9.2.2) for  $\varepsilon > 0$  small enough. A solution to this problem is given by the averaging theory.

We assume that there exists  $k \leq n$  such that  $M = M_k \times M_{n-k}$ , where  $M_k$  is a manifold of dimension  $k$  and  $M_{n-k}$  is a manifold of dimension  $n - k$ , and that the unperturbed system, namely system (9.2.1), contains an open set,  $V \subset M_k$ , such that  $\bar{V}$  is filled with periodic solutions all of them with the same period. Such a set is called *isochronous*.

Let  $x(t, z, \varepsilon)$  be the solution of system (9.2.2) such that  $x(0, z, \varepsilon) = z$ . We write the linearization of the unperturbed system (9.2.1) along the solution  $x(t, z, 0)$  as

$$\dot{y} = D_x F_0(t, x(t, z, 0))y, \tag{9.2.3}$$

and we denote by  $\mathcal{M}_z(t)$  the fundamental matrix of the linear differential system (9.2.3) such that  $\mathcal{M}_z(0)$  is the  $n \times n$  identity matrix, and by  $\xi : M = M_k \times M_{n-k} \rightarrow M_k$  the projection of  $M$  onto its first  $k$  coordinates, that is,  $\xi(x_1, \dots, x_n) = (x_1, \dots, x_k)$ .

The following results give sufficient conditions for the existence of limit cycles for a system of the form (9.2.2) bifurcating from the periodic orbits of system (9.2.1).

**Theorem 9.2.1.** *Let  $V \subset M_k$  be an open and bounded set, and let  $\beta_0 : \bar{V} \rightarrow M_{n-k}$  be a  $C^2$  function. Assume*

(i)  $\mathcal{Z} = \{z_\alpha = (\alpha, \beta_0(\alpha)) : \alpha \in \bar{V}\} \subset M$  and for each  $z_\alpha \in \mathcal{Z}$  the solution  $x(t, z_\alpha, 0)$  of system (9.2.1) is  $T$ -periodic.

(ii) For each  $z_\alpha \in \mathcal{Z}$ , there is a fundamental matrix  $\mathcal{M}_{z_\alpha}(t)$  of system (9.2.3) such that the matrix  $\mathcal{M}_{z_\alpha}^{-1}(0) - \mathcal{M}_{z_\alpha}^{-1}(T)$  has the  $k \times (n - k)$  zero matrix in the upper right corner, and a  $(n - k) \times (n - k)$  matrix  $\Delta_\alpha$  in the lower right corner with  $\det(\Delta_\alpha) \neq 0$ .

Consider the function  $\mathcal{F} : \bar{V} \rightarrow \mathbb{R}^k$  defined by

$$\mathcal{F}(\alpha) = \xi \left( \int_0^T \mathcal{M}_{z_\alpha}^{-1}(t) F_1(t, x(t, z_\alpha, 0)) dt \right).$$

If there exists  $a \in V$  with  $\mathcal{F}(a) = 0$  and with  $\det(\mathcal{DF}(a)) \neq 0$ , then there is a limit cycle  $x(t, \varepsilon)$  of period  $T$  of system (9.2.2) such that  $x(0, \varepsilon) \rightarrow z_a$  as  $\varepsilon \rightarrow 0$ .

The result given by Theorem 9.2.1 can be found in the books of Malkin [55] and Rosseau [62]. For a shorter proof, see [13]. There the result is proved in  $\mathbb{R}^n$ , but it can be easily extended to a smooth connected manifold  $M$ .

The next result allows to determine the existence of limit cycles in a system of the form (9.2.2) in the case when there exists an open set,  $V \subset M$ , such that for all  $z \in \bar{V}$ , the solution  $x(t, z, 0)$  is  $T$ -periodic.

**Theorem 9.2.2.** *Let  $V \subset M$  be an open and bounded set with  $\bar{V} \subset M$ , and assume that for all  $z \in \bar{V}$  the solution  $x(t, z, 0)$  of system (9.2.2) is  $T$ -periodic. Consider the function  $\mathcal{F} : \bar{V} \rightarrow \mathbb{R}^n$  defined by*

$$\mathcal{F}(z) = \int_0^T \mathcal{M}_z^{-1}(t) F_1(t, x(t, z, 0)) dt.$$

If there exists  $a \in V$  with  $\mathcal{F}(a) = 0$  and with  $\det(\mathcal{DF}(a)) \neq 0$ , then there is a limit cycle  $x(t, \varepsilon)$  of period  $T$  of system (9.2.2) such that  $x(0, \varepsilon) \rightarrow a$  as  $\varepsilon \rightarrow 0$ .

For the proof of Theorem 9.2.2 see Corollary 1 of [13].

