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Introduction

The well-known Hodge Decomposition Theorem states that the n-th Betti cohomology vector space with complex coefficients of every compact Kähler manifold admits a direct sum decomposition induced by the type of complex-valued differential forms. This result is the prototypical example of a *pure Hodge structure* of weight n, and it imposes strong topological restrictions for a compact complex manifold to be Kählerian. For instance, all Betti numbers of odd order must be even, and all Betti numbers of even order, from zero to twice the dimension, must be non-zero.

Influenced by Grothendieck's philosophy of *mixed motives*, and motivated by the Weil Conjectures, Deligne sought for a generalization of Hodge's theory to arbitrary complex algebraic varieties. His key idea was to foresee the existence of a natural increasing *weight filtration* on the Betti cohomology of algebraic varieties, in such a way that the successive quotients become pure Hodge structures of different weights. This led to the notion of *mixed Hodge structure*, first introduced in [**Del71a**]. Based on Hironaka's resolution of singularities and the logarithmic de Rham complex, Deligne [**Del71b**] proved that the n-th cohomology group of every smooth complex algebraic variety carries a functorial mixed Hodge structure, which for compact Kähler manifolds, coincides with the original pure Hodge structure. This result has important topological consequences, such as the theorem of the fixed part (see Theorem 4.1.1 of loc.cit). In [**Del74b**], Deligne introduced *mixed Hodge complexes* and extended his own results to singular varieties, using simplicial resolutions. As an alternative to simplicial resolutions, Guillén-Navarro developed the theory of cubical hyperresolutions. Its application to mixed Hodge theory is presented in [**GNPP88**].

Considerations related to the Weil Conjecture on the action of the Frobenius automorphism for l-adic cohomology in positive characteristic [**Del74a**] led to the expectation that, as a consequence of Hodge theory, triple Massey products of compact Kähler manifolds should vanish. In response to this problem, Deligne-Griffiths-Morgan-Sullivan [**DGMS75**] proved the Formality Theorem of compact Kähler manifolds, stating that the real homotopy type of every compact Kähler manifold is entirely determined by its cohomology ring. In particular, higher order Massey products are trivial.

Rational homotopy theory originated with the works of Quillen [**Qui69**] and Sullivan [**Sul77**]. First, Quillen established an equivalence between the homotopy category of simply connected rational spaces and the homotopy category of connected differential graded Lie algebras. Such equivalence is the composite of a long chain of intermediate equivalences, which make the construction quite complex. To better understand this mechanism, Sullivan introduced polynomial de Rham forms and proved that the rational homotopy type of every rational space is determined by a minimal model of its differential graded algebra of rational polynomial linear forms. Since the development of Sullivan's work, minimal models have found significant applications of both topological and geometric origin, one of the first and most striking being the Formality Theorem of compact Kähler manifolds.

Addressing homotopical aspects and multiplicative features of mixed Hodge theory, Morgan [**Mor78**] introduced *mixed Hodge diagrams of differential graded algebras* and proved the existence of functorial mixed Hodge structures on the rational homotopy type of smooth complex algebraic varieties. As an application, he obtained a formality result with respect to the first term of the spectral sequence associated with the weight filtration. In this line of work, Deligne [**Del80**] defined the $\mathbb{Q}_l$-homotopy type of an algebraic variety. Using the weights of the Frobenius action in the l-adic cohomology and his solution of the Riemann hypothesis, he proved a formality result of the $\mathbb{Q}_l$-homotopy type for smooth projective varieties defined over finite fields. Continuing the study of mixed Hodge structures on the rational homotopy type, Navarro [**Nav87**] introduced, in the context of sheaf cohomology of differential graded algebras, the *Thom-Whitney simple functor*, and used his construction to establish the functoriality of mixed Hodge diagrams associated with open smooth varieties, providing a multiplicative version of Deligne's theory. Thanks to this functoriality, and using simplicial hyperresolutions, he extended Morgan's result to possibly singular varieties. Alternatively, there is Hain's approach [**Hai87**] based on the bar construction and Chen's iterated integrals. Both extensions to the singular case depend on the initial constructions of Morgan.

One can interpret the theory of mixed Hodge diagrams of Morgan and his results on the existence of mixed Hodge structures on the rational homotopy type, as a multiplicative analogue of Beilinson's homotopy theory of Hodge complexes. Driven by motivic and Deligne cohomology, Beilinson [**Bei86**] introduced *absolute Hodge complexes*, which are related with the original mixed Hodge complexes of Deligne by a shift on the weight filtration, and studied their homotopy category. He proved formality for objects, showing that every absolute Hodge complex can be represented by the complex of mixed Hodge structures defined by taking cohomology, and established an equivalence with the derived category of mixed Hodge structures. This equivalence allows an interpretation of Deligne's cohomology in terms of extensions of mixed Hodge structures in the derived category. Though sufficient for its original purposes, in this sense Morgan's homotopy theory of mixed Hodge diagrams is incomplete, since it provides the existence of certain minimal models, but these are not shown to be cofibrant or minimal in any abstract categorical framework. Moreover, Morgan allows morphisms between diagrams to be homotopy commutative and does not claim any composition law. As a consequence, his results fall out of the realm of categories. This is one aspect that we intend to solve in this thesis.

The study of derived functors in duality theory led Grothendieck to the localization of a category of complexes with respect to the class of quasi-isomorphisms. The essential constructions were worked out by Verdier [**Ver96**], resulting in the theory of derived categories of abelian categories.

Simultaneously, and mimicking the idea of motives of Grothendieck, the study of spectra in algebraic topology led Quillen [**Qui67**] to the introduction of model categories. In [**BG76**], Bousfield-Gugenheim reformulated Sullivan's rational homotopy theory of differential graded algebras in the context of Quillen model categories. Following this line, it would be desirable to establish an analogous formulation for mixed Hodge complexes and mixed Hodge diagrams of differential graded algebras. Unfortunately, none of the contexts provided by the derived categories of Verdier and Quillen's model categories, considered nowadays as the standard basis of homological and homotopical algebra respectively, satisfy the needs to express the properties of diagram categories with filtrations.

Inspired by the original work of Cartan-Eilenberg [**CE56**] on derivation of additive functors between categories of modules, Guillén-Navarro-Pascual-Roig [**GNPR10**] introduced Cartan-Eilenberg categories, as a homotopical approach weaker than the one provided by Quillen model categories, but sufficient to study homotopy categories and to extend the classical theory of derived additive functors, to non-additive settings. In this context they introduced a notion of cofibrant minimal model, as an abstract characterization of the original minimal models of Sullivan. On the other hand, following Guillén-Navarro [**GN02**], we observe that it is advisable that the categories receiving functors defined over algebraic varieties are equipped, in addition to a model-type structure allowing to derive functors, with a cohomological descent structure, which provides the basis to extend some particular functors defined over smooth varieties, to singular varieties.

In the present work, we analyse the categories of mixed Hodge complexes and mixed Hodge diagrams of differential graded algebras in these two directions: we prove the existence of both a Cartan-Eilenberg structure, via the construction of cofibrant minimal models, and a cohomological descent structure. This allows to interpret the results of Deligne, Beilinson, Morgan and Navarro within a common homotopical framework.

In the additive context of mixed Hodge complexes we recover Beilinson's results. In our study we go a little further and show that the homotopy category of mixed Hodge complexes, and the derived category of mixed Hodge structures are equivalent to a third category whose objects are graded mixed Hodge structures and whose morphisms are certain homotopy classes, which are easier to manipulate. In particular, we obtain a description of the morphisms in the homotopy category in terms of morphisms and extensions of mixed Hodge structures, and recover the results of Carlson [**Car80**] in this area. As for the multiplicative analogue, we show that every mixed Hodge diagram can be represented by a mixed Hodge algebra which is Sullivan minimal, and establish a multiplicative version of Beilinson's Theorem. This provides an alternative to Morgan's construction. The main difference between the two approaches is that Morgan uses ad hoc constructions of models à la Sullivan, specially designed for mixed Hodge theory, while we follow the line of Quillen's model categories or Cartan-Eilenberg categories, in which the main results are expressed in terms of equivalences of homotopy categories, and the existence of certain derived functors. In particular, we obtain not only a description of mixed Hodge diagrams in terms of Sullivan minimal algebras, but we also have a description of the morphisms in the homotopy category in terms of certain homotopy classes, parallel to the additive case. In addition, our approach generalizes to broader settings, such as the study of compactificable analytic spaces, for which the Hodge and weight filtrations can be defined, but do not satisfy the properties of mixed Hodge theory.

Combining these results with Navarro's functorial construction of mixed Hodge diagrams, and using the cohomological descent structure defined via the Thom-Whitney simple, we obtain a more precise and alternative proof of that the rational homotopy type, and the rational homotopy groups of every simply connected complex algebraic variety inherit functorial mixed Hodge structures. As an application, and extending the Formality Theorem of Deligne-Griffiths-Morgan-Sullivan for compact Kähler varieties and the results of Morgan for open smooth varieties, we prove that every simply connected complex algebraic variety (possibly open and singular) and every morphism between such varieties is *filtered formal*: its rational homotopy type is entirely determined by the first term of the spectral sequence associated with the multiplicative weight filtration.

* * *

The categories of mixed Hodge complexes and mixed Hodge diagrams of algebras are examples of subcategories of a category of diagrams with variable vertices, defined as the category of sections of the projection of the Grothendieck construction. In order to study the homotopy theory of such diagram categories and, in particular, to build cofibrant minimal models, one must first prove the existence of cofibrant minimal models for the vertex categories, and second, rectify homotopy commutative morphisms of diagrams, taking into account that each arrow lives in a different category. Hence an essential preliminary step is to understand the homotopy theory of each of the vertex categories, which in our case, are given by filtered and bifiltered complexes of vector spaces and differential graded algebras, over the fields $\mathbb{Q}$ and $\mathbb{C}$.

The homotopy theory of filtered complexes was first studied by Illusie [**Ill71**], who defined the derived category of a filtered abelian category in an ad hoc scheme, studying the localization with respect to the class of weak equivalences defined by those morphisms inducing a quasi-isomorphism at the graded level. An alternative approach using exact categories is detailed in the work of Laumon [**Lau83**]. In certain situations, the filtrations under study are not well defined, and become a proper invariant only in higher stages of the associated spectral sequences. This is the case of the mixed Hodge theory of Deligne, in which the weight filtration of a variety depends on the choice of a hyperresolution, and is only well defined at the second stage. This circumstance is somewhat hidden by the degeneration of the spectral sequences, but it already highlights the interest of studying higher structures. In the context of rational homotopy, Halperin-Tanré [**HT90**] studied the class of weak equivalences defined by morphisms inducing an isomorphism at a certain stage of the associated spectral sequences and proved the existence of minimal models of filtered differential graded algebras with respect to this class of weak equivalences. Likewise, Paranjape [**Par96**] studied the existence of higher injective resolutions for filtered complexes of abelian categories.

In this thesis we show how all these homotopical approaches fit within the common framework of Cartan-Eilenberg categories and provide analogous results for bifiltered categories. In particular, we prove the existence of cofibrant minimal models in each of the above mentioned settings. In order to transfer these homotopical structures at the level of diagrams, we develop an abstract axiomatic which allows to rectify homotopy commutative morphisms of diagrams. This leads to the existence of a Cartan-Eilenberg structure on the diagram category, with level-wise weak equivalences and level-wise cofibrant minimal models.

We have structured our work into five interrelated chapters. We next detail our contributions regarding each of them.

Chapter 1. Homotopical Algebra and Diagram Categories. We develop an abstract axiomatic which allows to define level-wise cofibrant minimal models for a certain type of diagram categories.

Denote by $\Gamma\mathcal{C}$ the category of diagrams associated with a functor $\mathcal{C} : I \to \mathsf{Cat}$ (see Definition 1.3.1). A natural question in homotopy theory is if whether given compatible homotopical structures on the vertex categories $\mathcal{C}_i$, there exists an induced homotopical structure on $\Gamma\mathcal{C}$ with level-wise weak equivalences. For categories of diagrams $\mathcal{C}^I$ associated with a constant functor there are partial answers in terms of Quillen model structures: if $\mathcal{C}$ is cofibrantly generated, or I has a Reedy structure, then the category $\mathcal{C}^I$ inherits a level-wise model structure (see for example [**Hov99**], Theorem 5.2.5). It is also well known that if $\mathcal{C}$ is a Brown category of (co)fibrant objects [**Bro73**], then $\mathcal{C}^I$ inherits a Brown category structure, with weak equivalences and (co)fibrations defined level-wise. In this thesis we study the transfer of cofibrant minimal models in the context of Cartan-Eilenberg structures, and provide a positive answer for a certain type of diagram categories, whose vertex categories are endowed with a functorial path.

A *P-category* is a category $\mathcal{C}$ with a functorial path $P : \mathcal{C} \to \mathcal{C}$ and two classes of morphisms $\mathcal{F}$ and $\mathcal{W}$ of *fibrations* and *weak equivalences* satisfying certain axioms close to those of Brown categories of cofibrant objects, together with a homotopy lifting property with respect to trivial fibrations. Examples of P-categories are the category of differential graded algebras over a field, or the category of topological spaces.

We define a notion of cofibrant object via a lifting property with respect to trivial fibrations: an object C of a P-category $\mathcal{C}$ is called *$\mathcal{F}$-cofibrant* if any morphism $w : A \to B$ in $\mathcal{F} \cap \mathcal{W}$ induces a surjection $w_* : \mathcal{C}(C, A) \to \mathcal{C}(C, B)$. The functorial path defines a notion of homotopy between morphisms of $\mathcal{C}$, which becomes an equivalence relation for those morphisms whose source is $\mathcal{F}$-cofibrant. We prove that if C is $\mathcal{F}$-cofibrant, then every weak equivalence $w : A \to B$ induces a bijection $w_* : [C, A] \to [C, B]$ between homotopy classes of morphisms. In particular, $\mathcal{F}$-cofibrant objects are cofibrant in the sense of Cartan-Eilenberg categories, with the class $\mathcal{S}$ of homotopy equivalences defined by the functorial path, and the class $\mathcal{W}$ of weak equivalences.

We say that a P-category *has cofibrant models* if for any object A of $\mathcal{C}$ there exists an $\mathcal{F}$-cofibrant object C and a weak equivalence $C \to A$. Denote by $\mathcal{C}^{\mathcal{F}}_{cof}$ the full subcategory of $\mathcal{F}$-cofibrant objects, and by $\pi\mathcal{C}^{\mathcal{F}}_{cof}$ the quotient category defined modulo homotopy. We prove:

Theorem 1.2.30. *Let $(\mathcal{C}, P, \mathcal{W}, \mathcal{F})$ be a P-category with cofibrant models. Then the triple $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category with cofibrant models in $\mathcal{C}^{\mathcal{F}}_{cof}$. The inclusion induces an equivalence of categories*

$$\pi\mathcal{C}^{\mathcal{F}}_{cof} \xrightarrow{\sim} \mathcal{C}[\mathcal{W}^{-1}].$$

It is quite immediate, that if the vertices of a diagram category $\Gamma\mathcal{C}$ are endowed with compatible P-category structures, then the diagram category inherits a level-wise P-category structure. However, the existence of cofibrant and minimal models of diagrams is not straightforward, and requires a theory of rectification of homotopy commutative morphisms. We focus on diagrams indexed by a finite directed category of binary degree (see 1.3.4).

We call *ho-morphisms* those maps between diagrams that commute up to homotopy. In general, ho-morphisms cannot be composed. However, the level-wise functorial path of $\Gamma\mathcal{C}$ defines a notion of homotopy between ho-morphisms. Denote by $\Gamma\mathcal{C}_{cof}$ the full subcategory of $\Gamma\mathcal{C}$ defined by level-wise $\mathcal{F}_i$-cofibrant objects. Its objects, together with the homotopy classes of ho-morphisms define a category $\pi^h\Gamma\mathcal{C}_{cof}$.

Define a new class of strong equivalences of $\Gamma\mathcal{C}$ as follows. A morphism of $\Gamma\mathcal{C}$ is called a *ho-equivalence* if it has a homotopy inverse which is a ho-morphism. The class $\mathcal{H}$ defined by the closure by composition of ho-equivalences satisfies $\mathcal{S} \subset \mathcal{H} \subset \mathcal{W}$, where $\mathcal{S}$ denotes the class of homotopy equivalences defined by the natural path of $\Gamma\mathcal{C}$ and $\mathcal{W}$ denotes the class of level-wise weak equivalences. We prove:

Theorem 1.4.11. *Let $\Gamma\mathcal{C}$ be a diagram category indexed by a directed category I as in* 1.3.4. *Assume that for each $i \in I$, the categories $\mathcal{C}_i$ are P-categories with $\mathcal{F}_i$-cofibrant models, and that the functors $u_* : \mathcal{C}_i \to \mathcal{C}_j$ are compatible with the P-category structures preserving $\mathcal{F}_i$-cofibrant objects. Then $(\Gamma\mathcal{C}, \mathcal{H}, \mathcal{W})$ is a left Cartan-Eilenberg category with models in $\Gamma\mathcal{C}_{cof}$. The inclusion induces an equivalence of categories*

$$\pi^h\Gamma\mathcal{C}_{cof} \xrightarrow{\sim} \Gamma\mathcal{C}[\mathcal{W}^{-1}].$$

In particular, the vertices of a cofibrant model of a given diagram, are cofibrant models of its vertices. We prove an analogous result with minimal models (see Theorem 1.4.12), and a relative version concerning a full subcategory of a diagram category, closed by weak equivalences (see Lemma 1.4.13), useful in the applications to mixed Hodge theory.

Chapter 2. Filtered Derived Categories. We study filtered complexes within the framework of Cartan-Eilenberg categories. Although most of the contents of this chapter are possibly known, there seems to be a generalized lack of bibliography on the subject. Thus, this chapter is intended as a self-contained exposition of the main results on (bi)filtered complexes. This paves the way in two directions: the study of mixed Hodge complexes of Chapter 3, and the study of filtered differential graded algebras of Chapter 4.

The category $\mathbf{F}\mathcal{A}$ of filtered objects (with finite filtrations) of an abelian category $\mathcal{A}$ is additive, but not abelian in general. Consider the category $\mathbf{C}^+(\mathbf{F}\mathcal{A})$ of bounded below complexes of $\mathbf{F}\mathcal{A}$. For $r \geq 0$, denote by $\mathcal{E}_r$ the class of *E_r-quasi-isomorphisms*: these are morphisms of filtered complexes inducing a quasi-isomorphism at the E_r-stage of the associated spectral sequences. We are interested in the *r-derived category* defined by

$$\mathbf{D}^+_r(\mathbf{F}\mathcal{A}) := \mathbf{C}^+(\mathbf{F}\mathcal{A})[\mathcal{E}_r^{-1}].$$

The case $r = 0$ corresponds to the original filtered derived category, studied by Illusie in [**Ill71**]. There is a chain of functors

$$\mathbf{D}^+_0(\mathbf{F}\mathcal{A}) \to \mathbf{D}^+_1(\mathbf{F}\mathcal{A}) \to \cdots \to \mathbf{D}^+_r(\mathbf{F}\mathcal{A}) \to \cdots \to \mathbf{D}^+(\mathbf{F}\mathcal{A}),$$

where the rightmost category denotes the localization with respect to quasi-isomorphisms. Each of these categories keeps less and less information of the original filtered homotopy type.

In order to deal with the weight filtration, in [**Del71b**] Deligne introduced the décalage of a filtered complex, which shifts the associated spectral sequence of the original filtered complex by one stage. This defines a functor

$$\mathrm{Dec} : \mathbf{C}^+(\mathbf{F}\mathcal{A}) \longrightarrow \mathbf{C}^+(\mathbf{F}\mathcal{A})$$

which is the identity on morphisms and sends morphisms in $\mathcal{E}_{r+1}$ to morphisms in $\mathcal{E}_r$. The décalage does not admit an inverse, but it has a left adjoint S, defined by a shift of the filtration. Using this adjoint pair and the relation with the spectral sequences, we prove:

Theorem 2.2.15. *Deligne's décalage induces an equivalence of categories*

$$\mathrm{Dec} : \mathbf{D}^+_{r+1}(\mathbf{F}\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^+_r(\mathbf{F}\mathcal{A}),$$

for every $r \geq 0$.

The notion of homotopy between morphisms of complexes over an additive category is defined via a translation functor, and provides the homotopy category with a triangulated structure. In the filtered setting, we find that different choices of the filtration of the translation functor, lead to different notions of *r-homotopy*, suitable to the study of the r-derived category. The r-homotopy category is still triangulated, and for each $r \geq 0$ we obtain a class $\mathcal{S}_r$ of *r-homotopy equivalences* satisfying $\mathcal{S}_r \subset \mathcal{E}_r$.

As in the classical case, we address the study of the r-derived category of filtered objects $\mathbf{F}\mathcal{A}$ under the assumption that $\mathcal{A}$ has enough injectives. Denote by $\mathbf{C}_r^+(\mathbf{F}\mathrm{Inj}\mathcal{A})$ the full subcategory of those filtered complexes over injective objects of $\mathcal{A}$ whose differential satisfies $dF^p \subset F^{p+r}$, for all $p \in \mathbb{Z}$. In particular, the induced differential at the s-stage of the associated spectral sequence is trivial for all $s < r$. Its objects are called *r-injective complexes* and satisfy the classical property of fibrant objects: if I is an r-injective complex then every E_r-quasi-isomorphism $w : K \to L$ induces a bijection $w^* : [L, I]_r \to [K, I]_r$ between r-homotopy classes of morphisms.

We show that if $\mathcal{A}$ is an abelian category with enough injectives, then every filtered complex K has an *r-injective model*: this is an r-injective complex I, together with an E_r-quasi-isomorphism $K \to I$ (a similar result had been previously found by Paranjape in [**Par96**]). As a consequence, we have:

Theorem 2.2.26. *Let $\mathcal{A}$ be an abelian category with enough injectives, and let $r \geq 0$. The triple $(\mathbf{C}^+(\mathbf{F}\mathcal{A}), \mathcal{S}_r, \mathcal{E}_r)$ is a (right) Cartan-Eilenberg category. The inclusion induces an equivalence of categories*

$$\mathbf{K}_r^+(\mathbf{F}\mathrm{Inj}\mathcal{A}) \xrightarrow{\sim} \mathbf{D}_r^+(\mathbf{F}\mathcal{A})$$

between the category of r-injective complexes modulo r-homotopy and the r-derived category of filtered objects.

Note that for $r = 0$ we recover a result of Illusie (see [**Ill71**], Cor. V.1.4.7).

Consider the particular case in which $\mathcal{A}$ is the category of vector spaces over a field $\mathbf{k}$. In this case, every object of $\mathcal{A}$ is injective and the classical calculus of derived categories does not provide any additional information. However, we can consider minimal models: every complex K is quasi-isomorphic to its cohomology $K \to H(K)$. This gives an equivalence

$$\mathbf{G}^+(\mathbf{k}) \xrightarrow{\sim} \mathbf{D}^+(\mathbf{k})$$

between the category of non-negatively graded vector spaces and the derived category of vector spaces over a field $\mathbf{k}$. We provide an analogous result for (bi)filtered complexes of vector spaces defined over a field as follows.

A filtered complex of $\mathbf{C}^+(\mathbf{Fk})$ is called E_r-*minimal* if it is an object of $\mathbf{C}^+_{r+1}(\mathbf{Fk})$. That is, its differential satisfies $dF^p \subset F^{p+r+1}$, for all $p \in \mathbb{Z}$. We show that every E_r-quasi-isomorphism between E_r-minimal objects is an isomorphism, and that every filtered complex has an E_r-minimal model. As a consequence, we have:

Theorem 2.3.7. *Let $r \geq 0$. The triple $(\mathbf{C}^+(\mathbf{Fk}), \mathcal{S}_r, \mathcal{E}_r)$ is a Sullivan category, and $\mathbf{C}^+_{r+1}(\mathbf{Fk})$ is a full subcategory of minimal models.*

Note that for $r = 0$, the minimal models are those complexes whose differential is trivial at the graded level. This follows the pattern of the non-filtered case, in which the cohomology of a complex with the trivial differential, is a minimal model of the complex. The above result can be adapted to complexes having multiple filtrations. For the sake of simplicity and given our interests in mixed Hodge theory, in this thesis we only present the bifiltered case with respect to the classes $\mathcal{E}_{0,0}$ and $\mathcal{E}_{1,0}$ (see Theorem 2.4.12).

Chapter 3. Mixed Hodge Complexes. We study the homotopy theory of mixed Hodge complexes within the framework of Cartan-Eilenberg categories, via the construction of cofibrant minimal models.

A *mixed Hodge complex* over $\mathbb{Q}$ consists in a filtered complex $(K_{\mathbb{Q}}, W)$ over $\mathbb{Q}$, a bifiltered complex $(K_{\mathbb{C}}, W, F)$ over $\mathbb{C}$, together with a finite string of morphisms $\varphi : (K_{\mathbb{Q}}, W) \otimes \mathbb{C} \longleftrightarrow (K_{\mathbb{C}}, W)$ satisfying the following axioms:

(MHC_0) The comparison map φ is a string of E_1^W-quasi-isomorphisms.

(MHC_1) For all $p \in \mathbb{Z}$, the filtered complex $(Gr_p^W K_{\mathbb{C}}, F)$ is d-strict.

(MHC_2) The filtration F induced on $H^n(Gr_p^W K_{\mathbb{C}})$, defines a pure Hodge structure of weight $p+n$ on $H^n(Gr_p^W K_{\mathbb{Q}})$, for all n, and all $p \in \mathbb{Z}$.

The filtration W is known as the *weight filtration*, while F is called the *Hodge filtration.* A shift on the induced weight filtration endows the n-th cohomology of every mixed Hodge complex with mixed Hodge structures. Denote by **MHC** the category of mixed Hodge complexes over $\mathbb{Q}$.

To study the homotopy theory of mixed Hodge complexes it is more convenient to work with the category **AHC** of absolute Hodge complexes as introduced by Beilinson. The main advantage is that in this case, the spectral sequences associated with both W and F degenerate at the first stage and the cohomology is a graded mixed Hodge structure. We have functors

$$\mathbf{MHC} \stackrel{\mathrm{Dec}^W}{\longrightarrow} \mathbf{AHC} \stackrel{H}{\longrightarrow} \mathbf{G}^+(\mathsf{MHS}),$$

where Dec^W is the functor induced by décalage of the weight filtration.

Since the category of mixed Hodge structures is abelian, every graded mixed Hodge structure, and more generally, every complex of mixed Hodge structures is an absolute Hodge complex. We have a chain of full subcategories

$$\mathbf{G}^+(\mathsf{MHS}) \longrightarrow \mathbf{C}^+(\mathsf{MHS}) \longrightarrow \mathbf{AHC}.$$

The category of mixed (resp. absolute) Hodge complexes is a category of diagrams, whose vertices are filtered and bifiltered complexes. Hence the construction of minimal models involves a rectification of homotopy commutative morphisms of diagrams. We show that for every absolute Hodge complex is conected with its cohomology by a ho-morphism which is a levelwise quasi-isomorphism. This can be seen as the formality result for objects already stated by Beilinson in [**Bei86**]. However, morphisms are not formal.

Denote by $\pi^h\mathbf{G}^+(\mathsf{MHS})$ the category whose objects are non-negatively graded mixed Hodge structures and whose morphisms are homotopy classes of ho-morphisms. Denote by $\mathcal{H}$ the class of morphisms of absolute Hodge complexes that are homotopy equivalences as ho-morphisms, and by $\mathcal{Q}$ the class of quasi-isomorphisms of **AHC**. We prove:

Theorem 3.3.12. *The triple* $(\mathbf{AHC}, \mathcal{H}, \mathcal{Q})$ *is a Sullivan category, and* $\mathbf{G}^+(\mathsf{MHS})$ *is a full subcategory of minimal models. The inclusion induces* *an equivalence of categories*

$$\pi^h\mathbf{G}^+(\mathsf{MHS}) \xrightarrow{\sim} \mathrm{Ho}\,(\mathbf{AHC}) := \mathbf{AHC}[\mathcal{Q}^{-1}].$$

Note that although every absolute Hodge complex is quasi-isomorphic to its cohomology (which has trivial differentials), the full subcategory of minimal models has non-trivial homotopies. This reflects the fact that mixed Hodge structures have non-trivial extensions.

The above result allows to endow the category **MHC** with a Sullivan category structure via Deligne's décalage (see Theorem 3.3.13). We prove:

Theorem 3.3.14. *Deligne's décalage induces an equivalence of categories*

$$\mathrm{Dec}^W : \mathrm{Ho}(\mathbf{MHC}) \xrightarrow{\sim} \mathrm{Ho}(\mathbf{AHC}).$$

Using the equivalence of categories of Theorem 3.3.12 we recover Beilinson's result (see [**Bei86**], Thm. 3.2), providing an equivalence of categories

$$\mathbf{D}^+\,(\mathsf{MHS}) \xrightarrow{\sim} \mathrm{Ho}\,(\mathbf{AHC})$$

between the derived category of mixed Hodge structures and the homotopy category of absolute Hodge complexes. As an application of the above results, we read off the morphisms in the homotopy category of absolute and Hodge complexes, in terms of morphisms and extensions of mixed Hodge structures.

Theorem 3.3.17. *Let K and L be absolute Hodge complexes. Then*

$$\mathrm{Ho}(\mathbf{AHC})(K,L) = \bigoplus_n \left(\mathrm{Hom}_{\mathsf{MHS}}(H^nK, H^nL) \oplus \mathrm{Ext}^1_{\mathsf{MHS}}(H^nK, H^{n-1}L)\right).$$

In particular, we recover the results of Carlson [**Car80**] and Beilinson [**Bei86**] on extensions of mixed Hodge structures.

Chapter 4. Filtrations in Rational Homotopy. The category of filtered differential graded algebras (dga's for short) over a field $\mathbf{k}$ of characteristic 0 does not admit a Quillen model structure. However, the existence of filtered minimal models allows to define a homotopy theory in a non-axiomatic conceptual framework, as done by Halperin-Tanré [**HT90**]. We develop an alternative construction of filtered minimal models, which is an adaptation to the classical construction of Sullivan minimal models of dga's presented in [**GM81**]. The main advantage of this alternative method is that it is easily generalizable to differential algebras having multiple filtrations. Then, we study the homotopy theory of filtered dga's within the axiomatic framework of Cartan-Eilenberg categories.

As in the setting of filtered complexes, denote by $\mathcal{E}_r$ the class of E_r-quasi-isomorphisms of filtered dga's, and let

$$\mathrm{Ho}_r\left(\mathbf{F}\mathsf{DGA}(\mathbf{k})\right) := \mathbf{F}\mathsf{DGA}(\mathbf{k})[\mathcal{E}_r^{-1}]$$

denote the corresponding localized category. The localization with respect to $\mathcal{E}_0$ is the ordinary filtered category. There is a chain of functors

$$\mathrm{Ho}_0\left(\mathbf{F}\mathsf{DGA}(\mathbf{k})\right) \to \mathrm{Ho}_1\left(\mathbf{F}\mathsf{DGA}(\mathbf{k})\right) \to \cdots \to \mathrm{Ho}\left(\mathbf{F}\mathsf{DGA}(\mathbf{k})\right)$$

where the rightmost category denotes the localization with respect to the class of quasi-isomorphisms. The main invariant for an object of Ho is the cohomology algebra $H(A)$. In contrast, in Ho_r we have a family of invariants $E_s(A)$ with $s > r$, where $E_s(A)$ is an s-bigraded dga, the main invariant being $E_{r+1}(A)$. Analogously to the theory of filtered complexes, we prove:

Theorem 4.3.7. *Deligne's décalage induces an equivalence of categories*

$$\mathrm{Dec} : \mathrm{Ho}_{r+1}\left(\mathbf{F}\mathsf{DGA}(\mathbf{k})\right) \xrightarrow{\sim} \mathrm{Ho}_r\left(\mathbf{F}\mathsf{DGA}(\mathbf{k})\right).$$

for every $r \geq 0$.

To study the homotopy theory of filtered dga's we introduce a notion of r-homotopy via a weighted functorial r-path object. This defines a class $\mathcal{S}_r$ of *r-homotopy equivalences* satisfying $\mathcal{S}_r \subset \mathcal{E}_r$, and endows the category of filtered dga's with a P-category structure, for each $r \geq 0$.

Define a generalized notion of Sullivan minimal dga as follows. A *filtered KS-extension* of degree n and weight p of an augmented filtered dga (A, d, F) is a filtered dga $A \otimes_\xi \Lambda(V)$, where V is a graded vector space of degree n and pure weight p, and $\xi : V \to F^p A$ is a linear map of degree 1 satisfying $d\xi = 0$. The filtration on $A \otimes_\xi \Lambda(V)$ is defined by multiplicative extension.

Such an extension is said to be E_r-*minimal* if

$$\xi(V) \subset F^{p+r}(A^+ \cdot A^+) + F^{p+r+1}A,$$

where A^+ denotes the kernel of the augmentation. Define an E_r-*minimal dga* as the colimit of a sequence of E_r-minimal extensions, starting from the base field. In particular, every E_r-minimal dga A is free and augmented, and satisfies

$$d(F^pA) \subset F^{p+r}(A^+ \cdot A^+) + F^{p+r+1}A.$$

Note that for the trivial filtration, the notion of E_0-minimal dga coincides with the notion of a Sullivan minimal dga.

Every E_r-minimal dga M is E_r-*cofibrant*: the map $w_* : [A, M]_r \to [B, M]_r$ induced by any E_r-quasi-isomorphism $w : A \to B$ is bijective. Furthermore, any E_r-quasi-isomorphism between E_r-minimal dga's is an isomorphism.

An E_r-*minimal model* of a filtered dga A is an E_r-minimal dga M, together with an E_r-quasi-isomorphism $M \to A$. We prove the existence of such models for E_r-1-connected dga's (these are filtered dga's whose E_r-stage is a 1-connected bigraded dga).

Theorem 4.3.27 (cf. [**HT90**]). *Let $r \geq 0$. Every E_r-1-connected filtered dga has an E_r-minimal model.*

We prove an analogous result for bifiltered dga's (see Theorem 4.4.9). The homotopy theory of filtered dga's is summarized in the following theorem.

Theorem 4.3.28. *Let $r \geq 0$. The triple $(\mathbf{F}\mathsf{DGA}^1(\mathbf{k}), \mathcal{S}_r, \mathcal{E}_r)$ is a Sullivan category. The inclusion induces an equivalence of categories*

$$\pi_r\left(\mathsf{E_r}\text{-}\mathsf{min}^1(\mathbf{k})\right) \longrightarrow \mathrm{Ho}_r\left(\mathbf{F}\mathsf{DGA}^1(\mathbf{k})\right).$$

between the quotient category of 1-connected E_r-minimal dga's modulo r-homotopy equivalence, and the localized category of E_r-1-connected filtered dga's with respect to the class of E_r-quasi-isomorphisms.

The Sullivan category structure allows to define the E_r-*homotopy* of a filtered dga via the derived functor of the complex of indecomposables Q of augmented filtered dga's, parallel to the classical setting.

Theorem 4.3.47. *Let $r \geq 0$. The functor $Q : \mathbf{F}\mathsf{DGA}^1(\mathbf{k})_* \longrightarrow \mathbf{C}^+(\mathbf{Fk})$ admits a left derived functor*

$$\mathbb{L}_r Q : \mathrm{Ho}_r\left(\mathbf{F}\mathsf{DGA}^1(\mathbf{k})_*\right) \longrightarrow \mathbf{D}_r^+(\mathbf{Fk}).$$

The composition of functors

$$\mathrm{Ho}_r\left(\mathbf{F}\mathsf{DGA}^1(\mathbf{k})\right) \overset{\sim}{\longleftarrow} \mathrm{Ho}_r\left(\mathbf{F}\mathsf{DGA}^1(\mathbf{k})_*\right) \xrightarrow{\mathbb{L}_r Q} \mathbf{D}_r^+(\mathbf{Fk}) \xrightarrow{E_r} \mathbf{C}_{r+1}^+(\mathbf{Fk})$$

defines a functor

$$\pi_{E_r} : \mathrm{Ho}_r\left(\mathbf{F}\mathsf{DGA}^1(\mathbf{k})\right) \longrightarrow \mathbf{C}_{r+1}^+(\mathbf{Fk})$$

which associates to every object A, the E_r-minimal complex $\pi_{E_r}(A) = Q(M_A)$, where $M_A \to A$ is an E_r-minimal model of A.

The E_r-minimal model of a filtered dga is related to the bigraded minimal model of the E_r-stage of its associated spectral sequence. This gives a spectral sequence relating the E_r-homotopy $\pi_{E_r}(A)$ of a filtered dga A with its classical homotopy $\pi(A)$. Likewise, we have a notion of filtered formality, which generalizes the classical notion of formality. Let $r \geq 0$. A filtered dga (A, d, F) is said to be E_r*-formal* if there is an isomorphism

$$(A, d, F) \overset{\sim}{\longleftrightarrow} (E_{r+1}(A), d_{r+1}, F)$$

in the homotopy category $\mathrm{Ho}_r(\mathbf{F}\mathsf{DGA}(\mathbf{k}))$, where the filtration F on $E_{r+1}(A)$ is induced by the filtering degree. In particular, the E_0-formality of the Dolbeault algebra of forms of a complex manifold coincides with the notion of Dolbeault formality introduced by Neisendorfer-Taylor in [**NT78**].

Chapter 5. Mixed Hodge Theory and Rational Homotopy. In this last chapter we bring together the results of the previous chapters to study the homotopy theory of mixed Hodge diagrams, and their cohomological descent structure. We then provide applications to algebraic geometry.

The category **MHD** of mixed Hodge diagrams of dga's is defined analogously to that of mixed Hodge complexes, by replacing every complex by a dga. As in the additive case, to study the homotopy category of mixed Hodge diagrams it is more convenient to work with the shifted version **AHD** of absolute Hodge diagrams. Deligne's décalage with respect to the weight filtration induces a functor

$$\mathrm{Dec}^W : \mathbf{MHD} \longrightarrow \mathbf{AHD}.$$

The multiplicative analogue of a complex of mixed Hodge structures leads to the notion of *mixed Hodge dga*: this is a dga (A, d) such that each A^n is endowed with a mixed Hodge structure, and the differentials are morphisms of mixed Hodge structures. Denote by $\mathbf{MH}\mathsf{DGA}$ the category of mixed Hodge dga's over $\mathbb{Q}$. The cohomology of every absolute Hodge diagram is a mixed Hodge dga with trivial differential. We have a functor

$$\mathbf{AHD} \xrightarrow{H} \mathbf{MH}\mathsf{DGA}.$$

Conversely, since the category of mixed Hodge structures is abelian, every mixed Hodge dga is an absolute Hodge diagram. There is an inclusion functor

$$i : \mathbf{MH}\mathsf{DGA} \longrightarrow \mathbf{AHD}.$$

We show that every 1-connected absolute Hodge diagram is quasi-isomorphic to a mixed Hodge dga which is Sullivan minimal. More precisely, define a *mixed Hodge Sullivan minimal dga* as a Sullivan minimal dga $M = (\Lambda V, d)$ over $\mathbb{Q}$ such that each V^n is endowed with a mixed Hodge structure, and the differentials are compatible with the filtrations. In particular, the mixed Hodge structures on V^n, define a mixed Hodge structure on A^n. Hence every mixed Hodge Sullivan minimal dga is a mixed Hodge dga. We prove:

Theorem 5.1.17. *For every 1-connected absolute Hodge diagram A, there exists a 1-connected mixed Hodge Sullivan minimal dga M, together with a ho-morphism $\rho : M \rightsquigarrow A$, which is a quasi-isomorphism.*

Combining this result with the homotopy theory of diagram categories of Chapter 1 we prove the analogue of Theorem 3.3.12, which can be thought as a multiplicative version of Beilinson's Theorem.

Theorem 5.1.19. *The triple $(\mathbf{AHD}^1, \mathcal{H}, \mathcal{Q})$ is a Sullivan category. The category of mixed Hodge Sullivan minimal dga's is a full subcategory of minimal models. The inclusion induces an equivalence of categories*

$$\pi^h \mathbf{MH}\mathsf{DGA}^1_{min} \longrightarrow \mathbf{AHD}^1[\mathcal{Q}^{-1}]$$

between the category whose objects are 1-connected mixed Hodge Sullivan minimal dga's over $\mathbb{Q}$ and whose morphisms are classes of ho-morphisms modulo homotopy equivalence and the localized category of 1-connected absolute Hodge diagrams with respect to quasi-isomorphisms.

The above result allows to endow the category of mixed Hodge diagrams with a Sullivan category structure via Deligne's décalage. We prove:

Theorem 5.1.21. *Deligne's décalage induces an equivalence of categories*

$$\mathrm{Dec}^W : \mathrm{Ho}\left(\mathbf{MHD}^1\right) \xrightarrow{\sim} \mathrm{Ho}\left(\mathbf{AHD}^1\right).$$

As an application we define the homotopy of a mixed Hodge diagram via the derived functor of indecomposables.

Theorem 5.1.23. *The functor Q admits a left derived functor*

$$\mathbb{L}Q : \mathrm{Ho}\left(\mathbf{MHD}^1_*\right) \longrightarrow \mathrm{Ho}\left(\mathbf{MHC}\right).$$

The composition of functors

$$\mathrm{Ho}\left(\mathbf{MHD}^1\right) \xleftarrow{\sim} \mathrm{Ho}\left(\mathbf{MHD}^1_*\right) \xrightarrow{\mathbb{L}Q} \mathrm{Ho}\left(\mathbf{MHC}\right) \xrightarrow{H\circ\mathrm{Dec}^W} \mathbf{G}^+(\mathsf{MHS})$$

defines a functor

$$\pi : \mathrm{Ho}\left(\mathbf{MHD}^1\right) \longrightarrow \mathbf{G}^+(\mathsf{MHS})$$

which associates to every 1-connected mixed Hodge diagram A, the graded mixed Hodge structure $\pi(A) = Q(M_A)$, where $M_A \rightsquigarrow A$ is a minimal model of A.

The rational part of the graded mixed Hodge structure associated with a mixed Hodge diagram coincides with the classical homotopy of the rational part of the original diagram. As a consequence, the homotopy groups of the rational part of every 1-connected mixed Hodge diagram are endowed with functorial mixed Hodge structures.

Deligne's construction of functorial mixed Hodge structures can be restated as having a functor

$$\mathbb{H}dg : \mathbf{V}^2(\mathbb{C}) \longrightarrow \mathbf{MHC}$$

sending every smooth compactification $U \subset X$ of algebraic varieties over $\mathbb{C}$ with $D = X - U$ a normal crossings divisor, to a mixed Hodge complex, which computes the cohomology of U (see Theorem 5.3.3). Furthermore, the object $\mathbb{H}dg(X, U) \in \mathrm{Ho}\,(\mathbf{MHC})$ does not depend on X and is functorial in U. Inspired by the work of Deligne and Morgan and with the objective to extend Morgan's result to singular varieties, Navarro [**Nav87**] defined a multiplicative version of Deligne's functor

$$\mathbb{H}dg : \mathbf{V}^2(\mathbb{C}) \longrightarrow \mathbf{MHD}$$

with values in the category of mixed Hodge diagrams of dga's (see Theorem 5.3.6). Both functors are known to extend to functors defined over all complex algebraic varieties. We provide a proof via the extension criterion of [**GN02**], which is based on the assumption that the target category is a *cohomological descent category.* This is essentially a category $\mathcal{D}$ together with a saturated class $\mathcal{W}$ of weak equivalences and a simple functor $\mathbf{s}$ sending every cubical codiagram of $\mathcal{D}$ to an object of $\mathcal{D}$, and satisfying certain conditions analogous to those of the total complex of a double complex.

The primary example of a cohomological descent structure is given by the category of complexes $\mathbf{C}^+(\mathcal{A})$ of an abelian category $\mathcal{A}$ with the class of quasi-isomorphisms and the simple functor $\mathbf{s}$ given by the *total complex.* The choice of certain filtrations originally introduced by Deligne leads to a simple $\mathbf{s}_D$ for cubical codiagrams of mixed Hodge diagrams, defined levelwise. We restate the key Theorem 8.1.15 of Deligne [**Del74b**] as:

Theorem 5.2.20. *The category of mixed Hodge complexes* $\mathbf{MHC}$ *with the class* $\mathcal{Q}$ *of quasi-isomorphisms and the simple functor* $\mathbf{s}_D$ *is a cohomological descent category.*

An analogous result in the context of simplicial descent categories appears in [**Rod12b**]. Following Deligne's work, the main application of this result is the extension of Deligne's functor to possibly singular varieties.

Theorem 5.3.4. *There exists an essentially unique* Φ*-rectified functor*

$$\mathbb{H}dg' : \mathbf{Sch}(\mathbb{C}) \to \mathrm{Ho}\,(\mathbf{MHC})$$

extending the functor $\mathbb{H}dg : \mathbf{V}^2_{\mathbb{C}} \to \mathbf{MHC}$ *of Theorem* 5.3.3 *such that:*

(1) $\mathbb{H}dg'$ *satisfies the descent property (D) of Theorem* 5.2.7.

(2) The cohomology $H(\mathbb{H}dg'(X))$ is the mixed Hodge structure of the cohomology of X.

The Thom Whitney simple for strict cosimplicial dga's of Navarro [**Nav87**] adapts to the cubical setting to provide the category $\mathsf{DGA}(\mathbf{k})$ of dga's with a cohomological descent structure. The definition of certain filtrations on the Thom-Whitney simple leads to the construction of a simple $\mathbf{s}_{TW}$ for cubical codiagrams of mixed Hodge diagrams, defined level-wise. We have a quasi-isomorphism of simples $\mathbf{s}_{TW} \to \mathbf{s}_D$. Analogously to the additive case:

Theorem 5.2.30. *The category of mixed Hodge diagrams* **MHD** *with the class $\mathcal{Q}$ of quasi-isomorphisms and the Thom-Whitney simple functor $\mathbf{s}_{TW}$ is a cohomological descent category.*

Following Navarros's work, the main application of this result is the extension of Navarro's functor to possibly singular varieties.

Theorem 5.3.7. *There exists an essentially unique Φ-rectified functor*

$$\mathbb{H}dg' : \mathbf{Sch}(\mathbb{C}) \to \mathrm{Ho}\,(\mathbf{MHD})$$

extending the functor $\mathbb{H}dg : \mathbf{V}^2_{\mathbb{C}} \to \mathbf{MHD}$ of Theorem 5.3.6 such that:

(1) $\mathbb{H}dg'$ satisfies the descent property (D) of Theorem 5.2.7.

(2) The rational part of $\mathbb{H}dg'(X)$ is $A_X(\mathbb{Q}) = A_{Su}(X^{an})$.

(3) The cohomology $H(\mathbb{H}dg'(X))$ is the mixed Hodge structure of the cohomology of X.

As a consequence of Theorems 5.3.7 and 5.1.19, we recover the result of [**Nav87**], stating that the minimal model of the rational homotopy type of every simply connected complex algebraic variety is equipped with functorial mixed Hodge structures.

Furthermore, we prove the following formality theorem, which extends the results of [**Mor78**] concerning the filtered formality of the rational homotopy type of smooth complex varieties.

Theorem 5.3.9. *The rational homotopy type of every morphism of simply connected complex algebraic varieties is a formal consequence of the first term of the spectral sequence associated with the weight filtration, that is:*

(1) If X is a simply connected complex algebraic variety, there is a chain of quasi-isomorphisms

$$(A_X(\mathbb{Q}), d) \xleftarrow{\sim} (M_X, d) \xrightarrow{\sim} (E_1(A_X(\mathbb{Q}), W), d_1),$$

where (M_X, d) is a Sullivan minimal dga over $\mathbb{Q}$ and $A_X(\mathbb{Q})$ is the de Rham algebra of X over $\mathbb{Q}$.

(2) If $f : X \to Y$ is a morphism of simply connected complex algebraic varieties, there exists a diagram

$$\begin{tikzcd}
(A_X(\mathbb{Q}), d) \arrow[d, "f_{\mathbb{Q}}"] & (M_X, d) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, dotted] & (E_1(A_X(\mathbb{Q}), W), d_1) \arrow[d, "E_1(f_{\mathbb{Q}})"] \\
(A_Y(\mathbb{Q}), d) & (M_Y, d) \arrow[l, "\sim"'] \arrow[r, "\sim"] & (E_1(A_Y(\mathbb{Q}), W), d_1)
\end{tikzcd}$$

which commutes up to homotopy.

These results can be summarized as having an isomorphism of functors

$$U_{\mathbb{Q}} \circ \mathbb{H}dg' \cong E_1 \circ (U_{\mathbb{Q}} \circ \mathbb{H}dg') : \mathbf{Sch}^1(\mathbb{C}) \to \mathrm{Ho}_1(\mathbf{F}\mathsf{DGA}^1(\mathbb{Q})),$$

where $U_{\mathbb{Q}}$ denotes the forgetful functor sending every mixed Hodge diagram A to its rational part $(A_{\mathbb{Q}}, W)$.

CHAPTER 1

Homotopical Algebra and Diagram Categories

One of the main objectives of abstract Homotopy Theory is to address the problem of choosing a certain class of maps (called weak equivalences) in a category, and studying the passage to the *homotopy category*: this is the localized category obtained by making weak equivalences into isomorphisms. Originally inspired on the category of topological spaces, this is a problem of a very general nature, and central in many problems of algebraic geometry and topology. For example, the weak equivalences could be homology isomorphisms or homotopy equivalences in a certain algebraic setting, weak homotopy equivalences of topological spaces, or birational equivalences of algebraic varieties.

By formally inverting weak equivalences, one can always obtain the homotopy category, but in general, the resulting category does not behave in a controlled way. For example, the morphisms between two objects in the localized category might not even be a set. In addition, the understanding of the maps in the homotopy category can prove to be very difficult.

Quillen's model categories [**Qui67**] solve this problem: the verification of a set of axioms satisfied by three distinguished classes of morphisms (weak equivalences, fibrations and cofibrations) gives a reasonably general context in which it is possible to study homotopy theory. The axioms for Quillen's model categories are very powerful and they provide, not only a precise description of the maps in the homotopy category, but also other higher homotopical structures (such as the existence of homotopy (co)limits or mapping spaces). As a counterpart, in some cases it can be really hard to prove that a particular category is a model category. In addition, there exist interesting categories from the homotopical point of view, which do not satisfy all the axioms. Examples are the category of filtered complexes of an abelian category or the category of filtered dga's, both considered in this work.

A solution proposed by several authors consists in replacing the axioms of Quillen by a left- (or right-) handed version. This is the case of the categories of (co)fibrant objects introduced by Brown in [**Bro73**], or their stronger versions, such as the (co)fibration categories defined by Baues in [**Bau89**], or the Anderson-Brown-Cisinski categories presented in [**RB07**]. These alternatives are very close to Quillen's formulation.

The formalism of Cartan-Eilenberg categories was introduced in [**GNPR10**] by Guillén-Navarro-Pascual-Roig, as an alternative approach to model categories. Based on the initial data of two classes of morphisms (strong and weak equivalences), they define cofibrant objects and assume the existence of enough cofibrant models of objects. This provides the sufficient structure to study the homotopy category. An important observation is that in this setting, one can consider minimal models, as a particular case of cofibrant models, parallel to the theory of Sullivan [**Sul77**].

A desirable property of a homotopy theory is that its axiomatic is transferred to diagram categories, with level-wise weak equivalences. For categories of diagrams $\mathcal{C}^I$ associated with a constant functor there are partial answers in terms of Quillen model structures: if $\mathcal{C}$ is cofibrantly generated, or I has a Reedy structure, then the category $\mathcal{C}^I$ inherits a level-wise model structure (see for example [**Hov99**], Theorem 5.2.5). It is also well known that if $\mathcal{C}$ is a Brown category of (co)fibrant objects, then $\mathcal{C}^I$ inherits a Brown category structure, with weak equivalences and (co)fibrations defined level-wise.

In this chapter we study the homotopy theory of a certain type of diagram categories with vertices in variable categories within the axiomatic framework of Cartan-Eilenberg categories. We show that under certain hypothesis, the cofibrant minimal models of the vertices of a diagram define a cofibrant minimal model of the diagram. Hence the Cartan-Eilenberg structure transfers to diagram categories with level-wise weak equivalences and level-wise models.

In Section 2 we introduce *P-categories*. These are categories with a functorial path and two distinguished classes of morphisms, called fibrations and weak equivalences, satisfying a list of axioms similar to those of Brown categories of fibrant objects. We define a notion of cofibrant object in a P-category by means of a lifting property with respect to trivial fibrations, and prove that every P-category with enough cofibrant models is a Cartan-Eilenberg category with the same weak equivalences. An analogous result is obtained with cofibrant minimal models.

In Section 3 we study the *category of diagrams associated with a functor* whose target is the category of categories. Its objects are diagrams with vertices lying in variable categories. A diagram category $\mathcal{C}^I$ is a particular case for which the underlying functor is constant. It is quite immediate, that if the vertex categories are P-categories satisfying certain compatibility conditions, then the diagram category inherits a level-wise P-category structure. However, the existence of cofibrant and minimal models of diagrams is not straightforward, and requires a careful study of morphisms of diagrams.

In Section 4 we introduce a wider class of morphisms of diagrams, which make squares commute up to homotopy (we call them ho-morphisms for short), and show that if the index category of the diagram category is a directed category satisfying certain conditions, then every ho-morphism can be factored into a composition of morphisms in a certain localized category. In this way we can rectify ho-morphisms.

In the last section we use the rectification of ho-morphisms to prove that if the vertices of a diagram category are P-categories with cofibrant (minimal) models, then the diagram category is a Cartan-Eilenberg category with level-wise weak equivalences and level-wise cofibrant (minimal) models.

1.1. Preliminaries

In this section we provide the necessary background on homotopical algebra. We first recall some facts about localization of categories. Then, we give a basic overview of some of the distinct homotopical approaches existing in the literature: Quillen model categories, Brown categories of fibrant objects and Cartan-Eilenberg categories. We do not claim originality for any result stated in this preliminary section.

Localization of Categories. We collect, for further reference, some well-known facts about localization of categories.

Definition 1.1.1. A *category with weak equivalences* is a pair $(\mathcal{C}, \mathcal{W})$ where $\mathcal{C}$ is a category and $\mathcal{W}$ is a class of morphisms of $\mathcal{C}$, called *weak equivalences*, which contains all isomorphisms of $\mathcal{C}$ and is stable by composition.

Definition 1.1.2. Let $(\mathcal{C}, \mathcal{W})$ be a category with weak equivalences. A *localization of $\mathcal{C}$ with respect to $\mathcal{W}$* is a category $\mathcal{C}[\mathcal{W}^{-1}]$, together with a functor $\gamma : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ such that:

(i) The functor γ sends all maps in $\mathcal{W}$ to isomorphisms.
(ii) For any category $\mathcal{D}$ and any functor $F : \mathcal{C} \to \mathcal{D}$ sending maps in $\mathcal{W}$ to isomorphisms, there exists a unique functor $F' : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ such that $F' \circ \gamma = F$.

The second condition implies that, when it exists, the localization is uniquely defined up to isomorphism. The localization exists if $\mathcal{W}$ is small and, in general, it always exists in a higher universe.

Definition 1.1.3. A class of weak equivalences $\mathcal{W}$ of $\mathcal{C}$ is *saturated* if a morphism f of $\mathcal{C}$ is in $\mathcal{W}$ whenever $\gamma(f)$ is an isomorphism. The *saturation* $\overline{\mathcal{W}}$ of $\mathcal{W}$ is the pre-image by γ of the isomorphisms in $\mathcal{C}[\mathcal{W}^{-1}]$, and it is the smallest saturated class of morphisms of $\mathcal{C}$ which contains $\mathcal{W}$.

Some authors assume that the class $\mathcal{W}$ satisfies the usual two out of three property, or the stronger two out of six property, in which case, the pair $(\mathcal{C}, \mathcal{W})$ is said to be a *homotopical category*. We do not assume that $\mathcal{W}$ satisfies these conditions, but in any case, the saturation $\mathcal{W}$ always does.

Definition 1.1.4. A class of morphisms $\mathcal{W}$ of a category $\mathcal{C}$ satisfies the *two out of three property* if for all composable f, g of $\mathcal{C}$ we have that if two of the three morphisms f, g and gf are in $\mathcal{W}$, then so is the third.

We next describe the localization of categories using the Dwyer-Kan hammocks introduced in [**DK80**].

Definition 1.1.5. Let $(\mathcal{C}, \mathcal{W})$ be a category with weak equivalences and let X and Y be objects of $\mathcal{C}$. A $\mathcal{W}$*-zigzag* f from X to Y is a finite sequence of morphisms of $\mathcal{C}$, going in either direction, between X and Y,

$$X \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \cdots \text{ --- } \bullet \text{ --- } Y,$$

where the arrows going to the left are weak equivalences. Since each $\mathcal{W}$-zigzag is a diagram, it has a *type*, given by its index category.

Definition 1.1.6. A *hammock* between two $\mathcal{W}$-zigzags f and g of the same type is given by a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd}
& X_{11} \arrow[r, dash] \arrow[d, dash] & X_{12} \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & X_{1p} \arrow[d, dash] \arrow[ddr, dash] & \\
& X_{21} \arrow[r, dash] \arrow[d, dash] & X_{22} \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & X_{2p} \arrow[d, dash] \arrow[dr, dash] & \\
X \arrow[uur, dash] \arrow[ur, dash] \arrow[dr, dash] \arrow[ddr, dash] & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & Y \\
& X_{n-1,1} \arrow[r, dash] \arrow[d, dash] & X_{n-1,2} \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & X_{n-1,p} \arrow[d, dash] \arrow[ur, dash] & \\
& X_{n1} \arrow[r, dash] & X_{n,2} \arrow[r, dash] & \cdots \arrow[r, dash] & X_{n,p} \arrow[uur, dash] &
\end{tikzcd}$$

such that:

(i) in each column of arrows, all horizontal maps go in the same direction, and if they go to the left they are in $\mathcal{W}$ (any row is a $\mathcal{W}$-zigzag),

(ii) in each row of arrows, all vertical maps go in the same direction, and they are arbitrary maps in $\mathcal{C}$,

(iii) the top $\mathcal{W}$-zigzag is f and the bottom is g.

We next define an equivalence relation between $\mathcal{W}$-zigzags.

Definition 1.1.7. Two $\mathcal{W}$-zigzags f and g are *related* if there exist two $\mathcal{W}$-zigzags f' and g' of the same type, obtained from f and g by adding identities, together a hammock H between f' and g'.

Given a category with weak equivalences $(\mathcal{C}, \mathcal{W})$, consider the category $\mathcal{C}_{\mathcal{W}}$ whose objects are those of $\mathcal{C}$ and whose morphisms are equivalence classes of $\mathcal{W}$-zigzags, with the composition defined by juxtaposition of $\mathcal{W}$-zigzags.

Theorem 1.1.8 ([**DHKS04**], 33.10.)**.** *The category $\mathcal{C}_{\mathcal{W}}$, together with the obvious functor $\mathcal{C} \to \mathcal{C}_{\mathcal{W}}$ is a solution to the universal problem of the localized category $\mathcal{C}[\mathcal{W}^{-1}]$.*

In the cited reference there is a general hypothesis concerning the class $\mathcal{W}$, which is not necessary for this result.

In absence of additional hypothesis on the pair $(\mathcal{C}, \mathcal{W})$, working with $\mathcal{C}_{\mathcal{W}}$ is almost hopeless. However, there are some situations in which an easier description of the morphisms of a localized category is possible. An example is provided by categories with a congruence.

Definition 1.1.9. A *congruence* $\sim$ on a category $\mathcal{C}$ is an equivalence relation between morphisms of $\mathcal{C}$, which is compatible with the composition. This defines an *associated class of morphisms* $\mathcal{S}$ of $\mathcal{C}$: a morphism $f : X \to Y$ of $\mathcal{C}$ is in $\mathcal{S}$ if and only if there exists a morphism $g : Y \to X$ of $\mathcal{C}$ such that $fg \sim 1_Y$ and $gf \sim 1_X$.

The pair $(\mathcal{C}, \mathcal{S})$ is a category with weak equivalences, and we can consider the localization $\gamma : \mathcal{C} \to \mathcal{C}[\mathcal{S}^{-1}]$ of $\mathcal{C}$ with respect to $\mathcal{S}$. On the other hand, we have the *quotient category* $(\mathcal{C}/\sim)$, whose objects are those of $\mathcal{C}$, and whose morphisms are given by the equivalence classes of morphisms defined by the congruence.

Proposition 1.1.10 ([**GNPR10**], Prop. 1.3.3)**.** *Let $\sim$ be a congruence on a category $\mathcal{C}$ and let $\mathcal{S}$ be its associated class of equivalences. Assume that $\mathcal{S}$ and $\sim$ are compatible, that is, $f \sim g$ implies $\gamma f = \gamma g$. Then the categories $(\mathcal{C}/\sim)$ and $\mathcal{C}[\mathcal{S}^{-1}]$ are canonically isomorphic.*

A particular case in which the congruence is compatible with its associated class, occurs when the congruence is transitively generated by a cylinder or a path object, as we shall later see.

We next introduce the relative localization of a subcategory. This will be necessary in order to express the main results of Cartan-Eilenberg categories.

Definition 1.1.11. Let $(\mathcal{C}, \mathcal{W})$ be a category with weak equivalences, and let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. The *relative localization of $\mathcal{D}$ with respect to $\mathcal{W}$*, denoted by $\mathcal{D}[\mathcal{W}^{-1}, \mathcal{C}]$, is the full subcategory of $\mathcal{C}[\mathcal{W}^{-1}]$ whose objects are those of $\mathcal{D}$.

In general, the category $\mathcal{D}[\mathcal{W}^{-1}, \mathcal{C}]$ differs from the localization $\mathcal{D}[\mathcal{W}^{-1}]$. In particular, the relative localization need not be a localized category.

In a variety of examples that we shall consider, the relative localization is defined with respect to the class of morphisms associated with a compatible congruence. Then the relative localization is a quotient category.

Corollary 1.1.12. *Let $(\mathcal{C}, \sim)$ be a category with a congruence satisfying the hypothesis of Proposition* 1.1.10. *For any full subcategory $\mathcal{D}$ of $\mathcal{C}$, there is an equivalence of categories*

$$(\mathcal{D}/\sim) \xrightarrow{\sim} \mathcal{D}[\mathcal{S}^{-1}, \mathcal{C}],$$

where $\sim$ is the congruence induced on $\mathcal{D}$ by that of $\mathcal{C}$.

Proof. The quotient category $(\mathcal{D}/\sim)$ is a full subcategory of $(\mathcal{C}/\sim)$. By Proposition 1.1.10 any pair of objects A, B of $\mathcal{D}$ we have

$$\mathcal{D}[\mathcal{S}^{-1}, \mathcal{C}](A, B) = \mathcal{C}[\mathcal{S}^{-1}](A, B) \cong (\mathcal{C}/\sim)(A, B) = (\mathcal{D}/\sim)(A, B).$$

$\square$

Quillen Model Categories. We provide a basic introduction to Quillen's model categories. We refer to [**Hov99**] for details.

Definition 1.1.13. A *model category* is a category $\mathcal{C}$ together with three distinguished classes of morphisms $\mathcal{W}$, $\mathcal{C}of$ and $\mathcal{F}ib$, called *weak equivalences*, *cofibrations* and *fibrations* respectively. A (co)fibration is said to be *trivial* if it is also a weak equivalence. The following axioms must be satisfied:

(MC_1) The category $\mathcal{C}$ has finite limits and colimits.

(MC_2) The class $\mathcal{W}$ satisfies the two out of three property.

(MC_3) The three classes of maps $\mathcal{W}$, $\mathcal{C}of$ and $\mathcal{F}ib$ are closed under retracts: consider a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "1_A"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "1_B"'] & D \arrow[r] & B \end{tikzcd}$$

If g is a weak equivalence, cofibration or fibration, then so is f.

(MC_4) Given a solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

where i is a cofibration, p is a fibration, and where either i or p is trivial, the dotted arrow exists, making the triangles commute.

(MC_5) Any map f of $\mathcal{C}$ has two factorizations:
(1) $f = qi$, where i is a trivial cofibration and q is a fibration, and
(2) $f = pj$, where j is a cofibration and p is a trivial fibration.

If $\mathcal{C}$ is a model category, denote by 0 and 1 the initial and final objects.

Definition 1.1.14. An object A of a model category $\mathcal{C}$ is called *cofibrant* if the map $0 \to A$ is a cofibration. It is called *fibrant* if the map $A \to 1$ is a fibration.

Denote by $\mathcal{C}_c$ (resp. $\mathcal{C}_f$) the full subcategory of $\mathcal{C}$ of cofibrant (resp. fibrant) objects of $\mathcal{C}$. Denote by $\mathcal{C}_{cf}$ the subcategory of fibrant and cofibrant objects.

We next remark some properties of model categories, which follow directly from the axioms.

(1) Cofibrations and trivial cofibrations are closed under composition and push-out. Every isomorphism is a cofibration.
(2) Fibrations and trivial fibrations are closed under composition and pull-back. Every isomorphism is a fibration. In particular, the product of fibrant objects is fibrant.
(3) The lifting property (MC_4) implies that two of the three distinguished classes of maps determine the third.
(4) The axioms for a model category are self dual, in the sense that if $\mathcal{C}$ is a model category, then so is $\mathcal{C}^{op}$, and the roles of fibrations and cofibrations are interchanged.

Example 1.1.15 (see [**DS95**], Ex. 3.5)**.** The category **Top** of topological spaces has a model category structure, where a map is:

(i) a weak equivalence: if it is a weak homotopy equivalence,
(ii) a fibration: if it is a Serre fibration (see Definition 1.2.35).

Every object is fibrant, and the cofibrant objects are exactly those spaces which are retracts of generalized CW-complexes.

Example 1.1.16 (see [**DS95**], Ex. 3.7)**.** The category $\mathbf{C}^+(R)$ of bounded below cochain complexes over a ring R has a model structure, where:

(i) weak equivalences are quasi-isomorphisms of complexes,
(ii) cofibrations are level-wise monomorphisms with level-wise projective cokernels, and
(iii) fibrations are level-wise epimorphisms.

Example 1.1.17 (see [**BG76**], Thm. 4.3)**.** The category $\mathsf{DGA}(\mathbf{k})$ of commutative differential graded algebras over a field $\mathbf{k}$ of characteristic zero has the structure of a model category, where:

(i) weak equivalences are quasi-isomorphisms,
(ii) fibrations are level-wise surjections.

All dga's are fibrant, and all Sullivan dga's are cofibrant.

The axioms for a model category allow to define a cylinder (and dually, a path) object, giving rise to the notion of left- (and right-) homotopies of morphisms.

Definition 1.1.18. Let $\mathcal{C}$ be a model category. A *cylinder* of $A \in \mathcal{C}$ is an object $\mathrm{Cyl}(A)$ of $\mathcal{C}$, giving a factorization of the folding map

$$A \sqcup A \xrightarrow{\iota_A} \mathrm{Cyl}(A) \xrightarrow{p_A} A,$$

such that ι_A is a cofibration and p_A is a trivial fibration.

Given a cylinder object $\mathrm{Cyl}(A)$, we have maps $\iota_A^0, \iota_A^1 : A \to \mathrm{Cyl}(A)$, defined by $\iota_A^k = \iota_A \circ j_k$, where $j_0, j_1 : A \to A \sqcup A$ are the natural inclusions.

Definition 1.1.19. Two maps $f, g : A \to B$ are called *left-homotopic* if there exists a map $h : \mathrm{Cyl}(A) \to B$ such that $h\iota_A^0 = f$ and $h\iota_A^1 = g$.

If A is a cofibrant object in a model category $\mathcal{C}$, left-homotopy of morphisms defines an equivalence relation on $\mathcal{C}(A, X)$. Dually, one defines a path object to be a factorization of the diagonal map. This gives the corresponding notion of right-homotopy, and it induces an equivalence relation on $\mathcal{C}(A, X)$ for every fibrant object X.

If A is cofibrant and X is fibrant, the left and right homotopy relations on $\mathcal{C}(A, X)$ agree. Denote by $\pi\mathcal{C}_{cf}$ the quotient category of $\mathcal{C}_{cf}$ defined by this equivalence relation.

The factorization axiom (MC_5) implies that given an object X of $\mathcal{C}$, one can always find a *cofibrant replacement*: this is a cofibrant object X_c, together with a weak equivalence $X_c \to X$. Dually, a *fibrant replacement* for X is a fibrant object X_f, together with a weak equivalence $X \to X_f$. An important consequence is the following:

Theorem 1.1.20 ([**Qui67**], Thm. 1)**.** *There is an equivalence of categories*

$$\pi\mathcal{C}_{cf} \xrightarrow{\sim} \mathrm{Ho}(\mathcal{C}) = \mathcal{C}[\mathcal{W}^{-1}].$$

Observe that in a model category, the weak equivalences carry the fundamental homotopy theoretic information, while the cofibrations, fibrations, and the axioms they satisfy serve as tools for obtaining the desired constructions. This suggests that in defining a model category structure on a category, it is most important to focus on choosing the class of weak equivalences.

Brown Categories of Fibrant Objects. Introducing the notion of category of fibrant objects, Brown showed in [**Bro73**], how one can obtain a large part of Quillen's theory by using fibrant objects only.

Let $\mathcal{C}$ be a category with finite products and a final object e. Assume that $\mathcal{C}$ has two distinguished classes of maps $\mathcal{W}$ and $\mathcal{F}$ called *weak equivalences* and *fibrations* respectively. A map will be called a *trivial fibration* if it is both a weak equivalence and a fibration.

Definition 1.1.21. The triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a *Brown category of fibrant objects* if the following axioms are satisfied:

(BF_1) The classes $\mathcal{W}$ and $\mathcal{F}$ are closed under composition and contain all isomorphisms. The class $\mathcal{W}$ satisfies the two out of three property. The map $A \to e$ is a fibration for every object A of $\mathcal{C}$.

(BF_2) Given a diagram $A \overset{u}{\to} C \overset{v}{\twoheadleftarrow} B$, where v is a fibration, the fibre product $A \times_C B$ exists, and the projection $\pi : A \times_C B \to A$ is a fibration. If v is a trivial fibration, then π is so.

(BF_3) For every object A of $\mathcal{C}$ there exists a path object (not necessarily functorial in A). This is a factorization of the diagonal map

$$A \xrightarrow{\iota_A} P(A) \xrightarrow{(\delta^0_A, \delta^1_A)} A \times A \ ,$$

where ι_A is a weak equivalence, and $(\delta^0_A, \delta^1_A) : P(A) \to A \times A$ is a fibration. The maps δ^0_A and δ^1_A are necessarily trivial fibrations.

The basic result of Brown categories is the following.

Lemma 1.1.22 ([**Bro73**], Factorization Lemma)**.** *If $\mathcal{C}$ is a Brown category of fibrant objects, any morphism $f : A \to B$ in $\mathcal{C}$ factors as $f = q_f \iota_f$, where ι_f is right inverse to a trivial fibration and q_f is a fibration. In particular, the map ι_f is a weak equivalence.*

In Brown's categories there is no notion of fibrant replacement, since every object is already assumed to be fibrant. Therefore there is no construction of models involved. The counterpart is that the resulting description of the homotopy category might not be as satisfactory as in the case of Quillen's model categories. If $\mathcal{C}$ is a Quillen model category then $\mathcal{C}_f$ is a Brown category of fibrant objects.

To study the localization of $\mathcal{C}$ with respect to weak equivalences, introduce a relation on $\mathcal{C}$ as follows: let $f, g : A \to B$. Then $f \sim g$ if and only if there exists a weak equivalence $t : A' \to A$ such that $ft \simeq gt$, that is, there exists a map $h : A \to P(B)$ such that $\delta_B^0 h = ft$ and $\delta_B^1 h = gt$. This is an equivalence relation, which is compatible with the composition. Denote by $\pi\mathcal{C}$ the quotient category defined by this equivalence relation.

Given objects A, B of $\mathcal{C}$, define a new set

$$[A, B] := \varinjlim \pi\mathcal{C}(A', B),$$

where the direct limit is taken over the weak equivalences $t : A' \to A$.

Theorem 1.1.23 ([**Bro73**], Thm. 1)**.** *Let $\mathcal{C}$ be a Brown category of fibrant objects, and let A and B be objects of $\mathcal{C}$. There is a canonical isomorphism*

$$\mathrm{Ho}(\mathcal{C})(A, B) := \mathcal{C}[\mathcal{W}^{-1}](A, B) \cong [A, B].$$

In general, the sets $[A, B]$ defined above do not coincide with the usual homotopy classes of maps. This is exhibited in the following example.

Example 1.1.24. The category of complexes of abelian groups is a category of fibrant objects. Consider $\mathbb{Z}/\mathbb{Z}_2$ and $\mathbb{Z}$ as a complexes concentrated in degree 0 and with trivial differential. It is a well known fact that

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbb{Z})}(\mathbb{Z}/\mathbb{Z}_2, \mathbb{Z}[1]) = \mathrm{Ext}^1(\mathbb{Z}/\mathbb{Z}_2, \mathbb{Z}) \neq 0.$$

By the other hand, the homotopy classes of maps from $\mathbb{Z}/\mathbb{Z}_2$ to $\mathbb{Z}[1]$ are trivial, since the only map from $(\mathbb{Z}/\mathbb{Z}_2 \to 0)$ to $(0 \to \mathbb{Z})$ is the 0 map. Therefore in this case, Theorem 1.1.23 provides no information about the homotopy category.

In [**Bau89**], Baues introduced another method of generating a fibration structure on a category. By adding an axiom of existence of cofibrant models to Brown's categories, he defined fibration categories, which allow to handle situations in which objects are not necessarily fibrant. The axioms for Baues fibration categories are very similar to those of Anderson-Brown-Cisinski fibration categories, the latter including conditions relative to limits, such as closure of fibrations under transfinite compositions. The motivation behind these additional axioms lies in the construction of homotopy colimits indexed by small diagrams. We refer to [**Bau89**], [**Cis10**] and [**RB07**] for details.

Cartan-Eilenberg Categories. We next review the homotopical approach of Cartan-Eilenberg categories developed in [**GNPR10**]. The initial data consists in a category together with two classes of morphisms (strong and weak equivalences). From these classes one defines cofibrant objects by means of a lifting property analogous to the classical lifting property of projective modules. In this framework one can include minimal models as a particular type of cofibrant models, defined by the condition that weak equivalences between minimal objects are isomorphisms.

Definition 1.1.25. A *category with strong and weak equivalences* is a triple $(\mathcal{C},\mathcal{S},\mathcal{W})$, where $\mathcal{C}$ is a category and $\mathcal{S}$ and $\mathcal{W}$ are two classes of morphisms of $\mathcal{C}$, called *strong* and *weak equivalences* respectively, which contain all isomorphisms of $\mathcal{C}$, are stable by composition, and such that $\mathcal{S}\subset\overline{\mathcal{W}}$.

Given a category with strong and weak equivalences, we have canonical localization functors $\delta:\mathcal{C}\to\mathcal{C}[\mathcal{S}^{-1}]$ and $\gamma:\mathcal{C}\to\mathcal{C}[\mathcal{W}^{-1}]$. Since $\mathcal{S}\subset\overline{\mathcal{W}}$, the functor γ factors through δ as

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "\gamma"] \arrow[dr, "\delta"'] & & \mathcal{C}[\mathcal{W}^{-1}] \\
& \mathcal{C}[\mathcal{S}^{-1}] \arrow[ur, "\gamma'"'] &
\end{tikzcd}.$$

The approach of Cartan-Eilenberg categories consists in studying the localized category $\mathcal{C}[\mathcal{W}^{-1}]$, by means of the localization $\mathcal{C}[\mathcal{S}^{-1}]$.

The following is a notion of cofibrant object which is related (but not equivalent) to the notion of cofibrant object introduced by Quillen.

Definition 1.1.26. Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with strong and weak equivalences. An object C of $\mathcal{C}$ is said to be *cofibrant*, if for each weak equivalence $w : X \to Y$, the induced map

$$w_* : \mathcal{C}[\mathcal{S}^{-1}](C, X) \longrightarrow \mathcal{C}[\mathcal{S}^{-1}](C, Y) \ ; \ g \mapsto wg$$

is bijective.

Denote by $\mathcal{C}_{cof}$ the full subcategory of $\mathcal{C}$ of cofibrant objects. These are characterized as follows.

Proposition 1.1.27 ([**GNPR10**], Thm. 2.2.3)**.** *Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with strong and weak equivalences. An object C of $\mathcal{C}$ is cofibrant if and only if the map*

$$\gamma'_X : \mathcal{C}[\mathcal{S}^{-1}](C, X) \longrightarrow \mathcal{C}[\mathcal{W}^{-1}](C, X)$$

is bijective, for every object X of $\mathcal{C}$.

In particular, every weak equivalence between cofibrant objects is a strong equivalence.

Definition 1.1.28. Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with strong and weak equivalences. Let $\mathcal{M}$ be a full subcategory of $\mathcal{C}$ and let X be an object of $\mathcal{C}$. A *left model of X in $\mathcal{M}$* is an object M of $\mathcal{M}$, together with a morphism $\rho : M \to X$ in $\mathcal{C}[\mathcal{S}^{-1}]$, which is an isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$.

