UNIVERSITAT DE BARCELONA

Decision Analysis, Uncertainty Theories and Aggregation Operators in Financial Selection Problems

Binyamin Yusoff

Aquesta tesi doctoral està subjecta a la llicència ***Reconeixement- NoComercial – SenseObraDerivada 3.0. Espanya de Creative Commons***.

Esta tesis doctoral está sujeta a la licencia ***Reconocimiento - NoComercial – SinObraDerivada 3.0. España de Creative*** **Commons.**

This doctoral thesis is licensed under the ***Creative Commons Attribution-NonCommercial-NoDerivs 3.0. Spain License***.

PhD in Business

Decision Analysis, Uncertainty Theories and Aggregation Operators in Financial Selection Problems

Binyamin Yusoff

Universitat de Barcelona

PhD in Business

Thesis title:

Decision Analysis, Uncertainty Theories and Aggregation Operators in Financial Selection Problems

PhD student:
Binyamin Yusoff

Advisors:
Dr. David Ceballos Hornero
(Universitat de Barcelona)
Dr. José María Merigó Lindahl
(Universidad de Chile)

Date:
April 2017

UNIVERSITAT DE BARCELONA

Dedicated to my family

ABSTRACT

The complexity of financial analysis, particularly on selection process or decision making problems, has increased rapidly over several decades. As a result, much attention has been focused on developing and implementing the efficient mathematical models for supporting this kind of problems. Multiple criteria decision analysis, an advanced field of operations research provides analysts or decision makers a broad range of methodologies, which are all suited to the complexity of financial decision analysis. In the financial modeling, uncertainty problems are inevitable, owing to the fact that the consequences of events are not precisely known. In addition, human judgments as part of analysis also contribute to it intricacy. Correspondingly, many studies have been concentrated on integrating uncertainty theories in modeling the real financial problems. One area of interest is on the inclusion of the element of human behavior or attitudinal character of decision makers. Aggregation operator in this case can offer a wide spectrum of analysis or flexibility in modeling the human behavior in financial decision analysis.

In general, the main purpose of this work is on the study of financial selection problems from the perspective of decision analysis, uncertainty theories and aggregation operators. To be specific, the decision problems under a finite or discrete case and multidimensional factors are studied. The emphasis is given on the group decision making models, notably, the Dempster-Shafer theory (DST) of belief structure, the analytic hierarchy process (AHP) and the technique for order performance by similarity to ideal solution (TOPSIS). Moreover, the uncertainty theories based on fuzzy set theory and imprecise probability are employed, together with information fusion based on the ordered weighted average (OWA) operators. Quantitative and qualitative preferences, decision strategies based on the attitudinal character of decision makers, and majority concepts for group consensus are highlighted. The specific contributions of this work are summarized as the following.

The first contribution is on developing the multi-expert multi-criteria decision making (ME-MCDM) model with respect to two-stage aggregation processes. In specific, the aggregation of criteria is based on the integration of weighted arithmetic mean (WA) and OWA. The main attention is given to the proposed alternative OWAWA operator as an extension of immediate WA and OWAWA operators. Two approaches for modeling the majority opinion of experts are studied, in which based on the induced OWA (IOWA) operators. Some modifications to the support functions are suggested as to derive the order-inducing variables. The analysis of ME-MCDM model based

on these aggregation processes then is conducted. In this study the selection of investment strategy is used as to exemplify the model.

The weighted-selective aggregated majority-OWA operator may be considered as the second contribution. It is an extension of the SAM-OWA operator, where the reliability of information sources is considered. The WSAM-OWA then is generalized to the quantified WSAM-OWA by incorporating the concept of linguistic quantifier, mainly for the group fusion strategy. The QWSAM-IOWA with an ordering step is proposed for the individual fusion strategy. These aggregation operators are then implemented to the case of alternative scheme of heterogeneous group decision analysis, in particular for a selection of investment problem.

The third contribution is represented by the development of linguistic group decision making with Dempster-Shafer belief structure. Different type of linguistic aggregation operator such as the 2-tuple induced linguistic OWA operator is suggested. Specifically, it is based on order-inducing variables in which the ordering of the arguments and uncertain situations can be assessed with linguistic information. Then, by using the 2-TILOWA in the D-S framework, the belief structure-2-TILOWA operator can be formed. Some of its main properties are studied. This model is applied in a selection of financial strategies.

The extension of AHP for group decision making model is given as the fourth contribution, notably, based on the inclusion of IOWA operators. Two-stage aggregation processes used in the AHP-GDM model are extended. Firstly, a generalization of weighted maximal entropy OWA under the IOWA operator is proposed as to aggregate the criteria. Further, the majority concept based on the IOWA and Minkowski OWA-based similarity measure is suggested to determine a consensus among experts. This model provides a variant of decision strategies for analyzing the individual and the majority of experts. The application in investment selection problem is presented to test the reliability of the model.

The fifth contribution is on the integration of heavy ordered weighted geometric (HOWG) aggregation operators in AHP-GDM model. In the sense of heavy OWA operator (HOWA), the heavy weighted geometric (HWG) and HOWG are introduced as extensions of the normal weighted geometric mean (WG) and the OWG by relaxing the constraints on the associated weighting vector. These HWG and HOWG operators then are utilized in the aggregation process of AHP-GDM, specifically on the aggregation of individual judgments procedure. The main advantage of the model, besides the complete overlapping of information such in classical methods, is that it can also accommodate partial and non-overlapping information in the formulation. An investment selection problem is applied to demonstrate the model.

The extension of TOPSIS for group decision making model by the inclusion of majority concept may be considered as the sixth contribution. The majority concept is derived based on the induced generalized OWA (IGOWA) operators. Two fusion schemes in TOPSIS model are designed. First, an external fusion scheme to aggregate the experts' judgments with respect to the concept of majority opinion on each criterion is suggested. Then, an internal fusion scheme of ideal and anti-ideal solutions that represent the majority of experts is proposed using the Minkowski OWA distance measures. The comparison of the proposed model with some other TOPSIS models with respect to distance measures is presented. Here, a general case of selection problem is presented, specifically on the human resource selection problem.

Finally, the group decision making model based on conflicting bifuzzy sets (CBFS) is proposed. Precisely, the subjective judgments of experts, mainly from positive and negative aspects are considered simultaneously in the analysis. Moreover, the weighting method for the attribute (or sub-attribute) is subject to the integration of subjective and objective weights. The synthesis of CBFS in the model is naturally done by extending the fuzzy evaluation in parallel with the intuitionistic fuzzy set. A new technique to compute the similarity measure is proposed, in which, being the degree of agreement between the experts. The model then is applied in the case study of flood control project selection problem.

To sum up, the presented thesis dealt with the extension of multi-criteria decision analysis models for the financial selection problems (as a specific scope) and also the general selection problems with the inclusion of attitudinal character, majority concept, and fuzzy set theory. In particular, the group decision making model, Dempster-Shafer belief structure, AHP, and TOPSIS are proposed to overcome the shortcoming of the existing models, i.e., related to the financial decision analysis. The applicability and robustness of the developed models have been demonstrated and some sensitivity analyses are also provided. The main advantages of the proposed models are to provide more general and flexible models for a wider analysis of the decision problems.

RESUM

La complexitat de l'anàlisi financera, sobretot en els processos de selecció o en problemes de presa de decisions, s'ha incrementat molt en les darreres dècades. Un efecte d'això ha estat el major desenvolupament i implementació de models matemàtics eficients per donar suport a aquest tipus de problemes complexos. L'anàlisi de decisions multicriteri, un àmbit avançat de la investigació operativa, proporciona als analistes i als decisors una àmplia gamma de metodologies que s'adapten a la complexitat de l'anàlisi de decisions financeres. En els models financers, els problemes d'incertesa són inevitables, perquè els efectes i esdeveniments futurs no es coneixen amb precisió. A més, els judicis i opinions humans com a part de l'anàlisi també contribueixen a incrementar la complexitat de la decisió. En conseqüència, molts estudis s'han concentrat en la integració de les teories d'incertesa en el modelatge dels problemes financers de la vida real. Una àrea d'interès, dins d'aquesta integració, és la inclusió de l'element del comportament racional humà o del caràcter conductual dels decisors. En aquests casos, els operadors d'agregació poden oferir un ampli espectre d'anàlisi o flexibilitat en la modelització del comportament humà en l'anàlisi de decisions financeres.

En general, l'objectiu principal d'aquest treball és l'estudi dels problemes de selecció financera des de la perspectiva de l'anàlisi de decisions, les teories de la incertesa i els operadors d'agregació. En concret, s'estudien els problemes de decisió en virtut d'un conjunt finit d'alternatives (cas discret) i de factors multidimensionals. L'èmfasi se situa en els models de presa de decisions en grup i, en particular, en l'estructura de creences de Dempster-Shafer (D-S), el procés analític jeràrquic (AHP) i la tècnica d'ordre de preferència per similitud amb la solució ideal (TOPSIS). A més, es fan servir les teories d'incertesa basades en conjunts borrosos i de probabilitats imprecises juntament amb la fusió de la informació basada en operadors de mitjana amb pesos ordenats (OWA). També es destaquen les preferències quantitatives i qualitatives, les estratègies de decisió basades en el caràcter actitudinal dels decisors, i el concepte de majoria en el consens grupal. La recerca feta es pot sintetitzar en set aportacions específiques a l'*state-of-the-art* de l'anàlisi de decisions i els operadors d'agregació en els problemes de selecció financera. Es resumeixen a continuació:

La primera contribució té a veure amb el desenvolupament del model de presa de decisions multicriteri amb diversos experts (ME-MCDM) en els processos d'agregació en dues etapes. En particular, l'agregació dels criteris es basa en la integració de la mitjana aritmètica ponderada (WA) i els operadors de mitjana amb pesos ordenats (OWA). Es presta una atenció especial a

l'alternativa proposada de l'operador OWAWA com a extensió dels operadors immediats WA i OWAWA. S'estudien dos enfocaments per a la modelització de l'opinió de la majoria dels experts, els quals es basen en els operadors OWA induïts (IOWA). També se suggereixen algunes modificacions en les funcions de suport per derivar les variables d'ordre induït. A continuació, es porta a terme l'anàlisi del model ME-MCDM, basat en aquests processos d'agregació. La selecció de l'estratègia d'inversió es fa servir per exemplificar el model i la seva utilitat en els problemes de selecció financera.

L'operador OWA de majoria agregada selectiva ponderada es pot considerar la segona aportació. És com una extensió de l'operador SAM-OWA, en què es considera la fiabilitat de les fonts d'informació. L'operador WSAM-OWA es pot generalitzar en la forma quantificada de WSAM-OWA, mitjançant la incorporació del concepte de quantificador lingüístic, principalment en l'estratègia de fusió grupal. Es proposa l'operador QWSAM-IOWA amb una etapa d'ordenació per a l'estratègia de fusió individual. Aquests operadors d'agregació s'implementen per al cas d'un esquema alternatiu d'anàlisi de decisions en grups heterogenis, particularment en problemes de selecció d'inversions.

La tercera contribució es constata en el desenvolupament de la presa de decisions grupal amb quantificador lingüístic dins de l'estructura de creences Dempster-Shafer (D-S). Es suggereixen diferents tipus d'operadors d'agregació lingüística, com ara l'operador OWA induït de 2 tuples amb quantificador lingüístic (2-TILOWA). En concret, es basa en variables d'ordre induït en les quals l'ordre dels arguments i situacions d'incertesa es pot avaluar amb la informació lingüística. Així doncs, mitjançant l'ús del 2-TILOWA en el marc D-S, es pot formar l'operador 2-TILOWA amb l'estructura de creences. Se n'estudien algunes de les propietats principals i s'aplica aquest model a una selecció d'estratègies financeres.

L'extensió del model AHP per a la presa de decisions grupal, en particular, sobre la base de la inclusió dels operadors IOWA, configura la quarta aportació. Els processos d'agregació en dues etapes que es fan servir en el model AHP-GDM s'estenen per proporcionar un marc d'anàlisi de decisions més general. En primer lloc, es proposa com a agregació de criteris una generalització de l'operador OWA d'entropia màxima ponderada sota l'operador IOWA, amb la finalitat d'agregar els criteris. En segon lloc, se suggereix el concepte de majoria basat en la mesura de similitud dels operadors IOWA i OWA de Minkowski per tal de determinar el consens entre els experts. Aquest model proporciona un conjunt alternatiu d'estratègies de decisió dins de l'anàlisi de l'expert individual o de la majoria d'experts. Per provar la fiabilitat del model se'n presenta l'aplicació a un problema de selecció d'inversions.

La cinquena contribució es correspon a la integració d'alguns operadors d'agregació geomètrics ponderats ordenats pesats (HOWG) en el model AHP-GDM. Així, en el sentit de l'operador OWA pesat (HOWA), l'operador geomètric ponderat pesat (HWG) i l'operador HOWG s'introdueixen com a extensions de la mitjana normal geomètrica ponderada (WG) i l'OWG, a partir de relaxar les restriccions en el vector de pesos associats. Aquests operadors HWG i HOWG aleshores s'utilitzen en el procés d'agregació d'AHP-GDM, específicament en el procés d'agregació de judicis o opinions individuals. L'avantatge principal d'aquest model, a més de la superposició completa d'informació com en els mètodes clàssics, és que també pot adaptar informació parcial i no superposada en la formulació. Per demostrar la bondat del model s'aplica a un problema de selecció d'inversions.

L'extensió del model TOPSIS per a la presa de decisions grupal a través de la inclusió del concepte de majoria, es pot considerar la sisena aportació. El concepte de majoria es desenvolupa sobre la base dels operadors OWA generalitzats induïts (IGOWA). Es presenten dos esquemes de fusió en el model TOPSIS. En primer lloc, se suggereix un esquema de fusió externa per agregar els judicis o opinions dels experts pel que fa al concepte d'opinió majoritària en cada criteri. En segon lloc, es proposa un esquema de fusió interna de les solucions ideals i anti-ideals que representa la majoria dels experts fent servir les mesures de distància OWA de Minkowski. Es fa una comparació de les mesures de distància entre el model proposat i altres models TOPSIS. Novament, es fa un estudi aplicat a un problema de selecció, concretament a un problema de selecció de recursos humans.

Finalment, la setena aportació que es desenvolupa és la proposta d'un model de presa de decisions grupal basat en els conjunts borrosos en conflicte (CBFS). Específicament, l'anàlisi té en compte de manera simultània els judicis o opinions subjectius dels experts, principalment pel que fa a aspectes positius i negatius. Així mateix, el mètode de ponderació per a l'atribut (o atribut secundari) està subjecte a la integració de pesos subjectius i objectius. La síntesi del CBFS en el model es porta a terme de manera natural ampliant l'avaluació borrosa en paral·lel amb el conjunt borrós intuïcionista. Es proposa una nova tècnica per calcular la mesura de similitud, és a dir, establir el grau d'acord entre els experts. El model s'aplica a un cas de problema de selecció, en concret, a un problema de control d'inundacions.

En resum, la tesi desenvolupa una extensió dels models d'anàlisi de decisions multicriteri que es fan servir en la resolució dels problemes de selecció financers (com a àmbit específic), però també en els problemes de selecció generals, amb la inclusió del caràcter actitudinal, el concepte de majoria i la teoria dels conjunts borrosos. En particular, proposa els models AHP i TOPSIS, juntament amb el model de presa de decisions grupal i l'estructura de creences Dempster-Shafer, per tal de superar les deficiències dels models existents en relació amb l'anàlisi de decisions financeres. Es demostra l'aplicabilitat i la robustesa dels models desenvolupats, que també es reforça amb algunes anàlisis de sensibilitat. Els avantatges principals dels models proposats són que es tracta de models més generals i flexibles per a una anàlisi més àmplia dels problemes de decisió, i en particular dels de selecció financera.

ACKNOWLEDGEMENTS

This thesis would not have been possible without the help and support from many people. First and foremost, I am deeply indebted to my advisors David Ceballos Hornero and José Maria Merigó Lindahl, who introduced me to the exciting topics of financial decision analysis and aggregation operators. They guided me throughout the whole duration of my PhD project. Their critical questions and comments helped me to clarify many of the issues addressed in this thesis. They also gave me the freedom to develop my ideas and encouraged me to search for the solution. Without their enthusiasm and continuous guidance and support, the development of this work would not have been possible.

Secondly, I would like to express my gratitude to all academicians, staffs, and friends at the University of Barcelona for helping me, directly or indirectly, throughout my study period. The workshops, short-courses, and seminar organized by the department and faculty, provided many useful inputs which enriched my work. Especially, under the 'Research Group on Financial Analysis and Uncertainty' (IAFI), Department of Mathematics for Economics, Finance and Actuarial Sciences. My thanks also go to Ronald Yager, Radko Mesiar, Janusz Kacprzyk and Bernard De Baets as I have benefited a lot from their comments and suggestions during my participation in the international conferences. Thanks are also due to participants of the AEDEM congress 2012, the SIGEF conference 2012, the SSWIU Spring School 2014, the SSCI Summer School 2014, the IPMU conference 2014, the ISCAMI student conference 2015, the IEEE-FUZZ conference 2015, and the FIM-IMST conference 2015.

I would also like to thank David Martin Sebastian for his friendship and support, which helped me upon my first arrival in Barcelona. My special appreciation also goes out to all Malaysian friends and families in Barcelona, Valladolid, Zaragoza, Alicante and Madrid for their kind help and friendship, which supported me to enjoy the life and study in Spain for the past few years. I will never forget our numerous Spain adventures. Among them that I really need to mention are Zulkifly Jemaat, Akma Kamarzaman and Sobiah Yusoff. Many thanks for being someone that I could depend on for friendship, motivation, and wisdom.

My invaluable appreciation also goes to Abu Osman Md. Tap, Abdul Fatah Wahab, Sabri Ahmad, Che Mohd Imran Che Taib, Mohd Lazim Abdullah and all the staffs from University of Malaysia Terengganu (UMT) for their kind words, support, and encouragement during my PhD journey. Special thanks also go to the tutors and academic fellows, batch 2010-2012, for sharing aspiration and motivation to pursue PhD study. Needless to say, my eternal gratitude goes to all close friends who always there when I need them.

I would also like to acknowledge the support from Ministry of Higher Education Malaysia (MOHE) and UMT for providing me scholarships for my study. My deepest thanks go to all staffs of MOHE and UMT, especially the Study Leave Section of Registrar Office, for their kind assistance.

Last but not least, my heartfelt thanks are given to my parents, Yusoff Awang and Zaiton Ibrahim, my sisters, Nurul Farizan, Nur Yusniza, Nur Athirah, Najat Nasuha and my little brother, Bakhit Irfan for their support, encouragement, and understanding. May this small success will inspire you to pursue your dreams too. I would like to dedicate this thesis to them.

CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1

INTRODUCTION

1.1 General Background

Financial modeling is one of the main research areas in the field of finance or economics. The topic has been studied widely from various perspectives. Among the major interests in this topic are the financial analysis and decision making problems, such as portfolio selection problems, asset allocation problems, and selection of financial products, to name a few. The selection process, generally, is a complex procedure which involves decision under various conditions (e.g., certainty and uncertainty) and multidimensional aspects (e.g., multi-criteria and multi-expert judgments). Many mathematical models, methods or techniques have been developed to deal with such problems. Closely related to this topic is the subject of decision theory, which is under the field of operations research or management science.

Decision theory is fundamental in decision analysis or decision making models. Decision analysis can be explained as a careful analysis in deciding upon a course of action. The prominent work on this topic is due to von Neumann and Morgenstern (1944) in modeling human rational behavior, precisely in the realm of economics. In specific, it is based on a decision analysis under uncertainty (or risk) and is modeled using a probability theory, i.e., objectively defined. In this case, the alternatives to be analyzed are based on a set of uncertain states of the world (or nature) and the final decision is determined by the maximum expected utility. Basically, the maximum expected utility theory is subject to a one-dimensional set of consequences. The extended version of this model under the multidimensional factors is represented by a multi-attribute utility theory (MAUT), or a multi-attribute value theory (MAVT) in the case of certainty. The review on this topic is provided in Chapter 2.

The underlying concept in the existing decision making models is based on either the utility theory or the value theory, with respect to the preference relation approach. Lay upon the same concept, many other decision making models have been developed in the literature, such as the analytic hierarchy process (AHP), measuring attractiveness by a categorical based evaluation technique (MACBETH), technique for order performance by similarity to ideal solution (TOPSIS), preference disaggregation analysis, outranking models, e.g., elimination and choice expressing reality (ELECTRE) and preference ranking organization method for enrichment evaluation (PROMETHEE), among others (see Saaty, 1980; Bana e Costa & Vansnick, 1994; Hwang & Yoon, 1981; Jacquet-Lagréze & Siskos, 1982; Roy, 1991; Brans & Vincke, 1985). All these models are classified under the general class of decision analysis known as a multi-criteria decision analysis (MCDA). Besides, the models are also applicable for the case of multi-expert (or group) decision making problems.

Alternatively, to deal with uncertain decision problems, such as in the case of the absence of objective probability, Savage (1954) then proposed the subjective expected utility model which is based on a subjective probability theory. This type of probability is an alternative of the objective probability (i.e., random or chance phenomena), where it is derived from an individual's personal judgment (i.e., the degree of belief or perception) (see Ramsey, 1931; de Finetti, 1937). The Bayesian decision model (Berger, 1985; Savchuk & Tsokos, 2011) was the product of the subjective probability and later was extended by Dempster (1967) and Shafer (1976) for the new approach known as Dempster-Shafer belief structure, (i.e., the theory of evidence). This method can be considered as a general framework for reasoning with uncertainty, with the connections to the other frameworks such as probability, possibility and imprecise probability theories.

In addition to the probability theory, another well-known theory in modeling uncertainty problems is a fuzzy set theory by Zadeh (1965). This theory has been successfully applied in the domain of decision analysis. The essence of this theory is to deal with decision problem under the imprecision, vagueness and partially truth condition. Billot (1992) classifies the application of fuzzy set theory, specifically under the field of economics, into two categories. First, the direct application of the concept of fuzzy sets via the extension principle, which is from the world of crisp numbers and all its restrictions to the world of fuzzy numbers with flexibility in data representation. This includes fuzzy number arithmetic, ranking of fuzzy numbers, fuzzy preference relations, fuzzy mathematical programming, fuzzy multi-criteria decision problems, etc. The second is, the introduction of fuzzy measure or non-additive probability as a necessary tool to avoid well-known paradoxes, such as the Ellsberg paradox. This branch includes possibility theory, non-additive expected utility, and also the Dempster-Shafer theory.

Another important issue in decision analysis is the aggregation process or information fusion. Under the fuzzy measures, in particular Choquet and Sugeno integrals, a wide range of aggregation operators can be derived and integrated with decision making models (see Beliakov et al., 2007; Torra & Narukawa, 2007; Grabisch et al., 2009). The OWA operator (Yager, 1988; Yager & Kacprzyk, 1997) is a sub-class of these general operators of fuzzy measures. It provides a unified framework of mean-type aggregation operators such as the average, the maximum, the minimum, and a convex sum of the maximum and the minimum. With respect to the OWA operators, there are many other families, extensions and generalizations of aggregation operators, such as a generalized OWA, an induced generalized OWA, a weighted OWA and a majority additive-OWA, etc (see, for example, Yager, 1996; Merigó & Gil-Lafuente, 2009; Torra, 1997; Peláez & Doña, 2003). Recently, Yager (2003a) generalized the fuzzy measures to monitored heavy fuzzy measures which include a more general class of aggregation operators. In specific, these operators include both, the mean aggregation and the total aggregation operators. Heavy OWA, for instance, is a type of this general class aggregation operator. The general overview of aggregation operators can be referred in Chapter 3.

It can be noticed that the main development of decision analysis has been concentrated on the problems of multidimensional aspects, modeling uncertainty and the fusion of information. Highlighting on the aforementioned problems, this study is focused on extending and revising some of the decision making models, namely the general group decision making model, Dempster-Shafer belief structure, AHP and TOPSIS models with respect to the fuzzy set theory and the OWA-based aggregation operators. Then, analyses on the financial selection problems are studied for the specific scope of this research. In addition, other general applications such as human resource selection problems and flood control project selection are also presented to demonstrate the applicability of the developed models. In the subsequent section, the justification for the application of MCDA models in the domain of financial analysis is presented.

1.2 Multi-dimensional Aspects in Financial Decision Analysis

Financial modeling has demonstrated a huge amount of literature related to developing and integrating financial theory, tools of operations research and mathematical models for specific financial problems. For example, numerous studies have been devoted to portfolio analysis, particularly after the work by Markowitz (1952). The mean-variance model was proposed to construct an efficient frontier of optimal portfolios, offering the maximum possible expected return for a given level of risk, or vice versa.

The central directions of portfolio theory can be divided into two main approaches. They are the portfolio optimization and the portfolio selection problems. Portfolio optimization which stems from Markowitz's model is based on mathematical optimization techniques (see, for example, Focardi & Fabozzi, 2004; Mansini et al., 2015). On the other hand, portfolio selection is based on decision rules or decision strategies (see Grinold, 1999; Gosling, 2010, among others). The mean-variance model was introduced considering the bi-criteria evaluations (i.e., return and risk). The model was an extension of the classical one-dimensional approach of investment (i.e., focus exclusively on expected value or mean return). However, relying on the two factors of return and risk is not providing complete analysis for the portfolio performance. Spronk and Hallerbach (1997) stated that variance as a risk measure may miss its link with an investor's preference structure of the distributions of security and portfolio returns. Moreover, information concerning mean and variance is not always sufficient to adequately discriminate between investment alternatives. In this way, it can be said that the mean-variance framework deducts the multi-dimensional aspects that may be perceived by the investor. Hence, the portfolio management or selection process has been extended to the case of multidimensional factors to better model this kind of problem.

In the literature, there a substantial amount of MCDA models that have been applied in portfolio selection problems. Examples include, the AHP method (Saaty & Vargas, 1982; Tiryaki & Ahlatcioglu, 2005), the MACBETH (Bana e Costa & Soares, 2004; Hurson et al., 2012), the TOPSIS and outranking methods (Martel et al., 1988; Vetschera & de Almeida, 2012; Tavana et al., 2015). In addition to portfolio selection and management, other areas of finance, such as venture capital investments, bankruptcy prediction, financial planning, corporate mergers and acquisitions, country risk assessment, etc. have been studied under the realm of MCDA. The review of applications of MCDA models in the domain of finance can be referred to, for instance, in Zopounidis and Doumpos (2002), Spronk et al. (2005) and Dymowa (2011).

In other application, such as project assessments, the cost-benefit analysis (CBA) is another model that normally used in decision analysis. The model is mainly based on the net value, e.g., cash flow, etc. However, the same critique has been posed to the CBA model as the main consideration is bi-criteria on the monetary values (Munda, 1996; Bouyssou, 2000; Bana e Costa, 2004). This then limits the inclusion of the other qualitative factors (or non-monetary values) such as technical, social, environmental criteria. As a result, the MCDA models have been applied to provide a comprehensive analysis of the problem at hand.

In addition to the multi-criteria assessments, group decision problem is also vital in the financial decision analysis. According to Forsyth (2006) and Bonner et al. (2002), groups undeniably have advantages over decisions made by individuals. In general, it provides a potential of synergy and sharing of information. For instance, synergy can be interpreted as the idea that the whole group is better than the sum of its individual members. When a group makes a decision collectively, its judgment can be more incisive than that of any of its members. On the other hand, the sharing of information can be explained as taking into account a broader scope of information since each group member may contribute unique information and expertise. Some works on the group decision making in the finance research can be referred in Mottola and Utkus (2009), Baddeley and Parkinson (2012) and Huang et al. (2013), among others.

1.3 Problem Statements and Motivation

In the previous section, justifications for the application of MCDA models in the domain of finance have been discussed. In this section, the issues and main motivation related to the integration of OWA-based aggregation operators and uncertainty theories are presented.

Most of the MCDA models are developed based on the weighted arithmetic mean (WA) for the aggregation of criteria. The emphasis is only given on the weights concept in which all the criteria are associated with specific degrees of importance. However, in this respect, the analysis is very limited since there is no consideration for the behavioral or attitudinal character of experts. Thus, a broader analysis of the problem, such as the assessment with respect to different decision strategies (i.e., ranging from optimistic to pessimistic views of experts) cannot be performed. Alternatively, OWA operator (Yager, 1988) provides the inclusion of the attitudinal character of experts in the aggregation process. Its main focus is on the structure concept, for instance, the relationship between the criteria as perceived by the experts. In this case, different semantics can be associated to OWA operator by the construction of its weighting vector. Recently, OWA aggregation operators have been applied in the field of finance (see, for example, Engemann et al., 1996; Merigó & Gil-Lafuente, 2010; Belles-Sampera et al., 2013; Vigiera et al., 2016; Laengle et al., 2016).

In spite of that, OWA operator is only taken into account the case where the criteria are associated with equal degrees of importance (i.e., with respect to the weight concept). Then, this again not well representing the general decision making model where each of the criteria is associated with the different degree of importance. As a solution, the integration of WA and OWA operators has been proposed to include not only the degrees of importance but also, with the inclusion of the attitudinal character of experts (see Yager, 1988; Torra, 1997; Merigó, 2012, among others). Hence, the MCDA models for financial selection problems need to incorporate this type of aggregation operators as an additional feature to improve the modeling process.

As can be noted, the OWA is based on the permutation of the input vector according to the magnitude of its arguments. However, in some cases, it makes sense that the inputs be reordered by values different to those used in the calculation. In a more general framework, it can be represented as a two-tuple (or OWA pair), in which the argument is a component of a more complex object together with the order inducing variable. This particular aggregation operator is known as the induced OWA operator (Yager & Filev, 1999). The order inducing variable may be represented by the auxiliary variable or the function of argument values. Examples include the reliability of sources, nearest-neighbor rules, best-yesterday models, etc. (see Yager & Filev, 1999; Chiclana et al., 2007; Merigó & Gil-Lafuente, 2009; Beliakov & James, 2011). Analogously, IOWA operator can be integrated with the WA and then applied in the MCDA models to provide a general framework of the aggregation process.

It is clear that OWA is a mean type of aggregation operator where the estimation is respect to the spatial partition. Generally, it is based on the overlapping information of the same variable, as such, all the experts are evaluating the same space. Therefore, this provides freedom and flexibility to represent the information in the aggregation process. In addition to that, heavy OWA (Yager, 2002) as another extension of OWA is based on the totaling-type aggregation. The estimation is generated from the partition space or disjoint region. Hence, the information is non-overlapping and each space reflects different variable. In this case, all the information must be employed because they are totally independent. An interesting feature is, it provides a wider class of aggregation processes, in between the mean-type and totaling-type, specifically in the cases of partial-overlapping information. To deal with this type of problems in the financial analysis, the MCDA models can be integrated with the HOWA operators. Accordingly, this can accommodate more complex problems and wider decision strategies for the extensive analysis.

Another issue that may arise in MCDA problems is when dealing with the consensus measure of the experts. Similarly, WA or weighted geometric mean (WG) is applied as the group aggregator. In this situation, all experts are included in the evaluation process without an exception, even though there may be some biases of certain experts. Hence, this result will influence the validity of the overall judgment. However, in most cases, it is difficult to achieve a unanimous agreement when dealing with group evaluation. As an alternative, consensus measure among a majority of experts can be tolerated as a representative result. Thus, soft agreement to represent the majority concept can be conducted by specifying the appropriate semantics to the aggregation process. In the literature, there are some approaches which have been proposed recently to model the majority concept, notably, the IOWA-related models (Pasi & Yager 2006; Bordogna & Sterlacchini, 2014) and neat-OWA based models (Peláez & Doña, 2003a).

In general, the methodology for generating the consensus measure with respect to majority concept can be divided to several perspectives. First is dealing with the classes of group decision making model, either homogeneous case or heterogeneous case. For the homogeneous case, the majority can be achieved with respect to the most similar opinion of experts. On the other hand, in the case of heterogeneous group decision making, the integration of similarity of opinion and the degree of reliability is considered simultaneously. Another perspective in applying the majority concept is related to the decision scheme. Specifically, there are two different schemes of group decision model, namely, the classical scheme and the alternative scheme. Most of the GDM models rely heavily on the classical scheme, where the global judgment is referred to the final ranking of individual experts. Prior to that, the local judgment implicates the separate attitudinal character of each expert. On the contrary, for the alternative scheme, the global judgment is conducted with respect to each specific criterion of experts. Then, the final judgment is represented by the attitudinal character of the group of experts collectively. The development of MCDA models under group environment needs to be emphasized on these aspects for the holistic analysis. Then, the decision problems can be conducted for the specific purpose.

The uncertainty problems have been noticed as the main issue in the finance research over several decades. In reality, the future state of a system might not be known completely due to lack of information. Therefore, the investment analysis is often uncertain in various conditions. One way to deal with this uncertainty problem is by using the probability theory. There are a number of developments on the probability theory in the literature, such as objective, subjective and imprecise probability theories. However, it is assumed that not all uncertainty problems easily fit the probabilistic classification. The human judgment of events, for instance, may be significantly different based on individuals' subjective perceptions or personality tendencies. Thus human judgment is often vague and fuzzy. The source of uncertainty may also come from subjective attributes that urge the subjective evaluation of experts, as such, the preferences or estimation with respect to linguistic assessments is preferred than the exact assessments. For this purpose, fuzzy set theory (Zadeh, 1965) has been introduced as the alternative of probability theory to deal with this type of problem. There are many applications of fuzzy set theory in the finance domain, some of them can be referred in Terceño et al. (2003), Aliev et al. (2004), Gil-Lafuente (2005), Wang and Lee (2010) and Dymowa (2011).

As previously mentioned, the attitudinal character in OWA operators is characterized by specific semantics. The linguistic quantifiers (Zadeh, 1983) of fuzzy set theory provide tools in representing these semantics for the aggregation process. In addition, there are a lot of developments in the fuzzy linguistic variable towards computing with words or natural language. Many approaches have been proposed recently to model the linguistic information. One of them is by applying the 2-tuple linguistic representation model as proposed by Herrera and Martínez (2000a; 2000b). By using this approach, the linguistic analysis can be conducted without loss of information in the computing process. In other related work, Bordogna et al. (1997) proposed the conversion of linguistic labels to the numerical values to deal with the operations in the numerical environment. Accordingly, there are lots of developments on the linguistic aggregation operator, notably with respect to the OWA-based aggregation operators. Therefore, the inclusion of fuzzy set ideas into MCDA models need to focus on both expressions, i.e., the fuzzification of the preference information and the integration of soft aggregation processes.

1.4 Objectives

According to the aforementioned problems, in this study, some of the MCDA models, namely the general group decision making, Dempster-Shafer belief structure, AHP, and TOPSIS are proposed to overcome the shortcoming of the existing models. In order to make this research clear and consistent, the objectives of the research are defined as the followings.

i) To extend some of the MCDA models by the inclusion of attitudinal character of experts in the aggregation process. This can be done by incorporating the OWA-based aggregation operators, specifically:
   - the integration of WA and OWA for criteria aggregation,
   - generating the OWA using linguistic-functional specification (generalized quantifier) and characterizing feature (maximum entropy OWA),
   - applying the heavy OWA-based aggregation operators.

ii) To propose the majority concepts based on OWA and IOWA as the group aggregators for homogeneous and heterogeneous group decision making models. In particular:
   - majority concept based on induced OWA,
   - majority concept based on neat-OWA.

iii) To study and compare the group decision making models from two different perspectives, notably:
   - the classical scheme – aggregation of experts with respect to individual ranking of alternatives,
   - alternative scheme – aggregation of experts with respect to the specific criterion.

iv) To employ fuzzy linguistic information in representing the uncertainty of data, such as the imprecise knowledge, vague concepts and human subjective judgments (i.e., towards computing with words or natural language). In specific:
   - the 2-tuple linguistic approach,
   - the general linguistic labels.

v) To provide a complete analysis of the decision making problems by using the generalized version of OWA (GOWA) and IOWA (IGOWA). These provide a wide spectrum of results to be compared for the final decision making process.

vi) To implement or apply the developed models to the financial decision analysis and non-financial problems such as human resource selection problem and flood control project selection problem.

A list of contributions generated from this study is presented in the subsequent section.

1.5 Publications

Most ideas or contributions in this thesis have previously appeared in the following articles. Here is the list of chapters of this thesis to which they are related:

i)	OWA-based aggregation operations in multi-expert MCDM model – with José Maria Merigó Lindahl and David Ceballos Hornero. *Article accepted for publication in* Journal of Economic Computation and Economic Cybernetics Studies and Research (ECECSR), 2017. *The short version of this article has been published in* Proceedings of IMST 2015-FIM XXIV Conference on Interdisciplinary Mathematics, Statistics and Computational Techniques.	Chapter 4
ii)	Weighted selective aggregated majority-OWA operator and its application in linguistic group decision making model – with José Maria Merigó Lindahl and David Ceballos Hornero. *Article submitted to* International Journal of Intelligent Systems (IJIS), August 2016.	Chapter 4
iii)	Linguistic decision making with Dempster-Shafer theory and induced linguistic aggregation operators – with José Maria Merigó Lindahl, Montserrat Casanovas and Ligang Zhou. *Article accepted for publication in* Applied Mathematics and Information Sciences Journal (AMIS), 2017.	Chapter 4
iv)	Generalized AHP for group decision making model using induced OWA operators – with José Maria Merigó Lindahl and David Ceballos Hornero. *Article accepted for publication in* Kybernetes Journal, 2017. *The short version of this article has been published in* Lauren et al. (Eds.): IPMU 2014, Part 1 CCIS 442, pp. 476-485, 2014. (Chapter in book) Springer-Verlag.	Chapter 5

v) Heavy weighted geometric aggregation operators in AHP group decision making – with José Maria Merigó Lindahl and David Ceballos Hornero. *Article published in* Proceedings of International Conference on Fuzzy Systems, IEEE-FUZZ 2015. *The short version of this article has been published in* Book of Abstracts - International Student Conference on Applied Mathematics and Informatics (ISCAMI), April 2015. — Chapter 5

vi) TOPSIS model with the OWA-based aggregation – with José Maria Merigó Lindahl and David Ceballos Hornero. *Article submitted to* Computer and Industrial Engineering Journal, (CAIE), May 2016. *The short version of this article has been published in* Proceedings of IMST 2015-FIM XXIV Conference on Interdisciplinary Mathematics, Statistics and Computational Techniques (November 2015). — Chapter 6

In addition, the work produced from the master thesis in which generated as an output of another publication is also linked to some of the contents in the present thesis:

vii) Conflicting bifuzzy multi-attribute group decision making model with application to flood control project – with Che Mohd Imran Che Taib, Mohd Lazim Abdullah and Abdul Fatah Wahab. *Article published in* Group Decision and Negotiation (2016), 25(1), 157–180. — Chapter 6

I have been authorized by the co-authors of the stated publications, to compile all those ideas, tables and figures in the present work, and to be considered, to all effects, the author of this thesis.

1.6 Thesis Structure

This thesis is composed of seven chapters. The organization of this study is presented as the followings.

Chapter 1 – The first chapter covers research backgrounds, motivation and problem statements, objectives of the study and also publications resulted from this work.

Chapter 2 – This chapter is devoted to a general overview of decision analysis, specifically in the case of multi-criteria and multi-expert decision making problems. A summary of methods used in this study is outlined, such as general group decision making models, Dempster-Shafer belief structure, AHP and TOPSIS models. In addition, the theories for modeling uncertainty problems are also highlighted.

Chapter 3 – This chapter provides a review of some aggregation operators and their potential application in the realm of decision analysis. In particular, the emphasis is given on the OWA-based aggregation operators. The main properties, measures associated to OWA operators and also their variant of families are presented. Besides, the more general classes of aggregation operators are provided to complete this chapter, notably, fuzzy measures and monitored heavy fuzzy measures.

Next, the main contributions of the study are presented in Chapters 4, 5 and 6, respectively. With respect to the list of publications in the previous section, the specific methods and techniques applied in this study are explained.

Chapter 4 – This chapter is based on the compilation of three main articles. Firstly, the multi-expert MCDM model with the integration of WA and OWA operators is presented. Moreover, the analysis of majority aggregation operators under homogeneous and heterogeneous group decision making with respect to the classical and alternative schemes is conducted. Secondly, the neat OWA-based aggregation operators are studied, particularly for the alternative representation of majority aggregation operators. In this work, the selective aggregated majority-OWA has been extended and generalized to the case of weighted SAM-OWA and WSAM-IOWA operators. The linguistic group decision analysis then is developed based on these aggregation operators. Finally, the decision analysis based on Dempster-Shafer theory and linguistic group decision making models are demonstrated. For Dempster-Shafer belief structure model, the induced linguistic aggregation operators based on 2-tuple linguistic information and IOWA is developed. All these developed models then are applied in the case of financial selection problems.

Chapter 5 – This chapter is composed with respect to two main articles, in which devoted to the AHP model. Firstly, the induced weighted maximum entropy OWA (IWMEOWA) is proposed. Then, its generalization includes the induced generalized and induced quasi-WMEOWA are studied. In addition, the majority concept based on the IOWA operators is presented. The two-stage aggregation processes in the AHP-GDM model (i.e., the criteria and group stages) then is developed based on the aforesaid methods. Next, in the second article, the application of heavy WG and heavy OWG aggregation operators in the AHP-GDM model is considered and examined.

Chapter 6 – This chapter consist of two main articles related to the TOPSIS model. In specific, the TOPSIS with OWA-based aggregation operators is outlined in the first section. The model is applied in the human resource selection problem. In the second section, the TOPSIS model under conflicting bifuzzy condition for the selection of flood control project is presented.

Chapter 7 – Finally, the conclusions and recommendation for future research are presented in this chapter.

1.7 Summary

In summary, this chapter provides a basic guide to the direction of the overall study. This research will be driven towards achieving the objectives that have been stated and discussed earlier.

CHAPTER 2

DECISION ANALYSIS AND UNCERTAINTY THEORIES

2.1 Introduction

In this chapter, the overview of decision analysis is presented, specifically in the case of multidimensional aspects (multi-criteria and multi-experts assessments) and uncertain environments. Firstly, in Section 2.2 the general definition of decision making models is briefly outlined. Next, Section 2.3 provides a review of multi-criteria decision analysis (MCDA) as a basis for the later developments and extensions, in particular, the analytic hierarchy process (AHP) and the technique for order performance by similarity to ideal solution (TOPSIS). In Section 2.4, the decision under group environment is discussed. Further, Section 2.5 and Section 2.6 review the related uncertainty theories for modeling decision analysis, precisely, the probability theory and the fuzzy set theory, respectively. In addition, a decision analysis based on the generalization of probability theory which called as Dempster-Shafer theory of belief structure is provided. Finally, in Section 2.7 the summary is given to end this chapter.

2.2 General Definition of Decision Making Models

Decision analysis is a procedure used in dealing with complex decision making problems. In principle, it consists of a set of actions (or alternatives) to be selected, with a collection of criteria and/or a given amount of knowledge about the situation to be assessed. A number of authors have classified the decision making problems by several different ways. For instance, Luce and Raiffa (1957), French (1985) and Grabisch et al. (1995) have divided them into three main categories based on how much certainty there is about the outcomes of various actions. They are the decision making under certainty, decision making under uncertainty (or risk) and decision making under strict uncertainty (or ignorance). In general, the decision making problems can be defined in the following ways.

Definition 2.1 (Grabisch et al., 1995). A decision making problem is a 5-tuple $(A, \Theta, \phi, X, \succcurlyeq)$, where:

- A is a set of actions (or alternatives), among which the decision maker (or expert) must choose,
- X is a set of consequences or results, and these consequences come from the choice of action,
- Θ is the set of states of the world (or nature); according to the state of the world $\theta \in \Theta$ (usually unknown) and the consequences of the choice of an alternative $a \in A$ may differ (finite number of mutually exclusive states),
- φ is a map $A \times \Theta \to X$ which specifies for each state of the world θ and each alternative a the resulting consequence $x = \varphi(a, \theta)$,
- $\succcurlyeq$ is a weak preference relation on X, that is, a binary relation satisfying the properties of:
  i) Complete: $x \succcurlyeq y$ or $y \succcurlyeq x$, for $x, y \in X$, and
  ii) Transitivity: That is, $x \succcurlyeq y$ and $y \succcurlyeq z$ imply $x \succcurlyeq z$.

The notation $\succcurlyeq$ is the preference relation which characterizes the decision maker. More precisely, the ordinary preference relation on numbers, $x \succ y$ means that $x \succcurlyeq y$ satisfies, but not $y \succcurlyeq x$ (strict preference) and $x \sim y$ means that both $x \succcurlyeq y$ and $y \succcurlyeq x$ holds (indifference preference). This definition is mainly based on the utility theory, which underlies most of the classical decision making models. Basically, the idea behind the utility theory is to transform the weak preference relation $\succcurlyeq$ on X into a normal order $\geq$ on the real numbers by mean of a so-called utility function, $u: X \to \mathbb{R}$ whose the basic property is that $x \succ y$ if and only if $u(x) > u(y)$ (see, for example, Fishburn, 1970; Robert, 1985; Grabisch et al., 1995).

2.2.1 Decision making under certainty

In particular, the decision making under certainty can be given as the condition where the state of the world θ is always known. Thus φ is defined on A, such that, for each action A there is exactly one consequence X. However in practice, X is normally a multidimensional, that is, x is an *n*-tuple $(x_1, x_2, \dots, x_n), x_j \in X_j$, where the X_j are representing criteria. For each alternative a, the final decision is simply the result of linear weighted average (or weighted arithmetic mean) of the criteria, X_j (under some specific models). In the literature, this type of decision making problems is generally known as a multi-attribute value theory (MAVT) or multi-criteria decision analysis/making (MCDA/MCDM) (see Figueira et al., 2005). In this thesis, both MCDA and MCDM will be used interchangeably. Notice that, the term value theory is used to represent the decision under certainty, whilst the utility theory is meant for decision making under uncertainty as will be explained in the following.

2.2.2 Decision making under risk

The decision making under uncertainty (or risk) is a condition in which the (true) state of the world is unknown. In other words, the consequences of a decision are unpredictable. However, in this case, an uncertainty measure, such as a probability on Θ, is known. This model is rooted from the classical works of von Neumann and Morgenstern (1944), which is based on the maximum expected utility theory. The expected utility can be formalized as: $\sum_{j=1}^{n} p(\theta_j)\, u\left(\varphi(a, \theta_j)\right)$, where $p(\theta_j)$ is the probability that event θ_j occurs and $u\left(\varphi(a, \theta_j)\right)$ is the utility of consequence x_j. An action with the largest possible expected utility is preferred as the optimal decision (maximum expected utility). Note that, the von Neumann-Morgenstern model is mainly based on the objective probability theory. On the same basis, Savage (1954), proposed the maximum expected utility model by employing the subjective probability theory as an alternative approach. The details of different notions of probability theory will be presented in the succeeding section. Basically, the expected utility model was defined under the one-dimensional set of consequences. However, it is extendable to the case of multidimensional factors and in the literature this type of model is known as a multi-attribute utility theory (MAUT) (Keeney & Raiffa, 1976; Dyer, 2005).

2.2.3 Decision making under strict uncertainty

The decision making under strict uncertainty (or ignorance), in contrast to the decision making under risk, is based on an uncertainty measure with probability on Θ is unknown. In this case, there are several classical criteria (or rules) proposed to be used under this condition, such as:

- criterion of Laplace or the principle of insufficient reason – the alternative which maximizes the average utility:

$$\max_{a \in A} \frac{1}{n} \sum_{j=1}^{n} u\left(\varphi(a, \theta_j)\right),$$

- criterion of the max-min (Wald, 1950) – choose the best alternative under the assumption that the most unfavorable θ happens (known as a security level of a or pessimistic criterion of choice):

$$\max_{a \in A} \min_{\theta_j \in \Theta} u\left(\varphi(a, \theta_j)\right),$$

- criterion of the max-max (Wald, 1950) – choose the best alternative under the assumption that the most favorable θ happens (optimistic criterion of choice):

$$\max_{a \in A} \max_{\theta_j \in \Theta} u\left(\varphi(a, \theta_j)\right),$$

- criterion of Hurwicz (Hurwicz, 1951) – the criteria of the max-min (pessimist) and the max-max (optimist) are mixed in a ratio $\alpha \in [0,1]$, that is,

$$\max_{a \in A}\left(\alpha \max_{\theta_j \in \Theta} u\left(\varphi(a, \theta_j)\right) + (1-\alpha) \min_{a \in A} u\left(\varphi(a, \theta_j)\right)\right),$$

- criterion of minimax regret (Savage, 1954) – choose the best alternative with the minimum regret:

$$\min_{a_i \in A}\left\{\max_{a_i \in A}\{r_{ij}\}\right\},$$

  where $r_{ij} = \max_{a_i \in A}\left\{u\left(\varphi(a_i, \theta_j)\right)\right\} - u\left(\varphi(a_i, \theta_j)\right)$, the difference between the value resulting from the best action given that θ_j is the true state of the world and the value resulting from a_i under θ_j.

Moreover, the decision making models can be divided into two general forms of decision theory. They are the normative (or prescriptive) decision theory and the descriptive decision theory (Robert, 1985). These categories are fundamentally based on the axioms of the representation theorem. The normative interpretation looks at the axioms as the conditions of rationality. In specific, the normative statements assert about how things should be made or how to value them (judgments that satisfy the axioms). These types of models normally developed in the economic literature. For instance, the von Neumann-Morgenstern's expected utility model and the MAUT model are based on the normative approach. On the other hand, descriptive statements describe types of theories, beliefs or propositions; claim to be factual statements that attempt to describe reality. To be precise, it analyzes how decision maker actually makes decisions. This type of models is commonly established in the psychological literature. Examples of this category can be observed in most of the MCDM models. However, there are a number of models in which are developed, simultaneously based on the normative and descriptive approaches. The details on this subject can be referred to, for instance, in Robert (1985) and Peterson (2009). In the followings, the specific topics related to this study are presented.

2.3 Multi-Criteria Decision Making Categories

The MCDM as one of the well-known decision making methods can be defined as a process of ranking or evaluating alternatives with respect to multiple conflicting criteria. Others common names are multi-criteria decision analysis and multi-criteria decision aid. Based on the properties of the problem, MCDM can be characterized by several categories. Zeleny (1982), as well as Kahraman (2008) characterized MCDM into two categories which are multi-attribute decision making (MADM) and multi-objective decision making (MODM). The main distinction between these two categories is mainly based on the number of alternatives under consideration.

The MADM is referred to making a decision in the discrete decision spaces and focuses on how to select or to rank different predetermined alternatives (finite case). MADM has been developed depending on the type and the characteristic of the problem such as the Analytic Hierarchical Process (AHP), Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), Simple Additive Weighting (SAW), outranking methods like PROMETHEE and ELECTRE, etc. See Figueira et al. (2005) for the complete review of these models.

The MODM approach, in contrast to the MADM, is concentrated on continuous decision space where the realization of the best solution is done with respect to several objective functions. Solving MODM problems often involves modifying classical optimization methods for situations where multi-objectives must be satisfied. The well-known approaches in this category are based on mathematical programming models.

In this study, the main emphasis is on the MADM models. Hence, some related models, namely, AHP and TOPSIS models are briefly discussed here.

2.3.1 Analytic hierarchy process model

The AHP model as one of the MADM techniques was developed by Saaty (1980). The AHP is formulated to support the decision makers in some decision problems that are hard to conceptualize or even clearly defined. The AHP is based on the following two steps: structuring the decision as a hierarchical model and then, using a pairwise comparison of all attributes and alternatives to calculate the weight of each criterion and the score of each alternative. This approach allows decision makers to examine the complex problem in a detailed rational manner. The hierarchical representation helps in dealing with the decision problems, which are usually complex in nature. The decisions are made one level at a time, from the bottom up, to more aggregate strategic levels. The advantages of AHP include highly structured and more easily understood models and consistent decision making (or at least a measure of the level of consistency – the decision maker is always free to remain inconsistent in preferences and scores). Moreover, the AHP is one of the most popular MADM methods since it is generally perceived as being intuitive and flexible enough to help decision makers to address variety problems (Figueira et al., 2005; Brunelli, 2015).

2.3.2 Technique for order performance by similarity to ideal solution

The TOPSIS model was first proposed by Hwang and Yoon (1981) based on the concept of ideal solutions of alternatives. In specific, an ideal solution is a collection of ideal ratings for all attributes under the consideration. The composite of the best attribute ratings attainable is known as the positive ideal solution (PIS). Whilst the composite of all the worst attribute ratings attainable is the negative ideal solution (NIS). TOPSIS uses an evaluation index which measures each alternative's relative closeness to the PIS by scoring alternatives both in terms of closeness to the PIS and remoteness from the NIS. The Euclidean distance measure is used to calculate each alternative's distance from the PIS and NIS. However, it is extendable to other distance measures. Thus, TOPSIS simultaneously considers an alternative's best and worst characteristics with respect to identified decision attributes. The advantages of using this concept have been highlighted by: its intuitively appealing logic, its simplicity and comprehensibility, its computational efficiency, its ability to measure the relative performance of the alternatives with respect to individual or all evaluation criteria in a simple mathematical form, and its applicability in solving various practical MADM problems (Deng et al., 2000). This concept has been widely used and has been successfully applied to various decision contexts (Chen & Hwang, 1992; Olcer & Odabasi, 2005).

2.4 Decision under Group Decision Making

Another distinction to make aside from the uncertainty and multiple criteria problems is, either the decision analysis involves an individual or a group judgments. Clearly, the complexity of decision analysis encourages group decision as a way to combine interdisciplinary skills and improve management of the decision making process. Multi-expert decision making (MEDM) or group decision making (GDM) deals with the complex problem of identifying the most preferred alternative(s) for both individual decision makers and an aggregated group of individual decision makers (Chen & Hwang, 1992). Many times a group of experts needs to make a decision that represents the individual opinions and yet is mutually agreeable. GDM not only take into account the conflicting objectives/attributes and goals of individual decision makers, but also the conflicts that exist between the various members of a group. Decision analysis under GDM problems are often made more difficult because individual decision makers may not be able to evaluate alternatives using common criteria, depending on factors such as each individual's knowledge, experience, and the availability of data (Hwang & Lin, 1987; Jackson, 1999).

The issue of conflict between the individuals (or groups) and the solution for the collective decision has been studied by many authors (see Arrow, 1950; Luce & Raiffa, 1957; Fishburn, 1972, for the classical works on this topic). In addition, the recent developments on consensus measure in group decision making can be referred to Pasi and Yager (2006), Bordogna and Sterlacchini (2014) and Peláez and Doña (2016), among others. Note that, these latter developments of consensus measure will be studied in this study.

2.5 Probability Theory for Modeling Uncertainty in Decision Analysis

As explained in Section 2.2, the uncertainty theory traditionally used in decision analysis is based on the probability theory. In the literature, there are several categories of probability theory, namely the objective probability and subjective probability, as well as the classical notion of probability theory. The expected utility and the subjective expected utility were built based on the objective probability and the subjective probability, respectively. In addition, with respect to the subjective probability, the Bayesian approach has emerged as one of the tools for decision analysis. This model then has led to the development of the Dempster-Shafer theory of belief structure, in which, the subjective probability has been extended to the imprecise probability (interval-valued probability). This section provides a summary of probability theory and the generalization of probability to imprecise probability under the Dempster-Shafer theory.

As previously mentioned, there are three types of probability theory that have been reported in the literature. All these types of probability theory obey the general axioms of mathematical probability (or Kolmogorov's laws) as follows:

- $0 \leq P(\theta_j) \leq 1$ for every events θ_j,
- $P(\Theta) = 1$, and,
- if $\theta_j \cap \theta_k = \emptyset$ then $P(\theta_j \cup \theta_k) = P(\theta_j) + P(\theta_k)$.

In specific, these can be explained as i) the probability of any event is a nonnegative real number, ii) the assumption of unit measure: the probability of the entire sample space (or certain space) is one, and iii) if an event can occur in one of two mutually exclusive ways, then its probability is the sum of the probabilities of the mutually exclusive ways (French, 1985; Robert, 1985).

In the classical notion of probability, it is assumed that a partition of events is set as equally probable (or equally likely), e.g., a simple fair dice game. This also expressed as a principle of insufficient reason due to Laplace. Suppose that a particular event occurs a certain number times out of a total possible number of other events. The probability that the desired event will occur can be represented by $P(\theta) = n_\theta/n$, where, $P(\theta)$ is the probability of event θ occur, n_θ is the number of times event θ could occur, and n as the total number possible events. However, this classical notion has been criticized because of its restrictive condition. Practically, in most uncertain circumstances it is impossible to categorize the future into equally likely possibilities, such as the movement of the stock exchange index, etc. As in the simple dice games, this does not provide any problems, but, in the complex situation, ambiguous can arise.

Meanwhile, the objective probability is viewed as the long-run relative frequency with which a system is observed in a particular state (French, 1985). It can be explained as a likelihood of a specific occurrence, based on repeated random experiments and measurements. The probability that a particular event of the experiment will occur is given by the relative frequency as: $P(\theta) = \lim_{n\to\infty}(n_\theta/n)$. However, not all uncertainty problems can be modeled using the objective probability theory based on the relative frequency. For example, in making a decision about alternative sources of energy, the probabilities that various significant events will take place are not known exactly. Similarly, in the case of betting a horse race where the probability of a particular horse will win is not known. Most of the time, the decision makers have some ideas of how probable different outcomes are, or at least that one outcome is more probable than another. Hence, the degree of belief of decision maker is more appropriate in this case which can be modeled by the subjective probability.

The subjective probability is based on the experience or personal belief of decision maker, rather than theoretical or experimental work. This experience is then used to predict the probability of future events. In specific, $P(\theta)$ represents the decision maker's degree of belief that state θ will occur, such that, the stronger his belief, the greater is $P(\theta)$. In this case, for different observers, the different probabilities may assigned to the same event. Probability is therefore personal; it belongs to the observer or subjective. It can be interpreted as quantifying a personal degree of belief. Foundation works on this subject are due to de Finetti (1937), Ramsey (1931) and Savage (1954). For the detailed discussion on the objective probability and the subjective probability, they can be referred in Ramírez (1988) and Howson and Urbach (2006).

2.5.1 Dempster-Shafer theory of belief function

The Dempster-Shafer theory or evidence theory was first developed by Dempster (1967) on the concept of upper and lower probabilities and then extended by Shafer (1976) on belief functions. These works have laid the foundation for a new theory of probabilistic reasoning based on the generalization of classical probability. Central to this theory is its ability to model imprecision and also randomness. Hence, this often makes the D-S theory superior to Bayesian approaches in modeling knowledge of uncertainty problems. Since then, the applications of the model have been evolved, specifically in decision analysis (see Merigo & Casanovas, 2009; Yager & Alajlan, 2015). Review on the D-S theory can be referred in Yager and Liu (2008).

The D-S model can be explained as the following. A D-S belief structure m on the space X is defined via a collection of non-empty subsets of X, focal elements $B_1, B_2, \dots, B_n$, and a mapping $m(B_j) \in [0,1]$ such that $m(B_j) > 0$ and $\sum_{j=1}^{n} m(B_j) = 1$. Here $m(B_j)$ (a mass function or basic probability assignment) indicates an amount of probability allocated to the elements in B_j in some unknown matter. The D-S belief structure can be viewed as a probability distribution with imprecise probabilities. For example, the probability of x_j can be represented as a range of $[a_j, b_j]$ instead of p_j. Therefore, D-S can be viewed as a piece of information that contains two types of uncertainty, probabilistic, randomness, and imprecision in the parameters associated with the probability distribution (Yager & Alajlan, 2015). A belief structure has the ability to represent in a unified way many different types of information about this variable. Two important measures can be associated with a belief function m. These are called the measures of plausibility and belief (Shafer, 1976). Assume A is a subset of X then:

$$\mathrm{Pl}(A) = \sum_{j, A \cap B_j \neq \emptyset} m(B_j) \quad \text{(Plausibility), and}$$
$$\mathrm{Bel}(A) = \sum_{j, B_j \subseteq A} m(B_j) \quad \text{(Belief).}$$

It is known that $\mathrm{Pl}(A) \geq \mathrm{Bel}(A)$ for all A. And it is also known that $\mathrm{Pl}(X) = \mathrm{Bel}(X) = 1$ and $\mathrm{Pl}(\emptyset) = \mathrm{Bel}(\emptyset) = 0$. These measures are also monotonic, if $A \subset B$ then $\mathrm{Pl}(A) \leq \mathrm{Pl}(B)$ and $\mathrm{Bel}(A) \leq \mathrm{Bel}(B)$. Note that, Dempster (1967) referred to these as the upper and lower probabilities of A. In this perspective, $\mathrm{Bel}(A) \leq \mathrm{Prob}(A) \leq \mathrm{Pl}(A)$. Hence, it can be seen that a belief structure provides information about the probability of a set in an imprecise manner by giving the interval in which the probability lies.

2.6 Fuzzy Set Theory for Modelling Uncertainty in Decision Analysis

Fuzzy set theory was introduced by Zadeh in 1965. It was intended to improve the mathematical model, by developing a more stable and flexible model in order to solve real-world complex problems, i.e., involving human aspects. The theory of fuzzy set, unlike probability theory, represents imprecision by the fact that the certain objects (or certain classes of objects) have poorly or ill-defined boundaries. It is assumed that not all uncertainties easily fit the probabilistic classification (Bender & Simonovic, 2000). Fuzzy set theory and probabilistic approach can be considered as complement to each other, in which fuzzy set is used as an alternative way to deal with the uncertainty and imprecision that cannot be modeled by the probabilistic approach. In fuzzy set theory, the key elements in human thinking are not numbers but the labels of fuzzy sets (Zadeh, 1975). This condition makes the fuzzy set a powerful tool to handle imprecise data and fuzzy expressions that are more natural for humans than rigid mathematical rules and equations (Olcer & Odabasi, 2005). Since the effectiveness of fuzzy set theory in modeling imprecision, it has been applied in almost every field, including decision making.

Note that, the fuzzy set theory is an extension of the classical set. Traditional mathematics and logic assign a membership of '1' to items which are members of a set, and '0' to those which are not. In another way, it can be explained as objects either belong to or do not belong to a certain class. The type of a function that describes this is called a characteristic function. This is the dichotomy principle such a strong principle inevitably ran into philosophical problems. The definition of the classical set or crisp set can be given as follows:

Definition 2.2 (Dubois & Prade, 2000). Let X be a classical set of objects, called the universe, whose generic elements are denoted x. Membership in a classical subset A of X is often viewed as a characteristic function $\mu_A(x)$ from X to $\{0,1\}$ such that:

$$\mu_A(x) = \begin{cases} 1, iff\ x \in A, \\ 0, iff\ x \notin A, \end{cases}$$

where $\{0,1\}$ is called a valuation set. It should be noted that 'iff' is short for 'if and only if'.

