On the diagonals of a Rees algebra

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Als meus pares

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Introduction

The aim of this work is to study the ring-theoretic properties of the diagonals of a Rees algebra, which from a geometric point of view are the homogeneous coordinate rings of embeddings of blow-ups of projective varieties along a subvariety. First we are going to introduce the subject and the main problems. After that we shall review the known results about these problems, and finally we will give a summary of the contents and results obtained in this work.

Let A be a noetherian graded algebra generated over a field k by homogeneous elements of degree 1, that is, A has a presentation $A = k[X_1, \dots, X_n]/K = k[x_1, \dots, x_n]$, where K is a homogeneous ideal of the polynomial ring $k[X_1, \dots, X_n]$ with the usual grading. Given a homogeneous ideal I of A, let X be the projective variety obtained by blowing-up the projective scheme $Y = \text{Proj}\ (A)$ along the sheaf of ideals $\mathcal{I} = \widetilde{I}$, that is, $X = \mathcal{P}roj(\bigoplus_{n \geq 0} \mathcal{I}^n)$. For a given $c \in \mathbb{Z}$, let us denote by I_c the c-graded component of I. If I is generated by forms of degree less or equal than d, then $(I^e)_c$ corresponds to a complete linear system on X very ample for $c \geq de+1$ which embeds X in a projective space $X \cong \text{Proj}\ (k[(I^e)_c]) \subset \mathbb{P}_k^{N-1}$, with $N = \dim_k (I^e)_c$ [CH, Lemma 1.1].

Our main purpose is to study the arithmetic properties of the k-algebras $k[(I^e)_c]$, where c, e are positive integers and I is any homogeneous ideal of A. This problem was first started in the work by A. Gimigliano [Gi], A. Geramita and A. Gimigliano [GG], and A. Geramita, A. Gimigliano and B. Harbourne [GGH] who treated similar problems for the rational projective surfaces which arise as embeddings of blow-ups of a projective plane at a set of distinct points.

Let k be an algebraically closed field and $s = \binom{d+1}{2}$, $d \geq 2$. In [Gi] the particular case of the blow-up of $\mathbb{P}_k^2$ at a set of s different points $P_1, \dots, P_s$ which do not lie on a curve of degree $d-1$ and such that there is no subset of

d points on a line (if $d \geq 3$) is studied in detail. In this case, the defining ideal I of the set of points is generated by forms of degree d and the rational maps defined by the linear systems I_c give embeddings of the blow-up for $c \geq d$. In the case $c = d$ the surface obtained is called *White Surface*, and for $c = d + 1$ *Room Surface*. It is then shown that White surfaces are contained in $\mathbb{P}_k^d$ as surfaces of degree $\binom{d}{2}$ with defining ideal generated by the maximal minors of a $3 \times d$ matrix of linear forms. In particular, $k[I_d]$ is Cohen-Macaulay and it has a resolution given by the Eagon-Northcott complex [Gi, Proposition 1.1]. On the other hand, Room Surfaces are arithmetically Cohen-Macaulay [GG, Theorem B] with defining ideal generated by quadrics [GG, Theorem 1.2].

This detailed study of White and Room Surfaces is the first step to consider the following more general case. Let $P_1, \ldots, P_s$ be s distinct points in $\mathbb{P}_k^2$, with k an algebraically closed field, let I be its defining ideal and $d = \operatorname{reg}(I)$ the regularity of I. Assume that the points do not lie on a curve of degree $d-1$ and that there is no subset of d points on a line. Then the linear systems I_c give embeddings of the blowing-up of $\mathbb{P}_k^2$ at this set of points for $c \geq d$. The resultant surfaces are arithmetically Cohen-Macaulay [GG, Theorem B] and its defining ideal is defined by quadrics if $c \geq d+1$ [GG, Theorem 2.1].

Even more generally, A. Geramita, A. Gimigliano and Y. Pitteloud [GGP] consider the blow-up of $\mathbb{P}_k^n$ along an ideal of fat points, with k an algebraically closed field of characteristic zero. Given a set of points $P_1, \ldots, P_s \in \mathbb{P}_k^n$, let $\mathcal{P}_1, \ldots, \mathcal{P}_s \subset k[X_0, \ldots, X_n]$ be their defining ideals, and let us take ideals of the type $I = \mathcal{P}_1^{m_1} \cap \ldots \cap \mathcal{P}_s^{m_s}$, with $m_1, \ldots, m_s \in \mathbb{Z}_{\geq 1}$. Then one may study the projective varieties obtained by embeddings of the blow-up of $\mathbb{P}_k^n$ along $\mathcal{I}$ via the linear systems corresponding to the graded pieces of I, whenever these linear systems are very ample. Let $d = \operatorname{reg}(I)$, and let us assume that there are not d points on a line. Then the linear systems I_c are very ample for $c \geq d$, and the varieties obtained via these embeddings are projectively normal [GGP, Proposition 2.2] and arithmetically Cohen-Macaulay [GGP, Theorem 2.4].

A new point of view to treat these questions was introduced by A. Simis, N.V. Trung and G. Valla in [STV], and later followed by A. Conca, J. Herzog, N.V. Trung and G. Valla in [CHTV], to study the more general problem of the blow-up of a projective space along an arbitrary subvariety. If I is a homogeneous ideal of A, let us consider the Rees algebra $R_A(I) = \bigoplus_{n \geq 0} I^n \cong$

$A[It] \subset A[t]$ of I with the natural bigrading given by

$$R_A(I)_{(i,j)} = (I^j)_i.$$

The crucial point now is that all the algebras $k[(I^e)_c]$ are subalgebras of the Rees algebra in a natural way. To describe this relationship we need to introduce the diagonal functor.

Given positive integers c, e, the (c,e)-*diagonal* of $\mathbb{Z}^2$ is the set

$$\Delta := \{(cs, es) \mid s \in \mathbb{Z}\}.$$

For any bigraded algebra $S = \bigoplus_{(i,j)\in\mathbb{Z}^2} S_{(i,j)}$, the *diagonal subalgebra* of S along Δ is the graded algebra

$$S_\Delta := \bigoplus_{s\in\mathbb{Z}} S_{(cs,es)}.$$

Similarly we may define the diagonal of a bigraded S-module L along Δ as the graded S_Δ-module

$$L_\Delta := \bigoplus_{s\in\mathbb{Z}} L_{(cs,es)}.$$

So we have an exact functor

$$(\ )_\Delta : M^2(S) \to M^1(S_\Delta),$$

where $M^2(S)$, $M^1(S_\Delta)$ denote the categories of bigraded S-modules and graded S_Δ-modules respectively.

Now we may give a description of the rings $k[(I^e)_c]$ as diagonals of the Rees algebra in the following way: By taking Δ to be the (c,e)-diagonal of $\mathbb{Z}^2$, we have

$$R_A(I)_\Delta = \bigoplus_{s\geq 0} (I^{es})_{cs} = k[(I^e)_c].$$

This observation allows an algebraic approach to study the rings $k[(I^e)_c]$ via the diagonals of $R_A(I)$. This is the starting point in [STV] to study the case of homogeneous ideals of the polynomial ring generated by forms of the same degree, and later in [CHTV] to study arbitrary homogeneous ideals of the polynomial ring. By paraphrasing [STV]: *One is to believe that the algebraic approach via the diagonals of the Rees algebra may throw further light not only on the study of embedded rational surfaces obtained by blowing-up a set of points in $\mathbb{P}_k^2$ but also of the embedded rational n-folds obtained, more* *generally, by blowing-up $\mathbb{P}^n_k$ along some special smooth subvariety.* On the other hand, the diagonals of any standard bigraded algebra defined over a local ring have also been studied by E. Hyry [Hy] by using both an algebraic approach and a geometric approach. Finally, S.D. Cutkosky and J. Herzog [CH] have studied the diagonals of the Rees algebra of a homogeneous ideal in a general graded k-algebra.

Next we are going to expose the main results of those works.

The main contribution of A. Simis et al. [STV] to the problems considered by A. Geramita et al. is the algebraic approach via the diagonal of a bigraded algebra, a notion which generalizes the Segre product of graded algebras. Given algebraic varieties $V \subset \mathbb{P}^{n-1}_k$, $W \subset \mathbb{P}^{r-1}_k$ with homogeneous coordinate rings R_1, R_2, the image of $V \times W \subset \mathbb{P}^{n-1}_k \times \mathbb{P}^{r-1}_k$ under the Segre embedding

$$\mathbb{P}^{n-1}_k \times \mathbb{P}^{r-1}_k \hookrightarrow \mathbb{P}^{nr-1}_k$$

is a variety with homogeneous coordinate ring the Segre product of R_1 and R_2:

$$R_1 \underline{\otimes}_k R_2 = \bigoplus_{u \in \mathbb{N}} (R_1)_u \otimes_k (R_2)_u.$$

Given a standard bigraded k-algebra $R = \bigoplus_{(u,v) \in \mathbb{N}^2} R_{(u,v)}$, its diagonal R_Δ is defined as $R_\Delta = \bigoplus_{u \in \mathbb{N}} R_{(u,u)}$ (that is, the $(1,1)$-diagonal). By considering the tensor product $R = R_1 \otimes_k R_2$ bigraded by means of $R_{(u,v)} = (R_1)_u \otimes_k (R_2)_v$, we have that $R_\Delta = R_1 \underline{\otimes}_k R_2$. Classically R is taken to be the bihomogeneous coordinate ring of a projective subvariety of $\mathbb{P}^{n-1}_k \times \mathbb{P}^{r-1}_k$, and R_Δ is then the homogeneous coordinate ring of its image via the Segre embedding.

