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Heterogeneity and The Representative Agent in Macroeconomics

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Contents

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Chapter 1

Introduction

The thesis focuses on applications of the representative consumer to macroeconomics. The thesis includes three sections, all are joint work with Lilia Maliar.

The first chapter analyzes the possibility of aggregation in multi-good and multi-period economies. I relax the standard condition for aggregation that individual demand is linear in wealth and develop a notion of representative consumer by imposing restrictions on the demand differentials. In this way, I can extend the results from aggregation to some cases in which aggregate behavior of the multi-consumer economy depends on distributions. I derive restrictions on the individual preferences which are necessary for the existence of the representative consumer if the distribution of wealth is fixed and identify all strictly concave additive utilities for which such restrictions are satisfied.

The second chapter studies indivisible labor model with ex-ante heterogeneous agents. The standard indivisible labor model-economy with ex-ante identical agents behaves as if there is a representative agents whose preferences are linear in leisure. I address two questions: Does this result still holds if agents are ex-ante heterogeneous in such dimensions as initial endowments and productivity? What implications does such representative consumer have for distributions in the underlying heterogeneous agents economies? I show several example of economies with heterogeneous agents which admit the representative consumer whose preferences are linear in leisure. The equilibrium allocation in such economies can be both indeterminate and unique depending on the assumptions about the production function.

The third chapter analyzes a complete market neoclassical economy with heterogeneous agents. Agents have addilog preferences and receive idiosyncratic labor productivity shocks. I show that at the aggregate level, such an economy behaves as if there was a representative consumer who faces shocks to preferences and technology. This fact enables me to infer time-series properties of the model without specifying idiosyncratic shocks. Instead, I calibrate the shocks to preferences and technology in the model derived from aggregation. In contrast to the standard one-shock setup, the model with two types of shocks can generate the appropriate predictions with respect to labor markets.

Chapter 2

Aggregation in Dynamic Models

Joint with Lilia Maliar

2.1 Introduction

Gorman (1953) establishes that quasi-homotheticity of the individual preferences is both a necessary and sufficient conditions for the existence of the representative consumer. This is not a necessary condition for the existence of such consumer, however, if the distribution of wealth in the economy is fixed. Shafer (1977) shows an example when market demand in the multi-consumer economy where agents' preferences are not quasi-homothetic can be represented by demand of a single individual. According to Shafer and Sonnenschein (1982), necessary and sufficient conditions for the existence of a single consumer who rationalize market demand in the multi-consumer economy with a fixed distribution of wealth are unknown. This paper establishes a necessary condition and provides sufficient conditions under the assumption of additivity.

We analyze the possibility of aggregation in multi-good and multi-period economies under preferences which are additively separable in time. We develop an alternative notion of the representative consumer by using the restriction that the differential of demand is linear in commodities. We show that this restriction is a necessary condition for that the preferences of the economy as a whole can be constructed explicitly. The concept of a representative consumer described in the paper makes it possible to extend aggregation results to some cases when demand is not linear in wealth and when aggregate behavior of the multi-consumer economy depends on distributions.

There are two concepts of aggregation in the literature which are related to ours: Gorman's (1953) exact aggregation and Constantinides's (1982) aggregation in the equilibrium point. In the first case, a single agent reproduces aggregate demand in the multi-consumer economy for all distributions and prices, while in the second case, he does so only for distributions and prices that are a priori fixed. We show that the aggregation of demand differentials is a necessary condition for the existence of a representative consumer in the sense of Gorman (1953) and a sufficient condition for a representative consumer in the sense of Constantinides (1982) to be generalized for all distributions and prices. We identify all strictly concave additive utilities which lead to aggregation of the demand differentials.

The rest of the paper is organized as follows. In section 2, we give the formal definition of the aggregation concept based on the demand differentials and derive a necessary condition for this type of aggregation. In section 3, we construct some classes of utility functions which are consistent with the aggregation concept developed. Section 4 concludes.

2.2 Aggregation of demand differential

The economy is populated by a set of types of agents I. The share of agents of type $i \in I$ is μ^i; the total measure of agents is normalized to one, $\int_I d\mu^i = 1$. The variables of agent i are denoted by superscript i. Agents live for T periods. The timing is discrete, $t = 1, ..., T$, where T can be infinite.

All agents have identical preferences of type $\sum_{t=0}^{T} \beta^t U(X_t)$, where the momentary utility, $U : H \subseteq R_{++}^K \to R$, is thrice continuously differentiable, increasing and strictly concave for $\forall X \in H$, and the discount factor $\beta \in (0,1)$ is identical for all agents. The distribution of wealth across agents is given by the function $\{y^i\}_{i\in I} \in R_{++}^I$. Thus, the problem of agent $i \in I$ is

$$\max_{\{X_t^i\}} \sum_{t=0}^{T} \beta^t U\left(X_t^i\right) \mid \sum_{t=0}^{T} P_t' X_t^i = y^i, \tag{2.1}$$

where $X_t^i \equiv (x_{1t}^i, ..., x_{Kt}^i)$ is the agent's vector of commodities in period t and $P_t = (p_{1t}, ..., p_{Kt})$ is the corresponding vector of prices. The prices are strictly positive, $P_t \in R_{++}^K$ for $\forall t \in T$. We will refer to (2.1) as a multi-consumer or heterogeneous agents economy.

We assume that the solution to the utility maximization problem (2.1) of each consumer $i \in I$ exists, is interior and unique. The notation $U_k(\cdot)$ and $U_{kj}(\cdot)$ are used to denote the first and the second order partial derivatives of the function $U(\cdot)$ with respect to the arguments k and k, j respectively.

Let us define a set $\Lambda = \left\{\lambda^i\right\}_{i\in I} \in R_+^I$ such that $\int_I \lambda^i d\mu^i = 1$. The elements of Λ will be called the welfare weights. Define the function $V(X_t, \Lambda)$ by

$$V(X_t, \Lambda) = \max_{\{X_t^i\}} \left\{ \int_I \lambda^i U\left(X_t^i\right) d\mu^i \mid \int_I X_t^i d\mu^i = X_t \right\}. \tag{2.2}$$

Subsequently, consider the following problem

$$\max_{\{X_t\}} \sum_{t=0}^{T} \beta^t V(X_t, \Lambda) \mid \sum_{t=0}^{T} P_t' X_t = y, \tag{2.3}$$

where $y = \int_I y^i d\mu^i$ is aggregate wealth and $X_t = \int_I X_t^i d\mu^i$ is the aggregate vector of commodities in the multi-consumer economy.

Let us show that the problem (2.2) – (2.3) can reproduce the optimal allocations in the multi-consumer economy (2.1). The optimal allocations of each individual $i \in I$ in the multi-consumer economy has to satisfy the $FOCs$

$$U_k\left(X_t^i\right) = \eta_i p_{tk}, \qquad k \in K. \tag{2.4}$$

Definition (2.2) implies

$$\lambda^i U_k\left(X_t^i\right) = V_k\left(X_t, \Lambda\right), \qquad k \in K. \tag{2.5}$$

The $FOCs$ of the problem (2.3) are

$$V_k\left(X_t, \Lambda\right) = \eta p_{tk}, \qquad k \in K, \tag{2.6}$$

where η is the Lagrange multiplier. Therefore, if $\lambda^i = \eta/\eta_i$ then, condition (2.4) is equivalent to (2.5)–(2.6). The constraint in (2.3) obtains by summing of the individual budget constraints in (2.1) and thus, is to be satisfied in the multi-consumer economy as well. Thus, we have a version of the first fundamental theorem of welfare economics.

Proposition 1 *For any distribution of wealth $\{y^i\}_{i\in I}$ and a sequence of prices $\{P_t\}_{t\in T}$, there exists a set of weights Λ such that equilibrium in multi-consumer economy* (2.1) *is a solution to the problem* (2.2) – (2.3).

Definition We call the function $V(X_t, \Lambda)$ and the function $\sum_{t=0}^{T} \beta^t V(X_t, \Lambda)$ the (momentary) utility and the preferences of community, respectively.

If the community preferences are independent of the distribution of utility weights for all such distributions, then economy (2.1) has a representative consumer in the sense of Gorman (1953). A necessary condition for existence of such a representative consumer is that the preferences of the agents in a multi-consumer economy are quasi-homothetic. In this case, the preferences of the representative consumer are also quasi-homothetic and (up to a linear transformation) identical to the preferences of all agents.

An alternative definition of the representative consumer is employed by Constantinides (1982), Eichenbaum, Hansen and Singleton (1988), Huang (1987), Ogaki (1997) and Vilks (1988). These papers do not require that the community preferences are independent of distributions and interpret the constructed function V as the utility of a single representative consumer which changes from one equilibrium point to another. The application of this type of aggregation is limited because in general it is not known how the community preferences depend on distributions and prices. Therefore, our objective will be to find a set of restrictions which are sufficient to insure that the function V can be derived explicitly.

Definition We say that the economy (2.1) allows for aggregation if the community utility V can be constructed explicitly. Aggregation is perfect if V is identical to U up to a linear transformation, i.e. $V \sim U$. Otherwise, aggregation is called imperfect.

To characterize the function V, we employ the FOCs of (2.2)

$$\lambda^i \cdot U_k\left(X_t^i\right) = V_k\left(X_t, \Lambda\right) \mid \int_I X_t^i d\mu^i = X_t. \tag{2.7}$$

The standard approach in the literature to the problem of aggregation is to impose restriction on the individual utilities which are sufficient for the existence of a representative consumer. Given our objective to derive a necessary condition for the existence of such consumer, we will not analyze the $FOCs$

directly. Instead, we will study the properties of the first order differentials of the *FOCs*. The advantage of this approach consists in the fact that there always exists a closed form expression for the demand differentials, while in general, it is not known when demand can be derived explicitly.

Let us fix some particular distribution of wealth and a sequence of prices. This implies that the welfare weights are fixed as well. Introduce the function $\tilde{V}(X_t) = V(X_t, \Lambda)$ corresponding to the fixed set of weights. Eliminating $\tilde{V}_k(\cdot)$ by using (2.6), taking the logarithm of *FOC* (2.7) and totally differentiating the resulting equations, we get that for all $k \in K$

$$\sum_{l=1}^{K} \frac{U_{kl}(X_t^i)}{U_l(X_t^i)} dx_{lt}^i = d\log(\eta) \mid \int_I dX_t^i d\mu^i = dX_t. \tag{2.8}$$

In matrix notations the above relation can be written as

$$D\left(X_t^i\right) dX_t^i = d\log(\eta) \mid \int_I dX_t^i d\mu^i = dX_t,$$

where the matrix $D(X_t^i)$ is defined as

$$D\left(X_t^i\right) = \begin{bmatrix} U_{11}(X_t^i) U_1^{-1}(X_t^i) & \dots & U_{1K}(X_t^i) U_1^{-1}(X_t^i) \\ \dots & \dots & \dots \\ U_{K1}(X_t^i) U_k^{-1}(X_t^i) & \dots & U_{KK}(X_t^i) U_K^{-1}(X_t^i) \end{bmatrix}.$$

As the individual utility is a strictly concave, the matrix D has an inverse. This fact allows us to express demand differentials as

$$dX_t^i = d\log(\eta) D^{-1}\left(X_t^i\right) \times \mathbf{1},$$

where $\mathbf{1}$ is a unit vector of the dimensionality $1 \times K$. The resource constraint implies that integrating demand differentials across agents must give differentials of the corresponding aggregate quantities

$$dX_t = \int_I dX_t^i d\mu^i = d\log(\eta) \int_I D^{-1}\left(X_t^i\right) \times \mathbf{1} d\mu^i.$$

Aggregate demand differential can also be expressed in terms of the function $\widetilde{V}$

$$dX_t = d\log(\eta)\,\widetilde{D}^{-1}(X_t) \times \mathbf{1},$$

where the matrix $\widetilde{D}(X_t)$ is defined as

$$\widetilde{D}(X_t) \equiv \begin{bmatrix} V_{11}(X_t)\,V_1^{-1}(X_t) & \dots & V_{1K}(X_t)\,V_1^{-1}(X_t) \\ \dots & \dots & \dots \\ V_{K1}(X_t)\,V_K^{-1}(X_t) & \dots & V_{KK}(X_t)\,V_K^{-1}(X_t) \end{bmatrix}.$$

Therefore, the function $\widetilde{V}$ has to satisfy the restriction that

$$\widetilde{D}^{-1}(X_t) = \int_I D^{-1}\left(X_t^i\right) \times \mathbf{1} d\mu^i.$$

By definition, the function $\widetilde{V}(X_t)$ and consequently, the matrix $\widetilde{D}(X_t)$ have to be explicitly defined in terms of aggregate quantities. This could only be the case if each element inside the integral is given by a function which is linear in commodities. Precisely, it must be that

$$D^{-1}\left(X_t^i\right) \times \mathbf{1} = A + BX_t^i, \tag{2.9}$$

where A is the $1 \times K$ vector and B is the $K \times K$ matrix of arbitrary constants. However, if so, then it must also be that $\widetilde{D}(X_t) = D(X_t)$ and consequently,

$$d\log\left(\widetilde{V}_k(X_t)\right) = d\log(U_k(X_t)) \qquad \text{for } \forall k \in K.$$

Solving the above differential equations, we obtain a set of restrictions which identifies the first order partial derivatives of the function V

$$\widetilde{V}_k(X_t) = \xi_k U_k(X_t), \qquad \text{for } \forall k \in K. \tag{2.10}$$

where each ξ_k is the parameter from integration. Given that a choice of distribution of wealth and prices fixed to derive (2.10) is arbitrary, we have:

$$V(X_t, \Lambda) = \max_{\{X_t^i\}} \left\{ U\left(\int_I X_t^i d\mu^i\right) \mid \int_I X_t^i \xi_k^{-1} d\mu^i = X_t \right\}, \tag{2.11}$$

where the values of the parameters ξ_k's are determined by a given distribution of initial endowments and given prices in the multi-consumer economy, $\xi_k = \xi_k\left(\{P_t\}_{t\in T}, \{y_i\}_{i\in I}\right)$. The following proposition summarizes the results of the previous analysis and provides a condition which is necessary for the existence of the explicit preferences of community.