## 9.3 Existence theorems for limit cycles

### 9.3.1 Limit cycles on $\mathbb{S}^2 \times \mathbb{R}$

In this section, let  $M = \mathbb{S}^2 \times \mathbb{R}$  and consider the linear differential system in  $M$  given by

$$\dot{\theta} = 1, \quad \dot{\varphi} = 0, \quad \dot{r} = r - 1, \tag{9.3.1}$$

for  $r \in \mathbb{R}$ ,  $\theta \in [-\pi, \pi)$  and  $\varphi \in (-\pi/2, \pi/2)$ , and with  $\dot{\theta} = 0$  on the straight lines  $R_1 = \{\varphi = -\pi/2\}$  and  $R_2 = \{\varphi = \pi/2\}$ .

The general solution of system (9.3.1) is given by

$$\theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0, \quad r(t) = (r_0 - 1)e^t + 1.$$

Thus the sphere  $\{r = 1\}$  is an invariant manifold with two equilibrium points at the north and the south poles, and is foliated by periodic orbits of period  $2\pi$ , corresponding to the parallels of the sphere, except at the poles. Moreover the straight lines  $R_1$  and  $R_2$  are invariant.

We shall study the bifurcation of limit cycles when we perturb system (9.3.1) inside the class of all linear differential systems, and we shall see that one of the periodic orbits contained in the sphere  $\{r = 1\}$  may bifurcate to a limit cycle under certain hypotheses.

We consider the class of differential systems

$$\begin{aligned} \dot{\theta} &= 1 + \varepsilon(a_0 + a_1\theta + a_2\varphi + a_3r), \\ \dot{\varphi} &= \varepsilon(b_0 + b_1\theta + b_2\varphi + b_3r), \\ \dot{r} &= r - 1 + \varepsilon(c_0 + c_1\theta + c_2\varphi + c_3r). \end{aligned} \tag{9.3.2}$$

where  $a_i, b_i$  and  $c_i$ , for  $i = 0, \dots, 3$  are real numbers and with  $\varepsilon > 0$  being a small parameter. Note that this is a generic linear perturbation of system (9.3.1). For the class of systems (9.3.2) we have the following result.

**Theorem 9.3.1.** *For sufficiently small  $\varepsilon > 0$  the linear differential system (9.3.2) has a limit cycle bifurcating from a periodic orbit of system (9.3.1) provided that  $a_1b_2 - a_2b_1 \neq 0$ . Moreover this limit cycle bifurcates from the periodic orbit of system (9.3.1) parameterized by  $(\theta(t), \varphi(t), r(t)) = (\theta_0 + t, \varphi_0, 1)$ , with*

$$\begin{aligned} \theta_0 &= \frac{a_2(b_0 + b_3 + b_1\pi) - b_2(a_0 + a_3 + a_1\pi)}{a_1b_2 - a_2b_1}, \\ \varphi_0 &= \frac{b_1(a_0 + a_3 + a_1\pi) - a_1(b_0 + b_3 + b_2\pi)}{a_1b_2 - a_2b_1}. \end{aligned}$$

We remark that the existence of the limit cycle for system (9.3.2) does not depend on the perturbation of the  $\dot{r}$  equation.

As an example of the previous result, consider the system

$$\dot{\theta} = 1 + \varepsilon a\varphi, \quad \dot{\varphi} = \varepsilon b\theta, \quad \dot{r} = r - 1, \tag{9.3.3}$$

with  $a, b \in \mathbb{R}$  and  $\varepsilon > 0$ . In this case the sphere  $\{r = 1\}$  is still an invariant manifold. Applying Theorem 9.3.1 with  $a_2 = a$ ,  $b_1 = b$  and the rest of the coefficients of the perturbation being zero, we find that system (9.3.3) has a limit cycle bifurcating from the periodic orbit of system (9.3.1) parameterized by  $(\theta(t), \varphi(t), r(t)) = (-\pi + t, 0, 1)$ . That is, there is a limit cycle bifurcating from the periodic orbit corresponding to the equator of the sphere  $\{r = 1\}$  of system (9.3.1). Moreover this limit cycle is still contained in the sphere  $\{r = 1\}$ .

*Proof of Theorem 9.3.1.* We use the result from averaging theory given in Theorem 9.2.1 to deduce the existence of a limit cycle of system (9.3.2), for some  $\varepsilon > 0$  small enough, bifurcating from a periodic orbit of the same system with  $\varepsilon = 0$ .

Since the general solution of the differential system (9.3.1), corresponding to system (9.3.2) with  $\varepsilon = 0$ , is given by

$$\theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0, \quad r(t) = (r_0 - 1)e^t + 1,$$

it is clear that all the periodic solutions of that system are parameterized by

$$\theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0, \quad r(t) = 1,$$

with  $(\theta_0, \varphi_0) \in \mathbb{S}^2 \setminus \{\varphi_0 = \pm\pi/2\}$ . Then, all the periodic solutions have period  $2\pi$  and they fill the invariant sphere  $\{r = 1\}$  except for the poles, which are equilibrium points.

