UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH

b^m-*Symplectic manifolds: symmetries, classification and stability*

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DE CATALUNYA
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A thesis submitted in fulfillment of the requirements of the degree of Doctor of Philosophy in Mathematics at FME UPC

June 2020

Contents

Acknowledgements

The first person I want to thank for all her support is Eva, my advisor. *Gràcies Eva.* This thesis would not have been possible without all the support she gave me. She trusted me from the beginning, and helped me through a lot of difficulties far beyond what is expected from an advisor. Even when I did not take the path I was supposed to take, she was there. Even when she was extremely busy, she always found the time to meet. The meetings with her were always extremely useful. I cannot be thankful enough. *Mil gràcies!*

I also want to thank the members of the lab. Especially Cédric and Roisin with whom I shared really important moments throughout this journey. Also thanks to Amadeu, Anastasia, Joaquim, Juan, Pau, Robert and Rodrigo for some insightful talks, where I learned a lot of maths.

Thanks also to all the members of the jury for making the effort to be available for the defense. Thanks Amadeu Delshams, Marisa Fernandez, Jacques Féjoz, Ignasi Mundet and Geoff Scott.

I also want to thank my family for trusting me and supporting me from much before I started my PhD journey, and for encouraging me to make some of the decisions that have led me here. *Gràcies Mama, Papa, Anna i Helena!*

Lastly I want to give special thanks to the person that endured and supported me through all the late night working hours. Not to forget the weekend working hours. On the personal level I would not have been able to find the strenght to finish this thesis without her support.

I warmly remember all the moments she was by my side working or pushed me to work when I did not feel like it. For all this and a lot more, I have not enough words to thank her. *Gracias Chiqui!*

Abstract

This thesis explores classification and perturbation problems for group actions on a class of Poisson manifolds called b^m-Poisson manifolds. b^m-Poisson manifolds are manifolds which are symplectic away from a hypersurface along which they satisfy some transversality properties. They often model problems on symplectic manifolds with boundary such as the study of their deformation quantization and celestial mechanics.

One of the interesting properties of b^m-Poisson manifolds is that their study can be achieved considering the language of b^m-forms. That is to say, we can work with forms which are symplectic away from the critical set and admit a smooth extension as a *form over a Lie algebroid* generalizing De Rham forms as form over the standard Lie algebroid of the tangent bundle of the manifold. To consider b^m-forms the standard tangent bundle is replaced by the b^m-tangent bundle.

This thesis starts with the equivariant classification of b^m-Poisson structures investigating, in particular, the analogue of Moser's classification theorem for symplectic surfaces and their equivariant analogues. The classification invariants in the case of surfaces are encoded in a cohomology called b^m-cohomology which has been deeply studied by [1]. Mazzeo-Melrose type formula for b^m-cohomology decomposes it in two pieces which can be read off the De Rham cohomology of both the ambient manifold M and the critical hypersurface. As an outcome of this identification, the Poisson classification of these manifolds is given by the De Rham cohomology of the manifold and the hypersurface.

This classification is extended to the equivariant setting if we assume that the singular forms are preserved by the group action of a compact Lie group. These techniques can be extended to the classification of b^m-Nambu structures which are also considered in this thesis.

Group actions re-appear in the last chapters as integrable systems on these manifolds turn out to have associated Hamiltonian actions of tori in a neighbourhood of a Liouville torus. We use this Hamiltonian group action to prove existence of action-angle coordinates in a neighborhood of a Liouville torus. The action-angle coordinate theorem that we prove gives a semilocal normal form in the neighbourhood of a Liouville torus for the b^m-symplectic structure which depends on the modular weight of the connected component of the critical set in which the Liouville torus is lying and the modular weights of the associated toric action. This action-angle theorem allows us to identify a neighborhood of the Liouville torus with the b^m-cotangent lift of the action of a torus acting by translations on itself.

We end up this thesis proving a KAM theorem for b^m-Poisson manifolds which clearly refines and improves the one obtained for b-Poisson manifolds in [2]. As an outcome of this result together with the extension of the desingularization techniques of Guillemin-Miranda-Weitsman to the realm of integrable systems, we obtain a KAM theorem for folded symplectic manifolds where KAM theory has never been considered before. In the way, we also obtain a brand new KAM theorem for symplectic manifolds where the perturbation keeps track of a distinguished hypersurface. In celestial mechanics this distinguished hypersurface can be the line at infinity or the collision set.

Resumen

Esta tesis doctoral explora problemas de clasificación y perturbación para acciones de grupo en una clase particular de variedades de Poisson llamadas variedades de b^m-Poisson. Las variedades de b^m-Poisson son variedades que son simplécticas fuera de una hipersuperficie en la cual satisfacen ciertas propiedades de transversalidad. A menudo modelan problemas en variedades simplécticas con borde tales como el estudio de la cuantización por deformación o problemas de mecánica celeste.

Una de las propiedades interesantes de las variedades b^m-Poisson es que se pueden estudiar usando el lenguage de b^m-formas. Es decir, que podemos trabajar con formas que son simplécticas lejos un conjunto crítico y que admiten una extensión suave como forma sobre un algebroide de Lie generalizando formas de De Rham como formas sobre el algebroide de Lie del fibrado tangente de la variedad. Para considerar b^m-formas el fibrado tangente estándar debe reemplazarse por el fibrado b^m-tangente.

Esta tesis empieza con una clasificación equivariante de estructuras b^m-Poisson, investigando, en particular, el análogo del teorema de clasificación de Moser para superficies simplécticas y sus análogos equivariantes. La clasificación de invariantes en el caso de superfícies estan codificados en una cohomologia llamada b^m-cohomologia que ha sido estudiada en profundiad por [1]. Una fórmula del tipo de Mazzeo-Melrose para la b^m-cohomologia descompone en dos partes que pueden interpretarse como las cohomologias de De Rham tanto de la variedad ambiente M como de la hypersuperficie crítica. Como consecuencia

de esta identificación, la clasificación de Poisson de estas variedades viene dada por la cohomologia de De Rham de la variedad y de la hipersuperficie.

Esta clasificación se extiende al contexto equivariante si asumimos que las formas singulares son preservadas por la acción de un grupo de Lie compacto. Estas técnicas pueden ser extendidas a la clasificación de structuras de b^m-Nambu que se consideran también en esta tesis.

Las acciones de grupo reaparecen en los últimos capitulos ya que los sistemas integrables en estas variedades resulta que tienen asociadas acciones Hamiltonianas de toros en un entorno de un toro de Liouville. Usamos estas acciones Hamiltonianas de grupos para demostrar la existencia de coordenadas accion-ángulo en un entorno de un toro de Liouville. El teorema de acción-ángulo que demostramos da un teorema de formas normales semilocales en un entorno del toro de Liouville para la forma b^m-simpléctica que depende tanto del peso modular de la componente conexa de la hipersuperfície donde se encuentra el toro de Liouville como los pesos asociados a la acción tórica. Este teorema de accion-ángulo nos permite identificar un entorno del toro de Liouville como el b^m-cotangent lift de la acción de un toro actuano por translaciones sobre sí mismo.

Acabamos la tesis demostrando un teorema KAM para variedades de b^m-Poisson que claramente refina y mejora el teorema obtenido para variedades de b-Poisson en [2]. Como consecuencia de este resultado junto con la extension de las técnicas de desingularización de Guillemin-Miranda-Weitsman en el ambiente de los sistemas integrables, obtenemos un teorema KAM para variedad folded-simplécticas donde la teoria KAM nunca ha sido considerada con anterioridad. En el camino, también obtenemos un nuevo teorema KAM para variedades simplécticas dónde la perturbación permite seguir con detalle una hipersuperície concreta. En mecánica celeste esta hipersuperfície puede ser interpretada cómo la línea al infinito o el conjunto de colisión.

Chapter 1

Introduction

Both symplectic and Poisson geometry emerge from the study of classical mechanics. Both are broad fields widely studied and with powerful results. But the fact that Poisson structures are far more general than the symplectic ones imply that a lot of powerful results in symplectic geometry do not translate well to Poisson manifolds. Here is where b^m-Poisson structures come to play. b^m-Poisson structures (or b^m-symplectic structures) lie somewhere between these two worlds. They extend symplectic structures but in a really controlled way. Because of this reason, a lot of results that worked in symplectic geometry still work in b^m-symplectic geometry.

The study of b^m-Poisson geometry sparked from the study of symplectic manifold with boundary [3]. In the last years the interest in this field increased after the classification result for b-Poisson structures obtained in [4]. Later on, [5] translated these structures to the language of forms and started applying symplectic tools to study them. A lot of papers in the following years studied different aspects of these structures: [6], [5], [7], [8], [9] and [10] are some examples.

Inspired by the study of manifolds with boundary, we work on a pair of manifolds (M, Z) where Z is an hypersurface and call this pair b-manifold

In this context, [1] generalized the b-symplectic forms by allowing higher degrees of degeneracy of the Poisson structures. The b^m-symplectic structures inherited most of the properties of b-symplectic structures. This thesis studies different aspects of b^m-symplectic structures. First, we present some preliminary notions necessary to understand the core of the thesis. Then we illustrate the connection between b^m-symplectic structures and classical mechanics by providing several examples. After this we present a result that classifies all possible b^m-symplectic structures on surfaces up to b^m-symplectomorphisms. This classification comes encoded by some cohomology classes associated to those structures. We also present a theorem that determines when it is possible for a b-manifold to have a b^m-symplectic structure on it. In the next section we present similar results for b^m-Nambu structures, which are top-degree structures with similar singularities as the ones allowed for b^m-symplectic structures. We also give a classification result, in this case for manifolds of any dimension, as well as an existence result. Finally we study the analog of KAM theory in the b^m-setting. We present an action-angle theorem for b^m-Poisson structures. Finally we state and prove the KAM theory equivalent in manifolds with b^m-symplectic structures.

Arising from this thesis there have appeared three different publications, and we hope a fourth will follow from the two last chapters. The first publication is [11], and presents the results that appear at Chapter 4, about classification of b^m-symplectic structures. The second paper published was [12] and presents the results about classification of b^m-Nambu structures explained in chapter 5. The last paper published was a joint effort from the laboratory of geometry and dynamical systems. The results were published at [13]. Some of the examples on this publication are presented in chapter 3. Finally we are hoping to have a version of chapters 6 and 7 adapted for sending to a journal and have it published soon.

1.1 Structure and results of this thesis

1.1.1 Chapter 2: Preliminaries

In the preliminaries we give the basic notions that lead to the questions we are addressing in this thesis. In the first part we introduce the concept of b-Poisson manifolds, a type of Poisson manifold that comes from the study of manifolds with boundary. Next we talk about a generalization of these structures, that allow higher degree of degeneracy of the structure: the b^m-symplectic structures. These structures are the main focus of our study in this thesis. We also introduce the concept of desingularizing these manifolds. Finally we give a short introduction to KAM theory, a theory that will be generalized in the setting of b^m-manifolds in the last chapter.

1.1.2 Chapter 3: Examples of singular symplectic forms in celestial mechanics

In this section we give several examples of singular symplectic structures appearing naturally in classical problems of celestial mechanics. We also have a section where we present the difficulties of finding these examples, and the subtleties of dealing with these structures.

First we present a change of coordinates in the Kepler problem and how this change transforms the standard symplectic form to a degenerate form along a hypersurface given by two hyperplanes.

Then we present a change of coordinates made in the restricted elliptic 3-body problem, that sends the standard symplectic form to a b^3-symplectic structure.

Finally we present an example of a change of coordinates in the two body problem that leads to a b-symplectic manifold, while talking about why it is hard to find more naturally appearing examples.

1.1.3 Chapter 4: Existence and classification of b^m-symplectic structures

In this chapter we present the results published in [11]. We start by presenting some examples of b^m-symplectic structures in both orientable and non-orientable surfaces. Then we give an equivariant version of the Moser theorem for b^m-symplectic surfaces which lead to a classification of these structures on surfaces.

Theorem (A). *Suppose that S is a closed surface, let Z be a union of non-intersecting embedded curves. Consider the b^m-manifold given by (S, Z). Fix $m \in \mathbb{N}$ and let ω_0 and ω_1 be two b^m-symplectic structures on (S, Z) which are invariant under the action of a compact Lie group $\rho : G \times (S, Z) \longrightarrow (S, Z)$ and defining the same b^m-cohomology class, $[\omega_0] = [\omega_1]$. Then, there exists an equivariant b^m-diffeomorphism $\xi_1 : (S, Z) \to (S, Z)$, such that $\xi_1^* \omega_1 = \omega_0$.*

We also state an equivariant b^m-Moser theorem for higher dimensions, taking into account that we need a path joining the two b^m-symplectic structures.

Theorem (B). *Let (M, Z) be a closed b^m-manifold with m a fixed natural number and let ω_t for $0 \leq t \leq 1$ be a smooth family of b^m-symplectic forms on (M, Z) such that the b^m-cohomology class $[\omega_t]$ does not depend on t.*

Assume that the family of b^m-symplectic structures is invariant by the action of a compact Lie group G on M, then, there exists a family of equivariant b^m-diffeomorphisms $\phi_t : (M, Z) \to (M, Z)$, with $0 \leq t \leq 1$ such that $\phi_t^* \omega_t = \omega_0$.

After this we present three theorems that talk about conditions on the manifolds to allow b^m-symplectic structures.

Theorem (C). *If a closed surface admits a b^{2k}-symplectic structure* *then it is orientable.*

Theorem (D). *Given a b^m-manifold (S, Z) (fixed m) with S closed and orientable, there exists a b^m-symplectic structure whenever:*

1. $m = 2k$,
2. $m = 2k + 1$ *if only if the associated graph $\Gamma(S, Z)$ is 2-colorable.*

Theorem (E). *Let (S, Z) be a closed non-orientable b^{2k+1}-surface. Then, (S, Z) admits a b^{2k+1}-symplectic structure if and only if the following two conditions hold:*

1. *the graph of the covering $(\tilde{S}, \tilde{Z})$, $G(\tilde{S}, \tilde{Z})$ is 2-colorable and*
2. *the non-trivial deck transformation inverts colors of the graph obtained in the covering.*

Finally we have a section where we talk about desingularizing b^m-symplectic structures. And what happens to the classes of the structures when desingularized.

Theorem (F). *Let (S, Z, x), be a b^{2k}-manifold, where S is a closed orientable surface and let ω_1 and ω_2 be two b^{2k}-symplectic forms. Also let $\omega_{1\epsilon}$ and $\omega_{2\epsilon}$ be the f_ϵ-desingularizations of ω_1 and ω_2 respectively. If $[\omega_1] = [\omega_2]$ in b^{2k}-cohomology then $[\omega_{1\epsilon}] = [\omega_{2\epsilon}]$ in de Rham cohomology for any fixed ϵ.*

1.1.4 Chapter 5: Existence and classification of b^m-Nambu structures

In this chapter we follow the results in [12]. We first define the concept of b^m-Nambu structure. Then we present some examples. We then present a theorem relating b^m-Nambu structures and orientability.

Theorem (G). *A compact n-dimensional manifold M admitting a b^{2k}-Nambu structure is orientable.*

We finally present an equivariant theorem on classification of b^m-Nambu structures extending the results of [14].

Theorem (H). *Let Θ_0 and Θ_1 be two b^m-Nambu forms of degree n on a compact orientable manifold M^n and let $\rho : G \times M \longrightarrow M$ be a compact Lie group action preserving both b^m-forms. If $[\Theta_0] = [\Theta_1]$ in b^m-cohomology then there exists an equivariant diffeomorphism ϕ such that $\phi^*\Theta_1 = \Theta_0$.*

1.1.5 Chapter 6: An action-angle theorem for b^m-Poisson manifolds

In this section we define the concept of b^m-functions, b^m-integrable systems. We present several examples of b^m-integrable systems that come from classical mechanics. After all this we present a version of the action-angle theorem for b^m-symplectic manifolds.

Theorem (I). *Let (M, x, ω, F) a b^m-integrable system, where $F = (f_1 = a_0 \log(x) + \sum_{j=1}^{m-1} a_j \frac{1}{x^j}, f_2, \ldots, f_n)$. Let $m \in Z$ be a regular point, and such that the integral manifold through m is compact. Let $\mathcal{F}_m$ be the Liouville torus through m. Then, there exists a neighborhood U of $\mathcal{F}_m$ and coordinates $(\theta_1, \ldots, \theta_n, \sigma_1, \ldots, \sigma_n) : \mathcal{U} \to \mathbb{T}^n \times B^n$ such that:*

1. *We can find an equivalent integrable system $F = (f_1 = a'_0 \log(x) + \sum_{j=1}^{m-1} a'_j \frac{1}{x^j})$ such that $a'_0, \ldots, a'_{m-1} \in \mathbb{R}$,*

2. $$\omega|_{\mathcal{U}} = \left(\sum_{j=1}^{m} c'_j \frac{c}{\sigma_1^j} d\sigma_1 \wedge d\theta_n \right) + \sum_{i=2}^{n} d\sigma_i \wedge d\theta_i$$
   where c is the modular period and $c'_j = -(j-1)a'_{j-1}$, also

3. *the coordinates $\sigma_1, \ldots, \sigma_n$ depend only on $f_n, \ldots f_n$.*

1.1.6 Chapter 7: KAM theory on b^m-symplectic manifolds

In this chapter we give our version of a KAM theorem for b^m-symplectic manifolds. We begin by presenting the structure of the chapter. Then we give an outline of how to construct the b^m-symplectomorphism that will be the main protagonist in the proof of the theorem. After this we present some technical results that are needed for the proof, which are quite similar to the standard KAM equivalents, but there are some subtleties that need to be adressed. After all the preliminaries we state and prove the b^m-KAM theorem.

Theorem (J). *Let $\mathcal{G} \subset \mathbb{R}^n$, $n \geq 2$ be a compact set. Let $H(\phi, I) = \hat{h}(I) + f(\phi, I)$, where $\hat{h}$ is a b^m-function $\hat{h}(I) = h(I) + q_0 \log(I_1) + \sum_{i=1}^{m-1} \frac{q_i}{I_1^i}$ defined on $\mathcal{D}_\rho(G)$, with $h(I)$ and $f(\phi, I)$ analytic. Let $\hat{u} = \frac{\partial \hat{h}}{\partial I}$ and $u = \frac{\partial h}{\partial I}$. Assume $|\frac{\partial u}{\partial I}|_{G,\rho_2} \leq M$, $|u|_\xi \leq L$. Assume that u is μ non-degenerate ($|\frac{\partial u}{\partial I}| \geq \mu |v|$ for some $\mu \in \mathbb{R}^+$ and $I \in \mathcal{G}$. Take $a = 16M$. Assume that u is one-to-one on $\mathcal{G}$ and its range $F = u(\mathcal{G})$ is a D-set. Let $\tau > n-1, \gamma > 0$ and $0 < \nu < 1$. Let*

1. $$\varepsilon := \|f\|_{\mathcal{G},\rho} \leq \frac{\nu^2 \mu^2 \hat{\rho}^{2\tau+2}}{2^{4\tau+32} L^6 M^3} \gamma^2, \tag{1.1}$$

2. $$\gamma \leq \min\left(\frac{8LM\rho_2}{\nu \hat{\rho}^{\tau+1}}, \frac{L}{\mathcal{K}'}\right) \tag{1.2}$$

3. $$\mu \leq \min(2^{\tau+5} L^2 M, 2^7 \rho_1 L^4 K^{\tau+1}, \beta \nu^{\tau+1} 2^{2\tau+1} \rho_1^\tau), \tag{1.3}$$

where $\hat{\rho} := \min\left(\frac{\nu \rho_1}{12(\tau+2)}, 1\right)$. Define the set $\hat{G} = \hat{G}_\gamma := \{I \in \mathcal{G} - \frac{2\gamma}{\mu} | u(I)$ is $\tau, \gamma, c, \hat{q}-$Dioph.$\}$. Then, there exists a real continuous map $\mathcal{T} : \mathcal{W}_{\frac{\rho_1}{4}}(\mathbb{T}^n) \times \hat{G} \to \mathcal{D}_\rho(\mathcal{G})$ analytic with respect the angular variables such that

1. *For all* $I \in \hat{G}$ *the set* $\mathcal{T}(\mathbb{T}^n \times \{I\})$ *is an invariant torus of* H*, its frequency vector is equal to* $u(I)$*.*

2. *Writing* $\mathcal{T}(\phi, I) = (\phi + \mathcal{T}_\phi(\phi, I), I + \mathcal{T}_I(\phi, I))$ *with estimates*

$$|\mathcal{T}_\phi(\phi, I)| \leq \frac{2^{2\tau+15} ML^2}{\nu^2 \hat{\rho}^{2\tau+1}} \frac{\varepsilon}{\gamma^2}$$

$$|\mathcal{T}_I(\phi, I))| \leq \frac{2^{10+\tau} L(1+M)}{\nu \hat{\rho}^{\tau+1}} \frac{\varepsilon}{\gamma}$$

3. *meas*$[(\mathbb{T}^n \times \mathcal{G}) \setminus \mathcal{T}(\mathbb{T}^n \times \hat{G})] \leq C\gamma$ *where* C *is a really complicated constant depending on* n*,* μ*,* D*, diam*F*,* M*,* τ*,* ρ_1*,* ρ_2*,* K *and* L*.*

Also, we obtain a way to associate a standard symplectic integrable system or a folded integrable system to a b^m-integrable system, depending on the parity of m. This is done in such a way that the dynamics of the desingularized system are the same than the dynamics of the original one.

Theorem (K)**.** *The desingularization transforms a* b^m*-integrable system into an integrable system for* m *even on a symplectic manifold. For* m *odd the desingularization transforms it to a folded integrable system. The integrable systems are such that:*

$$X_{f_j}^{\omega} = X_{f_{j\epsilon}}^{\omega_\epsilon}.$$

Also this allows us to obtain two new KAM theorems using this desingularization in conjunction with our b^m-KAM theorem. The first of this theorems is a KAM theorem for standard symplectic manifolds, where the perturbation has a particular expression. This result is more restrictive than the standard KAM but in exchange we can ensure that the perturbations leave a given hypersurface invariant. This means that the tori belonging to that hypersurface remain there after the perturbation.

Theorem (L). *Consider a neighborhood of a Liouville torus of an integrable system F_ε as in 7.26 of a symplectic manifold (M, ω_ε) semilocally endowed with coordinates (I, ϕ), where ϕ are the angular coordinates of the torus, with $\omega_\varepsilon = c' dI_1 \wedge d\phi_i + \sum_{j=1}^n dI_j \wedge d\phi_j$. Let $H = (m-1)c_{m-1}c' I_1 + h(\tilde{I}) + R(\tilde{I}, \tilde{\phi})$ be a nearly integrable system where*

$$\begin{cases} \tilde{I}_1 &= c' \frac{I_1^{m+1}}{m+1}, \\ \tilde{\phi}_1 &= c' I_1^m \phi_1, \end{cases}$$

and

$$\begin{cases} \tilde{I} &= (\tilde{I}_1, I_2, \ldots, I_n), \\ \tilde{\phi} &= (\tilde{\phi}_1, \phi_2, \ldots, \phi_n). \end{cases}$$

Then the results for the b^m-KAM theorem 7.3.1 applied to $H_{sing} = \frac{1}{I_1^{2k-1}} + h(I) + R(I, \phi)$ hold for this desingularized system.

The second one is a KAM theorem for folded-symplectic manifolds, where KAM theory never was considered before.

Theorem (M). *Consider a neighborhood of a Liouville torus of an integrable system F_ε as in 7.27 of a folded symplectic manifold (M, ω_ε) semilocally endowed with coordinates (I, ϕ), where ϕ are the angular coordinates of the Torus, with $\omega_\varepsilon = 2cI_1 dI_1 \wedge d\phi_1 + \sum_{j=2}^m dI_j \wedge d\phi_j$. Let $H = (m-1)c_{m-1}cI_1^2 + h(\tilde{I}) + R(\tilde{I}, \tilde{\phi})$ a nearly integrable system with*

$$\begin{cases} \tilde{I}_1 &= 2c \frac{I_1^{m+2}}{m+2}, \\ \tilde{\phi}_1 &= 2c I_1^{m+1} \phi_1, \end{cases}$$

and

$$\begin{cases} \tilde{I} &= (\tilde{I}_1, I_2, \ldots, I_n), \\ \tilde{\phi} &= (\tilde{\phi}_1, \phi_2, \ldots, \phi_n). \end{cases}$$

Then the results for the b^m-KAM theorem 7.3.1 applied to $H_{sing} = \frac{1}{I_1^{2k}} + h(I) + R(I, \phi)$ hold for this desingularized system.

1.2 Publications resulting from this thesis

As stated previously the results of this thesis can be found in the following articles:

1. E. Miranda and A. Planas, "Equivariant classification of b^m-symplectic surfaces," *Regular and Chaotic Dynamics*, vol. 23, pp. 355–371, Jul 2018.

2. E. Miranda and A. Planas, "Classification of b^m-Nambu structures of top degree," *C. R. Math. Acad. Sci. Paris*, vol. 356, no. 1, pp. 92–96, 2018.

3. R. Braddell, A. Delshams, E. Miranda, C. Oms, and A. Planas, "An invitation to singular symplectic geometry," *International Journal of Geometric Methods in Modern Physics*, 05 2017.

4. E. Miranda and A. Planas, "A KAM theorem for b^m-symplectic manifolds," *Pre-print*.

Chapter 2

Preliminaries

Let M be a smooth manifold, a **Poisson structure** on M is a bilinear map $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ which is skew symmetric and satisfies both the Jacobi identity and the Leibniz rule. It is possible to express $\{f, g\}$ in terms of a bivector field via the following equality $\{f, g\} = \Pi(df \wedge dg)$ with Π a section of $\Lambda^2(TM)$. Π is the associated **Poisson bivector**. We will use indistinctively the terminology of Poisson structure when referring to the bracket or the Poisson bivector.

A *b-Poisson bivector field* on a manifold M^{2n} is a Poisson bivector such that the map

$$F : M \to \bigwedge^{2n} TM : p \mapsto (\Pi(p))^n \tag{2.1}$$

is transverse to the zero section. Then, a pair (M, Π) is called a ***b*-Poisson manifold** and the vanishing set Z of F is called the **critical hypersurface**. Observe that Z is an embedded hypersurface.

This class of Poisson structures was studied by Radko [4] in dimension two and considered in numerous papers in the last years: [6], [5], [7], [8], [9] and [10] among others.

2.1 b-Poisson manifolds

Next, we recall Radko's classification theorem and the cohomological re-statement presented in [5].

In what follows, (M, Π) will be a closed smooth surface with a b-Poisson structure on it, and Z its critical hypersurface.

Let h be the distance function to Z as in [9][1].

Definition 2.1.1. *The **Liouville volume of** (M, Π) is the following limit:* $V(\Pi) := \lim_{\epsilon \to 0} \int_{|h|>\epsilon} \omega^n$[2].

The previous limit exists and it is independent of the choice of the defining function h of Z (see [4] for the proof).

Definition 2.1.2. *For any (M, Π) oriented Poisson manifold, let Ω be a volume form on it, and let u_f denote the Hamiltonian vector field of a smooth function $f : M \to \mathbb{R}$. The **modular vector field** X^Ω is the derivation defined as follows:*

$$f \mapsto \frac{\mathcal{L}_{u_f}\Omega}{\Omega}.$$

Definition 2.1.3. *Given γ a connected component of the critical set $Z(\Pi)$ of a closed b-Poisson manifold (M, Π), the **modular period** of Π around γ is defined as:*

$$T_\gamma(\Pi) := \textit{period of } X^\Omega|_\gamma.$$

Remark 2.1.4. *The modular vector field X^Ω of the b-Poisson manifold (M, Z) does not depend at Z on the choice of Ω because for different choices for volume form the difference of modular vector fields is*

[1] Notice the difference with [4] where h is assumed to be a global defining function.

[2] For surfaces $n = 1$.

a Hamiltonian vector field. Observe that this Hamiltonian vector field vanishes on the critical set as Π *vanishes there too.*

Definition 2.1.5. *Let* $\mathcal{M}_n(M) = \mathcal{C}_n(M)/\sim$ *where* $\mathcal{C}_n(M)$ *is the space of disjoint oriented curves and* $\sim$ *identifies two sets of curves if there is an orientation-preserving diffeomophism mapping the first one to the second one and preserving the orientations of the curves.*

The following theorem classifies b-symplectic structures on surfaces using these invariants:

Theorem 2.1.6 (Radko [4]). *Consider two b-Poisson structures* Π, Π' *on a closed orientable surface* M*. Denote its critical hypersurfaces by* Z *and* Z'*. These two b-Poisson structures are globally equivalent (there exists a global orientation preserving diffeomorphism sending* Π *to* Π'*) if and only if the following coincide:*

- *the equivalence classes of* $[Z]$ *and* $[Z'] \in \mathcal{M}_n(M)$*,*
- *their modular periods around the connected components of* Z *and* Z'*,*
- *their Liouville volume.*

An appropriate formalism to deal with these structures was introduced in [6].

Definition 2.1.7. *A* b***-manifold***[3] *is a pair* (M, Z) *of a manifold and an embedded hypersurface.*

In this way the concept of b-manifold previously introduced by Mel-

[3]The 'b' of b-manifolds stands for 'boundary', as initially considered by Melrose (Chapter 2 of [15]) for the study of pseudo-differential operators on manifolds with boundary.

rose is generalized.

Definition 2.1.8. *A* b***-vector field*** *on a* b*-manifold* (M, Z) *is a vector field tangent to the hypersurface* Z *at every point* $p \in Z$*.*

Definition 2.1.9. *A* b***-map*** *from* (M, Z) *to* (M', Z') *is a smooth map* $\phi : M \to M'$ *such that* $\phi^{-1}(Z') = Z$ *and* ϕ *is transverse to* Z'*.*

Observe that if x is a local defining function for Z and $(x, x_1, \ldots, x_{n-1})$ are local coordinates in a neighborhood of $p \in Z$ then the $C^\infty(M)$-module of b-vector fields has the following local basis

$$\{x\frac{\partial}{\partial x}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}\}. \tag{2.2}$$

In contrast with [6], in this paper we are not requiring the existence of a global defining function for Z and orientability of M but we require the existence of a defining function in a neighborhood of each point of Z. By relaxing this condition the normal bundle of Z need not be trivial.

Given (M, Z) a b-manifold, [6] shows that there exists a vector bundle, denoted by ${}^bTM$ whose smooth sections are b-vector fields. This bundle is called the b**-tangent bundle** of (M, Z).

The b**-cotangent bundle** ${}^bT^*M$ is defined using duality. A b-**form** is a section of the b-cotangent bundle. Around a point $p \in Z$ the $C^\infty(M)$-module of these sections has the following local basis:

$$\{\frac{1}{x}dx, dx_1, \ldots, dx_{n-1}\}. \tag{2.3}$$

In the same way we define a b**-form of degree** k to be a section of the bundle $\bigwedge^k({}^bT^*M)$, the set of these forms is denoted ${}^b\Omega^k(M)$. Denoting by f the distance function[4] to the critical hypersurface Z, we may write the following decomposition as in [6] for any $\omega \in^b \Omega^k(M)$:

[4] Originally in [6] f stands for a global function, but for non-orientable manifolds we may use the distance function instead.

$$\omega = \alpha \wedge \frac{df}{f} + \beta, \text{ with } \alpha \in \Omega^{k-1}(M) \text{ and } \beta \in \Omega^k(M). \tag{2.4}$$

This decomposition allows to extend the differential of the de Rham complex d to ${}^b\Omega(M)$ by setting $d\omega = d\alpha \wedge \frac{df}{f} + d\beta$. The associated cohomology is called **b-cohomology** and it is denoted by ${}^bH^*(M)$.

Definition 2.1.10. *A b-**symplectic** form on a b-manifold (M^{2n}, Z) is defined as a non-degenerate closed b-form of degree 2 (i.e., ω_p is of maximal rank as an element of $\Lambda^2({}^bT_p^*M)$ for all $p \in M$).*

The notion of b-symplectic forms is dual to the notion of b-Poisson structures. The advantage of using forms is that symplectic tools can be 'easily' exported.

Radko's classification theorem [4] can be translated into this language. This translation was already formulated in [6]:

Theorem 2.1.11 (Radko's theorem in b-cohomological language, [5]). *Let S be a closed orientable surface and let ω_0 and ω_1 be two b-symplectic forms on (S, Z) defining the same b-cohomology class (i.e.,$[\omega_0] = [\omega_1]$). Then there exists a diffeomorphism $\phi : S \to S$ such that $\phi^*\omega_1 = \omega_0$.*

2.2 On b^m-Symplectic manifolds

2.2.1 Basic definitions

By relaxing the transversality condition allowing higher order singularities ([16] and [17]) we may consider other symplectic structures with singularities as done by Scott [1] with b^m-symplectic structures. Let m be a positive integer a **b^m-manifold** is a **b-manifold** (M, Z) together with a b^m-tangent bundle attached to it. The b^m-tangent bundle is (by

Serre-Swan theorem [18]) a vector bundle, ${}^{b^m}TM$ whose sections are given by,

$$\Gamma({}^{b^m}TM) = \{v \in \Gamma(TM) : v(x) \quad \text{vanishes to order } m \text{ at } Z\},$$

where x is a defining function for the critical set Z in a neighborhood of each connected component of Z and can be defined as $x : M \setminus Z \to (0, \infty), x \in C^\infty(M)$ such that:

- $x(p) = d(p)$ a distance function from p to Z for $p : d(p) \leq 1/2$
- $x(p) = 1$ on $M \setminus \{p \in M$ such that $d(p) < 1\}$.[5]

(This definition of x allows us to extend the construction in [1] to the non-orientable case as in [9].) We may define the notion of a b^m-map as a map in this category (see [1]). The sections of this bundle are referred to as **b^m-vector fields** and their flows define b^m-maps. In local coordinates the sections of the b^m-tangent bundle are generated by:

$$\{x^m \frac{\partial}{\partial x}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}\}. \tag{2.5}$$

Proceeding *mutatis mutandis* as in the b-case one defines the b^m-cotangent bundle (${}^{b^m}T^*M$), the b^m-de Rham complex and the b^m-symplectic structures.

A **Laurent Series** of a closed b^m-form ω is a decomposition of ω in a tubular neighborhood U of Z of the form

$$\omega = \frac{dx}{x^m} \wedge (\sum_{i=0}^{m-1} \pi^*(\alpha_i) x^i) + \beta \tag{2.6}$$

with $\pi : U \to Z$ the projection of the tubular neighborhood onto Z, α_i a closed smooth de Rham form on Z and β a de Rham form on M.

[5]Then a b^m-manifold will be a triple (M, Z, x), but for the sake of simplicity we refer to it as a pair (M, Z) and we tacitly assume the function x is fixed.

In [1] it is proved that in a neighborhood of Z, every closed b^m-form ω can be written in a Laurent form of type (2.6) having fixed a (semi)local defining function.

b^m-Cohomology is related to de Rham cohomology via the following theorem:

Theorem 2.2.1 (b^m**-Mazzeo-Melrose**, [1])**.** *Let (M, Z) be a b^m-manifold, then:*

$$ {}^{b^m}H^p(M) \cong H^p(M) \oplus (H^{p-1}(Z))^m. \tag{2.7} $$

The isomorphism constructed in the proof of the theorem above is non-canonical (see [1]).

The Moser path method can be generalized to b^m-symplectic structures:

Theorem 2.2.2 (**Moser path method**, [1])**.** *Let ω_0, ω_1 be two b^m-symplectic forms defining the same b^m-cohomology class $[\omega_0] = [\omega_1]$ on (M^{2n}, Z) with M^{2n} closed and orientable then there exist a b^m-symplectomorphism $\varphi : (M^{2n}, Z) \longrightarrow (M^{2n}, Z)$ such that $\varphi^*(\omega_1) = \omega_0$.*

An outstanding consequence of Moser path method is a global classification of closed orientable b^m-symplectic surfaces à la Radko in terms of b^m-cohomology classes.

Theorem 2.2.3 (**Classification of closed orientable b^m-surfaces,** [**1**])**.** *Let ω_0 and ω_1 be two b^m-symplectic forms on a closed orientable connected b^m-surface (S, Z). Then, the following conditions are equivalent:*

- *their b^m-cohomology classes coincide $[\omega_0] = [\omega_1]$,*
- *the surfaces are globally b^m-symplectomorphic,*
- *the Liouville volumes of ω_0 and ω_1 and the numbers*

$$ \int_\gamma \alpha_i $$

for all connected components $\gamma \subseteq Z$ and all $1 \leq i \leq m$ coincide (where α_i are the one-forms appearing in the Laurent decomposition of the two b^m-forms of degree 2, ω_0 and ω_1).

Definition 2.2.4. *The numbers $[\alpha_i] = \int_\gamma \alpha_i$ are called* modular weights *for the connected components $\gamma \subset Z$.*

A relative version of Moser path method is proved in [8] as a corollary we obtain the following local description of a b^m-symplectic manifold:

Theorem 2.2.5 (b^m**-Darboux theorem, [8]**)**.** *Let ω be a b^m-symplectic form on (M, Z) and $p \in Z$. Then we can find a coordinate chart $(U, x_1, y_1, \ldots, x_n, y_n)$ centered at p such that on U the hypersurface Z is locally defined by $x_1 = 0$ and*

$$\omega = \frac{dx_1}{x_1^m} \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

Remark 2.2.6. *For the sake of simplicity sometimes we will omit describing Z and we will talk directly about b^m-symplectic structures on manifolds M implicitly assuming that Z is the vanishing locus of Π^n where Π is the Poisson vector field dual to the b^m-symplectic form.*

Next we present two lemmas that allow us to talk about b^m-symplectic structures and b^m-Poisson as a single kind of structure. They are dual to each other and in one-to-one correspondence.

Lemma 2.2.7. *Let ω be a b^m-symplectic and Π its dual vector field, then Π is b^m-Poisson.*

Proof. The quickest way to do this is to take the inverse, which is a bivector field, and observe that it is a Poisson structure (because $d\omega = 0$ implies $[\Pi, \Pi] = 0$). To see that it is b^m-Poisson it is enough to take a point p on the critical set Z and because of our b^m-Darboux theorem $\omega = dx_1/x_1^m \wedge dy_1 + \sum_{i>1} dx_i \wedge dy_i$ This means that in the new coordinate system $\Pi = x_1^m \partial x_1 \wedge \partial y_1 + \sum_{i>1} \partial x_i \wedge \partial y_i$ and thus Π is b^m-Poisson. $\square$

Lemma 2.2.8. *Let Π be b^m-Poisson and ω its dual vector field, then ω is b^m-symplectic.*

Proof. If Π transverse à la Thom on Z with singularity of order m then because of Weinstein's splitting theorem we can locally write

$$\Pi = x_1^m \partial x_1 \wedge \partial y_1 + \sum_{i>1} \partial x_i \wedge \partial y_i$$

now its inverse is $\omega = dx_1/x_1^m \wedge dy_1 + \sum_{i>1} dx_i \wedge dy_i$ which is a b^m-symplectic form. $\square$

Hence we have a correspondence from b^m-symplectic to b^m-Poisson.

2.2.2 Desingularizing b^m-Poisson manifolds

In [8] Guillemin, Miranda and Weitsman presented a desingularization procedure for b^m-symplectic manifolds proving that we may associate a family of folded symplectic or symplectic forms to a given b^m-symplectic structure depending on the parity of m. Namely,

Theorem 2.2.9 (Guillemin-Miranda-Weitsman, [8]). *Let ω be a b^m-symplectic structure on a closed orientable manifold M and let Z be its critical hypersurface.*

- *If $m = 2k$, there exists a family of symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighborhood of Z and for which the family of bivector fields $(\omega_\epsilon)^{-1}$ converges in the C^{2k-1}-topology to the Poisson structure ω^{-1} as $\epsilon \to 0$.*

- *If $m = 2k+1$, there exists a family of folded symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighborhood of Z.*

As a consequence of Theorem 2.2.9, any closed orientable manifold that supports a b^{2k}-symplectic structure necessarily supports a symplectic structure.

In [8] explicit formulae are given for even and odd cases. Let us refer here to the even dimensional case as these formulae will be used later on.

Let us briefly recall how the desingularization is defined and the main result in [8]. Recall that we can express the b^{2k}-form as:

$$\omega = \frac{dx}{x^{2k}} \wedge \left(\sum_{i=0}^{2k-1} x^i \alpha_i \right) + \beta. \tag{2.8}$$

This expression holds on a ϵ-tubular neighborhood of a given connected component of Z. This expression comes directly from equation 2.6, to see a proof of this result we refer to [1].

Definition 2.2.10. *Let (S, Z, x), be a b^{2k}-manifold, where S is a closed orientable manifold and let ω be a b^{2k}-symplectic form. Consider the decomposition given by the expression (2.8) on an ϵ-tubular neighborhood U_ϵ of a connected component of Z.*

Let $f \in \mathcal{C}^\infty(\mathbb{R})$ be an odd smooth function satisfying $f'(x) > 0$ for all $x \in [-1, 1]$ and satisfying outside that

$$f(x) = \begin{cases} \frac{-1}{(2k-1)x^{2k-1}} - 2 & \text{for} \quad x < -1, \\ \frac{-1}{(2k-1)x^{2k-1}} + 2 & \text{for} \quad x > 1. \end{cases} \tag{2.9}$$

Let $f_\epsilon(x)$ be defined as $\epsilon^{-(2k-1)} f(x/\epsilon)$.

*The f_ϵ-**desingularization** ω_ϵ is a form that is defined on U_ϵ by the following expression:*

$$\omega_\epsilon = df_\epsilon \wedge \left(\sum_{i=0}^{2k-1} x^i \alpha_i \right) + \beta.$$

This desingularization procedure is also known as **deblogging** in the literature.

Remark 2.2.11. *Though there are infinitely many choices for f, we will assume that we choose one, and assume it fixed through the rest of the discussion. It would be interesting to discuss the existence of an isotopy of forms under a change of function f.*

Remark 2.2.12. *Because ω_ϵ can be trivially extended to the whole S in such a way that it agrees with ω (see [8]) outside a neighborhood of Z, we can talk about the f_ϵ-desingularization of ω as a form on S.*

2.3 A crash course on KAM theory

The last chapter of this thesis is entirely dedicated to prove a KAM theorem for b^m-symplectic structures. So the aim of this section is to give a quick overview on the traditional KAM theorem. The setting of the KAM theorem is a sympletic manifold with action angle coordinates and an integrable system in it. The theorem says that under small perturbations of the Hamiltonian "most" of the Liouville tori survive.

Consider $\mathbb{T}^n \times G \subset \mathbb{T}^n \times \mathbb{R}^n$ with action-angle coordinates in it $(\phi_1, \ldots, \phi_n, I_1, \ldots, I_n)$ and the standard symplectic form ω in it. And assume the Hamiltonian function of the system is given by $h(I)$ a function only depending on the action coordinates. Then the Hamilton equations of the system are given by

$$\iota_{X_h}\omega = dh$$

where X_h is the vector field generating the trajectories. Because h does not depend on ϕ the angular variables the system is really easy to solve, and the equations are given by

$$x(t) = (\phi(t), I(t)) = (\phi_0 + ut, I_0),$$

where $u = \partial h/\partial I$ is called the frequency vector. These motions for a fixed initial condition are inside a Liouville torus, and are called quasiperiodic.

The KAM theorem studies what happens to such a systems when a small perturbation is applied to the Hamiltonian function, i.e. we consider the evolution of the system given by the Hamiltonian $h(I) + R(I, \phi)$, where we think of the term $R(I, \phi)$ as the small perturbation in the system. With this in mind the Hamilton equations can be written as

$$\dot{\phi} = u(I) + \frac{\partial}{\partial I} R(I, \phi), \dot{I} = -\frac{\partial}{\partial \phi} R(I, \phi),$$

Another important concept to have in mind is the concept of *rational dependency*. A frequency u is rationally dependent if $\langle u, k \rangle = 0$ for some $k \in \mathbb{Z}^n$, if there exists no k satisfying the condition then the vector u is called rationally independent. There is a stronger concept to being rationally independent and that is the concept of being Diophantine. A vector u is γ,τ-diophantine if $\langle u, k \rangle \geq \frac{\gamma}{|k|_1^\tau}$ for all $k \in \mathbb{Z}^n \setminus \{0\}$. $\gamma > 0$ and $\tau > n - 1$.

The KAM theorem states that the Liouville tori with frequency vector satisfying the diophantine condition survive under the small perturbation $R(I, \phi)$. There are conditions relating the size of the perturbation with γ and τ. Also the set of tori satisfying the Diophantine condition has measure $1 - C\gamma$ for some constant C.

Now we give a proper statement of the theorem as was given in [19].