Definition 1.1.29. A *left Cartan-Eilenberg category* is a category with strong and weak equivalences $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ such that each object of $\mathcal{C}$ has a model in $\mathcal{C}_{cof}$.

Remark 1.1.30. Left Cartan-Eilenberg categories with cofibrant models have a dual counterpart, by defining right fibrant models of objects by means of a property analogous to that of injective modules.

The important result about Cartan-Eilenberg categories is the following:

Theorem 1.1.31 ([**GNPR10**], Thm. 2.3.2)**.** *Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a left Cartan-Eilenberg category. The inclusion induces an equivalence of categories*

$$\mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}] \xrightarrow{\sim} \mathrm{Ho}(\mathcal{C}) := \mathcal{C}[\mathcal{W}^{-1}].$$

The category $\mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}]$ in the above equivalence is the relative localization of $\mathcal{C}_{cof}$ with respect to $\mathcal{S}$ (see Definition 1.1.11).

In particular, if $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category, the localization functor γ' admits a left adjoint

$$\lambda : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{C}[\mathcal{S}^{-1}].$$

This allows to extend the classical theory of derived additive functors, to non-additive settings.

Proposition 1.1.32 ([**GNPR10**], Lemma 3.1.3)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor from a left Cartan-Eilenberg category $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ to an arbitrary category $\mathcal{D}$. If F sends morphisms in $\mathcal{S}$ to isomorphisms, then the left derived functor with respect to $\mathcal{W}$ exists, and*

$$\mathbb{L}_{\mathcal{W}} F = F' \circ \lambda \circ \gamma : \mathcal{C} \longrightarrow \mathcal{D},$$

where $F' : \mathcal{C}[\mathcal{S}^{-1}] \to \mathcal{D}$ is induced by F.

Given a Quillen model category, the full subcategory of its fibrant objects has a natural structure of a left Cartan-Eilenberg category: taking $\mathcal{S}$ as the class of left homotopy equivalences and $\mathcal{W}$ the class of weak equivalences. However, the theory of Cartan-Eilenberg categories differs from Quillen's theory in the following aspects. First, in Quillen's context, the class $\mathcal{S}$ appears as a consequence of the axioms, while fibrant/cofibrant objects are part of them. Second, cofibrant objects in this setting are homotopy invariant, in contrast with cofibrant objects in Quillen model categories. Actually, in a Quillen model category, an object is Cartan–Eilenberg cofibrant if and only if it is homotopy equivalent to a Quillen cofibrant one. Lastly, in Cartan-Eilenberg categories there are no cofibrations, but only cofibrant objects.

Example 1.1.33 (see Section 1.2)**.** The category **Top** of topological spaces has a left Cartan-Eilenberg category structure, where:

(i) strong equivalences are given by homotopy equivalences,
(ii) weak equivalences are given by weak homotopy equivalences.

Every topological space is weakly equivalent to a CW-complex, and CW-complexes are cofibrant.

Example 1.1.34 ([**GNPR10**], Ex. 2.3.5)**.** Let $\mathcal{A}$ be an abelian category with enough injective objects. The category $\mathbf{C}^+(\mathcal{A})$ of bounded below cochain complexes of $\mathcal{A}$ is a right Cartan-Eilenberg category, where:

(i) strong equivalences are given by homotopy equivalence,

(ii) weak equivalences are given by quasi-isomorphisms.

The category $\mathbf{C}^+(\mathrm{Inj}\mathcal{A})$ of complexes over injective objects of $\mathcal{A}$ is a full subcategory of fibrant models.

Recognizing cofibrant objects may prove difficult, as the definition is given in terms of a lifting property in $\mathcal{C}[\mathcal{S}^{-1}]$. In addition, in some situations, there is a distinguished subcategory of $\mathcal{C}_{cof}$ which serves as a category of cofibrant models. The next result gives sufficient conditions for the existence of such a subcategory.

Theorem 1.1.35 ([**GNPR10**], Thm. 2.3.4)**.** *Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with strong and weak equivalences, and let $\mathcal{M}$ be a full subcategory of $\mathcal{C}$. Assume that:*

(i) For any map $w : X \to Y$ in $\mathcal{W}$, and any object $M \in \mathcal{M}$ the map $w_* : \mathcal{C}[\mathcal{S}^{-1}](M, X) \longrightarrow \mathcal{C}[\mathcal{S}^{-1}](M, Y)$ is injective.

(ii) For any map $w : X \to Y$ and any map $f : M \to Y$, where $m \in \mathcal{M}$, there exists $g \in \mathcal{C}[\mathcal{S}^{-1}](M, Y)$ such that $wg = f$ in $\mathcal{C}[\mathcal{S}^{-1}]$.

(iii) Every object of $\mathcal{C}$ has a left model in $\mathcal{M}$.

Then:

(1) Every object in $\mathcal{M}$ is cofibrant.

(2) The triple $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg.

(3) The inclusion induces an equivalence of categories

$$\mathcal{M}[\mathcal{S}^{-1}, \mathcal{C}] \xrightarrow{\sim} \mathcal{C}[\mathcal{W}^{-1}].$$

A particular type of cofibrant models are the *minimal* models. Their abstract definition is based on the Sullivan minimal models of rational homotopy.

Definition 1.1.36. Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with strong and weak equivalences. An object M of $\mathcal{C}$ is called *minimal* if it is cofibrant and

$$End_{\mathcal{C}}(M) \cap \mathcal{W} = Aut_{\mathcal{C}}(M).$$

That is, any weak equivalence $w : M \to M$ of $\mathcal{C}$ is an isomorphism.

Denote by $\mathcal{C}_{min}$ the full subcategory of $\mathcal{C}_{cof}$ of minimal objects.

Definition 1.1.37. A *left Sullivan category* is a category with strong and weak equivalences $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ such that every object in $\mathcal{C}$ has a left minimal model.

A Sullivan category is a Cartan-Eilenberg category for which the canonical functor $\mathcal{C}_{min}[\mathcal{S}^{-1}, \mathcal{C}] \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ is an equivalence of categories.

Example 1.1.38. The category $\mathsf{DGA}^0(\mathbf{k})$ of cohomologically connected dga's over a field $\mathbf{k}$ of characteristic zero is a left Sullivan category, where:

(i) strong equivalences are given by homotopy equivalences,
(ii) weak equivalences are given by quasi-isomorphisms.

The full subcategory of minimal models is that of Sullivan minimal dga's.

1.2. P-categories with Cofibrant Models

In the present section we introduce *P-categories with cofibrant models.* These are categories with a functorial path and two distinguished classes of morphisms, called *fibrations* and *weak equivalences*, satisfying a list of axioms similar to those of Brown categories of fibrant objects. The functorial path defines a notion of homotopy, and therefore there is an associated class of homotopy equivalences. We provide a notion of cofibrant object in terms of a lifting property with respect to trivial fibrations and prove that every P-category with cofibrant models is a Cartan-Eilenberg category with homotopy equivalences as strong equivalences and the same weak equivalences. As a result, the localized category of a P-category with respect to weak equivalences, is equivalent to the quotient category of cofibrant objects modulo homotopy. Examples of P-categories are the category of commutative differential graded algebras over a field of characteristic zero, or the category of topological spaces.

Categories with a Functorial Path. Path and cylinder objects appear in almost every axiomatic of homotopy theory, either as a part of the axioms, or as a direct consequence. We next develop an abstract homotopy theory for a category with a functorial path. Every definition and result has its dual counterpart in terms of functorial cylinders. A basic reference for this section is [**KP97**], Section I.4.

Definition 1.2.1. A *functorial path* on a category $\mathcal{C}$ is a functor $P : \mathcal{C} \to \mathcal{C}$ together with natural transformations

$$\begin{tikzcd} A \arrow[r, "\iota_A"] & P(A) \arrow[r, shift left, "\delta_A^0"] \arrow[r, shift right, "\delta_A^1"'] & A \end{tikzcd}$$

such that $\delta_A^0 \iota_A = \delta_A^1 \iota_A = 1_A$, for every object A of $\mathcal{C}$.

The functorial path defines a notion of homotopy between morphisms of $\mathcal{C}$.

Definition 1.2.2. Let $(\mathcal{C}, P)$ be a category with a functorial path and let $f, g : A \to B$ be two morphisms of $\mathcal{C}$. A *homotopy from f to g* is a morphism $h : A \to P(B)$ of $\mathcal{C}$ such that $\delta_B^0 h = f$ and $\delta_B^1 h = g$. We use the notation $h : f \simeq g$ and say that *f is homotopic to g*.

Lemma 1.2.3. *The homotopy relation defined by a functorial path is reflexive and compatible with the composition.*

PROOF. Let $f : A \to B$ be a morphism of $\mathcal{C}$. A homotopy from f to itself is given by $\iota_B f : A \to P(B)$. Let $f, g : A \to B$ be two morphisms of $\mathcal{C}$ and let $h : f \simeq g$ be a homotopy from f to g. Given morphisms $f' : A' \to A$ and $g' : B \to B'$, the composition hf' is a homotopy from ff' to gf'. The naturality of δ_B^k makes $P(g')h$ into a homotopy from $g'f$ to $g'g$. □

Extra structure on the path will be necessary to develop a rich homotopy theory in the abstract sense. Which kind of structure is useful will depend on the particular objective one has in mind.

1.2.4. The notion dual to the functorial path is that of a functorial cylinder. The basic example of such construction lives in the category of topological spaces. The *cylinder* of a topological space X is the product $X \times I$ of X with the unit interval $I = [0, 1]$. Two maps $f, g : X \to Y$ between topological spaces are *homotopic* if there exists a map $\phi : X \times I \to Y$ such that $\phi(x, 0) = f(x)$ and $\phi(x, 1) = g(x)$, for all $x \in X$.

(1) The symmetry of I defined by $t \mapsto 1 - t$, makes the homotopy relation into a symmetric relation.
(2) There is an automorphism of $I^2 = I \times I$ given by the interchange of coordinates $(t, s) \mapsto (s, t)$.
(3) There is a product $I^2 \to I$, given by $(t, s) \mapsto ts$.
(4) There is diagonal map $\Delta : I \to I^2$, defined by $t \mapsto (t, t)$.

We next axiomatize these transformations in their dual abstract version.

Let $(\mathcal{C}, P)$ be a category with a functorial path. The natural transformations ι, δ^0 and δ^1 make P into a cubical object in the category of functors from $\mathcal{C}$ to $\mathcal{C}$: denote $P^0 = 1$, $P^1 = P$, $P^2 = PP$, $\cdots$. For all $0 \leq s \leq n$, we have natural transformations

$$P^n(A) \xrightarrow{\iota_A^{n,s}} P^{n+1}(A) \underset{(\delta^1)_A^{n,s}}{\overset{(\delta^0)_A^{n,s}}{\rightrightarrows}} P^n(A)\ , \begin{cases} \iota_A^{n,s} := P^s(\iota_{P^{n-s}(A)}) \\ (\delta^k)_A^{n,s} := P^s(\delta^k_{P^{n-s}(A)}) \end{cases} .$$

Definition 1.2.5. A *symmetry* of a functorial path P is a natural automorphism $\tau : P \to P$ such that $\tau_A \tau_A = 1_{P(A)}$, $\delta^0_A \tau_A = \delta^1_A$, $\delta^1_A \tau_A = \delta^0_A$ and $\tau_A \iota_A = \iota_A$.

Definition 1.2.6. A *coproduct* of a functorial path P is a natural transformation $c^0 : P \to P^2$ such that:

(a) The triple (P, δ^1, c^0) is a comonad, i.e. for every object A of $\mathcal{C}$ one has two commutative diagrams

$$\begin{tikzcd} P(A) \arrow[r, "c^0_A"] \arrow[d, "c^0_A"'] & P^2(A) \arrow[d, "c^0_{P(A)}"] \\ P^2(A) \arrow[r, "P(c^0_A)"] & P^3(A) \end{tikzcd} \quad , \quad \begin{tikzcd} P(A) & P^2(A) \arrow[l, "\delta^1_{P(A)}"'] \arrow[r, "P(\delta^1_A)"] & P(A) \\ & P(A) \arrow[ul, "1_{P(A)}"] \arrow[u, "c^0_A"] \arrow[ur, "1_{P(A)}"'] & \end{tikzcd} .$$

(b) The following diagrams commute

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[d, "\iota_A"'] & P(A) \arrow[d, "c^0_A"] \\ P(A) \arrow[r, "\iota_{P(A)}"] & P^2(A) \end{tikzcd} \quad , \quad \begin{tikzcd} P(A) & P^2(A) \arrow[l, "\delta^0_{P(A)}"'] \arrow[r, "P(\delta^0_A)"] & P(A) \\ & P(A) \arrow[ul, "\iota_A\delta^0_A"] \arrow[u, "c^0_A"] \arrow[ur, "\iota_A\delta^0_A"'] & \end{tikzcd} \quad .$$

The following result is straightforward.

Lemma 1.2.7. *The map c^0_A is a homotopy from $i_A\delta^0_A$ to 1_A.*

Definition 1.2.8. An *interchange transformation* of a functorial path P is a natural automorphism $\mu : P^2 \to P^2$ such that the following diagram commutes, for $k = 0, 1$.

$$\begin{tikzcd} P^2(A) \arrow[r, "\mu_A"] \arrow[dr, "P(\delta^k_A)"'] & P^2(A) \arrow[r, "\mu_A"] \arrow[d, "\delta^k_{P(A)}"] & P^2(A) \arrow[dl, "P(\delta^k_A)"] \\ & P(A) & \end{tikzcd}$$

Definition 1.2.9. A *folding map* of a path P is a natural transformation $\nabla : P^2 \to P$ such that $\delta^k_A\nabla_A = \delta^k_A\delta^k_{P(A)}$, for $k = 0, 1$, and $\nabla_A\iota_{P(A)} = 1_{P(A)}$.

The transformations defined so far, give rise to other useful transformations which will be needed in the sequel.

First, there is a transformation which is symmetric to the coproduct. This is the dual abstract version of the transformation $I^2 \to I$ of the unit interval defined by $(t, s) \mapsto t + s - st$.

Definition 1.2.10. Let $(\mathcal{C}, P)$ be a category with a functorial path, together with a symmetry τ and a coproduct c^0. Let $c^1 : P \to P^2$ be the natural transformation defined by

$$c^1_A := \tau_{P(A)}P(\tau_A)c^0_A\tau_A.$$

Lemma 1.2.11. *The following identities are satisfied:*

$$\begin{cases} (i) & \delta^0_{P(A)}c^1_A = P(\delta^0_A)c^1_A = 1_{P(A)} \\ (ii) & \delta^1_{P(A)}c^1_A = P(\delta^1_A)c^1_A = \iota_A\delta^1_A \\ (iii) & c^1_A\iota_A = \iota_{P(A)}\iota_A \end{cases} \quad .$$

In particular, c^1_A *is a homotopy from* $1_{P(A)}$ *to* $\iota_A\delta^1_A$.

PROOF. We use the naturality of each of the transformations involved:

$$\begin{aligned}
\delta^0_{P(A)}c^1_A &= \delta^0_{P(A)}\tau_{P(A)}P(\tau_A)c^0_A\tau_A = \delta^1_{P(A)}P(\tau_A)c^0_A\tau_A = \tau_A\delta^1_{P(A)}c^0_A\tau_A \\
&= \tau_A\tau_A = 1_{P(A)}. \\
P(\delta^0_A)c^1_A &= P(\delta^0_A)\tau_{P(A)}P(\tau_A)c^0_A\tau_A = \tau_A P(\delta^0_A\tau_A)c^0_A\tau_A = \tau_A P(\delta^1_A)c^0_A\tau_A \\
&= \tau_A\tau_A = 1_{P(A)}.
\end{aligned}$$

This proves (i). The identities of (ii) follow analogously. We prove (iii).

$$\begin{aligned}
c^1_A\iota_A &= \tau_{P(A)}P(\tau_A)c^0_A\tau_A\iota_A = \tau_{P(A)}P(\tau_A)c^0_A\iota_A = \tau_{P(A)}P(\tau_A)\iota_{P(A)}\iota_A \\
&= \tau_{P(A)}\iota_{P(A)}\tau_A\iota_A = \iota_{P(A)}\iota_A.
\end{aligned}$$

□

Definition 1.2.12. Define a natural transformation $c^2 : P^2 \to P^3$ by letting

$$c^2_A := c^1_{P(A)}c^0_A.$$

Lemma 1.2.13. *The following identities are satisfied:*

$$\begin{cases} (i) & \delta^0_{P^2(A)}c^2_A = P(\delta^0_{P(A)})c^2_A = c^0_A \\ (ii) & \delta^1_{P^2(A)}c^2_A = P(\delta^1_{P(A)})c^2_A = \iota_{P(A)} \end{cases} \qquad \begin{cases} (iii) & P^2(\delta^0_A)c^2_A = \iota_{P(A)}\iota_A\delta^0_A \\ (iv) & P^2(\delta^1_A)c^2_A = c^1_A \end{cases}$$

PROOF. Identities (i) and (ii) follow directly from the definitions and Lemma 1.2.11. For (iii) and (iv) we use the naturality of c^1:

$$\begin{aligned}
P^2(\delta^0_A)c^2_A &= P^2(\delta^0_A)c^1_{P(A)}c^0_A = c^1_A P(\delta^0_A)c^0_A = c^1_A\iota_A\delta^0_A = \iota_{P(A)}\iota_A\delta^0_A, \\
P^2(\delta^1_A)c^2_A &= P^2(\delta^1_A)c^1_{P(A)}c^0_A = c^1_A P(\delta^1_A)c^0_A = c^1_A.
\end{aligned}$$

□

We next deduce some important consequences of the existence of these transformations.

Lemma 1.2.14. *The homotopy relation defined by a functorial path with a symmetry, is a symmetric relation.*

PROOF. Let $f, g : A \to B$ be two morphisms, and let h be a homotopy from f to g. Then $h'\tau_B h$ is a homotopy from g to f. $\square$

Let $\mathcal{C}$ be a category with a functorial path, together with a symmetry. The homotopy relation is reflexive, symmetric and compatible with the composition, but is not transitive in general. Let $\sim$ denote the congruence of $\mathcal{C}$ transitively generated by the homotopy relation: $f \sim g$ if there is a finite family of morphisms f_i, for $1 \leq i \leq r$, such that

$$f \simeq f_1 \simeq f_2 \simeq \cdots \simeq f_r \simeq g.$$

Definition 1.2.15. A morphism $f : A \to B$ of $\mathcal{C}$ is a *homotopy equivalence* if there exists a morphism $g : B \to A$ satisfying $fg \sim 1_B$ and $gf \sim 1_B$.

Denote by $\mathcal{S}$ the class of homotopy equivalences. This is the class associated with the congruence $\sim$ (see Definition 1.1.9). This class is closed by composition and contains all isomorphisms.

Proposition 1.2.16. *Let $\mathcal{C}$ be a category with a functorial path, together with a symmetry and a coproduct.*

(1) There is an equivalence of categories

$$(\mathcal{C}/\sim) \xrightarrow{\sim} \mathcal{C}[\mathcal{S}^{-1}].$$

(2) For any full subcategory $\mathcal{D}$ of $\mathcal{C}$, there is an equivalence of categories

$$(\mathcal{D}/\sim) \xrightarrow{\sim} \mathcal{D}[\mathcal{S}^{-1}, \mathcal{C}].$$

PROOF. We first prove (1). In view of Proposition 1.1.10 it suffices to show that the congruence $\sim$ is compatible with $\mathcal{S}$. Let $f, g : A \to B$ be morphisms of $\mathcal{C}$ such that $h : f \simeq g$. There is a commutative diagram

$$\begin{tikzcd}
 & B & \\
A \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & P(B) \arrow[u, "\delta_B^0"'] \arrow[d, "\delta_B^1"] & B \arrow[l, "\iota_B"'] \arrow[ul, equal] \arrow[dl, equal] \\
 & B & .
\end{tikzcd}$$

By Lemma 1.2.7 the map ι_B is a homotopy equivalence. Hence the above diagram is a hammock between the $\mathcal{S}$-zigzags f and g. By Theorem 1.1.8 we have $f = g$ in $\mathcal{C}[\mathcal{S}^{-1}]$.

Assertion (2) follows from (1) and Corollary 1.1.12. □

Axioms for a P-category. Let $\mathcal{C}$ be a category with finite products and a final object e. Assume that $\mathcal{C}$ has a functorial path P, together with a symmetry τ, an interchange transformation μ, a coproduct c and a folding map ∇. Assume as well that $\mathcal{C}$ has two distinguished classes of maps $\mathcal{F}$ and $\mathcal{W}$ called *fibrations* and *weak equivalences* respectively. A map will be called a *trivial fibration* if it is both a fibration and a weak equivalence. As is customary, the symbol $\xrightarrow{\sim}$ will be used for weak equivalences, while $\twoheadrightarrow$ will denote a fibration.

We will make use of the following two constructions.

Definition 1.2.17. Let $f : A \to B$ be a morphism of $\mathcal{C}$.

(1) Assume that the pull-back diagram

$$\begin{tikzcd} \mathcal{P}(f) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner"] & P(B) \arrow[d, "\delta^0_B"] \\ A \arrow[r, "f"] & B \end{tikzcd} .$$

exists. Then $\mathcal{P}(f)$ is called the *mapping path of* f.

(2) Assume that the pull-back diagram

$$\begin{tikzcd} \mathcal{P}(f,f') \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner"] & P(B) \arrow[d, "(\delta^0_B,\delta^1_B)"] \\ A \times A' \arrow[r, "f\times f'"] & B \times B \end{tikzcd} .$$

exists. Then $\mathcal{P}(f,f')$ is called the *double mapping path of* f *and* f'.

With these notations we have $\mathcal{P}(f, 1_B) = \mathcal{P}(f)$.

Definition 1.2.18. The quadruple $(\mathcal{C}, P, \mathcal{F}, \mathcal{W})$ is called a *P-category* if the following axioms are satisfied:

(P_1) The classes $\mathcal{F}$ and $\mathcal{W}$ contain all isomorphisms and are closed by composition. The class $\mathcal{W}$ satisfies the two out of three property. The map $A \to e$ is a fibration for every object A of $\mathcal{C}$.

(P_2) For every object A of $\mathcal{C}$, the map $\iota_A : A \to P(A)$ is a weak equivalence and $(\delta^0_A, \delta^1_A) : P(A) \to A \times A$ is a fibration. The maps δ^0_A and δ^1_A are trivial fibrations.

(P_3) Given a diagram $A \overset{u}{\to} C \overset{v}{\twoheadleftarrow} B$, where v is a fibration, the fibre product $A \times_C B$ exists, and the projection $\pi_1 : A \times_C B \twoheadrightarrow A$ is a fibration. In addition, if v is a trivial fibration, then π_1 is so, and if u is a weak equivalence, then $\pi_2 : A \times_C B \to B$ is also a weak equivalence.

(P_4) The path preserves fibrations and weak equivalences and is compatible with the fibre product: $P(A \times_C B) = PA \times_{PC} PB$.

(P_5) For every fibration $v : A \twoheadrightarrow B$, the map $\overline{v}$ defined by the following diagram is a fibration:

$$
\begin{tikzcd}
P(A) \arrow[dr, dotted, "\overline{v}"] \arrow[drr, bend left, "P(v)"] \arrow[ddr, bend right, "(\delta^0_A,\delta^1_A)"'] & & \\
& \mathcal{P}(v,v) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P(B) \arrow[d, "(\delta^0_B,\delta^1_B)"] \\
& A \times A \arrow[r, "v\times v"'] & B \times B
\end{tikzcd}
$$

Remark 1.2.19. A category satisfying axioms (P_1) to (P_3) a Brown category of fibrant objects with a functorial path (see Definition 1.1.21).

Axiom (P_5) is dual to the *relative cylinder axiom* of Baues, and can be described as a certain kind of cubical homotopy lifting property in dimension 2 (see [**KP97**], pag. 86).

In [**Bau89**], Baues introduced P-categories in order to provide an abstract example of a fibration category. A P-category in the sense of Baues is a category with a functorial path and a class of fibrations, satisfying the analogue of axioms (P_1) to (P_5) obtained by forgetting the conditions on the class of weak equivalences, together with an extra axiom of lifting of homotopies with respect to fibrations (see I.3 of loc.cit).

Remark 1.2.20. Every P-category in the sense of Baues is a fibration category in which weak equivalences are defined by the homotopy equivalences associated with the functorial path, and every object is fibrant and cofibrant (see Theorem 3a.4 of [**Bau89**]).

Although our notion of a P-category differs substantially from the notion introduced by Baues, we borrow the same name, since Baues only uses P-categories as a particular example of a fibration category.

To describe the localized category of a P-category with respect to weak equivalences, we will consider an additional property: we will define $\mathcal{F}$-*cofibrant objects* as objects having a lifting property with respect to the trivial fibrations, and we will assume that every object is weakly equivalent to a cofibrant one. In this case, we will say that the *P-category has cofibrant models.*

Before introducing the notion of cofibrant object in this context, we prove some useful results that are a consequence of the axioms.

The first and most important property of P-categories gives a factorization of every map, as a homotopy equivalence whose inverse is a trivial fibration, followed by a fibration.

Lemma 1.2.21 (cf. [**Bro73**], Factorization Lemma)**.** *Let $f : A \to B$ be a morphism in a P-category category $\mathcal{C}$. Define maps $p_f := \pi_1$, $q_f := \delta^1_B \pi_2$, and $\iota_f := (1_A, \iota_B f)$, where $\pi_1 : \mathcal{P}(f) \to A$ and $\pi_2 : \mathcal{P}(f) \to P(B)$ denote the natural projections of the mapping path $\mathcal{P}(f)$. Then the diagram*

$$\begin{tikzcd}
A & \mathcal{P}(f) \arrow[l, "p_f"'] \arrow[r, "q_f"] & B \\
& A \arrow[ul, equal] \arrow[u, "\iota_f"] \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes. In addition:

(1) The map p_f is a trivial fibration, and q_f is a fibration. In particular, ι_f is a weak equivalence.
(2) The map ι_f is a homotopy equivalence with homotopy inverse p_f.
(3) If f is a weak equivalence, then q_f is a trivial fibration.

PROOF. Since δ_B^0 is a fibration, by (P_3), the mapping path $\mathcal{P}(f)$ exists. From the definitions it is immediate that the above diagram commutes.

Let us prove (1). Since δ_B^0 is a trivial fibration, by (P_2), the map p_f is a trivial fibration. The map q_f can be written as the composition

$$\mathcal{P}(f) \xrightarrow{(\pi_1, \delta_B^1 \pi_2)} A \times B \xrightarrow{\pi_2} B.$$

The morphism $(\pi_1, \delta_B^1 \pi_2)$ is a base extension of $(\delta_B^0, \delta_B^1) : P(B) \to B \times B$ by $f \times 1_B : A \times B \to B \times B$, and the projection $\pi_2 : A \times B \to B$ is a base extension of $A \to e$. By (P_3), both maps are fibrations, and hence q_f is a fibration. By the two out of three property, ι_f is a weak equivalence.

To prove (2), since $p_f \iota_f = 1_A$, it suffices to define a homotopy from $\iota_f p_f$ to the identity morphism $1_{\mathcal{P}(f)}$.

Let h be the morphism defined by the following pull-back diagram:

$$\begin{tikzcd}
\mathcal{P}(f) \arrow[dr, dotted, "h"] \arrow[drr, bend left, "c_B^0 \pi_2"] \arrow[ddr, bend right, "\iota_A \pi_1"'] & & \\
& P(\mathcal{P}(f)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P^2(B) \arrow[d, "P(\delta_B^0)"] \\
& P(A) \arrow[r, "P(f)"'] & P(B)
\end{tikzcd},$$

where c^0 is the coproduct of the path (see Definition 1.2.6), which satisfies $P(\delta_B^0) c_B^0 = \iota_B \delta_B^0$. From the naturality of ι we obtain:

$$P(\delta_B^0) c_B^0 \pi_2 = \iota_B \delta_B^0 \pi_2 = \iota_B f \pi_1 = P(f) \iota_A \pi_1.$$

Therefore the above solid diagram commutes, and the map h is well defined.

By (P_4), the fibre product $P(\mathcal{P}(f))$ is a path object of $\mathcal{P}(f)$, with

$$\delta^k_{\mathcal{P}(f)} = (\delta^k_A P(\pi_1), \delta^k_{P(B)} P(\pi_2)) : P(\mathcal{P}(f)) \to \mathcal{P}(f), \text{ for } k = 0, 1.$$

Therefore we have:

$$\delta^k_{\mathcal{P}(f)} h = (\delta^k_A P(\pi_1), \delta^k_{P(B)} P(\pi_2))(\iota_A \pi_1, c^0_B \pi_2) = (\pi_1, \delta^k_{P(B)} c^0_B \pi_2).$$

Since $\delta^0_{P(B)} c^0_B = \iota_B \delta^0_B$ and $\delta^1_{P(B)} c^0_B = 1_B$, it follows that

$$\begin{cases} \delta^0_{\mathcal{P}(f)} h = (\pi_1, \iota_B \delta^0_B \pi_2) = (1_A, \iota_B f)\pi_1 = \iota_f p_f, \\ \delta^1_{\mathcal{P}(f)} h = (\pi_1, \pi_2) = 1_{\mathcal{P}(f)}. \end{cases}$$

Hence h is a homotopy from $\iota_f p_f$ to the identity $1_{\mathcal{P}(f)}$, making ι_f into a homotopy equivalence.

Assertion (3) follows from (1) and the two out of three property of $\mathcal{W}$. $\square$

Axiom (P_5) states that for a fibration $v : A \twoheadrightarrow B$, the induced morphism $\overline{v} : P(A) \to \mathcal{P}(v, v)$ is a fibration. We prove an analogous statement for weak equivalences.

Lemma 1.2.22. *Let $w : A \xrightarrow{\sim} B$ be weak equivalence in a P-category $\mathcal{C}$. Then the induced map*

$$\overline{w} := ((\delta^0_A, \delta^1_A), P(w)) : P(A) \to \mathcal{P}(w, w)$$

is a weak equivalence.

Proof. We first prove that the map $w \times w : A \times A \to B \times B$ is a weak equivalence. Indeed, $w \times w$ can be written as the composition of $1_A \times w$ and $w \times 1_B$. Since $\mathcal{W}$ is closed by composition, it suffices to prove that these maps are weak equivalences. We have pull-back diagrams:

$$\begin{tikzcd} A \times A \arrow[r, "1_A \times w"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner"] & A \times B \arrow[d, "\pi_2"] \\ A \arrow[r, "w"] & B \end{tikzcd} , \quad \begin{tikzcd} A \times B \arrow[r, "w \times 1_B"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner"] & B \times B \arrow[d, "\pi_1"] \\ A \arrow[r, "w"] & B \end{tikzcd} .$$

Since w is a weak equivalence, by (P_3), both maps $1_A \times w$ and $w \times 1_B$ are weak equivalences. Therefore $w \times w$ is a weak equivalence.

Consider the commutative diagram

$$\begin{tikzcd}
 & P(A) \arrow[dl, "\overline{w}"'] \arrow[d, "P(w)"] \\
\mathcal{P}(w,w) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner"] & P(B) \arrow[d, "(\delta^0_B,\delta^1_B)"] \\
A \times A \arrow[r, "w\times w"] & B \times B
\end{tikzcd} .$$

Since $w \times w$ is a weak equivalence, by (P_3), the projection π_2 is a weak equivalence. Since the path preserves weak equivalences, $P(w)$ is a weak equivalence. By the two out of three property, $\overline{w}$ is so.

□

The following result is a consequence of the previous lemma, and will be used to lift homotopies.

Lemma 1.2.23. *The map π_A defined by the following pull-back diagram is a trivial fibration:*

$$\begin{tikzcd}
P^2(A) \arrow[dr, dotted, "\pi_A"] \arrow[drr, bend left, "P(\delta^1_A)"] \arrow[ddr, bend right, "\delta^0_{P(A)}"'] & & \\
 & \mathcal{P}(\delta^1_A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P(A) \arrow[d, "\delta^0_A"] \\
 & P(A) \arrow[r, "\delta^1_A"'] & A
\end{tikzcd}$$

PROOF. Define a map p_A via the commutative diagram:

$$\begin{tikzcd}
\mathcal{P}(\delta^1_A,\delta^1_A) \arrow[dr, dotted, "p_A"] \arrow[drr, bend left, "\pi_2"] \arrow[dd, "\pi_1"] & & \\
 & \mathcal{P}(\delta^1_A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P(A) \arrow[d, "\delta^0_A"] \\
P(A)\times P(A) \arrow[r, "\pi_1"] & P(A) \arrow[r, "\delta^1_A"'] & A
\end{tikzcd} .$$

This map is a base extension of the trivial fibration $\delta_A^1 : P(A) \to A$, by the map $\delta_A^1 \pi_2 : \mathcal{P}(\delta_A^1) \to A$. Therefore by (P$_3$) it is a trivial fibration. The map $\pi_A : P^2(A) \to \mathcal{P}(\delta_A^1)$ can be written as the composition

$$P^2(A) \xrightarrow{\overline{\delta_A^1}} \mathcal{P}(\delta_A^1, \delta_A^1) \xrightarrow{p_A} \mathcal{P}(\delta_A^1),$$

where $\overline{\delta_A^1}$ is a trivial fibration by (P$_5$) and Lemma 1.2.22. Therefore π_A is a trivial fibration. □

Cofibrant and Minimal Models. We next define $\mathcal{F}$-cofibrant objects by means of a lifting property with respect to trivial fibrations. The existence of $\mathcal{F}$-cofibrant models in a P-category $\mathcal{C}$ will allow a description of $\mathcal{C}[\mathcal{W}^{-1}]$ in terms of the quotient category of $\mathcal{F}$-cofibrant objects modulo homotopy.

Definition 1.2.24. An object C of a P-category $\mathcal{C}$ is called *$\mathcal{F}$-cofibrant* if for any solid diagram in $\mathcal{C}$

$$\begin{tikzcd} & A \arrow[d, "w", "\wr"'] \\ C \arrow[ur, dotted, "g"] \arrow[r, "f"] & B \end{tikzcd}$$

in which w is a trivial fibration, there exists a dotted arrow g making the diagram commute.

The following result is a homotopy lifting property for trivial fibrations with respect to $\mathcal{F}$-cofibrant objects.

Lemma 1.2.25. *Let C be an $\mathcal{F}$-cofibrant object of a P-category $\mathcal{C}$, and let $v : A \overset{\sim}{\twoheadrightarrow} B$ be a trivial fibration. For every commutative solid diagram of $\mathcal{C}$*

$$\begin{tikzcd} C \arrow[dr, dotted, "\widetilde{h}"] \arrow[drr, bend left, "(f_0,f_1)"] \arrow[ddr, bend right, "h"'] & & \\ & P(A) \arrow[r, "(\delta_A^0,\delta_A^1)"'] \arrow[d, "P(v)"] & A \times A \arrow[d, "v\times v"] \\ & P(B) \arrow[r, "(\delta_B^0,\delta_B^1)"'] & B \times B \end{tikzcd},$$

there exists a dotted arrow $\widetilde{h}$, making the diagram commute.

In other words: if v is a trivial fibration, every homotopy $h : vf_1 \simeq vf_1$, lifts to a homotopy $\widetilde{h} : f_0 \simeq f_1$ such that $P(v)\widetilde{h} = h$.

PROOF. The triple $H = (f_0, f_1, h)$ defines a morphism from C to the double mapping path of v. We have a solid diagram

$$\begin{tikzcd} & P(A) \arrow[d, "\overline{v}", "\wr"'] \\ C \arrow[ur, dotted, "\widetilde{h}"] \arrow[r, "H"] & \mathcal{P}(v, v) \end{tikzcd} .$$

By (P_5) and Lemma 1.2.22 the map $\overline{v} = ((\delta^0_A, \delta^1_A), P(v))$ is a trivial fibration. Since C is $\mathcal{F}$-cofibrant, there exists a dotted arrow $\widetilde{h}$ such that $\overline{v}\widetilde{h} = H$. Therefore $(\delta^0_A, \delta^1_A)\widetilde{h} = (f_0, f_1)$ and $P(v)\widetilde{h} = h$. $\square$

Proposition 1.2.26. *The homotopy relation in a P-category is an equivalence relation for morphisms whose source is $\mathcal{F}$-cofibrant.*

PROOF. By Lemma 1.2.3 the homotopy relation is reflexive. By Lemma 1.2.14 it is symmetric. We prove transitivity. Let A be an $\mathcal{F}$-cofibrant object, and let $f, f', f'' : A \to B$ be morphisms of $\mathcal{C}$, together with homotopies $h : f \simeq f'$ and $h' : f' \simeq f''$. We next define a homotopy from f to f''. Consider the solid diagram

$$\begin{tikzcd} & P^2(B) \arrow[d, "\pi_B", "\wr"'] \\ A \arrow[ur, dotted, "\mathcal{L}"] \arrow[r, "(h,h')"'] & \mathcal{P}(\delta^1_B) \end{tikzcd} .$$

By Lemma 1.2.23 the map $\pi_B = (\delta^0_{P(B)}, P(\delta^1_B))$ is a trivial fibration. Since A is $\mathcal{F}$-cofibrant, there exists a dotted arrow $\mathcal{L}$ such that $\pi_B\mathcal{L} = (h, h')$. Therefore $\delta^0_{P(B)}\mathcal{L} = h$ and $P(\delta^1_B)\mathcal{L} = h'$.

Define a morphism $h'' : A \to P(B)$ by letting $h'' := \nabla_B\mathcal{L}$, where ∇ is the folding map (see Definition 1.2.9), which satisfies $\delta^k_A\nabla_A = \delta^k_A\delta^k_{P(A)} = \delta^k_A P(\delta^k_A)$. Then $\delta^0_B h'' = f$ and $\delta^1_B h'' = f''$. Hence $h''f \simeq f''$. $\square$

Given an $\mathcal{F}$-cofibrant object C of a P-category $\mathcal{C}$, denote by

$$[C, A] := \mathcal{C}(C, A)/\simeq$$

the class of maps from C to A modulo homotopy. Denote by $\mathcal{C}^{\mathcal{F}}_{cof}$ the full subcategory of $\mathcal{F}$-cofibrant objects of $\mathcal{C}$ and by $\pi\mathcal{C}^{\mathcal{F}}_{cof}$ the quotient category defined by the homotopy equivalence relation.

Proposition 1.2.27. *Let $\mathcal{C}$ be a P-category and let C be an $\mathcal{F}$-cofibrant object of $\mathcal{C}$. Every weak equivalence $w : A \overset{\sim}{\longrightarrow} B$ of $\mathcal{C}$ induces a bijection*

$$w_* : [C, A] \longrightarrow [C, B],\ [f] \mapsto [wf].$$

Proof. We first prove surjectivity. Let $w : A \to B$ be a weak equivalence and let $f : C \to B$ be a map of $\mathcal{C}$. By Lemma 1.2.21 we have a solid diagram

$$\begin{tikzcd}
& A \arrow[d, shift left, "\iota_w"] \arrow[dd, bend left=60, "w"] \\
& \mathcal{P}(w) \arrow[u, shift left, "p_w"] \arrow[d, "q_w", "\wr"'] \\
C \arrow[r, "f"] \arrow[ur, dotted, "g'"] & B
\end{tikzcd},$$

where q_f is a trivial fibration, $q_w\iota_w = w$, and $\iota_w p_w \simeq 1_{\mathcal{P}(w)}$. Since C is $\mathcal{F}$-cofibrant, there exists a dotted arrow g' such that $q_w g' = f$. Let $g := p_w g'$. Then $wg = q_w\iota_w p_w g' \simeq q_w g' = f$. Therefore $[wg] = [f]$, and w_* is surjective.

To prove injectivity, let $f_0, f_1 : C \to B$ be two morphisms of $\mathcal{C}$ such that $h : wf_0 \simeq wf_1$. Let $H = (f_0, f_1, h)$ and consider the solid diagram

$$\begin{tikzcd}
& P(A) \arrow[d, "\overline{w}", "\wr"'] \\
C \arrow[r, "H"] \arrow[ur, dotted, "G"] & \mathcal{P}(w, w)
\end{tikzcd}.$$

By Lemma 1.2.22 the map $\overline{w} = ((\delta^0_A, \delta^1_A), P(w))$ is a weak equivalence. Since $\overline{w}_*$ is surjective, there exists a dotted arrow G such that $\overline{w}G \simeq H$. It follows that $f_0 \simeq \delta^0_A G \simeq \delta^1_A G \simeq f_1$, and hence $[f_0] = [f_1]$. $\square$

Definition 1.2.28. A P-category $\mathcal{C}$ is said to *have cofibrant models* if for every object A of $\mathcal{C}$ there is an $\mathcal{F}$-cofibrant object C, together with a weak equivalence $w : C \overset{\sim}{\to} A$.

Lemma 1.2.29. *The class $\mathcal{S}$ of homotopy equivalences of a P-category $\mathcal{C}$ is contained in the saturation $\overline{\mathcal{W}}$.*

PROOF. It suffices to prove that given two morphisms $f, g : A \to B$ of $\mathcal{C}$ such that $h : f \simeq g$, then $f = g$ in $\mathcal{C}[\mathcal{W}^{-1}]$. Consider the commutative diagram

$$\begin{tikzcd} & B & \\ A \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & P(B) \arrow[u, "\delta_B^0"'] \arrow[d, "\delta_B^1"] & B \arrow[l, "\iota_B"'] \arrow[ul, equal] \arrow[dl, equal] \\ & B & \end{tikzcd} .$$

Since ι_B is a weak equivalence, this is a hammock between the $\mathcal{W}$-zigzags f and g. Therefore $f = g$ in $\mathcal{C}[\mathcal{W}^{-1}]$. □

Theorem 1.2.30. *Let $(\mathcal{C}, P, \mathcal{W}, \mathcal{F})$ be a P-category with cofibrant models. The triple $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category with cofibrant models in $\mathcal{C}_{cof}^{\mathcal{F}}$. There are equivalences of categories*

$$\pi\mathcal{C}_{cof}^{\mathcal{F}} \xrightarrow{\sim} \mathcal{C}_{cof}^{\mathcal{F}}[\mathcal{S}^{-1}, \mathcal{C}] \xrightarrow{\sim} \mathcal{C}[\mathcal{W}^{-1}].$$

PROOF. By Lemma 1.2.29 the triple $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a category with strong and weak equivalences. By Proposition 1.2.27 every $\mathcal{F}$-cofibrant object is Cartan-Eilenberg cofibrant in $(\mathcal{C}, \mathcal{S}, \mathcal{W})$. By definition, every object has a model in $\mathcal{C}_{cof}^{\mathcal{F}}$. Therefore $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a Cartan-Eilenberg category. The equivalences of categories follow from Proposition 1.2.16 and Theorem 1.1.31 respectively. □

Definition 1.2.31. An object M of a P-category $\mathcal{C}$ is called *$\mathcal{F}$-minimal* if it is $\mathcal{F}$-cofibrant and every weak equivalence $w : M \to M$ is an isomorphism. An *$\mathcal{F}$-minimal model* of an object A of $\mathcal{C}$ is an $\mathcal{F}$-minimal object M, together with a weak equivalence $w : M \xrightarrow{\sim} A$.

Denote by $\mathcal{C}_{min}^{\mathcal{F}}$ the full subcategory of $\mathcal{F}$-minimal objects of $\mathcal{C}$.

Theorem 1.2.32. *Let $(\mathcal{C}, P, \mathcal{W}, \mathcal{F})$ be a P-category and assume that every object of $\mathcal{C}$ has an $\mathcal{F}$-minimal model. Then the triple $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a Sullivan* *category with minimal models in $\mathcal{C}^{\mathcal{F}}_{min}$. There is an equivalence of categories*

$$\pi\mathcal{C}^{\mathcal{F}}_{min} \xrightarrow{\sim} \mathcal{C}[\mathcal{W}^{-1}].$$

To end this section, we provide some important examples of P-categories.

Transfer of Structures. In a wide class of examples, one can obtain a P-category structure on a category $\mathcal{C}$ with a functorial path, by means of a functor $\psi : \mathcal{C} \to \mathcal{D}$ whose target is a P-category.

Let $\mathcal{C}$ be a category with finite products and a final object. Assume that $\mathcal{C}$ has a functorial path, together with a symmetry, an interchange transformation, a coproduct and a folding map.

Lemma 1.2.33. *Let $(\mathcal{D}, P, \mathcal{F}, \mathcal{W})$ be a P-category, and let $\psi : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that the following conditions are satisfied:*

(i) The functor ψ is compatible with the functorial path: for every object A of $\mathcal{C}$, $\psi(P(A)) = P(\psi(A))$, $\psi(\iota_A) = \iota_{\psi(A)}$, and $\psi(\delta^k_A) = \delta^k_{\psi(A)}$.

(ii) Given morphisms $A \xrightarrow{u} C \overset{v}{\twoheadleftarrow} B$ of $\mathcal{C}$, where $\psi(v)$ is a fibration, the fibre product exists, and satisfies

$$P(A \times_C B) = PA \times_{P(C)} P(A) \text{ and } \psi(A \times_C B) = \psi(A) \times_{\psi(C)} \psi(B).$$

Then the quadruple $(\mathcal{C}, P, \psi^{-1}(\mathcal{F}), \psi^{-1}(\mathcal{W}))$ is a P-category.

Proof. Axioms (P_1) and (P_5) are trivial. Axiom (P_2) follows from (i) and axioms (P_3), (P_4) and (P_5) follow from (ii). □

Another situation is that of a full subcategory of a P-category with enough $\mathcal{F}$-cofibrant models.

Lemma 1.2.34. *Let $(\mathcal{C}, P, \mathcal{F}, \mathcal{W})$ be a P-category and let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$ such that:*

(i) Given a weak equivalence $A \xrightarrow{\sim} B$ in $\mathcal{C}$, then A is an object of $\mathcal{D}$ if and only if B is so.

(ii) For every object D of $\mathcal{D}$ there is an object $C \in \mathcal{D}^{\mathcal{F}}_{cof} := \mathcal{D} \cap \mathcal{C}^{\mathcal{F}}_{cof}$, together with a weak equivalence $C \xrightarrow{\sim} D$.

Then the triple $(\mathcal{D}, \mathcal{S}, \mathcal{W})$ is a Cartan-Eilenberg category with cofibrant models in $\mathcal{D}^{\mathcal{F}}_{cof}$. There is an equivalence of categories

$$\pi\mathcal{D}^{\mathcal{F}}_{cof} \xrightarrow{\sim} \mathcal{D}[\mathcal{W}^{-1}].$$

PROOF. It suffices to show that the objects of $\mathcal{D}^{\mathcal{F}}_{cof}$ are Cartan-Eilenberg cofibrant in $(\mathcal{D}, \mathcal{S}, \mathcal{W})$. Let $C \in \mathcal{D}^{\mathcal{F}}_{cof}$ and let $w : A \to B$ be a weak equivalence in $\mathcal{D}$. By Proposition 1.2.27 w induces a bijection

$$\mathcal{C}[\mathcal{S}^{-1}](C, A) = [C, A] \longrightarrow \mathcal{C}[\mathcal{S}^{-1}](C, B) = [C, B].$$

Since A, B, C are in $\mathcal{D}$, this gives a bijection

$$\mathcal{D}[\mathcal{S}^{-1}, \mathcal{C}](C, A) \longrightarrow \mathcal{D}[\mathcal{S}^{-1}, \mathcal{C}](C, B).$$

By (i), the functorial path in $\mathcal{C}$ restricts to a functorial path in $\mathcal{D}$. By Proposition 1.2.16 we have equivalences

$$\pi\mathcal{D} \xrightarrow{\sim} \mathcal{D}[\mathcal{S}^{-1}] \xrightarrow{\sim} \mathcal{D}[\mathcal{S}^{-1}, \mathcal{C}].$$

Therefore every object of $\mathcal{D}^{\mathcal{F}}_{cof}$ is Cartan-Eilenberg cofibrant in $(\mathcal{D}, \mathcal{S}, \mathcal{W})$. By (ii), every object has a model in $\mathcal{D}^{\mathcal{F}}_{cof}$. Hence the result follows. □

There is an analogous version of Lemma 1.2.34 with cofibrant minimal objects.

Topological Spaces. Consider the category Top of topological spaces with continuous maps. Let $I = [0, 1] \subset \mathbb{R}$ be the unit interval. Given a topological space X, let $P(X) := X^I$ be the set of all maps $\sigma : I \to X$ with the compact open topology. There are maps

$$X \xrightarrow{\iota_X} P(X) \underset{\delta^1_X}{\overset{\delta^0_X}{\rightrightarrows}} X\ ,$$

given by $\iota_X(x)(t) = x$, and $\delta^k_X(\sigma) = \sigma(k)$, for $k = 0, 1$. This defines a functorial path $P : \mathsf{Top} \to \mathsf{Top}$.

The product topology for $X \times I$ and the compact open topology for X^I have the well-known property that a map $f : X \times I \to Y$ is continuous if and only if the adjoint map $\eta(f) : X \to Y^I$, defined by $\eta(f)(x)(t) = f(x,t)$ is continuous. This results in a bijection of sets

$$\mathsf{Top}(X, Y^I) \underset{\eta}{\overset{\epsilon}{\rightleftarrows}} \mathsf{Top}(X \times I, Y).$$

In particular, the structural maps for the functorial path P (symmetry, coproduct, interchange and folding map) are obtained via the corresponding adjoint maps defined in 1.2.4.

Definition 1.2.35. A map $v : X \to Y$ of topological spaces is called *Serre fibration* if for any commutative diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "i_0"'] & X \arrow[d, "v"] \\ U \times I \arrow[r, "G"'] \arrow[ur, dotted, "H"] & B \end{tikzcd} \qquad ,$$

where U is the unit disk of $\mathbb{R}^n$, a dotted arrow H exists, making the diagram commute, for every $n \geq 0$.

Definition 1.2.36. A map $w : X \to Y$ of topological spaces is called *weak homotopy equivalence* if the induced map $w_* : \pi_0(X) \to \pi_0(Y)$ is a bijection and $w_* : \pi_n(X, x) \to \pi_n(Y, w(x))$ is an isomorphism for every $x \in X$ and every $n \geq 1$.

Theorem 1.2.37. *The category* Top *of topological spaces with the classes* $\mathcal{F} = \{$*Serre fibrations*$\}$ *and* $\mathcal{W} = \{$*weak homotopy equivalences*$\}$*, and the functorial path defined by* $P(X) = X^I$*, is a P-category with cofibrant models.*

Proof. Axioms (P_1) to (P_4) are standard. The proof of (P_5) can be found in [**Bau77**], pag. 133. We next restate it. Assume that v is a Serre fibration. To prove that $\overline{v} : X^I \to \mathcal{P}(v,v)$ is a Serre fibration, we need to find a lifting for every diagram

$$\begin{tikzcd} U \times \{0\} \arrow[r, "f"] \arrow[d] & X^I \arrow[d, "\overline{v}"] \\ U \times I \arrow[r, "{(h_0,h_1,h)}"] \arrow[ur, dotted] & \mathcal{P}(w,w) \end{tikzcd} \qquad ,$$

where $h_i : U \times I \to X$ and $h : U \times I \to Y^I$ are such that $\delta^i h = v(h_i)$, for $k = 0, 1$. By adjunction we obtain the following commutative solid diagram:

$$\begin{tikzcd} U' \arrow[r, "{(h_0,h_1,\epsilon(f))}"] \arrow[d] & X \arrow[d, "v"] \\ U \times I \times I \arrow[r, "\epsilon(h)"] \arrow[ur, dotted] & Y \end{tikzcd},$$

where $U' = (U \times I \times \{0,1\}) \cup (U \times \{0\} \times I)$. There is a homeomorphism $g : U' \times I \to U \times I \times I$, which factors as

$$\begin{tikzcd} U' \times I \arrow[rr, "g"] & & U \times I \times I \\ & U' \times \{0\} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

This gives a solid diagram

$$\begin{tikzcd} U' \arrow[r, "{(h_0,h_1,\epsilon(f))}"] \arrow[d] & X \arrow[d, "v"] \\ U' \times I \arrow[r, "\epsilon(h)g"] \arrow[ur, dotted, "G'"] & Y \end{tikzcd}.$$

Since v is a Serre fibration, there exists a dotted arrow G' making the diagram commute. Let $G := G'g^{-1} : U \times I \times I \to X$. By adjunction we obtain the required lifting $\eta(G) : U \times I \to X^I$. Therefore ($P_5$) is satisfied.

Lastly, for every topological space X there exists a CW-complex C, together with a weak equivalence $C \xrightarrow{\sim} X$, and CW-complexes are $\mathcal{F}$-cofibrant (see for example [**Qui67**], [**DS95**] or [**Hov99**]). $\square$

Differential Graded Algebras. Consider the category $\mathsf{DGA}(\mathbf{k})$ of dga's over a field $\mathbf{k}$ of characteristic 0 (refer to Section 4.1 for the main definitions and results). The field $\mathbf{k}$ is the initial object, and 0 is the final object. Any diagram of dga's $A \xrightarrow{u} C \xleftarrow{v} B$ can be completed to a cartesian square

$$\begin{tikzcd} A \times_C B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "v"] \\ A \arrow[r, "u"] & C \end{tikzcd}$$

where

$$A \times_C B = \{(a,b) \in A \times B; u(a) = v(b)\},\ d(a,b) = (da, db).$$

The functorial path is defined by

$$P(A) = A[t,dt] = A \otimes (t,dt) \text{ and } P(f) = f \otimes 1,$$

together with structural maps $\iota_A = 1_A \otimes 1$, and $\delta_A^k(a(t)) = a(k)$, for $k = 0, 1$. The symmetry $\tau_A : A[t,dt] \to A[t,dt]$ is defined by $t \mapsto 1-t$. The iteration of the path object is defined by

$$P^n(A) = A[t_1, dt_1, \cdots, t_n dt_n] = A \otimes (t_1, dt_1) \otimes \cdots (t_n, dt_n),\ n \geq 1.$$

The interchange map $\mu_A : A[t,dt,s,ds] \to A[t,dt,s,ds]$ is defined by $t \mapsto s$ and $s \mapsto t$. The coproduct $c_A : A[t,dt] \to A[t,dt,s,ds]$ is defined by $t \mapsto ts$, and the folding map $\nabla_A : A[t,dt,s,ds] \to A[t,dt]$ by $t \mapsto t$ and $s \mapsto t$.

Proposition 1.2.38. *Let* $\mathbf{k}$ *be a field of characteristic 0. The category* $\mathsf{DGA}(\mathbf{k})$ *with the classes* $\mathcal{F} = \{$*surjections*$\}$ *and* $\mathcal{W} = \{$*quasi-isomorphisms*$\}$, *and the functorial path* $P(A) = A[t,dt]$, *is a P-category.*

Proof. The only non-trivial axiom for the P-category structure is (P_5). The double mapping path of a surjective map of dga's $v : A \to B$ is

$$\mathcal{P}(v.v) = \{(a_0, a_1, b(t)) \in A \times A \times B[t,dt]; b(i) = v(a_i)\},$$

and the map $\overline{v} : A[t,dt] \longrightarrow \mathcal{P}(v,v)$ is given by

$$\overline{v}(a(t)) = (a(0), a(1), (v \otimes 1)(a(t)).$$

Let $(a_0, a_1, b(t)) \in \mathcal{P}(v,v)$. Since $v \otimes 1$ is surjective, there exists an element $\widetilde{b}(t) \in A[t,dt]$ such that $(v \otimes 1)\widetilde{b}(t) = b(t)$. Let

$$a(t) := (a_0 - \widetilde{b}(0))(1-t) + (a_1 - \widetilde{b}(1))t + \widetilde{b}(t).$$

Then $\overline{v}(a(t)) = (a_0, a_1, b(t))$. Therefore $\overline{v}$ is surjective, and (P_5) is satisfied. $\square$

1.3. Diagrams Associated with a Functor

Let $\Gamma\mathcal{C}$ be the category of diagrams associated with a functor $\mathcal{C} : I \to \mathsf{Cat}$ (see Definition 1.3.1 below), and assume that for all $i \in I$, the category $\mathcal{C}(i)$ is equipped with a class $\mathcal{W}_i$ of weak equivalences. Our objective is to study of the localized category

$$\mathrm{Ho}\,(\Gamma\mathcal{C}) := \Gamma\mathcal{C}[\mathcal{W}^{-1}].$$

with respect to the class $\mathcal{W}$ of level-wise weak equivalences via the construction of level-wise cofibrant and minimal models.

We will show that if $\mathcal{C} : I \to \mathsf{Cat}$ is a functor whose source I is a directed category of a certain type (see 1.3.4), and for all $i \in I$, the category $\mathcal{C}(i)$ is a P-category with cofibrant models, whose structure is preserved by the functors $u_* : \mathcal{C}(i) \to \mathcal{C}(j)$, then the category of diagrams $\Gamma\mathcal{C}$ associated with $\mathcal{C}$ inherits a Cartan-Eilenberg structure. In particular, we will show that those objects that are level-wise $\mathcal{F}_i$-cofibrant in $\mathcal{C}(i)$, are Cartan-Eilenberg cofibrant in $\Gamma\mathcal{C}$, and that the category $\Gamma\mathcal{C}$ has enough models of such type.

Level-wise P-category Structure. We next define the category of diagrams associated with a functor and show that if the vertex categories are endowed with compatible level-wise P-category structures, then the diagram category inherits a level-wise P-category structure.

Definition 1.3.1. Let $\mathcal{C} : I \to \mathsf{Cat}$ be a functor from a small category I, to the category of categories Cat. For all $i \in I$, denote $\mathcal{C}_i := \mathcal{C}(i) \in \mathsf{Cat}$, and $u_* := \mathcal{C}(u) \in \mathsf{Fun}(\mathcal{C}_i, \mathcal{C}_j)$, for all $u : i \to j$. The *category* $\Gamma\mathcal{C}$ *of diagrams associated with the functor* $\mathcal{C}$ is defined as follows:

- An object A of $\Gamma\mathcal{C}$ is given by a family of objects $\{A_i \in \mathcal{C}_i\}$, for all $i \in I$, together with a family of morphisms $\varphi_u : u_*(A_i) \to A_j$, called *comparison morphisms*, for every map $u : i \to j$. Such an object is denoted as

$$A = \left(A_i \overset{\varphi_u}{\dashrightarrow} A_j\right).$$

- A morphism $f : A \to B$ of $\Gamma\mathcal{C}$ is a family of morphisms $\{f_i : A_i \to B_i\}$ of $\mathcal{C}_i$, for all $i \in I$, such that for every map $u : i \to j$ of I, the diagram

$$\begin{tikzcd} u_*(A_i) \arrow[r, "\varphi_u"] \arrow[d, "u_*(f_i)"'] & A_j \arrow[d, "f_j"] \\ u_*(B_i) \arrow[r, "\varphi_u"] & B_j \end{tikzcd}$$

  commutes in $\mathcal{C}_j$. Denote $f = (f_i) : A \to B$.

By an abuse of notation, we will omit the notation of the functors u_*, and write A_i for $u_*(A_i)$ and f_i for $u_*(f_i)$, whenever there is no danger of confusion.

Remark 1.3.2. The category of diagrams $\Gamma\mathcal{C}$ associated with $\mathcal{C}$ is the category of sections of the projection functor $\pi : \int_I \mathcal{C} \to I$, where $\int_I \mathcal{C}$ is the *Grothendieck construction* of the functor $\mathcal{C}$ (see [**Tho79**]).

Example 1.3.3. Assume that $\mathcal{C} : I \to \mathsf{Cat}$ is the constant functor $i \mapsto \mathcal{C}$, where $\mathcal{C}$ is a category, and that $\mathcal{C}(u : i \to j)$ is the identity functor of $\mathcal{C}$. Then $\Gamma\mathcal{C} = \mathcal{C}^I$ is the diagram category of objects of $\mathcal{C}$ under I.

1.3.4. We will restrict our study of diagram categories for which the index category I is a finite directed category whose degree function takes values in $\{0, 1\}$. That is: I is a finite category satisfying

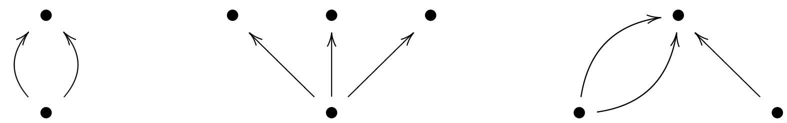
(I_1) There exists a *degree function* $|\cdot| : \mathrm{Ob}(I) \longrightarrow \{0, 1\}$ such that $|i| < |j|$ for every non-identity morphism $u : i \to j$ of I.

A finite category I satisfying (I_1) is a particular case of a Reedy category for which $I^+ = I$.

The main examples of such categories are given by finite zig-zags

$$\begin{tikzcd} & \bullet & & \bullet & & \bullet & \cdots & \bullet & & \bullet & \\ \bullet \arrow[ur] & & \bullet \arrow[ul] \arrow[ur] & & \bullet \arrow[ul] \arrow[ur] & & & & \bullet \arrow[ul] \arrow[ur] & & \bullet \arrow[ul] \end{tikzcd}$$

but other diagram shapes are admitted. For example:



All objects at the bottom of the diagrams have degree 0, while the objects at the top have degree 1.

For the rest of this section assume that $\Gamma\mathcal{C}$ is a diagram category indexed by I satisfying the following conditions:

(1) For all $i \in I$, the category $\mathcal{C}_i$ is equipped with a functorial path P, together with two classes of morphisms $\mathcal{F}_i$ and $\mathcal{W}_i$ of fibrations and weak equivalences, in such a way that the quadruple $(\mathcal{C}_i, P, \mathcal{F}_i, \mathcal{W}_i)$ is a P-category.
(2) For all $u : i \to j$ the functor $u_* : \mathcal{C}_i \to \mathcal{C}_j$ preserves path objects, fibrations, weak equivalences and fibre products.

Definition 1.3.5. A morphism $f : A \to B$ in $\Gamma\mathcal{C}$ is called *weak equivalence* (resp. *fibration*) if for all $i \in I$, the maps f_i are weak equivalences (resp. fibrations) of $\mathcal{C}_i$. Denote by $\mathcal{W}$ (resp. $\mathcal{F}$) the class of weak equivalences (resp. fibrations) of the diagram category $\Gamma\mathcal{C}$.

Definition 1.3.6. The *path object* $P(A)$ of a diagram A of $\Gamma\mathcal{C}$ is the diagram defined by

$$P(A) = \left(P(A_i) \overset{P(\varphi_u)}{\dashrightarrow} P(Aj) \right).$$

There are natural morphisms of diagrams

$$\begin{tikzcd} A & P(A) \arrow[l, "\delta_A^0"'] \arrow[r, "\delta_A^1"] & A \\ & A \arrow[ul, equal, "="] \arrow[u, "\iota_A"] \arrow[ur, equal, "="'] & \end{tikzcd},$$

where $(\delta_A^k)_i = (\delta_{A_i}^k)$, and $(\iota_A)_i = (\iota_{A_i})$. This defines a functorial path on $\Gamma\mathcal{C}$.

The functorial path defines a notion of homotopy between morphisms of $\Gamma\mathcal{C}$.

Definition 1.3.7. Let $f, g : A \to B$ be morphisms of $\Gamma\mathcal{C}$. A *homotopy from f to g* is a morphism $h = (h_i) : A \to P(B)$, where $h_i : A_i \to P(B_i)$ is a homotopy from f_i to g_i in $\mathcal{C}_i$ such that the diagram

$$\begin{tikzcd} A_i \arrow[r, "\varphi_u"] \arrow[d, "h_i"'] \arrow[dr, phantom, "="] & A_j \arrow[d, "h_j"] \\ P(B_i) \arrow[r, "P(\varphi_u)"'] & P(B_j) \end{tikzcd}$$

commutes in $\mathcal{C}_j$. Denote such a homotopy by $h : f \simeq g$.

Denote by $\sim$ the congruence of $\Gamma\mathcal{C}$ transitively generated by the homotopy relation, and let $\mathcal{S}$ denote the class of homotopy equivalences of $\Gamma\mathcal{C}$. If $f = (f_i)$ is in $\mathcal{S}$, then $f_i \in \mathcal{S}_i$. In particular, since $\mathcal{S}_i \subset \mathcal{W}_i$ and $\mathcal{W}$ is defined level-wise, we have $\mathcal{S} \subset \mathcal{W}$. Hence the triple $(\Gamma\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a category with strong and weak equivalences.

Let $A \stackrel{u}{\to} C \stackrel{v}{\leftarrow} B$ be a diagram of $\Gamma\mathcal{C}$, and assume that for all $i \in I$, the fibre product $A_i \times_{C_i} B_i$ exists. Then the fibre product $A \times_C B$ is determined level-wise by

$$(A \times_B C)_i = A_i \times_{B_i} C_i.$$

For a map $u : i \to j$, the comparison morphism

$$\psi_u : (A \times_C B)_i \to (A \times_C B)_j$$

is given by $\psi_u = (\varphi_u^A \pi_1, \varphi_u^B \pi_2)$, where φ_u^A and φ_u^B denote the comparison morphisms of A and B respectively. The following result is straightforward.

Proposition 1.3.8. *Let $\Gamma\mathcal{C}$ be a diagram category, and assume that for all $i \in I$, the category $\mathcal{C}_i$ has a P-category structure preserved by the functors $u_* : \mathcal{C}_i \to \mathcal{C}_j$. Then $\Gamma\mathcal{C}$ is a P-category with path objects, fibrations, weak equivalences and fibre products defined level-wise.*

Proof. The conditions of Lemma 1.2.33 are satisfied by the functor $\psi : \Gamma\mathcal{C} \to \Pi_{i \in I}\mathcal{C}_i$ induced by the inclusion $I_{dis} \to I$. $\square$

Corollary 1.3.9. *Every morphism $f : A \to B$ in $\Gamma\mathcal{C}$ fits into a commutative diagram*

$$\begin{tikzcd} A & \mathcal{P}(f) \arrow[l, "p_f"'] \arrow[r, "q_f"] & B \\ & A \arrow[ul, equal] \arrow[u, "\iota_f"] \arrow[ur, "f"'] \arrow[ul, phantom, "=" near start] \arrow[ur, phantom, "=" near start] & \end{tikzcd}$$

defined level-wise as in Lemma 1.2.21*. In particular, q_f is a fibration, p_f is a trivial fibration and ι_f is a homotopy equivalence. In addition, if f is a weak equivalence, then q_f is a trivial fibration.*

Let A be an object of $\Gamma\mathcal{C}$, and assume that for all $i \in I$, there exists an $\mathcal{F}_i$-cofibrant model $\rho_i : C_i \xrightarrow{\sim} A_i$. From the lifting property of $\mathcal{F}_i$-cofibrant objects, given the solid diagram

$$\begin{tikzcd} C_i \arrow[r, dotted, "\varphi'_u"] \arrow[d, "\rho_i"'] \arrow[dr, phantom, "\simeq"] & C_j \arrow[d, "\rho_j"] \\ A_i \arrow[r, "\varphi_u"] & A_j \end{tikzcd}$$

a dotted arrow φ'_u exits, and makes the diagram commute up to a homotopy of morphisms in $\mathcal{C}_j$. In order to have a true model, we need to rectify the above diagram, taking into account that each vertical morphism of the diagram lies in a different category. We will solve this problem by studying the factorization of homotopy commutative morphisms into the composition of morphisms in a certain localized category $\Gamma\mathcal{C}[\mathcal{H}^{-1}]$.

The following is a simple example illustrating the procedure that we will conduct in order to rectify homotopy commutative morphisms of diagrams.

Example 1.3.10 (Model of a morphism of dga's)**.** A morphism of dga's can be thought as an object of the diagram category of dga's indexed by $I = \{0 \to 1\}$. Let $\varphi : A_0 \to A_1$ be a morphism of dga's over a field of characteristic 0, and for $i = 0, 1$, let $f_i : M_i \xrightarrow{\sim} A_i$ be a Sullivan minimal model. By the lifting property of Sullivan dga's, there exists a morphism of dga's $\varphi' : M_0 \to M_1$, together with a homotopy $F : M_0 \to A_1[t, dt]$.

$$\begin{tikzcd} A_0 \arrow[r, "\varphi"] & A_1 \\ M_0 \arrow[u, "f_0"] \arrow[r, "\varphi'"'] \arrow[ur, Rightarrow, "F"] & M_1 \arrow[u, "f_1"'] \end{tikzcd}$$

For $i = 0, 1$, consider the mapping path

$$\mathcal{P}(f_i) = \{(m, a(t)) \in M_i \times A_i[t, dt]; f_i(m) = a(0)\},$$

and define morphisms $p_i : \mathcal{P}(f_i) \to M_i$ and $q_i : \mathcal{P}(f_i) \to A_i$ by letting $p_i(m, a(t)) = m$, and $q_i(m, a(t)) = a(i)$. The maps q_i and p_i are quasi-isomorphisms of dga's, for $i = 0, 1$. Define a morphism $\Psi : \mathcal{P}(f_0) \to \mathcal{P}(f_1)$ by letting $\Psi(m, a(t)) = (\varphi'(m), F(m))$. The diagram

$$\begin{tikzcd} A_0 \arrow[r, "\varphi"] & A_1 \\ \mathcal{P}(f_0) \arrow[u, "q_0"] \arrow[r, "\Psi"] \arrow[d, "p_0"'] & \mathcal{P}(f_1) \arrow[u, "q_1"'] \arrow[d, "p_1"] \\ M_0 \arrow[r, "\varphi'"] & M_1 \end{tikzcd}$$

commutes. The map $\varphi' : M_0 \to M_1$ is a Sullivan minimal model of f. Let us remark that the key part in the construction of the above diagram resides in the definition of the morphism Ψ (which depends on the lift φ' and the homotopy F, and only on the first variable), and the morphisms q_i (whose definition depends on whether the index i is a source or a target in the index category I).

Homotopy Commutative Morphisms. We next introduce homotopy commutative morphisms of diagrams (ho-morphisms for short) and define a notion of homotopy between them. This new class of maps does not define a category, since the composition is not well defined, but each ho-morphism will admit a factorization into morphisms in a certain localized category.

Definition 1.3.11. A *ho-morphism* $f : A \rightsquigarrow B$ between two diagrams of $\Gamma\mathcal{C}$ is pair of families $f = (f_i, F_u)$, where:

(i) $f_i : A_i \to B_i$ is a morphism in $\mathcal{C}_i$, for all $i \in I$, and

(ii) $F_u : A_i \to P(B_j)$ is a morphism in $\mathcal{C}_j$ satisfying $\delta^0_{B_j} F_u = f_j \varphi_u$ and $\delta^1_{B_j} F_u = \varphi_u f_i$, for every map $u : i \to j$ of I. Hence F_u is a homotopy of morphisms of $\mathcal{C}_j$ making the following diagram commute up to homotopy.

$$\begin{tikzcd} A_i \arrow[r, "\varphi_u"] \arrow[d, "f_i"'] \arrow[dr, Rightarrow, "F"] & A_j \arrow[d, "f_j"] \\ B_i \arrow[r, "\varphi_u"'] & B_j \end{tikzcd}$$

Given diagrams A, B of $\Gamma\mathcal{C}$, denote by $\Gamma\mathcal{C}^h(A, B)$ the set of ho-morphisms from A to B. Every morphism of diagrams $f = (f_i) : A \to B$ is trivially made into a ho-morphism $f = (f_i, F_u) : A \rightsquigarrow B$ by letting $F_u = \iota_{B_j}(f_j \varphi_u) = \iota_{B_j}(\varphi_u f_i)$. This defines an inclusion of sets

$$\Gamma\mathcal{C}(A, B) \subset \Gamma\mathcal{C}^h(A, B).$$

The composition of ho-morphisms is not well defined. This is due to the fact that the homotopy relation between objects of $\mathcal{C}_i$ is not transitive in general. However, we can compose ho-morphisms with morphisms.

Lemma 1.3.12. *Let $f : A \rightsquigarrow B$ be a ho-morphism, and let $g : A' \to A$ and $h : B \to B'$ be morphisms of $\Gamma\mathcal{C}$. There are ho-morphisms $fg : A' \rightsquigarrow B$ and $hf : A \rightsquigarrow B'$, given by*

$$fg = (f_i g_i, F_u g_i), \text{ and } hf = (h_i f_i, P(h_j) F_u).$$

If f is a morphism, then fg and hf coincide with the morphisms defined by the standard composition of morphisms of $\Gamma\mathcal{C}$.

PROOF. The homotopy relation between morphisms in $\mathcal{C}_i$ is compatible with the composition. The map $F_u g_i$ is a homotopy from $f_j g_j \varphi_u$ to $\varphi_u f_i g_i$, and $P(h_j) F_u$ is a homotopy from $h_j f_j \varphi_u$ to $\varphi_u h_i f_i$. $\square$

Definition 1.3.13. A ho-morphism $f : A \rightsquigarrow B$ is called *weak equivalence* if the maps f_i are weak equivalences for all $i \in I$.

Definition 1.3.14. Let $f, g : A \rightsquigarrow B$ be two ho-morphisms. A *homotopy from f to g* is a ho-morphism $h : A \rightsquigarrow P(B)$ such that $\delta^0_B h = f$ and $\delta^1_B h = g$. We use the notation $h : f \simeq g$.

Equivalently, such a homotopy is given by a family $h = (h_i, H_u)$ satisfying:

(i) $h_i : A_i \to P(B_i)$ is such that $\delta^0_{B_i} h_i = f_i$ and $\delta^1_{B_i} h_i = g_i$. Therefore h_i is a homotopy from f_i to g_i in $\mathcal{C}_i$.

(ii) $H_u : A_i \to P^2(B_j)$ is a morphism in $\mathcal{C}_j$ satisfying

$$\begin{cases} P(\delta^0_{B_j})H_u = F_u, \\ P(\delta^1_{B_j})H_u = G_u, \end{cases} \text{ and } \begin{cases} \delta^0_{P(B_j)}H_u = h_j\varphi_u, \\ \delta^1_{P(B_j)}H_u = \psi_u h_i \end{cases}.$$

The notion of homotopy between ho-morphisms allows to define a class of equivalences of $\Gamma\mathcal{C}$ as follows.

Definition 1.3.15. A morphism $f : A \to B$ of $\Gamma\mathcal{C}$ is said to be a *ho-equivalence* if there exists a ho-morphism $g : B \rightsquigarrow A$ together with chains of homotopies of ho-morphisms $gf \simeq \cdots \simeq 1_A$ and $fg \simeq \cdots \simeq 1_B$.

Denote by $\mathcal{H}$ the closure by composition of the class of ho-equivalences.

Lemma 1.3.16. *We have $\mathcal{S} \subset \mathcal{H} \subset \overline{\mathcal{W}}$. In particular, $(\Gamma\mathcal{C}, \mathcal{H}, \mathcal{W})$ is a category with strong and weak equivalences.*

Proof. If f and g are homotopic morphisms of $\Gamma\mathcal{C}$, then they are also homotopic as ho-morphisms. Therefore $\mathcal{S} \subset \mathcal{H}$. If f is a ho-equivalence, then f_i is a morphism of $\mathcal{S}_i$, for all $i \in I$. Since $\mathcal{S}_i \subset \mathcal{W}_i$, it follows that $\mathcal{H} \subset \overline{\mathcal{W}}$. □

Factorization of Ho-morphisms. We next define the mapping path of a ho-morphism. This will be used in Proposition 1.3.20 to define a factorization for ho-morphisms.

Definition 1.3.17. Let $f : A \rightsquigarrow B$ be a ho-morphism of diagrams. The *mapping path of f* is the diagram defined as

$$\mathcal{P}^h(f) = \left(\mathcal{P}(f_i) \overset{\psi_u}{\dashrightarrow} \mathcal{P}(f_j)\right),$$

where $\mathcal{P}(f_i)$ is the mapping path of f_i given by the fibre product

$$\mathcal{P}(f_i) = A_i \times_{B_i} P(B_i).$$

The comparison morphism $\psi_u : \mathcal{P}(f_i) \to \mathcal{P}(f_j)$ is defined as follows. Consider the commutative solid diagram

$$\begin{tikzcd}
\mathcal{P}(f_i) \arrow[r, "\pi_1"] & A_i \arrow[dr, dotted] \arrow[drr, bend left, "F_u"] \arrow[ddr, bend right, "\varphi_u"'] & & \\
& & \mathcal{P}(f_j) \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\lrcorner"] & P(B_j) \arrow[d, "\delta^0_{B_j}"] \\
& & A_j \arrow[r, "f_j"'] & B_j
\end{tikzcd}.$$

Then $\psi_u = (\varphi_u, F_u)\pi_1$.

Remark 1.3.18. Let $f : A \to B$ be a morphism of $\Gamma\mathcal{C}$. Since $\Gamma\mathcal{C}$ is a P-category, f has a mapping path $\mathcal{P}(f)$ (see Definition 1.2.17). On the other hand, we can consider f as a ho-morphism, by letting $F = \iota f_j \varphi_u$, and so it has an associated mapping path $\mathcal{P}^h(f)$. The comparison morphisms of $\mathcal{P}(f)$ and $\mathcal{P}^h(f)$ differ.