Proposition 2 *The multi-consumer economy* (2.1) *allows for aggregation only if the utility function U satisfies restriction* (2.9) *in which case the community momentary utility is given by* (2.11).

The results of the above proposition makes it possible to establish whether a given utility function U is consistent with aggregation and if so, to construct the corresponding utility of community. Observe that our aggregation results differ from the standard Gorman's (1953) aggregation in an important respect: according to (2.11), the preferences of the representative consumer can be affected by distribution of wealth. Our next objective will be to analyze when such imperfect aggregation can occur.

Consider the case of the individual utility U such that all elements of Hessian matrix are not equal to zero, $U_{kj} \neq 0$ for any $k, j \in K$. Then, it follows from equation (2.10) that for any $k, j \in K$ we have

$$\widetilde{V}_{kj}(X_t) = \xi_k U_{kj}(X_t) \qquad \text{and} \qquad \widetilde{V}_{jk}(X_t) = \xi_j U_{jk}(X_t)$$

These restriction can only be satisfied simultaneously if all ξ_k's are identical, i.e., $\xi_k = \xi$ for all $k \in K$. However, given that all heterogeneity parameters are identical, the function V is a linear transformation of the individual utility, $V \sim U$, which means that the aggregation is perfect.

In order to have imperfect aggregation, Hessian matrix must have a block diagonal structure. Consider a partition of the commodity vector X_t^i into subvectors $X_t^i = \left\{(X_t^i)^{(1)}, ..., (X_t^i)^{(M)}\right\}$ such that the utility $U(X_t)$ can be represented as a direct sum of M subfunctions

$$U\left(X_t^i\right) = \sum_{m=1}^{M} U^{(m)}\left(\left(X_t^i\right)^{(m)}\right). \tag{2.12}$$

We assume that the partition is maximal in the sense that each subfunction $U^{(m)}\left(X_t^{(m)}\right)$ cannot be subdivided into more additive peaces. According to result (2.11), the community utility can be represented as

$$V\left(X_t, \Lambda\right) = \sum_{m=1}^{M} \xi^{(m)} U^{(m)}\left(X_t^{(m)}\right), \tag{2.13}$$

where $\xi^{(m)}$ is the set of the heterogeneity parameters. Since a linear transformation of the utility has no effect on the optimal allocations, the number of the parameters can be reduced from M to $(M-1)$ by renormalizing the heterogeneity parameters such that $\xi^{(1)} = 1$. This yields the following result.

Proposition 3 *The multi-consumer economy* (2.1) *allows for aggregation if and only if the community utility can be represented in the form* (2.13) *in which case it depends on at most* $(M-1)$ *heterogeneity parameters.*

A straightforward implication of this proposition is that imperfect aggregation can occur only if the individual utility has the additive representation (2.12) with $M > 1$. Other utility functions can be consistent only with the representative consumer in the sense of Gorman (1953).

2.3 Sufficient conditions

In this section, we develop several classes of the utility functions which lead to the representative consumer in the sense of demand differentials. We begin from the case when there is a single commodity in the economy $x_t \equiv x_{t1}$. Therefore, restriction (2.9) can be rewritten as

$$u'(x_t)/u''(x_t) = \zeta_1 + \zeta_2 x_t, \tag{2.14}$$

where ζ_1 and ζ_2 are arbitrary constants. This differential equation is well-researched in the literature. Solutions to (2.14) are given by the Hyperbolic Absolute Risk Aversion (HARA) class of functions $\left\{u^c, u^{\ln}, u^{\exp}\right\}$

$$u_c(x) = a(\delta(x-b))^c, \quad \begin{array}{llll} \delta = 1, & c < 0, & a < 0, & x > b, \\ \delta = 1, & 0 < c < 1, & a > 0, & x > b, \\ \delta = -1, & c > 1, & a < 0, & x > b; \end{array}$$

$$u_{\ln}(x) = a\ln(x-b), \qquad a > 0,\ x > b;$$

$$u_{\exp}(x) = a\exp[-bx], \qquad a < 0, \quad b > 0.$$

The result of aggregation under the assumption of HARA utility function is also well-known. Gorman (1953) showed that if individual's preferences are quasi-homothetic, then the Engel curves for a given price vector are straight lines with the same slopes for all agents and consequently, the excess demand functions can be aggregated. Pollak (1971) demonstrated that in a one-period and multi-commodity economy, additive utility functions yielding demands which are linear in wealth are members of the generalized Bergson class

$$U^1(X) = \sum_{k=1}^{K} a_k (\delta(x_k - b_k))^c,$$

$$U^2(X) = \sum_{k=1}^{K} a_k \ln(x_k - b_k),$$

$$U^3(X) = \sum_{k=1}^{K} a_k \exp[-b_k x_k].$$

Finally, Rubinstein (1981) was first to mention that the multi-period economy with one commodity where agent's preferences are additively separable in time has the same aggregation implications as the multi-commodity economy existing only for one period. In the latter case, a single commodity in different periods is interpreted as Pollak's (1971) different commodities.

While a one commodity case is simple to analyze, in general not much can be said regarding the case when the number of commodities is larger then one. We provide as an example the sufficiency conditions in the case of two commodities

$$\frac{U_{11}U_2 - U_{12}U_1}{U_{11}U_{22} - U_{12}^2} = \zeta_1 + \zeta_2 x_{1t} + \zeta_3 x_{2t},$$

$$\frac{U_{22}U_1 - U_{12}U_2}{U_{11}U_{22} - U_{12}^2} = \vartheta_1 + \vartheta_2 x_{1t} + \vartheta_3 x_{2t},$$

where $\zeta_1 - \zeta_3$ and $\vartheta_1 - \vartheta_3$ are arbitrary constants. The only point of this example is to illustrate difficulties which arise when trying to derive sufficiency conditions. The above system of equations contains the partial derivatives of the utility function. With few exceptions, closed form solutions to such systems of equations are unknown.

2.3.1 Additive utility functions

Definition A utility $U(X)$ is called additive if there exist K functions $u^k(x_k)$ and a thrice differentiable function $F, F' > 0$ such that

$$U(X) = F\left(\sum_{k=1}^{K} u_k(x_k)\right). \tag{2.15}$$

If $F'' = 0$, then $U(X)$ is called strongly additive.
If $F'' \neq 0$, then $U(X)$ is called weakly additive.
If $\exists k$ such that $u_k'' = 0$, then $U(X)$ is called quasi-linear.

In the remainder of the section, we will identify strongly additive, weakly additive and quasi-linear utility functions which are consistent with the property of aggregation. As we have already analyzed one-commodity case, we assume that the number of commodities is larger then one, $K \geq 2$.

Strongly additive utility

In this section, we assume that the momentary utility is strongly additive which implies that the agent's preferences are given by

$$\sum_{t=1}^{T} U\left(X_t^i\right) = \sum_{t=1}^{T} \sum_{k=1}^{K} u_k\left(x_{tk}^i\right). \tag{2.16}$$

Under the assumption of strongly additive utility, the cross derivatives of the matrix D are zeros and therefore, inverse of matrix D can be written as

$$\mathbf{D} = \begin{bmatrix} u_1'(x_1)/u_1''(x_1) & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & u_K'(x_K)/u_K''(x_K) \end{bmatrix},$$

where for simplicity we omit the time subscript and agent's superscript.

Restriction (2.9) implies that each subfunctions $u_k(x_k)$ is to be a solution to differential equation (2.14) and thus be a member of the HARA class $\{u_c, u_{\ln}, u_{\exp}\}$. If a subfunction u_k is power, $u_k = u_c$, then FOC (2.7) is

$$\lambda^i a_k c_k \left(x_{kt}^i - b_k\right)^{c_k - 1} = \eta p_{tk}.$$

Expressing from the last condition x_{kt}^i and integrating across agents gives:

$$x_{kt} = b_k + \left(\frac{\eta p_{tk}}{a_k c_k}\right)^{1/(c_k-1)} \int_I \left(\lambda^i\right)^{1/(1-c_k)} d\mu^i.$$

Rearranging the terms, the last expression can be written as

$$\left(\int_I \left(\lambda^i\right)^{1/(1-c_k)} d\mu^i\right)^{1-c_k} a_k c_k \left(x_{kt}^i - b_k\right)^{c_k-1} = \eta p_{tk}.$$

Logarithmic and exponential subfunctions $u_{\ln}$and $u_{\exp}$ lead to the following *FOC*s

$$\lambda^i a_k / \left(x_{kt}^i - b_k\right) = \eta p_{tk},$$

$$-\lambda^i a_k b_k \exp\left(-b_k x_{kt}^i\right) = \eta p_{tk}.$$

Similarly, it can be shown that the aggregate quantities in these two cases are respectively

$$a_k / \left(x_{kt}^i - b_k\right) = \eta p_{tk},$$

$$-\exp\left(\int_I \ln \lambda^i d\mu^i\right) a_k b_k \exp\left(-b_k x_{kt}^i\right) = \eta p_{tk}.$$

To generalize the above results, let us associate to each of the functions u_c, $u_{\ln}$, $u_{\exp}$ the parameters $\xi(u_c)$, $\xi(u_{\ln})$, $\xi(u_{\exp})$ such that

$$\xi(u_c) = \left(\int_I \left(\lambda^i\right)^{1/(1-c)} d\mu^i\right)^{1-c}, \quad \xi(u_{\ln}) = 1, \quad \xi(u_{\exp}) = \exp\left(\int_I \ln \lambda^i d\mu^i\right).$$

Proposition 4 *If $U(X_t)$ is given by* (2.16), *then:*
a) multi-consumer economy (2.1) *allows for aggregation if and only if $u_k(x_k) \in \{u_c, u_{\ln}, u_{\exp}\}$ for $\forall k \in K$;*
b) the community utility has the form $V(X) = \sum_{k=1}^{K} u_k(x_k) \cdot \xi(u_k)$, where each $\xi(u_k)$ corresponds to a given subfunction u_k from $\{u_c, u_{\ln}, u_{\exp}\}$.

This proposition is a particular case of general aggregation result (2.13). Here, we derive explicitly the relation between the heterogeneity parameters and welfare weights. Notice that if $\xi(u_k) = \overline{\xi}$ for $\forall k \in K$, then the community utility is identical to the individual utility (up to a linear transformation), or in other words, the economy (2.1) allows for perfect aggregation. This is possible in three cases, precisely, when all subfuctions u_k are of (*i*) logarithmic, (*ii*) exponential and (*iii*) power classes such that the power is the same for all commodities. These three are the members of the generalized Bergson family by Pollak (1971).

Apart from known cases of Gorman's (1953) aggregation, the property that demand differentials are linear in commodities pins down imperfect aggregation which is the case when aggregate equilibrium allocations depend on distribution of wealth through the heterogeneity parameters $\xi(u_k)$. Such parameters affect the marginal rate of substitution between aggregate quantities in the equilibrium. Examples of imperfect aggregation are know to the literature since Shafer (1977). The main contribution of the above proposition, therefore, consists in establishing necessary and sufficient conditions for aggregation under the assumption of additivity.

Weakly additive utility

The weakly additive utility $F(U(X))$ is defined as an increasing non-linear transformation of a strongly additive utility $U(X)$. In a one-period economy, such a transformation does not affect the optimal allocation. In other words, both $U(X)$ and $F(U(X))$ lead to the same equilibrium. In a multi-period setup, however, the function F affects the intertemporal marginal rate of substitution between commodities so that the optimal allocation under the preferences $\sum_{t=0}^{T} \beta^t U(X_t)$ differs from that under $\sum_{t=0}^{T} \beta^t F(U(X_t))$. The results of the proposition 3 implies that this type of the utility may lead only to Gorman's (1953) representative consumer.