Therefore, for applying Theorem 9.2.1 we take  $M = \mathbb{S}^2 \times \mathbb{R}$  and

$$\begin{aligned} k &= 2, \quad n = 3, \\ M_k &= M_2 = \{(\theta, \varphi, r) \in M : r = 1\} \cong \mathbb{S}^2, \\ x &= (\theta, \varphi, r), \\ \alpha &= (\theta_0, \varphi_0), \\ \beta_0(\alpha) &= \beta_0(\theta_0, \varphi_0) = 1, \\ z_\alpha &= (\alpha, \beta_0(\alpha)) = (\theta_0, \varphi_0, 1), \\ V &= \{(\theta, \varphi, r) \in M : r = 1, \varphi \in (-\frac{\pi}{2} + \delta_0, \frac{\pi}{2} - \delta_0)\} \\ &\quad \text{with } \delta_0 > 0 \text{ small enough such that} \\ \varphi^* &:= \frac{b_1(a_0 + a_3 + a_1\pi) - a_1(b_0 + b_3 + b_2\pi)}{a_1b_2 - a_2b_1} \in (-\frac{\pi}{2} + \delta_0, \frac{\pi}{2} - \delta_0), \\ \mathcal{Z} &= \bar{V} \times \{r = 1\}, \\ x(t, z_\alpha, 0) &= (\theta_0 + t, \varphi_0, 1), \\ F_0(t, x) &= (1, 0, r - 1), \\ F_1(t, x) &= (a_0 + a_1\theta + a_2\varphi + a_3r, b_0 + b_1\theta + b_2\varphi + b_3r, c_0 + c_1\theta + c_2\varphi + c_3r), \\ F_2(t, x, \varepsilon) &= 0, \\ T &= 2\pi, \end{aligned} \tag{9.3.4}$$

where we took  $V \subset M_2$  as an open subset that contains the periodic orbit for which it bifurcates a limit cycle, as we shall see next.

The fundamental matrix  $\mathcal{M}_{z_\alpha}(t)$  with  $\mathcal{M}_{z_\alpha}(0) = Id$  of system (9.2.3) with  $F_0$  and  $x(t, z_\alpha, 0)$  described above is the matrix  $\mathcal{M}_{z_\alpha}(t) = \exp(D_x F_0 t)$ , *i.e.*

$$\mathcal{M}_{z_\alpha}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{pmatrix}.$$

Note that since  $F_0$  defines a linear differential system, the fundamental matrix  $\mathcal{M}_{z_\alpha}(t)$  is independent of the initial conditions  $z_\alpha$ .

We also have

$$\mathcal{M}_{z_\alpha}^{-1}(0) - \mathcal{M}_{z_\alpha}^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 - e^{-2\pi} \end{pmatrix},$$

and therefore, all the assumptions in the in the statement of Theorem 9.2.1 are satisfied.

With the described setting, the function  $\mathcal{F}(\alpha) = \mathcal{F}(\theta_0, \varphi_0)$  from the statement of Theorem 9.2.1 associated with system (9.3.2) is

$$\begin{aligned}\mathcal{F}(\theta_0, \varphi_0) &= \xi \left( \int_0^{2\pi} \mathcal{M}_{z_\alpha}^{-1}(t) F_1(\theta_0 + t, \varphi_0, 1) dt \right) \\ &= 2\pi(a_0 + a_1(\theta_0 + \pi) + a_2\varphi_0 + b_3, b_0 + b_1(\theta_0 + \pi) + b_2\varphi_0 + b_3).\end{aligned}$$

We have  $\det(D\mathcal{F}) = 4\pi^2(a_1b_2 - a_2b_1)$ , and therefore  $\det(D\mathcal{F}) \neq 0$  for all  $(\theta_0, \varphi_0) \in V$ . Thus, the only solution of  $\mathcal{F} = 0$  is given by

$$\begin{aligned}\theta_0 &= \frac{a_2(b_0 + b_3 + b_1\pi) - b_2(a_0 + a_3 + a_1\pi)}{a_1b_2 - a_2b_1}, \\ \varphi_0 &= \frac{b_1(a_0 + a_3 + a_1\pi) - a_1(b_0 + b_3 + b_2\pi)}{a_1b_2 - a_2b_1}.\end{aligned}\tag{9.3.5}$$

Note that such solution  $(\theta_0, \varphi_0)$ , where  $\varphi_0 = \varphi^*$ , is contained in the set  $V$  described in (9.3.4).