Theorem 2.3.1 (Isoenergetic KAM theorem)**.** *Let $\mathcal{G} \subset \mathbb{R}^n$, $n > 2$, a compact, and let $H(\phi, I) = h(I) + f(\phi, I)$ real analytic on $\mathcal{D}_\rho(\mathcal{G})$. Let $\omega = \partial h/\partial I$, and assume the bounds:*

$$\left|\frac{\partial^2 h}{\partial I^2}\right|_{\mathcal{G},\rho_2} \leq M, \quad |\omega|_{\mathcal{G}} \leq L \quad \textit{and} \quad |\omega_n(I)| \geq l \forall I \in \mathcal{G}.$$

Assume also that ω *is* μ*-isoenergetically nondegenerate on* $\mathcal{G}$*. For* $a = 16M/l^2$*, assume that the map* $\Omega = \Omega_{\omega,h,a}$ *is one-to-one on* $\mathcal{G}$*, and that its range* $F = \Omega(\mathcal{G})$ *is a D-set. Let* $\tau > n-1$*,* $\gamma > 0$ *and* $0 < \nu < 1$ *given, and assume:*

$$\varepsilon := \|f\|_{\mathcal{G},\rho} \leq \frac{\nu^2 l^6 \mu^2 \hat{\rho}^{2\tau+2}}{2^{4\tau+32} L^6 M^3} \cdot \gamma^2, \quad \gamma \leq \min\left(\frac{8LM\rho_2}{\nu l \hat{\rho}^{\tau+1}}, l\right),$$

where we write $\rho := \min\left(\frac{\nu\rho_1}{12(\tau+2)}, 1\right)$*. Define the set*

$$\hat{G} = \hat{G}_\gamma := \left\{ I \in \mathcal{G} - \frac{2\gamma}{\mu} : \omega(I) is \tau, \gamma - Diophantine \right\}.$$

Then, there exists a real continuous map $\mathcal{T} : \mathcal{W}_{\frac{\rho_1}{4}}(\mathbb{T}^n) \times \hat{G} \to \mathcal{D}_\rho(\mathcal{G})$*, analytic with respect to the angular variables, such that:*

1. *For every* $I \in \hat{G}$*, the set* $\mathcal{T}(\mathbb{T}^n \times \{I\})$ *is an invariant torus of* H*, its frequency vector is colinear to* $\omega(I)$ *and its energy is* $h(I)$*.*

2. *Writing*

$$\mathcal{T}(\phi, I) = (\phi + \mathcal{T}_\phi(\phi, I), I + \mathcal{T}_I(\phi, I)),$$

one has the estimates

$$|\mathcal{T}_\phi|_{\hat{G},(\frac{\rho_1}{4},0),\infty} \leq \frac{2^{2\tau+15} L^2 M}{\nu^2 l^2 \hat{\rho}^{2\tau+1}} \frac{\varepsilon}{\gamma^2}, \quad |\mathcal{T}_I|_{\hat{G},(\frac{\rho_1}{4},0)} \leq \frac{2^{\tau+16} L^3 M}{\nu l^3 \mu \hat{\rho}^{\tau+1}} \frac{\varepsilon}{\gamma}$$

3. *meas*$[(\mathbb{T}^n \times \mathcal{G}) \setminus \mathcal{T}(\mathbb{T}^n \times \hat{G})] \leq C\gamma$*, where* C *is a very complicated constant depending on* n*,* τ*, diam*F*,* D*,* $\hat{\rho}$*,* M*,* L*,* l*,* μ*.*

Remark 2.3.2. *This verion of the KAM theorem is the isoenergetic one, this version ensures that the energy of the Liouville Tori identified by the diffeomorphism after the perturbation remains the same as before the perturbation. Our version of the* b^m*-KAM is not isoenergetic for the sake of simplifying the computations.*

Also we have to remark that the KAM theorem has already been explored in singular symplectic manifolds before. In [2] the authors proved a KAM theorem for b-symplectic manifolds, for a particular kind of perturbations.

Theorem 2.3.3 (KAM Theorem for b-Poisson manifolds)**.** *Let $\mathbb{T}^n \times B_r^n$ be endowed with standard coordinates (φ, y) and the b-symplectic structure. Consider a b-function*

$$H = k \log |y_1| + h(y)$$

on this manifold, where h is analytic. Let y_0 be a point in B_r^n with first component equal to zero, so that the corresponding level set $\mathbb{T}^n \times \{y_0\}$ lies inside the critical hypersurface Z.

Assume that the frequency map

$$\tilde{\omega} : B_r^n \to \mathbb{R}^{n-1}, \quad \tilde{\omega}(y) := \frac{\partial h}{\partial \tilde{y}}(y)$$

has a Diophantine value $\tilde{\omega} := \tilde{\omega}(y_0)$ at $y_0 \in B^n$ and that it is non-degenerate at y_0 in the sense that the Jacobian $\frac{\partial \tilde{\omega}}{\partial \tilde{y}}(y_0)$ is regular.

Then the torus $\mathbb{T}^n \times \{y_0\}$ persists under sufficiently small perturbations of H which have the form mentioned above, i.e. they are given by ϵP, where $\epsilon \in \mathbb{R}$ and $P \in {}^b C^\infty(\mathbb{T}^n \times B_r^n)$ has the form

$$\begin{aligned} P(\varphi, y) &= k' \log |y_1| + f(\varphi, y) \\ f(\varphi, y) &= f_1(\tilde{\varphi}, y) + y_1 f_2(\varphi, y) + f_3(\varphi_1, y_1). \end{aligned}$$

More precisely, if $|\epsilon|$ is sufficiently small, then the perturbed system

$$H_\epsilon = H + \epsilon P$$

admits an invariant torus $\mathcal{T}$.

Moreover, there exists a diffeomorphism $\mathbb{T}^n \to \mathcal{T}$ close[6] *to the identity taking the flow γ^t of the perturbed system on $\mathcal{T}$ to the linear flow*

[6] By saying that the diffeomorphism is "ϵ-close to the identity" we mean that, for given H, P and r, there is a constant C such that $\|\psi - \mathrm{Id}\| < C\epsilon$.

on $\mathbb{T}^n$ *with frequency vector*

$$\left(\frac{k + \epsilon k'}{c}, \tilde{\omega}\right).$$

Chapter 3

Examples of singular structures in Celestial Mechanics

In this chapter we present several examples appearing in Celestial Mechanics where singular symplectic forms show up. Part of this chapter (not all) is contained in the article [13]. Most of the singularities appear as an outcome of *regularization* techniques. We invite the reader to consult the book [20] for a pedagogical approach to the study of regularization.

This list of examples is of special relevance for this thesis as the theoretical results that we obtain such as action-angle coordinates or KAM can be, de facto, applied to the list of problems considered below.

3.1 Transformations and Singular Symplectic Forms

Structures which are symplectic almost everywhere can arise as the result of a non-canonical changes of coordinates. Given configuration space $\mathbb{R}^2$ and phase space $T^*\mathbb{R}^2$ as is seen, for example, in the Kepler problem, the traditional (canonical) Levi-Civita transformation is the following: identify $\mathbb{R}^2 \cong \mathbb{C}$ so that $T^*\mathbb{R}^2 \cong T^*\mathbb{C} \cong \mathbb{C}^2$ and treat (q, p) as complex variables $(q_1 + iq_2 := u, p_1 + ip_2 := v)$. Take the following change of coordinates $(q, p) = (u^2/2, v/\bar{u})$, where $\bar{u}$ denotes the complex conjugation of u. The resulting coordinate change can easily be seen to be canonical. However this canonical change of coordinates can result in more difficult equations of motion, or a more difficult Hamiltonian, which can both obscure certain aspects of the dynamics of the system.

3.1.1 The Kepler Problem

In suitable coordinates in $T^*(\mathbb{R}^2 \setminus \{0\})$, the Kepler problem has Hamiltonian

$$H(q, p) = \frac{\|p\|^2}{2} - \frac{1}{\|q\|}. \tag{3.1}$$

With the canonical Levi-Civita transformation $(q, p) = (u^2/2, v/\bar{u})$, this becomes

$$H(u, v) = \frac{\|v\|^2}{2\|\bar{u}\|^2} - \frac{1}{\|u\|^2}. \tag{3.2}$$

Sometimes, as in this case, canonical changes lead to a more difficult system, so it may be desirable to leave the momentum unchanged and examine instead the transformation $(q, p) = (u^2/2, p)$ which can result in a simpler Hamiltonian. Now the transformation is not a symplectomorphism and the symplectic form on $T^*\mathbb{R}^2$ pulls back under the transformation to a two-form symplectic almost everywhere, but degenerate on a hypersurface of $T^*\mathbb{R}^2$.

Explicitly, the Liouville one-form $p_1 dq_1 + p_2 dq_2 = \Re(p d\bar{q})$ pulls back to

$$\begin{aligned} \theta = \Re\left(pd\left(\frac{\bar{u}^2}{2}\right)\right) &= \Re\left(p\bar{u}d\bar{u}\right) \\ &= p_1(u_1 du_1 - u_2 du_2) + p_2(u_2 du_1 + u_1 du_2) \end{aligned}$$

and computing $-d\theta$ we get the almost everywhere symplectic form

$$\omega = u_1 du_1 \wedge dp_1 - u_2 du_1 \wedge dp_2 + u_2 du_2 \wedge dp_1 + u_1 du_2 \wedge dp_2.$$

Wedging this form with itself we find

$$\omega \wedge \omega = (u_1^2 - u_2^2) du_1 \wedge dp_1 \wedge du_2 \wedge dp_2$$

which is degenerate along the hypersurface given by $u_1 = \pm u_2$.

3.1.2 The Problem of Two Fixed Centers

Related to the folded symplectic form found in the Levi-Civita transformation is the folded form associated with elliptic coordinates, employed while regularizing the problem of two fixed centers. This describes the motion of a satellite moving in a gravitational potential generated by two fixed massive bodies. We assume also that the motion of the satellite is restricted to the plane in $\mathbb{R}^3$ containing the two massive bodies. The Hamiltonian in suitable coordinates is given by

$$H = \frac{p^2}{2m} - \frac{\mu}{r_1} - \frac{1-\mu}{r_2} \tag{3.3}$$

where μ is the mass ratio of the two bodies (i.e. $\mu = \frac{m_1}{m_1+m_2}$).

Euler first showed the integrability of this problem using *elliptic* coordinates, where the coordinate lines are confocal ellipses and hyperbola. Explicitly, consider a coordinate system in which the two centers are placed at $(\pm 1, 0)$, in which the (Cartesian) coordinates are given by (q_1, q_2). Then the elliptic coordinates of the system are given by

$$q_1 = \sinh \lambda \cos \nu \tag{3.4}$$

$$q_2 = \cosh \lambda \sin \nu \tag{3.5}$$

for $(\lambda, \nu) \in \mathbb{R} \times S^1$. Thus lines of $\lambda = c$ and $\nu = c$ are given by confocal hyperbola and ellipses in the plane, respectively. Similar to the Levi-Civita transformation this results in a double branched covering with branch points at the centers of attraction.

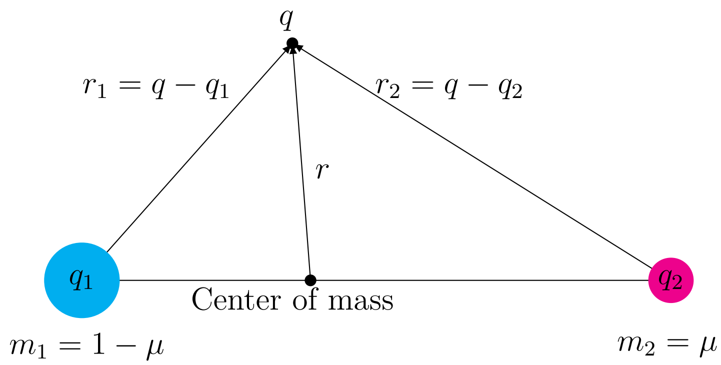


Figure 3.1: Scheme of the three body problem.

Pulling back the canonical symplectic structure $\omega = dq \wedge dp$ we find

$$\omega = \cosh\lambda\cos\nu(d\lambda\wedge dp_1 + d\nu\wedge dp_2) - \sinh\lambda\sin\nu(d\nu\wedge dp_1 + d\lambda\wedge dp_2) \tag{3.6}$$

which is degenerate along the hypersurface (λ, ν) satisfying $\cosh\lambda\cos\nu = \sinh\lambda\sin\lambda$.

3.2 Escape Singularities and b-symplectic forms

The restricted elliptic 3-body problem describes the behavior of a massless object in the gravitational field of two massive bodies, orbiting in elliptic Keplerian motion. The planar version assumes that all motion occurs in a plane. The associated Hamiltonian of the particle is given by

$$H(q,p) = \frac{\|p\|^2}{2} + \frac{1-\mu}{\|q-q_1\|} + \frac{\mu}{\|q-q_2\|} = T + U \tag{3.7}$$

where μ is the reduced mass of the system.

After making a change to polar coordinates $(q_1, q_2) = (r\cos\alpha, r\sin\alpha)$ and the corresponding canonical change of momenta we find the Hamiltonian

$$H(r,\alpha,P_r,P_\alpha)=\frac{P_r^2}{2}+\frac{P_\alpha^2}{2r^2}+U(r\cos\alpha,r\sin\alpha) \tag{3.8}$$

where P_r, P_α are the associated canonical momenta and

$$U(r\cos\alpha,r\sin\alpha)$$

is the potential energy of the system in the new coordinates.

The McGehee change of coordinates is traditionally employed to study the behavior of orbits near infinity, see also [21]. This non-canonical change of coordinates is given by

$$r=\frac{2}{x^2}. \tag{3.9}$$

The corresponding change for the canonical momenta is easily seen to be

$$P_r=-\frac{x^3}{4}P_x. \tag{3.10}$$

The Hamiltonian is then transformed to

$$H(r,\alpha,P_r,P_\alpha)=\frac{x^6P_x^2}{32}+\frac{x^4P_\alpha^2}{8}+U(x,\alpha). \tag{3.11}$$

By dropping the condition that the change is canonical and simply transforming the position coordinate (3.9), we are left with a simpler Hamiltonian, however the pull-back of the symplectic form under the non-canonical transformation is no longer symplectic, but rather b^3-symplectic:

$$\omega=\frac{4}{x^3}dx\wedge dP_r+d\alpha\wedge dP_\alpha. \tag{3.12}$$

3.3 Why is it so hard to find examples?

In this section we find another example of b-symplectic structure appearing quite naturally in physical dynamical systems. From this example it would seem natural that a collection of different examples for b^m-symplectic models or even b^m-folded models would follow. But one finds a major problem while pursuing these examples. We give an important remark to why this example does not extend to construct b^m-symplectic models of b^m-folded for any m.

First let us introduce te McGehee coordinate change.

The system of two particles moving under the influence of the generalized potential $U(x) = -|x|^{-\alpha}$, $\alpha > 0$, where $|x|$ is the distance between the two particles, is studied by McGehee in [22]. We fix the center of mass at the origin and hence can simplify the problem to the one of a single particle moving in a central force field.

The equation of motion writes down as

$$\ddot{x} = -\nabla U(x) = -\alpha|x|^{-\alpha-2}x \tag{3.13}$$

where the dot represents the derivative with respect to time. In the Hamiltonian formalism, this equation becomes

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\alpha|x|^{-\alpha-2}x. \end{aligned} \tag{3.14}$$

To study the behavior of this system, the following change of coordinates is suggested in [22]:

$$\begin{aligned} x &= r^{\gamma}e^{i\theta}, \\ y &= r^{-\beta\gamma}(v+iw)e^{i\theta} \end{aligned} \tag{3.15}$$

where the parameters β and γ are related with α in the following way:

$$\begin{aligned} \beta &= \alpha/2, \\ \gamma &= 1/(1+\beta). \end{aligned} \tag{3.16}$$

Identifying once more the plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, we can write the symplectic form of this problem as $\omega = \Re(dx \wedge d\overline{y})$.

Remark 3.3.1. *To check that a form ω is actually a b^m-symplectic form, it is no enough to check that the multi-vector field dual to $\omega \wedge \omega$ is a section of $\bigwedge^{2n}({}^{b^m}TM)$ which is transverse to the zero section. One has to check additionally that the Poisson structure dual to ω itself is a proper section of $\bigwedge^2({}^{b^m}TM)$.*

Proposition 3.3.2. *Under the coordinate change (3.15), the symplectic form ω is sent to a b-symplectic structure for $\alpha = 2$.*

Proof. The proof of this proposition is a straightforward computation. Observe that the change is not a smooth change, so we are not working with standard de Rham forms. But, we will see at the end of the computation that the form becomes a b-symplectic form and hence the computations are legitimate. If one does the change of variables, we obtain:

$$\begin{aligned}
\bar{y} &= r^{\beta\gamma}(v - iw)e^{-i\theta}. \\
dx &= \gamma r^{\gamma-1}e^{i\theta}dr + r^{\gamma}e^{i\theta}id\theta. \\
d\bar{y} &= r^{-\beta\gamma-1}(-\beta\gamma)(v - iw)e^{-i\theta}dr + r^{-\beta\gamma}e^{-i\theta}dv \\
&\quad +r^{\beta\gamma}(v - iw)e^{-i\theta}(-i)d\theta.
\end{aligned} \tag{3.17}$$

We wedge the previous two forms:

$$\begin{aligned}
dx \wedge d\bar{y} &= dr \wedge dv(\gamma r^{\gamma-1-\beta\gamma}) \\
&+ dr \wedge dw(\gamma r^{\gamma-1-\beta\gamma}) \\
&+ dr \wedge d\theta(\gamma r^{\gamma-1-\beta\gamma}(-iv - w)) \\
&+ d\theta \wedge dr(ir^{\gamma-1-\beta\gamma}(-\beta\gamma)(v - iw)) \\
&+ d\theta \wedge dv(ir^{\gamma-\beta\gamma}) \\
&+ d\theta \wedge dw(ir^{\gamma-\beta\gamma}(-i)).
\end{aligned} \tag{3.18}$$

Now we can take the real part of this form and use that $\gamma - 1 - \beta\gamma = -\alpha\gamma$. In the new coordinates, we obtain.

$$\begin{aligned}
\omega = \Re(dx \wedge d\bar{y}) &= \gamma r^{-\beta\gamma+\gamma-1}dr \wedge dv - \gamma(1-\beta)r^{-\beta\gamma+\gamma-1}wdr \wedge d\theta \\
&- r^{-\beta\gamma+\gamma}dw \wedge d\theta.
\end{aligned} \tag{3.19}$$

Moreover, we can use that $\gamma(1+\beta) = 1$ to further simplifly the previous expression to:

$$\omega = (dr\wedge dv+dr\wedge dw)\gamma r^{-\alpha\gamma}+dr\wedge d\theta(wr^{-\alpha\gamma})+d\theta\wedge dw(r^{-\alpha\gamma+1}). \tag{3.20}$$

We would like to know what kind of structure this form is. If we want to check this we have to wedge it with itself and look at the structure of the form in the singular set. Wedging this form, we obtain

$$\begin{aligned} \omega \wedge \omega &= -\gamma r^{-2\beta\gamma+2\gamma-1} dr \wedge dv \wedge d\theta \wedge dw \\ &= -\gamma r^{\frac{2-3\alpha}{2+\alpha}} dr \wedge dv \wedge d\theta \wedge dw. \end{aligned} \tag{3.21}$$

where we use (3.16). Let us set $f(\alpha) = \frac{2-3\alpha}{2+\alpha}$. We see that this function does not take values lower than -3 or higher than 1. We easily see that we obtain When $\alpha = 2$ this gives us a b-symplectic structure:

$$\omega \wedge \omega = -\gamma r dr \wedge dv \wedge d\theta \wedge dw.$$

The section of $\bigwedge^4({}^bTM)$ given by the dual structure of $\omega \wedge \omega$ is cleary transverse to the zero section.

On the other hand if $\alpha = 2$, then $\beta = 1$ and hence:

$$\omega = \gamma r^{-1} dr \wedge \omega \wedge dv,$$

and its dual Poisson structure is clearly also a proper section of $\bigwedge^2({}^bTM)$.

$\square$

Remark 3.3.3. *One may ask if for other values of α it is possible to obtain other kinds of b^m-symplectic structures. For example for $\alpha = 6$, $\omega \wedge \omega = -\gamma r^{-2} dr \wedge dv \wedge d\theta \wedge dw$ seems likely to be a b^2-symplectic form. But it actually is not. If one takes a look at the expression of ω it becomes clear that it is not a proper section of $\bigwedge^2({}^{b^2}T^*M)$*

Chapter 4

Existence and classification of b^m-symplectic structures

In this chapter we follow the article [11].

The motivation of this chapter comes from the theorem of classification of volume forms on a manifold. In the particular case of surfaces this corresponds to classification theorem on symplectic forms on a surface.

In his article [23], Moser proved the following theorem:

Theorem 4.0.1 (Classification of symplectic surfaces, [23]). *Let S be a compact oriented surface, and and let ω_0 and ω_1 be two symplectic forms on (M, Z) with $[\omega_0] = [\omega_1]$. Then there exists a diffeomorphism $\phi : M \to M$ such that $\phi^*\omega_1 = \omega_0$.*

Later on Radko classified b-symplectic forms on a surface. In this case, Z is a union of smooth curves and each point in these curves is a symplectic leaf of the symplectic foliation induced on Z. In [4], Radko describes the following invariant of b-Poisson surfaces:

- The set of curves $\gamma_1, \ldots, \gamma_n$ along which the Poisson structure vanishes.

- The periods along the curves $\gamma_1, \ldots, \gamma_n$ of a modular vector field on M associated to the volume form ω_Π, the two-form dual to Π, on the complement of Z.

- The regularized Liouville volume of (M, Π), which is a correction along Z of the natural volume associated to the Poisson structure which blows up at Z. It is defined by the integral

$$\int_M \omega = \lim_{\varepsilon \to 0} \int_{|f| > \varepsilon} \omega,$$

  where f is a defining function for Z. This limit exists and is independent of the choice of f.

The following classification holds:

Theorem 4.0.2 (Radko)**.** *The set of curves, modular periods and regularized Liouville volume completely determines, up to Poisson diffeomorphisms, the b-Poisson structure on a compact surface M.*

This was reproved in [5] by Guillemin-Miranda-Pires by identifying these invariants as determining the b-cohomology class. In this sense the theorem below is just Moser's classification theorem replacing the standard De Rham cohomology by b-cohomology.

Corollary 4.0.3 (**Classification of b-symplectic surface, [5]**)**.** *Let S be a compact orientable surface and and let ω_0 and ω_1 be two b-symplectic forms on (M, Z) defining the same b-cohomology class (i.e.,$[\omega_0] = [\omega_1]$). Then there exists a diffeomorphism $\phi : M \to M$ such that $\phi^*\omega_1 = \omega_0$.*

In this chapter we investigate constructions of b^m-surfaces using combinatorial invariants associated to the critical set and we prove an analogue of the theorem above by identifying the invariants of Scott in terms of b^m-cohomology considering also its equivariant counterparts.

4.1 Toy examples of b^m-symplectic surfaces

In this section we describe several examples of orientable and non-orientable b^m-symplectic surfaces.

1. **A b^m-symplectic structure on the sphere:** Consider the sphere $S^2 \subset \mathbb{R}^3$ with the equator $Z = \{(x_1, x_2, x_3) \in S^2 | x_3 = 0\}$ as critical set. Let $h = x_3$ the height function. Then (S^2, Z, h) is a b^m-manifold for any m. Consider $\omega = \frac{1}{h^m} dh \wedge d\theta$, where θ stands for the angular coordinate. This form is a b^m-symplectic form.

2. **A b^m-symplectic structure on the torus:** Consider $\mathbb{T}^2$ as quotient of the plane ($\mathbb{T}^2 = \{(x, y) \in (\mathbb{R}/\mathbb{Z})^2\}$). Let $\omega = \frac{1}{(\sin 2\pi x)^m} dx \wedge dy$ be a b^m-symplectic structure on $\mathbb{R}^2$. The action of $\mathbb{Z}^2$ leaves this form invariant and therefore this b^m-form descends to the quotient. Observe that this b^m-form defines $Z = \{x \in \{0, \frac{1}{2}\}\}$.

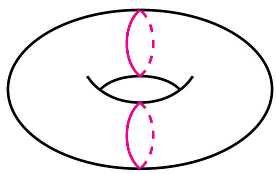


Figure 4.1: Example: b^m-symplectic structure in the torus.

3. **A b^{2k+1}-symplectic structure on the projective space:** Consider Example (1) and consider the quotient of S^2 by the antipodal action. Because this action leaves the critical set invariant, the b^m-manifold structure (S^2, Z) descends to $(\mathbb{RP}^2, \hat{Z})$ and gives a non-orientable b^m manifold. $\hat{Z}$ is the equator modulo the antipodal identification (thus diffeomorphic to $\mathbb{RP}^1 \cong S^1$). Moreover a neighborhood of Z is diffeomorphic to the Moebius band.

Observe that ω is invariant by the action for $m = 2k+1$, yielding a b^{2k+1}-symplectic form in $\mathbb{RP}^2$ with critical set $\hat{Z}$.

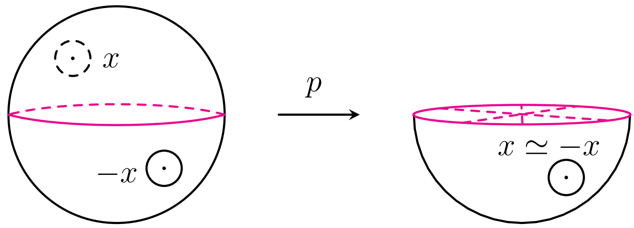


Figure 4.2: The b^{2k+1}-symplectic structure on the sphere S^2 that vanishes at the equator induces a b^{2k+1}-symplectic structure on the projective space $\mathbb{RP}^2$.

4. **A b^{2k+1}-symplectic structure on a Klein bottle:** Consider the torus with the structure given in Example (2).

   Consider $\mathbb{Z}/2\mathbb{Z}$ acting on $(x, y) \in \mathbb{T}^2$ by -Id$\cdot(x, y) = (1-x, y+1/2 \pmod 1)$. The orbit space by this action is the Klein bottle $\mathbb{K}$. Then the b^m-manifold $(\mathbb{R}^2, Z)$, descends to $(\mathbb{K}, \hat{Z})$ where $\hat{Z}$ is the quotient of Z by the action. It is easy to see that $\hat{Z} \cong S^1 \sqcup S^1$. Moreover the tubular neighborhood of each S^1 is isomorphic to the Moebius band.

   Thus, the b^m-symplectic form $\omega = \frac{1}{(\sin 2\pi x)^m} dx \wedge dy$ induces a b^m-symplectic structure in T if ω is invariant by the action of the group. It is easy to check that ω is invariant if and only if m is odd, in this case one obtains a b^m-symplectic structure on the Klein bottle.

Remark 4.1.1. *The previous examples only exhibit b^{2k+1}-symplectic structures on non-orientable surfaces. As we will see in Section 4.3 only orientable surfaces can admit b^{2k}-symplectic structures.*

4.2 Equivariant classification of b^m-surfaces and non-orientable b^m-surfaces.

In this section we give an equivariant Moser theorem for b^m-symplectic manifolds. This yields the classification of non-orientable surfaces thus extending the classification theorems of Radko and Scott for orientable surfaces (see Theorem 4.2.4).

We now extend the classification result (Theorem 2.2.2) for manifolds admitting a compact Lie group action leaving the b^m-symplectic structure invariant. The following theorem is a simple consequence of applying the equivariant tools to the Moser path method. We include the detail of the proof for the sake of completeness. Other applications of the equivariant tools in b-geometry can be found in [7] and [24].

Theorem 4.2.1 (Equivariant b^m-Moser theorem for surfaces). *Suppose that S is a closed surface, let Z be a union of non-intersecting embedded curves. Consider the b^m-manifold given by (S, Z). Fix $m \in \mathbb{N}$ and let ω_0 and ω_1 be two b^m-symplectic structures on (S, Z) which are invariant under the action of a compact Lie group $\rho : G \times (S, Z) \longrightarrow (S, Z)$ and defining the same b^m-cohomology class, $[\omega_0] = [\omega_1]$. Then, there exists an equivariant b^m-diffeomorphism $\xi_1 : (S, Z) \to (S, Z)$, such that $\xi_1^* \omega_1 = \omega_0$.*

Proof. Denote by ρ_g the induced diffeomorphism for a fixed $g \in G$. i.e., $\rho_g(x) := \rho(g, x)$. Consider the linear family of b^m-forms $\omega_s = s\omega_1 + (1-s)\omega_0$. Since the manifold is a surface, the fact that ω_0 and ω_1 are non-degenerate b^m-forms and of the same sign on $S \setminus Z$[1] (thus non-vanishing sections of $\Lambda^2({}^bT^*(S))$) implies that the linear path is non-degenerate too. We will prove that there exists a family $\xi_s : S \to S$,

[1]This is a consequence of Mazzeo-Melrose theorem and the determination of the Liouville volume from it.

with $0 \leq s \leq 1$ such that

$$\xi_s^* \omega_s = \omega_0. \tag{4.1}$$

We want to construct ξ_1 as the time-1 flow of a time-dependent Hamiltonian vector field X_s (as in the standard Moser trick).

Since the cohomology class of both forms coincide, $\omega_1 - \omega_0 = d\alpha$ for α a b^m-form of degree 1.

Therefore Moser's equation reads

$$\iota_{X_s}\omega_s = -\alpha. \tag{4.2}$$

This equation has a unique solution X_s because ω_s is b^m-symplectic and therefore it is non-degenerate. X_s depends smoothly on s because ω_s depends smoothly on s and ω_s defines a non-degenerate pairing between b^m-vector fields and b^m-forms. Furthermore, the solution is a b^m-vector field but this solution may not be compatible with the group action. From this solution we will construct an equivariant solution such that its s-dependent flow gives an equivariant diffeomorphism.

Since the forms ω_0 and ω_1 are G-invariant, we can find a G-invariant primitive $\tilde{\alpha}$ by averaging with respect to a Haar measure the initial form α: $\tilde{\alpha} = \int_G \rho_g^*(\alpha) d\mu$ and therefore the invariant vector field, $X_s^G = \int_G \rho_{g_*}(X_s) d\mu$ is a solution of the equation,

$$\iota_{X_s^G}\omega_s = -\tilde{\alpha}. \tag{4.3}$$

We can get an equivariant ξ_s^G by integrating X_s^G. This family satisfies $\xi_t^{G*}\omega_t = \omega_0$ and it is equivariant. Also observe that since X_t^G is a b^m-vector field ξ_t^G is a b^m-diffeomorphism of (S, Z).

$\square$

A non-orientable manifold can be seen as a pair $(\tilde{M}, \rho)$ with $\tilde{M}$ the orientable covering and ρ the action given by deck-transformations of $\mathbb{Z}/2\mathbb{Z}$ on $\tilde{M}$. This perspective is very convenient for classification issues because equivariant mappings on the orientable covering yield actual diffeomorphisms on the non-orientable manifolds. We adopt this point of view to provide a classification theorem for non-orientable b^m-surfaces in cohomological terms.

Remark 4.2.2. *Observe that the b^m-Mazzeo-Melrose allows us to determine whether a given b^m-cohomology of degree 2 is non-zero by reducing this question to de Rham cohomology.*

Corollary 4.2.3. *Let (S, Z) be a non-orientable b^m-manifold where Z is its critical set and let ω_1 and ω_2 be two b^m-symplectic forms such that $[\omega_1] = [\omega_2]$ in b^m-cohomology then (S, ω_1) is equivalent to (S, ω_2), i.e., there exists a b^m-diffeomorphism $\varphi : (S, Z) \to (S, Z)$ such that $\varphi^*\omega_2 = \omega_1$.*

Proof. Consider m fixed and assume $[\omega_1] = [\omega_2]$ in b^m-symplectic cohomology. Let $p : \tilde{S} \to S$ be a covering map, and $\tilde{S}$ the orientation double cover. $(\hat{S}, p^{-1}(Z))$ is a b^m-manifold and $p^*(\omega_1)$, $p^*(\omega_2)$ are b^m-symplectic structures on $(\hat{S}, p^{-1}(Z))$. By construction the previous two forms are invariant under the action by deck transformations of $\mathbb{Z}/2\mathbb{Z}$. The defining function of the critical set in the double cover is the pullback by p of the defining function in (S, Z). Since $[\omega_1] = [\omega_2]$ then $[p^*(\omega_1)] = [p^*(\omega_2)]$. By Theorem 4.2.1, there exists a $\mathbb{Z}/2\mathbb{Z}$-equivariant b^m-diffeomorhism $\tilde{\varphi} : (\hat{S}, p^{-1}(Z)) \to (\hat{S}, p^{-1}(Z))$ such that $\tilde{\varphi}^* p^*\omega_2 = p^*\omega_1$. Since $\tilde{\varphi}$ is $\mathbb{Z}/2\mathbb{Z}$-equivariant it descends to a map $\varphi : S \to S$. Moreover, because $\tilde{\varphi}(p^{-1}(Z)) = p^{-1}(Z)$, it follows that $\varphi(Z) = Z$. Since $\tilde{\varphi}$ is smooth and p is a submersion, then ϕ is smooth, (the same argument shows φ^{-1} is smooth). It follows that φ is a diffeomorphism and because $\varphi(Z) = Z$ it is also a b^m-diffeomorphism. Moreover, by construction, the condition $\tilde{\varphi}^* p^*\omega_2 = p^*\omega_1$ implies that $\varphi^*\omega_2 = \omega_1$.

□

A similar equivariant b^m-Moser theorem as theorem 4.2.1 holds for higher dimensions. In that case we need to require that there exists a path ω_t of b^m-symplectic structures connecting ω_0 and ω_1, which is not true in general [25]. The proof follows the same lines as Theorem 4.2.1. Such a result was already proved for b-symplectic manifolds (see Theorem 8 in [26]).

Theorem 4.2.4 (Equivariant b^m-Moser theorem). *Let (M, Z) be a closed b^m-manifold with m a fixed natural number and let ω_t for $0 \leq t \leq 1$ be a smooth family of b^m-symplectic forms on (M, Z) such that the b^m-cohomology class $[\omega_t]$ does not depend on t.*

Assume that the family of b^m-symplectic structures is invariant by the action of a compact Lie group G on M, then, there exists a family of equivariant b^m-diffeomorphisms $\phi_t : (M, Z) \to (M, Z)$, with $0 \leq t \leq 1$ such that $\phi_t^* \omega_t = \omega_0$.

4.3 Constructions and classification of b^m-symplectic structures

In this section we describe constructions of b^m-symplectic structures on closed surfaces. We obtain topological constraints on b^{2k}-symplectic surfaces as we will prove that the underlying closed surface needs to be orientable, see Theorem 4.3.1. Then we characterize the existence of b^m-symplectic forms depending on the parity of m and the colorability of an associated graph. We also obtain a result about non-orientable surfaces: if $m = 2k + 1$ we find necessary and sufficient conditions for a non-orientable b^m-surface to admit a b^m-symplectic structure (see Theorem 4.3.10).

4.3.1 b^{2k}-symplectic orientable surfaces

We start by proving that only orientable surfaces admit b^{2k}-symplectic structures:

Theorem 4.3.1. *If a closed surface admits a b^{2k}-symplectic structure then it is orientable.*

Proof. The proof consists in building a collar of b^{2k}-Darboux neighborhoods with compatible orientations (the local orientations on the complement of the critical hypersurface induced by the b^{2k}-Darboux charts agree) in a neighborhood of each connected component of Z. Indeed the proof does more, it constructs a symplectic structure in a neighborhood of Z which can be extended to S. This in particular will give an orientation on S.

Let (S, Z) be a closed b^{2k}-surface and let ω denote a b^{2k}-symplectic structure on (S, Z). Pick $(\tilde{S}, \tilde{Z})$ an orientable double cover of the b^{2k}-surface (S, Z), with $\rho : \mathbb{Z}/2\mathbb{Z} \times \tilde{S} \to \tilde{S}$ the action by deck transformations. For each point $q \in \tilde{Z}$, using Theorem 2.2.5, we can find a b^{2k}-Darboux neighborhood U_q (by shrinking the neighborhood if necessary) which does not contain other points identified by ρ ($\rho(U_q) \cong U_q$). Let us define $V_q := p(U_q)$, where p is the projection from $\tilde{S}$ to S. With the previous construction we have $\omega|_{U_q} = \frac{1}{x^{2k}} dx \wedge dy$.

Now we can use the desingularization formulae in Theorem 2.2.9 and Definition 2.2.10 in each U_q (because every U_q is orientable) to obtain a symplectic form $\omega_{\epsilon q}$ on each U_q. All these symplectic structures and hence the orientations on each U_q glue in a compatible manner because the function x is globally defined.

Since $\tilde{Z}$ is compact we can take a finite subcovering for U_q to define a collar U of symplectic and compatible orientations. Furthermore we can assume this covering to be symmetric as we can shrink further the neighborhoods and add the pre-images of all of them -for each U_q the image $\rho(U_q)$ is included in the covering.

Since ρ preserves ω, and the defining function is invariant by ρ, it also preserves the deblogged symplectic forms $\omega_{\epsilon q}$ and the compatible orientations and indeed the deblogged symplectic form descends to S, thus defining a symplectic form and an orientation on $V = p(U)$. Using the standard techniques of Radko [4] the symplectic structures on $V \backslash Z$ can be glued to define a compatible symplectic structure on the whole S. When Z has more than one connected component we may proceed in the same way by isolating collar neighborhoods of each component. Thus proving that S admits a symplectic structure and in particular it is oriented.

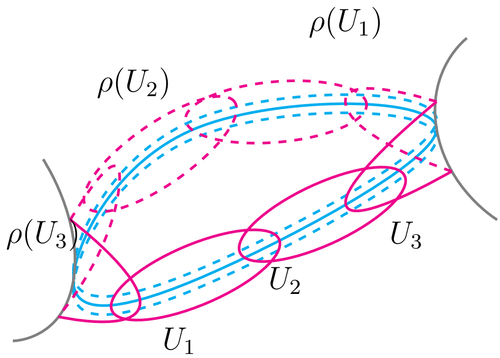


Figure 4.3: A collar of compatible neighborhoods.

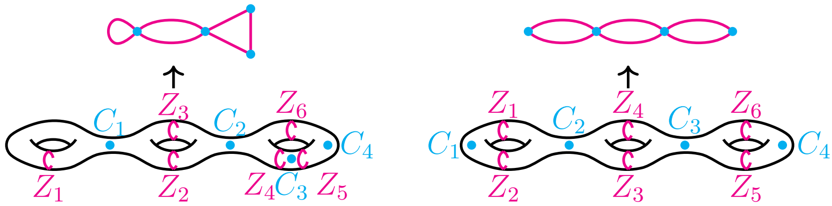
□

4.3.2 Associated graph of a b-manifold.

Let us introduce some definitions that will be needed in the next subsection.

Definition 4.3.2. *Let (M, Z) be a closed b-manifold. The associated graph $\Gamma(M, Z)$ to this b-manifold is defined as follows:*

1. *The set of vertices is in one-to-one correspondence with the connected components $(U_1, \ldots, U_n)$ of $M \setminus Z$.*



(a) Example of a non-colorable associated graph.

(b) Example of a colorable associated graph.

Figure 4.4: Examples of associated graphs.

2. *Let $(Z_1, \ldots, Z_n)$ be the connected components of Z. Two vertices (v_i, v_j), (represented by (U_i, U_j)) are connected by an edge if and only if for any tubular neighborhood of some Z_k, it intersects both U_i and U_j.*

Remark 4.3.3. *As observed in [1] (Section 3.2), associated to any b^m-manifold there is a canonical b-manifold, obtained by forgetting the distance function. The latter is henceforth said to be underlying the former. Using definition 4.3.2, the graph associated to a b^m-manifold is the graph associated to its underlying b-manifold.*

Remark 4.3.4. *Given an oriented closed b^m-manifold (M, Z), a b^m-symplectic structure induces a standard orientation on each connected component of $M \setminus Z$. Comparing this orientation with the fixed one determines a sign that can be attached to each vertex of $\Gamma(M, Z)$. A natural question to ask is whether adjacent vertices possess equal or opposite signs, thus yielding the following notions.*

Definition 4.3.5. *A* ***2-coloring of a graph*** *is a labeling (with only two labels) of the vertices of the graph such that no two adjacent vertices share the same label.*

Definition 4.3.6. *Since not every graph admits a 2-coloring, a graph is called* ***2-colorable*** *if it admits a 2-coloring.*

4.3.3 b^{2k+1}-symplectic orientable surfaces

Theorem 4.3.7. *Given a b^m-manifold (S, Z) (fixed m) with S a closed and orientable surface, there exists a b^m-symplectic structure whenever:*

1. *$m = 2k$,*
2. *$m = 2k + 1$ if and only if the associated graph $\Gamma(S, Z)$ is 2-colorable.*

Proof. (of Theorem 4.3.7)

Let $C_1, \dots, C_r$ be the connected components of $S \setminus Z$, let $Z_1, \dots, Z_s$ the connected components of Z and let $\mathcal{U}(Z_1), \dots, \mathcal{U}(Z_s)$ tubular neighborhoods of the connected components. Moreover, we denote the union of $\mathcal{U}(Z_1), \dots, \mathcal{U}(Z_s)$ by $\mathcal{U}(Z)$.

We assume there is an orientation defined by some symplectic form in S, that allows us to define a sign criterion.

The proof consists in 3-steps:

1. **Using Weinstein normal form theorem.** Fix $i \in \{1, \dots, s\}$, where s is the number of connected components. By virtue of Weinstein's normal form theorem for Lagrangian submanifolds (Corollary 6.2 in [27]) each tubular neighborhood $\mathcal{U}(Z_i)$ can be identified with the zero section of the cotangent bundle of Z_i. Now replace, the cotangent bundle of Z_i by the b^m-cotangent bundle of Z_i[2]. In this way the neighborhood of the zero section of the b^m-cotangent bundle has a b^m-symplectic structure that we will denote $\omega_{\mathcal{U}(Z_i)}$.

[2] This can be done after fixing a point in Z to define a b^m-structure on Z.

2. **Constructing compatible orientation using the graph.** For any $i = 1, \ldots, s$, $\mathcal{U}(Z_i) \setminus Z_i$ has two connected components (as S is orientable); to each such component, we assign the sign of the restriction of the b^m-symplectic form $\omega_{\mathcal{U}(Z_i)}$. Note that the sign does not change for m even, but it changes for k odd. Observe that we can apply Moser's trick to glue two rings that share some C_j (as done in Radko [4] to extend a symplectic form between the two rings) if and only if the sign of the two rings match on this component.

   Now, let us consider separately the odd and even cases:

   (a) For b^{2k} the color of adjacent vertices must coincide. And hence we have no additional constraint on the topology of the graph.

   (b) In the b^{2k+1} case the sign of two adjacent vertices must be different. Then, we have to impose the associated graph to be 2-colorable.

   These two conditions are necessary for the existence of the $b^{2k}-$ and $b^{2k+1}-$forms respectively.

3. **Gluing.** Now we may glue back this neighborhood to $S \setminus \mathcal{U}(Z)$ in such a way that the symplectic structures fit on the boundary (again using the standard techniques used in Radko [4] to extend with a symplectic form between the two rings), using the Moser's path method.

$\square$

Given a b^{2k+1}-symplectic structure ω on a b^{2k+1}-surface (S, Z) (where S is closed oriented) one can obtain a 2-coloring of the associated graph (by the local expression given by the b^m-Darboux theorem -see Theorem 2.2.5-, the sign has to change every time we cross a component

Z_i) by assigning to each connected component C_i of $S \setminus Z$ the 'color' $\text{sign}(\int_{C_i} \omega)$.

Remark 4.3.8. *Observe that any given* 2*-coloring has to be equivalent to the* 2*-coloring induced by a* b^{2k+1}*-symplectic form. This is due to the fact that for every connected component of the graph there exist only 2 possible* 2*-colorings of a graph (when it is 2-colorable). The difference between the two* 2*-colorings is only re-labeling of the signs. Then, if the* 2*-coloring induced by the* b^{2k+1}*-symplectic form does not correspond to the prescribed* 2*-coloring, it can be matched by changing the orientation of the underlying manifold at every connected component.*

Another way to construct b^{2k}-structures on a surface is to use decomposition theorem as connected sum of b^{2k}-spheres (1) and b^{2k}-torus (2). The drawback of this construction is that it is harder to adapt having fixed a prescribed Z.

4.3.4 b^{2k+1}-symplectic non-orientable surfaces

Definition 4.3.9. *Let* (S, Z) *be a closed orientable* b^{2k+1}*-surface and* $\Gamma(S, Z)$ *its associated graph. Fix the 2-coloring on* $\Gamma(S, Z)$ *given by by sign*$(\int_{C_i} \omega)$*. We say that a* b^{2k+1}*-map* φ ***inverts colors*** *of the associated graph if sign*$(\int_{C_i} \omega) = -$*sign*$(\int_{\varphi(C_i)} \omega)$*.*

Theorem 4.3.10. *Let* (S, Z) *be a closed non-orientable* b^{2k+1}*-surface. Then,* (S, Z) *admits a* b^{2k+1}*-symplectic structure if and only if the following two conditions hold:*

1. *the graph of some covering* $(\tilde{S}, \tilde{Z})$*,* $G(\tilde{S}, \tilde{Z})$ *is 2-colorable and*

2. *the non-trivial deck transformation inverts colors of the graph obtained in the covering*[3]*.*

[3] Observe that if a transformation inverts colors for a given coloring, then it

Proof. Let us assume the two conditions on the statement of the theorem hold. Apply Theorem 4.3.7 to endow the covering $(\tilde{S}, \tilde{Z})$ with a b^{2k+1}-symplectic structure, if the form obtained is invariant by the deck transformations, then it descends to the quotient, thus obtaining a b^{2k+1}-symplectic structure on $(\tilde{S}, \tilde{Z})$ and then we are done.

Now, let us assume that the b^{2k+1}-form ω obtained via theorem 4.3.7 is not invariant by deck transformations. We will note the deck transformation induced by $-\mathrm{Id}$ as ρ. Observe that

$$\operatorname{sign}\left(\int_{C_i} \rho^*\omega\right) = -\operatorname{sign}\left(\int_{\rho(C_i)} \omega\right) = +\operatorname{sign}\left(\int_{C_i} \omega\right). \tag{4.4}$$

The first equality is due to ρ changing orientations and the second one is due to ρ inverting colors. Then the pullback of ω has the same sign as ω, and hence $\omega + \rho^*(\omega)$ is a non-degenerate b^{2k+1}-form that is invariant under the action of ρ, and it descends to the quotient. Hence a b^{2k+1}-symplectic structure is obtained on (S, Z).

The other implication is easier. If we have a b^{2k+1}-symplectic form on (S, Z) we can pull it back to the double cover by means of the projection. Then we obtain a b^{2k+1}-form on the double cover, that induces a 2-coloring defined by the orientations. And since the b^{2k+1}-form on the double cover has to be invariant by the deck transformation, the deck transformation has to invert colors.

□

Example 4.3.11. *Let us illustrate what is happening in the previous proof with an example. Take the sphere having the equator as critical set and endowed with the b-symplectic form $\omega = \frac{1}{h} dh \wedge d\theta$. Let us call the north hemisphere C_1 and the south hemisphere C_2, and let ρ be the antipodal map. Look at the coloring of the graph (a path graph of*

inverts colors for all of them (there is only 2 possible 2-colorings when a graph is 2-colorable, and they correspond with the possible choices of orientation).

length two):

$$sign(C_1) = sign\left(\int_{C_1} \omega\right) = sign\left(\lim_{\epsilon \to 0} \int_{-\pi}^{\pi} \int_{\epsilon}^{1} \frac{1}{h} dh \wedge d\theta\right)$$
$$= sign(\lim_{\epsilon \to 0} -2\pi \log |\epsilon|) \tag{4.5}$$

which is positive. And

$$sign(C_2) = sign\left(\int_{C_2} \omega\right) = sign\left(\lim_{\epsilon \to 0} \int_{-\pi}^{\pi} \int_{-1}^{-\epsilon} \frac{1}{h} dh \wedge d\theta\right)$$
$$= sign(\lim_{\epsilon \to 0} 2\pi \log |\epsilon|) \tag{4.6}$$

which is negative. Then,

$$\int_{C_1} \rho^* \omega = \lim_{\epsilon \to 0} \int_{-\pi}^{\pi} \int_{\epsilon}^{1} \rho^* \left(\frac{1}{h} dh \wedge d\theta\right) = \lim_{\epsilon \to 0} \int_{-\pi}^{\pi} \int_{\epsilon}^{1} \frac{1}{-h} d(-h) \wedge d\theta$$
$$= \int_{C_1} \omega. \tag{4.7}$$

In this case ω was already invariant, but one can observe that if ρ inverts colors then the signs of the form and the pullback are the same.