If the momentary utility is weakly additive, then

$$\frac{U_{kk}}{U_k} = \frac{F''u_k' + F'u_k''}{F'u_k^i} = \frac{F''}{F'}u_k' + \frac{u_k''}{u_k'},$$

$$\frac{U_{kl}}{U_k} = \frac{F''}{F'}u_l',$$

where again, for convenience, we omit the time subscript and the agent's superscript. Consequently, the matrix D can be written as follows

$$D = \begin{bmatrix} \frac{F''}{F'}u_1' + \frac{u_1''}{u_1'} & \dots & \frac{F''}{F'}u_K' \\ \dots & \dots & \dots \\ \frac{F''}{F'}u_1' & \dots & \frac{F''}{F'}u_K' + \frac{u_K''}{u_K'} \end{bmatrix}.$$

According to (2.8) , we receive

$$\frac{u_k''}{u_k'}dx_k = d\log(\eta) - \frac{F''}{F'}\sum_{m=1}^{K} u_m' dx_m. \tag{2.17}$$

Using the last result, it can be shown that

$$dx_k = \frac{u_k'/u_k'' \cdot d\log(\eta)}{1 + \frac{F''}{F'}\sum_{m=1}^{K}(u_k'')^2/u_k''}.$$

In order integrability condition (2.9) holds, it is necessary that

$$\frac{F''}{F'}\sum_{m=1}^{K}\frac{(u_m')^2}{u_m''} = C, \tag{2.18}$$

where C is an arbitrary constant such that $C \neq -1$. The last equation has a solution only if the function U is one of U^1, U^2, U^3 from the generalized Bergson class. In these cases, we have respectively

$$\sum_{m=1}^{K}\frac{(u_m')^2}{u_m''} = \frac{c}{c-1}\sum_{m=1}^{K}u_m; \quad \sum_{m=1}^{K}\frac{(u_m')^2}{u_m''} = -2; \quad \sum_{m=1}^{K}\frac{(u_m')^2}{u_m''} = \sum_{m=1}^{K}u_m.$$

It is easy to check that in order to be consistent with the above results, the function F is to be power under the utilities U^1 and U^3 and exponential under the utility U^2. Consider the following family of functions

$$U^4(X) = \frac{\delta\left[\left(\delta U^1(X)\right)^{1-\sigma} - 1\right]}{1-\sigma}, \qquad \begin{array}{lll} c<0,\ c>1 & \delta=-1, & \sigma<1-1/c, \\ 0<c<1, & \delta=1, & \sigma>0, \end{array}$$

$$U^5(X) = \frac{\exp\left((1-\sigma)U^2(X)\right)-1}{1-\sigma} = \frac{\prod_{k=1}^{K}(x_k-b_k)^{(1-\sigma)a_k}-1}{1-\sigma}, \qquad \begin{array}{l} \sigma>0, \\ \sum a_k = 1. \end{array}$$

$$U^6(X) = \frac{\delta\left[\left(\delta\left(B+U^3(X)\right)\right)^{1-\sigma} - 1\right]}{1-\sigma}, \qquad \begin{array}{lll} \delta=1, & \sigma>0, & B+U^3(X)>0, \\ \delta=-1, & \sigma<0, & B+U^3(X)<0. \end{array}$$

If $\sigma \longrightarrow 1$, we obtain in the limit the log functions. The restrictions on the parameters δ and σ are such that the constructed utility are strictly concave.

Proposition 5 *If $U(X)$ is weakly additive, then:*
a) economy (2.1) allows for aggregation if and only if $U(X) \in \{U^4,\ U^5,\ U^6\}$;
b) aggregation is perfect, i.e. $V \sim U$.

Some of these results are known to the literature. Blackorby and Schworm (1993) showed that perfect aggregation obtains if an agent's utility is given by the constant elasticity of substitution function which corresponds to U^4 with $b_k = 0$ for $\forall k \in K$ and $\sigma = 1 - 1/c$, $0 < c < 1$, and its limiting case, the Cobb-Douglas function which obtains from U^5 if $b_k = 0$ for $\forall k \in K$. Proposition 4 extends the results of perfect aggregation under U^4 to include a large range of the parameters and shows that perfect aggregation obtains also under the utility U^6 which is a power transformation of U^3.

Quasi-linear utility

Consider the case when the utility is quasi-linear only in one commodity. Specifically, let the utility $U(\tilde{x}, X)$ be such that

$$U(\tilde{x}, X) = F(\tilde{x} + G(X)), \quad G(X) = \sum_{k=1}^{K} g_k(x_k), \tag{2.19}$$

where $g_k' > 0$ and $g_k'' < 0$ for all $k \in K$ and $F'' \neq 0$.

Under this utility, the FOCs are the following

$$\lambda^i F'\left(\tilde{x}_t^i + G\left(X_t^i\right)\right) = \eta \tilde{p}_t, \tag{2.20}$$

$$\lambda^i F'\left(\tilde{x}_t^i + G\left(X_t^i\right)\right) g'\left(x_{kt}^i\right) = \eta p_{tk}, \quad k \in K,$$

where $\tilde{p}_t$ is the price of the quasi-linear commodity in period $t \in T$. Dividing the second condition by the first implies $g'\left(x_{kt}^i\right) = p_{tk}/\tilde{p}_t$ and therefore, $x_{kt}^i = x_{kt}$ for all $k \in K$. Taking the logarithms of (2.20), differentiating and rearranging the terms, we obtain

$$d\tilde{x}_t^i = \frac{F'\left(\tilde{x}_t^i + G\left(X_t^i\right)\right)}{F''\left(\tilde{x}_t^i + G\left(X_t^i\right)\right)} d\log(\eta) - \sum_{k=1}^{K} g_k'\left(x_{kt}\right) dx_{kt}.$$

Integrability condition (2.9) implies that

$$\frac{F'\left(\tilde{x}_t^i + G\left(X_t^i\right)\right)}{F''\left(\tilde{x}_t^i + G\left(X_t^i\right)\right)} = \zeta + \varsigma\left(\tilde{x}_t^i + G\left(X_t^i\right)\right),$$

where ζ, ς are two arbitrary constants. The solutions to the above equations are power and exponential functions. Consider two classes of functions

$$U^7(X) = \frac{\delta\left[\left(\delta U(\tilde{x}, X)\right)^{1-\sigma} - 1\right]}{1-\sigma}, \quad \begin{array}{lll} \delta = 1, & \sigma > 0, & U(\tilde{x}, X) > 0, \\ \delta = -1; & \sigma < 0, & U(\tilde{x}, X) < 0, \end{array}$$

$$U^8(X) = \frac{\exp\left(\left(1-\sigma\right) U(\tilde{x}, X)\right) - 1}{1-\sigma}, \quad \sigma > 1.$$

When $\sigma \to 1$, U^7 becomes the log function, $\ln U(\tilde{x}, X)$. Again, the restrictions on the parameters δ and σ are imposed to ensure strict concavity.

Proposition 6 *If $U(X)$ is quasi-linear, then:*
a) economy (2.1) allows for aggregation if and only if $U(X) \in \{U^7, U^8\}$;
b) aggregation is perfect, i.e. $V \sim U$;

It can be shown that specification (2.19) is the only case when the quasi-linear utility is strictly concave. In other cases, such as $F'' = 0$ or when there are more then one commodity which enter linearly in the utility, the assumption of strict concavity is not satisfied.

2.4 Conclusion

We have analyzed the possibility of aggregation in multi-period and multi-good economies when community preferences are not restricted to coincide with preferences of each individual. The concept of aggregation developed in the paper allows us to construct a single agent that can replicate the behavior of some multi-consumer economies in which aggregate allocation depends on the distribution of wealth. The main result of the paper consists in providing a necessary condition for that the community preferences in the multi-consumer economy with the fixed distribution of wealth can be constructed explicitly.

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Chapter 3

Heterogeneity and Indivisible Labor

Joint with Lilia Maliar

3.1 Introduction

Hansen (1985) and Rogerson (1988) construct a neoclassical model in which the time allocated to the market job can take only two possible values, a fixed number of hours or zero hours. This model is referred to in the literature as an "indivisible labor model". If agents are ex-ante identical, the model admits a representative agent whose preferences are linear in leisure. In this paper, we address the following two questions: Does this result still holds if agents are ex-ante heterogeneous in such dimensions as initial endowments and productivity? What implications has the fact of the existence of such representative consumer for distributions in the underlying heterogeneous agents economies?

The assumption of indivisible labor implies that the agents' budget sets are not convex. The standard way to convexify the commodity space is to introduce employment lotteries. Hansen (1985) also assumes a perfect risk-sharing between agents. As a result, the agent makes the same investment decisions, independently of the realization of employment lottery, and therefore, the agent's records of employment have no effect on his current wealth. If agents are ex-ante identical, they will hold the same amount of wealth in each period. It turns out, however, that even ex-ante identical agents may have different sequences of employment probabilities. Indeed, the agent's preferences are linear in probability, and thus, the expected discounted leisure in different periods is perfect substitute. Therefore, the agent will be indifferent between any sequences of probabilities, which imply the same expected discounted sum of leisure. From the point of view of the whole economy, subdivisions of probabilities between different agents do not matter as long as such subdivisions lead to the same aggregate level of employment.

Such indeterminacy of the equilibrium does not create problems when individuals are ex-ante identical, however, it does so in the case of heterogeneous consumers. Specifically, if agents are ex-ante identical, the economy admits a representative consumer, and therefore, aggregate equilibrium allocation can be restored without distributive concerns. In order to decentralize such economy one can assume the "symmetric" equilibrium such that all decisions of the agents including those on sequences of employment probabilities are exactly identical. In a heterogeneous agents economy, however, the symmetric equilibrium does not exists; here, asymmetric equilibria are to be constructed.

There can be infinitely many asymmetric equilibria in the model. One example would be the following: the agents agree on arrangement when today one group leaves for vacations and another works with probability one, and tomorrow the two groups interchange. Another way to decentralize the aggregate economy is to assume sunspots instead of lotteries. Shell and Wright (1993), and Kehoe, Levine and Prescott (1998) constructs examples of exchange economies and indivisibilities in which the resource allocation process is governed entirely by extrinsic uncertainty. Coming back to our example, the corresponding sunspots equilibrium will be the following: the first group works when it is raining, and the other when it is sunny. Other related literature is Cho (1995) who extend the benchmark model to include temporary heterogeneity in skills and Prasad (1995) who assumes families where two members differ in productivity. The last two papers focus only on the models' predictions at the aggregate level.

The objective of this paper is to provide a systematic study of a general equilibrium model with indivisible labor and ex-ante heterogeneous agents. We focus on both distributive and aggregate implications of the model. We introduce ex-ante heterogeneity by assuming that the economy consists of a number of heterogeneous types of agents. Within each type, there is a continuum of ex-ante identical individuals. Across types agents differ in initial endowment of wealth and productivity of labor.

To convexify the consumption set, we employ the lotteries, however, we assume that there is a distinct employment lottery for each type of agents. Also, we assume the insurance company which sets type-specific prices and do not allow for across-types reselling of the lotteries. Such assumptions on lotteries and risk-sharing allow us to derive within-type aggregation. This result consist in that the behavior of each type of agents is reproduced by a single individual characterized by initial endowment and productivity. Although each agent within the type faces indivisible choice of labor, the agent which represents the type behaves as if he has divisible labor choice and the utility which is linear in leisure. This outcome is parallel to the one which obtains in Hansen's (1985).model with ex-ante identical agents for the economy as a whole.

Consequently, we study the possibility of aggregation across types. We derive restrictions on preferences and the production function which are necessary and sufficient for the existence of the representative consumer whose preferences are linear in leisure. Such restrictions guarantee that, at the aggregate level, the behavior of the heterogeneous economy is indistinguishable from the Hansen's (1985) model with ex-ante homogeneous agents. We describe several specifications of the model in which such restrictions are satisfied.[1] We show the possibility of aggregation not only for the economy with heterogeneity in endowments but also for the one with heterogeneity in endowments and productivity, including the case when differently skilled agents are not perfectly substitutable in production.

Regarding the distributive implications, the model with heterogeneity in endowments has indeterminate predictions for individual hours and counterfactual predictions for individual consumption, which does not depend on the level of wealth. The same is true for the case of heterogeneity in both endowments and skills, where differently skilled agents are perfectly substitutable in production. However, the assumption that individual efforts are imperfect substitutes allows us to overcome both of these problems. In the last model, working hours are uniquely determined and consumption depends on both endowment and productivity of the agent.

The rest of the paper is organized as follows. Section 2 describes the model economy in which individuals have indivisible choice of labor. Section 3 shows that at the level of types, this model is equivalent to the divisible labor model in which the representative agent of each type has the utility linear in leisure. Section 4 discusses the possibility of aggregation across types. Finally, Section 6 concludes.

[1]This result is in contrast with the existing mistaken view in the literature that the assumption of quasi-linear preferences is inconsistent with the property of aggregation, see, e.g., Blackorby and Schworm (1993).