Hence, by Theorem 9.2.1, if  $\varepsilon > 0$  is small enough, there is a periodic solution,  $(\theta(t, \varepsilon), \varphi(t, \varepsilon), r(t, \varepsilon))$ , of system (9.3.3), which is a limit cycle, and such that

$$(\theta(0, \varepsilon), \varphi(0, \varepsilon), r(0, \varepsilon)) \rightarrow (\theta_0, \varphi_0, 1),$$

when  $\varepsilon \rightarrow 0$ , and where  $\theta_0$  and  $\varphi_0$  are given in (9.3.5). □

### 9.3.2 Limit cycles on $(\mathbb{S}^2)^2 \times \mathbb{R}$

Next we consider linear differential systems defined on higher dimensional manifolds. In this section we take  $M = (\mathbb{S}^2)^2 \times \mathbb{R}$  and we consider the differential system

$$\dot{\theta} = 1, \quad \dot{\varphi} = 0, \quad \dot{\nu} = 1, \quad \dot{\phi} = 0, \quad \dot{r} = r - 1,\tag{9.3.6}$$

for  $(\theta, \varphi, \nu, \phi, r) \in M$ , with  $\theta, \nu \in [-\pi, \pi)$  and  $\varphi, \phi \in (-\pi/2, \pi/2)$ , and with  $\dot{\theta} = 0$  when  $\varphi = \pm\pi/2$  and  $\dot{\nu} = 0$  when  $\phi = \pm\pi/2$ .

The general solution of system (9.3.6) is

$$\theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0, \quad \nu(t) = \nu_0 + t, \quad \phi(t) = \phi_0, \quad r(t) = (r_0 - 1)e^t + 1,$$

and thus the product of spheres  $\{r = 1\} \cong (\mathbb{S}^2)^2$  is an invariant manifold foliated by periodic orbits of period  $2\pi$ , except for the four points  $\{r = 1, \varphi = \pm\pi/2, \phi = \pm\pi/2\}$ , which are equilibrium points.

We consider the most general perturbation of the differential system (9.3.6) inside the class of all linear differential systems, namely

$$\begin{aligned}\dot{\theta} &= 1 + \varepsilon(a_0 + a_1\theta + a_2\varphi + a_3\nu + a_4\phi + a_5r), \\ \dot{\varphi} &= \varepsilon(b_0 + b_1\theta + b_2\varphi + b_3\nu + b_4\phi + b_5r) \\ \dot{\nu} &= 1 + \varepsilon(c_0 + c_1\theta + c_2\varphi + c_3\nu + c_4\phi + c_5r), \\ \dot{\phi} &= \varepsilon(d_0 + d_1\theta + d_2\varphi + d_3\nu + d_4\phi + d_5r) \\ \dot{r} &= r - 1 + \varepsilon(e_0 + e_1\theta + e_2\varphi + e_3\nu + e_4\phi + e_5r),\end{aligned}\tag{9.3.7}$$

with  $a_i, b_i, c_i, d_i, e_i \in \mathbb{R}$  for  $i = 0, \dots, 5$ , and with  $\varepsilon > 0$  being a small parameter. In the following result we give sufficient conditions on the coefficients of system (9.3.7) in order that there is a limit cycle bifurcating from a periodic orbit of the corresponding unperturbed system.

**Theorem 9.3.2.** *For sufficiently small  $\varepsilon > 0$  the differential system (9.3.7) has a limit cycle bifurcating from a periodic orbit of system (9.3.6) provided that*

$$\det \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} \neq 0.$$

Moreover this limit cycle bifurcates from the periodic orbit of system (9.3.6) parameterized by  $(\theta(t), \varphi(t), \nu(t), \phi(t), r(t)) = (\theta_0 + t, \varphi_0, \nu_0 + t, \phi_0, 1)$ , where  $(\theta_0, \varphi_0, \nu_0, \phi_0)$  is the unique solution of the linear system

$$\begin{aligned} a_1\theta_0 + a_2\varphi_0 + a_3\nu_0 + a_4\phi_0 &= -a_0 - a_1\pi - a_3\pi - a_5, \\ b_1\theta_0 + b_2\varphi_0 + b_3\nu_0 + b_4\phi_0 &= -b_0 - b_1\pi - b_3\pi - b_5, \\ c_1\theta_0 + c_2\varphi_0 + c_3\nu_0 + c_4\phi_0 &= -c_0 - c_1\pi - c_3\pi - c_5, \\ d_1\theta_0 + d_2\varphi_0 + d_3\nu_0 + d_4\phi_0 &= -d_0 - d_1\pi - d_3\pi - d_5. \end{aligned}$$

*Proof.* We use the result from averaging theory given in Theorem 9.2.1 to prove that, for some  $\varepsilon > 0$  small enough, there exist a limit cycle of system (9.3.7) bifurcating from a periodic orbit of the same system with  $\varepsilon = 0$ .