Since the limitation condition of crisp set in modeling the real life problems, then, the fuzzy set theory was introduced as an extension of the classical set. Fuzzy set theory offers a logic which closely imitates the human thought process by allowing for possibilistic reasoning and vagueness. It allows a proposition to be neither fully true, nor fully false, but partly true and partly false to a given degree. Zadeh (1965) introduced fuzzy set to deal with uncertainty, vague or imprecise in concepts. The fuzzy set definition can be given as the followings.

Definition 2.3 (Zadeh, 1965). Let a set X be non-empty and finite. A fuzzy set A on X is an expression given as follows:

$$A = \{\langle x, \mu_A(x)\rangle | x \in X\},$$

where $\mu_A: X \to [0,1]$ is the membership function of the fuzzy set A, $x \in X$ in A.

Based on the fuzzy set definition, the closer the membership $\mu_A(x)$ is to '1', the more x belongs to A. Other properties fuzzy set can be referred to Dubois and Prade (1980).

2.6.1 Linguistic variables

The concept of linguistic variable is very useful in dealing with situations, which are too complex or ill-defined to be reasonably described in conventional quantitative expressions (Zadeh, 1975). Since its inception, the linguistic variables have been used extensively in the decision making process (see, for instance, Herrera et al., 1995; Herrera et al., 2008). According to Zadeh (1975), linguistic variables can be described as variables whose values are not numbers but words or sentences in a natural or artificial language; and these values of linguistic variables are called linguistic labels or linguistic terms. Each linguistic term is presented by fuzzy number. The definition of the linguistic variable can be given as the following.

Definition 2.4 (Zadeh, 1975). A linguistic variable is characterized by a quintuple (x, $T(x)$, U, G, M) in which x is the name of the variable; $T(x)$ denotes the term set of x, i.e., the set of names of linguistic values of x, which each value being a fuzzy variable denoted generically by x and ranging across a universe of discourse U which is associated with the base variable u; G is a syntactic rule (which usually takes the form of a grammar) for generating the names of values of x; and M is a semantic rule for associating its meaning with each x, $M(x)$, which is a fuzzy subset of U.

There are many extensions of the linguistic variables have been proposed in the literature. For example, Herrera and Martinez (2000) proposed the 2-tuple fuzzy linguistic representation model for computing with words, unbalanced fuzzy linguistic information (Herrera-Viedma & López-Herrera, 2007), linguistic aggregation operators (Bordogna et al., 1997; Merigó et al., 2010), multi-granular fuzzy linguistic information (Herrera et al., 2000), etc. Reviews on the latest development in fuzzy linguistic information can be referred to Xu (2012). In this study, the 2-tuple fuzzy linguistic representation (Herrera & Martinez, 2000) and the linguistic aggregation technique (Bordogna et al., 1997) will be implemented.

2.6.2 Fuzzy multi-criteria decision making

Much of the decision making problems take place in an environment in which the goals, the constraints and the consequences of possible actions are not known precisely (Bellman & Zadeh, 1970). In the preceding section, the MCDM or specifically the TOPSIS and the AHP models, are mainly based on the crisp data. The performance ratings and weights of the attributes, in specific, are given as crisp values as priori. However, under many conditions, crisp data are inadequate or insufficient to model real life decision problems. Since human judgments as preferences are always uncertain or imprecise in nature, then, it may not be appropriate to represent them by accurate numerical values. A more realistic approach could be to use linguistic variables to model human judgments that is, to suppose that the ratings and weights of the attributes in the decision making problem are assessed by means of linguistic variables (Zadeh, 1975). Hence, one of the most important aspects for a useful decision making model is to provide an ability to handle imprecise and vague information by fuzzy concepts. In addition, the theory is also extended and generalized by means of the theories of triangular norms and co-norms, and aggregation operators (see, for example, Dubois & Prade, 1988; Klir & Folder, 1988).

2.7 Summary

In this chapter, the general definition of decision making models has been given. This definition provides an overview of the main decision making problems in the literature, such as decision under certainty, risk and strict uncertainty. Afterward, the class of MCDM techniques is reviewed and the specific models namely, the AHP and the TOPSIS are presented. The general theories in modeling uncertainty then are given, such as the probability theory and the fuzzy set theory. The probabilistic model of Dempster-Shafer belief structure is outlined as one of the decision analysis approaches. In this study, the AHP, TOPSIS, D-S theory and together with the general GDM models will be extended using the fuzzy set concept. Besides, they are extended using the OWA-based aggregation operators that will be outlined in the next chapter. The application of fuzzy set theory in this study is demonstrated by two forms. The first one is the application of fuzzy linguistic variables or fuzzy numbers to represent subjective judgments or uncertain information. Then, the second one is by using the concept of linguistic variables for information fusion under the OWA-based aggregation operators. These aggregation operators can be explained as an extension of the classical WA, with several advantages. Among them are the inclusion of attitudinal character (or behavior) of expert(s) in aggregating the inputs, a flexibility for soft consensus (majority concept) of experts in group decision making models and analysis of complex decision making problems using order-inducing variables.

CHAPTER 3

AGGREGATION OPERATORS

3.1 Introduction

In this section, a review of some aggregation operators is provided, mainly on the OWA-based aggregation operators. For the scope of this study, the main consideration is focused on the means (or average) based on discrete space. Firstly, in Section 3.2, the classical mean-type aggregation operators are presented. Then, the OWA and its variants are provided in Section 3.3. The properties, characteristic measures, and families of OWA operators are discussed. In Section 3.4, a brief of more general classes of aggregation operators and their relation with OWA operators is given. Specifically, fuzzy measures and monitored heavy fuzzy measures with respect to the Choquet integrals are presented.

3.2 Classical Mean Type Aggregation Operators

Aggregation operators (or aggregation functions) have been extensively employed in decision analysis as a way to combine inputs or arguments of criteria (or experts) into a single representative value (Grabisch et al., 2009). By definition, assume that $\overline{\mathbb{R}} = [-\infty, +\infty]$ (or $\mathbb{R} \cup \{-\infty, +\infty\}$) is the extended real line and $\mathbb{I}$ be any type of non-empty real interval in $\overline{\mathbb{R}}$, bounded or not. Also, suppose that $n \in \mathbb{N}$ be any non-zero natural integer which represent the arity of aggregation function, such as, vector $\mathcal{A} = (a_1, a_2, \dots, a_n)$. The general aggregation operator can be defined as the following.

Definition 3.1 (Grabisch et al., 2009). An aggregation oprator in $\mathbb{I}^n$ is a function $F^{(n)}: \mathbb{I}^n \to \mathbb{I}$, that is non-decreasing in each variable and fulfills the boundary conditions, $\inf_{\mathcal{A}\in\mathbb{I}^n} (\mathcal{A}) = \inf \mathbb{I}$, $\sup_{\mathcal{A}\in\mathbb{I}^n} (\mathcal{A}) = \sup \mathbb{I}$ and $F^{(1)}(a) = a$, $\forall a \in \mathbb{I}$.

For instance, as in the case of $\mathbb{I} = [0,1]$, the boundary conditions are given as $F^{(n)}(0, \dots ,0) = 0$ and $F^{(n)}(1, \dots ,1) = 1$. Note that, the aggregation function can simply be expressed as F instead of $F^{(n)}$ when no confusion appear. The most basic aggregation operator that traditionally used in the literature is the arithmetic mean. Based on that, the generalized version called the quasi-arithmetic mean has been proposed which provide a unified version of mean type aggregation functions (see, for example, Aczél, 1948; Bullen, 2003; Grabisch et al., 2009). The definition of quasi-arithmetic mean is given as the following.

Definition 3.2 (Grabisch et al., 2009). Let $g: \mathbb{I} \to \mathbb{R}$ be a continuous and strictly monotonic function. The n-ary quasi-arithmetic mean generated by g is the function $F_g: \mathbb{I}^n \to \mathbb{I}$ defined as:

$$F_g(a_1, a_2, \dots, a_n) = g^{-1}\left(\frac{1}{n}\sum_{j=1}^{n} g(a_j)\right).$$

The function g is called generator of $F_g(a_1, a_2, \dots, a_n)$ and is determined up to a linear transformation (Aczél, 1948). In addition, it fulfills symmetric and idempotent properties. Examples of some basic aggregation functions are presented in Table 3.1.

Table 3.1. Examples of quasi-arithmetic means

$g(a)$	$F_g(\mathcal{A})$	Name
a	$\frac{1}{n}\sum_{j=1}^{n} a_j$	Arithmetic mean
a^2	$\left(\frac{1}{n}\sum_{j=1}^{n} a_j^2\right)^{1/2}$	Quadratic mean
$\log a$	$\left(\prod_{j=1}^{n} a_j\right)^{1/n}$	Geometric mean
a^{-1}	$\frac{1}{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{a_j}}$	Harmonic mean
$a^{\lambda}(\lambda \in \mathbb{R} \setminus \{0\})$	$\left(\frac{1}{n}\sum_{j=1}^{n} a_i^{\lambda}\right)^{1/\lambda}$	Root-mean-power
$e^{\lambda a}(\lambda \in \mathbb{R} \setminus \{0\})$	$\frac{1}{\lambda}\ln\left(\left(\frac{1}{n}\sum_{j=1}^{n} e^{\lambda a_j}\right)\right)$	Exponential mean

Source: Grabisch et al. (2009)

Afterward, the quasi-arithmetic means have been extended to the concept of weighted quasi-arithmetic means (also called as quasi-linear means). Its definition is given as follows.

Definition 3.3 (Grabisch et al., 2009). The function $F: \mathbb{I} \to \mathbb{R}$ is continuous, strictly increasing, idempotent and bounded if and only if there exists a continuous and strictly monotonic function $g: \mathbb{I} \to \mathbb{R}$ and real numbers $w_1, w_2, \ldots, w_n > 0$ satisfying $\sum_{j=1}^{n} w_j = 1$ such that:

$$F(a_1, a_2, \ldots, a_n) = g^{-1}\left(\sum_{j=1}^{n} w_j\, g(a_j)\right),$$

for all $a \in \mathbb{I}^n$.

However, as mentioned in Grabisch (1996), they are not stable under linear transformation (except for $g = \mathrm{Id}$, identity function). As can be noticed, the weighted arithmetic mean (WA) is simply a weighted quasi-arithmetic mean for which the generator g is the identity function. Table 3.2 provides some examples of weighted quasi-arithmetic means.

Table 3.2. Examples of weighted quasi-arithmetic means

$g(a)$	$F_g(\mathcal{A})$	Name
a	$\sum_{j=1}^{n} w_j a_j$	Weighted arithmetic mean
a^2	$\left(\sum_{j=1}^{n} w_j a_j^2\right)^{1/2}$	Weighted quadratic mean
$\log a$	$\left(\prod_{j=1}^{n} a_j\right)^{w_j}$	Weighted geometric mean
$a^\lambda (\lambda \in \mathbb{R} \setminus \{0\})$	$\left(\sum_{j=1}^{n} w_j a_j^\lambda\right)^{1/\lambda}$	Weighted root-mean power

Source: Grabisch et al. (2009)

In the context of decision analysis, the WA is fundamental in most of the decision making models, see for instance, in MAUT, AHP and TOPSIS models. In general, it is based on the association of degrees of importance (also called as relative importances, weights or priorities) with the criteria, so that its influence on overall aggregation process is considered. In addition, the ordered weighted average (OWA) operator is a different type of aggregation fusions that allowing the relationship (or structure) between the criteria as perceived by decision maker. In fact, this operator is taken into account the attitudinal character (also called behavior or tolerant) of decision maker for the overall decision. The OWA operator and its variant will be explained in the next sub-section.

3.3 Ordered Weighted Average Operators

The OWA operator was first introduced by Yager (1988) mainly for the application in multiple criteria decision analysis. In general, it provides a parameterized family of aggregation operators that include two extreme cases: the maximum and the minimum, and also the average criteria (arithmetic mean) as a special case. It can be defined as the following.

Definition 3.4 (Yager, 1988). An OWA operator of dimension n is a function $F_W: \mathbb{R}^n \to \mathbb{R}$ that has an associated weighting vector $W = (w_1, w_2 \dots, w_n)$ such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, given by:

$$F_W(a_1, a_2, \dots, a_n) = \sum\nolimits_{j=1}^{n} w_j \, a_{\sigma(j)},$$

where $a_{\sigma(j)}$ is the argument value a_j being ordered in non-increasing order, $a_{\sigma(1)} \geq a_{\sigma(2)} \geq \cdots \geq a_{\sigma(n)}$.

The prominent characteristic of the OWA operator is that the arguments are reordered based on their values or magnitudes, and the weights are associated with the ordered positions of the arguments rather than with a specific argument as in the case of WA.

It can be seen that different OWA operators are distinguished by their weight function. Three important special cases of OWA aggregation are as the following:

- F^*, in this case $W = W^* = [1,0,\dots,0]^{\mathrm{T}}$,
- F_*, in this case $W = W_* = [0,0,\dots,1]^{\mathrm{T}}$,
- F_{Ave}, in this case $W = W_{Ave} = [1/n, 1/n, \dots, 1/n]^{\mathrm{T}}$.

Moreover, from a generalized perspective of the reordering step, OWA can be distinguished between descending OWA (DOWA) and ascending OWA (AOWA) (Yager, 1993). Note that the weighting vectors are related by $w_j = w^*_{n+1-j}$ where w_j is the *j*th weight of the DOWA (or OWA) operator and w^*_{n+1-j} the *j*th weight of the AOWA operator.

In the similar way, the OWA operator can be generalized to the quasi-OWA operator as the following definition.

Definition 3.5 (Grabisch et al., 2009). A Quasi-OWA operator of dimension n is a mapping $F_{W,g}: \mathbb{I}^n \rightarrow \mathbb{R}$ that has an associated weighting vector W of dimension n such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, defined by:

$$F_{W,g}(a_1, a_2, \dots, a_n) = g^{-1}\left(\sum_{j=1}^{n} w_j \, g\left(a_{\sigma(j)}\right)\right),$$

where the generator $g: \mathbb{I} \rightarrow \mathbb{R}$ is a continuous strictly monotonic function and $a_{\sigma(j)}$ is the jth largest argument a_j.

Formerly, Yager (2004a) proposed the generalized OWA (GOWA) operator focuses only on the ordered weighted root-mean-power, where $g\left(a_{\sigma(j)}\right) = \left(a_{\sigma(j)}\right)^{\lambda}$ with ($\lambda \in \mathbb{R} \setminus \{0\}$).

3.3.1 Properties of OWA operators

In the following, the fundamental properties of OWA operators are presented.

Property 1 (Commutative). The OWA operator is commutative (or also called symmetry, neutrality or anonymity) where the indexing of the arguments is irrelevant. Let $\langle a_1, a_2, \dots, a_n \rangle$ be a bag of arguments and let $\langle a'_1, a_2', \dots, a'_n \rangle$ be a permutation of $a_j, \forall j = 1,2,\dots,n$. Then for any OWA operator,

$$F(a_1, a_2, \dots, a_n) = F(a'_1, a_2', \dots, a'_n).$$

Property 2 (Monotonicity). Assume that a_j and c_j are a collection of arguments, $j = 1,2, \dots, n$ such that $\forall j, a_j \geq c_j$. Then,

$$F(a_1, a_2, \dots, a_n) \geq F(c_1, c_2, \dots, c_n),$$

where F is some fixed weight OWA operator.

Property 3 (Idempotency). Third property associated with these operators is idempotency. If $\forall j, a_j = a$, then any OWA operator,

$$F(a, a, \dots, a) = a.$$

Property 4 (Boundedness). For any OWA operator F

$$\underset{j}{\text{Min}}[a_j] \leq F(a_1, a_2, \dots, a_n) \leq \underset{j}{\text{Max}}[a_j].$$

All these properties imply that the OWA operator is a class of mean type aggregation operator. By adjusting the weights of the weighting vector W, from the min (logical *and*) to the max (logical *or*), then a full spectrum of mean type aggregation operators can be derived. For example, in the decision analysis under certainty or uncertainty, the aggregation from one extreme of requiring '*all* the criteria' to the other extreme of requiring '*at least one* criterion' to be satisfied can be determined. Analogously, for the decision under strict uncertainty it provides a unified framework with different decision strategies, such as maximax (optimistic), maximin (pessimistic), equally likely (Laplace), and the Hurwicz procedure, where each is characterized by a specific OWA weighting vector. Thus, these processes delineate the attitudinal character of decision maker, either toward pessimistic or optimistic decision making.

3.3.2 Measures associated to OWA operators

Another important topic in the aggregation operators is the characteristic measure. There are various measures associated with OWA operators that can be found in the literature and the main ones are briefly explained here. A summary of these measures, their analytical expressions and references are provided in Table 3.3.

Table 3.3. Summary of measures associated with OWA operators

Measure	Analytical expression	Reference
Degree of orness	$\alpha(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right)$	Yager (1988)
Dispersion (Shannon entropy)	$Disp(W) = -\sum_{j=1}^{n} w_j \ln(w_j)$	Yager (1988)
Degree of balance	$Bal(W) = \sum_{j=1}^{n} w_j \frac{(n+1-2j)}{n-1}$	Yager (1996)
Divergence	$Div(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W)\right)^2$	Yager (2002)
Variance measure	$D^2(W) = \frac{1}{n}\sum_{j=1}^{n} w_j^2 - \frac{1}{n^2}$	Fuller and Majlender (2003)
Renyi entropy ($\alpha \neq 1$)	$H_\alpha(W) = \frac{1}{1-\alpha}\log_2\left(\sum_{j=1}^{n} w_j^\alpha\right)$	Majlender (2005)

Source: Belles-Sampera et al. (2014)

- *Degree of orness:* The degree of orness is a measure of attitudinal character of decision maker. It can be shown that $\alpha \in [0,1]$. In general, the more of the weight is concentrated near the top of W, the closer α approaches 1. On the other hand, the more of the weight is concentrated toward the bottom of W, the closer α approaches 0. Note that the tolerant (or optimistic) decision maker can be demonstrated as $\alpha(W^*) = 1$ and intolerant (or pessimistic) decision maker as $\alpha(W_*) = 0$. Whilst $\alpha(W_{Ave}) = 0.5$ as a neutral decision maker. Moreover, the measure of andness as a dual of orness measure can be defined as: $andness(W) = \bar{\alpha}(W) = 1 - \alpha(W)$.
- *Dispersion:* The dispersion is a measure of entropy. It is a well-known concept introduced by Shannon (1949) based on the information theory. Generally, it used to measure the amount of information given by a

vector of arguments that is used in the aggregation. In a certain sense, the more disperse the weighting vector W, the more of the information about the individual criteria is being used in the aggregation. For instance, if $w_j = 1/n$ for all j, then $Disp(W) = \ln n$, and the amount of information employed is maximum. If $w_j = 1$ for some j, then $Disp(W) = 0$, and the least amount of information is used.

- *Degree of balance:* Another measure used for the analysis of the weighting vector W is called the balance measure. It used to analyze the balance between favouring the arguments with high values or the arguments with low values. It can be shown that $Bal(W) \in [-1,1]$. To measure the degree of balance between favoring the higher-valued elements or lower-valued elements the degree of balance is introduced, where $Bal(W) = 1$ represents an optimist criteria, $Bal(W) = -1$ a pessimist criteria and $Bal(W) = 0$ is the Laplace criteria or arithmetic mean.

- *Divergence:* The divergence measure is useful in the case where the degree of orness and the dispersion measure are not enough to analyze the weighting vector W of an aggregation. For instance, let $n = 7$ be the number of inputs to aggregate with $W = (0.5, 0, 0, 0, 0, 0, 0.5)$ and $W' = (0, 0, 0.5, 0, 0.5, 0, 0)$ be the associated weighting vectors. In this case $\alpha(W) = \alpha(W') = 0.5$ and $Disp(W) = Disp(W') = \ln(2)$, then no significant information can be extracted from these measures. However, as can be noticed $Div(W) = 0.25$ and $Div(W') = 0.027$. Hence, the divergence measure provide a useful information in distinguishing between these weighting vectors.

- *Variance:* The variance measure is used to compute the variance of the weighting vector W where each input is considered equally probable. In general, it is used to determine the analytical expression of a minimum variability of OWA aggregation operator.

- *Renyi entropy:* Another characterization of OWA operator is the Rényi entropy (Rényi, 1961) as proposed by Majlender (2005). In specific, it is an extension or a generalization of the Shannon entropy measure. For any aggregation, $H_\alpha(W)$ can be considered as the Rényi entropy of degree α. Whenever $\lim_{\alpha \to 1} H_\alpha(W)$, then, it reduces to the Shannon entropy $H_S(W) = -\sum_{j=1}^{n} w_j \log_2(w_j)$ as can be proved by using l'Hôpital rule.

3.3.3 Families of OWA operators

An interesting feature of OWA operator is that it provides a parameterized family of aggregation operators between the maximum and the minimum. These families can be obtained by selecting a different manifestation in the weighting vector. In the following, a summary of some families of OWA operators is presented. More detail on this topic can be referred in Yager (1993).

- *Step-OWA:* The step-OWA (or order statistics) sets $w_k = 1$ and $w_j = 0, \forall j \neq k$. Note that, if $k = 1$, the step-OWA is transformed to the maximum operator, and it becomes the minimum operator if $k = n$.
- *Window-OWA:* The window OWA can be obtained when $w_j = 1/m$ for $k \leq j < k + m$ and $w_j = 0$ for $j \geq k + m$ and $j < k$. Note that k and m must be positive integers such that $k + m - 1 \leq n$, where n is the cardinality of the OWA aggregation.
- *Olympic-OWA:* The Olympic-OWA is generated when $w_1 = w_n = 0$, and for all others, $w_{j*} = 1/(n-2)$. The general form of the Olympic-OWA can be given as $w_{j*} = 1/(n-2k)$, where $k < n/2$, such that $w_j = 0$ for $j = 1,2,\dots,k,n,n-1,\dots,n-k+1$. Note that if $k = 1$, then this general form is reduced to the usual Olympic-OWA. If $k = (n-1)/2$, then this general form becomes the median-OWA aggregation. That is, if n is odd, then $w_{(n+1)/2} = 1$ is assigned, and $w_{j*} = 0$ for all other values. If n is even, then $w_{n/2} = w_{(n/2)+1} = 0.5$ are assigned and $w_{j*} = 0$ for all other values. In the similar way, the general form of Olympic-OWA can be derived for the case of $w_j = (1/2k)$.
- *S-OWA:* Another interesting family is the S-OWA. It can be subdivided into three classes: the 'or-like', the 'and-like' and the generalized S-OWA operators. The generalized S-OWA operator is obtained if $w_1 = (1/n)\big(1-(\alpha+\beta)\big)+\alpha$, $w_n = (1/n)\big(1-(\alpha+\beta)\big)+\beta$, and $w_j = (1/n)\big(1-(\alpha+\beta)\big)$ for $j = 2$ to $n-1$, where $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 1$. The generalized S-OWA operator becomes the 'and-like' S-OWA if $\alpha = 0$, and it becomes the 'or-like' S-OWA if $\beta = 0$. Also note that if $\alpha + \beta = 1$, then it is the Hurwicz criteria.
- *Centered OWA:* An OWA operator is defined as a centered aggregation operator (Yager, 2007) if it is symmetric, strongly decaying and inclusive. It is symmetric if $w_j = w_{j+n-1}$. It is strongly decaying when $i < j \leq (n+1)/2$, then $w_i < w_j$ and when $i > j \geq (n+1)/2$, then $w_i > w_j$.

It is inclusive if all the $w_j > 0$. Note that, it is possible to consider softening of the second condition by using $w_i \leq w_j$. A special type of centered OWA operator is the Gaussian-OWA (Yager, 2007), constructed by analogy to the Gaussion-OWA weights suggested by Xu (2005).

- *Neat OWA:* The neat OWA (or aggregate dependent weights) can be defined as $w_j = a_{\sigma(j)}^{\gamma} / \sum_{j=1}^{n} a_{\sigma(j)}^{\gamma}$, where $\gamma \geq 0$, and $a_{\sigma(j)}$ are the arguments a_j ordered in decreasing order. Note that, $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$. In specific, this function is known as BADD-OWA (basic defuzzification distribution). Similarly, the neat OWA can be represented as $w_j = \left(1 - a_{\sigma(j)}^{\gamma}\right) / \sum_{j=1}^{n}\left(1 - a_{\sigma(j)}^{\gamma}\right)$ and $w_j = \left(1/a_{\sigma(j)}^{\gamma}\right) / \sum_{j=1}^{n}\left(1/a_{\sigma(j)}^{\gamma}\right)$.

- *Monotone quantifiers:* Another useful approach to obtain the weights is by using the monotone quantifiers. This family of OWA operator can be summarized as follows. Let Q be a function $Q: \mathbb{I} \rightarrow \mathbb{I}$, where $\mathbb{I} = [0,1]$, such that $Q(0) = 0$, $Q(1) = 1$ and $Q(x) \geq Q(y)$ *if* $x > y$. This function is known as regular increasing monotone (RIM) quantifier (or also called basic unit interval monotonic (BUM) function). Based on RIM, the weights w_j, for $j = 1,2, \dots, n$ can be given as $w_j = Q(j/n) - Q((j-1)/n)$. Alternatively, the non-monotone quantifiers can be implemented which is based on the regular decreasing monotone quantifiers (see, Yager, 1993).

- *Maximum entropy-OWA:* The MEOWA (O'Hagan, 1988) can be achieved by solving the mathematical programming problem, which is maximize the dispersion, $Disp(W)$, with respect to the degree of orness, $\alpha(W)$ and the weights' constraints, $\sum_{j=1}^{n} w_j = 1$ with $w_j \in [0,1]$. The procedure is done by, first, select a desired value of orness (optimism or maxness), then, find the weights that maximize the dispersion.

3.4 Fuzzy Measures and Choquet Integral

In the preceding sections, the WA and OWA operators have been presented for the aggregations in the decision analysis. In this section, the more general classes of aggregation operators and their relation with the WA and OWA operators are discussed. Specifically, the concepts of fuzzy measure and monitored heavy fuzzy (MHF) measure under the Choquet integral (Choquet, 1953) are presented.

Basically, fuzzy measure theory is an extension of the classical measure theory in which the additive property is substituted by the weaker property of monotonicity. Other common names of fuzzy measure are capacities, non-additive measure and monotone measure. The definition of fuzzy measure can be given as the following. Let $X = (x_1, x_2, \dots, x_n)$ be the set of criteria and $\mathcal{P}(X)$ the power set of X, i.e., the set of all subsets of X.

Definition 3.6 (Sugeno, 1977). A fuzzy measure on the set X of criteria is a set function $\mu: \mathcal{P}(X) \rightarrow [0,1]$ satisfying the following axioms:

(i) $\mu(\emptyset) = 0, \mu(X) = 1$(boundary condition),
(ii) $\mu(S) \leq \mu(T)$ implies $S \subseteq T \subseteq X$ (monotonicity condition).

In this context, $\mu(S)$ represents the weight of importance of the set of criteria S. Hence, in addition to the usual weights on criteria taken separately, weights on any combination or subset of criteria are also defined. Some interesting cases of fuzzy measure can be given as follows:

- Additive property: A fuzzy measure is said to be additive if $\mu(S \cup T) = \mu(S) + \mu(T)$ whenever $S \cap T = \emptyset$,
- Non-additive property: It is said to be superadditive if $\mu(S \cup T) \geq \mu(S) + \mu(T)$ and subadditive $\mu(S \cup T) \leq \mu(S) + \mu(T)$ whenever $S \cap T = \emptyset$,
- Moreover, a fuzzy measure is said to be cardinal or symmetric if it depends only on the cardinality of sets, i.e., $\mu(S) = \mu(T)$ whenever $|S| = |T|$.

The fuzzy measure can be implemented in the Choquet integral as the general aggregation operators. Given that, $\mathcal{A} = (a_1, a_2, \dots, a_n)$ as the vector of arguments for the set of criteria X, the Choquet integral as an aggregation function over $\mathbb{I}^n$ can be defined as the following.

Definition 3.7 (Sugeno, 1977). The discrete Choquet integral with respect to a fuzzy measure μ is given by:

$$C_\mu(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} \left[a_{\tau(j)} - a_{\tau(j-1)}\right] \mu\left(S_{\tau(j)}\right),$$

where $\tau(j)$ is permutation of j elements such that $a_{\tau(1)} \leq a_{\tau(2)} \leq \cdots \leq a_{\tau(n)}$, with the convention $a_{\tau(0)} = 0$ and $S_{\tau(j)} = \left\{a_{\tau(j)}, \dots, a_{\tau(n)}\right\}$.

The equivalent formula can be given as:

$$C_\mu(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} a_{\tau(j)}\left[\mu\left(S_{\tau(j)}\right) - \mu\left(S_{\tau(j+1)}\right)\right],$$

where $a_{\tau(j)}$ is the argument value a_j being ordered in non-decreasing order, $a_{\tau(1)} \leq a_{\tau(2)} \leq \cdots \leq a_{\tau(n)}$ and $A_{\tau(n+1)} = \emptyset$.

The Choquet integrals are idempotent, continuous, monotonically non-decreasing operators and bounded (Grabisch, 2009). It can be demonstrated that, the Choquet integral is the general form of the WA and the OWA operator as given in the following:

- If the fuzzy measure is additive, then the Choquet integral reduces to a weighted arithmetic mean:

$$C_\mu(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} \mu\left(\{a_j\}\right)a_j.$$

- If the fuzzy measure is symmetric, then the Choquet integral reduces to an OWA operator:

$$C_\mu(a_1, a_2, \dots, a_n) = \sum\nolimits_{j=1}^{n} w_j\, a_{\tau(j)},$$

with weights w_j defined by $w_j = \mu\left(S_{n-j+1}\right) - \mu\left(S_{n-j}\right),\ j = 1,2, \dots, n$, where S_j denotes any set of j elements. Note that, the permutation with respect to non-decreasing order $a_{\tau(j)}$ is used here instead of non-increasing order $a_{\sigma(j)}$ as in the original OWA. However, it can be manipulated directly to the non-increasing order $a_{\sigma(j)}$, whenever $w_j = \mu\left(S_j\right) - \mu\left(S_{j-1}\right)$.

A related function to the Choquet integral is the Sugeno integral (Sugeno 1974), which similarly, is defined with respect to a fuzzy measure. Sugeno integrals are often used for ordinal data as they are able to operate on finite ordinal scales, whilst Choquet integral suitable for the cardinal scales. A complete review on these type of aggregation operators under discrete and continuous spaces, can be referred, for instance in Beliakov et al. (2007). Torra and Narukawa (2009) and Grabisch et al. (2009).

3.5 Monitored Heavy Fuzzy Measures and Choquet Integral

A MHF measure has been proposed by Yager (2003a) as an extension or a generalization of fuzzy measure. In specific, under the Choquet integral it allows a unified representation of mean type aggregation with totaling type aggregation operators. The definition of MHF measure can be given as the following.

Definition 3.8 (Yager, 2003a). Let X be finite set of criteria and let $q \in [1, n]$. A MHF measure on the set X is a set function $\hbar: \mathcal{P}(X) \to [0, q]$ satisfying the following conditions:

(i) $\hbar(\emptyset) = 0, \hbar(X) = q$ (boundary condition),
(ii) $\hbar(S) \leq \hbar(T)$ implies $S \subseteq T \subseteq X$ (monotonicity condition),
(iii) for any $x \in X$, then $\hbar(T \cup \{x\}) \leq \hbar(T) + 1$.

In general, these MHF measures contrast from the ordinary fuzzy measure in two cases. Firstly, the range of the measure is allowed to be 0 to q rather than 0 to 1. Secondly, these measures are constrained or monitored such that the addition of one element can at most increase the measure by one. Note that, the condition (iii) can alternatively be expressed as, if $S \subset T$ and $|T| = |S| + 1$ then $\hbar(T) \leq \hbar(S) + 1$, where $|S|$ is the cardinality of S.

Two important special classes of these MHF measures are worth pointing out here as follows:

- If $q = 1$, then $\hbar$ is reduced to an ordinary fuzzy measure μ. The condition (iii) is automatically satisfied for $q = 1$ since $\mu(T) \leq 1$,
- If $q = n$, then $\hbar(S) = |S|$ for all S. In this case, there exists only one MHF measure, which is the total measure.

Definition 3.9 (Yager, 2003a). A MHF measure $\hbar$ for which there exists some subset S and some element $x \in X$ such that $\hbar(S \cup \{x\}) - \hbar(S) = 1$ is said to be saturated.

Definition 3.10 (Yager, 2003a). Let X be a set of cardinality n and let $\hbar$ be a MHF measure on X of magnitude q, the beta value of $\hbar$ can be defined as $\beta = (q-1)/(n-1)$, such that $\beta \in [0,1]$.

It can be shown that, for $q = 1$, as the case of ordinary fuzzy measure, then value $\beta = 0$ is derived. On the other hand, for $q = n$, then value $\beta = 1$ is attained, which is the total measure. Hence, the fuzzy measure and the total measure provide the extreme cases of beta values. Another interesting measure is for $0 < \beta < 1$, which provide the characterization of the magnitude q in between these two extreme cases.

Analogously, the MHF measure can be implemented in the Choquet integral as a wider class of aggregation operators. It can be shown that the heavy Choquet integral under this measure is the general form of the WA and the OWA operator as given in the following:

- If the MHF measure is additive, then the heavy Choquet integral reduces to a heavy weighted arithmetic means and similarly, it reduces to a WA whenever $q = 1$.
- If the MHF measure is symmetric, then the heavy Choquet integral reduces to a HOWA operator and it is reformulated as the OWA if $q = 1$. Recently, Yager (2002) has proposed the HOWA operators in the decision making under uncertainty.

Thus, it can be noticed that the MHF measures provide a general form of aggregation operators, which consist of all the particular cases of mean type aggregation operators as previously discussed. Yager (2003) has also demonstrated the application of these measures in the decision making under uncertainty.

3.6 Summary

In this chapter, the review of the related concepts of aggregation operators is presented. In particular, the emphasis is given on the OWA operators. There are many extensions and generalization of the OWA operators that have been proposed in the literature. Among them are the integration of WA and OWA operators in the same formulation, the induced OWA operators and also the generalization of HOWA operators. In the next chapters, the main attention is given on these particular topics with the applications in the multi-criteria and multi-expert decision making models. The specific definition of those mentioned aggregation operators will be presented in the following Chapters.

CHAPTER 4

GROUP DECISION MAKING MODELS WITH OWA OPERATORS FOR FINANCIAL SELECTION PROBLEMS

4.1 Introduction

In this chapter, the extensions of group decision making models with OWA operators are presented. Specifically, in Section 4.2, the proposed method on OWA-based aggregation operators in multi-expert multi-criteria decision making model is put forward. Next, Section 4.3 provides the weighted selective aggregated majority-OWA operator and its application in linguistic group decision making model. Section 4.4 presents the method on the linguistic group decision making with Dempster-Shafer theory and induced linguistic aggregation operators. All these developed models then are applied in the case of financial selection problems. Finally, in Section 4.5, summary is given to conclude this chapter.

4.2 On OWA-based Aggregation Operations in ME-MCDM Model

Abstract. In this study, an analysis of multi-expert multi-criteria decision making (ME-MCDM) model based on the ordered weighted averaging (OWA) operators is presented. The main focus is given on the aggregation processes, specifically on the fusion of criteria and the fusion of experts' judgments. Firstly, two methods of modeling the majority opinion are studied as to aggregate the experts' judgments, in which based on the induced OWA (IOWA) operators. Some modifications to the support functions are suggested as to derive the order-inducing variables. Secondly, an overview of OWA operators with the inclusion of different degrees of importance or weighted arithmetic mean (WA) is provided for aggregating the criteria. An alternative OWA operator with a new weighting method then is proposed which termed as alternative OWAWA (AOWAWA) operator. Some extensions of ME-MCDM model with respect to two-stage aggregation processes are developed based on the classical and alternative schemes. A comparison of results of different decision schemes then is conducted. Moreover, with respect to the alternative scheme, a further comparison is given for different techniques in integrating the degrees of importance. A numerical example in the selection of investment strategy is used as to exemplify the model and for the analysis purpose.

A.1 Introduction

In the past, various multi-criteria decision making methods have been developed as tools for modeling human decision making and reasoning (see Figueira et al., 2005; Gal et al., 1999; Hwang & Yoon, 1981, for the state-of-art surveys). The methods have been extensively used in numerous applications to deal with the prioritizing, ranking and selection of option (or alternative). In complex decision making problems, normally a group of experts (or decision makers) involved in which each of them offsets and/or support the others for an exhaustive judgment. Since then, the expansion of such models to multi-expert multi-criteria decision making (ME-MCDM) problems has become the main focus in the literature of decision science (see Canfora & Troiano, 2004; Taib et al., 2016; Tsiporkova & Boeva, 2006).

Central to the ME-MCDM problems, aggregation process plays a crucial role in obtaining the final decision, either to aggregate the criteria or to aggregate the overall judgment of experts. An overview of the main aggregation operators and their properties can be referred, for example, in Beliakov et al. (2007), Grabisch et al. (2009), and Torra and Narukawa (2007). Weighted arithmetic mean (WA) and ordered weighted averaging (OWA) operators are among the most widely used aggregation operators in the decision making models. The OWA (Yager, 1988; Yager & Kacprzyk, 1997) provides a general class of mean-type aggregation operators which can be ranged from two extreme cases; a minimum operator (‘*and*’ – requiring all the criteria to be satisfied) to a maximum operator (‘*or*’ – requiring at least one of the criteria to be satisfied). The OWA operator modifies the basic aggregation process used in decision making model by applying the concept of fuzzy set theory, precisely, using the fuzzy linguistic quantifiers (Zadeh, 1983) for a soft aggregation process (Kacprzyk, 1986; Kacprzyk et al., 1992). In comparison to the WA which represents the degrees of importance (or relative weights) associated with particular criteria, the weights in OWA reflect the importance or satisfaction of values with respect to ordering (i.e., ordered weights). By appropriately selecting the weighting vector, different kinds of relationships between the criteria can be modeled, see (Yager, 1993) for the distinct families of OWA operators. In other different case, the WA is necessary in representing the MCDM problems. For example, some experts may prefer to associate a specific weight for each criterion based on its degree of importance. Hence, considering the advantages of both WA and OWA in modeling the real applications, Yager, (1988) then proposed the inclusion of unequal degrees of importance in OWA as an integrated approach. Consequently, a number of other techniques to deal with the same problem have been developed. According to Bordogna et al., (1997), the integration of these weighting methods has been formalized in two different approaches. In the first approach, the relative weights are only used to modify the argument values to be aggregated, specifically without the direct integration with ordered weights. Examples in this category include the method based on max-min and product (Yager, 1988), fuzzy system modeling (Yager, 1998) and hybrid weighted average (Xu & Da, 2003). On the other hand, in the second approach, the relative weights and ordered weights are directly integrated as a new set of weights, e.g., method based on linguistic quantifiers (Yager, 1996), weighted OWA (WOWA) (Torra, 1997), OWA-WA (Merigó, 2012) and immediate WA (IWA) (Llamazares, 2013).

Another important variant of OWA is an induced OWA (IOWA) operator (Yager & Filev, 1999). Generally, it is an extension of OWA which involves a pair of values, such as, the additional parameter (order-inducing variables) used to induce the argument values to be aggregated. Analogously, with respect to a group decision making, the majority agreement among experts can be implemented using the IOWA operators, which synthesizes the opinions of the majority of experts. In this case, the majority opinion refers to a consensual judgment of majority of experts who have similar opinions. In general, the OWA and IOWA operators provide a more flexible model for combining the information in decision making problems, specifically in the complex environment where the attitudinal character of experts is considered.

On the basis of previous discussion, the purpose of this study is on extending and analyzing the ME-MCDM model with respect to two-stage aggregation processes, i.e., the aggregation of criteria and the aggregation of experts' judgments. Firstly, two models based on majority concept for aggregating the experts' judgments are reviewed, particularly the methods as introduced by Pasi and Yager (2006) and its extension by Bordogna and Sterlacchini (2014). The differences between the two methods can be divided into three main categories, specifically, on assigning weights to the experts, the type of proximity measure employed to calculate the support between experts and the approach used in deriving the agreement between the majority of experts, i.e., either based on the classical scheme or the alternative scheme. Pasi and Yager (2006) proposed the method in case of the weights between experts are considered as identical (homogeneous group decision making) and used a support function based on distance measure to compute the majority agreement between experts. Besides, the support between experts is calculated with respect to the final rankings of options which derived primarily by each individual expert (classical scheme). On the contrary, Bordogna and Sterlacchini (2014) then extended this idea to include the case where the experts are assigned with different degrees of importance (heterogeneous group decision making) and utilized the similarity measure based on Minkowski OWA (MOWA) to calculate the majority support between experts. Instead of focusing on the individual ranking on options of each expert, they provide the similarity measure with respect to each specific criterion (alternative scheme). In this study, for the purpose of comparison, some modifications have been made to both methods. In specific, the extension of Pasi-Yager method from the classical scheme to the alternative scheme has been made. Likewise, the Bordogna-Sterlacchini method has been modified to deal with the classical scheme. Hence, these methods with the existing original methods are applied in the ME-MCDM model and then a comparison as to examine the results of different schemes is conducted.

Secondly, some methods based on the integration of OWA and WA for the purpose of aggregating the criteria are presented. In addition, an alternative OWAWA (AOWAWA) operator which combines the characteristics of OWA and OWAWA using the idea of geometric mean is proposed. As a comparison, the ME-MCDM model with respect to Bordogna-Sterlacchini approach on the alternative scheme is applied as to observe the results of distinct weighting techniques in the aggregation process. The outline of this study is as follows. In Section A.2 the definitions of OWA, IOWA and MOWA distance measures are presented. In Section A.3 the aggregation techniques for modeling the majority opinion are discussed and then Section A.4 reviews the integrated weighting methods based on WA and OWA as well as the proposed AOWAWA operator. In Section A.5, the general frameworks of ME-MCDM model based on classical and alternative schemes are outlined. Then, a numerical example in a selection of investment strategy is provided in section A.6.

A.2 Preliminaries

This section provides the definitions and basic concepts related to OWA, IOWA and MOWAD aggregation operators that will be used throughout the study.

A.2.1 OWA operator

Definition A.1 (Yager, 1988). An OWA operator of dimension n is a mapping $OWA: \mathbb{R}^n \rightarrow \mathbb{R}$ that has an associated weighting vector $W = (w_1, w_2, \dots, w_n)$ of dimension n, such that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, given by the following formula:

$$OWA_W(a_1, \dots, a_n) = \sum_{j=1}^{n} w_j a_{\sigma(j)} \qquad \text{(A.1)}$$

where $a_{\sigma(j)}$ is the argument value a_j being ordered in non-increasing order $a_{\sigma(1)} \geq a_{\sigma(2)} \geq \cdots \geq a_{\sigma(n)}$.

Note that, the reordering process makes the OWA operator is no longer a standard linear combination of weighted arguments, but it is rather a piecewise linear function (Beliakov & James, 2011). The OWA operators meet commutative, monotonic, bounded and idempotent properties.

Given that a function $Q: [0,1] \to [0,1]$ as a regular monotonically non-decreasing fuzzy quantifier and it satisfies: i) $Q(0) = 0$, ii) $Q(1) = 1$, iii) $a > b$ implies $Q(a) \geq Q(b)$, then the associated OWA weights can be derived using this function as follows (Yager, 1988):

$$w_j = Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right), \qquad j = 1,2,\dots,n \tag{A.2}$$

such that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

The linguistic quantifier Q (Zadeh, 1983) can be presented in the form of $Q(r) = r^{\gamma},\ \gamma > 0$ with the main characteristics such that: $\gamma \to 0$, then $W = W^*$, where $W^* = (1,0,\dots,0)$; $\gamma = 1$ then $W = W_{1/n}$, where $W_{1/n} = (1/n, 1/n, \dots, 1/n)$; and $\gamma \to \infty$ then $W = W_*$, where $W_* = (0,0,\dots,1)$. Moreover, Yager (1988) defined two measures, namely the orness measure and the entropy (or dispersion) measure to characterize the type of aggregation associated with a given weighting vector W.

Definition A.2 (Yager, 1988). Suppose that W is the associated weighting vector such that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, then the orness measure (or degree of optimism) of OWA can be given as the following:

$$\alpha(W) = \frac{1}{n-1}\sum_{j=1}^{n}(n-j)w_j. \tag{A.3}$$

It can be demonstrated that for any W, the value of $\alpha(W)$ lies in unit interval $[0,1]$. For instance: i) if $W = W_*$ then $\alpha(W_*) = 0$, ii) if $W = W_{1/n}$ then $\alpha(W_{1/n}) = 1/2$, and iii) if $W = W^*$ then $\alpha(W^*) = 1$.

Definition A.3 (Yager, 1988). Suppose that W is the associated weighting vector such that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, then the entropy of OWA can be given as follows:

$$E(W) = -\sum_{j=1}^{n} w_j ln(w_j). \tag{A.4}$$

The entropy is used to measure the degree of information that employed in the OWA aggregation. It can be shown that $0 \leq E(W) \leq ln(n)$, in which $E(W_*) = E(W^*) = 0$ and $E(W_{1/n}) = ln(n)$.

A.2.2 IOWA operator

Definition A.4 (Yager & Filev, 1999). An IOWA operator of dimension n is mapping $IOWA: \mathbb{R}^n \rightarrow \mathbb{R}$ that has an associated weighting vector W such that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, given by the following formula:

$$IOWA_W(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \dots, \langle u_n, a_n\rangle) = \sum_{j=1}^{n} w_j a_{\sigma(j)} \qquad \text{(A.5)}$$

where $a_{\sigma(j)}$ is the argument value of pair $\langle u_j, a_j\rangle$ of order-inducing variable u_j, reordered such that $u_{\sigma(1)} \geq u_{\sigma(2)} \geq \cdots \geq u_{\sigma(n)}$ and the convention that if z of the $u_{\sigma(j)}$ are tied, i.e., $u_{\sigma(j)} = u_{\sigma(j+1)} = \cdots = u_{\sigma(j+z-1)}$, then, the value $a_{\sigma(j)}$ is given as follow (Yager and Filev, 1999; Beliakov and James, 2011):

$$a_{\sigma(j)} = \frac{1}{z} \sum_{k=\sigma(j)}^{\sigma(j+z-1)} a_k \qquad \text{(A.6)}$$

The IOWA operators are all satisfying commutative, monotonic, bounded and idempotent properties.

A.2.3 Minkowski OWA distance

Definition A.5 (Merigó & Gil-Lafuente, 2008). A MOWAD operator of dimension n is a mapping $MOWAD: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ that has an associated weighting vector *W* of dimension *n* such that $\sum_{j=1}^{n} w_j = 1$ with $w_j \in [0,1]$ and the distance between two sets A and B is given as follows:

$$MOWAD_W(d_1, d_2, \dots, d_n) = \left(\sum_{j=1}^{n} w_j d_{\sigma(j)}^{\lambda}\right)^{1/\lambda}, \qquad \text{(A.7)}$$

where $d_{\sigma(j)}$ is the component of d_j being ordered in non-increasing order $d_{\sigma(1)} \geq d_{\sigma(2)} \geq \cdots \geq d_{\sigma(n)}$ and d_j is the individual distance between A and B, such that $d_j = |a_j - b_j|$ with λ is a parameter in a range $\lambda \in \mathbb{R} \setminus \{0\}$.

The MOWAD operators meet commutative, monotonic, bounded and idempotent properties. By setting different values for the norm parameter λ, some special distance measures can be derived. For example, if $\lambda = 1$, then the Manhattan OWA distance can be obtained, $\lambda = 2$ then the Euclidean OWA distance can be acquired, $\lambda = \infty$ then Tchebycheff OWA is derived, etc.

Equivalently, OWA and IOWA operators can be generalized in the similar way (see Merigó & Gil-Lafuente, 2009; Merigó & Yager, 2013; Yager, 2004; Yusoff & Merigó, 2014).

A.3 Aggregation Methods based on Majority Concept

In this section, the methods for aggregating experts' judgments by the inclusion of majority concept are presented. In particular, the method by Pasi and Yager, (2006) and its extension by Bordogna and Sterlacchini (2014) are studied.

A.3.1 Pasi-Yager approach

In the following, a brief description of the mentioned methods is given. Two fundamental steps in both methods are on determining the order-inducing variable and on deriving the associated weights of experts. The methodology used to obtain the majority opinion based on Pasi and Yager, (2006) can be expressed as the following. Figure A.1 illustrate the procedure of this approach.

Suppose that a set of individual opinions of h experts $(h = 1,2,\dots,k)$ is given as the vector $P_i^h = (p_i^1, p_i^2, \dots, p_i^k)$, i.e., with respect to each option $i, (i = 1,2,\dots,m)$. For a simple notation, P_h can be used instead of p_i^h since each option can be evaluated independently using the same formulation. For a single option, the similarity of each expert can be calculated using the support function as follows:

$$supp(p_l, p_h) = \begin{cases} 1 & if\ |p_l - p_h| < \beta, \\ 0 & otherwise. \end{cases} \tag{A.8}$$

The support function represents the similarity or dissimilarity between expert l with each of the other experts $h, (h = 1,2,\dots,k)$ (not include himself/herself), such that $l \in h$. Then the overall support for each individual expert l can be given as:

$$u_l = \sum_{\substack{h=1 \\ h \neq l}}^{k} supp(p_l, p_h), \tag{A.9}$$

where u_l constitute the values of order-inducing variable $U = (u_{\sigma(1)}, \dots, u_{\sigma(k)})$ which ordered in non-decreasing order, such that $u_{\sigma(1)} \leq u_{\sigma(2)} \leq \cdots \leq u_{\sigma(k)}$.

In consequence, to compute the weights of the weighting vector, define the values t_l based on an adjustment of the u_l values, such that: $t_l = u_l + 1$ (including himself/herself, $supp(p_l, p_l) = 1$). The t_l values are in non-decreasing order, $t_1 \leq t_2 \leq \cdots \leq t_k$. On the basis of t_l values, the weights are computed as follows:

$$w_l = \frac{Q(t_l/k)}{\sum_{l=1}^{k} Q(t_l/k)}. \quad \text{(A.10)}$$

The value $Q(t_l/k)$ denotes the degree to which a given member of the considered set of values represents the majority. The quantifier Q with semantic '*most*' for the majority opinion of experts can be given as follows:

$$Q(r) = \begin{cases} 1 & if\ r \geq 0.9, \\ 2r - 0.8 & if\ 0.4 < r < 0.9, \\ 0 & if\ r \leq 0.4, \end{cases} \quad \text{(A.11)}$$

where $r = t_l/k$. As can be seen, the weight of experts here is derived based on the arithmetic mean (AM) where each expert is considered as having an equal degree of importance or trust, e.g., reflect the average of the most of the similar values. Then, the final evaluation is determined using the IOWA operators. Note that, here the values of order-inducing variable are reordered in non-decreasing order instead of non-increasing order as in the original IOWA, such in Eq. (A.5). This type of ordering reflects the conformity of quantifier '*most*' as to model the majority concept (see Pasi & Yager, 2006) for detailed explanation. Note also that, the quantifier Q here is an alternative representation of $Q(r) = r^{\gamma}$. For representing the majority opinion of experts, this type of quantifier will be used throughout the study.

However, in some cases, the values of the vector $P_i^h = (p_i^1, p_i^2, \dots, p_i^k)$, which derived after the first stage of aggregation process show a very slight different between the values due to, for example, the normalization process. This case then leads to the values of $|p_l - p_h|$ less differentiable and cause a difficulty in assigning a value for β. Hence, in this study, a slight modification to the support function in Eq. (A.8) is suggested and the formulation is given as follows:

$$supp(p_l, p_h) = \begin{cases} 1 & if\ \dfrac{|p_l - p_h|}{\max\limits_{l \in h} |p_l - p_h|} < \beta, \\ 0 & otherwise. \end{cases} \quad \text{(A.12)}$$

where $\max\limits_{l \in h} |p_l - p_h|$ is the maximum distance between all experts.

Example A.1: Suppose that a set of individual opinion of experts is given as $P_h = (p_1, p_2, \dots, p_5) = (0.7, 0.86, 0.76, 0.72, 0.6)$ with respect to each option, A_i. Then, the final majority opinion of experts can be calculated as follows:

A_i / P_i^h	E_1 0.7	E_2 0.86	E_3 0.76	E_4 0.72	E_5 0.6		E_1 0.7	E_2 0.86	E_3 0.76	E_4 0.72	E_5 0.6
$supp_{1,h}$	-	0.16	0.06	0.02	0.1	divided by	-	0.62	0.23	0.08	0.39
$supp_{2,h}$	0.16	-	0.1	0.14	**0.26**	$\rightarrow$ $\max_h \lvert p_l - p_h \rvert$	0.62	-	0.39	0.54	1
$supp_{3,h}$	0.06	0.1	-	0.04	0.16		0.23	0.39	-	0.15	0.62
$supp_{4,h}$	0.02	0.14	0.04	-	0.12		0.08	0.54	0.15	-	0.46
$supp_{5,h}$	0.1	0.26	0.16	0.12	-		0.39	1	0.62	0.46	-

By setting $\beta = 0.4$, then the overall support for each expert can be calculated, such as: $s_1 = 3, s_2 = 1, s_3 = 3, s_4 = 2$, and $s_5 = 1$. In case of 'ties', the stricter β can be imposed, such as, $\beta = 0.1$ in this example to order the p_h values. The vector of order-inducing variable then can be given as $U = \left(u_{\sigma(1)}, \dots, u_{\sigma(5)}\right) = (1,1,2,3,3)$ and the weighting vector can be obtained as $W^{Maj} = (w_1, \dots, w_5) = (0, 0, 0.2, 0.4, 0.4)$. The final majority opinion of experts can be calculated as follows:

$$IOWA(\langle 1, 0.6\rangle, \langle 1, 0.86\rangle, \langle 2, 0.72\rangle, \langle 3, 0.76\rangle, \langle 3, 0.7\rangle) = (0 \times 0.6) + (0 \times 0.86) + (0.2 \times 0.72) + (0.4 \times 0.76) + (0.4 \times 0.7) = 0.73.$$

A.3.2 Bordogna-Sterlacchini approach

In the following, the method based on Bordogna and Sterlacchini, (2014) is presented. Contrary to the previous method, here the majority opinion of experts with respect to each specific criterion is considered (see Figure A.2). Suppose that a collection of judgment of h experts is given as vector $P_j^h = \left(p_j^1, p_j^2, \dots, p_j^k\right)$ for criterion $j, (j = 1,2, \dots, n)$. In this method, instead of using the support function based on distance measure, they used the Minkowski OWA-based similarity measure to obtain the $Q_{coherence}$ for an order-inducing variable. The $Q_{coherence}$ of each expert l can be defined as follows:

$$u_l = Q_{coherence}(P_l, P_h) = MOWA(s_1, \dots, s_k) = \left(\sum_{h=1}^{k} \omega_h s_{\sigma(h)}^{\lambda}\right)^{1/\lambda}, \tag{A.13}$$

where $s_l = s(p_l, p_h) = 1 - |p_l - p_h|$ is a similarity measure between expert l with each of the other experts h (includes himself), given that $l \in h$ and $s_{\sigma(h)}$ are ordering of $(s_1, \dots, s_k)$ in non-increasing order $\left(s_{\sigma(1)} \geq s_{\sigma(2)} \geq \cdots \geq s_{\sigma(k)}\right)$. Meanwhile ω_h are the ordered weights with the inclusion of importance degrees of experts $t_h, h = 1,2, \dots, k$, given as $\omega_h = Q\left(\sum_{i=1}^{h} t_{\sigma(i)}\right) - Q\left(\sum_{i=0}^{h-1} t_{\sigma(i)}\right)$, such that $\omega_h, t_h \in [0,1]$ and $\left(\sum_{h=1}^{k} \omega_h = \sum_{h=1}^{k} t_h = 1\right)$. The norm parameter $\lambda \in \mathbb{R} \setminus \{0\}$ provides a generalization of the model. Here the quantifier $Q(r) = r^{\gamma}$ is employed. The OWA weights ω_h will be explained in great detail in the next section.

With respect to the Eq. (A.13), the order inducing vector can be given as:

$$U = (u_1, \dots, u_k) = \left(Q_{coherence}(P_1, P_h), \dots, Q_{coherence}(P_k, P_h)\right), \tag{A.14}$$

Moreover, Q as generalized quantifiers can take any semantics to modify the weights of experts (or trust degrees) for different strategies. When $Q(t_h) = t_h$ as for $(\gamma = 1)$, then $Q_{coherence}$ is reduced to:

$$u_l = coherence(\, P_l, P_h) = \left(\sum_{h=1}^{k} t_h s_h^{\lambda}\right)^{1/\lambda}, \tag{A.15}$$

which is the Minkowski WA-based similarity measure. Formally, $Q_{coherence}$ can be ranged in between $Q_*(t_h)$ for $\gamma \to 0$, to $Q^*(t_h)$ for $\gamma \to \infty$.

Afterwards, the weights for the IOWA operator can be derived using the following formula:

$$m_h = \frac{argmin_h(u_1 \cdot t_1, \dots, u_k \cdot t_k)}{\sum_{h=1}^{k} argmin_i(u_1 \cdot t_1, \dots, u_k \cdot t_k)}, \tag{A.16}$$

where m_h are reordered in non-decreasing order. Analogously, given the quantifier Q as in Eq. (A.11) for the majority opinion, the weighting vector $W^{Maj} = (w_1, \dots, w_k)$ can be computed as follows:

$$w_h = \frac{Q(m_h)}{\sum_{h=1}^{k} Q(m_h)}. \tag{A.17}$$

Note that, the general weights w_h represent the quantification of majority of experts for the final agreement on each criterion, whilst the weights ω_h reflect $Q_{coherence}$ for deriving the order-inducing values.

Next, the overall aggregation process can be computed using the IOWA operator such in Eq. (A.5). Similarly, the non-decreasing inputs $\langle u_h, p_h \rangle$ is implemented as explained in previous sub-section. Moreover, it can be shown that, the coherence function Eq. (A.15) can be represented also as the dual of similarity measure, which is the distance measure:

$$coherence(P_l, P_h) = \left(\sum_{h=1}^{k} t_h (1 - |p_l - p_h|)^{\lambda} \right)^{1/\lambda}$$
$$= 1 - \left(\sum_{h=1}^{k} t_h |p_l - p_h|^{\lambda} \right)^{1/\lambda}, \tag{A.18}$$

such that for any p_l and p_h with $s(p_l, p_h) \in [0,1]$, the properties:

i) $s(p_l, p_l) = 1$ (reflexive) and,
ii) $s(p_l, p_h) = s(p_h, p_l)$ (symmetric) are fulfilled for each single value of l and h.

Similarly to the previous section, to more differentiate between values and to avoid the 'ties' problem, in this study a simple modification to the similarity measure is suggested as follows:

$$s(p_l, p_h) = 1 - \left(\frac{|p_l - p_h|}{\max_{l \in h} |p_l - p_h|} \right), \tag{A.19}$$

where $\max_{l \in h} |p_l - p_h|$ is the maximum distance between all experts.

Correspondingly, the weights for IOWA aggregation process Eq. (A.17) can also be modified to the following formula:

$$m_h = \frac{argmin_h(u_1 \cdot t_1, \dots, u_k \cdot t_k)}{Max_h(u_1 \cdot t_1, \dots, u_k \cdot t_k)}. \tag{A.20}$$

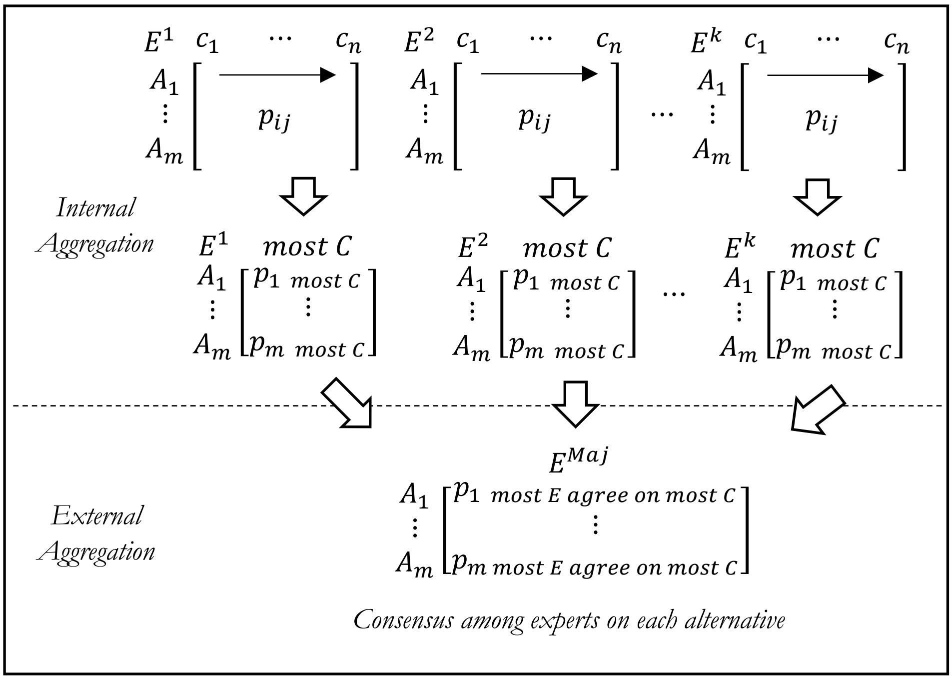

Figure A.1. Classical scheme of multi-expert decision making

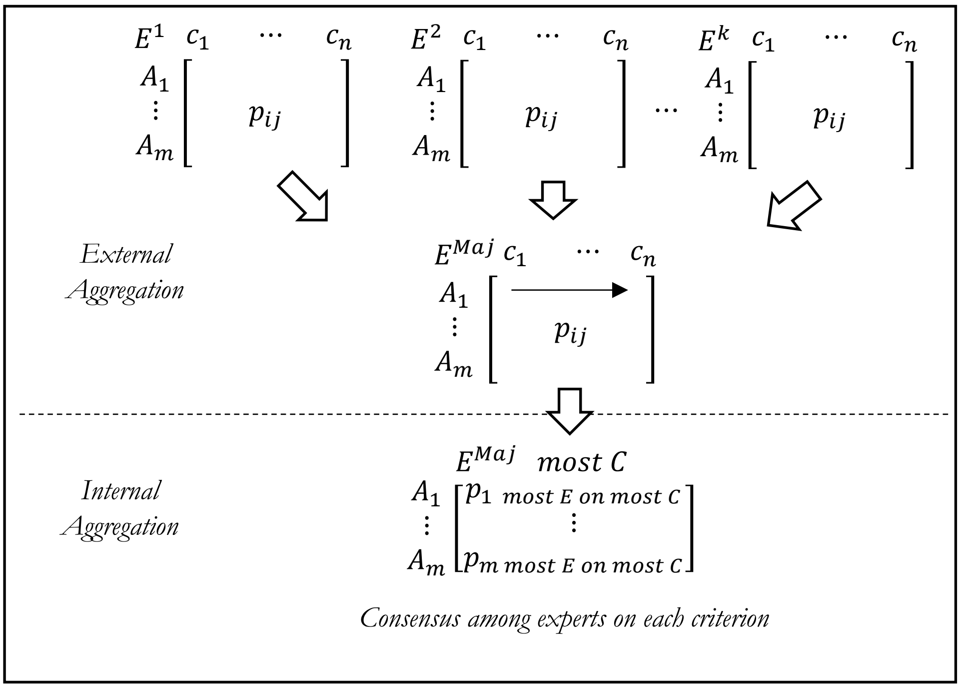

Figure A.2. Alternative scheme of multi-expert decision making

Example A.2: Suppose that a set of opinion of experts on a single criterion C_j is given as $P_j^h = (p_1, p_2, \dots, p_k) = (0.31, 0.34, 0.30, 0.28, 0.11)$. The majority agreement of experts can be calculated as follows:

C_j	E_1	E_2	E_3	E_4	E_5							
P_j^h	0.31	0.34	0.3	0.28	0.11		t_1	t_2	t_3	t_4	t_5	U
$supp_{1h}$	1	0.85	0.96	0.90	0.15		0.3	0.3	0.2	0.1	0.1	0.85
$supp_{2,h}$	0.85	1	0.87	0.75	0	$\xrightarrow{s_h \times t_h}$	0.3	0.3	0.2	0.1	0.1	0.79
$supp_{3,h}$	0.96	0.81		0.94	0.19		0.3	0.3	0.2	0.1	0.1	0.84
$supp_{4,h}$	0.9	0.75	0.94	1	0.26		0.3	0.3	0.2	0.1	0.1	0.81
$supp_{5,h}$	0.15	0	0.19	0.26	1		0.3	0.3	0.2	0.1	0.1	0.21

where $U = \sum_{h=1}^{k} s_h t_h$. In this case, for $Q(t_h) = t_h$ and by setting $\lambda = 1$, then, the vector of order-inducing variables can be derived, $U = \left(s_{\sigma(1)}, \dots, s_{\sigma(5)}\right) = (0.21, 0.79, 0.81, 0.84, 0.85)$. Next, by using the quantifier Q with semantics '*most*' for majority, the weighting vector $W^{Maj} = (w_1, \dots, w_5) = (0, 0, 0.20, 0.40, 0.40)$ can be obtained. The final majority opinion of experts can be calculated using the IOWA operator as follows:

$$IOWA(\langle 0.21, 0.11\rangle, \langle 0.79, 0.34\rangle, \langle 0.81, 0.28\rangle, \langle 0.84, 0.30\rangle, \langle 0.85, 0.31\rangle) = (0.20 \times 0.28) + (0.40 \times 0.30) + (0.40 \times 0.31) = 0.30.$$

A.4 OWA Operators with the Inclusion of Degrees of Importance

In this section, some OWA aggregation operators with their weighting methods are reviewed, in particular, the weighting methods based on the inclusion of the degrees of importance (WA). In addition, an alternative weighting method with its respective aggregation operator called as alternative OWAWA (AOWAWA) operator is proposed.