3.2 The model

The economy consists of S types of infinitely lived heterogeneous consumers, an output producing firm and an insurance company. We normalize the set of types to unity, $\int_S ds = 1$. Within each type there is a continuum of identical consumers with names on the unit interval.[2] The agents' types differ in skills and initial endowments. The skills are permanent characteristics of agents. We denote the skills of agents of type $s \in S$ by β^s, where $\beta^s > 0$ for $\forall s \in S$ and $\int_S \beta^s ds = 1$. The initial endowment of agents of this type is denoted by k_0^s. Time (t) is discrete and the horizon is infinite: $t \in T$, where $T = 0, 1, ..., \infty$. There is no uncertainty in the economy.[3]

The consumer of type $s \in S$ (further, agent, individual, etc.) maximizes expected life-time utility discounted at the rate $\delta \in (0, 1)$ by choosing the probability of being employed and consumption. In the beginning of each period, the agent is jobless. Job opportunities come at random, depending on the realization of employment lottery. The winning probability of the lottery is given by the probability of being employed chosen by the agent. If the agent wins the lottery, he gets a job and supplies a fixed number of hours, $\overline{n}$, in exchange for the efficiency wage. In the opposite case, the agent does not work. Before the outcome of the lottery is known, the agent can buy unemployment insurance which pays one unit of consumption if the agent is unemployed and zero otherwise. The agent is endowed with one unit of time; so that leisure in the employed and unemployed states is given by $1 - \overline{n}$ and 1 respectively. The agent owns the capital stock and rents it to the firm. Capital depreciates at the rate $d \in (0, 1]$. Therefore, the problem solved by the consumer is the following

[2] Alternatively, we could assume that there is a continuum of agents on the unit interval and the number of types is finite.

[3] All the results of the paper can be extended to the case of uncertainty, if we assume complete markets and introduce the Arrow-Debreu securities.

$$\max_{\{x_t^s\}} \sum_{t=0}^{\infty} \delta^t \left\{\varphi_t^s u\left(c_t^{s,e}, 1-\overline{n}\right) + \left(1-\varphi_t^s\right) u\left(c_t^{s,u}, 1\right)\right\} \tag{3.1}$$

$$\begin{aligned} \text{s.t.} \qquad & c_t^{s,e} + k_{t+1}^{s,e} + p_t^s y_t^s = k_t^s(1-d+r_t) + \overline{n}w_t^s, \\ & c_t^{s,u} + k_{t+1}^{s,u} + p_t^s y_t^s = k_t^s(1-d+r_t) + y_t^s, \end{aligned} \tag{3.2}$$

where $\{x_t^s\} = \left\{\varphi_t^s, c_t^{s,j}, k_{t+1}^{s,j}, y_t^s, \right\}_{t\in T}^{j\in\{e,u\}}$. Here, the superscript $j \in \{e,u\}$ refers to employed and unemployed states; $c_t^{s,j}$, $k_{t+1}^{s,j}$ denote consumption and capital chosen by the agent in state j. The variable y_t^s denotes individual holdings of unemployment insurance; the price of one unit of unemployment insurance is p_t^s. The prices of capital and labor are r_t and w_t^s. The variables φ_t^s and $(1-\varphi_t^s)$ denote the probabilities of employed and unemployed states respectively. The function $u(\cdot)$ is concave, strictly increasing in both arguments and twice continuously differentiable in consumption.

The representative firm rents capital, k_t, and hires labor, h_t, to maximize period-by-period profits. Capital input is given by $k_t = \int_S k_t^s ds$. Labor input is determined by a function $h(\cdot)$, which depends on both skills and working efforts of the consumers. Therefore, the problem solved by the firm is

$$\max_{k_t,\ \{n_t^s\}^{s\in S}} \pi_t^F = f(k_t, h_t) - r_t k_t - \int_S w_t^s n_t^s ds \tag{3.3}$$

$$\text{s.t.} \qquad h_t = h\left(\{n_t^s, \beta^s\}^{s\in S}\right), \tag{3.4}$$

where $n_t^s = \overline{n}\varphi_t^s$ is the total amount of labor of type s hired by the firm. The production function $f(\cdot)$ has constant returns to scale, is strictly concave, continuously differentiable, strictly increasing with respect to both arguments and satisfies the appropriate Inada conditions.

The insurance company maximizes period-by-period expected profits by each type

$$\max_{\{y_t^s\}^{s\in S}} \pi_t^{IC} = \int_S y_t^s p_t^s ds - \int_S (1 - \varphi_t^s)\, y_t^s ds. \tag{3.5}$$

Also, to insure no-arbitrage condition, we assume that trade in insurance contracts between agents is not allowed. The above insurance company can be viewed as an extension of Hansen's (1985) risk-sharing arrangement to the heterogeneous case.

An equilibrium in economy (3.1), (3.2), (3.3), (3.4) is defined as a sequence of consumers' allocation $\left\{c_t^s, \varphi_t^s, k_{t+1}^s\right\}_{t\in T}^{s\in S}$, the firm's allocation $\{k_t, n_t^s\}_{t\in T}^{s\in S}$ and the prices $\{r_t, w_t^s\}_{t\in T}^{s\in S}$ such that given the prices, the allocation of each agent $s \in S$ solves his utility maximization problem (3.1), (3.2), the allocation of the firm leads to zero-profit solution to (3.3), (3.4) for $\forall t \in T$, capital and labor markets clear and the economy's resource constraint is satisfied. Also, $c_t^s \geq 0$, and $1 \geq n_t^s \geq 0$ and $w_t^s, r_t, k_t \geq 0$ for $\forall s \in S$, $t \in T$.

3.3 Aggregation within types

In this section, we show that the behavior of each type of agents is reproduced by a single composed individual who behaves as if he has divisible labor choice and the utility which is linear in leisure. Such consumer is characterized by initial endowment and productivity.

To derive the individual FOCs, we use the value function representation of the agent's problem

$$\max_{\{x_t^s\}} V^s(k_t, k_t^s) = \tag{3.6}$$

$$\varphi_t^s \left[u\left(c_t^{s,e}, 1-\overline{n}\right) + \delta V^s\left(k_{t+1}, k_{t+1}^{s,e}\right)\right] + \left(1-\varphi_t^s\right)\left[u\left(c_t^{s,u}, 1\right) + \delta V^s\left(k_{t+1}, k_{t+1}^{s,u}\right)\right]$$

$$\text{s.t.} \qquad (3.2),$$

where $\{x_t^s\} = \left\{\varphi_t^s, c_t^{s,j}, k_{t+1}^{s,j}, y_t^s,\right\}_{t\in T}^{j\in\{e,u\}}$ and V^s is the value function of agent $s \in S$.

The first order conditions (FOCs) with respect to unemployment insurance holdings, capital in employed and unemployed states respectively are

$$\varphi_t^s p_t^s u_1(c_t^{s,e}, 1-\overline{n}) = (1-\varphi_t^s)(1-p_t^s)\, u_1(c_t^{s,u}, 1), \tag{3.7}$$

$$\begin{aligned} u_1\left(c_t^{s,e}, 1-\overline{n}\right) &= \delta \frac{\partial V^s\left(k_{t+1}, k_{t+1}^{s,e}\right)}{\partial k_{t+1}^{s,e}}, \\ u_1\left(c_t^{s,u}, 1\right) &= \delta \frac{\partial V^s\left(k_{t+1}, k_{t+1}^{s,u}\right)}{\partial k_{t+1}^{s,u}}, \end{aligned} \tag{3.8}$$

where u_1 is the derivative with respect to consumption.

Under the zero-profit assumption, the equilibrium price of insurance is $p_t^s = 1 - \varphi_t^s$. This result together with (3.8) gives the risk-sharing condition

$$u_1\left(c_t^{s,e}, 1-\overline{n}\right) = u_1\left(c_t^{s,u}, 1\right). \tag{3.9}$$

Equations (3.8) and (3.9) imply that the holdings of capital in both states are the same, i.e. $k_{t+1}^{s,e} = k_{t+1}^{s,u}$. Substituting this result into the state contingent constraints (3.2) gives the equilibrium holdings of unemployment insurance

$$y_t^s = \overline{n} w_t^s - c_t^{s,e} + c_t^{s,u}. \tag{3.10}$$

Finding $\partial V^s / \partial k_t^s$, updating it and combining the resulting condition with (3.8) and (3.9) , we obtain the standard intertemporal condition

$$u_1(c_t^{s,e}, 1-\overline{n}) = \delta\left[(1-d+r_{t+1})\, u_1\left(c_{t+1s}^{e}, 1-\overline{n}\right)\right]. \tag{3.11}$$

Using condition (3.10) and the result that $k_{t+1}^{s,e} = k_{t+1}^{s,u}$, we can replace the state contingent constraints (3.2) by a single one

$$\varphi_t^s c_t^{s,e} + (1-\varphi_t^s)\, c_t^{s,u} + k_{t+1}^{s,j} = k_t^s(1-d+r_t) + \varphi_t^s \overline{n} w_t^s. \tag{3.12}$$

Therefore, the agent faces the same constraint (3.12) independently of his employment status. Finally, maximization of (3.6) subject to (3.12) with respect to φ_t^s yields

$$u(c_t^{s,e}, 1-\overline{n}) - u(c_t^{s,u}, 1) + u_1(c_t^{s,e}, 1-\overline{n})\left(\overline{n}w_t^s - c_t^{s,e} + c_t^{s,u}\right) = 0. \quad (3.13)$$

In the paper, we assume that the utility of each agent $s \in S$ is separable in the consumption-leisure decisions[4]

$$u\left(c_t^s, 1-n_t^s\right) = v\left(c_t^s\right) + \varpi\left(1-\overline{n}\right).$$

Let us characterize now the behavior of individuals of type $s \in S$ as a group. Assuming that $B = \varpi\left(1-\overline{n}\right)/\overline{n}$ and $n_t^s = \overline{n}\varphi_t^s$, and taking into account that consumption in two states is equal (this follows by (3.9)), we can rewrite (3.1), (3.12) as follows (up to an additive constant in utility)

$$\max_{\left\{c_t^s, n_t^s, k_{t+1}^s\right\}_{t\in T}} \sum_{t=0}^{\infty} \delta^t \left\{v\left(c_t^s\right) + B\cdot\left(1-n_t^s\right)\right\} \quad (3.14)$$

$$\text{s.t.} \quad c_t^s + k_{t+1}^s = w_t^s n_t^s + k_t^s(1-d+r_t). \quad (3.15)$$

This problem describes the behavior of average quantities of type s. Therefore, (3.14), (3.15) is a problem solved by the representative consumer of type s (further, consumer, agent, etc.). As we see, at the level of type, this model is equivalent to the divisible labor model in which the representative agent of each type has the quasi-linear utility.

[4]This assumption is not necessary to achieve aggregation within the type, however, necessary in order the resulting model is identical to the standard divisible labor setup, see, Hansen and Prescott (1995).

3.4 Aggregation across types

In this section, we derive conditions under which the model allows for aggregation across types and discuss the distributive implications of the model.

The *FOC*s of the utility-maximization problem of the representative agent of type $s \in S$ with respect to capital, consumption and hours worked are the following

$$\mu_t = \delta\mu_{t+1}\left(1 - d + r_{t+1}\right), \tag{3.16}$$

$$v'\left(c_t^s\right) = \mu_t\mu^s, \tag{3.17}$$

$$B = \mu_t\mu^s w_t \frac{\partial h_t}{\partial n_t^s}, \tag{3.18}$$

where v' is the derivative of the function $v\left(\cdot\right)$. From the profit-maximization conditions of the firm $r_t = \partial f\left(k_t, h_t\right)/\partial k_t$ and $w_t^s = w_t \partial h_t/\partial n_t^s$ where $w_t = \partial f\left(k_t, h_t\right)/\partial h_t$ is the marginal product of labor input.

Here, μ_t^s is the Lagrange multiplier associated with the budget constraint of agent $s \in S$; μ_t^s can be represented as $\mu_t\mu^s$ because the ratio of marginal utilities of any two agents in the economy remains constant in all periods. In fact, if one formulates the associated planner's problem, μ_t will be the Lagrange multiplier associated with the economy's resource constraint and $1/\mu^s$ will be a welfare weight assigned by the planner to individual $s \in S$. Without loss of generality, we normalize $\{\mu^s\}^{s\in S}$ to unity, $\int_S 1/\mu^s ds = 1$.

Applying forward recursion to budget constraint (3.15) of each agent $s \in S$ and imposing the transversality condition, $\lim_{t\to\infty} \mu_t^s k_{t+1}^s = 0$, gives us the agent's life-time budget constraint

$$\sum_{\tau=0}^{\infty} \delta^\tau \frac{v_1\left(c_\tau^s\right)}{v_1\left(c_0^s\right)}\left(c_\tau^s - n_\tau^s w_\tau^s\right) = k_0^s. \tag{3.19}$$

Finally, the remaining equilibrium condition is the economy's resource constraint

$$c_t + k_{t+1} = (1-d)\,k_t + f\,(k_t, h_t)\,. \tag{3.20}$$

As is shown by Hansen (1985), a homogeneous agents economy with indivisible labor behaves at the aggregate level as if there exists a representative agent whose preferences are linear in leisure

$$\max_{\{c_t, h_t, k_{t+1}\}_{t\in T}} \sum_{t=0}^{\infty} \delta^t \left\{ \widetilde{v}\,(c_t) - \widetilde{B}\cdot(1-h_t) \right\} \qquad \text{s.t.} \quad RC, \tag{3.21}$$

where RC denotes the economy's resource constraint (4.3). Below we formulate the restrictions which are necessary and sufficient for the economy with heterogeneous agents to preserve the structure (3.21) at the aggregate level. However, we will allow for the case when the function $\widetilde{v}\,(\cdot)$ and the utility parameter $\widetilde{B}$ of the representative consumer differ from those of the individuals in the underlying multi-consumer economy.