Since the general solution of system (9.3.7) with  $\varepsilon = 0$  (that is, the one of system (9.3.6)), is

$$\theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0, \quad \nu(t) = \nu_0 + t, \quad \phi(t) = \phi_0, \quad r(t) = (r_0 - 1)e^t + 1,$$

then all the periodic solutions of that system are

$$\theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0, \quad \nu(t) = \nu_0 + t, \quad \phi(t) = \phi_0, \quad r(t) = 1,$$

with  $(\theta_0, \varphi_0, \nu_0, \phi_0) \in \mathbb{S}^2 \setminus \{\varphi_0 = \pm\pi/2\} \times \mathbb{S}^2 \setminus \{\varphi_0 = \pm\pi/2\}$ . That is, the periodic solutions fill the invariant manifold  $\{r = 1\}$  except for the four equilibrium points  $\{\varphi = \pm\pi/2, \phi = \pm\pi/2\}$ , and they have all period  $2\pi$ .



For applying Theorem 9.2.1 we take  $M = (\mathbb{S}^2)^2 \times \mathbb{R}$  and

$$\begin{aligned}
k &= 4, n = 5, \\
M_k &= M_4 = \{\theta, \varphi, \nu, \phi, r \in M : r = 1\} \cong (\mathbb{S}^2)^2, \\
x &= (\theta, \varphi, \nu, \phi, r), \\
\alpha &= (\theta_0, \varphi_0, \nu_0, \phi_0), \\
\beta_0(\alpha) &= \beta_0(\theta_0, \varphi_0, \nu_0, \phi_0) = 1, \\
z_\alpha &= (\alpha, \beta_0(\alpha)) = (\theta_0, \varphi_0, \nu_0, \phi_0, 1), \\
V &= \{(\theta, \varphi, \nu, \phi, r) \in M : r = 1, \varphi \in (-\frac{\pi}{2} + \delta_0, \frac{\pi}{2} - \delta_0)\} \\
&\quad \text{with } \delta_0 > 0 \text{ small enough such that } \varphi_0, \phi_0 \text{ satisfying (9.3.9) satisfy} \\
&\quad \varphi_0, \phi_0 \in (-\frac{\pi}{2} + \delta_0, \frac{\pi}{2} - \delta_0), \\
\mathcal{Z} &= \bar{V} \times \{r = 1\}, \\
x(t, z_\alpha, 0) &= (\theta_0 + t, \varphi_0, \nu_0 + t, \phi_0, 1), \\
F_0(t, x) &= (1, 0, 1, 0, r - 1), \\
F_1(t, x) &= \begin{pmatrix} a_0 + a_1\theta + a_2\varphi + a_3\nu + a_4\phi + a_5r \\ b_0 + b_1\theta + b_2\varphi + b_3\nu + b_4\phi + b_5r \\ c_0 + c_1\theta + c_2\varphi + c_3\nu + c_4\phi + c_5r \\ d_0 + d_1\theta + d_2\varphi + d_3\nu + d_4\phi + d_5r \\ e_0 + e_1\theta + e_2\varphi + e_3\nu + e_4\phi + e_5r \end{pmatrix}, \\
F_2(t, x, \varepsilon) &= 0, \\
T &= 2\pi,
\end{aligned} \tag{9.3.8}$$

where we chose  $V \subset M_4$  as an open subset that contains the periodic orbit for which it bifurcates a limit cycle, as we shall see next.

The fundamental matrix  $\mathcal{M}_{z_\alpha}(t)$  with  $\mathcal{M}_{z_\alpha}(0) = Id$ , of system (9.2.3) with  $F_0$  and  $x(t, z_\alpha, 0)$  described above is the matrix  $\mathcal{M}_{z_\alpha}(t) = \exp(D_x F_0 t)$ , *i.e.*,

$$\mathcal{M}_{z_\alpha}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^t \end{pmatrix}.$$

We also have

$$\mathcal{M}_{z_\alpha}^{-1}(0) - \mathcal{M}_{z_\alpha}^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 - e^{-2\pi} \end{pmatrix},$$

and therefore, all the assumptions in the statement of Theorem 9.2.1 are satisfied.

With the described setting, the function  $\mathcal{F}(\alpha) = \mathcal{F}(\theta_0, \varphi_0, \nu_0, \phi_0)$  in the statement of Theorem 9.2.1 associated with system (9.3.7) is

$$\mathcal{F}(\theta_0, \varphi_0, \nu_0, \phi_0) = \xi \left( \int_0^{2\pi} \mathcal{M}_{z_\alpha}^{-1}(t) F_1(\theta_0 + t, \varphi_0, \nu_0 + t, \phi_0, 1) dt \right) = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4),$$

with

$$\begin{aligned} \mathcal{F}_1 &= 2\pi(a_0 + a_1\theta_0 + a_1\pi + a_2\varphi_0 + a_3\nu_0 + a_3\pi + a_4\phi_0 + a_5), \\ \mathcal{F}_2 &= 2\pi(b_0 + b_1\theta_0 + b_1\pi + b_2\varphi_0 + b_3\nu_0 + b_3\pi + b_4\phi_0 + b_5), \\ \mathcal{F}_3 &= 2\pi(c_0 + c_1\theta_0 + c_1\pi + c_2\varphi_0 + c_3\nu_0 + c_3\pi + c_4\phi_0 + c_5), \\ \mathcal{F}_4 &= 2\pi(d_0 + d_1\theta_0 + d_1\pi + d_2\varphi_0 + d_3\nu_0 + d_3\pi + d_4\phi_0 + d_5). \end{aligned}$$