A.4.1 Some of the existing methods

Prior to the definition of integrated weighting methods, the general definition of WA is given as the following.

Definition A.6. Let $V = (v_1, v_2, \dots, v_n)$ be a weighting vector (degrees of importance) of dimension n such that $v_j \in [0,1]$ and $\sum_{j=1}^{n} v_j = 1$, then a mapping $WA: \mathbb{R}^n \to \mathbb{R}$ is a weighted arithmetic mean (WA) if $WA_V(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} v_j a_j$.

The WA satisfies monotonic, idempotent and bounded properties, but it is not commutative (Beliakov et al., 2007; Grabisch et al., 2009; Torra, 1997).

There are a number of methods in the literature which have been proposed for obtaining weights for OWA aggregation operators (see Xu, 2005). One of them is by using the linguistic quantifiers as defined in the preliminaries section, refer to Eq. (A.2). Throughout the study, the OWA weighting vector W is exclusively referred to this type of weights, specifically to be integrated with the weighting vector, V (except for the methods in Definitions A.10 and A.11 as will be explained later).

Definition A.7 (Yager, 1988). Let V and W be two weighting vectors of dimension n, then a mapping $OWA: \mathbb{R}^n \to \mathbb{R}$ is an OWA-MP operator of dimension n if:

$$OWA_{V,W}(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} w_j \, \breve{a}_{\sigma(j)}, \tag{A.21}$$

where $\breve{a}_{\sigma(j)}$ is the value $\breve{a}_j$ being ordered in non-increasing order $\breve{a}_{\sigma(1)} \geq \breve{a}_{\sigma(2)} \geq \cdots \geq \breve{a}_{\sigma(n)}$ such that $\breve{a}_j = H(a_j, v_j) = (v_j \vee \bar{\alpha}) \cdot (a_j)^{v_j \vee \alpha}$ and α is the orness measure and $\bar{\alpha} = 1 - \alpha$ is its complement.

This is the unified formulation of the methods which proposed earlier in Yager (1978) and Yager (1987), specifically based on the max-min and product approaches. In this study, it is denoted as OWA-MP. Notice that in the special cases: if $\alpha = 0$, then it can be reduced to a pure '*and*' operator. Specifically, given that $\breve{a}_j = a_j^{v_j}$ with $W = W_*$, then $\breve{a}_{\sigma(n)}$ is generated, which is the smallest value of $\breve{a}_{\sigma(j)}$. Conversely, if $\alpha = 1$, then it can be reduced to a pure '*or*' operator. Given that $\breve{a}_j = v_j a_j$ with $W = W^*$, then $\breve{a}_{\sigma(1)}$ is generated, which is the largest value of $\breve{a}_{\sigma(j)}$. The OWA-MP operators meet monotonic and idempotent properties, however they are not commutative as involve WA. Moreover they are also not bounded, as in the case of argument value, $a_j \in [0,1]$, the modified argument values $\breve{a}_j$ are always greater than or equal to the argument values, a_j.

Definition A.8 (Yager, 1998). Let V and W be two weighting vectors of dimension n, then a mapping $OWA: \mathbb{R}^n \rightarrow \mathbb{R}$ is an OWA-FSM operator of dimension n if:

$$OWA_{V,W}(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} w_j \hat{a}_{\sigma(j)}, \qquad \text{(A.22)}$$

where $\hat{a}_{\sigma(j)}$ is the value of $\hat{a}_j$ being ordered in non-increasing order $\hat{a}_{\sigma(1)} \geq \hat{a}_{\sigma(2)} \geq \cdots \geq \hat{a}_{\sigma(n)}$ given that $\hat{a}_j = H(a_j, v_j) = \bar{\alpha}\bar{v}_j + v_j a_j$ and $\bar{\alpha} = 1 - \alpha$, that is the complement of orness.

This method is based on fuzzy system modeling and is termed as OWA-FSM in this study. Notice that in the special cases: if $\alpha = 0$, then it reduces to a pure '*and*' operator. Specifically, given that $\hat{a}_j = \bar{v}_j + v_j a_j$ and $w_n = 1$, then $\hat{a}_{\sigma(n)}$ is generated, which is the smallest value of $\hat{a}_{\sigma(j)}$. Whilst, if $\alpha = 1$, then it is a pure '*or*' operator. Given that $\hat{a}_j = v_j a_j$ and $w_1 = 1$, then $\hat{a}_{\sigma(1)}$ is generated, which is the largest value of $\hat{a}_{\sigma(j)}$. The OWA-FSM operators meet monotonic and idempotent properties, but, they are not commutative as involve WA. Moreover, they are also not bounded, as in the case of $a_j \in [0,1]$, then $\hat{a}_j \geq a_j$.

Definition A.9 (Xu & Da, 2003). Let V and W be two weighting vectors of dimension n, then a mapping $HA: \mathbb{R}^n \rightarrow \mathbb{R}$ is a hybrid averaging (HA) operator of dimension n if:

$$HA_{V,W}(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} w_j \acute{a}_{\sigma(j)}, \qquad \text{(A.23)}$$

where $\acute{a}_{\sigma(j)}$ is the argument value $\acute{a}_j$ being ordered in non-increasing order $\acute{a}_{\sigma(1)} \geq \acute{a}_{\sigma(2)} \geq \cdots \geq \acute{a}_{\sigma(n)}$ given that $\acute{a}_j = n v_j a_j$ and n is the balancing coefficient.

It can be shown that when $W = (1/n, 1/n, \dots, 1/n)$, then HA operator reduces to WA, whilst when $V = (1/n, 1/n, \dots, 1/n)$, HA operator reduces to OWA (Llamazares, 2013). HA operators meet monotonic property, however, they are neither idempotent nor bounded. As can be seen, the Definitions A.7-A.9 are based on the approach where the degrees of importance, v_j are used to modify the argument values to be aggregated. In the following, the approaches based on the direct integration between v_j and w_j are presented.

Definition A.10 (Torra, 1997). Let V and W be two weighting vectors of dimension n, then a mapping $WOWA: \mathbb{R}^n \to \mathbb{R}$ is a weighted ordered weighted averaging (WOWA) operator of dimension n if:

$$WOWA_{V,W}(a_1, a_2, \dots, a_n) = \sum\nolimits_{j=1}^{n} \omega_j a_{\sigma(j)}, \tag{A.24}$$

where $a_{\sigma(j)}$ is the argument value of a_j being ordered in non-increasing order $a_{\sigma(1)} \geq a_{\sigma(2)} \geq \cdots \geq a_{\sigma(n)}$ and $\omega_j = f\left(\sum_{k=1}^{j} v_{\sigma(k)}\right) - f\left(\sum_{k=0}^{j-1} v_{\sigma(j)}\right)$ with f being a monotonic non-decreasing function that interpolates the points $\left((j/n), \sum_{k=1}^{j} w_j\right)$ together with the point $(0,0)$. The function f required to be a straight line when the points interpolated in this way.

Similarly, it can be demonstrated that when $W = (1/n, 1/n, \dots, 1/n)$, then WOWA operator reduces to WA, whilst when $V = (1/n, 1/n, \dots, 1/n)$, WOWA operator reduces to OWA (Llamazares, 2013). Moreover, they are monotonic, idempotent, and bounded. Equivalently, the WOWA operator can be transformed to the OWA operator with the inclusion of degrees of importance (Yager, 1996), if a regular monotonically non-decreasing fuzzy quantifier Q is used as the function f and it can be defined as the following.

Definition A.11 (Yager, 1996). Let V and W be two weighting vectors of dimension n, then a mapping $OWA: \mathbb{R}^n \to \mathbb{R}$ is an OWA operator of dimension n if :

$$OWA_{V,W}(a_1, a_2, \dots, a_n) = \sum\nolimits_{j=1}^{n} \omega_j a_{\sigma(j)}, \tag{A.25}$$

where $a_{\sigma(j)}$ is the argument value a_j being ordered in non-increasing order $a_{\sigma(1)} \geq a_{\sigma(2)} \geq \cdots \geq a_{\sigma(n)}$ and $\omega_j = Q\left(\sum_{k=1}^{j} v_{\sigma(k)}\right) - Q\left(\sum_{k=0}^{j-1} v_{\sigma(k)}\right)$ such that $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

Definition A.12 (Llamazares, 2013). Let V and W be two weighting vectors of dimension n, then a mapping $IWA: \mathbb{R}^n \to \mathbb{R}$ is an immediate weighted averaging (IWA) operator of dimension n if:

$$IWA_{V,W}(a_1, a_2, \dots, a_n) = \sum\nolimits_{j=1}^{n} \pi_j a_{\sigma(j)}, \tag{A.26}$$

where $a_{\sigma(j)}$ is the argument value a_j being ordered in non-increasing order $a_{\sigma(1)} \geq a_{\sigma(2)} \geq \cdots \geq a_{\sigma(n)}$ and $\pi_j = w_j v_j / \sum_{j=1}^{n} w_j v_j$.

As can be seen, the IWA is a manipulation of immediate probability (Engemann et al., 1996; Merigó, 2012; Yager et al., 1995) by using the WA instead of the probability distribution. IWA operators satisfy the generalization properties as $V = (1/n, 1/n, \dots, 1/n)$, it reduces to OWA and when $W = (1/n, 1/n, \dots, 1/n)$, IWA reduces to WA (Llamazares, 2013). IWA operators meet monotonic, idempotent, bounded properties.

Definition A.13 (Merigó, 2012). Let V and W be two weighting vectors of dimension n, then a mapping $OWAWA: \mathbb{R}^n \to \mathbb{R}$ is an ordered weighted averaging-weighted average (OWAWA) operator of dimension n if:

$$OWAWA_{V,W}(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} \varphi_j a_{\sigma(j)}, \quad \text{(A.27)}$$

where $a_{\sigma(j)}$ is the argument value of a_j being ordered in non-increasing order $a_{\sigma(1)} \geq a_{\sigma(2)} \geq \cdots \geq a_{\sigma(n)}$ and $\varphi_j = \beta w_j + (1-\beta) v_{\sigma(j)}$ with $\beta \in [0,1]$.

OWAWA operators satisfy monotonic, idempotent, bounded properties. Moreover, the value returned by the OWAWA operator lies between the values returned by the WA and OWA, and coincides with them when both are equal.

In addition, by taking the advantages of IWA and OWAWA operators, a new weighting method can be derived as in the next sub-section.

A.4.2 Alternative OWAWA operator

Definition A.14. Let V and W be two weighting vectors of dimension n, then a mapping $AOWAWA: \mathbb{R}^n \to \mathbb{R}$ is an alternative ordered weighted averaging-weighted average (AOWAWA) operator of dimension n if:

$$AOWAWA_{V,W}(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} \hat{\varphi}_j a_{\sigma(j)}, \quad \text{(A.28)}$$

where $a_{\sigma(j)}$ is the argument value of a_j being ordered in non-increasing order $a_{\sigma(1)} \geq \cdots \geq a_{\sigma(n)}$ and $\hat{\varphi}_j = \left(w_j{}^{\beta} \cdot v_{\sigma(j)}{}^{(1-\beta)}\right) / \sum_{j=1}^{n} \left(w_j{}^{\beta} \cdot v_{\sigma(j)}{}^{(1-\beta)}\right)$ with $\beta \in [0,1]$, by convention that $(0^0 = 0)$.

The AOWAWA operator are monotonic, bounded, idempotent. However, it is not commutative because the AOWAWA operator includes the WA. The AOWAWA operators generalized to WA and OWA when $\beta = 0$ and $\beta = 1$, respectively.

Theorem A.1 (Monotonicity) Assume that f is the AOWAWA operator, let $A = (a_1, a_2, \dots, a_n)$ and $B = (b_1, b_2, \dots, b_n)$ be two sets of arguments. If $a_j \geq b_j$, $\forall j \in (1,2, \dots, n)$, then:

$$f(a_1, a_2, \dots, a_n) \geq f(b_1, b_2, \dots, b_n).$$

Proof. It is straightforward and thus omitted.

Theorem A.2 (Idempotency) Assume f is the AOWAWA operator, if $a_j = a$, $\forall j \in (1,2, \dots, n)$, then:

$$f(a_1, a_2, \dots, a_n) = a.$$

Proof. It is straightforward and thus omitted.

Theorem A.3 (Bounded) Assume f is the AOWAWA operator, then:

$$Min\{a_j\} \leq f(a_1, a_2, \dots, a_n) \leq Max\{a_j\}$$

Proof. It is straightforward and thus omitted.

A.5 General Frameworks of ME-MCDM Model based on Different Decision Schemes

In this section, the general frameworks of ME-MCDM model based on the classical and alternative schemes are presented. In addition to the original methods by Pasi and Yager (2006) and Bordogna and Sterlacchini (2014), some extensions have been made as the following. First, the majority concept of Pasi-Yager method which is originally based on the classical scheme is extended to the case of alternative scheme. Secondly, the Bordogna-Sterlacchini method which is based on the alternative scheme is modified to the case of the classical scheme. These methods are used for the comparison purpose in the next section. The algorithms for the model are structured as in the following.

A.5.1 Classical scheme

- *Stage I: Internal aggregation* (*Local aggregation*)

Step 1: First, a decision matrix for each expert $D^h, h = 1,2, \dots, k$, is constructed as follows:

$$D^h = \begin{array}{c} \\ A_1 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & \dots & C_n \end{array} \\ \begin{pmatrix} a_{11}^h & \cdots & a_{1n}^h \\ \vdots & \ddots & \vdots \\ a_{m1}^h & \cdots & a_{mn}^h \end{pmatrix} \end{array}, \tag{A.29}$$

where A_i indicates the option/alternative i $(i = 1,2, \dots, m)$ and C_j denotes the criterion j $(j = 1,2, \dots, n)$. Meanwhile the a_{ij}^h represents the preference for option A_i with respect to criterion C_j, such that $a_{ij}^h \in [0,1]$.

Step 2: Next, determine the weighting vector for all the expert using one of the available methods, such as in Eqs. (A.21-A.28). Note that, in this case, the proportion of criteria to be considered is subject to the attitudinal character of individual experts. Hence, each expert can provide distinct decision strategies separately.

Step 3: Aggregate the judgment matrix of each expert by the weighting vector as determined in *Step 2*. At this stage, each expert derives the ranking of all options individually.

- *Stage II: External aggregation* (*Global aggregation*)

With respect to the type of aggregation methods, the consensus measure for the majority of experts can be calculated as follows:

(P-Y) The Pasi-Yager method (Homogeneous group decision making):*

Step 4: Determine the order-inducing variable using the Eqs. (A.8-A.9) or in the case where the argument values are very close to each other, use the modified support function such in Eq. (A.12).

Step 5: Calculate the weighting vector which represents the majority of experts using the Eq. (A.10) based on quantifier '*most*' as in Eq. (A.11). In this case, the weight of each expert is considered as equal (the same degrees of importance).

(B-S) The modified version of Bordogna-Sterlacchini method (Heterogeneous group decision making):*

Step 4: Determine the order-inducing variable using the Eqs. (A.13-A.15) or in the case where the argument values are very close to each other, then use the modified similarity measure such in Eq. (A.19).

Step 5: Calculate the weighting vector using the Eqs. (A.16-A.17). In this case, the weight or trust degree is associated to each expert.

A.5.2 Alternative scheme

- *Stage I: External aggregation*

Step 1: By the similar way, a decision matrix for each expert is constructed such in Eq. (A.29). Then, the aggregation based on majority concept can be implemented using one of the following methods:

*(B-S**) The Bordogna-Sterlacchini method (Heterogeneous group decision making):*

Step 2: Determine the order-inducing variable such in *Step 4(B-S*)* of the classical scheme. But, instead of aggregate the opinion of experts with respect to each option, in this step, the aggregation process is conducted on each criterion.

Step 3: Calculate the weighting vector such in *Step 5(B-S*)* of the classical scheme using the values of the order-inducing variable in the previous step.

*(P-Y**) The extension of Pasi-Yager method (Homogeneous group decision making):*

Step 2: Determine the order-inducing variable as in *Step 4(P-Y*)* of the classical scheme. But, instead of aggregate the opinion of experts with respect to each option, here, the aggregation process is conducted on each criterion.

Step 3: Calculate the weighting vector such in *Step 5(P-Y*)* of the classical scheme using the order-inducing variable derived in the previous step.

- *Stage II: Internal aggregation* (*Global aggregation*)

Step 4: Determine the weighting vector using one of the methods as shown in Eqs. (A.21-A.28).

Step 5: Finally, aggregate the judgment matrix of the majority of experts with respect to the weighting vector derived in *Step 4*. Note that here, the proportion of criteria is subject to the attitudinal character of the majority of experts.

A.6 Numerical Example

In the following, a numerical example is presented. In this case, an investment selection problem is studied where a group of experts or analysts are assigned for the selection of an optimal strategy. Assume that a company plans to invest some money in a region. Primarily, they consider five possible investment options as follows: A_1 = invest in the European market, A_2 = invest in the American market, A_3 = invest in the Asian market, A_4 = invest in the African market, A_5 = do not invest money. In order to evaluate these investments, the investor has brought together a group of experts. This group considers that each of investment options can be described with the following characteristics: C_1 = benefits in the short term, C_2 = benefits in the mid-term, C_3 = benefits in the long term, C_4 = risk of the investment, C_5 = other variables. The available investment strategies depending on the characteristic C_j and the option A_i for each expert are shown in Table A.1.

Table A.1. Available investment strategies of each expert, E_h

		E_1		
C_1	C_2	C_3	C_4	C_5
0.7	0.6	0.7	0.6	0.9
0.8	1	0.2	1	0.6
0.6	0.7	0.6	0.6	0.5
0.9	0.6	0.8	1	0.9
0.3	0.7	0.7	0.8	0.9

		E_2		
C_1	C_2	C_3	C_4	C_5
0.6	0.9	1	0.9	0.9
1	0.7	0.1	1	0.8
0.4	0.9	0.8	0.7	0.6
0.9	0.5	0.7	1	0.9
0.7	0.7	0.9	0.9	0.9

		E_3		
C_1	C_2	C_3	C_4	C_5
0.5	0.7	0.9	0.8	0.9
0.9	0.9	0.2	1	0.7
0.8	0.8	0.7	0.7	0.6
0.9	0.5	0.8	1	0.7
0.8	0.7	0.8	0.9	0.8

			E_4		
	C_1	C_2	C_3	C_4	C_5
A_1	0.4	0.7	0.9	0.8	0.8
A_2	0.9	0.7	0.1	0.9	0.6
A_3	0.6	0.6	0.5	0.8	0.4
A_4	0.7	0.5	0.7	0.7	0.9
A_5	0.4	0.6	0.7	0.8	0.9

			E_5		
	C_1	C_2	C_3	C_4	C_5
A_1	0.5	0.6	0.7	0.6	0.8
A_2	0.9	0.8	0.4	0.9	0.5
A_3	0.6	0.6	0.5	0.8	0.7
A_4	0.8	0.7	0.6	0.9	0.8
A_5	0.2	0.6	0.8	0.6	0.8

In this study, two analyses are conducted. First is to analyze the effect of different decision schemes for homogeneous and heterogeneous cases. The aggregated results of the analysis are presented in Table A.2. Note that, in this case, all the criteria are set to have equal degrees of importance. In addition, for the heterogeneous case (i.e., Bordogna-Sterlacchini method), the expert's weight is given as 0.3, 0.1, 0.1, 0.4, 0.1 for expert E_1, E_2, E_3, E_4 and E_5, respectively. As can be seen, there is a slight difference between the results which derived from both majority aggregation approaches (Pasi-Yager and Bordogna-Sterlacchini methods) with respect to different decision schemes. The majority opinion of experts with respect to the classical scheme provides A_4, A_2, A_1, A_5 and A_3 as the final ranking for both methods (ME-MCDM-PY* and ME-MCDM-BS*). Whilst the majority opinion of experts computed with respect to alternative scheme exhibits the ranking of A_4, A_1, A_5, A_2 and A_3 for ME-MCDM-PY** and ME-MCDM-BS**. Hence, the aggregated results demonstrated the effect on different decision schemes in ranking the options.

Table A.2. The aggregated results

	Homogeneous case, $t_h = 1/n$		Heterogeneous case, $t_h \neq 1/n$	
	ME-MCDM-PY*	ME-MCDM-PY**	ME-MCDM-BS*	ME-MCDM-BS**
A_1	0.7143 (R3)	0.7726 (R2)	0.7169 (R3)	0.7989 (R2)
A_2	0.7178 (R2)	0.6992 (R4)	0.7200 (R2)	0.6580 (R4)
A_3	0.6280 (R5)	0.6361 (R5)	0.5952 (R5)	0.6057 (R5)
A_4	0.7886 (R1)	0.8027 (R1)	0.7800 (R1)	0.8000 (R1)
A_5	0.7029 (R4)	0.7225 (R3)	0.6800 (R4)	0.6969 (R3)

Note: '*' refers to the classical scheme and '**' refers to the alternative scheme; R = ranking.

Secondly, as a further analysis, the method of ME-MCDM-BS** based on the integration of WA and OWA weights is conducted. Table A.3 shows the aggregated results of available financial strategies. The weights v_j (the degrees of importance) for the criteria are given as 0.1, 0.2, 0.3, 0.3, 0.1 and the ordered weights, w_j are represented as '*most*' $(\gamma = 10)$ "i.e., most of the criteria have to be satisfied". As can be noticed, the proposed AOWAWA operator with $\beta =$

0.5 indicates the similar ranking as the WOWA and IWA methods, A_1, A_4, A_3, A_5 and A_2. Concurrently, the rest weighting techniques show slightly different results.

Table A.3. The aggregated results with respect to ME-MCDM-BS** model

OWA (Q)	WOWA	IWA	OWA-WA	AOWA -WA	OWA (FSM)	OWA (MP)	HA
0.6957	0.6992	0.6972	0.7526	0.7076	0.9177	0.9053	0.3598
0.1543	0.1147	0.1207	0.3866	0.2124	0.7325	0.5319	0.167
0.4837	0.5080	0.4988	0.5547	0.5158	0.8564	0.8279	0.2455
0.5227	0.5217	0.5302	0.6563	0.5736	0.8791	0.8493	0.4504
0.4185	0.4946	0.4472	0.5685	0.5085	0.8926	0.8742	0.2215

Note that in this case, the decision strategy is subject to the attitudinal character of the majority of experts. By selecting any parameter γ to represent the linguistic quantifier, various decision strategies can be derived. Specifically for $\gamma \to 0$ (at least one criteria is considered), $\gamma = 1$ (averagely all) and $\gamma \to \infty$ (all criteria are considered). The aggregated results of AOWAWA operator with different decision strategies are presented in Tables A.4.

Table A.4. Decision strategies based on AOWAWA operator

At least one $\gamma \to 0$	Few $\gamma = 0.1$	Some $\gamma = 0.5$	Half (average) $\gamma = 1$	Many $\gamma = 2$	Most $\gamma = 10$	All $\gamma \to \infty$
0.8989	0.8463	0.8202	0.8038	0.7801	0.7076	0.6945
0.9976	0.8397	0.7188	0.6375	0.5249	0.2124	0.1000
0.6994	0.6628	0.6354	0.6169	0.5908	0.5158	0.4727
0.9986	0.9071	0.8379	0.7926	0.7320	0.5736	0.5000
0.8976	0.7871	0.7367	0.7084	0.6695	0.5085	0.3846

In addition, the rankings of AOWAWA operator with different values of β can be seen in Table A.5. These values show the effect of the selection WA and OWA in the final evaluation process. For example, if only WA is applied, then $\beta = 0$, whilst $\beta = 1$ implies only OWA is used.

Table A.5. Aggregated results of AOWAWA operator based on β values

$\beta = 0$	$\beta = 0.2$	$\beta = 0.4$	$\beta = 0.6$	$\beta = 0.8$	$\beta = 1$
0.6957	0.6975	0.7024	0.7160	0.7493	0.8094
0.1543	0.1699	0.1942	0.2376	0.3338	0.6190
0.4837	0.4925	0.5065	0.5270	0.5590	0.6257
0.5227	0.5370	0.5589	0.5916	0.6459	0.7900
0.4185	0.4426	0.4816	0.5410	0.6223	0.7185

4.3 Weighted Selective Aggregated Majority-OWA Operator and Its Application in Linguistic Group Decision Making Model

Abstract. This study focuses on the aggregation process in group decision making model based on the concept of majority opinion (neat-OWA-based method). The weighted-selective aggregated majority-OWA (WSAM-OWA) operator is proposed as an extension of the SAM-OWA operator, where the reliability of information sources is considered in the formulation. The WSAM-OWA operator is generalized to the quantified WSAM-OWA operator by including the concept of linguistic quantifier, mainly for the group fusion strategy. The QWSAM-IOWA operator, with an ordering step, is introduced to the individual fusion strategy. The proposed aggregation operators are then implemented to the case of alternative scheme of heterogeneous group decision analysis. The heterogeneous group includes the consensus of experts with respect to each specific criterion. The exhaustive multi-criteria group decision making model under the linguistic domain, which consists of two-stage aggregation processes, is developed in order to fuse the experts' judgments and to aggregate the criteria. The model provides a greater flexibility when analyzing the decision alternatives with a tolerance that considers the majority of experts and the attitudinal character of experts. A selection of investment problem is given to demonstrate the applicability of the developed model.

B.1 Introduction

An aggregation process is central in many applications which involve information processing, such as decision analysis, information retrieval, and pattern recognition. Group decision making (GDM), one of the research topics in multiple criteria decision analysis (MCDA), relies on the aggregation process to obtain a representative value for a group of experts. Two general frameworks or schemes that are normally used in GDM can be classified as classical and alternative schemes (Bordogna & Sterlacchini, 2014). These schemes, in general, have different approaches for aggregating the experts' judgments as the final group decision. In particular, the classical scheme refers to the consensus of experts for each ranking of alternative, whilst the alternative scheme deals with the consensus for each criterion. Principally, there are two main aggregation processes in GDM; they are the aggregation of criteria and the aggregation of experts. There are many aggregation functions that have been proposed as the fusion method in GDM models. One of the most commonly used aggregation operators is the ordered weighted averaging (OWA) operator introduced by Yager (1988). The OWA can be explained as a general class of aggregation functions that encompasses the operations between the min and max operators. The induced OWA (IOWA) operator, as another OWA extension, has also been applied to most of the GDM models. Recent development of OWA-related aggregation operators from theoretical and application perspectives can be referred to, for instance, in Yager and Kacprzyk (1997), Yager and Filev (1999), Merigó and Gil-Lafuente (2009), Merigó and Casanovas (2010) and Merigó and Yager (2013).

Fuzzy set theory (Zadeh, 1965), on the other hand, provides MCDA models with a flexibility in the representation and/or the aggregation of information. The information used in MCDA problems, in general, is either quantitative and/or qualitative. Quantitative information may be expressed by numerical values; whereas qualitative information may be represented by linguistic assessments in order to capture the vagueness and uncertainty of the information. Human judgments, for example, involve subjective evaluations that are more suitably and conveniently modeled by the fuzzy linguistic approach. They can be represented by linguistic values using linguistic variables, i.e., the variables whose values are not numbers but words or sentences in a natural or artificial language (Zadeh, 1975). This approach is adequate for qualifying phenomena related to human perception. Many approaches have been proposed recently to model linguistic information (see Bordogna et al., 1997; Delgado et al., 1993; Herrera & Herrera-Viedma, 2000; Merigó et al., 2010).

Fuzzy set theory is also useful in modelling the aggregation process. Soft aggregation processes can be implemented, specifically, by the inclusion of linguistic quantifiers in OWA operator (Yager, 1988; 1996). In this way, various decision strategies can be determined in order to provide a complete picture of the decision analysis. For example, considering a portion of criteria to be satisfied from '*at least one*' criterion (existential quantifier) to '*all*' criteria (universal quantifier). Analogously, with respect to the GDM, the soft majority agreement among experts can be modeled, for instance by using semantics such as '*at least 80%*' and '*most*'. However, the linguistic quantifiers used to represent the majority concept as a group consensus is manipulated differently than that of the regular quantifiers in the classical OWA. For instance, instead of defining "*Q* of the values need to be satisfied," where the argument values are seen as truth values or degrees of satisfaction and *Q* represents any semantic, alternatively "*Q* of the similar values" is used to model the meaning of majority (Pasi & Yager, 2006; Peláez et al., 2007).

In most cases, it is difficult to achieve a unanimous decision when dealing with a group of experts. As an alternative, agreement among a majority of experts can be tolerated. In the literature, there are some approaches which have been proposed to model the majority concept using OWA operators. Pasi and Yager (2006) proposed two approaches to deal with this issue. The first is based on the use of the IOWA operator, where the support function is applied to derive a set of order-inducing, scalar-valued variables, i.e., reordered based on the most similar opinions. While, the other approach is based on a fuzzy subset, that represents the majority opinion under the vague concept. Correspondingly, Bordogna and Sterlacchini (2014) extended the Pasi-Yager method, specifically based on the IOWA operator, by employing the Minkowski OWA-based similarity measure to obtain the order-inducing variables. Moreover, in their method, instead of synthesizing the consensus on each ranking of alternative (classical scheme), they proposed an alternative approach where the consensus measure on each specific criterion (alternative scheme) is implemented. Furthermore, they proposed to apply the importance degrees of experts to heterogeneous GDM.

In other related research, Peláez and Doña (2003a) proposed the majority additive OWA operator (MA-OWA) to aggregate the argument values that have cardinality greater than one. Particularly, this operator is an extension of the simple arithmetic mean (AM) since it is the 'arithmetic mean of arithmetic means'. Peláez and Doña (2003a) notes that for classical aggregation operators such as the AM, the aggregated value is not representative of the majority aggregation since the result is affected by the extreme values. This results in an aggregated value that is correlated to the symmetric tendency between the values. Even though the OWA operators can be implemented as an alternative approach, they have distribution problems when aggregating arguments with cardinalities (Peláez & Doña, 2003a). Hence, the MA-OWA can be used to treat this type of problem more effectively. Furthermore, in this case, the overall value of the majority opinion is determined without elimination of the minority opinion. In other words, all the information is employed in the aggregation process. Since its inception, some extensions of the MA-OWA operator have been proposed in the literature, such as: the linguistic aggregation MA-OWA, the majority multiplicative-OWA, the quantified MA-OWA and the work committee-OWA (Peláez & Doña, 2003b; Peláez et al., 2005; Peláez et al., 2007; La Red et al., 2011). Recently, Karanik et al. (2016) has proposed the selective MA-OWA (SMA-OWA) operator to deal with the problem of fast convergence of the associated weights. More precisely, when the difference between the cardinalities of the aggregated values is huge, then, only the argument value with the highest cardinality is taken into account, whilst the other may be excluded. As a solution, the cardinality relevance factor (CRF) was introduced as a degree of tolerance to modify the associated weights so that all the argument values can be included. In addition, Peláez et al. (2016) has proposed the selective aggregated majority OWA (SAM-OWA) operator where the cardinality is used to calculate the individual weight for each group of argument values. Previously, in the MA-OWA and SMA-OWA operators, the individual weights were set as equally important.

Nevertheless, the SAM-OWA operators are limited to the case of homogeneous GDM problems. Although the SAM-OWA is associated with a set of weights that are based on cardinalities, the argument values are still considered equally important. In addition, the information to be aggregated is not associated with the reliability of information sources as in the case of heterogeneous GDM problems. In the context of GDM, each expert has an associated degree of importance that reflects his/her expertise, knowledge, skill, etc. Motivated by the heterogeneous GDM problems, the inclusion of the reliability of information sources (or degree of importance) is suggested as the extension of SAM-OWA and it is denoted as the weighted SAM-OWA operator. Furthermore, by integrating over the linguistic quantifiers, the WSAM-OWA is extended to the quantified WSAM-OWA to provide a greater flexibility in the aggregation process, specifically for the group fusion strategy. While in the individual fusion strategy, QWSAM-IOWA is introduced to deal with the ordering problem and to better represent the majority opinion of experts. Finally, based on the proposed aggregation operators, the multi-expert GDM model with respect to the alternative scheme is developed under the linguistic domain. A selection of investment problem is given as an example of the applicability of the developed model. This study is structured as the followings. Section B.2 provides some preliminaries include the definitions and basic concepts of OWA, neat OWA, IOWA and linguistic labels. In Section B.3, a review of MA-OWA, SMA-OWA and SAM-OWA operators is provided. In section B.4, the proposed WSAM-OWA, QWSAM-OWA and QWSAM-IOWA are presented. Then, in Section B.5, the multi-criteria GDM model is developed based on the proposed aggregation operators and finally, in Section B.6, a numerical example is provided.

B.2 Preliminaries

In this section, some definitions and basic concepts related to the OWA, neat OWA and IOWA operators and also the linguistic labels are presented.

B.2.1 OWA, Neat-OWA and IOWA Operators

Definition B.1 (Yager, 1988). An OWA operator of dimension n is a mapping $F_{OWA}: \mathbb{R}^n \to \mathbb{R}$ that has an associated weighting vector $W = [w_1, w_2, \dots, w_n]$ such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, defined as:

$$F_{OWA}(a_1, a_2, \dots, a_n) = \sum_{i=1}^{n} w_i a_{\sigma(i)}, \qquad \text{(B.1)}$$

where $a_{\sigma(i)}$ is the argument value a_i being ordered in non-increasing order $a_{\sigma(i)} \geq a_{\sigma(i+1)}$.

As can be seen, the OWA is a nonlinear aggregation operator since it involves the ordering process. Moreover, it is a mean-type aggregation operator that meets all the commutative, monotonic, bounded and idempotent properties. The type of aggregation performed by OWA operator is mainly affected by the weighting vector W. It can be shown that a number of well-known aggregation operators are included in the OWA operator such as min and max operators, simple average, median, to name a few. Other families of OWA operators can be referred to Yager (1993) or Section 3.2.3 of Chapter 3.

Different approaches have been suggested to derive the weights for OWA operator, such as, using the linguistic quantifiers, maximum entropy, minimal variability, and learning method. See Xu (2005) for a complete review of the other approaches. In particular, Yager (1988) defined the OWA operator from the proportional linguistic quantifiers Q (i.e., based on monotonic non-decreasing function) by defining the weights in the following way:

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right), \qquad i = 1,2, \dots, n, \qquad \text{(B.2)}$$

where w_i represents the increase of satisfaction in getting i with respect to $i - 1$ criteria satisfied. In this case, all the criteria are associated with the identical degrees of importance, $w_i = 1/n$, as shown when $Q(x) = x$. However, in the case where each of the criteria c_i to be aggregated has an importance degree v_i associated with it, such that (v_i, c_i), the inclusion of importance degrees in OWA operators from Q can be defined as follows (Yager, 1996):

$$\omega_i = Q\left(\frac{\sum_{k=1}^{i} v_{\sigma(k)}}{T}\right) - Q\left(\frac{\sum_{k=0}^{i-1} v_{\sigma(k)}}{T}\right), \tag{B.3}$$

where $v_{\sigma(i)}$ are the degrees of importance associated with the criteria that has the ith largest satisfaction c_i, such as $(v_{\sigma(i)}, c_{\sigma(i)})$ and $T = \sum_{i=1}^{n} v_{\sigma(i)}$, the total sum of degrees of importance. The linguistic quantifiers Q can be presented in the form of (Zadeh, 1983):

$$Q(r) = \begin{cases} 0 & if \ \ r \leq a, \\ \dfrac{(r-a)}{(b-a)} & if \ a < r < b, \\ 1 & if \ \ r \geq b, \end{cases} \tag{B.4}$$

with $a, b, r \in [0,1]$. For example, the semantic '*most*', '*almost all*' and '*at least half*' can be given as parameters (a, b) with $(0.35, 0.7), (0, 0.5)$ and $(0.5, 1)$, respectively.

Alternatively, the associated weights for the OWA operator can be obtained directly from its argument values. This method is known as the neat OWA operator and it can be defined as the following.

Definition B.2 (Yager, 1993). Neat OWA or weight-dependent OWA operator is a function $F_{NOWA}: \mathbb{R}^n \rightarrow \mathbb{R}$, defined as:

$$F_{NOWA}(a_1, a_2, \dots, a_n) = \sum_{i=1}^{n} w_i\left(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)}\right) a_{\sigma(i)} \tag{B.5}$$

where $a_{\sigma(i)}$ is the argument value a_i with any permutation and the vector valued function $w: \mathbb{R}^n \rightarrow [0,1]^n$ is normalized such that $\sum_{i=1}^{n} w_i(a_1, a_2, \dots, a_n) = 1$.

The neat OWA meets the properties of idempotency, commutativity and boundedness. However the monotonicity property is generally lost. The arithmetic mean is one of the examples of neat OWA.

In addition, the induced OWA operator is another useful aggregation operator that deal with the different ordering step. Instead of ordering the arguments with respect to their magnitudes such in the OWA operator, the additional parameters called order-inducing variables are used to induce the arguments. The definition of IOWA can be given as follows.

Definition B.3 (Yager & Filev, 1999). An IOWA operator of dimension n is mapping $IOWA: \mathbb{R}^n \rightarrow \mathbb{R}$ that has an associated weighting vector W such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, given by the following formula:

$$I - F_{OWA}(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \ldots, \langle u_n, a_n\rangle) = \sum_{i=1}^{n} w_i a_{\sigma(i)} \qquad \text{(B.6)}$$

where $a_{\sigma(i)}$ is the argument value of pair $\langle u_i, a_i\rangle$ of order-inducing variable u_i, reordered such that $u_{\sigma(i)} \geq u_{\sigma(i+1)}$. The IOWA operators are all satisfying commutative, monotonic, bounded and idempotent properties.

B.2.2 Linguistic labels

The input of the decision analysis can be represented in various forms, such as in qualitative and quantitative forms. In the case of qualitative form, the linguistic labels are used to capture the information based on the subjective evaluation such as '*poor*', '*good*', '*very good*', etc. The general definition of linguistic labels can be given as follows:

Definition B.4 (Herrera & Herrera-Viedma, 2000). Let a set of linguistic labels, $S = \{s_0, s_1, \ldots, s_{max}\}$ be uniformly distributed on a scale, then, the ordering is defined as $(s_a, s_b) \in S, s_a < s_b \Leftrightarrow a < b$ with s_0 and s_{max} are the lowest and the highest elements, respectively. The max is given as $|S| - 1$, where $|S|$ denotes the cardinality of S.

As stated by Herrera and Herrera-Viedma (2000), the cardinality of S must be small enough so as not to impose useless precision on the experts and it must be rich enough in order to allow discrimination of the performances of each object in a limited number of grades. In the literature, there are many approaches which proposed to compute with the linguistic labels. In this study, the method by Bordogna et al. (1997) is applied, where the linguistic labels are converted directly to the numerical values to deal with the operations in numerical environment. Finally, the results based on numerical values are reconverted to the linguistic labels as the final ranking purpose.

Definition B.5 (Bordogna et al., 1997). The conversion of the linguistic labels to the numbers in unit interval $\mathbb{I} \in [0,1]$ can be conducted by using the function $Label^{-1}$ defined as: $Label^{-1}: S \rightarrow [0,1]$, $Label^{-1}(s_i) = \frac{i}{|S|-1}$ with $i = 0,1, \ldots, max$. Whilst, the retranslation from the numerical values into the linguistic labels can be given as: $Label(x) = s_i$ for $\frac{i}{|S|} \leq x < \frac{i+1}{|S|}$, $i = 0,1,\ldots,max$ and $Label\ (1) = S_{max}$.

B.3 Aggregation Functions based on Majority-Additive OWA

In this section, a review of the definitions and basic properties of MA-OWA, selective MA-OWA and selective aggregated majority-OWA operators are presented prior to the definitions of WSAM-OWA and QWSAM-OWA operators.

Definition B.6 (Peláez & Doña, 2003a). A MA-OWA operator is a function $F_{MA}: \mathbb{R}^n \times \mathbb{N}^N \to \mathbb{R}$ defined as:

$$F_{MA}(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} w_{i,N} b_{\sigma(i)}, \tag{B.7}$$

where $N = \max_{1 \leq i \leq n} m_i$ and σ denotes a permutation of group of argument b_i with respect to the cardinality m_i, such that $b_{\sigma(i)} \geq b_{\sigma(i+1)}$. The weights associated to the arguments are defined by the recurrence relations:

$$w_{i,1} = \frac{1}{u_1} = \frac{1}{n}: u_1 = n, \tag{B.8}$$

$$w_{i,k} = \frac{\gamma_{i,k} + w_{i,k-1}}{u_k}: \forall k, 2 \leq k \leq N, \tag{B.9}$$

where $u_k = 1 + \sum_{j=1}^{n} \gamma_{j,k}$, and $\sum_{i=1}^{n} w_{i,k} = 1$, for $k = N$, such that:

$$\gamma_{j,k} = \begin{cases} 1 & m_{\sigma(j)} \geq k, \\ 0 & otherwise. \end{cases} \tag{B.10}$$

Note that k factor represents the current cardinality considered at a moment in the aggregation process. The MA-OWA operators meet all the bounded, idempotent and commutative properties. However the monotonicity is preserved if only if the cardinality vector, m is exactly the same in both aggregate sets, i.e., $F_{MA,w}(b, m) \geq F_{MA,w}(d, m), b \geq d, \forall j$. Moreover, the MA-OWA reduces to arithmetic mean, $F_{MA}(a_1, a_2, \ldots, a_n) = F_{AM}(a_1, a_2, \ldots, a_n)$ if all cardinalities, $m_i = 1$ (Peláez & Doña, 2003a).

Example B.1. Assume that $A = \langle a_1, \dots a_i, \dots, a_n \rangle \in \mathbb{R}^n \times \mathbb{N}^n$ where $a_i = (b_i, m_i)$ represents the aggregate value b_i, and its cardinality $m_i > 0$. For $A = \{(0.6, 1), (0.2, 1), (0.1, 3)\}$, the MA-OWA can be computed as the following.

Table B.1. Values of $\gamma_{i,k}$ and u_k

	$b_{\sigma(1)}$ 0.6 $m_{\sigma(1)}$	$b_{\sigma(2)}$ 0.2 $m_{\sigma(2)}$	$b_{\sigma(3)}$ 0.1 $m_{\sigma(3)}$	$\delta = 1$
$\gamma_{i,k}$	1	1	3	u_k
$\gamma_{i,1}$	1	1	1	3
$\gamma_{i,2}$	0	0	1	2
$\gamma_{i,3}$	0	0	1	2

The cardinal-dependent weights can be given as:

$$w_{1,3} = \frac{1}{2}\left(0 + \frac{1}{2}\left(0 + \frac{1}{3}1\right)\right) = \frac{1}{12},$$

$$w_{2,3} = \frac{1}{2}\left(0 + \frac{1}{2}\left(0 + \frac{1}{3}1\right)\right) = \frac{1}{12},$$

$$w_{3,3} = \frac{1}{2}\left(1 + \frac{1}{2}\left(1 + \frac{1}{3}1\right)\right) = \frac{5}{6},$$

Then, the MA-OWA operator for $\delta = 1$ can be derived as:

$$F_{MA}(\{(0.6, 1), (0.2, 1), (0.1, 3)\}) = 0.6 \cdot \frac{1}{12} + 0.2 \cdot \frac{1}{12} + 0.1 \cdot \frac{5}{6} = 0.150.$$

Whilst for $\delta = 0.5$, the MA-OWA operator yields: $F_{MA} = F_{AM} = 0.220$.

As can be seen, the MA-OWA indicates the better result for the majority opinion than AM, as 80% of the argument values are equal and less than 0.2 and 60% is 0.1. Hence, the representative value should be in between these two values or closer to 0.1.

As mentioned earlier, the main goal of the MA-OWA operator is to determine a synthesized value with considering all the information, i.e., the majority opinion and the minority opinion. However in certain cases, the minority opinion is excluded in the aggregation process due to the huge different between the cardinalities of arguments. In this case, the weight $w_{i,N} = 0$ is obtained for the minority opinion, whilst $w_{i,N} = 1$ is given for the majority opinion. To deal with this problem, Karanik et al., (2016) proposed the selective MA-OWA operator where the cardinality relevance factor (CRF) is introduced to weaken the $\gamma_{j,k}$ values in MA-OWA as to obtain the weight, $w_{i,N} > 0$ for the minority opinion.

Definition B.7 (Karanik et al., 2016). A SMA-OWA operator is a function $F_{SMA}: \mathbb{R}^n \times \mathbb{N}^N \to \mathbb{R}$ defined as:

$$F_{SMA}(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} w_{i,N} b_{\sigma(i)}, \tag{B.11}$$

where $N = \max_{1 \le i \le n} m_i$ and σ denotes a permutation with respect to the cardinality m_i, such that $b_{\sigma(i)} \geq b_{\sigma(i+1)}$. Their weights are defined by the recurrence relations, such in Eq. (B.8) and Eq. (B.9), given that $u_k = 1 + \sum_{j=1}^{n} \gamma_{j,k}$ and $\sum_{i=1}^{n} w_{i,k} = 1$, for $k = N$, such that:

$$\gamma_{j,k} = \begin{cases} \delta & m_{\sigma(j)} \geq k, \\ 1 - \delta & otherwise. \end{cases} \tag{B.12}$$

The parameter δ is the cardinality relevance factor (CRF) with $\delta \in [0,1]$.

By assigning the appropriate value for CRF, the minority opinion can be included in the aggregation process, specifically, by increasing its associated weight, such that, $w_{i,N} > 0$. The behavior of CRF value can be explained as the following. For $\delta \to 1$, the opinion with the largest cardinality (majority of opinion) is more emphasized than the opinion with the smallest cardinality. Hence, it is given a higher weight than the others. On the contrary, if $\delta \to 0$, the opinion with the smallest cardinality is given more priority than the largest cardinality. Meanwhile, if $\delta = 0.5$, the AM of the arguments is obtained, $F_{SMA} = F_{AM}$ such that all the cardinalities of arguments are reduced to cardinality $m_i = 1$. It can be demonstrated that the properties of idempotency, commutativity and boundedness hold for the SMA-OWA. However, the monotonicity is preserved only if the cardinality vector is exactly the same in both aggregate sets (Karanik et al., 2016).

In other related work, Peláez et al. (2016) proposed the selective aggregated majority-OWA operator as a generalization of the SMA-OWA where weights are assigned to different group of arguments based on their cardinalities. The definition of SAM-OWA operator can be given as the following.

Definition B.8 (Peláez et al., 2016). A SAM-OWA operator is a function $F_{SAM}: \mathbb{R}^n \times \mathbb{N}^N \to \mathbb{R}$ defined as:

$$F_{SAM}(a_1, a_2, \dots, a_n) = \sum_{i=1}^{n} w_{i,N} b_{\sigma(i)}, \tag{B.13}$$

where $N = \max_{1 \le i \le n} m_i$ and σ denotes a permutation with respect to the cardinality m_i, such that $b_{\sigma(i)} \ge b_{\sigma(i+1)}$. The associated weights are defined by the recurrent relations:

$$w_{i,1} = w_i = \frac{m_i}{\sum_{j=1}^{n} m_j}, \tag{B.14}$$

$$w_{i,k} = \frac{w_i \gamma_{i,k} y_k + w_{i,k-1}}{z_k}, \tag{B.15}$$

$$y_1 = 1, y_k = \begin{cases} 1, & if \sum_{j=1}^{n} w_j \gamma_{j,k} = 0, \\ \dfrac{\sum_{j=1}^{n} \gamma_{j,k}}{\sum_{j=1}^{n} w_j \gamma_{j,k}}, & otherwise, \end{cases} \tag{B.16}$$

$$z_1 = 1, z_k = \begin{cases} 1, & if \sum_{j=1}^{n} w_j \gamma_{j,k} = 0, \\ 1 + \sum_{j=1}^{n} \gamma_{j,k}, & otherwise, \end{cases} \tag{B.17}$$

where $\gamma_{j,k}$ is defined in the similar way as Eq. (B.12), δ is the cardinality relevance factor (CRF) such that $\delta \in [0,1]$ and $1 \le i \le n$, $2 \le k \le N$.

Example B.2. Consider again the previous example where a set of aggregated values is given as $A = \{(0.6, 1), (0.2, 1), (0.1, 3)\}$. The weights $w_{i,1} = w_i$ then can be obtained as: $w_{1,1} = 1/5$, $w_{2,1} = 1/5$ and $w_{3,1} = 3/5$.

The final cardinal-dependent weights are derived as:

$$w_{1,3} = 0.050, w_{2,3} = 0.050, w_{3,3} = 0.900,$$

and the SAM-OWA operator for $\delta = 1$ yields:

$$F_{SAM}(\{(0.6, 1), (0.2, 1), (0.1, 3)\}) = 0.130.$$

However, as can be noticed, the individual weights in the MA-OWA and SMA-OWA operators are distributed uniformly to each group of arguments, i.e., $w_{i,1} = 1/u_1 = 1/n$. Thus, for each aggregated value a_i in (b_i, m_i), the weight can be given as $1/u_1 m_i = 1/nm_i$. On the contrary, for the SAM-OWA operator, the individual weights are distributed proportionally to each group of opinions, i.e., $w_{i,1} = w_i = m_i / \sum_{j=1}^{n} m_j$, such that, the weights are uniformly distributed to each argument a_i.

In general, the aggregated values in MA-OWA, SMA-OWA and SAM-OWA are independent of the degrees of importance or the reliability of information sources. In the context of group decision making, they can be considered as the homogenous GDM problems. However, under the heterogeneous GDM problems, each argument value is associated with the degree of importance as to reflect the knowledge, expertise or experience of each expert. Hence, in the next section, the weighted SAM-OWA operator is proposed as an extension of the SAM-OWA operator to deal with the mentioned problem. In addition, the quantified WSAM-OWA operator for the group fusion strategy and the QWSAM-Induced OWA for the individual fusion strategy are presented.

B.4 Weighted SAM-OWA Aggregation Functions

In this section, the WSAM-OWA operator is presented. In addition, the QWSAM-OWA and QWSAM-IOWA operators are proposed as its generalization and extension.

B.4.1 Weighted SAM-OWA operator

Definition B.9. A WSAM-OWA operator is a function $F_{WSAM}: \mathbb{R}^n \times \mathbb{N}^N \to \mathbb{R}$ that has an associated weighting vector V of dimension n such that $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, defined as:

$$F_{WSAM}(a_1, a_2, \dots, a_n) = \sum_{i=1}^{n} w_{i,N} b_{\sigma(i)}, \tag{B.18}$$

where $N = \max_{1 \le i \le n} m_i$ and σ denotes a permutation with respect to the cardinality m_i. The associated weights are defined by the recurrent relations:

$$w_{i,1} = \omega_i = \begin{cases} v_i, & if \;\; m_i = 1, \\ \sum_{i=1}^{m_i} v_i\,, & if \;\; m_i > 1, \end{cases} \tag{B.19}$$

and the cardinal-dependent weights are given as,

$$w_{i,k} = \frac{\omega_i \gamma_{i,k} y_k + w_{i,k-1}}{z_k}, \tag{B.20}$$

$$y_1 = 1, y_k = \begin{cases} 1, & if \sum_{j=1}^{n} \omega_j \gamma_{j,k} = 0, \\ \dfrac{\sum_{j=1}^{n} \gamma_{j,k}}{\sum_{j=1}^{n} \omega_j \gamma_{j,k}}, & otherwise, \end{cases} \tag{B.21}$$

$$z_1 = 1, z_k = \begin{cases} 1, & if \sum_{j=1}^{n} \omega_j \gamma_{j,k} = 0, \\ 1 + \sum_{j=1}^{n} \gamma_{j,k}, & otherwise, \end{cases} \tag{B.22}$$

where $\gamma_{j,k}$ is defined in the similar way as Eq. (B.12), the parameter δ is the cardinality relevance factor (CRF) and $1 \leq i \leq n$, $2 \leq k \leq N$.

Similarly, it can be demonstrated that the WSAM-OWA operator meets the bounded, idempotent and monotonic properties. However, they are not commutative as involve the importance degrees or weighted arithmetic mean (WA).

Property B.1: Boundedness

Let m_k is the cardinality of the lowest argument value of vector A, if $m_k \to \infty$ and $\delta \to 1$, then $F_{WSAM}\big((b_i, m_i)\big) = b_k$, $Min[a_i]$.

Let m_k is the cardinality of the highest argument value of vector A, if $m_k \to \infty$ and $\delta \to 1$, then $F_{WSAM}\big((b_i, m_i)\big) = b_k$, $Max[a_i]$.
Hence, it is bounded by $Min[a_i] \leq F_{WSAM}\big((b_i, m_i)\big) \leq Max[a_i]$.

Property B.2: Idempotency

An aggregation function F_{WSAM} is idempotent if, $F_{WSAM}\big((b, m)\big) = b$ for any δ and m.

Property B.3: Monotonicity

The monotonicity is preserved if and only if the cardinality vector is exactly the same in both aggregate sets, i.e., $F_{WSAM}(b_i, m) \geq F_{WSAM}(d_i, m), b_i \geq d_i$ for all $i = 1,2, \ldots, n$.

Property B.4: Commutativity

An aggregation function F_{WSAM} is commutative if and only if $v_i = 1/n$ for all $i = 1,2, \dots, n$. Otherwise, it is not commutative.

Remark B.1. It can be demonstrated that for $\delta = 0.5$, then WSAM-OWA is reduced to WA, $F_{WSAM} = F_{WAM}$. In addition, for $\delta \to 1$, a higher weight is given to the argument with greater cardinality (majority opinion) and if $\delta \to 0$, then a higher weight is given to the argument with lower cardinality (minority opinion).

Remark B.2. Conversely, when $\omega_i = w_i$ (or $v_i = 1/n$), then WSAM-OWA is reduced to SAM-OWA, $F_{WSAM} = F_{SAM}$.

The issue that may arise in WSAM-OWA operator is how to aggregate the argument values based on cardinality with respect to the inclusion of the degrees of importance. In WA, the degrees of importance reflect the reliability of information sources, for example, given more priority to the most skilled or experience person. Nevertheless, the majority of information which represents the highest degree of importance is not directly emphasized in the WA. Here, the WSAM-OWA can be used to include both characteristics, i.e., the degrees of importance and the majority concept. Note that in the SAM-OWA operators, the emphasis is directly given on cardinality or majority opinion since the degrees of importance are uniform. In WSAM-OWA, the CRF is suggested as a tolerant factor in considering the majority of similar values and the degrees of importance simultaneously. This value can be derived as the following formula (Karanik, et al., 2016):

$$\delta = 1 - \left(2 + s^2\left(m_{\sigma(i)}\right)\right)^{-1} \tag{B.23}$$

where $s^2\left(m_{\sigma(i)}\right)$ is the variance of cardinality values, such that $\delta \in [0,1]$. Notice that in Karanik et al. (2016) the expected value is calculated as $E\left(m_{\sigma(i)}\right) = \sum_{i=1}^{n} w_{i,1}\, m_{\sigma(i)}$, where $w_{i,1} = 1/n$.

For the case of WSAM-OWA, the degrees of importance, $w_{i,1} = \omega_i$ are used such in Eq. (B.19), then the variance can be given as $s^2\left(m_{\sigma(i)}\right) = \sum_{i=1}^{n} \omega_i \left(m_{\sigma(i)} - E\left(m_{\sigma(i)}\right)\right)^2$. Hence, by formulating in this way, the influence of the degrees of importance is taken into account in deriving the CRF value for the overall aggregation process. Should be noted that, in the case of WSAM-OWA, the CRF is applied to provide a compensation between the degrees of importance and the cardinalities of aggregated values instead of the obtaining the $w_{i,1} > 0$ for the minority opinion.

Remark B.3. It can be shown that for $\omega_k = 1$ and $\omega_i = 0$ for all $i \neq k$, then $F_{WSAM}\big((b_i, m_i)\big) = b_k$ for any $\delta = (0,1]$.

Example B.3: Given that $A = \langle 0.6, 0.2, 0.1, 0.1, 0.1\rangle$ and their associated weights are provided as $V = \langle 0.1,0.1,0.3,0.3,0.2\rangle$. For simplicity it can be represented as $A = \{(0.6, 1, 0.1), (0.2, 1, 0.1), (0.1, 3, 0.8)\}$, where $a_i = (b_i, m_i, \omega_i)$. Based on the cardinalities and degrees of importance, the CRF can be determined as follows:

$$E\big(m_{\sigma(i)}\big) = (0.1 \cdot 1) + (0.1 \cdot 1) + (0.8 \cdot 3) = 2.6,$$

$$s^2\big(m_{\sigma(i)}\big) = 0.1(1 - 2.6)^2 + 0.1(1 - 2.6)^2 + 0.8(3 - 2.6)^2 = 0.64.$$

$$\delta = 1 - (2 + 0.64)^{-1} = 0.621$$

Table B.2. Values of $\gamma_{i,k}$, u_k and y_k

	$b_{\sigma(1)}$ 0.6 $m_{\sigma(1)}$	$b_{\sigma(2)}$ 0.2 $m_{\sigma(2)}$	$b_{\sigma(3)}$ 0.1 $m_{\sigma(3)}$	$\delta = 0.621$	
$\gamma_{i,k}$	1	1	3	u_k	y_k
$\gamma_{i,1}$	1	1	1	3	
$\gamma_{i,2}$	0.379	0.379	0.621	2.379	2.407
$\gamma_{i,3}$	0.379	0.379	0.621	2.379	2.368

The cardinal-dependent weights are:

$$w_{1,3} = 0.064, w_{2,3} = 0.064, w_{3,3} = 0.872,$$

and the WSAM-OWA operator yields:

$$F_{WSAM}(\{(0.6, 1, 0.1), (0.2, 1, 0.1), (0.1, 3, 0.8)\}) = 0.138.$$

In this example, the WA is given as, $F_{WAM} = 0.160$. Similarly to the MA-OWA, in this case, the representative value is expected to be closer to 0.1 as the highest weight (the total sum of individual weights) is belong to the group of arguments $b_3 = 0.1$, which is the majority opinion.

Example B.4: Assume that $A = \{(0.6, 1, 0.6), (0.2, 1, 0.1), (0.1, 3, 0.3)\}$ where $V = \langle 0.6, 0.1, 0.1, 0.1, 0.1 \rangle$. In this example, the highest weight is associated with the minority opinion. Based on the cardinalities and the degrees of importance, the CRF is obtained as 0.648.

The cardinal-dependent weights are derived as:

$$w_{1,3} = 0.457, w_{2,3} = 0.076, w_{3,3} = 0.467,$$

and the WSAM-OWA operator yields:

$$F_{WSAM}(\{(0.6, 1, 0.6), (0.2, 1, 0.1), (0.1, 3, 0.3)\}) = 0.336.$$

In this example, the WA is given as, $F_{WAM} = 0.410$. As can be seen, this value is lower than WA which reflects the majority opinion with the relevancy of the degrees of importance.

B.4.2 Quantified weighted SAM-OWA operators

In the previous section, all the majority operators take into account not only the majority opinion but also the minority opinion in deriving the aggregated value. As mentioned by Peláez et al. (2007), this definition in general uses the majority semantics which consider '*all*' of the arguments, but it is not able to model the majority concepts like '*most*' or '*at least 80%*' of arguments. Hence, Peláez et al. (2007) proposed the inclusion of linguistic quantifiers as to generalize the MA-OWA operator. Two quantified weights in MA-OWA operators were introduced, namely the individual fusion strategy and the group fusion strategy. The individual fusion strategy can be explained as applying the semantics of quantifier on each individual weight of the aggregation process. Whilst, for the group fusion strategy, the semantics on each group of arguments (i.e., with respect to their cardinalities) is applied. Analogously, in this study, both decision strategies can be extended to the case of WSAM-OWA operator. The method for the group fusion strategy can be applied directly to the case of WSAM-OWA since the ordering of the group of cardinalities is not affecting the overall result. The definition of the group fusion strategy of WSAM-OWA is given as the following.