The FOCs of problem (3.21) with respect to consumption and hours worked are respectively

$$\widetilde{v}'\,(c_t) = \mu_t, \tag{3.22}$$

$$\widetilde{B} = \mu_t w_t, \tag{3.23}$$

where $\widetilde{v}'$ is the derivative of the function $\widetilde{v}\,(\cdot)$, μ_t is the Lagrange multiplier associated with the resource constraint and $w_t = \partial f\,(k_t, h_t)\,/\partial h_t$. The intertemporal FOC is the same as (3.16).

Dividing aggregate FOCs (3.22), (3.23) by corresponding individual FOCs (3.17), (3.18), we obtain

$$\widetilde{v}'\,(c_t) = \frac{v'\,(c_t^s)}{\mu^s}, \tag{3.24}$$

$$\frac{\partial h_t}{\partial n_t^s} = \frac{B}{\overline{\overline{B}}} \cdot \frac{1}{\mu^s}. \tag{3.25}$$

The individual utilities $v(\cdot)$ and the labor-input functions $h(\cdot)$ which satisfy these two conditions lead to the representative consumer of type (3.21).

Subsequently, we consider the following specifications of the model.

- Agents are heterogeneous in endowments, and labor input is given by a direct sum of hours worked by all agents.
- Agents differ in endowments and skills. We distinguish two cases.

(*i*) Individual labor efforts are perfectly substitutable in production.

(*ii*) Individual labor efforts are aggregated into the production input according to the CES aggregator.

3.4.1 Heterogeneous endowments

Consider an economy, in which all agents have identical skills, $\beta^s = 1$ for $\forall s \in S$, and differ only in initial endowments, $\{k_0^s\}^{s \in S}$. Labor input will be given by the sum of hours worked by all agents, $h_t = \int_S n_t^s ds$.

The result that such economy admits the representative consumer follows from Gorman's (1953) theorem, which postulates the existence of a representative consumer in the heterogeneous economy in which agents differ in endowments and have quasi-homothetic preferences. It is easy to check that the quasi-linear utility, which we use, belongs to the class of quasi-homothetic functions independently of the function $v(\cdot)$ assumed and, therefore, Gorman's (1953) perfect aggregation holds.

Indeed, in this case, condition (3.25) takes the form

$$1 = \frac{B}{\overline{\overline{B}}} \cdot \frac{1}{\mu^s}. \tag{3.26}$$

Since $\int_S 1/\mu^s ds = 1$, condition (3.26) implies that $B = \widetilde{B}$ and, therefore, $\mu^s = 1$. According to (3.24), all heterogeneous agents have equal consumption, $c_t^s = c_t$ for $\forall s \in S, t \in T$. Therefore, the model counterfactually predicts that consumption of agents is independent of agent's endowment. Wealth is only used to enjoy a high amount of leisure. The result that individual consumption is equal to aggregate consumption implies that any function $v(\cdot)$ is consistent with aggregation.

Given that aggregate equilibrium allocation can be calculated from the representative consumer model, all that remains is to restore the individual equilibrium quantities. This cannot be done, however.

The quasi-linear preferences are not strictly concave. As a result, restriction (3.26) is degenerate and a set of restrictions which identifies the working hours is missing. The only restriction which is to be satisfied for the agent's hours worked is the life-time budget constraint which in this case can be written as

$$\sum_{\tau=0}^{\infty} \delta^{\tau} n_{\tau}^{s} = -\frac{k_0^s}{w_0} + \sum_{\tau=0}^{\infty} \delta^{\tau} \frac{c_{\tau}}{w_{\tau}}. \tag{3.27}$$

This constraint restricts the discounted sum of hours worked by individual $s \in S$, however, does not determine how the working efforts are subdivided across time.[5]

The intuition behind this result is the following. The agent's preferences are linear in leisure, and thus, the discounted leisure in different periods is perfect substitute. Therefore, the agent will be indifferent between any sequences of probabilities, which imply the same expected discounted sum of leisure. From the point of view of the whole economy, subdivisions of probabilities between different agents do not matter as long as such subdivisions lead to the same aggregate level of employment.

[5]Note that there is no indeterminacy in a one-period economy.

Using either sunspots or lotteries, we can construct an employment allocation for each heterogeneous agent with a property that there are exactly optimal total number of agents working in a given period and that the lifetime budget constraint of each agent is satisfied. Although such mechanism would allow us to specify who works and who does not, there will be infinitely many of such reallocations among agents. Since the individual equilibrium allocation is constructed here artificially and not determined within the model, such model cannot be used to address distributive issues.

3.4.2 Heterogeneous endowments and skills

In this section, we assume that in addition to initial endowments, agents also differ with respect to skills. Suppose that individual working hours are aggregated into labor input according to the CES function

$$h_t = \left(\int_S \beta^s \left(n_t^s\right)^\varepsilon ds \right)^{1/\varepsilon}, \qquad \varepsilon \leq 1. \tag{3.28}$$

In (3.28), the parameter ε determines the rate of substitution between individual hours. If $\varepsilon = 1$, then different types of labor are perfect substitutes and labor input is $h_t = \int_S n_t^s \beta^s ds$. If $\varepsilon = 0$, individual hours are aggregated into labor input according to the Cobb-Douglas function; aggregate hours in the economy are given by $h_t = \exp\left(\int_S \beta^s \ln\left(n_t^s\right) ds\right)$, and β^s is the share of labor input of individual s in total hours worked. If $\varepsilon \to -\infty$, then efforts of different individuals are perfect compliments and the labor input function is of Leontieff type. Other values of ε correspond to different degrees of substitutability and complementarity between different types of labor.

We will assume that the function $v(\cdot)$ has the form[6]

[6] In order to derive explicitly the utility function of the representative consumer, the function $v(\cdot)$ must belong to the *HARA* class (see the appendix).

$$v\left(c_t^s\right) = \frac{\left(c_t^s\right)^{1-\gamma} - 1}{1-\gamma} \qquad 0 < \gamma < 1, \tag{3.29}$$

Under utility (3.29), condition (3.17) implies $c_t^s = \mu_t^{-1/\gamma}\left(\mu^s\right)^{-1/\gamma}$. Integrating across agents, we have $c_t = \mu_t^{-1/\gamma}\int_S \left(\mu^s\right)^{-1/\gamma} ds$, or $c_t^{-\gamma} = \mu_t\left(\int_S \left(\mu^s\right)^{-1/\gamma} ds\right)^{-\gamma}$. The last yields that the utility of the representative consumer $\widetilde{v}\left(\cdot\right)$

$$\widetilde{v}\left(c_t\right) = \left(\int_S \left(\mu^s\right)^{-1/\gamma} ds\right)^{\gamma} \cdot v\left(c_t\right). \tag{3.30}$$

This result together with individual *FOC* (3.17) allows also to express individual consumption in terms of aggregate consumption are related as

$$c_t^s = c_t \cdot \frac{\left(\mu^s\right)^{-1/\gamma}}{\int_S \left(\mu^s\right)^{-1/\gamma} ds}. \tag{3.31}$$

Below, we analyze the model's distributive implications for the cases of perfect $(\varepsilon = 1)$, and imperfect substitutes $(\varepsilon < 1)$.

Perfect substitutes

The standard approach to incorporate the heterogeneity in skills in the real business cycle literature is to assume that efforts of differently skilled workers are perfectly substitutable in production so that the differences in skills across agents are modelled in terms of efficiency labor, see, e.g., Garcia-Mila, Marcet and Ventura (1995), Krusell and Smith (1995), Kydland (1984), Ríos-Rull (1995). This corresponds to the case of CES under the assumption that $\varepsilon = 1$

$$h_t = \int_S n_t^s \beta^s ds. \tag{3.32}$$

Condition (3.25) for this specification of the labor-input function is

$$\beta^s = \frac{B}{\overline{B}} \cdot \frac{1}{\mu^s}. \tag{3.33}$$

Taking into account the normalization $\int_S 1/\mu^s ds = 1$ and integrating the above condition, we obtain that $B = \widetilde{B}$ and also, that for each agent $s \in S$, $\mu^s = 1/\beta^s$. According to (3.31), $c_t^s = c_t \cdot (\beta^s)^{1/\gamma} / \int_S (\beta^s)^{1/\gamma} ds$. Again, the model has the same undesirable prediction that agent's consumption is not affected by his endowment of wealth and that individual labor-leisure choice is not uniquely defined.

Imperfect substitutes

In this section, we demonstrate that introducing imperfect substitutability of labor can help us to overcome both problems which arise in the previously discussed economy. As we will see, the equilibrium allocations in the economy with imperfect substitutability of labor are uniquely defined and individual consumption depends on the agent'ss wealth.

Under (3.28), condition (3.25) becomes

$$h_t^{1-\varepsilon} \beta^s (n_t^s)^{\varepsilon-1} = \frac{B}{\widetilde{B}} \cdot \frac{1}{\mu^s}. \tag{3.34}$$

Expressing $\beta^s (n_t^s)^\varepsilon$ from this condition and integrating across agents yields

$$\widetilde{B} = B \left[\int_S (\mu^s \beta^s)^{\varepsilon/(1-\varepsilon)} \beta^s ds \right]^{(\varepsilon-1)/\varepsilon}. \tag{3.35}$$

The condition for individual hours follows after expressing n_t^s from (3.34) and substituting for $\widetilde{B}$

$$n_t^s = (\mu^s)^{1-\varepsilon} \beta^s \left[\int_S (\mu^s \beta^s)^{\varepsilon/(1-\varepsilon)} \beta^s ds \right]^{-1/\varepsilon} h_t. \tag{3.36}$$

The agents' welfare weights $\{\mu^s\}^{s \in S}$ are to be chosen such that they satisfy the agents' life-time budget constraints (3.19). After substituting (3.31), (3.34), (3.36) into (3.19), we get

$$\sum_{\tau=0}^{\infty} \delta^{\tau} \frac{c_{\tau}^{-\gamma}}{c_0^{-\gamma}} \left\{ \frac{(\mu^s)^{-1/\gamma}}{\int_S (\mu^s)^{-1/\gamma} ds} \cdot c_{\tau} - \frac{(\mu^s)^{\varepsilon/(1-\varepsilon)} (\beta^s)^{1/(1-\varepsilon)}}{\int_S (\mu^s \beta^s)^{\varepsilon/(1-\varepsilon)} \beta^s ds} w_{\tau} h_{\tau} \right\} = k_0^s. \quad (3.37)$$

In the appendix, we show that a set $\{\mu^s\}^{s \in S}$, which satisfies (3.37) of each agent $s \in S$, exists and is unique, if either $0 < \varepsilon < 1$ and γ is any, or $\varepsilon > 1$ and $\gamma \leq (\varepsilon - 1)/\varepsilon$.

In contrast to the previous cases, in which we had $\mu^s = 1$ and $\mu^s = \beta^s$, now, μ^s is endogenous parameter which is determined by the agent's skills and initial endowment. Both of these will affect the agent's consumption. Also, according to (3.36), individual hours worked are uniquely defined in the equilibrium. Indeterminacy of equilibrium is present in the previous cases because both constraint and utility are linear in hours. Here, the utility is still linear but the constraint is strictly convex. The individual hours worked are perfectly substitutable in the utility, however, they are not in the production function and as a result, the equilibrium is unique.

3.5 Concluding comments

This paper analyzes a dynamic general equilibrium model with indivisible labor and heterogeneous agents. We consider three specifications of the model: (*i*) agents are heterogeneous in endowments; (*ii*) they are heterogeneous in endowments and skills and different skill groups are perfectly substitutable in production; (*iii*) they are heterogeneous in endowments and skills but skill groups are imperfectly substitutable in production. In all of the cases considered, the model admits representative consumer with quasi-linear utility and generates the aggregate dynamics which is indistinguishable from those in Hansen's (1985) model with ex-ante identical agents. The distributive predictions differ, however, across the studied model economies. At the individual level, in the case of imperfect substitutes, the predictions are determinate and qualitatively correct. Parallel results do not hold for setups in which heterogeneous labor is perfectly substitutable in production or in which agents differ only in endowments.

3.6 Appendix

Proof: $v(\cdot)$ belongs to the HARA class of utilities.

(i) Consider aggregate FOC (3.22), which is $\tilde{v}'(c_t) = \mu_t$. Find full differential: $v''(c_t)\, dc_t = d\mu_t$. Divide the last by the original $FOC : \tilde{v}''(c_t)/\tilde{v}'(c_t)\, dc_t = d\mu_t/\mu_t$, or $dc_t = \tilde{v}''(c_t)/\tilde{v}'(c_t)\, d\log\mu_t$.

(ii) In a similar way, individual (3.17), which is $v'(c_t^s) = \mu_t\mu^s$, implies $dc_t^s = \tilde{v}(c_t^s)/v'(c_t^s)\, d\log\mu_t$.

(iii) The market clearing condition $\int_S c_t^s ds = c_t$ implies that $\int_S dc_t^s ds = dc_t$.

Combine the results of $(i) - (iii)$: $\tilde{v}''(c_t)/\tilde{v}'(c_t) = \int_S v''(c_t^s)/v'(c_t^s)\, ds$. To allow for aggregation, the term under the integral should be linear, i.e. $v''(c_t^s)/v'(c_t^s) = a + bc_t^s$, where a and b are some constants. All solutions to this equation are functions from the $HARA$ class which includes exponential, power and logarithmic functions.