Also, we have

$$\det(D\mathcal{F}) = 16\pi^4 \det \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} \neq 0,$$

by assumption. The initial conditions  $(\theta_0, \varphi_0, \nu_0, \phi_0)$  such that  $\mathcal{F}(\theta_0, \varphi_0, \nu_0, \phi_0) = 0$  are the solutions of the linear system

$$\begin{aligned} a_1\theta_0 + a_2\varphi_0 + a_3\nu_0 + a_4\phi_0 &= -a_0 - a_1\pi - a_3\pi - a_5, \\ b_1\theta_0 + b_2\varphi_0 + b_3\nu_0 + b_4\phi_0 &= -b_0 - b_1\pi - b_3\pi - b_5, \\ c_1\theta_0 + c_2\varphi_0 + c_3\nu_0 + c_4\phi_0 &= -c_0 - c_1\pi - c_3\pi - c_5, \\ d_1\theta_0 + d_2\varphi_0 + d_3\nu_0 + d_4\phi_0 &= -d_0 - d_1\pi - d_3\pi - d_5. \end{aligned} \tag{9.3.9}$$

Since  $\det(D\mathcal{F}) \neq 0$ , system (9.3.9) has a unique solution. Note that such solution  $(\theta_0, \varphi_0, \nu_0, \phi_0)$  is contained in the set  $V$  described in (9.3.8).

Hence, by Theorem 9.2.1, if  $\varepsilon > 0$  is small enough, there is a periodic solution,

$$(\theta(t, \varepsilon), \varphi(t, \varepsilon), \nu(t, \varepsilon), \phi(t, \varepsilon), r(t, \varepsilon)),$$

of system (9.3.7), which is a limit cycle, and such that

$$(\theta(0, \varepsilon), \varphi(0, \varepsilon), \nu(0, \varepsilon), \phi(0, \varepsilon), r(0, \varepsilon)) \rightarrow (\theta_0, \varphi_0, \nu_0, \phi_0, 1),$$

when  $\varepsilon \rightarrow 0$ , and where  $\theta_0, \varphi_0, \nu_0$ , and  $\phi_0$  are given by the unique solution of system (9.3.9).  $\square$

### 9.3.3 Limit cycles on $\mathbb{S}^2 \times \mathbb{R}^2$

In this section we consider the linear differential system defined in the manifold  $M = \mathbb{R}^2 \times \mathbb{S}^2$ , for  $(x, y, \theta, \varphi) \in \mathbb{R}^2 \times \mathbb{S}^2$ , with  $\theta \in [-\pi, \pi)$  and  $\varphi \in (-\pi/2, \pi/2)$ , given by

$$\dot{x} = -y, \quad \dot{y} = x, \quad \dot{\theta} = 1, \quad \dot{\varphi} = 0, \tag{9.3.10}$$

and with  $\dot{\theta} = 0$  in the planes  $P_1 = \{\varphi = -\pi/2\}$  and  $P_2 = \{\varphi = \pi/2\}$ , which are invariant. The general solution of system (9.3.10) is

$$x(t) = x_0 \cos t - y_0 \sin t, \quad y(t) = x_0 \sin t + y_0 \cos t, \quad \theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0,$$

and therefore the whole phase space is filled by periodic orbits of period  $2\pi$ , except for the two equilibrium points  $(x, y, \theta, \varphi) = (0, 0, \theta, -\pi/2)$  and  $(x, y, \theta, \varphi) = (0, 0, \theta, \pi/2)$ .

We consider a generic linear perturbation of system (9.3.10) and we study the existence of limit cycles bifurcating from the periodic orbits of system (9.3.10).

Let

$$\begin{aligned} \dot{x} &= -y + \varepsilon(a_0 + a_1x + a_2y + a_3\theta + a_4\varphi), \\ \dot{y} &= x + \varepsilon(b_0 + b_1x + b_2y + b_3\theta + b_4\varphi), \\ \dot{\theta} &= 1 + \varepsilon(c_0 + c_1x + c_2y + c_3\theta + c_4\varphi), \\ \dot{\varphi} &= \varepsilon(d_0 + d_1x + d_2y + d_3\theta + d_4\varphi), \end{aligned} \tag{9.3.11}$$

be the perturbed system, with  $a_i, b_i, c_i, d_i \in \mathbb{R}$  for  $i = 0, \dots, 4$ , and where  $\varepsilon > 0$  is a small parameter. For this linear differential system we have the following result.