Example 4.3.12. *One may ask why the condition of inverting colors is necessary. Next we provide an example where b^{2k+1}-structures can be exhibited on the double cover but cannot be projected to induce a b^{2k+1}-structure on the non-orientable surface.*

Consider the Example 3 in Section 4.1. If one translates the critical set in the h direction in the projective space, the double cover is still the sphere, but instead of Z being the equator, Z consists of different meridians $\{h = h_0\}$ and $\{h = -h_0\}$.

Observe that the associated graph of this double cover is a path graph of length 3, that can be easily 2-colored. Take a generic b^{2k+1}-form $\omega = f(h, \theta) dh \wedge d\theta$, and look at the poles N, S. $sign(f(N)) = sign(f(S))$ because of the 2-coloring of the graph. But $\rho^*(\omega)|_N = f(\rho(N)) d(-h) \wedge d\theta = -f(S) dh \wedge d\theta$. Then $sign(\omega) \neq sign(\rho^*(\omega))$, and hence ω can not be invariant for ρ.

4.4 Desingularization of closed b^{2k}-symplectic surfaces

In this section we only refer to the desingularization of b^{2k}-symplectic structures, because as we explained in section 2.2.2 the desingularization procedure, associates folded symplectic structures to b^{2k+1}-symplectic structures instead of symplectic structures. The goal of this section is to compare the classification schemes in the b^m-symplectic and symplectic realms.

The aim of this section is to use the desingularization formulas described in section 2.2.2 in the case of closed orientable surfaces. The main result of this section (Theorem 4.4.1) is that if $[\omega_1] = [\omega_2]$ (where ω_1 and ω_2 are two b^{2k}-symplectic forms on a b^{2k}-surface (S, Z)) in b^{2k}-cohomology, then the desingularization of the two forms also is in the same class $[\omega_{1\epsilon}] = [\omega_{2\epsilon}]$. But the converse is not true: it is possible to find different classes of b^{2k}-forms that go the same class when desingularized.

Next we apply our classification scheme and see how it behaves under the desingularization procedure.

Theorem 4.4.1. *Let (S, Z, x), be a b^{2k}-manifold, where S is a closed orientable surface and let ω_1 and ω_2 be two b^{2k}-symplectic forms. Also let $\omega_{1\epsilon}$ and $\omega_{2\epsilon}$ be the f_ϵ-desingularizations of ω_1 and ω_2 respectively. If $[\omega_1] = [\omega_2]$ in b^{2k}-cohomology then $[\omega_{1\epsilon}] = [\omega_{2\epsilon}]$ in de Rham cohomology for any fixed ϵ.*

Before proceeding to proving the theorem we will state some definitions in [1] that are necessary for the proof.

Definition 4.4.2. *Let (M, Z, x) be an n-dimensional b^m-manifold. Given ω a b^m-form of top degree with compact support, $\epsilon > 0$ small,* *and let U_ϵ an ϵ-tubular neighborhood*[4]*, then* $\mathrm{vol}_\epsilon(\omega)$ *is defined as:*

$$\mathrm{vol}_\epsilon(\omega) = \int_{M\setminus U_\epsilon} \omega$$

Theorem 4.4.3 (Theorem 4.3 in [1])**.** *For a fixed $[\omega]$ the b^m-cohomology class of a b^m-form ω, on a b^m-manifold (M, Z, x) with Z compact, there is a polynomial $P_{[\omega]}(t)$ for which*

$$\lim_{\epsilon\to 0}(P_{[\omega]}(1/\epsilon) - \mathrm{vol}_\epsilon(\omega)) = 0$$

for any ω representing $[\omega]$.

Definition 4.4.4. *The polynomial $P_{[\omega]}$ described in Theorem 4.4.3 is the* ***volume polynomial*** *of $[\omega]$. Its constant term $P_{[\omega]}(0)$ is the* ***Liouville volume*** *of $[\omega]$.*

Remark 4.4.5. *Let $U = [-1, 1] \times Z$ be a tubular neighborhood of Z containing U_ε. From the definition of the Liouville volume we may write:*

$$P_{[\omega]}(0) = \left(\int_{M\setminus U} \omega + \int_U \beta + \sum_{i=1}^{k} \left(\frac{-2}{2i-1}\right) \int_Z \alpha_{2i}\right). \tag{4.8}$$

Observe that in the proof of Theorem 5.3 in [1] the term $\int_{M\setminus U}\omega$ does not appear. This is because in [1] M is assumed to be U for the sake of simplicity. Adding this term is the way to extend this expression when $U \subsetneq M$.

Proof. (of Theorem 4.4.1) Our strategy for the proof is to show that the cohomology class of a desingularization of a b^{2k}-symplectic structure on a closed orientable surface (which is the cohomology class of a symplectic structure and hence it can be encoded by its signed area, i.e. the integral of itself over S), only depends on the b^{2k}-cohomology of

[4] the ϵ-tubular neighborhood is defined using the x from the b^m-manifold

the b^{2k}-symplectic structure (which, in its turn, can be encoded by the integral of the forms appearing in its Laurent series and its Liouville volume -Theorem 2.2.3-).

In order to compute the class of the desingularization we calculate the integral of the desingularized form over the whole manifold. We are going to proceed in two steps. Firstly we are going to compute the integral of the desingularization inside the ϵ-neighborhood U_ϵ of Z, and then we compute it outside.

Using the expression of ω_ϵ we compute:

$$\begin{aligned}
\int_{U_\epsilon} \omega_\epsilon &= \int_{U_\epsilon} df_\epsilon \wedge \left(\sum_{i=0}^{2k-1} x^i \alpha_i \right) + \int_{U_\epsilon} \beta \\
&= \epsilon^{-(2k-1)} \int_{U_\epsilon} df(x/\epsilon) \wedge \left(\sum_{i=0}^{2k-1} x^i \alpha_i \right) + \int_{U_\epsilon} \beta \\
&= \epsilon^{-2k} \int_{U_\epsilon} \frac{df(x/\epsilon)}{dx} dx \wedge \left(\sum_{i=0}^{2k-1} x^i \alpha_i \right) + \int_{U_\epsilon} \beta \\
&= \epsilon^{-2k} \sum_{i=0}^{2k-1} \int_{-\epsilon}^{+\epsilon} \frac{df(x/\epsilon)}{dx} x^i dx \int_Z \alpha_i + \int_{U_\epsilon} \beta.
\end{aligned}$$

Then, because f is an odd function, $df(x/\epsilon)/dx$ is even and hence the integral $\int_{-\epsilon}^{+\epsilon} \frac{df(x/\epsilon)}{dx} x^i dx$ is going to be different from 0 if i is even. Thus,

$$\int_{U_\epsilon} \omega_\epsilon = \epsilon^{-2k} \sum_{i=1}^{k-1} \int_{-\epsilon}^{+\epsilon} \frac{df(x/\epsilon)}{dx} x^{2i} dx \int_Z \alpha_{2i} + \int_{U_\epsilon} \beta.$$

Recall that outside the ϵ-neighborhood the desingularization ω_ϵ coincides with the b^{2k}-symplectic form ω. Moreover, let us define U a tubular neighborhood of Z containing U_ϵ, (assume $U = [-1, 1] \times Z$). Following the computations in [1] we obtain,

$$\begin{aligned}
\int_{M \setminus U_\epsilon} \omega_\epsilon &= \int_{M \setminus U_\epsilon} \omega \\
&= \int_{M \setminus U} \omega + \int_{U \setminus U_\epsilon} \omega \\
&= \int_{M \setminus U} \omega + \left(\int_{U \setminus U_\epsilon} \beta + \sum_{i=1}^{k} \frac{-2}{2i-1} \int_Z \alpha_{2i} \right) \\
&\qquad + \sum_{i=1}^{k} \left(\frac{2}{2i-1} \int_Z \alpha_{2i} \right) \epsilon^{2i-1}.
\end{aligned}$$

Now we may add the two terms in order to compute the integral over the whole surface M:

$$\begin{aligned}
\int_M \omega_\epsilon &= \epsilon^{-2k}\sum_{i=1}^{k-1}\int_{-\epsilon}^{+\epsilon}\frac{df(x/\epsilon)}{dx}x^{2i}dx\int_Z \alpha_{2i} + \int_{U_\epsilon}\beta \\
&+ \int_{M\setminus U}\omega + \left(\int_{U\setminus U_\epsilon}\beta + \sum_{i=1}^{k}\frac{-2}{2i-1}\int_Z \alpha_{2i}\right) \\
&\quad + \textstyle\sum_{i=1}^{k}\left(\frac{2}{2i-1}\int_Z \alpha_{2i}\right)\epsilon^{2i-1} \\
&= \epsilon^{-2k}\sum_{i=1}^{k-1}\int_{-\epsilon}^{+\epsilon}\frac{df(x/\epsilon)}{dx}x^{2i}dx\int_Z \alpha_{2i} \\
&+ \underbrace{\int_{M\setminus U}\omega + \left(\int_{U}\beta + \sum_{i=1}^{k}\frac{-2}{2i-1}\int_Z \alpha_{2i}\right)}_{=P_{[\omega]}(0) \text{ by the expression 4.8}} \\
&\quad + \textstyle\sum_{i=1}^{k}\left(\frac{2}{2i-1}\int_Z \alpha_{2i}\right)\epsilon^{2i-1}.
\end{aligned}$$

In a more compact way:

$$\int_M \omega_\epsilon = \sum_{i=1}^{k-1} a_i(\epsilon)\int_Z \alpha_{2i} + P_{[\omega]}(0) + \sum_{i=1}^{k} b_i(\epsilon)\int_Z \alpha_{2i} \tag{4.9}$$

This integral only depends on the classes $[\alpha_i]$ and the Liouville Volume $P_{[\omega]}(0)$, which are determined by (and determine) the class of $[\omega]$. So, two b^{2k}-forms on the same cohomology class, determine the same cohomology class when desingularized.

□

Remark 4.4.6. *This previous theorem asserts that, for b^{2k}-surfaces (S, Z) with S closed and orientable and f and ϵ fixed, equivalent b^{2k}-symplectic structures get mapped to equivalent symplectic structures under the desingularization procedure. Non-equivalent b^{2k}-symplectic structures might get mapped to equivalent symplectic structures via deblogging. It is easy to see that there are different classes of b^{2k}-forms that desingularize to the same class by looking at expression (4.9). We only have terms $[\alpha_i]$ with i even. As a consequence, if two forms differ only in the odd terms, they have the same desingularized forms (assuming the auxiliary function f in the desingularization process is the same). We compute a particular example below.*

Example 4.4.7. *Consider S^2 with coordinates (h,θ). Consider the b^2-manifold given by $(S^2,\{h=0\},h)$ with the following two b^2-symplectic structures:*

$$\omega_1 = \frac{1}{h^2} dh\wedge d\theta, \qquad \omega_2 = \left(\frac{1}{h}+\frac{1}{h^2}\right) dh\wedge d\theta = \frac{1}{h^2} dh\wedge (h d\theta + d\theta). \tag{4.10}$$

As before, assume f and ϵ fixed. Observe that for ω_1, the forms in the Laurent series are $\alpha_0^1 = d\theta$ and $\alpha_1^1 = 0$, while for ω_2 they are $\alpha_0^2 = d\theta$ and $\alpha_1^2 = d\theta$. Then $\int_Z \alpha_1^1 = 0 \neq \int_Z \alpha_1^2 = 2\pi$, and hence $[\alpha_1^1] \neq [\alpha_1^2]$ and $[\omega_1] \neq [\omega_2]$. The desingularized expressions of those forms are given by:

$$\omega_{1\epsilon} = \begin{cases} \dfrac{df_\epsilon(h)}{dh} dh\wedge d\theta & \text{if } |h|\leq \epsilon, \\ \omega_1 & \text{otherwise,} \end{cases} \tag{4.11}$$

and

$$\omega_{2\epsilon} = \begin{cases} \dfrac{df_\epsilon(h)}{dh} dh\wedge (h d\theta + d\theta) & \text{if } |h|\leq \epsilon, \\ \omega_2 & \text{otherwise.} \end{cases} \tag{4.12}$$

Let us compute the classes of $\omega_{1\epsilon}$ and $\omega_{2\epsilon}$.

$$\begin{aligned} \int_{S^2} \omega_{2\epsilon} &= \int_{S^2\setminus U_\epsilon} \omega_2 + \int_{U_\epsilon} \frac{df_\epsilon(h)}{dh}(h d\theta + d\theta) \\ &= \int_{S^2\setminus U_\epsilon} \frac{1}{h^2} dh\wedge(h d\theta + d\theta) + \int_{U_\epsilon} \frac{df_\epsilon(h)}{dh}(d\theta) \\ &\qquad + \underbrace{\int_{U_\epsilon} \frac{df_\epsilon(h)}{dh}(h d\theta)}_{=0} \\ &= \int_{S^2\setminus U_\epsilon} \frac{1}{h^2} dh\wedge d\theta + \underbrace{\int_{S^2\setminus U_\epsilon} \frac{1}{h} dh\wedge d\theta}_{=0} + \int_{U_\epsilon} \frac{df_\epsilon(h)}{dh}(d\theta) \\ &= \int_{S^2} \omega_{1\epsilon}. \end{aligned}$$

Let us consider the action of S^1 over S^2 given by $\phi : S^1\times S^2 \to S^2 : (t,(h,\theta)) \mapsto (h,\theta+t)$. Observe that both ω_1 and ω_2 are invariant under the previous action. Moreover, their desingularizations are also invariant.

Chapter 5

Existence and classification of b^m-Nambu structures

In this chapter we consider a natural generalization of b^m-Poisson structures by imposing transversality conditions on higher order multivector fields. Nambu mechanics is a generalization of Hamiltonian mechanics involving multiple Hamiltonian functions.

Nambu structures originally considered in [28] and a Nambu bracket can be defined attached to a set of Hamiltonian functions which inspired the following axiomatic definition to Takhtajan [29]:

Definition 5.0.1. *A **Nambu structure** of degree k on a smooth manifold M^n, where $k \leq n$, is an k-multilinear, skew-symmetric bracket,*

$$\{\cdot, \dots, \cdot\} \colon \underbrace{C^\infty(M) \times \cdots \times C^\infty(M)}_{k} \to C^\infty(M),$$

satisfying:

1. *The Leibniz rule:*

$$\{fg, f_1, \dots, f_{r-1}\} = f\{g, f_1, \dots, f_{k-1}\} + \{f, f_1, \dots, f_{k-1}\}g,$$

2. *The identity:*

$$\{f_1, \dots, f_{r-1}, \{g_1, \dots, g_k\}\}$$

$$= \sum_{i=1}^{k} \{g_1, \ldots, \{f_1, \ldots, f_{k-1}, g_i\}, \ldots, g_k\};$$

This is indeed a generalization of Poisson bracket and thanks to the Leibniz rule, it can be encoded in the language of multivectorfields. In that language a Nambu structure of order k is defined by the following simple criteria:

Definition 5.0.2. *A Nambu structure of order k on a smooth manifold M is a k-vector field Π on M such that for any point $p \in M$ such that $\Pi(p) \neq 0$, there is a local coordinate system $(x_1, \ldots, x_n)$ in a neighborhood of p such that*

$$\Pi = \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_k}$$

in that neighborhood.

We can think of the following definition above as an integrability condition. In this chapter we study normal forms of Nambu structures with a b^m-singularities which are of maximal degree. We study normal forms, existence and classification à la Moser in terms of b^m-cohomology in the same terms as the previous chapter.

The contents of this section are published in the *Comptes Rendus de l'Academie des Sciences de Paris* (joint paper with Eva Miranda) (see [12]).

5.1 Constructions and classification of b^m-Nambu structures

Nambu structures of b^m-type can be described using forms which are singular along a smooth hypersurface.

We now introduce b^m-Nambu structures of top degree,

Definition 5.1.1. *A b^m-Nambu structure of top degree on a pair*

$$(M^n, Z)$$

with Z a smooth hypersurface is given by a smooth n-multivector field Λ such that there exists a local system of coordinates for which

$$\Lambda = x_1^m \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n} \tag{5.1}$$

and Z is defined by $x_1 = 0$ in a neighborhood of Z.

Dualizing the local expression of the Nambu structure we obtain the form

$$\Theta = \frac{1}{x_1^m} dx_1 \wedge \ldots \wedge dx_n \tag{5.2}$$

(which is not a smooth de Rham form), but it is a b^m-form of degree n defined on a b^m-manifold. As it is done in [5], we can check that this dual form is non-degenerate. So we may define a b^m-Nambu form as follows.

Mimicking the same condition as for b^m-symplectic forms we can talk about non-degenerate b^m-forms of top degree. This means that seen as a section of $\Lambda^n({}^bT^*M)$ the form does not vanish.

Notation: We will denote by Λ the Nambu multivectorfield and by Θ its dual.

Definition 5.1.2. *A b^m-Nambu form is a non-degenerate b^m-form of top degree.*

We first include a collection of motivating examples, and then prove an equivariant classification theorem.

5.1.1 Examples

1. **b^m-symplectic surfaces:** Any b^m-symplectic surface is a b^m-Nambu manifold with Nambu structure of top degree.

2. **b^m-symplectic manifolds as b^m-Nambu manifolds:** Let a pair (M^{2n}, ω) be a b^m-symplectic manifold, then $(M^{2n}, \underbrace{\omega \wedge \ldots \wedge \omega}_{n})$ is automatically b^m-Nambu.

3. **Orientable manifolds:** Let (M^n, Ω) be any orientable manifold (with Ω a volume form) and let f be a defining function for Z, then $(1/f^m)\Omega$ defines a b^m-Nambu structure of top degree having Z as critical set.

   Any Nambu structure can be written in this way if the hypersurface can be globally described as the vanishing set of a smooth function.

4. **Spheres:** In [14], it was given special importance to the example $(S^n, \sqcup_i S_i^{(n-1)})$ because of the Schoenflies theorem[1], which imposes the associated graph to be a tree. The nice feature of this example is that $O(n)$ acts on the b^m-manifold $(S^n, S^{(n-1)})$, and it makes sense to consider its classification under these symmetries. This also works for other homogeneous spaces of type $(G_1/G_2, G_2/G_3)$ with G_2 and G_3 with codimension 1 in G_1 and G_2 respectively.

5.1.2 b^m-Nambu structures of top degree and orientability

We start proving:

[1] The nature of this theorem is purely topological in dimension equal or greater than four, and so is its construction.

Theorem 5.1.3. *A compact n-dimensional manifold M admitting a b^{2k}-Nambu structure is orientable.*

Proof: Consider a collar of charts for the b^{2k}-Nambu structure such that in local coordinates the Nambu structure can be written as $x_1^{2k}\frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n}$ with compatible orientations in a neighborhood of each connected component of Z.

Consider a 2:1 orientable covering $(\tilde{M}, \tilde{Z})$ of the manifold and denote by $\rho : \mathbb{Z}/2\mathbb{Z} \times \tilde{M} \to \tilde{M}$ the deck transformation. For each point $p \in \tilde{Z}$ take a neighborhood U_p which does not contain other points identified by ρ thus $U_p \cong \pi(U_p) =: V_p$, and $\Theta = \frac{1}{x^{2k}} dx_1 \wedge \ldots \wedge dx_n$. This form defines an orientation on $V_p \setminus \pi(Z)$. Take a symmetric covering of such neighborhoods to define a collar of Z with compatible orientations, and compatible with the covering. The compatible orientations and the symmetric coverings descend to (M, Z), thus defining an orientation in (M, Z). Thus, we have an orientation in $V \setminus Z$. By perturbing Θ in V we obtain a volume form on V, $\tilde{\omega}$, and thus an orientation in V. These can be glued to define an orientation via the volume form $\tilde{\Theta}$ on the whole M proving that M is oriented.

5.1.3 Classification of b^m-Nambu structures of top degree and b^m-cohomology

We present the definitions contained in [14] of modular period attached to the connected component of an orientable Nambu structure using the language of b^m-forms.

Let Θ be the dual to the multivectorfield Λ defining a Nambu structure. From the general decomposition of b^m-forms as it was set in Equation 2.8 we may write:

$$\Theta = \Theta_0 \wedge \frac{df}{f^m}$$

with $\Theta_0 \in \Omega^{n-1}(M)$.

This decomposition is valid in a neighborhood of Z whenever the defining function is well-defined. For non-orientable manifolds a similar decomposition can be proved by replacing the defining function f by an adapted distance (see [9]).

With this language in mind, the the **modular** $(n-1)$**-vector field** in [14] of Θ along Z is the dual of the form Θ_0 in the decomposition above which is indeed the **modular** $(n-1)$**-form** along Z in [14].

Recall from [14] in our language:

Definition 5.1.4. *The **modular period** T_Λ^Z of the component Z of the zero locus of Λ is*

$$T_\Lambda^Z := \int_Z \Theta_0 > 0.$$

In fact, this positive number determines the Nambu structure in a neighborhood of Z up to isotopy as it was proved in [14].

The following theorem gives a classification of b^m-Nambu structures.

Theorem 5.1.5. *Let Θ_0 and Θ_1 be two b^m-Nambu forms of degree n on a compact orientable manifold M^n. If $[\Theta_0] = [\Theta_1]$ in b^m-cohomology then there exists a diffeomorphism ϕ such that $\phi^*\Theta_1 = \Theta_0$.*

Proof: We will apply the techniques of [23] with the only difference that we work with b^m-volume forms instead of volume forms.

Since Θ_0 and Θ_1 are non-degenerate b^m-forms both of them are a multiple of a volume form and thus the linear path $\Theta_t = (1-t)\Theta_0 + t\Theta_1$ is a path of non-degenerate b^m-forms.

Because Θ_0 and Θ_1 determine the same cohomology class:

$$\Theta_1 - \Theta_0 = d\beta$$

with d the b^m-De Rham differential and β a b^m-form of degree $n-1$.

Now consider the Moser equation:

$$\iota_{X_t}\Theta_t = -\beta. \tag{5.3}$$

Observe that since β is a b^m-form and Θ_t is non-degenerate. The vector field X_t is a b^m-vector field. Let ϕ_t be the t-dependent flow integrating X_t.

The ϕ_t gives the desired diffeomorphism $\phi_t : M \to M$, leaving Z invariant (since X_t is tangent to Z) and $\phi_t^* \Theta_t = \Theta_0$.

In particular we recover the classification of b-Nambu structures of top degree in [14]:

Theorem 5.1.6 (Classification of b-Nambu structures of top degree, [14]). *A generic b-Nambu structure Θ is determined, up to orientation preserving diffeomorphism, by the following three invariants: the diffeomorphism type of the oriented pair (M, Z), the modular periods and the regularized Liouville volume.*

By Theorem 2.2.1,

$$^bH^n(M) \cong H^n(M) \oplus H^{n-1}(Z).$$

The first term on the right hand side is the Liouville volume image by the De Rham theorem, as it was done in [7] for b-symplectic forms. The second term collects the periods of the modular vector field. So if the three invariants coincide then they determine the same b-cohomology class.

In other words, the statement in [14] is equivalent to the following theorem in the language of b-cohomology.

Theorem 5.1.7. *Let Θ_1 and Θ_2 be two b-Nambu forms on an orientable manifold M. If $[\Theta_1] = [\Theta_2]$ in b-cohomology then there exists a diffeomorphism ϕ such that $\phi^* \Theta_1 = \Theta_2$.*

This global Moser theorem for b^m-Nambu structures admits an equivariant version,

Theorem 5.1.8. *Let Θ_0 and Θ_1 be two b^m-Nambu forms of degree n on a compact orientable manifold M^n and let $\rho : G \times M \longrightarrow M$ be a* *compact Lie group action preserving both b^m-forms. If $[\Theta_0] = [\Theta_1]$ in b^m-cohomology then there exists an equivariant diffeomorphism ϕ such that $\phi^*\Theta_1 = \Theta_0$.*

Proof: As in the former proof, write

$$\Theta_1 - \Theta_0 = d\beta$$

with d the b^m-De Rham differential and β a b^m-form of degree $n-1$. Observe that the path $\Theta_t = (1-t)\Theta_0 + t\Theta_1$ is a path of invariant b^m-forms.

Now consider Moser's equation:

$$\iota_{X_t}\Theta_t = -\beta. \tag{5.4}$$

Since Θ_t is invariant we can find an invariant $\tilde{\beta}$. For instance take $\tilde{\beta} = \int_G \rho_g^*(\beta) d\mu$ with μ a de Haar measure on G and ρ_g the induced diffeomorphism $\rho_g(x) := \rho(g, x)$.

Now replace β by $\tilde{\beta}$ to obtain,

$$\iota_{X_t^G}\Theta_t = -\tilde{\beta} \tag{5.5}$$

with $X_t^G = \int_G \rho_{g_*} X_t d\mu$. The vector field X_t^G is an invariant b-vector field. Its flow ϕ_t^G preserves the action and $\phi_t^{G*}\Theta_t = \Theta_0$.

Playing the equivariant b^m-Moser trick using the 2:1 cover of a non-orientable manifold and taking as G the group of deck transformations we obtain,

Corollary 5.1.9. *Let Θ_0 and Θ_1 be two b^m-Nambu forms of degree n on a manifold M^n (not necessarily oriented). If $[\Theta_0] = [\Theta_1]$ in b^m-cohomology then there exists a diffeomorphism ϕ such that $\phi^*\Theta_1 = \Theta_0$.*

Chapter 6

An action-angle theorem for b^m-Poisson manifolds

In this chapter we continue with the study of classification theorems. In this case we consider the semilocal classification for any b^m-Poisson manifold in a neighbourhood of an invariant compact submanifold. The compact submanifold which we will be considering are the compact invariant leaves of the distribution $\mathcal{D}$ generated by the Hamiltonian vector fields X_{f_i} of an integrable system. An integrable system is given by a set of n functions on a $2n$-dimensional symplectic manifold which we can order in a map $F = (f_1, \ldots, f_n)$. Historically, integrable systems where introduced to actually *integrate* Hamiltonian systems X_H using the *first-integrals* f_i and, classically, we identify $H = f_1$. It turns out that in the symplectic context the compact regular orbits of the distribution $\mathcal{D}$ coincide with the fibers $F^{-1}(F(p))$ for any point p on these orbits/fibers. The fact that the orbit coincides with the connected fiber is part of the magic of symplectic duality.

The same picture is reproduced for singular symplectic manifolds of b^m-type or b^m-Poisson manifolds as we will see in this chapter.

The study of action-angle coordinates has interest from this geometrical point of view of classification of geometric structures in a neighbourhood of a compact submanifold of a b^m-Poisson manifold, but also an interest from a dynamical point of view as these compact submanifolds now coincide with invariant subsets of the Hamiltonian system under consideration.

From a geometric point of view, the existence of action-angle coordinates determines a *unique* geometrical model for the b^m-Poisson (or b^m-symplectic) structure in a neighbourhood of the invariant set. From a dynamical point of view, the existence of action-angle coordinates provides a normal form theorem that can be used to study stability and perturbation problems of the Hamiltonian systems (as we will see in the last chapter of this thesis).

An important ingredient that makes our action-angle coordinate theorem *brand-new* from the symplectic perspective is that the system under consideration is more general than Hamiltonian, it is b^m-Hamiltonian as the first-integrals of the system can be b^m-functions which are not necessarily smooth functions. Dynamically, this means that we are adding to the set of Hamiltonian invariant vector fields, the *modular vector field* of the integrable system.

In contrast to the standard action-angle coordinates for symplectic manifolds, our action-angle theorem comes with m additional invariants associated to the modular vector field which can be interpreted in cohomological terms as the projection of the b^m-cohomology class determined by the modular vector field on the first cohomology group of the critical hypersurface under the Mazzeo-Melrose correspondence.

The strategy of the proof of the action-angle coordinate systems is the search of a toric action (so this takes us back to the motivation of the use of *symmetries* in this thesis). In contrast to the symplectic case, it is not enough that this action is Hamiltonian as then a direction of the Liouville torus would be missing. We need the toric action to be b^m-Hamiltonian. The structure of this proof looks like the one in [2] but encounters serious technical difficulties as in order to check that the *natural* action to be considered is b^m-Hamiltonian we need to go deeper inspired by [1] in the relation between the geometry of the modular vector field and the coefficients of the Taylor series c_i of one of the first-integrals. This enables to understand new connections between the geometry and analysis of b^m-Poisson structures not explored before.

Once we prove the existence of this b^m-Hamiltonian action the proof looks very close to the one in [2].

We end up this chapter restating the action-angle theorem as a cotangent lift theorem with the following mantra:

Every integrable system on a b^m-Poisson manifold looks like a b^m-cotangent lift in a neighborhood of a Liouville torus.

6.1 Basic definitions

6.1.1 On b^m-functions

The definition of the analogue of b-functions in the b^m-setting is somewhat delicate. The set of ${}^{b^m}\mathcal{C}^\infty(M)$ needs to be such that for all the functions $f \in^{b^m} \mathcal{C}^\infty(M)$, its differential df is a b-form, where d is the b^m-exterior differential. Recall that a form in ${}^{b^m}\Omega^k(M)$ can be locally written as

$$\alpha \wedge \frac{dx}{x^m} + \beta$$

where $\alpha \in \Omega^{k-1}(M)$ and $\beta \in \Omega^k(M)$. Recall also that

$$d\left(\alpha \wedge \frac{dx}{x^m} + \beta\right) = d\alpha \wedge \frac{dx}{x^m} + d\beta.$$

We need df to be a well-defined b^m-form fo degree 1. Let $f = g\frac{1}{x^{k-1}}$, then $df = dg\frac{1}{x^{k-1}} - g\frac{k-1}{x^k}dx$. This from can only be a b^m-form if and only if g only depends on x. If $f = g\log(x)$, then $dg\log(x) + g\frac{1}{x}dx$, which imposes $dg = 0$ and hence g to be constant.

With all this in mind, we make the following definition.

Definition 6.1.1. *The set of b^m-functions is defined recursively according to the formula*

$$^{b^m}\mathcal{C}^\infty(M) = x^{-(m-1)}\mathcal{C}^\infty(x) + {}^{b^{m-1}}\mathcal{C}^\infty(M)$$

with $\mathcal{C}^\infty(x)$ the set of smooth functions in the defining function x and

$$^{b}\mathcal{C}^\infty(M) = \{g \log|x| + h, g \in \mathbb{R}, h \in \mathcal{C}^\infty(M)\}.$$

Remark 6.1.2. *A ${}^{b^m}\mathcal{C}^\infty(M)$-function can be written as*

$$f = a_0 \log x + a_1 \frac{1}{x} + \ldots + a_{m-1}\frac{1}{x^{m-1}} + h$$

where $a_i, h \in \mathcal{C}^\infty(M)$.

Remark 6.1.3. *From this chapter on we are only considering b^m-manifolds (M, x, Z) with x defined up to order m. I.e. we can think of x as a jet of functions that coincide up to order m to some defining function. This is the original viewpoint of Scott in [1] which we adopt from now on. The difference with respect to the other chapters is that we do not fix an specific function (but the jet in this chapter).*

Definition 6.1.4. *We say that two b^m-integrable systems F_1, F_2 are equivalent if there exists φ, a b^m-symplectomorphism, i.e. a diffeomorphism preserving both ω and the critical set Z ("up to order m"[1]), such that $\varphi \circ F_1 = F_2$.*

Remark 6.1.5. *The Hamiltonian vector field associated to a b^m-function f is a smooth vector field. Let us compute it locally using the b^m-*

[1]I.e. it preserves the jet x

Darboux theorem:

$$\Pi = x_1^m \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{m} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} \text{ and } f = a_0 \log x_1 + \sum_{i=1}^{m-1} a_i \frac{1}{x_1^i} + h.$$

Then if we compute

$$\begin{aligned} df &= \overbrace{a_0}^{c_1} \frac{1}{x_1} + \sum_{i=1}^{m-1} \overbrace{(a_i' - (i-1)a_{i-1})}^{c_i} \frac{1}{x_1^i} dx_1 \\ &\quad - \overbrace{(m-1)a_{m-1}}^{c_m} \tfrac{1}{x_1^m} dx_1 + dh \\ &= \sum_{i=1}^{m} \frac{c_i}{x_1^i} dx_1 + dh. \end{aligned}$$

Then,

$$X_f = \Pi(df, \cdot) = \sum_{i=1}^{m} c_i x_1^{m-i} \frac{\partial}{\partial y_1} + \Pi(dh, \cdot), \tag{6.1}$$

we obtain a smooth vector field.

6.1.2 On b^m-integrable systems

In this section we present the definition of b^m-integrable system as well as some observations about these objects.

Definition 6.1.6. *Let (M^{2n}, Z, x) be a b^m-manifold, and let Π be a b^m-Poisson structure on it. $F = (f_1, \ldots, f_n)$[2] is a b^m-integrable system[3] if:*

1. *$df_1, \ldots, df_n$ are independent on a dense subset of M and in all the points of Z where independent means that the form $df_1 \wedge \ldots \wedge df_n$ is non-zero as a section of $\Lambda^n({}^{b^m}T^*(M))$,*

2. *the functions $f_1, \ldots, f_n$ Poisson commute pairwise.*

[2] f_i are b^m-functions.

[3] In this thesis we only consider integrable systems of maximal rank n.

Definition 6.1.7. *The points of M where $df_1, \ldots, df_n$ are independent are called **regular** points.*

The next remarks will lead us to a normal form for the first function f_1.

Remark 6.1.8. *Note that $df_1, \ldots, df_n$ are independent on a point if and only if $X_{f_1}, \ldots, X_{f_n}$ are independent at that point. This is because the map*

$$^{b^m}TM \rightarrow^{b^m} T^*M : u \mapsto \omega_p(u, \cdot)$$

is an isomorphism.

Remark 6.1.9. *The condition of $df_1, \ldots, df_n$ being independent must be understood as $df_1 \wedge \ldots \wedge df_n$ being a non-zero section of $\bigwedge^n(\ ^{b^m}T^*M)$.*

Remark 6.1.10. *By remark 6.1.8 the vector fields $X_{f_1}, \ldots, X_{f_n}$ have to be independent. This implies that one of the $f_1, \ldots, f_n$ has to be a singular (non-smooth) b^m-function with a singularity of maximal degree. If we write $f_i = c_{0,i} \log(x_1) + \sum_{j=1}^{m-1} \frac{c_{j,i}}{x_1^j} + \tilde{f}_1$*

$$X_{f_i} = \sum_{j=1}^{m} x_1^{m-j} \hat{c}_{j,i} \frac{\partial}{\partial y_1} + X_{\tilde{f}_i}$$

where $\hat{c}_{j,i}(x) = \frac{d(c_{j,i})}{dx} - (j-1)c_{j-1,i}$. If there is no b^m-function with a singularity of maximum degree all the terms in the $\partial/\partial y_1$ direction become 0 at Z. And hence $X_{f_1}, \ldots, X_{f_n}$ cannot have maximal rank at Z.

Lemma 6.1.11. *Let $F = (f_1, \ldots, f_n)$ a b^m-integrable system. If f_1 has a singularity of maximal degree, there exists an equivalent integrable* *system $F' = (f_1', \ldots, f_n')$ where f_1' has a singularity of maximal degree and **no other** f_i' has singularity of any degree.*

Proof. Let $f_i = \underbrace{c_{0,i}\log(x_1) + \sum_{j=1}^{m-1}\frac{c_{j,1}}{x_1^j}}_{\zeta_i(x_1)} + \tilde{f}_i = \zeta_i(x_1) + \tilde{f}_i$. By remark 6.1.10[4],

$$X_{f_i} = \underbrace{\sum_{i=1}^{m} x_1^{m-j}\hat{c}_{j,i}}_{g_i(x_1)}\frac{\partial}{\partial y_1} + X_{\tilde{f}_i} = g_i(x_1)\frac{\partial}{\partial y_1} + X_{\tilde{f}_i}.$$

Note that $g_i(x_1) = g_i(0) = \hat{c}_{m,i}$ at Z. Let us look at the distribution given by the Hamiltonian vector fields $X_{f_i} = g_i(x_1)\frac{\partial}{\partial y_1} + X_{\tilde{f}_i}$. This distribution is the same that the one given by:

$$\{X_{f_1}, X_{f_2} - \frac{g_2(x_1)}{g_1(x_1)}X_{f_1}, \ldots, X_{f_n} - \frac{g_n(x_1)}{g_1(x_1)}X_{f_1}\}. \tag{6.2}$$

Observe that for $i > 1$, $X_{f_i} - \frac{g_i(x_1)}{g_1(x_1)}X_{f_1} = X_{\tilde{f}_i} + \frac{g_2(x_1)}{g_1(x_1)}X_{\tilde{f}_1}$. Also $g_1(x_1)$ is different from 0 close to Z because at Z $g_1(x_1) = \hat{c}_{m,1}$. Since the distribution given by these vector fields is the same, an integrable system that has Hamiltonian vector fields 6.2 would be equivalent to F. From the expression 6.2 it is clear that the new vector fields commute. And it is also true that this new vector fields are Hamiltonian. Let us take F' the set of functions that have as Hamiltonian vector fields 6.2. $\square$

From now on we will assume the integrable system to have only one singular function and this function to be f_1.

Remark 6.1.12. *Because we asked $X_{f_1}, \ldots, X_{f_n}$ to be linearly independent at all the points of Z and using the previous remarks $c_m := c_{m,1} \neq 0$ at all the points of Z.*

Furthermore, we can assume f_1 to have a smooth part equal to zero as subtracting the smooth part of f_1 to all the functions gives an

[4]Here have used the b^m-Darboux theorem to do the computations.

equivalent system. Also, we can assume that c_m is 1 because dividing all the functions of the b^m-integrable system by c_m also gives us an equivalent system.

As a summary, we can assume $f_1 = a_0 \log(x) + a_1 1/x + \ldots + a_{m-2} 1/x^{m-2} + 1/x^{m-1}$ and $f_2, \ldots, f_n$ to be smooth, $a_0 \in \mathbb{R}$ and $a_1, \ldots, a_{m-2} \in \mathcal{C}^\infty(x)$.

Also we are going to state lemma 3.2 in [26], because we are going to use it later in this section. The result states that if we have a toric action on a b^m-symplectic manifold (which we will prove in a neighbourhood of a Liouville torus), then we can assume the coefficients $a_2, \ldots, a_{m-2}$ to be constants. More precisely

Lemma 6.1.13. *There exists a neighborhood of the critical set $U = Z \times (-\varepsilon, \varepsilon)$ where the moment map $\mu : M \to \mathfrak{t}^*$ is given by*

$$\mu = a_1 \log|x| + \sum_{i=2}^{m} a_i \frac{x^{-(i-1)}}{i-1} + \mu_0$$

with $a_i \in \mathfrak{t}_L^0$ and μ_0 is the moment map for the T_L-action on the symplectic leaves of the foliation.

6.1.3 Examples of b^m-integrable systems

The following example illustrates why it is necessary to use the definition of b^m-function as considered above. There are natural examples of changes of coordinates in standard integrable systems in symplectic manifolds that yield to b^m-symplectic manifolds but do not give b^m-integrable systems.

Example 6.1.14. *This example makes a time change in the two body problem, to obtain a b^2-integrable system. In the classical construction used to solve the 2-body problem we obtain the following two conserved quantities:*

$$\begin{aligned} f_1 &= \tfrac{\mu y^2}{2} + \tfrac{l^2}{2\mu r^2} - \tfrac{k}{r}, \\ f_2 &= l, \end{aligned}$$

with symplectic form $\omega = dr \wedge dy + dl \wedge d\alpha$, where r is the distance between the two masses and l is the angular momentum. We also know that l is constant along the trajectories. Because l is a constant of the movement, we can do a symplectic reduction on its level sets. The form on the symplectic reduction becomes $dr \wedge dy$. To simplify the notation we will note x instead of r. Then $\omega = dx \wedge dy$. With hamiltonian given by $f = \frac{\mu}{2}y^2 + \frac{l}{2\mu}\frac{1}{x^2} - k\frac{1}{x}$. Hence, the equations are:

$$\begin{aligned} \dot{x} &= \frac{\partial f}{\partial y}, \\ \dot{y} &= -\frac{\partial f}{\partial x}. \end{aligned}$$

Doing a time change $\tau = x^3 t$ then $\frac{dx}{d\tau} = \frac{1}{x^3}\frac{dx}{dt}$. With this time coordinate, the equations become:

$$\begin{aligned} \dot{x} &= \frac{1}{x^3}\frac{\partial f}{\partial y}, \\ \dot{y} &= -\frac{1}{x^3}\frac{\partial f}{\partial x}. \end{aligned}$$

These equations can be viewed as the motion equations given by a b^3-symplectic form $\omega = \frac{1}{x^3}dx \wedge dy$.

Let us check that this is actually a b^m-integrable system.

- *All the functions Poisson commute is immediate because we only have one.*
- *$df = \mu y dy + (\frac{k}{x^2} - \frac{l}{\mu}\frac{1}{x^3})dx$ is a b^3-form because the term with dx does not depend on y.*
- *All the functions are independent, this is true because df does not vanish as a b^3-form.*

Example 6.1.15. *In the paper [30] the author builds an action of $SL(2,\mathbb{R})$ over (P, ω_P) where $P = \{\xi \in \mathbb{C} | i(\bar{\xi} - \xi) > 0\}$ is the complex semi-plane, with moment map $J_P(\xi) = \frac{R}{\xi_{im}}((|\xi|^2+1), 2\xi_r, \pm(|\xi|^2+1))$, where the $\pm$ sign depends on the choice of the hemisphere projected by the stereographic projection. From now on we will take the sign $+$. Also the symplectic form ω_P has the following expression:*

$$\omega_P = \pm\frac{R}{\xi_{im}^2} d\xi_r \wedge d\xi_{im}$$

For sake of simplifying the notation we will identify P with the real half-plane $P = \{x, y \in \mathbb{R}^2 | y > 0\}$. With this identification, the moment map becomes $J_p(x, y) = \frac{R}{y}(x^2 + y^2 + 1, 2x, x^2 + y^2 + 1)$. Obviously, this moment map does not give an integrable system. The symplectic form writes as:

$$\omega_P = \frac{R}{y^2} dy \wedge dx.$$

Which can be viewed as a b^2-form if we extend P including the line $\{y = 0\}$ as its singular set. Let us consider only one of the components of J_P as b^m-function and let us see if it gives a b^m-integrable system. First we will try with $f_1 = \frac{R}{y}(x^2 + y^2 + 1)$ and then $f_2 = \frac{R}{y}(2x)$.

1. $f_1 = \frac{R}{y}(x^2 + y^2 + 1)$ *We have to check three things to see if this gives a b^2-integrable system.*

   (a) *All the functions Poisson commute is immediate because we only have one.*

   (b) *All the functions are b^m-functions. This point does not hold because $df_1 = \frac{R}{y^2}(2xydx + (y^2 - x^2 - 1)dy)$ and the first component does have no sense as a section of $\Lambda^1({}^{b^2}T^*M)$.*

   (c) *All the functions are independent. In this case we have to check that df_1 does not vanish, but since it is not a b^m-form it has no sense to be a non-zero section of $\Lambda^1({}^{b^2}T^*M)$.*

2. $f_2 = \frac{R}{y}(2x)$

   (a) *Same as before.*

   (b) *All the functions are b^m-functions. This point does not hold because $df_2 = \frac{2R}{y}dx - \frac{2Rx}{y^2}dy$ and the first component does have no sense as a section of $\Lambda^1({}^{b^2}T^*M)$.*

(c) Same as before.

Now we give a couple examples of b^m-integrable systems.

Example 6.1.16. *This example uses the product of b^m-integrable systems on a b^m-symplectic manifold with an integrable system on a symplectic manifold. Given $(M_1^{2n_1}, Z, x, \omega_1)$ a b^m-symplectic manifold with $f_1, \dots, f_{n_1}$ a b^m-integrable system and $(M_2^{2n_2}, \omega_2)$ a symplectic manifold with $g_1, \dots, g_{n_2}$ an integrable system. Then $(M_1 \times M_2, Z \times M_2, x, \omega_1 + \omega_2)$ is a b^m-symplectic manifold and $(f_1, \dots, f_{n_1}, g_1, \dots, g_{n_2})$ is a b^m-integrable system on it.*

Example 6.1.17. ***(From integrable systems on cosymplectic manifolds to b^m-integrable systems:)***

Using the extension theorem (Theorem 50) of [5] we can extend any integrable system $(f_2, \dots, f_n)$ to an integrable system in a neighbourhood of a cosymplectic manifold (Z, α, ω) by just adding a b^m-function f_1 to the integrable system so that the new integrable system is $(f_1, f_2, \dots, f_n)$ and considering the associated b^m-symplectic form:

$$\tilde{\omega} = p^*\alpha \wedge \frac{dt}{t^m} + p^*\omega. \tag{6.3}$$

(t is the defining function of Z).

6.2 Looking for a toric action

One of the main difficulties in comparison with the construction for b-symplectic structures it is that is not easy to prove that coefficients $a_1, \dots, a_n$ can be considered constants. This makes more difficult to prove the existence of a $\mathbb{T}^n$ action, but once we have it we can use the results in [8] to assume that the coefficients $a_1, \dots, a_n$ are constant.

Proposition 6.2.1. *Let (M, Z, x, ω) be a b^m-symplectic manifold such that Z is connected with modular period k. Let $\pi : Z \to S^1 \simeq \mathbb{R}/k\mathbb{Z}$ be the projection to the base of the corresponding mapping torus. Let $\gamma : S^1 = \mathbb{R}/k\mathbb{Z} \to Z$ be any loop such that $\pi \circ \gamma$ is positively oriented and has constant velocity 1. Then the following are equal:*

1. *The modular period of Z,*
2. *$\int_\gamma \iota_{\mathbb{L}}\omega$,*
3. *The value a_{m-1} for any ${}^{b^m}\mathcal{C}^\infty(M)$ function*
$$f = a_0 \log(x) + \sum_{j=1}^{m-1} a_j \frac{1}{x^j} + h$$
such that the hamiltonian vector field X_f has 1-periodic orbits homotopic in Z to some γ.