Proof: existence and uniqueness of $\{\mu^s\}^{s\in S}$.

$\{\mu^s\}^{s\in S}$ is a solution to the system of the life-time budget constraints (3.37). Let us show conditions under which $\{\mu^s\}^{s\in S}$, which solves (3.37), exists and is unique. Introduce new notations

$$f^s = (\mu^s)^{-1/\gamma} / \int_S (\mu^s)^{-1/\gamma} ds, \quad X = \left(\int_S (f^s)^{\gamma\varepsilon/(\varepsilon-1)} (\beta^s)^{1/(1-\varepsilon)} ds\right)^{(\varepsilon-1)/\varepsilon}.$$

Further, using these notations, introduce new functions

$$\phi(f^s) = f^s \cdot \left[\sum_{\tau=0}^{\infty} \delta^\tau \frac{c_\tau^{-\gamma}}{c_0^{-\gamma}} \cdot c_\tau \right],$$

$$\psi(f^s) = k_0^s + (f^s)^{\gamma\varepsilon/(\varepsilon-1)} \cdot \left[\sum_{\tau=0}^{\infty} \delta^\tau \frac{c_\tau^{-\gamma}}{c_0^{-\gamma}} (\beta^s)^{1/(1-\varepsilon)} X^{\varepsilon/(1-\varepsilon)} w_\tau h_\tau \right].$$

In terms of these functions, life-time budget constraint (3.37) can be written as $\phi(f^s) = \psi(f^s)$. The function $\phi(f^s)$ is such that $\phi(0) = 0$ and $\phi' > 0$ for $\forall f^s > 0$. The function $\psi(f^s)$ is such that $\psi(0) = k_0^s$ and its derivative is determined by the sign of the power $\gamma\varepsilon/(\varepsilon - 1)$. It is easy to check that for a unique crossing point to exist, it is necessary that $\gamma\varepsilon/(\varepsilon - 1) \leq 1$. If $0 < \varepsilon < 1$, then this inequality is always satisfied so that the solution is always unique. If $\varepsilon < 0$, then it must be the case that $\gamma \leq (\varepsilon - 1)/\varepsilon$. Therefore, whether a set $\{\mu^s\}^{s\in S}$ is unique will depend on the values of γ and ε.

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Chapter 4

Idiosyncratic Shocks, Aggregate Fluctuations and a Representative Consumer

Joint with Lilia Maliar

4.1 Introduction

The assumption of idiosyncratic shocks to earnings has been employed recently by many researchers for addressing various questions in computable general equilibrium models, e.g., Krusell and Smith (1995), Kydland (1995), Rios-Rull (1996), Castañeda, Diaz-Gimenez and Rios-Rull (1994), etc. All these papers have two features in common: first, they calibrate the studied models by assuming a particular process for shocks so that the models match microeconomic evidence; and second, they analyze the models' implications at the aggregate level by solving explicitly for the optimal allocations of all heterogeneous consumers.

In this paper, we investigate a model with idiosyncratic shocks where aggregate dynamics can be inferred, without making any explicit assumptions about the process for idiosyncratic shocks and without solving for the equilibrium allocations at the individual level. Specifically, we consider a complete market neoclassical economy where agents differ in initial endowments of wealth and receive idiosyncratic labor productivity shocks. We show that if the preferences of agents are of the addilog type, then at the aggregate level, such an economy behaves as if there was a representative consumer who faces shocks, to preferences and technology. In this case, particular assumptions about idiosyncratic uncertainty have no influence on the structure of the resulting macro model; they only affect the properties of shocks to preferences and technology at the aggregate level.

In fact, the shocks to technology and preferences of the representative consumer can be viewed as aggregate supply and demand shocks, respectively. Demand shocks have been thought for a long time to play an important role in economics, e.g., in Keynesian economics. However, some researchers have argued that this type of shocks is empirically implausible. Our results suggest that the assumption of demand shocks is not as artificial as it may seem to be. The only source of uncertainty in our heterogeneous economy are idiosyncratic shocks to individual productivity; however, at the aggregate level, it appears as if shocks affect the preferences of the representative consumer, or, in other words, aggregate demand.

The empirical part of the paper is motivated by the inability of the standard real business cycle (RBC) models to account for the behavior of labor markets. The model with homogeneous agents and technology shocks as the only source of impulses to business cycles implies that the return to working (measured either in average labor productivity or in real wage) must display a strong positive correlation with working hours. This is not the case in the real economies, where this statistic is close to zero or slightly negative (the Dunlop-Tarshis observation). Another closely related statistic which is significantly overstated in the standard RBC setup is the correlation between the return to working and output. A large body of economic research focuses on these "labor market puzzles".[1]

Maliar and Maliar (1999) analyze a heterogeneous-agent model, which is similar to that studied in this paper, but assuming that the level of agents' productivity do not change over time. That paper finds that under the assumption of addilog preferences, the model can generate the correlation between labor productivity and working hours which is close to the one in the data. This is possible, however, only if the intertemporal elasticities of consumption and leisure are substantially higher than one. The latter assumption has two undesirable side effects, specifically, the model's predictions are not robust to small changes in the intertemporal elasticities and the volatility of labor productivity becomes too low. The model presented in this paper overcomes both of these problems.

To calibrate the process for shocks to preferences and technology in the constructed representative-agent model, we use the time-series data on the U.S. economy. We assume that "aggregate" shocks follow a first-order Markov process with some joint transitional probabilities. We estimate the model's parameters, including the elements of the matrix of transitional probabilities and the variances of the error terms for aggregate shocks. Subsequently, we calibrate and simulate the model.

The key findings of the paper can be summarized as follows.

- The model with shocks to preferences and technology can reproduce the feature of the data that productivity and working hours as well as productivity and output are weakly correlated.

[1]For surveys of the literature see Christiano and Eichenbaum (1992) and Gomme and Greenwood (1995).

- The model's predictions are robust to changes in the values of the discount factor, the individual intertemporal elasticities of consumption and leisure, and the transitional probabilities of shocks.
- The remaining statistics, including the volatility of productivity, are in line with those in the data and in the standard representative-consumer model in which only technology shocks occur.

The paper is organized as follows. Section 2 describes the economy with heterogeneous agents and derives optimality conditions. Section 3 constructs the corresponding representative-consumer model. Section 4 outlines estimation and solution procedures. Section 5 discusses numerical results. Section 6 concludes.

4.2 The economy

The economy consists of a set of heterogeneous agents S and a representative firm. The timing is discrete, $t \in T$, where $T = 0, 1, ..., \infty$.

The measure of agent s in the set S is denoted by $d\omega^s$, where $\int_S d\omega^s = 1$. The agents differ in initial endowments and productivity levels. The productivity of an agent $s \in S$ in a period $t \in T$ is denoted by β_t^s. We denote the distribution of the productivities of agents in period t by $B_t \equiv \{\beta_t^s\}^{s \in S}$ and assume that B_t follows a first order Markov process with a transitional probability given by $\Pi\{B_{t+1} = B' \mid B_t = B\}_{B',B \in \Re}$, where $\Re \subseteq R_+^S$ is a bounded set. Note that this specification allows for correlation between idiosyncratic shocks to productivities of different individuals. The initial distribution of idiosyncratic shocks to productivities B_0 is given.

An infinitely-lived agent $s \in S$ seeks to maximize the expected sum of momentary utilities $u(c_t^s, l_t^s)$, discounted at the rate $\delta \in (0, 1)$, by choosing a path for consumption, c_t^s, and leisure, l_t^s. The utility function $u(\cdot)$ is continuously differentiable, strictly increasing in both arguments, and strictly concave. In period t the agent owns capital stock k_t^s and rents it to the firm at the rental price r_t. Also, he supplies to the firm n_t^s units of labor in exchange for income $n_t^s \beta_t^s w_t$, where w_t is the wage paid for one unit of efficiency labor. The total time endowment of the agent is normalized to one, $n_t^s + l_t^s = 1$. Capital depreciates at the rate $d \in (0,1]$. When making the investment decision, the agent faces uncertainty about the future returns on capital. We assume that markets are complete: the agent can insure himself against uncertainty by trading state contingent claims, $\{m_t^s(B)\}_{B \in \Re}$. The claim of type $B \in \Re$ costs $p_t(B)$ in period t and pays one unit of consumption good in period $t+1$ if the state B occurs and zero otherwise. Therefore, the problem solved by agent $s \in S$

$$\max_{\{c_t^s, n_t^s, k_{t+1}^s, m_{t+1}^s(B)\}_{B \in \Re, t \in T}} E_0 \sum_{t=0}^{\infty} \delta^t u\left(c_t^s, l_t^s, g^t\right) \tag{4.1}$$

$$c_t^s + k_{t+1}^s + \int_{\Re} p_t(B)\, m_{t+1}^s(B)\, dB = (1 - d + r_t)\, k_t^s + w_t g^t n_t^s \beta_t^s + m_t^s(B_t)\,, \tag{4.2}$$

where g denotes the rate of labor-augmenting technological progress. Initial holdings of capital and contingent claims, k_0^s and m_0^s, are given.

The production side of the economy consists of a representative firm. The firm owns a technology which allows to transform the inputs, capital k and labor h, into output. The production function $f(k,h)$ is strictly concave, continuously differentiable, strictly increasing with respect to both arguments, has constant return to scale, satisfies the appropriate Inada conditions, and is such that $f(k, zh) = \theta(z)\, f(k,h)$ for $\forall k, h, z \in R_+$. Given the prices, r_t and w_t, the firm rents capital k_t and hires labor h_t to maximize period-by-period profits:

$$\max_{k_t,\, h_t} \pi_t = f(k_t, h_t) - r_t k_t - w_t h_t \tag{4.3}$$

The choices of the consumers and the firm must satisfy the market clearing conditions for insurance payments

$$\int_S m_{t+1}^s (B)\, d\omega^s = 0 \qquad \text{for } \forall B \in \Re, \tag{4.4}$$

for capital and labor

$$k_t = \int_S k_t^s d\omega^s, \qquad h_t = g^t \int_S n_t^s \beta_t^s d\omega^s, \tag{4.5}$$

and the economy's resource constraint

$$c_t + k_{t+1} = (1-d)\, k_t + f\,(k_t, h_t)\,, \tag{4.6}$$

where $c_t = \int_S c_t^s d\omega^s$ is aggregate consumption.

The equilibrium is defined as a sequence of contingency plans for allocations of the consumers, for allocations of the firm and for the prices such that given the prices, the sequence of plans for the allocations solves the utility maximization problem of each consumer and the profit maximization problem of the firm and satisfies market clearing conditions. Moreover, the plans are such that $c_t^s \geq 0$, and $1 \geq n_t^s \geq 0$ for $\forall s \in S$, $t \in T$ and $w_t, r_t, k_t \geq 0$ for $\forall t \in T$. It is assumed that the equilibrium exists and is interior.

Let b_t^s be the normalized productivity of agent $s \in S$, $b_t^s = \beta_t^s / \int_S \beta_t^s d\omega^s$. We introduce a new variable n_t such that

$$n_t = \int_S n_t^s b_t^s d\omega^s.$$

Labor input, h_t, and the variable n_t are related as $h_t = g^t n_t \int_S \beta_t^s d\omega^s$. In what follows, n_t is referred to as the aggregate (efficiency) number of hours worked. In terms of n_t, the profit-maximization conditions of the firm are

$$r_t = \theta_t f_1 \left(k_t, g^t n_t\right), \qquad w_t = \frac{\theta_t f_2\,(k_t, g^t n_t)}{\int_S \beta_t^s d\omega^s},$$

where $\theta_t \equiv \theta\left(\int_S \beta_t^s d\omega^s\right)$ and $f_i(\cdot)$ denotes the first order partial derivative of the function $f(\cdot)$ with respect to the $i-$th argument. The parameter θ_t appears because the aggregate level of skills in the economy fluctuates. This parameter allows for the usual interpretation of technological innovations.

In terms of the variable n_t, the economy's resource constraints can be expressed as

$$c_t + k_{t+1} = (1-d)\,k_t + \theta_t f\left(k_t, g^t n_t\right). \tag{4.7}$$

With an interior solution, the First Order Conditions (FOCs) of the consumer's utility maximization problem (4.1), (4.2) with respect to insurance holdings, capital, consumption and hours worked, and the transversality condition are

$$\lambda_t p_t(B) = \delta\lambda_{t+1}(B)\cdot \Pi\left\{B_{t+1} = B' \mid B_t = B\right\}_{B',B\in\Re}, \tag{4.8}$$

$$\lambda_t = \delta E_t\left[\lambda_{t+1}\left(1 - d + r_{t+1}\right)\right], \tag{4.9}$$

$$\lambda_s u_1\left(c_t^s, l_t^s, g^t\right) = \lambda_t, \tag{4.10}$$

$$\lambda_s u_2\left(c_t^s, l_t^s, g^t\right) = \lambda_t w_t g^t b_t^s, \tag{4.11}$$

$$\lim_{t\to\infty} E_0\left[\delta^t\lambda_t\left(k_{t+1}^s + \int_{\Re} p_t(B)\, m_{t+1}^s(B)\, dB\right)\right] = 0. \tag{4.12}$$

Here, we can represent the Lagrange multiplier associated with the agent's budget constraint as $\lambda_t^s = \lambda_t/\lambda^s$ because due to market completeness, the ratio of marginal utilities of any two agents remains constant in all periods and states of nature. If one formulates the associated planner's problem, then the parameters $\{\lambda^s\}^{s\in S}$ and the variable λ_t will be the welfare weights and the Lagrange multiplier associated with the economy's resource constraint. Without loss of generality, we normalize the weights to unity, $\int_S \lambda^s d\omega^s = 1$.