In the first section of [STV], a relation between the presentations, the dimensions and the multiplicities of a standard bigraded k-algebra R and its diagonal R_Δ is obtained. The key for proving these results is the existence of the Hilbert polynomial of a standard bigraded k-algebra and the characterization of its degree, due to D. Katz et al. [KMV] and M. Herrmann et al. [HHRT] among others. Similarly to the graded case, one may define in this case the irrelevant ideal, the irrelevant primes and the biprojective scheme associated to a standard bigraded k-algebra.

After that, it is studied the behaviour of the normality and the Cohen-Macaulay property by taking diagonals. Since there is a Reynolds operator from R to R_Δ, one immediately gets that the normality of R will be inherited by its diagonal R_Δ. With respect to the Cohen-Macaulayness, the strategy is to reduce the problem to a special situation where the diagonal subalgebra becomes a Segre product, case in which it is known a criterion for the Cohen-Macaulayness.

These results are then applied to the study of the Rees algebra $R_A(I)$ of a homogeneous ideal $I \subset A = k[X_1, \ldots, X_n]$ generated by forms of the same degree d (*equigenerated* ideals). In this situation, the Rees algebra can be bigraded so that becomes standard by means of

$$R_A(I)_{(i,j)} = (I^j)_{i+dj},$$

and then $R_A(I)_\Delta = k[I_{d+1}]$. Mainly, two classes of ideals are then considered in detail: For complete intersection ideals generated by a regular sequence of r forms of degree d it is shown that $k[I_{d+1}]$ is a Cohen-Macaulay ring if $(r-1)d < n$, while $k[I_{d+1}]$ is not a Cohen-Macaulay ring if $(r-1)d > n$ [STV, Theorem 3.7]; for straightening closed ideals under some restrictions it is shown that $k[I_{d+1}]$ is a Cohen-Macaulay ring [STV, Theorem 3.13]. This second class of ideals includes for instance the determinantal ideals generated by the maximal minors of a generic matrix.

As a natural sequel of the work above, A. Conca et al. study in [CHTV] the diagonals R_Δ of a bigraded k-algebra R for $\Delta = (c, e)$, with c, e positive integers. The main problem considered there is to find suitable conditions on R such that certain algebraic properties of R are inherited by some diagonal R_Δ, mostly with respect to the Cohen-Macaulay property and the Koszul property. Their goal is to apply the results to the case of a standard bigraded k-algebra or the Rees algebra of any homogeneous ideal I of $A = k[X_1, \ldots, X_n]$. In the first case, R has a presentation as a quotient of a polynomial ring $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ endowed with the grading given by $\deg(X_i) = (1,0)$, $\deg(Y_j) = (0,1)$. As for the Rees algebra, if I is generated by forms $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$ respectively, we have a natural bigraded epimorphism

$$\begin{array}{ccc} S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r] & \longrightarrow & R = R_A(I) \\ X_i & \mapsto & X_i \\ Y_j & \mapsto & f_j t \end{array}$$

where $\deg(X_i) = (1,0)$, $\deg(Y_j) = (d_j, 1)$. Therefore, by working in the category of bigraded S-modules for $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ the polynomial ring with $\deg(X_i) = (1,0)$, $\deg(Y_j) = (d_j, 1)$, $d_1, \ldots, d_r \geq 0$, one may study both cases at the same time. Let us denote by $\mathcal{M}$ and $m = \mathcal{M}_\Delta$ the homogeneous maximal ideals of S and S_Δ respectively. Denoting by $d = \max\{d_1, \ldots, d_r\}$, we will consider diagonals $\Delta = (c, e)$ with $c \geq de + 1$.

Since the arithmetic properties of a module can be often characterized in terms of its local cohomology modules, it is of interest to study the local cohomology of the diagonals L_Δ of any finitely generated bigraded S-module L. This is done from the bigraded minimal free resolution of L over S: Let

$$0 \to D_l \to \ldots \to D_0 \to L \to 0$$

with $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$ be the bigraded minimal free resolution of L over S. By taking diagonals one gets a graded resolution of L_Δ

$$0 \to (D_l)_\Delta \to \ldots \to (D_0)_\Delta \to L_\Delta \to 0,$$

with $(D_p)_\Delta = \bigoplus_{(a,b)\in\Omega_p} S(a,b)_\Delta$. The first step is then the computation of the local cohomology modules of the S_Δ-modules $S(a,b)_\Delta$, which is done in the frame of a more general study about the local cohomology of the Segre product of two bigraded k-algebras. In particular, it is obtained a criterion for the Cohen-Macaulay property of $S(a,b)_\Delta$ by means of a, b and Δ. We say that the resolution of L is good if every module $(D_p)_\Delta$ is Cohen-Macaulay for large diagonals Δ. Then it is stated the following theorem:

Theorem [CHTV, Theorem 3.6, Lemma 3.8] *Assume $n \geq r$. For any finitely generated bigraded S-module L, there exists a canonical morphism*

$$\varphi_L^q : H_m^q(L_\Delta) \to H_\mathcal{M}^{q+1}(L)_\Delta, \ \forall q \geq 0$$

such that

(i) φ_L^q is an isomorphism for $q > n$.

(ii) φ_L^q is a quasi-isomorphism for $q \geq 0$.

(iii) If L has a good resolution, φ_L^q is an isomorphism for large diagonals.

As a corollary one gets necessary and sufficient conditions for the existence of Cohen-Macaulay or Buchsbaum diagonals L_Δ of L in terms of the graded pieces of the local cohomology modules of L.

Given a standard bigraded k-algebra R, one may define the graded k-subalgebras $\mathcal{R}_1 = \bigoplus_{i\in\mathbb{N}} R_{(i,0)}$, $\mathcal{R}_2 = \bigoplus_{j\in\mathbb{N}} R_{(0,j)}$. The following result gives a criterion for the Cohen-Macaulay property of the diagonals of R by means of $\mathcal{R}_1$ and $\mathcal{R}_2$. Namely,

Theorem [CHTV, Theorem 3.11] *Let R be a standard bigraded Cohen-Macaulay k-algebra. If the shifts in the resolutions of $\mathcal{R}_1$ and $\mathcal{R}_2$ are greater than $-n$ and $-r$ respectively, then R_Δ is Cohen-Macaulay for large Δ .*

In particular, they get the following corollary:

Corollary [CHTV, Corollary 3.12] *Let R be a standard bigraded Cohen-Macaulay k-algebra. If $\mathcal{R}_1$, $\mathcal{R}_2$ are Cohen-Macaulay with $a(\mathcal{R}_1), a(\mathcal{R}_2) < 0$, then R_Δ is Cohen-Macaulay for large Δ .*

This result applied to Rees algebras of equigenerated ideals gives a criterion for the Cohen-Macaulay property of their diagonals.

Furthermore, the study done in [STV] for the $(1,1)$-diagonal of the Rees algebra of an equigenerated complete intersection ideal is completed and extended to any complete intersection ideal and any diagonal, by determining exactly which are the Cohen-Macaulay diagonals. This is the only case where non equigenerated ideals are considered.

Theorem [CHTV, Theorem 4.6] *Let $I \subset A = k[X_1, \ldots, X_n]$ be a homogeneous complete intersection ideal minimally generated by r forms of degrees $d_1, \ldots, d_r$. Set $u = \sum_{j=1}^{r} d_j$. For $c \geq de + 1$, $k[(I^e)_c]$ is a Cohen-Macaulay ring if and only if $c > d(e-1) + u - n$.*

About the Cohen-Macaulay property of the diagonals of a Rees algebra is conjectured the following fact:

Conjecture *Let $I \subset A = k[X_1, \ldots, X_n]$ be a homogeneous ideal. If $R_A(I)$ is a Cohen-Macaulay ring, then there exists a diagonal Δ such that $R_A(I)_\Delta$ is a Cohen-Macaulay ring.*

With respect to the Gorenstein property, there is just one statement referred to the diagonals of the Rees algebra of a homogeneous ideal generated by a regular sequence of length 2.