Definition B.10. A QWSAM-OWA operator under the group fusion strategy is a function $F_{WSAM}: \mathbb{R}^n \times \mathbb{N}^N \to \mathbb{R}$ that has an associated weighting vector V of dimension n such that $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, defined as:

$$F_{QWSAM}(a_1, a_2, \dots, a_n) = \sum\nolimits_{i=1}^{n} w_i^{Q-G} b_{\sigma(i)} \tag{B.24}$$

where $N = \max_{1 \le i \le n} m_i$ and the weights are defined by the recurrent relations such in F_{WSAM}. The weights for the group fusion strategy can be presented as in the following expression (Peláez et al., 2007):

$$w_i^{Q-G} = \frac{\omega_i}{m_i} \cdot \sum\nolimits_{j=1}^{m_i} Q\left(\frac{j}{m_i}\right)$$
$$+ \left[\sum\nolimits_{j=1}^{m_i} Q\left(\frac{j}{m_i}\right) \cdot \frac{1 - \sum_{i=1}^{n} \frac{\omega_i}{m_i} \cdot \sum_{j=1}^{m_i} Q\left(\frac{j}{m_i}\right)}{\sum_{i=1}^{n} \sum_{j=1}^{m_i} Q\left(\frac{j}{m_i}\right)}\right], \tag{B.25}$$

where Q is the quantifier, n is the number of majority groups and m_i is the cardinality of the group i. This fusion strategy avoids the exclusion of any group in the aggregation process. Moreover, in this way it is possible to eliminate the distribution problems in the group decision making problems.

Example B.5: By extending the previous example (*Example B.4*), the group fusion strategy using the QWSAM-OWA operator can be implemented. Firstly, the cardinal-dependent weight vector is obtained, $W = [0.064, 0.064, 0.872]$ as in F_{WSAM}. After that, the value of the quantifier with semantics '*most*' such in Eq. (B.4) can be calculated for each group. The Q vectors for each majority group are obtained as:

Group with cardinality, $m = 1$: $[1]$,

Group with cardinality, $m = 1$: $[1]$,

Group with cardinality, $m = 3$: $[0, 0.633, 1]$.

Then, the quantified weight vector for the group fusion strategy is obtained as $W^{Q-G} = [0.173, 0.173, 0.653]$, where:

$$w_1^{Q-G} = \frac{0.064}{1} \cdot 1 + 1 \cdot \frac{1 - 0.603}{3.633} = 0.173,$$

$$w_2^{Q-G} = \frac{0.064}{1} \cdot 1 + 1 \cdot \frac{1 - 0.603}{3.633} = 0.173,$$

$$w_2^{Q-G} = \frac{0.872}{3} \cdot 1.633 + 1.633 \cdot \frac{1 - 0.603}{3.633} = 0.653.$$

Finally, the QWSAM-OWA operator for the group fusion strategy yields:

$$F_{QWSAM}(\{(0.6, 1, 0.3), (0.2, 1, 0.3), (0.1, 3, 0.4)\}) = 0.211.$$

B.4.3 Quantified weighted SAM-IOWA operators

For the individual fusion strategy, an extension of QMA-OWA to the QWSAM-IOWA is proposed as to deal with the issue of reordering process. As can be noticed, in this case each weight, $w_{i,N}$ is multiplied by the linguistic quantifier, $Q(i/n)$ of monotonically non-decreasing function. Peláez et al. (2007) suggests the reordering of arguments with respect to their cardinalities, i.e., in non-decreasing order such that, the greater the cardinality of argument, then the higher weight is associated to that argument. However, the problem may arise in the case where there are two or more arguments with identical cardinality, i.e., different order of these arguments may produce different results of the aggregation processes. For example, let say $(b_i, m_i) = (\langle 0.2,1 \rangle, \langle 0.3,1 \rangle, \langle 0,3 \rangle)$. The ordering of $(0.2,0.3,0,0,0)$ and $(0.3,0.2,0,0,0)$ then producing distinct results if the quantified weight vector is given as $V = [0,0.1,0.2,0.3,0.4]$. Hence, in this study, the extension of the individual fusion strategy to the case of IOWA operator is suggested, where the order-inducing variable reflects the similarity between arguments. Note that, in this case, both majority opinions and similarity between arguments are considered, but more emphasis is given to the most similar values. As can be seen, in this case, the order of $(0.3,0.2,0,0,0)$ is better represent the similarity between arguments. In the following, the definition of QWSAM-IOWA operator is presented.

Definition B.11. A QWSAM-IOWA operator of the individual fusion strategy is a function $I - F_{QWSAM}: \mathbb{R}^n \times \mathbb{N}^N \to \mathbb{R}$ that has an associated weighting vector V of dimension n such that $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, defined as:

$$I - F_{QWSAM}(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \dots, \langle u_n, a_n\rangle) = \sum_{i=1}^{n} w_i^{Q-I} a_{\sigma(i)} \tag{B.26}$$

where $a_{\sigma(i)}$ is the argument value of pair $\langle u_i, a_i\rangle$ of order-inducing variable u_i, with $u_{\sigma(i)} \leq u_{\sigma(i+1)}$, such that:

$$u_i = \left(\sum_{i=1}^{j} v_i s_i^{\lambda}\right)^{1/\lambda}, i = 1,2,\dots,n, \tag{B.27}$$

and $s_i = s(a_i, a_j) = 1 - |a_i - a_j|$ is a similarity measure between each argument a_i with respect to arguments a_j, $(j = 1,2,\dots,n)$, $i \in j$ and λ is a parameter in a range $\lambda \in \mathbb{R} \setminus \{0\}$. The individual fusion weight w_i^{Q-I} is obtained from the following equation:

$$w_i^{Q-I} = \frac{w_{i,N} \cdot v_i}{\omega_i} \cdot Q\left(\frac{i}{n}\right) + \left[Q\left(\frac{i}{n}\right) \cdot \frac{1 - \sum_{i=1}^{n}\left(\frac{w_{i,N} \cdot v_i}{\omega_i} \cdot Q\left(\frac{i}{n}\right)\right)}{\sum_{i=1}^{n} Q\left(\frac{i}{n}\right)}\right], \tag{B.28}$$

where $w_{i,N}$ is the weight determined by the recurrent relations such in F_{WSAM} for $N = \max_{1 \leq i \leq n} m_i$, Q is the linguistic quantifier and the expression in the bracket is the Q-normalization. It can be demonstrated that, the QWSAM-IOWA satisfies bounded, idempotent, monotonic properties. However, it is not commutative as it involves the WA.

As can be noticed, in this expression, some modifications have been made to the original QMA-OWA of individual fusion strategy where the weight, $w_{i,N}$ is multiplied by v_i/ω_i to decompose its individual weights proportionally with respect to their degrees of importance. In the original form (Peláez et. al., 2007), the weight $w_{i,N}$ is divided equally with respect to its cardinality. Moreover, the order-inducing variable is introduced to order the arguments with respect to their degrees of similarity and also resolve the issue of ordering problem.

Example B.6: Let $A = \langle 0.6, 0.2, 0.1, 0.1, 0.1 \rangle$ and its weight vector is provided as $V = \langle 0.1,0.1,0.3,0.3,0.2 \rangle$. The individual fusion strategy using the QWSAM-IOWA operator with semantics '*most*' can be computed as the following. Firstly, the order-inducing variable is computed for each argument:

$$u_1 = \sum_{i=1}^{5} v_i s_i = 0.1(1 - |0.6 - 0.6|) + 0.1(1 - |0.6 - 0.2|)$$
$$+0.3(1 - |0.6 - 0.1|) + 0.3(1 - |0.6 - 0.1|)$$
$$+0.2(1 - |0.6 - 0.1|) = 0.56.$$

Similarly, the rest of order-inducing variables can be determined, such that:

$$A = \langle (0.56,0.6), (0.88,0.2), (0.94,0.1), (0.94,0.1), (0.94,0.1) \rangle.$$

Secondly, the F_{WSAM} aggregation operator is applied to obtain the cardinal-dependent weighting vector, $W_N = [0.064, 0.064, 0.872]$. The individual weighting vector W is given as:

$$W = [0.064, 0.064, 0.327, 0.327, 0.218],$$

where $w_{3,N} = 0.872$ can be decomposed to: $w_{3,N}^1 = w_{3,N}^2 = (0.872 \times 0.3)/0.8 = 0.327$, $w_{3,N}^3 = (0.872 \times 0.2)/0.8 = 0.218$. Then, the individual weights w_i^{Q-I} are calculated using the above expression:

$$W^{Q-I} = [0, 0.017, 0.162, 0.389, 0.432]$$

Finally, the WSAM-OWA operator for individual fusion strategy yields:

$$I - F_{QWSAM}(\{(u_1, 0.6, 1, 0.1), (u_2, 0.2, 1, 0.1), (u_3, 0.1,0.3),$$
$$(u_4, 0.94,0.1), (u_5, 0.94,0.1)\}) = 0.102.$$

B.5 Multi-Criteria Group Decision Making Under Linguistic Domain

In this section, a multi-criteria group decision making model under the linguistic domain is developed. Two-stage aggregation processes are involved, in particular, the proposed WSAM-OWA operator and its extensions are used as group aggregators. On the other hand, the classical OWA operator with the inclusion of degrees of importance is applied to aggregate the criteria as the final ranking. The proposed model is based on the extension of Bordogna-Fedrizzi-Pasi model (Bordogna et al., 1997), specifically it is extended to the case of alternative scheme. The inputs provided by the experts are based on the linguistic labels. These inputs are then directly converted to the numeric values in unit interval $\mathbb{I} \in [0,1]$ to simplify the aggregation process. The algorithm of the proposed model is explained step by step as the following.

- *Stage 1: Majority aggregation for experts' judgments*

Step 1: Construct a decision matrix of dimension $M \times N$ for each expert, $D^h, (h = 1,2, \dots, k)$ as follows:

$$D^h = \begin{matrix} & C_1 & \dots & C_n \\ A_1 & a_{11}^h & \cdots & a_{1n}^h \\ \vdots & \vdots & \ddots & \vdots \\ A_m & a_{m1}^h & \cdots & a_{mn}^h \end{matrix}, \quad (B.29)$$

where A_i indicates the alternative i $(i = 1,2, \dots, m)$, C_j denotes the criterion j $(j = 1,2, \dots, n)$, and a_{ij}^h denotes the preferences for alternative A_i with respect to criterion C_j. The input value a_{ij}^h is the linguistic label provided by each expert based on the predefined linguistic scale, S.

Step 2: Determine the degree of importance (or trust) of each expert with respect to each criterion, such that, $T = \{t_1, t_2, \dots, t_k\}$. The degree of importance, t_h is drawn from the same linguistic scale, S.

Step 3: Transform the performance labels and the importance labels of all experts into the numeric values by applying the function $Label^{-1}: S \rightarrow [0,1]$. Then, the numeric value $t_h \in T$ is normalized to form $\hat{T} = \{\hat{t}_1, \hat{t}_2, \dots, \hat{t}_k\}$, where $\hat{t}_h = t_l / \sum_{h=1}^{k} t_h$, such that $\sum_{h=1}^{k} \hat{t}_h = 1$. With respect to each criterion, the transformed values (performance and importance labels of each expert) are used to determine the cardinality relevance factor (CRF), δ such in Eq. (B.23).

Step 4: Aggregate the experts' preferences using the WSAM-OWA operator to form a group decision matrix: Eqs. (B.18 – B.22). Note that, at this stage, the decision strategy (consensus on experts) can also be implemented by specifying the semantics '*most*' and manipulated either using the group fusion strategy: Eqs. (B.24 – B.25) or the individual fusion strategy: Eqs. (B.26 – B.28).

- *Stage 2: Aggregation of criteria and ranking process*

Step 5: Determine the importance degrees of criteria, $V = (v_1, v_2, \dots, v_n)$, such that v_j are drawn from the linguistic scale, S. Then, these weights are transformed to the numerical values using the function $Label^{-1}: S \rightarrow [0,1]$. At this stage, the OWA weights can be computed using the Eq. (B.3).

Step 6: Aggregate the judgment matrix of the majority of experts using the OWA operator such in Eq. (B.1) with respect to the weighting vector obtained in *Step 5*. Finally, rank the alternatives based on their values. Note that here, the proportion of criteria is subject to the attitudinal character of the majority of experts. Specifically, by assigning any semantics to the linguistic quantifiers, specifically in Eq. (B.3), various decision strategies can be obtained.

B.6 Numerical Example

In this section, an investment selection problem is studied where a group of experts or analysts are assigned for the judgment and selection of an optimal strategy. Assume that a company plans to invest some money in one or several available options (allocated proportionally based on their rankings). Primarily, five possible investment options are considered as follows: A_1 = hedge funds, A_2 = investment funds, A_3 = bonds, A_4 = stocks and A_5 = equity derivatives. These investment options are described with respect to the following characteristics: C_1 = benefits in the short term, C_2 = benefits in the long term, C_3 = risk of the investment, C_4 = social responsible investment and C_5 = difficulty of the investment.

In order to evaluate these options, the investor has brought together a group of experts which consist of five persons; with different backgrounds or areas of expertise. To enable the experts to formulate their judgments in a natural way, a set S of linguistic labels is supplied. For example, S can be defined so as its elements are uniformly distributed on a scale on which a total order is defined as:

$$S = \left\{ \begin{matrix} s_0 = none, \ s_1 = very\ low, \ s_2 = low, \ s_3 = medium, \\ s_4 = high, \ s_5 = very\ high, \ s_6 = perfect \end{matrix} \right\},$$

in which $s_a < s_b$ if and only if $a < b$. Based on this linguistic scale S, a decision matrix for each expert can be constructed for options A_i with respect to the characteristics C_j as shown in Table B.3 and the reliability of each expert on specific criterion is given in Table B.4.

Table B.3. Available investment strategies of each expert, E_h

E_1					E_2					E_3				
C_1	C_2	C_3	C_4	C_5	C_1	C_2	C_3	C_4	C_5	C_1	C_2	C_3	C_4	C_5
s_3	s_2	s_3	s_2	s_5	s_2	s_5	s_6	s_5	s_5	s_1	s_3	s_5	s_4	s_5
s_4	s_6	s_1	s_6	s_2	s_6	s_3	s_1	s_6	s_4	s_5	s_5	s_1	s_6	s_3
s_2	s_3	s_2	s_2	s_1	s_1	s_5	s_4	s_3	s_2	s_4	s_4	s_3	s_3	s_2
s_5	s_2	s_4	s_6	s_5	s_5	s_1	s_3	s_6	s_5	s_5	s_1	s_4	s_6	s_3
s_1	s_3	s_3	s_4	s_5	s_3	s_3	s_5	s_5	s_5	s_4	s_3	s_4	s_5	s_4

	E_4						E_5				
	C_1	C_2	C_3	C_4	C_5		C_1	C_2	C_3	C_4	C_5
A_1	s_1	s_3	s_5	s_4	s_4	A_1	s_1	s_2	s_3	s_2	s_4
A_2	s_5	s_3	s_2	s_5	s_2	A_2	s_5	s_4	s_1	s_5	s_1
A_3	s_2	s_2	s_1	s_4	s_1	A_3	s_2	s_2	s_1	s_4	s_3
A_4	s_3	s_1	s_3	s_3	s_5	A_4	s_4	s_3	s_2	s_5	s_4
A_5	s_2	s_2	s_3	s_4	s_5	A_5	s_1	s_2	s_4	s_2	s_4

Table B.4. Reliability of experts on each criterion

	E_1	E_2	E_3	E_4	E_5
C_1	s_5	s_4	s_5	s_3	s_3
C_2	s_4	s_5	s_3	s_4	s_4
C_3	s_3	s_3	s_5	s_4	s_5
C_4	s_4	s_4	s_5	s_4	s_3
C_5	s_3	s_4	s_4	s_5	s_4

At this stage, after transforming the preference labels and the importance labels into numbers in $\mathbb{I} = [0,1]$, the group aggregation based on majority concept can be implemented. For example, the computation for the majority aggregation of option A_1 with respect to characteristic C_1 can be shown as follows:

$$A_1 = \{E_1 = s_3, E_2 = s_2, E_3 = s_1, E_4 = s_1, E_5 = s_1\},$$

$$A_1 = \left\{ \begin{matrix} Label^{-1}(high), Label^{-1}(medium),\ Label^{-1}(low), \\ Label^{-1}(low), Label^{-1}(low) \end{matrix} \right\},$$

$$A_1 = \{0.667, 0.5, 0.333, 0.333, 0.333\} = \{(0.667,1), (0.5,1), (0.333,3)\}.$$

Similarly, weights are transformed to the numerical values:

$$T = \{E_1 = s_5, E_2 = s_4, E_3 = s_5, E_4 = s_3, E_5 = s_3\}$$

$$T = \{Label^{-1}(very\ high), Label^{-1}(high), Label^{-1}(very\ high), \\ Label^{-1}(medium), Label^{-1}(medium)\},$$

then $T = \{0.833, 0.667, 0.833, 0.5, 0.5\}$ and they are normalized so that the sum of all weights is one, $\hat{T} = \{0.25, 0.2, 0.25, 0.15, 0.15\}$.

Based on the cardinalities and the normalized degrees of importance, the CRF can determined and is given as $\delta = 0.666$. Then the resulted cardinal-dependent weights are:

$$w_{1,3} = 0.155, w_{2,3} = 0.124, w_{3,3} = 0.720,$$

and the WSAM-OWA operator on C_1 yields:

$$F_{WSAM}(\{(0.667, 1), (0.5, 1), (0.333, 3)\}) = 0.406,$$

The overall aggregated results of majority opinions based on WSAM-OWA are given in Table B.5.

Table B.5. Majority opinion based on WSAM-OWA

			E_{maj}		
	C_1	C_2	C_3	C_4	C_5
A_1	0.406	0.466	0.722	0.545	0.779
A_2	0.828	0.681	0.177	0.938	0.403
A_3	0.349	0.514	0.354	0.528	0.290
A_4	0.761	0.239	0.552	0.919	0.772
A_5	0.391	0.430	0.623	0.664	0.779

Having the decision matrix which represent the majority opinion of experts on each criteria, then the aggregation process to aggregate the final judgment or ranking of alternatives are conducted, where the weight of each criterion is provided as s_4, s_5, s_5, s_3, s_3, for each criterion C_1, C_2, C_3, C_4 and C_5, respectively. For example, the computation for A_1 can be given as the following:

$$I_{numeric} = [C_1 = I_4, C_2 = I_5, C_3 = I_5, C_4 = I_3, C_5 = I_3]$$

$$I_{numeric} = \{Label^{-1}(high), Label^{-1}(very\ high),\ Label^{-1}(very\ high), Label^{-1}(medium), Label^{-1}(medium)\},$$

$$I_{numeric} = \{I_4 = 0.667, I_5 = 0.833, I_5 = 0.833, I_3 = 0.5, I_3 = 0.5\}$$

The weight vector W_{most} is then obtained by applying the Eq. (3): $W_{most} = [0,0.2,0.3,0.5,0]$. The overall aggregation process can be determined using classical OWA operator, Eq. (B.1):

$$F_{OWA-W_{most}}(0.406, 0.466, 0.722, 0.545, 0.779) = 0.5411.$$

Finally, the linguistic overall performance value is obtained as:

$$Label(0.5411) = s_3 = medium.$$

The aggregated results for the entire alternatives are presented in Table B.6. In addition, the aggregated results based on SMA-OWA and SAM-OWA are also given as to see the results of the majority aggregation processes without the inclusion of the degrees of importance.

Table B.6. Overall aggregated results based on SMA-OWA, SAM-OWA and WSAM-OWA

	SMA-OWA	R	SAM-OWA	R	WSAM-OWA	R
A_1	S_3, 0.5372	4	S_3, 0.5544	4	S_3, 0.5411	3
A_2	S_4, 0.5738	2	S_4, 0.6034	2	S_3, 0.5619	2
A_3	S_2, 0.3557	5	S_2, 0.3975	5	S_2, 0.3842	5
A_4	S_4, 0.6837	1	S_4, 0.6646	1	S_4, 0.6391	1
A_5	S_3, 0.5708	3	S_3, 0.5672	3	S_3, 0.5069	4

*Note: R = ranking

In the case where only '*most*' of the experts are needed for the overall decision, then, the individual fusion strategy or the group fusion strategy can be implemented as given in the Table B.7 and Table B.8. Note that, the results of the individual fusion strategy are derived based on QWSAM-IOWA operator, whilst, the group fusion strategy is mainly based on QWSAM-OWA.

Table B.7. Majority opinion and overall aggregated results based on QWSAM-IOWA

			E_{maj}			Overall	
	C_1	C_2	C_3	C_4	C_5	Aggregation	Ranking
A_1	0.3384	0.4566	0.7523	0.5762	0.8273	S_3, 0.5102	2
A_2	0.8384	0.5991	0.1667	0.9935	0.3990	S_3, 0.4526	4
A_3	0.3435	0.4324	0.2756	0.6214	0.2952	S_2, 0.3243	5
A_4	0.8283	0.1770	0.6261	0.9952	0.8289	S_4, 0.5969	1
A_5	0.2955	0.4928	0.6261	0.7881	0.8273	S_3, 0.4672	3

Table B.8. Majority opinion and overall aggregated results based on QWSAM-OWA

			E_{maj}			Overall	
	C_1	C_2	C_3	C_4	C_5	Aggregation	Ranking
A_1	0.4510	0.5056	0.7470	0.5774	0.7688	S_3, 0.5458	1
A_2	0.8282	0.7160	0.2011	0.9287	0.4166	S_3, 0.5232	3
A_3	0.3639	0.5494	0.3931	0.5115	0.3089	S_2, 0.3971	5
A_4	0.7157	0.2843	0.5299	0.8553	0.7248	S_3, 0.5365	2
A_5	0.4176	0.4238	0.6422	0.6339	0.7668	S_3, 0.5072	4

4.4 Linguistic Group Decision Making Model with Dempster-Shafer Theory and Induced Linguistic Aggregation Operators

Abstract. In this study, a new approach for linguistic group decision making with Dempster-Shafer belief structure by applying the 2-tuple linguistic representation model is presented. By using this model, we are able to represent the D-S approach with linguistic information and without loss of information in the computing process. For doing so, it is suggested the use of different types of linguistic aggregation operators such as the 2-tuple induced linguistic ordered weighted averaging (2-TILOWA) operator. It is an extension of the OWA operator that uses complex attitudinal characters based on order-inducing variables in the reordering of the arguments and uncertain situations that can be assessed with linguistic information. By using the 2-TILOWA in the D-S framework, we form the belief structure - 2-tuple induced linguistic ordered weighted averaging (BS-2-TILOWA) operator. Some of its main properties are studied. The study ends with an application of the new approach in a decision making problem regarding selection of financial strategies.

C.1 Introduction

The Dempster-Shafer (D-S) theory of evidence (Dempster, 1967; Shafer, 1976) provides a unifying framework for representing uncertainty because it includes the cases of risk and ignorance in the same formulation. Usually, when using the D-S theory in decision making, it is assumed that the available information is numerical (Engemann et al., 1996; Merigó & Casanovas, 2009; Merigó et al., 2013; Yager, 1992). However, this may not be the real situation found in the decision making problem. Sometimes, the information is vague or imprecise and it is necessary to use another approach to assess it such as the use of linguistic variables. This problem has already been considered by Merigó et al. (2010) when the available information can be assessed with a linguistic model that computes with words directly following the ideas of (Herrera & Martínez, 2000a; 2000b; Martinez & Herrera, 2012; Merigó & Gil-Lafuente, 2013).

In order to do that, it is necessary to use an aggregation operator that aggregates the available information. The ordered weighted averaging (OWA) operator (Yager, 1988) is a very well-known aggregation operator for fusing the information that provides a parameterized family of aggregation operators between the minimum and the maximum. It has been used in a lot of applications (Beliakov et al., 2007; Belles-Sampera et al., 2013; Yager, 2004; Yager & Kacprzyk, 1997). Among these extensions and applications, some of them have focused on the use of linguistic information in the OWA operator (see, for example, Merigó et al., 2012; Xu, 2006; Zhang, 2013)

Another interesting extension is the induced OWA operator (Yager & Filev, 1999). It uses a more complex reordering process of the arguments by using order-inducing variables in the analysis. Then, we are able to assess more complex attitudinal characters rather than the degree of optimism. The IOWA operator has been studied by different authors, such as, Merigó and Gil-Lafuente (2009), Yager (2003b), Yager and Kacprzyk (1997), to name a few.

In this study, we further extend the analysis done in (Merigó et al., 2010) considering a situation where the available information cannot be assessed with numerical values but it is possible to use the 2-tuple linguistic representation model. Thus, we assume that we have a decision making problem where we assess the available probabilistic information with the D-S theory and the linguistic information using the 2-tuple linguistic model. Note that the 2-tuple linguistic approach was introduced by (Herrera & Martínez, 2000) to facilitate computing with words (CWW) processes.

The aggregation of the linguistic information, is carried out with different types of linguistic aggregation operators such as the ones described in (Jin et al., 2013; Li et al., 2008; Xu, 2008). The reason for doing this, is that we want to show that the linguistic decision making problem with D-S theory can be assessed in different ways depending on the interests of the decision maker. We will use the 2-tuple induced linguistic ordered weighted averaging (2-TILOWA) operator and all its particular cases such as the 2-tuple linguistic average (2-TLA) and the 2-tuple linguistic weighted average (2-TLWA). Then, we will get a new aggregation operator, the belief structure – 2-tuple LOWA (BS-2-TLOWA) operator.

We further generalize this approach by using generalized and quasi-arithmetic means in the analysis. Then, we get a more complete formulation of the analysis because we are able to consider a lot of other possibilities in the aggregation process. We use the 2-TILGOWA and the Quasi-2-TILOWA operator, obtaining the BS-2-TILGOWA and the BS-Quasi-2TILOWA operator, respectively. We also develop an application of the new approach in a linguistic decision making problem about selection of financial strategies. The main advantage of this approach is the possibility of considering a wide range of linguistic aggregation operators. Therefore, the decision maker gets a more complete view of the problem and he is able to select the alternative that it is in accordance with his interests.

The remainder of the study is organized as follows. In Section C.2, we briefly describe some basic concepts about the 2-tuple linguistic representation model and the D-S theory. In Section C.3, we introduce the new decision making approach. In Section C.4, we generalize the model by using generalized and quasi-arithmetic means. Then, in Section C.5, we present an application of the new model in financial selection problem.

C.2 Preliminaries

In this section, we briefly review some basic concepts about the 2-tuple linguistic representation model, the 2-TILOWA operator and the Dempster-Shafer theory of evidence.

C.2.1 The 2-tuple linguistic representation model

Normally, human activities and decisions are carried out in a quantitative setting, where the information is expressed by means of numerical values. However, many problems of the real world cannot be assessed in a quantitative form. Instead, it is possible to use a qualitative one, i.e., with vague or imprecise knowledge. In this case, a better approach may be the use of linguistic assessments instead of numerical values. The linguistic approach represents qualitative aspects as linguistic values by means of linguistic variables (Zadeh, 1975).

We have to select the appropriate linguistic descriptors for the term set and their semantics. One possibility for generating the linguistic term set consists in directly supplying the term set by considering all terms distributed on a scale on which a total order is defined. For example, a set of seven terms S could be given as follows:

$$S = \{s_1 = N, s_2 = VL, s_3 = L, s_4 = M, s_5 = H, s_6 = VH, s_7 = P\}.$$

Note that N = *None*, VL = *Very low*, L = *Low*, M = *Medium*, H = *High*, VH = *Very high*, P = *Perfect*. Usually, in these cases, it is required that in the linguistic term set there exists:

- A negation operator: neg(s_i) = s_j such that $j = g+1-i$.
- The set is ordered: $s_i \leq s_j$ if and only if $i \leq j$.
- Max operator: max(s_i, s_j) = s_i if $s_i \geq s_j$.
- Min operator: min(s_i, s_j) = s_i if $s_i \leq s_j$.

Different models have been presented in the literature for dealing with linguistic information such as (Herrera et al., 1995; Herrera et al., 2008; Xu, 2004a, 2004b). In (Herrera & Martínez, 2000a; 2000b), they presented a fuzzy linguistic representation model, which represents the linguistic information with a pair of values called 2-tuple, (s, α), where s is a linguistic label and α is a numerical value that represents the value of the symbolic translation. With this approach, it is possible to accomplish computing with words (CWW) processes without loss of information, solving one of the main problems of the previous linguistic computational models (Bonissone, 1982; Zadeh, 1975).

Definition C.1. Let β be the result of an aggregation of the indexes of a set of labels assessed in the linguistic label set $S = \{s_0, s_1, \ldots, s_g\}$, i.e., the result of a symbolic aggregation operation. $\beta \in [0, g]$, being $g + 1$ the cardinality of S. Let i = round(β) and $\alpha = \beta - i$ be two values, such that, $i \in [0, g]$ and $\alpha \in [-0.5, 0.5)$, then α is called a symbolic translation.

Note that the 2-tuple (s_i, α) that expresses the equivalent information to β is obtained with the following function:

$$\Delta : [0, g] \rightarrow S \times [-0.5, 0.5),$$

$$\Delta(\beta) = \begin{cases} s_i & i = round(\beta), \\ \alpha = \beta - i & \alpha \in [-0.5, 0.5). \end{cases} \tag{C.1}$$

where round is the usual round operation, s_i has the closest index label to β and α is the value of the symbolic translation. For more information on the 2-tuple linguistic representation models, they can be referred in Martinez and Herrera (2012), Wei (2011), and Xu and Wang (2011).

C.2.2 2-tuple linguistic aggregation operators

In the literature, we find a wide range of 2-tuple linguistic aggregation operators, for instance, Wang and Hao (2006), Wei (2009; 2011), Wei et al. (2013), Xu et al. (2013), Zeng et al. (2012), and Zhang (2013). In this study, we use the 2-tuple induced linguistic OWA (2-TILOWA) operator. It is a linguistic aggregation operator that uses the 2-tuple linguistic representation model and order-inducing variables in the OWA operator. It can be defined as follows.

Definition C.2. Let $\hat{S}$ be the set of the 2-tuples. A 2-TILOWA operator of dimension n is a mapping f: $\hat{S}^n \times \hat{S}^n \rightarrow \hat{S}$, which has an associated weighting vector W such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$f((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n)) = \Delta(\sum_{j=1}^{n} w_j \beta_j), \tag{C.2}$$

where $\beta_j = \Delta^{-1}(s_j, \alpha_j)$ are the argument values $\Delta^{-1}(s_i, \alpha_i)$ of the 2-TILOWA triplets (u_i, s_i, α_i) ordered in decreasing order of their u_i.

Note that it is possible to distinguish between descending (2-TDILOWA) and ascending (2-TAILOWA) orders. The weights of these operators are related by $w_j = w^*_{n+1-j}$, where w_j is the jth weight of the 2-TDILOWA (or 2-TILOWA) operator and w^*_{n+1-j} the jth weight of the 2-TAILOWA operator.

Remark C.1: If B is a vector corresponding to the ordered arguments β_j, we shall call this the linguistic ordered argument vector and W^T is the transpose of the weighting vector, then, the 2-TILOWA operator can be expressed as:

$$f((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n)) = W^T B. \tag{C.3}$$

Remark C.2: Note that if the weighting vector is not normalized, i.e., $W = \sum_{j=1}^{n} w_j \neq 1$, then, the 2-TILOWA operator can be expressed as:

$$f((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n)) = \frac{1}{W} \Delta(\sum_{j=1}^{n} w_j \beta_j). \tag{C.4}$$

By using a different weighting vector W, it is possible to study a wide range of families of 2-TILOWA operators such as the olympic-2-TILOWA, the S-2-TILOWA and centered-2-TILOWA. For further information, refer, i.e., to (Merigó & Gil-Lafuente, 2013; Yager, 1993).

The 2-TILOWA operator can be generalized by using generalized and quasi-arithmetic means. By using generalized means, we get the following definition ((Merigó & Gil-Lafuente, 2013).

Definition C.3. Let $\hat{S}$ be the set of the 2-tuples. A 2-TILGOWA operator of dimension n is a mapping f: $\hat{S}^n \times \hat{S}^n \rightarrow \hat{S}$, which has an associated weighting vector W such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$f((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n)) = \Delta\left(\sum_{j=1}^{n} w_j \beta_j^{\lambda}\right)^{1/\lambda}, \tag{C.5}$$

where $\beta_j = \Delta^{-1}(s_j, \alpha_j)$ are the argument values $\Delta^{-1}(s_i, \alpha_i)$ of the 2-TILGOWA triplets (u_i, s_i, α_i) ordered in decreasing order of their u_i, and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

And if we use quasi-arithmetic means, then, we get the following definition (Merigó and Gil-Lafuente, 2013). Note that the Quasi-2-TILOWA operator includes the 2-TILGOWA operator as a particular case.

Definition C.4. Let $\hat{S}$ be the set of the 2-tuples. A Quasi-2-TILOWA operator of dimension n is a mapping f: $\hat{S}^n \times \hat{S}^n \to \hat{S}$ that has an associated weighting vector W of dimension n such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, by a formula of the following form:

$$f((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n)) = g^{-1}\left(\sum_{j=1}^{n} w_j g(\beta_j)\right), \quad \text{(C.6)}$$

where $\beta_j = \Delta^{-1}(s_i, \alpha_j)$ are the argument values $\Delta^{-1}(s_i, \alpha_i)$ of the Quasi-2-TILOWA triplets (u_i, s_i, α_i) ordered in decreasing order of their u_i, and g is a strictly continuous monotonic function.

C.2.3 Dempster-Shafer Theory of Evidence

The D-S theory of evidence was introduced by (Dempster, 1967; Shafer, 1976). Since then, a lot of new developments have been developed (Le et al., 2007; Reformat & Yager, 2008; Srivastava & Mock, 2002; Yager & Liu, 2008). This type of formulation provides a unifying framework for representing uncertainty as it can include the cases of risk and ignorance as special situations of this framework. Obviously, the case of certainty is also included in this generalization as it can be seen as a particular situation of risk or ignorance. Apart from these traditional cases, the D-S framework allows to represent various other forms of information that a decision maker may have about the states of nature.

Definition C.5. A D-S belief structure defined on a space X consists of a collection of n non-null subsets of X, B_j for $j = 1,\ldots,n$, called focal elements and a mapping m, called the basic probability assignment, defined as, $m: 2^X \to [0, 1]$ such that:

- $m(B_j) \in [0, 1]$.
- $m(A) = 0, \ \forall A \neq B_j$.
- $\sum_{j=1}^{n} m(B_j) = 1$.

As mentioned before, the cases of risk and ignorance are included as special cases of belief structure in the D-S framework. For the case of risk, a belief structure is called Bayesian belief structure (Shafer, 1976), it consists of n focal elements such that $B_j = \{x_j\}$, where each focal element is a singleton. Then, we can see that we are in a situation of decision making under risk environment as $m(B_j) = P_j = \text{Prob}\{x_j\}$.

For the case of ignorance, the belief structure consists in only one focal element B, where $m(B)$ essentially is the decision making under ignorance environment as this focal element comprises all the states of nature. Thus, $m(B) = 1$. Other cases of belief structures are studied in (Shafer, 1976).

C.3 Linguistic Decision Making with D-S Theory and Induced Aggregation Operators

In this section, the decision making approach, the belief structures with 2-Tuple induced linguistic OWA operators and the families of BS-2-TILOWA operators are presented.

C.3.1 Decision making approach

Decision making with D-S belief structures has been studied by a wide range of authors (see, for instance, Casanovas & Merigó, 2012; Merigó & Casanovas, 2009; Yager, 1992). In these studies, the main assumption is that the available information is quantitative. However, many decision making problems cannot be assessed with numerical values because the decision maker's knowledge is vague and/or imprecise. Then, a better approach may be the use of linguistic assessments instead of numerical ones.

In this study, we develop an extension of this general approach for situations where the available information cannot be assessed with numerical values but it is possible to use the 2-tuple linguistic representation model. Thus, we are able to make computations with linguistic information without losing information in the problem. Moreover, we use induced aggregation operators in order to deal with complex reordering process that represent complex attitudinal characters of the decision-maker. We can summarize the approach as follows.

Assume we have a linguistic decision problem in which we have a collection of alternatives $\{A_1, \ldots, A_q\}$ with states of nature $\{N_1, \ldots, N_n\}$. (s_{hi}, α_{hi}) is the 2-tuple linguistic payoff to the decision maker if he selects alternative A_h and the state of nature is N_i. The knowledge of the state of nature is captured in terms of a belief structure m with focal elements $B_1, \ldots, B_r$ and associated with each of these focal elements is a weight $m(B_k)$. The objective of the problem is to select the alternative which best satisfies the 2-tuple linguistic payoff to the decision maker. In order to do so, we should follow the following steps:

Step 1: Calculate the 2-tuple linguistic payoff matrix.

Step 2: Calculate the belief function m about the states of nature and the decision makers degree of optimism.

Step 3: Calculate the collection of weights, *w*, to be used in the 2-TILOWA aggregation for each different cardinality of focal elements.

Step 4: Determine the 2-tuple linguistic payoff collection, M_{hk}, if we select alternative A_h and the focal element B_k occurs, for all the values of *h* and *k*. Hence $M_{hk} = \{(s_{hi}, \alpha_{hi}) \mid N_i \in B_k\}$.

Step 5: Calculate the linguistic aggregated payoff, V_{hk} = 2-TILOWA(M_{hk}), using Eq. (C.2), for all the values of *h* and *k*. Note that it is possible to use for each focal element a different type of 2-TILOWA operator.

Step 6: For each alternative, calculate the generalized 2-tuple linguistic expected value, (s_{hk}, α_{hk}), where:

$$(s_{hk}, \alpha_{hk}) = \sum_{k=1}^{r} V_{hk} m(B_k). \tag{C.7}$$

Step 7: Select the alternative with the largest (s_h, α_h) as the optimal.

Remark C.3: Sometimes it is better the use of the 2-TAILOWA operator in the D-S decision process instead of the 2-TILOWA operator. The main reason for this is that we have to distinguish between situations where the highest linguistic argument is the best result and situations where the smallest linguistic argument is the best result.

C.3.2 Belief structures with 2-tuple induced linguistic OWA operators

The aggregation in *Step 6* and *Step 7* can be integrated into a single equation that takes into account both processes. Thus, the result obtained is that the focal weights are aggregating the results obtained by using the 2-TILOWA operator. This process is called the belief structure-2-TILOWA (BS-2-TILOWA) aggregation and it can be defined as follows.

Definition C.6. A BS-2-TILOWA operator is defined by:

$$f((u_{1_1}, s_{1_1}, \alpha_{1_1}), \ldots, (u_{q_r}, s_{q_r}, \alpha_{q_r})) = \sum_{k=1}^{r} \sum_{j_k=1}^{q_k} m(B_k) w_{j_k} \beta^*_{j_k}, \tag{C.8}$$

where w_{j_k} is the weight of the *k*th focal element such that $\sum_{j=1}^{n} w_{j_k} = 1$ and $w_{j_k} \in [0, 1]$, $\beta_{j_k} = \Delta^{-1}(s_{j_k}, \alpha_{j_k})$ are the argument values $\Delta^{-1}(s_{i_k}, \alpha_{i_k})$ of the 2-TILOWA triplets $(u_{i_k}, s_{i_k}, \alpha_{i_k})$ ordered in decreasing order of their u_i, and $m(B_k)$ is the basic probability assignment.

Note that q_k refers to the cardinality of each focal element and r is the total number of focal elements.

The BS-2-TILOWA operator accomplishes the typical properties of the mean operators such as commutativity, monotonicity, boundedness and idempotency.

Theorem C.1 (Commutativity). Assume f is the BS-2-TILOWA operator, then:

$$f((u_{1_1}, s_{1_1}, \alpha_{1_1}),...,(u_{q_r}, s_{q_r}, \alpha_{q_r})) = f((u^*_{1_1}, s^*_{1_1}, \alpha^*_{1_1}),...,(u^*_{q_r}, s^*_{q_r}, \alpha^*_{q_r})), \tag{C.9}$$

where $((u^*_{1_1}, s^*_{1_1}, \alpha^*_{1_1}),...,(u^*_{q_r}, s^*_{q_r}, \alpha^*_{q_r}))$ is any permutation, for each focal element k, of $((u_{1_1}, s_{1_1}, \alpha_{1_1}),...,(u_{q_r}, s_{q_r}, \alpha_{q_r}))$.

Proof. It is trivial and thus omitted.

Theorem C.2 (Monotonicity). Assume f is the BS-2-TILOWA operator, if, $\beta_{j_k} \geq \beta^*_{j_k}$ $\forall$ i, then:

$$f((u_{1_1}, s_{1_1}, \alpha_{1_1}),...,(u_{q_r}, s_{q_r}, \alpha_{q_r})) \geq f((u^*_{1_1}, s^*_{1_1}, \alpha^*_{1_1}),...,(u^*_{q_r}, s^*_{q_r}, \alpha^*_{q_r})). \tag{C.10}$$

Proof. It is trivial and thus omitted.

Theorem C.3 (Boundedness). Assume f is the BS-2-TILOWA operator, then:

$$\min\{(u_{i_k}, s_{i_k}, \alpha_{i_k})\} \leq f((u_{1_1}, s_{1_1}, \alpha_{1_1}),...,(u_{q_r}, s_{q_r}, \alpha_{q_r})) \leq \max\{(u_{i_k}, s_{i_k}, \alpha_{i_k})\}. \tag{C.11}$$

Proof. It is trivial and thus omitted.

Theorem C.4 (Idempotency). Assume f is the BS-2-TILOWA operator, if $\beta_{j_k} = \beta$, $\forall\, j \in N$, then:

$$f((u_{1_1}, s_{1_1}, \alpha_{1_1}),...,(u_{q_r}, s_{q_r}, \alpha_{q_r})) = \beta. \tag{C.12}$$

Proof. It is trivial and thus omitted.

C.3.3 Families of BS-2-TILOWA operators

Different types of 2-TILGOWA operators are found in the aggregation by using a different weighting vector.

Remark C.4: For example, we can obtain the 2-tuple linguistic maximum, the 2-tuple linguistic minimum, the 2-tuple linguistic average (2-TLA), the 2-TLWA and the 2-TLOWA operator.

- The 2-tuple linguistic maximum is found if $w_1 = 1$ and $w_j = 0$, for all $j \neq 1$.
- The 2-tuple linguistic minimum is obtained if $w_n = 1$ and $w_j = 0$, for all $j \neq n$.
- More generally, if $w_k = 1$ and $w_j = 0$, for all $j \neq k$, we get the step-2-TILOWA operator.
- The 2-TLA is formed when $w_j = 1/n$, for all i.
- The 2-TLWA is obtained when the ordered position of i is the same as j.
- The 2-TLOWA is found if the ordered position of u_i is the same as the ordered position of the values of the a_i.

Remark C.5: Some other interesting families are the following. Note that they follow the same methodology as in the OWA version (Merigó & Gil-Lafuente, 2009; Yager, 1993; Yager & Kacprzyk, 1997):

- The S-2-TILOWA operator based on the S-OWA operator (Yager, 1993) can be subdivided in three classes:
  - The generalized S-2-TILOWA operator is obtained when $w_1 = (1/n)(1 - (\alpha + \beta)) + \alpha$, $w_n = (1/n)(1 - (\alpha + \beta)) + \beta$, and $w_j = (1/n)(1 - (\alpha + \beta))$ for $j = 2$ to $n - 1$ where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$.
  - If $\alpha = 0$, the generalized S-2-TILOWA operator becomes the "*and-like*" S-2-TILOWA
  - If $\beta = 0$, it becomes the "*or-like*" S-2-TILOWA.
  - Also note that if $\alpha + \beta = 1$, we get the 2-tuple induced linguistic Hurwicz criteria.
- The olympic-2-TILOWA is found when $w_1 = w_n = 0$, and for all others $w_{j*} = 1/(n - 2)$. Note that if $n = 3$ or $n = 4$, the olympic-2-TILOWA becomes the median-2-TILOWA and if $m = n - 2$ and $k = 2$, the window-2-TILOWA becomes the olympic-2-TILOWA.
- Note that it is possible to develop a general form of the olympic-2-TILOWA operator considering that $w_j = 0$ for $j = 1, 2, \ldots, k, n, n - 1, \ldots, n - k + 1$; and for all others $w_{j*} = 1/(n - 2k)$, where $k < n/2$. Note that if $k = 1$, then, this general form becomes the usual olympic-2-TILOWA. If $k = (n - 1)/2$, then, it becomes the median-2-TILOWA operator.
- It is also possible to develop the contrary case of the general olympic-2-TILOWA operator. In this case, $w_j = (1/2k)$ for $j = 1, 2, \ldots, k, n, n - 1, \ldots, n - k + 1$; and $w_j = 0$, for all others, where $k < n/2$. Note that if $k = 1$, then, we get the contrary case of the median-2-TILOWA.
- A 2-TILOWA operator is defined as a centered aggregation operator if it is symmetric, strongly decaying and inclusive.
  - It is symmetric if $w_j = w_{j+n-1}$.
  - It is strongly decaying when $i < j \leq (n + 1)/2$ then $w_i < w_j$ and when $i > j \geq (n + 1)/2$ then $w_i < w_j$.

- It is inclusive if $w_j > 0$.
- The nonmonotonic-2-TILOWA operator is found when at least one of the weights w_j is lower than 0 and $\sum_{j=1}^{n} w_j = 1$. Note that a key aspect of this operator is that it does not always accomplish the monotonicity property.

Remark C.6: Using a similar methodology, we could develop a lot of other families of 2-TILOWA weights in a similar way as it has been developed in a lot of studies for the OWA operator (Merigó & Gil-Lafuente, 2010, 2013; Yager, 1993; Zeng & Su, 2012; Zeng et al., 2012).

Remark C.7: Note that it is possible to use different families of 2-TILOWA operators for each focal element. If we strictly use only one case, then, we could refer to the aggregation as the BS-centered-2-TILOWA, BS-2-TLA, BS-2-TLWA and BS-2-TLOWA.

Remark C.8: Note that it is easy to apply these methods to the 2-TILOWA operator because the weights are not affected by the linguistic information. Obviously, it is also possible to develop more complex analysis where the weights are also linguistic variables but in this study we will not analyze this problem.

C.4 Generalized 2-TILOWA Operators in D-S Framework

Although we have already considered a wide range of 2-tuple linguistic aggregation operators that can be used in the D-S framework, it is interesting to present a general formulation that includes more types of 2-tuple linguistic aggregation operators. This formulation is carried out by using generalized and quasi-arithmetic means. The main advantage of using these operators is that they include a lot of linguistic aggregation operators. Therefore, the decision maker gets a more complete view of the decision problem because he is able to consider a lot of different situations and select the one that is in accordance with his interests. Thus, if we introduce this operator in decision making with D-S belief structure, we are able to develop a unifying framework that provides a general formulation with probabilities and different types of 2-TILOWAs.

In order to use this type of aggregation operator in D-S framework, we should make the following changes to the decision process explained in the previous section for the 2-TILOWA operator.

In *Step* 3, when calculating the collection of weights, *w*, we have to consider that we are using the 2-TILGOWA operator in the aggregation for each different cardinality of focal elements.

In *Step* 5, when calculating the linguistic aggregated payoff, we should use V_{hk} = 2-TILGOWA(M_{hk}), using Eq. (C.5) for all the values of h and k.

In this case, we could also formulate in one equation the whole aggregation process as follows. We call it the BS-2-TILGOWA operator.

Definition C.7. A BS-2-TILGOWA operator is defined by:

$$f((u_{1_1}, s_{1_1}, \alpha_{1_1}), \ldots, (u_{q_r}, s_{q_r}, \alpha_{q_r})) = \left(\sum_{k=1}^{r} \sum_{j_k=1}^{q_k} m(B_k) w_{j_k} \beta_{j_k}^{\lambda} \right)^{1/\lambda}, \qquad \text{(C.13)}$$

where w_{j_k} is the weighting vector of the kth focal element such that $\sum_{j=1}^{n} w_{j_k} = 1$ and $w_{j_k} \in [0, 1]$, where $\beta_{j_k} = \Delta^{-1}(s_{j_k}, \alpha_{j_k})$ are the argument values $\Delta^{-1}(s_{i_k}, \alpha_{i_k})$ of the 2-TILGOWA triplets $(u_{i_k}, s_{i_k}, \alpha_{i_k})$ ordered in decreasing order of their u_i, and λ is a parameter such that $\lambda \in (-\infty, \infty)$ and $m(B_k)$ is the basic probability assignment.

Note that q_k refers to the cardinality of each focal element and r is the total number of focal elements. As we can see, the focal weights are aggregating the results obtained by using the 2-TILGOWA operator.

Remark C.9: The BS-2-TILGOWA operator is commutative, monotonic, bounded and idempotent. Note that it is straightforward to prove these properties by looking at Theorems (C.1 – C.4).

Remark C.10: Note that it is also possible to distinguish between descending (BS-2-TDILGOWA) and ascending (BS-2-TAILGOWA) orders.

Remark C.11: When aggregating the collection of linguistic payoffs of each focal element with the 2-TILGOWA operator, it is also possible to use a wide range of families of 2-TILGOWA operators. Basically, we can distinguish between those cases found in the parameter λ and those found in the weighting vector W.

Remark C.12: By looking to the parameter λ, we find the following particular cases:

- The 2-TILOWA operator if λ = 1.
- The 2-TILOWG operator if λ approaches to 0.
- The 2-TILOWQA operator if λ = 2.
- The 2-TILOWHA operator if λ = –1.

Remark C.13: If we analyze the weighting vector W, then, we find the following cases:

- The 2-tuple linguistic maximum ($w_1 = 1$ and $w_j = 0$, for all $j \neq 1$).
- The 2-tuple linguistic minimum ($w_n = 1$ and $w_j = 0$, for all $j \neq n$).
- The 2-TLA ($w_j = 1/n$, for all K_i).
- The 2-TLWA (the ordered position of i is the same as the ordering established in u_i).
- The 2-TLGOWA (the ordered position of j is the same as the ordering established in u_i).
- The 2-TILGOWA with the Hurwicz criteria ($w_1 = \alpha$, $w_n = 1 - \alpha$ and $w_j = 0$, for all $j \neq 1, n$).
- The step-2-TILGOWA ($w_k = 1$ and $w_j = 0$, for all $j \neq k$).
- The olympic-2-TILGOWA operator ($w_1 = w_n = 0$, and $w_j = 1/(n - 2)$ for all others).
- The S-2-TILGOWA ($w_1 = (1/n)(1 - (\alpha + \beta)) + \alpha$, $w_n = (1/n)(1 - (\alpha + \beta) + \beta$, and $w_j = (1/n)(1 - (\alpha + \beta))$ for $j = 2$ to $n - 1$ where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$).
- The centered-2-TILGOWA (if it is symmetric, strongly decaying from the center to the maximum and the minimum, and inclusive).

Remark C.14: Note that it is also possible to use quasi-arithmetic means in the analysis in order to get a more general formulation of the problem. Then, instead of using the 2-TILGOWA operator in the decision process, we use the Quasi-2-TILOWA operator (Merigó & Gil-Lafuente, 2013) explained in Eq. (6). The general aggregation process formed in this case is the BS-Quasi-2-TILOWA operator and it can be defined as follows.

Definition C.8. A BS-Quasi-2-TILOWA operator is defined by:

$$f((u_{1_1}, s_{1_1}, \alpha_{1_1}), \ldots, (u_{q_r}, s_{q_r}, \alpha_{q_r})) = g^{-1}\left(\sum_{k=1}^{r} \sum_{j_k=1}^{q_k} m(B_k) w_{j_k} g(\beta_{j_k})\right), \quad \text{(C.14)}$$

where w_{j_k} is the weighting vector of the kth focal element such that $\sum_{j=1}^{n} w_{j_k} = 1$ and $w_{j_k} \in [0, 1]$, where $\beta_j = \Delta^{-1}(s_{j_k}, \alpha_{j_k})$ are the argument values $\Delta^{-1}(s_{i_k}, \alpha_{i_k})$ of the Quasi-2-TILOWA triplets $(u_{i_k}, s_{i_k}, \alpha_{i_k})$ ordered in decreasing order of their u_i, $\Delta^{-1}(s_{i_k}, \alpha_{i_k})$ is the argument variable, g is a strictly continuous monotonic function and $m(B_k)$ is the basic probability assignment.

Note that q_k refers to the cardinality of each focal element and r is the total number of focal elements.

Remark C.15: Note that all the properties and particular cases commented both in the BS-2-TILOWA and in the BS-2-TILGOWA are also applicable to the BS-Quasi-2-TILOWA operator. Thus, we could distinguish between descending and ascending orders and consider different particular cases.

C.5 Application in Linguistic Group Decision Making

Next, let us develop an illustrative example in order to clarify the procedures commented above. We analyze a decision making problem with D-S belief structure. We use different types of linguistic aggregation operators such as the 2-TLA, the 2-TLWA, the 2-TLOWA, the 2-TILOWA, the 2-TAILOWA and the 2-TILOWQA operator.

Step 1: Assume an enterprise is planning his financial strategy for the next year and they consider 5 alternatives.

1) A_1: Invest in the Asian market.
2) A_2: Invest in the American market.
3) A_3: Invest in the European market.
4) A_4: Invest in the African market.
5) A_5: Do not develop any investment.

Since the future states of nature are very imprecise, the experts cannot use numerical values in the payoff matrix. Instead, they use linguistic variables to calculate the future expected benefits of the enterprises depending on the state of nature that occurs in the future. They establish the following linguistic scale based on the 2-tuple approach.

$$S = \{s_1 = \textit{Extremely low}, s_2 = \textit{Very low}, s_3 = \textit{Low}, s_4 = \textit{Medium}, s_5 = \textit{High}, \\ s_6 = \textit{Very high}, s_7 = \textit{Extremely high}\}.$$

After careful analysis, the experts have considered five possible situations that could happen in the future: N_1 = Very bad, N_2 = Bad, N_3 = Regular, N_4 = Good, N_5 = Very good. Depending on different uncertain situations that could occur, the experts of the investment company establish the 2-tuple linguistic payoff matrix that represents the returns for the next year. The results that could happen in the future are shown in Table C.1.

Table C.1. Available investments

	N_1	N_2	N_3	N_4	N_5
A_1	$(s_5, -0.4)$	$(s_3, 0.2)$	$(s_3, -0.2)$	$(s_4, 0)$	$(s_5, -0.3)$
A_2	$(s_2, 0.4)$	$(s_3, 0)$	$(s_4, 0)$	$(s_3, 0.5)$	$(s_5, 0.2)$
A_3	$(s_4, 0)$	$(s_2, 0.4)$	$(s_3, 0.1)$	$(s_5, -0.4)$	$(s_5, -0.4)$
A_4	$(s_3, 0.4)$	$(s_2, 0.3)$	$(s_5, 0.2)$	$(s_4, 0.1)$	$(s_3, 0.3)$
A_5	$(s_2, 0.4)$	$(s_4, 0.2)$	$(s_4, 0.3)$	$(s_4, 0.1)$	$(s_3, 0.4)$

Step 2: The experts have obtained some empirical data that has permitted them to establish some probabilistic information about which state of nature will occur represented by the following belief structure.

Focal element
$B_1 = \{N_1, N_3\} = 0.3$
$B_2 = \{N_1, N_2, N_3\} = 0.3$
$B_3 = \{N_3, N_4, N_5\} = 0.4$

Step 3: Assume we have calculated the following weighting vectors for the 2-TILGOWA operator depending on the number of arguments used in the aggregation: $W_2 = (0.4, 0.6)$ and $W_3 = (0.3, 0.3, 0.4)$. Note that the reordering of the linguistic arguments is carried out with order-inducing variables when using induced aggregation operators. In this example, we use the order-inducing variables shown in Table C.2.

Table C.2. Order-inducing variables

	N_1	N_2	N_3	N_4	N_5
A_1	20	18	14	12	10
A_2	8	15	20	22	25
A_3	16	18	25	12	10
A_4	18	16	24	22	20
A_5	15	13	21	19	17

Step 4: Calculate the payoff collection, M_{ik}, if we select alternative A_i and the focal element B_k occurs, for all the values of i and k.

A_1: $M_{11} = \langle(s_4, 0.6), (s_3, -0.2)\rangle$; $M_{12} = \langle(s_4, 0.6), (s_3, 0.2), (s_3, -0.2)\rangle$;
$M_{13} = \langle(s_3, -0.2), (s_4, 0), (s_5, -0.3)\rangle$.

A_2: $M_{21} = \langle(s_2, 0.4), (s_4, 0)\rangle$; $M_{22} = \langle(s_2, 0.4), (s_3, 0), (s_4, 0)\rangle$;
$M_{23} = \langle(s_4, 0), (s_3, 0.5), (s_5, 0.2)\rangle$.

A_3: $M_{31} = \langle(s_4, 0), (s_3, 0.1)\rangle$; $M_{32} = \langle(s_4, 0), (s_2, 0.4), (s_3, 0.1)\rangle$;
$M_{33} = \langle(s_3, 0.1), (s_5, -0.4), (s_5, -0.4)\rangle$.

A_4: $M_{41} = \langle(s_3, 0.4), (s_5, 0.2)\rangle$; $M_{42} = \langle(s_3, 0.4), (s_2, 0.3), (s_5, 0.2)\rangle$;
$M_{43} = \langle(s_5, 0.2), (s_4, 0.1), (s_3, 0.3)\rangle$.

A_5: $M_{51} = \langle(s_2, 0.4), (s_4, 0.3)\rangle$; $M_{52} = \langle(s_2, 0.4), (s_4, 0.2), (s_4, 0.3)\rangle$;
$M_{53} = \langle(s_4, 0.3), (s_4, 0.1), (s_3, 0.4)\rangle$.

Step 5: Calculate the aggregated 2-tuple linguistic payoff, V_{hk}, using Eq. (C.5) (for the 2-TLA, 2-TLWA, 2-TLOWA and 2-TILOWA is also valid Eq. (C.2)). The results are shown in Table C.3.

Table C.3. Aggregated results

	2-TLA	2-TLWA	2-TLOWA	TILOWA	TILOWQA
V_{11}	$(s_4, -0.3)$	$(s_4, -0.48)$	$(s_4, -0.48)$	$(s_4, -0.48)$	$(s_4, -0.38)$
V_{12}	$(s_4, -0.47)$	$(s_3, 0.46)$	$(s_3, 0.46)$	$(s_3, 0.46)$	$(s_4, -0.46)$
V_{13}	$(s_4, -0.17)$	$(s_4, -0.08)$	$(s_4, -0.27)$	$(s_4, -0.08)$	$(s_4, -0.01)$
V_{21}	$(s_3, 0.20)$	$(s_3, 0.36)$	$(s_3, 0.04)$	$(s_3, 0.36)$	$(s_3, 0.45)$
V_{22}	$(s_3, 0.13)$	$(s_3, 0.22)$	$(s_3, 0.06)$	$(s_3, 0.12)$	$(s_3, 0.18)$
V_{23}	$(s_4, 0.23)$	$(s_4, 0.33)$	$(s_4, 0.16)$	$(s_4, 0.33)$	$(s_4, 0.39)$
V_{31}	$(s_4, 0.45)$	$(s_3, 0.46)$	$(s_3, 0.46)$	$(s_4, -0.36)$	$(s_4, -0.34)$
V_{32}	$(s_3, 0.16)$	$(s_3, 0.16)$	$(s_3, 0.09)$	$(s_3, 0.25)$	$(s_3, 0.31)$
V_{33}	$(s_4, 0.10)$	$(s_4, 0.15)$	$(s_4, 0.00)$	$(s_4, 0.15)$	$(s_4, 0.20)$
V_{41}	$(s_4, 0.30)$	$(s_4, 0.48)$	$(s_4, 0.12)$	$(s_4, 0.12)$	$(s_4, 0.21)$
V_{42}	$(s_4, -0.37)$	$(s_4, -0.21)$	$(s_3, 0.50)$	$(s_3, 0.50)$	$(s_4, -0.30)$
V_{43}	$(s_4, 0.20)$	$(s_4, 0.11)$	$(s_4, 0.11)$	$(s_4, 0.11)$	$(s_4, 0.18)$
V_{51}	$(s_3, 0.35)$	$(s_4, -0.46)$	$(s_3, 0.16)$	$(s_3, 0.16)$	$(s_3, 0.29)$
V_{52}	$(s_4, -0.37)$	$(s_4, -0.3)$	$(s_4, -0.49)$	$(s_4, -0.31)$	$(s_4, -0.22)$
V_{53}	$(s_4, -0.07)$	$(s_4, -0.12)$	$(s_4, -0.12)$	$(s_4, -0.12)$	$(s_4, -0.10)$

Step 6: For each alternative, calculate the linguistic generalized expected value, C_h, using Eq. (C.7). The results are shown in Table C.4.

Step 7: Select the best alternative for each 2-tuple linguistic aggregation operator. That is, select the investment with the highest linguistic expected value. As we can see, in this example we get that the optimal choice is always the investment A_4. Therefore, in this case, the decision is clear.

However, often we find that the optimal choice is different depending on the aggregation operator used. Thus, it is interesting to establish an ordering of the investments, a typical situation if we want to select more than one alternative. Note that $\succ$ means *preferred to*. The results are shown in Table C.5. As we can see, depending on the linguistic aggregation operator used, the results and decisions may be different. Therefore, the decision maker is able to consider a wide range of situations and select the one that is in accordance with his interests.

Table C.4. Linguistic generalized expected value

	2-TLA	2-TLWA	2-TLOWA	2-TILOWA	2-TILOWQA
A_1	$(s_4, -0.3)$	$(s_4, -0.34)$	$(s_4, -0.42)$	$(s_4, -0.34)$	$(s_4, -0.25)$
A_2	$(s_4, -0.41)$	$(s_4, -0.3)$	$(s_3, 0.49)$	$(s_4, -0.33)$	$(s_4, -0.26)$
A_3	$(s_4, -0.35)$	$(s_4, -0.36)$	$(s_4, -0.44)$	$(s_4, -0.28)$	$(s_4, -0.23)$
A_4	$(s_4, 0.05)$	$(s_4, 0.12)$	$(s_4, -0.07)$	$(s_4, -0.07)$	$(s_4, 0.04)$
A_5	$(s_4, -0.34)$	$(s_4, -0.28)$	$(s_4, -0.45)$	$(s_4, -0.4)$	$(s_4, -0.32)$

Table C.5. Ordering of the investments

	Ordering
2-TLA	$A_4 \succ A_1 \succ A_5 \succ A_3 \succ A_2$
2-TLWA	$A_4 \succ A_5 \succ A_2 \succ A_1 \succ A_3$
2-TLOWA	$A_4 \succ A_1 \succ A_3 \succ A_5 \succ A_2$
2-TILOWA	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$
2-TILOWQA	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$

4.5 Summary

To recap, in this chapter, the extensions of group decision making models with OWA-based aggregation operators have been discussed in great detail. Specifically, in Section 4.2, the analysis of OWA-based aggregation operators in ME-MCDM model has been studied. In the following section (Section 4.3), the WSAM-OWA operator and its application in the linguistic group decision making model has been proposed. Then, in Section 4.4, the method on the linguistic group decision making with Dempster-Shafer theory and induced linguistic aggregation operators has been presented. All these developed models then were applied in the case of financial selection problems.