4.3 Representative consumer

In this section, we construct a representative consumer for the heterogeneous-agent economy of section 4.2. For the remainder of the paper, we assume that the momentary utility function of each agent $s \in S$ is of the addilog type, i.e.[2]

$$u\left(c_t^s, l_t^s, g^t\right) = \frac{\left(c_t^s\right)^{1-\gamma} - 1}{1-\gamma} + Ag^{t(1-\gamma)}\frac{\left(l_t^s\right)^{1-\sigma} - 1}{1-\sigma}, \qquad \gamma, \sigma, A > 0. \quad (4.13)$$

Under such utility, FOCs $(4.10), (4.11)$ take the form

$$\lambda_s \left(c_t^s\right)^{-\gamma} = \lambda_t, \quad (4.14)$$

$$\lambda_s Ag^{t(1-\gamma)}(1 - n_t^s)^{-\sigma} = \lambda_t w_t g^t b_t^s, \quad (4.15)$$

Solving $(4.14), (4.15)$ with respect to c_t^s and $(1-n_t^s)b_t^s$ and integrating across agents, we obtain

$$c_t = \lambda_t^{-1/\gamma} \cdot \int_S \left(\lambda^s\right)^{1/\gamma} d\omega^s, \quad (4.16)$$

$$l_t = \left(\lambda_t w_t g^t\right)^{-1/\sigma} \left(Ag^{t(1-\gamma)}\right)^{1/\sigma} \cdot \int_S \left(\lambda^s\right)^{1/\sigma} \left(b_t^s\right)^{1-1/\sigma} d\omega^s. \quad (4.17)$$

where $l_t = 1 - n_t$. From equations $(4.9), (4.16), (4.17)$, we get

$$c_t^{-\gamma} = \delta E_t \left[c_{t+1}^{-\gamma}\left(1 - d + r_{t+1}\right)\right] \quad (4.18)$$

[2]Similar aggregation results obtain if the agents' preferences are quasi-homothetic.

$$AX_t g^{t(1-\gamma)} l_t^{-\sigma} = c_t^{-\gamma} w_t g^t \tag{4.19}$$

where the parameter X_t is given by

$$X_t = \frac{\left(\int_S (\lambda^s)^{1/\sigma} (b_t^s)^{1-1/\sigma} d\omega^s\right)^{-\sigma}}{\left(\int_S (\lambda^s)^{1/\gamma} d\omega^s\right)^{-\gamma}}.$$

Finally, integrating the individual transversality condition (4.12) across agents and imposing market clearing condition for claims (4.4), we have

$$\lim_{t\to\infty} E_0 \left[\delta^t \lambda_t k_{t+1}\right] = 0. \tag{4.20}$$

Using the above results, we formulate the representative-consumer model, which describes aggregate dynamics of the heterogeneous-agent economy

$$\max_{\{c_t,k_{t+1},n_t\}_{t\in T}} E_0 \sum_{t=0}^{\infty} \delta^t \left\{ \frac{c_t^{1-\gamma}-1}{1-\gamma} + AX_t g^{t(1-\gamma)} \frac{(1-n_t)^{1-\sigma}-1}{1-\sigma} \right\} \quad \text{s.t.} \quad RC, \tag{4.21}$$

where RC denotes the economy's resource constraint (4.7).

Proposition 7 *Under the addilog utility, the equilibrium sequence of contingency plans for aggregate quantities* $\{c_t, n_t, k_{t+1}\}_{t\in T}$ *in economy* (4.1) – (4.4) *is a solution to the representative-agent model* (4.21).

Proof. If a solution $\{c_t, n_t, k_{t+1}\}_{t\in T}$ to problem (4.21) exists and is interior, then it satisfies the FOCs, the transversality condition and the budget constraint. The FOCs of this problem are (4.18), (4.19). The transversality condition is equivalent to (4.20). Finally, by definition, resource constraint (4.7) is a necessary condition for the equilibrium. ||

Unless $\gamma = \sigma$, the addilog preferences are not quasi-homothetic and, therefore, they do not lead to a representative consumer in the sense of

Gorman (1953). The possibility of aggregation under the addilog utility is mentioned first by Shafer (1977). Note that, even if agents have identical time-invariant productivity levels and differ only in endowments, the parameter $X_t \equiv X_0$ does not vanish from problem (4.21). The value of this parameter depends on the particular distribution of endowments and affects the equilibrium marginal rate of substitution between aggregate quantities.

If productivities of agents are subhect to idiosyncratic shocks, the parameters θ_t and X_t vary with time. The parameters θ_t and X_t will be referred to as technology and preferences shocks, respectively. The parameter θ_t is exogenous both to the heterogeneous-agent model and the constructed representative-consumer setup. The parameter X_t is exogenous to the problem of the representative consumer, but endogenous to the economy with heterogeneous agents since it depends on the welfare weights, which in turn are determined by the decisions of all heterogeneous agents.

The existence of the representative consumer makes it possible to investigate the properties of the heterogeneous-agent economy without making any particular assumptions about the process for idiosyncratic shocks. Instead, one can assume some law of motion for the parameters θ_t and X_t and solve model (4.21). This model is sufficient to determine the sequence of contingency plans for aggregate allocations and prices.

Given that the idiosyncratic shocks to agents' productivities are assumed to follow a first-order Markov process, we presume that the aggregate shocks will also do so. Thus, the law of motion for shocks θ_t and X_t will be

$$\begin{bmatrix} \log \theta_t \\ \log X_t \end{bmatrix} = \begin{bmatrix} \rho_{\theta\theta} & \rho_{\theta X} \\ \rho_{X\theta} & \rho_{XX} \end{bmatrix} \begin{bmatrix} \log \theta_{t-1} \\ \log X_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_t^{\theta} \\ \varepsilon_t^{X} \end{bmatrix} \tag{4.22}$$

where $\varepsilon_t^{\theta} \sim N(0, \nu_{\theta}^2)$ and $\varepsilon_t^{X} \sim N(0, \nu_X^2)$. For the rest of the paper, we assume that the production function is of Cobb-Douglas type, $f(k, n) = k^{\alpha} n^{1-\alpha}$. In the remainder of the paper, we analyze quantitative implications of the constructed representative-consumer model.

4.4 Estimation and solution procedures

We now estimate the model's parameters and simulate the solutions. The estimation procedure plays an important role in our analysis as it allows us to evaluate the stochastic properties of shocks in (4.22).

To estimate the model's parameters, we use a version of Hansen's (1982) Generalized Method of Moments (GMM) procedure. The utility parameters, γ and σ, and the subjective discount factor, δ, are not estimated. In the baseline model, we presume $\gamma = \sigma = 1$ and $\delta = (1.03)^{-0.25}$. Later, to analyze the robustness of our results, we will also consider several alternative specifications for these parameters. The parameters under estimation are subdivided into two groups Ψ_1 and Ψ_2

$$\Psi_1 = \{\alpha, A, g, d\}, \qquad \Psi_2 = \left\{\rho_{\theta\theta}, \rho_{\theta X}, \rho_{X\theta}, \rho_{XX}, \nu_t^{\theta}, \nu_t^{X}\right\}.$$

The estimation procedure includes two steps. First, we estimate the parameters from the group Ψ_1 from the first moment conditions of model (4.21) and compute the residuals θ_t and X_t. Second, we estimate the parameters from the group Ψ_2 by using the computed residuals. As the stochastic properties of the processes for θ_t and X_t are not known, we compute the instrumental variable estimator at both steps of the estimation procedure. As instruments, we use 8 lags of consumption, capital, output and hours worked. The first-moment conditions, employed for estimating the parameters from Ψ_1 are given in the appendix. The parameters from Ψ_2 are estimated according to (4.22).

To estimate the parameters, we use quarterly data on the U.S. economy ranging from 1959 : 3 to 1998 : 3 . The variable consumption c_t in the model is defined as real personal expenditures on nondurables and services in the data. Investment i_t in the model is real personal consumption of durables and real fixed private investment in the data. Consequently, the series for output are constructed by adding up consumption and investment, $y_t = c_t + i_t$. The variable working hours n_t in the model is defined as level of the civilian employment premultiplied by average weakly hours worked in private nonagricultural establishments in the data. The average weakly hours were previously divided by 168, which is the total number of hours per week. Before computing the estimates, the constructed series are converted in per-capita terms by using the efficiency measure of the U.S. population. The data are taken from the Federal Reserve Bank of Saint-Louis data base (mnemonics $FPIC92$, $PCEDG92$, $PCENDC92$, $PCECS92$, $CE16OV$, $AWHNONAG$). The sources for these series are U.S. Department of Labor and U.S. Department of Commerce.

The estimates of the parameters from Ψ_1 in the baseline model are

$$\alpha = \underset{(0.0016)}{0.3341}, \qquad A = \underset{(0.008)}{3.317}, \qquad g = \underset{(0.0001)}{1.0047}, \qquad d = \underset{(0.0001)}{0.0209},$$

where the numbers in parenthesis are the standard deviations. These estimates are practically identical to those reported by Christiano and Eichenbaum (1992). We find that the resulting estimates are robust to modifications in the set of instruments and in the number of lags assumed. We will not report the estimates of the parameters from Ψ_1 under all considered values of (γ, σ, δ). However, we will report the first moments of the model for each set of the parameters (γ, σ, δ) under which the model is simulated. The estimates of the parameters from Ψ_2 will be reported in all the cases considered and discussed separately in the subsequent section.

We parametrize the model by using the values of the parameters, which are previously estimated by GMM and solve for the equilibrium. To compute numerical solutions, we employ the parametrized expectation algorithm, see, e.g., Marcet and Lorenzoni (1999). To approximate the conditional expectations, we use second order degree exponentiated polynomial. The length of simulations was 10000 and the iterations were performed until 5 digits precision in the polynomial coefficients was enforced.

In the last column of *Table* 1, we provide selected first and second moment of time series in the U.S. economy. The reported statistics are the sample averages of the variables provided in the first column of the table. The statistics σ_x and $corr\,(x, z)$ are the volatility of a variable x and the correlation between variables x and z respectively. In the remaining columns of the table, we report the first and second moments of time series generated by the model. The model's moments are sample averages of the statistics computed for each of 400 simulations. Each simulation has the length 157 periods, as do time series for the U.S. economy. Numbers in parentheses are sample standard deviations of these statistics. Before calculating the second moments, the corresponding variables for the U.S. and artificial economies were logged and detrended by using the Hodrick-Prescott filter.

As a measure of labor productivity (wage), we use the variable y_t/n_t. To check that the constructed measure of labor productivity behaves similarly to the one in the U.S. economy, we compared this measure to the CITIBASE variable *LBOUTU*, which is output per-hour of all persons in the nonagricultural business sector. We find that the properties of both measures are very similar. In such a way, if instead of y_t/n_t, we use the variable *LBOUTU*, then we have $\sigma_{y/n} = 1.023$, $corr\,(y/n, n) = 0.220$ and $corr\,(y/n, y) = 0.543$, which are close to the corresponding statistics reported in the table.

4.5 Findings

We begin from a baseline standard representative the model. This corresponds to the case when the process (4.22) is estimated under the restriction that only technology shocks can occur in the economy.

- Model 1. $\rho_{\theta\theta}$ is estimated under the restriction $\rho_{X\theta}, \rho_{\theta X}, \rho_{XX} \equiv 0$.

This version of the model is extensively studied in the literature, e.g., Hansen (1985), and Christiano and Eichenbaum (1992). These papers use different values for the coefficient of autocorrelation $\rho_{\theta\theta}$: the first assumes $AR(1)$ with $\rho_{\theta\theta} = 0.95$, while the second uses the random walk specification $\rho_{\theta\theta} = 1$. As it follows from the table, our own estimate is close to the latter.[3]

Comparing the results of Hansen (1985) and Christiano and Eichenbaum (1992) shows that the key properties of the model are not substantially affected by a variation in the coefficient of autocorrelation. Specifically, in either case, the model can generate most of the statistics in line with the data, except for those with respect to labor markets. The most serious failure of the model consists in its inability to account for the Dunlop-Tarshis observation, which consists in that productivity (wage) and hours worked in the real economies are not significantly correlated. In fact, the quantitative expression of the Dunlop-Tarshis observation varies substantially depending on time series used. For example, Christiano and Eichenbaum (1992) calculate $corr(y/n, n)$ for the U.S. economy by using the household and the establishment time series and obtain -0.2 and 0.16, respectively. According to Gomme and Greenwood (1995), if the real wages are used as a proxy for productivity, this statistic will be around -0.44. It turns out that the model cannot get close to any of the above numbers consistently predicting that $corr(y/n, n) \simeq 1$. In addition, it overstates considerably the correlation between productivity and hours worked and understates the volatilities of productivity and working hours compared to the data.