Proposition [CHTV, Corollary 4.7] *Let $I \subset A = k[X_1, \ldots, X_n]$ be a homogeneous complete intersection ideal minimally generated by two forms of degree $d_1 \leq d_2$. If $n \geq d_2 + 1$, $k[I_n]$ is a Gorenstein ring with a-invariant -1.*

Finally, it is shown that large diagonals of the Rees algebra are always Koszul:

Theorem [CHTV, Corollary 6.9] *Let $I \subset A = k[X_1, \ldots, X_n]$ be a homogeneous ideal generated by forms of degree $\leq d$. Then there exist integers a, b such that $k[(I^e)_{c+de}]$ is Koszul for all $c \geq a$ and $e \geq b$.*

Under a slightly different setting, E. Hyry [Hy] is concerned with comparing the Cohen-Macaulay property of the biRees algebra $R_A(I, J)$ with the Cohen-Macaulay property of the Rees algebra $R_A(IJ)$, where $I, J \subset A$ are ideals of positive height in a local ring. To this end, he studies the $\Delta = (1,1)$-diagonal of any standard bigraded ring R defined over a local ring. The main result [Hy, Theorem 2.5] gives necessary and sufficient conditions for the Cohen-Macaulayness of a standard bigraded ring R with negative a-invariants by means of the local cohomology of the modules $R(p,0)_\Delta$ and $R(0,p)_\Delta$ ($p \in \mathbb{N}$). In particular, it provides sufficient conditions on R so that the Cohen-Macaulay property is carried from R to R_Δ:

Theorem *Let R be a standard bigraded ring defined over a local ring. Suppose that* $\dim \mathcal{R}_1, \dim \mathcal{R}_2 < \dim R$ *and* $a^1(R), a^2(R) < 0$. *If R is Cohen-Macaulay, then so is R_Δ for $\Delta = (1,1)$.*

Now let A be a noetherian graded k-algebra generated in degree 1 and let $I \subset A$ be a homogeneous ideal. The general problem of studying the embeddings of the blow-up $X = \mathcal{P}roj(\bigoplus_{n \geq 0} \mathcal{I}^n)$ of the projective scheme $Y = \text{Proj}\ (A)$ along the sheaf of ideals $\mathcal{I} = \tilde{I}$ given by the graded pieces of I is treated by S.D. Cutkosky and J. Herzog [CH]. They are mainly concerned with the existence of an integer f such that $k[(I^e)_c]$ is Cohen-Macaulay for all $e > 0$ and $c \geq ef$. The first example considered is the blow-up of a smooth projective variety Y along a regular ideal in a field of characteristic zero, where the Kodaira Vanishing Theorem can be used to prove:

Theorem [CH, Theorem 1.6] *Suppose that k has characteristic zero, A is Cohen-Macaulay, Y is smooth, I is equidimensional and* Proj (A/I) *is smooth. Then there exists a positive integer f such that $k[(I^e)_c]$ is Cohen-Macaulay for all $e > 0$ and $c \geq ef$.*

Let $\pi : X \to Y$ be the blow-up morphism, $E = \mathcal{P}roj(\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1})$, and w_E its dualizing sheaf. The main result they obtain is the following general criterion:

Theorem [CH, Theorem 4.1] *Suppose that $I \subset A$ is a homogeneous ideal such that $I \not\subset \mathfrak{p}$, $\forall \mathfrak{p} \in \mathrm{Ass}\,(A)$, A is Cohen-Macaulay and X is a Cohen-Macaulay scheme. Suppose that $\pi_* \mathcal{O}_E(m) = \mathcal{I}^m/\mathcal{I}^{m+1}$ for $m \geq 0$, $R^i\pi_*\mathcal{O}_E(m)$=0 for $i > 0$ and $m \geq 0$, $R^i\pi_* w_E(m)$=0 for $i > 0$ and $m \geq 2$. Then there exists a positive integer f such that $k[(I^e)_c]$ is Cohen-Macaulay for $e > 0$ and $c \geq ef$.*

This result is applied there to the following classes of ideals:

Corollary [CH, Corollary 4.2] *Let $I \subset A$ be a homogeneous ideal such that $I \not\subset \mathfrak{p}$, $\forall \mathfrak{p} \in \mathrm{Ass}\,(A)$, A is Cohen-Macaulay and $I_\mathfrak{p}$ is a complete intersection ideal for any $\mathfrak{p} \in \mathrm{Proj}\,(A)$. Then there exists a positive integer f such that $k[(I^e)_c]$ is Cohen-Macaulay for $e > 0$, $c \geq ef$.*

Corollary [CH, Corollary 4.4] *Let $I \subset A$ be a homogeneous ideal such that $I \not\subset \mathfrak{p}$, $\forall \mathfrak{p} \in \mathrm{Ass}\,(A)$, A is Cohen-Macaulay and $I_{(\mathfrak{p})}$ is strongly Cohen-Macaulay with $\mu(I_{(\mathfrak{p})}) \leq \mathrm{ht}\,(\mathfrak{p})$ for any prime ideal $\mathfrak{p} \in \mathrm{Proj}\,(A)$ containing I. Then there exists a positive integer f such that $k[(I^e)_c]$ is Cohen-Macaulay for $e > 0$, $c \geq ef$.*

As a somehow unexpected by-product, the methods used to study the diagonals of a Rees algebra also allow to study the regularity of the powers of an ideal and their asymptotic properties. These problems have been previously handled by using other techniques. Let $A = k[X_1, \ldots, X_n]$ be a polynomial ring with the usual grading and let $I \subset A$ be a homogeneous ideal. I. Swanson [Swa] has shown that there exists an integer B such that $\mathrm{reg}\,(I^e) \leq Be$, $\forall e$. The problem is to make B explicit. In some particular cases, such B was already known. A. Geramita, A. Gimigliano and Y. Pitteloud [GGP] and K. Chandler [Cha] had proved that for ideals with $\dim(A/I) = 1$, $\mathrm{reg}\,(I^e) \leq \mathrm{reg}\,(I)\, e$. On the other hand, R. Sjögren [Sjo] had given another kind of bound: If I is an ideal generated by forms of degree $\leq d$ with $\dim(A/I) \leq 1$, $\mathrm{reg}\,(I^e) < (n-1)de$. Also A. Bertram, L. Ein and R. Lazarsfeld [BEL] have given a bound for the regularity of the powers of an ideal in terms of the degrees of its generators: If I is the ideal of a smooth complex subvariety X of $\mathbb{P}^{n-1}_{\mathbb{C}}$ of codimension c generated by forms of degrees $d_1 \geq d_2 \geq \ldots \geq d_r$, then

$$H^i(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathcal{I}^e(k)) = 0, \quad \forall i \geq 1, \forall k \geq ed_1 + d_2 + \ldots + d_c - (n-1).$$

Let $(A, \mathfrak{m}, k)$ be a local ring and let $I \subset A$ be an ideal. Concerning the asymptotic properties of the powers of I, a classical well known result of

M. Brodmann [Bro] says that depth A/I^j takes a constant asymptotic value C for $j >> 0$, and moreover $C \leq \dim A - l(I)$. This value C was determined by D. Eisenbud and C. Huneke [EH] for ideals under some restrictions: If I is an ideal of height greater than zero and $G_A(I)$ is Cohen-Macaulay, then $\inf\{\operatorname{depth} A/I^j\} = \dim A - l(I)$, and if $\operatorname{depth} A/I^s = \inf\{\operatorname{depth} A/I^j\}$, then $\operatorname{depth} A/I^{s+1} = \operatorname{depth} A/I^s$. Finally, V. Kodiyalam [Ko1] has shown that for any fixed nonnegative integer p and all sufficiently large j, the p-th Betti number $\beta_p^A(I^j) = \dim_k \operatorname{Tor}_p^A(I^j, k)$ and the p-th Bass number $\mu_A^p(I^j) = \dim_k \operatorname{Ext}_A^p(k, I^j)$ are polynomials in j of degree $\leq l(I) - 1$.

Now we are going to set and motivate the concrete problems and questions considered in this dissertation.

The restriction to Rees algebras of equigenerated ideals done by A. Simis et al. [STV] is due to the fact that in this case the Rees algebra can be endowed with a bigrading so that it becomes standard. For standard bigraded algebras one may define its biprojective scheme (see [STV], [Hy]) and there are also known results about its Hilbert polynomial (see [HHRT], [KMV]). If I is an ideal generated by forms $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$ respectively, the Rees algebra of I has a presentation as a quotient of $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ bigraded by setting $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (d_j, 1)$ which is non standard. Our first problem will be to extend the definitions and known results on bigraded modules over standard bigraded algebras to the category of bigraded S-modules.

Several arithmetic properties of a ring such as the Cohen-Macaulayness and the Gorenstein property can be characterized by means of its local cohomology modules. This is the reason why it is interesting and useful to study when the local cohomology modules and the diagonal functor commute, case in which we may conclude that certain arithmetic properties of the Rees algebra are inherited by its diagonals. The shifts (a, b) which arise in the bigraded minimal free resolution of the Rees algebra $R_A(I)$ over the polynomial ring S play an essential role in this problem as it was seen in [CHTV]. We will study and bound these shifts by relating them to the local cohomology of the Rees algebra. After that, we will focus on the obstructions for the local cohomology modules and the diagonal functor to commute.

Once we have done all those preliminaries, our main purpose will be to study the Cohen-Macaulayness of the rings $k[(I^e)_c]$. We will consider different questions such as the existence and the determination of the diagonals (c, e) for which $k[(I^e)_c]$ is Cohen-Macaulay, problems treated in [STV], [CHTV] and [CH]. Similarly, our next goal will be to study the Gorenstein property of the k-algebras $k[(I^e)_c]$. This has been only done in a very particular case in [CHTV].