CHAPTER 5

ANALYTIC HIERARCHY PROCESS MODELS FOR FINANCIAL SELECTION PROBLEMS

5.1 Introduction

In this chapter, the decision making models based on the analytic hierarchy process (AHP) are developed for the application in financial selection problems. In Section 5.2, the generalized AHP for group decision making model using the induced OWA operators is presented. Then, in Section 5.3, the heavy weighted geometric aggregation operators in AHP group decision making is given. Finally, Section 5.4 provides a summary to conclude this chapter.

5.2 Generalized Analytic Hierarchy Process for Group Decision Making Model using Induced OWA Operators

Abstract. This study proposes an extension of the analytic hierarchy process for group decision making model using the induced ordered weighted averaging (IOWA) operators. Two-stage aggregation processes used in the AHP-GDM model, particularly in aggregating the criteria and synthesizing a group of experts, are extended to provide a more general framework in the decision analysis. For the aggregation of criteria, a generalization of weighted maximal entropy OWA (WMEOWA) under the IOWA operator is proposed. This consists of the induced generalized and the induced quasi generalized WMEOWA operators. Then, the majority concept based on the IOWA and Minkowski OWA similarity measure is suggested to determine a consensus among experts. Two ways in computing the majority agreements for the AHP-GDM model are proposed in which measuring the similarity of experts with respect to the individual priorities of alternatives (classical scheme) and also on the individual preferences of criteria (alternative scheme). Based on the application of different decision schemes, distinct decision strategies of individual and majority of experts can be determined. The AHP-GDM model under the classical scheme is based on the individual decision strategies of experts. Whilst under the alternative scheme, the decision strategies are reflecting the majority of experts collectively. The final results obtained from both schemes shown slightly different rankings. With the inclusion of IOWA operator and its generalization, the AHP-GDM model provides a greater

flexibility in analyzing the alternatives with respect to different decision strategies. These include flexibility in evaluating each alternative based on different proportion of criteria (complex attitudinal character of experts), incorporating additional mechanisms for order-inducing process and allowing the majority of experts who have the most similar preferences as the representative results. The application in investment selection problem is presented to test the reliability of the proposed model.

D.1 Introduction

The analytic hierarchy process (Saaty, 1977; 1980) is one of the most widely used multiple criteria decision analysis techniques. The AHP model is popularly used in applications owing to several advantages. It has a hierarchy structure by reducing multiple criteria into pairwise comparison judgments and allows the use of quantitative and qualitative information in the evaluation process. Some reviews on application of the model can be referred, for example, in Golden et al. (1989), Forman and Gass (2001), Saaty (2013), and Vaidya and Kumar (2006). To deal with the complex decision making problems, particularly involving multiple experts' judgements, the AHP has been extended to the AHP group decision making model (AHP-GDM) (see Escobar & Moreno-Jimerez, 2007; Gargallo et al., 2007; Saaty, 1989, among others). Two-stage aggregation processes are involved in the AHP-GDM, specifically, the aggregation of criteria with respect to each alternative and the aggregation of a group of experts as the final ranking or priority of alternatives. Weighted arithmetic mean (WA) and weighted geometric mean (WG) are among the traditionally used synthesizing procedures in the AHP-GDM.

An ordered weighted averaging (OWA) operator (Yager, 1988) is a different type of aggregation operator. The OWA provides a general class of mean-type aggregation procedures, which comprises a family of functions that can be ranged between the '*and*' (min) and the '*or*' (max). A fundamental aspect of this operation is the reordering step. Specifically, an element or argument is not associated with a particular weight, but rather a weight is associated with a particular ordered position of argument (Yager, 1993). Implicitly, its meaning to some extent is based on the concept of fuzzy set theory (Bellman & Zadeh, 1970; Zadeh, 1983), allowing a flexibility in aggregation process. It provides a unified framework for decision analysis under uncertainty with different decision strategies, such as maximax (optimistic), maximin (pessimistic), equally likely (Laplace), and the Hurwicz procedure, where each is characterized by a specific OWA weighting vector. Yager (2004) generalized the OWA to include other types of means in the same formulation, for instance, the ordered weighted geometric (OWG), the ordered weighted harmonic (OWH), and the ordered weighted quadratic (OWQ), to name a few. In the literature, it has been known as the generalized OWA operators (GOWA). The induced OWA (Yager & Filev, 1999) is another extension of the OWA operator with a more general framework of synthesizing procedures. The main difference between OWA and IOWA is in the reordering process. Instead of directly ordering the arguments based on their magnitudes, such in OWA, the IOWA utilizes another mechanism called the order-inducing variable as a pair of argument. The main advantage of the IOWA operator is its ability in considering complex situations, in which the reordering process is done by other variables independently or by a function of arguments (Beliakov & James, 2011; Chiclana et al., 2004; Yager, 1999). Afterward, Merigó and Gil-Lafuente (2009) generalized the IOWA to the induced GOWA (IGOWA) to provide a unified framework for the aggregation process. The IOWA operators have been studied by many authors in recent years (see Chiclana et al., 2004; Merigó & Casanovas 2009; Merigó & Casanovas 2011; Xu, 2006; Yusoff & Merigó, 2014).

Recently, much attention has been given to the extension of MCDA models using the OWA operators. Yager and Kelman (1999) proposed the extension of AHP using the OWA operator. This approach generalizes the WA usually implemented in AHP by allowing flexibility in aggregating the criteria using the integration of the WA and OWA operators. Instead of taking '*averagely all*' criteria, the analysis can be made to include several cases in between two extreme conditions (i.e., '*at least*' one criterion must be satisfied and '*all*' criteria must be satisfied). This approach is mainly based on the inclusion of the fuzzy linguistic quantifiers (Yager, 1988, 1996). An alternative method to apply the OWA operator is using the maximum entropy OWA (MEOWA) approach. Initiated by O'Hagan (1988), MEOWA has been formulated as a constraint nonlinear optimization problem and has been applied in determining the weights of the OWA for the aggregation process. Filev and Yager (1995) and later Fuller and Majlender (2001) proposed the analytic approaches for MEOWA using the method of Lagrange multipliers and examined its analytic properties in great detail. Since then, MEOWA has been extensively studied and applied, specifically in MCDA models (Ahn, 2011; Chuu, 2009; Ma & Guo, 2013; Wang & Parkan, 2007). The main advantage of the MEOWA is that it can be used to model the behavior of decision makers or experts in facing with uncertain/risky decision problems, specifically by specifying any degree of optimism. With the inclusion of the relative importances of criteria, Yager (2009) then extended the MEOWA to the weighted MEOWA (WMEOWA) as the integrated approach. Other families of OWA operators can be referred (e.g., Xu, 2005; Yager, 1993).

Analogously, with respect to group decision making, the soft majority agreement among experts can be implemented using the OWA and IOWA operators to extend the classical WA or WG. For example, using the fuzzy linguistic quantifier with the semantics '*at least 80%*' or '*most*', the preferences of a group of experts can be determined. In this case, the majority opinion refers to a consensual judgment of experts who have similar preferences. This type of aggregation process is useful whenever a majority support among experts is adequate for the satisfactory decision, also to avoid bias in the decision analysis. In the literature, there are a number of works that have been done on this topic. Pelaez et al. (2003; 2005; 2016) and Karanik et al. (2016) for example, proposed the majority additive and majority multiplicative OWA operators to aggregate the arguments that have cardinality greater than one. On the other hand, Pasi and Yager (2006) proposed two approaches to deal with the majority concept. The first is based on the IOWA operator, where the support function (i.e., distance measure) is applied to obtain a set of scalar values to induce the opinions of experts as the majority agreement. The other approach is based on a fuzzy subset to represent the majority opinion under a vague concept. Correspondingly, Bordogna and Sterlacchini (2014) extended the Pasi-Yager method, specifically the first approach, by employing the Minkowski OWA similarity measure as a generalization of the support function. This method is used to measure the consensus among a group of experts with the inclusion of degrees of importance. Moreover, they proposed a consensus measure with respect to each specific criterion of experts (i.e., an alternative scheme) as an alternative approach instead of synthesizing the individual priorities of alternatives (a classical scheme).

The main focus of this study is to extend the two-stage of aggregation processes used in AHP-GDM by using the IOWA operators. This model is an extension of the AHP-based OWA model proposed by Yager and Kelman (1999) under the group decision making setting. Instead of using fuzzy linguistic quantifiers to aggregate the criteria of each alternative, the proposed method applies the MEOWA-based aggregation operators as an alternative approach to represent the decision strategies. Specifically, a generalization of WMEOWA under the IOWA aggregation functions is proposed, which consists of induced generalized and induced quasi generalized WMEOWA. Next, the classical means for aggregating the experts' judgments is extended to a majority-based aggregation process using the modified Bordogna-Sterlacchini method. In this setting, the fuzzy linguistic quantifier with the semantics '*most*' is applied in the IOWA operator. Two methods of computing the majority agreement of experts are studied. These include measuring the similarity of experts based on the individual priorities of alternatives (classical scheme) and on the individual preferences of criteria (alternative scheme). The proposed model with respect to these schemes is applied to the problem of selecting the optimal investment alternatives. The comparison of results obtained from the schemes under different decision strategies is conducted.

The structure of the study is organized as follows. In Section D.2, some preliminaries related to aggregation operators and the general frameworks for AHP are presented. Section D.3 briefly discusses the WMEOWA-based aggregation operators. Section D.4 examines the majority concept based on the

IOWA operator. In Section D.5, the proposed method on the extension of the AHP-GDM is presented. Finally, in Section D.6, an application in an investment selection problem is provided.

D.2 Preliminaries

In the following, the basic aggregation operators that are used in this study are briefly discussed. In consequence, a general framework for the AHP-GDM is reviewed.

D.2.1 Aggregation operators

Definition D.1 (Yager, 1988). An OWA operator of dimension n is a function $OWA_W: \mathbb{R}^n \rightarrow \mathbb{R}$ that has an associated weighting vector $W = (w_1, w_2 \dots, w_n)$ such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, given as the following formula:

$$OWA_W(a_1, a_2, \dots, a_n) = \sum\nolimits_{j=1}^{n} w_j \, a_{\sigma(j)}, \tag{D.1}$$

where $a_{\sigma(j)}$ is the jth largest a_j.

Definition D.2 (Yager & Filev, 1999). An IOWA operator of dimension n is a function $IOWA_W: \mathbb{R}^n \rightarrow \mathbb{R}$ that has an associated weighting vector W of dimension n such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, according to the following formula:

$$IOWA_W(\langle u_1, a_1 \rangle, \langle u_2, a_2 \rangle, \dots, \langle u_n, a_n \rangle) = \sum_{j=1}^{n} w_j a_{\sigma(j)}, \tag{D.2}$$

where $a_{\sigma(j)}$ is the a_j value of the IOWA pair $\langle u_j, a_j \rangle$ having the jth largest u_j, and the convention that if z of the $u_{\sigma(j)}$ are tied, i.e., $u_{\sigma(j)} = u_{\sigma(j+1)} = \cdots = u_{\sigma(j+z-1)}$, then, the value $a_{\sigma(j)}$ is given as follows (Beliakov & James, 2011; Yager & Filev, 1999):

$$a_{\sigma(j)} = \frac{1}{z} \sum_{k=\sigma(j)}^{\sigma(j+z-1)} a_k, \tag{D.3}$$

Definition D.3 (Merigó & Gil-Lafuente, 2009). An IGOWA operator of dimension n is a function $IGOWA: \mathbb{R}^n \rightarrow \mathbb{R}$ that has an associated weighting vector W of dimension n such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, given as the following formula:

$$(\langle u_1, a_1 \rangle, \langle u_2, a_2 \rangle, \ldots, \langle u_n, a_n \rangle) = \left(\sum_{j=1}^{n} w_j a_{\sigma(j)}^{\lambda} \right)^{1/\lambda}, \quad \text{(D.4)}$$

where $a_{\sigma(j)}$ is the a_j value of the IGOWA pair $\langle u_j, a_j \rangle$ having the jth largest u_j and λ is a parameter such that $\lambda \in \mathbb{R} \setminus \{0\}$.

With different values of λ, various type of aggregation functions can be derived. For instance, when $\lambda = -1$, IOWHA (harmonic) operator can be derived, when $\lambda \to 0$, then the IOWG (geometric) can be generated, for $\lambda = 2$, the IOWQA (quadratic) operator can be obtained, etc.

The OWA, the IOWA, and the IGOWA operators are all meet commutative, monotonic, bounded and idempotent properties (Beliakov & James, 2011; Grabisch et al., 2009; Merigó & Gil-Lafuente, 2009; Yager, 1988; Yager & Filev, 1999). Note that, the notation a_j is the argument variable and u_j is the order-inducing variable. In addition, Yager (1988) introduced two measures, namely the orness measure and the entropy (or dispersion) measure to characterize the type of aggregation associated with a given weighting vector, W.

Definition D.4 (Yager, 1988). Suppose that W is the associated weighting vector such that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, then the orness measure (or degree of optimism) of OWA can be given as the following:

$$\alpha(W) = \frac{1}{n-1} \sum_{j=1}^{n} (n-j) w_j. \quad \text{(D.5)}$$

It can be demonstrated that for any W, the value of $\alpha(W)$ lies in unit interval $[0,1]$. For example: i) if $W = W_*$ then $\alpha(W_*) = 0$ (pessimistic attitude), ii) if $W = W_{Ave}$ then $\alpha(W_{Ave}) = 1/2$ (neutral attitude, i.e., Laplace criterion), and iii) if $W = W^*$ then $\alpha(W^*) = 1$ (optimistic attitude), where $W_* = (0,0, \ldots, 1)$, $W_{Ave} = (1/n, 1/n, \ldots, 1/n)$ and $W^* = (1,0, \ldots, 0)$.

Definition D.5 (Yager, 1988). Suppose that W is the associated weighting vector such that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, then the entropy of OWA can be given as follows:

$$E(W) = -\sum_{j=1}^{n} w_j ln(w_j). \tag{D.6}$$

The entropy is used to measure the degree of information that employed in the OWA aggregation. It can be shown that it is bounded by $0 \leq E(W) \leq ln(n)$, in which $E(W_*) = E(W^*) = 0$ and $E(W_{Ave}) = ln(n)$.

D.2.2 Analytic hierarchy process model

The AHP is one of the existing preference relation methods, to be known as the multiplicative preference relation (Chiclana et al., 2004; Xu, 2007; Saaty, 1980). The general framework for AHP comprises of three major steps: i) developing the hierarchy (structuring complexity), ii) pairwise comparison of elements of the hierarchical structure (measurement on a ratio scale) and iii) constructing an overall priority rating (synthesis) (Forman & Gass, 2001; Saaty, 1980). The first step is the decomposition of the decision problem into a hierarchy, consists of the most related or important elements in the analysis, e.g., from the main goal (objective) to the level of criteria (sub-criteria) and the level of alternative. For the second step, the pairwise comparison matrix can be defined as the following:

Definition D.6 (Saaty, 1980). A multiplicative preference relation P on the set X (a discrete set of alternatives) is defined as a pairwise comparison matrix $A = [a_{i\bar{\imath}}]_{m\times m} \subset X \times X$ under the condition:

$$a_{i\bar{\imath}} > 0,\;\; a_{i\bar{\imath}} \cdot a_{\bar{\imath}i} = 1,\;\; a_{ii} = 1,\;\;\; \forall i, \bar{\imath} = 1,2,\dots,m, \tag{D.7}$$

where $a_{i\bar{\imath}}$ is the ratio of the preference intensity of alternative x_i to that of $x_{\bar{\imath}}$.

Specifically, it can be noticed that the matrix A is reciprocal, such that $a_{i\bar{\imath}} = a_{\bar{\imath}i}{}^{-1}$ for $(i \neq \bar{\imath})$ and all its diagonal elements are unity, $a_{i\bar{\imath}} = 1$, $(i = \bar{\imath})$. The intensity of preference $a_{i\bar{\imath}}$ basically is measured based on ratio-scale {1/9,…,9} as given by Saaty (1980), where: i) $a_{i\bar{\imath}} = 9$ means x_i absolutely preferred over $x_{\bar{\imath}}$; ii) $a_{i\bar{\imath}} = 1$ implies indifference between x_i and $x_{\bar{\imath}}$; iii) inversely $a_{i\bar{\imath}} = 1/9$ indicates that $x_{\bar{\imath}}$ absolutely preferred over x_i. Moreover, $a_{i\bar{\imath}} \cdot a_{\bar{\imath}i} = 1$ exhibits multiplicative reciprocal condition and $a_{i\bar{\imath}} = a_{ik} \cdot a_{k\bar{\imath}}$ signifies multiplicative transitivity. The indifference, reciprocal and transitivity are the main properties for multiplicative preference relation (Chiclana et al., 2004; Saaty, 1980; Xu, 2007).

Since human judgment is to some extent inconsistent in practice, Saaty (1980) then suggests a consistency index (CI) to measure the level of inconsistency associated with the pairwise comparison matrix as follows:

$$CI = \frac{\lambda_{max} - m}{m - 1}, \quad \text{(D.8)}$$

where λ_{max} is the biggest eigenvalue which obtained from its associated eigenvector and m is the number of columns of the matrix under consideration. Further, the consistency ratio (*CR*) can be calculated as follows:

$$CR = \frac{CI}{RI}, \quad \text{(D.9)}$$

where *RI* is the random index of a randomly generated pairwise comparison matrix. The value of *RI* depends on the number of elements being compared. For instance, $RI = 0$ for $m \leq 2$, $RI = 0.58$ for $m = 3$ and $RI = 0.9$ for $m = 4$, etc. The complete table of *RI* for $m > 4$ can referred in (Saaty, 1980) or Table G.4 in Chapter 6. The threshold value for consistency ratio (*CR*) is set as $CR < 0.10$ to indicate a reasonable level of consistency in the pairwise comparison.

The transformation of pairwise comparison matrix to a priority vector is known as the prioritization procedure, which is the third step of the AHP method. The eigenvector (EV) method and the row geometric mean method (RGMM) are regularly used procedures for local priorities (i.e., with respect to a single criterion). Equivalently, for a set of criteria, the overall evaluation scores (global priorities) for each alternative can be synthesized using the weighted arithmetic mean (WA) or the weighted geometric mean (WG). In general, the method based on the geometric means is the most commonly used in the literature as it is the only separable synthesizing function that meets the unanimity condition (Pareto principle), the homogeneity condition and reciprocal property (Aczel & Saaty, 1983; Escobar & Moreno-Jimenez, 2007).

In addition, under the group settings, there is an extra step involved, which is the aggregation of the experts' judgments as a group decision. Two approaches normally utilized in AHP-GDM can be categorized into: i) aggregation of individual judgments (AIJ) and ii) aggregation individual priorities (AIP) approaches (Forman & Peniwati, 1998; Ramanadhan & Ganesh, 1994; Saaty 1980). The AIJ can be explained as a group of experts which act together as a unit i.e., synergistic unit, whilst the AIP acts as separate individuals, i.e., a set of individual priorities/rankings of alternatives. For the AIJ, only geometric means-based methods are the most preferable, while for the AIP, both arithmetic and geometric means-based methods can be used (Forman & Peniwati, 1998). In this study, the AHP-GDM based on the AIP approach is employed and studied.

D.3 Maximum Entropy OWA-based Aggregation Functions

A MEOWA is developed based on the mathematical programming approach. It is used to generate a weighting vector by maximizing the entropy with subjects to the weight constraint and the degree of orness. It can be noticed that MEOWA weights used to spread the weights as uniformly as possible and at the same time satisfying the degree of orness. Note that, if unconstrained by the degree of orness, the solution would be to make $w_j = 1/n$.

In the context of decision making, the degree of orness reflects the attitudinal character of experts, either optimism or pessimism. Filev and Yager (1995) obtained an analytic solution for the MEOWA weights. The associated operator is called MEOWA aggregation operator which can be defined as follows.

Definition D.7 (Filev & Yager, 1995). Let W be a weighting vector of dimension n, then a mapping $MEOWA: \mathbb{R}^n \to \mathbb{R}$ is a MEOWA operator of dimension n if:

$$MEOWA_W(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} w_j a_{\sigma(j)}, \qquad \text{(D.10)}$$

where $a_{\sigma(j)}$ is the jth largest a_j and $w_j = \frac{e^{\beta((n-i)/(n-1))}}{\sum_{j=1}^{n} e^{\beta((n-i)/(n-1))}}$ is the MEOWA weights with $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$. Specifically, $\beta \in \mathbb{R}$ is a parameter dependent on the value α (degree of orness) and is given as $\beta = (n-1)\ln(h^*)$, where h^* is a positive solution of the polynomial equation $\sum_{j=1}^{n}[(n-j)/(n-1) - \alpha]h^{(n-j)} = 0$.

By specifying any value for parameter α, the MEOWA weights can be determined and the aggregation operation can be performed. The MEOWA operators are all meet the commutative, monotonic, bounded and idempotent properties.

Remark D.1. Some relationships between α and β can be given as follows: $\alpha \to 1$ induces large positive values for β; $\alpha \to 1/2$ induces values of β near zero; $\alpha \to 0$ induces large negative values for β.

Remark D.2. The main advantage of MEOWA is that the weights can be determined solely by specifying any orness value α in the unit interval [0,1] in which reflects the degree of optimism or pessimism of expert(s). For example, $\alpha \to 1$ provides the weighting vector as W^*, assigning $\alpha = 1/n$ produces W_{Ave} and for $\alpha \to 0$, weighting vector W_* is generated.

D.3.1 Weighted MEOWA aggregation functions

However, the weighting vector W generated from MEOWA is just a set of ordering weights (i.e., associated to arguments based on their magnitudes) without any inclusion of the relative importances of criteria. Yager (2009) extended the MEOWA to the weighted MEOWA (WMEOWA) to deal with this issue. To distinguish between the two types of weights, let specify $V = (v_1, v_2, \dots, v_n)$ as the vector of relative importances of criteria. The WMEOWA can be defined as the following:

Definition D.8 (Yager, 2009). Let V and W be two weighting vectors of dimension n, then a mapping $WMEOWA$: $\mathbb{R}^n \rightarrow \mathbb{R}$ is a WMEOWA operator of dimension n if:

$$WMEOWA_{V,W}(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} \omega_j a_{\sigma(j)}, \quad \text{(D.11)}$$

where $a_{\sigma(j)}$ is the jth largest of a_j and $v_{\sigma(j)}$ is the weight associated to the $a_{\sigma(j)}$. The WMEOWA weights, $\omega_j = h(S_j) - h(S_{j-1})$ is given, such that, $h(S_j) = \frac{1-e^{-\beta S_j}}{1-e^{-\beta}}$ with $S_j = \sum_{b=1}^{j} v_{\sigma(b)}$ (by convention $S_0 = 0$), satisfying $\sum_{j=1}^{n} w_j = 1$ and $\sum_{j=1}^{n} v_j = 1$, with $w_j, v_j \in [0,1]$. Specifically, $\beta \in \mathbb{R}$ is a parameter dependent on the value α and is specified as $\frac{1}{1-e^{-\beta}} - \frac{1}{\beta}$ where $\alpha \in [0,1]$.

By referring to Yager (2009), the formal relationship between β and α can be explained as the following. For the sake of simplicity, let say $x = S_j$, the attitudinal character α can be directly related to $h(x) = \frac{1-e^{-\beta x}}{1-e^{-\beta}}$ as follows:

$$\alpha = \int_0^1 h(x)dx = \frac{1}{1-e^{-\beta}} \int_0^1 1 - e^{-\beta x}$$

$$= \frac{1}{1-e^{-\beta}} \cdot \left[\left(1 + \frac{e^{-\beta}}{\beta}\right) - \frac{1}{\beta} \right] = \frac{1}{1-e^{-\beta}} \cdot \left[1 + \frac{\left(e^{-\beta} - 1\right)}{\beta} \right]$$

$$= \frac{1}{1-e^{-\beta}} + \frac{e^{-\beta} - 1}{\beta(-e^{-\beta} + 1)} = \frac{1}{1-e^{-\beta}} - \frac{1}{\beta}.$$

By assigning any value for α and using approximate value for β, then the WMEOWA weights can be derived and the aggregation operation can be performed. It can be shown that the WMEOWA operators satisfy monotonic, idempotent and bounded properties, but it is not commutative as WA is included. It is also important to note that when $V = (1/n, 1/n, \ldots, 1/n)$, the WMEOWA operator provides the identical result to MEOWA operator.

Remark D.3. Some correlations between α and β can be demonstrated as the following. For $\beta > 0$, hence $\alpha > 1/2$ is produced, for $\beta \to 0$, then $\alpha \to 1/2$, and $\beta < 0$ yields $\alpha < 1/2$.

In specific, the approximate β values in relation to α values can be given as follows: If $\alpha = 1$ then $\beta = 100$, $\alpha = 0.75$ then $\beta = 3.75$ and $\alpha = 0.5$ then $\beta \to 0$. On contrary, $\beta = -100$ provides $\alpha = 0$, $\beta = -3.75$ then $\alpha = 0.25$, refer to (Yager, 2009) for a complete table of the suggested values.

Example D.1: Assume that $A = (\langle 0.4, 0.2\rangle, \langle 0.6, 0.1\rangle, \langle 0.7, 0.4\rangle, \langle 0.5, 0.3\rangle)$ as a set of pairs $\langle a_j, v_j\rangle$. By reordering process $\langle a_{\sigma(j)}, v_{\sigma(j)}\rangle$ where $a_{\sigma(j)} \geq a_{\sigma(j+1)}$, then $\langle 0.7, 0.4\rangle$, $\langle 0.6, 0.1\rangle$, $\langle 0.5, 0.3\rangle$, $\langle 0.4, 0.2\rangle$ can be obtained. Having that, the WMEOWA weights can be determined as follows:

$$S_0 = 0,\ S_1 = 0.4,\ S_2 = 0.5,\ S_3 = 0.8,\ S_4 = 1.$$

For the specified value $\alpha = 0.75$ (moderately optimistic) and the approximate value $\beta = 3.75$, then,

$$h(S_0) = 0,\ h(S_1) = 0.7956,\ h(S_2) = 0.8670,$$
$$h(S_3) = 0.9731,\ h(S_4) = 1.$$

Next, based on $h(S_j)$ values, the weights can be obtained as:

$$\omega_1 = 0.7956,\ \omega_2 = 0.0715,\ \omega_3 = 0.1061,\ \omega_4 = 0.0269,$$

and using the WMEOWA operator, the overall evaluation can be derived as: $WMEOWA_{V,W}(A) = 0.6636$.

D.3.2 The proposed Induced WMEOWA based-aggregation operators

Notice that relative importances v_j are reordered with respect to arguments a_j and as for the case of WMEOWA, the argument values a_j are rearranged in non-increasing order based on their magnitudes. In the similar way, the WMEOWA can be extended to the case of IOWA operator, where different reordering process of arguments take place with respect to the order-inducing variables. In what follows, an extension of the WMEOWA operator to induced WMEOWA operator is proposed. Moreover, the induced generalized and the induced quasi generalized WMEOWA can be defined.

Definition D.9. Let V and W be two weighting vectors of dimension n, then a mapping $I-WMEOWA\colon \mathbb{R}^n \rightarrow \mathbb{R}$ is an induced WMEOWA operator of dimension n if:

$$I-WMEOWA_{V,W}(\langle u_1,a_1\rangle,\langle u_2,a_2\rangle,\dots,\langle u_n,a_n\rangle) = \sum_{j=1}^{n} \varphi_j a_{\sigma(j)}, \tag{D.12}$$

where $a_{\sigma(j)}$ is the a_j value of the pair $\langle u_j,a_j\rangle$ having the j-th largest u_j. The IWMEOWA weights $\varphi_j = h(S_j)-h(S_{j-1})$ is given, such that, $h(S_j)=\frac{1-e^{-\beta S_j}}{1-e^{-\beta}}$ with $S_j=\sum_{b=1}^{j} v_{\sigma(b)}$ satisfying $\sum_{j=1}^{n} w_j = 1$ and $\sum_{j=1}^{n} v_j = 1$ with $w_j, v_j \in [0,1]$, for $\alpha = -\frac{1}{\beta}+\frac{1}{1-e^{-\beta}}$, $\beta \in \mathbb{R}$.

Example D.2: Assume that a set of triple $\langle u_j, a_j, v_j\rangle$ as $A = (\langle 60,0.4,0.2\rangle,\langle 20,0.6,0.1\rangle,\langle 50,0.7,0.4\rangle,\langle 70,0.5,0.3\rangle)$. By reordering process $\langle u_{\sigma(j)},a_{\sigma(j)},v_{\sigma(j)}\rangle$ where $u_{\sigma(j)} \geq u_{\sigma(j+1)}$, then $\langle 70,0.5,0.3\rangle$, $\langle 60,0.4,0.2\rangle$, $\langle 50,0.7,0.4\rangle$, $\langle 20,0.6,0.1\rangle$ can be obtained. By the similar way, the IWMEOWA weights can be determined as follows:

$$S_0=0,\ \ S_1=0.3,\ \ S_2=0.5,\ \ S_3=0.9,\ \ S_4=1.$$

For the specified value $\alpha=0.75$ and the approximate value $\beta=3.75$, then,

$$h(S_0)=0,\ \ h(S_1)=0.7956,\ \ h(S_2)=0.8670,$$
$$h(S_3)=0.9731,\ \ h(S_4)=1.$$

Next, based on $h(S_j)$ values, the weights can be obtained as:

$$\varphi_1=0.6916,\ \ \varphi_2=0.1754,\ \ \varphi_3=0.1220,\ \ \varphi_4=0.0110,$$

and applying the IWMEOWA operator, the overall evaluation can be generated as: $I-WMEOWA_{V,W}(A)=0.6548$.

Furthermore, it is interesting also to present a general formulation that includes more types of aggregation operators. This formulation is carried out by using the generalized and quasi-arithmetic means. The main advantage of using these operators is that they include a wide range of aggregation operators. Hence, the analyst/expert gets a complete view of the decision problem under consideration and select the one that is in accordance with his/her interests. Thus, a unifying framework that provides a general formulation of different types of IWMEOWAs can be developed.

Definition D.10. An induced generalized WMEOWA operator of dimension n is a mapping $I-GWMEOWA\colon \mathbb{R}^n \to \mathbb{R}$ that has two associated weighting vectors V and W of dimension n such that $\sum_{j=1}^{n} w_j = 1,\ w_j \in [0,1]$ and $\sum_{j=1}^{n} v_j = 1,\ v_j \in [0,1]$, then:

$$I-GWMEOWA_{V,W}(\langle u_1,a_1\rangle,\langle u_2,a_2\rangle,\dots,\langle u_n,a_n\rangle) = \left(\sum_{j=1}^{n} \varphi_j a_{\sigma(j)}^{\lambda}\right)^{1/\lambda}, \tag{D.13}$$

where $a_{\sigma(j)}$ is the a_j value of the pair $\langle u_j, a_j\rangle$ having the j-th largest u_j and λ is a parameter such that $\lambda \in \mathbb{R} \setminus \{0\}$. Analogously, the IGWMEOWA weights $\varphi_j = h(S_j) - h(S_{j-1})$ can be derived in the similar way as the IWMEOWA weights.

Definition D.11. An induced quasi WMEOWA operator of dimension n is a mapping $I-QWMEOWA\colon \mathbb{R}^n \to \mathbb{R}$ that has two associated weighting vectors V and W of dimension n such that $\sum_{j=1}^{n} w_j = 1, w_j \in [0,1]$ and $\sum_{j=1}^{n} v_j = 1$, $v_j \in [0,1]$, then:

$$I-QWMEOWA_{V,W}(\langle u_1,a_1\rangle,\langle u_2,a_2\rangle,\dots,\langle u_n,a_n\rangle) = g^{-1}\left(\sum_{j=1}^{n} \varphi_j g\big(a_{\sigma(j)}\big)\right), \tag{D.14}$$

where $g\big(a_{\sigma(j)}\big)$ is a strictly continuous monotonic function. The IQWMEOWA weights $\varphi_j = h(S_j) - h(S_{j-1})$ can be derived in the same way as the IWMEOWA weights.

Notice that, when $u_1 = u_2 = \cdots = u_n$, the IWMEOWA operator is reduced to the case of WMEOWA operator. It can be demonstrated that the IWMEOWA, IGWMEOWA and IQWMEOWA operators meet monotonic, idempotent and bounded properties, but they are not commutative. The proofs are straightforward and thus, omitted in this study.

D.4 Majority Concept based on IOWA Operators

In this section, the majority concept based on Bordogna-Sterlacchini method is presented. Earlier, Pasi, and Yager (2006) proposed a method to determine the majority opinion of experts using the support function, mainly based on distance measure. The derived values are then used as order-inducing variables for the IOWA aggregation process. However, the suggested method is solely concerned with the homogeneous case where all experts are assumed to have equal degrees of importance (or trusts). Moreover, the measure of support between experts is concentrated only on the individual priorities of alternatives, i.e., based on the classical scheme.

On the contrary, Bordogna and Sterlacchini (2014) proposed an alternative approach, where the support function is measured with respect to each specific criterion of experts (i.e., alternative scheme) and developed mainly for the heterogeneous case or with the inclusion of degrees of trust. In addition, the similarity measure based on the Minkowski OWA is applied to determine the order-inducing variables. The method can be explained as the following.

D.4.1 Bordogna-Sterlacchini method for majority aggregation process

For each alternative, x_i and a single criterion, c_j, suppose that a set of preferences of experts $(h = 1,2, \dots, k)$ is given as a vector $P_j^{[h]} = (p_{ij}^1, p_{ij}^2, \dots, p_{ij}^k)$ and the associated degree of trust is given as $t_h \in [0,1]$ such that $\sum_{h=1}^{k} t_h = 1$. For simplicity, let $P_j^{[h]} = P_j = (p_1, p_2, \dots, p_k)$ and with respect to this vector, the order-inducing variable u_l based on $Q_{coherence}(P_l, P_h)$ can be defined as follows:

$$MOWA(s_1, \dots, s_k) = \left(\sum_{h=1}^{k} \theta_h s_{\sigma(h)}^{\lambda} \right)^{1/\lambda}, \qquad \text{(D.15)}$$

where $s_l = 1 - |p_l - p_h|$ as similarity measures between an expert $l \ (l \in h)$ with respect to all other experts h (including himself/herself) and $s_{\sigma(h)}$ are reordering of $s_1, \dots, s_k$ in non-increasing order, i.e., $s_{\sigma(1)} \geq s_{\sigma(2)} \geq \cdots \geq s_{\sigma(k)}$.

Moreover, the expression $\theta_h = Q\left(\sum_{b=1}^{h} t_{\sigma(b)}\right)^{\gamma} - Q\left(\sum_{b=0}^{h-1} t_{\sigma(b)}\right)^{\gamma}$ denotes the weights derived using a monotone non-decreasing linguistic quantifier, such that $t_{\sigma(h)}$ is associated with $s_{\sigma(h)}$ and $\theta_h \in [0,1], \sum_{h=1}^{k} \theta_h = 1$. It can be noted that, when $\gamma = 1$, Eq. (D.15) is reduced to the weighted Minkowski based-similarity measure with $\theta_h = t_h$.

The vector of order-inducing variables can be represented as:

$$U = (u_1, \dots, u_k) = \left(Q_{coherence}(P_1, P_h), \dots, Q_{coherence}(P_k, P_h)\right). \tag{D.16}$$

Afterwards, with these order-inducing variables, the weights for IOWA aggregation operator can be determined using the following formula:

$$m_h = \frac{argmin_h(u_1 \cdot t_1, \dots, u_k \cdot t_k)}{\sum_{h=1}^{k} argmin_h(u_1 \cdot t_1, \dots, u_k \cdot t_k)}, \tag{D.17}$$

where m_h are ordered in non-decreasing order. Further, given the quantifier Q with semantics '*most*' (to represent a majority of experts), such that:

$$Q(r) = \begin{cases} 1 \; if \; r \geq 0.9, \\ 2r - 0.8 \; if \; 0.4 < r < 0.9, \\ 0 \; if \; r \leq 0.4, \end{cases} \tag{D.18}$$

then, the weighting vector $W^{Maj} = (w_1, \dots, w_k)$ can be computed as follows:

$$w_h = \frac{Q(m_h)}{\sum_{h=1}^{k} Q(m_h)}. \tag{D.19}$$

Finally, the overall aggregation process can be calculated using the IOWA operator with non-decreasing inputs as the following:

$$IOWA(\langle u_1, p_1 \rangle, \langle u_2, p_2 \rangle, \dots, \langle u_k, p_k \rangle) = \sum_{h=1}^{k} w_h \, p_{\tau(h)}. \tag{D.20}$$

Note that, the weights and order-inducing variables are reordered in non-decreasing order instead of non-increasing order as in the original IOWA, see, Eq. (D.2). This type of ordering reflects the conformity of the quantifier '*most*' in modeling the majority concept (Pasi & Yager, 2006; Bordogna & Sterlacchini, 2014). The notation $\tau(h)$ denotes the ordering process with respect to the non-decreasing order.

D.4.2 Some modifications of the Bordogna-Sterlacchini method

In the context of AHP-GDM, the preferences of experts on each criterion are very close to each other (or less distinctive) due to the normalization/prioritization process. In what follows, a slight modification is suggested to differentiate between the preferences. The modified similarity measure can then be formulated as:

$$s(p_l, p_h) = 1 - \left(\frac{|p_l - p_h|}{\max_h |p_l - p_h|}\right), \forall h = 1,2, \dots, k. \tag{D.21}$$

where $\max_h |p_l - p_h|$ is the maximum distance between all experts h. It can be demonstrated that for any p_l and p_h with $s(p_l, p_h) \in [0,1]$, the properties i) $s(p_l, p_l) = 1$ (reflexive) and ii) $s(p_l, p_h) = s(p_h, p_l)$ (symmetric) are fulfilled. Analogously, the weights for the IOWA aggregation process in Eq. (D.17) can also be modified to the following formula:

$$m_h = \frac{argmin_h(u_1 \cdot t_1, \dots, u_k \cdot t_k)}{Max_h(u_1 \cdot t_1, \dots, u_k \cdot t_k)}. \tag{D.22}$$

In general, the aforementioned method can be directly applied to the case of classical scheme. Where, instead of comparing the preferences with respect to each criteria, the experts' rankings or individual priorities of alternatives can be compared.

In the next section, the proposed AHP-GDM with these methodologies is presented. Specifically, the synthesizing of experts' preferences based on two approaches is developed: i) the consensus on each criterion (alternative scheme) and ii) the consensus on individual priorities of alternatives (classical scheme).

Example D.3: Suppose that a set of experts' preferences on a single criterion is given as $P_j = (p_1, p_2, \dots, p_k) = (0.307, 0.343, 0.298, 0.283, 0.108)$. The preference of majority of experts (representative result) can be calculated as follows:

A_i	E_1	E_2	E_3	E_4	E_5							
P_j^h	0.307	0.343	0.298	0.283	0.108		t_1	t_2	t_3	t_4	t_5	U
$supp_{1h}$	1	0.847	0.960	0.899	0.154		0.3	0.3	0.2	0.1	0.1	0.851
$supp_{2,h}$	0.847	1	0.807	0.745	0	$\xrightarrow{s_h \times t_h}$	0.3	0.3	0.2	0.1	0.1	0.790
$supp_{3,h}$	0.960	0.807	1	0.938	0.193		0.3	0.3	0.2	0.1	0.1	0.843
$supp_{4,h}$	0.899	0.745	0.938	1	0.255		0.3	0.3	0.2	0.1	0.1	0.806
$supp_{5,h}$	0.154	0	0.193	0.255	1		0.3	0.3	0.2	0.1	0.1	0.210

where $U = \sum_{h=1}^{k} t_h s_h$. In this case, for $\theta_h = t_h$ (by setting $\lambda = 1$), the vector of order-inducing variable can be derived as $U = \left(s_{\sigma(1)}, \dots, s_{\sigma(5)}\right) = (0.210,\ 0.790,\ 0.806,\ 0.843,\ 0.851)$. Next, by using quantifier Q with semantics '*most*' to represent the majority, the weighting vector $W^{Maj} = (w_1, \dots, w_5) = (0, 0, 0.207, 0.397, 0.397)$ is obtained. Then, the final majority preference of experts can be calculated using the IOWA operator as follows:

$$IOWA(\langle 0.210, 0.108\rangle, \langle 0.790, 0.343\rangle, \langle 0.806, 0.283\rangle,$$
$$\langle 0.843, 0.298\rangle, \langle 0.851, 0.307\rangle) = 0.299$$

D.5 Generalized AHP-GDM Method

In this section, the generalization of AHP-GDM model is proposed based on the methods discussed in Section D.3.2, D.4.1 and D.4.2. In what follows, the proposed method is demonstrated step by step as in the subsequent algorithms. Assume that $x_i \in X, (x_i, i = 1,2, \dots, m)$ denotes a finite set of alternatives and that $c_j \in C, \left(c_j, j = 1,2, \dots, n\right)$ are the criteria under consideration. Let $E_h \in E$, $(E_h, h = 1,2, \dots, k)$, be a group of experts, where each expert E_h presents his/her preferences for rating the alternatives x_i and weighting the criteria c_j. Two cases of aggregating method for the majority of experts are given as follows.

- *Case 1: Majority opinion with respect to criteria (Alternative scheme)*

Step 1. Let $A^{[h]} = [a_{i\bar{\imath}}]_{m \times m}^{[h]}$ be the judgement matrix of h-th expert in comparing m alternatives $(i, \bar{\imath} = 1,2, \dots, m\ )$ with respect to each criterion c_j. The judgement matrix $A^{[h]}$ is constructed based on Eq. (D.7) and then the consistency-check is conducted using Eq. (D.8) and Eq. (D.9).

Step 2. The priority vector, $P^{[h]} = \left(p_{ij}\right)_{m \times 1}^{[h]}, \forall j = 1,2, \dots, n$, of h-th expert then can be derived using any of the prioritization methods, such as the eigenvector method (EV) or the row geometric means method (RGMM).

Step 3. With respect to each alternative, a set of experts' preferences on each criterion $P_j = \left(p_{ij}^1, p_{ij}^2, \dots, p_{ij}^k\right)$ can be obtained. The individual preferences of experts then are aggregated, such that θ_h is the weight assigned with the h-th expert in forming the group preference $(\theta_h > 0;\ \sum_{h=1}^{k} \theta_h = 1)$.

Step 4. The order-inducing variable for the majority opinion is determined based on Eq. (D.15) and Eq. (D.16), where the modified similarity measure is used as in Eq. (D.21). Based on the quantifier '*most*' in Eq. (D.18) and the weights obtained from Eq. (D.19) and Eq. (D.22), the majority opinion of experts can be derived using Eq. (D.20).

Step 5. Based on the previous steps, form a new decision matrix that represents the majority of experts by combining the priority vectors of all criteria, $P^{[Maj]} = \left[p_{ij}\right]_{m\times n}$.

Step 6. In the similar way, obtain the weighting vector $V = (v_1, v_2, \dots, v_n)$ of the criteria as in *Step 1*, such that $\left(v_j > 0;\ \sum_{j=1}^{n} v_j = 1\right)$. At this stage, the IWMEOWA weights φ_j can be determined. Finally, the overall ranking of alternatives can be generated using the IWMEOWA aggregation operator such in Eq. (D.13). Note that, the order-inducing variable for the aggregation of criteria is different than that of Eq. (D.16).

- *Case 2: Majority with respect to individual priorities of alternatives (classical scheme)*

Step 1. Construct the judgment matrix $A^{[h]} = [a_{i\bar{i}}]_{m\times m}^{[h]}$, such in Eq. (D.7) and then a consistency-check is conducted using Eq. (D.8) and Eq. (D.9).

Step 2. Further, the priority vector, $P^{[h]} = \left(p_{ij}\right)_{m\times 1}^{[h]}, \forall j = 1,2,\dots,n$, with respect to each criterion c_j can be derived using any of the prioritization methods, such as EV or RGMM.

Step 3. For a set of n criteria $(j = 1,2,\dots,n)$, the priority vectors $P^{[h]}$ can be represented as the matrix $\hat{P}^{[h]} = \left[p_{ij}\right]_{m\times n}^{[h]}$ for each individual of experts.

Step 4. Given the weighting vector $V = (v_1, v_2, \dots, v_n)$ for the criteria, such that $\left(v_j > 0;\ \sum_{j=1}^{n} v_j = 1\right)$, calculate the IWMEOWA weights φ_j for individual of experts based on the provided order-inducing variables u_j and the argument values as derived in *Step 2*.

Step 5. Applying the IWMEOWA operator, Eq. (D.13), the individual priorities of alternatives for each expert $P^{[h]} = \left(p_i^{[h]}\right)$ can be determined.

Step 6. Finally, let θ_h be the weight assigned to the h-th expert in forming the group decision $(\theta_h > 0;\ \sum_{h=1}^{k} \theta_h = 1)$. The final ranking of alternatives as a collective group decision (majority opinion of experts) can be derived using the IOWA operator as in Eq. (D.20).

D.6 Application in an Investment Selection Problem

In this section, an application of investment selection problem is provided to exemplify the proposed model. The main focus is on the analysis of results with respect to the classical and alternative schemes. The comparison then is conducted by specifying different degrees of optimism to reflect the decision strategies.

For this purpose, consider that an agency or a company is looking for an optimal investment and must conduct an analysis to achieve this objective. The following five possible alternatives are considered: x_1 is a computer company, x_2 is a chemical company, x_3 is a food company, x_4 is a car company, and x_5 is a television company. In order to evaluate these alternatives, a group of experts/analysts are selected to make a decision according to the following four criteria: C_1 is the risk analysis, C_2 is the growth analysis, C_3 is the social-political impact analysis, and C_4 is the environmental impact analysis. In this case, five experts are involved in the analysis and the associated degrees of importance (or trusts) of the experts are given as $t_h = (0.3, 0.3, 0.2, 0.1, 0.1)$ for $h = 1, 2, \ldots, 5$. Since the decision problem is complex, as it involves the preferences of different members of board directors, the additional parameter as the order-inducing variable then is taken into account to represent them, see Table D.1.

Table D.1. Order-inducing variables

	C_1	C_2	C_3	C_4
X_1	25	18	24	16
X_2	12	34	18	22
X_3	22	13	28	21
X_4	31	24	14	20
X_5	30	25	23	16

First, let all the experts with the guide of a moderator provide judgments for determining the relative importances of criteria. The final agreement of experts on pairwise comparisons is shown in Table D.2. Based on the EV prioritization procedure, the weight for each criterion can be derived, and the consistency ratio is conducted.

Table D.2. Pairwise comparison matrix and weights for criteria

	C_1	C_2	C_3	C_4	v_j	
C_1	1	0.5	2	4	0.3111	*CR*=0.036
C_2	2	1	2	3	0.4064	
C_3	0.5	0.5	1	2	0.1824	
C_4	0.25	0.3333	0.5	1	0.1001	

Next, in the same way, each expert provides preferences (or pairwise comparisons) for all alternatives with respect to each criterion in order to obtain the relative performance of alternatives. Using the EV method, the prioritization vectors for all experts with respect to a single criterion can be derived as shown in Tables D.3 – D.6, respectively. Then, based on these prioritization vectors, the consensus or aggregation of experts on each criterion is conducted (*case 1: alternative scheme*). The majority of experts' preferences is presented in Table D.7.

Table D.3 . Prioritization vectors for all experts with respect to C_1

	E_1	E_2	E_3	E_4	E_5
X_1	0.1866	0.1199	0.0890	0.1882	0.2434
X_2	0.3069	0.3431	0.2976	0.2831	0.1078
X_3	0.0573	0.1122	0.1579	0.0543	0.1540
X_4	0.3069	0.2553	0.2976	0.3739	0.0494
X_5	0.1422	0.1696	0.1579	0.1005	0.4454

Table D.4 . Prioritization vectors for all experts with respect to C_2

	E_1	E_2	E_3	E_4	E_5
X_1	0.2412	0.1618	0.1042	0.0805	0.0785
X_2	0.1353	0.2760	0.3902	0.1395	0.1351
X_3	0.0743	0.1054	0.0588	0.3552	0.2633
X_4	0.1353	0.0596	0.1505	0.2852	0.1271
X_5	0.4137	0.3971	0.2962	0.1395	0.3960

Table D.5 . Prioritization vectors for all experts with respect to C_3

	E_1	E_2	E_3	E_4	E_5
X_1	0.2618	0.0604	0.0743	0.2618	0.1062
X_2	0.0892	0.1382	0.1353	0.0892	0.1666
X_3	0.1528	0.3972	0.2412	0.1528	0.4377
X_4	0.0526	0.0954	0.1353	0.0526	0.0544
X_5	0.4436	0.3088	0.4137	0.4436	0.2352

Table D.6 . Prioritization vectors for all experts with respect to C_4

	E_1	E_2	E_3	E_4	E_5
X_1	0.2571	0.0890	0.0986	0.0743	0.0987
X_2	0.0881	0.1579	0.1611	0.1353	0.1574
X_3	0.1539	0.2976	0.4162	0.2412	0.3015
X_4	0.4129	0.2976	0.0624	0.1353	0.2949
X_5	0.0881	0.1579	0.2618	0.4137	0.1474

Table D.7. Aggregated performance values based on the majority concept

	C_1	C_2	C_3	C_4
X_1	0.1354	0.1281	0.0803	0.0973
X_2	0.2983	0.2078	0.1260	0.1582
X_3	0.0840	0.0838	0.1859	0.2985
X_4	0.2922	0.1354	0.0749	0.2964
X_5	0.1592	0.3988	0.4320	0.1438

Having the aggregated performance of the experts and the order-inducing variables in Table D.1, the weights v_j (WA) can be integrated with the ordered weights w_j (OWA weights) under the IWMEOWA weighting method. The new weights φ_j are shown in Table D.8, where α is the measure of orness or the attitudinal character of the majority experts. In this case, it is set as 0.75 (moderately optimistic), and the value of β is specified as 3.75.

Table D.8. The IWMEOWA weights φ_j

	ω_1	ω_2	ω_3	ω_4
X_1	0.7052	0.1579	0.1259	0.0110
X_2	0.8010	0.0698	0.0759	0.0533
X_3	0.5073	0.3559	0.0504	0.0865
X_4	0.7052	0.2494	0.0217	0.0236
X_5	0.7052	0.2494	0.0344	0.0110

Finally, the overall score of the alternatives can be derived using the IWMEOWA aggregation operator as depicted in the Table D.9. As in the case of $\alpha = 0.75$, the best alternative is X_4, followed by X_5, X_2, X_3, and X_1, respectively. Correspondingly, with respect to the attitudinal character α, variation of decision strategies can be attained, for example, $\alpha = 0$ as pessimistic, $\alpha = 0.25$ as slightly pessimistic, $\alpha = 0.5$ as neutral, and $\alpha = 1$ as an optimistic decision of the majority of experts.

Table D.9. The overall score with respect to majority opinion on criteria

	Overall score				
Alternative	$\alpha = 0$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
X_1	0.0973	0.1104	0.1186	**0.1254**	0.1354
X_2	0.2983	0.2604	0.2161	**0.2029**	0.2078
X_3	0.0838	0.1542	0.1240	**0.1465**	0.1859
X_4	0.0749	0.1354	0.1893	**0.2481**	0.2922
X_5	0.1438	0.2993	0.3084	**0.2281**	0.1592

In comparison, using the majority opinion based on individual priorities of alternatives (*Case 2: classical scheme*), different rankings of alternatives can be generated as given in Table D.10. There is a slight difference in the ranking for the case of $\alpha = 0.75$ as follows: X_4 as the best, followed by X_5, X_3, X_2, and X_1, respectively, compared to the alternative scheme.

Table D.10. The overall score with respect to majority opinion on individual priorities

Alternative	Overall score				
	$\alpha = 0$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
X_1	0.0918	0.1096	0.1320	**0.1613**	0.1825
X_2	0.3118	0.2604	0.2120	**0.2019**	0.2089
X_3	0.1350	0.1542	0.1786	**0.2122**	0.2772
X_4	0.1174	0.1354	0.1753	**0.2348**	0.2829
X_5	0.1934	0.2993	0.2832	**0.2173**	0.1592

As can be noticed, the results between the two schemes are different mainly due to the formation of decision strategies. Specifically in the *case 1 of alternative scheme*, the degree of optimism as the decision strategy is totally reflecting a group of expert, collectively. Whilst in the classical scheme, the decision strategy is provided by each expert individually. Hence, a comprehensive analysis on the decision problem can be conducted by applying the proposed model.

5.3 Heavy Weighted Geometric Aggregation Operators in Analytic Hierarchy Process-Group Decision Making

Abstract. In this study, some heavy weighted geometric aggregation operators in analytic hierarchy process under group decision making are proposed. First, in the sense of heavy ordered weighted average (HOWA) operator, the heavy weighted geometric (HWG) and heavy ordered weighted geometric (HOWG) are introduced as extensions of the normal weighted geometric mean and the ordered weighted geometric by relaxing the constraints on the associated weighting vector. These HWG and HOWG operators then are utilized in the aggregation process of AHP-GDM, specifically on the aggregation of individual judgments (AIJ) procedure. The main advantage of the model, besides the complete overlapping of information such in classical methods, is that it can also accommodate partial and non-overlapping information in the formulation. To show the applicability of the proposed method, a numerical example in an investment selection problem is provided.

E.1 Introduction

Analytic hierarchy process (AHP) is one of the available discrete-type of multiple criteria decision making models and was introduced by Saaty (1977; 1980) in the late 1970s. The methodology for solution is based on pairwise comparison matrix, specifically the multiplicative reciprocal to generate the priority of alternatives or degrees of importance. Since its inception, the model has been used extensively in numerous applications. There are two approaches traditionally employed to deal with AHP under group environments, which are the aggregation of individual judgments (AIJ) and the aggregation of individual priorities (AIP). AIJ can be elucidated as a group of decision makers or experts which act together as a unit i.e., synergistic unit, whilst AIP acts as separate individuals i.e., a collection of individuals (Forman & Peniwati, 1998). In this study, the AHP under the AIJ will be presented.

On the other hand, the ordered weighted averaging (OWA) operator is a family of multiple criteria aggregation procedures and was developed by Yager (1988). It provides a parameterized class of mean-type aggregation operators that lie between minimum and maximum, as well as average as the normal case. In general, this can also be explained as a fusion of decision making attitudes, e.g., pessimistic, neutral and optimistic. However, for this type of aggregation operators, the total sum of weights is always limited to one. In the context of group decision making problems, such condition in general means that the information is overlapping or redundant, so that there is a freedom on how to use or manipulate the information. For instance, consider a group decision making problem such in AIP procedure. All experts evaluate each alternative with respect to the same space of criteria and then derive the priorities individually. In that case, there is an option to present or take any decision of them (e.g., as average, minimum or maximum) since all experts are attached to the same problem.

In consequence, Yager (2002) introduced the heavy OWA (HOWA) operator as an extension of the OWA operator. The reason for the proposal is because there are situations where the available information is partially overlapping and/or non-overlapping from each other and this aspect needs to be considered in the aggregation. The example in this case can be explained as in AIJ procedure where every expert plays their role to evaluate certain criterion/criteria that the others do not count, i.e., there exist partition of criterion/criteria in group decision making problems. Note that, as in the case of OWA, the constraint represents minimum and maximum as extreme values which are ranged in zero and one (unit interval). Meanwhile, with the HOWA, a wider class of aggregation operators can be included from minimum to totaling operators. In relation to the AHP method, the HOWA concept can be implemented in case of the AIJ where the criteria or sub-criteria under evaluation are partitioned for specific experts based on their knowledge, experience and expertise. Some of the experts may consider a specific criterion, while some others may consider a combination of them. In the literature there are some works that have been done on the extension of HOWA, for instance Merigó and Casanovas (2011), Merigó et al. (2014a), Merigó et al. (2014b), among others.

It should be pointed out as well that the HOWA is a special type of a more general class of aggregation operators, called monitored heavy fuzzy measures (Yager, 2003), where it is based on the additive measure in discrete space. In specific, the monitored heavy fuzzy measure is an extension of fuzzy measure theory by Sugeno (1977). The fuzzy measure is a generalization of a classical measure theory in which the additive property is replaced by the weaker property of monotonicity, also called as non-additive measure. In the sense of multi-criteria decision making, Sugeno integral and Choquet integral are two general classes of aggregation operators based on the fuzzy measure. They take into consideration the interaction between criteria, ranging from redundancy, (e.g., negative interaction) to synergy (e.g., positive interaction). Concurrently, it is mentioned that the drawback of classical aggregation methods is that they count no interaction between criteria, i.e., independence and redundancy of criteria (Grabisch, 1996). The monitored heavy fuzzy measure, on the hand, generalizes the aggregation operators by considering the partition space of information under concern, providing independent yet non-redundant of information, instead of independent and redundant such in the classical methods. In addition, it is also consider the interaction between criteria such as the particular case of fuzzy measure. Hence, in general, this measure includes all possibility of information processes (additive and non-additive), specifically independence (redundancy and non-redundancy) of information and interaction or synergy of information. To limit the scope of the study, only HOWA in which the definition as given in (Yager, 2002) will be presented.

In the literature, there are a number of studies related to the integration of AHP with OWA operators. Commencing from Yager and Kelman (1999), an extension of the AHP using OWA operator has been proposed. They generalized the aggregation process used in the AHP by permitting more flexibility to combine information in hierarchies, specifically the determination of weights by linguistic quantifier and the determination of priority of preferences based on ordered position. Since the AHP is a part of preference relation models, which is based on multiplicative reciprocal, this method is also called as multiplicative preference relation model. Geometric-mean based methods are normally used in the aggregation process to consistently fit with multiplicative reciprocal conditions. Chiclana et al. (2004) have presented induced ordered weighted geometric operators and their use in the aggregation of multiplicative preference relations. In addition, some studies related to multiplicative preference relation models under group decision making problems have been presented (see Chiclana et al., 2007; Yusoff & Merigo, 2014).

The aim of this study is to introduce several extensions of geometric mean method, specifically HWG and HOWG operators as aggregation procedures. Furthermore, these operators are used to be integrated with AHP-GDM model under the AIJ procedure. The remainder of this study is organized as follows. Section E.2, briefly reviews some basic concepts related to OWA, OWG, HOWA and their properties. Section E.3, the HWG and HOWG operators are introduced and Section E.4, the definition of analytic hierarchy process and its properties are provided. Section E.5, presents the proposed method of AHP-GDM with HWG and HOWG. Then, Section E.6 provides a numerical example.

E.2 Preliminaries

This section provides some definitions and basic concepts related to OWA operator and its generalization that will be used throughout this study.

Definition E.1 (Yager, 1988). An OWA operator of dimension n is mapping $OWA: \mathbb{R}^n \rightarrow \mathbb{R}$ that has an associated weighting vector w of dimension n, such that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$OWA(a_1, \dots, a_n) = \sum_{j=1}^{n} w_j a_{\sigma(j)} , \tag{E.1}$$

where $\sigma(j)$ denotes the components of x being arranged in non-increasing order $a_{\sigma(j)} \geq a_{\sigma(j+1)} \geq \cdots \geq a_{\sigma(n)}$.

Definition E.2 (Xu & Da, 2002; Herrera, Herrera-Viedma, & Chiclana, 2003). An OWG operator of dimension n is mapping $OWG: \mathbb{R}^n \rightarrow \mathbb{R}$ that has an associated weighting vector w of dimension n, such that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$OWG(a_1, \dots, a_n) = \prod_{j=1}^{n} \left(a_{\sigma(j)}\right)^{w_j}, \tag{E.2}$$

where $\sigma(j)$ denotes the components of x being arranged in non-increasing order $a_{\sigma(j)} \geq a_{\sigma(j+1)} \geq \cdots \geq a_{\sigma(n)}$.

Definition E.3 (Yager, 2002). A HOWA operator of dimension n is mapping $HOWA: \mathbb{R}^n \rightarrow \mathbb{R}$ that has an associated weighting vector w such that $w_j \in [0,1]$ and $1 \leq \sum_{j=1}^{n} w_j \leq n$, such that:

$$HOWA(a_1, \dots, a_n) = \sum_{j=1}^{n} w_j a_{\sigma(j)}\,, \tag{E.3}$$

where $\sigma(j)$ denotes the components of x being arranged in non-increasing order $a_{\sigma(j)} \geq a_{\sigma(j+1)} \geq \cdots \geq a_{\sigma(n)}$.

If W_T is a weighting vector such that $w_j = 1$ for all j, then with $W = W_T$, so that $H(a_1, \dots, a_n) = Total(a_1, \dots, a_n)$ i.e., the case of non-overlapping or independent information. If $\sum_{j=1}^{n} w_j = 1$, then $H(a_1, \dots, a_n) = OWA(a_1, \dots, a_n)$, the overlapping case such in the original OWA operator. The new class of aggregation types lying between these two extremes $1 < \sum_{j=1}^{n} w_j < n$, i.e. partial redundancy. This variation can be seen by using the degree of totaling β of the vector W such that $\beta(W) = |W| - 1/n - 1$, as $|W| \in [1, n]$ then $\beta \in [0,1]$. For instance, $|W| = 1$, hence $\beta = 0$ (it turn out to be the ordinary OWA aggregation) and $|W| = n$, hence $\beta = 1$ (pure totaling aggregation). On the other hand, given a value for β and a dimension n of vector W then the magnitude $|W|$ can be derived as $|W| = \beta n + (1 - \beta)$. In addition, for $\rho = 1 - \beta, \rho = n - |W|/n - 1$, then magnitude $|W|$ can be represented as $|W| = n - \rho(n - 1)$, given that ρ as degree of redundancy.

The HOWA operators are all commutative, monotonic and idempotent, but they are not bounded by min and max. Instead, they are bounded by the min and the total operator which represents the sum of all the arguments. In addition, the measures for characterizing a weighting vector and the type of aggregation been performed in HOWA also been extended based on OWA operator.

Definition E.4 (Yager, 2002). Suppose that W is the weighting vector such that $w_i \in [0,1]$ and $1 \leq \sum_{i=1}^{n} w_i \leq n$, then the attitudinal character of HOWA can be given as follow:

$$\alpha(W) = \frac{1}{|W|} \sum_{j=1}^{n} \left(\frac{n - j}{n - 1}\right) w_j,\ \ \alpha(W) \in [0,1]\,. \tag{E.4}$$

Definition E.5 (Yager, 2002). Suppose that W is the weighting vector such that $w_i \in [0,1]$ and $1 \leq \sum_{i=1}^{n} w_i \leq n$, then the entropy or dispersion of HOWA can be given as follow:

$$H(W) = -\frac{1}{|W|} \sum_{j=1}^{n} w_j ln\left(\frac{w_j}{|W|}\right), \tag{E.5}$$

where $|W|$ is the magnitude of W and $|W| = \sum_{j=1}^{n} w_j\,,\ |W| \in [1, n]$. If $v_j = \frac{w_j}{|W|}$, then this can be reformulated as $H(W) = -\sum_{j=1}^{n} v_j ln(v_j)$.