We next provide a Brown Factorization Lemma for ho-morphisms, using the above mapping path.

1.3.19. Define morphisms $p_f : \mathcal{P}^h(f) \to A$ and $q_f : \mathcal{P}^h(f) \to B$, together with a ho-morphism $\iota_f : A \rightsquigarrow \mathcal{P}^h(f)$ as follows.

Let $i \in I$, and let $(p_f)_i = p_{f_i} = \pi_1 : \mathcal{P}(f_i) \to A_i$ be the first projection map. For $u : i \to j$ we have

$$p_{f_j}\psi_u = \pi_1(\varphi_u, F_u)\pi_1 = \varphi_u\pi_1 = \varphi_u p_{f_i}.$$

Therefore the family $p_f = (p_{f_i}) : \mathcal{P}^h(f) \to A$ is a morphism of diagrams.

Let $i \in I$, and let $q_{f_i} = \delta^{|i|}_{B_i}\pi_2 : \mathcal{P}(f_i) \to B_i$, where $|i| \in \{0, 1\}$ is the degree of i (see condition (I_1) of 1.3.4). For $u : i \to j$ we have

$$q_{f_j}\psi_u = \delta^1_{B_j}\pi_2(\varphi_u, F_u)\pi_1 = \delta^1_{B_j}F_u\pi_1 = \varphi_u f_i \pi_1 = \varphi_u \delta^0_{B_j}\pi_2 = \varphi_u q_{f_i}.$$

Therefore the family $q = (q_{f_i}) : \mathcal{P}^h(f) \to B$ is a morphism of diagrams.

The morphism q_f is not defined level-wise via the factorization Lemma 1.2.21 in which $q_f = \delta^1_B \pi_2$, but instead, we alternate between $\delta^0_B \pi_2$ and $\delta^1_B \pi_2$, depending on the degree of the index. This needs to be done in order to obtain a morphism, instead of a ho-morphism. As a result, q_f is not necessarily a level-wise fibration.

Let $\iota_{f_i} = (1_{A_i}, \iota_{B_i} f_i) : A_i \to \mathcal{P}(f_i)$. Then

$$\psi_u \iota_{f_i} = (\varphi_u, F_u), \text{ and } \iota_{f_j} \varphi_u = (\varphi_u, \iota_{A_j} f_j \varphi_u).$$

We next define a homotopy from $\psi_u \iota_{f_i}$ to $\iota_{f_j} \varphi_u$. Let J_{F_u} be the morphism defined by the following pull-back diagram:

$$\begin{tikzcd}
A_i \arrow[dr, dotted, "J_{F_u}"] \arrow[drr, bend left, "c^0_{B_j} F_u"] \arrow[ddr, bend right, "\iota_{A_j}\varphi_u"'] & & \\
& P(\mathcal{P}(f_j)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P^2(B_j) \arrow[d, "P(\delta^0_{B_j})"] \\
& P(A_j) \arrow[r, "P(f_j)"'] & P(B_j)
\end{tikzcd}.$$

The coproduct (see Definition 1.2.6) satisfies $P(\delta^0_{B_j}) c^0_{B_j} = \iota_{B_j} \delta^0_{B_j}$, and hence

$$P(\delta^0_{B_j}) c^0_{B_j} F_u = \iota_{B_j} \delta^0_{B_j} F_u = \iota_{B_j} f_j \varphi_u = P(f_j) \iota_{A_j} \varphi_u.$$

Therefore the solid diagram commutes, and the map J_{F_u} is well defined.

By (P_4), the fibre product $P(\mathcal{P}(f_j))$ is a path object of $\mathcal{P}(f_j)$, with

$$\delta^k_{\mathcal{P}(f_j)} = (\delta^k_{A_j} P(\pi_1), \delta^k_{P(B_j)} P(\pi_2)), \text{ for } k = 0, 1.$$

Therefore we have

$$\delta^k_{\mathcal{P}(f_j)} J_{F_u} = (\delta^k_{A_j} \iota_{A_j} \varphi_u, \delta^k_{P(B_j)} c^0_{B_j} F_u) = (\varphi_u, \delta^k_{P(B_j)} c^0_{B_j} F_u).$$

Since $\delta^0_{P(B_j)} c^0_{B_j} = \iota_{B_j} \delta^0_{B_j}$ and $\delta^1_{P(B_j)} c^0_{B_j} = 1_{B_j}$, it follows that

$$\begin{cases} \delta^0_{\mathcal{P}(f_j)} J_{F_u} = (\varphi_u, \iota_{B_j} \delta^0_{B_j} F_u) = \iota_{f_j} \varphi_u. \\ \delta^1_{\mathcal{P}(f_j)} J_{F_u} = (\varphi_u, F_u) = \psi_u \iota_{f_j}. \end{cases}$$

Therefore the family $\iota_f = (\iota_{f_i}, J_{F_u}) : A \rightsquigarrow \mathcal{P}(f)$ is a ho-morphism.

The main result of this section is the following.

Proposition 1.3.20. *Let $f : A \rightsquigarrow B$ be a ho-morphism. The diagram*

$$\begin{tikzcd}
A & \mathcal{P}^h(f) \arrow[l, "p_f"'] \arrow[r, "q_f"] & B \\
& A \arrow[ul, equal] \arrow[u, squiggly, "\iota_f"] \arrow[ur, squiggly, "f"'] &
\end{tikzcd}$$

commutes. In addition:

(1) The maps p_f and ι_f are weak equivalences.
(2) There is a homotopy of ho-morphisms between $\iota_f p_f$ and $1_{\mathcal{P}(f)}$, making p_f into a ho-equivalence.
(3) If f is a weak equivalence, then q_f is a weak equivalence.

PROOF. From the definitions it is straightforward that the above diagram commutes.

Let us prove (1). From axiom (P_3) of P-categories, the map p_f is a weak equivalence. By the two out of three property, it follows that ι_f is a weak equivalence as well.

To prove (2) we define a homotopy between $\iota_f p_f = (\iota_{f_i} p_{f_i}, J_{F_u} p_{f_i})$ and $1_{\mathcal{P}(f_i)}$ as follows.

For all $i \in I$, let $h_i : \mathcal{P}(f_i) \to P(\mathcal{P}(f_i))$ be the morphism of $\mathcal{C}_i$ defined by $h_i = (\iota_{A_i}\pi_1, c^0_{B_i}\pi_2)$. This is a homotopy from $\iota_i p_i$ to the identity morphism $1_{\mathcal{P}(f_i)}$ (see the proof of Lemma 1.2.21).

Let $\widetilde{H}_u$ be the morphism defined by the following pull-back diagram:

$$
\begin{tikzcd}
A_i \arrow[dr, dotted, "\widetilde{H}_u"] \arrow[drr, bend left, "c^2_{B_j} F_u"] \arrow[ddr, bend right, "\iota_{P(A_j)}\iota_{A_j}\varphi_u"'] & & \\
& P^2(\mathcal{P}(f_j)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P^3(B_j) \arrow[d, "P^2(\delta^0_{B_j})"] \\
& P^2(A_j) \arrow[r, "P^2(f_j)"'] & P^2(B_j)
\end{tikzcd}
,
$$

where c^2 is the transformation defined in 1.2.12. By Lemma 1.2.13 it satisfies $P^2(\delta^0_{B_j})c^2_{B_j} = \iota_{P(B_j)}\iota_{B_j}\delta^0_{B_j}$. Therefore the solid diagram commutes, and hence the map $\widetilde{H}_u$ is well defined.

Let $H_u := \widetilde{H}_u\pi_1 : \mathcal{P}(f_i) \to P^2(\mathcal{P}(f_j))$. By ($\mathrm{P}_4$), the fibre product $P^2(\mathcal{P}(f_j))$ is a double path object of $\mathcal{P}(f_j)$, with structural maps:

$$
\begin{cases}
\delta^k_{P(\mathcal{P}(f_j))} = (\delta^k_{P(A_j)} P^2(\pi_1), \delta^k_{P^2(B_j)} P^2(\pi_2)), \\
P(\delta^k_{\mathcal{P}(f_j)}) = (P(\delta^k_{A_j}) P^2(\pi_1), P(\delta^k_{P(B_j)}) P^2(\pi_2)),
\end{cases}
\quad \text{for } k = 0, 1.
$$

From the properties of c^2 (see Lemma 1.2.13) we have:

$$
\begin{cases}
\delta^0_{P(\mathcal{P}(f_j))} H_u = h_j \psi_u, & P(\delta^0_{\mathcal{P}(f_j)}) H_u = J_{F_u} p_{f_i} \\
\delta^1_{P(\mathcal{P}(f_j))} H_u = P(\psi_u) h_i, & P(\delta^1_{\mathcal{P}(f_j)}) H_u = \iota_{\mathcal{P}(f_j)}(\psi_u).
\end{cases}
$$

Therefore the family $h = (h_i, H_u)$ is a homotopy from $\iota_f p_f$ to $1_{\mathcal{P}(f)}$.

Let us prove (3). Assume that f is a weak equivalence. By (i), the map ι_f is a weak equivalence. By the two out of three property, q_f is a weak equivalence. $\square$

1.3.21. For every pair of objects A, B of $\Gamma\mathcal{C}$, define a map

$$\Phi_{A,B} : \Gamma\mathcal{C}^h(A, B) \longrightarrow \Gamma\mathcal{C}[\mathcal{H}^{-1}](A, B)$$

by letting

$$\Phi_{A,B}(f) := \left\{ A \xleftarrow{p_f} \mathcal{P}^h(f) \xrightarrow{q_f} B \right\} = \left\{ q_f p_f^{-1} \right\}.$$

By Proposition 1.3.20 the map p_f is a ho-equivalence, and hence this is a map with image in the localized category $\Gamma\mathcal{C}[\mathcal{H}^{-1}]$.

To end this section we collect some useful properties of this map.

Definition 1.3.22. Let $f : A \rightsquigarrow B$ and $g : C \rightsquigarrow D$ be two ho-morphisms. A *morphism from f to g* is given by a pair of morphisms $\alpha : A \to C$ and $\beta : B \to D$ of $\Gamma\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, squiggly, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\
C \arrow[r, squiggly, "g"] & D
\end{tikzcd}$$

commutes. Denote $(\alpha, \beta) : f \Rightarrow g$.

Lemma 1.3.23. *Let $(\alpha, \beta) : f \Rightarrow g$ be a morphism between ho-morphisms $f : A \rightsquigarrow B$ and $g : C \rightsquigarrow D$. There is an induced morphism*

$$(\alpha, \beta)_* : \mathcal{P}^h(f) \longrightarrow \mathcal{P}^h(g),$$

which is compatible with q_f, p_f, and ι_f.

PROOF. Let $(\alpha_i, \beta_i)_*$ be the morphism defined by the pull-back diagram:

$$\begin{tikzcd}
\mathcal{P}(f_i) \arrow[dr, dotted, "(\alpha_i,\beta_i)_*"] \arrow[drr, bend left, "P(\beta_i)\pi_2"] \arrow[ddr, bend right, "\alpha_i\pi_1"'] & & \\
& \mathcal{P}(g_i) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\lrcorner"] & P(D_i) \arrow[d, "\delta^0_{D_i}"] \\
& C_i \arrow[r, "g_i"] & D_i
\end{tikzcd} .$$

We have

$$\psi_u(\alpha_i, \beta_i)_* = (\varphi_u, G_u)\pi_1(\alpha_i\pi_1, P(\beta_i)\pi_2) = (\varphi_u\alpha_i, G_u\alpha_i)\pi_1,$$
$$(\alpha_j, \beta_j)_*\psi_u = (\alpha_j\pi_1, P(\beta_j)\pi_2)(\varphi_u, F_u)\pi_1 = (\alpha_j\varphi_u, P(\beta_j)F_u)\pi_1.$$

Since α is a morphism, $\varphi_u\alpha_i = \alpha_j\varphi_u$. Since $\beta f = g\alpha$, it follows that $P(\beta_j)F_u = G_u\alpha_i$. Therefore the diagram

$$\begin{tikzcd}
\mathcal{P}(f_i) \arrow[r, "\psi_u"] \arrow[d, "(\alpha_i,\beta_i)_*"'] & \mathcal{P}(f_j) \arrow[d, "(\alpha_j,\beta_j)^*"] \\
\mathcal{P}(g_i) \arrow[r, "\psi_u"] & \mathcal{P}(g_j)
\end{tikzcd}$$

commutes, and the family $(\alpha,\beta)_* := (\alpha_i,\beta_i)_*$ is a morphism of diagrams. From the definitions it follows that the diagram

$$\begin{tikzcd}
\mathcal{P}^h(f) \arrow[d, "(\alpha,\beta)_*"'] & A \arrow[l, squiggly, "\iota_f"'] \arrow[d, "\alpha"'] & \mathcal{P}^h(f) \arrow[l, "p_f"'] \arrow[r, "q_f"] \arrow[d, "(\alpha,\beta)_*"] & B \arrow[d, "\beta"] \\
\mathcal{P}^h(g) & C \arrow[l, squiggly, "\iota_g"] & \mathcal{P}^h(g) \arrow[l, "p_g"] \arrow[r, "q_g"'] & D
\end{tikzcd}$$

commutes. $\square$

Lemma 1.3.24. *Let $f : A \to B \in \Gamma\mathcal{C}$. Then $\Phi_{A,B}(f) = \{f\}$ in $\Gamma\mathcal{C}[\mathcal{H}^{-1}]$.*

Proof. Since f is a morphism, the map $\iota_f : A \to \mathcal{P}^h(f)$ is a morphism too. The diagram

$$\begin{tikzcd}
A \arrow[dr, phantom, "="] & \mathcal{P}^h(f) \arrow[l, "p_f"'] \arrow[r, "q_f"] \arrow[dr, phantom, "="] & B \\
& A \arrow[ul, equal] \arrow[u, "\iota_f"] \arrow[ur, "f"'] &
\end{tikzcd}$$

is a hammock between the $\mathcal{H}$-zigzags $q_f p_f^{-1}$ and f. $\square$

Lemma 1.3.25. *Let $f : A \rightsquigarrow B$ be a ho-morphism, and let $g : B \to C$ be a morphism of $\Gamma\mathcal{C}$. Then $\Phi_{B,C}(g) \circ \Phi_{A,B}(f) = \Phi_{A,C}(gf)$.*

Proof. The pair $(1_A, g)$ is a morphism of ho-morphisms $f \Rightarrow gf$. By Lemma 1.3.23 there is a morphism $(1_A, g)_* : \mathcal{P}^h(f) \to \mathcal{P}^h(gf)$, making the following diagram commute.

$$\begin{tikzcd}
& \mathcal{P}^h(f) \arrow[dl, "p_f"'] \arrow[r, "q_f"] \arrow[d, "(1_A{,}g)_*"] & B \arrow[d, "g"] \arrow[dr, "g"] & \\
A & \mathcal{P}^h(gf) \arrow[l, "p_{gf}"] \arrow[r, "q_{gf}"] & C \arrow[r, equal] & C
\end{tikzcd}$$

This is a hammock between $\mathcal{H}$-zigzags representing $\Phi_{A,C}(gf)$ and $\Phi_{B,C}(g) \circ \Phi_{A,B}(f)$. $\square$

Lemma 1.3.26. *Let $f, g : A \rightsquigarrow B$ be ho-morphisms and assume that there is a chain of homotopies $f \simeq \cdots \simeq g$. Then*

$$\Phi_{A,B}(f) = \Phi_{A,B}(g).$$

PROOF. It is sufficient to prove it for $f \simeq g$. Let $h : A \to P(B)$ be a homotopy of ho-morphisms from f to g. There are two morphisms

$$(1_A, \delta_B^0) : h \Rightarrow f, \text{ and } (1_A, \delta_B^1) : h \Rightarrow g.$$

By Lemma 1.3.23 these define morphisms

$$\mathcal{P}^h(f) \xleftarrow{\pi_f} \mathcal{P}^h(h) \xrightarrow{\pi_g} \mathcal{P}^h(g),$$

where $\pi_f := (1_A, \delta_B^1)_*$ and $\pi_g := (1_A, \delta_B^1)_*$. Consider the diagram

$$\begin{tikzcd}
 & \mathcal{P}^h(f) \arrow[dl, "p_f"'] \arrow[dr, "q_f"] & \\
A & \mathcal{P}^h(h) \arrow[l, "p_h"'] \arrow[u, "\pi_f"] \arrow[r, "q_f\pi_f"] \arrow[d, "\pi_g"] & B \\
 & \mathcal{P}^h(g) \arrow[ul, "p_g"] \arrow[ur, "q_g"'] &
\end{tikzcd}.$$

To see that $\Phi(f) = \Phi(g)$ in $\Gamma\mathcal{C}[\mathcal{H}^{-1}]$, it suffices to check that the four triangles are commutative in $\Gamma\mathcal{C}[\mathcal{S}^{-1}]$. By definition, every triangle is commutative in $\Gamma\mathcal{C}$, except for the lower-right triangle. We shall next build a homotopy of morphisms from $q_f\pi_f$ to $q_f\pi_g$.

For all $i \in I$, let $\theta_i := \delta^{|i|}_{P(B_i)}\pi_2 : \mathcal{P}(h_i) \to P(B_i)$. Then $\varphi_u\theta_i = \theta_j\psi_u$, and $\theta = (\theta_i) : \mathcal{P}^h(h) \to P(B)$ is a homotopy from $q_f\pi_f$ to $q_g\pi_g$. □

1.4. COFIBRANT MODELS OF DIAGRAMS

Denote by $\Gamma\mathcal{C}_{cof}$ the full subcategory of $\Gamma\mathcal{C}$ of those diagrams

$$C = (C_i \dashrightarrow C_j)$$

such that C_i is $\mathcal{F}_i$-cofibrant in $\mathcal{C}_i$, for all $i \in I$, and $u_*(C_i)$ is $\mathcal{F}_j$-cofibrant, for each $u : i \to j$ of I. In this section we will show that the objects of $\Gamma\mathcal{C}_{cof}$ are cofibrant diagrams of $(\Gamma\mathcal{C}, \mathcal{H}, \mathcal{W})$, and that every diagram has a model in $\Gamma\mathcal{C}_{cof}$. In particular, the triple $(\Gamma\mathcal{C}, \mathcal{H}, \mathcal{W})$ is a Cartan-Eilenberg category. In addition, we will show that the relative localization $\Gamma\mathcal{C}_{cof}[\mathcal{H}^{-1}, \Gamma\mathcal{C}]$ is equivalent to the category $\pi^h\Gamma\mathcal{C}_{cof}$ whose objects are those of $\Gamma\mathcal{C}_{cof}$, and whose morphisms are homotopy classes of ho-morphisms. These results lead to the equivalence of categories $\pi^h\Gamma\mathcal{C}_{cof} \xrightarrow{\sim} \Gamma\mathcal{C}[\mathcal{W}^{-1}]$.

Homotopy Classes of Ho-morphisms. We next show that the homotopy relation between ho-morphisms of diagrams is transitive for those morphisms whose source is level-wise cofibrant, and define a composition of homotopy classes of ho-morphisms.

Given two objects A, B of $\Gamma\mathcal{C}$, we will denote by

$$[A, B]^h := \Gamma\mathcal{C}^h(A, B)/\sim$$

the set of ho-morphisms from A to B modulo the equivalence relation transitively generated by the homotopy relation.

Lemma 1.4.1. *Let A be an object of $\Gamma\mathcal{C}_{cof}$. For every object B of $\Gamma\mathcal{C}$, the homotopy relation is an equivalence relation on the set of ho-morphisms from A to B. In particular,*

$$[A, B]^h = \Gamma\mathcal{C}^h(A, B)/\simeq .$$

Proof. Reflexivity and symmetry are trivial. We prove transitivity. Assume given ho-morphisms $f, f', f'' : A \rightsquigarrow B$, together with homotopies $h : f \simeq f'$ and $h' : f' \simeq f''$. For all $i \in I$, consider the solid diagram

$$\begin{tikzcd} & P^2(B_i) \arrow[d, "\pi_{B_i}"] \\ A_i \arrow[ur, dotted, "\mathcal{L}_i"] \arrow[r, "(h_i, h'_i)"] & \mathcal{P}(\delta^1_{B_i}). \end{tikzcd}$$

By Lemma 1.2.23 the map $\pi_{B_i} = (\delta^0_{P(B_i)}, P(\delta^1_{B_i}))$ is a trivial fibration. Since A_i is $\mathcal{F}_i$-cofibrant, there exists a dotted arrow $\mathcal{L}_i$ such that $\pi_{B_i}\mathcal{L}_i = (h_i, h'_i)$. In particular, $\delta^0_{P(B_i)}\mathcal{L}_i = h_i$ and $P(\delta^1_{B_i})\mathcal{L}_i = h'_i$. We let $h''_i := \nabla_{B_i}\mathcal{L}_i$, where ∇ is the folding map (see Definition 1.2.9). By construction, h''_i is a homotopy from f_i to f'_i.

Consider the commutative solid diagram:

$$\begin{tikzcd}
A_i \arrow[dr, dotted, "\mathcal{L}_u"] \arrow[drr, bend left, "(\mathcal{L}_j\varphi_u{,}\varphi_u\mathcal{L}_i)"] \arrow[ddr, bend right, "(H_u{,}H'_u)"'] & & \\
 & P^3(B_j) \arrow[r] \arrow[d, "P(\pi_{B_j})"] & P^2(B_j)\times P^2(B_j) \arrow[d, "\pi_{B_j}\times\pi_{B_j}"] \\
 & P(\mathcal{P}(\delta^1_{B_j})) \arrow[r] & \mathcal{P}(\delta^1_{B_j})\times\mathcal{P}(\delta^1_{B_j})
\end{tikzcd}.$$

Since π_{B_j} is a trivial fibration, by Lemma 1.2.25 there exists a dotted arrow $\mathcal{L}_u$, making the diagram commute. Let

$$H''_u := P(\nabla_{B_j})\mathcal{L}_u : A_i \to P^2(B_j).$$

The family $h'' = (h''_i, H''_u)$ is a homotopy of ho-morphisms from f to f''. Indeed, we have:

$$(\delta^0_{P(B_j)}, \delta^1_{P(B_j)})H''_u = \nabla_{B_j}(\delta^0_{P^2(B_j)}, \delta^1_{P^2(B_j)})\mathcal{L}_u = (h_j\varphi_u, \varphi_u h_i).$$
$$(P(\delta^0_{B_j}), P(\delta^1_{B_j}))H''_u = (P(\delta^0_{B_j})H_u, P(\delta^1_{B_j}))H'_u = (F_u, F''_u).$$

□

We define a composition of ho-morphisms subject to the condition that the source is a diagram of $\mathcal{F}_i$-cofibrant objects.

Lemma 1.4.2. *Let A be an object of $\Gamma\mathcal{C}_{cof}$. For every pair of objects B, C of $\Gamma\mathcal{C}$, there is a map*

$$[A,B]^h \times [B,C]^h \longrightarrow [A,C]^h$$

denoted by $([f],[g]) \mapsto [g]*[f]$ such that

(1) If either g or f are morphisms of $\Gamma\mathcal{C}$, then $[g]*[f] = [gf]$, where gf is the composition defined in Lemma 1.3.12.

(2) If h is a morphism and f, g are ho-morphisms, then

$$[h]*([g]*[f]) = [hg]*[f].$$

Proof. Let $[f] \in [A, B]^h$, and $[g] \in [B, C]^h$. Choose representatives $f = (f_i, F_u)$ and $g = (g_i, G_u)$ of $[f]$ and $[g]$ respectively. There is a chain of homotopies in $\mathcal{C}_j$,

$$g_j f_j \varphi_u \overset{P(g_j)F_u}{\simeq} g_j \varphi_u f_i \overset{G_u f_i}{\simeq} \varphi_u g_i f_i.$$

Consider the solid diagram

$$\begin{tikzcd} & P^2(C_j) \arrow[d, "\pi_{C_j}", "\wr"'] \\ A_i \arrow[ur, dotted, "\mathcal{L}_u"] \arrow[r, "\gamma_u"] & \mathcal{P}(\delta^1_{C_j}), \end{tikzcd}$$

where $\gamma_u := (P(g_j)F_u, G_u f_i)$. Since A_i is $\mathcal{F}_j$-cofibrant, there exists a dotted arrow $\mathcal{L}_u$ making the diagram commute. In particular,

$$K_u := \nabla_{C_j} \mathcal{L}_u : A_i \to P(C_j)$$

is a homotopy from $g_j f_j \varphi_u$ to $\varphi_u g_i f_i$. The family $g * f := (g_i f_i, K_u)$ is a ho-morphism from C to A.

Let f' be another representative of $[f]$, and let $h : f \simeq f'$ be a homotopy from f to f'. Assume that $g * f' = (g_i f'_i, K'_u)$, where $K'_u = \nabla_{C_j} \mathcal{L}'_u$, and $\mathcal{L}'_u$ a lifting of $\gamma'_u := (P(g_j)F'_u, G_u f'_i)$. We next show that there is a homotopy of ho-morphisms $g * f \simeq g * f'$.

Let Γ_u be the morphism defined by the following pull-back diagram:

$$\begin{tikzcd} A_i \arrow[dr, dotted, "\Gamma_u"] \arrow[drr, bend left, "P(G_u)h_i"] \arrow[ddr, bend right, "P^2(g_j)\mu_{B_j} H_u"'] & & \\ & P(\mathcal{P}(\delta^1_{C_j})) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P^2(C_j) \arrow[d, "P(\delta^0_{C_j})"] \\ & P^2(C_j) \arrow[r, "P(\delta^1_{C_j})"'] & P(C_j) \end{tikzcd} ,$$

where μ is the interchange transformation of the path (see Definition 1.2.8), and satisfies $\delta^k_{P(B_j)} \mu_{B_j} = P(\delta^k_{B_j})$ and $P(\delta^k_{B_j}) \mu_{B_j} = \delta^k_{P(B_j)}$. Consider the commutative solid diagram:

$$\begin{tikzcd}
A_i \arrow[dr, dotted, "\widetilde{K}_u"] \arrow[drr, bend left, "(\mathcal{L}_u{,}\mathcal{L}'_u)"] \arrow[ddr, bend right, "\Gamma_u"'] & & \\
& P^3(C_j) \arrow[r] \arrow[d, "P(\pi_{C_j})"] & P^2(C_j) \times P^2(C_j) \arrow[d, "\pi_{C_j}\times\pi_{C_j}"] \\
& P(\mathcal{P}(\delta^1_{C_j})) \arrow[r] & \mathcal{P}(\delta^1_{C_j}) \times \mathcal{P}(\delta^1_{C_j})
\end{tikzcd} .$$

Since π_{C_j} is a trivial fibration, by Lemma 1.2.25 there exists a dotted arrow $\widetilde{K}_u$, making the diagram commute. Let

$$H'_u := \mu_{C_j} P(\nabla_{C_j}) \widetilde{K}_u : A_i \to P^2(C_j).$$

The family $(P(g_i)h_i, H'_u)$ is a homotopy of ho-morphisms from $g * f$ to $g * f'$.

Analogously, given a representative g' of $[g]$, one proves that $g * f \simeq g' * f$. By Lemma 1.4.1 the homotopy relation between ho-morphisms for which the source is in $\Gamma\mathcal{C}_{cof}$, is transitive. Therefore the class $[g * f]$ does not depend on the chosen representatives and the chosen liftings, and the map $[g] * [f] := [g * f]$ is well defined.

Let us prove (1). Let $[f] \in [A, B]^h$, and let $g : B \to C$ be a morphism. Choose a representative f of $[f]$, and let $gf = (g_i f_i, P(g_i)F_u)$. By Lemma 1.3.12 this is a well defined ho-morphism from A to C. We next show that $[g]*[f] = [gf]$, when g is considered as a ho-morphism with $G_u = (\iota_{C_j} \varphi_u g_i)$. Consider the diagram

$$\begin{tikzcd}
& P^2(C_j) \arrow[d, "\pi_{C_j}"] \\
A_i \arrow[ur, "\mathcal{L}_u"] \arrow[r, "\gamma_u"] & \mathcal{P}(\delta^1_{C_j}),
\end{tikzcd}$$

where $\gamma_u = (P(g_j)F_u, \iota_{C_j} g_j \varphi_u f_i)$, and $\mathcal{L}_u := \iota_{P(C_j)} P(g_j)F_u$. By the naturality of δ^k and ι it follows that

$$\begin{cases} \delta^0_{P(C_j)} \mathcal{L}_u = \delta^0_{P(C_j)} \iota_{P(C_j)} P(g_j) F_u = P(g_j) F_u. \\ P(\delta^1_{C_j}) \mathcal{L}_u = \iota_{C_j} \delta^1_{C_j} P(g_j) F_u = \iota_{C_j} g_j \delta^1_{B_j} F_u = \iota_{c_j} g_j \varphi_u f_i. \end{cases}$$

Therefore the above diagram commutes. By definition, the folding map ∇ (see Definition 1.2.9) satisfies $\nabla_{C_j}\iota_{P(C_j)} = 1_{P(C_j)}$. It follows that

$$K_u := \nabla_{P(C_j)}\mathcal{L}_u = P(g_j)F_u.$$

Therefore $[g] * [f] = [gf]$. The proof for the other composition follows analogously.

Let us prove (2). Let $f : A \rightsquigarrow B$ and $g : B \rightsquigarrow C$ be ho-morphisms, and let $\gamma_u = (P(g_j)F_u, G_u f_i)$. We have $f * g = (f_i g_i, K_u)$, where $K_u = \nabla_{P(C_j)}\mathcal{L}_u$, and $\mathcal{L}_u$ is an arbitrary morphism satisfying $\pi_{C_j}\mathcal{L}_u = \gamma_u$. If $h : C \to D$ is a morphism, by (1) we have

$$[h] * ([g] * [f]) = [(h_i g_i f_i, P(h_j)K_u)].$$

On the other hand, let $\gamma'_u = (P(h_j g_j)F_u, P(h_j)G_u f_i)$, and define a morphism $\mathcal{L}'_u := P^2(h_j)\mathcal{L}_u$. Then $\pi_{D_j}\mathcal{L}'_u = \gamma'_u$. Therefore $hg * f = (h_j g_j f_j, K'_u)$, where $K'_u = \nabla_{D_j}\mathcal{L}'_u = P(h_j)K_u$. The identity $[hg]*[f] = [h]*([g]*[f])$ follows. $\square$

Localization with respect to Ho-equivalences. We next show that the relative localization $\Gamma\mathcal{C}_{cof}[\mathcal{H}^{-1}, \Gamma\mathcal{C}]$ is isomorphic to the category $\pi^h\Gamma\mathcal{C}_{cof}$, whose objects are those of $\Gamma\mathcal{C}_{cof}$ and whose morphisms are:

$$\pi^h\Gamma\mathcal{C}_{cof}(A, B) = [A, B]^h,$$

with the composition defined in Lemma 1.4.2.

Let A, B be arbitrary objects of $\Gamma\mathcal{C}$. By Lemma 1.3.26 the map defined in 1.3.21 induces a well defined map

$$\Phi_{A,B} : [A, B]^h \longrightarrow \Gamma\mathcal{C}[\mathcal{H}^{-1}](A, B).$$

We will see that if A an object of $\Gamma\mathcal{C}_{cof}$, the above map is a bijection of sets.

1.4.3. Let A be an object of $\Gamma\mathcal{C}_{cof}$. We next define a map

$$\Psi_{A,B} : \Gamma\mathcal{C}[\mathcal{H}^{-1}](A, B) \longrightarrow [A, B]^h,$$

for every object B of $\Gamma\mathcal{C}$.

A morphism $\{f\}$ of $\Gamma\mathcal{C}[\mathcal{H}^{-1}]$ can be represented by a zigzag $f : A \dashrightarrow B$

$$\begin{tikzcd}[column sep=small]
 & D_1 & & D_2 & & D_3 & \cdots & D_r & \\
A = C_0 \arrow[ur] & & C_1 \arrow[ul] \arrow[ur] & & C_2 \arrow[ul] \arrow[ur] & & \cdots & & C_r = B \arrow[ul]
\end{tikzcd}$$

where the arrows going to the left are morphisms of $\mathcal{H}$. We will define Ψ by sending each $\mathcal{H}$-zigzag with cofibrant source, to a homotopy class of ho-morphisms. We will proceed inductively over the length of the zigzag.

Let $\Psi(1_A) = [1_A]$ and assume that Ψ is defined for $\mathcal{H}$-zigzags of a given length. We consider two cases:

(1) Let $f = gf'$, where $f' : A \dashrightarrow C$ is an $\mathcal{H}$-zigzag and $g : C \to D$ is a morphism. Then

$$\Psi(f) := [g] * \Psi(f').$$

(2) Let $f = g^{-1}f'$, where $f' : A \dashrightarrow C$ is an $\mathcal{H}$-zigzag and $g : D \to C$ is a ho-equivalence. Let $h : C \rightsquigarrow D$ be a homotopy inverse of g. Then

$$\Psi(f) := [h] * \Psi(f').$$

Let h' be another homotopy inverse of g. Then $h' \simeq h'gh \simeq h$, and so $[h] = [h']$. Hence this does not depend on the chosen homotopy inverse.

Lemma 1.4.4. *Let A be an object of $\Gamma\mathcal{C}_{cof}$. The map*

$$\Psi_{A,B} : \mathcal{C}[\mathcal{H}^{-1}](A, B) \longrightarrow [A, B]^h$$

induced by $\{f\} \mapsto \Psi(f)$, is well defined for any object B of $\Gamma\mathcal{C}$.

PROOF. We need to prove that the definition does not depend on the chosen representative, that is, given a hammock between $\mathcal{H}$-zig-zags f and $\widetilde{f}$, then $\Psi(f) = \Psi(\widetilde{f})$. The proof is based on the fact that, given the commutative diagram on the left,

$$\begin{tikzcd}
D \arrow[d, "\alpha"'] \arrow[dr, phantom, "="] & C \arrow[l, "g"'] \arrow[d, "\beta"] & & D \arrow[r, rightsquigarrow, "h"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\simeq"] & C \arrow[d, "\beta"] \\
\widetilde{D} & \widetilde{C} \arrow[l, "\widetilde{g}"] & \Longrightarrow & \widetilde{D} \arrow[r, rightsquigarrow, "\widetilde{h}"'] & \widetilde{C}
\end{tikzcd},$$

where g and $\widetilde{g}$ is are ho-equivalences, then the diagram on the right commutes up to homotopy, where h and $\widetilde{h}$ are homotopy inverses of g and $\widetilde{g}$.

By induction, it suffices to consider the case when f and $\widetilde{f}$ are related by a hammock of height 1. Let

$$\begin{tikzcd}
f := A \arrow[r, "f_1"] \arrow[d, equal] & D_1 \arrow[d, "\alpha_1"] & C_1 \arrow[l, "g_1"'] \arrow[r, "f_2"] \arrow[d, "\beta_1"] & D_2 \arrow[d, "\alpha_2"] & C_2 \arrow[l, "g_2"'] \arrow[r, "f_3"] \arrow[d, "\beta_2"] & \cdots \arrow[r] & D_r \arrow[d, "\alpha_r"] & B \arrow[l, "g_r"'] \arrow[d, equal] \\
\widetilde{f} := A \arrow[r, "\widetilde{f}_1"] & \widetilde{D}_1 & \widetilde{C}_1 \arrow[l, "\widetilde{g}_1"'] \arrow[r, "\widetilde{f}_2"] & D_2 & \widetilde{C}_2 \arrow[l, "\widetilde{g}_2"'] \arrow[r, "\widetilde{f}_3"] & \cdots \arrow[r] & D_r & B \arrow[l, "\widetilde{g}_r"']
\end{tikzcd}$$

be a commutative diagram, where $g_k : C_k \to D_k$ and $\widetilde{g}_k : \widetilde{C}_k \to \widetilde{D}_k$ are compositions of ho-equivalences. For all $0 < k \leq r$, define

$$f(k) := g_k^{-1} f_k \cdots g_1^{-1} f_1, \text{ and } \widetilde{f}(k) := \widetilde{g}_k^{-1} \widetilde{f}_k \cdots \widetilde{g}_1^{-1} \widetilde{f}_1$$

as the $\mathcal{H}$-zigzags of length $2k$ defined by the first k roofs of f and $\widetilde{f}$ respectively. Let $f(0) = \widetilde{f}(0) = 1_A$.

Write $g_k = g_k^1 \cdots g_k^{n_k}$ and $\widetilde{g}_k = \widetilde{g}_k^1 \cdots \widetilde{g}_k^{\widetilde{n}_k}$, and let h_k^j and $\widetilde{h}_k^j$ be homotopy inverses of g_k^j and $\widetilde{g}_k^j$ respectively. With these notations we have

$$\Psi(f(k)) := [h_k^{n_k}] * (\cdots * ([h_k^2] * ([h_k^1] * ([f_k] * \Psi(f(k-1)))))),$$
$$\Psi(\widetilde{f}(k)) := [\widetilde{h}_k^{n_k}] * (\cdots * ([\widetilde{h}_k^2] * ([\widetilde{h}_k^1] * ([\widetilde{f}_k] * \Psi(\widetilde{f}(k-1)))))).$$

From the definition it follows that:

$$(p_k) \qquad [g_k] * \Psi(f(k)) = [f_k] * \Psi(f(k-1)).$$

We will now proceed by induction. Assume that for all $n < k$ we have

$$(h_n) \qquad \Psi(\widetilde{f}(n)) = [\beta_n] * \Psi(f(n)).$$

For the following identities we will constantly use properties (1) and (2) of Lemma 1.4.2. We have:

$$
\begin{aligned}
\Psi(\widetilde{f}(k)) &= [\widetilde{h}_k^{n_k}] * (\cdots * ([\widetilde{h}_k^1] * ([\widetilde{f}_k] * \Psi(\widetilde{f}(k\text{-}1))))) && (\text{ by } (h_{k-1}) \text{ }) \\
&= [\widetilde{h}_k^{n_k}] * (\cdots * ([\widetilde{h}_k^1] * ([\widetilde{f}_k \beta_{k-1}] * \Psi(f(k\text{-}1))))) && (\text{ } \widetilde{f}_k \beta_{k-1} = \alpha_k f_k \text{ }) \\
&= [\widetilde{h}_k^{n_k}] * (\cdots * ([\widetilde{h}_k^1] * ([\alpha_k f_k] * \Psi(f(k\text{-}1))))) && (\text{ by } (p_k) \text{ }) \\
&= [\widetilde{h}_k^{n_k}] * (\cdots * ([\widetilde{h}_k^1] * ([\alpha_k g_k] * \Psi(f(k)) && (\text{ } \alpha_k g_k = \widetilde{g}_k \beta_k \text{ }) \\
&= [\widetilde{h}_k^{n_k}] * (\cdots * ([\widetilde{h}_k^1] * ([\widetilde{g}_k \beta_k] * \Psi(f(k)) && (\text{ } \widetilde{g}_k = \widetilde{g}_k^1 \cdots \widetilde{g}_k^{n_k} \text{ }) \\
&= [\beta_k] * \Psi(f(k)).
\end{aligned}
$$

Since $\beta_r = 1_B$ we get $\Psi(\widetilde{f}) = \Psi(\widetilde{f}(r)) = \Psi(f(r)) = \Psi(f)$. $\square$

Proposition 1.4.5. *Let A be an object of $\Gamma\mathcal{C}_{cof}$. The maps*

$$\Phi_{A,B} : [A,B]^h \rightleftarrows \Gamma\mathcal{C}[\mathcal{H}^{-1}](A,B) : \Psi_{A,B}$$

are inverses to each other, for every object B of $\Gamma\mathcal{C}$.

PROOF. For the simplicity of notation, we omit the subscripts of both Ψ and Φ. Let $[f]$ be an element of $[A,B]^h$. Then

$$\Psi(\Phi([f])) = \Psi(\{q_f p_f^{-1}\}) = [q_f] * [\iota_f] = [q_f \iota_f] = [f].$$

For the other composition, we proceed by induction as follows. Assume that for every element $\{f\}$ of $\Gamma\mathcal{C}[\mathcal{H}^{-1}](A,C)$ we have $\Phi(\Psi(\{f\})) = \{f\}$. We will next show that:

(1) if $g : C \to D$ is a map, then $\Phi(\Psi(\{gf\})) = \{fg\}$, and
(2) if $g : D \to C$ is a ho-equivalence, then $\Phi(\Psi(\{g^{-1}f\})) = \{g^{-1}f\}$.

Let us prove (1). Since g is a map of $\Gamma\mathcal{C}$ we have $\Psi(\{gf\}) = [g] \circ \psi(\{f\})$ In addition, by Lemma 1.3.25 we have $\Phi([gf']) = \Phi([g]) \circ \Phi([f'])$, for every ho-morphism f'. We have:

$$\Phi(\Psi(\{gf\})) = \Phi([g] * \Psi(\{f\})) = \Phi([g]) \circ \{f\} = \{g\} \circ \{f\} = \{gf\}.$$

Let us prove (2). Let $h : C \rightsquigarrow D$ be a homotopy inverse of the ho-equivalence $g : D \to C$. Then $\Psi\{(g^{-1}f\}) = [h] * \Psi(\{f\})$. By Lemma 1.3.25 we can write $\{g\} = \Phi([g])$. Therefore

$$\{g\} \circ \Phi(\Psi(\{g^{-1}f\})) = \Phi([g]) \circ \Phi([h] * \Psi(\{f\})) = \Psi(\{f\}).$$

If we compose on the left by $\{g^{-1}\}$ we obtain

$$\Phi(\Psi(\{g^{-1}f\})) = \{g^{-1}\} \circ \{f\} = \{g^{-1}f\}.$$

$\square$

We next define a category $\pi^h\Gamma\mathcal{C}_{cof}$ having as objects the objects of $\Gamma\mathcal{C}_{cof}$ and as morphisms, the homotopy classes of ho-morphisms:

$$\pi^h\Gamma\mathcal{C}_{cof}(A, B) = [A, B]^h.$$

For $\pi^h\Gamma\mathcal{C}_{cof}(A, B)$ to be a category we need to prove that the operation $*$ defined in Lemma 1.4.2 is associative. We will use the bijection of sets

$$\Phi_{A,B} : [A, B]^h \rightleftarrows \mathcal{C}[\mathcal{H}^{-1}](A, B) : \Psi_{A,B}$$

to transfer the additivity of the composition of $\mathcal{C}[\mathcal{H}^{-1}]$.

Lemma 1.4.6. *Let A, B, C be objects of $\Gamma\mathcal{C}_{cof}$ and let $[f] \in [A, B]^h$ and $[g] \in [B, C]^h$ be homotopy classes of ho-morphisms. Then*

$$[g] * [f] = \Psi_{A,C} \left(\Phi_{B,C}([g]) \circ \Phi_{A,B}([f])\right).$$

Proof. Since A, B, C are objects of $\Gamma\mathcal{C}_{cof}$ the maps $\Psi_{A,-}$, $\Psi_{B,-}$ and $\Psi_{C,-}$ are well defined. For the rest of the proof we omit the subscripts of Ψ and Φ. By definition we have:

$$\Psi(\Phi([g]) \circ \Phi([f])) = \Psi(\{q_g p_g^{-1}\} \circ \{q_f p_f^{-1}\}) = [q_g] * ([\iota_g] * ([q_f] * [\iota_f])).$$

Since q_f and q_g are morphisms of $\Gamma\mathcal{C}$, we have $[q_g] * [\iota_g] = [q_g \iota_g] = [g]$, and $[q_f] * [\iota_f] = [q_f \iota_f] = [f]$. The result follows from (2) of Lemma 1.4.2. $\square$

Theorem 1.4.7. *The objects of $\Gamma\mathcal{C}_{cof}$ with the homotopy classes of ho-morphisms define a category $\pi^h\Gamma\mathcal{C}_{cof}$. There is an equivalence of categories*

$$\Phi : \pi^h\Gamma\mathcal{C}_{cof} \rightleftarrows \Gamma\mathcal{C}_{cof}[\mathcal{H}^{-1}, \Gamma\mathcal{C}] : \Psi$$

Proof. By Lemma 1.4.6 given ho-morphisms $f : A \rightsquigarrow B$, $g : B \rightsquigarrow C$ and $h_C \rightsquigarrow D$ between objects of $\Gamma\mathcal{C}_{cof}$, we have:

$$[h] * ([g] * [f]) = \Psi\left(\Phi([h]) \circ \Phi(\Psi(\Phi([g]) \circ \Phi([f])))\right).$$

By Proposition 1.4.5 we have $\Phi\Psi = 1$, and hence,

$$[h] * ([g] * [f]) = \Psi\left(\Phi([h]) \circ \Phi([g]) \circ \Phi([f])\right) = ([h] * [g]) * [f].$$

Therefore the composition of $\pi^h\Gamma\mathcal{C}_{cof}$ is associative. The equivalence of categories follows from Proposition 1.4.5. $\square$

A Cartan-Eilenberg Structure. We next prove that the objects of $\Gamma\mathcal{C}_{cof}$ are cofibrant in $(\Gamma\mathcal{C}, \mathcal{H}, \mathcal{W})$, and that every object has a left model in $\Gamma\mathcal{C}_{cof}$.

Lemma 1.4.8. *Let C be an object of $\Gamma\mathcal{C}_{cof}$. For every diagram*

$$\begin{tikzcd} & A \arrow[d, "w", "\wr"'] \\ C \arrow[ur, dotted] \arrow[r, squiggly, "f"] & B \end{tikzcd},$$

where w is a trivial fibration of $\Gamma\mathcal{C}$ and f is a ho-morphism, there exists a ho-morphism $g : C \rightsquigarrow A$ making the diagram commute.

PROOF. By the lifting property of $\mathcal{F}_i$-cofibrant objects, for each $i \in I$, there are morphisms $g_i : C_i \to A_i$ such that $w_i g_i = f_i$. We have

$$w_j g_j \varphi_u = f_j \varphi_u \overset{F_u}{\simeq} \varphi_u f_i = \varphi_u w_i g_i = w_j \varphi_u g_i.$$

Consider the commutative solid diagram

$$\begin{tikzcd} C_i \arrow[dr, dotted, "G_u"] \arrow[drr, bend left, "(g_j\varphi_u, \varphi_u g_i)"] \arrow[ddr, bend right, "F_u"'] & & \\ & P(A_j) \arrow[r, "(\delta^0_{A_j}, \delta^1_{A_j})"] \arrow[d, "P(w_j)"] & A_j \times A_j \arrow[d, "w_j \times w_j"] \\ & P(B_j) \arrow[r, "(\delta^0_{B_j}, \delta^1_{B_j})"'] & B_j \times B_j \end{tikzcd}.$$

Since w_j is a trivial filtration, by Lemma 1.2.25 there exists a dotted arrow G_u, making the diagram commute. The family $g = (g_i, G_u)$ is a ho-morphism, and $wg = f$. $\square$

Proposition 1.4.9. *Let C be an object of $\Gamma\mathcal{C}_{cof}$ and let $w : A \to B$ be a weak equivalence in $\Gamma\mathcal{C}$. The map*

$$w_* : [C, A]^h \longrightarrow [C, B]^h$$

defined by $[f] \mapsto [wf]$ is a bijection.

Proof. We first prove surjectivity. Let $f : C \rightsquigarrow B$ be a ho-morphism representing $[f] \in [C, B]^h$. By Corollary 1.3.9 the map w factors as a homotopy equivalence ι_w followed by a trivial fibration q_f, giving rise to a solid diagram

$$\begin{tikzcd} & A \arrow[d, shift left, "\iota_w"] \arrow[dd, bend left=60, "w"] \\ & \mathcal{P}(w) \arrow[u, shift left, "p_w"] \arrow[d, "q_w", "\wr"'] \\ C \arrow[ur, dotted, "g'"] \arrow[r, squiggly, "f"'] & B \end{tikzcd},$$

where q_w is a trivial fibration. By Lemma 1.4.8 there exists a ho-morphism $g' : C \rightsquigarrow \mathcal{P}(w)$ such that $q_w g' = f$. Let $g := p_w g'$. We have

$$wg = q_w \iota_w g = q_w \iota_w p_w g' \simeq q_w g' = f.$$

Therefore $[wg] = [f]$, and w_* is surjective.

To prove injectivity, let $g, g' : C \rightsquigarrow A$ be two ho-morphisms, representing $[g]$ and $[g']$ respectively and let $h : wg \simeq wg'$ be a homotopy. Let $\mathcal{P}(w, w)$ denote the double mapping path of w, defined by the fibre product

$$\begin{tikzcd} \mathcal{P}(w_i, w_i) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P(B_i) \arrow[d, "(\delta^0_{B_i}, \delta^1_{B_i})"] \\ A_i \times A_i \arrow[r, "w_i \times w_i"] & B_i \times B_i \end{tikzcd},$$

for each $i \in I$, together with the comparison morphism

$$\psi_u = ((\varphi_u \times \varphi_u)\pi_1, P(\varphi_u)\pi_2),$$

for each $u : i \to j$.

The triple (g, g', h) defines a ho-morphism $\gamma : C \rightsquigarrow \mathcal{P}(w, w)$. Indeed, for all $i \in I$, let γ_i be the map defined by the pull-back diagram:

$$\begin{tikzcd}
C_i \arrow[dr, dotted, "\gamma_i"] \arrow[drr, bend left, "h_i"] \arrow[ddr, bend right, "(g_i, g_i')"'] & & \\
& \mathcal{P}(w_i, w_i) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P(B_i) \arrow[d, "(\delta^0_{B_i}, \delta^1_{B_i})"] \\
& A_i \times A_i \arrow[r, "w_i \times w_i"'] & B_i \times B_i
\end{tikzcd},$$

and for all $u : i \to j$ let Γ_u be the map defined by the pull-back diagram:

$$\begin{tikzcd}
C_i \arrow[dr, dotted, "\Gamma_u"] \arrow[drr, bend left, "H_u"] \arrow[ddr, bend right, "(G_u, G_u')"'] & & \\
& P(\mathcal{P}(w_j, w_j)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P^2(B_j) \arrow[d, "(P(\delta^0_{B_j}), P(\delta^1_{B_j}))"] \\
& P(A_j) \times P(A_j) \arrow[r, "P(w_j \times w_j)"'] & P(B_j) \times P(B_j)
\end{tikzcd}.$$

Then the family $\gamma = (\gamma_i, \Gamma_u)$ is a ho-morphism of diagrams. Indeed,

$$\delta^0_{\mathcal{P}(w_j, w_j)} \Gamma_u = ((\delta^0_{A_j} G_u, \delta^0_{A_j} G_u'), \delta^0_{P(B_j)} H_u) = ((g_j \varphi_u, g_j' \varphi_u), h_j \varphi_u) = \gamma_j \varphi_u,$$
$$\delta^1_{\mathcal{P}(w_j, w_j)} \Gamma_u = ((\delta^1_{A_j} G_u, \delta^1_{A_j} G_u'), \delta^1_{P(B_j)} H_u) = ((\varphi_u g_i, \varphi_u g_i'), P(\varphi_u) h_i)) = \psi_u \gamma_i.$$

Consider the solid diagram

$$\begin{tikzcd}
& P(A) \arrow[d, "\overline{w}", "\wr"'] \\
C \arrow[ur, dotted, "\gamma'"] \arrow[r, squiggly, "\gamma"] & \mathcal{P}(w, w)
\end{tikzcd}.$$

By Lemma 1.2.22 the map $\overline{w}$ defined level-wise by $\overline{w_i} = ((\delta^0_{A_i}, \delta^1_{A_i}), P(w_i))$ is a weak equivalence. Hence $\bar{w}_*$ is surjective, and there exists a dotted arrow γ' such that $\overline{w}\gamma' \simeq \gamma$. It follows that $g \simeq \delta^0_A \gamma' \simeq \delta^1_A \gamma' \simeq g'$, and hence $[g] = [g']$. $\square$

Corollary 1.4.10. *The objects of* $\Gamma\mathcal{C}_{cof}$ *are cofibrant in* $(\Gamma\mathcal{C},\mathcal{H},\mathcal{W})$*, that is: for every weak equivalence* $w : A \to B$*, and every object* $C \in \Gamma\mathcal{C}_{cof}$ *the induced map*

$$w_* : \Gamma\mathcal{C}[\mathcal{H}^{-1}](C,A) \longrightarrow \Gamma\mathcal{C}[\mathcal{H}^{-1}](C,B)$$

is a bijection.

Proof. Let $w : A \to B$ be a weak equivalence, and let C be an object of $\Gamma\mathcal{C}_{cof}$. By Lemmas 1.3.24 and 1.3.25, and Proposition 1.4.5 the diagram

$$\begin{tikzcd} \Gamma\mathcal{C}[\mathcal{H}^{-1}](C,A) \arrow[r, "\{w\}\circ-"] \arrow[d, "\Psi"] & \Gamma\mathcal{C}[\mathcal{H}^{-1}](C,B) \\ {[C,A]^h} \arrow[r, "{[w]\circ-}"] & {[C,B]^h} \arrow[u, "\Phi"'] \end{tikzcd}$$

commutes, and the vertical arrows are bijective. By Proposition 1.4.9 the bottom arrow is a bijection. Therefore the top arrow is a bijection. $\square$

Theorem 1.4.11. *Let* $\Gamma\mathcal{C}$ *be a diagram category indexed by a directed category* I *as in* 1.3.4. *Assume that for each* $i \in I$*, the categories* $\mathcal{C}_i$ *are* P*-categories with* $\mathcal{F}_i$*-cofibrant models, and that the functors* $u_* : \mathcal{C}_i \to \mathcal{C}_j$ *are compatible with the* P*-category structures sending* $\mathcal{F}_i$*-cofibrant objects to* $\mathcal{F}_j$*-cofibrant objects. Then* $(\Gamma\mathcal{C},\mathcal{H},\mathcal{W})$ *is a Cartan-Eilenberg category. There is an equivalence of categories*

$$\pi^h\Gamma\mathcal{C}_{cof} \xrightarrow{\sim} \Gamma\mathcal{C}[\mathcal{W}^{-1}].$$

Proof. Let $\rho_i : C_i \to A_i$ be $\mathcal{F}_i$-cofibrant models in $\mathcal{C}_i$. By the lifting property, for each $u : i \to j$ there exists a morphism $\varphi_u : C_i \to C_j$, together with a homotopy $R_u : \varphi_u\rho_i \simeq \rho_j\varphi_u$. We obtain a diagram of $\Gamma\mathcal{C}_{cof}$

$$C = \left(C_i \overset{\varphi}{\dashrightarrow} C_j \right).$$

The family $\rho = (\rho_i, R_u)$ is a ho-morphism from C to A, which by construction is a weak equivalence. Then $\Phi_{C,A}(\rho) : C \to A$ is a (left) model of A. By Corollary 1.4.10, C is Cartan-Eilenberg cofibrant. The equivalences of categories

$$\pi^h\Gamma\mathcal{C}_{cof} \xrightarrow{\sim} \Gamma\mathcal{C}_{cof}[\mathcal{H}^{-1},\Gamma\mathcal{C}] \xrightarrow{\sim} \Gamma\mathcal{C}[\mathcal{W}^{-1}].$$

follow from Theorems 1.4.7 and 1.1.35 respectively. $\square$

Since weak equivalences in $\Gamma\mathcal{C}$ are defined level-wise, the following version with minimal models is straightforward. Denote by $\Gamma\mathcal{C}_{min}$ the full subcategory of $\Gamma\mathcal{C}$ of those diagrams A such that A_i is $\mathcal{F}_i$-minimal in $\mathcal{C}_i$, for all $i \in I$.

Theorem 1.4.12. *Let $\Gamma\mathcal{C}$ be a diagram category indexed by a directed category I as in* 1.3.4. *Assume that for each $i \in I$, the categories $\mathcal{C}_i$ are P-categories with $\mathcal{F}_i$-minimal models, and that the functors $u_* : \mathcal{C}_i \to \mathcal{C}_j$ are compatible with the P-category structures sending $\mathcal{F}_i$-minimal objects to $\mathcal{F}_j$-minimal objects. Then $(\Gamma\mathcal{C}, \mathcal{H}, \mathcal{W})$ is a Sullivan category. There is an equivalence of categories*

$$\pi^h\Gamma\mathcal{C}_{min} \xrightarrow{\sim} \Gamma\mathcal{C}[\mathcal{W}^{-1}].$$

To end this chapter we consider a situation in which the category under study is a full subcategory of a category of diagrams. This is a generalization of Lemma 1.2.34 and will be of use for the applications to mixed Hodge theory.

Lemma 1.4.13. *Assume that $\Gamma\mathcal{C}$ has a level-wise P-category structure. Denote by $\mathcal{W}$ the class of weak equivalences and let $\mathcal{H}$ denote the closure by composition of ho-equivalences. Let $\mathcal{D}$ be a full subcategory of $\Gamma\mathcal{C}$ such that:*

(i) Given a weak equivalence $A \overset{\sim}{\rightsquigarrow} B$ in $\Gamma\mathcal{C}$, then A is an object of $\mathcal{D}$ if and only if B is so.
(ii) For every object D of $\mathcal{D}$ there is an object $C \in \mathcal{D}_{cof} := \mathcal{D} \cap \Gamma\mathcal{C}_{cof}$, together with a ho-morphism $C \rightsquigarrow D$, which is a weak equivalence.

Then the triple $(\mathcal{D}, \mathcal{H}, \mathcal{W})$ is a Cartan-Eilenberg category with cofibrant models in $\mathcal{D}_{cof}$, and there are equivalences of categories

$$\pi^h\mathcal{D}_{cof} \longrightarrow \mathcal{D}_{cof}[\mathcal{H}^{-1}, \mathcal{D}] \xrightarrow{\sim} \mathcal{D}[\mathcal{W}^{-1}].$$

PROOF. By (i), the mapping path $\mathcal{P}^h(f)$ of a ho-morphism between objects of $\mathcal{D}$ is an object of $\mathcal{D}$. Hence by Proposition 1.3.20 the assignation $f \mapsto q_f p_f^{-1}$ gives rise to a well defined map

$$\Phi_{C,D} : \mathcal{D}^h(C, D) \longrightarrow \mathcal{D}[\mathcal{H}^{-1}](C, D),$$

which preserves weak equivalences. By Theorem 1.4.7 there is an equivalence of categories

$$\Phi : \pi^h\mathcal{D}_{cof} \rightleftarrows \mathcal{D}_{cof}[\mathcal{H}^{-1}, \mathcal{D}] : \Psi.$$

By Proposition 1.4.9 every object of $\mathcal{D}_{cof}$ is Cartan-Eilenberg cofibrant in $(\mathcal{D}, \mathcal{H}, \mathcal{W})$. Therefore, to prove that the triple $(\mathcal{D}, \mathcal{H}, \mathcal{W})$ is a Cartan-Eilenberg category, it suffices to prove that every object D of $\mathcal{D}$ has a model in $\mathcal{D}_{cof}$. By (ii), for every object D of $\mathcal{D}$, there exists a weak equivalence $\rho : C \rightsquigarrow D$, with $C \in \mathcal{D}_{cof}$. The morphism $\Phi_{C,D}(\rho) : C \to D$ of $\mathcal{D}[\mathcal{H}^{-1}]$ is an isomorphism in $\mathcal{D}[\mathcal{W}^{-1}]$. Therefore $(\mathcal{D}, \mathcal{H}, \mathcal{W})$ is a Cartan-Eilenberg category. $\square$

There is an analogous version of Lemma 1.4.13 with cofibrant minimal models.

CHAPTER 2

Filtered Derived Categories

The category of filtered objects $\mathbf{F}\mathcal{A}$ of an abelian category $\mathcal{A}$ is not abelian in general. Therefore the classical theory of derived categories of Verdier [**Ver96**] does not apply in this case. There have been several alternative approaches to address the study of filtered complexes. First, Illusie defined the derived category of a filtered abelian category in an ad hoc scheme, following the classical theory of abelian categories (see Chapter V of [**Ill71**]). The theory of exact categories of Quillen [**Qui73**] allows another approach, which is detailed in the work of Laumon [**Lau83**]. In this chapter we study the derived category of $\mathbf{F}\mathcal{A}$ within the axiomatic framework of Cartan-Eilenberg categories. This paves the way in two directions: the study of mixed Hodge complexes of Chapter 3, and the study of filtered differential graded algebras of Chapter 4.

In Section 1 we collect some preliminaries on homological algebra: we review the homotopy theory of an additive category, and the theory of exact categories. Then, we provide the main definitions and results regarding filtered objects, following mainly [**Del71b**]. Following [**Kel96**], we describe the filtered derived category of an abelian category in terms of exact categories, and interpret the main results in the context of Cartan-Eilenberg categories.

In Section 2 we study higher filtered derived categories. In order to deal with the weight filtration, in [**Del71b**] Deligne introduced the décalage of a filtered complex, which shifts the associated spectral sequence of the original filtered complex by one stage. We review the main properties of Deligne's décalage functor and use them to study the localized category of filtered complexes with respect to the class of E_r-quasi-isomorphisms via the construction of higher filtered injective cofibrant models.

In Section 3 we restrict our study to filtered complexes of vector spaces over a field. In this case, every object is injective and projective, and the classical calculus of filtered derived categories does not provide any additional information. However, we can consider filtered minimal models and study the r-derived category from the viewpoint of Sullivan categories. At the end of the section, we study d-strict filtered complexes, and some consequences of the degeneration of the spectral sequences on the minimal model, which will be of use in the applications to mixed Hodge Theory.

In the last section we generalize the results of the previous sections, to bifiltered complexes.

2.1. Preliminaries

In this preliminary section we review the basic notions and results of additive categories, exact categories and filtered abelian categories. Using the theory of exact categories, we provide a description of the filtered derived category of an abelian category.

Additive Categories. For the rest of this section let $\mathcal{A}$ be an additive category, and denote by $\mathbf{C}^\alpha(\mathcal{A})$ the category of cochain complexes of objects of $\mathcal{A}$, where α denotes the boundedness condition ($+$ and $-$ for bounded below and above respectively, b for bounded and $\emptyset$ for unbounded).

The following constructions will be useful to study the homotopy theory of complexes over $\mathcal{A}$ (see for example [**GM03**], Section III.3.2).

Definition 2.1.1. The *translation* of a complex K is the complex $K[1]$ defined by $K[1]^n = K^{n+1}$ with the differential $d^n_{K[1]} = -d^{n+1}_K$. This defines an autoequivalence

$$T : \mathbf{C}^+(\mathcal{A}) \longrightarrow \mathbf{C}^+(\mathcal{A}).$$

Definition 2.1.2. Let $f, g : K \to L$ be morphisms of complexes. A *homotopy* from f to g is a map $h : K \to L[-1]$ such that $dh + hd = g - f$. We denote $h : f \simeq g$, and say that f is *homotopic* to g.

The additive operation between morphisms of complexes makes the homotopy relation into an equivalence relation compatible with the composition.

Definition 2.1.3. Let $f : K \to L$ and $g : K \to M$ be two morphisms of complexes. The *double mapping cylinder of f and g* is the complex $\mathcal{C}yl(f, g)$ given by the direct sum

$$\mathcal{C}yl(f, g) = K[1] \oplus L \oplus M,$$

with the differential

$$D = \begin{pmatrix} -d & 0 & 0 \\ -f & d & 0 \\ g & 0 & d \end{pmatrix}.$$

The following result is straightforward.

Lemma 2.1.4. *Given morphisms $f : K \to L$ and $g : K \to M$, then*

$$\mathrm{Hom}(\mathcal{C}yl(f, g), X) = \{(\alpha, \beta, h);\ \alpha : L \to X, \beta : M \to X, h : \alpha f \simeq \beta g\}$$

for any complex X.

Definition 2.1.5. Let $f : K \to L$ be a morphism of complexes.

(1) The *mapping cylinder of f* is the complex defined by

$$\mathcal{C}yl(f) := \mathcal{C}yl(f, 1_K) = K[1] \oplus L \oplus K.$$

There is a commutative diagram of morphisms of complexes

$$\begin{tikzcd} K \arrow[r, "i_f"] \arrow[dr, "f"'] & \mathcal{C}yl(f) \arrow[d, "p_f"] & L \arrow[l, "j_f"'] \arrow[dl, equal] \\ & L & \end{tikzcd}$$

defined by $i_f(x) = (0, 0, x)$, $j_f(x) = (0, x, 0)$ and $p_f(x, y, z) = y + f(z)$.

(2) The *mapping cone of f* is the complex defined by

$$C(f) := \mathcal{C}yl(0, f) = K[1] \oplus L.$$

There is a short exact sequence

$$0 \to L \to C(f) \to K[1] \to 0.$$

Note that for every complex X,

$$\mathrm{Hom}(C(f), X) = \{(\beta, h);\ \beta : L \to X, h : 0 \simeq \beta g\}.$$

Definition 2.1.6. The *cylinder* of a complex K, is the complex defined by

$$\mathrm{Cyl}(K) := \mathcal{C}yl(1_K, 1_K) = \mathcal{C}yl(1_K).$$

The cylinder is functorial for morphisms of complexes. Denote by

$$\begin{tikzcd} K \arrow[r, "i_K"] \arrow[dr, equal] & \mathrm{Cyl}(K) \arrow[d, "p_K"] & K \arrow[l, "j_K"'] \arrow[dl, equal] \\ & K & \end{tikzcd}$$

the corresponding morphisms.

The following well known result states that notion of homotopy of Definition 2.1.2 coincides with the notion of homotopy defined by the functorial cylinder (see Definition 1.2.2 for a dual definition).

Corollary 2.1.7. *Let $f, g : K \to L$ be morphisms of complexes. A homotopy $h : K \to L[-1]$ from f to g is equivalent to a morphism of complexes $\overline{h} : \mathrm{Cyl}(K) \to L$ satisfying $\overline{h} j_K = f$ and $\overline{h} i_K = g$.*

PROOF. It is a consequence of Lemma 2.1.4. □

Denote by $[K, L]$ the set of equivalence classes of morphisms of complexes from K to L modulo homotopy, and let

$$\mathbf{K}^{\alpha}(\mathcal{A}) := \mathbf{C}^{\alpha}(\mathcal{A})/ \simeq$$

be the corresponding quotient category. Denote by $\mathcal{S}$ the class of *homotopy equivalences*: these are morphisms $f : K \to L$ such that there exists a morphism of complexes $g : L \to K$, together with homotopies $fg \simeq 1_L$ and $gf \simeq 1_K$. An important property of the cylinder is the following.

Proposition 2.1.8. *The morphism $p_K : \mathrm{Cyl}(K) \to K$ is a homotopy equivalence, for every complex K.*

Proof. A homotopy $h : \mathrm{Cyl}(K) \to \mathrm{Cyl}(K)[-1]$ from $j_K p_K$ to 1 is given by $(x, y, z) \mapsto (z, 0, 0)$. By definition, $p_K j_K = 1$. □

Corollary 2.1.9. *The inclusion induces an equivalence of categories*

$$\mathbf{K}^\alpha(\mathcal{A}) \xrightarrow{\sim} \mathbf{C}^\alpha(\mathcal{A})[\mathcal{S}^{-1}].$$

Proof. Let $f, g : K \to L$ be morphisms of complexes, and let $h : f \simeq g$ be a homotopy from f to g. By Corollary 2.1.7, this defines a morphism $\overline{h} : \mathrm{Cyl}(K) \to L$ such that the diagram

$$\begin{tikzcd} & K \arrow[dl, equal] \arrow[d, "j_K"] \arrow[dr, "f"] & \\ K & \mathrm{Cyl}(K) \arrow[l, "p_K"'] \arrow[r, "\overline{h}"] & L \\ & K \arrow[ul, equal] \arrow[u, "i_K"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes. By Proposition 2.1.8 the map p_K is a homotopy equivalence. Therefore this is a hammock between $\mathcal{S}$-zig-zags f and g. The result follows from Proposition 1.1.10. □

Remark 2.1.10. Both the notion of homotopy, and the double mapping cylinder, depend on the translation functor, which is an additive automorphism of the category of complexes. Different choices of this functor lead to distinct notions of homotopy. An example is provided by filtered categories, in which a shift by r on the filtration of the translation leads to the different notions of r-homotopy, suitable to study of the derived category with respect to E_r-quasi-isomorphisms (see Definition 2.2.16).

Exact Categories. We next introduce the notion of an exact category and review the main results regarding derived categories of exact categories. We mainly follow [**Büh10**] and [**Kel96**].

Definition 2.1.11. Let $\mathcal{A}$ be an additive category. A pair of morphisms

$$A \xrightarrow{i} B \xrightarrow{p} C$$

in $\mathcal{A}$ is *exact* if i is a kernel of p and p is a cokernel of i. The map i is said to be an *admissible monomorphism*, and p is an *admissible epimorphism.*

Definition 2.1.12. An *exact category* is an additive category $\mathcal{A}$, together with a class of exact pairs of $\mathcal{A}$, closed under isomorphisms, and satisfying the following axioms:

(E_0) For all objects A of $\mathcal{A}$, the identity morphism 1_A is an admissible monomorphism (resp. epimorphism).

(E_1) The class of admissible monomorphisms (resp. epimorphisms) is closed under composition.

(E_2) The push-out (resp. pull-back) of an admissible monomorphism (resp. epimorphism) always exists and is an admissible monomorphism (resp. epimorphism).

We shall also assume the following condition:

(E_3) Every morphism $f : A \to B$ of $\mathcal{A}$ has a kernel and a coimage. The sequence $\mathrm{Ker} f \to A \to \mathrm{Coim} f$ is an exact pair of $\mathcal{A}$.

Remark 2.1.13. Condition (E_3) is a strong assumption which makes calculation in exact categories significantly easier (see 1.3.0 of [**Lau83**]). In particular, it implies that the category is idempotent complete (see Definition 6.1 of [**Büh10**]).

Definition 2.1.14. A morphism $f : A \to B$ in an exact category $\mathcal{A}$ is called *admissible* if it factors as $f = ip$, where i is an admissible monomorphism and p is an admissible epimorphism.

Example 2.1.15. If $\mathcal{A}$ is abelian, the pairs of maps $A \to B \to C$ that fit into an exact sequence $0 \to A \to B \to C \to 0$, define an exact structure on $\mathcal{A}$. The admissible monomorphisms (resp. epimporphisms) are just the monomorphisms (resp. epimorphisms), and every morphism is admissible, with $A \xrightarrow{i} A/\mathrm{Ker} f \cong \mathrm{Im} f \xrightarrow{p} B$.

Definition 2.1.16 ([**Kel96**], Sect. 5)**.** An object I of an exact category $\mathcal{A}$ is said to be *injective* if for any admissible monomorphism $i : A \to B$, the induced morphism

$$i^* : \mathcal{A}(B, I) \longrightarrow \mathcal{A}(A, I)$$

is surjective. Denote by $\mathrm{Inj}\mathcal{A}$ the full subcategory of injective objects of $\mathcal{A}$.

Definition 2.1.17. An exact category $\mathcal{A}$ *has enough injective objects* if for any object A of $\mathcal{A}$, there exists an injective object I, together with an admissible monomorphism $A \to I$.

In abelian categories, weak equivalences are usually defined to be cochain maps inducing an isomorphism in cohomology. In a category of complexes of an arbitrary exact category there is no notion of cohomology. However, one can define weak equivalences, based on the notion of acyclic complex.

Definition 2.1.18. Let $\mathcal{A}$ be an exact category. A complex K of $\mathbf{C}^+(\mathcal{A})$ is called *acyclic* if the differentials $d^n : K^n \to K^{n+1}$ of K factor as

$$K^n \xrightarrow{p^n} Z^{n+1} \xrightarrow{i^{n+1}} K^{n+1}$$

in such a way that the sequence

$$Z^n \xrightarrow{i^n} K^n \xrightarrow{p^n} Z^{n+1}$$

is an exact pair of $\mathcal{A}$. In particular, d^n is an admissible morphism of $\mathcal{A}$.

Remark 2.1.19 (See Lemma 1.2.2 of [**Lau83**])**.** Condition (E_3) implies that a complex K is acyclic if and only if for all $n \geq 0$, the morphism

$$d^n : K^n \longrightarrow Z^{n+1}(K) := \mathrm{Ker}\ d^{n+1}$$

is an admissible epimorphism.

Definition 2.1.20. A morphism $f : K \to L$ of $\mathbf{C}^+(\mathcal{A})$ is called *weak equivalence* if its mapping cone $C(f)$ is an acyclic complex.

Denote by $\mathcal{W}$ the class of weak equivalences of $\mathbf{C}^+(\mathcal{A})$.

Lemma 2.1.21 ([**Büh10**], Prop. 10.9)**.** *If an exact category $\mathcal{A}$ satisfies (E_3), then every homotopy equivalence is a weak equivalence. In particular the triple $(\mathbf{C}^+(\mathcal{A}), \mathcal{S}, \mathcal{W})$ is a category with strong and weak equivalences.*

Propositions 2.1.22 and 2.1.23 below are generalized versions for exact categories, of the corresponding well-known results for the category of complexes of an abelian category (see for example Theorems 6.1 and 6.2 of [**Ive86**]).

Injective complexes satisfy the following fibrant-type property.

Proposition 2.1.22. *Let I be a complex of $\mathbf{C}^+(\mathrm{Inj}\mathcal{A})$. Every weak equivalence of complexes $w : K \xrightarrow{\sim} L$ induces a bijection*

$$w^* : [L, I] \longrightarrow [K, I]$$

between homotopy classes of maps.

PROOF. Consider the homotopy exact sequence induced by $[-, I]$

$$\cdots \to [C(w), I] \to [L, I] \to [K, I] \to [C(w)[-1], I] \to \cdots$$

If suffices to see that $[C(w), J] = 0$ for any injective complex J. Since $C(w)$ is acyclic, this follows from Lemma 4.1.a of [**Kel90**]. $\square$

In particular, every weak equivalence between bounded below complexes of injective objects, is a homotopy equivalence.

The existence of enough injective objects can be found in [**Kel90**], Lemma 4.1.b. See also the dual version in [**Büh10**], Theorem 12.7.

Proposition 2.1.23. *Let $\mathcal{A}$ be an exact category with enough injectives. For every complex K in $\mathbf{C}^+(\mathcal{A})$, there exists a complex $I \in \mathbf{C}^+(\mathrm{Inj}\mathcal{A})$, together with a weak equivalence $K \xrightarrow{\sim} I$.*

Corollary 2.1.24 (cf. [**Pas11**], Prop. 4.4.1)**.** *Let $\mathcal{A}$ be an exact category with enough injectives. The triple $(\mathbf{C}^+(\mathcal{A}), \mathcal{S}, \mathcal{W})$ is a (right) Cartan-Eilenberg category, and $\mathbf{C}^+(\mathrm{Inj}\mathcal{A})$ is a full subcategory of fibrant models. The inclusion induces an equivalence of categories*

$$\mathbf{K}^+(\mathrm{Inj}\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^+(\mathcal{A}) := \mathbf{C}^+(\mathcal{A})[\mathcal{W}^{-1}].$$

Filtered Abelian Categories. We outline some algebraic preliminaries about filtered objects and filtered complexes of an abelian category $\mathcal{A}$. The basic reference is [**Del71b**].

Definition 2.1.25. A *decreasing filtration* F of an object A of $\mathcal{A}$ is a sequence of sub-objects of A, indexed by the integers,

$$0 \subseteq \cdots \subseteq F^{p+1} \subseteq F^p A \subseteq \cdots \subseteq A.$$

An *increasing filtration* W of A is a sequence of sub-objects of A, indexed by the integers,

$$0 \subseteq \cdots \subseteq W_{p-1}A \subseteq W_pA \subseteq \cdots \subseteq A.$$

Given a decreasing filtration F we can define an increasing filtration by setting $F_pA = F^{-p}A$. Consequently, properties and results stated for one type of filtrations have obvious analogues for the other type.

We shall always assume that the filtrations are *finite*: for any filtered object (A, F) there exist $p, q \in \mathbb{Z}$ such that $F^pA = A$ and $F^qA = 0$.

Definition 2.1.26. A *filtered morphism* $f : (A, F) \to (B, F)$ is a morphism $f : A \to B$ in $\mathcal{A}$ which is *compatible with filtrations*:

$$f(F^pA) \subset F^pB, \text{ for all } p \in \mathbb{Z}.$$

The category $\mathbf{F}\mathcal{A}$ of finitely filtered objects of an abelian category $\mathcal{A}$ is additive, and has finite limits and colimits. Therefore kernel, images and their dual notions exist. However, it is not abelian in general.

A particular class of filtered morphisms plays an important role in filtered categories.

Definition 2.1.27. A morphism $f : (A, F) \to (B, F)$ is said to be *strictly compatible with the filtrations* or *strict* if the canonical morphism

$$\mathrm{Coim} f \longrightarrow \mathrm{Im} f$$

is an isomorphism of filtered objects. (cf. Proposition 2.1.31).

2.1.28. Let $j : X \hookrightarrow A$ be a sub-object of A. A filtration F on A induces a filtration on X by

$$F^pX = j^{-1}(F^pV) = F^pA \cap X.$$

This is the only filtration on X such that j is strictly compatible with the filtrations. Dually, the filtration on the quotient space A/X is given by

$$F^p(A/X) = \pi(F^pA) \cong (X + F^pA)/X \cong F^pA/(X \cap F^pA).$$

This is the only filtration on V/X such that the projection $\pi : A \to A/X$ strictly compatible with the filtrations.