[3] We find that the estimate of the autocorrelation coefficient $\rho_{\theta\theta}$ depends significantly on which particular time series are used as a proxy for working hours. If one uses aggregate working hours, as Hansen (1985) does, then the estimates for $\rho_{\theta\theta}$ will be about 0.95. However, if one uses the definition suggested by Cristiano and Eichebaum (1992) and adopted in this paper, the estimate for $\rho_{\theta\theta}$ will be close to one.

Next, we turn to the case when all uncertainty in the economy comes from shocks to preferences.

- Model 2. ρ_{XX} is estimated under the restriction $\rho_{X\theta}, \rho_{\theta X}, \rho_{\theta\theta} \equiv 0$.

This version of the model proves to be highly unsuccessful. It generates several serious failures such as very low volatility of consumption, output and investments and almost perfect negative correlation between productivity (output) and working hours. It is interesting to notice that in this case, the problem is exactly the opposite to the one that we had before: the productivity (output) and working hours in the model are too countercyclical compared to the U.S. data.

Next, we consider the model with two types of shocks.

- Model 3. $\rho_{\theta\theta}$, $\rho_{X\theta}$, $\rho_{\theta X}$, $\rho_{\theta\theta}$ are estimated without restrictions.

Once two sources of shocks are assumed, the model's performance improves considerably compared to Models 1 and 2. Model 3 generates weekly negative correlation between productivity and hours worked and therefore, accounts for the Dunlop-Tarshis observation. Further, the correlation between productivity and output in the model is close to that in the data. Finally, incorporating two shocks adds volatility to all model's variables except for investment. In particular, the volatility of working hours in Model 3 is more than twice as large as in Model 1 and becomes close to the empirical counterpart.

As we can see, the model with two types of shocks is remarkably successful in explaining the U.S. data. It is interesting to analyze, therefore, how robust our results are to modifications in the model's parameters. We begin with analyzing the role of the autocorrelation coefficients by considering the following experiment.

- Model 4. $\rho_{\theta\theta}, \rho_{XX} \equiv 0.95$ and $\rho_{X\theta}, \rho_{\theta X} \equiv 0$.

As it follows from the table, this modification not only does not worsen the positive features of the previous setup but improves model's performance with respect to the volatilities of investment, output and working hours. We have done several other experiments (not reported) and found that the model's implications are very robust to changes in the autocorrelation coefficients.

Consequently, we are left to explore how the model's properties are affected by changes in the values of the preference parameters (γ, σ, δ). Maliar and Maliar (1999) show an example of a heterogeneous-agent model where quantitative implications depend crucially on the intertemporal elasticities of consumption and leisure, $1/\gamma$ and $1/\sigma$. Thus, a sensitivity analysis with respect to the preference parameters is of potential interest. Below we report the results of experiments in which we vary the value of one of the parameters (γ, σ, δ), holding the remaining two parameters equal to the baseline values. In the remaining experiments, no prior restrictions are imposed on the values of autocorrelation coefficients; these are estimated from the data. Models 5-10 are the following.

- Models 5, 6: $\sigma = 1.0$, $\delta = 1.03^{-0.25}$ and $\gamma \in \{0.75, 1.5\}$.
- Models 7, 8: $\gamma = 1.0$, $\delta = 1.03^{-0.25}$ and $\sigma \in \{0.5, 2.0\}$.
- Models 9, 10. $\gamma = 1.0$, $\sigma = 1.0$ and $\delta \in \{1.05^{-0.25}, 1.015^{-0.25}\}$.

The results of this simulation exercise are reported in *Table* 2. First of all, let us notice that the fact that the estimated coefficients of the autocorrelation $\rho_{\theta\theta}$ in Models 6 and 8 are greater than one does not imply non-stationarity. In order for the process for shocks θ_t and X_t to be stationary, it is sufficient that both eigenvalues of the matrix constructed from the autocorrelation coefficients lie inside of the unit root cycle, see, e.g., Hamilton (1998). This restriction is satisfied in each of the models considered.

As we can see from the table, variations in the preference parameters (γ, σ, δ) inside a reasonable range do not significantly affect the properties of the model compared to the baseline case. An exception is Model 6 in which the correlation between productivity and working hours becomes too negative. However, even this model's prediction is not entirely inconsistent with the data as it is close to the correlation between real wages and working hours in the U.S. economy. In sum, the findings obtained for the baseline model are not considerably affected by changes in the model's parameters.

4.6 Conclusion

This paper describes an example of a general equilibrium model with idiosyncratic uncertainty where aggregate equilibrium allocation can be characterized in a simple and economic fashion. Our economy is populated by a number of individuals who differ in capital endowments and whose labor productivities fluctuate over time. At the aggregate level, however, it appears as if there exists a representative consumer who is hit by two types of shocks, to preferences and technology. The possibility of aggregation enables us to investigate the model's implications at the aggregate level without making explicit assumptions about unobservable idiosyncratic uncertainty.

The empirical finding of the paper is that taking into account the preference shocks can enhance considerably the performance of the RBC models. In contrast to the standard setup where fluctuations in technology is the only source of impulses to business cycles, the two-shock version of the model can successfully account for such labor market stylized facts as the Dunlop-Tarshis observation and the low correlation between productivity and output.

4.7 Appendix

The conditions used for GMM estimation:

The economy's resource constraint implies that the gross investment i_t is related to capital stock k_t as

$$E\{1-d+(i_{t+1}/k_t)-(k_{t+1}/k_t)\}=0.$$

The hypothesis of the balanced growth implies

$$E\{\log(y_t)-\log(y_{t-1})-\ln(g)\}=0,$$

$$E\{\log(c_t)-\log(c_{t-1})-\ln(g)\}=0,$$

$$E\{\log(k_t)-\log(k_{t-1})-\ln(g)\}=0.$$

The intertemporal condition of problem (4.21) is

$$E\{1-\delta(c_t/c_{t+1})[1-d+\alpha(y_t/k_t)]\}=0.$$

Taking the logarithm of *FOC* (4.19), we get

$$\ln(X_t)=-\gamma[\ln(c_t)-gt]+\sigma\ln(1-n_t)+\ln[(1-\alpha)y_t/n_t]-\ln(g)t-\ln(A).$$

From (4.7), the process for the parameter θ_t is

$$\ln(\theta_t)=\ln(y_t)-\alpha\ln(k_t)-(1-\alpha)\ln(n_t)-(1-\alpha)\ln(g)t-\ln(\theta),$$

where θ is the absolute level of technology.

Table 1. Baseline model: $\gamma = 1$, $\sigma = 1$, $\delta = 1.03^{-0.25}$

	Heterogeneous-agent model				U.S. economy
	Model 1	Model 2	Model 3	Model 4	
Parameters for the shocks					
$\rho_{\theta\theta}$	0.99456 (0.00939)	-	0.99602 (0.01588)	0.95000	-
$\rho_{\theta X}$	-	-	0.00594 (0.01551)	-	-
$\rho_{X\theta}$	-	-	0.10496 (0.01654)	-	-
ρ_{XX}	-	0.99886 (0.00997)	0.92671 (0.01538)	0.95000	-
v^2_θ	0.00652 (0.00055)	-	0.00598 (0.00058)	0.00669 (0.00055)	-
v^2_X	-	0.00622 (0.00051)	0.00585 (0.00046)	0.00697 (0.00045)	-
First moments					
c_t/y_t	0.750 (0.018)	0.748 (0.012)	0.750 (0.019)	0.751 (0.023)	0.745
k_t/y_t	10.365 (0.539)	10.311 (0.408)	10.359 (0.589)	10.351 (0.564)	10.237
n_t	0.211 (0.004)	0.222 (0.006)	0.212 (0.008)	0.211 (0.007)	0.213
Second moments					
σ_c	0.558 (0.067)	0.290 (0.033)	0.691 (0.081)	0.480 (0.063)	0.836
$\sigma_{y/n}$	0.679 (0.078)	0.274 (0.030)	0.764 (0.083)	0.721 (0.081)	1.011
σ_n	0.492 (0.057)	0.806 (0.090)	1.069 (0.122)	1.408 (0.153)	1.279
σ_i	3.025 (0.355)	1.304 (0.156)	2.789 (0.324)	5.220 (0.567)	4.793
σ_y	1.153 (0.127)	0.539 (0.061)	1.120 (0.130)	1.597 (0.174)	1.755
corr(y/n,n)	0.939 (0.046)	-0.983 (0.004)	-0.287 (0.138)	0.026 (0.145)	0.220
corr(c,y)	0.974 (0.006)	0.982 (0.004)	0.896 (0.031)	0.889 (0.018)	0.923
corr(y/n,y)	0.989 (0.003)	-0.963 (0.010)	0.402 (0.129)	0.471 (0.109)	0.715
corr(n,y)	0.979 (0.005)	0.996 (0.001)	0.753 (0.068)	0.890 (0.032)	0.830
corr(i,y)	0.989 (0.003)	0.990 (0.002)	0.938 (0.019)	0.986 (0.004)	0.979

Table 2. Sensitivity analysis with respect to the parameters (γ, σ, δ)

	Heterogeneous-agent model					
	Model 5	Model 6	Model 7	Model 8	Model 9	Model 10
	$\gamma = 0.75$ $\sigma = 1$ $\delta = 1.03^{-0.25}$	$\gamma = 1.5$ $\sigma = 1$ $\delta = 1.03^{-0.25}$	$\gamma = 1$ $\sigma = 0.5$ $\delta = 1.03^{-0.25}$	$\gamma = 1$ $\sigma = 2$ $\delta = 1\ 03^{-0.25}$	$\gamma = 1$ $\sigma = 1$ $\delta = 1.05^{-0.25}$	$\gamma = 1$ $\sigma = 1$ $\delta = 1.015^{-0.25}$
Parameters for the shocks						
$\rho_{\theta\theta}$	0.99447 (0.01242)	1.00052 (0.02037)	0.99213 (0.01561)	1.00247 (0.01645)	0.99923 (0.01587)	0.99210 (0.01614)
$\rho_{\theta x}$	0.00979 (0.01903)	0.00453 (0.01126)	0.00196 (0.01635)	0.01127 (0.01380)	0.00879 (0.01572)	0.00310 (0.01512)
$\rho_{x\theta}$	0.07405 (0.01273)	0.15957 (0.02232)	0.08744 (0.01476)	0.13987 (0.02074)	0.10291 (0.01651)	0.10639 (0.01685)
ρ_{xx}	0.92131 (0.01904)	0.93270 (0.01148)	0.93813 (0.01483)	0.90954 (0.01627)	0.92658 (0.01555)	0.92978 (0.01498)
v^2_{θ}	0.00602 (0.00057)	0.00597 (0.00059)	0.00595 (0.00058)	0.00604 (0.00057)	0.00599 (0.00059)	0.00599 (0.00057)
v^2_{x}	0.00599 (0.00045)	0.00600 (0.00048)	0.00535 (0.00042)	0.00700 (0.00055)	0.00583 (0.00046)	0.00582 (0.00046)
First moments						
c_t/y_t	0.760 (0.023)	0.743 (0.033)	0.751 (0.024)	0.750 (0.020)	0.749 (0.019)	0.752 (0.024)
k_t/y_t	9.958 (0.526)	11.030 (0.872)	10.358 (0.595)	10.382 (0.570)	10.346 (0.521)	10.382 (0.643)
n_t	0.212 (0.008)	0.208 (0.011)	0.213 (0.009)	0.211 (0.007)	0.213 (0.008)	0.212 (0.008)
Second moments						
σ_c	0.631 (0.078)	0.548 0.079	0.592 (0.084)	0.675 (0.078)	0.648 (0.075)	0.638 (0.080)
$\sigma_{y/n}$	0.663 (0.069)	0.862 (0.092)	0.698 (0.072)	0.770 (0.073)	0.759 (0.080)	0.751 (0.080)
σ_n	1.272 (0.142)	1.301 (0.430)	1.311 (0.192)	1.066 (0.132)	1.068 (0.121)	1.200 (0.128)
σ_i	4.136 (0.487)	4.494 (5.271)	4.165 (1.525)	3.103 (0.435)	2.881 (0.352)	3.534 (0.401)
σ_y	1.406 ((0.161)	1.122 (0.400)	1.359 (0.172)	1.144 (0153)	1.128 (0.134)	1.256 (0.148)
$corr(c,y)$	0.921 (0.019)	0.717 (0.081)	0.890 (0.042)	0.841 (0.047)	0.901 (0.030)	0.850 (0.043)
$corr(y/n,y)$	0.421 (0.136)	0.170 (0.141)	0.320 (0.156)	0.431 (0.122)	0.408 (0.129)	0.369 (0.124)
$corr(n,y)$	0.880 (0.035)	0.739 (0.082)	0.858 (0.048)	0.754 (0.066)	0.757 (0.069)	0.810 (0.053)
$corr(i,y)$	0.975 (0.006)	0.885 (0.189)	0.964 (0.039)	0.929 (0.022)	0.951 (0.015)	0.938 (0.019)
$corr(y/n,n)$	-0.051 (0.158)	-0.524 (0.117)	-0.197 (0.148)	-0.255 (0.143)	-0.274 (0.141)	-0.236 (0.142)

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