Some of the criteria we will obtain for the Cohen-Macaulayness of the k-algebras $k[(I^e)_c]$ are in terms of the local cohomology modules of the powers of the ideal I. This will lead us to study the a-invariants of the powers of a homogeneous ideal. We will then show how the bigrading defined in the Rees algebra can be used to study the a-invariants and the asymptotic properties of the powers of an ideal.

Summarizing, the main problems we have considered in this work are:

(1) To extend the definitions and results about the biprojective scheme and the Hilbert polynomial of finitely generated bigraded modules defined over standard bigraded k-algebras to finitely generated bigraded S-modules, for $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ the polynomial ring bigraded by $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (d_j, 1)$, $d_1, \ldots, d_r \geq 0$.

(2) To relate the shifts in the bigraded minimal free resolution of any finitely generated bigraded S-module to its a-invariants.

(3) To study the local cohomology modules of the diagonals of any finitely generated bigraded S-module.

(4) To study the Cohen-Macaulay property of the rings $k[(I^e)_c]$.

(5) To study the Gorenstein property of the rings $k[(I^e)_c]$.

(6) To study the a-invariants of the powers of a homogeneous ideal.

(7) To study the asymptotic properties of the powers of a homogeneous ideal.

Now we are ready to describe the results obtained in this work.

In **Chapter 1** we introduce the notations and definitions we will need throughout this work. We begin the chapter by defining the category of multigraded modules over a multigraded ring, and by recalling some well-known results about multigraded local cohomology and the canonical module mainly following M. Herrmann, E. Hyry and J. Ribbe [HHR] and S. Goto and K. Watanabe [GW1]. Then we define the multigraded a-invariants of a module and we study the relationship between these a-invariants and the shifts of its multigraded minimal free resolution. We will obtain a formula which extends [BH1, Example 3.6.15], where it was proved for Cohen-Macaulay modules in the graded case. This result will be a very useful device used all along this work. To precise it, let S be a d-dimensional $\mathbb{N}^r$-graded Cohen-Macaulay k-algebra with homogeneous maximal ideal $\mathcal{M}$ and let M be a finitely generated r-graded S-module of dimension m and depth ρ. For each $i = 0, \ldots, m$, we may associate to the i-th local cohomology module of M its multigraded a_i-invariant $\mathbf{a}_i(M) = (a_i^1(M), \ldots, a_i^r(M))$, where

$$a_i^j(M) = \max\{n \mid \exists \mathbf{n} = (n^1, \ldots, n^r) \in \mathbb{Z}^r \; s.t. \; \underline{\mathrm{H}}^i_{\mathcal{M}}(M)_{\mathbf{n}} \neq 0, n^j = n\}$$

if $\underline{\mathrm{H}}^i_{\mathcal{M}}(M) \neq 0$ and $a_i^j(M) = -\infty$ otherwise. Notice that $\mathbf{a}_m(M)$ coincides with the usual a-invariant, and so we will denote by $\mathbf{a}(M) = (a^1(M), \ldots, a^r(M)) = \mathbf{a}_m(M)$. Finally, the multigraded a_*-invariant of M is $\mathbf{a}_*(M) = (a_*^1(M), \ldots, a_*^r(M))$, where $a_*^j(M) = \max_{i=0,\ldots,m}\{a_i^j(M)\}$.

On the other hand, we may consider the r-graded minimal free resolution of M over S. Suppose that this resolution is finite:

$$0 \to D_l \to \ldots \to D_1 \to D_0 \to M \to 0,$$

with $D_p = \bigoplus_q S(a_{pq}^1, ..., a_{pq}^r)$. For every $p \in \{0, \ldots, l\}$, $j \in \{1, \ldots, r\}$, let us denote by

$$t_p^j(M) = \max_q\{-a_{pq}^j\},$$
$$t_*^j(M) = \max_{p,q}\{-a_{pq}^j\} = \max_p t_p^j(M),$$
$$\mathbf{t}_*(M) = (t_*^1(M), \ldots, t_*^r(M)).$$

Moreover, given a permutation σ of the set $\{1, ..., r\}$, let us consider $\leq_\sigma$ the order in $\mathbb{Z}^r$ defined by: $(u_1, ..., u_r) \leq_\sigma (v_1, ..., v_r)$ iff $(u_{\sigma(1)}, ..., u_{\sigma(r)}) \leq_{lex} (v_{\sigma(1)}, ..., v_{\sigma(r)})$, where $\leq_{lex}$ is the lexicographic order. Set $\mathbf{M}_p^\sigma = \max_{\leq_\sigma}\{\,(-a_{pq}^1, ..., -a_{pq}^r)\}$. Then we can relate the shifts and the a-invariants of M in the following way:

Theorem 1 [Theorem 1.3.4] *For every $j = 1, \ldots, r$,*

(i) $a_{d-p}^j(M) \leq t_p^j(M) + a^j(S)$, *for* $p = d - m, \ldots, d - \rho$.

(ii) Assume that for some p there exists σ s.t. $\sigma(1) = j$ and $M_p^\sigma >_\sigma M_{p+1}^\sigma$. Then $a_{d-p}^j(M) = t_p^j(M) + a^j(S)$.

(iii) $a_*^j(M) = t_*^j(M) + a^j(S)$. That is, $\mathbf{a}_*(M) = \mathbf{t}_*(M) + \mathbf{a}(S)$.

After that, we extend the definition and some of the results about the multiprojective scheme associated to a standard r-graded ring given by E. Hyry [Hy] and M. Herrmann et al. [HHRT] to rings endowed with a more general grading, which will also include the Rees algebra of a homogeneous ideal. Let S be a noetherian $\mathbb{N}^r$-graded ring generated over S_0 by homogeneous elements $x_{11}, \ldots, x_{1k_1}, \ldots, x_{r1}, \ldots, x_{rk_r}$ in degrees $\deg(x_{ij}) = (d_{ij}^1, \ldots, d_{ij}^{i-1}, 1, 0, \ldots, 0)$, with $d_{ij}^l \geq 0$. For every $j = 1, \ldots, r$, let I_j be the ideal of S generated by the homogeneous components of S of degree $\mathbf{n} = (n_1, \ldots, n_r)$ such that $n_j > 0, n_{j+1} = \ldots = n_r = 0$. The irrelevant ideal of S is $S_+ = I_1 \cdots I_r$. We may associate to S the r-projective scheme $\text{Proj}^{\,r}(S)$ which as a set contains all the homogeneous prime ideals $P \subset S$ such that $S_+ \not\subset P$. The relevant dimension of S is

$$\text{rel.dim}\, S = \begin{cases} r-1 & \text{if } \text{Proj}^{\,r}(S) = \emptyset \\ \max\{\dim S/P \mid P \in \text{Proj}^{\,r}(S)\} & \text{if } \text{Proj}^{\,r}(S) \neq \emptyset \end{cases}.$$

It can be proved that $\dim \text{Proj}^{\,r}(S) = \text{rel.dim}\, S - r$ by arguing as in [Hy, Lemma 1.2] where the standard r-graded case was considered. This result jointly with the isomorphism of schemes $\text{Proj}^{\,r}(S) \cong \text{Proj}\,(S_\Delta)$ that we have for certain diagonals allows to compute the dimension of S_Δ whenever S_0 is artinian, by extending [STV, Proposition 2.3] where this dimension was determined for the $(1,1)$-diagonal of a standard bigraded k-algebra by different methods.

Finally, we extend to the category of r-graded modules defined over the r-graded k-algebras introduced before the basic results concerning Hilbert functions and Hilbert polynomials. Some of them have been established in the standard r-graded case in [HHRT] and [KMV].

In **Chapter 2** we are concerned with the diagonal functor in the category of bigraded S-modules, where S is the polynomial ring $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ bigraded by setting $\deg X_i = (1,0)$, $\deg Y_j = (d_j, 1)$, $d_1, \ldots, d_r \geq 0$. In the first section, we compare the local cohomology modules of a finitely generated bigraded S-module L with the local cohomology modules of its diagonals. In particular, we will prove the main results in [CHTV] by a different and somewhat easier approach. In addition, this approach will provide more detailed information about several problems related to the behaviour of the local cohomology when taking diagonals. Set $d = \max\{d_1, \ldots, d_r\}$, and let $\Delta = (c, e)$ be a diagonal with $c \geq de + 1$. Let us consider the following subalgebras of S: $S_1 = k[X_1, \ldots, X_n]$, $S_2 = k[Y_1, \ldots, Y_r]$, with homogeneous maximal ideals $\mathfrak{m}_1 = (X_1, ..., X_n)$ and $\mathfrak{m}_2 = (Y_1, \ldots, Y_r)$. Let $\mathcal{M}_1$, $\mathcal{M}_2$ be the ideals of S generated by $\mathfrak{m}_1$, $\mathfrak{m}_2$ respectively, and let $\mathcal{M}$ be the homogeneous maximal ideal of S. Then:

Proposition 2 [Proposition 2.1.3] *Let L be a finitely generated bigraded S-module. There exists a natural exact sequence*

$$\ldots \to H^q_{\mathcal{M}}(L)_\Delta \to H^q_{\mathcal{M}_1}(L)_\Delta \oplus H^q_{\mathcal{M}_2}(L)_\Delta \to H^q_{\mathcal{M}_\Delta}(L_\Delta) \xrightarrow{\varphi^q_L} H^{q+1}_{\mathcal{M}}(L)_\Delta \to \ldots$$

In the rest of the section, we study the obstructions for φ^q_L to be an isomorphism. Firstly, we relate this question to the vanishing of the local cohomology with respect to $\mathcal{M}_1$ and $\mathcal{M}_2$ of the modules $S(a, b)$ which arise in the bigraded minimal free resolution of L over S. This allows us as said to recover the main results in [CHTV]. After that, we study the vanishing of the local cohomology modules of L with respect to $\mathcal{M}_1$ and $\mathcal{M}_2$ by themselves.