Lemma 2.1.29 ([**Del71b**], Lemma 1.1.9)**.**

(1) If $X_1 \subset X_2 \subset A$, and A has a filtration, then the two induced filtrations on the quotient X_2/X_1 agree.

(2) If $\Sigma : A \xrightarrow{f} B \xrightarrow{g} C$ is a 0-sequence and B has a filtration, then

$$H(\Sigma) = \mathrm{Ker}g/\mathrm{Im}f = \mathrm{Ker}(\mathrm{Coker}f \to \mathrm{Coim}g)$$

has a canonically induced filtration.

Definition 2.1.30. The *associated p-graded object $Gr_F^p A$* of a filtered object (A, F) is the object of $\mathcal{A}$ defined by the quotient

$$Gr_F^p A = F^p A / F^{p+1} A.$$

Every morphism of filtered objects $f : (A, F) \to (B, F)$ induces morphisms $Gr_F^p f : Gr_F^p A \to Gr_F^p B$ between their associated p-graded objects.

Proposition 2.1.31 ([**Del71b**], Prop. 1.1.11)**.**

(1) A morphism $f : (A, F) \to (B, F)$ is strict if and only if the sequence

$$0 \to Gr_F^p \mathrm{Ker}f \to Gr_F^p A \to Gr_F^p B \to Gr_F^p \mathrm{Coker}f \to 0$$

is exact for all $p \in \mathbb{Z}$.

(2) Let $\Sigma : (A, F) \to (B, F) \to (C, F)$ be a 0-sequence of filtered objects, in which both morphisms are strict. For all $p \in \mathbb{Z}$, there is a canonical isomorphism

$$H(Gr_F^p \Sigma) \cong Gr_F^p H(\Sigma).$$

Definition 2.1.32. A *filtered complex* is a pair (K, F), where K is a cochain complex, and F is a decreasing filtration of sub-complexes of K.

Since we assume that all filtrations are finite, every filtered complex we shall consider is biregularly filtered. Denote by $\mathbf{C}^+(\mathbf{F}\mathcal{A})$ the category of bounded below (biregularly) filtered complexes of $\mathcal{A}$.

By Lemma 2.1.29 the cohomology $H(K)$ of a filtered complex (K, F) receives a decreasing filtration induced from F:

$$F^p H(K) = \mathrm{Im}\left\{H(F^p K) \to H(K)\right\}.$$

Therefore $(H(K), F)$ is a filtered complex with trivial differential. However, as we shall next see, this is not the suitable object to detect the interesting weak equivalences.

Example 2.1.33. The *bête filtration* σ of a complex K is the decreasing filtration obtained by placing 0 in degrees $< p$, while keeping K^n in all other degrees,

$$\sigma^{\geq p}K = \{0 \to 0 \cdots \to 0 \to K^p \to K^{p+1} \to \cdots\}.$$

The p-graded complex associated with (K, σ) is K^p in weight p, and 0 elsewhere, so $Gr^p_\sigma K = K^p[-p]$.

Example 2.1.34. The *canonical filtration* τ of a complex K is the increasing filtration defined by truncation:

$$\tau_{\leq p}K = \{\cdots \to K^{p-1} \to \operatorname{Ker} d \to 0 \to 0 \to \cdots\}.$$

The p-graded complex associated with (K, τ) being

$$0 \to K^{p-1}/\operatorname{Ker} d \to \operatorname{Ker} d \to 0.$$

There is a quasi-isomorphism $Gr^\tau_p K \stackrel{\sim}{\longrightarrow} H^p(K)[-p]$.

Definition 2.1.35. A morphism $f : (K, F) \to (L, F)$ of filtered complexes is called a *filtered quasi-isomorphism* if, for all $p \in \mathbb{Z}$, the induced morphisms

$$H^n(Gr^p_F f) : H^n(Gr^p_F K) \to H^n(Gr^p_F L)$$

are isomorphisms.

Denote by $\mathcal{E}$ the class of filtered quasi-isomorphisms. Since the filtrations are biregular, every filtered quasi-isomorphism is a quasi-isomorphism. The converse is not true in general.

Since $\mathbf{F}\mathcal{A}$ is additive, there is a notion of filtered homotopy, defined by the filtered translation functor.

Definition 2.1.36. The *filtered translation* of a complex (K, F) is the filtered complex $(K[1], F)$ defined by $F^pK[1]^n = F^pK^{n+1}$. This defines an autoequivalence

$$T : \mathbf{C}^+(\mathbf{F}\mathcal{A}) \longrightarrow \mathbf{C}^+(\mathbf{F}\mathcal{A}).$$

Denote by $\mathcal{S}$ the class of filtered homotopy equivalences. We have $\mathcal{S} \subset \mathcal{E}$. Therefore the $(\mathbf{C}^+(\mathbf{F}\mathcal{A}), \mathcal{S}, \mathcal{E})$ is a category with strong and weak equivalences.

To end this preliminary section we introduce some notation and results regarding the filtered mapping cone, which will be of use in the sequel.

2.1.37. Let $f : (K, F) \to (L, F)$ be a map of filtered complexes. The filtered mapping cone of f is given by

$$F^pC(f) = F^pK^{n+1} \oplus F^pL^n.$$

By construction, the maps in the exact sequence

$$\Sigma : \ 0 \longrightarrow L \xrightarrow{i} C(f) \xrightarrow{\pi} K[1] \longrightarrow 0,$$

are all strictly compatible with filtrations, and for all $p \in \mathbb{Z}$, we have:

$$F^pC(f) = C(F^pf), \text{ and } Gr_F^pC(f) = C(Gr_F^pf).$$

Therefore the corresponding filtered and graded sequences $F^p\Sigma$ and $Gr_F^p\Sigma$ are exact. These exact sequences are, in turn, related via the exact sequences induced by

$$0 \longrightarrow F^{p+1} \xrightarrow{i} F^p \xrightarrow{\pi} Gr_F^p \longrightarrow 0.$$

We have a commutative diagram in which rows and columns are exact:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & Gr_F^pL \arrow[r] \arrow[u] & Gr_F^pC(f) \arrow[r] \arrow[u] & Gr_F^pK[1] \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & F^pL \arrow[r] \arrow[u] & F^pC(f) \arrow[r] \arrow[u] & F^pK[1] \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & F^{p+1}L \arrow[r] \arrow[u] & F^{p+1}C(f) \arrow[r] \arrow[u] & F^{p+1}K[1] \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

From the above diagram, it is straightforward that a morphism of filtered complexes $f : (K, F) \to (L, F)$ is a filtered quasi-isomorphism if and only if the complex $Gr_F^p C(f)$ is acyclic for all $p \in \mathbb{Z}$.

Lemma 2.1.38. *Let $f : (K, F) \to (L, F)$ be a morphism of filtered complexes, and let $p \in \mathbb{Z}$. The following are equivalent.*

(1) The map $F^q f : F^q A \to F^q B$ is a quasi-isomorphism for all $q > p$.
(2) The map $Gr_F^q f : Gr_F^q A \to Gr^q FB$ is a quasi-isomorphism for all $q > p$.
(3) The map $\pi : F^q C(f) \to Gr_F^q C(f)$ is a quasi-isomorphism, for all $q \geq p$.

Proof. (1) $\Rightarrow$ (2). If $F^q f$ is a quasi-isomorphism, then the complex $C(F^q f) = F^q C(f)$ is acyclic. From the short exact sequence

$$\Gamma_q := \left\{0 \to F^{q+1} C(f) \to F^q C(f) \to Gr_F^q C(f) \to 0\right\}$$

if follows that the complex $C(Gr_F^q f) = Gr_F^q C(f)$ is acyclic, and hence $Gr_F^q f$ is a quasi-isomorphism.

(2) $\Rightarrow$ (1). Assume that $Gr_F^q f$ is a quasi-isomorphism for all $q > p$. Since the filtrations are biregular, there exists an integer $r > p$ such that $F^r f = Gr_F^r f$ is a quasi-isomorphism. The result follows by induction, using the short exact sequence Γ_q.

This proves that (1) is equivalent to (2). That (1) is equivalent to (3) follows directly from the exact sequence Γ_q.

$\square$

The Filtered Derived Category. The category of filtered objects of an abelian category is the primary example of an exact category. The study of its exact structure appears in [**Kel90**]. We next give a detailed presentation of the main results, which will be needed in later sections.

Lemma 2.1.39 ([**Kel90**], 5.1)**.** *The category of filtered objects $\mathbf{F}\mathcal{A}$ of an abelian category $\mathcal{A}$ admits an exact category structure. The exact pairs are given by the sequences of filtered morphisms*

$$(A, F) \to (B, F) \to (C, F)$$

such that for all $p \in \mathbb{Z}$, the sequence

$$0 \to F^pA \to F^pB \to F^pC \to 0$$

is exact.

Remark 2.1.40. Consider a sequence of filtered morphisms

$$(A, F) \to (B, F) \to (C, F).$$

Since the filtrations are finite, it follows that the sequence

$$0 \to F^pA \to F^pB \to F^pB \to 0$$

is exact for all $p \in \mathbb{Z}$, if and only if the sequence

$$0 \to Gr_F^pA \to Gr_F^pB \to Gr_F^pB \to 0$$

is exact for all $p \in \mathbb{Z}$.

From Proposition 2.1.31 and since filtrations are finite, it follows that the admissible monomorphisms (resp. epimorphisms) of the exact structure of $\mathbf{F}\mathcal{A}$ are the strict monomorphisms (resp. epimorphisms). In particular, the admissible morphisms are the strict filtered morphisms.

Proposition 2.1.41 (cf. [**Kel96**], Ex. 5.5)**.** *Let $\mathcal{A}$ be an abelian category with enough injectives.*

(1) An object (I, F) is injective in $\mathbf{F}\mathcal{A}$ if and only if Gr_F^pI is an injective object of $\mathcal{A}$, for all $p \in \mathbb{Z}$.

(2) If $\mathcal{A}$ has enough injectives, then $\mathbf{F}\mathcal{A}$ has enough injectives.

PROOF. We prove (1). Let $f : (A, F) \to (I, F)$ be a morphism of filtered objects, and let $i : (A, F) \to (B, F)$ be a strict monomorphism. Assume that for $q > p$ we have morphisms $g_q : F^qB \to F^qI$ such that $g_q|_{F^rB} = g_r$ for all $r > q$ and $g_qF^qi = F^qf$. Consider the solid diagram

$$\begin{tikzcd} F^pA \arrow[r, "Gr_F^pf"] \arrow[d, "Gr_F^pi"'] & Gr_F^pI \\ Gr_F^pB \arrow[ur, dotted, "\widetilde{g}_p"'] & \end{tikzcd}$$

Assume that $Gr_F^p I$ is injective for all $p \in \mathbb{Z}$. Since $Gr_F^p i$ is a monomorphism, a dotted arrow $\widetilde{g}_p$ exists, making the diagram commute. Consider the solid diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & F^{p+1}B \arrow[r] \arrow[d, "g_{p+1}"] & F^pB \arrow[r] \arrow[d, dotted, "g_p"] & Gr_F^pB \arrow[r] \arrow[d, "\widetilde{g}_p"] & 0 \\ 0 \arrow[r] & F^{p+1}I \arrow[r] & F^pI \arrow[r] & Gr_F^pI \arrow[r] & 0 \end{tikzcd}$$

Since the filtrations are finite, the condition that $Gr_F^p I$ is injective for all $p \in \mathbb{Z}$ is equivalent to the condition that F^pI is injective for all $p \in \mathbb{Z}$ and the exact sequence at the bottom splits. In particular, $F^pI \cong F^{p+1}I \oplus Gr_F^p I$, and the dotted arrow g_p exists.

We prove (2). Let (A, F) be a filtered object, and assume that for all $p \in \mathbb{Z}$ there exists an injective object I_p, and a monomorphism $Gr_F^p A \to I_p$. Since the filtration is finite, we can define inductively over $p \in \mathbb{Z}$, a filtered injective object

$$F^pI := F^{p+1}I \oplus I_p,$$

together with a morphism $(A, F) \to (I, F)$, which by construction is a strict monomorphism. $\square$

The class of filtered quasi-isomorphisms of filtered complexes corresponds to the class of weak equivalences associated with the exact structure of $\mathbf{F}\mathcal{A}$ (see for example [**Hub95**], Lemma 3.1.6). We obtain:

Corollary 2.1.42. *Let $\mathcal{A}$ be an abelian category with enough injectives. The triple $(\mathbf{C}^+(\mathbf{F}\mathcal{A}), \mathcal{S}, \mathcal{E})$ is a (right) Cartan-Eilenberg category, and $\mathbf{C}^+(\mathbf{F}\mathrm{Inj}\mathcal{A})$ is a full subcategory of fibrant models. The inclusion induces an equivalence of categories*

$$\mathbf{K}^+(\mathbf{F}\mathrm{Inj}\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^+(\mathbf{F}\mathcal{A}) := \mathbf{C}^+(\mathbf{F}\mathcal{A})[\mathcal{E}^{-1}].$$

Proof. It follows from Proposition 2.1.41 and Corollary 2.1.24. $\square$

2.2. Deligne's Décalage Functor

In [**Del71b**], Deligne introduced the shift and the décalage of a filtered complex, and proved that the spectral sequences associated with these filtrations, are all related by a shift of indexing. Deligne's décalage functor is one of the key tools of mixed Hodge theory, being the responsible for endowing the cohomology of a mixed Hodge complex with a mixed Hodge structure. In this section we show how Deligne's décalage is already a key tool in the study higher filtered derived categories. We collect some main properties of the décalage which are probably known to experts, but which do not seem to have appeared in the literature. We introduce the r-derived category as the localization of the category of (bounded below) filtered complexes with respect to E_r-quasi-isomorphisms and, using Deligne's décalage functor, we provide results analogous to the classical setting.

Definitions and Properties. We first recall the definition of the shift, the décalage and its dual construction, and collect their main properties. For the rest of this section we let $\mathcal{A}$ be an abelian category.

Definition 2.2.1. The *shift* of a filtered complex (K, F) is the filtered complex (K, SF) defined by

$$SF^pK^n = F^{p-n}K^n.$$

This defines a functor

$$S : \mathbf{C}^+(\mathbf{F}\mathcal{A}) \longrightarrow \mathbf{C}^+(\mathbf{F}\mathcal{A})$$

which is the identity on morphisms.

The following result is straightforward.

Proposition 2.2.2. *There are isomorphisms of bigraded complexes*

$$E_{r+1}^{p,q}(SK) \cong E_r^{-q,p+2q}(K), \text{ for all } r \geq 0.$$

The shift functor does not admit an inverse, since the differentials of such would not be compatible with the filtrations. However, it has both a right and a left adjoint: these are the décalage and its dual construction.

Definition 2.2.3. The *décalage* of a filtered complex (K, F) is the filtered complex $(K, \mathrm{Dec}F)$ defined by

$$\mathrm{Dec}F^pK^n = F^{p+n}K^n \cap d^{-1}(F^{p+n+1}K^{n+1}).$$

The induced filtration on $H(K)$ is the filtration:

$$\mathrm{Dec}F^p(H^n(K)) = F^{p+n}(H^n(K)).$$

The dual to the décalage is the filtered complex (K, Dec^*F) defined by

$$\mathrm{Dec}^*F^pK^n = d(F^{p+n-1}K^{n-1}) + F^{p+n}K^n.$$

These filtrations define functors

$$\mathrm{Dec}, \mathrm{Dec}^* : \mathbf{C}^+(\mathbf{F}\mathcal{A}) \longrightarrow \mathbf{C}^+(\mathbf{F}\mathcal{A})$$

which are the identity on morphisms.

Example 2.2.4. Let G denote the trivial filtration $0 = G^1K \subset G^0K = K$ of a complex K. Then $\mathrm{Dec}G = \mathrm{Dec}^*G = \tau$ is the canonical filtration, while $SG = \sigma$ is the bête filtration.

The spectral sequences associated with a filtered complex and its décalage are related by a shift of indexing.

Proposition 2.2.5 ([**Del71b**], Prop. 1.3.4)**.** *The canonical maps*

$$E_0^{p,n-p}(\mathrm{Dec}K) \to E_1^{p+n,-p}(K) \to E_0^{p,n-p}(\mathrm{Dec}^*K)$$

are quasi-isomorphisms of bigraded complexes. The canonical maps

$$E_r^{p,n-p}(\mathrm{Dec}K) \to E_{r+1}^{p+n,-p}(K) \to E_r^{p,n-p}(\mathrm{Dec}^*K)$$

are isomorphisms for all $r \geq 1$.

The following result is a matter of verification.

Lemma 2.2.6. *The following identities are satisfied:*

(1) $\mathrm{Dec} \circ S = 1$, *and* $(S \circ \mathrm{Dec}F)^p = F^p \cap d^{-1}(F^{p+1})$,

(2) $\mathrm{Dec}^* \circ S = 1$, *and* $(S \circ \mathrm{Dec}^*F)^p = F^p + d(F^{p-1})$.

In particular, the identity defines natural transformations

$$S \circ \mathrm{Dec} \to 1 \text{ and } 1 \to S \circ \mathrm{Dec}^*.$$

As a consequence, we can easily prove that:

Proposition 2.2.7. *The functor S is left adjoint to* Dec *and right adjoint to* Dec*. *In particular:*

$$\mathrm{Hom}(SK, L) = \mathrm{Hom}(K, \mathrm{Dec}L),$$
$$\mathrm{Hom}(\mathrm{Dec}^*K, L) = \mathrm{Hom}(K, SL).$$

PROOF. The adjunction $S \dashv \mathrm{Dec}$ is given by the pair of transformations $\varepsilon : S \circ \mathrm{Dec} \to 1$ and $\eta : 1 \to \mathrm{Dec} \circ S$. Analogously, the adjunction $\mathrm{Dec}^* \dashv S$ is given by the pair $\varepsilon^* : \mathrm{Dec}^* \circ S \to 1$ and $\eta^* : 1 \to S \circ \mathrm{Dec}^*$. □

Equivalence of Derived Categories. Denote by $\mathcal{E}_0$ the class of filtered quasi-isomorphisms of $\mathbf{C}^+(\mathbf{F}\mathcal{A})$. Inductively over $r > 0$, define a class $\mathcal{E}_r$ of weak equivalences by letting

$$\mathcal{E}_r := \mathrm{Dec}^{-1}(\mathcal{E}_{r-1}) = (\mathrm{Dec}^*)^{-1}(\mathcal{E}_{r-1}).$$

Definition 2.2.8. Morphisms of $\mathcal{E}_r$ are called *E_r-quasi-isomorphisms* of filtered complexes.

In particular, an E_0-quasi-isomorphism is a filtered quasi-isomorphism. Note that by Proposition 2.2.5 we have $\mathrm{Dec}^{-1}(\mathcal{E}_r) = (\mathrm{Dec}^*)^{-1}(\mathcal{E}_r)$, for every $r \geq 0$, and hence the above formula makes sense.

Our objective is to study the localized category

$$\mathbf{D}_r^+(\mathbf{F}\mathcal{A}) := \mathbf{C}^+(\mathbf{F}\mathcal{A})[\mathcal{E}_r^{-1}]$$

of (bounded below) filtered complexes with respect to the class of E_r-quasi-isomorphisms, for $r \geq 0$ arbitrary.

The shift functor is compatible with the classes of E_r-quasi-isomorphisms:

Proposition 2.2.9. *Let $r \geq 0$. Then $\mathcal{E}_r = S^{-1}(\mathcal{E}_{r+1})$.*

PROOF. We prove both inclusions. Let $f \in \mathcal{E}_r$. By Lemma 2.2.6 we have $f = \mathrm{Dec}(Sf) \in \mathcal{E}_r$. By definition, this implies that $Sf \in \mathcal{E}_{r+1}$. Conversely, assume that $Sf \in \mathcal{E}_{r+1}$. Then $f = \mathrm{Dec}(Sf) \in \mathcal{E}_r$. □

We next show that, when restricted to certain subcategories, the shift and the décalage are inverses to each other.

2.2.10. Let $r \geq 0$, and denote by $\mathbf{C}^+_r(\mathbf{F}\mathcal{A})$ the full subcategory of $\mathbf{C}^+(\mathbf{F}\mathcal{A})$ of those complexes (K, F) such that

$$d(F^pK) \subset F^{p+r}K.$$

Obviously, there is a chain of full subcategories

$$\mathbf{C}^+_r(\mathbf{F}\mathcal{A}) \subset \mathbf{C}^+_{r-1}(\mathbf{F}\mathcal{A}) \subset \cdots \subset \mathbf{C}^+_0(\mathbf{F}\mathcal{A}) = \mathbf{C}^+(\mathbf{F}\mathcal{A}).$$

A simple verification shows that:

Lemma 2.2.11. *Let (K, F) be an object of $\mathbf{C}^+_1(\mathbf{F}\mathcal{A})$. Then*

$$\mathrm{Dec}F^pK^n = \mathrm{Dec}^*F^pK^n = F^{p+n}K^n.$$

Corollary 2.2.12. *Let $r \geq 0$. The functors*

$$\mathrm{Dec} = \mathrm{Dec}^* : \mathbf{C}^+_{r+1}(\mathbf{F}\mathcal{A}) \rightleftarrows \mathbf{C}^+_r(\mathbf{F}\mathcal{A}) : S$$

are inverses to each other.

Lemma 2.2.13. *Let $r \geq 0$, and consider the functor*

$$\mathcal{J}_r := (S^r \circ \mathrm{Dec}^r) : \mathbf{C}^+(\mathbf{F}\mathcal{A}) \longrightarrow \mathbf{C}^+_r(\mathbf{F}\mathcal{A}).$$

There is a natural transformation $\mathcal{J}_r \to 1$ such that for every filtered complex K, the morphism $\mathcal{J}_r(K) \to K$ is an E_r-quasi-isomorphism. In particular, there is an equivalence of categories

$$\mathcal{J}_r : \mathbf{D}^+_r(\mathbf{F}\mathcal{A}) \xrightarrow{\sim} \mathbf{C}^+_r(\mathbf{F}\mathcal{A})[\mathcal{E}_r^{-1}].$$

Proof. By Lemma 2.2.6 we have a natural transformation

$$\mathcal{J}_1 = S \circ \mathrm{Dec} \longrightarrow 1.$$

For every $r > 0$, this gives a natural transformation

$$\mathcal{J}_r = S^{r-1} \circ \mathcal{J}_1 \circ \mathrm{Dec}^{r-1} \longrightarrow S^{r-1} \circ 1 \circ \mathrm{Dec}^{r-1} = \mathcal{J}_{r-1}.$$

Let K be a filtered complex. For the morphism $\mathcal{J}_r(K) \to K$ to be an E_r-quasi-isomorphism it suffices to show that $\mathrm{Dec}^r(\mathcal{J}_r(K) \to K)$ is an E_0-quasi-isomorphism. Indeed, since $\mathrm{Dec}^r \circ S^r = 1$ we have

$$\mathrm{Dec}^r \circ \mathcal{J}_r K = \mathrm{Dec}^r \circ S^r \circ \mathrm{Dec}^r K = \mathrm{Dec}^r K.$$

$\square$

Remark 2.2.14. The functor $\mathcal{J}_r$ is idempotent. There are dual results for $1 \to \mathcal{J}_r^* := S^r \circ (\mathrm{Dec}^*)^r$.

We can now prove the main result of this section.

Theorem 2.2.15. *Deligne's décalage induces an equivalence of categories*

$$\mathrm{Dec} : \mathbf{D}^+_{r+1}(\mathbf{F}\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^+_r(\mathbf{F}\mathcal{A}),$$

for every $r \geq 0$.

PROOF. By Lemma 2.2.13 there is an equivalence of categories

$$\mathcal{J}_r : \mathbf{D}^+_r(\mathbf{F}\mathcal{A}) \xrightarrow{\sim} \mathbf{C}^+_r(\mathbf{F}\mathcal{A})[\mathcal{E}_r^{-1}].$$

By Corollary 2.2.12 we have an equivalence $\mathrm{Dec} : \mathbf{C}^+_{r+1}(\mathbf{F}\mathcal{A}) \xrightarrow{\sim} \mathbf{C}^+_r(\mathbf{F}\mathcal{A})$. Since $\mathcal{E}_{r+1} = \mathrm{Dec}^{-1}(\mathcal{E}_r)$, this induces an equivalence of localized categories

$$\mathbf{C}^+_{r+1}(\mathbf{F}\mathcal{A})[\mathcal{E}_{r+1}^{-1}] \xrightarrow{\sim} \mathbf{C}^+_r(\mathbf{F}\mathcal{A})[\mathcal{E}_r^{-1}].$$

Hence we have a commutative diagram of equivalences

$$\begin{tikzcd}
\mathbf{D}^+_{r+1}(\mathbf{F}\mathcal{A}) \arrow[r, dotted, "\sim"] \arrow[d, "\mathcal{J}_{r+1}", "\wr"'] & \mathbf{D}^+_r(\mathbf{F}\mathcal{A}) \arrow[d, "\mathcal{J}_r", "\wr"'] \\
\mathbf{C}^+_{r+1}(\mathbf{F}\mathcal{A})[\mathcal{E}_{r+1}^{-1}] \arrow[r, "\mathrm{Dec}", "\sim"'] & \mathbf{C}^+_r(\mathbf{F}\mathcal{A})[\mathcal{E}_r^{-1}].
\end{tikzcd}$$

□

Higher Injective Models. The notion of filtered homotopy between morphisms of filtered complexes generalizes to a notion of r-homotopy, suitable to the study of r-injective models with respect to E_r-quasi-isomorphisms.

Definition 2.2.16. Let $r \geq 0$. The *r-translation* is the autoequivalence

$$T_r : \mathbf{C}^+(\mathbf{F}\mathcal{A}) \longrightarrow \mathbf{C}^+(\mathbf{F}\mathcal{A})$$

which sends a filtered complex K to the filtered complex $K[1](r)$ defined by

$$F^pK[1](r)^n := F^{p+r}K^{n+1}.$$

The *inverse* T_r^{-1} of the r-translation is given by $K \mapsto K[-1](-r)$.

Definition 2.2.17. Let $f, g : K \to L$ be two maps of filtered complexes, and let $r \geq 0$. An *r-homotopy* from f to g is a morphism of filtered complexes $h : K \to L[-1](-r)$ such that $hd - dh = g - f$. We use the notation $h : f \underset{\mathrm{r}}{\simeq} g$, and say that f is *r-homotopic* to g.

Note that the condition that h is compatible with the filtrations is equivalent to the condition that for all $n \geq 0$ and all $p \in \mathbb{Z}$,

$$h(F^p K^n) \subset F^{p-r} L^{n-1}.$$

This coincides with the notion of r-homotopy introduced by [**CE56**], pag. 321. See also [**Ill71**], pag. 277. For $r = 0$ we recover the usual notion of filtered homotopy.

To control the effect of shift and décalage on r-homotopy equivalences it suffices to study its effect on the inverse of the r-translation functor. The following result is a matter of verification.

Lemma 2.2.18. *Let $r \geq 0$. The following identities are satisfied:*

$$\begin{aligned}
&\mathrm{Dec} \circ T_{r+1}^{-1} = T_r^{-1} \circ \mathrm{Dec},\\
&\mathrm{Dec}^* \circ T_{r+1}^{-1} = T_r^{-1} \circ \mathrm{Dec}^*,\\
&S \circ T_r^{-1} = T_{r+1}^{-1} \circ S.
\end{aligned}$$

Denote by $\mathcal{S}_r$ the class of r-homotopy equivalences. The following result is straightforward from Lemma 2.2.18.

Corollary 2.2.19. *Let $r \geq 0$. There are inclusions*

$$\mathrm{Dec}(\mathcal{S}_{r+1}), \mathrm{Dec}^*(\mathcal{S}_{r+1}) \subset \mathcal{S}_r \text{ and } S(\mathcal{S}_r) \subset \mathcal{S}_{r+1}.$$

Using the notion of r-cylinder defined via the r-translation, by Corollary 2.1.9 it follows that the quotient category

$$\mathbf{K}_r^+(\mathbf{F}\mathcal{A}) := \mathbf{C}^+(\mathbf{F}\mathcal{A}) / \underset{\mathrm{r}}{\simeq}$$

is canonically isomorphic to the localized category $\mathbf{C}^+(\mathbf{F}\mathcal{A})[\mathcal{S}_r^{-1}]$ with respect to r-homotopy equivalences.

Denote by $[K, L]_r$ the class of morphisms from K to L modulo r-homotopy.

Proposition 2.2.20 (cf. [**CE56**], Prop. 3.1)**.** *Let $r \geq 0$. Then every r-homotopy equivalence is an E_r-quasi-isomorphism.*

PROOF. By Lemma 2.1.21 we have $\mathcal{S}_0 \subset \mathcal{E}_0$. We proceed by induction:

$$\mathcal{S}_{r+1} = \mathrm{Dec}^{-1}(\mathcal{S}_r) \subset \mathrm{Dec}^{-1}(\mathcal{E}_r) = \mathcal{E}_{r+1}.$$

□

In particular, the triple $(\mathbf{C}^+(\mathbf{F}\mathcal{A}), \mathcal{S}_r, \mathcal{E}_r)$ is a category with strong and weak equivalences, for all $r \geq 0$.

To characterize fibrant objects we will use the following auxiliary Lemma, which reflects the behaviour of the décalage on the homotopy classes of certain morphisms. Since we will consider injective objects, we will use the functor Dec^*. Dual results for projective objects are obtained with Dec.

Lemma 2.2.21. *Let $r \geq 0$, and let I be an object of $\mathbf{C}^+_{r+1}(\mathbf{F}\mathcal{A})$. For every filtered complex K, there is a bijection*

$$\mathrm{Dec}^* : [K, I]_{r+1} \longrightarrow [\mathrm{Dec}^* K, \mathrm{Dec}^* I]_r$$

between homotopy classes of morphisms.

PROOF. By Corollary 2.2.12 we have $S \circ \mathrm{Dec}^* I = I$. Therefore

$$\mathrm{Hom}(K, I) = \mathrm{Hom}(K, S \circ \mathrm{Dec}^* I) = \mathrm{Hom}(\mathrm{Dec}^* K, \mathrm{Dec}^* I),$$

by the adjunction $\mathrm{Dec}^* \dashv S$. Therefore it suffices to show that every $(r+1)$-homotopy with respect to F, is in correspondence with an r-homotopy with respect to $\mathrm{Dec}^* F$. Indeed, by Lemma 2.2.18 we have

$$\mathrm{Hom}(K, T^{-1}_{r+1}(I)) = \mathrm{Hom}(K, T^{-1}_{r+1}(S \circ \mathrm{Dec}^* I)) = \mathrm{Hom}(\mathrm{Dec}^* K, T^{-1}_r(\mathrm{Dec}^* I)).$$

□

We will next show that the objects of

$$\mathbf{C}^+_r(\mathbf{F}\mathrm{Inj}\mathcal{A}) := \mathbf{C}^+_r(\mathbf{F}\mathcal{A}) \cap \mathbf{C}^+(\mathbf{F}\mathrm{Inj}\mathcal{A}).$$

are fibrant objects in the triple $(\mathbf{C}^+(\mathbf{F}\mathcal{A}), \mathcal{S}_r, \mathcal{E}_r)$, for all $r \geq 0$.

Definition 2.2.22. Objects of $\mathbf{C}^+_r(\mathbf{F}\mathrm{Inj}\mathcal{A})$ are called *r-injective complexes.*

The décalage and its inverse restrict to the full subcategory of r-injective complexes:

Lemma 2.2.23. *Let $r \geq 0$. The functors*

$$\mathrm{Dec} = \mathrm{Dec}^* : \mathbf{C}^+_{r+1}(\mathbf{F}\mathrm{Inj}\mathcal{A}) \rightleftarrows \mathbf{C}^+_r(\mathbf{F}\mathrm{Inj}\mathcal{A}) : S$$

are inverses to each other.

Proof. In view of Corollary 2.2.12 it suffices to see that both functors preserve injectives. Let I be an $(r+1)$-injective complex. By Lemma 2.2.11 we have that

$$Gr^p_{\mathrm{Dec}F} I^n = Gr^p_{\mathrm{Dec}^*F} I^n = Gr^{p+n}_F I^n$$

is injective for all $p \in \mathbb{Z}$ and all $n \geq 0$. Hence $\mathrm{Dec}I = \mathrm{Dec}^*I$ is r-injective. The converse is trivial. $\square$

We next show that r-injective complexes are fibrant.

Proposition 2.2.24. *Let $r \geq 0$, and let I be an r-injective complex. Every E_r-quasi-isomorphism $w : K \to L$ induces a bijection*

$$w^* : [L, I]_r \longrightarrow [K, I]_r$$

between r-homotopy classes of morphisms.

Proof. By Corollary 2.1.42, the statement is true for $r = 0$. We proceed by induction. Assume that I is $(r+1)$-injective. Consider the diagram

$$\begin{tikzcd}
{[L, I]_{r+1}} \arrow[r, "\mathrm{Dec}^*"] \arrow[d, "w^*"] & {[\mathrm{Dec}^*L, \mathrm{Dec}^*I]_r} \arrow[d, "w^*"] \\
{[K, I]_{r+1}} \arrow[r, "\mathrm{Dec}^*"] & {[\mathrm{Dec}^*K, \mathrm{Dec}^*I]_r}
\end{tikzcd}$$

By Lemma 2.2.23, Dec^*I is r-injective. Hence by induction hypothesis, the vertical arrow on the right is a bijection. By Lemma 2.2.21 the horizontal arrows are bijections. $\square$

Lastly, we prove the existence of enough r-injective complexes.

Proposition 2.2.25. *Let $\mathcal{A}$ be an abelian category with enough injectives, and let $r \geq 0$. For every complex K of $\mathbf{C}^+(\mathbf{F}\mathcal{A})$, there exists an r-injective complex I, together with an E_r-quasi-isomorphism $\rho : K \to I$.*

PROOF. By Corollary 2.1.42 the statement is true for $r = 0$. We proceed by induction. Let $r > 0$, and let $\rho : \mathrm{Dec}^*K \to I$ be an E_{r-1}-quasi-isomorphism, where I is $(r-1)$-injective. Then the adjunction $\mathrm{Dec}^* \dashv S$ gives an E_r-quasi-isomorphism $\rho : K \to SI$. By Lemma 2.2.23 SI is r-injective. $\square$

Theorem 2.2.26. *Let $\mathcal{A}$ be an abelian category with enough injectives, and let $r \geq 0$. The triple $(\mathbf{C}^+(\mathbf{F}\mathcal{A}), \mathcal{S}_r, \mathcal{E}_r)$ is a (right) Cartan-Eilenberg category. The inclusion induces an equivalence of categories*

$$\mathbf{K}_r^+(\mathbf{F}\mathrm{Inj}\mathcal{A}) \xrightarrow{\sim} \mathbf{D}_r^+(\mathbf{F}\mathcal{A})$$

between the category of r-injective complexes modulo r-homotopy, and the localized category of filtered complexes with respect to E_r-quasi-isomorphisms.

PROOF. By Proposition 2.2.24 every r-injective complex is a fibrant object. By Proposition 2.2.25 every filtered complex has an r-injective model. The equivalence of categories follows from Theorem 1.1.35. $\square$

2.3. Filtered Complexes of Vector Spaces

Consider the category $\mathbf{C}^+(\mathbf{k})$ of bounded below complexes of vector spaces over a field $\mathbf{k}$. In this case, every object is injective, and the classical calculus of derived categories does not provide any additional information. The minimal objects of $\mathbf{C}^+(\mathbf{k})$ are those complexes with trivial differential. In addition, every complex K is homotopically equivalent to $H(X)$, regarded as a complex with trivial differential. This provides $\mathbf{C}^+(\mathbf{k})$ with the structure of a Sullivan category. A well known corollary of this fact is the equivalence

$$\mathbf{G}^+(\mathbf{k}) \xrightarrow{\sim} \mathbf{D}^+(\mathbf{k})$$

between the category of non-negatively graded vector spaces and the bounded below derived category of vector spaces over a field $\mathbf{k}$. This simple example exhibits the utility of minimal models and Sullivan categories.

In this section we study minimal models of filtered complexes of vector spaces. We show that a filtered complex is minimal if and only if the differential on each associated graded complex is trivial, and that any bounded below filtered complex with biregular filtrations has a model of such type. As a consequence, the category of filtered complexes of vector spaces inherits a Sullivan category structure.

We remark that if a filtered complex is minimal in the category of filtered complexes, it need not be minimal as a complex, when we forget the filtrations. This will only be the case if the differential of the complex is strictly compatible with the filtration.

The results obtained in this section are easily extended to complexes having multiple filtrations. However, to keep notations clear, and given our interests in mixed Hodge theory, we will only state such results for bifiltered complexes, at the end of this chapter.

Filtered Minimal Models. Denote by $\mathbf{C}^+(\mathbf{Fk})$ the category of bounded below filtered complexes of vector spaces over a field $\mathbf{k}$, with biregular filtrations. We will first study the ordinary filtered derived category. We let $\mathcal{S}$ and $\mathcal{E}$ denote the classes of filtered homotopy equivalences and filtered quasi-isomorphisms respectively.

Recall that $\mathbf{C}^+_1(\mathbf{Fk})$ is the full subcategory of $\mathbf{C}^+(\mathbf{Fk})$ of those filtered complexes (K, F) such that $d(F^pK) \subset F^{p+1}K$, for all $p \in \mathbb{Z}$.

Proposition 2.3.1. *Every object of* $\mathbf{C}^+_1(\mathbf{Fk})$ *is minimal in* $(\mathbf{C}^+(\mathbf{Fk}), \mathcal{S}, \mathcal{E})$.

Proof. Since vector spaces are injective, every filtered complex is fibrant. It suffices to show that every filtered quasi-isomorphism $f : K \to K$ of $\mathbf{C}^+_1(\mathbf{Fk})$ is an isomorphism. Indeed, since $dGr^p_F K = 0$ for all $p \in \mathbb{Z}$, it follows that $Gr^p_F K = H(Gr^p_F K)$. Therefore the map $Gr^p_F f$ is an isomorphism for all $p \in \mathbb{Z}$. Since the filtrations are biregular, the map f is an isomorphism. $\square$

We next prove the existence of enough minimal models. The construction is made inductively over the weight of the filtration, by adding at each step a graded vector space of pure weight. If K is a complex with the trivial filtration $0 = F^1K \subset F^0K = K$, the construction reduces to the classical case, and the minimal model is just $H(K) \to K$.

Theorem 2.3.2. *For every filtered complex K in $\mathbf{C}^+(\mathbf{Fk})$ there exists an object M of $\mathbf{C}_1^+(\mathbf{Fk})$, and a filtered quasi-isomorphism $\rho : M \to K$.*

PROOF. We construct, by a decreasing induction on $p \in \mathbb{Z}$, a family of filtered complexes M_p, together with morphisms $\rho_p : M_p \to K$ such that:

(a_p) $M_p = M_{p+1} \oplus V_p$, where V_p is a graded vector space of pure weight p satisfying $dV_p \subset F^{p+1}M_{p+1}$. The map ρ_p extends ρ_{p+1}.

(b_p) $H^n(F^qC(\rho_p)) = 0$ for all $n \geq 0$ and all $q \geq p$.

Since the filtration of K is bounded below, we can take $M_r = 0$ for $r >> 0$ as a base case of our induction. The above conditions are trivially satisfied.

Assume that for all $q > p$ we constructed M_q as required. Let V_p be the graded vector space of weight p given by

$$V_p^n = H^n(Gr_F^p C(\rho_{p+1})).$$

Define a filtered graded vector space by taking the direct sum

$$M_p = M_{p+1} \oplus V_p.$$

Since $H^i(F^{p+1}C(\rho_{p+1})) = 0$ for all $i \geq 0$, we have

$$H^n(F^pC(\rho_{p+1})) \cong H^n(Gr_F^p C(\rho_{p+1})).$$

Define a differential $d : V_p \to M_{p+1}$ and a map $\rho_p : M_p \to K$ extending ρ_{p+1} by taking a section of the projection

$$Z^n(F^pC(\rho_{p+1})) \twoheadrightarrow H^n(F^pC(\rho_{p+1})) \cong V_p^n.$$

Since M_{p+1} is generated by elements of weight $> p$, the differential of V_p satisfies $d(V_p^n) \subset F^{p+1}M_{p+1}$. Hence condition (a_p) is satisfied.

Let us prove (b_p). If $q > p$ we have $F^qM_p = F^qM_{p+1}$. Therefore

$$H^n(F^qC(\rho_p)) = H^n(F^qC(\rho_{p+1})) = 0, \text{ for all } n \geq 0.$$

Consider the exact sequence

$$\Sigma := \{0 \to C(\rho_{p+1}) \to C(\rho_p) \to V_p[1] \to 0\}.$$

Since the morphisms are strict, the sequence $F^p\Sigma$ is exact. We obtain an exact sequence

$$V_p^n \xrightarrow{\cong} H^n(F^pC(\rho_{p+1})) \longrightarrow H^n(F^pC(\rho_p)) \longrightarrow 0.$$

Hence $H^n(F^pC(\rho_{p+1})) = 0$, and (b_n) is satisfied.

Let

$$\rho := \varinjlim \rho_p : \left(M := \varinjlim M_p = \bigoplus_p M_p \right) \longrightarrow K.$$

Since M_p satisfies (a_p) for all $p \in \mathbb{Z}$, we have $d(F^pM) \subset F^{p+1}M$. Hence M is an object of $\mathbf{C}_1^+(\mathbf{Fk})$. By construction, for every $p \in \mathbb{Z}$ we have

$$H^n(F^pC(\rho)) = H^n(F^pC(\rho_p)) = 0.$$

By Lemma 2.1.38, ρ is a filtered quasi-isomorphism. □

Corollary 2.3.3. *The triple* $(\mathbf{C}^+(\mathbf{Fk}), \mathcal{S}, \mathcal{E})$ *is a Sullivan category, and* $\mathbf{C}_1^+(\mathbf{Fk})$ *is a full subcategory of minimal models. The inclusion induces an equivalence of categories*

$$\left(\mathbf{C}_1^+(\mathbf{Fk})/\simeq\right) \xrightarrow{\sim} \mathbf{D}^+(\mathbf{Fk}).$$

We next generalize this result to study higher order filtered derived categories. For convenience, we introduce the following:

Definition 2.3.4. Let $r \geq 0$. A filtered complex K is called E_r*-minimal* if $d(F^pK) \subset F^{p+r+1}K$, for all $p \in \mathbb{Z}$.

According to the previous notations, the full subcategory of E_r-minimal complexes is $\mathbf{C}_{r+1}^+(\mathbf{Fk})$.

Proposition 2.3.5. *Let* $r \geq 0$. *Every* E_r*-minimal complex is a minimal object of* $(\mathbf{C}^+(\mathbf{Fk}), \mathcal{S}_r, \mathcal{E}_r)$.

Proof. By proposition 2.3.1 the case $r = 0$ is true. For $r > 0$, let $f : K \to K$ be an E_r-quasi-isomorphism between E_r-minimal objects. By Corollary 2.2.12 the morphism $f : \mathrm{Dec}K \to \mathrm{Dec}K$ is an E_{r-1}-quasi-isomorphism between E_{r-1}-minimal objects. By induction hypothesis $\mathrm{Dec}f$ is an isomorphism. Hence f is an isomorphism. □

Proposition 2.3.6. *For every complex $K \in \mathbf{C}^+(\mathbf{Fk})$ there exists an E_r-minimal complex M, together with an E_r-quasi-isomorphism $\rho : M \to K$.*

PROOF. The case $r = 0$ follows from Corollary 2.3.3. Assume that $r > 0$. Let $M \to \mathrm{Dec}K$ be an E_{r-1}-minimal model. The morphism $SM \to K$ given by the adjunction $S \dashv \mathrm{Dec}$ is an E_r-quasi-isomorphism, and SM is E_r-minimal. □

Theorem 2.3.7. *Let $r \geq 0$. The triple $(\mathbf{C}^+(\mathbf{Fk}), \mathcal{S}_r, \mathcal{E}_r)$ is a Sullivan category, and $\mathbf{C}^+_{r+1}(\mathbf{Fk})$ is a full subcategory of minimal models. The inclusion induces an equivalence of categories*

$$\left(\mathbf{C}^+_{r+1}(\mathbf{Fk})/ \underset{r}{\simeq}\right) \xrightarrow{\sim} \mathbf{D}^+_r(\mathbf{Fk}).$$

Strict Complexes. To end this section we collect some properties of strict complexes. These results will be particularly useful in the applications to mixed Hodge theory of Chapters 3 and 5.

Definition 2.3.8. A filtered complex (K, F) is called *d-strict* if its differential is strictly compatible with the filtration, that is,

$$d(F^pK) = d(K) \cap F^pK, \text{ for all } p \in \mathbb{Z}.$$

The following lemmas are straightforward from the definition.

Lemma 2.3.9. *A filtered complex (K, F) is d-strict if and only if the morphism $H^*(F^pK) \longrightarrow H^*(K)$ is injective for all $p \in \mathbb{Z}$.*

Lemma 2.3.10. *Let (K, F) be a d-strict filtered complex.*
(1) Every class in $F^pH^n(K)$ has a representative in F^pK.
(2) If x is a coboundary in F^pK, then $x = dy$, with $y \in F^pK$.

The strictness of the differential of a filtered complex is related to the degeneration of its associated spectral sequence.

Proposition 2.3.11. *[**[Del71b]**, Prop. 1.3.2] Let K be a (biregularly) filtered complex. Then $E_1(K) = E_\infty(K)$ if and only if K is d-strict.*

We next provide sufficient conditions for a quasi-isomorphism of filtered complexes to be an E_r-quasi-isomorphism.

Lemma 2.3.12. *Let $f : K \to L$ be a morphism of filtered complexes satisfying the following conditions:*

(i) f is a quasi-isomorphism.
(ii) The map $f^* : H(K) \to H(L)$ is strictly compatible with filtrations.
(iii) $E_{r+1}(K) = E_\infty(K)$ and $E_{r+1}(L) = E_\infty(L)$.

Then f is an E_r-quasi-isomorphism.

Proof. Since f^* is strictly compatible with filtrations

$$f^*(F^pH(K)) = f^*(H(K)) \cap F^pH(L).$$

Since f^* is an isomorphism, we obtain $f^*(F^pH(K)) \cong F^pH(L)$. Therefore

$$E_{r+1}(K) = Gr^\bullet_F H(K) \cong Gr^\bullet_F H(L) = E_{r+1}(L).$$

□

Proposition 2.3.13. *Let $\rho : M \to K$ be an E_r-minimal model of a filtered complex K. If $E_{r+1}(K) = E_\infty(K)$. Then $dM = 0$, hence M is minimal.*

Proof. Since ρ is an E_r-quasi-isomorphism we have $E_{r+1}(M) = E_\infty(M)$. Assume that $r = 0$. By Proposition 2.3.11, M is d-strict. Since M is E_r-minimal, it satisfies $d(F^pM) \subset F^{p+1}M$. Hence $d(F^pM) \subset F^{p+1}M \cap dM = d(F^{p+1}M)$. Since filtrations are biregular, it follows that $dM = 0$. The result follows by induction, using décalage. □

2.4. Bifiltered Complexes

In this last section we extend the definitions and results of the previous sections, to bifiltered complexes. Given our interests in Hodge theory, and for the sake of simplicity, we shall only study the derived category defined with respect to the class of $E_{r,0}$-quasi-isomorphisms, with $r \in \{0, 1\}$. We first study bifiltered complexes over an abelian category, by means of injective models. Then, we treat the particular case of bifiltered complexes of vector spaces over a field, via the existence of minimal models. Lastly, we provide some definitions and results concerning d-bistrict complexes.

Bifiltered Abelian Categories. Given an abelian category $\mathcal{A}$, denote by $\mathbf{F}^2\mathcal{A}$ the category of bifiltered objects of $\mathcal{A}$: these are triples (A, W, F) such that both (A, W) and (A, F) are objects of $\mathbf{F}\mathcal{A}$. We will denote

$$W^pF^qA := W^pA \cap F^qA,$$

for all $p, q \in \mathbb{Z}$. Morphisms of $\mathbf{F}^2\mathcal{A}$ are those morphisms $f : A \to B$ of $\mathcal{A}$ such that $f(W^pF^qA) \subset W^pF^qB$, for all $p, q \in \mathbb{Z}$.

The bigraded objects $Gr^p_W Gr^q_F A$ and $Gr^q_F Gr^p_W A$ associated with a bifiltered object (A, W, F) are canonically isomorphic, and equal to

$$\begin{array}{rcl} Gr^p_W Gr^q_F A & \cong & W^pF^qA/(W^{p+1}F^qA + W^pF^{q+1}A) \\ & & \| \\ Gr^q_F Gr^p_W A & \cong & F^qW^pA/(F^{q+1}W^pA + F^qW^{p+1}A) \end{array}.$$

Lemma 2.1.39 is also valid when $\mathcal{A}$ is an exact category. As a consequence:

Lemma 2.4.1. *The category of bifiltered objects $\mathbf{F}^2\mathcal{A}$ of an abelian category $\mathcal{A}$ admits an exact category structure. The exact pairs are given by the sequences of bifiltered morphisms*

$$(A, W, F) \to (B, W, F) \to (C, W, F)$$

such that the sequence

$$0 \to W^pF^qA \to W^pF^qB \to W^pF^qC \to 0$$

is exact for all $p, q \in \mathbb{Z}$.

In particular, the admissible monomorphisms (resp. epimorphisms) are those morphisms of filtered objects $f : A \to B$ such that, for all $p, q \in \mathbb{Z}$, the morphism W^pF^qf is a monomorphism (resp. epimorphism).

Injective objects are characterized as follows.

Proposition 2.4.2. *Let $\mathcal{A}$ be an abelian category with enough injectives.*

(1) An object (I, W, F) is injective in $\mathbf{F}^2\mathcal{A}$ if and only if $Gr^p_W Gr^q_F I$ is an injective object of $\mathcal{A}$, for all $p, q \in \mathbb{Z}$.

(2) If $\mathcal{A}$ has enough injectives, then $\mathbf{F}^2\mathcal{A}$ has enough injectives.

PROOF. The proof is analogous to that of Proposition 2.1.41. $\square$

Definition 2.4.3. A morphism $f : (K, W, F) \to (L, W, F)$ of bifiltered complexes is called *bifiltered quasi-isomorphism* or $E_{0,0}$*-quasi-isomorphism* if for all $p, q \in \mathbb{Z}$, the morphism $Gr_F^p Gr_W^q f$ is a quasi-isomorphism of complexes. Denote by $\mathcal{E}_{0,0}$ the class of $E_{0,0}$-quasi-isomorphisms of $\mathbf{C}^+(\mathbf{F}^2\mathcal{A})$.

Décalage with respect to the weight filtration defines a functor

$$\mathrm{Dec}^W : \mathbf{C}^+(\mathbf{F}^2\mathcal{A}) \longrightarrow \mathbf{C}^+(\mathbf{F}^2\mathcal{A}).$$

Define a new class of weak equivalences by

$$\mathcal{E}_{1,0} := (\mathrm{Dec}^W)^{-1}(\mathcal{E}_{0,0}).$$

Morphisms of $\mathcal{E}_{1,0}$ are called $E_{1,0}$*-quasi-isomorphisms.*

Definition 2.4.4. A bifiltered complex (I, W, F) is called $(r, 0)$*-injective* if:

(i) The complex $Gr_W^p Gr_F^q K$ is an object of $\mathbf{C}^+(\mathrm{Inj}\mathcal{A})$, and
(ii) $d(W^p F^q K) \subset W^{p+r} F^q K$, for all $p, q \in \mathbb{Z}$.

Lemma 2.4.5. *If* K *is* $(1, 0)$*-injective then* $\mathrm{Dec}^W K$ *is* $(0, 0)$*-injective.*

Proof. The proof is analogous to that of Lemma 2.2.23. □

A notion of $(r, 0)$-homotopy between morphisms of bifiltered complexes is defined via the $(r, 0)$-translation functor, sending each bifiltered complex K, to the bifiltered complex $K[1](r, 0)$ defined by

$$W^p F^q K[1](r, 0) := W^{p+r} F^q K^{n+1}.$$

Denote by $\mathcal{S}_{r,0}$ the class of $(r, 0)$-homotopy equivalences.

Corollary 2.4.6. *Let* $r \in \{0, 1\}$*. Let* $\mathcal{A}$ *be an abelian category with enough injectives. The triple* $(\mathbf{C}^+(\mathbf{F}^2\mathcal{A}), \mathcal{S}_{r,0}, \mathcal{E}_{r,0})$ *is a (right) Cartan-Eilenberg category. The full subcategory of* $(r, 0)$*-injective complexes is a full subcategory of fibrant models. The inclusion induces an equivalence of categories*

$$\mathbf{K}^+_{r,0}(\mathbf{F}^2\mathrm{Inj}\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^+_{r,0}(\mathbf{F}^2\mathcal{A}) := \mathbf{C}^+(\mathbf{F}^2\mathcal{A})[\mathcal{E}_{r,0}^{-1}].$$

Proof. The case $r = 0$ follows from Proposition 2.4.2 and Corollary 2.1.24. The case $r = 1$ follows by décalage and Lemma 2.4.5. □

Bifiltered Complexes of Vector Spaces. We prove the existence of minimal models for bifiltered complexes of vector spaces over a field $\mathbf{k}$.

The following characterization of bifiltered quasi-isomorphisms will be useful in the construction of minimal models.

2.4.7. Given a bifiltered complex (K, W, F), we have a diagram of exact rows and columns

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & W^{p+1}Gr_F^q K \arrow[r] \arrow[u] & W^pGr_F^qK \arrow[r] \arrow[u] & Gr_W^pGr_F^qK \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & W^{p+1}F^qK \arrow[r] \arrow[u] & W^pF^qK \arrow[r] \arrow[u] & Gr_W^pF^qK \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & W^{p+1}F^{q+1}K \arrow[r] \arrow[u] & W^pF^{q+1}K \arrow[r] \arrow[u] & Gr_W^pF^{q+1}K \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Lemma 2.4.8. *Let $f : K \to L$ be a morphism of bifiltered complexes, and let $r \in \mathbb{Z}$. The following are equivalent:*

(1) The map W^pF^qf is a quasi-isomorphism for all $p + q > r$.
(2) The map $Gr_W^pGr_F^qf$ is a quasi-isomorphism for all $p + q > r$.
(3) The map $\pi : W^pF^qC(f) \to Gr_W^pGr_F^qC(f)$ is a quasi-isomorphism, for all $p + q \geq r$.

PROOF. The proof is analogous to that of Proposition 2.1.38 using the above diagram of exact sequences. □

Definition 2.4.9. A bifiltered complex (K, W, F) is called $E_{r,0}$*-minimal* if

$$d(W^pF^qK) \subset W^{p+r+1}F^qK + W^{p+r}F^{q+1}K, \text{ for all } p, q \in \mathbb{Z}.$$

Proposition 2.4.10. *Let $r \in \{0, 1\}$. Every $E_{r,0}$-quasi-isomorphism between $E_{r,0}$-minimal complexes is an isomorphism.*

Proof. For $r = 0$, the proof is analogous to that of Proposition 2.3.5: given a bifiltered quasi-isomorphism $f : K \to K$, we show that it is an isomorphism by noting that

$$H^n(Gr^p_W Gr^q_F K) = Gr^p_W Gr^q_F K^n,$$

and using the fact that both filtrations are biregular. For $r = 1$ the proof follows by décalage. □

Theorem 2.4.11. *Let $r \in \{0, 1\}$. For every bifiltered complex K there is an $E_{r,0}$-minimal complex M, and an $E_{r,0}$-quasi-isomorphism $\rho : M \to K$.*

Proof. We first prove the case $r = 0$. We build by decreasing induction over $r \in \mathbb{Z}$, a family of bifiltered complexes M_r together with morphisms $\rho_r : M_r \to K$ satisfying:

(a_r) $M_r = M_{r+1} \oplus V_r$, where $V_r = \bigoplus_{p+q=r} V_{p,q}$, and $V_{p,q}$ is a graded vector space of pure biweight (p, q) satisfying

$$d(V_{p,q}) \subset W^{p+1} F^q M_{r+1} + W^p F^{q+1} M_{r+1}.$$

The map ρ_r extends ρ_{r+1}.

(b_r) $H^n(W^p F^q C(\rho_r)) = 0$ for all n, and all $p + q \geq r$.

Assume that we constructed M_{r+1}. For all $p, q \in \mathbb{Z}$ such that $p + q = r$, let $V_{p,q}$ be the graded vector space of pure biweight (p, q) defined by

$$V^n_{p,q} = H^n(Gr^p_W Gr^q_F C(\rho_{r+1})).$$

Define a bifiltered complex as

$$M_r = M_{r+1} \oplus \left(\bigoplus_{p+q=r} V_{p,q} \right).$$

Since $H^i(W^{p+1} F^q C(\rho_{r+1})) = 0$ and $H^i(W^p F^{q+1} C(\rho_{r+1})) = 0$ for all $i \geq 0$, it follows from the exact diagram of 2.4.7 that

$$H^n(W^p F^q C(\rho_{r+1})) \cong H^n(Gr^p_W Gr^q_F C(\rho_{r+1})) = V_{p,q}.$$

Define a differential $d : V_r \to M_{r+1}$ and a map $\rho_r : M_r \to K$ extending ρ_{r+1} by taking sections of the projections

$$Z^n(W^p F^q C(\rho_{r+1})) \twoheadrightarrow H^n(W^p F^q C(\rho_{r+1})) \cong V^n_{p,q}.$$

The proof follows analogously to Theorem 2.3.2, using Lemma 2.4.8. Hence the case $r = 0$ is completed.

The case $r = 1$ follows by décalage of the weight filtration: given a bifiltered complex K, take an $E_{0,0}$-minimal model $\rho : M \to \mathrm{Dec}K$. Then the morphism $\rho : S^W M \to K$ is an $E_{1,0}$-minimal model, where S^W denotes the shift with respect to the weight filtration. $\square$

Theorem 2.4.12. *Let $r \in \{0, 1\}$. The triple $(\mathbf{C}^+(\mathbf{F}^2\mathbf{k}), \mathcal{S}_{r,0}, \mathcal{E}_{r,0})$ is a Sullivan category. The category of $E_{r,0}$-minimal complexes is a full subcategory of minimal models.*

Bistrict Complexes. To end this section we study the main definitions and properties regarding complexes with bistrict differentials. The results of this section will be most important in the setting of mixed Hodge theory.

Definition 2.4.13. A bifiltered complex (K, W, F) is said to be *d-bistrict* if for all $p, q \in \mathbb{Z}$:

(i) $d(W^pF^qK) = d(K) \cap W^pF^qK$.
(ii) The filtered complexes $(Gr^p_W K, F)$ and $(Gr^q_F K, W)$ are d-strict

Proposition 2.4.14. *A bifiltered complex (K, W, F) is d-bistrict if and only if the four spectral sequences*

$$\begin{tikzcd}
E_1(Gr^\bullet_W K, F) \arrow[r, Rightarrow] \arrow[dd, equal] & E_1(K, W) \arrow[dr, Rightarrow] & \\
 & & H(K) \\
E_1(Gr^\bullet_F K, W) \arrow[r, Rightarrow] & E_1(K, F) \arrow[ur, Rightarrow] &
\end{tikzcd}$$

degenerate at E_1.

PROOF. Since the filtrations are biregular, a bifiltered complex (K, W, F) is d-strict if and only if the filtered complexes (K, W), (K, F), $(Gr^p_W K, F)$ and $(Gr^q_F K, W)$ are d-strict. By Proposition 2.3.11 this is equivalent to the degeneration at the first stage. $\square$

In particular, if (K, W, F) is d-bistrict then both (K, F) and (K, W) are d-strict, but the converse is not true in general (see also A.2 of [**Sai00**]).

Proposition 2.4.15 (cf. [**Del74b**], 7.2.8)**.** *Let (K, W, F) be a biregular d-bistrict bifiltered complex. Then*

$$W^pF^qH^n(K) = \mathrm{Im}\{H^n(W^pF^qK) \to H^n(K)\}.$$

Proof. By definition we have

$$W^pF^qH(K) = \mathrm{Im}\{H(W^pK) \xrightarrow{i^*} H(K)\} \cap \mathrm{Im}\{H(F^qK) \xrightarrow{j^*} H(K)\}.$$

Since K is d-strict, both morphisms i^* and j^* are injective. In particular we have a short exact sequence

$$0 \to W^pH^n(F^qK) \xrightarrow{j_p^*} W^pH^n(K) \xrightarrow{\pi_p^*} W^pH^n(K/F^qK) \to 0.$$

It suffices to note the identities

$$\mathrm{Im}\{H^n(F^pW^qK) \to H^n(K)\} = \mathrm{Im}(j_p^*) \text{ and } F^pW^qH^n(K) = \mathrm{Ker}(\pi_p^*).$$

□

The following is a consequence of Proposition 2.4.15, and generalizes Lemma 2.3.10. This result will be used in the study of mixed Hodge complexes. For convenience, we next consider F to be a decreasing filtration, and W to be increasing.

Lemma 2.4.16. *Let (K, W, F) be a d-bistrict bifiltered complex, and define:*

$$R^{p,q}H^n(K) := W_{p+q}F^pH^n(K),$$

$$R^{p,q}K := W_{p+q}F^pK.$$

Then

(1) Every class in $R^{p,q}H^n(K)$ has a representative in $R^{p,q}K$.
(2) If x is a coboundary in $R^{p,q}K$, then $x = dy$, with $y \in R^{p,q}K$.

The following is the analogous of Lemma 2.3.12 for bifiltered complexes, and gives sufficient conditions for a quasi-isomorphism of bifiltered complexes to be a bifiltered quasi-isomorphism.

Lemma 2.4.17. *Let $f : (K, W, F) \to (L, W, F)$ be a morphism of bifiltered complexes such that:*

(i) f is a quasi-isomorphism.
(ii) The map $f^* : H(K) \to H(L)$ is strictly compatible with W and F.

(iii) The complexes K and L are d-bistrict.

Then f is a bifiltered quasi-isomorphism.

PROOF. Since f^* is strictly compatible with filtrations we have

$$f^*(W^pF^qH(K)) = f^*(H(K)) \cap W^pF^qH(L).$$

Since f^* is an isomorphism, $f^*(W^pF^qH(K)) \cong W^pF^qH(L)$. Therefore

$$H(Gr_W^p Gr_F^q K) = Gr_W^p Gr_F^q H(K) \cong Gr_W^p Gr_F^q H(L) = H(Gr_W^p Gr_F^q L).$$

$\square$

Proposition 2.4.18. *Let $\rho : M \to K$ be an $E_{0,0}$-minimal model of a bifiltered complex K. If K is d-bistrict then $dM = 0$.*

PROOF. Since ρ is an $E_{0,0}$-quasi-isomorphism, if K is d-bistrict then M is d-bistrict. In particular, the complex $(Gr_W^p M, F)$ is d-strict and E_0-minimal. By Proposition 2.3.13 we have $dGr_W^p M = 0$ for all $p \in \mathbb{Z}$. Hence (M, W) is E_0-minimal and d-strict. Therefore we have $dM = 0$. $\square$

CHAPTER 3

Mixed Hodge Complexes

In this chapter we study the homotopy category of mixed Hodge complexes of vector spaces over the field of rational numbers. The main result is that the category of mixed Hodge complexes can be endowed with a Sullivan category structure, where the weak equivalences are the quasi-isomorphisms. In particular, we show that there exists a finite string of quasi-isomorphisms between a mixed Hodge complex and its cohomology, so that mixed Hodge complexes are formal. We also provide a description of the morphisms of mixed Hodge complexes in the homotopy category, in terms of morphisms and extensions of mixed Hodge structures, recovering the results of Carlson [**Car80**] and Beilinson [**Bei86**] on extensions of eixed Hodge structures, and provide an alternative proof of Beilinson's Theorem on the derived category of mixed Hodge structures (see [**Bei86**], Theorem 3.2).

The category of mixed Hodge complexes is a category of diagrams, whose vertices are filtered or bifiltered complexes. Hence the construction of minimal models involves a rectification of homotopy commutative morphisms of diagrams. This fits within the framework of P-categories developed in Chapter 1. However, since here the categories involved are categories of complexes of additive categories, the homotopy theory carries stronger properties. In general, the homotopy relation between morphisms in a P-category is not an equivalence relation. However, it becomes transitive for objects whose source is cofibrant. In particular, homotopy commutative morphisms of diagrams can not be composed. Thanks to the additive properties of complexes, this problematic vanishes, resulting in a much simpler homotopy theory for diagram categories. In Section 2 we develop such theory for diagrams of complexes of additive categories.

In Section 3 we recall the definition of mixed and absolute Hodge complexes respectively. Both definitions are related by Deligne's décalage functor. We use the abstract homotopy theory of the previous section to show that the graded mixed Hodge structure given by the cohomology of an absolute Hodge complex, defines a minimal model for the complex. This allows to define minimal models of mixed Hodge complexes, via Deligne's décalage. As an application, at the end of the chapter we read off the morphisms of the homotopy category of absolute or mixed Hodge complexes, in terms of morphisms and extensions of mixed Hodge structures.

3.1. Preliminaries

We give a brief survey on mixed Hodge theory. Most of the results of this section can be found in [**Del71b**].

Pure Hodge Structures. The primary example of a Hodge structure of weight n is that of the n-th cohomology of a compact Kähler manifold: this is a complex hermitian manifold such that the associated metric form is closed. Examples are given by any projective manifold equipped with its Fubini-Study metric. The condition on the metric has deep consequences on the geometry of the manifold. If X is a compact Kähler manifold and $H^{p,q}(X)$ denotes the space of cohomology classes of differential forms whose harmonic representative is of type (p,q), then the *Hodge decomposition Theorem* (see [**Hod41**] or e.g. [**Wel80**]) gives the following direct sum decomposition of the de Rham cohomology:

$$H^n_{dR}(X;\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X),$$

and $H^{p,q}(X) = \overline{H^{q,p}(X)}$. According to the formalism of Deligne, this is a Hodge structure of weight n of the real de Rham cohomology $H^n_{dR}(X;\mathbb{R})$.

For the rest of this section we let $\mathbf{k} \subset \mathbb{R}$ be a field.

Definition 3.1.1. A *Hodge structure of weight n* on a finite-dimensional vector space V defined over $\mathbf{k}$ is a direct sum decomposition of the complex vector space $V_{\mathbb{C}} = V \otimes_{\mathbf{k}} \mathbb{C}$ by a finite bigrading,

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}, \text{ with } V^{p,q} = \overline{V}^{q,p}.$$

The *Hodge numbers* of V are defined by $h^{p,q}(V) := \dim V^{p,q}$.

Definition 3.1.2. A *morphism of Hodge structures* $f : V_1 \to V_2$ is a morphism of $\mathbf{k}$-vector spaces whose complexification is a bigraded morphism.

Denote by $\mathsf{HS}(n)$ the category of Hodge structures of weight n over $\mathbf{k}$. Equivalently, a Hodge structure is given by a filtration F on $V_{\mathbb{C}}$.

Definition 3.1.3. Two decreasing filtrations W and F on a vector space V are said to be *n-opposed* if

$$Gr^p_W Gr^q_G V = 0 \text{ for all } p+q \neq n.$$

Given a Hodge structure of weight n on V, we define the *Hodge filtration* of $V_{\mathbb{C}}$ by

$$F^p V_{\mathbb{C}} = \bigoplus_{i \geq p} V^{i,n-i}.$$

Then F is n-opposed to its complex conjugate $\overline{F}$. By definition, any morphism of Hodge structures of weight n is compatible with the associated Hodge filtrations. Actually, any morphism of Hodge structures $f : V_1 \to V_2$ is strictly compatible with the filtrations.

Conversely, given a filtration F on $V_{\mathbb{C}}$ satisfying the n-opposed condition, we obtain a Hodge structure of weight n on V by

$$V^{p,q} := F^p V_{\mathbb{C}} \cap \overline{F}^q V_{\mathbb{C}}.$$

This gives an equivalence between Hodge structures of weight n and filtrations that are n-opposed to their complex conjugates.

Mixed Hodge Structures.

Definition 3.1.4. A *mixed Hodge structure* on a vector space V defined over $\mathbf{k} \subset \mathbb{R}$ consists in a bounded below increasing filtration W on V, called *weight filtration*, and a decreasing filtration F on $V_{\mathbb{C}}$, called *Hodge filtration*, such that for all n, the filtration induced by F on $Gr_n^W(V_{\mathbb{C}})$ is finite and n-opposed to its complex conjugate.

Given a mixed Hodge structure $\{V, W, F\}$, then on $Gr_n^W(V)$ we have a Hodge structure of weight n. This gives a decomposition

$$Gr_n^W(V)_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}.$$

Define $h^{p,q}(V) = \dim_{\mathbb{C}}(V^{p,q})$. These are the *Hodge numbers* of the mixed Hodge structure. If we have a mixed Hodge structure on V such that $h^{p,q}(V) = 0$ for all $p + q \neq n$, then it is identical to a Hodge structure of weight n on V.

Definition 3.1.5. A *morphism of mixed Hodge structures* is a $\mathbf{k}$-linear map $f : V_1 \to V_2$ which is compatible with both filtrations W and F (and therefore it is automatically compatible with $\overline{F}$). It induces morphisms $Gr_n^W f$ of Hodge structures of weight n.

Denote by MHS the category of mixed Hodge structures over $\mathbf{k}$. The key properties of mixed Hodge structures are summarized in the following theorem of Deligne.

Theorem 3.1.6 ([**Del71b**], Thm 2.3.5)**.**

(1) The category of mixed Hodge structures is abelian; the kernels and cokernels of morphisms of mixed Hodge structures are endowed with the induced filtrations.
(2) Every morphism of mixed Hodge structures is strictly compatible with both the weight filtration and the Hodge filtration.
(3) The functor $Gr_n^W : \mathsf{MHS} \to \mathsf{HS}(n)$ is exact.
(4) The functor $Gr_F^p : \mathbf{C}^+(\mathbf{F}\mathbb{C}) \to \mathbf{C}^+(\mathbb{C})$ is exact.

Deligne's proof uses a splitting. This is a global decomposition for any given mixed Hodge structure, which generalizes the decomposition of pure Hodge structures. Since we will make extensive use of it, we next recall its definition.

Lemma 3.1.7 (See [**GS75**], Lemma 1.12)**.** *Let $\{V, W, F\}$ be a mixed Hodge structure. There is a direct sum decomposition $V_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q}$, given by*

$$I^{p,q} := (F^p W_{p+q}) \cap (\overline{F}^q W_{p+q} + \sum_{i \geq 2} \overline{F}^{q+1-i} W_{p+q-i}),$$

and such that

$$F^p = \bigoplus_{r \geq p} I^{r,q}, \text{ and } W_m = \bigoplus_{p+q \leq m} I^{p,q}.$$

This decomposition is functorial for morphisms of mixed Hodge structures.

It follows from the above lemma that we have a congruence $I^{p,q} \equiv \overline{I}^{p,q} \mod W_{p+q-2}$. This congruence explains why every mixed Hodge structure with a weight filtration of length two splits over $\mathbb{R}$ into a sum of pure Hodge structures.

3.2. Diagrams of Complexes

In this section we develop a homotopy theory for diagrams whose vertices are categories of complexes of additive categories, parallel to the homotopy theory developed in Chapter 1. We will apply this theory to the particular case in which the vertex categories are categories of filtered and bifiltered complexes in the next section.

For the rest of this section I is a finite directed category whose degree function takes values in $\{0, 1\}$ (see 1.3.4).

3.2.1. Let $\mathcal{C} : I \to \mathsf{Cat}$ be a functor from I to the category of categories. Denote $\mathcal{C}(i) = \mathcal{C}_i$, for all $i \in I$ and $\mathcal{C}(u) = u_* : \mathcal{C}_i \to \mathcal{C}_j$, for all $u : i \to j$. Assume that $\mathcal{C}$ satisfies the following conditions:

- (D_1) For all $i \in I$, $\mathcal{C}_i = \mathbf{C}^+(\mathcal{A}_i)$ is the category of complexes of an additive category $\mathcal{A}_i$.
- (D_2) There is a class of weak equivalences $\mathcal{W}_i$ of $\mathcal{C}_i$ making the triple $(\mathcal{C}_i, \mathcal{S}_i, \mathcal{W}_i)$ into a category with strong and weak equivalences, where $\mathcal{S}_i$ denotes the class of homotopy equivalences.
- (D_3) For all $u : i \to j$, the functor u_* is additive and preserves fibrant objects and strong and weak equivalences.

Objects of $\Gamma\mathcal{C}$ (see Definition 1.3.1) will be called *diagrams of complexes*: recall that a diagram of complexes is given by a family of objects $\{X_i \in \mathcal{C}_i\}$, for all $i \in I$, together with morphisms $\varphi_u : u_*(X_i) \to X_j$, for all $u : i \to j$. We will often omit the notation of the functors u_*, when there is no danger of confusion. Such a diagram will be denoted as

$$X = \left(X_i \overset{\varphi_u}{\dashrightarrow} X_j \right).$$

The category of diagrams $\Gamma\mathcal{C}$ has a class $\mathcal{W}$ of weak equivalences defined level-wise: a morphism $f : X \to Y$ of $\Gamma\mathcal{C}$ is in $\mathcal{W}$ if and only if $f_i \in \mathcal{W}_i$ for all $i \in I$. In addition, the constructions of complexes of additive categories of Section 2.1 extend naturally to the category of diagrams. In particular we have a translation functor, defined level-wise. This gives a notion of homotopy between morphisms of diagrams of complexes. Denote by $\mathcal{S}$ the class of homotopy equivalences of $\Gamma\mathcal{C}$. Note that if $f = (f_i) \in \mathcal{S}$, then $f_i \in \mathcal{S}_i$ for all $i \in I$, but the converse is not true in general. Since $\mathcal{S}_i \subset \mathcal{W}_i$ for all $i \in I$, it follows that $\mathcal{S} \subset \mathcal{W}$. Therefore the triple $(\Gamma\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a category with strong and weak equivalences.

Homotopy Commutative Morphisms. We next introduce a new category $\Gamma\mathcal{C}^h$ which has the same objects of those in $\Gamma\mathcal{C}$, but in which morphisms between diagrams are homotopy commutative.

Definition 3.2.2. Let X and Y be two objects of $\Gamma\mathcal{C}$. A *pre-morphism of degree n from X to Y* is pair of families $f = (f_i, F_u)$, where

(i) $f_i : X_i \to Y_i[n]$ is a map of degree n in $\mathcal{C}_i$, for all $i \in I$.
(ii) $F_u : X_i \to Y_j[n]$ is a map of degree $n-1$ in $\mathcal{C}_j$, for $u : i \to j \in I$.

Let $\underline{\mathrm{Hom}}^n(X, Y)$ denote the set of pre-morphisms of degree n from X to Y. Define the differential of $f = (f_i, F_u) \in \underline{\mathrm{Hom}}^n(X, Y)$ as

$$Df = (df_i - (-1)^n f_i d, F_u d + (-1)^n dF_u + (-1)^n (f_j \varphi_u - \varphi_u f_i)).$$

With this definition, $(\underline{\mathrm{Hom}}^*(X, Y), D)$ is a cochain complex.

Definition 3.2.3. A *ho-morphism* $f : X \rightsquigarrow Y$ is a pre-morphism of degree 0 such that $Df = 0$. Therefore it is given by pairs $f = (f_i, F_u)$ such that:

(i) $f_i : X_i \to Y_i$ is a morphism of complexes, for all $i \in I$.

(ii) $F_u : X_i \to Y_j[-1]$ satisfies $dF_u + F_u d = \varphi_u f_i - f_j \varphi_u$. Hence F_u is a homotopy from $f_j \varphi_u$ to $\varphi_u f_i$ in $\mathcal{C}_j$, making the diagram

$$\begin{tikzcd} X_i \arrow[r, "\varphi_u"] \arrow[d, "f_i"'] \arrow[dr, Rightarrow, "F_u"] & X_j \arrow[d, "f_j"] \\ Y_i \arrow[r, "\varphi_u"'] & Y_j \end{tikzcd}$$

commute up to homotopy in $\mathcal{C}_j$, for all $u : i \to j$.

The composition of two ho-morphisms $X \overset{f}{\rightsquigarrow} Y \overset{g}{\rightsquigarrow} Z$ is given by

$$gf = (g_i f_i, G_u f_i + g_j F_u) : X \rightsquigarrow Z.$$

The identity ho-morphism is $1_X = (1_{X_i}, 0)$. A ho-morphism $f : X \rightsquigarrow Y$ is invertible if and only if f_i are. In such case,

$$f^{-1} = (f_i^{-1}, -f_j^{-1} F_u f_i^{-1}).$$

Denote by $\Gamma\mathcal{C}^h$ the category whose objects are those of $\Gamma\mathcal{C}$, and whose morphisms are ho-morphisms. Every morphism $f = (f_i)$ can be made into a ho-morphism by setting $F_u = 0$. Hence $\Gamma\mathcal{C}$ is a subcategory of $\Gamma\mathcal{C}^h$.

Definition 3.2.4. A ho-morphism $f = (f_i, F_u)$ is said to be a *weak equivalence* if f_i are weak equivalences in $\mathcal{C}_i$. Denote by $\mathcal{W}^h$ the class of weak equivalences of $\Gamma\mathcal{C}^h$.