Proof. Let us prove separately that (1)=(2) and later (2)=(3).

(1)=(2) Lets denote by $\mathcal{V}_{mod}$ the modular vector field. Recall from [8] that $\iota_{\mathbb{L}}(\mathcal{V}_{mod})$ is the constant function 1. Let $s : [0, k] \to Z$ be the trajectory of the modular vector field. Because the modular period is k, $s(0)$ and $s(k)$ are in the same leaf $\mathcal{L}$. Let $\hat{s} : [0, k+1] \to Z$ a smooth extension of s such that $s|_{[k,k+1]}$ is a path in $\mathcal{L}$ joining $\hat{s}(k) = s(k)$ to $\hat{s}(k+1) = s(0)$. This way $\hat{s}$ becomes a loop. Then,

$$k = \int_0^k 1dt = \int_S \iota_{\mathbb{L}}\omega = \int_{\hat{s}} \iota_{\mathbb{L}}\omega = \int_\gamma \iota_{\mathbb{L}}\omega$$

(2)=(3) Let $r : [0, 1] \mapsto Z$ be the trajectory of X_f the hamiltonian vector field of f. Recall that X_f satisfies

$$\iota_{X_f}\omega = \sum_{j=1}^{m} c_j \frac{dx}{x^i} + dh.$$

Let $x^m\frac{\partial}{\partial x}$ be a generator of the linear normal bundle $\mathbb{L}$. We know that X_f is 1-periodic and its trajectory is homotopic to γ. Hence,

$$\begin{aligned} k = \textstyle\int_r \iota_{\mathbb{L}}\omega &= \int_0^1 \iota_{x^m\frac{\partial}{\partial x}}\omega(X_f|_{r(t)})dt \\ &= \int_0^1 -(\sum_{j=1}^m c_i\frac{dx}{x^i} + dh)\cdot(x^m\frac{\partial}{\partial x})|_{r(t)}dt \\ &= -c_m = -a_{m-1} \end{aligned}$$

□

Theorem 6.2.2 (Darboux-Carathéodory (b^m-version))**.** *Let*

$$(M^{2n}, x, Z\omega)$$

be a b^m-symplectic manifold and m be a point on Z. Let $f_1,\dots,f_n$ be a b^m-integrable system. Then there exist b^m-functions $(q_1,\dots,q_n)$ around m such that

$$\omega = \sum_{i=1}^n df_i\wedge dq_i$$

and the vector fields $\{X_{f_i}, X_{q_j}\}_{i,j}$ commute. If f_1 is not smooth (recall $f_1 = a_0\log(x) + \sum_{j=1}^{m-1} a_j\frac{1}{x^i}$ with $a_n \neq 0$ on Z and $a_0\in\mathbb{R}$) the q_i can be chosen to be smooth functions, and $(x, f_2,\dots,f_n,q_1,\dots,q_n)$ is a system of local coordinates.

Proof. The first part of this proof is exactly as in [2]. Assume now $f_1 = a_0\log(x) + \sum_{j=1}^{m-1} a_j\dfrac{1}{x^i}$. We modify the induction requiring also that μ_i (in addition to be in K_i) is also in $T^*M \subseteq^b T^*M$. We can also ask this extra condition while asking $\mu_i(X_{f_i}) = 1$, we only have to check that X_{f_i} does not vanish in TM. This is clear because X_{f_i} does not vanish at ${}^bTM$ and

$$0 = \{f_n, f_i\} = \left(\sum_{i=1}^m \tilde{a}_i\frac{dx}{x^i}\right)(X_{f_i}) = \left(\frac{dx}{x^m}\sum_{i=1}^m a_i x^i\right)(X_{f_i}).$$

Where the last expression becomes 0 for each and every term except for the one of degree m.

Then dx/x^m is in the kernel of X_{f_i}, hence X_{f_i} does not vanish on TM and the q_i can be chosen smooth.

$\{X_x, X_{f_2}, \dots, X_{f_n}, X_{q_1}, \dots X_{q_n}\}$ commute because $\{X_{f_i}, X_{q_i}\}_{i,j}$ commute. Then

$$dx \wedge df_2 \dots \wedge df_n \wedge dq_1 \wedge \dots \wedge dq_n$$

is a non-zero section of $\bigwedge^n({}^bTM)$. And hence

$$(x, f_2, \dots, f_{n-1}, q_1, \dots, q_n)$$

are local coordinates.

□

Lemma 6.2.3 (Topological Lemma)**.** *Let $m \in Z$ be a regular point of a b^m-integrable system (M, x, Z, ω, F). Assume that the integral manifold $\mathcal{F}_m$ through m is compact. Then there exists a neighborhood U of $\mathcal{F}_m$ and a diffeomorphism*

$$\phi : U \simeq \mathbb{T}^n \times B^n$$

which takes the foliation $\mathcal{F}$ to the trivial foliation $\{\mathbb{T}^n \times \{b\}\}_{b \in B^n}$.

Proof. As in the proof of [2] we follow the steps of [31]. In this case, the only extra step that must be checked is that the foliation given by the b^m-hamiltonian vector fields of $F = (f_1, f_2, \dots, f_n)$ is the same as the one given by the level sets of $\tilde{F} := (x, f_2, \dots, f_n)$. In our case $f_1 = a_0 \log(x) + \sum_{u=1}^{m-1} a_i \frac{1}{x^i}$, where $a_0 \in \mathbb{R}$, $a_i \in \mathcal{C}^\infty(x)$, $a_{m-1} = 1$. Hence the foliations are the same. Then as in [31], we take an arbitrary Riemannian metric on M and this defines a canonical projection $\psi : U \to \mathcal{F}_m$. Let us define $\phi := \psi \times \tilde{F}$. We obtain the commutative diagram (Figure 6.1).

$$\begin{tikzcd} U \arrow[r, dashed, "\phi"] \arrow[dr, "\tilde{F}"'] & \mathbb{T}^n \times B^n \arrow[d, "p"] \\ & B^n \end{tikzcd}$$

Figure 6.1: Commutiative diagram of the construction of the isomorphism of b^m-integrable systems.

which provides the necessary isomorphism of b^m-integrable systems.

□

6.2.1 Action-angle coordinates on b^m-symplectic manifolds

In a neighbourhood of one of our Liouville tori all we can assume about the form of our b^m-symplectic structure is that is given by the Laurent series defined in [1].

That is to say we can assume that in a tubular neighborhood U of Z

$$\omega = \sum_{j=1}^{m-1} \frac{dx}{x^i}\pi^*(\alpha_i) + \beta,$$

where $\pi : U \to Z$ is the projection of the tubular neighborhood onto Z, α_i are closed smooth de Rham forms on Z and β a de Rham form on M.

Theorem 6.2.4 (Action-angle coordinates for b^m-symplectic manifolds)**.** *Let (M, x, ω, F) be a b^m-integrable system, where $F = (f_1 = a_0 \log(x) + \sum_{j=1}^{m-1} a_j \frac{1}{x^j}, \ldots, f_n)$ with a_j for $j > 1$ functions in x. Let $m \in Z$ be a regular point and let us assume that the integral manifold of the distribution generated by the X_{f_i} through m is compact. Let $\mathcal{F}_m$ be the Liouville torus through m. Then, there exists a neighborhood U of $\mathcal{F}_m$ and coordinates $(\theta_1, \ldots, \theta_n, \sigma_1, \ldots, \sigma_n) : \mathcal{U} \to \mathbb{T}^n \times B^n$ such that:*

1. *We can find an equivalent integrable system $F = (f_1 = a_0' \log(x)+$*

$\sum_{j=1}^{m-1} a'_j \frac{1}{x^j}, \ldots, f_n)$ *such that the coefficients* $a'_0, \ldots, a'_{m-1}$ *of* f_1 *are constants* $\in \mathbb{R}$,

2.
$$\omega|_{\mathcal{U}} = \left(\sum_{j=1}^{m} c'_j \frac{c}{\sigma_1^j} d\sigma_1 \wedge d\theta_1 \right) + \sum_{i=2}^{n} d\sigma_i \wedge d\theta_i$$
where c *is the modular period and* $c'_j = -(j-1)a'_{j-1}$, *also*

3. *the coordinates* $\sigma_1, \ldots, \sigma_n$ *depend only on* $f_1, \ldots f_n$.

Proof. The idea of this proof is to construct an equivalent b^m-integrable system whose fundamental vector fields define a $\mathbb{T}^n$-action on a neighborhood of $\mathbb{T}^n \times \{0\}$. It is clear that all the vector fields $X_{f_1}, \ldots, X_{f_n}$ define a torus action on each Liouville tori $\mathbb{T}^n \times \{b\}$ where $b \in B^n$, but this does not guarantee that their flow defines a toric action on all $\mathbb{T}^n \times B^n$. The proof is structured in three steps. The first one is the uniformization of the periods, i.e. we define an $\mathbb{R}^n$-action on a neighborhood of $\mathbb{T}^n \times \{0\}$ such that the lattice defined by its kernel at every point is constant. This allows to induce an actual action of a torus (as the periods are constant) of rank n: A $\mathbb{T}^n$ action by taking quotients. The second step consists in checking that this action is actually b^m-Hamiltonian. And in the final step we apply theorem 6.2.2 to obtain the expression of ω.

1. Uniformization of periods.

   Let $\Phi^s_{X_F}$ be defined as the joint flow by the Hamiltonian vector fields of the action:
$$\begin{array}{rcl} \Phi : \mathbb{R}^n \times (\mathbb{T}^n \times B^n) & \to & (\mathbb{T}^n \times B^n) \\ ((s_1, \ldots, s_n), (x, b)) & \mapsto & \Phi^{s_1}_{X_{f_1}} \circ \cdots \circ \Phi^{s_n}_{X_{f_n}}((x, b)) \end{array} \tag{6.4}$$
   this defines an $\mathbb{R}^n$-action on $\mathbb{T}^n \times B^n$. For each $b \in B^n$ at a single orbit $\mathbb{T}^n \times \{b\}$ the kernel of this action is a discrete subgroup of $\mathbb{R}^n$. We will denote the lattice given by this kernel Λ_b. Because

the orbit is compact, the rank of Λ_b is maximal i.e. n. This lattice is known as the period lattice of $\mathbb{T}^n \times \{b\}$ as we know by standard arguments in group theory that the lattice has to be of maximal rank so as to have a torus as a quotient. In general we can not assume that Λ_b does not depend on b. The process of uniformization of the periods modifies the action 6.4 in such a way that $\Lambda_b = \mathbb{Z}^n$ for all b. Let us consider the following Hamiltonian vector field $\sum_{i=1}^n k_i X_{f_i}$. The b^m-function that generates this Hamiltonian vector field is:

$$k_1 \left(a_0 \log(x) + \sum_{j=1}^{m-1} a_j \frac{1}{x^j} \right) + \sum_{i=2}^{n} k_i f_i$$

where recall that a_{m-1} is constant equal 1. Observe that the coefficient multiplying $1/x^{m-1}$ is k_1. By proposition 6.2.1 $k_1 = c$ the modular period. In this case $c = [\alpha_m]$.

Hence, for $b \in B^{n-1} \times \{0\}$ the lattice Λ_b is contained in $\mathbb{R}^{n-1} \times c\mathbb{Z} \subseteq \mathbb{R}^n$. Pick $(\lambda_1, \ldots, \lambda_n) : B^n \to \mathbb{R}^n$ such that:

- $(\lambda_1(b), \ldots, \lambda_n(b))$ is a basis of Λ_b for all $b \in B^n$,
- λ_i^n vanishes along $B^{n-1} \times \{0\}$ at order m for $i < n$ and λ_i is equal to c along $B^{n-1} \times \{0\}$.

In the previous points, λ_i^j denotes the j-th component of λ_i. The first condition can be satisfied by using the implicit function theorem. That is because $\Phi(\lambda, m) = m$ is regular with respect to the s coordinates. The second condition is automatically true because $\Lambda_b \subseteq \mathbb{R}^{n-1} \times c\mathbb{Z}$. We define the uniformed flow as:

$$\begin{array}{rcl} \tilde{\Phi} : \mathbb{R}^n \times (\mathbb{T}^n \times B^n) & \to & (\mathbb{T}^n \times B^n) \\ ((s_1, \ldots, s_n), (x, b)) & \mapsto & \Phi(\sum_{i=1}^n s_i \lambda_i, (x, b)) \end{array} \tag{6.5}$$

2. The $\mathbb{T}^n$ action is b^m-Hamiltonian. The objective of this step is to find b^m-functions $\sigma_1, \ldots, \sigma_n$ such that X_{σ_i} are the fundamental vector fields of the $\mathbb{T}^n$-action $Y_i = \sum_{j=1}^n \lambda_i^j X_{f_j}$.

By using the Cartan formula for a b^m-symplectic form:

$$\begin{aligned}\mathcal{L}_{Y_i}\mathcal{L}_{Y_i}\omega &= \mathcal{L}_{Y_i}(d(\iota_{Y_i}\omega)+\iota_{Y_i}d\omega)\\ &= \mathcal{L}_{Y_i}(d(-\textstyle\sum_{j=1}^n \lambda_i^j df_i))\\ &= -\mathcal{L}_{Y_i}(\textstyle\sum_{j=1}^n d\lambda_i^j \wedge df_j) = 0\end{aligned}$$

Note that λ_i^j are constant on the level sets of F because $\Phi(\lambda, m) = m$ is solved in the definition of λ and the level sets of F are invariant by Φ.

Recall that if Y is a complete periodic vector field and P is a bivector such that $\mathcal{L}_Y\mathcal{L}_Y P = 0$, then $\mathcal{L}_Y P = 0$. So,the vector fields Y_i are Poisson. To show that each $\iota_{Y_i}\omega$ has a ${}^{b^m}\mathcal{C}^\infty$ primitive we will see that $[\iota_{Y_i}\omega] = 0$ in the b^m-cohomology.

One one hand, if $i > 1$, $\iota_{Y_i}\omega$ vanishes at Z. This holds because Y_i has not any component $\partial/\partial Y$.

Recall Proposition 6 from [5]

Proposition 6.2.5. *If $\omega \in^b \Omega(M)$ with $\omega|_Z = 0$, then $\omega \in \Omega(M)$.*

The same holds for b^m-forms thus,

Proposition 6.2.6. *If $\omega \in^{b^m} \Omega(M)$ with $\omega|_Z$ vanishing up to order m, then $\omega \in \Omega(M)$.*

Thus as $\iota_{Y_i}\omega$ vanishes at Z, the b^m-forms $\iota_{Y_i}\omega$ are indeed smooth. Thus we can now apply the standard Poincaré lemma and as these forms are closed they are locally exact. This proves that all the vector fields Y_i with $i > 1$ are indeed Hamiltonian.

On the other hand, the fact that $\iota_{Y_1}\omega = cdf_1$ is obvious.

Then, because we have a toric action that is Hamiltonian, we can use lemma 3.2 in [26], and we get an equivalent system such that a_i are all constant and moreover $\langle a_i', X\rangle = \alpha_i(X^\omega)$. Note that by dividing by a_{m-1}', we can still assume $a_{m-1}' = 1$ to be consistent with our notation, but we then have to multiply $f_1 \cdot c$ in the next step.

3. Apply Darboux-Carathéodory.

   The construction above gives us some candidates $\sigma_1 = cf_1, \sigma_2, \ldots, \sigma_n$ for the action coordinates.

   We now apply the Darboux-Carathéodory theorem and express the form in terms of x:

$$\omega = \left(\sum_{j=1}^{m} c\frac{c_j}{x^j} dx \wedge dq_1 \right) + \sum_{i=2}^{n} d\sigma_i \wedge dq_i.$$

   Since the vector fields $X_{\sigma_i} = \frac{\partial}{\partial q_i}$ are fundamental fields of the $\mathbb{T}^n$-action the flow 6.5 gives a linear action on the q_i coordinates.

   Observe that the coordinate system is only defined in $\mathcal{U}$. It may not be valid at points outside $\mathcal{U}$ that may be in the orbit of points in $\mathcal{U}$. Let us see that we can extend the coordinates to these points.

   Define $\mathcal{U}'$ the union of all tori that intersect $\mathcal{U}$. We will see that the coordinates are valid at $\mathcal{U}'$.

   Let $\{p_i, \theta_j\}$ be the extension of $\{\sigma_i, q_j\}$. It is clear that $\{p_i, \theta_j\} = \delta_{ij}$ by its construction in the Darboux-Carathéodory theorem.

   To see that $\{\theta_i, \theta_j\} = 0$ we take the flows by X_{p_k} and extend the expression to the whole $\mathcal{U}'$:

$$X_{p_k}(\{\theta_i, \theta_j\}) = \{\{\theta_i, \theta_j\}, p_k\} = \{\theta_i, \delta_{ij}\} - \{\theta_j, \delta_{jk}\} = 0.$$

The fact that ω is preserved is obvious because X_{p_k} are hamiltonian vector fields and thus they preserve the b^m-symplectic forms. Moreover $t, \theta_1, p_2, \theta_2, \ldots, p_n, \theta_n$ are independent on $\mathcal{U}'$ and hence are a coordinate system in a neighbourhood of the torus.

$\square$

Remark 6.2.7. *In the proof we have seen that there exists an equivalent integrable system where the coefficients of the singular function are indeed constant. From now on when considering a b^m-integrable system we are going to make this assumption.*

6.3 Reformulating the action-angle coordinate via cotangent lifts

The action-angle theorem for symplectic manifolds (also known as action-angle coordinate theorem) can be reformulated in terms of a cotangent lift.

Recall that given a Lie group action on any manifold its cotangent lifted action is automatically Hamiltonian. By considering the action of a torus on itself by translations this action can be lifted to its cotangent bundle and give a semilocal normal form theorem as the Arnold-Liouville-Mineur theorem for symplectic manifolds. If we now replace this cotangent lift to the cotangent bundle to a lift to the b^m-cotangent bundle we obtain the semilocal normal form of the main theorem of this chapter.

Let us recall this from the article [32].

6.3.1 Cotangent lifts and Arnold-Liouville-Mineur in Symplectic Geometry

Let G be a Lie group and let M be any smooth manifold. Given a group action $\rho : G \times M \longrightarrow M$, we define its cotangent lift as the action on T^*M given by $\hat{\rho}_g := \rho^*_{g^{-1}}$ where $g \in G$. We then have a commuting diagram

$$\begin{tikzcd} T^*M \arrow[r, "\hat{\rho_g}"] \arrow[d, "\hat{\pi}"] & T^*M \arrow[d, "\pi"] \\ M \arrow[r, "\rho_g"] & M \end{tikzcd}$$

Figure 6.2: Commutiative diagram of the construction of the isomorphism of b^m-integrable systems.

where π is the canonical projection from T^*M to M.

The cotangent bundle T^*M is a symplectic manifold endowed with the exact symplectic form given by the differential of the Liouville one-form $\omega = -d\lambda$. The Lioville one-form can be defined intrinsically:

$$\langle \lambda_p, v \rangle := \langle p, (\pi_p)_*(v) \rangle \tag{6.6}$$

with $v \in T(T^*M), p \in T^*M$.

A standard argument (see for instance [33]) shows that the cotangent lift $\hat{\rho}$ is Hamiltonian with moment map $\mu : T^*M \to \mathfrak{g}^*$ given by

$$\langle \mu(p), X \rangle := \langle \lambda_p, X^{\#}|_p \rangle = \langle p, X^{\#}|_{\pi(p)} \rangle,$$

where $p \in T^*M$, X is an element of the Lie algebra $\mathfrak{g}$ and we use the same symbol $X^{\#}$ to denote the fundamental vector field of X generated by the action on T^*M or M. This construction is known as the **cotangent lift**.

In the special case where the manifold M is a torus $\mathbb{T}^n$ and the group is $\mathbb{T}^n$ acting by translations, we obtain the following explicit structure: Let $\theta_1, \dots, \theta_n$ be the standard (S^1-valued) coordinates on $\mathbb{T}^n$ and let

$$\underbrace{\theta_1, \dots, \theta_n}_{=:\theta}, \underbrace{t_1, \dots, t_n}_{=:t} \tag{6.7}$$

be the corresponding chart on $T^*\mathbb{T}^n$, i.e. we associate to the coordinates (6.7) the cotangent vector $\sum_i t_i d\theta_i \in T^*_\theta \mathbb{T}^n$. The Liouville one-form is given in these coordinates by

$$\lambda = \sum_{i=1}^n t_i d\theta_i$$

and its negative differential is the standard symplectic form on $T^*\mathbb{T}^n$:

$$\omega_{can} = \sum_{i=1}^n d\theta_i \wedge dt_i. \tag{6.8}$$

Denoting by τ_β the translation by $\beta \in \mathbb{T}^n$ on $\mathbb{T}^n$, its lift to $T^*\mathbb{T}^n$ is given by

$$\hat{\tau}_\beta : (\theta, t) \mapsto (\theta + \beta, t).$$

The moment map $\mu_{can} : T^*\mathbb{T}^n \to \mathfrak{t}^*$ of the lifted action with respect to the canonical symplectic form is

$$\mu_{can}(\theta, t) = \sum_i t_i d\theta_i, \tag{6.9}$$

where the θ_i on the right hand side are understood as elements of $\mathfrak{t}^*$ in the obvious way. Even simpler, if we identify $\mathfrak{t}^*$ with $\mathbb{R}^n$ by choosing the standard basis $\frac{\partial}{\partial \theta_i}$ of $\mathfrak{t}$ then the moment map is just the projection onto the second component of $T^*\mathbb{T}^n \cong \mathbb{T}^n \times \mathbb{R}^n$. Note that the components of μ naturally define an integrable system on $T^*\mathbb{T}^n$.

We can rephrase Arnold-Liouville-Mineur theorem in terms of the symplectic cotangent model:

Theorem 6.3.1. *Let $F = (f_1, \dots, f_n)$ be an integrable system on the symplectic manifold (M, ω). Then semilocally around a regular Liouville torus the system is equivalent to the cotangent model $(T^*\mathbb{T}^n)_{can}$ restricted to a neighbourhood of the zero section $(T^*\mathbb{T}^n)_0$ of $T^*\mathbb{T}^n$.*

6.3.2 The case of b^m-symplectic manifolds

Let us now go to the case of b^m-symplectic manifolds.

Let start introducing what is called the twisted b^m-cotangent model for torus actions. This model has additional invariants: the modular vector field of the connected component of the critical set and the modular weights of the associated toric action. Consider $T^*\mathbb{T}^n$ be endowed with the standard coordinates (θ, t), $\theta \in \mathbb{T}^n$, $t \in \mathbb{R}^n$ and consider again the action on $T^*\mathbb{T}^n$ induced by lifting translations of the torus $\mathbb{T}^n$. We will now view this action as a b^m-Hamiltonian action with respect to a suitable b^m-symplectic form. In analogy to the classical Liouville one-form we define the following non-smooth one-form away from the hypersurface $Z = \{t_1 = 0\}$:

$$\left(cc_1 \log |t_1| + \sum_{i=2}^{m} cc_i \frac{t_1^{-(i-1)}}{-(i-1)} \right) d\theta_1 + \sum_{i=2}^{n} t_i d\theta_i.$$

When differentiating this form we obtain a b^m-symplectic form on $T^*\mathbb{T}^n$ which we call (after a sign change) the **twisted b^m-symplectic form** on $T^*\mathbb{T}^n$ with invariants $(cc_1, \ldots, cc_m)$:

$$\omega_{tw,c} := \left(\sum_{j=1}^{m} c_j \frac{c}{t_1^j} dt_1 \wedge d\theta_1 \right) + \sum_{i=2}^{n} dt_i \wedge d\theta_i, \tag{6.10}$$

where c is the modular period. The moment map of the lifted action is then given by

$$\mu_{tw,q_0,\ldots,q_{m-1})} := (q_0 \log |t_1| + \sum_{i=2}^{m} q_i t_1^{-(i-1)}, t_2, \ldots, t_n), \tag{6.11}$$

where we are identifying $\mathbf{t}^*$ with $\mathbb{R}^n$ and $c_j = -(j-1)q_{j-1}$.

We call this lift together with the b^m-symplectic form 6.10 the **twisted b^m-cotangent lift** with modular period c and invariants $(c_1, \ldots, c_m)$. Note that the components of the moment map define a b^m-integrable system on $(T^*\mathbb{T}^n, \omega_{tw,(cc_1,\ldots,cc_m)})$.

The model of twisted b^m-cotangent lift allows us to express the action-angle coordinate theorem for b^m-integrable systems in the following way:

Theorem 6.3.2. *Let $F = (f_1, \dots, f_n)$ be a b^m-integrable system on the b^m-symplectic manifold (M, ω). Then semilocally around a regular Liouville torus $\mathbb{T}$, which lies inside the critical hypersurface Z of M, the system is equivalent to the cotangent model $(T^*\mathbb{T}^n)_{tw,(cc_1,\dots,cc_m)}$ restricted to a neighbourhood of $(T^*\mathbb{T}^n)_0$. Here c is the modular period of the connected component of Z containing $\mathbb{T}$ and the constants $(c_1, \dots, c_m)$ are the invariants associated to the integrable system and its associated toric action.*

Chapter 7

KAM theory on b^m-symplectic manifolds

The KAM theorem explains how integrable systems behave under small perturbations. More precisely, it studies how an integrable system in action-angle coordinates responds to a small perturbation on its hamiltonian. The trajectories of an integrable system in action-angle coordinates can be seen as linear trajectories over a torus. The KAM theorem fins a way to transform these original trajectories to other linear trajectories over some transformed torus. The KAM theorem states that most of these tori, and the linear solutions of the system on these tori, survive if the perturbation is small enough.

The objective of this chapter is to give a construction of KAM theory in the setting of b^m-symplectic manifolds and with b^m-integrable systems. The core of the chapter is the construction of the proper statement and the proof of the equivalent of the KAM theorem on b^m-symplectic manifolds.

This chapter is divided in 5 sections:

1. **On the structure of the proof.** On this section we are going to present the main ideas that are going to appear in the proper statement and proof of the main theorem. The idea of the theo-

rem is to build a sequence of b^m-symplectomorphisms such that its limit transforms the hamiltonian to only depend on the action coordinates.

2. **Technical results and definitions.** On this section we present some technical results and definitions that are key for the proof of the main theorem.

3. **KAM theorem on b^m-symplectic manifolds.** On this section we present the statement and the proof of the main result of this chapter. The proof is structured in 6 parts. In the first part we define the parameters that are going to be used to define the sequence of b^m-symplectomorphisms. In the second part we build precisely this sequence of b^m-symplectomorphisms. In the third part we see that the sequence of frequency maps of the transformed Hamiltonian functions at every step converges. In the fourth part we wee that the sequence of b^m-symplectomorphisms converges. In the fifth part we obtain results on the stability of the trajectories under the original perturbation. In the sixth part we find bounds to explain how close the invariant tori are from the unperturbed. In the last part we obtain a bound for the measure of the set of invariant tori.

4. **Desingularization of b^m-integrable systems.** We present a way to use the desingularization of b^m-symplectic manifolds presented in [8] to construct standard smooth integrable systems from b^m-integrable systems. This desingularized integrable system is uniquely defined.

5. **Desingularization of the KAM theorem on b^m-symplectic manifolds.** In this section we use the desingularization of b^m-integrable systems in conjunction with the KAM theorem for b^m-symplectic manifolds to deduce the original KAM theorem as

well as a completely new KAM theorem for folded symplectic forms.

7.1 On the structure of the proof

The first thing we do is to prove that we can reduce our study to the case the perturbation is not a b^m-function but an analytic one. This is because any purely singular perturbation only has effect on the component in the direction of the modular vector field and can be easily controlled.

The idea of the proof is really similar to the classical KAM case. We want to build a diffeomorphism such that its transformed hamiltonian only depends on the action coordinates. But it is not possible to build this diffeomorphism in one step. What we do, as it is done in the classical case, it is to build a sequence of diffeomorphisms such that the part of the hamiltonian depending on the angular variables decreases at every step. The idea is to remove the first K terms of its Fourier expression at every step while making K rapidly increase. This is done by assuming the diffeomorphism comes as the flow at time 1 generated by a Hamiltonian function. In this way one can use the Lie Series in conjunction with the Fourier series to find the expression for the hamiltonian function that generates our diffeomorphism. The final diffeomorphism will be the composition of all the diffeomorphisms obtained at each step. One of the main difficulties of the proof, as in the classical case, is to prove that these diffeomorphisms converge and to prove some bounds of its norm.

We also note that for our b^m-symplectic setting, the diffeomorphisms we consider leave the defining function of the critical set invariant up to order m, this will have an important role later. Also observe in particular that the critical set can not be transformed by any perturbation given by a b^m-function.

Next we give some technical definitions and results. We define the norms we are going to use to do all the estimates. We set the notation for the proof and the statement of the theorem. We define the notion of non-resonance for a neighborhood of the critical set of the b^m-symplectic manifold. We study the set of all possible non-resonant vectors. And we state the inductive lemma, which gives us estimates and constructions for every step of our sequence of diffeomorphisms.

After all this discussion we are in conditions to properly state the b^m-version of the KAM theorem. One important difference to the classical KAM theorem is that we have to guarantee that at Z the set of non-resonant vectors does not become the whole set of frequencies. This condition can be understood as the perturbation being smaller than some constant multiplied by the inverse of the modular period.

The proof of the theorem is done in six different steps by following the structure on [19]. Since we are going to use the inductive lemma at every step, first we define the parameters and sets to which we are going apply such lemma. Then we check that we can actually apply the lemma and obtain some extra estimates for the results of the lemma. After this we see that the sequence of frequency vectors converges. We do the same with the sequence of canonical transformations. Then we get some bounds for the size of the components of the final diffeomorphism. Next we characterize the tori that survive by the perturbation. Finally we give some estimates for the measure of the set of these tori.

Note that our version of the b^m-KAM theorem improves the one in [2] in several ways. Firstly it is applicable to b^m-symplectic structures not only for b-symplectic. Also we give several estimates that are not obtained in [2], this estimates have sense in a neighborhood of the critical set Z, while [2] only studied the behavior at Z. Finally the type of perturbation we consider is far more general, since we do not have any condition of the form of the perturbation but only on its size.

7.1.1 Reducing the problem to an analytical perturbation.

In the standard KAM, we assume to have an analytic Hamiltonian $h(I)$ depending only on the action coordinates and we add to it a small analytical perturbation $R(\phi, I)$. This perturbed system receives the name of *nearly integrable system*. And then find a new coordinate system such that $h(I)+R(\phi,I) = \tilde{h}(\tilde{I})$ where most of the quasi-periodic orbits are preserved and can be mapped to the unperturbed quasi-periodic orbits by means of the coordinate change.

In our setting we may assume $h(I)$ to not be analytical and be a b^m-function. Also the perturbation $R(\phi, I)$ may as well be considered a b^m-function. In the following lines we justify without loss of generality that actually we can assume the perturbation to be analytical.

Let us state this more precisely. Let (M, x, Z, ω, F) be a b^m-manifold with a b^m integrable system F on it. Consider action angle coordinates on a neighborhood of Z. Then we can assume the expressions:

$$\omega = \left(\sum_{j=1}^{m} \frac{c_j}{I_1^j}\right) dI_1 \wedge d\phi_1 + \sum_{i=2}^{n} dI_i \wedge d\phi_i, \text{ and}$$

$$F = (q_0' \log I_1 + \sum_{i=1}^{m-1} q_i' \frac{1}{I_1^i} + h(I), f_2, \ldots, f_n)$$

where $h, f_2, \ldots, f_n$ are analytical.

Let the Hamiltonian function of our system be the first component of the moment map $\hat{h}' = q_0' \log I_1 + \sum_{i=1}^{m-1} q_i' \frac{1}{I_1^i} + h = \zeta' + h$, where $\zeta' := q_0' \log I_1 + \sum_{i=1}^{m-1} q_i' \frac{1}{I_1^i}$. Note that $d\zeta' = \sum_{i=1}^{m} \hat{q}_i' \frac{1}{I_1^i}$, where $\hat{q}_i' = -(i-1)q_{i-1}'$. Note that by the result of the previous chapter $c_j/\hat{q}_j' = \mathcal{K}$ the modular period. In particular $c_m/\hat{q}_m' = \mathcal{K}$.

The hamiltonian system given by $\hat{h}'$ can be easily solved by $\phi = \phi_0 + u't, I = I_0$ where u' is going to be defined in the following sections. Consider now a perturbation of this system: $\hat{H}' = \hat{h}'(I) = \hat{R}(I, \phi)$, where $\hat{R}$ is a b^m-function $\hat{R}(I, \phi) = R_\zeta(I_1) + R(I, \phi)$ where

$R_\varsigma(I_1) = (r_0 \log I_1 + \sum_{i=1}^{m-1} r_i \frac{1}{I_1^i})$ is the singular part. Then we can consider the perturbations $R_\varsigma(I_1)$ and $R(I,\phi)$ separately. This way, we may consider $R_\varsigma(I)$ as part of $\hat{h}'(I)$. Then we have a new hamiltonian

$$\hat{h}(I) = (q_0' + r_0)\log I_1 + \sum_{i=1}^{m-1}(q_i' + r_i)\frac{1}{I_1^i} + h = q_0 \log I_1 + \sum_{i=1}^{m-1} q_i \frac{1}{I_1^i} + h.$$

Now, instead of the identity $\mathcal{K}\hat{q}_j' = c_j$ we will have $\mathcal{K}(\hat{q}_j - \hat{r}_j) = c_j$, which implies $\mathcal{K}\left(1 - \frac{\hat{r}_j}{\hat{q}_j' + \hat{r}_j}\right) = \frac{c_j}{\hat{q}_j}$. In particular

$$\mathcal{K}\left(1 - \frac{\hat{r}_m}{\hat{q}_m' + \hat{r}_m}\right) = \frac{c_m}{\hat{q}_m}$$

Let us define $\mathcal{K}' = \mathcal{K}\left(1 - \frac{\hat{r}_m}{\hat{q}_m' + \hat{r}_m}\right)$. So from now on we assume $\hat{h} = q_0 \log I_1 + \sum_{i=1}^{m-1} q_i \frac{1}{I_1^i} + h$, that the perturbation $R(\phi, I)$ is analytical, and we have the condition $\frac{c_m}{\hat{q}_m} = \mathcal{K}'$. Observe that this system with only the singular perturbation is still easy to solve in the same way that the system previous to this perturbation was.

7.1.2 Looking for a b^m-symplectomorphism

Assume we have a Hamiltonian function $H = \hat{h}(I) + R(\phi, I)$ in action-angle coordinates. Where $\hat{h}(I)$ is the singular component of the b^m-integrable system, i.e.

$$\hat{h}(I) = h(I) + q_0 \log(I_1) + \sum_{i=1}^{m-1} q_i \frac{1}{I_1^i}, \tag{7.1}$$

where $h(I)$ is analytical[1]. Assume also that the b^m-symplectic form ω[2] in these coordinates is expressed as:

$$\omega = \left(\sum_{j=1}^{m} \frac{c_j}{I_1^j}\right) dI_1 \wedge d\phi_1 + \sum_{i=2}^{n} dI_i \wedge d\phi_i. \tag{7.2}$$

[1] If another component of the moment map is chosen to be the hamiltonian of the system, the result still holds: the computations can be replicated assuming $\hat{h}(I) = h(I)$.

[2] In classical KAM, ω is used to denote the frequency vector $\frac{\partial h}{\partial I}$. We need ω to denote the b^m-symplectic form so we are going to use u to denote the frequency vector.

And finally, the expression for the frequency vector is:

$$\hat{u} = \frac{\partial \hat{h}}{\partial I} = \frac{\partial(h(I) + q_0 \log(I_1) + \sum_{i=1}^{m-1} q_i \frac{1}{I_1^i})}{\partial I}$$

$$= \left(u_1 + \sum_{i=1}^{m} \frac{\hat{q}_i}{I_1^i}, u_2, \ldots, u_n\right),$$

where $\hat{q}_1 = q_0$ and $\hat{q}_{i-1} = -iq_i$ if $i \neq 0$.

The objective is to follow the steps of the usual KAM construction (the steps followed are highly inspired in [19]) replacing the standard symplectic form for ω and taking as hamiltonian the b^m-function $\hat{h}$.

Remark 7.1.1. *The objective of the construction is to find a diffeomorphism (actually a b^m-symplectomorphism) ψ such that $H \circ \psi = h(\tilde{I})$. This is done inductively, by taking $H \circ \psi = H \circ \phi_1 \circ \ldots \circ \phi_q \circ \ldots$, while trying to make $R(\phi, I)$ smaller at every step.*

Let us focus in one single step

Recall the classical formula:

Lemma 7.1.2. *See [19].*

$$f \circ \phi_t = \sum_{j=0}^{\infty} \frac{t^j}{j!} L_W^j f, \quad L_W^j f = \{L_W^{j-1} f, W\}$$

Where W is the Hamiltonian that generates the flow ϕ_t, and $\{\cdot, \cdot\}$ is the corresponding Poisson bracket.

We will denote $r_k(H, W, t) = \sum_{j=k}^{\infty} \frac{t^j}{j!} L_W^j H$.

$$
\begin{aligned}
H\circ\phi = H\circ\phi|_{t=1} &= \left.\sum_{j=0}^{\infty}\frac{t^j}{j!}\,L_W^j\underbrace{H}_{\hat h+R}\right|_{t=1}\\
&= \hat h + R\{\hat h + R, W\} + r_2(H,W,1)\\
&= \hat h + R + \{\hat h, W\} + \{R,W\} + r_2(\hat h, W, 1)\\
&\quad + r_2(R,W,1)\\
&= \hat h + \underbrace{R + \{\hat h, W\}}_{\substack{\text{We want to cancel}\\ \text{this term as}\\ \text{fast as we can}}} + r_2(\hat h, W, 1) + r_2(R,W,1)
\end{aligned}
\tag{7.3}
$$

We want $\{\hat h, W\} + R_{\leq k} = 0$, equivalently $\{W, \hat h\} = R_{\leq k}$, where $R_{\leq k}$ means the Fourier expression of R up to order K:

$$R_{\leq k} = \sum_{\substack{k\in\mathbb{R}^n\\ |k|_1\leq K}} R_k(I)e^{ik\cdot\phi}$$

Let us impose the condition $\{W, \hat h\} = R_{\leq K}$. Let us write the expression of the Poisson bracket associated to the b^m-symplectic form.

$$
\begin{aligned}
\{W,\hat h\} &= \left(\frac{1}{\sum_{j=1}^m \frac{c_j}{I_1^j}}\right)\left(\frac{\partial W}{\partial\phi_1}\frac{\partial\hat h}{\partial I_1} - \frac{\partial W}{\partial I_1}\frac{\partial\hat h}{\partial\phi_1}\right)\\
&\quad + \sum_{i=2}^{n}\left(\frac{\partial W}{\partial\phi_i}\frac{\partial\hat h}{\partial I_i} - \frac{\partial W}{\partial I_i}\frac{\partial\hat h}{\partial\phi_i}\right)
\end{aligned}
$$

Because $\hat h$ depends only on I, $\frac{\partial\hat h}{\partial\phi_i} = 0$ for all i. Moreover, the singular part of the b^m-function only depends on I_1 and hence its derivatives with respect to the other variables are also 0. Using that $\frac{\partial\hat h}{\partial I} = u + \sum_{i=1}^m \frac{\hat q_i}{I_1^i}$ the previous expression can be simplified:

$$\{W,\hat{h}\} = \left(\frac{u_1+\sum_{i=1}^{m}\frac{\hat{q}_i}{I_1^i}}{\sum_{j=1}^{m}\frac{c_j}{I_1^j}}\right)\frac{\partial W}{\partial \phi_1}+\sum_{i=2}^{n}\frac{\partial W}{\partial \phi_i}u_i$$

To expand the expression further we develop W in its Fourier expression: $W=\sum_{\substack{k\in\mathbb{R}^n\\|k|_1\leq K}} W_k(I)e^{ik\phi}$. The Fourier expansion is added up to order K, because it is only necessary for the expressions to agree up to order K. With this notations the condition becomes:

$$\begin{aligned}
\{W,\hat{h}\}_{\leq K} &= \left(\frac{u_1+\sum_{i=1}^{m}\frac{\hat{q}_i}{I_1^i}}{\sum_{j=1}^{m}\frac{c_j}{I_1^j}}\right)\frac{\partial}{\partial \phi_1}\left(\sum_{\substack{k\in\mathbb{R}^n\\|k|_1\leq K}} W_k(I)e^{ik\phi}\right)\\
&\qquad +\sum_{j=2}^{n}u_j\frac{\partial}{\partial \phi_j}\left(\sum_{\substack{k\in\mathbb{R}^n\\|k|_1\leq K}} W_k(I)e^{ik\phi}\right)\\
&= \left(\frac{u_1+\sum_{i=1}^{m}\frac{\hat{q}_i}{I_1^i}}{\sum_{j=1}^{m}\frac{c_j}{I_1^j}}\right)\left(\sum_{\substack{k\in\mathbb{R}^n\\|k|_1\leq K}} W_k(I)e^{ik\phi}ik_1\right)\\
&\qquad +\sum_{j=2}^{n}u_j\left(\sum_{\substack{k\in\mathbb{R}^n\\|k|_1\leq K}} W_k(I)e^{ik\phi}ik_j\right)\\
&= \sum_{\substack{k\in\mathbb{R}^n\\|k|_1\leq K}} W_k(I)e^{ik\phi}\cdot\left(ik_1\left(\frac{u_1+\sum_{i=1}^{m}\frac{\hat{q}_i}{I_1^i}}{\sum_{j=1}^{m}\frac{c_j}{I_1^j}}\right)+\sum_{j=2}^{n}ik_ju_j\right)\\
&= R_{\leq K}
\end{aligned}$$

Then, it is possible to make the two sides of the equation equal by imposing the condition term by term:

$$\begin{aligned} W_k(I) &= R_k(I)\frac{1}{i\left(k_1\left(\frac{u_1+\sum_{i=1}^m \frac{\hat{q}_i}{I_1^i}}{\sum_{j=1}^m \frac{c_j}{I_1^j}}\right)+\sum_{j=2}^n k_j u_j\right)} \\ &= R_k(I)\frac{1}{i\left(k_1\left(\frac{u_1+\sum_{i=1}^m \frac{\hat{q}_i}{I_1^i}}{\sum_{j=1}^m \frac{c_j}{I_1^j}}\right)+\bar{k}\bar{u}\right)}, \end{aligned} \tag{7.4}$$

where we adopted the notation $\sum_{j=2}^n k_j u_j = \bar{k}\bar{u}$.

Remark 7.1.3. *Observe that the expression 7.4 has no sense when $k = \vec{0}$ and hence $\{W, h\}_0 = R_0$*[3] *can not be solved. Let $W_0(I) = 0$, then $\{h, W\}_{\leq K} = R_{\leq K} - R_0$.*

Plugging the results above into the equation 7.3, one obtains:

$$H \circ \phi = \hat{h} + R_0 + R_{\geq K} + r_2(\hat{h}, W, 1) + r_1(R, W, 1)$$

With this construction the diffeomorphism ϕ is found. But this only makes for one of the steps that must be done. If q denotes the number of the iteration of this procedure, in general, we obtain:

$$\begin{aligned} H^{(q)} = H^{(q-1)} \circ \phi^{(q)} &= \hat{h}^{(q-1)} + R_0^{(q-1)} + R_{\geq K}^{(q-1)} \\ &\quad + r_2(h^{(q-1)}, W^{(q)}, 1) + r_1(R^{(q-1)}, W^{(q)}, 1), \end{aligned} \tag{7.5}$$

and at every step:

$$\begin{cases} \hat{h}^{(q)} = \hat{h}^{(q-1)} + R_0^{(q-1)} \\ R^{(q)} = R_{>K}^{(q-1)} + r_2(\hat{h}^{(q-1)}, W^{(q)}, 1) + r_1(R^{(q-1)}, W^{(q)}, 1) \end{cases} \tag{7.6}$$

[3] The zero term of the Fourier series can be seen as the angular average of the function

7.1.3 On the change of the defining function under b^m-symplectomorphisms

Note that since we are in a b^m-manifold it only has sense to consider I_1 up to order m, see [1]. When talking about defining functions we are interested in $[I_1]$ its jet up to order m. By definition b^m-maps preserve I_1 up to order m and b^m-vector fields X are such that $\mathcal{L}_X(I_1) = g \cdot I_1^m$ for $g \in \mathcal{C}^\infty(M)$.

Lemma 7.1.4. *Let ϕ_t be the integral flow of X a b^m-vector field, then ϕ_t is a b^m-map.*

Proof. We want

$$I_1 \circ \phi_t = I_1 + I_1^m \cdot g$$

fo some $g \in \mathcal{C}^\infty(M)$. We will use 7.1.2.

$$I_1 \circ \phi_t = \sum_{j=0}^{\infty} \frac{t^j}{j!} L_X^j I_1 = I_1 + \mathcal{L}_X(I_1) + \sum_{j=2}^{\infty} \frac{t^j}{j!} L_X^j I_1$$

$$= I_1 + I_1^m + \sum_{j=2}^{\infty} \frac{t^j}{j!} L_X^j I_1.$$

On the other hand, let us prove by induction $L_X^k I_1 = g^{(k)} I_1^m$. The base case is obvious, assume the case k holds and let us prove the case $k+1$.

$$\begin{aligned} L_X^{k+1} I_1 &= \{L_X^k I_1, X\} \\ &= \{g^{(k)} I_1^m, X\} \\ &= (L_X g^{(q)}) I_1^m + g^{(k)} \cdot m I_1^{m-1} L_X I_1 \\ &= (L_X g^{(k)} + g^{(k)} \cdot m \cdot I_1^{m-1} \cdot g) I_1^m \\ &= g^{(k+1)} I_1^m \end{aligned}$$

where $g^{(k+1)} = L_X g^{(k)} + g^{(k)} \cdot m \cdot I_1^{m-1} \cdot g$.

□

Lemma 7.1.5. *The Hamiltonian vector flow of some smooth hamiltonian function h is a b^m-vector field.*

Proof. At each point of Z the following identity holds $\mathcal{L}_{X_h} I_1 = I_1^m \frac{\partial f}{\partial \phi_1}$. The result can be extended at a neighborhood of Z. $\square$

Observe that combining the two previous results we get that the hamilonian flow of a function preserves I_1 up to order m.

