SUMMARY OF "ULRICH BUNDLES AND VARIETIES OF WILD REPRESENTATION TYPE", BY JUAN FRANCISCO PONS LLOPIS

Given a projective variety $X \subseteq \mathbb{P}^n$ it is usual to understand its complexity in terms of the associated category of its (semi)-stable vector bundles, which is known to behave well and, in particular, there exists a nice moduli space parameterizing them. It is also possible to pay attention to another property of a vector bundle $\mathcal{E}$: the fact of having cohomology as simple as possible, i.e., $\mathrm{H}^i(X, \mathcal{E}(l)) = 0$ for all $l \in \mathbb{Z}$ and $i = 1, \ldots, \dim(X) - 1$. The vector bundles holding this property are called *Arithmetically Cohen-Macaulay (ACM)* vector bundles. It is well-known that, on an ACM variety X there exists a bijection between ACM vector bundles $\mathcal{E}$ on X and Maximally Cohen Macaulay modules $\bigoplus_t \mathrm{H}^0(\mathcal{E}(t))$ over the coordinate ring R_X. In our thesis, we made a contribution to this problem by showing that the two following families of ACM varieties are of wild representation type, namely, Fano varieties (i.e., varieties for which the anticanonical divisor is ample) obtained as the blow-up of points on $\mathbb{P}^n$, $n \geq 2$; and general surfaces $X \subseteq \mathbb{P}^3$ of degree $3 \leq d \leq 9$. In particular, we proved:

Theorem. Let $X = Bl_Z\mathbb{P}^n$ be a Fano blow-up of points in $\mathbb{P}^n$, $n \geq 3$ and let $r \geq n$.

(i) If n is even, fix $c \in \{0, \ldots, n/2-1\}$ such that $c \equiv r \bmod n/2$ and set the number $u := \frac{2(r-c)}{n}$. Then there exists a family of rank r simple (hence, indecomposable) ACM vector bundles of dimension $\frac{(n+2)n-4}{4}u^2 - cu - c^2 + 1$.

(ii) If n is odd, fix $c \in \{0, \ldots, n-1\}$ such that $c \equiv r \bmod n$ and set $u := \frac{(r-c)}{n}$. Then there exists a family of rank r simple (hence, indecomposable) ACM vector bundles of dimension $((n+2)n-4)u^2 - 2cu - c^2 + 1$.

In particular, Fano blow-ups are varieties of wild representation type.

In the particular case of surfaces, i.e., in the case of del Pezzo surfaces, we moreover proved that the ACM bundles that we constructed share a particular interesting property, namely the fact of having the maximal possible number of global sections. This kind of vector bundles are known under the name of Ulrich sheaves. We proved:

Theorem. Let $X \subseteq \mathbb{P}^d$ be a del Pezzo surface of degree d. Assume that X is not the smooth quadric embedded in $\mathbb{P}^8$ via the anticanonical divisor $-K_X$. Then for any $r \geq 2$ there exists a family of dimension $r^2 + 1$ of simple initialized Ulrich vector bundles of rank r with Chern classes $c_1 = rH$ and $c_2 = \frac{dr^2+(2-d)r}{2}$. Moreover, they are μ-semistable with respect to the polarization $H = 3e_0 - \sum_{i=1}^{9-d} e_i$ and μ-stable with respect to $H_n := (n-3)e_0 + H$ for $n \gg 0$. In particular, del Pezzo surfaces are of wild representation type.

A possible approach to the construction of ACM and Ulrich vector bundles on a given projective surface $X \subseteq \mathbb{P}^n$ is offered by the well-known *Serre correspondence*. It is therefore a meaningful problem to find out the shape of the minimal free resolution of the coordinate ring R_Z of a general set of points Z lying on a given surface, or, more generally, on a given variety X. We recall Mustaţă's version of the Minimal Resolution Conjecture:

Conjecture. Let $X \subset \mathbb{P}^n$ be a projective variety with $d = \dim(X) \geq 1$, $\text{reg}(X) = m$ and with Hilbert polynomial P_X. Let $s \in \mathbb{Z}$ be an integer such that $P_X(r-1) \leq s < P_X(r)$ for some $r \geq m+1$. The *Minimal Resolution Conjecture (MRC for short)* holds for the value s if for every set Z of s general distinct points we have

$$b_{i+1,r-1}(Z)b_{i,r}(Z) = 0 \quad \text{for} \quad i = 1, \cdots, n-1.$$

where $b_{i,j}(Z)$ stands for the Betti graded numbers of Z.

In our thesis we obtained the following contribution to this problem:

Theorem. Let $X \subseteq \mathbb{P}^d$ be an ACM quasi-minimal surface. Assume that X is not the anticanonical model of $\mathrm{F}_2 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ or a complete intersection of two quadrics on $\mathbb{P}^4$ with a double line. Let r be an integer such that $r \geq reg(X)+1 = 4$. Then for any general set of distinct points Z on X such that $P_X(r-1) \leq |Z| \leq m(r)$ or $n(r) \leq |Z| \leq P_X(r)$ the Minimal Resolution Conjecture is true.