In Section 2.2 we will focus on standard bigraded k-algebras. Given a standard bigraded k-algebra R, let us consider the graded subalgebras $\mathcal{R}_1 = \bigoplus_{i \in \mathbb{N}} R_{(i,0)}$, $\mathcal{R}_2 = \bigoplus_{j \in \mathbb{N}} R_{(0,j)}$. By using Theorem 1, we obtain a characterization for R to have a good resolution in terms of the a_*-invariants of $\mathcal{R}_1$ and $\mathcal{R}_2$ which, in particular, provides a criterion for the Cohen-Macaulay property of its diagonals. We also find necessary and sufficient conditions on the local cohomology of $\mathcal{R}_1$ and $\mathcal{R}_2$ for the existence of Cohen-Macaulay diagonals of a Cohen-Macaulay standard bigraded k-algebra R. This result extends [CHTV, Corollary 3.12].

Proposition 3 [Proposition 2.2.7] *Let R be a standard bigraded Cohen-Macaulay k-algebra of relevant dimension δ. There exists Δ such that R_Δ is Cohen-Macaulay if and only if $H^q_{\mathfrak{m}_1}(\mathcal{R}_1)_0 = H^q_{\mathfrak{m}_2}(\mathcal{R}_2)_0 = 0$ for any $q < \delta - 1$.*

Now let us consider a standard bigraded ring R defined over a local ring with $a^1(R), a^2(R) < 0$. In [Hy, Theorem 2.5] it is shown that if R is Cohen-Macaulay then the $\Delta = (1, 1)$-diagonal of R has also this property. This result can be extended to any diagonal of a standard bigraded k-algebra:

Proposition 4 [Proposition 2.2.6] *Let R be a standard bigraded Cohen-Macaulay k-algebra with $a^1(R), a^2(R) < 0$. Then R_Δ is Cohen-Macaulay for any diagonal Δ.*

At the end of the chapter, we apply the results about bigraded k-algebras to the Rees algebra of a homogeneous ideal. Let A be a noetherian graded k-algebra generated in degree 1 of dimension $\overline{n}$ and let $\mathfrak{m}$ be the homogeneous maximal ideal of A. Given a homogeneous ideal I of A, the Rees algebra $R = R_A(I)$ of I is bigraded by $R_A(I)_{(i,j)} = (I^j)_i$. If I is generated in degree $\leq d$, for any diagonal $\Delta = (c, e)$ with $c \geq de + 1$ we have:

$$R_A(I)_\Delta = k[(I^e)_c].$$

The diagonals $k[(I^e)_c]$ are graded k-algebras of dimension $\overline{n}$ if no associated prime of A contains I. In the sequel we will always assume such hypothesis. We can relate the local cohomology modules of the k-algebras $k[(I^e)_c]$ and those of the powers of I. Denoting by m the homogeneous maximal ideal of $k[(I^e)_c]$, we have:

Proposition 5 [Corollary 2.3.5] *For any $c \geq de + 1$, $e > a_*^2(R)$, $s > 0$, we have isomorphisms*

$$H^q_m(k[(I^e)_c])_s \cong H^q_{\mathfrak{m}}(I^{es})_{cs}, \forall q \geq 0.$$

In the particular case where $A = k[X_1, \ldots, X_n]$, A. Conca et al. [CHTV] conjectured that if the Rees algebra of a homogeneous ideal I of A is Cohen-Macaulay, then there exists a Cohen-Macaulay diagonal. The results proved for standard bigraded k-algebras provide an affirmative answer for equigenerated homogeneous ideals. In fact, we can give a full answer to this conjecture.

Theorem 6 [Theorem 2.3.12] *Let I be a homogeneous ideal of the polynomial ring $A = k[X_1, \ldots, X_n]$. If $R_A(I)$ is a Cohen-Macaulay ring, then $R_A(I)$ has a good resolution. In particular, $k[(I^e)_c]$ is Cohen-Macaulay for $c \gg e \gg 0$.*

Furthermore, we obtain sufficient and necessary conditions on the ring A for the existence of Cohen-Macaulay diagonals of a Rees algebra $R_A(I)$ with this property. Namely,

Theorem 7 [Theorem 2.3.13] *If $R_A(I)$ is Cohen-Macaulay, then the following are equivalent:*

(i) There exist c, e *such that* $k[(I^e)_c]$ *is Cohen-Macaulay.*

(ii) $H^i_{\mathfrak{m}}(A)_0 = 0$ *for* $i < \overline{n}$.

In **Chapter 3** we study in detail the Cohen-Macaulay property of the rings $k[(I^e)_c]$. We consider the problem of the existence of Cohen-Macaulay diagonals of the Rees algebra. Once studied this problem, we will try to determine the diagonals with this property. The following isomorphisms will play an important role:

Proposition 8 [Proposition 3.1.2] *Let* X *be the blow-up of* Proj (A) *along* $\mathcal{I} = \widetilde{I}$, *where* I *is a homogeneous ideal of* A *generated by forms of degree* $\leq d$. *For any* $c \geq de + 1$, *there are isomorphisms of schemes*

$$X \cong \text{Proj}\ ^2(R_A(I)) \cong \text{Proj}\ (k[(I^e)_c]).$$

First of all, these isomorphisms will be used to give a criterion for the existence of diagonals $k[(I^e)_c]$ which are generalized Cohen-Macaulay modules, thereby solving a conjecture of [CHTV].

Proposition 9 [Proposition 3.2.6] *The following are equivalent:*

(i) $H^i_{\mathcal{M}}(R_A(I))_{(p,q)}$=0 *for* $i < \overline{n} + 1$, $p \ll q \ll 0$.

(ii) $k[(I^e)_c]$ *is a generalized Cohen-Macaulay module for* $c \gg e \gg 0$.

(iii) There exist c, e *such that* $k[(I^e)_c]$ *is generalized Cohen-Macaulay.*

(iv) $k[(I^e)_c]$ *is a Buchsbaum ring for* $c \gg e \gg 0$.

(v) There exist c, e *such that* $k[(I^e)_c]$ *is a Buchsbaum ring.*

(vi) There exist q_0, t *such that* $H^i_{\mathcal{M}}(R_A(I))_{(p,q)} = 0$ *for* $i < \overline{n}+1$, $q < q_0$ *and* $p < dq + t$.

After that, we use Proposition 8 to give necessary and sufficient conditions for a Rees algebra to have Cohen-Macaulay diagonals. Namely,

Theorem 10 [Theorem 3.2.3, Corollary 3.2.5] *The following are equivalent:*

(i) There exist c, e *such that* $k[(I^e)_c]$ *is a Cohen-Macaulay ring.*

(ii) *(1) There exist* $q_0, t \in \mathbb{Z}$ *such that* $H^i_{\mathcal{M}}(R_A(I))_{(p,q)} = 0$ *for all* $i < \overline{n}+1$, $q < q_0$ *and* $p < dq + t$.

(2) $H^i_{R_A(I)_+}(R_A(I))_{(0,0)} = 0$ *for all* $i < \overline{n}$.

(iii) *(1)* X *is a Cohen-Macaulay scheme.*

(2) $\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = k$, $H^i(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = 0$ *for* $0 < i < \overline{n} - 1$.

In this case, $k[(I^e)_c]$ *is a Cohen-Macaulay ring for* $c \gg e \gg 0$.

By using this theorem, we can exhibit some general situations in which we can ensure the existence of Cohen-Macaulay coordinate rings for X. For instance,

Proposition 11 [Proposition 3.3.3] *Let* X *be the blow-up of* $\mathbb{P}^{n-1}_k$ *along a closed subscheme, where* k *has* char$k = 0$. *Assume that* X *is smooth or with rational singularities. Then* X *is arithmetically Cohen-Macaulay.*

Our next goal in the chapter will be to determine the Cohen-Macaulay diagonals once we know its existence. This is a difficult problem which has been completely solved only for complete intersection ideals in the polynomial ring [CHTV, Theorem 4.6]. For equigenerated ideals, we can give a criterion for the Cohen-Macaulayness of a diagonal in terms of the local cohomology modules of the powers of the ideal by just assuming that the Rees algebra is Cohen-Macaulay. Namely,

Proposition 12 [Proposition 3.4.1] *Let* $I \subset A$ *be an ideal generated by forms of degree* d *whose Rees algebra is Cohen-Macaulay. For any* $c \geq de+1$, $k[(I^e)_c]$ *is Cohen-Macaulay if and only if*

(i) $H^i_{\mathfrak{m}}(A)_0 = 0$ *for* $i < \overline{n}$.