Definition 3.2.5. Let $f, g : X \rightsquigarrow Y$ be two ho-morphisms. A *homotopy from f to g* is a pre-morphism h of degree -1 such that $Dh = g - f$. Therefore $h = (h_i, H_u)$ is a pair of families such that:

(i) $h_i : X_i \to Y_i[-1]$ satisfies $dh_i + h_i d = g_i - f_i$. Hence h_i is a homotopy of complexes from f_i to g_i.

(ii) $H : X_i \to Y_j[-2]$ satisfies $H_u d - dH_u = G_u - F_u + h_j \varphi_u - \varphi_u h_i$.

Denote such a homotopy as $h : f \simeq g$.

Lemma 3.2.6. *The homotopy relation between ho-morphisms is an equivalence relation, compatible with the composition.*

PROOF. Symmetry and reflexivity are trivial. For transitivity, consider ho-morphisms $f, f', f'' : X \rightsquigarrow Y$ such that $h : f \simeq f'$, and $h' : f' \simeq f''$. A homotopy from f to f'' is then given by

$$h'' = h + h' = (h_i + h'_i, H_u + H'_u).$$

Let $g : Y \rightsquigarrow Z$ be a ho-morphism, and assume that $h : f \simeq f'$ is a homotopy from f to f'. A homotopy from gf to gf' is given by

$$gh = (g_i h_i, G_u h_i + g_j H_u).$$

Let $g' : W \rightsquigarrow X$ be a ho-morphism, and assume that $h : f \simeq f'$ is a homotopy from f to f'. A homotopy from fg' to $f'g'$ is given by

$$hg = (h_i g_i, H_u g_i + h_j G_u).$$

□

We will denote by $[X, Y]^h$ the class of ho-morphisms from X to Y modulo homotopy. Note that

$$[X, Y]^h = H^0(\underline{\mathrm{Hom}}^*(X, Y), D).$$

Denote by $\pi^h \Gamma\mathcal{C} := \Gamma\mathcal{C}^h / \simeq$ the corresponding quotient category.

Definition 3.2.7. Let $f : X \rightsquigarrow Y$ and $g : X \rightsquigarrow Z$ be ho-morphisms. The *double mapping cylinder of f and g* is the diagram of complexes given by

$$\mathcal{C}yl(f, g) = \left(\mathcal{C}yl(f_i, g_i) \overset{\psi_u}{\dashrightarrow} \mathcal{C}yl(f_j, g_j) \right),$$

where $\mathcal{C}yl(f_i, g_i) = X_i[1] \oplus Y_i \oplus Z_i$ is the double mapping cylinder of f_i and g_i (see Definition 2.1.3), with differential

$$D = \begin{pmatrix} -d & 0 & 0 \\ -f_i & d & 0 \\ g_i & 0 & d \end{pmatrix}.$$

For all $u : i \to j$, the comparison morphism is given by

$$\psi_u(x, y, z) = (\varphi_u(x), \varphi_u(y) + F_u(x), \varphi_u(z) + G_u(x)).$$

Remark 3.2.8. If $f : X \to Y$ and $g : X \to Z$ are morphisms of diagrams, since $F_u = 0$ and $G_u = 0$, we recover the level-wise definition of the double mapping cylinder of morphisms in $\Gamma\mathcal{C}$.

Definition 3.2.9. The *mapping cylinder* of a ho-morphism $f : X \rightsquigarrow Y$ is the diagram of complexes given by

$$\mathcal{C}yl(f) = \mathcal{C}yl(f, 1_X) = \left(\mathcal{C}yl(f_i) \overset{\psi}{\dashrightarrow} \mathcal{C}yl(f_j) \right),$$

where $\mathcal{C}yl(f_i) = X_i[1] \oplus Y_i \oplus X_i$ is the mapping cylinder of $f_i : X_i \to Y_i$. The comparison morphism is given by

$$\psi_u(x, y, z) = (\varphi_u(x), \varphi_u(y) + F(x), \varphi_u(z)).$$

Definition 3.2.10. Let $f : X \rightsquigarrow Y$ be a ho-morphism. The *mapping cone of* f is the diagram defined by $C(f) = \mathcal{C}yl(0, f)$.

Lemma 3.2.11. *Let $w : X \rightsquigarrow Y$ be a ho-morphism, and Z a diagram. A ho-morphism $\tau : C(w) \rightsquigarrow Z$ is equivalent to a pair (f, h), where $f : Y \rightsquigarrow Z$ is a ho-morphism and $h : X \rightsquigarrow Z[-1]$ is a homotopy from* 0 *to* fw.

Proof. Let $\tau = (\tau_i, T_u) : C(w) \rightsquigarrow Z$ be a ho-morphism. Define a ho-morphism $f : Y \rightsquigarrow Z$ and a homotopy $h : X \rightsquigarrow Z[-1]$ by:

$$\begin{cases} f_i(y) = \tau_i(0, y), & F_u(y) = T_u(0, y). \\ h_i(x) = \tau_i(x, 0), & H_u(x) = T_u(x, 0), \end{cases}$$

Conversely, given a ho-morphism $f : Y \rightsquigarrow Z$, and a homotopy $h : 0 \simeq fw$, define a ho-morphism $\tau : C(w) \rightsquigarrow Z$ by

$$\tau_i(x, y) = h_i(x) + f_i(y), \text{ and } T_u(x, y) = H_u(x) + F_u(y).$$

$\square$

Factorization of Ho-morphisms. The notion of homotopy between ho-morphisms allows to define a new class of strong equivalences of $\Gamma\mathcal{C}$.

Definition 3.2.12. A morphism $f : X \to Y$ of diagrams of complexes is said to be a *ho-equivalence* if there exists a ho-morphism $g : Y \rightsquigarrow X$, together with homotopies of ho-morphisms $gf \simeq 1_X$ and $fg \simeq 1_Y$.

Denote by $\mathcal{H}$ the class of ho-equivalences of $\Gamma\mathcal{C}$. This class is closed by composition, and satisfies $\mathcal{S} \subset \mathcal{H} \subset \mathcal{W}$, where $\mathcal{S}$ denotes the class of homotopy equivalences of $\Gamma\mathcal{C}$.

Consider the solid diagram of functors

$$\begin{tikzcd} \Gamma\mathcal{C} \arrow[r] \arrow[d, "\gamma"'] & \pi^h\Gamma\mathcal{C} \\ \Gamma\mathcal{C}[\mathcal{H}^{-1}] \arrow[ur, dotted, "\Psi"'] & \end{tikzcd}$$

Since every morphism of $\mathcal{H}$ is an isomorphism in $\pi^h\Gamma\mathcal{C}$, by the universal property of the localizing functor γ, the dotted functor exists. Our next objective is to prove that Ψ defines an equivalence of categories. We will do this by defining an inverse functor.

We first show that ho-morphisms satisfy a Brown factorization Lemma via the mapping cylinder of a ho-morphism.

Let $i_f : X \to \mathcal{C}yl(f)$ and $j_f : Y \to \mathcal{C}yl(f)$ be the maps defined level-wise by

$$i_{f_i}(x) = (0, 0, x) \text{ and } j_{f_i}(x) = (0, x, 0).$$

Define a ho-morphism $p_f = (p_{f_i}, P_{F_u}) : \mathcal{C}yl(f) \rightsquigarrow Y$ by means of the level-wise morphisms

$$p_{f_i}(x, y, z) = y + f_i(z),$$

together with the homotopies $P_{F_u} : \mathcal{C}yl(f_i) \to X_j[-1]$ given by

$$P_{F_u}(x, y, z) = F_u(z).$$

Proposition 3.2.13. *Let $f : X \rightsquigarrow Y$ be a ho-morphism of $\Gamma\mathcal{C}$. The diagram*

$$\begin{tikzcd} X \arrow[r, "i_f"] \arrow[dr, squiggly, "f"'] & \mathcal{C}yl(f) \arrow[d, squiggly, "p_f"] & Y \arrow[l, "j_f"'] \arrow[dl, equal] \\ & Y & \end{tikzcd}$$

commutes. In addition,

(1) The maps j_f and p_f are weak equivalences.

(2) There is a homotopy of ho-morphisms between $j_f p_f$ and $1_{\mathcal{C}yl(f)}$.

(3) If f is a weak equivalence, then i_f is a weak equivalence.

Proof. It is a matter of verification that the diagram commutes. Since weak equivalences are defined level-wise, (1) and (3) are straightforward. We prove (2). Let $h_i : \mathcal{C}yl(f_i) \to \mathcal{C}yl(f_i)[-1]$ be defined by

$$h_i(x, y, z) = (z, 0, 0).$$

Then h_i is a homotopy from $1_{\mathcal{C}yl(f_i)}$ to $j_f p_f$. Indeed,

$$(dh_i + h_i d)(x, y, z) = (-x, f_i(z), -z) = j_{f_i} p_{f_i}(x, y, z) - (x, y, z).$$

We have

$$(h_j \psi_u - \psi_u h_i)(x, y, z) = (0, -F_u(z), 0) = -j_{f_j} P_{F_u}(x, y, z).$$

Therefore the pair of families $h = (h_i, H_u)$ with $H_u = 0$, is a homotopy of ho-morphisms from $1_{\mathcal{C}yl(f)}$ to $\iota_f p_f$. $\square$

3.2.14. Given arbitrary diagrams X, Y of $\Gamma\mathcal{C}$, define a map

$$\Phi_{X,Y} : \Gamma\mathcal{C}^h(X, Y) \longrightarrow \Gamma\mathcal{C}[\mathcal{H}^{-1}](X, Y)$$

as follows. Let $f : X \rightsquigarrow Y$ be a ho-morphism. By Proposition 3.2.13 we can write $f = p_f i_f$, where i_f is a morphism of $\Gamma\mathcal{C}$ and p_f is ho-morphism with homotopy inverse j_f. In particular, j_f is a ho-equivalence. We let

$$\Phi_{X,Y}(f) := \{j_f^{-1} i_f\} \in \Gamma\mathcal{C}[\mathcal{H}^{-1}].$$

We will need the following technical lemmas.

Lemma 3.2.15. *If $f : X \to Y \in \Gamma\mathcal{C}$, then $\Phi_{X,Y}(f) = \{f\}$.*

Proof. If f is a morphism, then p_f is also a morphism. The diagram

$$\begin{tikzcd} X \arrow[r, "i_f"] \arrow[dr, "f"'] & \mathcal{C}yl(f) \arrow[d, "p_f"] & Y \arrow[l, "j_f"'] \arrow[dl, equal] \\ & X & \end{tikzcd}$$

is a hammock between the $\mathcal{H}$-zigzags $j_f^{-1} i_f$ and f. $\square$

Lemma 3.2.16. *If $h : f \simeq g$ then $\Phi_{X,Y}(f) = \Phi_{X,Y}(g)$.*

PROOF. Define a ho-morphism $\widetilde{h} = (\widetilde{h}_i, \widetilde{H}_u) : \mathcal{C}yl(f) \to \mathcal{C}yl(g)$ by

$$\widetilde{h}_i(x, y, z) = (x, y + h_i(x), z).$$

Then $d\widetilde{h}_i = \widetilde{h}_i d$, and $(\psi_u \widetilde{h}_i - \widetilde{h}_j \psi_u)(x, y, z) = (0, (\varphi_u h_i - h_j \varphi_u + G_u - F_u)x, 0)$. Define $\widetilde{H}_u : \mathcal{C}yl(f_i) \to \mathcal{C}yl(g_j)$ by

$$\widetilde{H}_u(x, y, z) = (0, H_u(x), 0).$$

Then $d\widetilde{H}_u + \widetilde{H}_u d = \psi_u \widetilde{h}_i - \widetilde{h}_j \psi_u$ and so $\widetilde{h} = (\widetilde{h}_i, \widetilde{H})$ is a ho-morphism. Consider the diagram of morphisms

$$\begin{tikzcd}
 & \mathcal{C}yl(f) \arrow[d, "i_{\widetilde{h}}"] & \\
X \arrow[ur, "i_f"] \arrow[r, "i_{\widetilde{h}} i_f"] \arrow[dr, "i_g"'] & \mathcal{C}yl(\widetilde{h}) & Y \arrow[ul, "j_f"'] \arrow[l, "j_{\widetilde{h}} j_g"'] \arrow[dl, "j_g"] \\
 & \mathcal{C}yl(g) \arrow[u, "j_{\widetilde{h}}"'] &
\end{tikzcd}$$

To see that $\Phi_{X,Y}(f) = \Phi_{X,Y}(g)$ it suffices to show that the above diagram is commutative in $\Gamma\mathcal{C}[\mathcal{S}^{-1}]$. By definition, the upper-left and the bottom-right triangles are commutative in $\Gamma\mathcal{C}$. Let $k_{\widetilde{h}} : \mathcal{C}yl(f) \to \mathcal{C}yl(\widetilde{h})[-1]$ be defined level-wise by $k_{\widetilde{h}_i}(x) = (x, 0, 0)$ with $K_{\widetilde{H}_u} = 0$. Then $k_{\widetilde{h}}$ is a homotopy from $i_{\widetilde{h}}$ to $j_{\widetilde{h}}\widetilde{h}$. On the other hand, note that $\widetilde{h} i_f = i_g$ and $\widetilde{h} j_f = j_g$. Therefore

$$i_{\widetilde{h}} i_f \simeq j_{\widetilde{h}} \widetilde{h} i_f = j_{\widetilde{h}} i_g, \text{ and } i_{\widetilde{h}} j_f \simeq j_{\widetilde{h}} \widetilde{h} j_f = j_{\widetilde{h}} j_g.$$

Hence the upper-right and the bottom-left triangles commute in $\Gamma\mathcal{C}[\mathcal{S}^{-1}]$. $\square$

Lemma 3.2.17. *Let $f : X \rightsquigarrow Y$ be a ho-morphism, and $g : Z \to X$ a morphism. Then $\Phi_{Z,Y}(fg) = \Phi_{X,Y}(f) \circ \Phi_{Z,X}(g)$.*

PROOF. Let $h_i : \mathcal{C}yl(f_i g_i) \to \mathcal{C}yl(f_i)$ be defined by

$$h_i(x, y, z) = (g_i(x), y, g_i(z)).$$

These define a morphism $h : \mathcal{C}yl(fg) \to \mathcal{C}yl(f)$, since

$$(\psi_u h_i - h_j \psi_u)(x, y) = \psi_u(g_i(x), y, g_i(z)) - h_j(\varphi_u x, \varphi_u y + F_u g_i(x), \varphi_u z) = 0.$$

The diagram

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, equal] & X \arrow[r, "i_f"] & \mathcal{C}yl(f) & Y \arrow[l, "j_f"'] \arrow[d, equal] \\
Z \arrow[r, equal] & Z \arrow[u, "g"] \arrow[r, "i_{fg}"] & \mathcal{C}yl(fg) \arrow[u, "h"] & Y \arrow[l, "j_{fg}"']
\end{tikzcd}$$

is a hammock between the $\mathcal{H}$-zigzags $j_f^{-1} i_f g$ and $j_{fg}^{-1} i_{fg}$. □

Theorem 3.2.18. *There is an equivalence of categories*

$$\Psi : \Gamma\mathcal{C}[\mathcal{H}^{-1}] \xrightarrow{\sim} \pi^h \Gamma\mathcal{C}.$$

Proof. By Lemma 3.2.16 the map

$$\Phi_{X,Y} : \pi^h \Gamma\mathcal{C}(X,Y) \longrightarrow \Gamma\mathcal{C}[\mathcal{H}^{-1}](X,Y)$$

given by $[f] \mapsto \{j_f^{-1} i_f\}$ is well defined, for any pair of objects X and Y. For the rest of the proof we omit the subscripts of Φ.

Let $f : X \rightsquigarrow Y$ be a ho-morphism. Then

$$\Psi(\Phi([f])) = \Psi\{j_f^{-1} i_f\} = [p_f i_f] = [f].$$

For the other composition, it suffices to show that if $g : X \to Y$ is a ho-equivalence, then $\Phi(\Psi(g^{-1})) = g^{-1}$. Let $h : Y \rightsquigarrow X$ be a homotopy inverse of g. Then

$$g \circ \Phi(\Psi(g^{-1})) = [g] \circ \Phi(h) = \Phi(gh) = 1.$$

If we compose on the left by g^{-1} we have: $\Phi(\Psi(g^{-1})) = g^{-1}$. □

Fibrant Models of Diagrams. Denote by $\Gamma\mathcal{C}_f$ the full subcategory of $\Gamma\mathcal{C}$ consisting of those diagrams

$$Q = \left(Q_i \overset{\varphi}{\dashrightarrow} Q_j \right)$$

such that for all $i \in I$, Q_i is fibrant in $(\mathcal{C}_i, \mathcal{S}_i, \mathcal{W}_i)$, that is: every weak equivalence $f : X \to Y$ in $\mathcal{C}_i$ induces a bijection $w^* : [Y, Q_i] \to [X, Q_i]$. Condition ($\mathrm{D}_3$) of 3.2.1 implies that for all $u : i \to j$, the object $u_*(Q_i)$ is fibrant in $(\mathcal{C}_j, \mathcal{S}_j, \mathcal{W}_j)$.

Proposition 3.2.19. *Let Q be an object of $\Gamma\mathcal{C}_f$. Then every weak equivalence $w : X \rightsquigarrow Y$ induces a bijection*

$$w^* : [Y, Q]^h \longrightarrow [X, Q]^h,\ [f] \mapsto [fw]$$

PROOF. We first prove surjectivity. Let $f : X \rightsquigarrow Q$ be a ho-morphism. Since Q_i is fibrant in $\mathcal{C}_i$, there exists a morphism $g_i : Y_i \to Q_i$, together with a homotopy $h_i : g_i w_i \simeq f_i$, for all $i \in I$. We have a chain of homotopies

$$g_j \varphi_u w_i \overset{-g_j W_u}{\simeq} g_j w_j \varphi_u \overset{h_j \varphi_u}{\simeq} f_j \varphi_u \overset{F_u}{\simeq} \varphi_u f_i \overset{-\varphi_u h_i}{\simeq} \varphi_u g_i w_i.$$

This gives a homotopy

$$G'_u := h_j \varphi_u - \varphi_u h_i + F_u - g_j W_u : g_j \varphi_u w_i \simeq \varphi_u g_i w_i.$$

Since w_i is a weak equivalence in $\mathcal{C}_i$ and $u_*(Q_i)$ is fibrant in $\mathcal{C}_j$, there exists a homotopy $G_u : g_j \varphi_u \simeq \varphi_u g_i$, together with a second homotopy $H_u : G'_u \simeq G_u w_i$. Then the pair of families $g = (g_i, G_u)$ is a ho-morphism, and $H = (h_i, H_u)$ is a homotopy from gw to f.

To prove injectivity, it suffices to show that if $f : Y \rightsquigarrow Q$ is a ho-morphism such that $0 \simeq fw$, then $0 \simeq f$. Let $h : 0 \simeq fw$ be a homotopy of ho-morphisms from 0 to fw. By Lemma 3.2.11 this defines a ho-morphism $\tau : C(w) \rightsquigarrow Q$. Consider the solid diagram

$$\begin{tikzcd} C(w) \arrow[r, rightsquigarrow, "\tau"] \arrow[d, rightsquigarrow] & Q \\ C(1_Y) \arrow[ur, dotted, "\tau'"'] & \end{tikzcd}$$

Since the map w is a weak equivalence, the induced map

$$(w \otimes 1)^* : [C(1_Y), Q]^h \longrightarrow [C(w), Q]^h$$

is surjective. This means that there exists a ho-morphism $\tau' : C(1_Y) \rightsquigarrow Q$ such that $h' : \tau' w \simeq \tau$. By Lemma 3.2.11 this defines a ho-morphism $f' : Y \rightsquigarrow F$ such that $0 \simeq f$. Since $\tau' w \simeq \tau$, it follows that $f' \simeq f$. Lastly, since the homotopy of ho-morphisms is transitive we have $f \simeq 0$. Therefore the map w^* is injective. $\square$

Proposition 3.2.20. *Let Q be an object of $\Gamma\mathcal{C}_f$, then every weak equivalence $w : X \to Y$ induces a bijection*

$$w^* : \Gamma\mathcal{C}[\mathcal{H}^{-1}](Y,Q) \longrightarrow \Gamma\mathcal{C}[\mathcal{H}^{-1}](X,Q).$$

PROOF. Given a weak equivalence $w : X \to Y$, consider the diagram

$$\begin{tikzcd} \Gamma\mathcal{C}[\mathcal{H}^{-1}](Y,Q) \arrow[r, "-\circ\{w\}"] \arrow[d, "\Psi"] & \Gamma\mathcal{C}[\mathcal{H}^{-1}](X,Q) \\ {[Y,Q]^h} \arrow[r, "-\circ[w]"] & {[X,Q]^h} \arrow[u, "\Phi"'] \end{tikzcd}$$

Let $f \in \Gamma\mathcal{C}[\mathcal{H}^{-1}](Y,Q)$. Then by Lemmas 3.2.17 and 3.2.15, we have

$$\Phi\Psi(f) \circ [w]) = \Phi(\Psi(f)) \circ \Phi([w]) = \Phi(\Psi(f)) \circ \{w\}.$$

By Theorem 3.2.18 the vertical arrows are bijections, and the diagram commutes. The result follows from Proposition 3.2.19. □

We next prove the existence of enough fibrant models.

Proposition 3.2.21. *Let $\Gamma\mathcal{C}$ be a category of diagrams satisfying the hypothesis of 3.2.1, and assume that every object of $\mathcal{C}_i$ has a fibrant model in $(\mathcal{C}_i, \mathcal{S}_i, \mathcal{W}_i)$. Then for every object X of $\Gamma\mathcal{C}$ there is an object $Q \in \Gamma\mathcal{C}_f$, together with a ho-morphism $K \rightsquigarrow Q$, which is a weak equivalence.*

PROOF. Let $\rho_i : X_i \to Q_i$ be fibrant models in $(\mathcal{C}_i, \mathcal{S}_i, \mathcal{W}_i)$. Since $u_*(Q_i)$ is fibrant in $\mathcal{C}_j$, for every solid diagram

$$\begin{tikzcd} X_i \arrow[r, "\varphi_u"] \arrow[d, "\rho_i"] & X_j \arrow[d, "\rho_j"] \\ Q_i \arrow[r, dotted, "\varphi'_u"] & Q_j \end{tikzcd}$$

there exists a dotted arrow φ'_u, together with a homotopy $R_u : \rho_j\varphi_u - \varphi'_u\rho_i$. This defines a diagram of $\Gamma\mathcal{C}_f$

$$Q = \left(Q_i \overset{\varphi'_u}{\dashrightarrow} Q_j \right).$$

The pair of families $\rho = (\rho_i, R_u) : X \rightsquigarrow Q$ is a ho-morphism of $\mathcal{W}^h$. □

The main result of this section is the following.

Theorem 3.2.22. *Let* $\Gamma\mathcal{C}$ *be a category of diagrams of complexes satisfying the hypothesis of* 3.2.1. *Let* $\mathcal{D}$ *be a full subcategory of* $\Gamma\mathcal{C}$ *such that:*

(i) Given a weak equivalence $X \overset{\sim}{\rightsquigarrow} Y$ *in* $\Gamma\mathcal{C}$, *then* X *is an object of* $\mathcal{D}$ *if and only if* Y *is so.*

(ii) For every object D *of* $\mathcal{D}$ *there is an object* $Q \in \mathcal{D}_f := \mathcal{D} \cap \Gamma\mathcal{C}_f$, *together with a ho-morphism* $D \rightsquigarrow Q$, *which is a weak equivalence.*

Then the triple $(\mathcal{D}, \mathcal{H}, \mathcal{W})$ *is a right Cartan-Eilenberg category with models in* $\mathcal{D}_f$. *There are equivalences of categories*

$$\pi^h\mathcal{D}_f \xrightarrow{\sim} \mathcal{D}_f[\mathcal{H}^{-1}, \mathcal{D}] \xrightarrow{\sim} \mathcal{D}[\mathcal{W}^{-1}].$$

PROOF. Let $f : X \rightsquigarrow Y$ be a ho-morphism between objects of $\mathcal{D}$. Since the map $j_f : Y \longrightarrow \mathcal{C}yl(f)$ is a weak equivalence, by (i), the mapping cylinder $\mathcal{C}yl(f)$ is an object of $\mathcal{D}$. As a consequence, there is a map

$$\Phi_{X,Y} : \mathcal{D} \cap \Gamma\mathcal{C}^h(X, Y) \longrightarrow \mathcal{D}[\mathcal{H}^{-1}](X, Y),$$

for every pair of objects X, Y of $\mathcal{D}$, defined as in 3.2.14. The proper variant of Theorem 3.2.18 gives an equivalence of categories

$$\Psi : \mathcal{D}[\mathcal{H}^{-1}] \xrightarrow{\sim} \pi^h\mathcal{D}.$$

By (ii), for every object D of $\mathcal{D}$ there exists a fibrant object Q and a weak equivalence $\rho : D \rightsquigarrow Q$. Then $\Phi_{D,Q}(\rho) : D \to Q$ is a fibrant model. $\square$

3.3. HOMOTOPY THEORY OF HODGE COMPLEXES

Diagrams of Filtered Complexes. For the rest of this chapter we let $\mathbf{k} = \mathbb{Q}$ be the field of Rational numbers, and we let I be the category

$$I = \{0 \to 1 \leftarrow 2 \to \cdots \leftarrow s\}.$$

We next define the category of diagrams of filtered complexes. This is a diagram category of fixed type I, whose vertices are categories of filtered and bifiltered complexes. Additional assumptions on the behaviour of the filtrations will lead to the notion of mixed and absolute Hodge complexes.

Definition 3.3.1. Let $\mathbf{C} : I \to \mathsf{Cat}$ be the functor defined by

$$\begin{tikzcd} 0 \arrow[r, "u"] \arrow[d, mapsto] & 1 \arrow[d, mapsto] & \cdots \arrow[l] \arrow[r] & s-1 \arrow[d, mapsto] & s \arrow[l, "v"'] \arrow[d, mapsto] \\ \mathbf{C}^+(\mathbf{Fk}) \arrow[r, "u_*"] & \mathbf{C}^+(\mathbf{F}\mathbb{C}) & \cdots \arrow[l, "Id"'] \arrow[r, "Id"] & \mathbf{C}^+(\mathbf{F}\mathbb{C}) & \mathbf{C}^+(\mathbf{F}^2\mathbb{C}) \arrow[l, "v_*"'] \end{tikzcd}$$

where u_* is defined by extension of scalars

$$u_*(K_{\mathbf{k}}, W) := (K_{\mathbf{k}}, W) \otimes \mathbb{C},$$

and v_* is defined by forgetting the second filtration

$$v_*(K_{\mathbb{C}}, W, F) := (K_{\mathbb{C}}, W).$$

All intermediate functors are defined to be the identity.

The category of diagrams $\Gamma\mathbf{C}$ associated with the functor $\mathbf{C}$ is called the *category of diagrams of filtered complexes over* $\mathbf{k}$.

Objects and morphisms in $\Gamma\mathbf{C}$ are defined as follows:

- A *diagram of filtered complexes* consists in
  (i) a filtered complex $(K_{\mathbf{k}}, W)$ over $\mathbf{k}$,
  (ii) a bifiltered complex $(K_{\mathbb{C}}, W, F)$ over $\mathbb{C}$, together with
  (iii) a morphism $\varphi_u : (K_i, W) \to (K_j, W)$ of filtered complexes over $\mathbb{C}$, for each $u : i \to j$ of I, with $(K_0, W) = (K_{\mathbf{k}}, W) \otimes \mathbb{C}$ and $K_s = (K_{\mathbb{C}}, W)$.

Such a diagram is denoted as

$$K = \Big((K_{\mathbf{k}}, W) \overset{\varphi}{\longleftarrow\!\!\dashrightarrow} (K_{\mathbb{C}}, W, F) \Big).$$

- A *morphism of diagrams of filtered complexes* $f : K \to L$ consists in
  (i) a morphism of filtered complexes $f_{\mathbf{k}} : (K_{\mathbf{k}}, W) \to (L_{\mathbf{k}}, W)$,
  (ii) a morphism of bifiltered complexes $f_{\mathbb{C}} : (K_{\mathbb{C}}, W, F) \to (L_{\mathbb{C}}, W, F)$, and
  (iii) a family of morphisms of filtered complexes $f_i : (K_i, W) \to (L_i, W)$ with $f_0 = f_{\mathbf{k}} \otimes \mathbb{C}$, and $f_s = f_{\mathbb{C}}$, making the following diagrams commute.

$$\begin{tikzcd} (K_i, W) \arrow[r, "\varphi_u"] \arrow[d, "f_i"'] & (K_j, W) \arrow[d, "f_j"] \\ (L_i, W) \arrow[r, "\varphi_u"] & (L_j, W) \end{tikzcd}$$

Definition 3.3.2. A morphism $f : K \to L$ of diagrams of filtered complexes is said to be a *quasi-isomorphism* if the maps $f_{\mathbf{k}}$, $f_{\mathbb{C}}$ and f_i are quasi-isomorphisms: that is, the induced maps $H(f_{\mathbf{k}})$, $H(f_{\mathbb{C}})$ and $H(f_i)$ are isomorphisms.

Denote by $\mathcal{Q}$ the class of quasi-isomorphisms of $\Gamma\mathbf{C}$.

For the rest of this section we fix $r \in \{0, 1\}$. We introduce two subclasses of weak equivalences of $\mathcal{Q}$, defined level-wise by E_r-quasi-isomorphisms and $E_{r,0}$-quasi-isomorphisms (see the corresponding definitions in Chapter 2).

Definition 3.3.3. A morphism $f : A \to B$ of diagrams of filtered complexes is called $E_{r,0}$-*quasi-isomorphism* if $f_{\mathbf{k}}$ and f_i are E_r-quasi-isomorphisms of filtered complexes, and $f_{\mathbb{C}}$ is an $E_{r,0}$-quasi-isomorphism of bifiltered complexes.

Denote by $\mathcal{E}_{r,0}$ the class of $E_{r,0}$-quasi-isomorphisms of $\Gamma\mathbf{C}$. Since the filtrations are biregular, we have $\mathcal{E}_{0,0} \subset \mathcal{E}_{1,0} \subset \mathcal{Q}$. Hence we have functors

$$\mathrm{Ho}_{0,0}(\Gamma\mathbf{C}) \longrightarrow \mathrm{Ho}_{1,0}(\Gamma\mathbf{C}) \longrightarrow \mathrm{Ho}(\Gamma\mathbf{C})$$

relating the localizations with respect to $\mathcal{E}_{0,0}$, $\mathcal{E}_{0,1}$ and $\mathcal{Q}$ respectively.

Deligne's décalage with respect to the weight filtration defines a functor

$$\mathrm{Dec}^W : \Gamma\mathbf{C} \longrightarrow \Gamma\mathbf{C}$$

which sends a diagram of complexes K, to the diagram

$$\mathrm{Dec}^W K := \Big((K_{\mathbf{k}}, \mathrm{Dec}W) \overset{\varphi}{\dashleftarrow\dashrightarrow} (K_{\mathbb{C}}, \mathrm{Dec}W, F) \Big).$$

Analogously to Theorem 2.2.15 for filtered complexes we have:

Theorem 3.3.4. *Deligne's décalage induces an equivalence of categories*

$$\mathrm{Dec}^W : \mathrm{Ho}_{1,0}(\Gamma\mathbf{C}) \longrightarrow \mathrm{Ho}_{0,0}(\Gamma\mathbf{C}).$$

PROOF. The proof is analogous to that of Theorem 2.2.15. We only explain the main differences. The shift with respect to the weight filtration defines a functor S^W which is left adjoint to Dec^W, and there is an equivalence of categories $\mathrm{Dec}^W : \Gamma\mathbf{C}_1 \rightleftarrows \Gamma\mathbf{C} : S^W$, where $\Gamma\mathbf{C}_1$ denotes the full subcategory of $\Gamma\mathbf{C}$ of those diagrams such that $d_0 = 0$ on the associated spectral sequences. Since $\mathrm{Dec}^W(\mathcal{E}_{1,0}) \subset \mathcal{E}_{0,0}$, this induces an equivalence between the corresponding localizations. The adjunction $S^W \dashv \mathrm{Dec}^W$ gives an equivalence $\mathrm{Ho}_{1,0}(\Gamma\mathbf{C}_1) \to \mathrm{Ho}_{1,0}(\Gamma\mathbf{C})$. See Theorem 2.2.15 for details. $\square$

Hodge Complexes. We next recall the main definitions and properties regarding mixed and absolute Hodge complexes.

Definition 3.3.5 ([**Del74b**], 8.1.5)**.** A *mixed Hodge complex* is a diagram of filtered complexes

$$K = \Big((K_{\mathbf{k}}, W) \xleftarrow[\sim]{\varphi} \dashrightarrow (K_{\mathbb{C}}, W, F)\Big),$$

satisfying the following conditions:

(MHC$_0$) The comparison map φ is a string of E_1^W-quasi-isomorphisms.

(MHC$_1$) For all $p \in \mathbb{Z}$, the filtered complex $(Gr_p^W K_{\mathbb{C}}, F)$ is d-strict.

(MHC$_2$) The filtration F induced on $H^n(Gr_p^W K_{\mathbb{C}})$, defines a pure Hodge structure of weight $p+n$ on $H^n(Gr_p^W K_{\mathbf{k}})$, for all n, and all $p \in \mathbb{Z}$.

Denote by **MHC** the category of mixed Hodge complexes.

The following is an important result concerning the degeneration of each of the spectral sequences associated with a mixed Hodge complex.

Lemma 3.3.6 ([**Del74b**], Scholie 8.1.9)**.** *Given a mixed Hodge complex*

$$K = \Big((K_{\mathbf{k}}, W) \xleftarrow[\sim]{\varphi} \dashrightarrow (K_{\mathbb{C}}, W, F)\Big) \in \mathbf{MHC},$$

(1) the spectral sequence of $(K_{\mathbb{C}}, F)$ degenerates at E_1,

(2) the spectral sequences of $(K_{\mathbf{k}}, W)$ and $(Gr_F^p K_{\mathbb{C}}, W)$ degenerate at E_2.

For our convenience, we consider a shifted version of mixed Hodge complexes, in which both associated spectral sequences degenerate at the first stage.

Definition 3.3.7. An *absolute Hodge complex* is a diagram of filtered complexes

$$K = \Big((K_{\mathbf{k}}, W) \xleftarrow[\sim]{\varphi} \dashrightarrow (K_{\mathbb{C}}, W, F)\Big),$$

satisfying the following conditions:

(AHC_0) The comparison map φ is a string of E_0^W-quasi-isomorphisms.
(AHC_1) For all $p \in \mathbb{Z}$, the bifiltered complex $(K_{\mathbb{C}}, W, F)$ is d-bistrict.
(AHC_2) The filtration F induced on $H^n(Gr_p^W K_{\mathbb{C}})$, defines a pure Hodge structure of weight p on $H^n(Gr_p^W K_{\mathbf{k}})$, for all n, and all $p \in \mathbb{Z}$.

Denote by **AHC** the category of absolute Hodge complexes. The definition of absolute Hodge complex given here corresponds to the notion of mixed Hodge complex given by Beilinson in [**Bei86**] (see also pag. 273 of Levine's book on Mixed Motives [**Lev05**], or the appendix [**Hai87**], in which Hain defines shifted mixed Hodge complexes in a similar way).

By Lemma 3.3.6 Deligne's décalage with respect to the weight filtration sends every mixed Hodge complex to an absolute Hodge complex. Note however that the shift functor with respect to the weight filtration of an absolute Hodge complex is not in general a mixed Hodge complex. Therefore in this case, décalage does not have a left adjoint. Moreover, the cohomology of every absolute Hodge complex is an absolute Hodge complex with trivial differentials. We have functors

$$\mathbf{MHC} \xrightarrow{\mathrm{Dec}} \mathbf{AHC} \xrightarrow{H} \mathbf{G}^+(\mathsf{MHS}).$$

Conversely, since the category of mixed Hodge structures is abelian, every graded mixed Hodge structure and more generally, every complex of mixed Hodge structures is an absolute Hodge complex. We have full subcategories

$$\mathbf{G}^+(\mathsf{MHS}) \longrightarrow \mathbf{C}^+(\mathsf{MHS}) \longrightarrow \mathbf{AHC}.$$

We will prove that Hodge complexes are formal: every Hodge complex is quasi-isomorphic to the graded mixed Hodge structure given by its cohomology.

The following result is essentially due to Deligne's Theorem 3.1.6, which states that morphisms of mixed Hodge structures are strictly compatible with filtrations.

Lemma 3.3.8. *Denote by $\mathcal{Q}$ and $\mathcal{E}_{r,0}$ the classes of quasi-isomorphisms and $E_{r,0}$-quasi-isomorphisms of $\Gamma\mathbf{C}$.*

(1) The classes of maps $\mathcal{Q}$ and $\mathcal{E}_{0,0}$ coincide in **AHC**.

(2) The classes of maps $\mathcal{Q}$ and $\mathcal{E}_{1,0}$ coincide in **MHC**.

PROOF. We first prove (1). Let $f : K \to L$ be a quasi-isomorphism of absolute Hodge complexes. Then the induced morphism $f^* : H(K) \to H(L)$ is a morphism of graded mixed Hodge structures. By Theorem 3.1.6 the morphisms $f^*_{\mathbf{k}}$, f^*_i and $f^*_{\mathbb{C}}$ are strictly compatible with filtrations. Hence by Lemma 2.3.12, $f_{\mathbf{k}}$ and f_i are E_0-quasi-isomorphisms. Likewise, by Lemma 2.4.17, $f_{\mathbb{C}}$ is an $E_{0,0}$-quasi-isomorphism. The converse is trivial.

Let us prove (2). Let $f : K \to L$ be a quasi-isomorphism of mixed Hodge complexes. Then $\mathrm{Dec}^W f$ is a quasi-isomorphism of absolute Hodge complexes. The result follows from (1), and the fact that $\mathcal{E}_{1,0} = (\mathrm{Dec}^W)^{-1}(\mathcal{E}_{0,0})$. □

Minimal Models. The following technical lemma will be of use for the construction of minimal models.

Lemma 3.3.9. *Let K be an absolute Hodge complex.*

(1) There are sections $\sigma^n_{\mathbf{k}} : H^n(K_{\mathbf{k}}) \to Z^n(K_{\mathbf{k}})$ and $\sigma^n_i : H^n(K_i) \to Z^n(K_i)$ of the projection, which are compatible with W.

(2) There exists a section $\sigma^n_{\mathbb{C}} : H^n(K_{\mathbb{C}}) \to Z^n(K_{\mathbb{C}})$ of the projection, which is compatible with both filtrations W and F.

PROOF. The first assertion follows directly from Lemma 2.3.10. To prove the second assertion, note that by Lemma 3.1.7 the cohomology of $K_{\mathbb{C}}$ admits a splitting $H^n(K_{\mathbb{C}}) = \bigoplus I^{p,q}$, where

$$I^{p,q} := (W_{p+q}F^pH^n(K_{\mathbb{C}})) \cap (W_{p+q}\overline{F}^qH^n(K_{\mathbb{C}}) + \sum_{i\geq 2} W_{p+q-i}\overline{F}^{q+1-i}H^n(K_{\mathbb{C}})).$$

Therefore it suffices to define sections $\sigma^{p,q} : I^{p,q} \to Z^n(K_{\mathbb{C}})$. Let

$$R^{p,q}H^n(K_{\mathbb{C}}) := W_{p+q}F^pH^n(K_{\mathbb{C}}), \text{ and } R^{p,q}K_{\mathbb{C}} := W_{p+q}F^pK_{\mathbb{C}}.$$

We then have $I^{p,q} \subset R^{p,q}H^n(K_{\mathbb{C}})$. By Lemma 2.4.16 the morphisms $\sigma^{p,q}$ satisfy $\sigma^{p,q}(I^{p,q}) \subset R^{p,q}K_{\mathbb{C}}$. Define $\sigma^n_{\mathbb{C}} = \bigoplus \sigma^{p,q} : H^n(K_{\mathbb{C}}) \to K_{\mathbb{C}}$. Using the fact that

$$F^pH^n(K_{\mathbb{C}}) = \bigoplus_{p' \geq p} I^{p',q}$$

we obtain

$$\sigma^n_{\mathbb{C}}(F^pH^n(K_{\mathbb{C}})) = \bigoplus_{p'\geq p}\sigma^{p',q}(I^{p',q}) \subset \sum_{p'\geq p} R^{p',q}K_{\mathbb{C}} \subset \sum_{p'\geq p} F^{p'}K_{\mathbb{C}} \subset F^pK_{\mathbb{C}}.$$

Therefore $\sigma^n_{\mathbb{C}}$ is compatible with F. For the weight filtration we have

$$W_mH^n(K_{\mathbb{C}}) = \bigoplus_{p+q\leq m} I^{p,q}.$$

Then

$$\sigma^n_{\mathbb{C}}(W_mH^n(K_{\mathbb{C}})) = \bigoplus_{p+q\leq m}\sigma^{p,q}(I^{p,q}) \subset \sum_{p+q\leq m} R^{p,q}K_{\mathbb{C}} \subset W_mK_{\mathbb{C}}.$$

Therefore $\sigma^n_{\mathbb{C}}$ is compatible with W. □

We next show that the minimal model of every absolute Hodge diagram is given by its cohomology. In particular, the objects of **AHC** are formal (cf. pag. 47 of [**Bei86**]).

Theorem 3.3.10. *Let K be an absolute Hodge complex, and let*

$$H(K) := \left((H^*(K_{\mathbf{k}}), W) \overset{\varphi^*}{\underset{\cong}{\leftarrow\!\!\dashrightarrow}} (H^*(K_{\mathbb{C}}), W, F)\right)$$

be the absolute Hodge complex defined by the cohomology of K with the induced filtrations. There is a ho-morphism $\rho : K \rightsquigarrow H(K)$, which is a quasi-isomorphism.

PROOF. By Lemma 3.3.9 we can find sections $\sigma_{\mathbf{k}} : H^*(K_{\mathbf{k}}) \to K_{\mathbf{k}}$ and $\sigma_i : H^*(K_i) \to K_i$ compatible with the filtration W, together with a section $\sigma_{\mathbb{C}} : H^*(K_{\mathbb{C}}) \to K_{\mathbb{C}}$ compatible with W and F. By definition, all maps are quasi-isomorphisms. Let $\varphi_u : K_i \to K_j$ be a component of the quasi-equivalence φ of K. The diagram

$$\begin{tikzcd} H^*(K_i) \arrow[r, "\varphi_u^*"] \arrow[d, "\sigma_i"'] & H^*(K_j) \arrow[d, "\sigma_j"'] \\ K_i \arrow[r, "\varphi_u"] & K_j \end{tikzcd}$$

is not necessarily commutative, but for any element $x \in H^*(K_i)$, the difference $(\sigma_j\varphi_u^* - \varphi_u\sigma_j)(x)$ is a coborder. By Lemma 2.3.10 there exists a linear map

$$\Sigma_u : H^*(K_i) \to K_j[-1]$$

compatible with the weight filtration W, and such that

$$\sigma_j \varphi_u^* - \varphi_u \sigma_i = \Sigma_u d.$$

The above diagram commutes up to a filtered homotopy, and hence the morphisms $\sigma_{\mathbf{k}}$, σ_i and $\sigma_{\mathbb{C}}$, together with the homotopies Σ_u, define a ho-morphism $\sigma : H(K) \rightsquigarrow K$, which is a quasi-isomorphism by construction. Since every object of **AHC** is fibrant, by Proposition 3.2.19 this lifts to a quasi-isomorphism $\rho : K \rightsquigarrow H(K)$. □

Lemma 3.3.11. *Let $f : K \to L$ be a morphism of $\Gamma\mathbf{C}$.*

(1) If $f \in \mathcal{E}_{0,0}$, then K is an absolute Hodge complex if and only if L is so.
(2) If $f \in \mathcal{E}_{1,0}$, then K is a mixed Hodge complex if and only if L is so.

Proof. We first prove (1). Let $f : K \to L$ be an $E_{0,0}$-quasi-isomorphism of diagrams of filtered complexes. Let us check (AHC_0). Consider the diagram

$$\begin{tikzcd}
(K_i, W) \arrow[r, "\varphi_u^K"] \arrow[d, "f_i"'] & (K_j, W) \arrow[d, "f_j"] \\
(L_i, W) \arrow[r, "\varphi_u^L"] & (L_j, W)
\end{tikzcd} .$$

By assumption, the maps f_i and f_j are E_0-quasi-isomorphisms. By the two out of three property, it follows that φ_u^K is an E_0-quasi-isomorphism if and only if φ_u^L is so. Condition (AHC_1) follows from the fact that d-bistrictness is preserved by $E_{0,0}$-quasi-isomorphisms $f_{\mathbb{C}} : (K_{\mathbb{C}}, W, F) \to (L_{\mathbb{C}}, W, F)$. Condition ($\mathrm{AHC}_2$) is a consequence of the following isomorphisms:

$$H^n(Gr_p^W Gr_F^q K_{\mathbb{C}}) \cong H^n(Gr_p^W Gr_F^q L_{\mathbb{C}}), \text{ and } H^n(Gr_p^W K_{\mathbf{k}}) \cong H^n(Gr_p^W L_{\mathbf{k}}).$$

The proof of (2) follows analogously. □

Denote by $\pi^h\mathbf{G}^+(\mathsf{MHS})$ the category whose objects are non-negatively graded mixed Hodge structures and whose morphisms are homotopy classes of ho-morphisms. Denote by $\mathcal{H}$ the class of morphisms of absolute Hodge complexes that are homotopy equivalences as ho-morphisms.

Theorem 3.3.12. *The triple* $(\mathbf{AHC}, \mathcal{H}, \mathcal{Q})$ *is a Sullivan category, and* $\mathbf{G}^+(\mathsf{MHS})$ *is a full subcategory of minimal models. The inclusion induces an equivalence of categories*

$$\pi^h\mathbf{G}^+(\mathsf{MHS}) \xrightarrow{\sim} \mathrm{Ho}\,(\mathbf{AHC})\,.$$

PROOF. By Theorem 3.3.10 and Lemma 3.3.11 the conditions of Theorem 3.2.22 are satisfied. In addition, by Lemma 3.3.8 we have $\mathcal{H}_{0,0} = \mathcal{H}$ and $\mathcal{E}_{0,0} = \mathcal{Q}$. Hence the result follows. $\square$

Note that while objects of **AHC** are formal, its morphisms are not formal, since the category of minimal models has non-trivial homotopies.

Theorem 3.3.13. *The triple* $(\mathbf{MHC}, \mathcal{H}_{1,0}, \mathcal{Q})$ *is a Sullivan category. The minimal models are those mixed Hodge complexes* M *with trivial differential such that* $\mathrm{Dec}^W M$ *is a graded mixed Hodge structure.*

PROOF. By Lemma 3.3.11 condition (i) of Theorem 3.2.22 is satisfied for mixed Hodge complexes. Given a mixed Hodge complex K, by Theorem 3.3.10 there exists a quasi-isomorphism $\sigma : M := H(\mathrm{Dec}^W K) \rightsquigarrow K$. Recall that at the level of diagrams of filtered complexes we have an adjunction $S^W \dashv \mathrm{Dec}^W$. This gives a quasi-isomorphism $S^W M \rightsquigarrow K$. It remains to show that $S^W M$ is a mixed Hodge complex. The only non-trivial condition is (MHC_2). By Proposition 2.2.2 we have

$$H^n(Gr_p^{SW} M_{\mathbf{k}}) \cong Gr_{n+p}^W M_{\mathbf{k}}^n.$$

By construction, M is a graded mixed Hodge structure. In particular, for each $p \in \mathbb{Z}$, the vector space $Gr_{n+p}^W M_{\mathbf{k}}^n$ is endowed with a pure Hodge structure of weight $p+n$. Therefore M satisfies (MHC_2). Hence condition (ii) is satisfied. Since $\mathrm{Dec}^W S^W = 1$, by construction $\mathrm{Dec}^W S^W M = H(\mathrm{Dec}^W K)$ is a graded mixed Hodge structure. $\square$

Theorem 3.3.14. *Deligne's décalage induces an equivalence of categories*

$$\mathrm{Dec}^W : \mathrm{Ho}(\mathbf{MHC}) \xrightarrow{\sim} \mathrm{Ho}(\mathbf{AHC}).$$

PROOF. If suffices to note that when restricted to Hodge complexes with trivial differentials, the functor Dec^W has an inverse functor S^W. $\square$

Beilinson's Theorem. We next provide an alternative proof to Beilinson's Theorem on absolute Hodge complexes over the field of rational numbers and study further properties of the morphisms of absolute Hodge complexes in the homotopy category.

Theorem 3.3.15 (cf. [**Bei86**], Theorem. 3.2)**.** *The inclusion functor induces an equivalence of categories*

$$\pi^h\mathbf{G}^+(\mathsf{MHS}) \xrightarrow{\sim} \mathbf{D}^+(\mathsf{MHS}) \xrightarrow{\sim} \mathrm{Ho}\,(\mathbf{AHC})\,.$$

Proof. By Theorem 3.3.12 the triple $(\mathbf{AHC}, \mathcal{H}, \mathcal{Q})$ is a Sullivan category, and the minimal models are graded mixed Hodge structures. Furthermore, we have a chain of full subcategories

$$\mathbf{G}^+(\mathsf{MHS}) \subset \mathbf{C}^+(\mathsf{MHS}) \subset \mathbf{AHC}.$$

Since the category of mixed Hodge structures is abelian, the mapping cylinder of a ho-morphism of complexes of mixed Hodge structures is a complex of mixed Hodge structures. We have equivalences of categories

$$\mathbf{D}^+(\mathsf{MHS}) \xleftarrow{\sim} \mathbf{G}^+(\mathsf{MHS})[\mathcal{H}^{-1}, \mathbf{AHC}] \xrightarrow{\sim} \mathrm{Ho}\,(\mathbf{AHC})\,.$$

□

Every mixed Hodge structure can be identified with a complex of mixed Hodge structures concentrated in degree 0. With this identification, and since the category MHS is abelian, given mixed Hodge structures H and H' over a field $\mathbf{k}$, one can compute their extensions as

$$\mathrm{Ext}^n(H, H') = \mathbf{D}^+(\mathsf{MHS})(H, H'[n]).$$

Given filtered (resp. bifiltered) vector spaces X and Y over $\mathbf{k}$, denote by $\mathrm{Hom}^W(X,Y)$ (resp. $\mathrm{Hom}^W_F(X,Y)$ the set of morphisms from X to Y that are compatible with the filtration W (resp. the filtrations W and F).

We next recover a result of Carlson [**Car80**] regarding extensions of mixed Hodge structures, by studying the morphisms in the homotopy category of absolute Hodge complexes.

Proposition 3.3.16 (cf. [**Mor78**], Prop. 8.1)**.** *Let H and H' be mixed Hodge structures. Then*

$$\mathrm{Ext}^1(H,H') = \frac{\mathrm{Hom}^W(H_{\mathbb{C}},H'_{\mathbb{C}})}{\mathrm{Hom}^W(H_{\mathbf{k}},H'_{\mathbf{k}}) + \mathrm{Hom}^W_F(H_{\mathbb{C}},H'_{\mathbb{C}})},$$

and $\mathrm{Ext}^n(H,H') = 0$ *for all* $n > 1$.

PROOF. By Theorem 3.3.15 we have an equivalence of categories

$$\pi^h\mathbf{G}^+(\mathsf{MHS}) \xrightarrow{\sim} \mathbf{D}^+(\mathsf{MHS}).$$

we have

$$\mathbf{D}^+(\mathsf{MHS})(H,H'[n]) = H^0\left(\underline{\mathrm{Hom}}(H,H'[n])\right),$$

where $\underline{\mathrm{Hom}}^m(-,-)$ is the set of pre-morphisms of degree m in **AHC** (see Definition 3.2.2): a pre-morphism $f \in \underline{\mathrm{Hom}}^m(H,H'[n])$ is given by a triple $f = (f_{\mathbf{k}}, f_{\mathbb{C}}, F)$, where

(i) $f_{\mathbf{k}} : H_{\mathbf{k}} \to H'_{\mathbf{k}}[n+m]$ is compatible with W,
(ii) $f_{\mathbb{C}} : H_{\mathbb{C}} \to H'_{\mathbb{C}}[n+m]$ is compatible with W and F, and
(iii) $F : H_{\mathbb{C}} \to H'_{\mathbb{C}}[n+m-1]$ is compatible with W.

The differential of f is given by $Df = (0,0,(-1)^m(f_{\mathbb{C}} - f_{\mathbf{k}} \otimes \mathbb{C}))$.

For $n > 1$ we have $\underline{\mathrm{Hom}}^0(H,H'[n]) = 0$, and hence $\mathrm{Ext}^n(H,H') = 0$.

Let $f \in \underline{\mathrm{Hom}}^0(H,H'[1])$. Then $f_{\mathbf{k}} = 0$, and $f_{\mathbb{C}} = 0$. Therefore $Df = 0$, and

$$Z^0(\underline{\mathrm{Hom}}(H,H'[1])) = \mathrm{Hom}^W(H_{\mathbb{C}},H'_{\mathbb{C}}).$$

A morphism $f = (0,0,F) \in Z^0(\underline{\mathrm{Hom}}(H,H'[n]))$ is a coborder if and only if there exists a pair $h = (h_{\mathbf{k}}, h_{\mathbb{C}})$ where

(i) $h_{\mathbf{k}} : H_{\mathbf{k}} \to H'_{\mathbf{k}}$ is compatible with W,
(ii) $h_{\mathbb{C}} : H_{\mathbb{C}} \to H'_0\mathbb{C}$ is compatible with W and F,

and such that $F = h_{\mathbf{k}} \otimes \mathbb{C} - h_{\mathbb{C}}$. Therefore

$$B^0(\underline{\mathrm{Hom}}(H,H'[1])) = \mathrm{Hom}^W(H_{\mathbf{k}},H'_{\mathbf{k}}) + \mathrm{Hom}^W_F(H_{\mathbb{C}},H'_{\mathbb{C}}).$$

$\square$

Lastly, the morphisms in the homotopy category of absolute Hodge diagrams are characterized as follows.

Theorem 3.3.17. *Let K and L be absolute Hodge complexes. Then*

$$\mathrm{Ho}(\mathbf{AHC})(K,L) = \bigoplus_n \left(\mathrm{Hom}_{\mathsf{MHS}}(H^nK, H^nL) \oplus \mathrm{Ext}^1_{\mathsf{MHS}}(H^nK, H^{n-1}L)\right).$$

Proof. By Theorem 3.3.12 there is a bijection

$$\mathrm{Ho}(\mathbf{AHC})(K,L) \longrightarrow H^0(\underline{\mathrm{Hom}}(H(K), H(L))).$$

An element $f \in Z^0(\underline{\mathrm{Hom}}(H(K), H(L)))$ is given by:

(i) a morphism $f^*_{\mathbf{k}} : H^*(K_{\mathbf{k}}) \to H^*(L_{\mathbf{k}})$ compatible with W,

(ii) a morphism $f^*_{\mathbb{C}} : H^*(K_{\mathbb{C}}) \to H^*(L_{\mathbb{C}})$ compatible with W and F, such that $f_{\mathbf{k}} \otimes \mathbb{C} \cong f_{\mathbb{C}}$, together with

(iii) a morphism $F^* : H^*(K_{\mathbb{C}}) \to H^*(L_{\mathbb{C}})[-1]$ compatible with W.

Such a map is a coboundary if $f = Dh$, for some $h \in \underline{\mathrm{Hom}}^{-1}(H(K), H(L))$. This implies that $f_{\mathbf{k}} = 0$ and $f_{\mathbb{C}} = 0$, and that there exist:

(i) a morphism $h^*_{\mathbf{k}} : H^*(K_{\mathbf{k}}) \to H^*(L_{\mathbf{k}})[-1]$ compatible with W,

(ii) a morphism $h^*_{\mathbb{C}} : H^*(K_{\mathbb{C}}) \to H^*(L_{\mathbb{C}})[-1]$ compatible with W and F,

such that $F \cong h_{\mathbf{k}} \otimes \mathbb{C} - h_{\mathbb{C}}$. The result follows from Proposition 3.3.16. □

CHAPTER 4

Filtrations in Rational Homotopy

From Sullivan's theory, we know that the de Rham algebra of a manifold determines all its real homotopy invariants. In addition, the Formality Theorem of [**DGMS75**], exhibits the use of rational homotopy in the study of complex manifolds. For instance, it provides homotopical obstructions for the existence of Kähler metrics. Bearing these results in mind, and with the objective to study complex homotopy invariants, Neisendorfer and Taylor define in [**NT78**] the Dolbeault homotopy groups of a complex manifold by means of a bigraded model of its Dolbeault algebra of forms. Not only interesting in themselves, these new invariants prove to be useful in the computation of classical invariants such as the real homotopy or the cohomology of the manifold.

The Frölicher spectral sequence associated with complex manifolds provides a connection between Dolbeault and de Rham models, and indicates an interplay between models and spectral sequences. In [**HT90**], Halperin and Tanré analyse this issue in the abstract setting, by constructing models of filtered dga's and establishing a relationship with the bigraded minimal models of each stage of their associated spectral sequences. This allows the study of any spectral sequence coming from a filtration of geometric nature. The Dolbeault homotopy theory of Neisendorfer and Taylor fits naturally in this wider context. As an application, Tanré studies in [**Tan94**], the Borel spectral sequence associated with an holomorphic fibration, and constructs a Dolbeault model of the total space from those of the fibre and the base.

The construction of Halperin and Tanré is a generalization of the construction of bigraded models developed by Halperin and Stasheff in [**HS79**]. Their chief technique is to construct a filtered minimal model for a filtered dga, by perturbing the differential of a bigraded minimal model of the r-stage of its associated spectral sequence.

The category of filtered dga's does not admit a Quillen model structure. However, the existence of filtered minimal models allows to define a homotopy theory in a non-axiomatic conceptual framework, as done by Halperin-Tanré. In this chapter we develop an alternative construction of filtered minimal models, which is an adaptation to the classical construction of Sullivan minimal models of dga's presented in [**GM81**]. The main advantage of this alternative method is that it is easily generalizable to differential algebras having multiple filtrations. Then, we study the homotopy theory of filtered dga's within the axiomatic framework of Sullivan categories.

The first section is devoted to the fundamentals on classical homotopy theory of dga's and rational homotopy of simply connected manifolds, with a special attention to the homotopy groups of a dga, and their relation with the derived functor of the indecomposables.

As in the case of filtered complexes, the study of the homotopy theory of filtered dga's is done in two stages. In section 2 we introduce filtered minimal dga's, and prove that every 1-connected filtered dga has an filtered minimal model, providing the category of 1-connected filtered dga's with a Sullivan category structure. This is without doubt, the most important and laborious result of this chapter.

We study the higher homotopy theories in Section 3. We prove the existence of cofibrant minimal models by induction, using the results of the previous section, together with Deligne's décalage fucntor. As applications, we define the E_r-homotopy of a filtered dga via the derived functor of the complex of indecomposables with respect to E_r-quasi-isomorphisms, and show that it has an associated spectral sequence, converging to the classical homotopy of the underlying dga. We also introduce the notion of E_r-formality as a generalization to the classical formality of dga's.

In the last section of this chapter we extend the results of the previous sections to bifiltered dga's.

4.1. Preliminaries

This section constitutes a review of the theory of Sullivan relating differential graded algebras and rational homotopy theory. The theory is valid for differential algebras over any field $\mathbf{k}$ containing the rational numbers $\mathbb{Q}$. Most of the results of this section can be found in [**BG76**]. We also refer to the book [**FHT01**] on the subject, or the very comprehensive monograph [**GM81**], for further details.

Differential Graded Algebras. We begin with a summary of the main definitions and results on commutative differential graded algebras.

Definition 4.1.1. A *(non-negatively) graded vector space over* $\mathbf{k}$ is a family of vector spaces $V = \{V^n\}_{n\geq 0}$ over $\mathbf{k}$, indexed by the non-negative integers. Elements belonging to V^n are called *homogeneous elements of degree* n, and we denote $|x| = n$ if $x \in V^n$. We say that V is of *finite type* if each V^n is finite dimensional.

Definition 4.1.2. A *commutative differential graded algebra* (A, d) *over* $\mathbf{k}$ is a graded vector space $A = \{A^i\}_{i\geq 0}$ over $\mathbf{k}$, together with a linear differential $d : A^i \to A^{i+1}$, an associative product $A^i \times A^j \to A^{i+j}$ with a unit $\eta : \mathbf{k} \to A^0$ satisfying:

(i) Graded commutativity: $a \cdot b = (-1)^{|a|\cdot|b|} b \cdot a$.
(ii) Graded Leibnitz: $d(a \cdot b) = da \cdot b + (-1)^{|a|} a \cdot db$.

We use the notation *dga* for commutative differential graded algebras.

Definition 4.1.3. A *morphism of dga's* is a $\mathbf{k}$-linear map $f : A \to B$ of degree 0, preserving the differential, the product and the unit.

Denote by $\mathsf{DGA}(\mathbf{k})$ the category of dga's over $\mathbf{k}$. The field $\mathbf{k}$ is considered as a graded algebra of homogeneous degree 0 with trivial differential. The unit $\eta : \mathbf{k} \to A$ of a dga A is a morphism of dga's. The field $\mathbf{k}$ is the initial object, and 0 is the final object of $\mathsf{DGA}(\mathbf{k})$.

Examples of dga's are the de Rham algebra $\mathcal{A}^*_{dR}(X)$ of differential forms over $\mathbb{R}$ or $\mathbb{C}$ of a smooth manifold X, or the de Rham algebra $\mathcal{A}^*_{PL}(K)$ of piecewise linear forms of a simplicial complex K.

The cohomology $H^*(A)$ of a dga A is defined in the standard way and naturally inherits a grading and a product, making it into a dga on its own with trivial differential.

Definition 4.1.4. A dga (A, d) over $\mathbf{k}$ is called *0-connected* if the unit $\eta : \mathbf{k} \to A$ induces an isomorphism $\mathbf{k} \cong H^0(A)$. It is called *1-connected* if, in addition, $H^1(A) = 0$.

Denote by $\mathsf{DGA}^0(\mathbf{k})$ and $\mathsf{DGA}^1(\mathbf{k})$ the categories of 0-connected and 1-connected dga's over $\mathbf{k}$ respectively.

Definition 4.1.5. A morphism of dga's $f : A \to B$ is said to be a *quasi-isomorphism* if the induced map $f^* : H^*(A) \to H^*(B)$ in cohomology is an isomorphism.

Definition 4.1.6. An *augmented dga* is a dga (A, d), together with a morphism $\varepsilon : A \to \mathbf{k}$. The morphism ε is called an *augmentation* of A. Denote by A^+ the kernel of ε.

Denote by $\mathsf{DGA}(\mathbf{k})_*$ the category of augmented dga's over $\mathbf{k}$, whose morphisms are compatible with the augmentations.

For instance, the choice of a point x in a manifold X defines, by evaluation at x, an augmentation $\varepsilon_x : \mathcal{A}^*_{dR}(X) \longrightarrow \mathbb{R}$.

Remark 4.1.7. If a dga (A, d) satisfies $A^0 = \mathbf{k}$, then it admits a unique augmentation, and $A^+ = \bigoplus_{i>0} A^i$.

Definition 4.1.8. The *complex of indecomposables* of an augmented dga (A, d, ε) is the complex of vector spaces defined by the quotient

$$Q(A) = A^+/(A^+ \cdot A^+),$$

together with the induced differential. This defines a functor

$$Q : \mathsf{DGA}(\mathbf{k})_* \longrightarrow \mathbf{C}^+(\mathbf{k}).$$

Given a graded vector space V, denote by ΛV the free graded commutative algebra generated by V. It can be written as the tensor product of the symmetric algebra on V^{even} with the exterior algebra of V^{odd},

$$\Lambda V = S[V^{\text{even}}] \otimes E[V^{\text{odd}}].$$

A linear map of degree 0 from V to a commutative graded algebra A extends to a unique morphism of graded algebras $\Lambda V \to A$. By the Leibnitz rule, a linear map $V \to \Lambda(V)$ of degree 1 extends to a unique differential in ΛV.

Definition 4.1.9. A dga (A, d) is said to be *free*, if $A = \Lambda V$ as a graded algebra, where V is a graded vector space.

Every free dga $(\Lambda V, d)$ has a canonical augmentation defined by $\varepsilon(V) = 0$. The inclusion $V \to \Lambda V$ induces an isomorphism $V \cong Q(\Lambda V)$ of graded vector spaces.

Denote by $\Lambda(t, dt)$ the free dga generated by t and dt of degree 0 and 1 respectively.

Definition 4.1.10. The *path* of a dga A is the dga given by

$$P(A) = A[t, dt] := A \otimes \Lambda(t, dt).$$

There is a map of evaluation of forms $\delta_A^k : P(A) \to A$, for $k \in \mathbf{k}$, defined by $t \mapsto k$ and $dt \mapsto 0$. The inclusion $\iota_A : A \to P(A)$ is defined by $a \mapsto a \otimes 1$.

Definition 4.1.11. Let $f, g : A \to B$ be morphisms of dga's. A *homotopy from f to g* is a morphism of dga's $h : A \to P(B)$ such that the diagram

$$\begin{tikzcd} B & P(B) \arrow[l, "\delta_B^0"'] \arrow[r, "\delta_B^1"] & B \\ & A \arrow[ul, "f"] \arrow[u, "h"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes. We use the notation $h : f \simeq g$.

The homotopy relation between morphisms of dga's is symmetric, reflexive and compatible with the composition. The path is functorial for morphisms, and defines a P-category structure on $\mathsf{DGA}(\mathbf{k})$ (see Proposition 1.2.38).

A particular kind of algebras will be of special interest to us. These are the Sullivan (minimal) dga's, which we introduce next.

The tensor product $A \otimes B$ of dga's is a dga. The following is a special type of a twisted tensor product, of an augmented dga, by a free graded algebra.

Definition 4.1.12. A *KS-extension* of a dga (A, d) of degree n, is a dga of the form $A \otimes_\xi \Lambda V$, where V is a finite dimensional vector space of homogeneous degree n and $\xi : V \to A$ is a linear map of degree 1 such that $d\xi = 0$. By the Leibnitz rule, the differential on the full algebra is determined by $d|_A \otimes 1$ and $1 \otimes d|_V = \xi$.

Let $f : A \to B$ be a morphism of dga's. A morphism $A \otimes_\xi \Lambda V \to B$ extending f is uniquely determined by a linear map $\varphi : V \to B$ of degree 0 satisfying $d\varphi = f\xi$.

Definition 4.1.13. A KS-extension $A \otimes_\xi \Lambda V$ of an augmented dga A is said to be *decomposable* if $dV \subset A^+ \cdot A^+$.

Definition 4.1.14. A *Sullivan dga over* $\mathbf{k}$ is the colimit of a sequence of KS-extensions starting from $\mathbf{k}$. A *Sullivan minimal dga* is a Sullivan dga A such that $\eta : \mathbf{k} \cong A^0$, and all the extensions are decomposable.

In particular, every Sullivan minimal dga (A, d) has a unique augmentation, and its differential satisfies $dA \subset A^+ \cdot A^+$. Hence the differential on $Q(A)$ is trivial.

The prototypical example of a Sullivan dga which is not minimal is the exterior algebra $\Lambda(t, dt)$ generated by t of degree 0 and dt of degree 1.

For the 1-connected case, there is a simple characterization of Sullivan minimal dga's.

Proposition 4.1.15 ([**BG76**] Prop. 7.4 and [**FHT01**], Prop. 12.8)**.** *Let (A, d) be a 1-connected dga. Then A is Sullivan minimal if and only if A is free with $A^1 = 0$ and satisfies $dA \subset A^+ \cdot A^+$.*

Unfortunately, this proposition does not hold for 0-connected dga's. For instance, if $A = \Lambda(x, y)$ is the exterior algebra on 1-dimensional generators x and y, with $dx = xy$ and $dy = 0$. Then $dA \subset A^+ \cdot A^+$, but A is not Sullivan minimal.

The important result for 0-connected dga's is the following.

Theorem 4.1.16 ([**BG76**], Prop. 7.7)**.** *Every 0-connected dga A has a Sullivan minimal model: this is a Sullivan minimal dga M, together with a quasi-isomorphism $M \to A$. If A is 1-connected, then $M^1 = 0$.*

The homotopy relation between morphisms of dga's is an equivalence relation when restricted to maps in which the source is a Sullivan dga (see [**BG76**], Prop 6.3). If M is a Sullivan dga, denote by $[M, A]$ the class of morphisms from M to A modulo homotopy.

Sullivan dga's satisfy the characteristic property of cofibrant objects.

Proposition 4.1.17 ([**BG76**], Prop 6.4)**.** *Let M be a Sullivan dga, and let $w : A \to B$ be a quasi-isomorphism of dga's. Then w induces a bijection*

$$w_* : [M, A] \longrightarrow [M, B].$$

As a consequence, a formal Whitehead Theorem is satisfied: any quasi-isomorphism between Sullivan dga's is a homotopy equivalence.

If the dga's are Sullivan minimal, then the implication is stronger.

Proposition 4.1.18 ([**BG76**], Prop. 7.6)**.** *Every quasi-isomorphism between Sullivan minimal dga's is an isomorphism.*

As a consequence, the Sullivan minimal model of a dga is uniquely defined up to an isomorphism, which is well defined up to homotopy.

Theorem 4.1.19. *Let $\alpha \in \{0, 1\}$. The category $\mathsf{DGA}^{\alpha}(\mathbf{k})$ with the classes $\mathcal{S}$ and $\mathcal{W}$ of homotopy equivalences and quasi-isomorphisms, is a Sullivan* *category. The category* $\mathsf{Smin}^\alpha(\mathbf{k})$ *of Sullivan minimal dga's is the full subcategory of minimal models. The inclusion induces an equivalence of categories*

$$\pi(\mathsf{Smin}^\alpha(\mathbf{k})) := (\mathsf{Smin}^\alpha(\mathbf{k})/\simeq) \xrightarrow{\sim} \mathrm{Ho}\,(\mathsf{DGA}^\alpha(\mathbf{k})) := \mathsf{DGA}^\alpha(\mathbf{k})[\mathcal{W}^{-1}].$$

PROOF. By Proposition 4.1.17 Sullivan dga's are cofibrant, and by Proposition 4.1.18, Sullivan minimal dga's are minimal. By Theorem 4.1.16 every α-connected dga has a Sullivan minimal model in $\mathsf{DGA}^\alpha(\mathbf{k})$. The equivalence of categories follows follows from Theorem 1.1.35. $\square$

Remark 4.1.20. For the non-connected case, Bousfield-Gugenheim define a Quillen model structure on $\mathsf{DGA}(\mathbf{k})$, for which Sullivan dga's are cofibrant (see Theorem 4.3 of [**BG76**]).

Homotopy and Indecomposables. Following the approach of Cartan-Eilenberg categories of [**GNPR10**], we show that the homotopy groups of an augmented 1-connected dga are given by the derived functor of the functor of indecomposables. Although most of the results of this section are well known to the experts on the subject, we provide detailed proofs, since we will later extend these results to the filtered case.

Definition 4.1.21. Let (A,d) be a 1-connected dga, and let $\rho : M_A \to A$ be a Sullivan minimal model of (A,d). For all $n>0$, the *n-homotopy group* of (A,d) is given by

$$\pi^n(A) = Q(M_A)^n.$$

We next check that this definition is correct, in the sense that it is functorial, and does not depend on the chosen minimal model.

Remark 4.1.22. Since every 0-connected dga has a Sullivan minimal model, one could think that the homotopy $\pi^n(A)$ of a 0-connected dga A can be defined in the same manner, by choosing a Sullivan minimal model $M_A \to A$ and letting $\pi^n(A) = Q(M_A)^n$. However, in this case the homotopy is not functorial for morphisms of dga's. This is due to the fact that homotopic morphisms of dga's need not induce the same morphism of indecomposables, unless the homotopy is augmented (see Definition 4.1.23). A main difference between 0-connected and 1-connected dga's is that, for the latter case, every homotopy of augmented morphisms is augmented (see Proposition 4.1.24), and hence, the homotopy is independent of the augmentation.

The category $\mathsf{DGA}(\mathbf{k})_*$ can be viewed as a category of diagrams over $\mathsf{DGA}(\mathbf{k})$. Hence it inherits a Quillen model structure. In [**BG76**], homotopy is defined for every augmented dga as $\pi^n(A) = H^n(Q(C_A))$, where $C_A \to A$ is a Sullivan model of A. For 1-connected augmented dga's, both definitions coincide, since every Sullivan minimal dga M satisfies $H^n(Q(M)) = Q^n(M)$.

The duality between pointed spaces and augmented dga's leads naturally to the notion of augmented homotopy: a pointed homotopy between morphisms $f, g : (Y, y_0) \to (X, x_0)$ of pointed topological spaces is a homotopy $h : Y \times I \to X$ from f to g which is constant at the base point y_0. Equivalently, it is given by a commutative diagram

$$\begin{tikzcd} X & * \arrow[l] \\ Y \times I \arrow[u, "h"] & * \times I \arrow[l] \arrow[u] \end{tikzcd}$$

where $h(y, 0) = f(y)$ and $h(y, 1) = g(y)$. Dually, we have:

Definition 4.1.23 (See [**GM81**], p.147)**.** Let $f, g : A \to B$ be morphisms of augmented dga's. A homotopy $h : A \to P(B)$ from f to g is said to be *augmented* if the diagram

$$\begin{tikzcd} A \arrow[r, "\varepsilon"] \arrow[d, "h"'] & \mathbf{k} \arrow[d, "\iota"] \\ P(B) \arrow[r, "P(\varepsilon)"] & P(\mathbf{k}) \end{tikzcd}$$

commutes.

Denote by $[A, B]_*$ the class of morphisms from A to B modulo augmented homotopy. As we stated earlier, every Sullivan minimal dga admits a unique augmentation. Likewise, every morphism $f : A \to B$ of Sullivan minimal dga's is augmented. This gives an equivalence of categories

$$\mathsf{Smin}(\mathbf{k})_* \xrightarrow{\sim} \mathsf{Smin}(\mathbf{k}).$$

In contrast, a homotopy $h : A \to P(B)$ between morphisms of Sullivan minimal dga's is not augmented in general, unless the dga's are 1-connected.

Proposition 4.1.24 (cf. [**GM81**], Lemma 12.5)**.** *Let $f, g : A \to B$ be two morphisms of Sullivan dga's such that $\mathbf{k} \cong A^0$, $\mathbf{k} \cong B^0$ and $A^1 = 0$. Let $h : A \to P(B)$ be a homotopy from f to g. Then*

$$h|_{A^0} = f|_{A^0} \otimes 1 = g|_{A^0} \otimes 1,$$

and h is augmented. In particular, $[A, B]_* = [A, B]$.

PROOF. Since $A^0 \cong \mathbf{k}$ and $B^0 \cong \mathbf{k}$, both A and B admit unique canonical augmentations, and the morphisms f and g are augmented.

For $a \in A^n$, $h(a)$ can be uniquely written as

$$h(a) = \sum_{i \geq 0} a_i t^i + \sum_{i \geq 0} b_i t^i dt,$$

where $a_i \in B^n$ and $b_i \in B^{n-1}$.

Assume that $|a| = 0$. Then $|b_i| = -1$ and hence $b_i = 0$ for all $i \geq 0$. In addition, since $A^1 = 0$ we have $da = 0$, and

$$0 = h(da) = dh(a) = \sum_{i \geq 0} (da_i) t^i + \sum_{i \geq 0} i a_i t^{i-1} dt.$$

Therefore $a_i = 0$ for all $i > 0$. Since $\delta^0_B h = f$ and $\delta^1_B h = g$, we find that $a_0 = f(a) = g(a)$. Therefore

$$h(a) = f(a) \otimes 1 = g(a) \otimes 1.$$

This proves that

$$h|_{A^0} = f|_{A^0} \otimes 1 = g|_{A^0} \otimes 1.$$

Since f is augmented, it satisfies $\varepsilon f = \varepsilon$. This implies that for $a \in A^0$,

$$P(\varepsilon)h(a) = P(\varepsilon)(f(a) \otimes 1) = \varepsilon f(a) \otimes 1 = \varepsilon(a) \otimes 1 = \iota\varepsilon(a).$$

Assume that $|a| > 1$. Then $|a_i| = |a| > 1$ and $|b_i| \geq 1$. Hence $\varepsilon(a) = 0$, and $\varepsilon(a_i) = \varepsilon(b_i) = 0$ for all $i \geq 0$. It follows that

$$P(\varepsilon)(h(a)) = 0 = \iota\varepsilon(a).$$

Hence h is augmented. $\square$

Corollary 4.1.25. *The forgetful functor induces an equivalence of categories*

$$\pi_*\mathsf{Smin}^1(\mathbf{k})_* \xrightarrow{\sim} \pi\mathsf{Smin}^1(\mathbf{k}).$$

This result already suggests the independence of the base point of the homotopy groups of 1-connected dga's.