**Theorem 9.3.3.** *For sufficiently small  $\varepsilon > 0$  the linear differential system (9.3.11) has a limit cycle bifurcating from a periodic orbit of system (9.3.10) provided that*

$$\det \begin{pmatrix} b_2 + a_1 & a_2 - b_1 \\ b_1 - a_2 & b_2 + a_1 \end{pmatrix} \neq 0 \quad \text{and} \quad \det \begin{pmatrix} c_3 & c_4 \\ d_3 & d_4 \end{pmatrix} \neq 0.$$

Moreover this limit cycle bifurcates from the periodic orbit of system (9.3.10) passing through the point  $(x_0, y_0, \theta_0, \varphi_0)$  where

$$\begin{aligned} x_0 &= \frac{(2b_2 + 2a_1)b_3 - 2a_3b_1 + 2a_2a_3}{b_2^2 + b_1^2 + a_2^2 + a_1^2 + 2a_1b_2 - 2a_2b_1}, \\ y_0 &= -\frac{(2b_1 - 2a_2)b_3 + 2a_3b_2 + 2a_1a_3}{b_2^2 + b_1^2 + a_2^2 + a_1^2 + 2a_1b_2 - 2a_2b_1}, \\ \theta_0 &= -\frac{(\pi c_3 + c_0)d_4 - \pi c_4d_3 - c_4d_0}{c_3d_4 - c_4d_3}, \\ \varphi_0 &= \frac{c_0d_3 - c_3d_0}{c_3d_4 - c_4d_3}. \end{aligned}$$

As an example consider the system

$$\dot{x} = -y + \varepsilon ay, \quad \dot{y} = x + \varepsilon bx, \quad \dot{\theta} = 1 + \varepsilon c\varphi, \quad \dot{\varphi} = \varepsilon d\theta, \tag{9.3.12}$$

with  $a, b, c, d \in \mathbb{R}$ , and  $\varepsilon > 0$ . Applying Theorem 9.3.3 with  $a_2 = a$ ,  $b_1 = b$ ,  $c_4 = c$ ,  $d_3 = d$  and the rest of the coefficients of the perturbation being zero, we obtain that system (9.3.12) has a limit cycle bifurcating from the periodic orbit of system (9.3.10) passing through the point  $(x_0, y_0, \theta_0, \varphi_0) = (0, 0, -\pi, 0)$ , provided that  $(a - b)cd \neq 0$ . That is, here the limit cycle bifurcates from the periodic orbit corresponding to the equator of the invariant sphere  $\{x = y = 0\}$  of system (9.3.10).

*Proof of Theorem 9.3.3.* Since the general solution of system (9.3.11) with  $\varepsilon = 0$  is given by

$$x(t) = x_0 \cos(t) - y_0 \sin(t), \quad y(t) = x_0 \sin(t) + y_0 \cos(t), \quad \theta(t) = \theta_0 + t, \quad \varphi(t) = \varphi_0,$$

the whole phase space is filled by periodic solutions, except from the equilibrium points  $(x, y, \theta, \varphi) = (0, 0, \theta, -\pi/2)$  and  $(x, y, \theta, \varphi) = (0, 0, \theta, \pi/2)$ . Hence, the periodic solutions of the differential system (9.3.10) fill an open set of the phase space  $M = \mathbb{R}^2 \times \mathbb{S}^2$ .

To prove Theorem 9.3.3 we use the result given in Theorem 9.2.2 to deduce that there exist a limit cycle of system (9.3.11), for some  $\varepsilon > 0$  small enough, bifurcating from the periodic orbits of the same system with  $\varepsilon = 0$ .

To clarify the notation, here the solution  $x(t, z, 0)$  from the statement of Theorem 9.2.2 will be denoted by  $\mathbf{x}(t, z, 0)$ , and  $x$  will denote the first variable in the phase space.

To apply Theorem 9.2.2 we take  $M = \mathbb{R}^2 \times \mathbb{S}^2$  and

$$\begin{aligned} \mathbf{x} &= (x, y, \theta, \varphi), \\ z &= (x_0, y_0, \theta_0, \varphi_0), \\ \mathbf{x}(t, z, 0) &= (x(t), y(t), \theta(t), \varphi(t)) \text{ given by (9.3.3)} \\ F_0(t, x) &= (-y, x, 1, 0), \\ F_1(t, x) &= \begin{pmatrix} a_0 + a_1x + a_2y + a_3\theta + a_4\varphi \\ b_0 + b_1x + b_2y + b_3\theta + b_4\varphi \\ c_0 + c_1x + c_2y + c_3\theta + c_4\varphi \\ d_0 + d_1x + d_2y + d_3\theta + d_4\varphi \end{pmatrix}, \end{aligned} \tag{9.3.13}$$