7.2 Technical results

As the non-singular part of our functions we will be considering analytic functions on $\mathbb{T} \times G$, $G \subset \mathbb{R}^n$. The easiest way to work with these functions is to consider them as holomorphic functions on some complex neighborhood. Let us define formally this neighborhood.

$$\mathcal{W}_{\rho_1}(\mathbb{T}^n) := \{\phi : \Re\phi \in \mathbb{T}^n, |\Im\phi|_\infty \leq \rho_1\},$$

$$\mathcal{V}_{\rho_2}(G) := \{I \in \mathbb{C}^n : |I - I'| \leq \rho_2 \text{ for some } I' \in G\},$$

$$\mathcal{D}_\rho(G) := \mathcal{W}_{\rho_1}(\mathbb{T}^n) \times \mathcal{V}_{\rho_2}(G),$$

where $|\cdot|_\infty$ denotes the maximum norm and $|\cdot|_2$ denotes de Euclidean norm. Now it is necessary to clarify the norms that are going to be used on these sets.

Definition 7.2.1. *Let f be an action function (only depending on the I coordinates), and F an action vector field.*

$$|f|_{G,\eta} := \sup_{I\in\mathcal{V}_\eta(G)} |f(I)|, \quad |f|_G := |f|_{G,0}$$
$$|F|_{G,\eta,p} := \sup_{I\in\mathcal{V}_\eta(G)} |F(I)|_p, \quad |F|_{G,\eta} := |F|_{G,\eta,2}$$

Now, assume $f(I,\phi)$ to be an action-angle function written in its Fourier expansion as $\sum_{k\in\mathbb{Z}^n} f_k(I)e^{ik\cdot\phi}$, and F to be an action-angle vector field.

$$|f|_{G,\rho} := \sup_{(\phi,I)\in\mathcal{D}_\rho(G)} |f(I)|, \quad \|f\|_{G,\rho} := \sum_{k\in\mathbb{Z}^n} |f_k|_{G,\rho_2} e^{|k|_1\rho_1}$$
$$|F|_{G,\rho,p} := \sum_{k\in\mathbb{Z}^n} |F_k|_{G,\rho_2,p} e^{|k|_1\rho_1}, \qquad \|F\|_{G,\rho} = \|F\|_{G,\rho,2}$$

Lemma 7.2.2 (Cauchy Inequality).

$$\left\|\frac{\partial f}{\partial \phi}\right\|_{G,(\rho_1-\delta_1,\rho_2),1} \leq \frac{1}{e\delta_1}\|f\|_{G,\rho}$$

$$\left\|\frac{\partial f}{\partial I}\right\|_{G,(\rho_1,\rho_2-\delta_2),\infty} \leq \frac{1}{\delta_2}\|f\|_{G,\rho}$$

Definition 7.2.3. *If* $Df = (\frac{\partial f}{\partial \phi}, \frac{\partial f}{\partial I})$,

$$\|Df\|_{G,\rho,c} := \max\left(\|\frac{\partial f}{\partial \phi}\|_{G,\rho,1}, c\|\frac{\partial f}{\partial I}\|_{G,\rho,\infty}\right)$$

Definition 7.2.4. *To simplify notation let us define:*

$$\mathcal{A}(I_1) = \frac{\sum_{j=1}^{m}\frac{\hat{q}_j}{I_1^j}}{\sum_{j=1}^{m}\frac{c_j}{I_1^j}} \quad \textit{and} \quad \mathcal{B}(I_1) = \frac{1}{\sum_{j=1}^{m}\frac{c_j}{I_1^j}}.$$

Remark 7.2.5. *With this notation, equation 7.4 can be written as:*

$$W_k(I) = \frac{R_k(I)}{i(k_1\mathcal{B}(I_1)u_1 + \bar{k}\bar{u} + k_1\mathcal{A}(I_1))}$$

Observe that $\mathcal{A}(I_1)$ and $\mathcal{B}(I_1)$ are analytic (holomorphic in the complex extended domain) where the denominator does not vanish. We can assume that this does not happen by shrinking the domain G in the direction of I_1. Observe in particular that when $I_1 \to 0$, $\mathcal{A}(I_1) \to \hat{q}_m/c_m = 1/\mathcal{K}'$ the inverse of the modular period and $\mathcal{B}(I_1) \to 0$. In this way, we can define the norms of $\mathcal{A}(I_1)$ and $\mathcal{B}(I_1)$ are bounded and well defined. We will denote this norms $K_{\mathcal{A}}$ and $K_{\mathcal{B}}$ respectively. Also, since $\mathcal{A}(I_1)$ and $\mathcal{B}(I_1)$ are analytic, their derivatives will also be bounded, and we will denote the norms of this derivatives as $K_{\mathcal{A}'}$ and $K_{\mathcal{B}'}$.

To further simplify notation in the next computations we introduce the following definition:

Definition 7.2.6.

$$\bar{\mathcal{A}} = \begin{pmatrix} \mathcal{A} \\ 0 \end{pmatrix} \quad \textit{and} \quad \bar{\mathcal{B}} = \begin{pmatrix} \mathcal{B} & 0 \\ 0 & Id_{n-1,n-1} \end{pmatrix}$$

Remark 7.2.7. *With this notation, equation 7.4 can be written as:*

$$W_k(I) = \frac{R_k(I)}{i(k\bar{\mathcal{B}}(I_1)u + k\bar{\mathcal{A}}(I_1))} \tag{7.7}$$

Definition 7.2.8. *Having fixed ω a b^m-symplectic form (as in equation 7.2) and $\hat{h}$ a b^m-function (as in equation 7.1) as a hamiltonian. Given an integer K and $\alpha > 0$, $F \subset \mathbb{R}^n$ (or $\mathbb{C}^n$) the space of frequencies is said to be α, K-nonresonant with respect to $(c_1, \ldots, c_m)$ and $(\hat{q}_1, \ldots, \hat{q}_m)$ if*

$$|k\bar{\mathcal{B}}(I_1)u + k\bar{\mathcal{A}}(I_1)| \geq \alpha, \forall k \in \mathbb{Z} \setminus \{0\}, |k|_1 \leq K, \forall u \in F.$$

We are going to use the notation $\alpha, K, c, \hat{q}$-nonresonant.

Remark 7.2.9. *The non-resonance condition is established on $u = \partial h/\partial I$, not on $\hat{u} = \partial \hat{h}/\partial I$, because our non-resonance condition already takes into account the singularities. In this way we can use the analyticity of u.*

Remark 7.2.10. *If $\left|\frac{\partial u}{\partial I}\right|_{G,\rho_2}$ is bounded by M', then $\left|\frac{\partial}{\partial I}\left(\bar{\mathcal{B}}u + \bar{\mathcal{A}}\right)\right|_{G,\rho_2}$ is also bounded:*

$$\begin{aligned} \left|\frac{\partial}{\partial I}\left(\bar{\mathcal{B}}u + \bar{\mathcal{A}}\right)\right|_{G,\rho_2} &\leq \left|\frac{\partial \bar{\mathcal{B}}}{\partial I}u + \bar{\mathcal{B}}\frac{\partial u}{\partial I} + \frac{\partial \bar{\mathcal{A}}}{\partial I}\right|_{G,\rho_2} \\ &\leq K_{\mathcal{B}'}|u|_{G,\rho_2} + K_{\mathcal{B}}M' + K_{\mathcal{A}} =: M. \end{aligned} \tag{7.8}$$

Remark 7.2.11. *When we consider the standard KAM theorem, the* *frequency vector u is relevant because the solution to the Hamilton equations of the unperturbed problem has the form:*

$$I = I_0, \quad \phi = \phi_0 + ut.$$

Let us see what plays the role of u in our b^m-KAM theorem. Let us find the coordinate expression of the solution to $\iota_{X_{\hat{h}}}\omega = d\hat{h}$, where ω is a b^m-symplectic form in action-angle coordinates.

$$X_{\hat{h}} = \dot{I}_1 \frac{\partial}{\partial I_1} + \ldots + \dot{I}_n \frac{\partial}{\partial I_n},$$

where $\dot{I}_1, \ldots, \dot{I}_n$ are the functions we want to find.

$$d\hat{h} = \left(\sum_{j=1}^{m} \hat{q}_i \frac{1}{I_1^j} \right) dI_1 + dh,$$

and hence,

$$X_{\hat{h}} = \Pi(d\hat{h}, \cdot) = \frac{\sum_{i=1}^m \frac{\hat{q}_i}{I_1^i}}{\sum_{i=1}^m \frac{c_j}{I_1^j}} \frac{\partial}{\partial \phi_i} + X_h.$$

Hence $\phi = \phi_0 + (\underbrace{\bar{\mathcal{B}}u + \bar{\mathcal{A}}}_{u'})t$. So the frequency vector that we are going to be concerned about is going to be u' and not $\hat{u} = \frac{\partial}{\partial I}\hat{h}$.

Lemma 7.2.12. *If u is one-to-one from $\mathcal{G}$ to its image then $u' = \bar{\mathcal{B}}u + \bar{\mathcal{A}}$ is also one-to-one from $\mathcal{G}'$ to its image in a neighborhood of Z, while at Z it is the projection of u such that the first coordinate is sent to $\frac{\hat{q}_m}{c_m} = 1/\mathcal{K}'$ the inverse of the modular period, were $\mathcal{G}' \subseteq \mathcal{G}$.*

Proof. Because

$$u' = \left(\frac{1}{\sum_{j=1}^m \frac{c_j}{I_1^j}} u_1 + \frac{\sum_{j=1}^m \frac{\hat{q}_j}{I_1^j}}{\sum_{j=1}^m \frac{c_j}{I_1}}, u_2, \ldots, u_n \right),$$

and $\mathcal{B}$ is invertible outside $I_1 = 0$, shrinking $\mathcal{G}$ if necessary in the first dimension the map is one-to-one. But at the critical set $\{I_1 = 0\}$, u'

is a projection of u where the first component is sent to the constant value $\frac{\hat{q}_m}{c_m} = \frac{1}{\mathcal{K}'}$. $\square$

Lemma 7.2.13. *If $u(G)$ is $\alpha, K, c, \hat{q}$-nonresonant, then $u(\mathcal{V}_{\rho_2}(G))$ is $\frac{\alpha}{2}, K, c, \hat{q}$-nonresonant, assuming that $\rho_2 \leq \frac{\alpha}{2MK}$ and $\left|\frac{\partial u}{\partial I}\right|_{G,\rho_2} \leq M'$*

Proof. Fix $k \in \mathbb{Z} \setminus \{0\}$, we want to bound $|k\bar{\mathcal{B}}(I_1)v + k\bar{\mathcal{A}}(I_1)|$ where $v \in u(\mathcal{V}_{\rho_2}(G))$ as a function on v. Given $v \in u(\mathcal{V}_{\rho_2}(G))$ one must ask if there is any bound for the distance to some $v' \in u(G)$.

$v \in u(\mathcal{V}_{\rho_2}(G)) \Rightarrow v = u(x), x \in \mathcal{V}_{\rho_2}(G) \Rightarrow \exists y \in G$ such that $|x-y| \leq \rho_2$.

Take $v' = u(y)$.

$$|v-v'| \leq |x-y| \left|\frac{\partial u}{\partial I}\right|_{G,\rho_2} \leq \rho_2 M' \leq \rho_2 M/K_{\mathcal{B}} \leq \frac{\alpha}{2MK} M/K_{\mathcal{B}} = \frac{\alpha}{2KK_{\mathcal{B}}}.$$

Where we used equation 7.8 in the third inequality.

$$\begin{aligned}
|k_1\mathcal{B}(I_1)v_1 + \bar{k}\bar{v} + k_1\mathcal{A}(I_1)| &\geq \underbrace{|k_1\mathcal{B}(I_1)v_1' + \bar{k}\bar{v}' + k_1\mathcal{A}(I_1)|}_{\geq\alpha} \\
&\quad -|k_1\mathcal{B}(I_1)(v_1 - v_1') + \bar{k}(\bar{v} - \bar{v}')| \\
&\geq \alpha - K_{\mathcal{B}} \underbrace{|k \cdot (v - v')|}_{\leq K\alpha/(2KK_{\mathcal{B}})} \\
&\geq \alpha - \alpha/2 = \alpha/2
\end{aligned}$$

$\square$

Proposition 7.2.14. *Let $\hat{h}(I)$ be a b^m-function as in equation 7.1. Assume $h(I)$ and $R(\phi, I)$ be real analytic on $\mathcal{D}_\rho(G)$, $u(G) = \frac{\partial h}{\partial I}(G)$ is $\alpha, K, c, \hat{q}$-nonresonant. Assume also that $|\frac{\partial}{\partial I}u|_{G,\rho_2} \leq M'$ and $\rho_2 \leq \frac{\alpha}{2MK}$. Let $c > 0$ given. Then $R_0(\phi, I)$, $W_{\leq K}(\phi, I)$ given by the previous construction are both real analytic on $\mathcal{D}_\rho(G)$ and the following bounds hold*

1. $||DR_0||_{G,\rho,c} \le ||DR||_{G,\rho,c}$

2. $||D(R - R_0)||_{G,\rho,c} \le ||DR_0||_{G,\rho,c}$

3. $||DW||_{G,\rho,c} \le \frac{2A}{\alpha}||DR_0||_{G,\rho,c}$

Where $A = 1 + \frac{2Mc}{\alpha}$

Proof. Inequalities 1 and 2 are obvious because of the Fourier expression. Let us prove inequality 3. Let us expand $R(\phi, I)$ and $W(\phi, I)$ in their Fourier expression:

$$R = \sum_{k\in\mathbb{R}^n} R_k(I)e^{ik\cdot\phi}, \quad W = \sum_{k\in\mathbb{R}^n} W_k(I)e^{ik\cdot\phi}.$$

We will bound this expression finding term-by-term bounds.

$$\frac{\partial R}{\partial \phi} = \sum_{k\in\mathbb{R}^n} R_k(I)e^{ik\cdot\phi}ik.$$

Hence, if we denote $[\frac{\partial R}{\partial \phi}]_k$ the k-th term of the Fourier expansion of $\frac{\partial R}{\partial \phi}$, we have:

$$\left[\frac{\partial R}{\partial \phi}\right]_k = R_k ik.$$

Let us compute the derivative of W_k with respect to the I variables:

$$\begin{aligned}
\frac{\partial W_k}{\partial I} &= \frac{\partial}{\partial I}\left(\frac{R_k}{i(k\bar{\mathcal{B}}(I_1)u + k\bar{\mathcal{A}}(I_1))}\right) \\
&= \frac{\partial R_k/\partial I}{i(k\bar{\mathcal{B}}(I_1)u + k\bar{\mathcal{A}}(I_1)))} - \frac{R_k i\frac{\partial}{\partial I}(k\bar{\mathcal{B}}(I_1)u + k\bar{\mathcal{A}}(I_1)))}{[i(k\bar{\mathcal{B}}(I_1)u + k\bar{\mathcal{A}}(I_1)))]^2} \\
&= \frac{\partial R_k/\partial I}{i(k\bar{\mathcal{B}}(I_1)u + k\bar{\mathcal{A}}(I_1)))} + \frac{R_k ik\frac{\partial}{\partial I}(\bar{\mathcal{B}}(I_1)u + \bar{\mathcal{A}}(I_1)))}{[(k\bar{\mathcal{B}}(I_1)u + k\bar{\mathcal{A}}(I_1)))]^2} \\
&= \frac{\partial R_k/\partial I}{i(k\bar{\mathcal{B}}(I_1)u + k\bar{\mathcal{A}}(I_1)))} + \frac{[\frac{\partial R_k}{\partial \phi}]_k\frac{\partial}{\partial I}(\bar{\mathcal{B}}(I_1)u + \bar{\mathcal{A}}(I_1)))}{[(k\bar{\mathcal{B}}(I_1)u + k\bar{\mathcal{A}}(I_1)))]^2}.
\end{aligned}$$

Then, we take norms ($|\cdot|_{G,\rho_2,\infty}$) at each side of the equation.

$$\begin{aligned}\left|\frac{\partial W_k}{\partial I}\right|_{G,\rho_2,\infty} &\leq \frac{2}{\alpha}\left|\frac{\partial R_k}{\partial I}\right|_{G,\rho_2,\infty}+\frac{4M}{\alpha^2}\left|\left[\frac{\partial R_k}{\partial \phi}\right]_k\right|_{G,\rho_2,\infty}\\ &\leq \frac{2}{\alpha}\left|\frac{\partial R_k}{\partial I}\right|_{G,\rho_2,\infty}+\frac{4M}{\alpha^2}\left|\left[\frac{\partial R_k}{\partial \phi}\right]_k\right|_{G,\rho_2,1}.\end{aligned}$$

Taking the supremum at the whole domain:

$$\left\|\frac{\partial W_k}{\partial I}\right\|_{G,\rho_2,\infty} \leq \frac{2}{\alpha}\left\|\frac{\partial R_k}{\partial I}\right\|_{G,\rho_2,\infty}+\frac{4M}{\alpha^2}\left\|\left[\frac{\partial R_k}{\partial \phi}\right]_k\right\|_{G,\rho_2,1}.$$

Moreover,

$$\begin{aligned}\frac{\partial W(I)}{\partial \phi} &= \frac{\partial}{\partial \phi}\left(\sum_{k\in\mathbb{R}^n} W_k(I)e^{ik\cdot\phi}\right)\\ &= \frac{\partial}{\partial \phi}\left(\sum_{k\in\mathbb{R}^n} ikW_k(I)e^{ik\cdot\phi}\right).\end{aligned}$$

Hence, the k-th term of the Fourier series of $\frac{\partial W}{\partial \phi}$ is

$$\begin{aligned}\left[\frac{\partial W}{\partial \phi}\right]_k = W_k ik &= \frac{R_k}{i(k\bar{\mathcal{B}}(I_1)u+k\bar{\mathcal{A}}(I_1)))}ik\\ &= \frac{1}{i(k\bar{\mathcal{B}}(I_1)u+k\bar{\mathcal{A}}(I_1)))}\left[\frac{\partial R}{\partial \phi}\right]_k.\end{aligned}$$

Taking norms ($\|\cdot\|_{G,\rho,1}$) at each side:

$$\left\|\frac{\partial W}{\partial \phi}\right\|_{G,\rho,1} \leq \frac{2}{\alpha}\left\|\frac{\partial W}{\partial \phi}\right\|_{G,\rho,1}.$$

Then,

$$\|DW\|_{G,\rho,c} = \max\left(\left\|\frac{\partial W}{\partial \phi}\right\|_{G,\rho,1}, c\left\|\frac{\partial W}{\partial I}\right\|_{G,\rho,\infty}\right)$$

$$
\begin{aligned}
&\leq \max\left(\frac{2}{\alpha}\left\|\frac{\partial R}{\partial \phi}\right\|_{G,\rho,1}, c\frac{2}{\alpha}\left\|\frac{\partial R}{\partial I}\right\|_{G,\rho_2,\infty} + c\frac{4M}{\alpha^2}\left\|\frac{\partial R}{\partial \phi}\right\|_{G,\rho_2,1}\right) \\
&\leq \max\left(\frac{2}{\alpha}\left\|\frac{\partial R}{\partial \phi}\right\|_{G,\rho,1}, \frac{2}{\alpha}\|DR\|_{G,\rho_2,c} + c\frac{4M}{\alpha^2}\|DR\|_{G,\rho_2,c}\right) \\
&= \max\left(\frac{2}{\alpha}\left\|\frac{\partial R}{\partial \phi}\right\|_{G,\rho,1}, \frac{2}{\alpha}\left(1+\frac{2M}{\alpha}c\right)\|DR\|_{G,\rho_2,c}\right) \\
&\leq \frac{2}{\alpha}\left(1+\frac{2M}{\alpha}c\right)\|DR\|_{G,\rho_2,c} \\
&\leq \frac{2}{\alpha}A\|DR\|_{G,\rho_2,c},
\end{aligned}
$$

where A is as desired.

$\square$

Recall the Cauchy inequalities, see [34]:

$$
\begin{aligned}
\left\|\frac{\partial f}{\partial \phi}\right\|_{G,(\rho_1,\rho_2),1} &\leq \frac{1}{e\delta_1}\|f\|_{G,\rho} \\
\left\|\frac{\partial f}{\partial I}\right\|_{G,(\rho_1,\rho_2-\delta_2),\infty} &\leq \frac{1}{\delta_2}\|f\|_{G,\rho}
\end{aligned}
\tag{7.9}
$$

Lemma 7.2.15. *Let f, g be analytic functions on $\mathcal{D}_\rho(G)$, where $0 < \delta = (\delta_1, \delta_2) < \rho = (\rho_1, \rho_2)$ and $c > 0$. Define $\hat{\delta}_c := \min(c\delta_1, \delta_2)$. The following inequalities hold:*

1. $\|Df\|_{G,\rho-\delta,c} \leq \frac{c}{\hat{\delta}_c}\|f\|_{G,\rho}$
2. $\|\{f,g\}\|_{G,\rho} \leq \frac{2}{c}\|Df\|_{G,\rho,c} \cdot \|Dg\|_{G,\rho,c}$
3. $\|D(f_{>K})\|_{G,(\rho-\delta_1,\rho_2),c} \leq e^{-K\delta_1}\|Df\|_{G,\rho,c}$

Proof. Let us prove each point separately.

1. Using the Cauchy inequalities one obtains the following:

$$\begin{aligned}
\left\|\frac{\partial f}{\partial \phi}\right\|_{G,\rho-\delta,1} &= \left\|\frac{\partial f}{\partial \phi}\right\|_{G,(\rho_1-\delta_1,\rho_2-\delta_2),1} \\
\leq \left\|\frac{\partial f}{\partial \phi}\right\|_{G,(\rho_1-\delta_1,\rho_2),1} &\leq \frac{1}{e\delta_1}\|f\|_{G,\rho}, \\
\left\|\frac{\partial f}{\partial I}\right\|_{G,\rho-\delta,\infty} &= \left\|\frac{\partial f}{\partial I}\right\|_{G,(\rho_1-\delta_1,\rho_2-\delta_2),\infty} \\
\leq \left\|\frac{\partial f}{\partial I}\right\|_{G,(\rho_1,\rho_2-\delta_2),\infty} &\leq \frac{1}{\delta_1}\|f\|_{G,\rho}.
\end{aligned}$$

Putting the two inequalities inside the definition of the norm:

$$\begin{aligned}
\|Df\|_{G,\rho-\delta,c} &= \max\left\{\left\|\frac{\partial f}{\partial \phi}\right\|_{G,\rho-\delta,1}, c\left\|\frac{\partial f}{\partial I}\right\|_{G,\rho-\delta,\infty}\right\} \\
&\leq \max\left\{\frac{1}{e\delta_1}\|f\|_{G,\rho}, \frac{c}{\delta_2}\|f\|_{G,\rho}\right\} \\
&\leq \max\left\{\frac{1}{e\delta_1}\frac{c}{c}, \frac{c}{\delta_2}\right\}\|f\|_{G,\rho} \\
&\leq \max\left\{\frac{c}{e\hat{\delta}_c}, \frac{c}{\hat{\delta}_c}\right\}\|f\|_{G,\rho},
\end{aligned}$$

where the last inequality holds because $\hat{\delta}_c = \min(c\delta_1, \delta_2)$.

2. Let us find the expression of $\{f, g\}$ for a b^m-symplectic structure. $\{f, g\} = \omega(X_f, X_g)$ where X_f and X_g are such that $\iota_{X_f}\omega = df$ and $\iota_{X_g}\omega = dg$. Let restrict the computions only to f.

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial \phi_1} d\phi_1, \quad X_f = \sum_{i=1}^{n} a_i \frac{\partial}{\partial \phi_i} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial \phi_i}.$$

Where a_i and b_i are coefficients to be determined by imposing the following condition:

$$\iota_{X_f}\omega = \left(\sum_{j=1}^{m}\frac{c_j}{I_1^j}\right)(a_1 dI_1 - b_1 d\phi_1) + \sum_{i=2}^{n}(a_i dI_i - b_i d\phi_i) = df.$$

Then, solving for the coefficients the following expressions are obtained:

$$a_1 = \frac{1}{\left(\sum_{j=1}^{m}\frac{c_j}{I_1^j}\right)}\frac{\partial f}{\partial \phi_1} \quad \text{and} \quad a_i = \frac{\partial f}{\partial \phi_i} \text{ for } i \neq 1,$$

$$b_1 = -\frac{1}{\left(\sum_{j=1}^{m}\frac{c_j}{I_1^j}\right)}\frac{\partial f}{\partial \phi_1} \quad \text{and} \quad b_i = -\frac{\partial f}{\partial \phi_i} \text{ for } i \neq 1.$$

Hence, the expression for the hamiltonian vector fields becomes:

$$X_f = \frac{1}{\left(\sum_{j=1}^{m}\frac{c_j}{I_1^j}\right)}\left(\frac{\partial f}{\partial \phi_1}\frac{\partial}{\partial \phi_1} - \frac{\partial f}{\partial I_1}\frac{\partial}{\partial I_1}\right) + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial \phi_i}\frac{\partial}{\partial \phi_i} - \frac{\partial f}{\partial I_i}\frac{\partial}{\partial I_i}\right),$$

$$X_g = \frac{1}{\left(\sum_{j=1}^{m}\frac{c_j}{I_1^j}\right)}\left(\frac{\partial g}{\partial \phi_1}\frac{\partial}{\partial \phi_1} - \frac{\partial g}{\partial I_1}\frac{\partial}{\partial I_1}\right) + \sum_{i=1}^{n}\left(\frac{\partial g}{\partial \phi_i}\frac{\partial}{\partial \phi_i} - \frac{\partial g}{\partial I_i}\frac{\partial}{\partial I_i}\right).$$

Then the Poisson bracket applied to the two functions:

$$\begin{aligned}\{f, g\} = \omega(X_f, X_g) &= \frac{1}{\left(\sum_{j=1}^{m}\frac{c_j}{I_1^j}\right)}\left(\frac{\partial f}{\partial I_1}\frac{\partial g}{\partial \phi_1} - \frac{\partial f}{\partial \phi_1}\frac{\partial g}{\partial I_1}\right) \\ &\quad + \sum_{i=2}^{n}\left(\frac{\partial f}{\partial I_i}\frac{\partial g}{\partial \phi_i} - \frac{\partial f}{\partial \phi_i}\frac{\partial g}{\partial I_i}\right).\end{aligned}$$

And hence the norm of the Poisson bracket becomes:

$$
\begin{aligned}
\|\{f,g\}\|_{G,\rho} &= \left\| \frac{1}{\left(\sum_{j=1}^m \frac{c_j}{I_1^j}\right)} \left(\frac{\partial f}{\partial I_1}\frac{\partial g}{\partial \phi_1} - \frac{\partial f}{\partial \phi_1}\frac{\partial g}{\partial I_1} \right) \right. \\
&\qquad \left. + \sum_{i=2}^n \left(\frac{\partial f}{\partial I_i}\frac{\partial g}{\partial \phi_i} - \frac{\partial f}{\partial \phi_i}\frac{\partial g}{\partial I_i} \right) \right\|_{G,\rho} \\
&\leq \left\| \sum_{i=1}^n \left(\frac{\partial f}{\partial I_i}\frac{\partial g}{\partial \phi_i} - \frac{\partial f}{\partial \phi_i}\frac{\partial g}{\partial I_i} \right) \right\|_{G,\rho}
\end{aligned}
$$

Where we assumed $\left|\sum_{j=1}^m \frac{c_j}{I_1^j}\right| \geq 1$. This assumption makes sense, because we are interested in the behaviour close the critical set Z. Close enough to the critical set this expression holds. Then,

$$
\begin{aligned}
\|\{f,g\}\|_{G,\rho} &\leq \sum_{i=1}^n \left\|\frac{\partial f}{\partial I_i}\right\|_{G,\rho} \left\|\frac{\partial g}{\partial \phi_i}\right\|_{G,\rho} + \sum_{i=1}^n \left\|\frac{\partial f}{\partial \phi_i}\right\|_{G,\rho} \left\|\frac{\partial g}{\partial I_i}\right\|_{G,\rho} \\
&\leq \left|\frac{\partial f}{\partial I}\right|_{G,\rho,\infty} \left|\frac{\partial g}{\partial I}\right|_{G,\rho,1} + \left|\frac{\partial f}{\partial I}\right|_{G,\rho,1} \left|\frac{\partial g}{\partial I}\right|_{G,\rho,\infty} \\
&\leq \frac{1}{c}|Df\|_{G,\rho,c}\|Dg\|_{G,\rho,c} + \frac{1}{c}|Df\|_{G,\rho,c}\|Dg\|_{G,\rho,c} \\
&\leq \frac{2}{c}\|Df\|_{G,\rho,c}\|Dg\|_{G,\rho,c}.
\end{aligned}
$$

3. Lastly,

$$
\begin{aligned}
&\|D(f_{>K})\|_{G,(\rho_1-\delta_1,\rho_2),1} \\
&= \max\left\{ \left\|\frac{\partial f_{>K}}{\partial \phi}\right\|_{G,(\rho_1-\delta_1,\rho_1),1}, c\left\|\frac{\partial f_{>K}}{\partial I}\right\|_{G,(\rho_1-\delta_1,\rho_1),\infty} \right\}.
\end{aligned}
$$

We will proceed by bounding each term separately. On one hand:

$$
\left\|\frac{\partial f}{\partial \phi}\right\|_{G,(\rho_1,\rho_2),1} = \left\| \sum_{k\in\mathbb{Z}^n} ikf_k(I)e^{ik\phi} \right\|_{G,(\rho_1,\rho_2),1}
$$

$$\begin{aligned}
&\geq \sum_{k\in\mathbb{Z}^n} k \left\| f_k(I) \right\|_{G,\rho_2,1} e^{|k|_1\rho_1} \\
&\geq \sum_{\substack{k\in\mathbb{Z}^n\\|k|_1>K}} k \left\| f_k(I) \right\|_{G,\rho_2,1} e^{|k|_1(\rho_1+\delta_1-\delta_1)} \\
&\geq e^{K\delta_1} \sum_{\substack{k\in\mathbb{Z}^n\\|k|_1>K}} k \left\| f_k(I) \right\|_{G,\rho_2,1} e^{|k|_1(\rho_1-\delta_1)} \\
&= e^{K\delta_1} \left\| \frac{\partial f_{>K}}{\partial \phi} \right\|_{G,(\rho_1-\delta_1,\rho_2),1}.
\end{aligned}$$

On the other hand:

$$\begin{aligned}
\left\| \frac{\partial f}{\partial I} \right\|_{G,(\rho_1,\rho_2),\infty} &= \left\| \sum_{k\in\mathbb{Z}^n} \frac{\partial f_k(I)}{\partial I} e^{ik\phi} \right\|_{G,(\rho_1,\rho_2),\infty} \\
&\geq \sum_{k\in\mathbb{Z}^n} \left\| \frac{\partial f_k(I)}{\partial I} \right\|_{G,\rho_2,\infty} e^{|k|_1\rho_1} \\
&\geq \sum_{\substack{k\in\mathbb{Z}^n\\|k|_1>K}} \left\| \frac{\partial f_k(I)}{\partial I} \right\|_{G,\rho_2,\infty} e^{|k|_1(\rho_1+\delta_1-\delta_1)} \\
&\geq e^{K\delta_1} \sum_{\substack{k\in\mathbb{Z}^n\\|k|_1>K}} \left\| \frac{\partial f_k(I)}{\partial I} \right\|_{G,\rho_2,\infty} e^{|k|_1(\rho_1-\delta_1)} \\
&\geq e^{K\delta_1} \left\| \frac{\partial f_{>K}}{\partial I} \right\|_{G,(\rho_1-\delta_1,\rho_2),\infty}.
\end{aligned}$$

Hence $\|D(f_{>k})\|_{G,(\rho_1-\delta_1,\rho_2),c} \leq e^{-K\delta_1}\|Df\|_{G,\rho,c}$.

□

Now we define a norm that indicates how close a map Φ is to the identity.

Definition 7.2.16. *Let $x = (\phi, I) \in \mathbb{C}^{2n}$, then*

$$|x|_c := \max(|\phi|_1, c|I|_\infty)$$

Definition 7.2.17. *For a map $\Upsilon : \mathcal{D}_\rho(G) \to \mathbb{C}^{2n}$ its norm and the norm of its derivative its defined as:*

$$|\Upsilon|_{G,\rho,c} := \sup_{x\in\mathcal{D}_\rho(G)} |\Upsilon(x)|_c,$$

$$|D\Upsilon|_{G,\rho,c} := \sup_{x\in\mathcal{D}_\rho(G)} |D\Upsilon(x)|_c,$$

where $|D\Upsilon(x)|_c = \sup_{\substack{y\in\mathbb{R}^{2n}\\ |y|_c=1}} |D\Upsilon(x)\cdot y|_c$

Lemma 7.2.18. *If Υ is analytic on $\mathcal{D}_\rho(G)$, then $|D\Upsilon|_{G,\rho-\delta,C} \leq \frac{|\Upsilon|_{G,\rho,c}}{\hat{\delta}_c}$*

Proof. Observe that if we have $\|.\|$ any norm on $\mathbb{C}^n$ and we have a matrix A of size $n\times n$, and $\|A\|$ defines the induced norm of matrices i.e.

$$\|A\| = \sup_{\substack{y\in\mathbb{C}^{2n}\\ \|y\|=1}} \|A\cdot y\|$$

then one has that $\|(\|a_1\|', \ldots, \|a_n\|')\| \leq \|A\|$ where a_j denotes the j-th row of A. Also note that $\|\cdot\|'$ can be a any norm consider the infinity norm. This can be easily proven in the following way:

$$\|A\cdot y\| = \left\| \begin{pmatrix} a_1\cdot y \\ \vdots \\ a_n \cdot y \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} \|a_1\|'\|y\|' \\ \vdots \\ \|a_n\|'\|y\|' \end{pmatrix} \right\|$$

Where $\forall y \in \mathbb{C}^n$ such that $\|y\| = 1$. Let a_j be the rows of $D\Upsilon(x)$,

$$a_j = \left(\frac{\partial \Upsilon_j}{\partial\phi}, \frac{\partial \Upsilon j}{\partial I}\right),$$

and be $\|a_j\|'$ its norm. With this property in mind we proceed as follows:

$$
\begin{aligned}
|D\Upsilon|_{G,\rho-\delta,c} &= \sup_{x\in\mathcal{D}_{\rho-\delta}(G)} |D\Upsilon(x)|_c \\
&\leq \sup_{x\in\mathcal{D}_{\rho-\delta}(G)} |(|a_1|_\infty,\dots,|a_n|_\infty)|_c \\
&\leq \left|\left(\sup_{x\in\mathcal{D}_{\rho-\delta}} \|D\Upsilon_1\|_\infty,\dots,\sup_{x\in\mathcal{D}_{\rho-\delta}} \|D\Upsilon_{2n}\|_\infty\right)\right|_c \\
&= \left|\left(\|D\Upsilon_1\|_{G,\rho-\delta,\infty},\dots,\|D\Upsilon_{2n}\|_{G,\rho-\delta,\infty}\right)\right|_c \\
&\leq \left|\left(\tfrac{1}{\delta_1}\|\Upsilon_1\|_{G,\rho},\dots,\tfrac{1}{\delta_1}\|\Upsilon_{2n}\|_{G,\rho}\right)\right|_c \\
&\leq \tfrac{1}{\hat{\delta}_c}\left|\|\Upsilon_1\|_{G,\rho},\dots,\|\Upsilon_{2n}\|_{G,\rho}\right|_c \\
&= \tfrac{1}{\hat{\delta}_c}\sup_{x\in\mathcal{D}_\rho(G)} |\Upsilon_1,\dots,\Upsilon_{2n}|_c = \tfrac{1}{\hat{\delta}_c}\sup_{x\in\mathcal{D}_\rho(G)} |\Upsilon|_c \\
&= \tfrac{1}{\hat{\delta}_c}|\Upsilon|_{G,\rho,c}
\end{aligned}
$$

□

Lemma 7.2.19. *Let W be an analytic function on $\mathcal{D}_\rho(G)$, $\rho > 0$ and let Φ_t be its Hamiltonian flow at time t ($t > 0$). Let $\delta = (\delta_1, \delta_2) > 0$ and $c > 0$ given. Assume that $\|DW\|_{G,\rho,c} \leq \hat{\delta}_c$. Then, Φ_t maps $\mathcal{D}_{\rho-t\delta}(G)$ into $\mathcal{D}_\rho(G)$ and one has:*

1. *$|\Phi_t - Id|_{G,\rho-t\delta,c} \leq t\|DW\|_{G,\rho,c}$,*

2. *$\Phi(\mathcal{D}_\rho(G)) \supset \mathcal{D}_{\rho-t\delta}(G)$ for $\rho' \leq \rho - t\delta$,*

3. *Assuming that $\|DW\|_{G,\rho,c} < \hat{\delta}_c/2e$, for any given function f analytic on $\mathcal{D}_\rho(G)$, and for any integer $m \geq 0$, the following bound*

holds:

$$\begin{aligned}
&\|r_m(f,W,t)\|_{G,\rho-t\delta} \\
&\leq \sum_{l=0}^{\infty}\left[\frac{1}{\binom{l+m}{m}}\cdot\left(\frac{2e\|DW\|_{G,\rho,c}}{\hat{\delta}_c}\right)^l\right]\frac{t^m}{m!}\|L_W^m f\|_{G,\rho} \\
&= \gamma_m\left(\frac{2e\|DW\|_{G,\rho,c}}{\hat{\delta}_c}\right)\cdot t^m\|L_W^m f\|_{G,\rho},
\end{aligned}$$

where for $0 \leq x \leq 1$ *we define*

$$\gamma_m(x) := \sum_{l=0}^{\infty}\frac{l!}{(l+m)!}x^l$$

Proof. During the proof we are going to denote $\Phi_s(\phi_0, I_0)$ by $(\phi(s), I(s))$.

Let us find the coordinate expression of the hamiltonian flow for the expression 7.2 of a b^m-symplectic form. Recall that the equation for the hamiltonian flow is $\frac{d}{ds}\phi_i(s) = \{\phi_i, W\}$ and $\frac{d}{ds}I_i(s) = \{I_i, W\}$.

$$\begin{aligned}
\{\phi_i, W\} = &\frac{1}{\left(\sum_{j=1}^m \frac{c_j}{I_1^j}\right)}\left(\frac{\partial\phi_i}{\partial I_1}\cdot\frac{\partial W}{\partial\phi_1} - \frac{\partial\phi_i}{\partial\phi_1}\cdot\frac{\partial W}{\partial I_1}\right) \\
&+\sum_{j=2}^{n}\left(\frac{\partial\phi_i}{\partial I_j}\cdot\frac{\partial W}{\partial\phi_j} - \frac{\partial\phi_i}{\partial\phi_j}\cdot\frac{\partial W}{\partial I_j}\right).
\end{aligned}$$

Hence,

$$\frac{d}{ds}\phi_i(s) = -\frac{1}{\left(\sum_{j=1}^m \frac{c_j}{I_1^j}\right)}\frac{\partial W}{\partial I_1} \text{ if } i=1 \text{ and } \frac{d}{ds}\phi_i(s) = -\frac{\partial W}{\partial I_i} \text{ if } i\neq 1.$$

On the other side,

$$\begin{aligned}
\{I_i, W\} = &\frac{1}{\left(\sum_{j=1}^m \frac{c_j}{I_1^j}\right)}\left(\frac{\partial I_i}{\partial I_1}\cdot\frac{\partial W}{\partial\phi_1} - \frac{\partial I_i}{\partial\phi_1}\cdot\frac{\partial W}{\partial I_1}\right) \\
&+\sum_{j=2}^{n}\left(\frac{\partial I_i}{\partial I_j}\cdot\frac{\partial W}{\partial\phi_j} - \frac{\partial I_i}{\partial\phi_j}\cdot\frac{\partial W}{\partial I_j}\right).
\end{aligned}$$

Hence,

$$\frac{d}{ds}I_i(s) = \frac{1}{\left(\sum_{j=1}^m \frac{c_j}{I_1^j}\right)}\frac{\partial W}{\partial \phi_1} \text{ if } i=1 \text{ and } \frac{d}{ds}I_i(s) = \frac{\partial W}{\partial \phi_i} \text{ if } i \neq 1.$$

1. Assume now that $0 < s_0 \leq t$. Then,

$$\begin{aligned}
|\phi(s_0) - \phi_0|_\infty &\leq s_0 \sup_{0<s\leq s_0} |\phi'(s)|_\infty \\
&= s_0 \sup_{0<s\leq s_0} (\max(|\phi_1'(s)|, \dots, |\phi_n'(s)|)) \\
&= s_0 \sup_{0<s\leq s_0} \left(\max\left(\left|\frac{1}{\left(\sum_{j=1}^m \frac{c_j}{I_1^j}\right)}\frac{\partial W}{\partial I_1}\right|, \left|\frac{\partial W}{\partial I_2}\right|, \dots, \left|\frac{\partial W}{\partial I_n}\right|\right)\right) \\
&\leq s_0 \sup_{0<s\leq s_0} \left|\frac{\partial W}{\partial I}\right|_\infty \leq s_0 \left\|\frac{\partial W}{\partial I}\right\|_{G,\rho,\infty}
\end{aligned}$$

Where we have used again that on the domain $\mathcal{D}_\rho(G)$ the inequality $\left|\sum_{j=1}^m \frac{c_j}{I_1^j}\right| \geq 1$ holds. Similarly, $|I(s_0) - I_0| \leq s_0 \|\frac{\partial W}{\partial \phi}\|_{G,\rho,1}$, and hence $|\Phi_t - \mathrm{Id}|_{G,\rho-t\delta,c} \leq t\|DW\|_{G,\rho,c}$.

Because

$$\begin{array}{ll}
|\phi(s) - \phi_0|_\infty \leq t\|\frac{\partial W}{\partial I}\|_{G,\rho,\infty} \leq t\frac{\hat{\delta}_c}{c} \leq t\delta_1\frac{c}{c} = t\delta_1 & \forall 0 < s \leq s_0 \\
|I(s) - I_0|_1 \leq t\|\frac{\partial W}{\partial \phi}\|_{G,\rho,1} \leq t\hat{\delta}_c \leq t\delta_2 & \forall 0 < s \leq s_0
\end{array} \tag{7.10}$$

hence, $(\phi(s), I(s)) \in \mathcal{D}_{\rho-t\delta+t\delta}(G) = \mathcal{D}_\rho(G)$ for all $0 < s \leq s_0$.

2. Repeat the same argument as in 7.10 with $\phi(-s)$. If $(\phi_0, I_0) \in \mathcal{D}_{\rho'-t\delta}$, then $(\phi(-s), I(-s)) \in \mathcal{D}_{\rho'-t\delta+t\delta}(G) = \mathcal{D}_{\rho'}$. Hence,

$$\mathcal{D}_{\rho'}(G) \supset \Phi^{-1}(\mathcal{D}_{\rho'-t\delta}(G)),$$

then $\Phi(\mathcal{D}_{\rho'}(G)) \supset \mathcal{D}_{\rho'-t\delta}(G)$.

3. Consider f an analytical function. By the previous construction $f \circ \Phi_t$ is defined in $\mathcal{D}_{\rho-t\delta}(G)$. Because W is analytic we also

have that $f\circ\Phi_t$ is analytic and we can expand its Lie series. Let $m\in\mathbb{Z}, l\geq m+1, j=m+1,\dots,l$ then

$$\begin{aligned}\|L^j_W f\|_{G,\rho-(j-m)t\eta} &\leq \tfrac{2}{c}\|D(L^{j-1}_W f)\|_{G,\rho-(j-m)t\eta,c}\|DW\|_{G,\rho,c}\\ &\leq \tfrac{2}{t\hat\eta_c}\|L^{j-1}_W f\|_{G,\rho-(j-1-m)t\eta}\|DW\|_{G,\rho,c},\end{aligned}$$

where we used lemma 7.2.15 and defined $\eta=\frac{\delta}{(l-m)}$ and $\hat\eta_c=\min(c\eta_1,\eta_2)$.

Then,

$$\begin{aligned}\|L^l_W f\|_{G,\rho-t\delta} &\leq \left(\tfrac{2\|DW\|_{G,\rho,c}}{t\hat\eta_c}\right)^{l-m}\\ &\leq e^{l-m}\cdot(l-m)!\left(\tfrac{2\|DW\|_{G,\rho,c}}{\hat\delta_c}\right)^{l-m}\|L^m_W f\|_{G,\rho},\end{aligned}$$

where we used that $\hat\eta_c=\frac{\hat\delta_c}{l-m}$ and $(l-m)^{(l-m)}\leq e^{l-m}\cdot(l-m)!$ And hence, the bound for $\|r_m(f,W,t)\|_{G,\rho-t\delta}$ is

$$\sum_{l=m}^{\infty}\frac{t^l}{l!}\|L^l_W f\|_{G,\rho-t\delta}\leq\left[\sum_{l=m}^{\infty}\frac{(l-m)!}{l!}\left(\frac{2e\|DW\|_{G,\rho,c}}{\hat\delta_c}\right)^{l-m}\right]\cdot t^m\|L^m_W f\|_{G,\rho}$$

and this series converges if $\|DW\|_{G,\rho,c}\leq\frac{\hat\delta_c}{2e}$.