Proposition 4.1.26. *Let (A, d, ε) be an augmented 1-connected dga, and let $\rho : M \to A$ be a Sullivan minimal model of A. Then ρ is augmented.*

Proof. By Proposition 4.1.17 the solid diagram

$$\begin{tikzcd} M \arrow[r, dotted, "\varepsilon'"] \arrow[d, "\rho"'] \arrow[dr, dotted, "h"] & \mathbf{k} \arrow[d, equal] \\ A \arrow[r, "\varepsilon"] & \mathbf{k} \end{tikzcd}$$

can be completed with a morphism of dga's $\varepsilon' : M \to \mathbf{k}$, modulo a homotopy $h : M \to P(\mathbf{k}) = \Lambda(t, dt)$. Since A is 1-connected, its minimal model satisfies $M^1 = 0$. In addition, $\Lambda(t, dt)^{\geq 2} = 0$. Therefore $h(M^{\geq 1}) = 0$, and hence $\varepsilon'|_{M^{\geq 1}} = \varepsilon\rho|_{M^{\geq 1}} = 0$. By Proposition 4.1.24, h is augmented, and $h|_{M^0} = \varepsilon'|_{M^0} \otimes 1 = \varepsilon\rho|_{M^0} \otimes 1$. Therefore h is a constant homotopy, and the diagram commutes. $\square$

Theorem 4.1.27. *The category $\mathsf{DGA}^1(\mathbf{k})_*$ with the classes $\mathcal{S}$ and $\mathcal{W}$ of augmented homotopy equivalences and augmented quasi-isomorphisms, is a Sullivan category. The category $\mathsf{Smin}^1(\mathbf{k})$ of Sullivan minimal dga's is a full subcategory of minimal models. The inclusion induces an equivalences of categories*

$$\pi(\mathsf{Smin}^1(\mathbf{k})) \xrightarrow{\sim} \mathrm{Ho}\left(\mathsf{DGA}^1(\mathbf{k})_*\right).$$

Proof. By Corollary 4.1.25 Sullivan minimal dga's are minimal cofibrant objects in $\mathsf{DGA}^1(\mathbf{k})_*$. By Proposition 4.1.26 every 1-connected augmented dga has a Sullivan minimal model $\rho : M \to A$ which is augmented. $\square$

Corollary 4.1.28. *There is an equivalence of categories*

$$\mathrm{Ho}\left(\mathsf{DGA}^1(\mathbf{k})_*\right) \xrightarrow{\sim} \mathrm{Ho}\left(\mathsf{DGA}^1(\mathbf{k})\right).$$

PROOF. It follows from Theorems 4.1.19 and 4.1.27, and Corollary 4.1.25. $\square$

To ensure the existence of a right derived functor of $Q : \mathsf{DGA}^1(\mathbf{k})_* \to \mathbf{C}^+(\mathbf{k})$, in view of the derivability criterion of Proposition 1.1.32 together with Theorem 4.1.27, it suffices to check that Q sends augmented homotopy equivalences to quasi-isomorphisms of complexes.

For a dga A, consider the homogeneous linear map of degree -1

$$\int_0^1 : P(A) \longrightarrow A$$

defined by (see [**GM81**], X.10.3)

$$a \otimes t^i \mapsto 0, \text{ and } a \otimes t^i dt \mapsto (-1)^{|a|}\frac{a}{i+1}.$$

Proposition 4.1.29. *Every augmented homotopy of morphisms of augmented dga's $h : A \to P(B)$ induces a homotopy of morphisms of complexes*

$$\widetilde{h} := \int_0^1 h : Q(A) \to Q(B)[-1].$$

PROOF. It follows from the definition of $\int_0^1$ that

$$d\int_0^1 h + \int_0^1 dh = g - f.$$

Therefore the map $\int_1^0 h : A \to B[-1]$ is a homotopy of complexes.

Since h is augmented, the homotopy of complexes $\int_0^1 h$ satisfies

$$(\int_0^1 h)(A^+) \subset B^+, \text{ and } (\int_0^1 h)(A^+ \cdot A^+) \subset B^+ \cdot B^+.$$

Therefore it induces a homotopy of morphisms of complexes

$$\int_0^1 h : Q(A) \longrightarrow Q(B)[-1].$$

$\square$

Theorem 4.1.30. *The functor $Q : \mathsf{DGA}^1(\mathbf{k})_* \to \mathbf{C}^+(\mathbf{k})$ admits a left derived functor*

$$\mathbb{L}Q : \mathrm{Ho}\left(\mathsf{DGA}^1(\mathbf{k})_*\right) \longrightarrow \mathbf{D}^+(\mathbf{k}).$$

The composition of functors

$$\mathrm{Ho}\left(\mathsf{DGA}^1(\mathbf{k})\right) \xleftarrow{\sim} \mathrm{Ho}\left(\mathsf{DGA}^1(\mathbf{k})_*\right) \xrightarrow{\mathbb{L}Q} \mathbf{D}^+(\mathbf{k}) \xrightarrow{H} \mathbf{G}^+(\mathbf{k})$$

defines a functor

$$\pi : \mathrm{Ho}\left(\mathsf{DGA}^1(\mathbf{k})\right) \longrightarrow \mathbf{G}^+(\mathbf{k})$$

which associates to every object A, the graded vector space $\pi(A) = Q(M_A)$, where $M_A \to A$ is a Sullivan minimal model of A,

Proof. By Proposition 4.1.29 the functor Q preserves strong equivalences. Therefore it induces a functor $Q' : \mathsf{DGA}^1(\mathbf{k})_*[\mathcal{S}^{-1}] \to \mathbf{C}^+(\mathbf{k})[\mathcal{W}^{-1}]$. By Theorem 4.1.27 and Proposition 1.1.32, Q admits a left derived functor

$$\mathbb{L}Q : \mathrm{Ho}\left(\mathsf{DGA}^1(\mathbf{k})_*\right) \longrightarrow \mathbf{C}^+(\mathbf{k})[\mathcal{W}^{-1}].$$

The functor

$$\pi : \mathrm{Ho}\left(\mathsf{DGA}^1(\mathbf{k})\right) \longrightarrow \mathbf{G}^+(\mathbf{k})$$

follows from the existence of $\mathbb{L}Q$ and the equivalence of categories

$$\mathrm{Ho}\left(\mathsf{DGA}^1(\mathbf{k})_*\right) \xrightarrow{\sim} \mathrm{Ho}\left(\mathsf{DGA}^1(\mathbf{k})\right).$$

$\square$

Rational Homotopy of Simply Connected Manifolds. The decomposition of every Sullivan minimal dga into decomposable KS-extensions is dual to the rational Postnikov tower of a simply connected simplicial complex, and gives rise to the following important result, connecting the homotopy groups of the algebra of forms on a manifold, to the rational homotopy of the space.

Theorem 4.1.31 ([**GM81**], Cor. 11.6)**.** *Let X be a 1-connected manifold and let $M_X \to \mathcal{A}^*(X)$ be a minimal model of its algebra of rational differential forms. There is a natural isomorphism of vector spaces*

$$\pi_*(X) \otimes \mathbb{Q} \cong \mathrm{Hom}_{\mathbb{Q}}(Q(M_X), \mathbb{Q}) = \pi^*(\mathcal{A}^*(X))^\vee.$$

With an appropriate definition of the rational differential forms, the theorem is applicable to simplicial complexes.

Example 4.1.32 (Rational homotopy of S^{2n-1})**.** The de Rham cohomology of the odd sphere S^{2n-1} is an exterior algebra on one generator of degree $2n-1$. Hence a minimal model for S^{2n-1} is $M = \Lambda(x)$, with $|x| = 2n-1$

and $dx = 0$. The map $\rho : M \to \mathcal{A}^*_{dR}(S^{2n-1})$ is given by $x \mapsto w$, where w is a volume form on S^{2n-1}. Therefore

$$\pi_n(S^{2n-1}; \mathbb{Q}) = \begin{cases} \mathbb{Q} & , n = 2n-1. \\ 0 & , \text{otherwise.} \end{cases}$$

Example 4.1.33 (Rational homotopy of S^{2n})**.** The de Rham cohomology of the even sphere S^{2n} is $\mathbb{R}[x]/(x^2)$, where $|x| = 2n$. A minimal model for S^{2n} is $M = \Lambda(x, y)$, with $|x| = 2n$, $|y| = 4n-1$, $dx = 0$ and $dy = x^2$. We define a map $\rho : M \to \mathcal{A}^*_{dR}(S^{2n})$ by $x \mapsto w$, where w is a volume form on S^{2n}, and $y \mapsto 0$. Therefore

$$\pi_n(S^{2n}; \mathbb{Q}) = \begin{cases} \mathbb{Q} & , n = 2n, 4n-1. \\ 0 & , \text{otherwise.} \end{cases}$$

Example 4.1.34 (Rational homotopy of $\mathbb{P}^n_{\mathbb{C}}$)**.** The cohomology of the complex projective space is $\mathbb{R}[x]/(x^{n+1})$. A minimal model for $\mathbb{P}^n_{\mathbb{C}}$ is $M = \Lambda(x, y)$, where $|x| = 2$, $|y| = 2n+1$ and the differential is defined by $dx = 0$ and $dy = x^{n+1}$. Therefore

$$\pi_i(\mathbb{P}^n_{\mathbb{C}}; \mathbb{Q}) = \begin{cases} \mathbb{Q} & , i = 2, 2n+1. \\ 0 & , \text{otherwise.} \end{cases}$$

Differential Bigraded Algebras. To end this preliminary section we recall the main definitions and properties of differential bigraded algebras.

Definition 4.1.35. Let $r \geq 0$. An *r-bigraded dga over* $\mathbf{k}$ is a dga (A, d) over $\mathbf{k}$, together with a direct sum decomposition $A = \bigoplus A^{p,q}$ such that

$$d(A^{p,q}) \subset A^{p+r,q-r+1}, \text{ and } A^{p,q} \cdot A^{p',q'} \subset A^{p+p',q+q'}.$$

The *bidegree* of $x \in A^{p,q}$ is $|x| = (p, q)$, and its *total degree* is $|x| = p + q$.

The base field $\mathbf{k}$ is considered as an r-bigraded dga of bidegree $(0, 0)$.

Definition 4.1.36. A *morphism of r-bigraded dga's* is a morphism of dga's $f : (A, d) \to (B, d)$ of bidegree $(0, 0)$. That is $f(A^{p,q}) \subset B^{p,q}$, for all $p, q \in \mathbb{Z}$.

Denote by $\mathsf{DG^2A}_r(\mathbf{k})$ the category of r-bigraded dga's over the field $\mathbf{k}$.
The *total degree functor* $t : \mathsf{DG^2A}_r(\mathbf{k}) \longrightarrow \mathsf{DGA}(\mathbf{k})$ sends elements of bidegree (p, q) to elements of degree $p + q$.

The cohomology $H(A)$ of an r-bigraded dga (A,d) admits a bigrading

$$H(A) = \bigoplus H^{p,q}(A),$$

where

$$H^{p,q}(A) := \frac{\mathrm{Ker}(d : A^{p,q} \to A^{p+r,q-r+1})}{\mathrm{Im}(d : A^{p-r,q+r-1} \to A^{p,q})}.$$

Therefore $H(A)$ is bigraded dga on its own with trivial differential. A morphism $f : (A,d) \to (B,d)$ of bigraded dga's induces morphisms

$$H^{p,q}(f) : H^{p,q}(A) \to H^{p,q}(B).$$

Definition 4.1.37. A morphism of r-bigraded dga's $f : (A,d) \to (B,d)$ is a *quasi-isomorphism* if $H^{p,q}(f)$ is an isomorphism for all $p,q \in \mathbb{Z}$.

The cohomology functor factors as

$$\begin{tikzcd}
\mathsf{DG^2A}_r(\mathbf{k}) \arrow[r, "H"] \arrow[d, "t"] & \mathsf{G^2A}(\mathbf{k}) \arrow[d, "t"] \\
\mathsf{DGA}(\mathbf{k}) \arrow[r, "H"] & \mathsf{GA}(\mathbf{k}).
\end{tikzcd}$$

The total degree functor sends quasi-isomorphisms of r-bigraded dga's to quasi-isomorphisms of dga's.

Definition 4.1.38. An *r-bigraded minimal model* of an r-bigraded dga (A,d) is a quasi-isomorphism of r-bigraded dga's $(M,d) \to (A,d)$ such that $t(M,d)$ is a Sullivan minimal dga.

Theorem 4.1.39 ([**FOT08**], Thm. 4.53)**.** *Let $r \geq 0$. Every 0-connected r-bigraded dga has an r-bigraded minimal model.*

We will provide a proof of this result in the general setting of filtered dga's.

Corollary 4.1.40. *The homotopy of an r-bigraded dga (A,d) is bigraded:*

$$\pi^n(A) = \bigoplus_{p+q=n} \pi^{p,q}(A).$$

An example of a 0-bigraded dga is given by the Dolbeault algebra of forms of a complex manifold. As a consequence of Theorem 4.1.39 the Dolbeault homotopy groups of simply connected complex manifolds are bigraded (see [**NT78**]).

4.2. Homotopy Theory of Filtered Algebras

In this section we study the localized category of filtered dga's with respect to filtered quasi-isomorphisms: these are morphisms of filtered dga's, inducing quasi-isomorphisms at the graded level. We first provide the main definitions and results regarding the homotopy theory of filtered dga's, and prove that the category of filtered dga's admits a P-category structure in which the weak equivalences are the filtered quasi-isomorphisms. We introduce filtered cofibrant and filtered minimal extensions, and prove that iterated extensions starting from the base field, give rise to cofibrant and minimal filtered dga's respectively. We then prove the existence of enough filtered minimal models for 1-connected filtered dga's. This provides the category of filtered dga's with the structure of a Sullivan category.

Filtered Differential Graded Algebras. The notion of filtered dga arises from the compatible combination of a filtered complex with the multiplicative structure of a dga. As in the case of filtered complexes, we will restrict to dga's with biregular filtrations indexed by the integers. All dga's are non-negatively graded and defined over a field $\mathbf{k}$ of characteristic 0.

Definition 4.2.1. A *filtered dga* (A, d, F) is a dga (A, d) together with a decreasing filtration $\{F^pA\}$ indexed by the integers and satisfying:

(i) $F^{p+1}A \subset F^pA$, $d(F^pA) \subset F^pA$, and $F^pA \cdot F^qA \subset F^{p+q}A$.

(ii) The filtration is biregular: for any $n \geq 0$ there exist integers $p, q \in \mathbb{Z}$ such that $F^pA^n = 0$ and $F^qA^n = A^n$.

If $a \in A$, the *weight* $w(a)$ of a is the largest integer p such that $a \in F^pA$.

The following properties are satisfied:

$$w(da) \geq w(a),\ w(a \cdot b) = w(a) + w(b) \text{ and } w(a + b) \geq \max\{w(a), w(b)\}.$$

Definition 4.2.2. A *morphism of filtered dga's* $f : (A, d, F) \to (B, d, F)$ is a morphism of dga's such that $f(F^pA) \subset F^pB$, for all $p \in \mathbb{Z}$.

Denote by $\mathsf{FDGA}(\mathbf{k})$ the category of filtered dga's over $\mathbf{k}$. The base field $\mathbf{k}$ is considered as a filtered dga concentrated in weight 0, and the unit map

$\eta : \mathbf{k} \to A$ is a morphism of filtered dga's.

The cohomology algebra $H(A)$ of a filtered dga (A, d, F) inherits a filtration compatible with its multiplicative structure:

$$F^p H^n(A) = \mathrm{Im}\{H^n(F^p A) \to H^n(A)\}.$$

Therefore $(H(A), F)$ is a filtered dga with trivial differential.

Definition 4.2.3. A filtered dga (A, d, F) is called *0-connected* if $\eta : \mathbf{k} \to A$ induces an isomorphism $H^0(Gr_F^0 A) = \mathbf{k}$, and $H^0(Gr_F^p A) = 0$ for all $p \neq 0$. It is called *1-connected* if, in addition, $H^1(Gr_F^p A) = 0$ for all $p \in \mathbb{Z}$.

Denote by $\mathbf{FDGA}^0(\mathbf{k})$ and $\mathbf{FDGA}^1(\mathbf{k})$ the categories of 0-connected and 1-connected filtered dga's respectively. Since the filtrations are biregular, every 0-connected (resp. 1-connected) filtered dga is a 0-connected (resp. 1-connected) dga.

Definition 4.2.4. A morphism of filtered dga's $f : (A, d, F) \to (B, d, F)$ is called *filtered fibration* if the induced morphism

$$Gr_F^p f : Gr_F^p A \to Gr_F^p B$$

is surjective for all $p \in \mathbb{Z}$.

Since the filtrations are biregular, this is equivalent to the condition that the morphism $F^p f : F^p A \to F^p B$ is surjective for all $p \in \mathbb{Z}$. In particular, every filtered fibration (of biregularly filtered dga's) is surjective.

Definition 4.2.5. A morphism of filtered dga's $f : (A, d, F) \to (B, d, F)$ is called *filtered quasi-isomorphism* if it is a quasi-isomorphism of filtered complexes, that is, the induced morphism

$$H^n(Gr_F^p f) : H^n(Gr_F^p A) \to H^n(Gr_F^p B)$$

is an isomorphism for all $p \in \mathbb{Z}$.

Since the filtrations are biregular, this is equivalent to the condition that the morphisms $H^n(F^p A) \to H^n(F^p B)$ are isomorphisms for all $p \in \mathbb{Z}$ and all $n \geq 0$. In particular, every filtered quasi-isomorphism (of biregularly filtered dga's) is a quasi-isomorphism of dga's.

Definition 4.2.6. Let (V, F) be a biregularly filtered non-negatively graded module. The *free filtered ga* defined by (V, F) is the free ga ΛV endowed with the multiplicative filtration induced by the filtration of V. If it has a differential compatible with its multiplicative filtration, then it is called a *free filtered dga*.

Definition 4.2.7. The *filtered path* of a filtered dga (A, d, F) is the filtered dga $P(A)$ with the filtration defined by

$$F^pP(A) = F^pA \otimes \Lambda(t, dt).$$

This is the multiplicative filtration defined from the filtration F of A and the trivial filtration of $\Lambda(t, dt)$.

Definition 4.2.8. We will call *filtered homotopy* the notion of homotopy defined by the filtered path object. The associated homotopy equivalences will be called *filtered homotopy equivalences.* Denote by $\mathcal{S}$ the class of filtered homotopy equivalences.

Proposition 4.2.9. *The category* $\mathbf{F}\mathsf{DGA}(\mathbf{k})$ *with the filtered path object, and the classes* $\mathcal{F}$ *and* $\mathcal{E}$ *of filtered fibrations and filtered quasi-isomorphisms is a P-category.*

PROOF. By Proposition 1.2.38 the category of dga's over $\mathbf{k}$ admits a P-category structure, with the classes of surjections and quasi-isomorphisms as fibrations and weak equivalences respectively. We will show that the graded functor

$$Gr^\bullet = \bigoplus_p Gr^p : \mathbf{F}\mathsf{DGA}(\mathbf{k}) \to \mathsf{DGA}(\mathbf{k})$$

satisfies the conditions of Lemma 1.2.33, to conclude that the P-category structure of $\mathsf{DGA}(\mathbf{k})$ is transferred to the category of filtered dga's.

Let $p \in \mathbb{Z}$. Since $F^pP(A) = F^pA \otimes (t, dt)$, we have

$$Gr_F^pP(A) = P(Gr_F^pA) = Gr_F^pA \otimes \Lambda(t, dt).$$

Therefore

$$Gr_F^\bullet P(A) = \bigoplus_p Gr_F^pP(A) = \bigoplus_p Gr_F^pA \otimes \Lambda(t, dt) = P(Gr_F^\bullet A).$$

Hence the functor $Gr^\bullet$ is compatible with the filtered path.

Consider a sequence of filtered morphisms $(A,F) \xrightarrow{u} (C,F) \overset{v}{\twoheadleftarrow} (B,F)$, where v is a filtered fibration. Then

$$A \times_C B = \operatorname{Ker}\left(A \times B \xrightarrow{u-v} C\right).$$

Since $Gr^p v$ is surjective for all $p \in \mathbb{Z}$, by Proposition 2.1.31, v is a strict morphism, and hence $u - v$ is so. Therefore

$$Gr^\bullet \operatorname{Ker}(u-v) = \operatorname{Ker} Gr^\bullet(u-v).$$

Therefore $Gr^\bullet$ is compatible with fibre products. □

Definition 4.2.10. An *augmented filtered dga* is a filtered dga (A,d,F), together with a morphism of filtered dga's $\varepsilon : A \to \mathbf{k}$. Denote by $A^+ = \ker \varepsilon$ the filtered complex defined by the kernel of the augmentation.

Denote by $\mathbf{FDGA}(\mathbf{k})_*$ the category of augmented filtered dga's with the evident morphisms.

Definition 4.2.11. The *filtered complex of indecomposables* of an augmented filtered dga (A,d,F) is the filtered complex given by

$$Q(A) = A^+/A^+ \cdot A^+,$$

together with the induced filtration, and the induced differential. This defines a functor

$$Q : \mathbf{FDGA}(\mathbf{k})_* \longrightarrow \mathbf{C}^+(\mathbf{Fk}).$$

Proposition 4.2.12. *An augmented filtered homotopy $h : A \to P(B)$ of augmented filtered dga's induces a filtered homotopy*

$$\int_0^1 h : Q(A) \to Q(B)[-1]$$

of filtered complexes.

Proof. By Proposition 4.1.29 the homotopy $h : A \to P(B)$ induces a homotopy of complexes

$$\int_0^1 h : Q(A) \to Q(B)[-1].$$

Since $F^p P(B) = P(F^p B)$, the morphism $\int_0^1 : P(B) \to B$ is compatible with filtrations. Therefore $\int_0^1 h$ is filtered. □

In particular, every augmented filtered homotopy equivalence of augmented filtered dga's induces a filtered homotopy equivalence between their filtered complexes of indecomposables.

Cofibrant and Minimal Extensions. We now define filtered cofibrant (resp. minimal) extensions of a filtered dga, and prove that iterated filtered cofibrant (resp. minimal) extensions of the base field $\mathbf{k}$ give $\mathcal{F}$-cofibrant (resp. $\mathcal{F}$-minimal) objects in the category of filtered dga's.

Definition 4.2.13. Let (A, d, F) be a filtered dga. A *filtered KS-extension of A of degree n and weight p* is a filtered dga $A \otimes_\xi \Lambda V$, where V is a filtered graded module concentrated in pure degree n and pure weight p, and $\xi : V \to F^p A$ is a linear map of degree 1 such that $d\xi = 0$. The filtration on $A \otimes_\xi \Lambda V$ is defined by multiplicative extension.

Definition 4.2.14. A *filtered cofibrant dga* is the colimit of a sequence of filtered KS-extensions, starting by the base field $\mathbf{k}$.

In particular, every filtered cofibrant dga is a Sullivan dga.

Proposition 4.2.15. *Let C be a filtered cofibrant dga. For every solid diagram*

$$\begin{tikzcd} & A \arrow[d, "w", "\wr"'] \\ C \arrow[ur, dotted, "g"] \arrow[r, "f"] & B \end{tikzcd},$$

in which $w \in \mathcal{F} \cap \mathcal{E}$, there exists a dotted arrow g, making the diagram commute. In particular, every filtered cofibrant dga is $\mathcal{F}$-cofibrant.

PROOF. The proof is an adaptation of the classical Lifting Lemma for dga's (see [**FHT01**], Lemma 12.4): Assume that $C = C' \otimes_\xi \Lambda V$ is a filtered KS-extension of C' of degree n and weight p, and that we have constructed a filtered morphism $g' : C' \to A$ satisfying $wg' = f'$, where $f' : C' \to B$ denotes the restriction of f to C'. Consider the solid diagram

$$\begin{tikzcd} & Z^n(F^pC(1_A)) \arrow[d, "1\oplus w"] \\ V \arrow[ur, dotted] \arrow[r, "(g'\xi, f|_V)"] & Z^n(F^pC(w)) \end{tikzcd}.$$

Since $F^p w$ is a surjective quasi-isomorphism, this is well defined, and $1 \oplus w$ is surjective. Therefore there exists a dotted arrow $g|_V$, and satisfies $wg|_V = f|_V$ and $dg|_V = g'\xi$. This defines a filtered morphism $g : C \to A$ such that $wg = f$.

□

By Proposition 1.2.26 the filtered homotopy defines an equivalence relation for those maps of filtered dga's whose source is filtered cofibrant.

Corollary 4.2.16. *Let (C, d, F) be a filtered cofibrant dga. Any filtered quasi-isomorphism $w : (A, d, F) \to (B, d, F)$ induces a bijection*

$$w_* : [C, A] \longrightarrow [C, B]$$

between the classes of maps defined by filtered homotopy equivalence.

PROOF. If follows from Propositions 1.2.27 and 4.2.15. □

Definition 4.2.17. Let (A, d, F) be an augmented filtered dga. A *filtered minimal extension of A of degree n and weight p* is a filtered KS-extension $A \otimes_\xi \Lambda V$ of degree n and weight p such that

$$\xi(V) \subset F^p(A^+ \cdot A^+) + F^{p+1}A.$$

Definition 4.2.18. A *filtered minimal dga over* $\mathbf{k}$ is the colimit (A, d, F) of a sequence of filtered minimal extensions, starting from the base field $\mathbf{k}$ such that $\eta : \mathbf{k} \cong A^0$.

In particular, every filtered minimal dga (A, d, F) admits a unique augmentation, with $A^+ = \bigoplus_{i>0} A^i$, and satisfies

$$d(F^p A) \subset F^p(A^+ \cdot A^+) + F^{p+1}A.$$

The following result is straightforward.

Proposition 4.2.19. *Let (A, d, F) be a filtered minimal dga. Then the filtered complex $(Q(A), d, F)$ is minimal: $dF^pQ(A) \subset F^{p+1}Q(A)$.*

Proposition 4.2.20. *Let (A, d, F) be a filtered minimal dga. If it is 1-connected then $A^1 = 0$.*

PROOF. We adapt the proof of Proposition 12.8.(ii) of [**FHT01**]. Assume that $A = \Lambda V$, where $V = \bigcup V_n$ satisfies

$$d(F^p V_n) \subset F^p \Lambda^{\geq 2}(V_{n-1}) + F^{p+1} V_{n-1}.$$

Assume inductively that $V^1_{n-1} = 0$. Then $d(Gr^p_F V^1_n) = 0$. Since $d(Gr^p_F V)$ is decomposable, no element of $Gr^p_F V^1_n$ is a coboundary. Since $H^1(Gr^p_F A) = 0$, it follows that $Gr^p_F V^1_n = 0$, for all $p \in \mathbb{Z}$. $\square$

Proposition 4.2.21. *Every filtered quasi-isomorphism between 1-connected filtered minimal dga's is an isomorphism. In particular, every 1-connected filtered minimal dga is $\mathcal{F}$-minimal.*

PROOF. Let $f : A \to B$ be a filtered quasi-isomorphism between filtered minimal dga's. Since A and B are filtered cofibrant, by Lemma 4.2.16, $f : A \to B$ is a filtered homotopy equivalence. By Proposition 4.2.20 we have $A^1 = B^1 = 0$. Hence by Proposition 4.1.24, f is an augmented homotopy equivalence. Consequently, it induces a homotopy equivalence $Q(f) : Q(A) \to Q(B)$, by Proposition 4.2.12. Since both $Q(A)$ and $Q(B)$ are filtered minimal complexes, it follows that $Q(f)$ is an isomorphism. Since A and B are free as dga's, it follows that f is an isomorphism (see Lemma 10.10 of [**GM81**]). $\square$

Filtered Minimal Models. We next prove the existence of filtered minimal models. Our proof is an adaptation of the classical proof for the existence of Sullivan minimal models of 1-connected dga's (see of Theorem. 9.5 of [**GM81**]), to the filtered setting: we will construct a filtered minimal model step by step, performing filtered minimal extensions, starting from the base field.

Definition 4.2.22. A *filtered cofibrant* (resp. *filtered minimal*) *model* of a filtered dga A is a filtered cofibrant (resp. filtered minimal) dga M, and a filtered quasi-isomorphism $\rho : M \to A$.

Let (M, d) be a 1-connected Sullivan minimal dga. Since $dM \subset M^+ \cdot M^+$, the differential of an element of degree n is a linear combination of generators of degree $< n$. It is therefore reasonable that the construction of Sullivan minimal models works inductively, by performing decomposable extensions of increasing degree. In contrast, let (M, d, F) be a filtered minimal dga satisfying $M^1 = 0$. Since $d(F^pM) \subset F^p(M^+ \cdot M^+) + F^{p+1}M$, the differential of an element of degree n and weight p, is a linear combination of generators of degree $< n$ and arbitrary weights $p \in \mathbb{Z}$, plus generators of degree $n+1$ and weights $> p$. The construction of filtered minimal models will be done inductively over the ordinal

$$\cdots \leq (n, p) \leq (n, p-1) \leq \cdots (n, -\infty) = (n+1, +\infty) \leq \cdots \leq (n+1, p),$$

and at each inductive step we will perform a series of filtered minimal extensions of degree $n+1$ and weights $> p$, followed by an extension of degree n and weight p.

Theorem 4.2.23 (cf. [**HT90**], Thm. 4.4)**.** *Every 1-connected filtered dga over* $\mathbf{k}$ *has a filtered minimal model.*

Proof. Given a 1-connected filtered dga (A, d, F) we will define, inductively over $n \geq 1$, a sequence of free filtered dga's M_n together with filtered morphisms $\rho_n : M_n \to A$, with $M_1 = \mathbf{k}$, satisfying the following conditions:

(a_n) The algebra M_n is a composition of filtered minimal extensions of M_{n-1} of degrees n and $n+1$. The map ρ_n extends ρ_{n-1}.

(b_n) $H^i(Gr_F^p C(\rho_n)) = 0$ for all $i \leq n$ and all $p \in \mathbb{Z}$.

Then the filtered morphism

$$\rho = \bigcup_n \rho_n : M = \bigcup_n M_n \to A$$

will be a filtered minimal model of A. Indeed, the condition that (a_n) is satisfied for all $n \geq 0$, implies that M is filtered minimal, and that $M^n = M_k^n$ for all $k \geq n$. From (b_{n+1}), it follows that

$$H^n(Gr_F^p C(\rho)) = H^n(Gr_F^p C(\rho_{n+1})) = 0, \text{ for all } p \in \mathbb{Z}.$$

Therefore ρ is a filtered quasi-isomorphism.

Let $M_0 = M_1 = \mathbf{k}$, concentrated in degree 0 and with pure weight 0, and define $\rho_1 : M_1 \to A$ to be the unit map. Condition (a_1) is trivially satisfied. Since $H^0(Gr_F^0 A) = \mathbf{k}$, $H^0(Gr_F^p A) = 0$ for all $p \neq 0$, and $H^1(Gr_F^p A) = 0$ for all $p \in \mathbb{Z}$, (b_1) is satisfied.

Assume that for all $1 < i < n$ we have defined $\rho_i : M_i \to A$ as required. We will define, by a decreasing induction over the weight $p \in \mathbb{Z}$, a sequence of filtered dga's $M_{n,p}$, together with filtered morphisms $\rho_{n,p} : M_{n,p} \to A$ satisfying the following conditions:

($a_{n,p}$) The algebra $M_{n,p}$ is a composition of filtered minimal extensions of $M_{n,p+1}$ of degree n and weight p, and degree $n+1$ and weight $> p$. The map $\rho_{n,p}$ extends $\rho_{n,p+1}$.

($b_{n,p}$) $H^i(Gr_F^q C(\rho_{n,p})) = 0$ for $i < n$ and $q \in \mathbb{Z}$, or $i = n$ and $q \geq p$.

Since the filtrations of A and M_{n-1} are biregular, we can choose a sufficiently large integer r such that $F^r A^n = 0$ and $F^r M_{n-1}^{n+1} = 0$. We then take $M_{n,r} = M_{n-1}$ and $\rho_{n,r} = \rho_{n-1}$ as base case for our induction. Condition (a_{n-1}) implies condition ($a_{n,r}$). Condition (b_{n-1}) implies that $H^i(Gr_F^q C(\rho_{n,r})) = 0$ for all $i < n$ and all $q \in \mathbb{Z}$. If $q \geq r$, then

$$H^n(Gr_F^q A) = 0 \text{ and } H^{n+1}(Gr_F^q M_{n-1})) = 0.$$

It follows that $H^n(Gr_F^q C(\rho_{n,r})) = 0$. Therefore ($b_{n,r}$) is satisfied.

Assume that for each q, with $r > q > p$, we have constructed $\rho_{n,q} : M_{n,q} \to A$ satisfying ($a_{n,q}$) and ($b_{n,q}$). We will define $M_{n,p}$ in two steps. In the first step we will perform a finite number of filtered minimal extensions of $M_{n,p+1}$ of fixed degree $n+1$, and decreasing weights $> p$, while the second step will consist in a single extension of degree n and weight p.

To simplify notation, let $M := M_{n,p+1}$ and $\rho := \rho_{n,p+1}$. By Lemma 4.2.24 below, there exists a filtered morphism $\widetilde{\rho} : \widetilde{M} \to A$ satisfying the following conditions:

(1) The algebra $\widetilde{M}$ is a composition of filtered minimal extensions of M of degree $n+1$ and weights $> p$. The map $\widetilde{\rho}$ extends ρ.
(2) $H^i(Gr_F^q C(\widetilde{\rho})) = H^i(Gr_F^q C(\rho))$ for all $i \leq n$ and all $q \in \mathbb{Z}$.
(3) The map $\pi_* : H^n(F^p C(\widetilde{\rho})) \to H^n(Gr^p C(\widetilde{\rho}))$ is surjective.

In particular, by (1) and (2), and since M satisfies ($a_{n,p+1}$) and ($b_{n,p+1}$), the algebra $\widetilde{M}$ satisfies ($a_{n,p+1}$) and ($b_{n,p+1}$).

Define a graded algebra $M_{n,p} = \widetilde{M} \otimes \Lambda(V_{n,p})$, where

$$V_{n,p} = H^n(Gr_F^p C(\widetilde{\rho})) = H^n(Gr_F^p C(\rho_{n,p+1}))$$

is a graded vector space of degree n and weight p. Since the map

$$\pi^* : H^n(F^p C(\widetilde{\rho})) \to H^n(Gr_F^p C(\widetilde{\rho})) = V_{n,p}$$

is surjective, to define a differential d on $V_{n,p}$ and a map $\rho_{n,p}$ extending $\widetilde{\rho}$ we take a splitting of the composition

$$Z^n(F^p C(\widetilde{\rho})) \twoheadrightarrow H^n(F^p C(\widetilde{\rho})) \twoheadrightarrow V_{n,p}.$$

Since $dV_{n,p} \subset F^p \widetilde{M}^{n+1}$, and by construction the generators of $\widetilde{M}$ of degree $n+1$ are of weight $> p$, condition $(a_{n,p})$ is satisfied.

We prove $(b_{n,p})$. Let $Q = M_{n,p}/\widetilde{M}$. There is a short exact sequence of complexes

$$\Sigma := \{0 \longrightarrow C(\widetilde{\rho}) \longrightarrow C(\rho_{n,p}) \longrightarrow Q[1] \longrightarrow 0\},$$

such that $F^q\Sigma$ and $Gr_F^q\Sigma$ are exact for all $q \in \mathbb{Z}$. We have

$$Q^n = Gr_F^p Q^n = V_{n,p},\ Q^k = 0 \text{ for all } k < n \text{ and } Q^{n+1} = 0.$$

From the long exact sequence induced by $Gr_F^q\Sigma$ we have

$$H^i(Gr_F^q C(\rho_{n,p})) = H^i(Gr_F^q C(\widetilde{\rho})) = 0$$

for all $(i, q) \neq (n, p)$, with $i \leq n$ and $q \in \mathbb{Z}$. In addition, the connecting morphism δ of the long exact sequence induced by $Gr_F^r\Sigma$ is the identity. Furthermore, the sequence

$$V_{n,p} \xrightarrow{\delta} H^n(Gr_F^p C(\widetilde{\rho})) \longrightarrow H^n(Gr_F^p C(\rho_{n,p})) \longrightarrow 0$$

is exact. Hence $H^n(Gr_F^p C(\rho_{n,p})) = 0$, and $(b_{n,p})$ is satisfied.

Since the filtrations are biregular, there exists a sufficiently small integer s such that $(b_{n,s})$ implies (b_n). We then take

$$\rho_n = \rho_{n,s} : M_n = M_{n,s} \to A.$$

Condition $(a_{n,s})$ trivially implies (a_n). $\square$

Lemma 4.2.24. *Let $n \geq 1$ and $p \in \mathbb{Z}$. Let $\rho : M \to A$ be a filtered morphism of dga's such that M is freely generated in degrees $\leq n+1$, with $\mathbf{k} \cong M^0$ and $M^1 = 0$. Then there exists a filtered morphism $\widetilde{\rho} : \widetilde{M} \to A$ such that:*

(1) The algebra $\widetilde{M}$ is a composition of filtered minimal extensions of M of degree $n+1$ and weights $> p$. The map $\widetilde{\rho}$ extends ρ.
(2) $H^i(Gr_F^q C(\widetilde{\rho})) = H^i(Gr_F^q C(\rho))$ for all $i \leq n$ and all $q \in \mathbb{Z}$.
(3) $H^{n+1}(F^{p+1}C(\widetilde{\rho})) = 0$.

PROOF. We first establish some notations. Given a morphism of filtered dga's $f : (B, F) \to (C, F)$, we have a short exact sequence of complexes

$$\Gamma_r[f] := \left\{0 \to F^r C(f) \to F^{p+1} C(f) \to F^{p+1} C(f)/F^r C(f) \to 0\right\},$$

for all $r > p$. This induces a long exact sequence in cohomology. Denote by

$$\delta_r[f] : H^n(F^{p+1} C(f)/F^r C(f)) \longrightarrow H^{n+1}(F^r C(f))$$

the connecting morphism, and by

$$i_r[f] : H^{n+1}(F^r C(f)) \longrightarrow H^{n+1}(F^{p+1} C(f))$$

the morphism induced by the inclusion.

We will define, by a decreasing induction over $r > p$, a family of morphisms of filtered dga's $\rho_r : M_r \to A$ satisfying the following conditions:

(1_r) The algebra M_r is a filtered minimal extension of M_{r+1} of degree $n+1$ and weight r. The map ρ_r extends ρ_{r+1}.
(2_r) $H^i(Gr_F^q C(\rho_r)) = H^i(Gr_F^q C(\rho))$ for all $i \leq n$ and all $q \in \mathbb{Z}$.
(3_r) The map $i_r[\rho_r] : H^{n+1}(F^r C(\rho_r)) \to H^{n+1}(F^{p+1} C(\rho_r))$ is 0.

Since the filtrations are biregular, there exists a sufficiently large integer s such that $H^{n+1}(F^s C(\rho)) = 0$. We then take $\rho_s = \rho : M_s = M \to A$ as the base case for the induction. Assuming that $\rho_{r+1} : M_{r+1} \to A$ satisfies (1_{r+1}), (2_{r+1}) and (3_{r+1}), we will define $\rho_r : M_r \to A$, with $r > p$.

Let U_r be the filtered graded vector space of homogeneous degree $n+1$ and pure weight r given by

$$U_r := \mathrm{Im} i_r[\rho_{r+1}],$$

and define a filtered graded algebra by

$$M_r = M_{r+1} \otimes \Lambda U_r.$$

To extend the differential on M_r, by the Leibnitz rule we need only to define a linear map $\xi : U_r \to M_{r+1}$ of degree 1, subject to the condition that $d\xi = 0$. To define $\rho_r : M_r \to A$ extending ρ_{r+1}, it suffices to define a filtered map $\eta : U_r \to A$ subject to the condition $d\eta = \rho_{r+1}\xi$. Both maps are defined by splitting the composition

$$Z^{n+1}(F^r C(\rho_{r+1})) \twoheadrightarrow H^{n+1}(F^r C(\rho_{r+1})) \twoheadrightarrow U_r.$$

This gives a morphism of filtered dga's $\rho_r : M_r \to A$

Since M_{r+1} is generated in degrees $\leq n+1$, we have $M^{n+2}_{r+1} \subset M^+_{r+1} \cdot M^+_{r+1}$. Since the degree of U_r is $n+1$, it follows that

$$d(U_r) \subset M^+_{r+1} \cdot M^+_{r+1}.$$

Therefore M_r is a filtered minimal extension of degree $n+1$ and weight r of M_{r+1}, and (1_r) is satisfied.

We prove (2_r). Let $Q := M_r/M_{r+1}$. There is an exact sequence of filtered complexes

$$\Sigma := \{0 \longrightarrow C(\rho_{r+1}) \longrightarrow C(\rho_r) \longrightarrow Q[1] \longrightarrow 0\}.$$

Since the morphisms are strict, the sequences $F^q\Sigma$ and $Gr^q_F\Sigma$ are exact for all $q \in \mathbb{Z}$. We have

$$Q^{n+1} = Gr^r_F Q^{n+1} = U_r,\ Q^k = 0 \text{ for all } k \leq n.$$

In addition, (1_{r+1}) and the condition that $M^1_{r+1} = 0$, we have $Q^{n+2} = 0$.

From the long exact sequence associated with $Gr^q_F\Sigma$ we have

$$H^i(Gr^q_F C(\rho_r)) = H^i(Gr^q_F C(\rho_{r+1}))$$

for all $(i,q) \neq (n,r)$, with $i \leq n$ and $q \in \mathbb{Z}$, and the sequence

$$0 \to H^n(Gr^r_F C(\rho_{r+1})) \to H^n(Gr^r_F C(\rho_r)) \to U_r \xrightarrow{\varphi} H^{n+1}(Gr^r_F C(\rho_{r+1})).$$

is exact. We next show that φ is a monomorphism.

From the long exact sequence induced by the sequence $\Gamma_r[\rho_{r+1}]$, we have

$$U_r = \frac{H^{n+1}(F^r(C(\rho_{r+1}))}{\mathrm{Im}\delta_r[\rho_{r+1}]}.$$

Since φ is induced by the morphism

$$\pi_* : H^{n+1}(F^r C(\rho_{r+1})) \longrightarrow H^{n+1}(Gr_F^r C(\rho_{r+1})),$$

in order to prove that φ is a monomorphism, it suffices to show that

$$\mathrm{Ker}\pi_* \subset \mathrm{Im}\delta_r[\rho_{r+1}].$$

Consider the commutative diagram (to ease notation we let $\rho = \rho_{r+1}$)

$$\begin{tikzcd}
 & H^n(F^{p+1}C(\rho)/F^r C(\rho)) \arrow[d, "\delta_r[\rho]"] & \\
H^{n+1}(F^{r+1}C\rho) \arrow[r, "j_*"] \arrow[dr, "i_{r+1}[\rho]"'] & H^{n+1}(F^r C(\rho)) \arrow[r, "\pi_*"] \arrow[d, "i_r[\rho]"] & H^{n+1}(Gr_F^r C(\rho)) \\
 & H^{n+1}(F^{p+1}C(\rho)) &
\end{tikzcd}$$

By induction hypothesis we have $i_r[\rho] \circ j_* = i_{r+1}[\rho] = 0$. Therefore

$$\mathrm{Ker}\pi_* = \mathrm{Im}j_* \subset \mathrm{Ker}i_r[\rho] = \mathrm{Im}\delta_r[\rho].$$

Hence φ is a monomorphism, and $H^n(Gr_F^r C(\rho_r)) = H^n(Gr_F^r C(\rho_{r+1}))$. This proves (2_r).

Let us prove (3_r). Consider the commutative diagram with exact rows (to ease notation we write $\rho = \rho_{r+1}$ and $\widetilde{\rho} = \rho_r$)

$$\begin{tikzcd}
U_r = H^{n+1}(F^r Q) \arrow[r] \arrow[d] & H^{n+1}(F^r C(\rho)) \arrow[r, "\mu"] \arrow[d, "i_r[\rho]"] & H^{n+1}(F^r C(\widetilde{\rho})) \arrow[r] \arrow[d, "i_r[\widetilde{\rho}]"] & 0 \arrow[d] \\
U_r = H^{n+1}(F^{p+1}Q) \arrow[r] & H^{n+1}(F^{p+1}C(\rho)) \arrow[r, "\nu"] & H^{n+1}(F^{p+1}C(\widetilde{\rho})) \arrow[r] & 0
\end{tikzcd}$$

Since the morphism μ is surjective, to see that $i_r[\widetilde{\rho}] = 0$, it suffices to see that the composition $i_r[\rho] \circ \nu = i_r[\widetilde{\rho}] \circ \mu$ is null. Since the image of $i_r[\rho]$ is U_r, and $\nu(U_r) = 0$, it follows that $i_r[\rho] \circ \nu = 0$, and hence $i_r[\widetilde{\rho}] \circ \mu = 0$. This proves (3_r).

The morphism $\widetilde{\rho} = \rho_{p+1} : \widetilde{M} = M_{p+1} \to A$ satisfies the properties of the Lemma. Indeed, (1) and (2) follow directly from (1_{p+1}) and (2_{p+1}) respectively. Since $i_{p+1}[\rho_{p+1}]$ is the identity, (3_{p+1}) implies (3). □

Remark 4.2.25. The extension of the previous result to the 0-connected case, can be performed analogously to Theorem V.4.11 of [**GM03**].

Corollary 4.2.26. *The triple* $(\mathbf{F}\mathsf{DGA}^1(\mathbf{k}), \mathcal{S}, \mathcal{E})$ *is a Sullivan category. The inclusion induces an equivalence of categories*

$$\pi\left(\mathbf{F}\mathrm{min}^1_{\mathbf{k}}\right) \longrightarrow \mathrm{Ho}\left(\mathbf{F}\mathsf{DGA}^1(\mathbf{k})\right).$$

between the category of 1-connected filtered minimal dga's modulo filtered homotopy, and the localized category of 1-connected filtered dga's with respect to filtered quasi-isomorphisms.

Proof. By Corollary 4.2.16 every filtered dga is cofibrant. By Proposition 4.2.21, 1-connected filtered minimal dga's are minimal. By Theorem 4.2.23 every 1-connected dga has a filtered minimal model. The equivalence of categories follows from Theorem 1.1.35. □

4.3. Spectral Sequences and Models

Décalage of Filtered Algebras. Every filtered dga (A, d, F) has an associated spectral sequence, each of whose stages $(E_r(A, F), d_r)$ is an r-bigraded dga. Likewise, every morphism $f : A \to B$ of filtered dga's induces morphisms between their associated spectral sequences $E_r(f) : E_r(A) \to E_r(B)$.

Definition 4.3.1. Let $r \geq 0$. A morphism $f : A \to B$ of filtered dga's is called E_r*-fibration* if the induced morphism $E_r(f) : E_r(A) \to E_r(B)$ of r-bigraded dga's is surjective.

Definition 4.3.2. Let $r \geq 0$. A morphism $f : A \to B$ of filtered dga's is called an E_r*-quasi-isomorphism* if the morphism $E_r(f) : E_r(A) \to E_r(B)$ is a quasi-isomorphism of r-bigraded dga's (that is, the morphism $E_{r+1}(f)$ is an isomorphism).

Denote by $\mathcal{F}_r$ the class of E_r-fibrations, and by $\mathcal{E}_r$ the class of E_r-quasi-isomorphisms. Note that for $r = 0$ we recover the classes $\mathcal{F}$ and $\mathcal{E}$ of filtered fibrations and filtered quasi-isomorphisms. Since the filtrations are biregular, every E_r-fibration (resp. E_r-quasi-isomorphism) is a fibration (resp. quasi-isomorphism).

It is easy to check that both the shift and the décalage of a filtered complex (see Definitions 2.2.1 and 2.2.3 respectively) preserve multiplicative structures. Therefore we have a pair of endofunctors S and Dec, defined on the category of filtered dga's. As in the case of filtered complexes, these functors play a very important role in the study of the localized category of filtered dga's with respect to E_r-quasi-isomorphisms. Let us recall the corresponding definitions in the context of dga's.

Definition 4.3.3. The *shift* of a filtered dga $A = (A, d, F)$ is the filtered dga $SA = (A, d, SF)$ defined by

$$(SF)^p A^n = F^{p-n} A^n.$$

This defines a functor

$$S : \mathbf{FDGA}(\mathbf{k}) \longrightarrow \mathbf{FDGA}(\mathbf{k})$$

which is the identity on morphisms.

The following is a consequence of Proposition 2.2.2.

Proposition 4.3.4. *Let $r \geq 0$. Then $\mathcal{E}_r = S^{-1}(\mathcal{E}_{r+1})$ and $\mathcal{F}_r = S^{-1}(\mathcal{F}_{r+1})$.*

Definition 4.3.5. The *décalage* of a filtered dga $A = (A, d, F)$ is the filtered dga $\mathrm{Dec}A = (A, d, \mathrm{Dec}F)$ defined by

$$(\mathrm{Dec}F)^p A^n = \{x \in F^{p+n} A^n \,;\, dx \in F^{p+n+1} A^{n+1}\},$$

This defines a functor

$$\mathrm{Dec} : \mathbf{FDGA}(\mathbf{k}) \longrightarrow \mathbf{FDGA}(\mathbf{k})$$

which is the identity on morphisms. We have an adjunction of functors $S \dashv \mathrm{Dec}$ (see Proposition 2.2.7).

The following is a consequence of Proposition 2.2.5.

Proposition 4.3.6. *Let $r \geq 0$. Then*

$$\mathcal{E}_{r+1} = \mathrm{Dec}^{-1}(\mathcal{E}_r) \text{ and } \mathcal{F}_{r+1} = \mathrm{Dec}^{-1}(\mathcal{F}_r).$$

Analogously to Theorem 2.2.15 for filtered complexes we have:

Theorem 4.3.7. *Deligne's décalage induces an equivalence of categories*

$$\mathrm{Dec} : \mathrm{Ho}_{r+1}\left(\mathbf{FDGA}(\mathbf{k})\right) \xrightarrow{\sim} \mathrm{Ho}_r\left(\mathbf{FDGA}(\mathbf{k})\right).$$

for every $r \geq 0$.

Proof. The proof is parallel to that of Theorem 2.2.15. We first define a family of auxiliary categories: for $r \geq 0$, let $\mathcal{C}_r$ denote the full subcategory of $\mathbf{FDGA}(\mathbf{k})$ of those filtered dga's such that $d(F^pA) \subset F^{p+r}A$. We have a chain of full subcategories

$$\mathcal{C}_r \subset \mathcal{C}_{r-1} \subset \cdots \subset \mathcal{C}_1 \subset \mathcal{C}_0 = \mathbf{FDGA}(\mathbf{k}).$$

The key property of these subcategories is that if $A \in \mathcal{C}_1$, then

$$\mathrm{Dec}F^pA^n = F^{p+n}A^n.$$

A simple verification shows that the functors $\mathrm{Dec} : \mathcal{C}_{r+1} \rightleftarrows \mathcal{C}_r : S$ are inverses to each other, for any $r \geq 0$ (cf. Corollary 2.2.12). By Propositions 4.3.4 and 4.3.6 we have $\mathrm{Dec}(\mathcal{E}_{r+1}) \subset \mathcal{E}_r$ and $S(\mathcal{E}_r) \subset \mathcal{E}_{r+1}$. Therefore this induces an equivalence between the corresponding localized categories

$$\mathrm{Dec} : \mathcal{C}_{r+1}[\mathcal{E}_{r+1}^{-1}] \rightleftarrows \mathcal{C}_r[\mathcal{E}_r^{-1}] : S.$$

By Lemma 2.2.13 we have a functor

$$\mathcal{J}_r := (S^r \circ \mathrm{Dec}^r) : \mathcal{C}_0 \longrightarrow \mathcal{C}_r,$$

and the morphism $\mathcal{J}_r(A) \to A$ is an E_r-quasi-isomorphism, for every filtered dga A. This gives a commutative diagram of equivalences of categories

$$\begin{tikzcd}
\mathcal{C}_0[\mathcal{E}_{r+1}^{-1}] \arrow[r, dotted, "\sim"] \arrow[d, "\mathcal{J}_{r+1}", "\wr"'] & \mathcal{C}_0[\mathcal{E}_r^{-1}] \arrow[d, "\mathcal{J}_r", "\wr"'] \\
\mathcal{C}_{r+1}[\mathcal{E}_{r+1}^{-1}] \arrow[r, "\mathrm{Dec}", "\sim"'] & \mathcal{C}_r[\mathcal{E}_r^{-1}]
\end{tikzcd}$$

for all $r \geq 0$. Since $\mathcal{C}_0 = \mathbf{FDGA}(\mathbf{k})$ the result follows. $\square$

We next introduce the r-path associated with a filtered dga. This will provide the notion of r-homotopy suitable to the study of the localized category of filtered dga's with respect to E_r-quasi-isomorphisms.

Let $\Lambda(t, dt)$ be the free dga with generators t and dt of degree 0 and 1 respectively. For $r \geq 0$, define a decreasing filtration σ_r on $\Lambda(t, dt)$ by letting $w(t) = 0$ and $w(dt) = r$, and extending multiplicatively.

Definition 4.3.8. The *r-path* of a filtered dga (A, d, F) is the filtered dga

$$P_r(A, F) = (P(A), F_r) := (A \otimes \Lambda(t, dt), F * \sigma_r),$$

where $F_r := F * \sigma_r$ is the multiplicative filtration defined by:

$$F_r^p P(A) = \bigoplus_q F^{p-q} A \otimes \sigma_r^q \Lambda(t, dt) = (F^p A \otimes \mathbf{k}[t]) \oplus (F^{p-r} A \otimes \mathbf{k}[t]dt).$$

Definition 4.3.9. We will call *r-homotopy* the notion of homotopy defined by the r-path object. The associated homotopy equivalences will be called *r-homotopy equivalences*. Denote by $\mathcal{S}_r$ the class of r-homotopy equivalences.

Note that the filtration σ_0 is the trivial filtration, and hence for $r = 0$ we recover the notions of filtered path and filtered homotopy introduced in the previous section.

Lemma 4.3.10. *Let $r \geq 0$, and let A be a filtered dga. Then:*

$$\mathrm{Dec}(P_{r+1}(A)) = P_r(\mathrm{Dec}A).$$

PROOF. An easy computation shows that

$$(\Lambda(t, dt), \mathrm{Dec}\sigma_{r+1}) = (\Lambda(t, dt), \sigma_r), \text{ for } r \geq 0.$$

To prove the general case, let

$$a(t) = \sum_{i \geq 0} a_i t^i + \sum_{i \geq 0} b_i t^i dt \in P_{r+1}(A, F)^n.$$

Since $|t| = 0$, and $|dt| = 1$, it follows that $|a_i| = n$ and $|b_i| = n - 1$. The conditions for $a(t)$ to be an element of $\mathrm{Dec}F_r^p P(A)^n$ are that

$$a(t) \in F_r^{p+n} P(A)^n, \text{ and } da(t) \in F_r^{p+n+1} P(A)^{n+1}.$$

From the first condition, since $w(t) = 0$ and $w(dt) = r + 1$, it follows that

$$a_i \in F^{p+n}A^n, \text{ and } b_i \in F^{p+n-r-1}A^{n-1}, \forall\, i \geq 0.$$

From the second condition, and since

$$da(t) = \sum_{i\geq 0} da_i t^i + \sum_{i\geq 0} (a_{i+1}(i+1) + db_i)\, t^i dt,$$

we find:

$$da_i \in F^{p+n+1}A^{n+1}, \text{ and } (a_{i+1}(i+1) + db_i) \in F^{p+n-r}A^n, \forall\, i \geq 0.$$

Since $a_i \in F^{p+n}A^n \subseteq F^{p+n-r}A^n$ it follows that $db_i \in F^{p+n-r}A^n$. Therefore

$$a_i \in \mathrm{Dec}F^pA^n, \text{ and } b_i \in \mathrm{Dec}F^{p-r}A^{n-1}.$$

Therefore $a(t) \in P_r(\mathrm{Dec}A)$. The converse follows analogously. $\square$

Corollary 4.3.11. *Let $r \geq 0$, and let $f, g : A \to B$ be morphisms of filtered dga's. If $f \underset{r+1}{\simeq} g$ then $\mathrm{Dec}f \underset{r}{\simeq} \mathrm{Dec}g$. In particular $\mathrm{Dec}(\mathcal{S}_{r+1}) \subset \mathcal{S}_r$.*

Proof. Let $h : A \to P_{r+1}(B)$ be an $(r+1)$-homotopy. By Lemma 4.3.10 we have $\mathrm{Dec}h : \mathrm{Dec}A \to \mathrm{Dec}P_{r+1}B = P_r(\mathrm{Dec}B)$. Hence $\mathrm{Dec}h$ is an r-homotopy. $\square$

Proposition 4.3.12. *Let $r \geq 0$. The category $\mathbf{FDGA}(\mathbf{k})$ with the r-path object, and the classes $\mathcal{F}_r$ and $\mathcal{E}_r$ of E_r-fibrations and E_r-quasi-isomorphisms is a P-category.*

Proof. The case $r = 0$ follows from Proposition 4.2.9. Assume inductively that the Proposition is true for $0 \leq r' < r$. To prove it for r, it suffices to show that the décalage functor $\mathrm{Dec} : \mathbf{FDGA}(\mathbf{k}) \to \mathbf{FDGA}(\mathbf{k})$ satisfies the properties of Lemma 1.2.33. Indeed, since Dec has a left adjoint, it is compatible with fibre products. By Lemma 4.3.10 the functor Dec is compatible with the functorial paths. By Proposition 4.3.6 we have $\mathcal{E}_{r+1} = \mathrm{Dec}^{-1}(\mathcal{E}_r)$ and $\mathcal{F}_{r+1} = \mathrm{Dec}^{-1}(\mathcal{F}_r)$. The result follows from Lemma 1.2.33. $\square$

Higher Cofibrant and Minimal Models.

Definition 4.3.13. A filtered KS-extension $A \otimes_\xi \Lambda V$ of degree n and weight p is called *E_r-cofibrant* if $\xi(V) \subset F^{p+r}A$.

Definition 4.3.14. An *E_r-cofibrant dga* is the colimit of a sequence of E_r-cofibrant extensions, starting from the base field $\mathbf{k}$.

The following properties are straightforward from the definition.

Lemma 4.3.15. *Let $r \geq 0$, and let (A, d, F) be an E_r-cofibrant dga. Then:*
(1) $d(F^pA) \subset F^{p+r}A$ for all $p \in \mathbb{Z}$.
(2) $(Gr_F A, 0) = (E_0(A, F), d_0) = \cdots = (E_{r-1}(A, F), d_{r-1})$.

Denote by $\mathsf{E_r\text{-}cof_{\mathbf{k}}}$ the full subcategory of $\mathbf{F}\mathsf{DGA}(\mathbf{k})$ of E_r-cofibrant dga's. Note that for $r \geq 0$ we have $\mathsf{E_{r+1}\text{-}min}(\mathbf{k}) \subset \mathsf{E_r\text{-}min}(\mathbf{k})$.

Lemma 4.3.16. *Let $r \geq 0$. The functors*

$$\mathrm{Dec} : \mathsf{E_{r+1}\text{-}cof_{\mathbf{k}}} \rightleftarrows \mathsf{E_r\text{-}cof_{\mathbf{k}}} : S$$

are inverses to each other.

PROOF. We work inductively as follows. Let A be an E_{r+1}-cofibrant dga. Since $d(F^pA) \subset F^{p+1}A$, it follows that $\mathrm{Dec}F^pA^n = F^{p+n}A^n$. Therefore $(S \circ \mathrm{Dec})A = A$. Assume inductively that $\mathrm{Dec}A$ is E_r-cofibrant. A careful study of the multiplicative filtrations implies that if $B = A \otimes_\xi \Lambda(V)$ is an E_{r+1}-cofibrant extension of degree n and weight p of A, then

$$\mathrm{Dec}B = \mathrm{Dec}(A \otimes_\xi \Lambda(V)) = \mathrm{Dec}A \otimes_\xi \Lambda(\mathrm{Dec}V)$$

is an E_r-cofibrant extension of $\mathrm{Dec}A$ degree n and weight $p-n$. The converse follows analogously. $\square$

Proposition 4.3.17. *Let $r \geq 0$, and let C be an E_r-cofibrant dga. For every solid diagram*

$$\begin{tikzcd} & A \arrow[d, "w", "\wr"'] \\ C \arrow[ur, dotted, "g"] \arrow[r, "f"] & B \end{tikzcd},$$

in which $w \in \mathcal{F}_r \cap \mathcal{E}_r$, there exists a dotted arrow g, making the diagram commute. In particular, E_r-cofibrant dga's are $\mathcal{F}_r$-cofibrant.

Proof. The case $r = 0$ follows from Proposition 4.2.15. Assume the statement is true for $0 \leq r - 1$. For each solid diagram as above, we have a solid diagram

$$\begin{tikzcd} & \mathrm{Dec}A \arrow[d, "w", "\wr"'] \\ \mathrm{Dec}C \arrow[ur, dotted, "g"] \arrow[r, "f"] & \mathrm{Dec}B \end{tikzcd} .$$

By Lemma 4.3.16, Dec$C$ is E_{r-1}-cofibrant, and by Proposition 4.3.6 we have $w \in \mathcal{F}_{r-1} \cap \mathcal{E}_{r-1}$. By induction hypothesis, there exists $g : \mathrm{Dec}C \to \mathrm{Dec}A$ such that $wg = f$. Since C is E_r-cofibrant, by Lemma 4.3.16 we have $S \circ \mathrm{Dec}C = C$. The adjunction $S \dashv \mathrm{Dec}$ gives a morphism $g : C \to A$. $\square$

By Proposition 1.2.26 the r-homotopy relation is transitive for those morphisms with E_r-cofibrant source. We obtain the following important result.

Proposition 4.3.18. *Let $r \geq 0$, and let C be an E_r-cofibrant dga. Any E_r-quasi-isomorphism $w : A \to B$ induces a bijection*

$$w_* : [C, A]_r \longrightarrow [C, B]_r$$

between the classes of maps defined by r-homotopy equivalence.

Definition 4.3.19. Let $r \geq 0$. A filtered KS-extension $A \otimes_\xi \Lambda V$ of degree n and weight p is called *E_r-minimal* if

$$\xi(V) \subset F^{p+r}(A^+ \cdot A^+) + F^{p+r+1}A.$$

Definition 4.3.20. An *E_r-minimal dga* is the colimit (A, d, F) of a sequence of E_r-minimal extensions, starting from the base field such that $\eta : \mathbf{k} \cong Gr_F^0 A^0$ and $Gr_F^p A^0 = 0$ for $p \neq 0$.

Note that every E_r-minimal dga is, in particular, an E_r-cofibrant dga. The following result is straightforward from the definition.

Lemma 4.3.21. *Let $r \geq 0$, and let (M, d, F) be an E_r-minimal dga. The differentials of its associated spectral sequence satisfy $d_0 = \cdots = d_{r-1} = 0$, and d_r is decomposable. In particular, $Gr_F^p M = E_0^p(M) = \cdots = E_r^p(M)$, and $(E_r(M), d_r)$ is a minimal r-bigraded dga.*

Denote by $\mathsf{E_r\text{-}min}(\mathbf{k})$ the full subcategory of $\mathbf{FDGA}(\mathbf{k})$ of E_r-minimal dga's. The proof of the following result is analogous to the proof of Lemma 4.3.16.

Lemma 4.3.22. *Let $r \geq 0$. The functors*

$$\mathrm{Dec} : \mathsf{E_{r+1}}\text{-}\mathsf{min}(\mathbf{k}) \rightleftarrows \mathsf{E_r}\text{-}\mathsf{min}(\mathbf{k}) : S$$

are inverses to each other.

Definition 4.3.23. Let $r \geq 0$. A filtered dga (A, d, F) is *E_r-0-connected* if $E_r(A)$ is a 0-connected bigraded algebra, that is, $E_{r+1}^{p,-p}(A) = 0$ for all $p \neq 0$, and $E_{r+1}^{0,0}(A) = \mathbf{k}$. It is *$E_r$-1-connected* if, in addition, the bigraded algebra $E_r(A)$ is 1-connected, that is, $E_{r+1}^{p+1,-p}(A) = 0$ for all $p \in \mathbb{Z}$.

Note that an E_r-0-connected (resp. E_r-1-connected) filtered dga is 0-connected (resp. 1-connected), for any $r \geq 0$. For E_r-minimal dga's we have:

Proposition 4.3.24. *If an E_r-minimal dga A is E_r-1-connected, then $A^1 = 0$.*

PROOF. It follows from Proposition 4.2.20 and an induction using the décalage functor. □

Proposition 4.3.25. *Let $r \geq 0$. Every E_r-quasi-isomorphism between E_r-1-connected E_r-minimal dga's is an isomorphism.*

PROOF. The case $r = 0$ follows from Proposition 4.2.21. Assume that the theorem is true for $0 \leq r - 1$. We next prove it for r. Let f be an E_r-quasi-isomorphism between E_r-minimal dga's. By Proposition 4.3.6 and Lemma 4.3.22, the morphism $\mathrm{Dec}f$ is an E_{r-1}-quasi-isomorphism between E_{r-1}-minimal dga's. By induction hypothesis, $\mathrm{Dec}f$ is an isomorphism. Hence f is an isomorphism. □

Definition 4.3.26. An *E_r-cofibrant* (resp. *E_r-minimal*) *model* of a filtered dga A is an E_r-cofibrant (resp. E_r-minimal) dga M, together with an E_r-quasi-isomorphism $\rho : M \to A$.

Theorem 4.3.27. *Let $r \geq 0$. Every E_r-1-connected filtered dga has an E_r-minimal model.*

PROOF. We use induction over $r \geq 0$. By Theorem 4.2.23 every E_0-1-connected dga has an E_0-minimal model. Let $r > 0$. Given a filtered dga A, take an E_{r-1}-minimal model $\rho : M \to \mathrm{Dec}A$ of its décalage. The adjunction

$$\mathrm{Hom}(SM, A) = \mathrm{Hom}(M, \mathrm{Dec}A)$$

gives a morphism $\rho : SM \to A$. By Lemma 4.3.22, SM is E_r-minimal, and ρ is an E_r-quasi-isomorphism. $\square$

Theorem 4.3.28. *Let $r \geq 0$. The triple $(\mathbf{F}\mathsf{DGA}^1(\mathbf{k}), \mathcal{S}_r, \mathcal{E}_r)$ is a Sullivan category. The inclusion induces an equivalence of categories*

$$\pi_r\left(\mathsf{E_r}\text{-}\mathsf{min}^1(\mathbf{k})\right) \longrightarrow \mathrm{Ho}_r\left(\mathbf{F}\mathsf{DGA}^1(\mathbf{k})\right).$$

between the quotient category of 1-connected E_r-minimal dga's modulo r-homotopy equivalence, and the localized category of E_r-1-connected filtered dga's with respect to the class of E_r-quasi-isomorphisms.

Remark 4.3.29. The E_r-minimal model of a (1-connected) filtered dga A is defined in two steps. First, take an E_0-minimal model $M \to \mathrm{Dec}^r A$ of the r-th composition of its décalage. Second, the r-th shift gives an E_r-minimal model $S^r M \to A$ of A.

Filtered Formality. We begin by studying under which conditions an E_r-minimal model of a filtered dga is a Sullivan minimal model.

Proposition 4.3.30 (cf. [**HT90**], Thm. 4.4)**.** *Let $\rho : (M, d, F) \to (A, d, F)$ be an E_r-minimal model of a filtered dga (A, d, F). Then the induced morphism*

$$E_r(\rho) : (E_r(M), d_r) \to (E_r(A), d_r)$$

is an r-bigraded minimal model of $(E_r(A), d_r)$, and $\rho : (M, d) \to (A, d)$ is a Sullivan model of (A, d).

Proof. If follows from the definitions and Lemma 4.3.21. $\square$

The following example shows that, contrary to the case of filtered complexes of vector spaces, the degeneration of the spectral sequence of a filtered dga at a certain stage r is not a sufficient condition for the E_r-minimal dga, to be a Sullivan minimal dga.

Example 4.3.31. Let $A = \Lambda(x, y, z)$, with $dx = y + z^2$, and $dy = dz = 0$. Define a filtration F on (A, d) by setting the weights on the generators to be $w(x) = w(z) = 0$, and $w(y) = 1$. Then $E_1(A) = E_\infty(A) = H(A)$. The filtered dga (A, d, F) is E_0-minimal while (A, d) is not Sullivan minimal.

Definition 4.3.32. An *r-splitting* for a filtered dga (A, d, F) is the structure of an r-bigraded dga $A = \bigoplus A^{p,q}$ such that $F^p A = \bigoplus_{j \geq p} A^{p,*}$. In particular, $E_r^{p,q}(A) = A^{p,q}$, $(A, d) \cong (E_r(A), d_r)$ and $E_{r+1}(A) = E_\infty(A)$.

The spectral sequence associated with a filtered dga (A, d, F) has a natural filtration

$$F^p E_r(A) = \bigoplus_{q \geq p} E_r^q(A).$$

Therefore $(E_r(A), d, F)$ is a filtered dga on its own, for all $r \geq 0$. Its associated spectral sequence satisfies $(E_i(E_r(A)), d_i) = (E_r(A), 0)$, for all $i < r$, and $(E_r(E_r(A)), d_r) = (E_r(A), d_r)$.

Proposition 4.3.33. *Let $(M, d, F) \to (A, d, F)$ be an E_r-minimal model. If (A, d, F) admits an r-splitting, then (M, d, F) admits an r-splitting, and $(M, d) \to (A, d)$ is a Sullivan minimal model of (A, d).*

PROOF. Since (A, d, F) admits an r-splitting, there is an isomorphism of filtered dga's $(A, d, F) \cong (E_r(A), d_r, F)$. By Proposition 4.3.30 the induced morphism $(E_r(M), d, F) \to (E_r(A), d_r, F)$ is an E_r-minimal model. We have a chain of E_r-quasi-isomorphisms

$$(E_r(M), d_r, F) \xrightarrow{\sim} (E_r(A), d_r, F) \cong (A, d, F) \xleftarrow{\sim} (M, d, F).$$

Since both $(E_r(M), d_r, F)$ and (M, d, F) are E_r-minimal dga's, there is an isomorphism

$$(E_r(M), d_r) \cong (M, d).$$

Therefore (M, d, F) admits an r-splitting $M = \bigoplus E_r^{p,q}(M)$, and since $d = d_r$ is decomposable, (M, d) is a Sullivan minimal dga. $\square$

We introduce the following notion of formality.

Definition 4.3.34. A filtered dga (A, d, F) is *E_r-formal* if there is a chain of E_r-quasi-isomorphisms

$$(A, d, F) \longleftrightarrow (E_{r+1}(A), d_{r+1}, F).$$

If (A, d, F) is E_r-formal, then $E_{r+2}(A) = E_\infty(A)$.

Definition 4.3.35. A filtered dga (A, d, F) is *strictly E_r-formal* if it is E_r-formal and $E_{r+1}(A) = E_\infty(A)$.

Proposition 4.3.36. *Let (A, d, F) be a strictly E_r-formal filtered dga, and let $(M, d, F) \to (A, d, F)$ be an E_r-minimal model. Then (M, d) is a Sullivan minimal dga, and there is a diagram of quasi-isomorphisms*

$$\begin{tikzcd} (H(A),0) & (M,d) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\sim"] & (A,d) \\ & (E_r(A),d_r) & \end{tikzcd}$$

In particular (A, d) and $(E_r(A), d_r)$ are formal dga's.

Proof. Since (A, d, F) is E_r-formal we have E_r-quasi-isomorphisms

$$(E_{r+1}(A), d_r, F) \leftarrow (M, d) \to (A, d, F).$$

By Proposition 4.3.30 we also have E_r-quasi-isomorphisms

$$(E_{r+1}(A), 0, F) = (E_r(E_{r+1}(A), d_r, F) \leftarrow (E_r(M), d_r, F) \to (E_r(A), d_r, F).$$

Since $(E_{r+1}(A), d_{r+1}, F) = (H(A), 0, F)$ it follows that (M, d, F) is E_r-quasi-isomorphic to $(E_r(M), d_r, F)$. Since both dga's are E_r-minimal, it follows that

$$(M, d, F) \cong (E_r(M), d_r, F).$$

In particular (M, d, F) admits an r-splitting, and by Lemma 4.3.33, it is a Sullivan minimal dga. □

Example 4.3.37 (See [**NT78**])**.** The de Rham algebra of a compact Kähler manifold is strictly E_0-formal with respect to the Hodge filtration.

Example 4.3.38. Let $X = S^3 \times S^3$, and let $(\mathcal{A}_{dR}(X), d, F)$ denote its de Rham algebra with the Hodge filtration. Then $E_2(\mathcal{A}_{dR}(X)) = E_\infty(\mathcal{A}_{dR}(X))$. The algebras $(\mathcal{A}_{dR}(X)), d)$ and $(E_0(\mathcal{A}_{dR}(X)), d_0)$ are both formal as dga's, and $(\mathcal{A}_{dR}(X), d, F)$ is E_0-formal but not strictly E_0-formal.

Example 4.3.39. Let $(\mathcal{A}_{dR}(X), \overline{\partial}, F)$ be the Dolbeault algebra of a complex manifold X, together with the Hodge filtration. Then its associated spectral sequence degenerates at E_1, and $(\mathcal{A}_{dR}(X), \overline{\partial}, F)$ is strictly E_0-formal if and only if X is Dolbeault formal in the sense of [**NT78**].

Example 4.3.40. Let (A, d) be a dga, and let F be the trivial filtration $F^1A = 0$ and $F^0A = A$. Then $(E_0(A), d_0) = (A, d)$, and $E_1(A) = H(A)$. The filtered dga (A, d, F) is E_0-formal if and only if (A, d) is formal.

Example 4.3.41. Let (A, d, F) be the filtered dga of example 4.3.31. Then $E_1(A) = E_\infty(A)$. The dga's (A, d) and $(E_0(A), d_0)$ are formal, but (A, d, F) is not E_0-formal.

Homotopy Spectral Sequence.

Definition 4.3.42. Given an E_r-1-connected filtered dga A, let $\rho : M \to A$ be an E_r-minimal model. The *E_r-homotopy of A* is the E_r-minimal filtered complex

$$\pi_r(A) := Q(M).$$

We next check that this definition is correct, in the sense that it is functorial, and does not depend on the chosen minimal model. We will follow a process parallel to that of Theorem 4.1.30 for dga's.

We first show that the category $\mathbf{\mathsf{FDGA}}^1(\mathbf{k})_*$ of 1-connected augmented filtered dga's admits a Sullivan category structure.