(ii) $H^i_{\mathfrak{m}}(I^{es})_{cs} = 0$ *for* $i < \overline{n}$, $s > 0$.

For arbitrary homogeneous ideals, we can also prove a criterion for the Cohen-Macaulayness of a diagonal by means of the local cohomology of the powers of the ideal and the local cohomology of the graded pieces of the canonical module of the Rees algebra. Let us denote by $K = K_{R_A(I)} = \bigoplus_{(i,j)} K_{(i,j)}$ the canonical module of the Rees algebra, and for each $e \in \mathbb{Z}$, let us consider the graded A-module $K^e = \bigoplus_i K_{(i,e)}$. Then we have:

Theorem 13 [Theorem 3.4.3] *Let* I *be a homogeneous ideal of* A *generated by forms of degree* $\leq d$ *whose Rees algebra is Cohen-Macaulay. For any* $c \geq de+1$, $k[(I^e)_c]$ *is Cohen-Macaulay if and only if*

(i) $H^i_{\mathfrak{m}}(A)_0 = 0$ *for* $i < \overline{n}$.

(ii) $H^i_{\mathfrak{m}}(I^{es})_{cs} = 0$ *for* $i < \overline{n}$, $s > 0$.

(iii) $H^{\overline{n}-i+1}_{\mathfrak{m}}(K^{es})_{cs} = 0$ *for* $1 \leq i < \overline{n}$, $s > 0$.

If the form ring is quasi-Gorenstein we can express the criterion above only in terms of the local cohomology of the powers of the ideal.

Theorem 14 [Corollary 3.4.4] *Let I be a homogeneous ideal of A generated by forms of degree $\leq d$. Assume that $R_A(I)$ is Cohen-Macaulay, $G_A(I)$ is quasi-Gorenstein. Set $a = -a^2(G_A(I))$, $b = -a(A)$. For any $c \geq de + 1$, $k[(I^e)_c]$ is Cohen-Macaulay if and only if*

(i) $H^i_{\mathfrak{m}}(A)_0 = 0$ *for* $i < \overline{n}$.

(ii) $H^i_{\mathfrak{m}}(I^{es})_{cs} = 0$ *for* $i < \overline{n}$, $s > 0$.

(iii) $H^i_{\mathfrak{m}}(I^{es-a+1})_{cs-b} = 0$ *for* $1 < i \leq \overline{n}$, $s > 0$.

We can use Theorem 14 to determine exactly the Cohen-Macaulay diagonals of the Rees algebra of a complete intersection ideal in any Cohen-Macaulay ring. In particular, we get a new proof of [CHTV, Theorem 4.6] where the case $A = k[X_1, \ldots, X_n]$ was studied.

These criteria will be also applied in the Chapter 5, once we have studied in detail the local cohomology modules of the powers of several families of ideals, such as equimultiple ideals or strongly Cohen-Macaulay ideals.

Furthermore, the results and methods used up to now allow us to show the behaviour of the a_*-invariant of the powers of a homogeneous ideal. The following statement has been obtained independently by S.D. Cutkosky, J. Herzog and N. V. Trung [CHT] and V. Kodiyalam [Ko2] by different methods.

Theorem 15 [Theorem 3.4.6] *Let L be a finitely generated bigraded S-module. Then there exists α such that*

$$a_*(L^e) \leq de + \alpha, \ \forall e.$$

After that, we use the bound on the shifts of the bigraded minimal free resolution of the Rees algebra obtained in Theorem 1 to determine a family of Cohen-Macaulay diagonals of a Cohen-Macaulay Rees algebra.

Theorem 16 [Theorem 3.4.12] *Let I be a homogeneous ideal of A generated by r forms of degree $d_1 \leq \ldots \leq d_r = d$. Assume that $H^i_{\mathfrak{m}}(A)_0 = 0$ for $i < \overline{n}$. Set $u = \sum_{j=1}^{r} d_j$. If the Rees algebra is Cohen-Macaulay, then*

(i) $k[(I^e)_c]$ is Cohen-Macaulay for $c > \max\{d(e-1) + u + a(A), d(e-1) + u - d_1(r-1)\}$.

(ii) If I is generated by forms in degree d, $k[(I^e)_c]$ is Cohen-Macaulay for $c > d(e - 1 + l(I)) + a(A)$.

Our results can be also applied to study the embedddings of the blow-up of a projective space along an ideal I of fat points via the linear systems $(I^e)_c$ whenever these linear systems are very ample, slightly extending [GGP, Theorem 2.4] where only the linear systems I_c were considered.

Theorem 17 [Theorem 3.4.15] *Let $I \subset A = k[X_1, \ldots, X_n]$ be an ideal of fat points, with k a field of characteristic zero. Then*

(i) $k[(I^e)_c]$ is Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(I^{es})_{cs} = 0$ for $s > 0$, $i < n$.

(ii) For $c > \operatorname{reg}(I)e$, $k[(I^e)_c]$ is Cohen-Macaulay with $a(k[(I^e)_c]) < 0$. In particular, $\operatorname{reg}(k[(I^e)_c]) < n - 1$.

The chapter finishes by studying sufficient conditions for the existence of a positive integer f such that $k[(I^e)_c]$ is a Cohen-Macaulay ring for all $c \geq ef$ and $e > 0$, a question that has been treated by S.D. Cutkosky and J. Herzog. Our main result, which improves [CH, Corollaries 4.2, 4.3 and 4.4], is the following:

Theorem 18 [Theorem 3.5.3] *Let I be a homogeneous ideal of A such that $R_{A_{\mathfrak{p}}}(I_{\mathfrak{p}})$ is Cohen-Macaulay for any prime ideal $\mathfrak{p} \in \operatorname{Proj}(A)$. Assume that $H^i_{\mathfrak{m}}(A)_0 = 0$ for $i < \overline{n}$. Then there exists an integer α such that $k[(I^e)_c]$ is Cohen-Macaulay for all $c \geq de + \alpha$ and $e > 0$.*

The aim of **Chapter 4** is to study the Gorenstein property of the k-algebras $k[(I^e)_c]$. About the Cohen-Macaulay property, we have already proved that if there exists a Cohen-Macaulay diagonal then there are infinitely many with this property. We show that the behaviour of the Gorenstein property is totally different. For instance, by considering the polynomial ring $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ with $\deg X_i = (1, 0)$, $\deg Y_j = (d_j, 1)$,

$d_1, \ldots, d_r \geq 0$, we have that S_Δ is Cohen-Macaulay for any diagonal Δ but there is just a finite set of Gorenstein diagonals.

Proposition 19 [Proposition 4.1.1] *S_Δ is Gorenstein if and only if $\frac{r}{e} = \frac{n+u}{c} = l \in \mathbb{Z}$. Then $a(S_\Delta) = -l$.*

To determine the rings $k[(I^e)_c]$ which are Gorenstein, we will compare the canonical module of the Rees algebra with the canonical module of each diagonal. For complete intersection ideals of the polynomial ring, it was proved in [CHTV, Proposition 4.5] that the canonical module and the diagonal functor commute. This result can be extended to more general situations.

Proposition 20 [Proposition 4.1.4 and Remark 4.1.5] *Let $A = k[X_1, \ldots, X_n]$ be the polynomial ring, $n \geq 2$, and let I be a homogeneous ideal of A with $\mu(I) \geq 2$.*

(i) If $\mu(I) \leq n$, $K_{R_\Delta} \cong (K_R)_\Delta$.

(ii) If I is equigenerated and R is Cohen-Macaulay, $K_{R_\Delta} \cong (K_R)_\Delta$.

Although this isomorphism can be extended to a more general class of rings, we will restrict our attention to the above two cases. This will suffice to study the rational surfaces obtained by blowing-up the projective plane at a set of points.

Next we study the behaviour of the Gorenstein property of the Rees algebra when we take diagonals. If the Rees algebra is Gorenstein then the form ring is also Gorenstein. Under this assumption on the form ring, which is less restrictive, we can determine exactly for which c, e the algebra $k[(I^e)_c]$ is quasi-Gorenstein. Namely,

Theorem 21 [Theorem 4.1.9] *Let $I \subset A = k[X_1, \ldots, X_n]$ be a homogeneous ideal with $1 < \mathrm{ht}(I) < n$ whose form ring $G_A(I)$ is Gorenstein. Set $a = -a^2(G_A(I))$. Then $k[(I^e)_c]$ is a quasi-Gorenstein ring if and only if $\frac{n}{c} = \frac{a-1}{e} = l_0 \in \mathbb{Z}$. In this case, $a(k[(I^e)_c]) = -l_0$.*

For homogeneous non principal ideals I of height 1, the ring $k[(I^e)_c]$ is never Gorenstein. If I has height n, then the diagonals determined in the theorem are always Gorenstein, but the converse is not true. As a corollary of this result we can solve the problem of determining completely the Gorenstein diagonals for complete intersection ideals or determinantal ideals generated by the maximal minors of a generic matrix.