$\square$

Theorem 7.2.20. *[Iterative Lemma] $H(\phi,I)=\hat h(I)+R(\phi,I)$ where $\hat h(I)$ is as in equation 7.1 defined on $\mathcal{D}_\rho(G)$. Let $\hat u=\frac{\partial\hat h}{\partial I}$ and $u=\frac{\partial h}{\partial I}$, and assume u is $\alpha,K,c,\hat q$-non-resonant. Assume that $\left|\frac{\partial}{\partial I}u\right|_{G,\rho_2}\leq M'$. Let $\delta<\rho$ and $c>0$, $A=1+\frac{2Mc}{\alpha}$. Assume that $\rho_2\leq\frac{\alpha}{2MK}$, $\|DR\|_{G,\rho,c}\leq\frac{\alpha\hat\delta_c}{74A}$. Then, there exists a real analytic map $\Phi:\mathcal{D}_{\rho-\frac{\delta}{2}}(G)\to\mathcal{D}_\rho(G)$, such that $H\circ\Phi=\hat h+\tilde R$,with:*

1. $\|D\tilde R\|_{G,\rho-\delta,c}\leq e^{-K\delta_1}\|DR\|_{G,\rho,c}+\frac{14A}{\alpha\hat\delta_c}\|DR\|^2_{G,\rho,c}$,

2. $|\Phi - Id|_{G,\rho-\frac{\delta}{2},c} \leq \frac{2A}{\alpha}\|DR\|_{G,\rho,c}$,

3. $\Phi(\mathcal{D}_{\rho'}(G)) \supset \mathcal{D}_{\rho'-\frac{\delta}{2}}(G)$ *for* $\rho' \leq \rho - \frac{\delta}{2}$

Proof. Recall that $\left|\frac{\partial}{\partial I}u\right|_{G,\rho_2} \leq M'$ implies $\left|\frac{\partial}{\partial I}(\bar{\mathcal{B}}u + \bar{\mathcal{A}})\right|_{G,\rho_2} \leq M$ by equation 7.8. By equation 7.6

$$R^{(q)} = R^{(q-1)}_{>K} + r_2(\hat{h}^{(q-1)}, W^{(q)}, 1) + r_1(R^{(q-1)}, W^{(q)}, 1).$$

To simplify the notation we are going to omit the index of the iteration:

$$\tilde{R} = R_{>K} + r_2(\hat{h}W, 1) + r_1(R, W, 1). \tag{7.11}$$

Where W is defined in terms of its Fourier expressions by equation 7.7:

$$W_k(I) = \frac{R_k(I)}{i(k\bar{\mathcal{B}}(I_1)u + k\bar{\mathcal{A}}(I_1))}$$

By proposition 7.2.14: $\|DW\|_{G,\rho,c} \leq \frac{2A}{\alpha}\|DR\|_{G,\rho,c} \leq \frac{2A}{\alpha}\frac{\alpha\hat{\delta}_c}{74A} = \frac{\hat{\delta}_c}{37}$. And Φ is defined as in lemma 7.2.15: $\Phi : \mathcal{D}_{\rho-\frac{\delta}{2}}(G) \to \mathcal{D}_\rho(G)$.

1. Differentiating equation 7.11 we obtain:

$$D\tilde{R} = DR_{>K} + Dr_2(\hat{h}, W, 1) + Dr_1(R, W, 1).$$

Taking norms at every side of the expression:

$$\begin{aligned}
\|D\tilde{R}\|_{G,\rho-\delta,c} &= \|DR_{>K} + Dr_2(\hat{h}, W, 1) + Dr_1(R, W, 1)\|_{G,\rho-\delta,c} \\
&\leq \|DR_{>K}\|_{G,\rho-\delta,c} + \|Dr_2(\hat{h}, W, 1)\|_{G,\rho-\delta,c} \\
&\quad + \|Dr_1(R, W, 1)\|_{G,\rho-\delta,c} \\
&\leq e^{-K\delta_1}\|DR\|_{G,\rho,c} \\
&\quad + \frac{2c}{\hat{\delta}_c}\left(\|r_2(\hat{h}, W, 1)\|_{G,\rho-\frac{\delta}{2},c} + \|r_1(R, W, 1)\|_{G,\rho-\frac{\delta}{2},c}\right)
\end{aligned}$$

Let us further develop the two last terms of the previous expression, by using lemma 7.2.19:

$$\begin{aligned}
\|r_2(\hat{h},W,1)\|_{G,\rho-\frac{\delta}{2},c} &\leq \gamma_2\left(\frac{2e\|DW\|_{G,\rho,c}}{\hat{\delta}_c/2}\right)\|L_W^2 h\|_{G,\rho}\\
&\leq \gamma_2\left(\frac{4e\|DW\|_{G,\rho,c}}{\hat{\delta}_c}\right)\|\{\{h,W\},W\}\|_{G,\rho},
\end{aligned}$$

$$\begin{aligned}
\|r_1(\hat{h},W,1)\|_{G,\rho-\frac{\delta}{2},c} &\leq \gamma_1\left(\frac{2e\|DW\|_{G,\rho,c}}{\hat{\delta}_c/2}\right)\|L_W^1 R\|_{G,\rho}\\
&\leq \gamma_1\left(\frac{4e\|DW\|_{G,\rho,c}}{\hat{\delta}_c}\right)\|\{R,W\}\|_{G,\rho}.
\end{aligned}$$

Then, using the second statement of lemma 7.2.15 and that $\{W,h\} = R_{\leq K}$:

$$\|\{R,W\}\|_{G,\rho} \leq \frac{2}{c}\|DR\|_{G,\rho,c}\|DW\|_{G,\rho,c}, \text{ and}$$

$$\begin{aligned}
|\{\{h,W\},W\}\|_{G,\rho} &= \|\{R_{\leq K},W\}\|_{G,\rho}\\
&\leq \frac{2}{c}\|DR_{\leq K}\|_{G,\rho,c}\|DW\|_{G,\rho,c}\\
&\leq \frac{2}{c}\|DR\|_{G,\rho,c}\|DW\|_{G,\rho,c}.
\end{aligned}$$

Moreover, it is easy to see that $\gamma_1(x) = \frac{-\log(1-x)}{x}$ and $\gamma_2(x) = \frac{x+(1-x)\log(1-x)}{x^2}$. Observe that these functions are monotonously increasing in x. Recall that $\|DW\|_{G,\rho,c} \leq \frac{2A}{\alpha}\|DR\|_{G,\rho,c}$. Then,

$$\begin{aligned}
&\|r_1(\hat{h},W,1)\|_{G,\rho-\frac{\delta}{2},c}\\
&+\|r_2(\hat{h},W,1)\|_{G,\rho-\frac{\delta}{2},c} \leq \gamma_1\left(\frac{4e\|DW\|_{G,\rho,c}}{\hat{\delta}_c}\right)\|\{R,W\}\|_{G,\rho}\\
&\qquad +\gamma_2\left(\frac{4e\|DW\|_{G,\rho,c}}{\hat{\delta}_c}\right)\|\{\{h,W\},W\}\|_{G,\rho}\\
&\leq \gamma_1\left(\frac{4e\|DW\|_{G,\rho,c}}{\hat{\delta}_c}\right)\frac{2}{c}\|DR\|_{G,\rho,c}\|DW\|_{G,\rho,c}\\
&\qquad +\gamma_2\left(\frac{4e\|DW\|_{G,\rho,c}}{\hat{\delta}_c}\right)\frac{2}{c}\|DR\|_{G,\rho,c}\|DW\|_{G,\rho,c}\\
&\leq \gamma_1\left(\frac{4e\|DW\|_{G,\rho,c}}{\hat{\delta}_c}\right)\frac{2}{c}\frac{2A}{\alpha}\|DR\|^2_{G,\rho,c}\\
&\qquad +\gamma_2\left(\frac{4e\|DW\|_{G,\rho,c}}{\hat{\delta}_c}\right)\frac{2}{c}\frac{2A}{\alpha}\|DR\|^2_{G,\rho,c}\\
&\leq \frac{2}{c}[\gamma_1(\tfrac{4e}{37})+\gamma_2(\tfrac{4e}{37})]\frac{2A}{\alpha}\|DR\|^2_{G,\rho,c}\\
&= \frac{4A}{\alpha c}[\gamma_1(\tfrac{4e}{37})+\gamma_2(\tfrac{4e}{37})]\|DR\|^2_{G,\rho,c}.
\end{aligned}$$

Moreover $\gamma_1(\frac{4e}{37}) + \gamma_2(\frac{4e}{37}) \approx 1.741\ldots < \frac{7}{4}$.

Then,

$$\begin{aligned}\|D\tilde{R}\|_{G,\rho-\delta,c} &\leq e^{-K\delta_1}\|DR\|_{G,\rho,c} + \tfrac{2c}{\hat{\delta}_c}\tfrac{4A}{\alpha c}\tfrac{7}{4}\|\|^2_{G,\rho,c} \\ &\leq e^{-K\delta_1}\|DR\|_{G,\rho,c} + \tfrac{14A}{\hat{\delta}_c\alpha}\|DR\|^2_{G,\rho,c},\end{aligned}$$

as we wanted to prove.

2. Direct from lemma 7.2.19:

$$|\Phi - \mathrm{Id}|_{G,\rho.\frac{\delta}{2},c} \leq \|DW\|_{G,\rho,c} \leq \frac{2A}{\alpha}\|DR\|_{G,\rho,c}$$

3. Also direct from lemma 7.2.19:

$$\Phi(\mathcal{D}_\rho(G)) \supset \mathcal{D}_{\rho'-\frac{\delta}{2}}(G), \text{ for } \rho' \leq \rho - \delta/2$$

□

Definition 7.2.21. $\Delta_{c,\hat{q}}(k,\alpha) = \{J \in \mathbb{R}^n$ *such that* $|k\bar{\mathcal{B}}(I_1)J + k\bar{\mathcal{A}}(I_1)| < \alpha\}$

Lemma 7.2.22. *With the previous definitions we have the following bounds.*

Outside of Z*:*

$$meas\,(F \cap \Delta_{c,\hat{q}}(k,\alpha)) \leq (\mathrm{diam}F)^{n-1}\frac{2\alpha}{|k|_{2,\omega}}.$$

At Z*:*

$$meas\,(F \cap \Delta_{c,\hat{q}}(k,\alpha)) \begin{cases} = 0 & \text{if } \alpha \leq \frac{|k_1|}{\mathcal{K}'} \\ \leq (diamF)^n & \text{if } \alpha > \frac{|k_1|}{\mathcal{K}'} \end{cases}$$

Proof. It is important to understand the geometry of the set $\Delta_{c,\hat{q}}(k,\alpha)$. Recall that $k\bar{\mathcal{B}}(I_1)J = k_1\mathcal{B}(I_1)J_1 + \bar{k}\bar{J}$, hence this part of the expression can be interpreted as the scalar product of the vector J with the vector

$(k_1\mathcal{B}(I_1), k_2, \dots, k_n)$. Then the set $\{J \in \mathbb{R}^n$ such that $|k\bar{\mathcal{B}}(I_1)J| < \alpha\}$ is the space between two hyperplanes orthogonal to $(k_1\mathcal{B}(I_1), k_2, \dots, k_n)$. Adding the term $k\bar{\mathcal{A}}(I_1)$ only applies a transition to the previous set. Let us find what is the separation between the hyperplanes. Assume J is parallel to $(k_1\mathcal{B}(I_1), k_2, \dots, k_n)$ with lengths a:

$$J = a\frac{(k_1\mathcal{B}(I_1), k_2, \dots, k_n)}{|k|_{2,\omega}},$$

where $|k|_{2,\omega} = \sqrt{B(I_1)^2k_1^2 + k_2^2 + \dots k_n^2}$. Then,

$$\begin{aligned} J \cdot (B(I_1), k_1, \dots, k_n) &= c(B(I_1)k_1^2 + k_2^2 + \dots k_n^2)\tfrac{1}{|k|_{2,\omega}} \\ &= a|k|_{2,\omega} \leq \alpha \Leftrightarrow a \leq \tfrac{\alpha}{|k|_{2,\omega}}. \end{aligned}$$

And finally,

$$\text{meas}\,(F \cap \Delta_{c,\hat{q}}(k,\alpha)) \leq (\text{diam}F)^{n-1}\frac{2\alpha}{|k|_{2,\omega}}.$$

The previous formula can not be applied if when we are at Z and

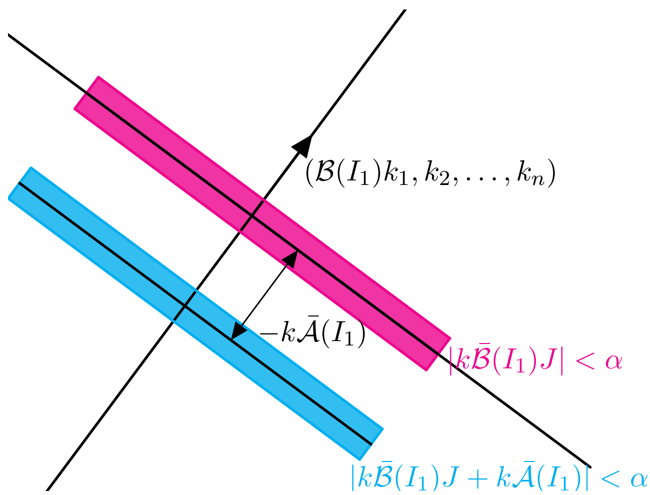


Figure 7.1: Graphical representation of the set $\Delta_{c,\hat{q}}(\alpha)$

$k = (k_1, 0, \dots, 0)$. At Z,

$$\Delta_{c,\hat{q}}(K,\alpha) = \{J \in \mathbb{R}^n \text{ such that } |\bar{K}\bar{J} + k_1\frac{\hat{q}_m}{c_m}| < \alpha\}.$$

And if $k = (k_1, 0, \dots, 0)$ then

$$\Delta_{c,\hat{q}}(K, \alpha) = \{J \in \mathbb{R}^n \text{ such that } |k_1 \frac{\hat{q}_m}{c_m}| < \alpha\}.$$

Then

$$\Delta_{c,\hat{q}}(k, \alpha) = \begin{cases} \mathbb{R}^n & \text{if } |k_1| < \alpha \frac{c_m}{\hat{q}_m} = \alpha \mathcal{K}', \\ \{\emptyset\} & \text{if } |k_1| \geq \alpha \frac{c_m}{\hat{q}_m} = \alpha \mathcal{K}'. \end{cases}$$

Using this last identity, the statement we wanted to prove is immediate.

□

Definition 7.2.23. *$G - b := \{I \in G$ such that $\mathcal{U}_b(I) \subset G\}$, where $\mathcal{U}_b(I)$ is the ball of radius b centered at I.*

Definition 7.2.24. *F is a D-set if* $\operatorname{meas}[(F - b_1) \setminus (F - b_2)] \leq D(b_2 - b_1)$.

Lemma 7.2.25. *Let $F \subset \mathbb{R}^n$ be a D-set for $d \geq 0$, $\tau > 0$, $\beta \geq 0$ and $k \geq 0$ an integer. Consider the set*

$$F(d, \beta, K) := (F - d) \setminus \bigcup_{\substack{k \in \mathbb{Z}^n \setminus \{0\} \\ |k|_1 \leq K}} \Delta_{c,\hat{q}} \left(k, \frac{\beta}{|k|_1^\tau} \right).$$

Then, outside of Z:

1. *If $d' \geq d$, $\beta' \geq \beta$, $k' \geq k$, then*

$$\operatorname{meas}[F(d, \beta, k) \setminus F(d', \beta', k')] \leq$$

$$D(d' - d) + 2(\operatorname{diam} F)^{n-1} \left(\sum_{\substack{k \in \mathbb{Z}^n \setminus \{0\} \\ |k|_1 \leq K}} \frac{\beta' - \beta}{|k|_1^\tau |k|_{2,\omega}} + \sum_{\substack{k \in \mathbb{Z}^n \setminus \{0\} \\ 0 < |k|_1 \leq K}} \frac{\beta'}{|k|_1^\tau |k|_{2,\omega}} \right)$$

2. *For every $b \geq 0$*

$$\operatorname{meas}[F(d, \beta, K) \setminus (F(d, \beta, K) - b)] \leq (D + 2^{n+1} (\dim F)^{n-1} K^n) b$$

And inside of Z, if we assume $\beta \leq \frac{1}{K'}$, the equation 1 holds adding only the terms $\bar{k} \neq 0$ and 2 holds without any change.

Proof. Recall that

$$\Delta_{c,\hat{q}}\left(k, \frac{\beta}{|k|_1^\tau}\right) = \left\{J \in \mathbb{R}^n \text{ such that } \left|k\bar{\mathcal{B}}(I_1)J + k\bar{\mathcal{A}}(I_1)\right| < \frac{\beta}{|k|_1^\tau}\right\}.$$

First we will prove the results outside of Z and then

1. Let us expand the expression of meas$[F(d,\beta,k) \setminus F(d',\beta',k')]$:

$$\left[(F-d) \setminus \bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\} \\ |k|_1\leq K}} \Delta_{c,\hat{q}}\left(k, \frac{\beta}{|k|_1^\tau}\right)\right] \setminus \left[(F-d) \setminus \bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\} \\ |k|_1\leq K}} \Delta_{c,\hat{q}}\left(k, \frac{\beta}{|k|_1^\tau}\right)\right].$$

Now we use the following property on the previous expression:

$$\begin{aligned}(A \setminus B) \setminus (C \setminus D) &= [(A \setminus B) \setminus C] \cup [(A \setminus B) \cap D] \\ &\subset (A \setminus C) \cup [(A \setminus B) \cap D] \\ &= (A \setminus C) \cup (A \cap (D \setminus B)),\end{aligned}$$

where the last equality is true because $D \supset B$. Using this property we have that meas$[F(d,\beta,k) \setminus F(d',\beta',k')]$ is included in

$$[(F-d) \setminus (F-d')] \;\cup\; \left[(F-d) \cap \left[\left(\bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\} \\ |k|_1\leq K'}} \Delta_{c,\hat{q}}\left(k, \frac{\beta'}{|k|_1}\right)\right)\right.\right.$$
$$\left.\left.\setminus \left(\bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\} \\ |k|_1\leq K}} \Delta_{c,\hat{q}}\left(k, \frac{\beta}{|k|_1}\right)\right)\right]\right].$$

And this expression is equivalent to:

$$[(F-d)\setminus(F-d')] \quad \cup \bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\|k|_1\le K}} \left((F-d)\cap\left(\Delta_{c,\hat q}\left(k,\frac{\beta'}{|k|_1^\tau}\right)\right.\right.$$
$$\left.\left.\setminus\Delta_{c,\hat q}\left(k,\frac{\beta}{|k|_1^\tau}\right)\right)\right)$$
$$\cup \bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\K<|k|_1\le K'}} \left((F-d)\cap\Delta_{c,\hat q}\left(k,\frac{\beta'}{|k|_1^\tau}\right)\right).$$

Now, using lemma 7.2.22 we obtain:

$$\mathrm{meas}(F(d,\beta,K)\setminus F(d',\beta',K'))\le$$
$$\le D(d'-d)+(\mathrm{diam}F)^{n-1}\left(\sum_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\|k|_1\le K}}\frac{2(\beta'-\beta)}{|k|_1^\tau|k|_{2,\omega}}+\sum_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\K<|k|_1\le K'}}\frac{2\beta'}{|k|_1^\tau|k|_{2,\omega}}\right)$$

2. Observe that:

$$\begin{aligned}F(d,\beta,K)-b &= \left[(F-d)\setminus\bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\|k|_1\le K}}\Delta_{c,\hat q}\left(k,\frac{\beta}{|k|_1^\tau}\right)\right]-b\\ &\supset (F-(d+b))\setminus\bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\|k|_1\le K}}\Delta_{c,\hat q}\left(k,\frac{\beta}{|k|_1^\tau}+b|k|_{2,\omega}\right).\end{aligned}$$

Then,

$$\begin{aligned}&\mathrm{meas}[(F(d,\beta,K))\setminus(F(d,\beta,K)-b)]\\ \le\ &\mathrm{meas}\left[\left((F-d)\setminus\textstyle\bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\|k|_1\le K}}\Delta_{c,\hat q}\left(k,\frac{\beta}{|k|_1^\tau}\right)\right)\setminus\right.\\ &\left.\left((F-(d+b))\setminus\textstyle\bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\|k|_1\le K}}\Delta_{c,\hat q}\left(k,\frac{\beta}{|k|_1^\tau}\right)\right)\right]\\ \le\ &\mathrm{meas}\left[(F-d)\setminus(F-(d+b))\cup\right.\end{aligned}$$

$$
\begin{array}{rl}
& \bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ |k|_1\leq K}} \left((F-d)\cap\left(\Delta_{c,\hat{q}}\left(k,\frac{\beta}{|k|_1^\tau}+b|k|_{2,\omega}\right)\right)\right)\setminus\left(\Delta_{c,\hat{q}}\left(k,\frac{\beta}{|k|_1^\tau}\right)\right)\Bigg] \\
\leq & Db+\sum_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ |k|_1\leq K}}(\mathrm{diam}F)^{n-1}\frac{2b|k|_{2,\omega}}{|k|_{2,\omega}} \\
\leq & Db+2^nK^n(\mathrm{diam}F)^{n-1}\cdot 2 = Db+2^{n+1}K^n(\mathrm{diam}F)^{n-1},
\end{array}
$$

where in the last inequality we used that the number of vectors k such that $|k|_1\leq K$ is less or equal than 2^nK^n.

The previous identities worked outside of Z. Let us understand the set $F(d,\beta,K)$ when we are ate Z.

$$
\begin{array}{rcl}
F(d,\beta,K) & := & (F-d)\setminus\bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ |k|_1\leq K}}\Delta_{c\hat{q}}(k,\frac{\beta}{|k|_1^\tau}) \\
& = & (F-d)\setminus\left[\left(\bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ |k|_1\leq K\\ \bar{k}\neq 0}}\Delta_{c\hat{q}}(k,\frac{\beta}{|k|_1^\tau})\right)\right. \\
& & \left.\cup\left(\bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ |k|_1\leq K\\ \bar{k}=0}}\Delta_{c\hat{q}}(k,\frac{\beta}{|k|_1^\tau})\right)\right] \\
& = & (F-d)\setminus\left[\left(\bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ |k|_1\leq K\\ \bar{k}\neq 0}}\Delta_{c\hat{q}}(k,\frac{\beta}{|k|_1^\tau})\right)\right. \\
& & \left.\cup\left(\bigcup_{\substack{k_1\in\mathbb{Z}\setminus\{0\}\\ |k|_1\leq\frac{\beta}{|k_1|^\tau}\mathcal{K}'}}\mathbb{R}^n\right)\right].
\end{array}
$$

Note that if for some $k_1\in\mathbb{Z}\setminus\{0\}$, $|k|_1\geq\frac{\beta}{|k|_1^\tau}\mathcal{K}'$, we take out all the possible frequencies. Then seems natural to ask $|k|_1\geq\frac{\beta}{|k|_1^\tau}\mathcal{K}'$ for all $k_1\in\mathbb{Z}\setminus\{0\}$, which holds if and only if $|k_1|^{1+\tau}\geq\beta K'$ for all $k_1\in\mathbb{Z}\setminus\{0\}$ or simply $\beta\leq\frac{1}{\mathcal{K}'}$ which we assumed. Then

$$
F(d,\beta,K) := (F-d)\setminus\bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ |k|_1\leq K\\ \bar{k}\neq 0}}\Delta_{c\hat{q}}(k,\frac{\beta}{|k|_1^\tau}).
$$

Hence we can replicate the proof of 1 only with the terms $\bar{k}\neq 0$. And the bound of 2 can be slightly improved by using that the number of vectors $k \in \mathbb{Z}^n \setminus \{0\}$ such that $|k|_1 \leq K$ and $|\bar{k}| \neq 0$ is bounded by $2^n K^n - K$, but since it is not a big improve, for the sake of simplicity we assume the bound 2 at Z.

□

Lemma 7.2.26. *Let $G \subset \mathbb{R}^n$ be compact. $u, \tilde{u} : G \to \mathcal{R}^n$ maps of class $\mathcal{C}^2$. $|\tilde{u} - u| \leq \varepsilon$. Assume that u is one-to-one on G, let $F = u(G)$. Consider the following bounds:*

$$\left|\frac{\partial u}{\partial I}\right|_G \leq M, \left|\frac{\partial u}{\partial I}(I) \cdot v\right| \geq \mu|v| \quad \forall v \in \mathbb{R}^n, \forall I \in G,$$

$$\left|\frac{\partial \tilde{u}}{\partial I}\right|_G \leq \tilde{M}, \left|\frac{\partial \tilde{u}}{\partial I^2}\right|_G \leq \tilde{M}_2, \left|\frac{\partial \tilde{u}}{\partial I}(I)v\right| \geq \tilde{\mu}|v| \quad \forall v \in \mathbb{R}^n, \forall I \in G,$$

$\tilde{\mu} < \mu$ and $\tilde{M} < M$. Assume $\varepsilon \leq \tilde{m}u^2/(4\tilde{M}_2)$. Then, given a subset $\tilde{F} \subset F - \frac{4M\varepsilon}{\tilde{\mu}}$ and writing $\tilde{G} = (\tilde{u})^{-1}(\tilde{F})$, the map $\tilde{u}$ is one-to-one from $\tilde{G}$ to $\tilde{F}$ and

$$\tilde{G} \subset G - \frac{2\epsilon}{\tilde{\mu}}, \quad u(\tilde{G}) \supset \tilde{F} - \varepsilon.$$

Moreover,

$$|(\tilde{u})^{-1} - u^{-1}|_{\tilde{F}} \leq \frac{\varepsilon}{\mu}$$

Proof. The statement is not any different than the classical one, so we are not going to prove it in here. A proof can be found in [19]. □

Lemma 7.2.27 (Inductive lemma)**.** *Let $G \subset \mathbb{R}^n$ be a compact.*

$$H(\phi, I) = \hat{h}(I) + R(\phi, I)$$

where $\hat{h}$ is defined as in 7.1 in the domain $\mathcal{D}_\rho(G)$,and $R(\phi, I)$ analytic on the same domain. Let $\hat{u} = \frac{\partial \hat{h}}{\partial I}$ and $u = \frac{\partial h}{\partial I}$. Assume that $|\frac{\partial}{\partial I} u|_{G,\rho_2} \leq M'$ and $|u|_G \leq L$. Also, assume that u is non-degenerate:

$$\left|\frac{\partial u}{\partial I} v\right| \geq \mu|v| \quad \forall I \in \mathcal{G}.$$

Let $\tilde{M} > M$, $\tilde{L} > L$ and $\tilde{\mu} < \mu$. Assume u is one-to-one on G and denote $F = u(G)$. Assume $\tau > 0$, $0 < \beta \leq 1$ and K given. Assume also that

$$F \cap \Delta_{c,\hat{q}}\left(K, \frac{\beta}{|k|_1^\tau}\right) = \emptyset, \quad \forall k \in \mathbb{Z}^n, |k|_1 \leq K, k \neq 0.$$

Denote $\epsilon := \|DR\|_{G,\rho,c}$, $\eta := |R_0|_{G,\rho_2}$ and $\xi := \left|\frac{\partial R_0}{\partial I}\right|_{G,\rho_2}$.

1. $\rho_2 \leq \frac{\beta}{2MK^{\tau+1}}$
2. $\epsilon \leq \min\left(\frac{\beta\hat{\delta}_c}{74AK^\tau}, \frac{\tilde{\mu}^2(\rho_2-\delta_2)}{4\tilde{M}}\right)$
3. $\xi \leq \min\left((\tilde{M}-M)\delta_2/\mathcal{R}, (\mu-\tilde{\mu})\rho_2\right)$

Then there exists a real canonical transformation

$$\Phi : \mathcal{D}_{\rho-\frac{\delta}{2}}(G) \to \mathcal{D}_\rho(G)$$

and a decomposition $H \circ \Phi = \tilde{\hat{h}}(I) + \tilde{R}(\phi, I)$. Writing $\tilde{u} = \frac{\partial}{\partial I}\tilde{h}$ one has.

1. $|\tilde{u} - u|_{G,\rho_2} = \xi, \quad |\tilde{h} - h|_{G,\rho_2} = \eta,$
2. $\tilde{\epsilon} := \|D\tilde{R}\|_{G,\rho-\delta,c} \leq e^{-K\delta_1}\epsilon + \frac{14AK^\tau}{\beta\hat{\delta}_c}\epsilon^2,$
3. $\tilde{\eta} := |\tilde{R}_0|_{G,\rho_2-\frac{\delta_2}{2}} \leq \frac{7AK^\tau}{c\beta}\epsilon^2,$
4. $|\Phi - Id|_{G,\rho-\frac{\delta}{2},c} \leq \frac{2AK^\tau}{\beta}\epsilon,$
5. $\left|\frac{\partial}{\partial I}\tilde{u}\right|_{G,\rho_2} \leq \tilde{M}'$, $|\tilde{u}|_G \leq \tilde{L},$
6. $|\frac{\partial\tilde{u}}{\partial I}v| \geq \tilde{\mu}|v| \quad \forall I \in \mathcal{G},$
7. *Given a subset $\tilde{F} \subset F - \frac{4M\epsilon}{\tilde{\mu}}$, $\tilde{G}(\tilde{u})^{-1}(\tilde{F})$ the map $\tilde{u}$ is one-to-one from $\tilde{G}$ to $\tilde{F}$, $\tilde{G} \subset G - \frac{2\epsilon}{\tilde{\mu}}$, $u(\tilde{G}) \supset \tilde{F} - \epsilon$. Moreover $|\tilde{u}^{-1} - u^{-1}|_{\tilde{F}} \leq \epsilon/\mu$.*

Proof. The set $u(I)$ is $\beta/K^\tau, K$-non-resonant with respect to ω. This implies that

$$|k_1\mathcal{B}(I_1)u_1 + \bar{k}\bar{u} + \mathcal{A}(I_1)u_1| \geq \beta/K^\tau. \geq \frac{\beta}{|k|_1^\tau} \geq \frac{\beta}{K^\tau}. \tag{7.12}$$

Then $\rho_2 \leq \frac{\beta/K^\tau}{2MK} = \frac{\beta}{2MK^{\tau+1}}$, $\|DR\|_{G,\rho,c} \leq \frac{\beta/K^\tau \hat{\delta}_c}{74A} = \frac{\beta\hat{\delta}_c}{74AK^\tau}$. We apply the iterative lemma (Theorem 7.2.20) to obtain $\Phi : \mathcal{D}_{\rho-\frac{\delta}{2}}(G) \to \mathcal{D}_\rho(G)$, such that $H \circ \Phi = \tilde{h} + \tilde{R}$ where $\tilde{h} = h + R_0$.

We have taken out the points that are not $\beta/K^\tau, K$-non-resonant with respect to ω. Because of conditions 1 and 2 we can apply the Iterative lemma. Now let us prove each of the points in the statement.

1. We know by definition that $\tilde{u} = \frac{\partial\tilde{h}}{\partial I} = \frac{\partial(h+R_0)}{\partial I} = \frac{\partial h}{\partial I} + \frac{R_0}{\partial I}$, hence:

$$|\tilde{u} - u|_{G,\rho_2} = |\frac{\partial h}{\partial I} + \frac{\partial R_0}{\partial I} - \frac{\partial h}{\partial I}|_{G,\rho_2} = |\frac{\partial R_0}{\partial I}|_{G,\rho_2} = \xi$$

$$\tilde{h} = h + R_0 \Rightarrow |\tilde{h} - h|_{G,\rho_2} = |h + R_0 - h|_{G,\rho_2} = |R_0|_{G,\rho_2} = \eta$$

2. By the iterative lemma:

$$\begin{aligned} \|D\tilde{R}\|_{G,\rho-\delta,c} &\leq e^{-K\delta_1}\|DR\|_{G,\rho,c} + \tfrac{14A}{\alpha\hat{\delta}_c}\|DR\|_{G,\rho,c} \\ &\leq e^{-K\delta_1}\varepsilon + \tfrac{14A}{\alpha\hat{\delta}_c}\varepsilon^2 \\ &= e^{-K\delta_1}\varepsilon + \tfrac{14AK^\tau}{\beta\hat{\delta}_c}\varepsilon^2, \end{aligned}$$

where we have used that $\alpha = \frac{\beta}{K^\tau}$.

3. At this point we use an inequality used in the proof of the iterative Lemma (theorem 7.2.20).

$$\begin{aligned} |\tilde{R}_0|_{G,\rho_2-\delta_2/2} &\leq |r_2(h,W,1) + r_1(R,W,1)|_{G,\rho_2-\delta_2/2} \\ &\leq \tfrac{7A}{\alpha c}\|DR\|^2_{G,\rho,c} = \tfrac{7AK^\tau}{\beta}\varepsilon^2. \end{aligned}$$

4. Also using the the iterative Lemma:

$$|\Phi - \mathrm{id}|_{G,\rho-\delta/2,c} \leq \frac{2A}{\alpha}\|DR\|_{G,\rho,c} = \frac{2AK^\tau}{\beta}\|DR\|_{G,\rho,c}.$$

5. Recall that $|\frac{\partial}{\partial I}A^\omega\tilde{u}|_{G,\rho_2-\delta_2} \leq \tilde{M}$, $|\tilde{u}|_G \leq \tilde{L}$, $\tilde{h} = h+R_0$, $|\frac{\partial}{\partial I}A^\omega u|_{G,\rho_2} \leq M$, $|u|_G \leq L$. Note that $\mathcal{A}(I_1) \leq m \cdot \max_j(q_j)/\min_j(c_j)$ and $\mathcal{B}(I_1) \leq 1/\min_j(c_j)$. Hence $\mathcal{A}(I_1)+\mathcal{B}(I_1) \leq \max_j(q_j)/\min_j(c_j)+1/\min_j(c_j) := \mathcal{R}$, and we have that $|A^\omega| \leq \mathcal{R}$.

$$\begin{aligned}
|\tfrac{\partial}{\partial I}A^\omega\tilde{u}|_{G,\rho_2-\delta_2} &= |\tfrac{\partial}{\partial I}A^\omega\tilde{u} + \tfrac{\partial}{\partial I}A^\omega u - \tfrac{\partial}{\partial I}A^\omega u|_{G,\rho_2-\delta_2} \\
&\leq |\tfrac{\partial}{\partial I}A^\omega(\tilde{u}-u)|_{G,\rho_2-\delta_2} + |\tfrac{\partial}{\partial I}A^\omega u|_{G,\rho_2-\delta_2} \\
&\leq |\tfrac{\partial}{\partial I}A^\omega R_0|_{G,\rho_2-\delta_2} + M \\
&\leq \tfrac{|A^\omega|_{G,\rho_2}|R_0|_{G,\rho}}{\delta_2} + M \\
&\leq \tfrac{|A^\omega|_{G,\rho_2}\cdot\xi}{\delta_2} + M \\
&\leq \tfrac{\mathcal{R}\xi}{\delta_2} + M \\
&\leq \frac{(\frac{(\tilde{M}-M)\delta_2}{\mathcal{R}})\mathcal{R}}{\delta_2} + M \leq \tilde{M} - M + M = \tilde{M},
\end{aligned}$$

where $\xi \leq (\tilde{M}-M)\delta_2/\mathcal{R}$.

6. We know $|\frac{\partial u}{\partial I}(I)v| \geq \mu|v|$ for all $I \in \mathcal{G}$, then $|\frac{\partial u}{\partial I}(I)v|_G \geq \mu|v|$. We want to find $|\frac{\partial\tilde{u}}{\partial I}(I)v|_G \geq \mu'|v|$ if $\mu' < \mu$.

$$\begin{aligned}
|\tfrac{\partial\tilde{u}}{\partial I}v|_G &= |(\tfrac{\partial\tilde{u}}{\partial I} + \tfrac{\partial u}{\partial I} - \tfrac{\partial u}{\partial I})v|_G \\
&= |(\tfrac{\partial^2 R_0}{\partial I^2} + \tfrac{\partial u}{\partial I})v|_G \\
&\geq -|\tfrac{\partial^2 R_0}{\partial I^2}v|_G + |\tfrac{\partial u}{\partial I}v|_G \\
&\geq \mu|v| - |\tfrac{\partial^2 R_0}{\partial I^2}|_G|v| \\
&\geq \mu|v| - |\tfrac{\partial R_0}{\partial I}|_G\tfrac{1}{\delta_2}|v| \\
&\geq \mu|v| - \tfrac{\xi}{\rho_2}|v| = (\mu - \xi/\rho_2)|v| \geq \mu'|v|,
\end{aligned}$$

where we have used that $|\frac{\partial^2 R_0}{\partial I^2}|_G \leq |\frac{\partial R_0}{\partial I}|\frac{1}{\rho_2}$, and also that $\mu' < \mu - \xi/\rho_2$, hence $\xi \leq (\mu-\mu')\rho_2$.

7. To apply lemma 7.2.26 we only need to check that $\varepsilon \leq \frac{\tilde{\mu}^2}{\tilde{M}_2}$. $\tilde{M}_2$ can be chosen such that $|\frac{\partial^2 u}{\partial I^2}|_G \leq \tilde{M}_2$. Note that $|\frac{\partial^2 u}{\partial I^2}|_G \leq |\frac{\partial^2 u}{\partial I^2}|_{G,\rho_2-\delta_2}$.

$$\begin{aligned}
|\tfrac{\partial u}{\partial I}|_{G,\rho_2-\delta_2} \leq \tilde{M} &\Rightarrow |\tfrac{\partial^2 u}{\partial I^2}|_{G,\rho_2-\delta_2}(\rho_2-\delta_2) \leq |\tfrac{\partial u}{\partial I}|_{G,\rho_2-\delta_2} \leq \tilde{M} \\
&\Rightarrow |\tfrac{\partial^2 u}{\partial I^2}|_{G,\rho_2-\delta_2} \leq \tfrac{\tilde{M}}{\rho_2-\delta_2} = \tilde{M}_2 \\
&\Rightarrow |\tfrac{\partial^2 u}{\partial I^2}|_G \leq \tilde{M}_2
\end{aligned}$$

Then $\varepsilon \leq \frac{\tilde{M}}{4\tilde{M}_2}$ if and only if $\varepsilon \leq \mu^2/(4\frac{\tilde{M}}{(\rho_2-\delta_2)})$ if and only if $\varepsilon \leq \frac{\mu^2(\rho_2-\delta_2)}{4\tilde{M}}$ which it is assumed in the statement.

□

7.3 KAM theorem on b^m-symplectic manifolds

Theorem 7.3.1 (b^m-KAM theorem)**.** *Let $\mathcal{G} \subset \mathbb{R}^n$, $n \geq 2$ be a compact set. Let $H(\phi, I) = \hat{h}(I) + f(\phi, I)$, where $\hat{h}$ is a b^m-function $\hat{h}(I) = h(I) + q_0 \log(I_1) + \sum_{i=1}^{m-1} \frac{q_i}{I_1^i}$ defined on $\mathcal{D}_\rho(G)$, with $h(I)$ and $f(\phi, I)$ analytic. Let $\hat{u} = \frac{\partial \hat{h}}{\partial I}$ and $u = \frac{\partial h}{\partial I}$. Assume $|\frac{\partial u}{\partial I}|_{G,\rho_2} \leq M$, $|u|_{\mathcal{G}} \leq L$. Assume that u is μ non-degenerate ($|\frac{\partial u}{\partial I} v| \geq \mu |v|$ for some $\mu \in \mathbb{R}^+$ and $I \in \mathcal{G}$. Take $a = 16M$. Assume that u is one-to-one on $\mathcal{G}$ and its range $F = u(\mathcal{G})$ is a D-set. Let $\tau > n-1, \gamma > 0$ and $0 < \nu < 1$. Let*

1.
$$\varepsilon := \|f\|_{\mathcal{G},\rho} \leq \frac{\nu^2 \mu^2 \hat{\rho}^{2\tau+2}}{2^{4\tau+32} L^6 M^3} \gamma^2, \tag{7.13}$$

2.
$$\gamma \leq \min\left(\frac{8LM\rho_2}{\nu \hat{\rho}^{\tau+1}}, \frac{L}{\mathcal{K}'}\right) \tag{7.14}$$

3.
$$\mu \leq \min(2^{\tau+5} L^2 M, 2^7 \rho_1 L^4 K^{\tau+1}, \beta \nu^{\tau+1} 2^{2\tau+1} \rho_1^\tau), \tag{7.15}$$

where $\hat{\rho} := \min\left(\frac{\nu \rho_1}{12(\tau+2)}, 1\right)$. Define the set $\hat{G} = \hat{G}_\gamma := \{I \in \mathcal{G} - \frac{2\gamma}{\mu} | u(I)$ is $\tau, \gamma, c, \hat{q}-$Dioph.$\}$. Then, there exists a real continuous map $\mathcal{T} : \mathcal{W}_{\frac{\rho_1}{4}}(\mathbb{T}^n) \times \hat{G} \to \mathcal{D}_\rho(\mathcal{G})$ analytic with respect the angular variables such that

1. *For all $I \in \hat{G}$ the set $\mathcal{T}(\mathbb{T}^n \times \{I\})$ is an invariant torus of H, its frequency vector is equal to $u(I)$.*

2. *Writing* $\mathcal{T}(\phi, I) = (\phi + \mathcal{T}_\phi(\phi, I), I + \mathcal{T}_I(\phi, I))$ *with estimates*

$$|\mathcal{T}_\phi(\phi, I)| \leq \frac{2^{2\tau+15} M L^2}{\nu^2 \hat{\rho}^{2\tau+1}} \frac{\varepsilon}{\gamma^2}$$
$$|\mathcal{T}_I(\phi, I))| \leq \frac{2^{10+\tau} L(1+M)}{\nu \hat{\rho}^{\tau+1}} \frac{\varepsilon}{\gamma}$$

3. *meas*$[(\mathbb{T}^n \times \mathcal{G}) \setminus \mathcal{T}(\mathbb{T}^n \times \hat{G})] \leq C\gamma$ *where* C *is a really complicated constant depending on* n*,* μ*,* D*, diam*F*,* M*,* τ*,* ρ_1*,* ρ_2*,* K *and* L*.*

Proof. This proof, as the one in [19] is going to be structured in six sections. First we define the parameters used in each iteration while building the diffeomorphism. After that, we prove that we can apply the inductive lemma 7.2.27 and we prove some bound that hold using the results of the inductive lemma. After that we find that the succession of frequency vectors and the succession of diffeomorphisms we built actually converge. Then we find estimates of the components of the canonical transformation we have built. Then we find a way to identify the invariant tori and finally we give a bound for the measure of the set fo invariant tori.

1. Choice of parameters

   We are going to make iterative use of proposition 7.2.27. So we need to properly define all the parameters in the statement for every iteration. Let:

$$\begin{cases} M_q &= (2 - \frac{1}{2^q})M, \\ L_q &= (2 - \frac{1}{2^q})L, \\ \mu_q &= (1 + \frac{1}{2^q})\frac{\mu}{2}. \end{cases}$$

   Note that M_q, L_q monotonically increase from M to $2M$ and L to $2L$ when $q \to \infty$. On the other hand μ_q monotonically decreases from μ to $\mu/2$. Also, let:

$$\begin{cases} K_0 &= 0, \\ K_q &= K \cdot q^{q-1}, q \geq 1, \end{cases}$$

where K is the minimum natural number such that is greater or equal than $1/\hat{\rho}$ and greater or equal than $(\frac{\nu\beta}{\mu 2^{2\tau+12}})^{1/\tau}$. Moreover $\beta := \gamma/L \leq 1$, and

$$\begin{cases} \rho^{(q)} &= (\rho_1^{(q)}, \rho_2^{(q)}), \\ \rho_1^{(q)} &= (1 + \frac{1}{2^{\nu q}})\frac{\rho_1}{4}, \\ \rho_2^{(q)} &= \frac{\nu\beta}{32MK_{q+1}^{\tau+1}}. \end{cases}$$

Notice that $\rho_1^{(q)}$ decreases monotonically from $\rho_1/2$ to $\rho_1/4$. Also, $\rho_2^{(q)}$ decreases to 0. We also denote:

$$\begin{cases} \delta_1^{(q)} &= \rho_1^{(q-1)} - \rho_1^{(q)}, \\ \delta_2^{(q)} &= \rho_2^{(q-1)} - \rho_2^{(q)}, \\ c_q &= \frac{\delta_2^{(q)}}{\delta_1^{(q)}}. \end{cases}$$

Note that

$$\begin{aligned} \delta_1^{(q)} &= \left(1 + \frac{1}{2^{\nu(q-1}}\right)\frac{\rho_1}{4} - \left(1 + \frac{1}{2^{\nu q}}\right)\frac{\rho_1}{4} \\ &= \left(\frac{1}{2^{\nu(q-1)}} - \frac{1}{2^{\nu q}}\right)\frac{\rho_1}{4} \\ &= \frac{1-1/2^\nu}{2^{\nu(q-1)}}\frac{\rho_1}{4}. \end{aligned}$$

Also, since $0 < \nu < 1$ then $\nu/2 \leq 1 - 1/2^\nu \leq \nu$. Using this in the previous equation we obtain:

$$\frac{\nu\rho_1}{2^{\nu(q-1)}8} \leq \delta_1^{(q)} \leq \frac{\nu\rho_1}{2^{\nu(q-1)}4}. \tag{7.16}$$

Also,

$$\begin{aligned} \delta_2^{(q)} &= \frac{\nu\beta}{32MK_q^{\tau+1}} - \frac{\nu\beta}{32MK_{q+1}^{\tau+1}} \\ &= \frac{\nu\beta}{32M(K2^{q-1})^{\tau+1}} - \frac{\nu\beta}{32M(K2^q)^{\tau+1}} \\ &= \frac{\nu\beta}{32M(K2^{q-1})^{\tau+1}}\left(1 - \frac{1}{2^{\tau+1}}\right). \end{aligned}$$

Also, since $\tau > 0$ then $1/2 \leq (1 - 1/2^{\tau+1}) \leq 1$. Using this in the previous equation:

$$\frac{\nu\beta}{64MK_q^{\tau+1}} \leq \delta_2^{(q)} \leq \frac{\nu\beta}{32MK_q^{\tau+1}}. \tag{7.17}$$

Using equations 7.16 and 7.17 we find bounds for c_q

$$\begin{cases} c_q & \leq & \dfrac{\left(\frac{\nu\beta}{32MK_q^{\tau+1}}\right)}{\left(\frac{\nu\rho_1}{2^{\nu(q-1)}}\right)} = \dfrac{\beta 2^{\nu(q-1)}}{4MK_q^{\tau+1}\rho_1}, \\ c_q & \geq & \dfrac{\left(\frac{\nu\beta}{64MK_q^{\tau+1}}\right)}{\left(\frac{\nu\rho_1}{2^{\nu(q-1)}4}\right)} = \dfrac{\beta 2^{\nu(q-1)}}{16MK_q^{\tau+1}\rho_1}. \end{cases}$$

Then, we also define

$$\begin{cases} \beta_q & = & (1 - \frac{1}{2^{\nu q}})\beta, \\ \beta'_q & = & \frac{\beta_q + \beta_{q+1}}{2}. \end{cases}$$

Observe that both β_q and β'_q tend to β. Also observe that $\beta'_q \geq \frac{\nu}{4}\beta$, because:

$$\begin{array}{rcl} \beta'_q & = & \frac{\beta_q + \beta_{q+1}}{2} \\ & = & \frac{\left(1 - \frac{1}{2^{\nu q}}\right) + \left(1 - \frac{1}{2^{\nu(q+1)}}\right)}{2}\beta \\ & = & \left(1 - \left(\frac{1 + \frac{1}{2^\nu}}{2^{\nu q}}\right)\frac{1}{2}\right)\beta \\ & \geq & \left(1 - (1 - 1/2^\nu)\frac{1}{2}\right)\beta \geq \frac{\nu}{4}\beta. \end{array}$$

Because K is the minimal natural number such that $K \geq 1/\hat{\rho}$ then $K \leq 2/\hat{\rho}$. Hence $\hat{\rho} \leq \frac{2}{K}$. Also

$$\frac{1}{\hat{\rho}^{\tau+1}} \geq \left(\frac{K}{2}\right)^{\tau+1}.$$

Recall that $\hat{\rho} = \min(\frac{\nu\rho_1}{12(\tau+2)}, 1)$ and, in particular, $\hat{\rho} \leq \nu\rho_1$ and $\hat{\rho} \leq 1$.