Based on the definitions above, when $|W| = 1$, the orness and the entropy of HOWA operator reduce to the usual definitions such in OWA operator. Meanwhile when the totaling operator $|W| = n$ is used, then $w_j = 1$ for all j and hence $\alpha(W) = 0.5$ and $H(W) = -\ln n$.

E.3 Heavy Weighted Average and Heavy Ordered Weighted Average Operators

This section presents the definition of HWG and HOWG operators. These operators are based on the extension of HOWA as introduced in Yager (2002).

Definition E.6. A HWG operator of dimension *n* is a mapping $HWG: \mathbb{R}^n \rightarrow \mathbb{R}$ that has an associated weighting vector w with $w_j \in [0,1]$, such that $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$HWG(a_1, \dots, a_n) = \prod_{j=1}^{n} (a_j)^{w_j}, \tag{E.6}$$

where $w = (w_1, w_2, \dots, w_n)^T$ and $1 \leq \sum_{j=1}^{n} w_j \leq n$.

Definition E.7. A HOWG operator of dimension *n* is a mapping $HOWG: \mathbb{R}^n \rightarrow \mathbb{R}$ that has an associated weighting vector w with $w_j \in [0,1]$ and $1 \leq \sum_{j=1}^{n} w_j \leq n$, such that:

$$HOWG(a_1, \dots, a_n) = \prod_{j=1}^{n} (a_{\sigma(j)})^{w_j}, \tag{E.7}$$

where $\sigma(j)$ denotes the components of x being arranged in non-increasing order $a_{\sigma(j)} \geq a_{\sigma(j+1)} \geq \cdots \geq a_{\sigma(n)}$.

The HOWG operator is monotonic, commutative and bounded by the minimum and total operators. It should be noted that HOWG provides a wider class of aggregation operator by allowing the weighting vector between the OWG and total operator, include geometric mean, WG, HWG, etc. From a generalized perspective of the reordering step, HOWG can be distinguished between descending HOWG (DHOWG) and ascending HOWG (AHOWG). In the similar fashion, the properties of HWG can equivalently be defined. The proof is straightforward and thus, omitted in this study.

E.4 Analytic Hierarchy Process

In this section, the definition of analytic hierarchy process and its properties are presented.

Definition E.8 (Saaty, 1977; 1980). A multiplicative preference relation P on a set of alternatives X is defined as a reciprocal matrix $P = \left(p_{ij}\right)_{n\times n} \subset X \times X$ with the condition as follow:

$$p_{ij} > 0, \quad p_{ij} \cdot p_{ji} = 1, \quad p_{ii}, \qquad \forall i,j = 1,2,\dots,n, \tag{E.8}$$

where p_{ij} is interpreted as the ratio of the preference intensity of the alternative x_i over x_j.

In general, the intensity of preference p_{ij} is measured based on ratio-scale {1/9,…,9}, where $p_{ij} = 9$ means x_i absolutely preferred over x_j; $p_{ij} = 1$ implies indifference between x_i and x_j; $p_{ij} = 1/9$ indicates that x_j absolutely preferred over x_i. In addition to that, $p_{ij} \cdot p_{ji} = 1$ indicates multiplicative reciprocal condition and $p_{ij} = p_{ik} \cdot p_{kj}$ implies multiplicative transitivity. Indifference, reciprocal and transitivity are the main properties for multiplicative preference relation. However AHP is usually inconsistent in practice. Saaty (1980) suggested a consistency index (*CI*) as follow:

$$CI = \frac{\lambda_{max} - n}{n - 1}, \tag{E.9}$$

where λ_{max} is the largest eigenvalue of P and a consistency ratio (*CR*) can be calculated as follow:

$$CR = \frac{CI}{RI}, \tag{E.10}$$

where *RI* is the random index, the consistency index of a randomly generated pairwise comparison matrix. The *RI* depends on the number of elements being compared. The consistency ratio $CR < 0.1$ indicates an acceptable inconsistency in pairwise comparison. Then, the overall evaluation score of alternative can be calculated as follow:

$$D_i = \sum\nolimits_{i=1}^{n} w_i p_i, \tag{E.11}$$

The evaluation process in the AHP uses a simple weighted average to calculate the scores or priorities of each alternative and $\sum_{i=1}^{n} w_i = 1$.

In addition to the weighted arithmetic mean as aggregation technique used in AHP, different aggregation procedures have been proposed in the literature, specifically on arithmetic and geometric mean based methods, e.g. geometric mean, weighted geometric mean (WG), ordered weighted average (OWA), ordered weighted geometric (OWG), etc. Most commonly used method is the geometric mean, which is the only separable synthesizing function that satisfies the unanimity condition (i.e. Pareto principle), the homogeneity condition (e.g. if all individuals judge a ratio *t* times as large as other ratio, then the synthesized judgments should be *t* times as large), and reciprocal property (Aczel & Saaty, 1983; Forman & Peniwati, 1998; Escobar & Moreno-Jimenez, 2007). In the group decision settings, these types of aggregation procedures depend on the category of group decision making processes, i.e., the AIJ and the AIP as defined in the following.

Definition E.9 (Forman & Peniwati, 1998). If a set of decision makers or experts, $E_h = \{E_1, E_2, \ldots, E_r\}$ with $h = 1,2, \ldots, r$ provides preference $\{P^1, P^2, \ldots, P^n\}$ about a set of alternatives $X = \{x_1, x_2, \ldots, x_m\}, k = 1,2, \ldots, m$ and each have importance degree $\beta(E_h) \in [0,1], \sum_{=h=1}^{r} \beta(E_h) = 1$, then, the AIJ can be defined as a collective pairwise comparison judgment matrix for the group, such that, $X^{[G]} = p_{ij}^{[G]} = \prod_{h=1}^{r} \left(p_{ij}^{[h]}\right)^{\beta_h}$, and the priority vector $\overline{w}^G$ is derived from $X^{[G]}$ using one of the prioritization methods.

Definition E.10 (Forman & Peniwati, 1998). If a set of decision makers or experts, $E_h = \{E_1, E_2, \ldots, E_r\}$ with $h = 1,2, \ldots, r$ provides preference $\{P^1, P^2, \ldots, P^n\}$ about a set of alternatives $X = \{x_1, x_2, \ldots, x_m\}, k = 1,2, \ldots, m$ and each have importance degree $\beta(E_h) \in [0,1], \sum_{=h=1}^{r} \beta(E_h) = 1$, then, the AIP can be defined as a collective priorities of the alternatives for the group, such that, $\widehat{w}_k^G = \prod_{h=1}^{n} \left(w_k^{[h]}\right)^{\beta_h}, k = 1,2, \ldots m$, where the priority vector $w^{[h]} = \left(w_k^{[h]}\right)$ is derived from each individual DM or expert.

As for the AIJ procedure, the geometric mean based methods are more preferable than the arithmetic mean based methods since they satisfy the unanimity condition, homogeneity condition and multiplicative reciprocal property as mentioned before. While for the AIP, both methods can be used to aggregate the individuals' priorities. For the purpose of this study, the AIJ procedure is put forward to be integrated with heavy aggregation operators, specifically HWG and HOWG as defined in the previous section.

E.5 Analytic Hierarchy Process under Group Decision Making with Heavy OWG

In this section, the AHP based on AIJ procedure with HOWG is presented. As mentioned earlier, the only difference of the proposed method with the classical approach is by relaxing the constraints of the associated weighting vector and providing the possibility of partial and non-overlapping information, instead of fully overlap or redundant information. In what follows, the proposed method is presented step by step as in the subsequent algorithms.

Assume $x_k \in X, (x_k, k = 1,2, \dots, m)$ comprise a finite set of alternatives. Let $c_l \in C, (c_l, l = 1,2, \dots, n)$ and $c_p^l \in C$, $(c_p^l, l = 1,2, \dots, n;\ p = 1,2, \dots, q)$ are the criteria and sub-criteria under consideration, respectively. Then, let $E_h \in E$, $(E_h, h = 1,2, \dots, r)$, be a group of experts with each expert E_h presenting his/her intensity preferences for rating the alternatives x_k and weighting the criteria c_l (or sub-criteria c_p^l) with respect to the AIJ procedure. Based on the above concepts, the algorithm for the AHP-GDM-HOWG consists of the following steps.

Step 1: Each decision maker or expert $(E_h, h = 1,2, \dots, r)$, compares the m alternatives on each criterion (or a cluster of criteria) and provides a pairwise comparison matrix as follows:

$$P_l^{[h]} = \left[p_{ij}\right]_{m\times m}^{[h]}, i, j = 1,2, \dots, m, \tag{E.12}$$

with $(\forall i = j,\ p_{ij} = 1)$ and $(\forall i \neq j,\ p_{ij} = \left[p_{ji}\right]^{-1})$. Then, the pairwise comparison matrices $P_l^{[h]}$ of all experts E_h $(E_h, h = 1,2, \dots, r)$, are aggregated with respect to each criterion $c_l, l = 1,2, \dots, n$, where $\beta(E_h)$ is the importance degree of expert. By using the weighted geometric mean method (WGM), this process can be calculated as follows:

$$x^{c_l} = p_{ij}^{c_l} = \prod_{h=1}^{r} \left(p_{ij}^{[h]}\right)^{\beta_h}, c_l, l = 1,2, \dots, n, \tag{E.13}$$

At this stage, each criterion represents the collective judgment of experts (or only single judgment of expert in case of total independence or non-overlapping).

Step 2: Analogous to the *Step 1*, next, calculate the pairwise comparison matrix for the criteria $c_l \in C, (c_l,\ l = 1,2, \dots, n)$ and sub-criteria $c_p^l \in C, \left(c_p^l, l = 1,2, \dots, n; p = 1,2, \dots, q\right)$. Then, derive the criteria weights ω_j and sub-criteria weights ω_k^j.

Step 3: Afterwards, calculate the composite weights of the criteria c_j and sub-criteria c_k^j as follows:

$$w_l = \omega_l \times \omega_p^l, \quad \text{(E.14)}$$

Step 4: Finally, compute the overall collective judgment matrix for all criteria c_l of a group by using HOWG operator, with w_l as the weighting vector that agreed by all experts $(E_h, h = 1,2, \dots, r)$ as explained in *Step 2*:

$$X^{[G]} = p_{ij}^{[G]} = \prod_{l=1}^{n} \left(x_{\sigma(j)}^{c_l}\right)^{w_l}, \quad \text{(E.15)}$$

where $\sigma(j)$ denotes the components of x^{c_l} being arranged in non-increasing order $x_{\sigma(j)} \geq x_{\sigma(j+1)} \geq \cdots \geq x_{\sigma(n)}$, with $w = (w_1, w_2, \dots, w_n)^T$, $w_l \in [0,1]$ and $1 \leq \sum_{j=1}^{n} w_j \leq n$.

Remark E.1. When no ordering process involve for the x^{c_l} values, the proposed method is reduced to the AHP-GDM based on HWG aggregation operator.

Remark E.2. In case of fully overlapping information, such that, $\sum_{l=1}^{n} w_l = 1$, then, the proposed AHP-GDM can be reduced to the cases of OWG, WG and GM.

E.6 Numerical Example

A numerical example is given to implement the methodology discussed in the previous sections. For this purpose, consider an investment selection problem where a company is looking for an optimal investment. There are five possible alternatives to be considered as follows: x_1 is a computer company; x_2 is a chemical company; x_3 is a food company; x_4 is a car company; x_5 is a TV company. In order to evaluate these alternatives, a group of experts must make a decision according to the following four criteria: C_1 = risk analysis; C_2 = growth analysis; C_3 = social-political impact analysis; and C_4 = environmental impact analysis.

In this case, assume that five experts involved which are categorized based on their related areas of expertise. For instance, the criterion C1 is evaluated by both experts E1 and E2, the criterion C2 judged by experts E1, E3 and E4, the criterion C3 by experts E3 and E4, and finally for the criterion C4, experts E4 and E5 are responsible for this criterion. In addition, let say that for all criteria C_1, C_2, C_3, and C_4, there are partial redundancy information provided by each expert in each category. Based on experts' agreements, 50% of each of the criteria C_1, C_2, C_3 and C_4 will be increased to reflect this issue. In this case, the original weighting vector for the criteria is given as w_l = (0.311, 0.406, 0.1824, 0.100) which is derived from pairwise comparison matrix. Then, for criteria C_1, C_2, C_3 and C_4, 50% of each value will be increased. Hence the final weight vector is given as w_l = (0.467, 0.610, 0.243, 0.150).

To demonstrate this problem, first, let all the experts provide pairwise comparison matrices as comparison of m alternatives on each of the specific criterion. Assume that the degrees of importance $\beta(E_h)$ are distributed equally for all experts in the evaluation process (homogeneous case). The consistency ratio is then computed to check the consistency of the pairwise comparison matrix for each expert.

Henceforth, using the formulation in *Step 1*, the collective judgment matrices for the criteria can be derived as shown in Tables E.1-E.4.

Table E.1. Collective Judgment Matrix for Criterion C_1

C_1	A_1	A_2	A_3	A_4	A_5
A_1	1	0.354	1.732	0.408	1
A_2	2.828	1	4.472	1.414	2.449
A_3	0.577	0.223	1	0.258	0.408
A_4	2.449	0.707	3.873	1	2
A_5	0.5	0.408	2.449	0.5	1

Table E2. Collective Judgment Matrix for Criterion C_2

C_1	A_1	A_2	A_3	A_4	A_5
A_1	1	1	2.449	2.449	0.408
A_2	1	1	2.449	2.236	0.408
A_3	0.408	0.408	1	1	0.258
A_4	0.408	0.447	1	1	0.258
A_5	0.249	2.449	3.872	3.872	1

Table E.3. Collective Judgment Matrix for Criterion C_3

C_1	A_1	A_2	A_3	A_4	A_5
A_1	1	1.442	0.928	1.077	0.721
A_2	0.693	1	0.437	1	0.342
A_3	1.077	2.289	1	1.587	1.100
A_4	0.928	1	0.630	1	0.575
A_5	1.386	2.924	0.909	1.738	1

Table E.4. Collective Judgment Matrix for Criterion C_4

C_1	A_1	A_2	A_3	A_4	A_5
A_1	1	0.354	0.250	2.828	0.333
A_2	2.828	1	0.333	3	0.5
A_3	4	3	1	5	2
A_4	0.354	0.333	0.2	1	0.25
A_5	3	2	0.5	4	1

The next step after this stage is the determination of weight for the final ranking of alternatives. As in this case no sub-criteria are considered, hence, the weight for each criterion can be directly derived as in the *Step 2*. Then, the final step is to compute the overall collective judgment matrix for all alternatives x_k with respect to criteria c_l for the group decision making. By using HOWG operator, with w_l as the weighting vector, then the final aggregated results and ranking can be derived using the formula in *Step 4*.

For the comparison purpose, different type of geometric mean based aggregation operators are used with respect to different weighting vector and ordering of argument values. The aggregated results are demonstrated in Table E.5 and for the ranking of investments, it is shown in Table E.6.

Table E.5. Aggregated Results

HOWG	HWG	OWG	WG	GM
0.144	0.165	0.165	0.180	0.167
0.223	0.253	0.217	0.240	0.221
0.180	0.103	0.193	0.132	0.182
0.105	0.128	0.135	0.152	0.139
0.348	0.350	0.291	0.297	0.291

Table E.6. Ranking of Investments

AHP-GDM	Ranking
HOWG	$A_5 \succ A_2 \succ A_3 \succ A_1 \succ A_4$
HWG	$A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$
OWG	$A_5 \succ A_2 \succ A_3 \succ A_1 \succ A_4$
WG	$A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$
GM	$A_5 \succ A_2 \succ A_3 \succ A_1 \succ A_4$

5.4 Summary

In this chapter, the decision making models based on the analytic hierarchy process (AHP) have been discussed for the application in financial selection problems. In specific, Section 5.2 presented the proposed method on the generalization of AHP for group decision making model using the induced OWA operators. Then, in Section 5.3, the heavy weighted geometric aggregation operators in AHP group decision making has been proposed.

CHAPTER 6

SOME EXTENSIONS OF TOPSIS MODEL FOR GROUP DECISION MAKING PROBLEMS

6.1 Introduction

In this chapter, the technique for order performance by similarity to ideal solution (TOPSIS) is studied, mainly for the group decision making problems. In Section 6.2, the TOPSIS model with induced generalized OWA operators is presented. The model then is applied to the case of human resource selection problem. Secondly, in Section 6.3, the integration of TOPSIS with AHP method under the conflicting bifuzzy condition is proposed for the selection of flood control project. To end up the chapter, a summary is given in Section 6.4.

6.2 Induced Generalized OWA Operators in TOPSIS for Majority Group Decision Making Model

Abstract. This study suggests an extension of TOPSIS for group decision making model by the inclusion of a concept of majority opinion. This concept is derived based on the induced generalized OWA operators. To achieve this objective, two fusion schemes in TOPSIS model are designed. First, an external fusion scheme to aggregate the experts' judgments is suggested, specifically with respect to the concept of majority opinion on each criterion. Then, an internal fusion scheme of ideal and anti-ideal solutions that represents the majority of experts is proposed using the Minkowski OWA distance measures. The advantages of the proposed model include, a consideration of soft majority concept as a group aggregator and a flexibility in applying the decision strategies of criteria for analysing the decision making process. In addition, instead of calculate the majority opinion with respect to the individual ranking of alternatives, the proposed method takes into account the majority of experts on each criterion, in which reflects the consensus on specific criteria for the overall decision. A numerical example in human resource selection problem is provided to demonstrate the applicability of the proposed model and the comparison is conducted with some other TOPSIS models with respect to the distance measures.

F.1 Introduction

Multiple criteria decision analysis (MCDA) is one of the active topics in the field of operations research. MCDA deals with the problem of selecting, prioritizing or ranking a finite number (or discrete set) of courses of action. There are a number of techniques in the literature which were developed to deal with different types of MCDA problems (see Hwang & Yoon, 1981; Figueira et al., 2005; Behzadian et al., 2012, for the state-of-art surveys of MCDA techniques). The technique for order performance by similarity to ideal solution (TOPSIS) is one of the known methods and was first proposed by Hwang and Yoon (1981) based on the concept of ideal distances of alternatives. The ranking of alternatives refers to the shortest distance from the positive ideal solution and the farthest from the negative ideal solution.

The TOPSIS is a flexible method, in which it can be integrated with other techniques as an extension model, for example, in the case of group decision making problems (Olson, 2004; Shih et al., 2007; Afful-Dadzie et al., 2015; Taib et al., 2016). Recently, much attention has been given on the aggregation of preferences among the group of experts. The ordered weighted averaging (OWA) operator as proposed by Yager (1988) is one of the approaches normally used for the aggregation of preferences in MCDA models. The OWA provides a parameterized class of mean-type aggregation operators, such as the min, arithmetic average and max, with a flexibility for the inclusion of linguistic quantifiers (Yager & Kacprzyk, 1997; Torra & Narukawa, 2007). With such characteristics, it can be interpreted as a generalization of the original decision making model as suggested by Bellman and Zadeh (1970). In addition, Yager and Filev (1999) proposed the induced OWA (IOWA) operator as an extension of the OWA operator. In general, it has an additional feature, namely the vector of order-inducing variables for the complex decision making process. Subsequently, Merigó and Gil-Lafuente (2009) generalized the IOWA operator to include some other types of mean operators such as the induced ordered weighted geometric average (IOWGA), the induced ordered weighted harmonic average (IOWHA), the ordered weighted quadratic average (IOWQA) operators, to name a few. In the literature, the OWA and IOWA operators have been successfully applied in some of the MCDA models (see Chen et al., 2011; Kacprzyk et al., 2011; Liu et al., 2015; Yager et al., 2011; Yusoff & Merigó 2014, 2015).

Recently, there are a number of studies that have been done on the TOPSIS for group decision making model. Shih et al. (2007) for example, provides an analysis of TOPSIS model under group decision environment (TOPSIS-GDM). The aggregation of the individual experts' judgments as the overall group decision is generated using either the arithmetic or geometric means with some distance normalization methods, e.g., the Manhattan distance and the Euclidean distance. Later, Chen et al. (2011) extended the

TOPSIS-GDM model with the inclusion of the OWA operator. Three fusion schemes were suggested where one deals with the local judgment (or internal aggregation) of each expert and the rest defined as the global judgments (or external aggregations) which deal with the fusion of individual experts' judgments as the overall group decision. In internal aggregation, the OWA is used to provide flexibility in the selection of ideal and anti-ideal values as specified by different experts. Specifically, the ideal and anti-ideal values are given directly and independently by each expert. On the other hand, the external aggregations allow a tolerance in the selection of individual experts' judgments, such as, either to consider the total or partial compensation of experts. In other related research, Islam et al. (2013) proposed an integrated approach of fuzzy TOPSIS-OWA and geographic information system (GIS) for evaluating the water quality problems. In their method, the aggregation of criteria is based on the OWA operators, in which providing a flexibility for considering the decision strategies on criteria (e.g., either total or partial compensation of criteria). But the aggregation of experts as the overall group decision is conducted based on the arithmetic means.

As can be noticed, most of the group aggregators of the previous methods is mainly based on the arithmetic or geometric means, in which just the average of all the opinions of experts. In such cases, there is no flexibility for considering the majority concept as to represent the group decision. Even though Chen et al. (2011) proposed the OWA operator as to provide a flexibility in the group aggregation process, but, they did not take into consideration the support (or similarity) between experts as a consensus measure. Pasi and Yager (2006) then proposed the concept of majority opinion by utilizing the IOWA operator and linguistic quantifiers as a tool for the consensus measure where the opinions of the experts supporting each other on each alternative are taken into account. Consequently, based on Pasi-Yager method, Hajimirsadeghi and Lucas (2009) proposed the inclusion of majority concept in the TOPSIS-GDM model where the consensus measure is explicitly included. In addition, Boroushaki and Maczewski (2010) utilized the concept of fuzzy majority for GIS-based multi-criteria group decision making.

Nevertheless, the method as proposed by Pasi and Yager (2006) is simply focused on the proximity measure (Yager, 2004) on individual ranking of experts on each alternative (classical scheme of group decision making process) with disregard the conflicts or incoherence between the performance judgments on each single criterion. In this case, two experts may produce the same performance judgment for an alternative even their single performance judgments on the criteria are completely different. In consequence, Bordogna and Sterlacchini (2014) proposed an extension of Pasi-Yager method by considering the consensus on each criterion (or called as an alternative scheme) instead of on each alternative and calculate the consensus among experts using the Minkowski OWA distance measures. This alternative approach has some advantages which include considering the degrees of trust of experts, providing the uniformity in reflecting the behaviour of the majority of experts regarding the proportion of criteria to consider and obtaining a more robust decision by determine the performance judgments on each specific criterion.

In this study, the integration of TOPSIS for group decision making model with the concept of majority opinion and the induced generalized OWA aggregation operators is proposed. This model is an extension of the methods proposed by Hajimirsadeghi and Lucas (2009) by considering the majority concept with respect to each criterion instead of consensus on each alternative. Some modifications to the concept of majority opinion introduced by Pasi and Yager (2006) with the idea proposed in Bordogna and Sterlacchini (2014) is put forward to be applied in the TOPSIS-GDM. However, the focus in this study is just limited to the case of homogeneous group decision making where each expert is associated with an equal degree of importance for each criterion. The rest of the study is structured as the following. In Section F.2, some preliminaries related to the definitions and concepts used in this study are presented. In Section F.3, the general framework of the classical TOPSIS method is given; Section F.4, discussed the concept of majority opinion based on Pasi-Yager method and its extension. In Section F.5, the proposed model on IGOWA-TOPSIS based on majority concept is explained. Then, in Section F.6, a numerical example in human resource selection problem is provided and some comparisons with other TOPSIS models with respect to the distance measures are conducted.

F.2 Preliminaries

In the following, the basic aggregation operators that are used in this study are briefly discussed, such as the OWA, IOWA and IGOWA operators as well as the Minkowski OWA distance measures.

F.2.1 OWA, IOWA and IGOWA operators

Definition F.1 (Yager, 1988). An OWA operator of dimension n is a mapping $OWA: \mathbb{R}^n \rightarrow \mathbb{R}$ that has an associated weighting vector $W = (w_1, w_2, \dots, w_n)$ of dimension n such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, then:

$$OWA(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} w_j a_{\sigma(j)} , \qquad \text{(F.1)}$$

where $a_{\sigma(j)}$ is the jth largest of argument value a_j.

Definition F.2 (Yager & Filev, 1999). An IOWA operator of dimension n is a mapping $IOWA: \mathbb{R}^n \to \mathbb{R}$ that has an associated weighting vector W of dimension n such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, then:

$$IOWA(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \dots, \langle u_n, a_n\rangle) = \sum_{j=1}^{n} w_j a_{\sigma(j)} \, , \tag{F.2}$$

where $a_{\sigma(j)}$ is the argument value a_j of the IOWA pair $\langle u_j, a_j\rangle$ having the jth largest of order-inducing variable u_j.

Note that, in case of 'ties' between the inducing variables, the procedure as suggested by Yager and Filev (1999) will be implemented in which each argument of tied IOWA pair is replaced by their average. For instance, if z of the $u_{\sigma(j)}$ are tied, i.e., $u_{\sigma(j)} = u_{\sigma(j+1)} = \cdots = u_{\sigma(j+z-1)}$, then, the value $a_{\sigma(j)}$ is given as follow (Yager & Filev, 1999; Beliakov & James, 2011):

$$a_{\sigma(j)} = \frac{1}{z} \sum_{k=\sigma(j)}^{\sigma(j+z-1)} a_k \, . \tag{F.3}$$

Definition F.3 (Merigó & Gil-Lafuente, 2009). An IGOWA operator of dimension n is a mapping $IGOWA: \mathbb{R}^n \to \mathbb{R}$ that has an associated weighting vector W of dimension n such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, then:

$$IGOWA(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \dots, \langle u_n, a_n\rangle) = \left(\sum_{j=1}^{n} w_j a_{\sigma(j)}^{\lambda} \right)^{1/\lambda} \, , \tag{F.4}$$

where $a_{\sigma(j)}$ is the argument variable a_j of the IGOWA pair $\langle u_j, a_j\rangle$ having the jth largest of order-inducing variable u_j and λ is a parameter such that $\lambda \in \mathbb{R} \setminus \{0\}$.

With different values of λ, various type of averaging operators can be derived. For example, when $\lambda = -1$, the IOWHA operator can be obtained, when $\lambda \to 0$, then the IOWG is generated, for $\lambda = 2$, the IOWQA operator is derived, etc. The OWA, IOWA and IGOWA operators are all meet the commutative, monotonic, bounded and idempotent properties (Yager, 1988; Yager & Filev, 1999; Merigó & Gil-Lafuente, 2009). Note that, the notation $\sigma(j)$ denotes the ordering process with respect to non-increasing order.

F.2.2 Minkowski OWA distance measures

Definition F.4 (Merigó & Gil-Lafuente, 2008). A Minkowski OWAD operator of dimension n is a mapping $MOWAD: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ that has an associated weighting vector *W* of dimension *n* such that $\sum_{j=1}^{n} w_j = 1$ with $w_j \in [0,1]$ and the distance between two sets A and B is given as follows:

$$MOWAD(d_1, d_2, \dots, d_n) = \left(\sum_{j=1}^{n} w_j d_{\sigma(j)}^{\lambda}\right)^{1/\lambda}, \qquad \text{(F.5)}$$

where $d_{\sigma(j)}$ is the jth largest of the d_j and d_j is the individual distance between A and B, such that $d_j = |a_j - b_j|$ with λ is a parameter in a range $\lambda \in \mathbb{R} \setminus \{0\}$.

By setting different values for the norm parameter λ, some special distance measures can be derived. For example, if $\lambda = 1$, then the Manhattan OWA distance can be obtained, $\lambda = 2$ then the Euclidean OWA distance can be acquired, $\lambda = \infty$ then Tchebycheff OWA is derived, etc.

F.2.3 OWA operators with the inclusion of linguistic quantifiers

The linguistic quantifier was first introduced by Zadeh (1983) as a generalization of the existential '*at least one*) and universal (*all*) quantifiers of classical logic. Linguistic quantifiers are expressed by terms, for example, *most, many, half, some, few* to indicate an approximate way a quantity of the elements belonging to a reference set (or the universe of discourse). In general, there are two types of quantifiers, in which termed as absolute and proportional quantifiers. For the absolute quantifier, its function is given as (Zadeh, 1983):

$$Q: \mathbb{R}^{+} \rightarrow \mathbb{I}, \textit{ satisfies } Q(0) = 0, \exists x \in \mathbb{I} \ni Q(x) = 1, \qquad \text{(F.6)}$$

where the quantifier Q is assumed to be a fuzzy in the unit interval, $\mathbb{I} = [0,1]$ and $\mathbb{R}^{+}$ is a set of positive real numbers. While for a proportional quantifier, its function is denoted as (Zadeh, 1983):

$$Q: \mathbb{I} \rightarrow \mathbb{I}, \textit{ satisfies } Q(0) = 0, \exists x \in \mathbb{I} \ni Q(x) = 1, \qquad \text{(F.7)}$$

such that if Q is a fuzzy subset corresponding to a proportional linguistic quantifier, then for any value x in the unit interval $\mathbb{I}$ the membership grade $Q(x)$ corresponds to the compatibility of the value x with the concept in which Q is representing. In addition, there are two kinds of fuzzy quantified propositions as defined by Zadeh (1983). First, "$Q\ X\ are\ Y$", i.e., Q elements of set X satisfy the fuzzy predicate Y. The other proposition is "$Q\ B\ X\ are\ Y$", i.e., Q elements of set X which satisfy the fuzzy predicate B also satisfy the fuzzy predicate Y.

There are some categories exist for quantifiers Q such as regular increasing monotone (RIM), regular decreasing monotone (RDM) and regular normalized unimodal (RUM) quantifiers. But in the context of MCDA, the regular increasing monotone (RIM) quantifier is sufficient, as one wants to represent the fact that the larger the number of satisfied criteria the more satisfied the solution is (Yager, 1988). RIM quantifier is defined as the following:

$$Q(0) = 0, Q(1) = 1 \text{ and } Q(x) \geq Q(y) \textit{ if } x > y. \qquad \text{(F.8)}$$

Specifically, the proportional linguistic quantifiers Q of RIM can be represented as the parameterized fuzzy subset in the form:

$$Q(r) = r^{\alpha}, \alpha > 0\,, \qquad \text{(F.9)}$$

where parameter α indicates the degree of inclusion for different elements and $r \in [0,1]$. The main characteristics can be represented as: for $\alpha \to 0$, the existential quantifier is obtained, for $\alpha = 1$, the unitor quantifier is attained $Q(r) = r$ and for $\alpha \to \infty$, the universal quantifier is acquired. In natural language many additional semantics can be demonstrated, for instance, given $\alpha = 0.1$ and $\alpha = 10$ then '*few*' and '*most*' can be obtained. Alternatively, the proportional linguistic quantifiers Q can be represented as in the following definition:

$$Q(r) = \begin{cases} 0 & if\ \ r < a, \\ \dfrac{(r-a)}{(b-a)} & if\ a \leq r \leq b, \\ 1 & if\ \ r > b, \end{cases} \qquad \text{(F.10)}$$

with $a, b, r \in [0,1]$. For example, the semantic *most*, *almost all* and *at least half* can be represented as parameters $(a, b) = (0.3, 0.8), (0, 0.5),\ (0.5, 1)$, respectively (Zadeh, 1983).

In the context of OWA, Yager (1988) then defined the OWA aggregation from Q by defining the weights in the following way:

$$w_j = Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right), \qquad j = 1,2,\dots,n, \tag{F.11}$$

where w_j represents the increase of satisfaction in getting j with respect to $j-1$ criteria satisfied. By changing the α values, different decision strategies then can be derived. For example, $\alpha \to 0$, then $W = W^*$, where $W^* = (1,0,\dots,0)$, $\alpha = 1$ then $W = W_{1/n}$, where $W_{1/n} = (1/n, 1/n, \dots, 1/n)$ and $\alpha \to \infty$ then $W = W_*$, where $W_* = (0,0,\dots,1)$.

In the case where the criteria c_j to be aggregated have relative importances v_j associated with them (v_j, c_j), the inclusion of degrees of importance in OWA operators from Q can be defined as follows (Yager, 1996):

$$\omega_j = Q\left(\frac{\sum_{k=1}^{j} v_{\sigma(k)}}{T}\right) - Q\left(\frac{\sum_{k=0}^{j-1} v_{\sigma(k)}}{T}\right), \tag{F.12}$$

where $v_{\sigma(j)}$ is the degrees of importance associated with the criteria that has the *j*th largest satisfaction c_j such as $(v_{\sigma(j)}, c_{\sigma(j)})$ and $T = \sum_{j=1}^{n} v_{\sigma(j)}$, the total sum of importances. For example, if $c_g =$ *j*th largest of the c_j, then $c_{\sigma(1)} = c_g$ and $v_{\sigma(1)} = v_g$, then $(v_g, c_g) = (v_{\sigma(1)}, c_{\sigma(1)})$.

F.3 TOPSIS Model under Group Decision Making

In this section, the classical TOPSIS under group decision making is presented prior to its extension to the proposed model.

The classical TOPSIS under group decision making procedure (Shih et al., 2007) can be summarized as the following steps. First, a decision matrix for each expert $D^h, h = 1,2,\dots,k$, is constructed as follows:

$$D^h = \begin{array}{c} \\ A_1 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & \cdots & C_n \end{array} \\ \begin{pmatrix} a_{11}^h & \cdots & a_{1n}^h \\ \vdots & \ddots & \vdots \\ a_{m1}^h & \cdots & a_{mn}^h \end{pmatrix} \end{array}, \tag{F.13}$$

where A_i indicates the alternative i $(i = 1,2,\dots,m)$ and C_j denotes the criterion j $(j = 1,2,\dots,n)$, and a_{ij}^h denotes the preferences on consequence data space (original and raw information) for alternative A_i with respect to criterion C_j.

Then, each decision matrix D^h which represents each expert is normalized to $\overline{D}^h$ using the vector normalization method:

$$x_{ij}^h = \frac{a_{ij}^h}{\sqrt{\sum_{i=1}^m \left(a_{ij}^h\right)^2}}, \forall h = 1,2, \dots, k, \tag{F.14}$$

Alternatively, some other normalization methods can be used instead of the Eq. (F.14), see for example in Hwang and Yoon (1981). For the next step, instead of directly construct the weighted normalized decision matrix as in the original TOPSIS (Hwang & Yoon, 1981), motivated by Shipley et al. (1991), the integration of weights in the separation measure is suggested by Shih et al. (2007) as the following.

Determine the ideal and anti-ideal solutions X^{h+} and X^{h-} for each expert. The ideal and anti-ideal are obtained as follows:

$$\begin{aligned} X^{h+} &= \{x_1^{h+}, \dots, x_n^{h+}\} \\ &= \left\{\left(max_i x_{ij}^h | j \in J\right), \left(min_i x_{ij}^h | j \in J'\right)\right\}, \end{aligned} \tag{F.15}$$

$$\begin{aligned} X^{h-} &= \{x_1^{h-}, \dots, x_n^{h-}\} \\ &= \left\{\left(min_i x_{ij}^h | j \in J\right), \left(max_i x_{ij}^h | j \in J'\right)\right\}, \end{aligned} \tag{F.16}$$

where J is associated with the set of benefit criteria and J' is associated with the set of cost criteria.

Further, calculate the separation measures from the ideal and anti-ideal solutions for the group. The manipulation for Minkowski's L_p metric as the distance measure is described as follows:

$$S_i^{h+} = \left\{\sum_{j=1}^n w_j^h \left(x_{ij}^h - x_j^{h+}\right)^p\right\}^{1/p}, \ \forall i = 1,2, \dots, m, \tag{F.17}$$

$$S_i^{h-} = \left\{\sum_{j=1}^n w_j^h \left(x_{ij}^h - x_j^{h-}\right)^p\right\}^{1/p}, \ \forall i = 1,2, \dots, m, \tag{F.18}$$

where $p \geq 1$ and w_j^h is the weight for the criterion j and expert h and $\sum_{j=1}^n w_j^h = 1$. Note that with $p = 1$, then S_i^{h+} and S_i^{h-} provide the Manhattan distance, whilst the metric with $p = 2$ is the Euclidean distance. At this stage, the ranking of alternatives is obtained individually by each expert (i.e., internal aggregation).

In the next stage, the consensus of experts (i.e., external aggregation) is computed using the group aggregator, such as the arithmetic mean (AM) or the geometric mean (GM):

$$S_i^{G+} = S_i^{1+} \circledast \dots \circledast S_i^{k+} , \quad \forall i = 1,2, \dots , m , \qquad \text{(F.19)}$$

$$S_i^{G-} = S_i^{1-} \circledast \dots \circledast S_i^{k-} , \quad \forall i = 1,2, \dots , m , \qquad \text{(F.20)}$$

where the operators $\circledast$ are either AM or GM.

Finally, calculate the relative closeness RC_i^G to the ideal solution. The relative closeness can be calculated according to the following formula:

$$RC_i^G = \frac{S_i^{G-}}{S_i^{G-} + S_i^{G+}}, \qquad \forall i = 1,2, \dots m. \qquad \text{(F.21)}$$

Note that the larger the value of RC_i^G denotes the better performance of the alternative.

F.4 The Concept of Majority Opinion in Group Decision Making

As previously mentioned, the group aggregator in the classical TOPSIS-GDM, in general, is based on the arithmetic mean (AM) or geometric mean (GM). However, the AM or GM as a group aggregator does not take into account the support (or similarity) between experts as a consensus measure. Even though it can be extended to the OWA or OWG operators, but such operator is not sufficient to represent the majority concept. In this section, the overview of the method for aggregating the majority opinions of experts based on the IOWA operator (Pasi & Yager, 2006) is provided. Then, the extension of Pasi-Yager method from the classical scheme to the alternative scheme of group decision making process is suggested to be integrated in the TOPSIS-GDM model.

F.4.1 Majority concept based on the classical scheme

Normally, in group decision making the unanimous agreement is not easy to achieve due to some factors, such as conflict interest, different background and experiences among the decision makers or experts. Hence, the concept of majority is crucial as it is required to find a soft agreement that satisfies the opinions, for example, *most* of the experts. The OWA aggregation operator with the regular linguistic quantifier is not ideal for modelling the concept of majority, as it produces a value that reflects the satisfaction of the proposition "*most* of the criteria have to be satisfied" instead of "satisfaction value of *most* of the criteria" (Pasi & Yager, 2006). Therefore, a mechanism based on IOWA operators is proposed by Pasi and Yager (2006) to model the majority opinion. The methodology used to obtain the majority opinion is described as the following.

The order-inducing variables are obtained by means of a function of support or proximity measure (Yager, 2004) between pairs of the values to be aggregated. A support function is a binary function that used to compute a value $supp(x, y)$, which expresses support from x to y. In this case, the more similar or close the two values then the more they support to each other. The support function for the group of experts can be given as follows:

$$supp(p_l, p_h) = \begin{cases} 1 & if \ |p_l - p_h| < \beta, \\ 0 & otherwise. \end{cases} \tag{F.22}$$

This function is defined to measure the support for each expert l with respect to all the other experts h in the group (not include himself/herself), where $l \in h$ for $(h = 1,2, \dots, k)$. The overall support s_l for an expert l can be directly calculated by the sum of all supports as follows:

$$s_l = \sum_{\substack{h=1 \\ h \neq l}}^{k} supp(p_l, p_h). \tag{F.23}$$

By the same process, the overall support values for the other experts $l \in h$ can be derived. These values $s_1, s_2 \dots, s_k$ are used as the order-inducing variables and they are reordered such that $U = \left(s_{\tau(1)}, s_{\tau(2)}, \dots, s_{\tau(k)}\right)$ with $s_{\tau(1)} \leq s_{\tau(2)} \leq \cdots \leq s_{\tau(k)}$. The notation $\tau(h)$ denotes the ordering process with respect to non-decreasing order.

Next, to compute the weights, define the values t_l based on a modification of the s_l values, such that: $t_l = s_l + 1$ (the similarity of p_l with himself/herself, e.g., $supp(p_l, p_l) = 1$). The t_l values are in non-decreasing order $t_1 \leq \cdots \leq t_k$. Then, on the basis of t_l values, the weights are computed as follows:

$$w_l = \frac{Q(t_l/k)}{\sum_{l=1}^{k} Q(t_l/k)}, \qquad \forall l \in h, h = 1,2, \dots, k, \tag{F.24}$$

where $Q(t_l/k)$ denotes the degree to which a given member of the considered set of values represent the '*most*' such as in Eq. (F.10). Finally, the final evaluation is derived using the IOWA operator as in Eq. (F.2). However, the order-inducing variables s_h and weight w_h are reordered in non-decreasing order.

F.4.2 Majority concept based on alternative scheme

Should be noted that, the method as proposed by Pasi and Yager (2006) is mainly based on the classical scheme of group decision making process where the result of consensus measure is determined according to the support on each alternative of individual experts. In general, the classical scheme can be divided into two stages of aggregation processes, namely internal and external aggregations. The internal aggregation involves the fusion of criteria for each expert, either full or partial compensation. At this stage, the ranking of alternatives for each expert is obtained. Then, as regard to these rankings, in the external aggregation, the soft majority concept is implemented to find the final ranking which reflects the majority opinion of experts on alternative.

In addition to the classical scheme, there is another type of group decision making process, in which called as the alternative scheme. For this approach, instead of dealing with internal aggregation at the first step, where the ranking of alternatives of each expert is derived, this method is initiated with the external aggregation to aggregate the majority opinions with respect to each criterion (Bordogna & Sterlacchini, 2014). At this stage, a new decision matrix which represents the soft majority of experts is obtained. Then, the internal aggregation to fuse the criteria is performed with flexibility for decision strategy to obtain the final decision.

The proposal of this study is to deal with the Pasi-Yager method with a slight modification is made to be adapted in the alternative scheme. In addition, the IOWA operator used in Pasi and Yager (2006) is then generalized to the IGOWA operators to provide a greater flexibility in considering other type of aggregation operators. Here, the Eq. (F.22) can be directly implemented by focusing on each criterion of experts' judgments instead of on each alternative.

Example F.1. Suppose that a collection of individual opinion of experts, h is given as $X = \left(x_j^1, x_j^2, \dots, x_j^5\right) = (0.40, 0.70, 0.60, 0.65, 0.30)$, where j is a criterion under consideration. Then, the final majority opinion of experts can be calculated as follows:

A_i	E_1	E_2	E_3	E_4	E_5		E_1	E_2	E_3	E_4	E_5	
X_j^h	0.40	0.70	0.60	0.65	0.30		0.40	0.70	0.60	0.65	0.30	s_l
$supp_{1,h}$	-	0.30	0.20	0.25	0.10		-	0	0	0	1	1
$supp_{2,h}$	0.30	-	0.1	0.14	0.40		0	-	1	1	0	2
$supp_{3,h}$	0.20	0.1	-	0.04	0.30	→	0	1	-	1	0	2
$supp_{4,h}$	0.25	0.14	0.04	-	0.35		0	1	1	-	0	2
$supp_{5,h}$	0.10	0.40	0.30	0.35	-		1	0	0	0	-	1

where → means $supp(p_l, p_h)$. By setting $\beta = 0.2$, then the overall support for each expert s_l, $(l = 1,2, \ldots, 5)$ can be obtained. In case of 'ties', stricter β can be imposed $(\beta = 0.1)$ in this example to order x_j^h values, then $s_1 = 1$, $s_2 = 3, s_3 = 3, s_4 = 4$, and $s_5 = 1$ is derived. The vector of order-inducing variable can be given as $U = \left(s_{\tau(1)}, s_{\tau(2)}, \ldots, s_{\tau(5)}\right) = (1,1\ 3,3,4)$ and the weighting vector $W = (w_1, \ldots, w_5) = (0, 0, 0{,}333, 0.333, 0.333)$ can be generated. The final majority opinion of experts can be calculated as follows:

$$IOWA\,(\langle 1{,}0.30\rangle, \langle 1{,}0.40\rangle, \langle 3{,}0.60\rangle, \langle 3{,}0.70\rangle, \langle 4{,}0.65\rangle) = (0 \times 0.30) + (0 \times 0.40) + (0.333 \times 0.60) + (0.333 \times 0.70) + (0.333 \times 0.65) = 0.65.$$

F.5 Induced Generalized OWA-TOPSIS based on Majority Concept

In this section, the algorithm for the proposed model is explained. Two stages of external and internal fusion schemes are presented, where external fusion scheme deals with the aggregation of the majority opinions of experts and internal fusion scheme deals with the implementation of decision strategy, i.e., the proportion of criteria to consider.

F.5.1 External fusion scheme: Inclusion of majority concept for group aggregator

Step 1: Construct the decision matrix for each expert $D^h, h = 1,2, \ldots, k$ as in Eq. (F.13). Here, instead of normalize the data using the vector normalization method, the following normalization procedure is used:

$$x_{ij}^h = \frac{a_{ij}^h - a_{min}}{a_{max} - a_{min}}, \tag{F.25}$$

where a_{max} is the maximum value and a_{min} is minimum value with respect to each criterion. This normalization technique will maintain the measurement scale and thus measuring support between experts for majority opinion can be implemented. Note that, the vector normalization method does not maintain the measurement scale due to non-linear scale transformation.

Step 2: At this stage, the group aggregation for the majority opinion of experts can be conducted with respect to each criterion C_j. First, calculate the support of each expert l, $\forall l = 1,2, \ldots, k$, with all the other experts h on each criterion C_j in evaluating an alternative A_i

using the Eq. (F.22) and Eq. (F.23). The supported values as the order-inducing variables then can be given as:

$$(s_1, \dots, s_k) = \left(\sum_{h=2}^{k} supp_j(x_1, x_h), \dots, \sum_{h=1}^{k-1} supp_j(x_k, x_h)\right), \tag{F.26}$$

Step 3: To compute the non-decreasing weights of the weighting vector, then define the values $t_l, l = 1,2, \dots, k$ based on a modification of the s_l support values, $t_l = s_l + 1$ such in Eq. (F.24).

Step 4: Then, the aggregation of the experts' judgments $\overline{D}^h = [x_{ij}^h]$ with respect to each criterion C_j using the concept of majority opinion can be given as follows:

$$IGOWA(\langle s_1, x_1\rangle, \dots, \langle s_k, x_k\rangle) = \left(\sum_{h=1}^{k} w_h x_{\tau(h)}^{\lambda}\right)^{1/\lambda}, \tag{F.27}$$

where $x_{\tau(h)}$ is the x_h value of the IGOWA pair $\langle s_h, x_h\rangle$ reordered such that $s_{\tau(1)} \leq s_{\tau(2)} \leq \dots \leq s_{\tau(h)}$ and λ is parameter for the generalization of aggregation operator, $\lambda \in (-\infty, \infty) \setminus \{0\}$. At this stage, a new decision matrix which represents the majority of experts on all criteria is derived as follows:

$$D^{Maj} = \begin{matrix} \\ A_1 \\ \vdots \\ A_m \end{matrix} \begin{matrix} \begin{matrix} C_1 & \dots & C_n \end{matrix} \\ \begin{pmatrix} x_{11}^{Maj} & \cdots & x_{1n}^{Maj} \\ \vdots & \ddots & \vdots \\ x_{m1}^{Maj} & \cdots & x_{mn}^{Maj} \end{pmatrix} \end{matrix}, \tag{F.28}$$

where x_{ij}^{Maj} represent the majority opinion of experts with respect to each criterion C_j for the alternative A_i. Note that if $\lambda = 1$, the IGOWA becomes the IOWA operator. If $\lambda = 2$, the IOWQ operator and if $\lambda \to 0$, the IOWG operator. Furthermore, it is possible to extend this approach into a more general framework by using quasi-arithmetic means forming the Quasi-IOWA operator (Merigó & Gil-Lafuente, 2009).

Step 5: Then, to make the proposed method comparable with the classical TOPSIS-GDM method as in Section F.3, convert the x_{ij}^{Maj} data to the $\bar{x}_{ij}^{Maj}$ data using the vector normalization function, $\bar{x}_{ij}^{Maj} = f\left(x_{ij}^{Maj}\right)$ for the decision matrix D^{Maj} as in the Eq. (F.14).

F.5.2 Internal fusion scheme: Inclusion of decision strategies on criteria

Step 6: Further, determine the ideal and anti-ideal solutions X^+ and X^- for the majority of experts using the Eq. (F.15) and Eq. (F.16).

Step 7: Calculate the separation measures from the ideal and anti-ideal solutions using the Minkowski OWA distance such in Eq. (F.5). This step can be divided into the following steps.

Compute the separation measures from ideal solution X^+, where the argument variables as distance measures are reordered in non-decreasing order; the shortest distance to the ideal solution is the best as follows:

$$S_i^+ = MOWAD(d_1, \dots, d_n)^+ = \left\{\sum_{j=1}^{n} \omega_j^+ \left(d_{\tau(j)}^+\right)^\lambda\right\}^{1/\lambda}, \qquad \text{(F.29)}$$

where $d_{\tau(j)}^+$ is the jth smallest of the d_j and d_j is the individual distance between x_j and x_j^+, such that $d_j = \left|x_j - x_j^+\right|$.

Similarly, compute the separation measure for anti-ideal solution, where the argument variables as distance measures are reordered in non-increasing order; as the farthest distance to the anti-ideal solution is the best as follows:

$$S_i^- = MOWAD(d_1, \dots, d_n)^- = \left\{\sum_{j=1}^{n} \omega_j^- \left(d_{\sigma(j)}^-\right)^\lambda\right\}^{1/\lambda}, \qquad \text{(F.30)}$$

where $d_{\sigma(j)}^-$ is the jth largest of the d_j and d_j is the individual distance between x_j and x_j^-, such that $d_j = \left|x_j - x_j^-\right|$.

Observe that if $\lambda = 1$, the Minkowski OWAD operator becomes the Manhattan OWAD operator and if $\lambda = 2$, then the Euclidean OWAD operator can be derived (see Xu & Chen, 2008; Merigó & Gil-Lafuente, 2010).

Step 8: Derive the weighting vectors for ideal and anti-ideal solutions using RIM (quantifier guided aggregation) for criteria C_j, where the $v_j, j = 1,2, \dots, n$ is relative importance associated with criterion C_j such in Eq. (F.12). The weights of ideal solution can be calculated as follows:

$$\omega_j^+ = Q\left(\sum_{k=1}^{j} v_{\tau(k)}\right) - Q\left(\sum_{k=0}^{j-1} v_{\tau(k)}\right), \qquad \text{(F.31)}$$

where $v_{\tau(j)}$ is the importance associated with the criterion that has the *j*th smallest d_j, such as $\left(v_{\tau(j)}, d_{\tau(j)}\right)$. The weight ω_j^+ is the inclusion of relative importance associated with linguistic quantifier for the criterion C_j, and $\sum_{j=1}^{n} \omega_j^+ = 1$.

Equivalently, the weights of anti-ideal solution can be calculated as follow:

$$\omega_j^- = Q\left(\sum_{k=1}^{j} v_{\sigma(k)}\right) - Q\left(\sum_{k=0}^{j-1} v_{\sigma(k)}\right), \qquad \text{(F.32)}$$

where $v_{\sigma(j)}$ is the importance associated with the criterion that has the *j*th greatest d_j, such as $\left(v_{\sigma(j)}, d_{\sigma(j)}\right)$. The weight ω_j^- is the inclusion of relative importance associated with anti-ideal solutions, $\sum_{j=1}^{n} \omega_j^- = 1$.

Step 9: Calculate the relative closeness RC_i^{Maj} to the ideal solution for the group. The relative closeness can be calculated as follows:

$$RC_i^{Maj} = \frac{S(a^i)^-}{S(a^i)^- + S(a^i)^+}, \qquad i = 1,2, \dots m, \qquad \text{(F.33)}$$

where the alternatives are ranked in descending order. Note that the larger the value of RC_i^{Maj} denotes the better performance of the alternative.

F.6 Illustrative Example

In this section, the case study of human resource selection problem for a local chemical company as described by Shih et al. (2007) is implemented. There are 17 candidates (alternatives) and four decision makers considered for the evaluation, and each of candidates is evaluated through a number of objective and subjective tests. The basic data for this experiment is demonstrated in Table F.1 and Table F.2. However, in order to consort with the context of this study, a slight modification is made to the original data regarding the weights associated to the decision makers. In this case, a homogenous type of problem is considered by associating the equal weights of criteria for each expert as follows: language test (C_1), professional test (C_2), safety rule test (C_3), professional skills (C_4), computer skills (C_5), panel interview (C_6) and 1-on-1 interview (C_7) as 0.066, 0.196, 0.066, 0.130, 0.130, 0.216 and 0.196, respectively.

Table F.1. Decision matrix of human resource selection problem – Objective attribute

Alternative	Knowledge tests			Skill tests	
	Language	Professional	Safety	Professional	Computer
1	80	70	87	77	76
2	85	65	76	80	75
3	78	90	72	80	85
4	75	84	69	85	65
5	84	67	60	75	85
6	85	78	82	81	79
7	77	83	74	70	71
8	78	82	72	80	78
9	85	90	80	88	90
10	89	75	79	67	77
11	65	55	68	62	70
12	70	64	65	65	60
13	95	80	70	75	70
14	70	80	79	80	85
15	60	78	87	70	66
16	92	85	88	90	85
17	86	87	80	70	72

Table F.2. Decision matrix of human resource selection problem – Subjective attribute

Alternative	DM 1		DM 2		DM 3		DM 4	
	Panel	One	Panel	One	Panel	One	Panel	One
1	80	75	85	80	75	70	90	85
2	65	75	60	70	70	77	60	70
3	90	85	80	85	80	90	90	95
4	65	70	55	60	68	72	62	72
5	75	80	75	80	50	55	70	75
6	80	80	75	85	77	82	75	75
7	65	70	70	60	65	72	67	75
8	70	60	75	65	75	67	82	85
9	80	85	95	85	90	85	90	92
10	70	75	75	80	68	78	65	70
11	50	60	62	65	60	65	65	70
12	60	65	65	75	50	60	45	50
13	75	75	80	80	65	75	70	75
14	80	70	75	72	80	70	75	75
15	70	65	75	70	65	70	60	65
16	90	95	92	90	85	80	88	90
17	80	85	70	75	75	80	70	75

Note: (Panel = Panel interview, One = One-on-one interview)

Here, the comparison is made between the proposed model and the classical TOPSIS-GDM model (Shih et al., 2007) and the TOPSIS-GDM with majority opinion based on classical scheme (Hajimirsadeghi & Lucas, 2009). In particular, the comparison with respect to the distance measures (i.e., the Manhattan and the Euclidean) is conducted and the results of all the models are shown in Table F.3. In this case, the semantic *half* ($\alpha = 1$) or averagely all of the criteria is used as the decision strategy.

As can be seen, the rankings generated from all the models show slightly different results, especially on the first two candidates: a_9 and a_{16}. The classical TOPSIS method for both distance measures, rank a_{16} as the best and then a_9 as the second best. On the contrary, the Hajimirsadeghi-Lucas model ranks a_9 as the top ranking and then followed by a_{16} for both distance measures. While the proposed method provides different results for each of the distance measures, such as, the best and the second best is given as a_9 and a_{16} for the Manhattan distance, and for vice versa for the Euclidean distance.

Table F.3. Final distance performance and rankings of the aggregation

Classical TOPSIS-GDM with Arithmetic Mean				TOPSIS-GDM with Majority Opinion (Classical Scheme)				The proposed model (alternative scheme)			
Manhattan Distance		Euclidean Distance		Manhattan OWAD		Euclidean OWAD		Manhattan OWAD		Euclidean OWAD	
RC	R	RC	R	RC	R	RC	R	RC	R	RC	R
0.6169	7	0.6122	5	0.6065	7	0.5916	7	0.5569	7	0.5506	7
0.4393	14	0.4338	14	0.4703	13	0.4587	14	0.3414	15	0.3649	15
0.8212	3	0.7767	3	0.8830	3	0.8278	3	0.8263	3	0.7794	3
0.4603	12	0.4645	13	0.5036	11	0.4722	13	0.4238	13	0.4466	13
0.4559	13	0.4651	12	0.4695	14	0.4746	12	0.4860	10	0.4837	11
0.6677	4	0.6596	4	0.6884	4	0.6812	4	0.6528	4	0.6440	4
0.4655	11	0.4732	11	0.4958	12	0.4764	11	0.4295	12	0.4493	12
0.5779	8	0.5755	8	0.5715	8	0.5857	8	0.5037	9	0.5147	9
0.9103	**2**	**0.8729**	**2**	**0.9531**	**1**	**0.9150**	**1**	**0.9329**	**1**	**0.8902**	**2**
0.5187	10	0.5167	10	0.5259	10	0.5276	10	0.4844	11	0.4881	10
0.1715	16	0.2145	16	0.1436	16	0.2078	16	0.1048	17	0.1425	17
0.1561	17	0.1838	17	0.1396	17	0.1549	17	0.1104	16	0.1524	16
0.5654	9	0.5626	9	0.5687	9	0.5581	9	0.5181	8	0.5229	8
0.6195	5	0.6063	7	0.6521	5	0.6429	5	0.5786	5	0.5736	5
0.4079	15	0.4223	15	0.3784	15	0.3964	15	0.3510	14	0.3876	14
0.9104	**1**	**0.8899**	**1**	**0.9277**	**2**	**0.9128**	**2**	**0.9183**	**2**	**0.8957**	**1**
0.6190	6	0.6081	6	0.6498	6	0.6135	6	0.5606	6	0.5597	6

Note: (RC = Relative Closeness, R = Ranking)

Table F.4. Confidence measures for different methods

	Classical TOPSIS-GDM with Arithmetic Mean		TOPSIS-GDM with Majority Opinion (Classical Scheme)		The Proposed method (Alternative Scheme)	
Measure	Manhattan distance	Euclidean distance	Manhattan OWAD	Euclidean OWAD	Manhattan OWAD	Euclidean OWAD
1	0.7543	0.7061	0.8135	0.7601	0.8281	0.7532
2	0.0001	0.0006	0.0008	0.0018	0.0016	**0.0027**
3	[0.1561, 0.9104]	[0.1838, 0.8899]	[0.1396, 0.9531]	[0.1549, 0.9150]	[0.1048, 0.9329]	[0.1425, 0.8957]

In Table F.4, the confidence measures for all the models are provided. These measures include: 1) the sum of absolute difference between relative closeness of the consecutive alternatives, 2) the minimum of the absolute difference between relative closeness of the consecutive alternatives, and 3) the range (in unit interval) of calculated relative closeness for the alternatives. For all the measures: the higher the value (or the bigger the interval), the better the result. In general, as can be noticed, the TOPSIS-GDM model under the majority concept (either based on classical scheme or alternative scheme) exhibits the better results compared to the classical TOPSIS-GDM model. Specifically, with respect to the results of the second measure, the proposed method under the Euclidean distance indicates the highest difference (0.0027) between the relative closeness of the consecutive alternatives compared to the rest of the models. Hence, the results imply that the conclusive decision can be made by the decision makers when the distinction values between each alternative are greater.

In addition, a series of rankings of alternatives can be determined with respect to the different decision strategies. For example, the aggregation on specific criteria can be easily adjusted which represent the attitudinal character of the majority of experts. Table F.5 shows the results of different decision strategies for the proposed TOPSIS-GDM model based on the Euclidean OWA distance measure.

As can be seen, with respect to semantics *half*, *many*, *most* and *all* as the decision strategies, the alternative a_{16} is ranked as the best alternative, followed by the alternative a_9 in the second position. On the contrary, for the decision strategy *some* to *at least one*, the alternative a_9 is ranked as the best and a_{16} comes the second. For the rest alternatives there are slightly changes in ranking for the different strategies under consideration. This analysis can provide a complete picture for the stakeholders or decision makers in analysing the possible alternative for the best decision. In particular, the attitudinal character of the majority or group of decision makers are considered in the evaluation process, such as the semantics *at least one* (max) reflects the degree of optimism, whilst the semantics *all* (min) represents the degree of pessimism.

Table F.5. Rankings of the proposed method with different strategies

TOPSIS-GDM with Majority Opinion (Euclidean OWAD)													
At least one (Max) $\alpha = 0.001$		Few $\alpha = 0.1$		Some $\alpha = 0.5$		Half (Average) $\alpha = 1$		Many $\alpha = 2$		Most $\alpha = 10$		All (Min) $\alpha = 1000$	
RC	R	RC	R	RC	R	RC	R	RC	R	RC	R	RC	R
0.9168	5	0.7890	6	0.6248	8	0.5506	7	0.4949	5	0.4316	5	0.4242	5
0.7062	15	0.6449	15	0.4780	15	0.3649	15	0.2538	15	0.0730	15	0.0013	15
0.9765	3	0.9278	3	0.8420	3	0.7794	3	0.7008	3	0.5131	4	0.4264	4
0.8367	8	0.7512	9	0.5725	12	0.4466	13	0.2978	14	0.1109	13	0.1027	11
0.8211	10	0.7439	11	0.5830	11	0.4837	11	0.3890	11	0.2503	9	0.0136	13
0.7947	12	0.7666	8	0.6919	4	0.6440	4	0.5971	4	0.5316	3	0.5246	3
0.7906	13	0.7249	14	0.5638	13	0.4493	12	0.3247	12	0.2156	11	0.2094	9
0.7663	14	0.7257	13	0.6061	9	0.5147	9	0.4102	9	0.1629	12	0.0535	12
0.9890	1	0.9656	1	0.9228	1	0.8902	2	0.8459	2	0.7338	2	0.6781	2
0.8180	11	0.7472	10	0.5867	10	0.4881	10	0.3932	10	0.2238	10	0.1761	10
0.3298	16	0.3008	16	0.2099	16	0.1425	17	0.0757	17	0.0091	16	0.0002	16
0.2900	17	0.2710	17	0.2066	17	0.1524	16	0.0876	16	0.0067	17	0.0001	17
0.9363	4	0.8175	5	0.6348	7	0.5229	8	0.4225	8	0.3382	6	0.3297	6
0.8254	9	0.7717	7	0.6516	6	0.5736	5	0.4845	6	0.3069	8	0.2671	8
0.8949	7	0.7369	12	0.5450	14	0.3876	14	0.3146	13	0.0985	14	0.0040	14
0.9874	2	0.9614	2	0.9202	2	0.8957	1	0.8709	1	0.8378	1	0.8333	1
0.8997	6	0.8213	4	0.6662	5	0.5597	6	0.4445	7	0.3181	7	0.2857	7

Note: (RC = Relative Closeness, R = Ranking)

6.3 Conflicting Bifuzzy Multi-attribute Group Decision Making Model with Application to Flood Control Project

Abstract. In this paper, we propose a multi-attribute group decision making model based on conflicting bifuzzy sets (CBFS). Specifically, the evaluations are bi-valued in accordance to the subjective judgment of experts with respect to the positive and negative views. This study discusses the weighting methods for particular attribute and sub-attribute with emphasis is given to the unification of subjective and objective weights. The integration of CBFS in the model is naturally done by extending the fuzzy evaluation in parallel with the intuitionistic fuzzy. We introduce a new technique to compute the similarity measure, being the degree of agreement between experts. We end up the study by demonstrating the applicability of the proposed model to the empirical case of flood control project, one of the project selection problems.