Proposition 4.3.43. *There is an equivalence of categories*

$$\pi_{*r}\left(\mathsf{E_r}\text{-}\mathsf{min}^1(\mathbf{k})_*\right) \xrightarrow{\sim} \pi_r\left(\mathsf{E_r}\text{-}\mathsf{min}^1(\mathbf{k})\right).$$

PROOF. Every E_r-minimal dga admits a unique augmentation. Likewise, morphisms of augmented E_r-minimal dga's are augmented. This gives an equivalence of categories between E_r-minimal dga's and augmented E_r-minimal dga's. By Proposition 4.1.24 every r-homotopy between augmented morphisms of E_r-minimal dga's is augmented. Hence the corresponding quotient categories are equivalent. □

Proposition 4.3.44. *Let A be an augmented E_r-1-connected filtered dga, and let $\rho : M \to A$ be an E_r-minimal model. Then ρ is augmented.*

PROOF. The proof is analogous to that of Proposition 4.1.26. □

Theorem 4.3.45. *The category $\mathbf{FDGA}^1(\mathbf{k})_*$ with the classes $\mathcal{S}_r^*$ of augmented r-homotopy equivalences and $\mathcal{E}_r^*$ of E_r-quasi-isomorphisms is a Sullivan category, and $\mathsf{E_r}\text{-}\mathsf{min}^1(\mathbf{k})$ is the full subcategory of minimal models. There is an equivalence of categories*

$$\pi_r\left(\mathsf{E_r}\text{-}\mathsf{min}^1(\mathbf{k})\right) \xrightarrow{\sim} \mathbf{FDGA}^1(\mathbf{k})_*[\mathcal{E}_r^{-1}].$$

Proof. By Proposition 4.3.43 E_r-minimal dga's are minimal cofibrant objects in $\mathbf{FDGA}^1(\mathbf{k})_*$. By Theorem 4.3.27 every 1-connected augmented dga has an E_r-minimal model, which by Proposition 4.3.44 is augmented. □

Corollary 4.3.46. *There is an equivalence of categories*

$$\mathrm{Ho}_r\left(\mathbf{FDGA}^1(\mathbf{k})_*\right) \xrightarrow{\sim} \mathrm{Ho}_r\left(\mathbf{FDGA}^1(\mathbf{k})\right).$$

Proof. It follows from Theorem 4.3.28, Theorem 4.3.45 and Proposition 4.3.43. □

Theorem 4.3.47. *Let $r \geq 0$. The functor $Q : \mathbf{FDGA}^1(\mathbf{k})_* \longrightarrow \mathbf{C}^+(\mathbf{Fk})$ admits a left derived functor*

$$\mathbb{L}_r Q : \mathrm{Ho}_r\left(\mathbf{FDGA}^1(\mathbf{k})_*\right) \longrightarrow \mathbf{D}_r^+(\mathbf{Fk}).$$

The composition of functors

$$\mathrm{Ho}_r\left(\mathbf{FDGA}^1(\mathbf{k})\right) \xleftarrow{\sim} \mathrm{Ho}_r\left(\mathbf{FDGA}^1(\mathbf{k})_*\right) \xrightarrow{\mathbb{L}_r Q} \mathbf{D}_r^+(\mathbf{Fk}) \xrightarrow{E_r} \mathbf{C}_{r+1}^+(\mathbf{Fk})$$

defines a functor

$$\pi_r : \mathrm{Ho}_r\left(\mathbf{FDGA}^1(\mathbf{k})\right) \longrightarrow \mathbf{C}_{r+1}^+(\mathbf{Fk})$$

which associates to every object A, the E_r-minimal complex $\pi_r(A) = Q(M_A)$, where $M_A \to A$ is an E_r-minimal model of A,

Proof. By Proposition 4.2.12 the functor Q preserves strong equivalences. By Theorem 4.3.45 and Proposition 1.1.32, Q admits a left derived functor

$$\mathbb{L}_r Q : \mathrm{Ho}_r\left(\mathbf{FDGA}^1(\mathbf{k})_*\right) \longrightarrow \mathbf{D}_r^+(\mathbf{Fk}).$$

The functor π_r follows from $\mathbb{L}Q$ and the equivalence of categories

$$\mathrm{Ho}_r\left(\mathbf{FDGA}^1(\mathbf{k})_*\right) \xrightarrow{\sim} \mathrm{Ho}_r\left(\mathbf{FDGA}^1(\mathbf{k})\right).$$

□

Proposition 4.3.48. *Let $r \geq 0$. Let A be a E_r-1-connected filtered dga. The spectral sequence associated with the filtered complex $\pi_r(A)$ satisfies*

$$E_{r+1}^{*,*}(\pi_r(A)) = \pi^{*,*}(E_r A) \Rightarrow \pi^*(A).$$

PROOF. Let $M \to A$ be an E_r-minimal model. Then $\pi_r(A) = Q(M)$, and $\pi^{*,*}(E_r(A)) = Q(E_r(M))$. Since M is a Sullivan dga, we have $\pi(A) = H(Q(M))$. Since $Q(M)$ is an E_r-minimal complex, its associated spectral sequence satisfies $d_0 = \cdots = d_r = 0$. Therefore

$$Gr_F Q(M) = E_0(\pi_r(A)) = \cdots E_{r+1}(\pi_r(A)).$$

Since $Q \circ Gr_F = Gr_F \circ Q$, it follows that

$$E_{r+1}(\pi_r(A)) = Gr_F Q(M) = QGr_F M = Q(E_0(M)).$$

In addition, since M is E_r-minimal, $E_0(M) = E_r(M)$, and hence

$$E_{r+1}(\pi_r(A)) = Q(E_r(M)) = \pi(E_r(A)).$$

□

4.4. BIFILTERED DIFFERENTIAL GRADED ALGEBRAS

We extend the results of the previous sections, to bifiltered dga's.

Denote by $\mathbf{F}^2\mathsf{DGA}(\mathbf{k})$ the category of bifiltered dga's over $\mathbf{k}$. It's objects are given by (A, d, W, F) such that both (A, d, W) and (A, d, F) are objects of $\mathbf{F}\mathsf{DGA}(\mathbf{k})$. Its morphisms are assumed to be compatible with both filtrations. Given a bifiltered dga (A, d, W, F) we will denote

$$W^p F^q A := W^p A \cap F^q A.$$

We next provide the corresponding definitions of bifiltered quasi-isomorphism and bifiltered fibrations, and define a P-category structure on $\mathbf{F}^2\mathsf{DGA}(\mathbf{k})$. As in the case of filtered complexes, and given out interests in Hodge theory, we shall only develop the theory of minimal models for the homotopy categories $\mathrm{Ho}_{0,0}$ and $\mathrm{Ho}_{1,0}$ defined as follows.

Definition 4.4.1. A morphism $f : A \to B$ of bifiltered dga's is called $E_{0,0}$-*quasi-isomorphism* if the induced morphism

$$E_1(E_0(f,F),W) = H^n(Gr^p_W Gr^q_F f) \cong H^n(Gr^q_F Gr^p_W f) = E_1(E_0(f,W),F)$$

is an isomorphism for all $n \geq 0$ and all $p,q \in \mathbb{Z}$. Denote by $\mathcal{E}_{0,0}$ the class of $E_{0,0}$-quasi-isomorphisms. Define a new class of morphisms

$$\mathcal{E}_{1,0} = (\mathrm{Dec}^W)^{-1}(\mathcal{E}_{0,0}).$$

Morphisms in $\mathcal{E}_{1,0}$ are called $E_{1,0}$-*quasi-isomorphisms.*

Definition 4.4.2. A morphism $f : (A,d,W,F) \to (B,d,W,F)$ of bifiltered dga's is called $E_{0,0}$-*fibration* if the induced morphism

$$Gr^q_W Gr^p_F f \cong Gr^p_F Gr^q_W f$$

is surjective for all $p,q \in \mathbb{Z}$. Define $E_{1,0}$-*fibrations* as in the previous definition, by décalage. Denote by $\mathcal{F}_{r,0}$ the class of $E_{r,0}$-fibrations.

Definition 4.4.3. Let $r \in \{0,1\}$. The $(r,0)$-*path object* of a bifiltered dga A is the bifiltered dga defined by

$$P_{r,0}(A) = (A[t,dt], W * \sigma_r, F * \sigma_0).$$

Lemma 4.4.4. *Let* (A,W,F) *be a filtered dga. Then*

$$\mathrm{Dec}^W(P_{1,0}(A)) = P_{0,0}(\mathrm{Dec}^W A).$$

Proof. The proof is analogous to that of Lemma 4.3.10. □

Proposition 4.4.5. *Let* $r \in \{0,1\}$. *The category of bifiltered dga's with the* $(r,0)$-*path object and the classes* $\mathcal{F}_{r,0}$ *and* $\mathcal{E}_{r,0}$ *is a P-category.*

Proof. For $r = 0$ it suffices to check that the functor

$$\Psi := Gr^\bullet_W Gr^\bullet_F : \mathbf{F}^2\mathsf{DGA}(\mathbf{k}) \longrightarrow \mathsf{DGA}(\mathbf{k})$$

satisfies the conditions of Lemma 1.2.33. The proof is analogous to that of Proposition 4.2.9. For $r = 1$, it suffices to check that the functor

$$\mathrm{Dec}^W : \mathbf{F}^2\mathsf{DGA}(\mathbf{k}) \longrightarrow \mathbf{F}^2\mathsf{DGA}(\mathbf{k})$$

satisfies the conditions of Lemma 1.2.33. The proof is analogous to that of Proposition 4.3.12, using Lemma 4.4.4. □

Definition 4.4.6. Let (A, d, W, F) be a bifiltered dga. A *bifiltered KS-extension* of A of degree n and biweight (p, q) is a bifiltered dga $A \otimes_\xi \Lambda V$, where V is a bifiltered graded module concentrated in pure degree n and pure biweight (p, q), and $\xi : V \to W^p F^q A$ is a linear map of degree 1 such that $d\xi = 0$. The filtrations on $A \otimes_\xi \Lambda V$ are defined by multiplicative extension.

Definition 4.4.7. Let $r \in \{0, 1\}$. An *$E_{r,0}$-minimal extension* of an augmented bifiltered dga A is a bifiltered KS-extension $A \otimes_\xi \Lambda V$ of degree n and biweight (p, q) such that

$$\xi(W^p F^q V) \subset W^p F^q(A^+ \cdot A^+) + W^{p+r+1} F^q A + W^{p+1} F^{q+1} A.$$

A bifiltered dga is called *$E_{r,0}$-minimal* if it is the colimit of $E_{r,0}$-minimal extensions, starting from the base field.

Denote by $\mathsf{E_{r,0}}\text{-}\mathsf{min}(\mathbf{k})$ the full subcategory of $E_{r,0}$-minimal dga's.

Lemma 4.4.8. *The functors*

$$\mathrm{Dec}^W : \mathsf{E_{1,0}}\text{-}\mathsf{min}(\mathbf{k}) \rightleftarrows \mathsf{E_{0,0}}\text{-}\mathsf{min}(\mathbf{k}) : S^W$$

are inverses to each other.

PROOF. The proof is analogous to that of Lemma 4.3.22. □

Theorem 4.4.9. *Let $r \in \{0, 1\}$. For every 1-connected bifiltered dga A there is an $E_{r,0}$-minimal dga M, together with an $E_{r,0}$-quasi-isomorphism $\rho : M \to A$.*

PROOF. For $r = 0$ the proof is analogous to that of Theorem 4.2.23, so we only indicate the main changes. Given a 1-connected bifiltered dga A we will define, inductively over $n \geq 1$, a sequence of free bifiltered dga's M_n together with bifiltered morphisms $\rho_n : M_n \to A$, with $M_1 = \mathbf{k}$, satisfying the following conditions:

- (a_n) The algebra M_n is a composition of $E_{0,0}$-minimal extensions of M_{n-1} of degrees n and $n+1$. The map ρ_n extends ρ_{n-1}.
- (b_n) $H^i(Gr_W^p Gr_F^q C(\rho_n)) = 0$ for all $i \leq n$ and all $p, q \in \mathbb{Z}$.

Define $M_0 = M_1 = \mathbf{k}$ concentrated in degree 0 and with pure biweight $(0,0)$, and let $\rho_1 : M_1 \to A$ be the unit map. This is the base case for our induction. Assume inductively that we have defined $\rho_{n-1} : M_{n-1} \to A$ satisfying (a_{n-1}) and (b_{n-1}). We will construct $\rho_n : M_n \to A$ inductively over decreasing $r \in \mathbb{Z}$ as follows. We will define a family of bifiltered morphisms of dga's $\rho : M_{n,r} \to A$ satisfying the following conditions:

$(a_{n,r})$ The algebra $M_{n,p}$ is a composition of $E_{0,0}$-minimal extensions of $M_{n,r+1}$ of degree n and total weight $p+q=r$, and of degree $n+1$ and total weight $p+q>r$. The map $\rho_{n,r}$ extends $\rho_{n,r+1}$.

$(b_{n,r})$ $H^i(Gr_W^p Gr_F^q C(\rho_{n,r})) = 0$ whenever $i < n$ and $p,q \in \mathbb{Z}$, or $i = n$ and p,q are such that $p+q \geq r$.

Assume that we have constructed $\rho_{n,r+1} : M_{n,r+1} \to A$. To simplify notation, let $M := M_{n,r+1}$ and $\rho := \rho_{n,r+1}$.

By Lemma 4.2.24 applied to the dga's W^pM and F^qM, for all p,q such that $p+q=r$, there exists a bifiltered morphism $\widetilde{\rho} : \widetilde{M} \to A$ satisfying the following conditions:

(1) The algebra $\widetilde{M}$ is a composition of $E_{0,0}$-minimal extensions of M of degree $n+1$ and total weight $p+q>r$. The map $\widetilde{\rho}$ extends ρ.

(2) $H^i(Gr_W^{p'} Gr_F^{q'} C(\widetilde{\rho})) = H^i(Gr_W^{p'} Gr_F^{q'} C(\rho))$ for all $i \leq n$ and all $p', q' \in \mathbb{Z}$.

(3) The map $\pi_* : H^n(W^pF^qC(\widetilde{\rho})) \to H^n(Gr_W^p Gr_F^q C(\widetilde{\rho}))$ is surjective whenever $p+q=r$.

Consider the graded vector space of degree n and biweight (p,q) defined by

$$V_{n,p,q} = H^n(Gr_W^p Gr_F^q C(\widetilde{\rho})).$$

Define a graded algebra $M_{n,r} = \widetilde{M} \otimes \Lambda(V_{n,r})$, where

$$V_{n,r} = \bigoplus_{p+q=r} V_{n,p,q}.$$

The proof now follows analogously to that of Theorem 4.2.23. The case $r=1$ follows by décalage of the weight filtration and Lemma 4.4.8. $\square$

Corollary 4.4.10. *Let $r \in \{0,1\}$. The triple $(\mathbf{F}^2\mathsf{DGA}^1(\mathbf{k}), \mathcal{S}_{r,0}, \mathcal{E}_{r,0})$ is a Sullivan category. The subcategory of $E_{r,0}$-minimal dga's is a full subcategory of minimal models. The inclusion induces an equivalence of categories*

$$\pi_{r,0}\left(\mathsf{E_{r,0}\text{-}min}^1(\mathbf{k})\right) \xrightarrow{\sim} \mathrm{Ho}_{r,0}\left(\mathbf{F}^2\mathsf{DGA}^1(\mathbf{k})\right).$$

CHAPTER 5

Mixed Hodge Theory and Rational Homotopy of Algebraic Varieties

In this last chapter we bring together the results of the previous chapters to study the homotopy theory of mixed Hodge diagrams, and their cohomological descent structure. We then provide applications to algebraic geometry.

The main result of Section 1 is the existence of minimal models of mixed Hodge diagrams, endowing $\mathbf{MHD}^1$ with a Sullivan category structure. More specifically, we prove, using the results of Chapters 1 and 4, that every mixed Hodge diagram is quasi-isomorphic to a mixed Hodge dga which is Sullivan minimal. This result allows to define the homotopy of a mixed Hodge diagram as in the classical case of dga's, via the derived functor of indecomposables. The homotopy associates to every mixed Hodge diagram, a graded mixed Hodge structure, whose rational part coincides with the classical homotopy of the rational part of the original diagram. Hence the homotopy of every mixed Hodge diagram is endowed with a functorial mixed Hodge structure. We also show that the minimal model of the rational part of a mixed Hodge diagram can be computed from the first stage of the spectral sequence associated with the weight filtration.

Section 2 is devoted to the theory of cohomological descent. We recall the Thom-Whitney simple of [**Nav87**] defined over strict cosimplicial dga's and extend its definition to filtered and bifiltered dga's. This allows to define a simple functor for cubical mixed Hodge diagrams, providing **MHD** with a cohomological descent structure. This descent structure endows the category **MHD** with realizable homotopy colimits of diagrams indexed by finite categories (see [**Rod12a**]).

In Section 3 we recall the Hodge-Deligne [**Del71b**] theory for open smooth varieties, as well as the multiplicative version of [**Nav87**]. Using the cohomological descent structure, and the extension criterion of [**GN02**], we provide an extension of these constructions to singular varieties. As an application, we obtain a proof of that the rational homotopy type of every simply connected algebraic variety over $\mathbb{C}$ is equipped with a functorial mixed Hodge structure (see [**Nav87**] and [**Hai87**] where the same result is proved, using the initial constructions of Morgan [**Mor78**]). We also show that the rational homotopy type of every simply connected complex algebraic variety is a formal consequence of the first term its associated weight filtration. This result is also true for morphisms of varieties.

5.1. Homotopy Theory of Mixed Hodge Diagrams

Diagrams of Filtered Algebras. As a preliminary step to the study of mixed Hodge diagrams we consider a more general situation of diagrams of filtered dga's, which occur for compactificable analytic spaces. Indeed, Guillén-Navarro defined a Hodge filtration (see [**GN02**], Section 4) for every compactificable analytic space. Likewise, it is also possible to define a weight filtration (analogously to the theory of motives of ibid. Section 5). These filtrations define a diagram of filtered dga's, which for algebraic spaces, becomes a mixed Hodge diagram. The Hodge and the weight filtrations are well defined up to E_0- and E_1-quasi-isomorphisms respectively, but contrary to the case of algebraic varieties, for general compactificable analytic spaces the associated spectral sequences need not degenerate at any specific stage, and hence the two filtrations W and F do not define a mixed Hodge structure. Thanks to the theory developed in Chapters 1 and 4, we are able to provide the basic definitions to treat this general case, although we will not delve into all the consequences. For the rest of this chapter we let $\mathbf{k} = \mathbb{Q}$ be the field of rational numbers.

We next define the category of diagrams of filtered dga's. This is a diagram category (see Definition 1.3.1) of fixed type

$$I = \{0 \to 1 \leftarrow 2 \to \cdots \leftarrow s\},$$

whose vertices are categories of filtered and bifiltered dga's. Additional assumptions on the behaviour of the filtrations will lead to the notion of mixed and absolute Hodge diagrams of dga's.

Definition 5.1.1. Let $\mathbf{A} : I \to \mathsf{Cat}$ be the functor defined by

$$\begin{tikzcd} 0 \arrow[r, "u"] \arrow[d, mapsto] & 1 \arrow[d, mapsto] & \cdots \arrow[l] \arrow[r] & s-1 \arrow[d, mapsto] & s \arrow[l, "v"'] \arrow[d, mapsto] \\ \mathbf{F}\mathsf{DGA}(\mathbf{k}) \arrow[r, "u_*"] & \mathbf{F}\mathsf{DGA}(\mathbb{C}) & \cdots \arrow[l, "Id"'] \arrow[r, "Id"] & \mathbf{F}\mathsf{DGA}(\mathbb{C}) & \mathbf{F}^2\mathsf{DGA}(\mathbb{C}) \arrow[l, "v_*"'] \end{tikzcd}$$

where u_* is defined by extension of scalars

$$u_*(A_{\mathbf{k}}, W) := (A_{\mathbf{k}}, W) \otimes \mathbb{C},$$

and v_* is defined by forgetting the second filtration

$$v_*(A_{\mathbb{C}}, W, F) := (A_{\mathbb{C}}, W).$$

All intermediate functors are defined to be the identity.

The category of diagrams $\Gamma\mathbf{A}$ associated with the functor $\mathbf{A}$ is called the *category of diagrams of filtered dga's over* $\mathbf{k}$. Objects and morphisms in $\Gamma\mathbf{A}$ are defined as follows:

- A *diagram of filtered dga's* consists in
  (i) a filtered dga $(A_{\mathbf{k}}, W)$ over $\mathbf{k}$,
  (ii) a bifiltered dga $(A_{\mathbb{C}}, W, F)$ over $\mathbb{C}$, together with
  (iii) a morphism $\varphi_u : (A_i, W) \to (A_j, W)$ of filtered dga's over $\mathbb{C}$, for each $u : i \to j$ of I, with $A_0 = A_{\mathbf{k}} \otimes \mathbb{C}$ and $A_s = A_{\mathbb{C}}$.

Such a diagram is denoted as

$$A = \Big((A_{\mathbf{k}}, W) \overset{\varphi}{\longleftrightarrow} (A_{\mathbb{C}}, W, F) \Big).$$

- A *morphism of diagrams of filtered dga's* $f : A \to B$ consists in
  (i) a morphism of filtered dga's $f_{\mathbf{k}} : (A_{\mathbf{k}}, W) \to (B_{\mathbf{k}}, W)$,
  (ii) a morphism of bifiltered dga's $f_{\mathbb{C}} : (A_{\mathbb{C}}, W, F) \to (B_{\mathbb{C}}, W, F)$, and

(iii) a family of morphisms of filtered dga's $f_i : (A_i, W) \to (B_i, W)$ with $f_0 = f_{\mathbf{k}} \otimes \mathbb{C}$, and $f_s = f_{\mathbb{C}}$, making the following diagrams commute.

$$\begin{tikzcd} (A_i, W) \arrow[r, "\varphi_u"] \arrow[d, "f_i"'] & (A_j, W) \arrow[d, "f_j"] \\ (B_i, W) \arrow[r, "\varphi_u"] & (B_j, W) \end{tikzcd}$$

Definition 5.1.2. A morphism $f : A \to B$ of diagrams of filtered dga's is said to be a *quasi-isomorphism* if the maps $f_{\mathbf{k}}$, $f_{\mathbb{C}}$ and f_i are quasi-isomorphisms: the maps $H(f_{\mathbf{k}})$, $H(f_{\mathbb{C}})$ and $H(f_i)$ are isomorphisms.

Denote by $\mathcal{Q}$ the class of quasi-isomorphisms of $\Gamma\mathbf{A}$.

The P-category structures on $\mathbf{F}\mathsf{DGA}(\mathbf{k})$, $\mathbf{F}\mathsf{DGA}(\mathbb{C})$ and $\mathbf{F}^2\mathsf{DGA}(\mathbb{C})$ define a P-category structure on $\Gamma\mathbf{A}$ as follows. Given our interests in Hodge theory, we shall only study $E_{r,0}$-homotopy structures, with $r \in \{0, 1\}$.

Definition 5.1.3. A morphism $f : A \to B$ of diagrams of filtered dga's is called $E_{r,0}$-*fibration* if the morphisms $f_{\mathbf{k}}$ and f_i are E_r-fibrations of filtered dga's, and the morphism $f_{\mathbb{C}}$ is an $E_{r,0}$-fibration of bifiltered dga's.

Denote by $\mathcal{F}_{r,0}$ the class of $E_{r,0}$-fibrations of $\Gamma\mathbf{A}$.

Definition 5.1.4. A morphism $f : A \to B$ of diagrams of filtered dga's is called $E_{r,0}$-*quasi-isomorphism* if $f_{\mathbf{k}}$ and f_i are E_r-quasi-isomorphisms of filtered dga's, and $f_{\mathbb{C}}$ is an $E_{r,0}$-quasi-isomorphism of bifiltered dga's.

Denote by $\mathcal{E}_{r,0}$ the class of $E_{r,0}$-quasi-isomorphisms of $\Gamma\mathbf{A}$. Since the filtrations are biregular, we have $\mathcal{E}_{0,0} \subset \mathcal{E}_{1,0} \subset \mathcal{Q}$. Hence we have functors

$$\mathrm{Ho}_{0,0}(\Gamma\mathbf{A}) \longrightarrow \mathrm{Ho}_{1,0}(\Gamma\mathbf{A}) \longrightarrow \mathrm{Ho}(\Gamma\mathbf{A})$$

relating the localizations with respect to $\mathcal{E}_{0,0}$, $\mathcal{E}_{0,1}$ and $\mathcal{Q}$ respectively.

Definition 5.1.5. The $(r, 0)$-*path object* of a diagram of filtered dga's A is the diagram defined by:

$$P_{r,0}(A) := \left(P_r(A_{\mathbf{k}}, W) \overset{P(\varphi)}{\underset{\sim}{\longleftarrow\!\!\dashrightarrow}} P_{r,0}(A_{\mathbb{C}}, W, F) \right),$$

where $P_r(A_{\mathbf{k}}, W)$ and $P_{r,0}(A_{\mathbb{C}}, W, F)$ are the r-path and the $(r,0)$-path of the dga's $(A_{\mathbf{k}}, W)$ and $(A_{\mathbb{C}}, W, F)$ respectively (see Definitions 4.3.8 and 4.4.3), and $P(\varphi_u)$ is induced by φ_u.

Proposition 5.1.6. *The $(r,0)$-path object with the classes $\mathcal{F}_{r,0}$ and $\mathcal{E}_{r,0}$ define a level-wise P-category structure on $\Gamma\mathbf{A}$.*

Proof. By Proposition 4.2.9 the category of filtered dga's (over $\mathbf{k}$ or $\mathbb{C}$), with the r-path object and the classes $\mathcal{F}_r$ and $\mathcal{E}_r$ is a P-category. Likewise, by Proposition 4.4.5 the category of bifiltered dga's over $\mathbb{C}$, with the $(r,0)$-path object and the classes $\mathcal{E}_{r,0}$ and $\mathcal{F}_{r,0}$ is a P-category. Both the functor $-\otimes_{\mathbf{k}}\mathbb{C} : \mathbf{F}\mathsf{DGA}(\mathbf{k}) \to \mathbf{F}\mathsf{DGA}(\mathbb{C})$, and the forgetful functor $\mathbf{F}^2\mathsf{DGA}(\mathbb{C}) \to \mathbf{F}\mathsf{DGA}(\mathbb{C})$ defined by forgetting F are compatible with such P-category structures. The result follows from Proposition 1.3.8. $\square$

Deligne's décalage with respect to the weight filtration defines a functor

$$\mathrm{Dec}^W : \Gamma\mathbf{A} \longrightarrow \Gamma\mathbf{A}$$

which is the identity on morphisms, and satisfies $\mathcal{E}_{1,0} = (\mathrm{Dec}^W)^{-1}(\mathcal{E}_{0,0})$. Likewise, the shift with respect to the weight filtration defines a functor

$$S^W : \Gamma\mathbf{A} \longrightarrow \Gamma\mathbf{A}$$

which is left adjoint to Dec^W. Analogously to the case of diagrams of filtered complexes we have:

Theorem 5.1.7. *Deligne's décalage induces an equivalence of categories*

$$\mathrm{Dec}^W : \mathrm{Ho}_{1,0}(\Gamma\mathbf{A}) \longrightarrow \mathrm{Ho}_{0,0}(\Gamma\mathbf{A}).$$

Proof. The proof is analogous to that of Theorem 4.3.7. $\square$

Let us now turn to the construction of level-wise minimal models.

By an abuse of notation, we will denote by $\Gamma\mathbf{A}^1$ the full subcategory of diagrams of filtered dga's A such that: $A_{\mathbf{k}}$ and A_i are E_r-1-connected and $A_{\mathbb{C}}$ is $E_{r,0}$-1-connected. Consider a diagram of $\Gamma\mathbf{A}^1$

$$A = \Big((A_{\mathbf{k}}, W) \overset{\varphi}{\dashleftarrow\dashrightarrow} (A_{\mathbb{C}}, W, F)\Big).$$

By Theorem 4.3.27 there exist E_r-minimal models $\rho_{\mathbf{k}} : (M_{\mathbf{k}}, W) \to (A_{\mathbf{k}}, W)$, and $\rho_i : (M_i, W) \to (A_i, W)$, for all $0 < i < r$. Likewise, by Theorem 4.4.9 there exists an $E_{r,0}$-minimal model $\rho_{\mathbb{C}} : (M_{\mathbb{C}}, W, F) \to (A_{\mathbb{C}}, W, F)$. Let $(M_0, W) = (M_{\mathbf{k}}, W) \otimes \mathbb{C}$, and $(M_r, W) = (M_{\mathbb{C}}, W)$. Note that $(M_{\mathbb{C}}, W)$ is not E_r-minimal as a filtered dga over $\mathbb{C}$, but instead, it is E_r-cofibrant. For every $u : i \to j$ we have a solid diagram

$$\begin{tikzcd}
(M_i, W) \arrow[r, dotted, "\varphi'_u"] \arrow[d, "\rho_i"] \arrow[dr, phantom, "\simeq"] & (M_j, W) \arrow[d, "\rho_j"] \\
(A_i, W) \arrow[r, "\varphi_u"] & (A_j, W).
\end{tikzcd}$$

in which the elements of the top row are E_r-cofibrant dga's, and the vertical arrows are E_r-quasi-isomorphisms. Therefore the diagram can be completed with a dotted arrow φ'_u, and commutes up to an r-homotopy of filtered dga's. This leads to the notion of ho-morphism (see Definition 1.3.11). We next recall its definition in the context of diagrams of filtered dga's.

Definition 5.1.8. An $(r, 0)$-*ho-morphism* $f : A \rightsquigarrow B$ between two diagrams of filtered dga's of type I is given by:

(i) a morphism of filtered dga's $f_{\mathbf{k}} : (A_{\mathbf{k}}, W) \to (B_{\mathbf{k}}, W)$,
(ii) a morphism of bifiltered dga's $f_{\mathbb{C}} : (A_{\mathbb{C}}, W, F) \to (B_{\mathbb{C}}, W, F)$,
(iii) a morphism of filtered dga's $f_i : (A_i, W) \to (B_i, W)$, for each $i \in I$, such that $f_0 = f_{\mathbf{k}} \otimes \mathbb{C}$, and $f_r = f_{\mathbb{C}}$, together with
(iv) an r-homotopy of filtered dga's $F_u : (A_i, W) \to P_r(A_j, W)$ making the diagram

$$\begin{tikzcd}
(A_i, W) \arrow[r, "\varphi_u"] \arrow[d, "f_i"'] \arrow[dr, Rightarrow, "F_u"] & (A_j, W) \arrow[d, "f_j"] \\
(B_i, W) \arrow[r, "\varphi_u"] & (B_j, W)
\end{tikzcd}$$

commute, for each $u : i \to j \in I$.

The notion of $(r, 0)$-homotopy between $(r, 0)$-ho-morphisms (see Definition 1.3.15) allows to define a class of equivalences of $\Gamma\mathbf{A}$ as follows:

Definition 5.1.9. A morphism of filtered dga's $f : A \to B$ is said to be an $(r, 0)$-*ho-equivalence* if there exists an $(r, 0)$-ho-morphism $g : B \rightsquigarrow A$, together with $(r,0)$-homotopies

$$gf \simeq \cdots \simeq 1_A \text{ and } fg \simeq \cdots \simeq 1_B.$$

Denote by $\mathcal{H}_{r,0}$ the closure by composition of the class of ho-equivalences. We have $\mathcal{S}_{r,0} \subset \mathcal{H}_{r,0} \subset \mathcal{E}_{r,0}$. In particular the triple $(\Gamma\mathbf{A}, \mathcal{H}_{r,0}, \mathcal{E}_{r,0})$ is a category with strong and weak equivalences.

Denote by $\Gamma\mathsf{E}_{\mathsf{r},\mathsf{0}}\text{-}\mathsf{min}^1$ the full subcategory of $\Gamma\mathbf{A}^1$ of diagrams

$$M = \Big((M_{\mathbf{k}}, W) \overset{\varphi}{\longleftarrow\!\!\dashrightarrow} (M_{\mathbb{C}}, W, F)\Big)$$

such that $M_{\mathbf{k}}$ and M_i are E_r-minimal, and $M_{\mathbb{C}}$ is $E_{r,0}$-minimal.

Let $\pi^h\left(\Gamma\mathsf{E}_{\mathsf{r},\mathsf{0}}\text{-}\mathsf{min}^1\right)$ denote the category with the same objects, and whose morphisms are $(r,0)$-ho-morphisms modulo $(r,0)$-homotopy. We can now state the main result of this section.

Theorem 5.1.10. *The triple* $(\Gamma\mathbf{A}^1, \mathcal{H}_{r,0}, \mathcal{E}_{r,0})$ *is a Sullivan category, and* $\Gamma\mathsf{E}_{\mathsf{r},\mathsf{0}}\text{-}\mathsf{min}^1$ *is a full subcategory of minimal models. There is an equivalence of categories*

$$\pi^h_{r,0}\left(\Gamma\mathsf{E}_{\mathsf{r},\mathsf{0}}\text{-}\mathsf{min}^1\right) \xrightarrow{\sim} \mathrm{Ho}_{r,0}\left(\Gamma\mathbf{A}^1\right).$$

Proof. By Proposition 5.1.6 the category $\Gamma\mathbf{A}$ inherits a level-wise P-category structure. Furthermore, the condition of being 1-connected is preserved by $E_{r,0}$-quasi-isomorphisms. The result follows from Lemma 1.4.13, together with the existence of minimal models of (bi)filtered dga's of Theorems 4.3.27 and 4.4.9 respectively. $\square$

Hodge Diagrams of Algebras.

Definition 5.1.11. A *mixed Hodge diagram* is a diagram of filtered dga's

$$A = \Big((A_{\mathbf{k}}, W) \underset{\sim}{\overset{\varphi}{\longleftarrow\!\!\dashrightarrow}} (A_{\mathbb{C}}, W, F)\Big),$$

satisfying the following conditions:

(MHD_0) The comparison map φ is a string of E_1-quasi-isomorphisms.

(MHD_1) For all $p \in \mathbb{Z}$, the filtered complex $(Gr_p^W A_{\mathbb{C}}, F)$ is d-strict.

(MHD$_2$) The filtration F induced on $H^n(Gr_p^W A_{\mathbb{C}})$, defines a pure Hodge structure of weight $p+n$ on $H^n(Gr_p^W A_{\mathbf{k}})$, for all n, and all $p \in \mathbb{Z}$.

Denote by **MHD** the category of mixed Hodge diagrams of a fixed type.

The spectral sequence associated with the Hodge filtration F of a mixed Hodge diagram degenerates at the first stage, while the spectral sequence associated with the weight filtration W, degenerates at the second stage. As in the case of mixed Hodge complexes, it is more convenient to work with a shifted version of mixed Hodge diagrams, in which both associated spectral sequences degenerate at the first stage.

Definition 5.1.12. An *absolute Hodge diagram* is a diagram of filtered dga's

$$A = \left((A_{\mathbf{k}}, W) \overset{\varphi}{\underset{\sim}{\dashleftarrow\dashrightarrow}} (A_{\mathbb{C}}, W, F)\right),$$

satisfying the following conditions:

(AHD$_0$) The comparison map φ is a string of E_0-quasi-isomorphisms.
(AHD$_1$) For all $p \in \mathbb{Z}$, the bifiltered complex $(A_{\mathbb{C}}, W, F)$ is d-bistrict.
(AHD$_2$) The filtration F induced on $H^n(Gr_p^W A_{\mathbb{C}})$, defines a pure Hodge structure of weight p on $H^n(Gr_p^W A_{\mathbf{k}})$, for all n, and all $p \in \mathbb{Z}$.

Denote by **AHD** the category of absolute Hodge diagrams of a fixed type. The décalage with respect to the weight filtration induces a functor

$$\mathrm{Dec}^W : \mathbf{MHD} \longrightarrow \mathbf{AHD}.$$

The following is a direct consequence of Lemma 3.3.8.

Lemma 5.1.13. *Denote by $\mathcal{Q}$ and $\mathcal{E}_{r,0}$ the classes of quasi-isomorphisms and $E_{r,0}$-quasi-isomorphisms of $\Gamma\mathbf{A}$ respectively.*

(1) The classes of maps $\mathcal{Q}$ and $\mathcal{E}_{1,0}$ coincide in **MHD**.
(2) The classes of maps $\mathcal{Q}$ and $\mathcal{E}_{0,0}$ coincide in **AHD**.

Minimal Models. For the construction of minimal models we will restrict to the subcategory $\mathbf{MHD}^1$ of 1-connected mixed Hodge diagrams:

Definition 5.1.14. A mixed (resp. absolute) Hodge diagram A is called *0-connected* if the unit map $\eta : \mathbf{k} \to A_{\mathbf{k}}$ induces an isomorphism $\mathbf{k} \cong H^0(A_{\mathbf{k}})$. It is called *1-connected* if, in addition, $H^1(A_{\mathbf{k}}) = 0$.

Definition 5.1.15. A *mixed Hodge dga over* $\mathbf{k}$ is a dga (A, d) over $\mathbf{k}$ such that each A^n is endowed with a mixed Hodge structure, and the differential $d : A^n \to A^{n+1}$ is a morphism of mixed Hodge structures.

Denote by $\mathbf{MH}\mathsf{DGA}$ the category of mixed Hodge dga's over $\mathbf{k}$.

The cohomology of every absolute Hodge diagram is a mixed Hodge dga with trivial differential. We have functors

$$\mathbf{MHD} \xrightarrow{\mathrm{Dec}^W} \mathbf{AHD} \xrightarrow{H} \mathbf{MH}\mathsf{DGA}.$$

Conversely, since the category of mixed Hodge structures is abelian, every mixed Hodge dga is an absolute Hodge diagram in which the comparison morphisms are identities. There is an inclusion functor

$$i : \mathbf{MH}\mathsf{DGA} \longrightarrow \mathbf{AHD}.$$

We will prove that every 1-connected absolute Hodge diagram is quasi-isomorphic to a 1-connected mixed Hodge dga which is Sullivan minimal.

Definition 5.1.16. A *mixed Hodge Sullivan minimal dga* is a Sullivan minimal dga $M = \Lambda V$ over $\mathbf{k}$ such that each V^n is endowed with a mixed Hodge structure $\{(V^n, W), (V^n \otimes \mathbb{C}, W, F)\}$, and the differentials are compatible with the filtrations.

In particular, the mixed Hodge structures on V^n, define a mixed Hodge structure on A^n. Therefore every mixed Hodge Sullivan minimal dga is a mixed Hodge dga. Denote by $\mathbf{MH}\mathsf{DGA}_{min}$ the category of mixed Hodge Sullivan minimal dga's.

To construct minimal models for 1-connected absolute Hodge diagrams, we adapt the classical construction of minimal models for 1-connected dga's of [**GM81**], to absolute Hodge diagrams.

Theorem 5.1.17. *For every 1-connected absolute Hodge diagram A, there exists a 1-connected mixed Hodge Sullivan minimal dga M, together with a ho-morphism $\rho : M \rightsquigarrow A$, which is a quasi-isomorphism.*

PROOF. Inductively over $n \geq 0$, assume that we have constructed a 1-connected mixed Hodge Sullivan minimal dga

$$M = \{(M_{\mathbf{k}}, W), (M_i, W), (M_{\mathbb{C}}, W, F)\},$$

with $(M_{\mathbf{k}}, W) \otimes \mathbb{C} \cong (M_i, W) \cong (M_{\mathbb{C}}, W)$, together with a ho-morphism $\rho : M \rightsquigarrow A$ such that:

(a_n) $M_{\mathbf{k}}$ is freely generated in degrees $\leq n$, and $dM_{\mathbf{k}} \subset M_{\mathbf{k}}^+ \cdot M_{\mathbf{k}}^+$.
(b_n) For $i \leq n$, the maps $H^i(\rho_{\mathbf{k}})$, $H^i(\rho_i)$ and $H^i(\rho_{\mathbb{C}})$ are isomorphisms.
(c_n) The maps $H^{n+1}(\rho_{\mathbf{k}})$, $H^{n+1}(\rho_i)$ and $H^{n+1}(\rho_{\mathbb{C}})$ are monomorphisms.

Define $V_{\mathbf{k}} = H^n(C(\rho_{\mathbf{k}}))$, $V_i = H^n(C(\rho_i))$, and $V_{\mathbb{C}} = H^n(C(\rho_{\mathbb{C}}))$. The vector spaces $V_{\mathbf{k}}$ and V_i are equipped with weight filtrations W, while $V_{\mathbb{C}}$ is bifiltered by W and F. There is a chain of filtered isomorphisms

$$(V_{\mathbf{k}}, W) \otimes \mathbb{C} \to \cdots \leftarrow (V_i, W) \to \cdots \leftarrow (V_{\mathbb{C}}, W),$$

which endows $V_{\mathbf{k}}$ with a mixed Hodge structure. Such isomorphisms are defined in the following way: let $\varphi_u : A_i \to A_j$ be a component of the quasi-equivalence φ of A. Since the map $\rho : M \rightsquigarrow A$ is a ho-morphism of diagrams, there is a filtered homotopy $R_u : \varphi_u \rho_i \simeq \rho_j$. The pair of maps (φ_u, R_u) defines a filtered quasi-isomorphism $f_u = (\varphi_u, R_u) : (C(\rho_i), W) \to (C(\rho_j), W)$, inducing the isomorphisms $(V_i, W) \cong (V_j, W)$.

By Lemma 3.3.11 the mapping cone $C(\rho)$ is an absolute Hodge complex. Therefore by Proposition 3.3.9 there are sections

$$\sigma_{\mathbf{k}} : V_{\mathbf{k}} \to Z^n(C(\rho_{\mathbf{k}})) \text{ and } \sigma_i : V_i \to Z^n(C(\rho_i))$$

compatible with W, and a section

$$\sigma_{\mathbb{C}} : V_{\mathbb{C}} \to Z^n(C(\rho_{\mathbb{C}})),$$

compatible with both filtrations F and W. Define filtered dga's

$$\widetilde{M}_{\mathbf{k}} = M_{\mathbf{k}} \otimes \Lambda(V_{\mathbf{k}}) \text{ and } \widetilde{M}_i = M_i \otimes \Lambda(V_i),$$

together with a bifiltered dga

$$\widetilde{M}_{\mathbb{C}} = M_{\mathbb{C}} \otimes \Lambda(V_{\mathbb{C}}).$$

The corresponding filtrations are defined by multiplicative extension. The sections $\sigma_{\mathbf{k}}$, σ_i and $\sigma_{\mathbb{C}}$ allow to define differentials and maps $\widetilde{\rho}_{\mathbf{k}} : V_{\mathbf{k}} \to A_{\mathbf{k}}$, $\widetilde{\rho}_i : V_i \to A_i$ and $\widetilde{\rho}_{\mathbb{C}} : V_{\mathbb{C}} \to A_{\mathbb{C}}$, compatible with the corresponding filtrations. By construction, condition (a_{n+1}) is satisfied.

Since $(\widetilde{M}_i, W)$ is $\mathcal{F}_i$-cofibrant in $\mathbf{F}\mathsf{DGA}(\mathbb{C})$, given the solid diagram

$$\begin{tikzcd} \widetilde{M}_i \arrow[r, dotted, "\widetilde{\varphi}_u"] \arrow[d, "\widetilde{\rho}_i"'] & \widetilde{M}_j \arrow[d, "\widetilde{\rho}_j"] \\ A_i \arrow[r, "\varphi_u"] & A_j \end{tikzcd}$$

there exists a dotted arrow $\widetilde{\varphi}_u$ making the diagram commute up to a filtered homotopy $\widetilde{R}_u$. By the two out of three property, the map $\widetilde{\varphi}_u$ is a quasi-isomorphism. Since $\widetilde{M}_i$ are Sullivan minimal dga's, it follows that $\widetilde{\varphi}_u$ is an isomorphism, which is strictly compatible with filtrations.

The chain of isomorphisms

$$(\widetilde{M}_{\mathbf{k}}, W) \otimes \mathbb{C} \to \cdots \leftarrow (\widetilde{M}_i, W) \to \cdots \leftarrow (\widetilde{M}_{\mathbb{C}}, W),$$

defines a mixed Hodge Sullivan minimal dga

$$\widetilde{M} := \{(\widetilde{M}_{\mathbf{k}}, W), (\widetilde{M}_i, W), (\widetilde{M}_{\mathbb{C}}, W, F)\}.$$

The ho-morphism $\widetilde{\rho} = (\widetilde{\rho}_{\mathbf{k}}, \widetilde{\rho}_i, \widetilde{\rho}_{\mathbb{C}}, \widetilde{R}_u) : \widetilde{M} \rightsquigarrow A$ satisfies (b_{n+1}) and (c_{n+1}). $\square$

The following result is straightforward from Lemma 3.3.11.

Lemma 5.1.18. *Let $f : K \to L$ be a morphism of $\Gamma\mathbf{A}$.*

(1) If $f \in \mathcal{E}_{0,0}$, then K is an absolute Hodge diagram if and only if L is so.
(2) If $f \in \mathcal{E}_{1,0}$, then K is a mixed Hodge diagram if and only if L is so.

We can now prove the main result of this section.

Theorem 5.1.19. *The triple* $(\mathbf{AHD}^1, \mathcal{H}, \mathcal{Q})$ *is a Sullivan category. The category of mixed Hodge Sullivan minimal dga's is a full subcategory of minimal models. The inclusion induces an equivalence of categories*

$$\pi^h \mathsf{MHDGA}^1_{min} \longrightarrow \mathrm{Ho}\left(\mathbf{AHD}^1\right) := \mathbf{AHD}^1[\mathcal{Q}^{-1}]$$

between the category whose objects are 1-connected mixed Hodge Sullivan minimal dga's over $\mathbb{Q}$ *and whose morphisms are classes of ho-morphisms modulo homotopy equivalence, and the localized category of 1-connected absolute Hodge diagrams with respect to quasi-isomorphisms.*

PROOF. It follows from the analogue of Lemma 1.4.13 with minimal models, together with Lemma 5.1.18 and Theorem 5.1.17. □

Theorem 5.1.20. *The triple* $(\mathbf{MHD}^1, \mathcal{H}_{1,0}, \mathcal{Q})$ *is a Sullivan category. The minimal models are those mixed Hodge diagrams* M *such that* $\mathrm{Dec}M$ *is a 1-connected mixed Hodge Sullivan minimal dga. In particular,* $M_{\mathbf{k}}$ *is a Sullivan minimal dga and* $\{M_{\mathbf{k}}, W[n], F\}$ *is a mixed Hodge structure for all* $n \geq 0$.

PROOF. By Lemmas 5.1.13 and 5.1.18, condition (i) of Lemma 1.4.13 is satisfied for mixed Hodge diagrams, with respect to the P-category structure of $\Gamma\mathbf{A}$ associated with the class of $E_{0,1}$-quasi-isomorphisms. Therefore it suffices to show that for every mixed Hodge diagram A, there exists a mixed Hodge diagram M such that $\mathrm{Dec}M$ is a mixed Hodge Sullivan minimal dga, together with a quasi-isomorphism $\rho' : M \rightsquigarrow A$.

Let A be a 1-connected mixed Hodge diagram. By Theorem 5.1.17 there exists a quasi-isomorphism $\rho : M \rightsquigarrow \mathrm{Dec}^W A$, where M is a mixed Hodge Sullivan minimal dga. Note that at the level of diagrams of filtered dga's we have the adjunction $S^W \dashv \mathrm{Dec}^W$. This defines a quasi-isomorphism $\rho : S^W M \rightsquigarrow K$. It remains to show that $S^W M$ is a mixed Hodge diagram. The proof is analogous to that of Theorem 3.3.13. □

Theorem 5.1.21. *Deligne's décalage induces an equivalence of categories*

$$\mathrm{Dec}^W : \mathrm{Ho}\left(\mathbf{MHD}^1\right) \xrightarrow{\sim} \mathrm{Ho}\left(\mathbf{AHD}^1\right).$$

Proof. By Theorems 5.1.19 and 5.1.20 it suffices to show the equivalence between the corresponding subcategories of minimal models. For that it suffices to note that when restricted to mixed Hodge Sullivan dga's, the functor Dec^W has an inverse S^W defined by shifting the weight filtration. $\square$

Homotopy of Mixed Hodge Diagrams. The functor of indecomposables for filtered and bifiltered dga's defines a functor

$$Q : \mathbf{MHD}^1 \longrightarrow \mathbf{MHC}$$

sending every 1-connected mixed Hodge diagram A to the mixed Hodge complex defined by

$$Q(A) = \left((Q(A_{\mathbf{k}}), W) \overset{Q(\varphi)}{\longleftarrow\dashrightarrow} (Q(A_{\mathbb{C}}), W, F) \right).$$

Note that if A is a mixed Hodge Sullivan minimal dga, then $Q(A)$ is a graded mixed Hodge structure.

We will need the following result.

Proposition 5.1.22. *Every augmented homotopy $h : A \rightsquigarrow P(B)$ between ho-morphisms of mixed Hodge diagrams induces a homotopy*

$$\int_0^1 h : Q(A) \rightsquigarrow Q(B)[-1].$$

between ho-morphisms of mixed Hodge complexes.

Proof. The proof is analogous to that of Proposition 4.1.29, so we only indicate the main differences. A homotopy $h : A \rightsquigarrow P(B)$ from $f = (f_i, F_u)$ to $g = (g_i, G_u)$ is given by a family of homotopies $h_i : A_i \to P(B_i)$ from f_i to g_i, together with second homotopies $H_u : A_i \to P^2(B_j)$ satisfying the conditions of Definition 1.3.14. It follows from the definition of $\int_0^1$ that

$$d\int_0^1 h_i + \int_0^1 dh_i = g_i - f_i.$$

Therefore the map $\int_1^0 h_i : A_i \to B_i[-1]$ is a homotopy of complexes. Likewise, we find that

$$\int_0^1\int_0^1 H_u d - d\int_0^1\int_0^1 H_u = \int_0^1 G_u - \int_0^1 F_u + \int_0^1 h_j\varphi_u - \varphi_u\int_0^1 h_i.$$

Hence (by Definition 3.2.5) the family of pairs

$$\int_0^1 h := (\int_0^1 h_i, \int_0^1 \int_0^1 H_u)$$

is a homotopy of ho-morphisms of mixed Hodge complexes from f to g. Since h is augmented, it induces a homotopy

$$\int_0^1 h : Q(A) \rightsquigarrow Q(B)[-1].$$

□

Theorem 5.1.23. *The functor Q admits a left derived functor*

$$\mathbb{L}Q : \mathrm{Ho}\left(\mathbf{MHD}^1_*\right) \longrightarrow \mathrm{Ho}\left(\mathbf{MHC}\right).$$

The composition of functors

$$\mathrm{Ho}\left(\mathbf{MHD}^1\right) \xleftarrow{\sim} \mathrm{Ho}\left(\mathbf{MHD}^1_*\right) \xrightarrow{\mathbb{L}Q} \mathrm{Ho}\left(\mathbf{MHC}\right) \xrightarrow{H\circ \mathrm{Dec}^W} \mathbf{G}^+(\mathsf{MHS})$$

defines a functor

$$\pi : \mathrm{Ho}\left(\mathbf{MHD}^1\right) \longrightarrow \mathbf{G}^+(\mathsf{MHS})$$

which associates to every 1-connected mixed Hodge diagram A, the graded mixed Hodge structure $\pi(A) = Q(M_A)$, where $M_A \rightsquigarrow A$ is a minimal model of A.

PROOF. By proposition 1.1.32, we need to check that Q sends strong equivalences in $\mathbf{MHD}^1_*$ to weak equivalences in $\mathbf{MHC}$. Indeed, the class $\mathcal{H}$ of strong equivalences is defined as the class of morphisms of mixed Hodge diagrams which are homotopy equivalences as ho-morphisms. The result follows from Proposition 5.1.22. The remaining of the proof follows analogously to that of Theorem 4.3.45. □

Corollary 5.1.24 (cf. [**Mor78**], Thm. 8.6)**.** *Let A be a 1-connected mixed Hodge diagram of dga's.*

(1) The Sullivan minimal model $M_{\mathbf{k}} \to A_{\mathbf{k}}$ of its rational part is equipped with functorial mixed Hodge structures, which are unique up to isomorphisms homotopic to the identity, and are functorial for morphisms of diagrams.

(2) The homotopy groups of $A_{\mathbf{k}}$ are endowed with functorial mixed Hodge structures.

Proof. Assertion (1) follows directly from Theorem 5.1.20, since given a minimal model $M \rightsquigarrow A$ of a mixed Hodge diagram A, then $M_{\mathbf{k}} \to A_{\mathbf{k}}$ is a Sullivan minimal model of $A_{\mathbf{k}}$, and $M^n_{\mathbf{k}}$ has a mixed Hodge structure for all $n \geq 0$. Assertion (2) follows from Theorem 5.1.23, since $\pi^n(A_{\mathbf{k}}) = Q(M_{\mathbf{k}})^n$, and the homotopy $\pi(A)$ is a graded mixed Hodge structure. □

We next study the formality of mixed Hodge diagrams and morphisms of mixed Hodge diagrams. Following [**DGMS75**] we pose the following definitions:

Definition 5.1.25. Let (A, d, W) be a filtered dga. The homotopy type of (A, d) is a *formal consequence of* $E_1(A)$ if there is a chain of quasi-isomorphisms $(A, d) \xleftarrow{\sim} (M, d) \xrightarrow{\sim} (E_1(A), d_1)$, where (M, d) is a Sullivan minimal dga.

Note that this is a notion of formality weaker that the notion of E_0-formality for filtered dga's (see Definition 4.3.34).

Definition 5.1.26. Let $f : (A, d, W) \to (B, d, W)$ be a morphism of filtered dga's, and assume that the homotopy type of A (resp. B) is a formal consequence of $E_1(A)$ (resp. $E_1(B)$). We say that the homotopy type of f *is a formal consequence of* $E_1(f)$ if there exists a diagram

$$\begin{array}{ccccc} (A,d) & \longleftarrow & (M_A,d) & \longrightarrow & (E_1(A),d_1) \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow & & \Big\downarrow{\scriptstyle E_1(f)} \\ (B,d) & \longleftarrow & (M_B,d) & \longrightarrow & (E_1(B),d) \end{array}$$

which commutes up to a homotopy of dga's.

The first term of the spectral sequence associated with the trivial filtration is the cohomology algebra. Hence in this case we recover the classical notion of formality of [**DGMS75**]. We remark that in general, the formality of objects does not imply formality of morphisms (see [**FT88**]).

We will prove that the homotopy type of the rational part of both mixed Hodge diagrams and morphisms of mixed Hodge diagrams is a formal consequence of E_1. This result is done in two steps: first, we prove formality over $\mathbb{C}$. Second, we use the descent of formality from $\mathbb{C}$ to $\mathbb{Q}$.

The descent of formality of dga's from $\mathbb{C}$ to $\mathbb{Q}$ is proved in Theorem 12.1 of [**Sul77**]. Based on the Sullivan formality criterion of Theorem 1 of [**FT88**], a descent of formality for morphisms of dga's is proved in Theorem 3.2 of [**Roi94**]. The proof does not depend on the particular construction of minimal models, but rather on abstract properties of formalizability and minimality. An adaptation of this result gives:

Lemma 5.1.27. *Let $\mathbf{k} \subset \mathbf{K}$ be a field extension, and let $f : A \to B$ be a morphism of filtered dga's over $\mathbf{k}$. The homotopy type of f is a formal consequence of $E_1(f)$ if and only if the homotopy type of $f_{\mathbf{K}} := f \otimes_{\mathbf{k}} \mathbf{K}$ is a formal consequence of $E_1(f_{\mathbf{K}})$.*

PROOF. The descent of formality with respect to E_1 from $\mathbb{C}$ to $\mathbb{Q}$ reduces to lifting a grading (see Theorem 12.7 of [**Sul77**]). Hence the Lemma follows from the proofs of Theorem 12.1 of loc.cit for objects and Theorem 3.2 of [**Roi94**] for morphisms respectively. $\square$

Proposition 5.1.28 (cf. [**Mor78**], Thm. 10.1)**.** *Let $f : A \to A'$ be a morphism of 1-connected mixed Hodge diagrams. The homotopy type of $f_{\mathbb{Q}} : A_{\mathbb{Q}} \to A'_{\mathbb{Q}}$ is a formal consequence of $E_1(f_{\mathbb{Q}})$.*

PROOF. Let A be a 1-connected mixed Hodge diagram of dga's. Since $\mathrm{Dec}^W A$ is a 1-connected absolute Hodge diagram, by Theorem 5.1.17 there exists a quasi-isomorphism $M \rightsquigarrow \mathrm{Dec}^W A$, where $(M, \widetilde{W}, F)$ is a mixed Hodge Sullivan minimal dga. Hence we have a filtered quasi-isomorphism $(M_{\mathbb{C}}, \widetilde{W}) \to (A_{\mathbb{C}}, \mathrm{Dec}W)$. By Lemma 3.1.7 the algebra $M^n_{\mathbb{C}}$ admits a splitting:

$$M^n_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q}_n, \text{ with } \widetilde{W}_m M^n_{\mathbb{C}} = \bigoplus_{p+q \leq m} I^{p,q}_n.$$

Since the differential is a morphism of mixed Hodge structures, it satisfies $d(I^{p,q}_n) \subset I^{p,q}_{n+1}$, for all $n \geq 0$. Let

$$M^{p,q}_{\mathbb{C}} := \bigoplus_r I^{-p-r,r}_{p+q}.$$

We next check that this defines a 0-splitting for the filtered dga $(M_{\mathbb{C}}, d, \widetilde{W})$ (see Definition 4.3.32). We have:

(1) $dM_{\mathbb{C}}^{p,q} = \bigoplus_r dI_{p+q}^{-p-r,r} \subset \bigoplus_r dI_{p+q+1}^{-p-r,r} = M_{\mathbb{C}}^{p,q+1}$.
(2) $\widetilde{W}_p M_{\mathbb{C}}^n = \bigoplus_{i+j\leq p} I_n^{i,j} = \bigoplus_{q\geq -p} I_n^{-q-r,r} = \bigoplus_{q\geq -p} M_{\mathbb{C}}^{q,n-q}$.
Therefore is follows that

$$(E_0(M_{\mathbb{C}}, \widetilde{W}), d_0) \cong (M_{\mathbb{C}}, d).$$

A minimal model for the mixed Hodge diagram A is defined by shifting the weight filtration $\widetilde{W}$ of M. Hence we have

$$(E_1(M_{\mathbb{C}}, S\widetilde{W}), d_1) \cong (M_{\mathbb{C}}, d).$$

Since $(E_1(M_{\mathbb{C}}, S\widetilde{W}), d_1)$ is a bigraded minimal model of $(E_1(A_{\mathbb{C}}, W), d_1)$, we have E_1-quasi-isomorphisms

$$(A_{\mathbb{C}}, d, W) \xleftarrow{\sim} (M_{\mathbb{C}}, d, S\widetilde{W}) \xrightarrow{\sim} (E_1(A_{\mathbb{C}}, W), d_1, W).$$

In particular, the homotopy type of $A_{\mathbb{C}}$ is a formal consequence of $E_1(A_{\mathbb{C}}, W)$.

Let $f : A \to A'$ be a morphism of 1-connected mixed Hodge diagrams. Consider the solid diagram of filtered dga's

$$\begin{tikzcd} (A_{\mathbb{C}}, d, W) \arrow[d, "f_{\mathbb{C}}"'] & (M_{\mathbb{C}}, d, S\widetilde{W}) \arrow[l, "\rho"'] \arrow[d, dotted, "\widetilde{f}_{\mathbb{C}}"] \\ (A'_{\mathbb{C}}, d, W) & (M'_{\mathbb{C}}, d, S\widetilde{W}). \arrow[l, "\rho'"'] \end{tikzcd}$$

Since $(M'_{\mathbb{C}}, d, S\widetilde{W})$ is E_1-cofibrant and ρ' is an E_1-quasi-isomorphism, the dotted arrow $\widetilde{f}_{\mathbb{C}}$ exists, and makes the diagram commute up to a 1-homotopy of filtered dga's. As a consequence, the induced diagram at the E_1-stage of the associated spectral sequences commutes up to a homotopy of dga's. Since the splitting of Lemma 3.1.7 is functorial for morphisms of mixed Hodge structures, the diagram

$$\begin{tikzcd} (A_{\mathbb{C}}, d) \arrow[d, "f_{\mathbb{C}}"] & (M_{\mathbb{C}}, d) \arrow[l, "\rho"'] \arrow[d, dotted, "\widetilde{f}_{\mathbb{C}}"] & (E_1(M_{\mathbb{C}}, S\widetilde{W}), d_1) \arrow[l, "\cong"'] \arrow[r, "E_1(\rho)"] \arrow[d, dotted, "E_1(\widetilde{f}_{\mathbb{C}})"] & (E_1(A_{\mathbb{C}}, W), d_1) \arrow[d, "E_1(f_{\mathbb{C}})"] \\ (A'_{\mathbb{C}}, d) & (M'_{\mathbb{C}}, d) \arrow[l, "\rho'"'] & (E_1(M'_{\mathbb{C}}, S\widetilde{W}), d_1) \arrow[l, "\cong"'] \arrow[r, "E_1(\rho')"] & (E_1(A'_{\mathbb{C}}, W), d_1) \end{tikzcd}$$

commutes up to a homotopy. Hence the homotopy type of $f_{\mathbb{C}}$ is a formal consequence of $E_1(f_{\mathbb{C}}, W)$. The result follows from Lemma 5.1.27. □

The following is a formality result for the forgetful functor

$$U_{\mathbb{Q}} : \mathrm{Ho}\left(\mathbf{MHD}^1\right) \longrightarrow \mathrm{Ho}_1\left(\mathbf{F}\mathsf{DGA}^1(\mathbb{Q})\right).$$

Corollary 5.1.29. *There is an isomorphism of functors $E_1 \circ U_{\mathbb{Q}} \cong U_{\mathbb{Q}}$.*

5.2. Cohomological Descent

The theory of cubic hyperresolutions [**GNPP88**] allows to replace a singular variety by a cubic diagram of smooth varieties. This replacement is constructive, and relies on Hironaka's Theorem of resolutions of singularities for algebraic varieties over a field of characteristic 0. In [**GN02**], Guillén and Navarro developed a general descent theory which, aided by the theory of cubic hyperresolutions, allows to extend some particular contravariant functors defined on the category of smooth schemes, to the category of all schemes. The extension criterion of Guillén-Navarro is based on the assumption that the target category is a *cohomological descent category.* This is essentially a category $\mathcal{D}$, together with a saturated class of morphisms, and a functor $\mathbf{s}$, sending every cubical codiagram of $\mathcal{D}$ to an object of $\mathcal{D}$, and satisfying certain axioms. From [**Rod12a**] it follows that the simple functor of a cubical descent category is essentially the homotopy limit, and allows to define realizable homotopy limits for diagrams indexed by finite categories.

In this section we show that the categories **MHC** and **MHD** of mixed Hodge complexes and mixed Hodge diagrams of dga's are equipped with cohomological descent structures. For the additive case of mixed Hodge complexes, an analogous result in the context of simplicial descent categories appears in [**Rod12b**]. Using the above results, and the extension criterion of [**GN02**], we provide a proof of that the functor $\mathbb{H}dg : \mathbf{V}^2(\mathbb{C}) \longrightarrow \mathbf{MHD}$ of Theorem 5.3.6 extends to a functor defined on the category of all schemes over $\mathbb{C}$, whose target is the homotopy category of mixed Hodge diagrams.

Preliminaries. We next recall the main features of descent categories and descent functors. We refer to [**GN02**] for the precise definitions and proofs.

Given a set $\{0, \cdots, n\}$, with $n \geq 0$, the set of its non-empty parts, ordered by the inclusion, defines the category $\square_n$. Likewise, any non-empty finite set S defines the category $\square_S$. Denote by $\square_S^+$ the category defined by including the empty set. Every injective map $u : S \to T$ between non-empty finite sets induces a functor $\square_u : \square_S \to \square_T$ defined by $\square_u(\alpha) = u(\alpha)$.

Denote by Π the category whose objects are finite products of categories $\square_S$ and whose morphisms are the functors associated to injective maps in each component.

Definition 5.2.1. Let $\delta : \square \to \square'$ be a morphism of Π. The *inverse image of* δ is the functor $\delta^* : Fun(\square', \mathcal{D}) \longrightarrow Fun(\square, \mathcal{D})$ defined by $\delta^*(F) := F \circ \delta$.

Definition 5.2.2. Let $\mathcal{D}$ be an arbitrary category. A *cubical codiagram* of $\mathcal{D}$ is a pair $(X, \square)$, where $\square$ is an object of Π and X is a functor $X : \square \to \mathcal{D}$. A *morphism* $(X, \square) \to (Y, \square')$ between cubical codiagrams is given by a pair (a, δ) where $\delta : \square' \to \square$ is a morphism of Π and $a : \delta^* X \to Y$ is a natural transformation.

Denote by $CoDiag_\Pi \mathcal{D}$ the category of cubical codiagrams of $\mathcal{D}$.

Definition 5.2.3. Let $\delta : \square \to \square'$ be a morphism of Π. The *direct image of* δ is the functor $\delta_* : Fun(Fun(\square, \mathcal{D}), \mathcal{D}) \longrightarrow Fun(Fun(\square', \mathcal{D}), \mathcal{D})$ defined by $F \mapsto \delta_*(F) := F \circ \delta^*$.

Definition 5.2.4 ([**GN02**], Def. 1.5.3)**.** A *cohomological descent category* is given by the data $(\mathcal{D}, \mathcal{E}, \mathbf{s})$, where:

- (CD_1) $\mathcal{D}$ is a cartesian category with initial object 0.
- (CD_2) $\mathcal{E}$ is a saturated class of morphisms of $\mathcal{D}$, called *weak equivalences*, which is stable by products.
- (CD_3) $\mathbf{s} : CoDiag_\Pi \mathcal{D} \to \mathcal{D}$ is a contravariant functor such that for any morphism $\delta : \square \to \square'$ of Π and any codiagram $(X, \square)$ of $\mathcal{D}$, the morphism $\mathbf{s}_{\square'}(\delta_* X) \to \mathbf{s}_{\square}(X)$ induced by $\delta_* X \to X$ is in $\mathcal{E}$.

The following list of axioms must be satisfied:

(CD_4) *Additivity:* for every object $\square$ of Π, the unity map $\mathbf{s}_\square(\square \times 1) \to 1$, and the Künneth map $\mathbf{s}_\square(X \times Y) \to \mathbf{s}_\square(X) \times \mathbf{s}_\square(Y)$ are in $\mathcal{E}$.

(CD_5) *Exactness:* let $f : X \to Y$ be a morphism of codiagrams of $\mathcal{D}$. If f_α is in $\mathcal{E}$ for every $\alpha \in \square$, then $\mathbf{s}_\square(f) : \mathbf{s}_\square(X) \to \mathbf{s}_\square(Y)$ is in $\mathcal{E}$.

(CD_6) *Factorization:* Let $\square, \square'$ be objects of Π. For any cubical codiagram $X = (X_{\alpha\beta}) : \square \times \square' \to \mathcal{D}$ there is an isomorphism $\mu : \mathbf{s}_{\alpha\beta} X_{\alpha\beta} \to \mathbf{s}_\alpha \mathbf{s}_\beta X_{\alpha\beta}$.

(CD_7) *Acyclicity:* let X^+ be a $\square_n^+$-diagram, and denote by X the cubical codiagram obtained by restriction to $\square_n$. The augmentation morphism $\lambda_\varepsilon : X_0 \to \mathbf{s}_\square X$ is a weak equivalence if and only if the canonical morphism $0 \to \mathbf{s}_{\square^+} X^+$ is a weak equivalence.

Remark 5.2.5. The transformations μ and λ of axioms (CD_6) and (CD_7) are part of the data of a descent structure.

Given a field $\mathbf{k}$ of characteristic 0, denote by $\mathbf{Sch}(\mathbf{k})$ the category of reduced schemes, that are separated and of finite type over $\mathbf{k}$. Denote by $\mathbf{Sm}(\mathbf{k})$ the full subcategory of smooth schemes.

Definition 5.2.6. A cartesian diagram of $\mathbf{Sch}(\mathbf{k})$

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "j"] \arrow[d, "g"'] & \widetilde{X} \arrow[d, "f"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

is said to be an *acyclic square* if i is a closed immersion, f is proper, and the induced morphism $\widetilde{X} \setminus \widetilde{Y} \to X \setminus Y$ is an isomorphism. It is an *elementary acyclic square* if, in addition, all the objects in the diagram are irreducible smooth schemes of $\mathbf{Sm}(\mathbf{k})$, and f is the blow-up of X along Y. In the latter case, the map f is said to be an *elementary proper modification.*

Theorem 5.2.7 (**[GN02]**, Thm. 2.1.5)**.** *Let $\mathcal{D}$ be a cohomological descent category and let $G : \mathbf{Sm}(\mathbf{k}) \longrightarrow \mathrm{Ho}\mathcal{D}$ be a contravariant Φ-rectified functor satisfying:*

(F1) $G(\emptyset) = 0$, and $G(X \sqcup Y) \to G(X) \times G(Y)$ is an isomorphism.

(F2) If $X^\bullet$ is an elementary acyclic square, then $\mathbf{s}G(X^\bullet)$ is acyclic.

Then there is a Φ-rectified functor $G' : \mathbf{Sch}(\mathbf{k}) \longrightarrow \mathrm{Ho}\mathcal{D}$ satisfying the descent condition:

(D) If $X^\bullet$ is an elementary acyclic square, then $\mathbf{s}G'(X^\bullet)$ is acyclic.

In addition, this extension is essentially unique: If G'' is another extension of G, satisfying (D), then there exists a unique isomorphism of Φ-rectified functors $G' \Rightarrow G''$.

We next state a relative version of Theorem 5.2.7. Denote by $\mathbf{Sch}(\mathbf{k})^2_{\mathbf{Comp}}$ the category of pairs (X, U), where X is a proper scheme over $\mathbf{k}$ and U is an open subscheme of X. Denote by $\mathbf{V}^2(\mathbf{k})$ the full subcategory of $\mathbf{Sch}(\mathbf{k})^2_{\mathbf{Comp}}$ of those pairs (X, U), where X is smooth projective and $D = X - U$ is a divisor with normal crossings.

Definition 5.2.8. A commutative diagram of $\mathbf{Sch}(\mathbf{k})^2_{\mathbf{Comp}}$

$$\begin{array}{ccc} (\widetilde{Y}, \widetilde{U} \cap \widetilde{Y}) & \xrightarrow{j} & (\widetilde{X}, \widetilde{U}) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ (Y, U \cap Y) & \xrightarrow{i} & (X, U) \end{array}$$

is said to be an *acyclic square* if $f : \widetilde{X} \to X$ is proper, $i : Y \to X$ is a closed immersion, the diagram of the first components is cartesian, $f^{-1}(U) = \widetilde{U}$ and the diagram of the second components is an acyclic square of $\mathbf{Sch}(\mathbf{k})$.

Definition 5.2.9. A morphism $f : (\widetilde{X}, \widetilde{U}) \to (X, U)$ in $\mathbf{V}^2(\mathbf{k})$ is called *proper elementary modification* if $f : \widetilde{X} \to X$ is the blow-up of X along a smooth centre Y which has normal crossings with the complementary D of U in X, and if $\widetilde{U} = f^{-1}(U)$.

Definition 5.2.10. An acyclic square of objects of $\mathbf{V}^2(\mathbf{k})$ is said to be an *elementary acyclic square* if the map $f : (\widetilde{X}, \widetilde{U}) \to (X, U)$ is a proper elementary modification, and the diagram of the second components is an elementary acyclic square of $\mathbf{Sm}(\mathbf{k})$.

Theorem 5.2.11 ([**GN02**], Thm. 2.3.6)**.** *Let $\mathcal{D}$ be a cohomological descent category, and $G : \mathbf{V}^2(\mathbf{k}) \longrightarrow \mathrm{Ho}\mathcal{D}$ a contravariant Φ-rectified functor satisfying conditions $F1$ and $F2$ of Theorem* 5.2.7*. Then there exists a contravariant Φ-rectified functor $G' : \mathbf{Sch}(\mathbf{k}) \longrightarrow \mathrm{Ho}\mathcal{D}$ satisfying the descent* *condition (D) of Theorem 5.2.7, and such that $G'(U) \cong G(X, U)$, for every pair $(X, U) \in \mathbf{V}^2(\mathbf{k})$.*

Simple Functor for Complexes. The primary example of a cohomological descent structure is given by the category of complexes $\mathbf{C}^+(\mathcal{A})$ of an abelian category $\mathcal{A}$ with the class of quasi-isomorphisms and the simple functor $\mathbf{s}$ given by the *total complex*. We next recall its definition, and provide a simple for filtered and bifiltered complexes. This will enable us define a simple functor for cubical mixed Hodge complexes, endowing the category **MHC** with a cohomological descent structure.

Define the simple $\mathbf{s}$ for a cubic codiagram of complexes $K = (K^\alpha)$ of type $\square$, by

$$\mathbf{s}_\square(K) = \int_\alpha C_\alpha \otimes K^\alpha,$$

where C_α is the cochain complex associated with $\Delta|\alpha|$. For example, if $\square = \square_n$, then $\mathbf{s}_n(K)$ is the complex given by

$$\mathbf{s}_n(K) := \mathbf{s}_\alpha K^\alpha = \bigoplus_{\alpha \in \square_n} K^\alpha[-|\alpha|],$$

together with the differential defined by an alternating sum of the differentials of the complexes K^α and the transition morphisms $K^\alpha \to K^{\delta_{i*}\alpha}$.

For every pair of objects $\square, \square'$ of Π, let $\mu_{\square,\square'} : \mathbf{s}_{\square \times \square'} \to \mathbf{s}_\square \circ \mathbf{s}_{\square'}$ denote the isomorphism corresponding to the iterated end, defined from the isomorphisms $C_\alpha \otimes C_\beta \cong C_{(\alpha,\beta)}$, for each $(\alpha, \beta) \in \square \times \square'$. Likewise, for every object $\square$ of Π, and every complex K, let

$$\lambda_\square(K) : K \longrightarrow \mathbf{s}_\square(\square^{op} \times K) \cong C^*(\square) \otimes K$$

be the map induced by the coaugmentation $\mathbb{Z} \to C^*(\square)$.

Proposition 5.2.12 ([**GN02**], 1.7.2)**.** *Let $\mathcal{A}$ be an abelian category. The category $\mathbf{C}^+(\mathcal{A})$ with the class of quasi-isomorphisms and the simple functor $\mathbf{s}$ together with data (μ, λ) is a cohomological descent category.*

Note that the definition of the simple functor depends on the translation functor. To generalize this construction to the filtered setting it suffices to consider the r-translation functor (see Definition 2.2.16).

Definition 5.2.13. Let $r \in \{0,1\}$ and let (K,F) be a codiagram of filtered complexes. The *r-simple* of (K,F) is the filtered complex

$$\mathbf{s}^r(K,F) := (\mathbf{s}(K), F_r)$$

defined by

$$(F_r)^p \mathbf{s}(K) = \int_\alpha C_\alpha \otimes F^{p-r|\alpha|} K^\alpha.$$

Note that $\mathbf{s}^0$ and $\mathbf{s}^1$ correspond to the filtered total complexes defined via the convolution with the trivial and the bête filtrations respectively, introduced by Deligne in [**Del74b**].

The morphisms $\mu_{\Box,\Box'}$ and $\lambda_\Box$ defined for the non-filtered case are compatible with filtrations, so we have the data (μ,λ) associated with $\mathbf{s}^r$. In addition:

Proposition 5.2.14. *Let (K,F) be a codiagram of filtered complexes. Then*

$$\mathrm{Dec}\left(\mathbf{s}^1(K,F)\right) = \mathbf{s}^0(K, \mathrm{Dec}F).$$

Proof. The category $\mathbf{C}^+(\mathbf{F}\mathcal{A})$ complete. Furthermore, since the décalage has a left adjoint (see Proposition 2.2.7), it commutes with pull-backs. Hence we have

$$\mathrm{Dec} \int_\alpha C_\alpha \otimes K^\alpha = \int_\alpha \mathrm{Dec}(C_\alpha \otimes K^\alpha).$$

By Lemma 2.2.18, the décalage commutes with the r-translation functor. Hence for all $p \in \mathbb{Z}$ we have

$$\int_\alpha \mathrm{Dec}(C_\alpha \otimes F^{p-|\alpha|} K^\alpha) = \int_\alpha C_\alpha \otimes DecF^p K^\alpha.$$

$\Box$

Proposition 5.2.15. *Let $r \in \{0,1\}$. The category $\mathbf{C}^+(\mathbf{F}\mathcal{A})$ with the class $\mathcal{E}_r$ of E_r-quasi-isomorphisms and the r-simple functor $\mathbf{s}^r$ together with data (μ,λ) is a cohomological descent category.*

PROOF. For $r = 0$, the proof follows from Prop. 1.7.5 of [**GN02**], via the functor $Gr^\bullet : \mathbf{C}^+(\mathbf{F}\mathcal{A}) \longrightarrow \mathbf{C}^+(\mathcal{A})$ defined by sending every filtered complex to its associated graded object. Let $r = 1$. By Proposition 5.2.14 the décalage $\mathrm{Dec} : \mathbf{C}^+(\mathbf{F}\mathcal{A}) \to \mathbf{C}^+(\mathbf{F}\mathcal{A})$ commutes with the r-simple. The result follows from Prop. 1.5.12 of [**GN02**]. $\square$

The previous results extend to bifiltered complexes as follows.

Definition 5.2.16. Let $r \in \{0, 1\}$, and let (K, W, F) be a codiagram of bifiltered complexes. The $(r, 0)$-*simple* of (K, W, F) is the bifiltered complex defined by

$$\mathbf{s}^{r,0}(K, W, F) := (\mathbf{s}(A), W_r, F_0)\,.$$

Proposition 5.2.17. *Let $r \in \{0, 1\}$. The category $\mathbf{C}^+(\mathbf{F}^2\mathcal{A})$ with the class $\mathcal{E}_{r,0}$ of $E_{r,0}$-quasi-isomorphisms and the simple functor $\mathbf{s}^{r,0}$ together with data (μ, λ), is a cohomological descent category.*

PROOF. The proof is analogous to that of Proposition 5.2.15. $\square$

We next define a simple functor for mixed Hodge complexes.

Definition 5.2.18. Let K be a cubical codiagram of mixed Hodge complexes. The *simple* of K is the diagram of complexes

$$\mathbf{s}_D(K) = \left(\mathbf{s}^1_D(K_{\mathbf{k}}, W) \overset{\mathbf{s}_D(\varphi)}{\longleftarrow\!\dashrightarrow} \mathbf{s}^{1,0}_D(K_{\mathbb{C}}, W, F)\right).$$

Proposition 5.2.19. *The simple of a cubical codiagram of mixed Hodge complexes, is a mixed Hodge complex.*

PROOF. It suffices to prove that the associated functor of cosimplicial objects is a mixed Hodge complex. This follows from Theorem 8.1.15 of [**Del74b**]. $\square$

Theorem 5.2.20. *The category of mixed Hodge complexes* **MHC** *with the class $\mathcal{Q}$ of quasi-isomorphisms and the simple functor $\mathbf{s}_D$ is a cohomological descent category.*

PROOF. Consider the forgetful functor

$$\psi : \mathbf{MHC} \longrightarrow \mathbf{C}^+(\mathbf{Fk}) \times \mathbf{C}^+(\mathbf{F}^2\mathbb{C}),$$

defined by sending every mixed Hodge complex K to the pair of complexes $(K_{\mathbf{k}}, W)$ and $(K_{\mathbb{C}}, W, F)$. By Proposition 5.2.17 we have cohomological descent structures $(\mathbf{C}^+(\mathbf{Fk}), \mathcal{E}_1, \mathbf{s}^1)$ and $(\mathbf{C}^+(\mathbf{F}^2\mathbb{C}), \mathcal{E}_{1,0}, \mathbf{s}^{1,0})$. By Proposition 3.3.8 we have $\mathcal{Q} = \psi^{-1}(\mathcal{E}_1, \mathcal{E}_{1,0})$. Since the simple $\mathbf{s}_D$ is defined level-wise, it commutes with the functor ψ. The result follows from Prop. 1.5.12 of [**GN02**]. $\square$

Thom-Whitney Simple. The Thom-Whitney simple functor defined by Navarro in [**Nav87**] for strict cosimplicial dga's is easily adapted to the cubical setting. We next recall its definition, and provide a Thom-Whitney simple for filtered and bifiltered dga's. This will enable us define a simple functor for cubical mixed Hodge diagrams, endowing the category **MHD** with a cubical cohomological descent structure.