By definition $\gamma \leq \frac{8LM\rho_2}{\nu\hat{\rho}^{\tau+1}}$. And because $\beta = \gamma/L$:

$$\beta L \leq \frac{8LM\rho_2}{\nu\hat{\rho}^{\tau+1}} \leq \frac{8LM\rho_2 K^{\tau+1}}{\nu}.$$

Because we assumed $\varepsilon \leq \frac{\nu^2\mu^2\hat{\rho}^{2\tau+2}}{2^{4\tau+32}L^6M^3}\gamma^2$ then, using that $\gamma = L\beta$ and $\hat{\rho} \leq 2/K$:

$$\varepsilon \leq \frac{\nu^2\mu^2\left(\frac{2}{K}\right)^{2\tau+2}}{2^{4\tau+32}L^6M^3} \leq \frac{\nu^2\mu^2\beta^2}{2^{4\tau+30}L^4M^3K^{2\tau+2}}. \tag{7.18}$$

Also using again the assumption that $\varepsilon \leq \frac{\nu^2\mu^2\hat{\rho}^{2\tau+2}}{2^{4\tau+32}L^6M^3}\gamma^2$ we want to prove that

$$\varepsilon \leq \frac{\nu^3\rho_1\beta^2}{2^{2\tau+22}MK^{2\tau+1}}. \tag{7.19}$$

It is enough to check that:

$$\frac{\nu^2\mu^2\hat{\rho}^{2\tau+2}L^2\beta^2}{2^{4\tau+32}L^6M^3} \leq \frac{\nu^3\rho_1\beta^2}{2^{2\tau+22}MK^{2\tau+1}}$$

where we used $\gamma = L\beta$. Now using $\hat{\rho} \leq \nu\rho_1$ it suffices to see

$$\frac{\nu^2\mu^2\nu^{2\tau+2}\rho_1^{2\tau+2}L^2\beta^2}{2^{4\tau+32}L^6M^3} \leq \frac{\nu^3\rho_1\beta^2}{2^{2\tau+22}MK^{2\tau+1}},$$

which simplifies to

$$\frac{\mu^2\rho_1^{2\tau+2}}{2^{4\tau+10}L^4M^2} \leq \frac{1}{K^{2\tau+1}}.$$

Using that $K \geq 1/(\nu\rho_1)$ is enough to see that

$$\frac{\mu^2\rho_1^{2\tau+1}\nu^{2\tau+2}}{2^{2\tau+12}L^4M^2} \leq (\nu\rho_1)^{2\tau+1},$$

which holds if and only if $\mu \leq 2^{\tau+5}L^2M$ as we assumed.

2. Induction

Let us take $G_0 = \mathcal{G}$. The objective now is to construct a decreasing sequence of compacts $G_q \subset \mathcal{G}$ and a sequence of real analytic canonical transformations

$$\Phi^{(q)} : \mathcal{D}_{\rho^{(q)}}(G_q) \to \mathcal{D}_{\rho^{(q-1)}}(G_{q-1}), \quad q \geq 1.$$

Denoting $\Psi^{(q)} = \Phi^1 \circ \cdots \circ \Phi^{(q)}$ the transformed Hamiltonian functions will be noted by $H^{(q)} = H \circ \Psi^{(q)} = \hat{h}^{(q)}(I) + R^{(q)}(\phi, I)$. Moreover, $u^{(q)} = \frac{\partial h^{(q)}}{\partial I}$ and $\hat{u}^{(q)} = \frac{\partial \hat{h}^{(q)}}{\partial I}$.

We are going to show that the following bounds hold for all $q \geq 0$:

(a) $\varepsilon_q := \|DR^{(q)}\|_{G_q,\rho^{(q)},c_{q+1}} \leq \frac{8\varepsilon}{\nu\rho_1 2^{(2\tau+2)q}}$,

(b) $\eta_q := |R_0^{(q)}|_{G_q,\rho_2^{(q)}} \leq \frac{\varepsilon}{2^{(2\tau+3)q}}$ and $\xi_q := |\frac{\partial R_0^{(q)}}{\partial I}|_{G_q,\rho_2^{(q)}} \leq \frac{4MK^{\tau+1}\varepsilon}{\nu\beta 2^{(\tau+2)q}}$,

(c) $|\frac{\partial^2 h^{(q)}}{\partial I^2}|_{G_q,\rho_2^{(q)}} \leq M_q, \quad |u^{(q)}| \leq L_q \quad \forall I \in G_q$,

(d) $u^{(q)}$ is μ_q-non-degenerate on G_q,

(e) $u^{(q)}$ is one-to-one on G_q, and $u^{(q)}(G_q) = F_q$ where we define:

$$F_q := (F - \beta_q) \setminus \bigcup_{\substack{k \in \mathbb{Z}^n \setminus \{0\} \\ |k|_1 \leq K}} \Delta_{c_q,\hat{q}}(K, \frac{\beta_q}{|k|_1^\tau})$$

To prove this we proceed by induction. For $q = 0$:

$$\begin{cases} G_0 = \mathcal{G}, \\ h^{(0)} = h, \hat{h}^{(0)} = \hat{h}, \\ R^{(0)} = f. \end{cases}$$

Using the definitions from the previous point:

$$\begin{cases} \rho_1^{(0)} &= (1+1)\frac{\rho_1}{4} = \rho_1/2, \\ \rho_2^{(0)} &= \frac{\nu\beta}{32MK^{\tau+1}} \leq \frac{\rho_2}{2}, \end{cases}$$

where in the last inequality we have used that $\beta \leq \frac{8M\rho_2 K^{\tau+1}}{\nu}$ and hence $\rho_2 \geq \frac{\beta\nu}{8MK^{\tau+1}}$.

Then,

$$\varepsilon_0 = \|Df\|_{G,\rho(0),c_1} = \|Df\|_{G,\rho(1)+\delta(1)}.$$

Now, let us use that $|D\Upsilon|_{G,\rho-\delta,c} \leq \frac{2|\Upsilon|_{G,\rho,c}}{\hat{\delta}_c}$ while having in mind that $\hat{\delta}^{(1)}_{c_1} = \min(c_1\delta^{(1)}_1, \delta^{(1)}_2)$. Then,

$$\|Df\|_{G,\rho(1),c_1} \leq \frac{c_1|f|_{G,\rho(0)}}{\hat{\delta}_{c_1}} \leq \frac{|f|_{G,\rho(0)}}{\delta^{(1)}_1} \leq \frac{|f|_{G,\rho(0)}8}{\nu\rho_1} = \frac{8\varepsilon}{\nu\rho_1},$$

where we have used $\delta^{(1)}_1 \geq \frac{\nu\rho_1}{8\cdot 2^{\nu(1-1)}} = \frac{\nu\rho_1}{8}$. This proves the base case for 2a).

Let us prove now the base case for 2b). On one side $\eta_0 = |R^{(0)}_0|_{G_0,\rho_2(0)} \leq \frac{\varepsilon}{2^{(2\tau+3)0}} = \varepsilon$, which holds because $|R^{(0)}_0|_{G_0,\rho^{(0)}_2} \leq |R^{(0)}|_{\mathcal{G},\rho^{(0)}} = |f|_{\mathcal{G},\rho(0)} = \epsilon$. On the other hand $\xi_0 = |\frac{\partial R^{(0)}_0}{\partial I}|_{G_0,\rho^{(0)}_2} \leq |\frac{\partial R^{(0)}_0}{\partial I}|_{G_0,\rho_2-\rho_2/2} \leq \frac{1}{\rho_2/2}\|R_0\|_{G,\rho} \leq \frac{2\varepsilon}{\rho_2} \leq \frac{\varepsilon}{\rho^{(0)}_2}$, where we used that $\rho_2(0) \leq \rho_2/2 = \rho_2 - \rho_2/2$.

The base case of 2c) is immediate because $|\frac{\partial^2 h^{(0)}}{\partial I^2}|_{G_0,\rho^{(0)}_2} \leq |\frac{\partial^2 h}{\partial I^2}|_{\mathcal{G},\rho_2} = M = M_0$ and also $|u^{(0)}|_{G_0} = |u|_{\mathcal{G}} \leq L = L_0$.

The base case of 2d holds because $u^{(0)} = u$ is μ non-degenerate in $\mathcal{G} = G_0$.

The base case of 2e holds because $u^{(0)} = u$ is one-to-one in $G_0 = \mathcal{G}$ by hypothesis. $u^{(0)}(G_0) = F_0$ where $F_0 = (F - \beta_0) \setminus \{\emptyset\} = F$ because $K_0 = 0$ and $\beta_0 = 0$.

For $q \geq 1$, we assume the statements true for $q-1$ and we prove it for q. Let us apply proposition 7.2.27 (Inductive Lemma) to $H^{(q-1)} = h^{q-1} + R^{q-1}$ with K_q instead of K.

We have to be careful with the condition $F \cap \Delta_{c,\hat{q}}(k, \frac{\beta}{|k|^{\tau}_1}) = \emptyset$

$\forall k \in \mathbb{Z}^n, |k|_1 \leq K_q, k \neq 0$ and with the definition

$$F_{q-1} := (F - \beta_{q-1}) \setminus \bigcup_{\substack{k \in \mathbb{Z}^n \setminus \{0\} \\ |k|_1 \leq K}} \Delta_{c,\hat{q}}(k, \frac{\beta_{q-1}}{|k|_1^\tau}),$$

because the resonances have to be removed up to order K_q, not K_{q-1}. Let us define

$$F'_{q-1} := (F - \beta_{q-1}) \setminus \bigcup_{\substack{k \in \mathbb{Z}^n \setminus \{0\} \\ |k|_1 \leq K}} \Delta_{c,\hat{q}}(k, \frac{\beta'_{q-1}}{|k|_1^\tau})$$

where we simply replaced β_{q-1} for β'_{q-1} because $\Delta_{c,\hat{q}}(k, \frac{\beta_{q-1}}{|k|_1^\tau})$ makes no sense when $q = 1, \beta_{q-1} = 0$.

Accordingly let us define $G'_{q-1} := (u^{(q-1)})^{-1}(F'_{q-1})$. The conditions in proposition 7.2.27 are going to be satisfied with F'_{q-1}, β'_{q-1}, K_q, M_{q-1}, L_{q-1}, μ_{q-1},$\rho^{(q-1)}$, $\delta^{(q)}$, c_q, M_q, L_q, μ_q replacing $F, \beta, K, M, L, \mu, \rho, \delta, c, \tilde{M}, \tilde{L}, \tilde{\mu}$. And also $a = 16M \geq 8M_q$.

We are now going to check that 1, 2 and 3 are satisfied so we can apply proposition 7.2.27.

- 1 We want to see that $\rho_2^{(q-1)} \leq \frac{\beta'_{q-1}}{2M_q K_q^{\tau+1}}$. By definition $\rho_2^{q-1} = \frac{\nu\beta}{32MK_q^{\tau+1}} \leq \frac{4\beta'_{q-1}}{32MK_q^{\tau+1}} \leq \frac{\beta'_{q-1}}{8M_q K_q^{\tau+1}} \leq \frac{\beta_{q-1}}{2M_q K_q^{\tau+1}}$, where we used that $M_q \geq M$.
- 2 We want to see that $\varepsilon_{q-1} \leq \min\left(\frac{\beta_{q-1}\hat{\rho}_c^{(q)}}{74A_q K_{q-1}^\tau}, \frac{\mu_q^\tau(\rho_2^{(q-1)} - \delta_c^{(q-1)})}{4M_q}\right)$, where $A_q := 1 + \frac{2M_{q-1}c_q K_q^\tau}{\beta'_{q-1}}$.

  Notice that:

$$\begin{aligned}
A_q &:= 1+\tfrac{2M_{q-1}c_qK_q^\tau}{\beta'_{q-1}}\\
&\leq 1+\tfrac{8M_{q-1}c_qK_q^\tau}{\nu\beta}\\
&\leq 1+\tfrac{8M_{q-1}\beta 2^{\nu(q-1)}K_q^\tau}{4MK_q^{\tau+1}\rho_1\nu\beta}\\
&= 1+\tfrac{2M_{q-1}2^{\nu(q-1)}}{MK_q\rho_1\nu}\\
&= 1+\tfrac{2M_{q-1}2^{\nu(q-1)}}{MK2^{q-1}\rho_1\nu}\\
&\leq 1+\tfrac{4M2^{q-1}}{MK2^{q-1}\rho_1\nu}\\
&= 1+\tfrac{4}{K\rho_1\nu}\leq 1+4=5
\end{aligned}$$

First we will check that $\varepsilon_{q-1}\leq\frac{\beta_{q-1}\hat{\rho}_c^{(q)}}{74A_qK_{q-1}^\tau}$.
By induction hypothesis we know that $\varepsilon_{q-1}\leq\frac{8\varepsilon}{\nu\rho_1 2^{(2\tau+2)(q-1)}}$.
Hence it is enough to see

$$\frac{8\varepsilon}{\nu\rho_1 2^{(2\tau+2)(q-1)}}\leq\frac{\beta'_{q-1}\delta_2^{(q)}}{75\cdot 5K_q^\tau}.$$

Notice that

$$\begin{aligned}
\tfrac{\beta'_{q-1}\delta_2^{(q)}}{379K_q^\tau} &\geq \tfrac{\nu\beta}{4}\tfrac{\nu\beta}{64MK_q^{\tau+1}}\tfrac{1}{37K_q^\tau}\\
&= \tfrac{\nu^2\beta^2}{4\cdot 64\cdot 370}\tfrac{1}{MK_q^{2\tau+1}}\\
&= \tfrac{\nu^2\beta^2}{4\cdot 64\cdot 370\cdot M2^{(q-1)(2\tau+1)}K^{2\tau+1}}.
\end{aligned}$$

And this holds if the following is true:

$$\frac{8\varepsilon}{\nu\rho_1}\leq\frac{\nu^2\beta^2}{4\cdot 64\cdot 379\cdot K^{2\tau+1}M}\Leftrightarrow\varepsilon\leq\frac{\nu^3\beta^2\rho_1}{2^{12}135MK^{2\tau+1}}.$$

Which is true because in the previous section we have seen that $\varepsilon\leq\frac{\nu^3\rho_1\beta^2}{2^{2\tau+30}MK^{2\tau+1}}$
Let us now prove that $\varepsilon\leq\frac{\mu_q^2(\rho_2^{(q-1)}-\delta_2^{(q-1)})}{2M_q}$. First of all observe that $(\rho_2^{(q-1)}-\delta_2^{(q)})=\rho_2^{(q)}$. So, what we want to prove is equivalent to proving $\varepsilon_{q-1}\leq\frac{\mu_q^2\rho_2^{(q)}}{2M_q}$.
On the other hand, we know that $\varepsilon_{q-1}\leq\frac{8\varepsilon}{\nu\rho_1 2^{(2\tau+2)(q-1)}}$. And observe also that $\frac{\mu_q^2\rho_2^{(q)}}{2M_q}\geq\frac{(\mu/2)^2\frac{\nu\beta}{32MK^{\tau+1}}}{2M}=\frac{\mu^2\nu\beta}{2^8M^2K^{\tau+1}}$.

If are able to check that $\frac{8\varepsilon}{\nu\rho_1 2^{(2\tau+2)(q-1)}} \leq \frac{\mu\nu\beta}{2^8 M^2 K^{\tau+1}}$ we would be fine. The previous equation holds if and only if the following holds,

$$\varepsilon \leq \frac{\mu\nu^2\beta\rho_1 2^{(2\tau+2)(q-1)}}{2^{11}M^2K^{\tau+1}}.$$

If we knew beforehand that $\varepsilon \leq \frac{\mu\nu^2\beta\rho_1 2^{-(2\tau+2)}}{2^{11}M^2K^{\tau+1}} = \frac{\mu\nu^2\beta\rho_1}{2^{2\tau+13}M^2K^{\tau+1}}$ we would be done.

But we also know that

$$\epsilon \leq \frac{\nu^2\mu^2\beta^2}{2^{2\tau+30}L^4M^3K^{2\tau+2}}.$$

Then it is enough to check that

$$\frac{\nu^2\mu^2\beta^2}{2^{2\tau+30}L^4M^3K^{2\tau+2}} \leq \frac{\mu\nu^2\beta\rho_1}{2^{2\tau+13}M^2K^{\tau+1}}.$$

And this holds because $\mu \leq 2^7\rho_1 L^4 K^{\tau+1}$

– 3 Lastly we want to see that

$$\xi_{q-1} \leq \min((M_q - M_{q-1})\frac{\delta_2^{(q)}}{\mathcal{R}}, (\mu_{q-1} - \mu_q)\rho_2^{(q-1)}).$$

Observe that $\mathcal{R}$ does not depend on q because at each iteration $\hat{h}^{(q)}$ singular part is not modified. $\hat{h}^{(q)} = \hat{h}^{(q)} + R_0^{(q)}$ and R_0 is analytic depending only on the action coordinates. By induction hypothesis we know that

$$\xi_{q-1} = |\frac{\partial R_0^{(q)}}{\partial I}|_{G_q,\rho_2^{(q)}} \leq \frac{4MK^{\tau+1}\varepsilon}{\nu\beta 2^{(\tau+2)q}}.$$

We are going to check the two different inequalities separately

(a) $\xi_{q-1} \leq (M_q - M_{q-1})\frac{\delta_2^{(q)}}{\mathcal{R}}$. Note that $M_q = (2 - \frac{1}{2^q})M$, then $M_q - M_{q-1} = \frac{M}{2^q}$.

$$\delta_2^{(q)} \geq \frac{\nu\beta}{64M(K2^{q-1})^{\tau+1}} \geq \frac{\nu\beta}{64M(K2^q)^{\tau+1}}$$

$$= \frac{\nu\beta}{64MK^{\tau+1}}\frac{1}{2^{q\tau+q}}.$$

We deduce

$$(M_q - M_{q-1})\delta_2^{(q)} \geq \frac{\nu\beta}{64K^{\tau+1}}\frac{1}{2^{\tau q+2q}}.$$

Hence we only need to check that

$$\frac{4MK^{\tau+1}\varepsilon}{\nu\beta 2^{(\tau+2)q}} \leq \frac{\nu\beta}{64K^{\tau+1}}\frac{1}{2^{\tau q+2q}}.$$

The previous condition holds if and only if

$$\frac{4MK^{\tau+1}\varepsilon}{\nu\beta} \leq \frac{\nu\beta}{2^6K^{\tau+1}} \Leftrightarrow \varepsilon \leq \frac{\nu^2\beta^2}{2K^{2\tau+2}M}.$$

On the other hand, let us use again $\varepsilon \leq \frac{\nu^2\mu^2\beta^2}{2^{2\tau+30}L^4M^3K^{2\tau+2}}$. If we apply the condition $\mu \leq 2^{\tau+6}L^2M$ in the last expression we obtain:

$$\varepsilon \leq \frac{\nu^2\beta^2 2^{2\tau+12}L^4M^2}{2^{2\tau+30}L^4M^3K^{2\tau+2}} = \frac{\nu^2\beta^2}{2^8K^{2\tau+2}M}.$$

(b) $\xi_{q-1} \leq (\mu_{q-1} - \mu_q)\rho_2^{(q-1)}$.
Observe that

$$\mu_q = (1 + \frac{1}{2^q})\frac{\mu}{2},$$

$$(\mu_{q-1} - \mu_q) = ((1 + \frac{1}{2^{q-1}}) - (1 + \frac{1}{2^q}))\frac{\mu}{2} = (\frac{1}{2^{q-1}} - \frac{1}{2^q})\frac{\mu}{2}$$
$$= \left(\frac{2-1}{2^q}\right)\frac{\mu}{2} = \frac{1}{2^q}\frac{\mu}{2} = \frac{\mu}{2^{q+1}}$$

Also,

$$\rho_2^{(q-1)} = \frac{\nu\beta}{32MK_q^{\tau+1}} = \frac{\nu\beta}{32M(K2^{q-1})^{\tau+1}}$$

$$\geq \frac{\nu\beta}{32MK^{\tau+1}2^{q(\tau+1)}}.$$

Then,

$$(\mu_{q-1}-\mu_q)\rho_2^{(q-1)} \geq \frac{\mu}{2^{q+1}}\frac{\nu\beta}{32MK^{\tau+1}2^{q(\tau+1)}}.$$

Then we only have to check that

$$\begin{aligned} \frac{4MK^{\tau+1}\varepsilon}{\nu\beta 2^{\tau q+2q-2}} &\leq \frac{\mu}{2^{q+1}}\frac{\nu\beta}{32MK^{\tau+1}2^{q(\tau+1)}} \\ &= \frac{\mu}{2^{\tau q+2q+1}}\frac{\nu\beta}{32MK^{\tau+1}}. \end{aligned}$$

Which holds if and only if

$$\frac{MK^{\tau+1}\varepsilon}{\nu\beta 2^{-2}} \leq \frac{\mu}{2}\frac{\nu\beta}{32MK^{\tau+1}}.$$

Then,

$$\varepsilon \leq \frac{\mu 2^{-2}}{2}\frac{\nu^2\beta^2}{32M^2K^{2\tau+2}} = \frac{\mu\nu^2\beta^2}{2^8M^2K^{2\tau+2}}.$$

But we know

$$\begin{aligned} \varepsilon &\leq \frac{\nu^2\mu^2\beta^2}{2^{\tau+30}L^4M^3K^{2\tau+2}} \\ &\leq \frac{\nu^2\mu 2^{\tau+5}L^4M\beta^2}{2^{\tau+30}L^4M^3K^{2\tau+2}} \\ &= \frac{\nu^2\mu\beta^2}{2^25M^2K^{2\tau+2}} \\ &\leq \frac{\mu\nu^2\beta^2}{2^8M^2K^{2\tau+2}} \end{aligned}$$

as we wanted. In the second inequality we used that $\mu \leq 2^{\tau+5}L^4M$.

So, finally we can apply the inductive lemma 7.2.27 with the parameters mentioned previously in this section. Hence we obtain a canonical transformation $\Phi^{(q)}$ and a transformed hamiltonian $H^{(q)} = h^{(q)}+R^{(q)}$. The new domains $G_q \subset G'_{q-1}$ are going to be specified in the following lines. So now we are going to prove 2a,2b,2c,2d,2e.

- 2a. We want to see $\varepsilon_q := \|DR^{(q)}\|_{G_q,\rho^{(q)},c_{q+1}} \leq \frac{8\varepsilon}{\nu\rho_1 2^{(2\tau+2)q}}$. By the second result of proposition 7.2.27 we have:

$$\varepsilon_q \le e^{-K_q\delta_1^{(q)}}\varepsilon_{q-1} + \frac{14A_qK_q^\tau}{\beta'_{q-1}\delta_2^{(q)}}\varepsilon_{q-1}^2. \tag{7.20}$$

Now we are going to bound each term of the right hand of the expression at a time.
Recall that $\delta_1^{(q)} \ge \frac{\nu\rho_1}{82^{\nu(q-1)}}$.

$$\begin{aligned} K_q\delta_1^{(q)} &\ge K2^{q-1}\tfrac{\nu\rho_1}{8}2^{-\nu(q-1)} \\ &= \tfrac{\nu\rho_1}{8}K2^{(1-\nu)(q-1)} \\ &\ge \tfrac{12(\tau+2)}{8}2^{(1-\nu)(q-1)} \\ &= (3/2\tau+3)2(1-\nu)(q-1) \\ &\ge (2\tau+3)\tfrac{3}{4} \ge (2\tau+3)\ln 2, \end{aligned}$$

where we used that $K\hat{\rho} \ge 1$ and hence $K \ge \frac{12(\tau+2)}{\nu\rho_1}$. So we conclude that $e^{-K_q\delta_1^{(q)}} \le \frac{1}{2^{2\tau+3}}$, and we have bounded the first term of 7.20. Let us bound the second one.
On one hand we have that

$$\frac{14A_qK_q^\tau}{\beta'_{q-1}} \le \frac{14\cdot 5K_q^\tau}{\frac{\nu\beta}{4}} \le \frac{2^9K_q^2}{\nu\beta}$$

where we have used that $\beta'_q \ge \frac{\nu\beta}{4}$ and $A_q \le 5$.
Now we are going to apply that $\varepsilon_{q-1} \le \frac{8\varepsilon}{\nu\rho_1 2^{(2\tau+2)(q-1)}}$, $\delta_2^{(q)} \ge \frac{\nu\beta}{64MK_q^{\tau+1}}$ and $\epsilon \le \frac{\nu^3\rho_1\beta^2}{2^{2\tau+22}MK^{2\tau+1}}$ to obtain

$$\begin{aligned} \frac{14A_qK_q^\tau}{\beta'_{q-1}\delta_2^{(q)}}\varepsilon_{q-1} &= \frac{14A_qK_q^\tau}{\beta'_{q-1}}\frac{1}{\delta_2^{(q)}}\varepsilon_{q-1} \\ &\le \frac{2^9K_q^\tau}{\nu\beta}\frac{64MK_q^{\tau+1}}{\nu\beta}\frac{8\varepsilon}{\nu\rho_1 2^{(2\tau+2)(q-1)}} \\ &\le \frac{2^{18}MK_q^{2\tau+1}}{\nu^3\beta^2\rho_1 2^{(2\tau+2)(q-1)}}\frac{\nu^3\rho_1\beta^2}{2^{2\tau+22}MK^{2\tau+1}} \\ &\le 2^{18}2^{(q-1)(2\tau+1)-(2\tau+2)(q-1)-(2\tau+22)} \\ &= 2^{(1-q)}2^{-2\tau-4} = \frac{1}{2^{2\tau+3}2^{q-1}}. \end{aligned}$$

This gives us the bound of the second term of 7.20. Now we put both bounds together:

$$\varepsilon_q \leq \frac{1}{2^{2\tau+3}}\varepsilon_{q-1} + \frac{1}{2^{2\tau+3}}\frac{1}{2^{q-1}}\varepsilon_{q-1} \leq \frac{1}{2^{2\tau+2}}\varepsilon_{q-1}.$$

That implies $\varepsilon_q \leq \frac{\epsilon}{2^{(2\tau+2)(q-1)}}$ as we wanted. Because we can assume $\nu\rho_1 \leq 1$.

– 2b

Let us write $\sigma_2^{(q)} = \rho_2^{(q-1)} - \delta_2^{(q)}/2 = \rho_2^{(q)} + \delta_2^{(q)}/2 \geq \rho_2^{(q)}$, then $\eta_q = |R_0^{(q)}|_{G_q,\rho_2^{(q)}} \leq |R_0^{(q)}|_{G_q,\sigma_2^{(q)}}$.

By the inductive lemma 7.2.27:

$$\begin{aligned}
\eta_q &\leq \frac{7A_qK_q^\tau}{c_q\beta'_{q-1}}\varepsilon_{q-1}^2 \\
&\leq \frac{7A_qK_q^\tau}{\beta'_{q-1}}\varepsilon_{q-1}^2\frac{\delta_1^{(q)}}{\delta_2^{(q)}} \\
&= \frac{14A_qK_q^\tau}{\beta'_{q-1}\delta_2^{(q)}}\varepsilon_{q-1}^2\frac{\delta_1^{(q)}}{2} \\
&\leq \frac{1}{2^{2\tau+3}2^{q-1}}\varepsilon_{q-1}\frac{\delta_1^{(q)}}{2} \\
&\leq \frac{1}{2}\frac{\delta_1^{(q)}}{2^{2\tau+3}2^{q-1}}\frac{\varepsilon}{\nu\rho_1 2^{(2\tau+2)(q-1)}} \\
&\leq \frac{1}{2}\frac{\nu\rho_1}{4\cdot 2^{\nu(q-1)}}\frac{1}{2^{2\tau+3}2^{q-1}}\frac{8\varepsilon}{\nu\rho_1 2^{(2\tau+2)(q-1)}} \\
&\leq \frac{\varepsilon}{2^{(2\tau+3)q}}.
\end{aligned}$$

For the second part we only need to apply Cauchy inequalities:

$$\xi_q \leq \frac{2}{\delta_2^{(q)}}|R_0^q|_{G_q,\rho_2^{(q)}} \leq \frac{2}{\delta_2^{(q)}}\frac{\varepsilon}{2^{(2\tau+3)q}}.$$

– 2c and 2d are direct from lemma 7.2.27.

– 2e We need to consider again the results from lemma 7.2.27 with F_q as F. We have to check the condition $F_q \subset F'_{q-1} - \frac{4M_{q-1}\varepsilon_{q-1}}{\mu_q}$. Let us define $d_q := \frac{\beta_q-\beta_{q-1}}{2K_q^{\tau+1}}$. Using that $F'_{q-1} := (F-\beta_{q-1}) \setminus \bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ |k|_1\leq K}} \Delta_{c,\hat{q}}(k.\frac{\beta'_{q-1}}{|k|_1^\tau})$ we have

$$F'_{q-1}-d_q \supset (F-(\beta_{q-1}+d_q))\setminus \bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ |k|_1\leq K}} \Delta_{c,\hat{q}}(k,\frac{\beta'_{q-1}}{|k|_1^\tau}+|k|d_q).$$

Moreover,

$$\begin{cases} \beta_{q-1} + d_q & \leq & \beta_q, \text{ and} \\ \frac{\beta'_{q-1}}{|k|_1^\tau} + |k| d_q & = & \frac{\beta'_{q-1} + |k|_1^\tau |k| \frac{\beta_q - \beta_{q-1}}{2K_q^{\tau+1}}}{|k|_1^\tau} \\ & \leq & \frac{\beta'_{q-1} + K_q^{\tau+1} \frac{\beta_q - \beta_{q-1}}{2K_q^{\tau+1}}}{|k|_1^\tau} = \frac{\beta'_{q-1} + \frac{\beta_q}{2} - \frac{\beta_{q-1}}{2}}{|k|_1^\tau} = \frac{\beta_q}{|k|_1^\tau}. \end{cases}$$

Now if we see that $\frac{4M_{q-1}\varepsilon_{q-1}}{\mu_q} \leq d_q$ we will have the inclusion we want. Observe that $\frac{4M_{q-1}}{\mu_q} \leq \frac{4 \cdot 2M}{\mu/2} = \frac{16M}{\mu}$. So, it is enough to check that $\frac{16M}{\mu}\varepsilon_{q-1} \leq d_q$.

$$\begin{aligned} \varepsilon_{q-1} & \leq \frac{8\varepsilon}{\nu\rho_1 2^{(2\tau+2)q}} \\ & \leq \frac{8\nu^3\rho_1\beta^2}{\nu\rho_1 2^{(2\tau+2)q} 2^{(2\tau+22)} M K^{2\tau+1}} \\ & \leq \frac{8\nu^2\beta^2}{2^{(2\tau+2)q+2\tau+20} M K^{2\tau+1}} \\ & \leq \frac{8\nu^2\beta \frac{2(\beta_q - \beta_{q-1})}{\nu}}{2^{(\tau+1)q+(\tau+1)q+2\tau+20} M K^{2\tau+1}} \\ & = \frac{8\nu\beta 2(\beta_q - \beta_{q-1})}{2^{(\tau+1)+(\tau+1)q+2\tau+20} M K_q^{\tau+1} K^\tau} \\ & = \frac{8\nu\beta 2}{2^{(\tau+1)+(\tau+1)q+2\tau+19} M K^\tau} \frac{(\beta_q - \beta_{q-1})}{2K_q^{\tau+1}} \\ & = \frac{\nu\beta}{2^{(\tau+1)+(\tau+1)q+2\tau+15} M K^\tau} d_q \\ & \leq \frac{\nu\beta}{2^{3\tau+16} M K^\tau} dq. \end{aligned}$$

Hence, it is enough to prove the following:

$$\frac{16M}{\mu} \frac{\nu\beta}{2^{3\tau+16} M K^\tau} d_q \leq d_q.$$

Wich holds if an only if

$$\frac{16M}{\mu} \frac{\nu\beta}{2^{\tau+16} M K^\tau} \leq 1 \Leftrightarrow K^\tau \geq \frac{\nu\beta}{\mu 2^{\tau+12}},$$

which we assumed when choosing K.

3. Convergence of diffeomorphisms

   Now we are going to prove the convergence of the successive maps

   $u^{(q)} : G_q \to F_q$

i.e. we want to see that exist proper sets G^*, F^* and an analytical map u^* such that $u^{(q)} : G_q \to F_q$ converge to $u^* : G^* \to F^*$.

Let us use lemma 7.2.27 as before.

For $q \geq 1$ we obtain

$$|u^{(q)} - u^{(q-1)}|_{G_q} \leq \xi_q \quad \text{and} \quad |(u^{(q)})^{-1} - (u^{(q-1)})^{-1}|_{F_q} \leq \frac{\varepsilon_q}{\mu_q}.$$

Now, because the following two inequalities hold

$$\begin{cases} \xi_q & \leq & \frac{4MK^{\tau+1}\varepsilon}{\nu\beta 2^{(\tau+2)q}} \\ \frac{\varepsilon_q}{\mu_q} & \leq & \frac{8\varepsilon}{\nu\rho_1 2^{2\tau+2}q} \frac{1}{(1+\frac{1}{2^q})\frac{\mu}{2}} = \frac{8\varepsilon 2^{q-1}}{\nu\beta 2^{(2\tau+2)q}(2^q+1)\mu} \end{cases}$$

the sequences u^q and $(u^{(q)})^{-1}$ converge to maps u^* and Υ respectively. This maps are defined on the following sets:

$$\begin{array}{rcl} G^* & := & \bigcap_{q\geq 0} G_q, \\ F^* & := & \bigcap_{q\geq 0} F_q = (F-\beta) \setminus \bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\} \\ |k|_1\leq K}} \Delta_{c,\hat{q}}(k, \frac{\beta}{|k|_1^\tau}). \end{array}$$

The second equality holds because F^* is a compact for being intersection of compact sets. We can now deduce that

$$\begin{array}{rcl} |u^* - u^{(q)}|_{G^*} & \leq & \sum_{s\geq q} |u^{(q)} - u^{(q-1)}|_{G^*} \\ & \leq & \sum_{s\geq q} |u^{(q)} - u^{(q-1)}|_{G} \\ & \leq & \sum_{s\geq q} \xi_q. \end{array}$$

with the same argument we see that $|\Upsilon - (u^{(q)})^{-1}|_{F^*} \leq \ldots \leq \sum_{s\geq q} \frac{\varepsilon_q}{\mu_q}$.

The next steps are going to be to prove that $G_q \subset G_{q-1} - \frac{2\varepsilon_{q-1}}{\mu_{q-1}}$ and $F_q \subset F_{q-1} - \frac{4M_{q-1}\varepsilon_{q-1}}{\mu_{q-1}}$. If we check it and we take the limit we would have:

$$G^* \subset G_q - \sum_{s\geq q} \frac{2\varepsilon_q}{\mu_q} \quad \text{and} \quad F^* \subset F_q - \sum_{s\geq q} \frac{4M_q\varepsilon_q}{\mu_q}.$$

Let us first check $F_q \subset F_{q-1} - \frac{4M_{q-1}\varepsilon_{q-1}}{\mu_{q-1}}$. Let us define $x := \frac{4M_{q-1}}{\mu_{q-1}}$.

$$\begin{aligned} F_{q-1} - x &\supset (F - (\beta_{q-1} + x)) \setminus \bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\} \\ |k|_1\leq K}} \Delta_{c,\hat{q}}(k, \tfrac{\beta_{q-1}}{|k|_q^\tau} + |k|x) \\ &\supset (F - (\beta_{q-1} + x)) \setminus \bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\} \\ |k|_1\leq K}} \Delta_{c,\hat{q}}(k, \tfrac{\beta_{q-1}+K_q^{\tau+1}x}{|k|_q^\tau}). \end{aligned}$$

To have the inclusion we want, we have to check that:

(a) $\beta_{q-1} + x \leq \beta_q$.

(b) $\frac{\beta_{q-1}+K_q^{\tau+1}x}{|k|_1^\tau} \leq \frac{\beta_q}{|k|_1^\tau} \Leftrightarrow \beta_{q-1} + K_q^{\tau+1}x.$

Since the second one implies the first we will only check the second one.

$$\begin{aligned} \beta_{q-1} + K_q^{\tau+1}x &= \beta_{q-1} + K_q^{\tau+1}\tfrac{4M_{q-1}\varepsilon_{q-1}}{\mu_{q-1}} \\ &\leq \beta_{q-1} + K_q^{\tau+1}\tfrac{16M\varepsilon_{q-1}}{\mu} \\ &\leq \beta_{q-1} + K_q^{\tau+1}d_q \\ &= \beta_{q-1} + K_q^{\tau+1}\tfrac{\beta_q-\beta_{q-1}}{2K_q^{\tau+1}} \\ &= \beta_{q-1} - \beta_{q-1}/2 + \beta_q/2 \\ &= \tfrac{\beta_{q-1}+\beta_q}{2} \\ &= \beta_q \end{aligned}$$

Where we have used that $16M\varepsilon_q/\mu \leq d_q$ and that β_q is monotonically increasing with q.

The inclusion $G_q \subset G_{q-1} - \frac{2\varepsilon_{q-1}}{\mu_{q-1}}$ is given as a result of the lemma 7.2.27.

So we proved what we wanted. We are now going to see that u^*

is one-to-one on G^* and taht $u^*(G^*) = F^*$.

Tale $I \in G^*$, we have that $u^{(q)}(I) \in F_q$ for every q. Hence $u^*(I) \in F^*$, and we deduce that $u^*(G^*) \subset F^*$. With the same argument we see $\Upsilon(F^*) \subset G^*$. Let us prove that $\Upsilon(u^*(I)) = I$.

$$\begin{aligned}
|\Upsilon(u^*(I)) - I| &\leq |\Upsilon(u^*(I)) - (u^{(q)})^{-1}(u^*(I)) \\
&\quad +(u^{(q)})^{-1}(u^*(I)) - (u^{(q)})^{-1}(u^{(q)}(I))| \\
&\leq |\Upsilon(u^*(I)) - (u^{(q)})^{-1}(u^*(I))| \\
&\quad +|(u^{(q)})^{-1}(u^*(I)) - (u^{(q)})^{-1}(u^{(q)}(I))| \\
&\leq |\Upsilon - (u^{(q)})^{-1}|_{F^*} + \frac{1}{\mu_q}|u^* - u^{(q)}|_{G^*}.
\end{aligned}$$

Where to bound the second term we used the mean value theorem, i.e. $|u^{(q)}(x) - u^{(q)}(y)|_{G_q} \leq |\frac{\partial}{\partial I}u^{(q)}|_{G_q}|x-y|$, and the fact that because of the μ_q-nondegeneracy, $|\frac{\partial u^{(q)}}{\partial I}| \geq \mu_q|v|$, $\forall v \in \mathbb{R}^n$ and $\forall I' \in G_q$. Note that we can use the mean value theorem because $u^*(I) - u^{(q)}(I)$ belongs to F_q because $\frac{4M_q\varepsilon_q}{\mu_q} \geq \xi_q$. Let us prove this inequality. If we want to see $\frac{4M_q\varepsilon_q}{\mu_q} \geq \xi_q$, it is enough to see $\frac{4M\varepsilon_q}{\mu} \geq \xi_q$.

$$\begin{aligned}
\xi_q &\leq \frac{2}{\delta_2^{(q)}}|R_0^{(q)}|_{G_q,\sigma_2^{(q)}} \\
&\leq \frac{2}{\delta_2^{(q)}}\frac{\delta_1^{(q)}\varepsilon_{q-1}}{2}\frac{1}{2^{2\tau+3}2^{q-1}} \\
&= \frac{1}{c_q}\frac{1}{2^{2\tau+2}2^{q-1}}\varepsilon_{q-1} \leq \frac{4M}{\mu}\varepsilon_{q-1}
\end{aligned}$$

The last inequality is true if and only if

$$\begin{aligned}
\mu &\leq \frac{\beta 2^{\nu(q-1)}2^{2\tau+3}2^{q-1}}{K_q^{\tau+1}\rho_1 4} \\
&= \frac{\beta 2^{\nu(q-1)}2^{2\tau+3}2^{q-1}}{K^{\tau+1}2^{(\tau+1)(q-1)}\rho_1 4} \\
&\leq \frac{\beta 2^{2\tau+3}}{K^{\tau+1}\rho_1 4} \\
&= \frac{\beta 2^{2\tau+1}}{K^{\tau+1}\rho_1} \\
&\leq \frac{\beta 2^{2\tau+1}}{(\frac{1}{\nu\rho_1})^{\tau+1}\rho_1} = \beta\nu^{\tau+1}2^{2\tau+1}\rho_1^t au
\end{aligned}$$

as we assumed at the statement of the theorem. Since the bound obtained tends to 0, we have $\Upsilon(u^*(I)) = I$ and hence u^* is one-to-one. Analogously we obtain $u^*(\Upsilon(J)) = J \quad \forall J \in F^*$. Finally u^* is one-to-one and $u^*(G^*) = F^*$. Note also that from the inductive lemma we obtain $|h^{(q)} - h^{(q-1)}|_{G_q,\rho_2^{(q-1)}} \leq \eta_{q-1}$. Also observe the following bound that we are going to use in the next sections.

$$|u^* - u^{(q)}|_{G^*} \leq \sum_{s\geq q} \frac{4MK^{\tau+1}\varepsilon}{\nu\beta 2^{(\tau+2)s}}.$$

4. Convergence of the canonical transformations

Let $\sigma^{(q)} = \rho^{(q-1)} - \delta_2^{(q)}/2$. Observe that this definition implies that $\sigma^{(q)} - \rho^{(q)} = \delta_2^{(q)}$ and $\sigma^{(q)} - \delta_2^{(q)} = \rho^{(q)}$. Observe that applying the inductive lemma 7.2.27:

$$\begin{aligned}
|\Phi^{(q)} - \mathrm{id}|_{G_q,\sigma^{(q)},c_q} &\leq \frac{2A_{q-1}K_q^\tau}{\beta'_{q-1}}\varepsilon_{q-1} \\
&\leq \frac{2{\cdot}5{\cdot}4}{\nu\beta}\frac{8\varepsilon}{\nu\rho_1 2^{(2\tau+2)(q-1)}} \\
&\leq \frac{2^9K^\tau\varepsilon}{\nu^2\rho_1\beta 2^{(\tau+2)(q-1)}} \\
&\leq \frac{2^9K^\tau\nu^3\rho_1\beta^2}{\nu^2\rho_1\beta 2^{(\tau+2)(q-1)}2^{2\tau+22}MK^{2\tau+1}} \\
&\leq \frac{2^9\nu\beta}{2^{(\tau+2)(q-1)}2^{2\tau+20}MK^{\tau+1}} \\
&= \frac{\nu\beta}{2^6M(K2^{q-1})^{\tau+1}}\frac{2^9}{2^{(q-1)}2^{2\tau+14}} \\
&\leq \delta_2^{(q)}\frac{1}{2^{(q-1)}2^{2\tau+5}} \\
&\leq \frac{\delta_2^{(q)}}{2^{(q-1)}32},
\end{aligned}$$

where we have used that $\delta_2^{(q)} \geq \frac{\nu\beta}{84MK_qP\tau+1}$, $\varepsilon \leq \frac{\nu^3\rho_1\beta^2}{2^{2\tau+20}MK^{2\tau+1}}$, $\beta \leq \frac{8MK^{\tau+1}\rho_2}{\nu}$ and $\beta'_{q-1} \geq \frac{\nu\beta}{4}$.

Now, recall that $\hat{\delta}_c = \min(c\delta_1, \delta_2)$, then $\hat{\delta}_{c_q} = \min(c_q\delta_1^{(q)}, \delta_2^{(q)}) = \min(\delta_2^{(q)}, \delta_2^{(q)}) = \delta_2^{(q)}$.

Now using that $|D\Upsilon|_{G,\rho-\delta,c} \leq \frac{|\Upsilon|_{G,\rho,c}}{\hat{\delta}_c}$, we can obtain:

$$\begin{aligned}
|D\Phi^{(q)} - \mathrm{Id}|_{G_q,\rho^{(q)},c_q} &= |D(\Phi^{(q)}) - \mathrm{id}|_{G_q,\rho^{(q)},c_q} \\
&\leq |D(\Phi^{(q)}) - \mathrm{id}|_{G_q,\sigma^{(q)}-\delta_2^{(q)},c_q}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{|\Phi^{(q)} - \mathrm{id}|_{G_q,\sigma^{(q)},c_q}}{\hat{\delta}_{c_q}} \\
&\leq \frac{|\Phi^{(q)} - \mathrm{id}|_{G_q,\sigma^{(q)},c_q}}{\delta_2^{(q)}} \\
&\leq \frac{2|\Phi^{(q)} - \mathrm{id}|_{G_q,\sigma^{(q)},c_q}}{\delta_2^{(q)}} \\
&\leq \frac{2}{\delta_2^{(q)}} \frac{\delta_2^{(q)}}{2^{(q-1)} \cdot 32} \leq \frac{1}{2^{q-1}16} \leq \frac{1}{2^{(q-1)}4}
\end{aligned}
$$

Let x, y be such that the segment joining them is contained in $\mathcal{D}_{\rho^{(q)}}(G_q)$. Using the mean value theorem one can deduce the following bound:

$$
|\Phi^q(x) - \Phi^q(y)|_{c_q} \leq |D\Phi^{(q)}|_{G_q,\rho^{(q)},c_q} \cdot |x - y|_{c_q}.
$$

By 7.22, in particular $|\Phi^{(q)}(x) - x|_{c_q} \leq \delta_2^q$ and $|\Phi^{(q)}(y) - y|_{c_q} \leq \delta_2^q$. Then the segment that join $\Phi^{(q)}(x)$ and $\Phi^{(q)}(y)$ is contained in $\mathcal{D}_{\rho^{(q-1)}}(G_{q-1}) = \mathcal{D}_{\rho^{(q)} + \delta^{(q)}}$, because $G_q \subset G_{q-1} - \frac{2\varepsilon_{q-1}}{\mu_{q-1}}$ and because $\rho^{(q)} - \rho^{(q-1)} \leq \delta_2^{(q)}$ because $\rho^{(q)} - \rho^{(q-1)} = \delta_2^{(q)}$.