G.1 Introduction

Multi-attribute group decision making (MAGDM) is a well-known model used for choosing the best candidate from a set of possible options under the evaluation of a group of experts. It is admissible that a complex decision problem requires an integration of various expertise in which the lack of knowledge or experience of an expert can be offset by the others. Due to its ability in solving the decision problem with the presence of conflict and agreement among experts, the MAGDM has been successfully applied in various applications (see, for example, in Figueira et al., 2005; Gal et al., 1999).

Most of the time, the evaluation of attributes or criteria is vague, ambiguous or imprecise. As a result, the numerical measurement (or crisp data) may not provide the best assessment to the problem. Thus, the rating of alternatives with respect to this kind of attributes can be better represented using the linguistic approach. Many authors have applied the theory of fuzzy set (FS) as introduced by Zadeh (1965) to interpret or present the attributes in linguistic variable (or by means of fuzzy number). For instance, a study by Chen (2000) has demonstrated the ability of FS theory to solve the fuzziness in the technique for order preference by similarity to ideal solution (TOPSIS). In Tiryaki and Ahlatcioglu (2005), a ranking system has been developed with the use of fuzzy analytic hierarchy process (AHP) in the stock selection problem. In other contribution, Langroudi et al. (2013) has employed an extended version of FS theory, referred as type-2 fuzzy set, in the TOPSIS method. The general review on the applications of FS theory in multi-criteria decision making can be referred in Kahraman (2008).

Recently, the intuitionistic fuzzy sets (IFS) founded by Atanassov (1986) has played its role in multi-attribute decision making process (see, for example, Liu

& Wang 2007; Xu & Yager 2008). It is one of the extensions of FS theory that has been proposed in the literature. For example, Ye (2013) has described the weights in the MAGDM model under the intuitionistic fuzzy setting. The intuitionistic fuzzy provides both the membership and non-membership functions, implies that there are two-sided evaluation. In comparison, the FS only emphasizes on the membership function or single-sided judgment. We argue that, since the sum of membership and non-membership values must be less or equal to one as in the case of IFS, the data are rather restricted. For example, we may rate the candidate in an interview as '*good*' with membership 0.75 and '*bad*' with membership 0.25, which is complementary as in the state of FS. The sum of good (membership) and bad (non-membership) may less than one (as in IFS) and may possibly exceed one in some circumstances. In this study, we focus on the problem of the sum of two contradict evaluations exceed one by resorting to the so-called conflicting bifuzzy sets (CBFS). Our aim in this study is to propose a conflicting bifuzzy MAGDM model.

The conflicting bifuzzy sets was first proposed by Tap (2006) to deal with the conflicting conditions where the constraint of IFS has been slightly released. He then demonstrated the potential application in the decision making problems. Taib et al. (2008) and Zamali et al. (2010) have applied the conflicting bifuzzy concept in the analytic hierarchy process model to deal with the selection process in waste management. Xu and Yan (2011) then has employed the bifuzzy evaluation in the multi-objective decision making model for a vendor selection problem. In the similar way, in this study we construct the decision matrix for certain attributes and sub-attributes using the conflicting bifuzzy evaluation. There is no restriction imposed to the evaluation process to ensure that all data are significant and considered in the decision analysis. This led us to a fair and better decision.

However, the lack of knowledge and experience of experts may affect the decision making process. Brought them together may offset or compensate each other for the better decision (heterogeneous case). Hence, the weighting process is extremely importance to specify the expertise of experts on each specific criterion. For doing so, we weight the experts according to their depth of knowledge and experiences. In weighting the attributes and sub-attributes, we follow the works done by Liu and Kong (2005) and Wang and Lee (2009) using the integrated fuzzy subjective and objective weights. The fuzzy AHP approach (see Saaty, 1980) is used to measure the subjective weight, while the objective weight is obtained from the entropy method (see Shannon, 1949; Zeleny, 1982). The main advantage of fuzzy AHP is that it can deal with the quantitative and qualitative judgment provided directly by experts. On the contrary, the entropy method does not require a direct involvement of the experts, but the weights can be derived directly from the rating table or decision matrices. The entropy method has been used by many researchers for the stipulation of weights. Examples of this literature can be referred in Pomerol & Barba-Romero (2000), Li et al. (2014), and Chen et al. (2014). In this study, we propose the entropy method to deal with the conflicting bifuzzy conditions as the objective weights. Then, we integrate the subjective and objective weights using the Hurwicz's criterion which reflects the subjective judgment of the decision maker and objective information obtained from the solution of a mathematical model. In addition, we propose a similarity measure to quantify the consensus between experts in deriving the overall group decision. For the ranking procedure, the TOPSIS method is put forward for the final evaluation.

In exemplify the proposed model, we take the selection of flood control projects as our empirical case. According to Maragoudaki and Tsakiris (2005), most of the previous evaluation and selection of flood control projects are focused mainly on the technical and economic factors. For instance, in the technical aspect, the selection of flood control projects are merely based on the relationship between the flood magnitudes (e.g., flood depth, flood velocity, flood flow rate, etc.), together with the anticipated flood damage. On the other hand, the cost-benefit analysis (CBA) approach is usually employed in the economic domain (see Morris-Oswald, 2001). The CBA focuses on the implementation and maintenance costs of the selected alternatives, besides the direct and indirect benefits of total change in income from the project. However, recent approaches in the selection of flood control project recognize the fact that these types of projects interact with various sectors including social, politic, economic and environmental aspects. Hence, the application of multi-attribute decision making technique in the flood management is clearly significant. Moreover, since the effect and side-effect of every decision is greatly important, then the consideration of conflicting bifuzzy concept is prominent in this selection process.

For this purpose, we have collected a set of data of experts' evaluations for the possible alternatives of flood control projects. Specifically, the case study has been conducted in the state of Kelantan, Malaysia in which the flood is regularly hit the state almost once in a year. Malaysia is the country with two seasons in general. The dry season is usually ranging from March to October and rainy season from November to February. With the poor flood management system, the rainy season will be worst to some residents especially farmers. This would be a highly-justified issue since they will lose their income as a result of crop damages from flooding, as well as the infrastructure. We have consulted three different groups of expert namely specialize engineers, local authority and a non-governmental organization (environmentalist). These group of experts will be explained in detail in the consequence section. The rest of the study is organized as follows. Section G.2 discusses the theoretical part of model consisting the definition of FS and IFS, together with the introduction of CBFS. Later on, we design our MAGDM model with CBFS concept in Section G.3. Finally, we demonstrate the application of our proposed model in Section G.4.

G.2 Preliminaries

In this section, we state the theoretical parts of fuzzy set and intuitionistic fuzzy sets towards the introduction of conflicting bifuzzy evaluation.

Definition G.1 (Zadeh, 1965). Let X be a finite and non-empty set. A fuzzy set A on X is characterized by:

$$A = \{\langle x, \mu_A(x)\rangle | x \in X\}, \tag{G.1}$$

where $\mu_A: X \to [0,1]$ is the membership function of the fuzzy set A.

In a fuzzy set A, an element x which belongs to a finite set X is given the membership value represents how much x belongs to A. It is clear that fuzzy in all circumstances describing element with a single value.

It is of special interest to have a look at fuzzy number. We will explain a triangular fuzzy number (this should be countered most in this study) rather than other fuzzy numbers which can be referred in Kaufmann and Gupta (1991). A triangular fuzzy number can be expressed as $A = (a_1, a_2, a_3)$. For each fuzzy number, the membership value is computed using the formula:

$$\mu_A(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x < a_2, \\ 1, & x = a_2, \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 < x \leq a_3, \\ 0, & otherwise. \end{cases}$$

For $a_1 = a_2 = a_3$, a triangular fuzzy number gives a crisp value and it is known as a special case of fuzzy number.

Definition G.2. Given two fuzzy numbers $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$, and let λ be any real number. Some operations on fuzzy numbers can be express as:

$$\begin{gathered} A \oplus B = (a_1 + a_2, b_1 + b_2, c_1 + c_2), \\ A \otimes B = (a_1 \times b_1, a_2 \times b_2, a_3 \times b_3), \\ A \oslash B = (a_1/b_3, a_2/b_2, a_3/b_1), \\ \lambda \otimes A = (\lambda \times a_1, \lambda \times a_2, \lambda \times a_3), \\ A^\lambda = \left(a_1^\lambda, a_2^\lambda, a_3^\lambda\right). \end{gathered} \tag{G.2}$$

We now proceed to the definition of intuitionistic fuzzy set.

Definition G.3 (Atanassov, 1986). Let X be a finite and non-empty set. An intuitionistic fuzzy set A in X is expressed as:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}, \quad \text{(G.3)}$$

where $\mu_A: X \to [0,1]$ and $\nu_A: X \to [0,1]$ are the representative membership and non-membership functions of the fuzzy set A with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all x in X.

As proposed by Atanassov (1986), there exist an intuitionistic index of x in A which can be formulated as $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ where $0 \leq \pi_A(x) \leq 1$. It turns out that every fuzzy set A can be represented as the following intuitionistic fuzzy set:

$$A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in X\},$$

that proves the absence of hesitancy degree in fuzzy set since:

$$\pi_A(x) = 1 - \mu_A(x) - \left(1 - \mu_A(x)\right) = 0.$$

The $\nu_A(x)$ does not always be $1 - \mu_A(x)$. Therefore, the sum of membership and non-membership degrees can be less than one in IFS. However, if we let those degrees varies within the range $[0,1]$ for each, then the sum can take any values within the range $[0,2]$.

Next, we state the definition of conflicting bifuzzy sets retrieved from Tap (2006) (see Zamali et al., 2008).

Definition G.4. Let X be a finite and non-empty set. If $\{A^+, A^-\}$ be two fuzzy sets with conflicting characteristic contained in A, then A is called a conflicting bifuzzy set which can precisely defined as:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}, \quad \text{(G.4)}$$

where $\mu_A: X \to [0,1]$ and $\nu_A: X \to [0,1]$ represent the degree of positivity and negativity of fuzzy set A respectively for all x in X.

The IFS condition is reformulated to be:

$$0 < \mu_A(x) + \nu_A(x) \leq 1 + \varepsilon,$$

where ε is a small nonnegative value, $\varepsilon \in [0, \frac{1}{2})$. If we simultaneously consider the positivity and negativity, only one aspect will be dominant at one time, either positive or negative. There will be no possible situation where both appear to be dominant. Thus, this implies that the sum of the degrees cannot exceed 3/2.

In order to integrate the positivity and negativity of attribute for a possible alternative, one should have a combination operator to deal with. Such combination operators are the geometric mean, arithmetic mean and multiplicative operator (see Zamali et al., 2008; Gau & Buehrer, 1993; Kaufmann & Gupta, 1991). For simplicity, we consider the arithmetic mean combination operator defined as follows:

$$\phi(\mu_A(x), \eta_A(x)) = \frac{\mu_A(x) + \eta_A(x)}{2}, \tag{G.5}$$

where $\eta_A(x) = 1 - \nu_A(x)$ is the nonnegativity degree. Here, the degree of nonnegativity would rather be a special interest in solving the decision making problem. We show some examples of calculations for different combination operators in Table G.1.

Table G.1. Examples of result for different combination operator

Operator	Formula	$(\mu_A(x), \nu_A(x))$		
		(0.7,0.1)	(0.7,0.3)	(0.7,0.5)
Geometric mean	$\phi(\mu_A(x), \eta_A(x)) = \sqrt{\mu_A(x) \times \eta_A(x)}$	0.79	0.70	0.59
Arithmetic mean	$\phi(\mu_A(x), \eta_A(x)) = \frac{\mu_A(x) + \eta_A(x)}{2}$	0.80	0.70	0.60
Multiplicative	$\phi(\mu_A(x), \eta_A(x)) = \mu_A(x) \times \eta_A(x)$	0.63	0.49	0.35

G.3 Conflicting Bifuzzy Multi-attribute Group Decision Making Model

This section presents a group decision making model that demonstrates the applicability of CBFS. We deal with positive and negative aspects concurrently resulting to a fair decision. This model involves three stages which generally classified as rating, aggregation and selection. Figure G.1 shows a general framework of the proposed model.

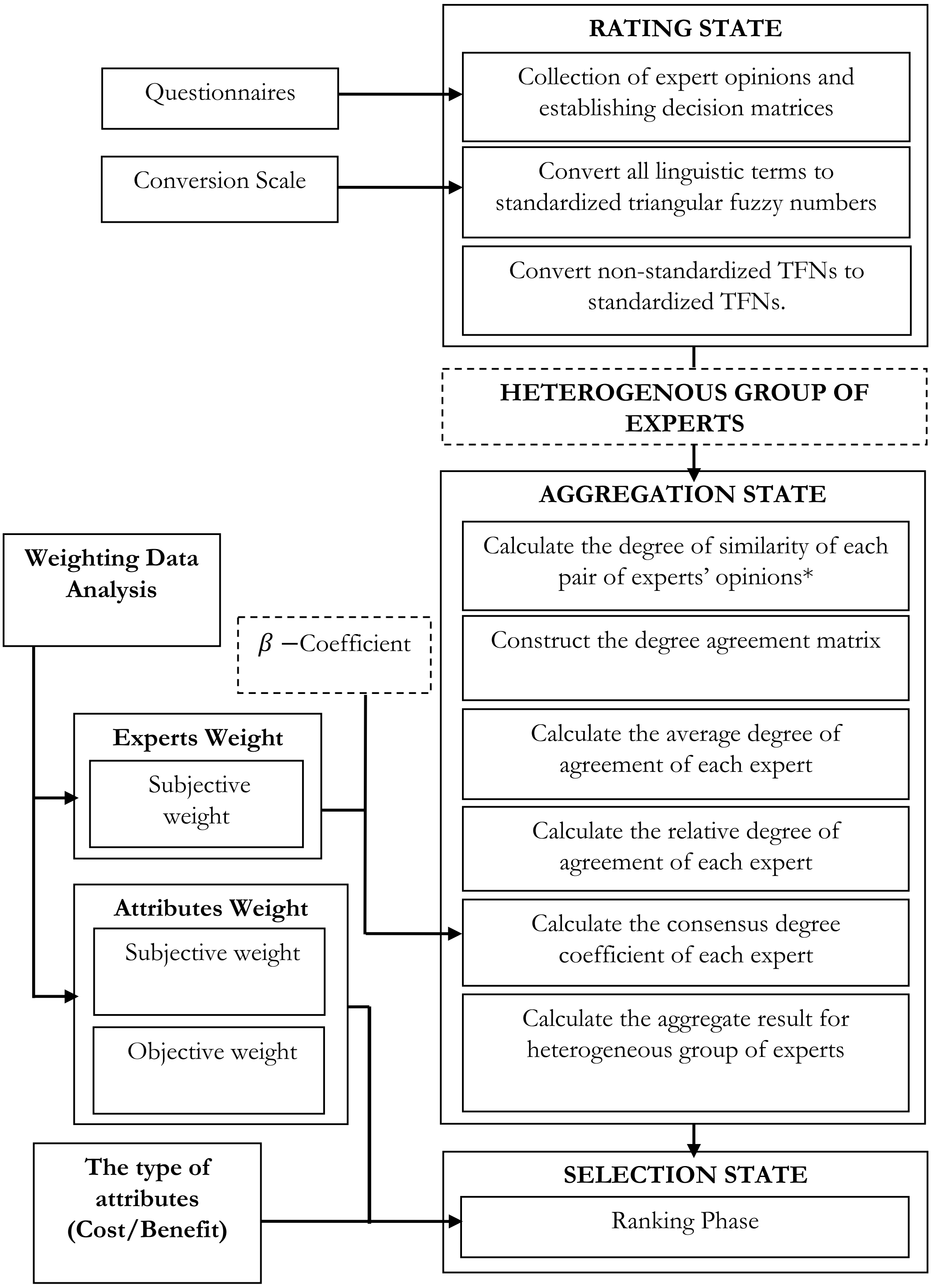

Figure G.1. General framework of the proposed model

At the rating stage, the evaluations of alternatives are provided by experts with respect to the objective and subjective attributes. The objective attributes are quantifiable (e.g., monetary terms, exact measurements, etc.) but the subjective attributes are not. Here, the subjective attribute is expressed in linguistic variable and is directly computed by converting it into the fuzzy number. For our decision making model, we use the linguistic variables described in Table G.2 as adopted from Chen and Hwang (1992) and Wang and Lee (2009).

Table G.2. Linguistic variables for rating alternative

Level of importance	Abbreviation	Fuzzy number
Very Poor/Very Low	VP/VL	(0, 0, 0.2)
Poor/Low	P/L	(0.05,0.2,0.35)
Medium Poor/Medium Low	MP/ML	(0.2,0.35,0.5)
Medium	M	(0.35,0.5,0.65)
Medium Good/Medium High	MG/MH	(0.5,0.65,0.8)
Good/High	G/H	(0.65,0.8,0.95)
Very Good/Very High	VG/VH	(0.8, 1, 1)

On the other hand, the aggregation phase involves the setting of weights to the experts based their expertise and specifying the degrees of importance for the criteria. Then, these weights together with the ratings of alternative under the multiple criteria are aggregated. The aggregated results obtained at this stage will be used for the ranking process in the selection phase. The procedure of fuzzy TOPSIS will be fully utilized in the final stage. The specific procedures at each stage are presented as the following.

G.3.1 Determination of weights

G.3.1.1 Weighting an expert

We use the simple weighted evaluation technique (WET) see (Olcer & Odabasi, 2005; Chiclana et al., 2004) to estimate the weight of each decision maker or expert. Let $w(e_k)$ be a priority degree of expert $e_k (k = 1,2, \ldots, n)$ where $w(e_k) \in [0,1]$ and $\sum_{k=1}^{n} w(e_k) = 1$. We first take an expert with the highest priority as proxy and assign value one to him, $r(e_k) = 1$. The relative priority for the expert-l, $r(e_l)(l = 1,2,\dots,n-1)$ is directly obtained by comparing him to the proxy regarding to his priority in the group of experts. Hence, we have $max\{r(e_1), r(e_2), \dots, r(e_2)\} = 1$ and $min\{r(e_1), r(e_2), \dots, r(e_2)\} > 0$. The weight of the decision maker $w(e_k)$ is defined as:

$$w(e_k) = \frac{r(e_k)}{\sum_{k=1}^{n} r(e_k)}. \tag{G.6}$$

If we let the priority of n experts are equal, then $w(e_k) = 1/n$ for $k = 1,2,\dots,n$. In the following, an example is given to clearly demonstrate the weighting method.

Example G.1 Consider three experts e_1, e_2 and e_3 are involved. Assume the expert e_1 has absolute knowledge in evaluating an attribute (let say A_1), thus he is assigned as proxy, given the priority $r(e_1) = 1$. Based on how depth is the expertise of the other two experts, the priority is given, for instance, $r(e_2) = 0.5$ and $r(e_3) = 0.25$. Using Eq. (G.6), we then obtain the experts' weights:

$$w(e_1) = 0.571, w(e_2) = 0.286, w(e_3) = 0.143, \text{ and } \sum_{k=1}^{3} w(e_k) = 1.$$

G.3.1.2 The weight of attribute and sub-attribute

In this study, we integrate the subjective and objective weights for the final degree of importance of attribute. The subjective weight is respected to the subjective judgment of the expert where the weight of attribute is directly given, such as the fuzzy AHP method. The objective weight is based on the objective information obtained by solving a mathematical model automatically.

In a normal procedure, the determination of weight relies heavily on the expert's knowledge and experience which typically characterized as the subjective evaluation. But, this procedure does not consider the relationship between the evaluated objects. Hence, applying fuzzy AHP as a subjective weight is inadequate to capture the priority in the assessment of alternative (see Wang et al., 2008). The subjective approach will be more consistent with the integration of objective approach and the integration method is more desirable in the computation of weight. Therefore, the integrated weight based on the fuzzy AHP and entropy method is implemented here. We use the linguistic terms described in Table G.3 to form a pairwise comparison matrix for fuzzy AHP where the evaluation is based on its corresponding mean of fuzzy number.

Table G.3. Linguistic variable for the weight of attribute and its corresponding fuzzy number

Linguistic terms	The mean of fuzzy number	Triangular Fuzzy Number
Equally important	$\tilde{1}$	(1,1,1)
Intermediate values between $\tilde{1}$ and $\tilde{3}$	$\tilde{2}$	(1,2,3)
Moderately important	$\tilde{3}$	(2,3,4)
Intermediate values between $\tilde{3}$ and $\tilde{5}$	$\tilde{4}$	(3,4,5)
Essentially important	$\tilde{5}$	(4,5,6)
Intermediate values between $\tilde{5}$ and $\tilde{7}$	$\tilde{6}$	(5,6,7)
Very vital important	$\tilde{7}$	(6,7,8)
Intermediate values between $\tilde{7}$ and $\tilde{9}$	$\tilde{8}$	(7,8,9)
Extremely vital important	$\tilde{9}$	(9,9,9)

The procedure starts with the determination of weight for attribute-i, w_i^{att}. Assume that a set of m attributes $A_i, i = 1,2,\ldots,m$ is given. A fuzzy reciprocal judgment matrix for attributes is defined as:

$$D = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1m} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \cdots & \tilde{a}_{mm} \end{bmatrix},$$

where $\tilde{a}_{ij} = \tilde{1} = (1,1,1)$ for all $i = j$ $(i,j = 1,2,\ldots m)$ and $\tilde{a}_{ij} = 1/\tilde{a}_{ji}$ for $i \neq j$ (reciprocal of $\tilde{a}_{ij}$). By applying the fuzzy synthetic extent, we obtained the corresponding weight for each attribute as:

$$w_i = \left[\sum_{i=1}^{m} \tilde{a}_{ij} \otimes \left[\sum_{i=1}^{m} \sum_{j=1}^{m} \tilde{a}_{ij}\right]^{-1}\right],\ i = 1,2,,\ldots,m. \quad \text{(G.7)}$$

The weights w_i^{att} are in normalized fuzzy numbers. Note that Eq. (G.7) may result from fuzzy arithmetic or it can be derived from the extension principle.

The attribute A_i normally has k sub-attributes. Thus, it is important to determine the relative importance of sub-attribute, w_{ij}^{sub} to that particular attribute. We define the fuzzy judgment matrix for k sub-attributes with respect to attribute A_i as:

$$D_i = \begin{bmatrix} \tilde{a}_{1_i1_i} & \tilde{a}_{1_i2_i} & \cdots & \tilde{a}_{1_ik_i} \\ \tilde{a}_{2_i1_i} & \tilde{a}_{2_i2_i} & \cdots & \tilde{a}_{2_ik_i} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{k_i1_i} & \tilde{a}_{k_i2_i} & \cdots & \tilde{a}_{k_ik_i} \end{bmatrix},$$

where $\tilde{a}_{u_iv_i}$ for $u, v = 1,2, \dots, k_i$ is evaluated using Table G.3.

By multiplying sub-attribute's weight to the respective attribute weight in Eq. (G.7), we derive the final weight for sub-attribute through the aggregation of weights at two consecutive levels as follows:

$$w_{ij}^{agg} = w_i^{att} \otimes w_{ij}^{sub}, \text{ for } i, j = 1,2, \dots, m, \tag{G.8}$$

where w_{ij}^{agg} is the aggregated fuzzy weight of sub-attribute and,

$$w_{ij}^{sub} = \left[\sum_{i=1}^{k_i} \tilde{a}_{ij} \otimes \left[\sum_{i=1}^{k_i} \sum_{j=1}^{k_i} \tilde{a}_{ij}\right]^{-1}\right].$$

Hence, the entries of the subjective weight vector, notated as w_{ij}^{subj} with length k, is given as:

$$w_{ij}^{subj} = \left(w_{11}^{agg}, w_{12}^{agg}, \dots, w_{1k_1}^{agg}; w_{21}^{agg}, \dots, w_{2k_2}^{agg}; w_{m1}^{agg}, \dots, w_{mk_m}^{agg}\right).$$

As part of the procedure in AHP method, the determination of consistency index (*CI*) seems compulsory as it prescribes the acceptance level of the pairwise comparison matrix. To obtain *CI*, we first multiply the matrix with its priority vector (with respect to the mean of fuzzy number):

$$\begin{vmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \cdots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \cdots & \frac{w_2}{w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \cdots & \frac{w_n}{w_n} \end{vmatrix} \times \begin{vmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{vmatrix} = \begin{vmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{vmatrix}.$$

Then, we divide $r_i (i = 1,2, \dots, m)$ with its corresponding priority vector:

$$\begin{vmatrix} r_1 / w_1 \\ r_2 / w_2 \\ \vdots \\ r_n / w_n \end{vmatrix} = \begin{vmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{vmatrix}.$$

The consistency index can now be computed using:

$$CI = \frac{(\lambda_{max} - m)}{m - 1}, \tag{G.9}$$

where $\lambda_{max} = \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_m}{m}$. Finally, we calculate the consistency ratio (CR) using:

$$CR = \frac{CI}{RI}, \tag{G.10}$$

where *RI* represents the random index (i.e., the consistency index of a randomly generated pairwise comparison matrix). The *RI* depends on the number of elements/criteria, m being compared as presented in Table G.4. The detailed of the consistency ratio can be referred to Saaty (1980).

Table G.4. Consistency index of a randomly generated reciprocal matrix

m	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
RI	0.00	0.00	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49

We now turn to the procedure of obtaining the objective weight, w_{ij}^{obj}. Suppose we have a decision matrix for n-alternatives and m-attributes, $\overline{D} = (x_{ij})_{m\times n}$, $i = 1,2,\dots,m; j = 1,2,\dots,n$. By normalizing this decision matrix we obtain a matrix $\overline{\overline{D}} = (z_{ij})$, where $z_{ij} \in [0,1]$. Among these attributes, to which the bigger the better is:

$$z_{ij} = \frac{x_{ij} - \min_j\{x_{ij}\}}{\max_j\{x_{ij}\} - \min_j\{x_{ij}\}}, \tag{G.11}$$

while, the smaller the better is:

$$z_{ij} = \frac{\max_j\{x_{ij}\} - x_{ij}}{\max_j\{x_{ij}\} - \min_j\{x_{ij}\}}. \tag{G.12}$$

Next, we calculate the values z_{ij} using the entropy formula:

$$E = -k\sum_{i=1}^{m} f_{ij}\ \ln f_{ij}, i = 1,2,\dots,m\,, \qquad \text{(G.13)}$$

where $f_{ij} = z_{ij} / \sum_{j=1}^{n} z_{ij}, k = 1/\ln n,$ by convention $f_{ij} = 0, f_{ij}\ \ln f_{ij} = 0.$ The objective weight then is defined as:

$$w_{ij}^{obj} = \frac{1-E}{m - \sum_{i=1}^{m} E}. \qquad \text{(G.14)}$$

The sum of w_{ij}^{obj} is equal to one and $w_{ij}^{obj} \in [0,1]$.

Finally, the integration of subjective weight with the objective weight to obtain the fuzzy integrated weight Ω_{ij} is done by using the following formula (see Liu & Kong, 2005; Wang et al., 2008):

$$\Omega_{ij} = \frac{\left(w_{ij}^{subj}\right)^{\lambda} \otimes \left(w_{ij}^{obj}\right)^{1-\lambda}}{\sum_{i=1}^{m}\left(\left(w_{ij}^{subj}\right)^{\lambda} \otimes \left(w_{ij}^{obj}\right)^{1-\lambda}\right)}, \qquad \text{(G.15)}$$

where λ represents the relative importance of the subjective and objective weights to expert(s). Note that, the value of subjective weight is in form of fuzzy number and the objective weight is in crisp value. Therefore, the fuzzy integrated weight is a multiplication of fuzzy number and a scalar. The weight is an indicator that does not only show how important an attribute is, but also indicate the level of difference of attribute for various alternatives (see Liu & Kong, 2005).

G.3.2 Rating phase

Assume that we have n-alternatives and m-attributes. The CBFS decision matrix is given by:

$$\overline{D}_{CBFS} = \begin{bmatrix} (R_{11}^{+}, R_{11}^{-}) & (R_{12}^{+}, R_{12}^{-}) & \cdots & (R_{1n}^{+}, R_{1n}^{-}) \\ (R_{21}^{+}, R_{21}^{-}) & (R_{22}^{+}, R_{22}^{-}) & & (R_{2n}^{+}, R_{2n}^{-}) \\ \vdots & & \ddots & \vdots \\ (R_{m1}^{+}, R_{m1}^{-}) & (R_{m2}^{+}, R_{m2}^{-}) & \cdots & (R_{mn}^{+}, R_{mn}^{-}) \end{bmatrix},$$

where $R_{ij}^{+} = \left(a_{ij}^{+}, b_{ij}^{+}, c_{ij}^{+}\right)$ and $R_{ij}^{-} = \left(a_{ij}^{-}, b_{ij}^{-}, c_{ij}^{-}\right)$ are ratings for the positive and negative parts with respect to ith-alternative and jth-attribute being described by the triangular fuzzy number. The rating is based on linguistic variable defined in Table G.2.

Some modification should be made to those fuzzy numbers which are not standardized as will be explained the following. Assume that we have a positive triangular fuzzy number $R_{ij} = (a_{ij}, b_{ij}, c_{ij})$ of rating for alternative with respect to subjective attribute where $0 \le a_{ij} \le b_{ij} \le c_{ij} \le h$. The fuzzy number is converted to a new normalized fuzzy number using:

$$\bar{R}_{ij} = \left(\frac{a_{ij}}{h}, \frac{b_{ij}}{h}, \frac{c_{ij}}{h}\right) = (\bar{a}_{ij}, \bar{b}_{ij}, \bar{c}_{ij}), \tag{G.16}$$

where $0 \le \bar{a}_{ij} \le \bar{b}_{ij} \le \bar{c}_{ij} \le 1$ and h is the maximum value of non-standardize fuzzy number.

G.3.3 Aggregation phase

It is crucial to find a similarity degree for heterogeneous group of experts where different evaluations are given to each alternative. For a finite k of experts, we obtain the similarity degree of each pair of experts (e_u, e_v) for $u, v = 1,2, \dots, k$ and $u \ne v$ by computing the similarity measure $S_{uv}(R_u, R_v)$. Let $M = R_u = (R_{ij}^+, R_{ij}^-)_u$ and $N = R_v = (R_{ij}^+, R_{ij}^-)_v$, then the similarity measure can be calculated using:

$$S_{uv}(M, N) = 1 - \left(\sum_{i=1}^{n} \frac{\varphi_1(x_i) + \varphi_2(x_i)}{2n}\right), \tag{G.17}$$

where,

$$\varphi_1(x_i) = \left|\left(\frac{\mu_M(x_i) + (1 - \nu_M(x_i))}{2}\right) - \left(\frac{\mu_N(x_i) + (1 - \nu_N(x_i))}{2}\right)\right|,$$

$$\varphi_2(x_i) = \left|\frac{\mu_M(x_i) - \mu_N(x_i)}{2}\right| + \left|\left(\frac{1 - \nu_M(x_i)}{2}\right) - \left(\frac{1 - \nu_N(x_i)}{2}\right)\right|.$$

The similarity degree measures how similar is M to N for uv-pair of experts. The higher value of $S_{uv}(M, N)$ indicates that M is more similar to N. In other words, if $S_{uv}(M, N) = 1$, then M is equivalent to N. It is worth noted that $S_{uv}(M, N) = S_{uv}(N, M)$. Further, we construct the agreement matrix as:

$$AM = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1k} \\ S_{21} & S_{22} & & S_{2k} \\ \vdots & & \ddots & \vdots \\ S_{k1} & S_{k2} & \cdots & S_{kk} \end{bmatrix},$$

where $S_{uv}(M,N) = S_{uv}$ for $u \neq v$ and $S_{uv} = 1$ for $u = v$. We take average of the similarity degree for the expert e_u by computing:

$$\bar{S}(e_u) = \frac{1}{k-1} \sum_{\substack{u,v=1 \\ u \neq v}}^{k} S_{uv} \,. \tag{G.18}$$

Next, we find the relative similarity degree $\tilde{S}(e_u)$ as follows:

$$\tilde{S}(e_u) = \frac{\bar{S}(e_u)}{\sum_{u=1}^{k} \bar{S}(e_u)} \,. \tag{G.19}$$

By using a relaxation factor $\beta, (0 \leq \beta \leq 1)$ and the relative similarity degree $\tilde{S}(e_u)$, the consensus coefficient $\widehat{CC}(e_u)$ is calculated as:

$$\widehat{CC}(e_u) = \beta w(e_u) + (1-\beta)\tilde{S}(e_u), \tag{G.20}$$

where $w(e_u)$ is the weight of the expert u obtained from Eq. (G.6). The last step is to compute the aggregated fuzzy evaluation using, R_{agg}:

$$R_{agg} = \left[\widehat{CC}(e_1) \otimes \hat{R}_1 \oplus \widehat{CC}(e_2) \otimes \hat{R}_2 \oplus \ldots \oplus \widehat{CC}(e_k) \otimes \hat{R}_k\right] \tag{G.21}$$

where,

$$\hat{R}_i = \frac{R_i^+ \oplus (1 - R_i^-)}{2} \tag{G.22}$$

$$\hat{R}_i = \left(\frac{a_{ij}^+ + \left(1 - a_{ij}^-\right)}{2}, \frac{b_{ij}^+ + \left(1 - b_{ij}^-\right)}{2}, \frac{c_{ij}^+ + \left(1 - c_{ij}^-\right)}{2}\right) \tag{G.23}$$

The aggregated fuzzy evaluation will be used to rank alternatives in the next stage.

G.3.4 Selection phase

We present a general idea of fuzzy TOPSIS and the detailed procedure can be referred in Chen (2000). According to benefit-cost related attributes, we initially obain the normalized fuzzy decision matrix $\tilde{R} = \left[\tilde{r}_{ij}\right]_{m \times n}$ by normalizing R_{agg} using:

$$\tilde{r}_{ij} = \left\{\frac{a_{ij}}{c_j^+}, \frac{b_{ij}}{c_j^+}, \frac{c_{ij}}{c_j^+}\right\}, c_j^+ = \max_i c_{ij} \quad if\ j \in B, i = 1,2, \dots, m \tag{G.24}$$

$$\tilde{r}_{ij} = \left\{\frac{a_j^-}{c_{ij}}, \frac{a_j^-}{b_{ij}}, \frac{a_j^-}{a_{ij}}\right\}, a_j^- = \min_i c_{ij} \quad if\ j \in B, i = 1,2, \dots, m \tag{G.25}$$

where B and C are the set of benefit criteria and the set of cost criteria, respectively.

Next, we calculate the overall performance evaluation of alternative by multiplying the weight to each normalized attribute, $\tilde{v}_{ij} = \tilde{r}_{ij} \otimes \Omega_i$ for $i = 1,2, \dots, n; j = 1,2, \dots, m$, yielding:

$$\tilde{V} = \left[\tilde{v}_{ij}\right]_{m \times n}. \tag{G.26}$$

The positive ideal solution $\tilde{v}^+$ and negative ideal solution $\tilde{v}^-$ will then be computed as:

$$\tilde{v}^+ = \max(\tilde{v}_1^+, \tilde{v}_2^+, \dots, \tilde{v}_m^+), \quad \tilde{v}^- = \min(\tilde{v}_1^-, \tilde{v}_2^-, \dots, \tilde{v}_m^-). \tag{G.27}$$

We calculate the distance of the fuzzy decision $\tilde{v}_{ij}$ to the positive ideal solution $\tilde{v}^+ = (a^+, b^+, c^+)$ and the negative ideal solution $\tilde{v}^- = (a^-, b^-, c^-)$ using:

$$d\left(\tilde{v}_{ij}, \tilde{v}^{(\cdot)}\right) = \sqrt{\frac{1}{3}\left[\left(a_{ij} - a^{(\cdot)}\right)^2 + \left(b_{ij} - b^{(\cdot)}\right)^2 + \left(c_{ij} - c^{(\cdot)}\right)^2\right]}, \tag{G.28}$$

where,

$$d_i^+ = \sum_{j=1}^{n} d\left(\tilde{v}_{ij}, \tilde{v}^+\right), \quad d_i^- = \sum_{j=1}^{n} d\left(\tilde{v}_{ij}, \tilde{v}^-\right). \tag{G.29}$$

Finally, the alternatives are ranked by computing its closeness coefficient:

$$CC = \frac{d_i^-}{d_i^+ + d_i^-}. \tag{G.30}$$

The alternative with the highest closeness coefficient will be selected as the best alternative.

G.3.5 Algorithm for CBFS-MAGDM model

Step 1: Establish a CBFS decision matrix for each expert.
Step 2: Transform the bifuzzy data into a normalized positive triangular fuzzy number using Eq. (G.16).
Step 3: Assign the relative importance or weight for experts and attributes using Eqs. (G.6 – G.15).
Step 4: Measure the similarity degree using Eq. (G.17). Construct the agreement matrix, the average degree of agreement, the relative degree of agreement and the consensus coefficient by using Eqs. (G.18 – G.20). Then, aggregate all experts' fuzzy evaluations for each alternative using Eqs. (G.21 – G.23).
Step 5: Construct the normalized rating and weighted normalized rating using Eqs. (G.24 – G.26).
Step 6: Calculate the positive-ideal solution, the negative-ideal solution and compute the distance of fuzzy decision to the positive and negative ideal solution using Eqs. (G.27 – G.29).
Step 7: Calculate the closeness coefficient (CC) using Eq. (G.30). Rank the alternative according to the value of its closeness coefficient.

G.4 Selection of Flood Control Project

In this section, we applied our model to the selection problem of flood control project. There are four alternatives to be considered namely reservoir (X_1), channel improvement (X_2), diversion scheme (X_3) and dikes (X_4). Each alternative is evaluated based on four attributes, namely the economic factor (A_1), social factor (A_2), environmental factor (A_3) and technical factor (A_4). These attributes together with their corresponding sub-attributes are listed in Table G.5. The evaluations of alternatives with respect to attribute and sub-attribute are bi-valued except for the monetary term factors, A_{11} and A_{12} and the timeframe-based factor, A_{41}. We choose three experts in the evaluation process, notably the specialized engineers in the Department of Drainage and Irrigation, Kelantan (e_1), Kelantan's local authority (e_2) and Malaysian non-governmental organization (e_3). First, we establish a CBFS decision matrix for the rating of alternatives with respect to the given attributes (sub-attributes). The rating provided by experts is presented in Table G.6. We see that the ratings are in linguistic variables being described in Table G.2, except for the project cost, the operation and maintenance and the lifetime criteria. As an example, for the reservoir X_1, the experts e_2 and e_3 provided the same rating '*medium high*' to the positive effect of soil impact A_{33}, but different rating for the negative effect (or side effect). Expert e_2 felt that reservoir will result '*moderate low*' negative impact compared to '*low*' negative soil impact for expert e_3.

Table G.5. List of attributes and corresponding sub-attributes

Attribute	Sub-attribute
Economic (A_1)	Project cost (A_{11})
	Operation and maintenance cost (A_{12})
	Project benefit (A_{13})
	Reliability economic parameter (A_{14})
Social (A_2)	Social acceptability (A_{21})
	Effect on demographic (A_{22})
	Effect on structure (A_{23})
	Recreation activity (A_{24})
Environmental (A_3)	Water quality impact (A_{31})
	Nature conservation (A_{32})
	Soil impact (A_{33})
	Landscape (A_{34})
	Sanitary condition (A_{35})
Technical (A_4)	Lifetime (A_{41})
	Adaptability (A_{42})
	Level of protection (A_{43})
	Technical complexity (A_{44})
	Flexibility (A_{45})

The experts agreed that the costs of running all the alternatives are very high, approximately ranging from 0.6 to 2.1 billion Ringgit Malaysia (reflects the overall cost for project and also the cost for operation and maintenance). Furthermore, different alternative has different lifetime frame. Reservoir can retain up to 100 years while the rest can operates just in the 10 years. At first glimpse, we see that the reservoir is not economically efficient as it requires the very high running cost, even though the contribution to the social and environmental is positively high. The channel improvement is the most admissible if the budget is limited, but the positive impacts to all factors are considerably moderate. The other two alternatives have moderate impact to all factors.

Some sub-attributes are in monetary term and timeframe-based which need be normalized. We refer to *Step 2* in Section G.3.5 to normalize the rating for the sub-attributes A_{11}, A_{12} and A_{41}. While the other factor remains the same. We show the normalized decision matrix for those three sub-attributes in Table G.7.

Table G.6. Decision matrix of rating alternatives

Attribute / Sub-attribute		X_1			X_2			X_3			X_4		
		e_1	e_2	e_3	e_1	e_2	e_3	e_1	e_2	e_3	e_1	e_2	e_3
A_1	A_{11}	2			1			0.5			0.5		
	A_{12}	0.1			0.5			0.1			0.5		
	A_{13}	(VH,L)	(VH,L)	(VH,L)	(M,M)	(MH,ML)	(M,M)	(MH,ML)	(H,ML)	(H,ML)	(H,L)	(MH,L)	(H,L)
	A_{14}	(H,L)	(MH,L)	(MH,ML)	(M,ML)	(M,ML)	(M,ML)	(MH,ML)	(M,ML)	(H,ML)	(H,L)	(MH,L)	(MH,L)
A_2	A_{21}	(H,L)	(MH,ML)	(M,L)	(VH,VL)	(MH,L)	(MH,L)	(MH,ML)	(H,ML)	(MH,ML)	(MH,L)	(MH,L)	(MH,L)
	A_{22}	(L,M)	(H,ML)	(MH,ML)	(H,VL)	(H,L)	(H,L)	(M,M)	(MH,M)	(MH,M)	(MH,ML)	(H,ML)	(H,ML)
	A_{23}	(H,L)	(MH,M)	(H,ML)	(M,VL)	(M,L)	(M,L)	(M,L)	(MH,ML)	(M,ML)	(MH,L)	(H,L)	(H,L)
	A_{24}	(VH,L)	(H,L)	(H,L)	(VL,M)	(L,M)	(VL,M)	(VH,ML)	(H,ML)	(H,ML)	(VH,L)	(MH,L)	(MH,L)
A_3	A_{31}	(H,L)	(MH,L)	(MH,L)	(M,M)	(MH,ML)	(H,M)	(M,L)	(MH,L)	(M,L)	(MH,L)	(MH,L)	(H,L)
	A_{32}	(MH,L)	(M,ML)	(M,L)	(M,M)	(L,H)	(VL,H)	(ML,M)	(M,ML)	(ML,M)	(M,L)	(M,ML)	(M,L)
	A_{33}	(MH,L)	(MH,ML)	(MH,L)	(M,M)	(M,M)	(ML,MH)	(MH,L)	(M,ML)	(MH,M)	(M,L)	(M,L)	(M,ML)
	A_{34}	(VH,L)	(VH,L)	(H,L)	(VL,H)	(L,M)	(L,MH)	(VH,L)	(H,L)	(H,L)	(VH,L)	(H,VL)	(H,L)
	A_{35}	(MH,L)	(MH,ML)	(H,ML)	(VL,H)	(L,H)	(VL,H)	(MH,ML)	(M,L)	(MH,ML)	(MH,L)	(M,L)	(MH,ML)
A_4	A_{41}	100-yr			10-yr			10-yr			10-yr		
	A_{42}	(VH,L)	(H,ML)	(H,L)	(M,M)	(M,ML)	(M,ML)	(MH,ML)	(H,ML)	(H,ML)	(H,L)	(MH,L)	(MH,ML)
	A_{43}	(VH,L)	(H,ML)	(H,L)	(ML,H)	(M,L)	(ML,MH)	(MH,ML)	(H,L)	(H,L)	(H,L)	(MH,L)	(H,L)
	A_{44}	(H,L)	(H,L)	(MH,L)	(M,ML)	(MH,L)	(M,ML)	(M,MH)	(MH,M)	(H,M)	(H,L)	(MH,ML)	(MH,ML)
	A_{45}	(VH,L)	(H,L)	(VH,L)	(M,M)	(M,ML)	(M,ML)	(H,ML)	(MH,L)	(H,ML)	(MH,L)	(M,L)	(M,L)

Table G.7. The normalized decision matrix for the three sub-attributes

	X_1	X_2	X_3	X_4
A_{11}	(0.818,0.909,1.000)	(0.362,0.445,0.545)	(0.136,0.227,0.318)	(0.136,0.227,0.318)
A_{12}	(0.091,0.182,0.273)	(0.818,0.909,1.000)	(0.091,0.182,0.273)	(0.818,0.909,1.000)
A_{41}	(0.091,0.182,0.273)	(0.073,0.091,0.109)	(0.073,0.091,0.109)	(0.073,0.091,0.109)

Due to the lack of knowledge of experts for certain attributes, we then define the weight of each expert as in *Step 3*. We admit that the specialized engineer, e_1 has the highest priority in this decision making processes. Therefore, we choose e_1 to be the proxy and the weight of experts will be determined according to the degree of importance as calculated using Eq. (G.6). In parallel, we calculate the weight for attribute (and sub-attribute) using fuzzy AHP for subjective weight and entropy method for objective weight. The results for subjective and objective weights are shown in Tables G.8 and G.9 respectively.

Table G.8. The subjective weight for attributes and sub-attributes and the aggregated weight

Attribute	w_i^{att}	Sub	w_{ij}^{sub}	w_{ij}^{agg}
A_1	(0.304,0.460,0.687)	A_{11}	(0.140,0.239,0.391)	(0.042,0.110,0.269)
		A_{12}	(0.147,0.220,0.343)	(0.045,0.101,0.236)
		A_{13}	(0.319,0.489,0.742)	(0.097,0.225,0.509)
		A_{14}	(0.040,0.052,0.072)	(0.012,0.024,0.049)
A_2	(0.237,0.353,0.534)	A_{21}	(0.256,0.471,0.840)	(0.061,0.166,0.449)
		A_{22}	(0.083,0.164,0.330)	(0.020,0.058,0.176)
		A_{23}	(0.139,0.278,0.540)	(0.033,0.098,0.288)
		A_{24}	(0.056,0.087,0.165)	(0.013,0.031,0.088)
A_3	(0.085,0.137,0.213)	A_{31}	(0.178,0.334,0.637)	(0.015,0.047,0.136)
		A_{32}	(0.076,0.137,0.262)	(0.006,0.019,0.056)
		A_{33}	(0..145,0.278,0.524)	(0.012,0.038,0.112)
		A_{34}	(0.037,0.056,0.097)	(0.003,0.008,0.021)
		A_{35}	(0.092,0.185,0.375)	(0.008,0.025,0.080)
A_4	(0.038,0.050,0.071)	A_{41}	(0.185,0.293,0.458)	(0.007,0.015,0.033)
		A_{42}	(0.142,0.225,0.366)	(0.005,0.011,0.026)
		A_{43}	(0.230,0.367,0.576)	(0.009,0.019,0.041)
		A_{44}	(0.042,0.075,0.126)	(0.002,0.004,0.009)
		A_{45}	(0.029,0.041,0.068)	(0.001,0.002,0.005)

Table G.9. The objective weight and integrated weight

Attribute	Sub	Entropy	w_{ij}^{obj}	Ω_{ij}
A_1	A_{11}	0.781	0.040	(0.030,0.078,0.193)
	A_{12}	0.500	0.091	(0.047,0.113,0.274)
	A_{13}	0.753	0.045	(0.048,0.118,0.282)
	A_{14}	0.763	0.043	(0.017,0.038,0.086)
A_2	A_{21}	0.646	0.064	(0.046,0.121,0.317)
	A_{22}	0.588	0.075	(0.028,0.077,0.215)
	A_{23}	0.728	0.049	(0.029,0.082,0.223)
	A_{24}	0.791	0.038	(0.016,0.040,0.108)
A_3	A_{31}	0.762	0.043	(0.019,0.053,0.143)
	A_{32}	0.766	0.042	(0.012,0.033,0.091)
	A_{33}	0.785	0.039	(0.016,0.045,0.124)
	A_{34}	0.792	0.038	(0.008,0.020,0.052)
	A_{35}	0.792	0.038	(0.013,0.036,0.103)
A_4	A_{41}	0.000	0.181	(0.026,0.061,0.144)
	A_{42}	0.773	0.041	(0.011,0.025,0.061)
	A_{43}	0.790	0.038	(0.013,0.031,0.073)
	A_{44}	0.726	0.050	(0.007,0.016,0.040)
	A_{45}	0.721	0.047	(0.005,0.012,0.028)

For the sake of simplicity, we use the mean of fuzzy number as in Table G.3 to calculate the consistency ratio. We obtain the consistency ratio equal 0.059 accepting the validity of our pairwise comparison matrix. Expert's rating are then aggregated using Eqs. (G.17 – G.23). We refer to *Step 4* in Section G.3.5. The value β is set to 0.4 which represent the expert dominance for this problem. We report the aggregated fuzzy rating in Table G.10. Next, the normalized ratings and weighted normalized ratings of the matrices are constructed using Eqs. (G.24 – G.26) as in *Step 5*. Tables G.11 and G.12 present the fuzzy normalized rating and the weighted fuzzy normalized rating, respectively. Note that, Table G.12 is obtained by multiplying the integrated weight reported in Table G.9 with the fuzzy normalized rating.

Table G.10. The aggregated fuzzy rating for heterogeneous group of experts

Attribute	Sub	X_1	X_2	X_3	X_4
A_1	A_{11}	(0.818,0.909,1.000)	(0.364,0.455,0.545)	(0.136,0.227,0.318)	(0.136,0.227,0.318)
	A_{12}	(0.091,0.182,0.273)	(0.818,0.909,1.000)	(0.091,0.182,0.273)	(0.818,0.909,1.000)
	A_{13}	(0.725,0.900,0.975)	(0.408,0.558,0.708)	(0.548,0.698,0.848)	(0.687,0.770,0.920)
	A_{14}	(0.587,0.737,0.887)	(0.425,0.575,0.725)	(0.487,0.637,0.787)	(0.603,0.753,0.903)
A_2	A_{21}	(0.572,0.722,0.872)	(0.652,0.820,0.918)	(0.530,0.680,0.830)	(0.575,0.725,0.875)
	A_{22}	(0.436,0.586,0.736)	(0.695,0.860,0.965)	(0.400,0.550,0.700)	(0.550,0.700,0.850)
	A_{23}	(0.558,0.708,0.858)	(0.524,0.682,0.808)	(0.470,0.620,0.770)	(0.626,0.776,0.926)
	A_{24}	(0.674,0.832,0.958)	(0.192,0.318,0.476)	(0.559,0.757,0.883)	(0.621,0.779,0.906)
A_3	A_{31}	(0.602,0.752,0.902)	(0.446,0.596,0.746)	(0.518,0.668,0.818)	(0.605,0.755,0.905)
	A_{32}	(0.523,0.673,0.823)	(0.123,0.242,0.402)	(0.318,0.468,0.618)	(0.478,0.628,0.778)
	A_{33}	(0.557,0.707,0.857)	(0.408,0.558,0.708)	(0.509,0.659,0.809)	(0.500,0.065,0.800)
	A_{34}	(0.701,0.868,0.967)	(0.529,0.686,0.816)	(0.641,0.798,0.927)	(0.725,0.900,0.975)
	A_{35}	(0.552,0.702,0.852)	(0.032,0.130,0.297)	(0.500,0.650,0.800)	(0.530,0.680,0.830)
A_4	A_{41}	(0.818,0.909,1.000)	(0.073,0.091,0.109)	(0.073,0.091,0.109)	(0.073,0.091,0.109)
	A_{42}	(0.680,0.840,0.960)	(0.395,0.545,0.695)	(0.545,0.695,0.845)	(0.605,0.755,0.905)
	A_{43}	(0.656,0.816,0.936)	(0.259,0.409,0.559)	(0.592,0.742,0.892)	(0.623,0.776,0.926)
	A_{44}	(0.632,0.782,0.932)	(0.477,0.627,0.777)	(0.384,0.534,0.684)	(0.558,0.708,0.858)
	A_{45}	(0.698,0.864,0.966)	(0.395,0.545,0.695)	(0.575,0.725,0.875)	(0.315,0.680,0.830)

Table G.11. Fuzzy normalized ratings for heterogeneous group of experts

Attribute	Sub	X_1	X_2	X_3	X_4
A_1	A_{11}	(0.136,0.150,0.166)	(0.249,0.299,0.374)	(0.427,0.598,1.000)	(0.427,0.598,1.000)
	A_{12}	(0.333,0.500,1.000)	(0.091,0.100,0.111)	(0.333,0.500,1.000)	(0.091,0.100,0.111)
	A_{13}	(0.744,0.923,1.000)	(0.419,0.573,0.726)	(0.562,0.716,0.870)	(0.705,0.790,0.944)
	A_{14}	(0.650,0.816,0.982)	(0.471,0.637,0.803)	(0.539,0.705,0.871)	(0.668,0.834,1.000)
A_2	A_{21}	(0.623,0.787,0.950)	(0.711,0.893,1.000)	(0.577,0.740,0.904)	(0.626,0.790,0.953)
	A_{22}	(0.451,0.607,0.762)	(0.720,0.892,1.000)	(0.414,0.570,0.725)	(0.570,0.725,0.881)
	A_{23}	(0.603,0.765,0.927)	(0.565,0.736,0.872)	(0.507,0.669,0.831)	(0.676,0.838,1.000)
	A_{24}	(0.703,0.868,1.000)	(0.201,0.332,0.497)	(0.625,0.790,0.922)	(0.649,0.813,0.946)
A_3	A_{31}	(0.665,0.831,0.997)	(0.493,0.659,0.825)	(0.572,0.738,0.904)	(0.668,0.834,1.000)
	A_{32}	(0.635,0.818,1.000)	(0.149,0.294,0.489)	(0.387,0.569,0.751)	(0.581,0.763,0.945)
	A_{33}	(0.650,0.825,1.000)	(0.476,0.651,0.826)	(0.594,0.769,0.944)	(0.583,0.758,0.933)
	A_{34}	(0.719,0.891,0.992)	(0.543,0.704,0.837)	(0.658,0.818,0.951)	(0.744,0.923,1.000)
	A_{35}	(0.648,0.824,1.000)	(0.038,0.152,0.349)	(0.587,0.763,0.939)	(0.622,0.799,0.975)
A_4	A_{41}	(0.818,0.909,1.000)	(0.073,0.091,0.109)	(0.073,0.091,0.109)	(0.073,0.091,0.109)
	A_{42}	(0.708,0.875,1.000)	(0.412,0.568,0.724)	(0.568,0.724,0.881)	(0.631,0.787,0.943)
	A_{43}	(0.701,0.872,1.000)	(0.276,0.436,0.597)	(0.632,0.792,0.952)	(0.666,0.829,0.990)
	A_{44}	(0.678,0.839,1.000)	(0.512,0.673,0.834)	(0.412,0.573,0.734)	(0.599,0.760,0.921)
	A_{45}	(0.723,0.895,1.000)	(0.409,0.565,0.720)	(0.595,0.750,0.906)	(0.326,0.703,0.859)

Table G.12. Weighted fuzzy normalized ratings for group of experts

Attribute	Sub	X_1	X_2	X_3	X_4
A_1	A_{11}	(0.004,0.012,0.032)	(0.007,0.023,0.072)	(0.013,0.047,0.193)	(0.013,0.047,0.193)
	A_{12}	(0.016,0.056,0.274)	(0.004,0.011,0.030)	(0.016,0.056,0.274)	(0.004,0.011,0.030)
	A_{13}	(0.036,0.109,0.282)	(0.021,0.070,0.210)	(0.027,0.084,0.245)	(0.039,0.092,0.264)
	A_{14}	(0.011,0.031,0.086)	(0.008,0.024,0.069)	(0.009,0.026,0.073)	(0.011,0.031,0.086)
A_2	A_{21}	(0.028,0.094,0.298)	(0.033,0.110,0.317)	(0.026,0.090,0.287)	(0.028,0.095,0.300)
	A_{22}	(0.013,0.047,0.163)	(0.020,0.068,0.215)	(0.012,0.044,0.156)	(0.016,0.056,0.189)
	A_{23}	(0.018,0.063,0.209)	(0.017,0.060,0.194)	(0.015,0.054,0.184)	(0.020,0.069,0.223)
	A_{24}	(0.012,0.035,0.108)	(0.003,0.013,0.054)	(0.010,0.032,0.100)	(0.011,0.033,0.102)
A_3	A_{31}	(0.012,0.044,0.142)	(0.009,0.034,0.116)	(0.010,0.038,0.127)	(0.012,0.044,0.143)
	A_{32}	(0.008,0.027,0.091)	(0.002,0.009,0.043)	(0.005,0.019,0.068)	(0.007,0.026,0.087)
	A_{33}	(0.011,0.037,0.124)	(0.008,0.030,0.103)	(0.010,0.035,0.117)	(0.009,0.034,0.114)
	A_{34}	(0.006,0.018,0.052)	(0.004,0.014,0.043)	(0.005,0.016,0.049)	(0.006,0.018,0.052)
	A_{35}	(0.008,0.030,0.103)	(0.000,0.005,0.035)	(0.007,0.028,0.096)	(0.008,0.029,0.100)
A_4	A_{41}	(0.021,0.055,0.144)	(0.002,0.006,0.016)	(0.002,0.006,0.016)	(0.002,0.006,0.016)
	A_{42}	(0.008,0.022,0.061)	(0.004,0.014,0.044)	(0.006,0.018,0.054)	(0.007,0.020,0.058)
	A_{43}	(0.009,0.026,0.074)	(0.004,0.013,0.043)	(0.008,0.024,0.069)	(0.009,0.026,0.073)
	A_{44}	(0.004,0.014,0.040)	(0.003,0.011,0.033)	(0.002,0.009,0.028)	(0.004,0.012,0.037)
	A_{45}	(0.004,0.010,0.028)	(0.002,0.006,0.020)	(0.003,0.009,0.026)	(0.002,0.008,0.025)

The positive ideal solution and negative ideal solution are then calculated using Eq. (G.27) and the distance measure is computed using Eq. (G.29) as in *Step 6*. We simply determine the positive ideal solution by taking the element with the highest value for the benefit attribute and the element with the lowest value for the cost attribute. In contrast, the negative ideal solution is determined by taking the element with opposite values of benefit and cost attributes. The result are reported in Table G.13. Next, Table G.14 shows the distance measure to the positive and negative ideal solutions and its corresponding closeness coefficient as in *Step 7*.

As a result, we found that $X_1 \succ X_3 \succ X_4 \succ X_2$, which simply mean the best alternative for the flood control project is reservoir and the worst is channel improvement. Reservoir is the highest cost project which was initially seems inefficient as it budget sensitive. However, it has a longer lifetime and a very high positive rating for its technicality (on average). Furthermore, it conserves nature and has a high impact to the society. Even though the channel improvement uses less money but it should be maintained for estimated every 10 years. In addition, we see that it has (on average) the moderate impact to social and environment. Thus, the experts have selected reservoir to be the best alternative in controlling the flood in the problem area.

Table G.13. The positive ideal solution and negative ideal solution

Attribute	Sub-attribute	$\tilde{v}^+$	$\tilde{v}^-$
A_1	A_{11}	(0.013,0.047,0.193)	(0.004,0.012,0.032)
	A_{12}	(0.016,0.056,0.274)	(0.004,0.011,0.030)
	A_{13}	(0.039,0.109,0.282)	(0.021,0.070,0.210)
	A_{14}	(0.011,0.031,0.086)	(0.008,0.024,0.069)
A_2	A_{21}	(0.033,0.110,0.317)	(0.026,0.090,0.287)
	A_{22}	(0.020,0.068,0.215)	(0.012,0.044,0.156)
	A_{23}	(0.020,0.069,0.223)	(0.015,0.054,0.184)
	A_{24}	(0.012,0.035,0.108)	(0.003,0.013,0.054)
A_3	A_{31}	(0.012,0.044,0.143)	(0.009,0.034,0.116)
	A_{32}	(0.008,0.027,0.091)	(0.002,0.009,0.043)
	A_{33}	(0.011,0.037,0.124)	(0.008,0.030,0.103)
	A_{34}	(0.006,0.018,0.052)	(0.004,0.014,0.043)
	A_{35}	(0.008,0.030,0.103)	(0.000,0.005,0.035)
A_4	A_{41}	(0.021,0.055,0.144)	(0.002,0.006,0.016)
	A_{42}	(0.008,0.022,0.061)	(0.004,0.014,0.044)
	A_{43}	(0.009,0.027,0.074)	(0.004,0.013,0.043)
	A_{44}	(0.004,0.014,0.040)	(0.002,0.009,0.028)
	A_{45}	(0.004,0.010,0.028)	(0.002,0.006,0.020)

Table G.14. Distance measure to the positive ideal solution and negative ideal solution using CBFS-MAGDM

	X_1	X_2	X_3	X_4
d_j^+	0.152	0.552	0.249	0.286
d_j^-	0.492	0.092	0.395	0.359
CC	0.764	0.142	0.613	0.556
Ranking	1	4	2	3

In comparison, we provide the ranking based on the fuzzy MAGDM in Table G.15. In general, the CC values for all alternatives are greater for CBFS-MAGDM except for the channel improvement where the earlier method gives CC's value 0.300. Hence, the results justify the effect or influence of two-sided judgment. In the case of complementary (FS), the sum of positive and negative membership values is equal to one. Thus, the decision process may not influenced by the other side of evaluation (side-effect). Therefore, fuzzy approach is adequate in this case. For the no complementary case, the result may different. If the sum of positive and negative evaluation is less than one, we will see a greater result for each alternative. While, if the sum of positive and negative aspects is greater than one, then one will have a lower result.

Table G.15. Distance measure to the positive ideal solution and negative ideal solution using fuzzy MAGDM

	X_1	X_2	X_3	X_4
d_j^+	0.313	0.666	0.441	0.470
d_j^-	0.638	0.285	0.511	0.482
CC	0.671	0.300	0.537	0.506
Ranking	1	4	2	3

Another important point that should be noted is the result of attribute (sub-attribute) weight for both models. The objective weight for the fuzzy MAGDM which derived from the entropy method is slightly different from the case of CBFS-MAGDM since the analysis of data is based on the single-side and double-sided evaluation of experts. Hence, the final weight (an integration of the subjective and objective weights) of CBFS-MAGDM method also produces different results and affects the final decision.