$$F_2(t, x, \varepsilon) = 0,$$

$$T = 2\pi,$$

$$V = \{(x, y, \theta, \varphi) \in M : \|(x, y)\| < 1 + \kappa, \varphi \in (-\frac{\pi}{2} + \delta_0, \frac{\pi}{2} - \delta_0)\},$$

$$\text{with } \kappa = \frac{2\sqrt{a_3^2 + b_3^2}}{\sqrt{a_1^2 + a_2^2 + b_1^2 + b_2^2 + 2a_1b_2 - 2a_2b_1}},$$

and with  $\delta_0 > 0$  small enough such that

$$\varphi^* := \frac{c_0d_3 - c_3d_0}{c_3d_4 - c_4d_3} \in (-\frac{\pi}{2} + \delta_0, \frac{\pi}{2} - \delta_0).$$

where we chose  $V \subset M$  as an open subset that contains the periodic orbit for which it bifurcates a limit cycle, as we shall see next.

The fundamental matrix  $\mathcal{M}_z(t)$  of system (9.2.3) with  $\mathcal{M}_z(0) = Id$  and with  $F_0$  and  $\mathbf{x}(t, z, 0)$  described in (9.3.13) is given by

$$M_z(t) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 & 0 \\ \sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore all the assumptions in the statement of Theorem 9.2.2 are satisfied.

With the described setting the function  $\mathcal{F}(z) = \mathcal{F}(x_0, y_0, \theta_0, \varphi_0)$  in the statement of Theorem 9.2.2 associated with system (9.3.11), namely,

$$\mathcal{F}(x_0, y_0, \theta_0, \varphi_0) = \int_0^{2\pi} M_z^{-1}(t) F_1(t, \mathbf{x}(t, x, 0)) dt,$$

is given by  $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)$ , which after some straightforward computations can be written as

$$\begin{aligned}\mathcal{F}_1 &= (\pi a_2 - \pi b_1)y_0 + (\pi b_2 + \pi a_1)x_0 - 2\pi b_3, \\ \mathcal{F}_2 &= (\pi b_2 + \pi a_1)y_0 + (\pi b_1 - \pi a_2)x_0 + 2\pi a_3, \\ \mathcal{F}_3 &= 2\pi c_3\theta_0 + 2\pi c_4\varphi_0 + 2\pi^2 c_3 + 2\pi c_0, \\ \mathcal{F}_4 &= 2\pi d_3\theta_0 + 2\pi d_4\varphi_0 + 2\pi^2 d_3 + 2\pi d_0.\end{aligned}$$

Assuming that

$$\det(\mathcal{DF}) = \det \begin{pmatrix} \pi(b_2 + a_1) & \pi(a_2 - b_1) & 0 & 0 \\ \pi(b_1 - a_2) & \pi(b_2 + a_1) & 0 & 0 \\ 0 & 0 & 2\pi c_3 & 2\pi c_4 \\ 0 & 0 & 2\pi d_3 & 2\pi d_4 \end{pmatrix} \neq 0, \quad (9.3.14)$$

the linear system  $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4) = (0, 0, 0, 0)$  has a unique solution, given by

$$\begin{aligned}x_0 &= \frac{(2b_2 + 2a_1)b_3 - 2a_3b_1 + 2a_2a_3}{b_2^2 + b_1^2 + a_2^2 + a_1^2 + 2a_1b_2 - 2a_2b_1}, \\ y_0 &= -\frac{(2b_1 - 2a_2)b_3 + 2a_3b_2 + 2a_1a_3}{b_2^2 + b_1^2 + a_2^2 + a_1^2 + 2a_1b_2 - 2a_2b_1}, \\ \theta_0 &= -\frac{(\pi c_3 + c_0)d_4 - \pi c_4d_3 - c_4d_0}{c_3d_4 - c_4d_3}, \\ \varphi_0 &= \frac{c_0d_3 - c_3d_0}{c_3d_4 - c_4d_3}.\end{aligned} \quad (9.3.15)$$

Note that such solution  $(x_0, y_0, \theta_0, \varphi_0)$ , where  $\varphi_0 = \varphi^*$ , is contained in the set  $V$  described in (9.3.13).

The condition (9.3.14) is clearly satisfied for all  $(x_0, y_0, \theta_0, \varphi_0) \in V$  taking into account the assumptions in the statement of Theorem 9.3.3.

Hence, by Theorem 9.2.2, there is a periodic solution  $(x(t, \varepsilon), y(t, \varepsilon), \theta(t, \varepsilon), \varphi(t, \varepsilon))$  of system (9.3.11), which is a limit cycle, and such that

$$(x(0, \varepsilon), y(0, \varepsilon), \theta(0, \varepsilon), \varphi(0, \varepsilon), r(0, \varepsilon)) \rightarrow (x_0, y_0, \theta_0, \varphi_0)$$

when  $\varepsilon \rightarrow 0$ , and where  $x_0, y_0, \theta_0$  and  $\varphi_0$  are given in (9.3.15).  $\square$

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