Corollary 22 [Corollary 4.1.12] *Let $I \subset A = k[X_1, \dots, X_n]$ be a complete intersection ideal minimally generated by r forms of degree $d_1 \leq \dots \leq d_r = d$, with $r < n$. For any $c \geq de + 1$, $k[(I^e)_c]$ is a Gorenstein ring if and only if $\frac{n}{c} = \frac{r-1}{e} = l_0 \in \mathbb{Z}$. In this case, $a(k[(I^e)_c]) = -l_0$.*

Corollary 23 [Example 4.1.13] *Let $\mathbf{X} = (X_{ij})$ denote a matrix of indeterminates, with $1 \leq i \leq n, 1 \leq j \leq m$ and $m \leq n$. Let $I \subset A = k[\mathbf{X}]$ denote the ideal generated by the maximal minors of $\mathbf{X}$, where k is a field. Then:*

(i) If $m < n$, then $k[(I^e)_c]$ is Gorenstein if and only if $\frac{nm}{c} = \frac{n-m}{e} \in \mathbb{Z}$.

(ii) If $m = n$, then $\Delta = (n(n+1), 1)$ is the only Gorenstein diagonal.

We have shown that if the form ring is Gorenstein there is just a finite set of Gorenstein diagonals. This fact also holds under the general assumptions of the chapter. Namely,

Proposition 24 [Proposition 4.2.1] *There is a finite set of diagonals $\Delta = (c, e)$ such that $k[(I^e)_c]$ is quasi-Gorenstein.*

If the Rees algebra is Cohen-Macaulay, then we can bound the diagonals $\Delta = (c, e)$ for which $k[(I^e)_c]$ is Gorenstein.

Proposition 25 [Proposition 4.2.2] *Assume that $\operatorname{ht}(I) \geq 2$ and $R_A(I)$ is Cohen-Macaulay. Let $a = -a^2(G_A(I))$. If $k[(I^e)_c]$ is quasi-Gorenstein, then $e \leq a - 1$ and $c \leq n$. Moreover, if $\dim(A/I) > 0$ then $\lceil \frac{a}{e} \rceil - 1 = \frac{n}{c} = l \in \mathbb{Z}$. In particular, if $a = 1$ there are no diagonals (c, e) such that $k[(I^e)_c]$ is quasi-Gorenstein.*

Finally, we show that in some cases the existence of a diagonal (c, e) such that $k[(I^e)_c]$ is quasi-Gorenstein forces the form ring to be Gorenstein. It may be seen as a converse of Theorem 21 for those cases.

Theorem 26 [Theorem 4.2.3] *Assume that $R_A(I)$ is Cohen-Macaulay, $\operatorname{ht}(I) \geq 2$, $l(I) < n$ and I is equigenerated. If there exists a diagonal (c, e) such that $k[(I^e)_c]$ is quasi-Gorenstein then $G_A(I)$ is Gorenstein.*

We finish the chapter by applying the previous results to recover the fact that the Del Pezzo sestic surface in $\mathbb{P}^6$ is the only Room surface which is Gorenstein.

In **Chapter 5** we study the a-invariant and the regularity of any finitely generated bigraded S-module L, for $S = k[X_1, \dots, X_n, Y_1, \dots, Y_r]$ the polynomial ring with $\deg X_i = (1,0)$, $\deg Y_j = (0,1)$. This class of modules includes for instance any standard bigraded k-algebra R.

Given a finitely generated bigraded S-module L, let us consider the bigraded minimal free resolution of L over S

$$0 \to D_l \to \ldots \to D_0 \to L \to 0,$$

with $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$. The bigraded regularity of L is $\mathbf{reg}(L) = (\operatorname{reg}_1 L, \operatorname{reg}_2 L)$, where

$$\operatorname{reg}_1 L = \max_p\{-a-p \mid (a,b) \in \Omega_p\},$$

$$\operatorname{reg}_2 L = \max_p\{-b-p \mid (a,b) \in \Omega_p\}.$$

For each $e \in \mathbb{Z}$, we may define the graded S_1-module $L^e = \bigoplus_{i\in\mathbb{Z}} L_{(i,e)}$ and the graded S_2-module $L_e = \bigoplus_{j\in\mathbb{Z}} L_{(e,j)}$. Our first result gives a new description of the a_*-invariant $\mathbf{a}_*(L)$ of L and the regularity $\mathbf{reg}(L)$ of L in terms of the a_*-invariants and the regularities of the graded modules L^e and L_e. Namely,

Theorem 27 [Theorem 5.1.1, Theorem 5.1.2] *Let L be a finitely generated bigraded S-module. Then:*

(i) $a_*^1(L) = \max_e\{a_*(L^e)\} = \max_e\{a_*(L^e) \mid e \le a_*^2(L) + r\}$.

(ii) $a_*^2(L) = \max_e\{a_*(L_e)\} = \max_e\{a_*(L_e) \mid e \le a_*^1(L) + n\}$.

(iii) $\operatorname{reg}_1 L = \max_e\{\operatorname{reg}(L^e)\} = \max_e\{\operatorname{reg}(L^e) \mid e \le a_*^2(L) + r\}$.

(iv) $\operatorname{reg}_2 L = \max_e\{\operatorname{reg}(L_e)\} = \max_e\{\operatorname{reg}(L_e) \mid e \le a_*^1(L) + n\}$.

This result will be used to study the a_*-invariant and the regularity of the powers of a homogeneous ideal I in the polynomial ring $A = k[X_1, \ldots, X_n]$. According to Theorem 15, there exists an integer α such that $a_*(I^e) \le de + \alpha$, $\forall e$. The first aim is to determine such an α explicitly, and this will be done for any equigenerated ideal by means of a suitable a-invariant of the Rees algebra. For a homogeneous ideal I, we will denote by R, G and F the Rees algebra of I, its form ring and the fiber cone respectively. If I is an ideal generated by forms in degree d, let us denote by R^φ the Rees algebra endowed with the bigrading $[R^\varphi]_{(i,j)} = (I^j)_{i+dj}$. Then we have

Theorem 28 [Theorem 5.2.1] *Let I be a homogeneous ideal of A generated by forms in degree d. Set $l = l(I)$. Then* *(i)* $a^1_*(R^\varphi) = \max_e\{\, a_*(I^e) - de\} = \max\{\, a_*(I^e) - de \mid e \leq a^2_*(R) + l\}$.

(ii) $\operatorname{reg}_1(R^\varphi) = \max_e\{\operatorname{reg}(I^e) - de\} = \max\{\operatorname{reg}(I^e) - de \mid e \leq a^2_*(R) + l\}$.

Therefore, we need to study $a^1_*(R^\varphi)$ to get concrete bounds for the a_*-invariant of the powers of several families of ideals. If the Rees algebra is Cohen-Macaulay we have

Proposition 29 [Proposition 5.2.5] *Let I be a homogeneous ideal generated by forms in degree d whose Rees algebra is Cohen-Macaulay. Set $l = l(I)$. Then*

$$-n + d(-a^2(G) - 1) \leq \max_{e \geq 0}\{a_*(I^e) - de\} \leq -n + d(l-1).$$

The a_*-invariants of the powers of a complete intersection ideal are well-known, and in this case the inequalities above are sharp. Next we compute explicitly $a^1_*(R^\varphi) = \max_{e\geq 0}\{a_*(I^e) - de\}$ for other families of ideals. First we consider equimultiple ideals.

Proposition 30 [Proposition 5.2.8] *Let I be an equimultiple ideal equigenerated in degree d and set $h = \operatorname{ht}(I)$. If the Rees algebra is Cohen-Macaulay,*

(i) $a(I^e/I^{e+1}) = de + a(A/I)$. *In particular,* $a^1(G^\varphi) = a(A/I)$.

(ii) $a_{n-h+1}(I^e) = d(e-1) + a(A/I)$. *In particular,* $a^1(R^\varphi) = a(A/I) - d$.

For ideals whose form ring is Gorenstein we can also compute explicitly $\max_{e\geq 0}\{a_*(I^e) - de\}$, and then we get that the lower bound given by Proposition 29 is sharp.

Proposition 31 [Proposition 5.2.9] *Let I be a homogeneous ideal equigenerated in degree d whose form ring is Gorenstein. Set $l = l(I)$. Then*

(i) $\max_{e\geq 0}\{a_*(I^e) - de\} = d(-a^2(G) - 1) - n$.

(ii) For $e > a^2(G) - a(F)$, $\operatorname{depth}(A/I^e) = n - l$ *and* $a_*(I^e) = a_{n-l}(A/I^e) = d(e - a^2(G) - 1) - n$.

For instance, we may apply this result to determinantal ideals generated by the maximal minors of a generic matrix as well as to strongly Cohen-Macaulay ideals satisfying condition $(\mathcal{F}_1)$.