G.4.1 Sensitivity of coefficient

Sensitivity analysis is performed to see the effect of coefficient β to the final ranking. The β takes value in the interval [0, 1]. Table G.16 shows the closeness coefficient computed for several β. Since the CC values do not much deviate and not change the order ranking of alternatives, we may conclude that this case is not β sensitive. In this case, we found that the ranking stays at relatively the same level. Thus, the model is not sensitive.

Table G.16. Estimated CC for different β

β	X_1	X_2	X_3	X_4
0	0.765	0.141	0.621	0.558
0.1	0.765	0.141	0.619	0.557
0.2	0.764	0.142	0.617	0.557
0.3	0.764	0.142	0.615	0.557
0.4	0.764	0.142	0.613	0.556
0.5	0.764	0.143	0.612	0.556
0.6	0.764	0.143	0.610	0.556
0.7	0.763	0.143	0.608	0.555
0.8	0.763	0.143	0.607	0.555
0.9	0.762	0.144	0.605	0.555
1	0.761	0.144	0.603	0.554

6.4 Summary

In this chapter, the technique for order performance by similarity to ideal solution (TOPSIS) for the group decision making problems has been discussed. To recap, in Section 6.2, the TOPSIS model with induced generalized OWA operators has been presented. The model then was applied to the case study of human resource selection problem. Lastly, in Section 6.3, the integration of TOPSIS with the AHP method has been developed based on the conflicting bifuzzy condition. The model then was applied in the case study of flood control project selection problem.

CHAPTER 7

CONCLUSIONS AND FUTURE RESEARCH

7.1 Introduction

In this chapter, the general and specific conclusions of this research are presented. In addition, rooms for the future research are also highlighted. Lastly, the summary ends this chapter.

7.2 General Conclusions

The main focus of this study has been given to the decision analysis (i.e., multi-dimensional aspects), uncertainty theories and aggregation operators. The scope of the study is limited to the general problems in financial decision making and some other applications like human resource and flood control project selection problems.

In specific, a paradigm shift of financial modeling tools from bi-criteria to multi-dimensional analysis has been discussed as the start-of-art of the study. Moreover, the issue of human behavior has been considered as the additional features in the current decision analysis models. This encompasses the subjectivity in human preferences and also the inclusion of the attitudinal character of the decision maker(s) for a comprehensive analysis (i.e., from optimistic to pessimistic views).

The emphasis has been given on the two-stage aggregation processes in MCDA models, namely the aggregation of criteria and the aggregation of experts' judgments. In this work, the group decision making models, notably, the Dempster-Shafer theory (DST) of belief structure, the analytic hierarchy process (AHP) and the technique for order performance by similarity to ideal solution (TOPSIS) are put forward for the specific analysis. The ordered weighted average (OWA) aggregation operators are employed as to extend the existing models, such as using the monotone quantifiers, maximum entropy OWA, neat-OWA, induced OWA, heavy OWA, and their generalizations. In addition to that, some of the models are developed based on the linguistic-type of data assessments (i.e., 2-tuple linguistic approach and general linguistic labels). Thus, the inclusion of fuzzy set ideas to model human behavior in MCDA problems is the main motivation of this study.

Quantitative and qualitative preferences, decision strategies based on the attitudinal character of the decision maker(s), and majority concepts for group consensus are highlighted. These contributions allow the modeling of financial decision problems with more complete and relevant information. Besides, a wide spectrum of risk and uncertainty analyses can be conducted for the detail assessments. As a conclusion, the proposed models provide some added values in the analysis of financial selection problems by considering the various issues, namely, multidimensional aspects, uncertainty theories, and soft aggregation processes. Thus, this justifies the problem statements, objectives and methodologies developed in this study. The specific conclusions based on the contributions of this work are summarized in the next section.

7.3 Specific Conclusions based on the Main Contributions

As already mentioned, this research is developed based on MCDA models, uncertainty theories, and OWA-based aggregation operators. The proposed models have some advantages over the existing models as will be explained in the following sub-sections. Beforehand, the methods and techniques used in the proposed models are recapitulated

7.3.1 On OWA-based aggregation operations in ME-MCDM model

This work is focused on the extension and analysis of group decision making model with respect to the OWA-based aggregation operators. The synthesis of experts' judgments and the fusion of criteria are studied. Firstly, the analysis of majority concepts based on the induced OWA and linguistic quantifier is presented as the group aggregators. This consists of the classical and alternative schemes of group decision making model. In specific, the methods by Pasi and Yager (2006) and Bordogna and Sterlacchini (2014) are examined. Some modifications to the support functions are suggested as to derive the vector of order-inducing variables. Secondly, the aggregation operators based on the integration of WA and OWA are studied for the fusion of criteria. Correspondingly, the alternative OWAWA operator is proposed as the new approach. In particular, it is a modification or an alternative representation of the immediate weighted average (WA) and OWAWA operators.

Based on these two-stage aggregation processes, then the multi-expert MCDM model is developed for the heterogeneous case. A comparison is conducted to see the effect of different weighting techniques in aggregating the criteria and the results of using different decision schemes for the fusion of majority opinion of experts. A numerical example based on the investments selection problem is used in the analysis. As for the results, it can be demonstrated that the selection of decision schemes (either classical scheme or alternative scheme) as well as weighting methods employed in the aggregation process shown slightly different rankings of the alternatives. Specifically, each of the decision schemes reflects different decision strategy in deriving the final ranking. The classical scheme is based on the aggregation of the individual decision strategy of experts. In contrast, the alternative scheme is the result of the group decision strategy collectively. Thus, this orientation in manipulating the consensus of experts has produced distinct results.

The alternative scheme demonstrates more specificity in term of comparison (i.e., with respect to each criterion of experts) than the classical scheme (i.e., the final ranking of individual experts). The procedure based on the alternative scheme is more consistent in the spirit of cooperative-group decision making as the attitudinal character is respected to the group, not the individual. Hence, in certain cases of financial decision making problems, the analysis can be conducted in this way in achieving the final decision. However, the classical scheme can be applied in the cases where the decision based on the independent attitudinal characters of experts is needed in the analysis.

7.3.2 Weighted selective aggregated majority-OWA operator and its application in linguistic group decision making model

The different approach of majority concept as the group aggregation is discussed here. Particularly, the majority additive-OWA (Peláez & Doña, 2003), the selective MA-OWA (Karanik et al., 2016) and the selective aggregated majority-OWA (Peláez et al., 2016) operators are studied and analyzed. These aggregation operators are only applicable in the case of homogeneous GDM problems. The weighted SAM-OWA (WSAM-OWA) operator then is proposed as the extension of the SAM-OWA to deal with the heterogeneous case. In particular, it is formulated with the inclusion of the reliability of information sources (or the degrees of importance). Integrated with the linguistic quantifiers, the WSAM-OWA is extended to the quantified WSAM-OWA operator, mainly for the group fusion strategy. Moreover, the QWSAM-IOWA operator is introduced for the individual fusion strategy. The similarity between experts' opinions as the order-inducing variables is included to present the majority. This is done by specifying specific semantics for the linguistic quantifier. The multi-criteria GDM model under the linguistic domain then is developed where the proposed aggregation operators can be implemented as the group aggregator and the weighted OWA operator is applied to derive the final ranking of alternatives. The investment selection problem is provided to demonstrate the applicability of the developed model.

The proposed aggregation operators for majority concept not only take into account the most similar values of experts but also consider the degrees of importance for the heterogeneous case. Moreover, these operators are consistent in aggregating the arguments with cardinalities to avoid the distribution problems. In general, the proposed model offers a greater flexibility in analyzing the financial selection problems with the degrees of tolerance in the aggregation processes. Therefore, a wide spectrum of analysis can be performed for the final decision that suits the decision maker's tendency.

7.3.3 Linguistic Group Decision Making Model with Dempster-Shafer Theory and Induced Linguistic Aggregation Operators

This work has presented a new model for linguistic group decision making under Dempster-Shafer framework based on the 2-tuple linguistic approach and the use of induced linguistic aggregation operators. By using this model, the decision maker gets a complete view of the decision problem because he can consider a wide range of aggregation operators between the minimum and maximum and select the one that is in accordance with his interests. The main advantage of using the 2-tuple linguistic approach is that the uncertain environments can be assessed without losing of information in the computation process.

The model has been presented by using the 2-TILOWA operator. Thus, this formulation has produced the BS-2-TILOWA as the general aggregation process of this approach. Some of its main properties and particular cases are studied. This model has been extended by using generalized (2-TILGOWA) and quasi-arithmetic (Quasi-2-TILOWA) means in the aggregation of the linguistic information. As a result, the BS-2-TILGOWA and the BS-Quasi-2-TILOWA operator are obtained. Then, an application of the new approach in a linguistic group decision making problem about the selection of financial strategies has been developed with respect to the aforementioned aggregation operators.

The main advantage of using these generalizations is that they provide more complete representation of the financial decision problems since a broad range of linguistic aggregation operators can be conducted. Moreover, it is more realistic and closer to the real life problems as the experts normally provide the preferences using the subjective judgments instead of the exact figures or values.

7.3.4 Generalized AHP for group decision making model using induced OWA operators

The extension of the aggregation operations in the AHP-GDM using the IOWA operators is the main focus here. The induced WMEOWA has been proposed to determine the weighting vector for aggregating the criteria in the AHP-GDM. Furthermore, the aggregation of experts' judgments by the inclusion of the majority concept is implemented with respect to the priorities of alternatives and the individual preferences of criteria.

The main advantages of the proposed model are the ability to deal with the complex attitudinal character of the decision makers and the aggregation of the information with a particular reordering process. Therefore, the decision makers attain a complete view of the problem and are able to select the alternative in accordance with their interests. In addition, the integration of the majority concept based on the modified of Bordogna-Sterlacchini method has some advantages, including providing a uniformity in reflecting the behavior of the majority of experts and a robust decision by determining the performance judgments on each specific criterion or alternative. Moreover, the result of the majority of experts based on individual priorities of the alternatives is also provided. The application in the investment selection problem has been given to exemplify the feasibility of the proposed method. The comparison of different schemes of group aggregation with respect to different degrees of optimism has also been conducted.

7.3.5 Heavy weighted geometric aggregation operators in AHP group decision making

In this study, the heavy weighted geometric (HWG) and heavy ordered weighted geometric (HOWG) operators have been introduced. These operators are used as aggregation methods in the analytic hierarchy process (AHP) under group decision making. The aggregation of individual judgments (AIJ) procedure of AHP is implemented as the extension model. For the AIJ procedure, the geometric mean-based methods are preferable as they satisfy the unanimity condition, homogeneity condition, and multiplicative reciprocal property. Hence, the inclusion of HWG and HOWG aggregation operators justify the extension model.

The main advantage of the proposed model is, it not only consider the overlapping of information but also takes into account the partial and non-overlapping information in the aggregation process. Moreover, the integration of AHP-GDM with the HOWG provides a wider class of aggregation operators from the minimum to the total operators.

7.3.6 TOPSIS model with the OWA-based aggregation operators

The central focus is given on the TOPSIS for homogeneous group decision-making model. The two stages of aggregation processes in the TOPSIS-GDM model are designed, which are the external aggregation (i.e., the inclusion of majority concept as the consensus measure of experts) and the internal aggregation (i.e., the aggregation of criteria for the ranking process). In internal aggregation, the inclusion of the attitudinal character (or behavior) of experts is also included, such as considering the pessimistic or optimistic case. For this purpose, some modifications on separation measures are made using the Minkowski-OWA distance measures. In general, the advantages of the proposed model include: provide the uniformity in modeling the behavior of the majority of experts regarding the proportion of criteria to consider and establish a more robust decision by taking into account the consensus of experts on each criterion instead of on each alternative.

The analysis of the human resources selection problem is implemented to test the reliability of the proposed model and the comparisons are made between some other models, for example, the classical TOPSIS-GDM by Shih et al., (2007) and the TOPSIS-GDM with majority opinion based on the classical scheme by Hajimirsadeghi & Lucas (2009). Specifically, a comparison of the models is conducted with respect to the Manhattan distance and the Euclidean distance. The confidence measures are established to test the results of different TOPSIS-GDM models. In general, the inclusion of majority concept in the TOPSIS-GDM model (either classical scheme or alternative scheme) showed a more convincing result than the classical TOPSIS-GDM in term of discrimination between the alternatives. This is because the classical TOPSIS-GDM does not consider the similarity between experts as to derive the final ranking.

The aggregation operator based arithmetic mean or geometric mean is affected by the extreme values. Hence, the aggregated result is correlated to the symmetric tendency between the values rather than the consensus between the similar values. On the other hand, the group aggregator with the majority concept provides flexibility to consider the most similar opinions and exclude the less similar opinion to avoid the aggregation on the extreme values. However, based on the analysis of case study, there are slight differences in the results obtained from each model. This is due to the fact that there are only two subjective criteria considered in this case for evaluating the majority opinion of experts and the rest are based on the objective criteria. However, this analysis provides a promising tool for analyzing the data which focuses on majority opinions.

7.3.7 Conflicting Bifuzzy Multi-attribute Group Decision Making Model with Application to Flood Control Project

In this work, a group decision making model based on conflicting bifuzzy set approach has been constructed, namely CBFS-MAGDM model. The developed model considers two conflicting perspectives (i.e. positive and negative views of experts) in the evaluation process. These two-sided judgments are not limited to the complementary condition as in fuzzy set approach or restricted by the intuitionistcic fuzzy condition. Hence, this approach can be considered as the generalization of the FS and IFS approaches. With respect to this concept, the conflicting bifuzzy similarity measure then is proposed to compute the degree of agreement between experts. The result of the group aggregation process is further evaluated using the integration of fuzzy TOPSIS and fuzzy AHP models. The fuzzy TOPSIS is applied for the ranking process. While, fuzzy AHP is employed for the weighting process, together with entropy method. In specific, the subjective and objective weights are integrated for the attribute and sub-attribute which reflects the subjective rating of experts and objective information obtained from a mathematical model respectively. The fuzzy AHP method computes the subjective weight, and the entropy method is employed to calculate the objective weight.

To demonstrate the feasibility of the proposed model, a case study of selection flood control project is conducted. In comparison, the fuzzy MAGDM model is put forward as the reference for the CBFS-MAGDM model. Even though, the FMAGDM and CBFS-MAGDM models produce similar ranking of alternatives, but, the value of the closeness coefficient is different for both models. In specific, the CBFS-MAGDM gives a slightly lower or greater closeness coefficient due to the effect of non-complement double-sided judgments. Thus, in this case, the CBFS-MAGDM has some advantages as it is more realistic and closer to the real life problems.

The application of the proposed model is not limited to the selection of flood control projects. The model can be applied to other research domains having the similar features, notably, the conflicting condition, incomplete information and non-standard data structure. One of the potential areas is in the domain of finance. The model is designed to solve the conflicting judgment between the experts, thus provides a better policy for the decision analysis. With the aim to minimizing the cost and maximizing the profit, the conflicting condition and incomplete information for each option can be carefully analyzed and revised.

7.4 Future Research

For the future research, a lot of studies can be conducted related to the extensions of decision analysis models (MCDA and GDM) by the integration of uncertainty theories and aggregation operators. In the domain of finance, the MCDA and GDM models can be applied to many other financial problems (see, for examples, Ceballos, 2001; Ceballos & Sorrosal, 2002; Ramírez et al., 2008). Generally, the applications of these models are not limited to the realm of finance. Other decision problems having the similar features, namely the multi-dimensional aspects, uncertainty and/or aggregation issues can be applied.

There are many other MCDA models in the literature which can be extended using the aforementioned concepts and theories. Among them, include, the analytic network process (Saaty, 2013), MACBETH (preference relation-based method), outranking methods: ELECTRE (Roy, 1991) and PROMETHEE (Brans & Vincke, 1985), multi-objective decision making models (Hwang & Masud, 1979), etc. The hybrid model can be developed as well by integrating the different part of MCDA models, specifically for deriving the weighting vector and the ranking process. In addition, further development can be made by the inclusion of the evolutionary algorithm techniques like genetic algorithm (Melanie, 1996), particle swarm optimization (Kennedy & Eberhart, 1995), ant colony techniques (Dorigo, 1992), etc. These methodologies can be used to simulate the human behavior via intelligent machines to perform well and better than humans.

Other direction of research can be focused on the representation of imprecise data using the higher order fuzzy sets. These include the type-2 fuzzy set (Zadeh, 1975), intuitionistic fuzzy set (Attanassov, 1986), conflicting bifuzzy set (Abu Osman, 2004), Z-number (Zadeh, 2011), neutrosophic set (Smarandache, 1999), etc. Recently, much attention has been given on the bi-value evaluations (bipolar or bicapacities), notably, the positive and negative sides of data. This provides the analysis become more general and powerful as two-sided judgments of effect and side-effect are considered simultaneously. This concept has been applied in some of the MCDA models, but there are still rooms that can be improved in the model developments, especially with the application in the domain of finance.

Analogously, the extension of aggregation operators can be conducted as well to deal with the higher order fuzzy sets. The main focus can be given on the extension of OWA operators to the higher order of OWA operators. Recently, the OWA has been extended to the type-1 OWA (Zhou et al., 2008) and type-2 OWA (Zhou et al., 2010) to cope with the uncertainty data. The extension can be further generalized to the other variant of OWA operators, such as induced-OWA, heavy OWA, etc.

Prior to that, the Choquet and Sugeno integrals have been applied in some of MCDA models. The extension using the concept of higher order fuzzy sets also can be made to these more general aggregation operators such as fuzzy measures (Sugeno, 1977) and monitored heavy fuzzy measures (Yager, 2003a). Thus, the interaction between data (criteria or experts' judgments) can be included, then providing the dependency (synergy positive or synergy negative) for the detailed and extensive analysis.

7.5 Summary

To sum up, the presented thesis dealt with the extension of multi-criteria decision analysis models for the financial selection problems (as a specific scope) and also the general selection problems (human resource and flood control project selection problems) with the inclusion of the attitudinal character, majority concept, and fuzzy set theory. In particular, the group decision making model, Dempster-Shafer belief structure, AHP, and TOPSIS are proposed to overcome the shortcoming of the existing models related to the financial decision analysis. The applicability and robustness of the developed models have been demonstrated and some sensitivity analyses are also provided. The main advantages of the proposed models are to provide a generality and flexibility of models for a wider analysis of the decision making problems.

BIBLIOGRAPHY

Aczél, J. (1948). On mean values. *Bulletin American Mathematical Society*, 54, 392-400.

Aczél, J. & Saaty, T.L. (1983). Procedures for synthesizing ratio judgments. *Journal of Mathematical Psychology*, 27(1), 93-102.

Afful-Dadzie, A., Afful-Dadzie, E., & Turkson, C. (2015). A TOPSIS extension framework for re-conceptualizing sustainability measurement. *Kybernetes*, 45, 70-86.

Ahn, B.S. (2011). Compatible weighting method with rank order centroid: Maximum entropy ordered weighted averaging approach. *European Journal of Operational Research*, 212(3), 552-559.

Aliev, R.A., Fazlollahi, B., & Aliev, R.R. (2004). *Soft Computing and its Applications in Business and Economics*. Berlin, Heidelberg: Springer-Verlag.

Arrow, K.J. (1950). A difficulty in the concept of social welfare. *Journal of Political Economy*, 58(4), 328-346.

Atanassov, K. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 110, 87-96.

Baddeley, M. & Parkinson, S. (2012). Group decision-making: an economic analysis of social influence and individual difference in experimental juries. *Journal of Socio-Economics*, 41(5), 558-573.

Bana e Costa, C.A. & Vansnick, J.C. (1994). MACBETH – An interactive path towards the construction of cardinal value functions. *International Transactions in Operational Research*, 1(4), 489-500.

Bana e Costa, C.A. & Soares, J.O. (2004). A multi-criteria model for portfolio management. *European Journal of Finance*, 10(3), 198-211.

Behzadian, M., Otaghsara, S.K., Yazdani, M., & Ignatius, J. (2012). A state-of-the-art survey of TOPSIS applications. *Expert Systems with Applications*, 39, 13051-13069.

Beliakov, G. & James, S. (2011). Induced ordered weighted averaging operators. In R.R. Yager, J. Kacprzyk, & G. Beliakov (Eds.), *Recent Developments in the OWA Operators* (pp. 29-47). Berlin Heidelberg: Springer-Verlag.

Beliakov, G., Pradera, A., & Calvo, T. (2007). *Aggregation Functions: A Guide for Practitioners*. Berlin: Springer-Verlag.

Belles-Sampera, J. Merigó, J.M., Guillén, M., & Santolino, M. (2013). The connection between distortion risk measures and ordered weighted averaging operators. *Insurance: Mathematics and Economics*, 52, 411-420.

Belles-Sampera, J., Merigó, J.M., Guillén, M., & Santolino, M. (2014). Indicators for the characterization of discrete Choquet integrals. *Information Sciences*, 267, 201-216.

Bellman, R.E. & Zadeh, L.A. (1970). Decision making in a fuzzy environment. *Management Science*, 17, 141-164.

Bender, M. & Simonovic, S. P. (2000). A fuzzy compromise approach to water resource systems planning under uncertainty. *Fuzzy Sets and Systems*, 115, 35-44.

Berger, J.O. (1985). *Statistical Decision Theory and Bayesian Analysis*. New York: Springer Verlag.

Billot, A. (1992). *Economic Theory of Fuzzy Equilibria: An Axiomatic Analysis*. Berlin, Heidelberg: Springer-Verlag.

Bonissone, P.P. (1982). A fuzzy sets based linguistic approach: Theory and applications. In M.M. Gupta & E. Sanchez (Eds.), *Approximate Reasoning in Decision Analysis* (pp. 329-339). North-Holland.

Bonner, B.L., Baumann, M.R., & Dalal, R.S. (2002). The effects of member expertise on group decision making and performance. *Organizational Behaviour and Human Decision Processes*, 88, 719–36.

Bordogna, G. & Sterlacchini, S. (2014). A Multi-criteria group decision making process based on the soft fusion of coherent evaluations of spatial alternatives. In L.A. Zadeh, A.M. Abbasov, R.R. Yager, S.N. Shahbazova, M.Z. Reformat (Eds.), *Recent Developments and New Directions on Soft Computing, Studies in Fuzziness and Soft Computing (*pp. 65-79). Switzerland: Springer.

Bordogna, G., Fedrizzi, M., & Pasi, G. (1997). A linguistic modelling of consensus in group decision making based on OWA operators. *IEEE Transaction on Systems, Man Cybernetic*, *Part A: System and Human*, 27, 126-132.

Boroushaki, S. & Malczewski, J. (2010). Using the fuzzy majority approach for GIS-based multi-criteria group decision-making. *Computers and Geosciences*, 36, 302-312.

Brans, J.P. & Vincke, A. (1985). A preference ranking organization method: The PROMETHEE method for MCDM. *Management Science*, 31(6), 647-656.

Brunelli, M. (2015). *Introduction to the Analytic Hierarchy Process*. Springer.

Bullen, P.S. (2003). *Handbook of Means and their Inequalities*. Dordrecht: Kluwer Academic Publishers Group.

Bouyssou, D., Marchant, T., Pirlot, M., Perny, P., Tsoukias, A., & Vincke, P. (2000). *Evaluation and Decision Models: A Critical Perspective*. Springer.

Canfora, G. & Troiano, L. (2004). A model for opinion agreement and confidence in multi-expert multi-criteria decision making. *Mathware Soft Computing*, 11, 67-82.

Casanovas, M. & Merigó, J.M. (2012). Fuzzy aggregation operators in decision making with Dempster-Shafer belief structure. *Expert Systems with Applications,* 39, 7138-7149.

Ceballos, D. (2001). *An approach to Financial Time*. The 4th Italian-Spanish Conference on Financial Mathematics, Alghero, 112-126.

Ceballos, D. & Sorrosal, M.T. (2002). *Time Aggregation problems in Financial Time Series*. MS'2002 International Conference on Modelling and Simulation in Technical and Social Sciences*,* Girona, 243-252.

Chen, C.T. (2000). Extensions of the TOPSIS for group decision making under fuzzy environment. *Fuzzy Sets and Systems*, 114, 1-9.

Chen, C.T. & Hung, W.Z. (2009). A new decision making method for stock portfolio selection based on computing with linguistic assessment. *Journal of Applied Mathematics and Decision Sciences*, 1-20.

Chen, S.J. & Hwang, C.L. (1992). *Fuzzy Multiple Attribute Decision Making*. Lecture notes in Economics and Mathematical Systems. Berlin: Springer.

Chen, Y., Li, K.W., & Liu, S.F. (2011). An OWA-TOPSIS method for multiple criteria decision analysis. *Expert Systems with Applications*, 38, 5205-5211.

Chen, T., Jin, Y., Qiu, X., & Chen, X. (2014). A hybrid fuzzy evaluation method for safety assessment of food-waste feed based on entropy and the analytic hierarchy process methods. *Expert Systems with Applications*, 41(16), 7328-7337.

Chiclana, F., Herrera, F., & Herrera-Viedma, E. (2004). Rationality of induced ordered weighted averaging operators based on the reliability of the source of information in group decision making. *Kybernetika*, 40,121-142.
Chiclana, F., Herrera-Viedma, E., Herrera, F., & Alonso, S. (2004). Induced ordered weighted geometric operators and their use in the aggregation of multiplicative preference relations. *International Journal of Intelligent Systems*, 19(3), 233-255.
Chiclana, F., Herrera-Viedma, E., Herrera, F., & Alonso, S. (2007). Some induced ordered weighted averaging operators and their use for solving group decision making problems based on fuzzy preference relations. *European Journal of Operational Research*, 182, 383–399.
Choquet, G. (1953). Theory of Capacities. *Annales de l'Institut Fourier*, 131-295.
Chuu, S.J. (2009). Group decision making model using fuzzy multiple attributes analysis for the evaluation of advanced manufacturing technology. *Fuzzy Sets and Systems*, 160(5), 586-602.
de Finetti, B. (1937). Foresight: its logical laws, its subjective sources. In H.E. Kyburg & H.E. Smokler (Eds.), *Studies in Subjective Probability* (pp. 55-118). New York: Krieger.
Delgado, M., Verdegay, J.L., & Vila, A. (1993). On aggregation operational of linguistic labels. *International Journal of Intelligent Systems*, 8, 351-370.
Dempster, A.P. (1967). Upper and lower probabilities induced by a multi-valued mapping. *Annals of Mathematical Statistics*, 38, 325-339.
Deng, H., Yeh, C.H., & Willis, R.J. (2000). Inter-company comparison using modified TOPSIS with objective weights. *Computers and Operations Research*, 27(10), 963-973.
Dorigo, M. (1992). *Optimization, Learning and Natural Algorithms*. PhD Thesis, Politecnico di Milano, Italy.
Dubois, D. and Prade, H. (1988). *Fuzzy Sets and Systems*. Academic Press, New York.
Dubois, D. and Prade, H. (2000). *Fundamental of Fuzzy Sets*, New York: Springer.
Dyer, J.S. (2005). Multi-attribute utility theory. In J.S. Figueira, S. Greco, M. Ehrgott (Eds.). *Multiple Criteria Decision Analysis: State of the Art Surveys*. New York: Springer.
Dymowa, L. (2011). *Soft Computing in Economics and Finance*. Berlin Heidelberg: Springer-Verlag.
Engemann, K.J., Miller, H.E. & Yager, R.R. (1996). Decision making with belief structures: An application in risk management. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 4, 1-26.
Engemann, K.J., Filev, D.P. & Yager, R.R. (1996). Modelling decision making using immediate probabilities. *International Journal of General Systems*, 24, 281-284.
Escobar, M.T. & Moreno-Jimenez, J.M. (2007). Aggregation of individual preference structures in AHP-group decision making. *Group Decision and Negotiation*, 16(4), 287-301.
Figueira, J.R., Greco, S., & Ehrgott, M. (2005). *Multiple Criteria Decision Analysis: State of the Art Surveys*. New York: Springer.
Filev, D.P. & Yager, R.R. (1995). Analytic properties of maximum entropy OWA operators. *Information Sciences*, 85(1-3) 11-27.
Fishburn, P.C. (1970). *Utility Theory for Decision Making*. John Wiley.
Fishburn, P.C. (1972). *Mathematics of Decision Theory*. Mouton.
Fuller, R. & Majlender, P. (2001). An analytic approach for obtaining maximal entropy OWA operator weights. *Fuzzy Sets and Systems*, 124(1), 53–57.

Fuller, R. & Majlender, P. (2003). On obtaining minimal variability OWA operator weights. *Fuzzy Sets and Systems*, 136, 203–215.
Forman, E.H. & Gass, S.I. (2001). The analytic hierarchy process - An exposition. *Operational Research*, 49(4), 469-486.
Forman, E.H. & Peniwati, K. (1998). Aggregation individual judgments and priorities with the analytic hierarchy process. *European Journal of Operational Research*, 108(1), 165-169.
Focardi, S.M. & Fabozzi, F.J. (2004). *The Mathematics of Financial Modeling and Investment Management*. John Wiley.
Forsyth, D.R. (2006). *Group Dynamics* (4th ed.). Belmont, CA.: Thomson Higher Education.
French, S. (1985). *Decision Theory: An Introduction to the Mathematics of Rationality*. Ellis Horwood Limited.
Gal, T., Steward, T., & Hanne, T. (1999). *Multi-criteria Decision Making: Advances in MCDM Models, Algorithms, Theory and Applications*. Dordrecht: Springer.
Gargallo, P., Moreno-Jimenez, J.M., & Salvador, M. (2007). AHP-group decision making: A Bayesian approach based on mixtures for group pattern identification. *Group Decision and Negotiation*, 16(6), 485-506.
Gau, W.L. & Buehrer, D.J. (1993). Vague sets. *IEEE Transaction on Systems, Man, Cybernetics*, 23(2), 610-614.
Gil-Lafuente, A.M. (2005). *Fuzzy Logic in Financial Analysis*. Berlin, Heidelberg: Springer-Verlag.
Golden, B.L., Wasil, E.A., & Harker, P.T. (1989). *The Analytic Hierarchy Process: Application and Studies*. New York: Springer-Verlag.
Gosling, S. (2010). A scenarios approach to asset allocation. *Journal of Portfolio Management*, 37, 53-66.
Grabisch, M. (1996). The application of fuzzy integrals in multicriteria decision making. *European Journal of Operational Research*, 89, 445-456.
Grabisch, M., Nguyen, H.T., & Walker, E.A. (1995). *Fundamentals of Uncertainty Calculi with Applications to Fuzzy Inference*. Springer.
Grabisch, M., Marichal, J.L., Mesiar, R. & Pap, E. (2009). *Aggregation Functions*. Cambridge: Cambridge University Press.
Grinold, R.C. (1999). Mean-variance and scenario-based approaches to portfolio selection. *Journal of Portfolio Management*, 25, 10-22.
Hajimirsadeghi, H. & Lucas, C. (2009). Extended TOPSIS for multi-criteria group decision making with linguistic quantification and concept of majority opinion. *Final project for Fuzzy Systems course*, School of Electrical and Computer Engineering, University of Tehran, Iran.
Herrera, F. & Herrera-Viedma, E. (2000). Linguistic decision analysis: steps for solving decision problems under linguistic information. *Fuzzy Sets and Systems*, 115, 67-82.
Herrera, F. & Martínez, L. (2000a). A 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Transactions on Fuzzy Systems*, 8, 746-752.
Herrera, F. & Martínez, L. (2000b). An approach for combining numerical and linguistic information based on the 2-tuple fuzzy linguistic representation model in decision making. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 8, 539-562.

Herrera, F., Herrera-Viedma, E., & Verdegay, J.L. (1995). A sequential selection process in group decision making with a linguistic assessment approach. *Information Sciences*, 85, 223-239.
Herrera, F., Herrera-Viedma, E., & Martínez, L. (2000) A fusion approach for managing multi-granularity linguistic term sets in decision making. *Fuzzy Sets and Systems*, 114, 43-58.
Herrera, F., Herrera-Viedma, E., & Chiclana, F. (2003). A study of the origin and uses of the ordered weighted geometric operator in multicriteria decision making. *International Journal of Intelligent Systems*, 18, 689-707.
Herrera, F., Herrera-Viedma, E., & Martínez, L. (2008). A fuzzy linguistic methodology to deal with unbalanced linguistic term sets. *IEEE Transactions on Fuzzy Systems*, 16, 354-370.
Herrera-Viedma, E. & López-Herrera, A.G. (2007). A model of an information retrieval system with unbalanced fuzzy linguistic information. *International Journal of Intelligent Systems*, 22(11), 1197–1214.
Howson, C. & Urbach, P. (2006). *Scientific Reasoning: The Bayesian Approach*. Open Court Publisher.
Huang, C.H., Padmanabhan, P., & Wenqing, Z. (2013). Mitigating the impact of managerial anchoring: The case for the management by committee for major corporate financial decisions. *Multinational Finance Journal*, 17, 341-369.
Hurson, C., Mastorakis, K., & Siskos, Y. (2012). Application of a synergy of MACBETH and MAUT multi-criteria methods to portfolio selection in Athens stock exchange. *International Journal of Multi-criteria Decision Making*, 2(2), 113-127.
Hurwicz, L. (1951). Some specification problems and applications to econometric methods. *Econometrica*, 19, 343-344.
Hwang, C.L. & Masud, A.B. (1979). *Multiple objective decision making, methods and applications: A state-of-the-art survey*. University of California: Springer-Verlag.
Hwang, C.L. & Yoon, K. (1981). *Multiple Attribute Decision Making*. Berlin: Springer-Verlag.
Hwang, C.L. & Lin, M. (1987). *Group Decision-Making Under Multiple Criteria: Methods and Applications*. Springer-Verlag.
Islam, M.S., Sadiq, R., Rodriguez, M.J. Najjaran, H., Francisque, A., & Hoorfar, M. (2013). Evaluating water quality failure potential in water distribution systems: A fuzzy -TOPSIS-OWA-based methodology. *Water Resources Management*, 27, 2195-2216.
Jackson, H.V.J. (1999). A Structured Approach for Classifying and Prioritizing Product Requirements. Ph.D Thesis, North Carolina State University.
Jacquet-Lagréze, E. & Siskos, Y. (1982). Assessing a set of additive utility functions for multi-criteria decision making, the UTA method. *European Journal of Operational Research*, 10, 151-164.
Jin, F., Liu, P.D., & Zhang, X. (2013). The multi-attribute group decision making method based on the interval grey linguistic variables weighted harmonic aggregation operators. *Technological and Economic Development of Economy*, 19, 409-430.
Kacprzyk, J. (1986). Group decision making with a fuzzy linguistic majority. *Fuzzy Sets and Systems*, 18, 105-118.

Kacprzyk, J., Fedrizzi, M., & Nurmi, H. (1992). Group decision making and consensus under fuzzy preferences and fuzzy majority. *Fuzzy Sets and Systems*, 49, 21-31.
Kacprzyk, J., Zadrozny, S., & Nurmi, H. (2011). The role of the OWA operators as a unification tool for the representation of collective choice sets. In R.R. Yager, J. Kacprzyk, G. Beliakov (Eds.), *Recent Developments in the OWA Operators, Studies in Fuzziness and Soft Computing* (pp.149-166). Switzerland: Springer.
Kahraman, C. (2008). *Fuzzy Multi-Criteria Decision Making: Theory and Applications with Recent Developments*. US: Springer-Verlag.
Kaufmann, A. & Gupta, M.M. (1991). *Introduction to Fuzzy Arithmetic: Theory and Applications*. New York: Van Nostrand Reinhold.
Karanik, M., Peláez, J.I., & Bernal, R. (2016). Selective majority additive ordered weighted averaging operator. *European Journal of Operational Research*, 250, 816-826.
Keeney, R.L. & Raiffa, H. (1976). *Decisions with Multiple Objectives: Preferences and Value Trade-offs*. John Wiley.
Kennedy, J. & Eberhart, R. (1995). Particle Swarm Optimization. *Proceedings of IEEE International Conference on Neural Networks* IV, 1942–1948.
Klir, G.J. & Folder, T.A. (1988). *Fuzzy Sets, Uncertainty and Information*. Prentice Hall, Englewood Cliffs.
La Red, D.L., Doña, J.M., Peláez, J.I., & Fernandez, E.B. (2011). WKC-OWA – A new neat OWA operator to aggregate information in democratic decision problems. *International Journal of Uncertainty and Fuzziness*, 19, 759-779.
Laengle, S., Loyola, G., & Merigó, J.M. (2016). Mean-variance portfolio selection with the ordered weighted average. *IEEE Transactions on Fuzzy Systems*, 99.
Langroudi, M.Z.A., Mustafa, A., Ignatius, J., & Emrouznejad, A. (2013). Type-2 TOPSIS: a group decision problem when ideal values are not extreme points. *Group Decision and Negotiation*, 22(5), 851-866.
Le, C.A., Huynh, V.N., Shimazu, A., & Nakamori, Y. (2007). Combining classifiers for word sense disambiguation based on Dempster-Shafer theory and OWA operators. *Data and Knowledge Engineering*, 63, 381-396.
Li, L., Liu, F., & Li, C. (2014). Customer satisfaction evaluation method for customized product development using entropy weight and analytic hierarchy process. *Computer and Industrial Engineering*, 77, 80-87.
Li, X., Ruan, D., Liu, J., & Xu, Y. (2008). A linguistic-valued weighted aggregation operator to multiple attribute decision making with quantitative and qualitative information. *International Journal of Computational Intelligence Systems*, 1, 274-284.
Liu, H. & Kong, F. (2005). A new MADM algorithm based on fuzzy subjective and objective integrated weights. *International Journal of Information and System Sciences*, 3-4(1), 420-427.
Liu, H.W. & Wang, G.J. (2007). Multi-criteria decision making based on intuitionistic fuzzy sets. *European Journal of Operational Research*, 179, 220-233.
Liu, F., Shang, Y.F., & Pan, L.H. (2015). A modified TOPSIS method for obtaining the associated weights of the OWA-type operators. *International Journal of Intelligent Systems*, 30, 1101-1116.
Llamazares, B. (2013). An analysis of some functions that generalizes weighted means and OWA operators. *International Journal of Intelligent Systems*, 28(4), 380-393.
Luce, R. & Raiffa, H. (1957). *Games and Decisions*. John Wiley.

Ma, F.M. & Guo, Y.J. (2013). Determination of attitudinal character by self-evaluation for the maximum entropy OWA approach. *International Journal of Intelligent Systems*, 28(11), 1089-1098.

Martel, J.M., Khoury, N.T., & Bergeron, M. (1988). An application of a multi-criteria approach to portfolio comparisons. *Journal of the Operational Research Society*, 39(7), 617–628.

Munda, G. (1996). Cost-benefit analysis in integrated environmental assessment: some methodological issues. *Ecological Economics*, 19(2), 157-168.

Majlender, P. (2005). OWA operators with maximal Rényi entropy. *Fuzzy Sets and Systems*, 155(3), 340-360.

Mansini, R., Ogryczak, W. & Speranza, M.G. (2015). *Linear and mixed integer programming for portfolio optimization*. EURO Advanced Tutorials on Operational Research. Switzerland: Springer.

Maragoudaki, R. & Tsakiris, G. (2005). Flood Mitigation Planning using PROMETHEE. *European Water*, 9, 51–58.

Markowitz, H.M. (1952). Portfolio Selection. *Journal of Finance*. 7(1), 77–91.

Martinez, L. & Herrera, F. (2012). An overview on the 2-tuple linguistic model for computing with words in decision making: Extensions, applications and challenges. *Information Sciences*, 207, 1-18.

Melanie, M. (1996). *An Introduction to Genetic Algorithms*. Cambridge, MA: MIT Press.

Merigó, J.M. (2012). OWA operators in the weighted average and their application in decision making. *Control and Cybernetics*, 41(3), 605-643.

Merigó, J.M. & Casanovas, M. (2009). Induced aggregation operators in decision making with the Dempster-Shafer belief structure. *International Journal of Intelligent Systems*, 24(8), 934-954.

Merigó, J.M. & Casanovas, M. (2010). The fuzzy generalized OWA operator and its application in strategic decision making. *Cybernetics and Systems*, 41, 359-370.

Merigó, J.M. & Casanovas, M. (2011a). Induced aggregation operators in the Euclidean distance and its application in financial decision making. *Expert Systems with Applications*, 38(6), 7603-7608.

Merigó, J.M. & Casanovas, M. (2011b). Induced and uncertain heavy OWA operators. *Computer and Industrial Engineering*, 60, 106-116.

Merigó, J.M. & Casanovas, M. (2011c). A new Minkowski distance based on induced aggregation operators. *International Journal of Computational Intelligence Systems*, 4, 123-133.

Merigó, J.M. & Casanovas, M. (2011d). Decision making with distance measures and induced aggregation operators. *Computers and Industrial Engineering*, 60, 66-76.

Merigó, J.M. & Casanovas, M. (2012). Decision making with uncertain aggregation operators and Dempster-Shafer belief structure. *International Journal of Innovative Computing, Information and Control*, 8, 1037-1062.

Merigó, J.M. & Gil-Lafuente, A.M. (2008). Using OWA operator in the Minkowski distance. *International Journal Social, Humanities and Science*, 2, 564-572.

Merigó, J.M. & Gil-Lafuente, A.M. (2009). The induced generalized OWA operator. *Information Sciences*, 179, 729-741.

Merigó, J.M. & Gil-Lafuente, A.M. (2010). New decision-making techniques and their application in the selection of financial products. *Information Sciences*, 180, 2085-2094.

Merigó, J.M. & Gil-Lafuente, A.M. (2013a). Induced 2-tuple linguistic generalized aggregation operators and their application in decision-making, *Information Sciences*, 236, 1-16.

Merigó, J.M. & Gil-Lafuente, A.M. (2013b). A general method for decision making based on generalized OWA operators. *International Journal of Intelligent Systems*, 28, 453-473.

Merigó, J.M. & Yager, R.R. (2013). Generalized moving averages, distance measures and OWA operators. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 21, 533-559.

Merigó, J.M., Casanovas, M., & Martínez, L. (2010). Linguistic aggregation operators for linguistic decision making based on the Dempster-Shafer theory of evidence. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 18, 287-304.

Merigó, J.M., Engemann, K.J., & Palacios-Marqués, D. (2013). Decision making with Dempster-Shafer theory and the OWAWA operator. *Technological and Economic Development of Economy*, 19, S194-S212.

Merigó, J.M., Casanovas, M., & Zeng, S.Z. (2014). Distance measures with heavy aggregation operators. *Applied Mathematical Modelling*, 38(13), 3142-3153.

Merigó, J.M., Casanovas, M., & Liu, P. (2014). Decision making with fuzzy induced heavy ordered weighted averaging operators. *International Journal of Fuzzy Systems*, 16(3), 277-289.

Merigó, J.M., Gil-Lafuente, A.M., Zhou, L.G., & Chen, H.Y. (2012). Induced and linguistic generalized aggregation operators and their application in linguistic group decision making. *Group Decision and Negotiation*, 21, 531-549.

Mottola, G.R. & Utkus, S.P. (2009). Group decision making: Implications for investment committees. *Vanguard Investment Counseling & Research*. *The Vanguard Group*.

Morris-Oswald, T. (2001). *Social criteria for multi-criteria decision analysis in flood plain management*. Master Thesis, University of Manitoba, Canada

O'Hagan, M. (1988). Aggregating template or rule antecedents in real-time expert systems with fuzzy set logic. *IEEE Asilomar Conference on Signals, Systems and Computers*, 2, 681-689.

Olcer, A.I. & Odabasi, A.Y. (2005). A new fuzzy multiple attributive group decision making methodology and its application to propulsion/maneuvering system selection problem. *European Journal of Operational Research*, 166, 93-114.

Olson, D.L. (2004). Comparison of weights in TOPSIS models. *Mathematical and Computer Modelling,* 40, 721-727.

Pasi, G. & Yager, R.R. (2006). Modelling the concept of majority opinion in group decision making. *Information Sciences*, 176, 390-414.

Peláez, J.I. & Doña, J.M. (2003a). LAMA: A linguistic aggregation majority additive operator. *International Journal of Intelligent Systems*, 18 809-820.

Peláez, J.I. & Doña, J.M. (2003b). Majority additive-ordered weighted averaging: A new neat ordered weighted averaging operator based on the majority process. *International Journal of Intelligent Systems*, 18, 469-481.

Peláez, J.I., Doña, J.M., & Mesas, A. (2005). Majority multiplicative ordered weighted geometric operators and their use in the aggregation of multiplicative preference relations. *Mathware and Soft Computing*, 12, 107-120.

Peláez, J.I., Doña, J.M., & Gomez-Ruiz, J.A. (2007) Analysis of OWA operators in decision making for modelling the majority concept. *Applied Mathematics and Computation*, 186, 1236-1275.

Peláez, J.I., Bernal, R., & Karanik, M. (2016). Majority OWA operator for opinion rating in social media. *Soft Computing*, 20(3) 1047-1055.

Peterson, M. (2009). *An Introduction to Decision Theory*. Cambridge University Press.

Pomerol, J.C. & Barba-Romero, S. (2000). *Multi-criterion decision in management: principle and practice*. Netherlands: Kluwer Academic Publisher Group.

Ramanathan, R. & Ganesh, L.S. (1994). Group preference aggregation methods employed in AHP: An evaluation and intrinsic process for deriving members' weights. *European Journal of Operational Research*, 79(2), 249-265.

Ramsey, F.P. (1931). *Truth and Probability*. London: John Wiley & Sons.

Ramírez, D. (1988). Methodological foundations for economic analysis in the context of uncertainty. PhD Tesis, Universitat de Barcelona (In Spanish).

Ramírez, D., Ceballos, D., Cortez, K., & Rodríguez, M. (2008). *Financial crisis: Theory and Practice*. Book based on Workshop IAFI at Financial Crisis, University of Barcelona.

Reformat, M. & Yager, R.R. (2008). Building ensemble classifiers using belief functions and OWA operators. *Soft Computing*, 12, 543-558.

Rényi, A. (1961). On measures of entropy and information. *Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability*, 547-561.

Robert, F.S. (1985). *Measurement Theory with Applications to Decision-making, Utility and Social Sciences*. Cambridge University Press.

Roy, B. (1991). The outranking approach and the foundations of ELECTRE methods. *Theory and Decision*, 31, 49-73.

Saaty, T.L. (1977). A scaling method for priorities in hierarchical structures. *Journal of Mathematical Psychology*, 15(3), 234-281.

Saaty, T.L. (1980). *The Analytic Hierarchy Process*. New York: McGraw Hill.

Saaty, T.L. & Vargas, L.G. (1982). The logic of Priorities: Applications in Business, Energy, Health, and Transportation. Netherlands: Springer.

Saaty, T.L. (1989). Group decision making and the AHP. In B.L Golden, E.A. Wasil, P.T. Harker (Eds.), *The Analytic Hierarchy Process: Application and Studies* (pp. 59-67). New York: Springer-Verlag.

Saaty, T.L. (2013). The modern science of multi-criteria decision making and its practical applications: The AHP/ANP approach. *Operational Research*, 61(5), 1101-1118.

Savage, L.J. (1954). *The Foundations of Statistics*. New York: Wiley.

Savchuk, V. & Tsokos, C.P. (2011). *Bayesian Theory and Methods with Applications*. Atlantis Press.

Shafer, G. (1976). *Mathematical Theory of Evidence*. Princeton, N.J.: Princeton University Press.

Shannon, C.E. (1949). Communication theory of secrecy systems. *Bell System Technical Journal*, 128(4), 656–715.

Shih, H.S., Shyur, H.J., & Lee, E.S. (2007). An extension of TOPSIS for group decision making. *Mathematical and Computer Modelling*, 45, 801-813.

Shipley, M.F., Korvin D.K., & Obit, R. (1991). A decision making model for multi-attribute problems incorporating uncertainty and bias measures. *Computer and Operational Research*, 18, 335–342.

Smarandache, F. (1999). *A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability*. Rehoboth, NM: American Research Press.

Spronk, J. & Hallerbach, W. (1997). Financial modelling: Where to go? With an illustration for portfolio management. *European Journal of Operational Research*, 99, 113-125.

Spronk, J., Steuer, R.E., & Zopounidis, C. (2005). Multi-criteria decision aid/analysis in finance. In J.S. Figueira, S. Greco, M. Ehrgott (Eds.). *Multiple Criteria Decision Analysis: State of the Art Surveys*. New York: Springer.

Srivastava, R.P., & Mock, T. (2002). *Belief Functions in Business Decisions*. Physica-Verlag, Heidelberg.

Sugeno, M. (1977). Fuzzy measures and fuzzy integral: a survey. In M.M. Gupta, Saridis, G.N., Gaines, B.R. (Eds.), *Fuzzy Automata and Decision Process* (pp. 89-102). North-Holland, Amsterdam.

Taib, C.M.I.C., Naim, N.S.M., & Abdullah, M.L. (2008). New condition for conflicting bifuzzy sets based on intuitionistic evaluation. *International Journal Computational Mathematics and Sciences*, 2(4), 161-165.

Taib, C.M.I.C., Yusoff, B., Abdullah, M.L., & Wahab, A.F. (2016). Conflicting bifuzzy multi-attribute group decision making model with application to flood control project. *Group Decision and Negotiation*, 25, 157-180.

Tap A.O.M. (2006). Conflicting bifuzzy evaluation. *Proceedings of Computer Science and Mathematics Symposium, KUSTEM, Terengganu, Malaysia.*

Tavana, M., Keramatpour, M., Santos-Arteaga, F.J., & Ghorbaniane, E. (2015). A fuzzy hybrid project portfolio selection method using data envelopment analysis, TOPSIS and integer programming. *Expert Systems With Applications*, 42, 8432–8444.

Terceño, A., Andrés, J.D., Barberà, G., & Lorenzana, T. (2003). Using fuzzy set theory in the choice of capital investment. *International Journal of Uncertainty, Fuzziness and Knowledge-based System*, 11(3), 263-281.

Tiryaki, F. & Ahlatcioglu, M. (2005). Fuzzy stock selection using a new fuzzy ranking and weighting algorithm. *Applied Mathematics and Computation*, 170, 144–157.

Torra, V. (1997). The weighted OWA operator. *International Journal of Intelligent Systems*, 12, 153-166.

Torra, V. & Narukawa, Y. (2007). *Modelling Decisions: Information Fusion and Aggregation Operators*. Berlin: Springer-Verlag.

Tsiporkova, E. & Boeva, V. (2006). Multi-step ranking of alternatives in multi-criteria and multi-expert decision making environment. *Information Sciences*, 176, 2673-2697.

Vaidya, O.S. & Kumar, S. (2006). Analytic hierarchy process: An overview of applications. *European Journal of Operational Research*, 169(1), 1-29.

Vetschera, R. & de Almeida, A.T. (2012). A PROMETHEE-based approach to portfolio selection problems. *Computers & Operations Research*, 39(5), 1010-1020.

Vigiera, H.P., Schergerb, V., & Terceño, A. (2016). An application of OWA operators in fuzzy business diagnosis. *Applied Soft Computing* (In press).

von Neumann, J. & Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. New York: John Wiley & Sons, Inc.

Wald, A. (1950). *Statistical Decision Functions*. New York: John Wiley & Sons, Inc.

Wang, J.H. & Hao, J. (2006). A new version of 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Transactions on Fuzzy Systems*, 14, 435-445.

Wang, T.C. & Lee, H.D. (2009). Developing a fuzzy TOPSIS approach based on subjective weights and objective weights. *Expert Systems with Applications,* 36, 8980-8985.

Wang, S.Y. & Lee, C.F. (2010). Application of fuzzy set theory to finance research: Method and application. In C.F. Lee & J. Lee (Eds.), *Handbook of Quantitative Finance and Risk Management* (pp. 1183-1199). US: Springer.

Wang, Y.M. & Parkan, C. (2007). A preemptive goal programming method for aggregating OWA operators weights in group decision making. *Information Sciences*, 177(8), 1867-1877.

Wang, Z.F., Hong, W.M., Xun, J.Z., & Yan, B. (2008). Improved multi-attribute fuzzy comprehensive evaluation in project delivery decision making. *4th International Conference on Wireless Communications, Networking & Mobile Computing*, 1-5.

Wei. G.W. (2009). Uncertain linguistic hybrid geometric mean operator and its application to group decision making under uncertain linguistic environment. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 17, 251-267.

Wei, G.W. (2011). Some generalized aggregating operators with linguistic information and their application to multiple attribute group decision making. *Computers and Industrial Engineering,* 61, 32-38.

Wei, G.W., Zhao, X., & Lin, R. (2013). Some hybrid aggregating operators in linguistic decision making with Dempster-Shafer belief structure. *Computers and Industrial Engineering*, 65, 646-651.

Wierzbicki, A.P. (1980). The use of reference objectives in multi-objective optimization. In G. Fandel & T. Gal (Eds.) *Multiple criteria decision making theory and applications* (pp 468-486). Berlin: Springer.

Wierzbicki, A.P., Makowski, M., & Wessels, J. (2000). *Model-based decision support methodology with environmental applications*. Dordrecht: Kluwer.

Xu, Z.S. (2004). A method based on linguistic aggregation operators for group decision making with linguistic preference relations. *Information Sciences*, 166, 19-30.

Xu, Z.S. (2004). EOWA and EOWG operators for aggregating linguistic labels based on linguistic preference relations. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 12, 791-810.

Xu, Z.S. (2005). An overview of methods for determining OWA weights. *International Journal of Intelligent Systems*, 20(8), 843-865.

Xu, Z.S. (2005). An overview of methods for determining OWA weights. *International Journal of Intelligent Systems*, 20, 843-865.

Xu, Z.S. (2006). Induced uncertain linguistic OWA operators applied to group decision making. *Information Fusion*, 7(2) 231-238.

Xu, Z.S. (2007). A survey of preference relations. *International Journal of General Systems*, 36(2), 179-203.

Xu, Z.S. (2007). Linguistic aggregation operators: An overview. In H. Bustince, F. Herrera, & Montero, J. (Eds.), *Fuzzy Sets and Their Extensions: Representation, Aggregation and Models* (pp. 163-181). Berline: Springer.

Xu, Z.S. (2008). Group decision making based on multiple types of linguistic preference relations. *Information Sciences*, 178, 452-467.

Xu, Z.S. (2012). *Linguistic Decision Making: Theory and Methods*. Berlin: Springer.

Xu, Z.S. & Da, Q.L. (2002). The ordered weighted geometric averaging operators. *International Journal Intelligent Systems*, 17, 709-716.

Xu, Z.S. & Da, Q.L. (2003). An overview of operators for aggregating information. *International Journal of Intelligent Systems*, 18, 953-969.

Xu, Z.S. & Chen, J. (2008). Ordered weighted distance measure. *Journal of Systems Science and Systems Engineering*, 17, 432-445.

Xu, Z.S. & Yager, R.R. (2008). Dynamic intuitionistic fuzzy multi-attribute decision making. *International Journal of Approximate Reasoning*, 1, 246-262.

Xu, J. & Yan, F. (2011). A multi-objective decision making model for the vendor selection problem in a bifuzzy environment. *Expert Systems with Application*, 38, 9684-9695.

Xu, Y.J. & Wang, H. (2011). Approaches based on 2-tuple linguistic power aggregation operators for multiple attribute group decision making under linguistic environment. *Applied Soft Computing*, 11, 3988-3997.

Xu, Y.J., Shi, P., Merigó, J.M., & Wang, H.M. (2013). Some proportional 2-tuple geometric aggregation operators for linguistic decision making. *Journal of Intelligent and Fuzzy Systems,* 25, 833-843.

Xu, Y.J., Ma, F, Tao, F.F., & Wang, H.M. (2014). Some methods to deal with unacceptable incomplete 2-tuple fuzzy linguistic preference relations in group decision making. *Knowledge-Based Systems*, 56, 179-190.

Yager, R.R. (1978). Fuzzy decision making including unequal objectives. *Fuzzy Sets and Systems*, 1, 87-89.

Yager, R.R. (1987). A note on weighted queries in information retrieval systems. *Journal of the American Society for Information Science*, 38, 23-24.

Yager, R.R. (1988). On ordered weighted averaging aggregation operators in multi-criteria decision making. *IEEE Transactions on Systems and Man Cybernetics,* 18, 183-190.

Yager, R.R. (1992). Decision making under Dempster-Shafer uncertainties. *International Journal of General Systems*, 20, 233-245.

Yager, R.R. (1993). Families of OWA operators. *Fuzzy Sets and Systems*, 59(2), 125-148.

Yager, R.R. (1996). Quantifier guided aggregation using OWA operators. *International Journal Intelligent Systems*, 11, 49-73.

Yager, R.R. (1998). Including importances in OWA aaggregations using fuzzy systems modelling. *IEEE Transactions on Fuzzy Systems*, 6(2), 286-294.

Yager, R.R. (2002). Heavy OWA operators. *Fuzzy optimization and Decision Making*, 1, 379-397.

Yager. R.R. (2003a). Monitored heavy fuzzy measures and their role in decision making under uncertainty. *Fuzzy Sets and Systems*, 139, 491-513.

Yager, R.R. (2003b). Induced aggregation operators. *Fuzzy Sets and Systems*, 137, 59-69.

Yager, R.R. (2004a). Generalized OWA aggregation operators. *Fuzzy Optimization and Decision Making*, 3(1) 93-107.

Yager, R.R. (2004b). A framework for multi-source data fusion. *Information Sciences*, 163, 175-200.
Yager, R.R. (2004c). Choquet aggregation using order inducing variables. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 12, 69-88.
Yager, R.R. (2007). Centered OWA operators. *Soft Computing*, 11, 631-639.
Yager, R.R. (2009). Weighted maximum entropy OWA aggregation with applications to decision making under risk. *IEEE Transactions on Systems, Man and Cybernetics-Part A*, 39(3), 555-564.
Yager, R.R. (2010). Norms induced from OWA operators. *IEEE Transactions on Fuzzy Systems*, 18, 57-66.
Yager, R.R. & Kacprzyk, J. (1997). *The Ordered Weighted Averaging Operators: Theory and Applications*. US: Kluwer, Norwell, MA.
Yager, R.R. & Filev, D.P. (1999). Induced ordered weighted averaging operators. *IEEE Transactions on Systems and Man Cybernetics. - Part B, Cybernetics*, 29, 141-150.
Yager, R.R. & Kelman, A. (1999). An extension of the analytical hierarchy process using OWA operators. *Journal of Intelligent and Fuzzy Systems*, 7(4), 401-417.
Yager, R.R. and Liu, L. (2008). *Classic Works of the Dempster-Shafer Theory of Belief Functions*. Berline: Springer-Verlag.
Yager, R.R. & Alajlan, N. (2015). Dempster-Shafer belief structures for decision making under uncertainty. *Knowledge-Based Systems*, 80, 58-66.
Yager, R.R., Engemann, K.J., & Filev, D.P. (1995). On the concept of immediate probabilities. *International Journal of Intelligent Systems*, 10, 373-397.
Yager, R.R., Kacprzyk, J., & Beliakov, G. (2011). *Recent Developments on the Ordered Weighted Averaging Operators: Theory and practice*. Berlin: Springer-Verlag.
Ye, J. (2013). Multiple attribute group decision-making methods with completely unknown weights in intuitionistic fuzzy setting and interval-valued intuitionistic fuzzy setting. *Group Decision and Negotiation*, 22(2), 173–188.
Yusoff, B. & Merigó, J.M. (2014). Analytic hierarchy process under group decision making with some induced aggregation operators. In A. Laurent, O. Strauss, B. Bouchon-Meunier, R.R. Yager (Eds.), *Information Processing and Management of Uncertainty in Knowledge-Based Systems* (Part I, CCIS 442, pp. 476-485). Switzerland: Springer.
Yusoff, B. & Merigó, J.M. (2015). Heavy weighted geometric aggregation operators in analytic hierarchy process-group decision making. *Proceedings of International Conference on Fuzzy Systems, Istanbul, Turkey.*
Zadeh, L.A. (1965). Fuzzy sets. *Information Control*, 8, 338–353.
Zadeh, L.A. (1975). The concept of a linguistic variable and its application to approximate reasoning. *Information Sciences* – (Part 1), 8, 199-249; (Part 2), 8, 301-357; (Part 3), 9, 43-80.
Zadeh, L.A. (1983). A computational approach to fuzzy quantifiers in natural languages. *Computing and Mathematics Applications*, 9, 149-184.
Zadeh, L.A. (2011). A note on Z – numbers. *Information Sciences*, 181, 2923–2932.
Zamali, T., Abdullah M.L., & Tap A.O.M. (2008). An introduction to conflicting bifuzzy set theory. *International Journal of Mathematical Statistics*, 3(8), 86-95.
Zamali, T., Abdullah, M.L., & Tap, A.O. (2010). A conflicting bifuzzy evaluation approach in multi-criteria decision making. *International Journal of Mathematical Statistics*, 6(10), 79-92.

Zeleny, M. (1982). *Multiple Criteria Decision Making*. New York: McGraw Hill.
Zeng, S.Z. & Su, W. (2012). Linguistic induced generalized aggregation distance operators and their application to decision making. *Economic Computation and Economic Cybernetics Studies and Research*, 46, 155-172.
Zeng, S.Z., Balezentis, T., & Zhang, C.H. (2012). A method based on OWA operator and distance measures for multiple attribute decision making with 2-tuple linguistic information. *Informatica*, 23, 665-681.
Zeng, S.Z. Su, W., & Le, A. (2012). Fuzzy generalized ordered weighted averaging distance operator and its application to decision making. *International Journal of Fuzzy Systems*, 14, 402-412.
Zeng, S.Z., Merigó, J.M., & Su, W.H. (2013). The uncertain probabilistic OWA distance operator and its application in group decision making. *Applied Mathematical Modelling*, 37, 6266-6275.
Zhang, H.M. (2013). Some interval-valued 2-tuple linguistic aggregation operators and application in multi-attribute group decision making. *Applied Mathematical Modelling*, 37, 4269-4282.
Zhang, N. (2013). Method for aggregating correlated interval grey linguistic variables and its application to decision making. *Technological and Economic Development of Economy*, 19, 189-202.
Zhou, L.G. & Chen, H.Y. (2013). On compatibility of uncertain multiplicative linguistic preference relations and its application to group decision making. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems,* 21, 9-28.
Zhou, S.M., Chiclana, F., John R.I., Garibaldi, J.M. (2008). Type-1 OWA operators for aggregating uncertain information with uncertain weights induced by type-2 linguistic quantifiers. *Fuzzy Sets and Systems*, 159(24), 3281-3296.
Zhou, S.M, John, R.I., Chiclana, F., & Garibaldi, J.M. (2010). On aggregating uncertain information by type-2 OWA operators for soft decision making. *International Journal of Intelligent Systems*, 25(6), 540-558.
Zopounidis, C. and Doumpos, M. (2002). Multi-criteria decision aid in financial decision making: Methodologies and literature review. *Journal of Multi-Criteria Decision Analysis*, 11, 167-186.