Given a non-empty finite set S, denote by L_S the dga over $\mathbf{k}$ of smooth differential forms over the hyperplane of the affine space $\mathbb{A}^S_{\mathbf{k}}$, defined by the equation $\sum_{s\in S} t_s = 1$.

Given an object $\square = \Pi_{i\in I}\square_{S_i}$ of Π, we let $L_\alpha = \otimes_i L_{\alpha_i}$, for every $\alpha = (\alpha_i) \in \square$. This defines a functor $L : \square^{op} \longrightarrow \mathsf{DGA}(\mathbf{k})$. For a codiagram of dga's $A = (A^\beta)$ of type $\square$, we let

$$\mathbf{s}_\square(A) := \int_\alpha L_\alpha \otimes A^\alpha$$

denote the end of the functor $\square^{op} \times \square \to \mathsf{DGA}(\mathbf{k})$ given by $(\alpha, \beta) \mapsto L_\alpha \otimes A^\beta$. Since $\mathbf{s}_\square$ is functorial with respect to $\square$, this defines a functor

$$\mathbf{s}_{TW} : CoDiag_\Pi(\mathsf{DGA}(\mathbf{k})) \longrightarrow \mathsf{DGA}(\mathbf{k}).$$

For every pair of objects $\square, \square'$ of Π, let $\mu_{\square,\square'} : \mathbf{s}_{\square\times\square'} \to \mathbf{s}_\square \circ \mathbf{s}_{\square'}$ denote the isomorphism corresponding to the iterated end, defined from the isomorphisms $L_\alpha \otimes L_\beta \cong L_{(\alpha,\beta)}$, for each $(\alpha, \beta) \in \square \times \square'$. Likewise, for every object $\square_S$ of Π, and every dga (A, d), let

$$\lambda_{\square_S}(A) : A \longrightarrow \mathbf{s}_{\square_S}(\square_S^{op} \times A) \cong L_S \otimes A$$

be the map induced by the structural map $\mathbf{k} \to L_S$.

Proposition 5.2.21 ([**GN02**], Prop 1.7.4)**.** *The category* $\mathsf{DGA}(\mathbf{k})$ *with the class of quasi-isomorphisms, and the simple functor* $\mathbf{s}_{TW}$ *together with data* (μ,λ) *defined above, is a cohomological descent category.*

Define a family of filtrations of L_S, for every non-empty finite set S as follows. For $r \geq 0$, let σ_r be the decreasing filtration of L_S defined by $w(t_s)=0$, and $w(dt_s)=r$, for every generator t_s of degree 0 of L_S, and extending multiplicatively. Note that σ_0 is the trivial filtration, while σ_1 is the bête filtration of L_S.

Given a filtered dga (A,d,F), we have a family of filtered dga's

$$L_S^r(A) = (L_S \otimes A, F_r) := (L_S \otimes A, \sigma_r * F),$$

where $F_r = \sigma_r * F$ is the multiplicative filtration defined by

$$F_r^p(L_S \otimes A) = \bigoplus_q \sigma_r^q L_S \otimes F^{p-q} A.$$

Lemma 5.2.22. *With the previous notations,* $\mathrm{Dec}(L_S^1(A)) = L_S^0(\mathrm{Dec}A)$.

PROOF. The proof follows from Lemma 4.3.10, and an induction over the cardinal of S. $\square$

Definition 5.2.23. Let $r \in \{0,1\}$, and let (A,F) be a codiagram of filtered dga's. The *r-Thom-Whitney simple* of (A,F) is the filtered dga

$$\mathbf{s}_{TW}^r(A,F) := (\mathbf{s}_{TW}(A), F_r)\,,$$

defined by the end

$$F_r^p \mathbf{s}_{TW}(A) = \int_\alpha F_r^p(L_\alpha \otimes A^\alpha)$$

of the functor

$$(\alpha,\beta) \mapsto F_r^p(L_\alpha \otimes A^\alpha) = \bigoplus_q \left(\sigma_r^q L_\alpha \otimes F^{p-q} A^\beta\right).$$

The morphisms $\mu_{\square,\square'}$ and $\lambda_\square$ defined for the non-filtered case, are compatible with filtrations, so we have the data (μ,λ) associated with $\mathbf{s}_{TW}^r$.

Proposition 5.2.24. *Let* (A,F) *be a codiagram of filtered dga's. Then*

$$\mathrm{Dec}\left(\mathbf{s}_{TW}^1(A,F)\right) = \mathbf{s}_{TW}^0(A, \mathrm{Dec}F).$$

Proof. Since $\mathbf{F}\mathsf{DGA}(\mathbf{k})$ is a complete category, and the décalage commutes with pull-backs, we have

$$\operatorname{Dec}\int_\alpha L_\alpha\otimes A^\alpha=\int_\alpha \operatorname{Dec}(L_\alpha\otimes A^\alpha).$$

The result follows from Lemma 5.2.22. □

Proposition 5.2.25. *Let $r\in\{0,1\}$. The category $\mathbf{F}\mathsf{DGA}(\mathbf{k})$ with the class $\mathcal{E}_r$ of E_r-quasi-isomorphisms and the simple functor $\mathbf{s}^r_{TW}$ together with data (μ,λ) is a cohomological descent category.*

Proof. Consider the functor $Gr^\bullet:\mathbf{F}\mathsf{DGA}(\mathbf{k})\to\mathsf{DGA}(\mathbf{k})$ defined by sending every filtered dga to its associated graded object. It is clear that this functor commutes with the simple functor. The result follows from Prop. 1.5.12 of loc. cit. Let $r=1$. By Proposition 5.2.24 the décalage $\operatorname{Dec}:\mathbf{F}\mathsf{DGA}(\mathbf{k})\to\mathbf{F}\mathsf{DGA}(\mathbf{k})$ is compatible with the Thom-Whitney simple. Again, the result follows from Prop. 1.5.12 of loc. cit. □

The previous results extend to bifiltered dga's as follows.

Definition 5.2.26. Let $r\in\{0,1\}$, and let (A,W,F) be a codiagram of bifiltered dga's. The $(r,0)$-*Thom-Whitney simple* of (A,W,F) is the bifiltered dga defined by

$$\mathbf{s}^{r,0}_{TW}(A,W,F):=(\mathbf{s}_{TW}(A),W_r,F_0)\,.$$

Proposition 5.2.27. *Let $r\in\{0,1\}$. The category $\mathbf{F}^2\mathsf{DGA}(\mathbf{k})$ with the class $\mathcal{E}_{r,0}$ of $E_{r,0}$-quasi-isomorphisms and the simple functor $\mathbf{s}^{r,0}_{TW}$ together with data (μ,λ), is a cohomological descent category.*

Proof. The proof is analogous to that of Proposition 5.2.25. □

We next define the Thom-Whitney simple for mixed Hodge diagrams.

Definition 5.2.28. Let A be a cubical codiagram of mixed Hodge diagrams. The Thom-Whitney simple of A is the diagram of dga's

$$\mathbf{s}_{TW}(A)=\left(\mathbf{s}^1_{TW}(A_{\mathbf{k}},W)\overset{\mathbf{s}(\varphi)}{\longleftarrow\!\!\dashrightarrow}\mathbf{s}^{1,0}_{TW}(A_{\mathbb{C}},W,F)\right).$$

Proposition 5.2.29. *The Thom-Whitney simple of a cubical codiagram of mixed Hodge diagrams, is a mixed Hodge diagram.*

PROOF. It suffices to prove that the associated functor of strict cosimplicial objects is a mixed Hodge diagram. This follows from 7.11 of [**Nav87**]. $\square$

Theorem 5.2.30. *The category of mixed Hodge diagrams* **MHD** *with the class* $\mathcal{Q}$ *of quasi-isomorphisms and the Thom-Whitney simple functor* $\mathbf{s}_{TW}$ *is a cohomological descent category.*

PROOF. The proof is analogous to that of Theorem 5.2.20, using the corresponding cohomological descent structures of $\mathbf{F}\mathsf{DGA}(\mathbf{k})$ and $\mathbf{F}^2\mathsf{DGA}(\mathbb{C})$. Alternatively, one can use the forgetful functor $\mathbf{MHD} \longrightarrow \mathbf{MHC}$, together with the quasi-isomorphism of simples $\mathbf{s}_{TW} \to \mathbf{s}_D$. $\square$

5.3. Application to Complex Algebraic Varieties

Hodge-Deligne Theory. The first fundamental result by Deligne after defining mixed Hodge structures was to construct a mixed Hodge structure on cohomology of an arbitrary algebraic variety over $\mathbb{C}$. We next recall Deligne's Theorem for open non-singular varieties. All proofs are to be found in Section 3 of [**Del71b**]. Another basic reference is [**PS08**].

Let U be a smooth complex algebraic variety. From Hironaka's Theorem on resolution of singularities one may find a compactification $j : U \hookrightarrow X$ with X smooth and compact, and such that the complement $D = X - U$ is a normal crossings divisor. This means that the irreducible components of D are smooth, and that every point of D has a neighbourhood which looks like a collection of hyperplanes meeting at the origin.

Denote by $\Omega^*_X(\log D)$ the holomorphic logarithmic complex. This is the subcomplex of Ω^*_X of holomorphic differential forms w that have logarithmic poles along D.

For any continuous map $f : X \to Y$, denote $\mathbb{R}f_* = f_*\mathcal{C}^\bullet_{Gdm}$, where $\mathcal{C}^\bullet_{Gdm}$ is the Godement resolution.

Proposition 5.3.1. *The natural morphisms*

$$\Omega^*_X(logD) \to j_*\Omega^*_U \to \mathbb{R}j_*\Omega \leftarrow \mathbb{R}j_*\mathbb{C}_U$$

are quasi-isomorphisms of sheaves, inducing a canonical isomorphism

$$H^i(U;\mathbb{C}) \cong \mathbb{H}^i(X, \Omega^*_X(logD)).$$

Define two filtrations of $\Omega^*_X(\log D)$ as follows:

• The *weight filtration* W of $\Omega^*_X(\log D)$ is the non-negative increasing filtration defined by restricting the order of the poles:

$$W_p\Omega^n_X(\log D) = \Omega^{n-p}_X \wedge \Omega^p_X(\log D),\ 0 \leq p \leq n.$$

• The *Hodge filtration* F of $\Omega^*_X(\log D)$ is the decreasing filtration defined by the *bête* filtration:

$$F^p\Omega^*_X(\log D) : 0 \to \Omega^p_X(\log D) \xrightarrow{d} \Omega^{p+1}_X(\log D) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_X(\log D) \to \cdots .$$

The weight filtration of the holomorphic logarithmic complex is related to the canonical filtration τ (see Example 2.1.34), and allows to define a filtration over $\mathbb{Q}$. We have:

Proposition 5.3.2 ([**Del71b**], see also [**PS08**], Prop. 4.11)**.** *There is a chain of filtered quasi-isomorphisms of sheaves*

$$\begin{tikzcd}[column sep=small]
 & (\mathbb{R}j_*\Omega^*_U, \tau) & & (\Omega^*_X(logD), W) \\
(\mathbb{R}j_*\underline{\mathbb{Q}}_U, \tau)\otimes\mathbb{C} \arrow[ur] & & (\Omega^*_X(logD), \tau) \arrow[ul] \arrow[ur] &
\end{tikzcd}.$$

Denote by $\mathcal{H}dg(X,U)$ the pair of filtered complexes of sheaves $(\mathbb{R}j_*\underline{\mathbb{Q}}_U, \tau)$ and $(\Omega^*_X(\log D), W, F)$, together with the above diagram of filtered quasi-isomorphisms. By the adjoint formula (see Proposition 5.2.1 of [**Hub95**]) the above constructions are functorial. As a consequence, Deligne's result can be restated as follows.

Theorem 5.3.3 ([**Del71b**])**.** *Let U be a smooth complex algebraic variety. Let $j : U \to X$ be a smooth compactification, where $D = X - U$ is a normal crossings divisor. The assignation*

$$(X,U) \longmapsto \mathbb{R}\Gamma(X, \mathcal{H}dg(X,U))$$

defines a functor

$$\mathbb{H}dg : \mathbf{V}^2(\mathbb{C}) \longrightarrow \mathbf{MHC}.$$

In particular, the cohomology of U has a functorial mixed Hodge structure

$$\{(H(U;\mathbb{Q}), \mathrm{Dec}W), (H(U;\mathbb{C}), \mathrm{Dec}W, F)\},$$

obtained by décalage of the induced weight filtration.

In [**Del74b**] Deligne extended this result to possibly singular complex algebraic varieties, using simplicial hypercoverings of varieties. An alternative approach based on cubical hyperresolutions is presented in [**GNPP88**]. See also Section 5.3 of [**PS08**]. We provide a proof within the framework of cohomological descent categories, via the extension criterion of [**GN02**].

Theorem 5.3.4. *There exists an essentially unique Φ-rectified functor*

$$\mathbb{H}dg' : \mathbf{Sch}(\mathbb{C}) \to \mathrm{Ho}\,(\mathbf{MHC})$$

extending the functor $\mathbb{H}dg : \mathbf{V}^2_{\mathbb{C}} \to \mathbf{MHC}$ of Theorem 5.3.3 such that:

(1) $\mathbb{H}dg'$ satisfies the descent property (D) of Theorem 5.2.7.

(2) The cohomology $H(\mathbb{H}dg'(X))$ is the mixed Hodge structure of the cohomology of X.

PROOF. By Theorem 5.2.20 the category **MHC** is a cohomological descent category. It suffices to prove that the functor

$$\mathbf{V}^2_{\mathbb{C}} \xrightarrow{\mathbb{H}dg} \mathbf{MHC} \xrightarrow{\gamma} \mathrm{Ho}\,(\mathbf{MHC})$$

satisfies the hypothesis of Theorem 5.2.11.

Condition F1 is trivial. To prove F2, it suffices to show that for every elementary acyclic diagram

$$\begin{tikzcd}
(\widetilde{Y}, \widetilde{U} \cap \widetilde{Y}) \arrow[r, "j"] \arrow[d, "g"'] & (\widetilde{X}, \widetilde{U}) \arrow[d, "f"] \\
(Y, U \cap Y) \arrow[r, "i"] & (X, U)
\end{tikzcd}$$

of $\mathbf{V}^2(\mathbb{C})$, the mixed Hodge complex $\mathbb{H}dg(X,U)$ is quasi-isomorphic to the simple of the mixed Hodge diagrams associated with the remaining vertices.

This follows from the fact that $\mathbb{H}(\Omega_X(\log D)) \cong H(U)$. Hence (1) and (3) are proven. Assertion (2) is a consequence of Theorem 3.2 of [**GN02**]. $\square$

Mixed Hodge Structures and Rational Homotopy. To study the multiplicative aspects of mixed Hodge theory, and using the Thom-Whitney simple functor of dga's, Navarro defined a functor

$$\mathbb{R}_{TW} f_* : A(X, \mathbb{Q}) \to A(Y, \mathbb{Q})$$

between the categories of sheaves of $\mathbb{Q}$-algebras over X and Y respectively, for every map $f : X \to Y$ of topological spaces. This is essentially equivalent, when forgetting the multiplicative structure, to the common additive derived functor $\mathbb{R}f_*$. In analogy to Proposition 5.3.2, we have:

Proposition 5.3.5 ([**Nav87**], Prop. 8.4)**.** *There is a chain of filtered quasi-isomorphisms of sheaves of filtered dga's over X*

$$\begin{tikzcd}[column sep=small] & (\mathbb{R}_{TW} j_* \mathcal{A}_U^*, \tau) & & (\mathcal{A}_X^*(logD), W) \\ (\mathbb{R}_{TW} j_* \underline{\mathbb{Q}}_U, \tau) \otimes \mathbb{C} \arrow[ur] & & (\mathcal{A}_X^*(logD), \tau) \arrow[ul] \arrow[ur] & \end{tikzcd} ,$$

Denote by $\mathcal{H}dg(X, U)$ the pair $(\mathbb{R}_{TW} j_* \underline{\mathbb{Q}}_U, \tau)$ and $(\mathcal{A}_X^*(\log D), W, F)$, together with the above diagram of filtered quasi-isomorphisms. This allows to define a functor with values in the category of mixed Hodge diagrams. Applying the functor $\mathbb{R}_{TW}\Gamma$ we obtain:

Theorem 5.3.6 ([**Nav87**], Thm. 8.15)**.** *Let U be a smooth complex algebraic variety. Let $j : U \to X$ be a smooth compactification, where $D = X - U$ is a normal crossings divisor. The assignation*

$$(X, U) \longmapsto \mathbb{R}_{TW}\Gamma(X, \mathcal{H}dg(X, U))$$

defines a functor $\mathbb{H}dg : \mathbf{V}^2(\mathbb{C}) \longrightarrow \mathbf{MHD}$.

Using the cohomological descents structure of **MHD** we provide a proof of that Navarro's functor extends to all complex algebraic varieties.

Theorem 5.3.7. *There exists an essentially unique Φ-rectified functor*

$$\mathbb{H}dg' : \mathbf{Sch}(\mathbb{C}) \to \mathrm{Ho}\,(\mathbf{MHD})$$

extending the functor $\mathbb{H}dg : \mathbf{V}^2_{\mathbb{C}} \to \mathbf{MHD}$ *of Theorem* 5.3.6 *such that:*

(1) $\mathbb{H}dg'$ *satisfies the descent property (D) of Theorem* 5.2.7.
(2) The rational part of $\mathbb{H}dg'(X)$ *is* $A_X(\mathbb{Q}) = \mathcal{A}_{Su}(X^{an};\mathbb{Q})$.
(3) The cohomology $H(\mathbb{H}dg'(X))$ *is the mixed Hodge structure of the cohomology of* X.

PROOF. By Theorem 5.2.30 the category **MHD** is a cohomological descent category. It suffices to prove that the functor

$$\mathbf{V}^2_{\mathbb{C}} \xrightarrow{\mathbb{H}dg} \mathsf{MHS} \xrightarrow{\gamma} \mathrm{Ho}\,(\mathbf{MHD})$$

satisfies the hypothesis of Theorem 5.2.11. The proof follows analogously to that of Theorem 5.3.4, using the quasi-isomorphism of simples $\mathbf{s}_{TW} \to \mathbf{s}_D$. $\square$

Corollary 5.3.8. *The rational homotopy functor* $\pi : \mathbf{Sch}^1(\mathbb{C}) \longrightarrow \mathbf{G}^+(\mathbb{Q})$ *defined by sending every simply connected complex algebraic variety* X *to the complex of indecomposables of a Sullivan minimal model of its algebra of rational forms, lifts to a functor*

$$\pi : \mathbf{Sch}^1(\mathbb{C}) \longrightarrow \mathbf{G}^+(\mathsf{MHS}).$$

In particular, the rational homotopy groups of every simply connected algebraic variety over $\mathbb{C}$ *are endowed with functorial mixed Hodge structures.*

PROOF. By Theorems 5.1.23 and 5.3.7 we have functors

$$\mathbf{Sch}^1(\mathbb{C}) \xrightarrow{\mathbb{H}dg'} \mathrm{Ho}(\mathbf{MHD}) \xrightarrow{\pi} \mathbf{G}^+(\mathsf{MHS})$$

whose composite with the forgethful functor $\mathbf{G}^+(\mathsf{MHS}) \to \mathbf{G}^+(\mathbb{Q})$ gives the classical rational homotopy functor. $\square$

Theorem 5.3.9. *The rational homotopy type of every morphism of simply connected complex algebraic varieties is a formal consequence of the first term of the spectral sequence associated with the weight filtration, that is:*

(1) If X *is a simply connected complex algebraic variety, there is a chain of quasi-isomorphisms*

$$(A_X(\mathbb{Q}), d) \xleftarrow{\sim} (M_X, d) \xrightarrow{\sim} (E_1(A_X(\mathbb{Q}), W), d_1),$$

where (M_X, d) *is a Sullivan minimal dga over* $\mathbb{Q}$ *and* $A_X(\mathbb{Q})$ *is the de Rham algebra of* X *over* $\mathbb{Q}$.

(2) If $f : X \to Y$ is a morphism of simply connected complex algebraic varieties, there exists a diagram

$$\begin{tikzcd}
(A_X(\mathbb{Q}), d) \arrow[d, "f_{\mathbb{Q}}"] & (M_X, d) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, dotted] & (E_1(A_X(\mathbb{Q}), W), d_1) \arrow[d, "E_1(f_{\mathbb{Q}})"] \\
(A_Y(\mathbb{Q}), d) & (M_Y, d) \arrow[l, "\sim"'] \arrow[r, "\sim"] & (E_1(A_Y(\mathbb{Q}), W), d_1)
\end{tikzcd}$$

which commutes up to homotopy.

Proof. The composition of the functor $\mathbb{H}dg' : \mathbf{Sch}(\mathbb{C}) \longrightarrow \mathrm{Ho}(\mathbf{MHD})$ of Theorem 5.3.7 with the forgetful functor $\mathbf{MHD} \longrightarrow \mathsf{DGA}(\mathbb{Q})$ gives the rational Sullivan de Rham functor $X \mapsto A_X(\mathbb{Q})$. By Proposition 5.1.28, if X is 1-connected, then the homotopy type of $A_X(\mathbb{Q})$ is a formal consequence of $E_1(A_X(\mathbb{Q}), W)$, and if $f : X \to Y$ is a morphism of 1-connected complex algebraic varieties, then the homotopy type of $A_f(\mathbb{Q})$ is a formal consequence of $E_1(A_f(\mathbb{Q}), W)$. $\square$

The previous result can be restated as a generalization of the formality Theorem 3.2.3 of [**GNPR05**].

Corollary 5.3.10. *There is an isomorphism of functors*

$$U_{\mathbb{Q}} \circ \mathbb{H}dg \cong E_1 \circ (U_{\mathbb{Q}} \circ \mathbb{H}dg) : \mathbf{Sch}^1(\mathbb{C}) \to \mathrm{Ho}_1(\mathbf{F}\mathsf{DGA}^1(\mathbb{Q})),$$

where $U_{\mathbb{Q}}$ denotes the forgetful functor sending every mixed Hodge diagram A to its rational part $(A_{\mathbb{Q}}, W)$.

Resum en Català

El Teorema de Descomposició de Hodge estableix que l'n-èssim espai vectorial de cohomologia de Betti amb coeficients complexos de tota varietat Kähler compacta admet una descomposició en suma directa induïda pel tipus de les formes diferencials complexes. Aquest resultat és un exemple primari d'*estructura de Hodge pura de pes n*, i imposa certes restriccions per tal que una varietat complexa sigui Kähleriana. Per exemple, els nombres de Betti d'ordre senar han de ser parells, i els nombres de Betti d'ordre parell, des del zero fins a dues vegades la dimensió han de ser no nuls.

Influenciat per la filosofia dels motius mixtos de Grothendieck, i motivat per les Conjectures de Weil, Deligne busca una generalització de la teoria de Hodge per a varietats algebraiques complexes arbitràries. La seva idea principal és preveure l'existència d'una filtració natural per al pes en la cohomologia de Betti de les varietats algebraiques, de manera que els quocients successius esdevinguin estructures de Hodge pures de pesos diferents. Aquesta idea dóna lloc a la noció d'*estructura de Hodge mixta*, introduïda a [**Del71a**]. Basant-se en la teoria de resolució de singularitats d'Hironaka i en el complex de de Rham logarítmic, Deligne [**Del71b**] demostra que l'n-èssim grup de cohomologia de tota varietat algebraica llisa definida sobre els complexos, està dotada d'una estructura de Hodge mixta functorial, que en el cas Kähler compacte, coincideix amb l'estructura de Hodge pura original. Aquest resultat té conseqüències topològiques importants, com per exemple el teorema de la part fixa (veure Teorema 4.1.1 de loc. cit). Per tal de tractar el cas general, a [**Del74b**], Deligne introdueix els complexos de Hodge mixtos i estén els seus propis resultats al cas singular, mitjançant resolucions simplicials de varietats. Com una via alternativa a les resolucions simplicials, Guillén-Navarro introdueixen les hiperresolucions cúbiques. La seva aplicació a la teoria de Hodge-Deligne apareix a [**GNPP88**].

Consideracions relacionades amb la Conjectura de Weil sobre l'acció de l'automorfisme de Frobenius per a la cohomologia l-àdica en característica positiva [**Del74a**] porten a pensar que, com a conseqüència de la teoria de Hodge, els productes triples de Massey de les varietats Kähler compactes són nuls. En resposta a aquest problema, Deligne-Griffiths-Morgan-Sullivan [**DGMS75**] proven el Teorema de Formalitat de les varietats Kähler compactes, afirmant que la homotopia real de tota varietat Kähler compacta està determinada per l'anell de cohomologia de la varietat. En particular, els productes de Massey d'ordre superior són trivials.

La teoria d'homotopia racional s'origina amb els treballs de Quillen [**Qui69**] i Sullivan [**Sul77**]. En primer lloc, Quillen estableix una equivalència entre la categoria homotòpica dels espais racionals simplement connexos i la categoria homotòpica de les àlgebres de Lie diferencials graduades connexes. Aquesta equivalència és la composició d'una llarga cadena d'equivalències intermèdies, que compliquen força la construcció. Per tal d'entendre millor aquest mecanisme, Sullivan introdueix les formes polinòmiques de de Rham, demostrant que el tipus d'homotopia de tot espai racional queda determinat per al model minimal de la seva àlgebra diferencial graduada de formes polinòmiques definida sobre els racionals. D'ençà la seva aparició, els models minimals han trobat aplicacions molt significatives tant d'origen topològic com geomètric. Una de les aplicacions inicials més sorprenents és el Teorema de Formalitat de les varietats Kähler compactes.

Per a tractar els aspectes homotòpics i les propietats multiplicatives de la teoria de Hodge mixta, Morgan [**Mor78**] introdueix els diagrames de Hodge mixtos d'àlgebres diferencials graduades, i prova l'existència d'estructures de Hodge mixtes functorials en el tipus d'homotopia de les varietats llises complexes. Com a aplicació, obté un resultat de formalitat respecte el primer temre de la successió espectral associada a la filtració pel pes. En la mateixa línia, Deligne [**Del80**] defineix el $\mathbb{Q}_l$-tipus d'homotopia d'una varietat algebraica. Usant els pesos de l'acció de Frobenius en la cohomologia

l-àdica i la seva solució a la hipòtesi de Riemann, obté un resultat de formalitat del $\mathbb{Q}_l$-tipus d'homotopia per a varietats llises projectives definides sobre cossos finits. Continuant l'estudi de la teoria de Hodge mixta en homotopia racional, Navarro [**Nav87**] introdueix, en el context de la cohomologia de feixos, el simple de Thom-Whitney, per tal d'establir la functorialitat dels diagrames de Hodge mixtos d'àlgebres associats a les algebraiques varietats llises, donant una versió multiplicativa de la teoria de Deligne. Gràcies a aquesta functorialitat estén els resultats de Morgan a les varietats singulars, usant hiperresolucions singulars. De forma independent, Hain [**Hai87**] dóna una extensió alternativa basada en la construcció barra i les integrals iterades de Chen. Ambdues extensions al cas singular es basen en les construccions inicials de Morgan.

Hom pot interpretar la teoria dels diagrames de Hodge mixtos de Morgan, i els seus resultats sobre l'existència d'estructures de Hodge mixtes en el tipus d'homotopia, com una versió multiplicativa de la teoria d'homotopia de Beilinson per als complexos de Hodge mixtos. Impulsat per la cohomologia motívica de Deligne, Beilinson [**Bei86**] introdueix els complexos de Hodge absoluts, relacionats amb els complexos de Hodge mixtos originals de Deligne mitjançant un desplaçament de la filtració per al pes, i n'estudia la teoria d'homotopia. Demostra una formalitat per a objectes, provant que tot complex de Hodge absolut es pot representar mitjançant el complex definit per la seva cohomologia, i estableix una equivalència amb la categoria derivada de les estructures de Hodge mixtes. Aquesta equivalència permet interpretar la cohomologia de Deligne en termes d'extensions d'estructures de Hodge mixtes en la categoria derivada. Tot i que suficient per als seus propòsits inicials, en aquest sentit la teoria d'homotopia de Morgan resulta incompleta, doncs dóna una existència de certs models minimals, però no es demostra que aquests siguin cofibrants o minimals en cap marc categòric abstracte. D'altra banda, Morgan permet que els morfismes entre diagrames siguin homotòpicament commutatius, i no imposa cap llei de composició. Aquest fet fa que la seva teoria s'escapi de l'àmbit de la teoria de categories. Aquest és un aspecte que pretenem solucionar en aquesta tesi.

L'estudi dels functors derivats en la teoria de dualitat porta a Grothendieck a estudiar la localització de la categoria de complexos respecte la classe dels quasi-isomorfismes. Les construccions essencials són dutes a terme per Verdier [**Ver96**], donant lloc a la teoria de les categories derivades d'una categoria abeliana. Simultàniament, i imitant la idea dels motius de Grothendieck, l'estudi dels espectres en topologia algebraica porta a Quillen [**Qui67**] a la introducció de les categories de models. En [**BG76**], Bousfield-Gugenheim reformulen la teoria d'homotopia racional de Sullivan en el marc de les categories de models de Quillen. En aquesta línia, seria desitjable obtenir una formulació equivalent per als diagrames de Hodge. Malauradament, cap dels dos contextos proporcionats per les categories derivades de Verdier i les categories de models de Quillen, considerats avui dia com els pilars de l'àlgebra homològica i homotòpica respectivament, satisfan les necessitats per a expressar les propietats de les categories de diagrames amb filtracions.

Inspirats en els treballs originals de Cartan-Eilenberg [**CE56**] sobre derivació de functors additius entre categories de mòduls, Guillén-Navarro-Pascual-Roig [**GNPR10**] introdueixen les categories de Cartan-Eilenberg, com un enfocament a la teoria d'homotopia més dèbil que el proporcionat per les categories de models de Quillen, però suficient per a estudiar les categories homotòpiques, i per a estendre la teoria clàssica dels functors derivats, al cas no additiu. En aquest context, introdueixen una noció de model cofibrant minimal, com una caracterització abstracta dels models minimals de Sullivan. D'altra banda, seguint Guillén-Navarro [**GN02**], observem que és recomanable demanar que les categories receptores de functors definits sobre les varietats algebraiques estiguin dotades, a més d'una estructura de models que permeti derivar functors, d'una estructura de descens cohomològic, que permet estendre certs functors definits sobre les varietats llises, a varietats singulars.

En aquest treball, analitzem les categories de complexos de Hodge mixtos i de diagrames de Hodge d'àlgebres diferencials graduades en aquestes dues direccions: provem l'existència d'una estructura de Cartan-Eilenberg, via la construcció de models cofibrants minimals, i d'una estructura de descens cohomològic. Aquest estudi permet interpretar els resultats de Deligne, Beilinson, Morgan i Navarro en un marc homotòpic comú.

En el context additiu dels complexos de Hodge mixtos recuperem els resultats de Beilinson. En el nostre estudi anem una mica més enllà, i provem que tant la categoria homotòpica dels complexos de Hodge mixtes, com la categoria derivada d'estructures de Hodge mixtes són equivalents a una tercera categoria en que els objectes són estructures de Hodge mixtes graduades i els morfismes són certes classes d'homotopia, més fàcils de manipular. En particular, obtenim una descripció dels morfismes de complexos de Hodge mixtos en la categoria homotòpica en termes de morfismes i extensions d'estructures de Hodge mixtes, recuperant resultats de Carlson [**Car80**] en aquest àmbit. En quant a l'anàleg multiplicatiu, provem que tot diagrama de Hodge mixt d'àlgebres es pot representar mitjançant una àlgebra dg de Hodge mixta que és minimal de Sullivan, establint una versió multiplicativa del Teorema de Beilinson. Aquest resultat ofereix una via alternativa a les construccions de Morgan. La principal diferència entre les dues vies és que Morgan utilitza construccions de models minimals ad hoc à la Sullivan, especialment definits en el marc de la teoria de Hodge, mentre que nosaltres seguim les línies generals de les categories de models de Quillen o de Cartan-Eilenberg, en tant que els resultats principals es donen en termes d'equivalències de categories i d'existència de certs functors derivats. En particular, obtenim, no tan sols una descripció dels objectes en termes d'àlgebres de Sullivan minimals, sinó que també tenim una descripció dels morfismes en la categoria homotòpica, en termes de certes classes d'homotopia, anàlogament al cas additiu. A més, el nostre enfocament generalitza a contextos més amplis, com per exemple l'estudi dels espais analítics compactificables, en que les filtracions de Hodge i per al pes es poden definir, però aquestes no satisfan les propietats de la teoria de Hodge mixta.

Combinant aquests resultats amb la construcció functorial de Navarro de diagrames de Hodge mixtos, i usant l'estructura de descens cohomològic definida a partir del simple de Thom-Whitney, obtenim una prova més precisa i alternativa al fet que el tipus d'homotopia, i els grups d'homotopia de tota varietat algebraica complexa simplement connexa estan dotats d'estructures de Hodge mixtes functorials. Com a aplicació, i estenent el Teorema de Formalitat de Deligne-Griffiths-Morgan-Sullivan per a varietats Kähler compactes, i els resultats de Morgan per a varietats llises, provem que tota varietat algebraica complexa simplement connexa, i tot morfisme entre aquestes varietats, és formal filtrada: el seu tipus d'homotopia racional està determinat pel primer terme de la successió espectral associada a la filtració per al pes.

* * *

Les categories de complexos de Hodge mixtos i de diagrames de Hodge mixtos d'àlgebres dg són exemples de subcategories d'una categoria de diagrames amb vèrtexs variables, definida mitjançant la categoria de seccions de la projecció de la construcció de Grothendieck. Per tal d'estudiar la teoria d'homotopia d'aquestes categories de diagrames, i en particular, per a construir models cofibrants minimals, cal en primer lloc provar l'existència de models per a les categories dels vèrtexs, i en segon lloc, rectificar diagrames homotòpicament commutatius, tenint en compte que cada morfisme prové d'una categoria deferent. Per tant, un pas preliminar essencial és el d'entendre la teoria d'homotopia de les categories dels vèrtexs, que en el nostre cas són categories de complexos d'espais vectorials i àlgebres dg amb (bi)filtracions, sobre $\mathbb{Q}$ i $\mathbb{C}$.

La teoria d'homotopia dels complexos filtrats va ser iniciada per Illusie [**Ill71**], el qual va definir la categoria derivada d'una categoria abeliana filtrada seguint un esquema ad hoc, estudiant les localitzacions respecte la classe d'equivalències dèbils definida per aquells morfismes que indueixen un quasi-isomorfisme a nivell graduat. Una via alternativa usant categories exactes es detalla en el treball de Laumon [**Lau83**]. En certes situacions, les filtracions sota estudi no estan ben definides, i esdevenen un invariant adequat en termes superiors de la successió espectral associada. Aquest és el cas de la teoria de Hodge mixta de Deligne, en que la filtració per al pes d'una varietat depèn de l'elecció d'una hiperresolució, i està ben definida en el segon terme. Aquesta circumstància està en certa manera amagada per la degeneració de les successions espectrals, però ja posa de manifest l'interès d'estudiar estructures superiors. En el context de la homotopia racional, Halperin-Tanré [**HT90**] estudien la classe d'equivalències dèbils definida per aquells morfismes que indueixen un isomorfisme en un cert estadi de la successió espectral, provant l'existència de models minimals de les àlgebres dg filtrades respecte aquesta classe d'equivalències. Així mateix, Paranjape [**Par96**] estudia l'existència de resolucions injectives superiors per als complexos filtrats de categories abelianes.

En aquest treball mostrem que tots aquests enfocaments homotòpics encaixen en el marc de les categories de Cartan-Eilenberg, i donem resultats anàlegs per a categories bifiltrades. En particular, provem l'existència de models minimals cofibrants en cadascun dels contextos mencionats anteriorment. Per tal de transferir l'estructura homotòpica a nivell de diagrames, desenvolupem una axiomàtica abstracta que permet rectificar diagrames homotòpicament commutatius. Això condueix a l'existència d'una estructura de Cartan-Eilenberg en la categoria de diagrames, amb equivalències dèbils i models minimals cofibrants definits nivell a nivell.

Hem estructurat el nostre treball en cinc capítols relacionats entre sí. A continuació detallem les contribucions de cada capítol.

Capítol 1. Àlgebra Homotòpica i Categories de Diagrames. Desenvolupem una axiomàtica abstracta que permet definir models minimals cofibrants nivell a nivell per a cert tipus de categories de diagrames.

Denotem per $\Gamma\mathcal{C}$ la categoria de diagrames associada al functor $\mathcal{C} : I \to \mathsf{Cat}$ (Definició 1.3.1). Una qüestió natural en àlgebra homotòpica és si donades estructures homotòpiques compatibles en les categories $\mathcal{C}_i$ dels vèrtexs, existeix una estructura homotòpica induïda en la categoria $\Gamma\mathcal{C}$, amb equivalències dèbils definides nivell a nivell. Per a categories de diagrames $\mathcal{C}^I$ associades al functor constant hi ha respostes parcials en termes de les categories de models de Quillen: si $\mathcal{C}$ és cofibrantment generada, o bé I té una estructura de Reedy, aleshores $\mathcal{C}^I$ hereta una estructura de models definida a nivells (veure per exemple [**Hov99**], Teorema 5.2.5). És també ben sabut, que si $\mathcal{C}$ és una categoria d'objectes (co)fibrants de Brown [**Bro73**] aleshores $\mathcal{C}^I$ hereta una estructura de Brown, definida a nivells. En aquesta tesi estudiem la transferència de models cofibrants minimals en el context de les categories de Cartan-Eilenberg, i proporcionem una resposta positiva per a certs tipus de categories de diagrames, en que les categories dels vèrtexs estan dotades d'un objecte camí functorial.

Una *P-categoria* és una categoria $\mathcal{C}$ amb un camí functorial $P : \mathcal{C} \to \mathcal{C}$ i dues classes de morfismes $\mathcal{F}$ i $\mathcal{W}$ de *fibracions* i *equivalències dèbils* que satisfan certs axiomes similars als de les categories de Brown, juntament amb una propietat d'aixecament d'homotopies respecte les fibracions trivials. Exemples de P-categories són la categoria d'àlgebres dg, o la categoria dels espais topològics.

Definim una noció d'objecte cofibrant en termes d'una propietat d'aixecament respecte fibracions trivials: diem que un objecte C d'una P-categoria $\mathcal{C}$ és $\mathcal{F}$*-cofibrant* si tot morfisme $w : A \to B$ de $\mathcal{F} \cap \mathcal{W}$ indueix un morfisme exhaustiu $w_* : \mathcal{C}(C, A) \to \mathcal{C}(C, B)$. El camí functorial defineix una relació d'homotopia entre els morfismes de $\mathcal{C}$, que esdevé d'equivalència per a morfismes amb origen $\mathcal{F}$-cofibrant. Provem que si $\mathcal{C}$ és $\mathcal{F}$-cofibrant, aleshores tota equivalència dèbil $w : A \to B$ indueix una bijecció $w_* : [C, A] \to [C, B]$ entre classes d'homotopia de morfismes. En particular, els objectes $\mathcal{F}$-cofibrants són cofibrants en el sentit de les categories de Cartan-Eielnberg, respecte les classes d'equivalències homotòpiques $\mathcal{S}$ i equivalències dèbils $\mathcal{W}$.

Diem que una P-categoria *té models cofibrants* si per tot objecte A de $\mathcal{C}$ hi ha un objecte $\mathcal{F}$-cofibrant C, juntament amb una equivalència dèbil $C \to A$. Denotem per $\mathcal{C}^{\mathcal{F}}_{cof}$ la subcategoria plena d'objectes $\mathcal{F}$-cofibrants de $\mathcal{C}$, i per $\pi\mathcal{C}^{\mathcal{F}}_{cof}$ la categoria quocient definida mòdul homotopia. Demostrem:

Teorema 1.2.30. *Sigui* $(\mathcal{C}, P, \mathcal{W}, \mathcal{F})$ *una P-categoria amb models cofibrants. La terna* $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ *és una categoria de Cartan-Eilenberg amb models cofibrants en* $\mathcal{C}^{\mathcal{F}}_{cof}$. *La inclusió indueix una equivalència de categories*

$$\pi\mathcal{C}^{\mathcal{F}}_{cof} \xrightarrow{\sim} \mathcal{C}[\mathcal{W}^{-1}].$$

És força immediat que si les categories dels vèrtexs d'una categoria de diagrames $\Gamma\mathcal{C}$ estan dotades d'estructures de P-categoria compatibles, aleshores la categoria de diagrames hereta una estructura de P-categoria. Per contra, la transferència de models cofibrants i minimals de diagrames no és immediata, i requereix una teoria de rectificació de morfismes de diagrames homotòpicament commutatius. Ens reduïm al cas en que els diagrames estan indexats per una categoria finita dirigida de grau binari (veure 1.3.4).

Anomenem *ho-morfismes* aquells morfismes entre diagrames que commuten mòdul homotopia. En general, els ho-morfismes no són componibles. No obstant això, el camí functorial de $\Gamma\mathcal{C}$ defineix una noció d'homotopia entre ho-morfismes. Denotem per $\Gamma\mathcal{C}_{cof}$ la subcategoria plena de $\Gamma\mathcal{C}$ definida per aquells objectes que són $\mathcal{F}_i$-cofibrants nivell a nivell. Els seus objectes, juntament amb les classes d'homotopia d'ho-morfismes, defineixen una categoria $\pi^h\Gamma\mathcal{C}_{cof}$.

Definim una nova classe d'equivalències de $\Gamma\mathcal{C}$ de la manera següent. Un morfisme de $\Gamma\mathcal{C}$ s'anomena *ho-equivalència* si té una inversa homotòpica que és un ho-morfisme. La classe $\mathcal{H}$ definida per la clausura per composició de les ho-equivalències satisfà $\mathcal{S} \subset \mathcal{H} \subset \mathcal{W}$, on $\mathcal{S}$ denota la classe d'equivalències homotòpiques definides pel camí functorial de $\Gamma\mathcal{C}$, i $\mathcal{W}$ denota la classe d'equivalències dèbils definides nivell a nivell. Demostrem:

Teorema 1.4.11. *Sigui* $\Gamma\mathcal{C}$ *una categoria de diagrames indexada per una categoria dirigida* I *com a* 1.3.4. *Assumim que per tot* $i \in I$, *les categories* $\mathcal{C}_i$ *són P-categories amb models* $\mathcal{F}_i$*-cofibrants, i els functors* $u_* : \mathcal{C}_i \to \mathcal{C}_j$ *són compatibles amb les estructures de P-categoria, preservant objectes* $\mathcal{F}_i$*-cofibrants. Aleshores* $(\Gamma\mathcal{C}, \mathcal{H}, \mathcal{W})$ *és una categoria de Cartan-Eilenberg amb models en* $\Gamma\mathcal{C}_{cof}$. *La inclusió indueix una equivalència de categories*

$$\pi^h\Gamma\mathcal{C}_{cof} \xrightarrow{\sim} \Gamma\mathcal{C}[\mathcal{W}^{-1}].$$

En particular, els vèrtexs d'un model cofibrant d'un diagrama donat són models cofibrants dels seus vèrtexs. Provem un resultat anàleg amb models minimals (Teorema 1.4.12), i una versió relativa en que es tracta una subcategoria tancada per equivalències dèbils (Lema 1.4.13), útil en l'estudi de la teoria de Hodge.

Capítol 2. Categories Derivades Filtrades Estudiem els complexos filtrats en el marc de les categories de Cartan-Eilenberg. Tot i que molts dels resultats d'aquest capítol són possiblement coneguts, sembla que hi ha una manca generalitzada de bibliografia sobre el tema. Així, el propòsit d'aquest capítol és el de proporcionar una exposició auto-continguda sobre els principals resultats dels complexos (bi)filtrats.

La categoria $\mathbf{F}\mathcal{A}$ d'objectes filtrats (amb filtracions finites) d'una categoria abeliana $\mathcal{A}$ és additiva, però en general no és abeliana. Considerem la categoria $\mathbf{C}^+(\mathbf{F}\mathcal{A})$ de complexos acotats inferiorment sobre $\mathbf{F}\mathcal{A}$. Per a $r \geq 0$, denotem per $\mathcal{E}_r$ la classe d'E_r-*quasi-isomorfismes*: són aquells morfismes que indueixen un quasi-isomorfisme al terme E_r de la successió espectral associada. Ens interessa estudiar la *categoria* r-*derivada* definida per

$$\mathbf{D}_r^+(\mathbf{F}\mathcal{A}) := \mathbf{C}^+(\mathbf{F}\mathcal{A})[\mathcal{E}_r^{-1}].$$

El cas $r = 0$ correspon a la categoria derivada filtrada original, estudiada per Illusie [**Ill71**]. Tenim una cadena de functors

$$\mathbf{D}_0^+(\mathbf{F}\mathcal{A}) \to \mathbf{D}_1^+(\mathbf{F}\mathcal{A}) \to \cdots \to \mathbf{D}_r^+(\mathbf{F}\mathcal{A}) \to \cdots \to \mathbf{D}^+(\mathbf{F}\mathcal{A}),$$

on la categoria de més a la dreta denota la localització respecte quasi-isomorfismes. Cadascuna d'aquestes categories manté menys informació que l'anterior sobre el tipus d'homotopia filtrat original.

Per a tractar amb la filtració per al pes, Deligne [**Del71b**] introdueix el décalage d'un complex filtrat, que trasllada en un terme la successió espectral associada. El décalage defineix un functor

$$\mathrm{Dec} : \mathbf{C}^+(\mathbf{F}\mathcal{A}) \longrightarrow \mathbf{C}^+(\mathbf{F}\mathcal{A})$$

que és la identitat en morfismes, i envia morfismes de $\mathcal{E}_{r+1}$ a morfismes de $\mathcal{E}_r$. El décalage no té un functor invers, però admet un adjunt per l'esquerra, definit per una translació en la filtració. Usant aquest fet, juntament amb la relació entre les successions espectrals associades obtenim:

Teorema 2.2.15. *El functor de décalage de Deligne indueix una equivalència de categories*

$$\mathrm{Dec}: \mathbf{D}^+_{r+1}(\mathbf{F}\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^+_r(\mathbf{F}\mathcal{A}),$$

per a tot $r \geq 0$.

La noció d'homotopia entre morfismes de complexos d'una categoria additiva es defineix mitjançant un functor de translació, i dota la categoria homotòpica d'una estructura triangulada. En el cas filtrat, observem que diferents eleccions en la filtració del functor de translació, donen lloc a les diferents nocions d'r-homotopia, adequades per a l'estudi de la categoria r-derivada. La categoria r-homotòpica és triangulada, i per a tot $r \geq 0$ obtenim una classe $\mathcal{S}_r$ definida per les r-equivalències homotòpiques que satisfà $\mathcal{S}_r \subset \mathcal{E}_r$.

Com en el cas clàssic, estudiem la categoria r-derivada de $\mathbf{F}\mathcal{A}$ assumint l'existència de suficients injectius en $\mathcal{A}$. Denotem per $\mathbf{C}^+_r(\mathbf{F}\mathrm{Inj}\mathcal{A})$ la subcategoria plena d'aquells complexos filtrats sobre objectes injectius de $\mathcal{A}$ tals que la seva diferencial satisfà $dF^p \subset F^{p+r}$, per a tot $p \in \mathbb{Z}$. Els seus objectes s'anomenen *complexos r-injectius*, i satisfan la propietat clàssica dels objectes fibrants: si I és un complex r-injectiu aleshores tot E_r-quasi-isomorfisme $w : K \to L$ indueix una bijecció $w^* : [L, I]_r \to [K, I]_r$ entre classes d'r-homotopia.

Provem que si $\mathcal{A}$ és una categoria abeliana amb suficients injectius, aleshores tot complex filtrat K té un *model r-injectiu*: és a dir, hi ha un complex r-injectiu I, juntament amb un E_r-quasi-isomorfisme $K \to I$ (un resultat similar ha estat provat per Paranjape [**Par96**]). Com a conseqüència, tenim:

Teorema 2.2.26. *Sigui $\mathcal{A}$ una categoria abeliana amb suficients injectius, i sigui $r \geq 0$. La terna $(\mathbf{C}^+(\mathbf{F}\mathcal{A}), \mathcal{S}_r, \mathcal{E}_r)$ és una categoria de Cartan-Eilenberg. La inclusió indueix una equivalència de categories*

$$\mathbf{K}_r^+(\mathbf{F}\mathrm{Inj}\mathcal{A}) \xrightarrow{\sim} \mathbf{D}_r^+(\mathbf{F}\mathcal{A})$$

entre la categoria de complexos r-injectius mòdul r-homotopia i la categoria r-derivada d'objectes filtrats.

Per a $r = 0$ recuperem un resultat d'Illusie (veure [**Ill71**], Cor. V.1.4.7).

Considerem el cas particular en que $\mathcal{A} = \mathrm{Vect}_{\mathbf{k}}$ és la categoria d'espais vectorials sobre un cos $\mathbf{k}$. En aquest cas, tot object és injectiu, i el càlcul clàssic de categories derivades no proporciona informació addicional. No obstant, podem considerar models minimals: tot complex K és quasi-isomorf al complex definit per la seva cohomologia $K \to H(K)$. Obtenim una equivalència

$$\mathbf{G}^+(\mathbf{k}) \xrightarrow{\sim} \mathbf{D}^+(\mathbf{k})$$

entre la categoria d'espais vectorials graduats sobre $\mathbf{k}$ i la categoria derivada. En el cas filtrat, obtenim resultats anàlegs com segueix.

Diem que un complex filtrat de $\mathbf{C}^+(\mathbf{Fk})$ és *E_r-minimal* si pertany a la categoria $\mathbf{C}_{r+1}^+(\mathbf{Fk})$. És a dir, la seva diferencial satisfà $dF^p \subset F^{p+r+1}$, per a tot $p \in \mathbb{Z}$. Provem que tot E_r-quasi-isomorfisme entre complexos E_r-minimals és un isomorfisme, i que tot complex filtrat té un model E_r-minimal. Obtenim:

Teorema 2.3.7. *Sigui $r \geq 0$. La terna $(\mathbf{C}^+(\mathbf{Fk}), \mathcal{S}_r, \mathcal{E}_r)$ és una categoria de Sullivan, i $\mathbf{C}_{r+1}^+(\mathbf{Fk})$ és la subcategoria plena d'objectes minimals.*

Observem que per a $r = 0$, els models minimals són aquells complexos tals que la seva diferencial és nul·la a nivell d'objectes graduats. Aquest fet segueix el patró del cas sense filtrar, en que els models minimals són aquells complexos amb diferencial trivial. El resultat anterior es pot adaptar a complexos amb múltiples filtracions. Per simplicitat, i donat el nostre interès per a la teoria de Hodge, en aquesta tesi només detallem el cas bifiltrat, respecte la classe $\mathcal{E}_{r,0}$, amb $r \in \{0, 1\}$ (veure Teorema 2.4.12).

Capítol 3. Complexos de Hodge Mixtos. Estudiem la teoria d'homotopia dels complexos de Hodge mixtos en el marc de les categories de Cartan-Eilenberg, via la construcció de models cofibrants minimals.

Un *complex de Hodge mixt* sobre $\mathbb{Q}$ consisteix en un complex filtrat $(K_{\mathbb{Q}}, W)$ sobre $\mathbb{Q}$, un complex bifiltrat $(K_{\mathbb{C}}, W, F)$ sobre $\mathbb{C}$, juntament amb una cadena finita de complexos filtrats $\varphi : (K_{\mathbb{Q}}, W) \otimes \mathbb{C} \longleftrightarrow (K_{\mathbb{C}}, W)$. Els següents axiomes es compleixen:

(MHC$_0$) El morfisme φ és una cadena d'E_1^W-quasi-isomorfismes.
(MHC$_1$) Per a tot $p \in \mathbb{Z}$, el complex filtrat $(Gr_p^W K_{\mathbb{C}}, F)$ és d-estricte.
(MHC$_2$) La filtració F induïda en $H^n(Gr_p^W K_{\mathbb{C}})$, defineix una estructura de Hodge pura de pes $p+n$ en $H^n(Gr_p^W K_{\mathbb{Q}})$, per tot n, i tot $p \in \mathbb{Z}$.

La filtració W és la *filtració per al pes*, metre que F s'anomena *filtració de Hodge*. L'n-èssim grup de cohomologia de tot complex de Hodge mixt hereta una estructura de Hodge mixta, mitjançant una translació de la filtració per al pes.

Per tal d'estudiar la teoria d'homotopia dels complexos de Hodge mixtos resulta convenient treballar amb la categoria **AHC** de complexos de Hodge absoluts, tal i com els defineix Beilinson. L'avantatge principal és que en aquest cas, les successions espectrals associades a les filtracions W i F degeneren al primer terme. A més, la cohomologia és una estructura de Hodge mixta graduada. Tenim functors

$$\mathbf{MHC} \xrightarrow{\mathrm{Dec}^W} \mathbf{AHC} \xrightarrow{H} \mathbf{G}^+(\mathsf{MHS}),$$

on Dec^W denota el functor induït per décalage de la filtració per al pes.

Donat que la categoria d'estructures de Hodge mixtes és abeliana, tota estructura de Hodge mixta graduada, i més generalment, tot complex d'estructures de Hodge mixtos és un complex de Hodge absolut. Hi ha una cadena de subcategories plenes

$$\mathbf{G}^+(\mathsf{MHS}) \longrightarrow \mathbf{C}^+(\mathsf{MHS}) \longrightarrow \mathbf{AHC}.$$

Tot complex de Hodge absolut està relacionat amb la seva cohomologia mitjançant una cadena de quasi-isomorfismes.

Denotem per $\pi^h\mathbf{G}^+(\mathsf{MHS})$ la categoria que té per objectes les estructures de Hodge mixtes graduades, i per morfismes les classes d'homotopia d'homorfismes. Denotem per $\mathcal{H}$ la classe dels morfismes que són equivalències homotòpiques com a ho-morfismes. Provem:

Teorema 3.3.12. *La terna* $(\mathbf{AHC}, \mathcal{H}, \mathcal{Q})$ *és una categoria de Sullivan, i* $\mathbf{G}^+(\mathsf{MHS})$ *és una subcategoria plena de models minimals. La inclusió indueix una equivalència de categories*

$$\pi^h\mathbf{G}^+(\mathsf{MHS}) \xrightarrow{\sim} \mathrm{Ho}\,(\mathbf{AHC}) := \mathbf{AHC}[\mathcal{Q}^{-1}].$$

Observem que tot i que els objectes de la categoria són formals, la subcategoria plena de models minimals té homotopies no trivials. Això reflecteix el fet que les estructures de Hodge mixtes tenen extensions no trivials.

El resultat anterior permet dotar la categoria **MHC** d'una estructura de categoria de Sullivan, via el functor de décalage de Deligne (Teorema 3.3.13). Provem:

Teorema 3.3.14. *El functor décalage de Deligne indueix una equivalència de categories*

$$\mathrm{Dec}^W : \mathrm{Ho}(\mathbf{MHC}) \xrightarrow{\sim} \mathrm{Ho}(\mathbf{AHC}).$$

Usant l'equivalència de categories del Teorema 3.3.12 recuperem el resultat de Beilinson, que dóna una equivalència

$$\mathbf{D}^+\,(\mathsf{MHS}) \xrightarrow{\sim} \mathrm{Ho}\,(\mathbf{AHC})$$

entre la categoria derivada de les estructures de Hodge mixtes i la categoria homotòpica dels complexos de Hodge absoluts. Com a aplicació els resultats anteriors, estudiem els morfismes de la categoria homotòpica, en termes de morfismes i extensions d'estructures de Hodge mixtes.

Teorema 3.3.17. *siguin K i L complexos de Hodge absoluts. Aleshores*

$$\mathrm{Ho}(\mathbf{AHC})(K, L) = \bigoplus_n \left(\mathrm{Hom}_{\mathsf{MHS}}(H^nK, H^nL) \oplus \mathrm{Ext}^1_{\mathsf{MHS}}(H^nK, H^{n-1}L)\right).$$

En particular, recuperem resultats de Carlson [**Car80**] i Beilinson [**Bei86**] sobre les extensions de les estructures de Hodge mixtes.

Capítol 4. Filtracions en Homotopia Racional. La categoria d'àlgebres dg filtrades sobre un cos $\mathbf{k}$ de característica 0 no admet una estructura de models de Quillen. No obstant, l'existència de models minimals permet definir una teoria d'homotopia en un marc conceptual sense axiomatitzar, tal i com es fa en [**HT90**]. Aquí desenvolupem una construcció alternativa de models minimals filtrats, que és una adaptació del cas clàssic de la construcció de models minimals de Sullivan presentada a [**GM81**]. El principal avantatge és que aquesta construcció és fàcilment generalitzable a àlgebres dg amb múltiples filtracions. Després, estudiem la teoria d'homotopia de les àlgebres dg filtrades en el marc de les categories de Cartan-Eielnberg.

Com en el cas dels complexos, denotem per $\mathcal{E}_r$ la classe definida per als E_r-quasi-isomorfismes d'àlgebres dg filtrades, i denotem

$$\mathrm{Ho}_r\left(\mathbf{F}\mathsf{DGA}(\mathbf{k})\right) := \mathbf{F}\mathsf{DGA}(\mathbf{k})[\mathcal{E}_r^{-1}]$$

la categoria localitzada corresponent. La localització respecte $\mathcal{E}_0$ és la categoria ordinària filtrada. Hi ha una cadena de functors

$$\mathrm{Ho}_0\left(\mathbf{F}\mathsf{DGA}(\mathbf{k})\right) \to \mathrm{Ho}_1\left(\mathbf{F}\mathsf{DGA}(\mathbf{k})\right) \to \cdots \to \mathrm{Ho}\left(\mathbf{F}\mathsf{DGA}(\mathbf{k})\right)$$

on la categoria de més a la dreta és la localització respecte quasi-isomorfismes. L'invariant principal en Ho és la cohomologia $H(A)$. En Ho_r tenim famílies d'invariants $E_s(A)$, amd $s > r$. L'invariant principal és $E_{r+1}(A)$. Anàlogament a la teoria dels complexos filtrats tenim:

Teorema 4.3.7. *El functor décalage de Deligne indueix una equivalència de categories*

$$\mathrm{Dec} : \mathrm{Ho}_{r+1}\left(\mathbf{F}\mathsf{DGA}(\mathbf{k})\right) \xrightarrow{\sim} \mathrm{Ho}_r\left(\mathbf{F}\mathsf{DGA}(\mathbf{k})\right).$$

per a tot $r \geq 0$.

Introduïm una noció d'r-homotopia mitjançant un objecte camí amb pesos. Denotem per $\mathcal{S}_r$ la classe d'r-equivalències d'homotòpiques, que satisfà $\mathcal{S}_r \subset \mathcal{E}_r$. L'objecte camí, juntament amb les equivalències dèbils $\mathcal{E}_r$ defineix una estructura de P-categoria.

Definim una noció generalitzada d'àlgebra de Sullivan com se segueix. Una *KS-extensió filtrada de grau n i pes p* d'una àlgebra dg augmentada filtrada (A, d, F) és una àlgebra dg filtrada $A \otimes_\xi \Lambda(V)$, on V és un espai vectorial de grau n i pes pur p, i $\xi : V \to F^P A$ és una aplicació lineal de grau 1 tal que $d\xi = 0$. La filtració en $A \otimes_\xi \Lambda(V)$ es defineix per extensió multiplicativa. Diem que l'extensió és E_r-minimal si

$$\xi(V) \subset F^{p+r}(A^+ \cdot A^+) + F^{p+r+1}A,$$

on A^+ denota el nucli de l'augmentació. Definim una àlgebra dg *E_r-minimal* com el colímit d'una successió d'extensions E_r-minimals, començant per al cos base. En particular, tota àlgebra E_r-minimal és lliure i satisfà

$$d(F^pA) \subset F^{p+r}(A^+ \cdot A^+) + F^{p+r+1}A.$$

Observem que per la filtració trivial, la noció d'E_0-minimal coincideix amb la noció d'àlgebra minimal de Sullivan.

Tota àlgebra E_r-minimal és *E_r-cofibrant*: l'aplicació $w_* : [A, M]_r \to [B, M]_r$ induïda per a un $E_:r$-quasi-isomorfisme $w : A \to B$ és bijectiva. A més, tot E_r-quasi-isomorfisme entre àlgebres E_r-minimals és un isomorfisme.

Un *model E_r-minimal* d'una àlgebra dg filtrada A és una àlgebra dg E_r-minimal, juntament amb un E_r-quasi-isomorfisme $M \to A$. Provem l'existència de models per a àlgebres dg filtrades 1-connexes.

Teorema 4.3.27 (cf. [**HT90**]). *Sigui $r \geq 0$. Tota àlgebra dg filtrada 1-connexa té un model E_r-minimal.*

Provem un resultat anàleg per a àlgebres bifiltrades (Teorema 4.4.9). La teoria d'homotopia de les àlgebres dg filtrades es resumeix en el següent teorema.

Teorema 4.3.28. *Sigui $r \geq 0$. La terna $(\mathbf{FDGA}^1(\mathbf{k}), \mathcal{S}_r, \mathcal{E}_r)$ és una categoria de Sullivan. La inclusió indueix una equivalència de categories*

$$\pi_r\left(\mathsf{E_r}\text{-}\mathsf{min}^1(\mathbf{k})\right) \longrightarrow \mathrm{Ho}_r\left(\mathbf{FDGA}^1(\mathbf{k})\right).$$

L'estructura de categoria de Sullivan permet definir la E_r*-homotopia* de les àlgebres filtrades mitjançant el functor derivat dels indescomponibles.

Teorema 4.3.47. *Sigui* $r \geq 0$. *El functor* $Q : \mathbf{F}\mathsf{DGA}^1(\mathbf{k})_* \longrightarrow \mathbf{C}^+(\mathbf{Fk})$ *admet un functor derivat per l'esquerra*

$$\mathbb{L}_r Q : \mathrm{Ho}_r\left(\mathbf{F}\mathsf{DGA}^1(\mathbf{k})_*\right) \longrightarrow \mathbf{D}_r^+(\mathbf{Fk}).$$

La composició de functors

$$\mathrm{Ho}_r\left(\mathbf{F}\mathsf{DGA}^1(\mathbf{k})\right) \xleftarrow{\sim} \mathrm{Ho}_r\left(\mathbf{F}\mathsf{DGA}^1(\mathbf{k})_*\right) \xrightarrow{\mathbb{L}_r Q} \mathbf{D}_r^+(\mathbf{Fk}) \xrightarrow{E_r} \mathbf{C}_{r+1}^+(\mathbf{Fk})$$

defineix un functor

$$\pi_{E_r} : \mathrm{Ho}_r\left(\mathbf{F}\mathsf{DGA}^1(\mathbf{k})\right) \longrightarrow \mathbf{C}_{r+1}^+(\mathbf{Fk})$$

que associa a cada objecte A, *el complex* E_r*-minimal* $\pi_{E_r}(A) = Q(M_A)$, *on* $M_A \to A$ *és un model* E_r*-minimal de* A.

El model E_r-minimal es relaciona amb el model bigraduat del terme E_r de la successió espectral associada. Això dóna una relació entre la E_r-homotopia d'una àlgebra dg filtrada i la seva homotopia clàssica. Així mateix, tenim una noció de formalitat filtrada, que generalitza la noció clàssica. Sigui $r \geq 0$. Una àlgebra dg filtrada (A, d, F) és E_r*-formal* si hi ha una cadena d'E_r-quasi-isomorfismes

$$(A, d, F) \overset{\sim}{\longleftrightarrow} (E_{r+1}(A), d_{r+1}, F).$$

En particular, la E_0-formalitat de l'àlgebra de Dolbeault d'una varietat llisa complexa coincideix amb la formalitat de Dolbeault introduïda per Neisendorfer-Taylor en [**NT78**].

Capítol 5. Teoria de Hodge Mixta en Homotopia Racional. En aquest darrer capítol usem els resultats dels capítols anteriors per a estudiar la teoria d'homotopia dels diagrames de Hodge mixtos d'àlgebres dg, i la seva estructura de descens cohomològic. Després, donem aplicacions a la geometria algebraica.

La categoria **MHD** dels diagrames de Hodge mixtos d'àlgebres dg es defineix anàlogament a la categoria **MHC**, substituïnt els complexos dels vèrtexs per a àlgebres dg. Com en els cas dels complexos de Hodge mixtos, per a estudiar-ne teoria d'homotopia és convenient treballar amb la versió traslladada de diagrames de Hodge absoluts. El functor de décalage de Deligne respecte la filtració per al pes defineix un functor

$$\mathrm{Dec}^W : \mathbf{MHD} \longrightarrow \mathbf{AHD}.$$

L'anàleg multiplicatiu d'un complex d'estructures de Hodge mixtes és el d'*àlgebra de Hodge mixta*: consisteix en una àlgebra dg (A, d), tal que cada A^n està dotat d'una estructura de Hodge mixta, i les diferencials són morfismes compatibles. Denotem per $\mathbf{MH}\mathsf{DGA}$ la categoria d'àlgebres de Hodge mixtes sobre $\mathbb{Q}$. La cohomologia de tot diagrama de Hodge absolut és una àlgebra de Hodge mixta amb diferencial trivial. Tenim un functor

$$\mathbf{AHD} \xrightarrow{H} \mathbf{MH}\mathsf{DGA}.$$

Recíprocament, donat que la categoria d'estructures de Hodge mixtes és abeliana, tota àlgebra de Hodge mixta és un diagrama de Hodge absolut. Disposem d'un functor d'inclusió

$$i : \mathbf{MH}\mathsf{DGA} \longrightarrow \mathbf{AHD}.$$

Provem que tot diagrama de Hodge d'àlgebres 1-connex és quasi-isomorf a una àlgebra de Hodge mixta, que és minimal de Sullivan. Més precisament, definim una *àlgebra de Hodge mixta minimal de Sullivan* com una àlgebra $M = (\Lambda V, d)$ sobre $\mathbb{Q}$ tal que cada V^n està dotat d'una estructura de Hodge mixta, i les diferencials són compatibles amb les filtracions. Demostrem:

Teorema 5.1.17. *Per a tot diagrama de Hodge absolut 1-connex A existeix una àlgebra de Hodge mixta minimal de Sullivan M, juntament amb un ho-morfisme $M \rightsquigarrow A$ que és un quasi-isomorfisme.*

Combinant aquest resultat amb la teoria d'homotopia de diagrames desenvolupada al capítol 1 obtenim un resultat anàleg al Teorema 3.3.12, que es pot entendre com una versió multiplicativa del Teorema de Beilinson.

Teorema 5.1.19. *La terna $(\mathbf{AHD}^1, \mathcal{H}, \mathcal{Q})$ és una categoria de Sullivan. La categoria de les àlgebres de Hodge mixtes minimals de Sullivan és una* *subcategoria plena de models minimals. La inclusió indueix una equivalència de categories*

$$\pi^h \mathbf{MH}\mathsf{DGA}^1_{min} \longrightarrow \mathbf{AHD}^1[\mathcal{Q}^{-1}]$$

entre la categoria que té per objectes les àlgebres de Hodge mixtes minimals de Sullivan 1-connexes, i per morfismes les classes d'homotopia d'homorfismes, i la categoria localitzada de diagrames de Hodge absoluts 1-connexos respecte dels quasi-isomorfismes.

Usant el décalage de Deligne obtenim:

Teorema 5.1.21. *El functor décalage de Deligne indueix una equivalència de categories*

$$\mathrm{Dec}^W : \mathrm{Ho}\left(\mathbf{MHD}^1\right) \xrightarrow{\sim} \mathrm{Ho}\left(\mathbf{AHD}^1\right).$$

Com a aplicació definim la homotopia d'un diagrama de Hodge mixt via el functor derivat dels indescomponibles.

Teorema 5.1.23. *El functor Q admet un derivat per l'esquerra*

$$\mathbb{L}Q : \mathrm{Ho}\left(\mathbf{MHD}^1_*\right) \longrightarrow \mathrm{Ho}\left(\mathbf{MHC}\right).$$

La composició de functors

$$\mathrm{Ho}\left(\mathbf{MHD}^1\right) \xleftarrow{\sim} \mathrm{Ho}\left(\mathbf{MHD}^1_*\right) \xrightarrow{\mathbb{L}Q} \mathrm{Ho}\left(\mathbf{MHC}\right) \xrightarrow{H\circ \mathrm{Dec}^W} \mathbf{G}^+(\mathsf{MHS})$$

defineix un functor

$$\pi : \mathrm{Ho}\left(\mathbf{MHD}^1\right) \longrightarrow \mathbf{G}^+(\mathsf{MHS})$$

que associa a cada diagrama de Hodge mixt 1-connex A, l'estructura de Hodge mixta graduada $\pi(A) = Q(M_A)$, on $M_A \rightsquigarrow A$ és un model de A.

La part racional de l'estructura de Hodge mixta graduada associada a cada diagrama de Hodge mixt coincideix amb la homotopia de la part racional del diagrama. Com a conseqüència, els grups d'homotopia racional de tot diagrama de Hodge mixt 1-connex estan dotats d'estructures de Hodge mixtes functorials i multiplicatives.

El Teorema de Deligne es pot reformular mitjançant l'existència d'un functor

$$\mathbb{H}dg : \mathbf{V}^2(\mathbb{C}) \longrightarrow \mathbf{MHC}$$

assignant a cada compactificació llisa $U \subset X$ de varietats algebraiques sobre $\mathbb{C}$, amb $D = X - U$ un divisor amb encreuaments normals, un complex de Hodge mixt, que calcula la cohomologia de U (veure Teorema 5.3.3). Inspirat en els treballs de Deligne i Morgan, i amb l'objectiu d'estendre els resultats de Morgan al cas singular, Navarro [**Nav87**] defineix una versió multiplicativa del functor de Deligne

$$\mathbb{H}dg : \mathbf{V}^2(\mathbb{C}) \longrightarrow \mathbf{MHD}$$

amb valors en la categoria de diagrames de Hodge mixtos d'àlgebres dg (veure Teorema 5.3.6). Ambdós functors s'estenen a functors definits sobre les varietats llises. Donem una prova via el criteri d'extensió de [**GN02**], que es basa en la hipòtesi de que la categoria d'arribada és de descens cohomològic. Essencialment, és una categoria $\mathcal{D}$ amb una classe d'equivalències saturada $\mathcal{W}$ i un functor simple $\mathbf{s}$ que assigna a cada diagrama cúbic de $\mathcal{D}$, un objecte de la categoria, i que satisfà certes propietats de compatibilitat anàlogues a les propietats del complex total d'un diagrama de complexos.

L'exemple primari de categoria de descens cohomològic és la categoria $\mathbf{C}^+(\mathcal{A})$ de complexos d'una categoria abeliana, amb la classe dels quasi-isomorfismes i el functor simple definit pel complex total. L'elecció de certes filtracions originalment introduïdes per Deligne donen lloc a un simple $\mathbf{s}_D$ per a diagrames cúbics de Hodge mixtos, definit nivell a nivell. El Teorema 8.1.15 de Deligne [**Del74b**] es por enunciar de la següent manera:

Teorema 5.2.20. *La categoria de complexos de Hodge mixtos* $\mathbf{MHC}$ *amb la classe* $\mathcal{Q}$ *de quasi-isomorfismes i el simple* $\mathbf{s}_D$ *és de descens cohomològic.*

Un resultat anàleg en el context de les categories de descens simplicial ha estat provat per [**Rod12b**]. Seguin la línia de Deligne, l'aplicació principal d'aquest resultat és l'extensió del functor de Deligne a varietats singulars.

Teorema 5.3.4. *Existeix un functor essencialment únic i* Φ*-rectificat*

$$\mathbb{H}dg' : \mathbf{Sch}(\mathbb{C}) \to \mathrm{Ho}\,(\mathbf{MHC})$$

que estén el functor $\mathbb{H}dg : \mathbf{V}^2_{\mathbb{C}} \to \mathbf{MHC}$ *del Teorema* 5.3.3 *tal que:*

(1) $\mathbb{H}dg'$ *satisfà la condició de descens (D) del Teorema* 5.2.7.

(2) La cohomologia $H(\mathbb{H}dg'(X))$ és l'estructura de Hodge mixta de la cohomologia de X.

El simple de Thom-Whitney per a àlgebres cosimplicials estrictes de Navarro [**Nav87**] s'adapta al cas cúbic per a donar lloc a una estructura de descens cohomològic sobre la categoria d'àlgebres dg. L'elecció de certes filtracions sobre aquest simple, dóna lloc a un simple de Thom-Whitney per a diagrames cúbics de diagrames de Hodge mixtos d'àlgebres dg. Hi ha un quasi-isomorfisme de simples $\mathbf{s}_{TW} \to \mathbf{s}_D$. Anàlogament al cas additiu obtenim:

Teorema 5.2.30. *La categoria de diagrames de Hodge d'àlgebres dg* **MHD** *amb la classe $\mathcal{Q}$ de quasi-isomorfismes i el simple de Thom-Whitney $\mathbf{s}_{TW}$ és de descens cohomològic.*

Seguint els treballs de Navarro, la principal aplicació d'aquest resultat és l'extensió del functor de Navarro a les varietats singulars.

Teorema 5.3.7. *Existeix un functor essencialment únic i Φ-rectificat*

$$\mathbb{H}dg' : \mathbf{Sch}(\mathbb{C}) \to \mathrm{Ho}\,(\mathbf{MHD})$$

que estén el functor $\mathbb{H}dg : \mathbf{V}^2_{\mathbb{C}} \to \mathbf{MHD}$ del Teorema 5.3.6 i tal que:

(1) $\mathbb{H}dg'$ satisfà la propietat de descens (D) del Teorema 5.2.7.

(2) La part racional de $\mathbb{H}dg'(X)$ és $A_{\mathbb{Q}} = A_{Su}(X^{an})$.

(3) La cohomologia $H(\mathbb{H}dg'(X))$ és l'estructura de Hodge mixta de la cohomologia de X.

Com a conseqüència dels Teoremes 5.3.7 i 5.1.19, recuperem el resultat de [**Nav87**], que dóna estructures de Hodge mixtes functorials en el tipus d'homotopia de les varietats algebraiques complexes simplement connexes. A més, provem el següent resultat de formalitat, que estén els resultats de Morgan [**Mor78**] sobre la formalitat filtrada de les varietats llises.

Teorema 5.3.9. *El tipus d'homotopia de tot morfisme de varietats algebraiques complexes simplement connexes és una conseqüència formal del primer terme de la successió espectral associada a la filtració per al pes. És a dir:*

(1) Si X és una varietat algebraica complexa simplement connexa, hi ha una cadena de quasi-isomorfismes

$$(A_X(\mathbb{Q}), d) \xleftarrow{\sim} (M_X, d) \xrightarrow{\sim} (E_1(A_X(\mathbb{Q}), W), d_1),$$

on (M_X, d) és una àlgebra minimal de Sullivan sobre $\mathbb{Q}$ i $A_X(\mathbb{Q})$ l'àlgebra de de Rham de X sobre $\mathbb{Q}$.

(2) Si $f : X \to Y$ és un morfisme de varietats, hi ha un diagrama

$$\begin{tikzcd}
(A_X(\mathbb{Q}), d) \arrow[d, "f_{\mathbb{Q}}"] & (M_X, d) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, dotted] & (E_1(A_X(\mathbb{Q}), W), d_1) \arrow[d, "E_1(f_{\mathbb{Q}})"] \\
(A_Y(\mathbb{Q}), d) & (M_Y, d) \arrow[l, "\sim"'] \arrow[r, "\sim"] & (E_1(A_Y(\mathbb{Q}), W), d_1)
\end{tikzcd}$$

que commuta mòdul homotopia.

Aquests resultats es poden resumir mitjançant l'existència d'un isomorfisme de functors

$$U_{\mathbb{Q}} \circ \mathbb{H}dg' \cong E_1 \circ (U_{\mathbb{Q}} \circ \mathbb{H}dg') : \mathbf{Sch}^1(\mathbb{C}) \to \mathrm{Ho}_1(\mathbf{F}\mathsf{DGA}^1(\mathbb{Q})),$$

on $U_{\mathbb{Q}}$ denota el functor oblit que envia tot diagrama de Hodge mixt A, a la seva part racional $(A_{\mathbb{Q}}, W)$.

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