The computation of the a_*-invariants of the powers of these families of ideals is then applied to determine the Cohen-Macaulay diagonals of a Rees algebra. For equimultiple ideals, we have

Proposition 32 [Proposition 5.2.20] *Let I be an equimultiple ideal generated in degree d whose Rees algebra is Cohen-Macaulay. For any $c \geq de+1$, $k[(I^e)_c]$ is Cohen-Macaulay if and only if $c > d(e-1) + a(A/I)$.*

For strongly Cohen-Macaulay ideals, we have

Proposition 33 [Proposition 5.2.21] *Let I be a strongly Cohen-Macaulay ideal such that $\mu(I_\mathfrak{p}) \leq \mathrm{ht}(\mathfrak{p})$ for any prime ideal $\mathfrak{p} \supseteq I$. Assume that I is minimally generated by forms of degree $d = d_1 \geq \ldots \geq d_r$, and let $h = \mathrm{ht}(I)$. For $c > d(e-1) + d_1 + \ldots + d_h - n$, $k[(I^e)_c]$ is Cohen-Macaulay.*

If the Rees algebra is Cohen-Macaulay, we have proved the existence of an integer α such that $k[(I^e)_c]$ is a Cohen-Macaulay ring for any $c > de + \alpha$ and $e > 0$ by Theorem 16. For equigenerated ideals we had $\alpha = d(l-1)$ as an upper bound. We can determine the best α.

Proposition 34 [Proposition 5.2.15, Corollary 5.2.16] *Let I be an ideal in the polynomial ring $A = k[X_1, \ldots, X_n]$ generated by forms in degree d whose Rees algebra is Cohen-Macaulay. Set $l = l(I)$. For $\alpha \geq 0$, the following are equivalent*

(i) $k[(I^e)_c]$ *is CM for* $c > de + \alpha$.

(ii) $a_i(I^e) \leq de + \alpha$, $\forall i$, $\forall e$.

(iii) $a_i(I^e) \leq de + \alpha$, $\forall i$, $\forall e \leq l-1$.

(iv) $H^{n+1}_{\mathcal{M}}(R_A(I))_{(p,q)} = 0$, $\forall p > dq + \alpha$, *that is,* $\alpha \geq a^1(R^\varphi)$.

(v) The minimal bigraded free resolution of $R_A(I)$ *is good for any diagonal* $\Delta = (c, e)$ *such that* $c > de + \alpha$.

If the form ring is Gorenstein, these conditions are equivalent to

(vi) $\alpha \geq d(-a^2(G) - 1) - n$.

Up to now, we have used Theorem 27 to bound the a_*-invariants of the powers of an ideal, which has been applied to study the Cohen-Macaulayness of the diagonals. In the last section, we use this theorem to prove a bigraded version of the Bayer-Stillman theorem which characterizes the bigraded regularity of a homogeneous ideal of S by means of generic homogeneous forms. Next, similarly to the graded case, we define the generic initial ideal $\mathbf{gin}I$ of a homogeneous ideal I of S and we establish its basic properties. In particular, we may use the Bayer-Stillman theorem to compute the regularity of a Borel-fix ideal in S when k has characteristic zero. For $j = 1, 2$, let us denote by $\delta_j(I)$ the maximum of the j-th component of the degrees in a minimal homogeneous system of generators of I. Then we have

Proposition 35 [Proposition 5.3.10] *Let $I \subset S$ be a Borel-fix ideal. If $\operatorname{char} k = 0$, then*

$$\operatorname{reg}_1(I) = \delta_1(I),$$
$$\operatorname{reg}_2(I) = \delta_2(I).$$

This result has been also proved by A. Aramova et al. [ACD] by different methods. In the graded case, D. Bayer and M. Stillman [BaSt] also proved the existence of an order in the polynomial ring $A = k[X_1, \ldots, X_n]$ (the reverse lexicographic order) such that $\operatorname{reg} I = \operatorname{reg}(\operatorname{gin} I)$ for any homogeneous ideal I of A. We finish the chapter by showing that the analogous bigraded result does not hold because we can find a homogeneous ideal I of S such that for any order $\mathbf{reg}(I) \neq \mathbf{reg}(\mathbf{gin}I)$.

In **Chapter 6** we study the asymptotic properties of the powers of a homogeneous ideal I in the polynomial ring $A = k[X_1, \ldots, X_n]$. We will show how the bigraded structure of the Rees algebra provides information about the Hilbert polynomials, the Hilbert series and the graded minimal free resolutions of the powers of I. This grading of the Rees algebra will be also useful to study the mixed multiplicities of the Rees algebra and the form ring of an equigenerated ideal.

Theorem 36 [Theorem 6.1.1] *Let I be a homogeneous ideal of A. Set $c = a_*^2(R_A(I))$, $h = \operatorname{ht}(I)$. Then there are polynomials $e_0(j), \ldots, e_{n-h-1}(j)$ with integer values such that for all $j \geq c+1$*

$$P_{A/I^j}(s) = \sum_{k=0}^{n-h-1} (-1)^{n-h-1-k} e_{n-h-1-k}(j) \binom{s+k}{k}.$$

Furthermore, $\deg e_{n-h-1-k}(j) \leq n-k-1$ *for all k.*

In particular, this result says that a finite set of Hilbert polynomials of the powers of an ideal allows to compute the Hilbert polynomials of its Rees algebra and its form ring, without needing an explicit presentation of these bigraded algebras. For equigenerated ideals, we may also compute the multiplicities of their Rees algebras and form rings.

Corollary 37 [Corollary 6.1.8] *Let I be a homogeneous ideal in A. Let $c = a_*^2(R_A(I))$, $h = \mathrm{ht}\,(I)$. Then the Hilbert polynomials of I^j for $c+1 \leq j \leq c+n$ determine*

(i) The polynomials $e_{n-h-1-k}(j)$ for $k = 0, \ldots, n-h-1$.

(ii) The Hilbert polynomials of A/I^j for $j > c+n$.

(iii) The Hilbert polynomial of $R_A(I)$ and the Hilbert polynomial of $G_A(I)$.

(iv) If I is equigenerated and not $\mathfrak{m}$-primary, the mixed multiplicities of $R_A(I)$ and $G_A(I)$.

A similar result can be proved for the Hilbert series of the powers of a homogeneous ideal. Namely,

Proposition 38 [Theorem 6.2.1, Proposition 6.2.7] *Let I be a homogeneous ideal. Set $r = \mu(I)$, $l = l(I)$, $c = a_*^2(R_A(I))$. Then:*

(i) The Hilbert series of I^j for $j \leq c+r$ determine the Hilbert series of I^j for $j > c+r$.

(ii) If I is an equigenerated ideal, the Hilbert series of I^j for $c+1 \leq j \leq c+l$ determine the Hilbert series of I^j for $j > c+l$.

Next we study the behaviour of the projective dimension of the powers of an ideal. As a by-product, we recover the classic result of M. P. Brodmann [Bro] which says that the depth of the powers of an ideal becomes constant asymptotically, and a result of D. Eisenbud and C. Huneke [EH] which precises this asymptotic value under some restrictions. Moreover, for ideals whose form ring is Gorenstein we may determine exactly the powers of the ideal for which the projective dimension takes the asymptotic value. Namely,

Proposition 39 [Proposition 6.3.2] *Let I be a homogeneous ideal in A and set $l = l(I)$. If G is Gorenstein,* $\mathrm{proj.dim}_A(I^j) \leq l-1$ *for all j, and* $\mathrm{proj.dim}_A I^j = l-1$ *if and only if $j > a^2(G) - a(F)$.*

Finally, we show that the graded minimal free resolutions of the powers of an ideal also have a uniform behaviour. For equigenerated ideals, we can prove that the shifts which arise in the minimal resolutions are linear functions asymptotically and the Betti numbers are polynomial functions asymptotically. More explicitly,

Proposition 40 [Proposition 6.3.6] *Let I be a homogeneous ideal generated in degree d. Set $l = l(I)$, $s = n - \mathrm{depth}_{(\mathfrak{m}R)}(R)$. Then there is a finite set of integers $\{\alpha_{pi} \mid 0 \leq p \leq s, 1 \leq i \leq k_p\}$ and polynomials $\{Q_{\alpha_{pi}}(j) : 0 \leq p \leq s, 1 \leq i \leq k_p\}$ of degree $\leq l-1$ such that the graded minimal free resolution of I^j for j large enough is*

$$0 \to D_s^j \to \ldots \to D_0^j \to I^j \to 0\ ,$$

with $D_p^j = \bigoplus_i A(-\alpha_{pi} - dj)^{\beta_{pi}^j}$ and $\beta_{pi}^j = Q_{\alpha_{pi}}(j)$.

From this result, we may deduce that a finite number of the graded minimal free resolutions of the powers of an ideal determine the rest of them. This finite set of resolutions can be found for ideals with a very particular behaviour. For instance, we get

Proposition 41 [Proposition 6.3.10] *Let I be an equigenerated homogeneous ideal, and $b = a_*^2(R_A(I)) + l(I)$. If the graded minimal free resolutions of $I, I^2, \ldots, I^b$ are linear, then the graded minimal free resolutions of I^j are also linear for any j. Furthermore, the minimal free resolutions of $I, I^2, \ldots, I^b$ determine the minimal graded free resolutions of I^j for any j.*

Some parts of this work have already appeared published in:

- O. Lavila–Vidal, *On the Cohen-Macaulay property of diagonal subalgebras of the Rees algebra*, manuscripta math. 95 (1998), 47–58.

- O. Lavila–Vidal, S. Zarzuela, *On the Gorenstein property of the diagonals of the Rees algebra*, Collect. Math. 49, 2-3 (1998), 383–397.

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