Therefore we can apply the mean value theorem once again:

$$
\begin{aligned}
&|\Phi^{(q-1)}(\Phi^{(q)}(x)) - \Phi^{(q-1)}(\Phi^{(q)}(y))|_{c_{q-1}} \\
&\leq |D\Phi^{(q-1)}|_{G_{q-1},\rho^{q-1},c_{q-1}} |\Phi^{(q)}(x) - \Phi^{(q)}(y)|_{c_{q-1}} \\
&\leq 2^{\tau+1-\nu} |D\Phi^{(q-1)}|_{G_{q-1},\rho^{q-1},c_{q-1}} |\Phi^{(q)}(x) - \Phi^{(q)}(y)|_{c_q},
\end{aligned}
$$

where we have used that $c_{q-1}/c_q = \frac{\delta_2^{(q-1)}/\delta_1^{(q-1)}}{\delta_2^{(q)}/\delta_1^{(q)}} = \frac{\delta_2^{(q-1)}}{\delta_2^{(q)}} \frac{\delta_1^{(q)}}{\delta_1^{(q-1)}} = 2^{\tau+1} \frac{1}{2^\nu} = 2^{\tau+1-\nu}$.

Using the previous bounds and iterating by q, we obtain the following:

$$
\begin{aligned}
&|\Psi^{(q)}(x) - \Psi^{(q)}(y)|_{c_1} \\
&\leq 2^{(\tau+1-\nu)(q-1)} |D\Phi^{(1)}|_{G_1,\rho^{(1)},c_1} \cdot \ldots \cdot |D\Phi^{(q)}|_{G_q,\rho^{(q)},c_q} |x - y|_{c_q} \\
&\leq 2^{(\tau+1-\nu)(q-1)} (1 + \tfrac{1}{4})(1 + \tfrac{1}{4\cdot 2}) \cdot \ldots \cdot (1 + \tfrac{1}{4\cdot 2^{q-1}}) |x - y|_{c_q} \\
&\leq 2^{(\tau+1-\nu)(q-1)} e^{1/2} |x - y|_{c_q} \leq 2^{(\tau+1-\nu)(q-1)} \cdot 2|x - y|_{c_q}.
\end{aligned}
$$

Which holds for $q \geq 1$ and for every x, y such that the segment joining them is contained in $\mathcal{D}_{\rho^{(q)}}(G_q)$. Now, given $q \geq 2$ and $x \in \mathcal{D}_{\rho^{(q)}}(G_q)$ let $y = \Phi^{(q)}(x)$:

$$\begin{aligned}
|\Psi^{(q)}(x) - \Psi^{(q-1)}(x)|_{c_1} &= |\Psi^{(q-1)}(\Phi^{(q)}(x)) - \Psi^{(q-1)}(x)|_{c_1} \\
&\leq 2^{(\tau+1+\nu)(q-2)} 2 |\Phi^{(q)}(x) - x|_{c_{q-1}} \\
&\leq 2^{(\tau+1+\nu)(q-1)} 2 |\Phi^{(q)}(x) - x|_{c_q} \\
&\leq 2^{(\tau+1+\nu)(q-1)} 2 \delta_2^{(q)} \\
&\leq 2^{(\tau+1+\nu)(q-1)} 2 \frac{2^8 K^\tau \varepsilon}{\nu^2 \rho_1 \beta 2^{(\tau+2)(q-1)}} \\
&= \frac{2^9 K^\tau \varepsilon}{\nu^2 \rho_1 \beta 2^{(1+\nu)(q-1)}}.
\end{aligned}$$

Which holds even for $q = 1$ by setting $\Psi^{(0)} = \mathrm{id}$ by 7.22. Hence 7.25 implies that $\Psi^{(q)}$ converges to a map

$$\Psi^* : D_{(\rho_1/4,0)}(G^*) = \mathcal{W}_{\frac{\rho_1}{4}}(\mathbb{T}^n) \times G^* \to \mathcal{D}_\rho(G).$$

And we deduce for every $q \geq 0$ that

$$|\Psi^* - \Psi^{(q)}|_{G^*,(\frac{\rho_1}{4},0),c_1} \leq \frac{2^1 0 K^\tau \varepsilon}{\nu^2 \rho_1 \beta 2^{(1+\nu)q}}.$$

Moreover by taking the limit to the equation

$$H \circ \Psi^{(q)} = h^{(q)} + R^{(q)}$$

we see that $H \circ \Psi^* = h^*(I)$ on $\mathcal{D}_{(\frac{\rho_1}{4},0)}(G^*)$.

5. Stability estimates

   Next we see that for $q \to \infty$, the motions associated to the transformed hamiltonian $\hat{H}^{(q)} = \hat{h}^{(q)} + R^{(q)}$ and the quasiperiodic motions of $\hat{h}^{(q)}$ become closer and closer.

   Let us denote

$$\begin{cases} x^{(q)}(t) = (\phi^{(q)}(t), I^{(q)}(t)) & \text{the trajectory of } H^{(q)}, \\ \hat{x}^{(q)}(t) = (\hat{\phi}^{(q)}(t), \hat{I}^{(q)}(t)) & \text{the trajectory of } \hat{H}^{(q)} \end{cases}$$

corresponding to a given initial condition $x^{(q)}(0) = x_0^* = (\phi_0^*, I_0^*) \in \mathbb{T}^n \times G_q$. Let

$$\begin{cases} \tilde{x}^{(q)}(t) & := & (\tilde{\phi}^{(q)}(t), I_0^*) = (\phi_0^* + u^{(q)}(I_0^*))t, I_0^*, \\ \hat{\tilde{x}}^{(q)}(t) & := & (\hat{\tilde{\phi}}^{(q)}(t), I_0^*) = (\phi_0^* + u'^{(q)}(I_0^*))t, I_0^* \end{cases}$$

the corresponding trajectories of the integrable parts of $h^{(q)}$ and $\tilde{h}^{(q)}$ respectively. Recall that $\hat{h}^{(q)}(I) = h^{(q)}(I) + \zeta^{(q)}(I_1) = h^{(q)}(I) + q_0 \log(I_1) + \sum_{i=1}^{m-1} q_i \frac{1}{I_1^i}$ and $u'^{(q)} = \bar{\mathcal{B}} u^{(q)} + \bar{\mathcal{A}}(I_1)$. It is clear that $\tilde{x}^{(q)}(t)$ and $\hat{\tilde{x}}^{(q)}(t)$ are defined for all $t \in \mathbb{R}$.

Let us denote:

$T_q = \inf\{t > 0 : |I^{(q)}(t) - I_0^*| > \delta_2^{(q+1)}$ or $|\phi^{(q)}(t) - \tilde{\phi}^{(q)}(t)|_\infty > \delta_1^{(q+1)}\}$. $\hat{T}_q = \inf\{t > 0 : |\hat{I}^{(q)}(t) - I_0^*| > \delta_2^{(q+1)}$ or $|\hat{\phi}^{(q)}(t) - \hat{\tilde{\phi}}^{(q)}(t)|_\infty > \delta_1^{(q+1)}\}$.

Observe that $x^{(q)}(t)$ and $\hat{x}^{(q)}(t)$ are defined and belong do $\mathcal{D}_{\rho^{(q)}}(G_q)$, for $0 \leq t \leq T_q$ and $0 \leq t \leq \hat{T}_q$ respectively, because $\delta^{(q)} \leq \rho^{(q)}$. Also recall the Hamiltonian equations. Let us first state the motion equations for our Hamiltonian function $\hat{H}^{(q)}$:

$$\iota_{X_{\hat{H}^{(q)}}} \omega = d\hat{H}^{(q)}, \quad \text{or} \quad X_{\hat{H}^{(q)}} = \Pi(d\hat{H}^{(q)}, \cdot).$$

Let us write

$$X_{\hat{H}^{(q)}} = \dot{\hat{I}}_1^{(q)} \frac{\partial}{\partial I_1} + \ldots \dot{\hat{I}}_n^{(q)} \frac{\partial}{\partial I_n} + \dot{\hat{\phi}}_1^{(q)} \frac{\partial}{\partial \phi_1} + \ldots + \dot{\hat{\phi}}_n^{(q)} \frac{\partial}{\partial \phi_n}.$$

Moreover

$$
\begin{aligned}
d\hat{H}^{(q)} &= d\hat{h}^{(q)} + dR^{(q)} \\
&= d\zeta^{(q)} + dh^{(q)} + dR^{(q)} \\
&= \textstyle\sum_{i=1}^{n} \frac{\partial \zeta^{(q)}}{\partial I_i} + \underbrace{\displaystyle\sum_{i=1}^{n} \frac{\partial \zeta^{(q)}}{\partial \phi_i}}_{=0} + \textstyle\sum_{i=1}^{n} \frac{\partial h^{(q)}}{\partial I_i} \\
&\qquad + \underbrace{\displaystyle\sum_{i=1}^{n} \frac{\partial h^{(q)}}{\partial \phi_i}}_{=0} + \textstyle\sum_{i=1}^{n} \frac{\partial R^{(q)}}{\partial I_i} + \textstyle\sum_{i=1}^{n} \frac{\partial R^{(q)}}{\partial \phi_i}.
\end{aligned}
$$

Recall

$$
\omega = \left(\sum_{j=1}^{m} \frac{c_j}{I_1^j} \right) dI_1 \wedge d\phi_1 + \sum_{i=2}^{n} dI_i \wedge d\phi_i,
$$

$$
\Pi = \frac{1}{\left(\sum_{j=1}^{m} \frac{c_j}{I_1^J} \right)} \frac{\partial}{\partial I_1} \wedge \frac{\partial}{\partial \phi_1} + \sum_{i=2}^{n} \frac{\partial}{\partial I_i} \wedge \frac{\partial}{\partial \phi_i}.
$$

Then:

$$
\begin{cases}
\dot{\hat{I}}_j^{(q)} &= -\frac{\partial R^{(q)}}{\partial \phi_j}(\hat{x}^{(q)}(t)), \quad \text{if } j \neq 1 \text{ and} \\
\dot{\hat{I}}_1^{(q)} &= -\frac{1}{\left(\sum_{i=1}^{m} \frac{c_i}{I_1^i} \right)} \frac{\partial R^{(q)}}{\partial \phi_j}(\hat{x}^{(q)}(t)) = -\mathcal{B}(I_1) \frac{\partial R^{(q)}}{\partial \phi_1}(\hat{x}^{(q)}(t)).
\end{cases}
$$

Observe that

$$
|\dot{\hat{I}}_1^{(q)}(t)| \leq \left| \frac{\partial R^{(q)}}{\partial \phi_1}(\hat{x}^{(q)}(t)) \right|. \tag{7.21}
$$

Moreover,

$$
\begin{cases}
\dot{\hat{\phi}}_j^{(q)} &= \hat{u}_j^{(q)}(\hat{I}^{(q)}(t)) + \frac{\partial R^{(q)}}{\partial I_j}(\hat{x}^{(q)}(t)) \\
&= u_j^{(q)}(\hat{I}^{(q)}) + \frac{\partial R^{(q)}}{\partial I_j}(\hat{x}^{(q)}(t)) \quad \text{if } j \neq 1 \\
\dot{\hat{\phi}}_1^{(q)} &= \underbrace{(\mathcal{B}(I_1) u_1^{(q)} + \mathcal{A}(I_1))}_{u_1^{(q)}}(\hat{I}^{(q)}(t)) + \mathcal{B}(I_1) \frac{\partial R^{(q)}}{\partial I_1}(\hat{x}^{(q)}(t)),
\end{cases}
$$

where we have used that $\hat{u}_j^{(q)} = u_j'^{(q)}$ if $j \neq 1$. Using 7.21 we obtain

$$|\dot{\hat{I}}^{(q)}(t)| \leq \left\| \frac{\partial R^{(q)}}{\partial \phi} \right\|_{G_q, \rho^{(q)}} \leq \varepsilon_q.$$

Hence,

$$\begin{array}{rcl} |\dot{\hat{\phi}}^{(q)} - u'^{(q)}(I_0^*)|_\infty & = & |u'^{(q)}(\hat{I}^{(q)}(t)) + \bar{\mathcal{B}}\frac{\partial R^{(q)}}{\partial I_1}(\hat{x}^{(q)}(t)) - u'^{(q)}(I_0^*)|_\infty \\ & \leq & |u'^{(q)}(\hat{I}^{(q)})(\hat{I}^{(q)}(t)) - u'^{(q)}(I_0^*)|_\infty + |\frac{\partial R^{(q)}}{\partial I}(\hat{x}^{(q)}(t))|_\infty \\ & \leq & M_q'|\hat{I}^{(q)}(t) - I_0^*| + \|\frac{\partial R^{(q)}}{\partial I}\|_{F_q, \rho^{(q)}, \infty} \\ & \leq & M_q|\hat{I}^{(q)}(t) - I_0^*| + \frac{\varepsilon_q}{c_{q+1}} \\ & \leq & 2M\delta_2^{(q+1)} + \frac{\varepsilon_q}{c_{q+1}} \leq 3M\delta_2^{(q+1)}. \end{array}$$

Where in the last bound we used that

$$\frac{\varepsilon_q}{c_{q+1}} \leq M\delta_2^{(q+1)}, \tag{7.22}$$

that holds because:

$$\begin{array}{rcl} \frac{\varepsilon_q}{c_{q+1}} & \leq & \frac{16MK_{q+1}^{\tau+1}\rho_1}{\beta 2^{\nu q}} \frac{8\varepsilon}{\nu\rho_1 2^{(2\tau+2)q}} \\ & \leq & \frac{16MK_{q+1}^{\tau+1}\rho_1}{\beta e^{\nu q}} \frac{8}{\nu\rho_1 2^{(2\tau+2)q}} \frac{\nu^2\mu^2\beta^2}{2^{\tau+30}L^4M^2K^{2\tau+2}} \\ & \leq & \frac{2^7K_{q+1}^{\tau+1}\nu\mu^2\beta}{2^{(2\tau+2)q+\nu q+\tau+30}L^4M^2K^{2\tau+2}} \\ & \leq & \frac{\nu\beta\mu^2}{K_{q+1}^{\tau+1}2^{\nu q+\tau+23}L^4M^2} \\ & = & \frac{\mu^2}{2^{\nu q+\tau+17}L^4M} \frac{\nu\beta}{2^6MK_{q+1}^{\tau+1}} \\ & \leq & \frac{\mu^2}{2^{\tau+17+\nu q}L^4M}\delta_2^{(q+1)} \\ & \leq & \frac{2^{2\tau+12}L^4M^2}{2^{\tau+17+\nu q}L^4M}\delta_2^{(q+1)} \\ & \leq & 2^{\tau-5-\nu q}\delta_2^{(q+1)} \\ & \leq & \frac{2^\tau}{2^{5+\nu q}}M\delta_2^{(q+1)} \\ & \leq & M\delta_2^{(q+1)} \quad \text{if } q \text{ is large enough.} \end{array}$$

Thus, since one of the inequalities defining $\hat{T}_q$ has to be an equality for $t = T_q$ we have that

$$\begin{array}{rcl} \delta_2^{(q+1)} & = & |\hat{I}^{(q)}(T_q) - I_0^*| \le T_q \varepsilon_q, \quad \text{or} \\ \delta_1^{(q+1)} & = & |\hat{\phi}^{(q)}(T_q) - \hat{\hat{\phi}}^{(q)}(T_1)|_\infty \le T_q 3M \delta_2^{(q+1)}. \end{array}$$

Hence, $\hat{T}_q \ge \min(\frac{\delta_2^{(q+1)}}{\varepsilon_q}, \frac{\delta_1^{(q+1)}}{3M\delta_2^{(q+1)}}) \ge \frac{1}{3Mc_{q+1}}$, where we used again 7.22.

Let us denote $T_q' := \frac{1}{3Mc_{q+1}}$, then $\hat{T}_q \ge T_q'$. This implies

$$|\hat{x}^{(q)}(t) - \hat{\hat{x}}^{(q)}(t)|_{c_{q+1}} \le \delta_2^{(q+1)} \quad \text{for } |t| \le T_q'.$$

Since $\hat{H}^{(q)} = \hat{H} \circ \Psi^{(q)}$ and $\Psi^{(q)}$ is canonical it turns out that $\Psi^{(q)}(\hat{x}^{(q)}(t))$ is a trajectory of $\hat{H}$ defined for $t \le T_q'$. It is important to observe that for q big enough this trajectory remains near the torus $\Psi^{(q)}(\mathbb{T}^n \times \{I_0^*\})$. Moreover T_q' tends to infinity when $q \to \infty$.

6. Invariant tori

Assume now that $x_0^* \in \mathbb{T}^n \times G^*$ and let us write

$$\left\{ \begin{array}{rcl} x^*(t) & = & (\phi_0^* + u^*(I_0^*)t, I_0^*) \\ \hat{x}^*(t) & = & (\phi_0^* + u'^*(I_0^*)t, I_0^*) \end{array} \right. \quad \text{for } t \in \mathbb{R}.$$

Note that

$$\begin{array}{rcl} |\hat{\hat{x}}^{(q)}(t) - \hat{x}^*(t)|_{c_{q+1}} & \le & c_{q+1}|u'^{(q)}(I_0^*) - u'^*(I_0^*)|_\infty |t| \\ & \le & c_{q+1}|u'^{(q)} - u'^*|_{G^*,\infty}|t|. \end{array}$$

And observe that if $|t| \le \frac{\delta_1^{(q+1)}}{|u'^{(q)} - u'^*|_{G^*,\infty}} =: T_q''$ then,

$$\begin{array}{rcl} |\hat{\hat{x}}^{(q)}(t) - \hat{x}^*(t)|_{c_{q+1}} & \le & c_{q+1}|u'^{(q)} - u'^*|_{G^*,\infty} \frac{\delta_1^{(q+1)}}{|u'^{(q)} - u'^*|_{G^*,\infty}} \\ & \le & \frac{\delta_2^{q+1}}{\delta_1^{q+1}} \delta_1^{q+1} = \delta_2^{q+1}. \end{array}$$

Observe that

$$|u'^* - u'^{(q)}|_{G^*} = |\bar{\mathcal{B}}u^* + \bar{\mathcal{A}} - \bar{\mathcal{B}}u^{(q)} - \bar{\mathcal{A}}|_{G^*}$$

$$= |\mathcal{B}(u^* - u^{(q)})|_{G^*} \leq |u^* - u^{(q)}|_{G^*},$$

close enough to Z.

Hence the bound obtained for $|u^* - u^{(q)}|_{G^*}$ also holds for $|u'^* - u'^{(q)}|_{G^*}$.

$$|u'^* - u'^{(q)}|_{G^*} \leq \sum_{s \geq q} \frac{4MK^{\tau+1}\varepsilon}{\nu\beta 2^{(\tau+2)s}} \leq \frac{8MK^{\tau+1}\varepsilon}{\nu\beta 2^{(\tau+2)q}}.$$

Using this bound, we see that T''_q tends to infinity because

$$T''_q \geq \left(\frac{\nu\rho_1}{8 \cdot 2^{\nu q}}\right)\left(\frac{\nu\beta 2^{(\tau+2)q}}{8MK^{\tau+1}\varepsilon}\right) = \frac{\nu^2\beta\rho_1}{64MK^{\tau+1}\varepsilon}2^{(\tau+2-\nu)q}.$$

Then

$$|\hat{x}^{(q)}(t) - \hat{x}^*(t)|_{c_{q+1}} \leq |\hat{x}^{(q)}(t) - \hat{\hat{x}}^{(q)}(t)|_{c_{q+1}} + |\hat{\hat{x}}^{(q)}(t) - \hat{x}^*(t)|_{c_{q+1}} \leq 2\delta_2^{(q+1)}.$$

when $t \leq T'''_q := \min(T'_q, T''_q)$.

Next, we see that the trajectory $\Psi^{(q)}(x^{(q)}(t))$ is very close to $\Psi^*(x^*(t))$ for large values of q. This is true because, when $|t| \leq T'''_q$.

$$\begin{aligned}
&|\Psi^{(q)}(\hat{x}^{(q)} - \Psi^*(\hat{x}^*(t)))|_{c_1} \\
&\leq |\Psi^{(q)}(\hat{x}^{(q)}(t)) - \Psi^{(q)}(\hat{x}^*(t))|_{c_1} + |\Psi^{(q)}(\hat{x}^*(t)) - \Psi^*(\hat{x}^*(t))|_{c_1} \\
&\leq 2^{(\tau+1-\nu)(q-1)} \cdot 2|\hat{x}^{(q)}(t) - \hat{x}^*(t)|_{c_q} + |\Psi^{(q)} - \Psi^*|_{G^*,(\rho_1/4,0),c_1} \\
&\leq 2^{(\tau+1-\nu)(q-1)} \cdot 4\delta_2^{(q+1)} + |\Psi^{(q)} - \Psi^*|_{G^*,(\rho_1/4,0),c_1} \\
&\leq 2^{(\tau+1-\nu)(q-1)} \cdot 4\delta_2^{(q+1)} + \tfrac{2^{10}K^\tau\varepsilon}{\nu^2\rho_1\beta 2^{(1+\nu)q}} \\
&\leq \tfrac{c_1}{c_{q+1}} \tfrac{4\delta_2^{(q+1)}}{2^{(\tau+1-\nu)}} + \tfrac{2^{10}K^\tau\varepsilon}{\nu^2\rho_1\beta 2^{(1+\nu)q}} \\
&\leq \tfrac{c_1 4}{2^{(\tau+1-\nu)}} \tfrac{\delta_1^{(q+1)}}{\delta_2^{(q+1)}} \delta_2^{(q+1)} + \tfrac{2^{10}K^\tau\varepsilon}{\nu^2\rho_1\beta 2^{(1+\nu)q}} \\
&\leq \tfrac{c_1 4}{2^{(\tau+1-\nu)}} \delta_1^{(q+1)} + \tfrac{2^{10}K^\tau\varepsilon}{\nu^2\rho_1\beta 2^{(1+\nu)q}}
\end{aligned}$$

where we used that $c_{q-1}/c_q = 2^{\tau+1-\nu}$ then $c_1/c_{q+1} = 2^{(\tau+1-\nu)q}$.

The bound 7.28 tends to zero. So we deduce, for every fixed t, $\Psi^{(q)}(\hat{x}^{(q)}(t))$ exits or q large enough and its limit is $\Phi^*(\hat{x}^*(t))$. This fact and the continuity of the flow of $\hat{H}$ imply that $\Psi^*(\hat{x}^*(t))$ is also a trajectory of $\hat{H}$, which is defined for all $t \in \mathbb{R}$.

This holds for every initial condition $x_0^* = (\phi_0^*, I_0^*) \in \mathbb{T}^n \times G^*$ for this reason $\Psi^*(\mathbb{T}^n \times \{I_0^*\})$ is an invariant torus of $\hat{H}$, with frequency vector $u'^*(I_0^*)$. Observe that the energy on the torus is $\hat{H}(\Psi^*(\phi_0^*, I_0^*)) = h^*(I_0^*)$.

The preserved invariant tori are completely determined by the transformed actions $I_0^* \in G^*$. We are now going to characterize the preserved tori by the original action coordinates.

First, let us see that $u(\hat{G}) \subset F^*$. Recall that:

$$\Delta_{c,\hat{q}}(k,\alpha) = \{J \in \mathbb{R} \text{ such that } |k\bar{\mathcal{B}}u(I) + k\bar{\mathcal{A}}| < \alpha\},$$
$$\hat{G} = \{I \in \mathcal{G} - \frac{2\gamma}{\mu} \text{ such that } |k\bar{\mathcal{B}}u(I) + k\bar{\mathcal{A}}| < \frac{\beta}{|k|_1^\tau}\}.$$

With this definition is obvious that if $I \in \hat{G}$ then $u(I)$ is $\frac{\beta}{|k|_1^\tau}, K, c, \hat{q}$-nonresonant. Hence $u(I) \notin \Delta_{c,\hat{q}}(k, \frac{\beta}{|k|_1^\tau})$ for all $k \neq 0$. Then $u(\hat{G}) \subset F^*$.

We want to find a correspondence between the invariant tori of $\hat{h}$ and the invariant tori of the perturbed system $\hat{H} = \hat{h} + R$, or in the new coordinates $\hat{h}^*$.

Recall

$$u' = \bar{\mathcal{B}}u + \bar{\mathcal{A}},$$
$$u'^* = \bar{\mathcal{B}}u^* + \bar{\mathcal{A}} = (\frac{1}{\sum_{i=1}^m \frac{c_i}{I_1^i}} u_1^* + \frac{\sum_{i=1}^m \frac{\hat{q}_i}{I_1^i}}{\sum_{i=1}^m \frac{c_i}{I_1^i}}, u_2^*, \ldots, u_n^*).$$

Observe $u'^*(0, I_2, \ldots, I_n) = \frac{\hat{q}_m}{c_m} = \frac{1}{K'}$ the inverse of the modular period, hence u'^* and u' are not one-to-one at Z because they project the first component of u^* and u to $\frac{1}{K'}$.

$$
\begin{tikzcd}
\mathcal{D}_\rho(G) = \mathcal{W}_{\rho_1}(\mathbb{T}^n \times \mathcal{V}_{\rho_1}(G)) & & \mathcal{W}_{\rho_1/4}(\mathbb{T}^n) \times G^* \arrow[ll, "\Psi^*"'] \arrow[dddd, shift left, "u^*"] \\
\mathcal{W}_{\rho_1/4}(\mathbb{T}^n) \times G \arrow[u, hook, "i"] & & \\
\mathcal{W}_{\rho_1/4}(\mathbb{T}^n) \times \hat{G} \arrow[u, hook, "i"] \arrow[d, "\pi"] & & \\
\hat{G} \arrow[d, "u|_{\hat{G}}"] \arrow[uuu, bend right=90, "\mathcal{T}"'] & & \\
u(\hat{G}) \subset F \arrow[rr, hook, "i"] & & F^* \arrow[uuuu, shift left, "(u^*)^{-1}"]
\end{tikzcd}
$$

Figure 7.2: Diagram of the different maps and sets used in the proof.

Let us define $I_0^* = (u^*)^{-1}(u(I_0))$, recall that u and u^* are indeed one-to-one even though u' and u'^* are not, so I_0^* is properly defined.

With this definition $u^*(I_0^*) = u(I_0)$ and this implies $u'^*(I_0^*) = u'(I_0)$. Now, let us define $\mathcal{T}(\phi_0, I_0) = \Psi^*(\phi_0, I_0^*)$.

We obtain 7.13 because the set $\mathcal{T}(\mathbb{T}^n \times \{I_0\})$ is an invariant torus of the hamiltonian flow of $\hat{H}$ with frequency vector $u'^*(I_0^*)$ because $\mathbb{T}^n \times \{I_0^*\}$ is an invariant torus for the hamiltonian flow of $\hat{h}^*$. And we have seen that $u'^*(I_0^*) = u'(I_0)$. In a nutshell, the original frequencies (of the unperturbed system) $u(I_0)$ for $I_0 \in \hat{G}$ are in F^* and hence can be seen as frequencies of the unperturbed system in the new coordinates $u^*(I_0^*)$. Hence we can conclude that for this $I_0 \in \hat{G}$ its new (perturbed) solution is also linear in a torus $(\phi_0 + u'^* t, I_0^*) \in \Psi^*(\mathbb{T}^n \times \{I_0^*\}) = \mathcal{T}(\mathbb{T}^n \times \{I_0\})$. And the new frequency vector u'^* is such that $u'^* = u'$.

Let us now prove 7.14. Let us write, for $(\phi_0, I_0^*) \in \mathcal{W}_{\frac{\rho_1}{4}}(\mathbb{T}^n) \times G^*$.

$$\Psi^*(\phi_0, I_0^*) = (\phi_0 + \Psi^*_\phi(\phi_0, I_0^*), I_0^* + \Psi^*_I(\phi_0, I_0^*)).$$

And for $(\phi_0, I_0) \in \mathcal{W}_{\frac{\rho_1}{4}(\mathbb{T}^n)\times\hat{G}}$.

$$\mathcal{T}(\phi_0, I_0) = (\phi_0 + \mathcal{T}_\phi(\phi_0, I_0), I_0 + \mathcal{T}_I(\phi_0, I_0)).$$

Then, for $(\phi_0, I_0) \in \mathcal{W}_{\frac{\rho_1}{4}(\mathbb{T}^n)\times\hat{G}}$:

$$\mathcal{T}_\phi(\phi_0, I_0) = \Psi^*_\phi(\phi_0, I_0^*), \quad \text{and} \quad \mathcal{T}_I(\phi_0, I_0) = \Psi^*_I(\phi_0, I_0^*) + I_0 - I_0^*.$$

Let us bound the norms of these terms:

$$\begin{aligned} |\Psi^*_\phi(\phi_0, I_0^*)|_\infty &\leq \tfrac{1}{c_1}|\Psi^* - \mathrm{id}|_{G^*,(\frac{\rho_1}{4},0),c_1} \\ &\leq \tfrac{16MK^{\tau+1}\rho_1}{\beta}\tfrac{2^{10}K^\tau\varepsilon}{\nu^2\rho_1\beta} \\ &\leq \tfrac{2^{14}MK^{2\tau+1}\varepsilon}{\nu^2\beta^2}, \end{aligned}$$

where we used that $c_1 \geq \frac{\beta}{16MK^{\tau+1}\rho_1}$. Then,

$$\begin{aligned} \Psi^*_I(\phi_0, I_o^*) &\leq |\Phi^* - \mathrm{id}|_{G^*,(\frac{\rho_1}{4},0,c_1)} \\ &\leq \tfrac{2^{10}k^\tau\varepsilon}{\nu^2\rho_1\beta}. \end{aligned}$$

Now it only remains the term $I_0^* - I_0$:

$$|I_0^* - I_0| \leq |(u^*)^{(-1)} - (u)^{(-1)}|_{F^*} \leq \sum_{s\geq 0}\xi_s$$
$$\leq \sum_{s\geq 0}\frac{4MK^{\tau+1}\varepsilon}{\nu\beta 2^{(\tau+2)s}} \leq \frac{8MK^{\tau+1}\varepsilon}{\nu\beta 2^{(\tau+2)}}.$$

Let us put everything together and use $\hat{\rho} \leq \nu\rho_1$, $K \leq 2/\hat{\rho}$ and $\beta = \gamma/L$.

$$\begin{aligned} |\Psi^*_\phi(\phi_0, I_0^*)|_\infty &\leq \frac{2^{14}M(\frac{2}{\hat{\rho}})^{2\tau+1}\varepsilon}{\nu^2(\frac{\gamma}{L})^2} \\ &\leq \frac{2^{2\tau+15}ML^2}{\nu^2\hat{\rho}^{2\tau+1}}\frac{\varepsilon}{\gamma^2} \end{aligned}$$

$$\begin{aligned}|\Psi^*(\phi_0, I_0^*)| + |I_0^* - I_0| &\leq \tfrac{2^{10}(\frac{2}{\hat\rho})^\tau \varepsilon}{\nu\hat\rho(\frac{\gamma}{L})} + \tfrac{8M(\frac{2}{\hat\rho})^{\tau+1}\varepsilon}{\nu(\frac{\gamma}{L})2^{(\tau+2)}}\\ &= \tfrac{2^{10+\tau}L\varepsilon}{\nu\hat\rho^{\tau+1}\gamma} + \tfrac{8M2^{\tau+1}L\varepsilon}{\nu\hat\rho^{\tau+1}\gamma 2^{(\tau+2)}}\\ &\leq \tfrac{2^{20+\tau}L\varepsilon + M2^{\tau+4}L\varepsilon}{\nu\hat\rho^{\tau+1}\gamma} \leq \tfrac{2^{10+\tau}L(1+M)}{\nu\hat\rho^{\tau+1}}\tfrac{\varepsilon}{\gamma}\end{aligned}$$

7. Estimate of the measure

Finally we carry out the estimate of part 3. Let us write

$$\hat G^* = (u^*)^{-1}(u(\hat G)).$$

The invariant tori fill the set

$$\mathcal{T}(\mathbb{T}^n \times \hat G) = \Psi^*(\mathbb{T}^n \times \hat G^*)$$

i.e. all the tori inside $\mathcal{T}(\mathbb{T}^n \times \hat G)$ are invariant although there are more of them. Because $\Psi^{(q)}$ are hamiltonian transformations, in particular preserve the volumes:

$$\mathrm{meas}[\Psi^{(q)}(\mathbb{T}^n \times \hat G^*)] = \mathrm{meas}(\mathbb{T}^n \times \hat G^*) = (2\pi)^n \mathrm{meas}(\hat G^*).$$

Now, let us consider the measure of the limit:

$$\mathrm{meas}[\Psi^*(\mathbb{T}^n \times \hat G^*)].$$

To do this we use the superior limit of sets:

$$\bigcap_{n=q}^{\infty}\bigcup_{j=q}^{\infty}(\Psi^{(j)}(\mathbb{T}^n \times \hat G^*)).$$

Because $\Psi^{(j)}(\mathbb{T}^n \times \hat G^*)$ are compact and we have the bound

$$|\Psi^* - \Psi^{(q)}|_{G^*,(\frac{\rho_1}{4},0),c_1} \leq \frac{2^{10}K^\tau\varepsilon}{\nu^2\rho_1\beta 2^{(1+\nu)q}},$$

$\bigcup_{j=q}^{\infty}(\Psi^{(j)}(\mathbb{T}^n \times \hat G^*))$ is also compact. All the measures are well defined and we can say that

$$\text{meas}[\Psi^*(\mathbb{T}^n \times \hat{G}^*)] \geq (2\pi)^n \text{meas}(\hat{G}^*).$$

Then, to bound the measure of the complement of the invariant set it is enough to bound the measure of $\mathcal{G} \setminus \hat{G}^*$.

But first we are going to define some auxiliary sets. Let $\tilde{\beta} = \frac{2\gamma M}{\mu}$, $\tilde{\beta}_q = (1 - \frac{1}{2^{\nu q}})\tilde{\beta}$. Note that $\tilde{\beta} \geq \beta$ if and only if $\mu \leq 2ML$ and we assumed $\mu \leq 2^{\tau+6}L^2M$.

Then, for $q \geq 0$ we define

$$\tilde{F}_q = (F - \tilde{\beta}_q) \setminus \bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\} \\ |k|_1 \leq K}} \Delta_{c,\hat{q}}(k, \frac{\tilde{\beta}_q}{|k|_1^\tau}), \quad \tilde{G}_q = (u^{(q)})^{-1}(\tilde{F}_q)$$

and

$$\tilde{F}^* = \bigcap_{q\geq 0} \tilde{F}_q = (F - \tilde{\beta}) \setminus \bigcup_{\substack{k\in\mathbb{Z}^n\setminus\{0\} \\ |k|_1 \leq K}} \Delta_{c,\hat{q}}(k, \frac{\tilde{\beta}}{|k|_1^\tau}), \quad \tilde{G}^* = \bigcap_{q\geq 0} \tilde{G}_q.$$

In order to prove the bounds, we need to prove previously the inclusions $\tilde{G}^* \subset \hat{G}^*$ and $\tilde{G}_0 \subset \mathcal{G}$.

(a) $\mathcal{G} \supset \tilde{G}_0 = (u^{(0)})^{-1}(\tilde{F}_0) = (u)^{-1}(F - \tilde{\beta})$, but we know $u(\mathcal{G}) = F$.

(b) $\tilde{G}^* \subset \hat{G}^*$. Take $I \in \tilde{G}^*$, then $I \in \tilde{G}_q \forall q \geq 0$. Hence $\exists J \in \tilde{F}_q \forall q$ such that $u^{(q)}(I)$. Then $\exists J \in \tilde{F}^*$ such that $u^*(J) = I$. If we check that $J \in u(\tilde{G})$ the we will have that $(u^*)^{-1}(J) = I \in \hat{G}^*$ and we will be done. We want $\tilde{F}^* \subset u(\hat{G})$. Because we take out all the resonances in $\tilde{F}^*$ it is enough to see $(F - \tilde{\beta}) \subset u(\mathcal{G} - \frac{2\gamma}{\mu})$. We only need to use that $|\frac{\partial u}{\partial I}|_{\mathcal{G},\rho_2} \leq M$. Then $F - \tilde{\beta} \subset u(\mathcal{G} - \frac{2\gamma}{\mu})$. This holds if and only if $\frac{\tilde{\beta}}{M} \leq \frac{2\gamma}{\mu}$ which is true because $\hat{\beta} \leq \frac{2\gamma M}{\mu}$.

Then, we proceed as follows

$$\begin{aligned}\mathrm{meas}(\mathcal{G}\setminus \hat{G}^*) &\leq \mathrm{meas}(\mathcal{G}\setminus \tilde{G}^*)\\ &\leq \mathrm{meas}(\tilde{G}_0\setminus \tilde{G}^*)\\ &\leq \textstyle\sum_{q=1}^{\infty}\mathrm{meas}(\tilde{G}_{q-1}\setminus \tilde{G}_q).\end{aligned}$$

For $q\geq 1$ we obtain the following estimate:

$$\mathrm{meas}(\tilde{G}_{q-1}\backslash\tilde{G}_q)\leq \frac{1}{|\det(\frac{\partial u^{(q-1)}}{\partial I}(I))|}\mathrm{meas}(\overbrace{\tilde{F}_{q-1}}^{u^{(q-1)}(\tilde{G}_{q-1})}\backslash\overbrace{(\tilde{F}_q-\varepsilon_{q-1})}^{u^{(q-1)}(\tilde{G}_q)}).$$

Where we have used lemma 7.2.27. Also $\det(\frac{\partial u^{(q-1)}}{\partial I}(I))\geq \mu_{q-1}^n$ because of the μ_{q-1}-nondegeneracy condition all the eigenvalues have to be greater than μ_{q-1}.

$$\begin{aligned}\mathrm{meas}(\tilde{G}_{q-1}\setminus\tilde{G}_q) &\leq \tfrac{1}{\mu_{q-1}^n}\mathrm{meas}(\tilde{F}_{q-1}\setminus(\tilde{F}_q-\varepsilon_{q-1}))\\ &\leq \tfrac{2^n}{\mu^n}\mathrm{meas}(\tilde{F}_{q-1}\setminus(\tilde{F}_q-\varepsilon_{q-1})).\end{aligned}$$

Now, we are going to apply lemma 7.2.25 with

$$\tilde{F}_{q-1}=F(\tilde{\beta}_{q-1},\tilde{\beta}_{q-1},K_q-1)$$

and $\tilde{F}_q=F(\tilde{\beta}_q,\tilde{\beta}_q,K_q)$.

Applying the lemma:

$$\begin{aligned}&\mathrm{meas}(\tilde{F}_{q-1}\setminus\tilde{F}_q)\leq D(\tilde{\beta}_q-\tilde{\beta}_{q-1})\\ &+2(\mathrm{diam}F)^{n-1}\left(\sum_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ |k|_1\leq K_{q-1}}}\frac{\tilde{\beta}_q-\tilde{\beta}_{q-1}}{|k|_1^{\tau}|k|_{2,\omega}}+\sum_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ K_{q-1}\leq|k|_1\leq K_q}}\frac{\tilde{\beta}_q}{|k|_1^{\tau}|k|_{2,\omega}}\right)\end{aligned}$$

and

$$\text{meas}(\tilde{F}_q \setminus (\tilde{F}_q - \varepsilon_q)) \leq (D + 2^{n+1}(\text{diam}F)^{n-1}K^n)\varepsilon_q.$$

Putting everything together (and using that $\tilde{\beta}_0 = 0$), we get

$$\begin{aligned}\text{meas}(\mathcal{G} \setminus \hat{G}^*) \;\; \leq \;\; & \frac{2^n}{\mu^n}\Bigg(D\tilde{\beta} + 2(\text{diam}F)^{n-1}\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\frac{\tilde{\beta}}{|k|_1^\tau|k|_{2,\omega}} \\ & + D\sum_{q=1}^{\infty}\varepsilon_{q-1} + 2^{n+1}(\text{diam}F)^{n-1}\sum_{q=1}^{\infty}K_q^n\varepsilon_{q-1}\Bigg).\end{aligned} \tag{7.23}$$

We now only have to check that the series converge in the previous expression converge. Let us check that they converge at Z first and then outside of Z. Recall that at Z we take the vectors $\bar{k} \neq 0$.

$$\begin{aligned}\textstyle\sum_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ \bar{k}\neq 0}}\frac{1}{|k|_1^\tau|k|_{2,\omega}} \;\; & \leq \;\; \textstyle\sum_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ \bar{k}\neq 0}}\frac{1}{|k|_1^\tau|\bar{k}|} \\ & \leq \;\; \textstyle\sum_{\substack{\bar{k}\in\mathbb{Z}^{n-1}\setminus\{0\}\\ \bar{k}\neq 0}}\sum_{k_n\in\mathbb{Z}}\frac{\sqrt{n}}{(|\bar{k}|_1+|k_n|)^\tau|\bar{k}|_1} \\ & \leq \;\; \textstyle\sqrt{n}2^{n-1}\sum_{j=1}^{\infty}\sum_{k_n\in\mathbb{Z}}\frac{j^{n-3}}{j+|k_n|)^\tau}\end{aligned}$$

where we used that the number of vectors $\bar{k} \in \mathbb{Z}^{n-1}$ with $|\bar{k}|_1 = j \geq 1$ can be bounded by $2^{n-1}j^{n-2}$. This series can be bounded by comparing it to an integral:

$$\begin{aligned}\sum_{k_n\in\mathbb{Z}}\frac{1}{(j+|k_n|)^\tau} &\leq \frac{1}{j^\tau} + 2\int_0^\infty \frac{dx}{(j+x)^\tau} \\ &= \frac{1}{j^\tau} + \frac{2}{(\tau+1)j^{\tau-1}} \leq \frac{\tau+1}{\tau-1}\frac{1}{j^{\tau+1}}.\end{aligned}$$

Where we used that $\tau > 1$ because $n \geq 2$. Then

$$\sum_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ \bar{k}\neq 0}}\frac{1}{|k|_1^\tau|k|_{2,\omega}} \leq \frac{\sqrt{n}2^{n-1}(\tau+1)}{\tau-1}\sum_{j=1}^{\infty}\frac{1}{j^{\tau-n+2}}$$

which converges by the condition $\tau > n-1$.

Now let us check that it converges outside of Z.

$$\begin{array}{rcl} \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{|k|_1^\tau |k|_{2,\omega}} & = & \sum_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ \bar{k}\neq 0}} \frac{1}{|k|_1^\tau |k|_{2,\omega}} + \sum_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ \bar{k}=0}} \frac{1}{|k|_1^\tau |k|_{2,\omega}} \\ & = & \sum_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ \bar{k}\neq 0}} \frac{1}{|k|_1^\tau |\bar{k}|} + \sum_{k_1\in\mathbb{Z}} \frac{1}{|k|_1^\tau |k_1^2 \mathcal{B}(I_1)^2|} \end{array}$$

We have seen before that the first term converges. The second term:

$$\sum_{k_1\in\mathbb{Z}} \frac{1}{|k_1|^\tau |k_1^\tau \mathcal{B}(I_1)^2|} = \frac{1}{\mathcal{B}(I_1)^2} \sum_{k_1\in\mathbb{Z}} \frac{1}{k_1^{\tau+2}},$$

which converges $\forall I_1 \neq 0$, i.e. outside of Z.

Now we go back to the expression 7.23. The other terms of that expression can be bounded simultaneouslty inside and outside Z. Now we only have to check that the third series converges, because if the third converges so does the second. We only have to check that $\sum_{q=1}^\infty K_q^n \varepsilon_{q-1}$ converges. We will use that $\varepsilon \leq \frac{8\varepsilon}{\nu\rho_1 2^{(2\tau+2)(q-1)}}$.

$$\begin{array}{rcl} \sum_{q=1}^\infty K_q^n \varepsilon_{q-1} & = & K^n \sum_{q=1}^\infty 2^{n(q-1)} \varepsilon_{q-1} \\ & = & K^n \sum_{q=1}^\infty \frac{8\varepsilon 2^{n(q-1)}}{\nu\rho_1 2^{(2\tau+2)(q-1)}} \\ & = & K^n \frac{8\varepsilon}{\nu\rho_1} \sum_{q=1}^\infty \frac{1}{2^{(2\tau+2-n)(q-1)}}. \end{array}$$

Which converges if and only if $2\tau + 2 - n \geq 1$. And we are done because $2\tau \geq n-1$ since $\tau \geq n-1$ by hypothesis.

Putting everything together:

$$\begin{aligned}
&\operatorname{meas}(\mathcal{G} \setminus \hat{G}^*) \\
&\leq \frac{2^n}{\mu^n}\Bigg(D^2 \frac{2\gamma M}{\mu} + 2(\mathrm{diam} F)^{n-1} \frac{2\gamma M}{\mu} \frac{\sqrt{n} 2^{n-1}(\tau+1)}{\tau - 1} \sum_{j=1}^{\infty} \frac{1}{j^{\tau-n+2}} \\
&D \frac{8\varepsilon}{\nu\rho_1} \sum_{q=1}^{\infty} \frac{1}{2(2\tau+2)(q-1)} \\
&+2^{n+1}(\mathrm{diam} F)^{n-1} K^n \frac{8\varepsilon}{\nu\rho_1} \sum_{q=1}^{\infty} \frac{1}{2^{(2\tau+2-n)(q-1)}} \Bigg)
\end{aligned}$$

Now using that

$$\varepsilon \leq \frac{\nu^2\mu^2\beta^2}{2^{\tau+30}L^4M^3K^{2\tau+1}} \leq \frac{2^{\tau-18}\cdot 8MK^{\tau+1}\rho_2}{LMK^{2\tau+2}}\gamma \leq \frac{2^{\tau-15}\rho_2}{LK^{\tau+1}}\gamma$$

We can write $\operatorname{meas}(\mathcal{G} \setminus \hat{G}^*) \leq C'\gamma$ where C' depends only on n, μ, D, $\mathrm{diam}F$, M, τ, ρ_1, ρ_2, L, K and if we efine $C = (2\pi)^n C'$. Hence,

$$\operatorname{meas}[(\mathbb{T}^n \times \mathcal{G}) \setminus \mathcal{T}(\mathbb{T}^n \hat{G})] \leq C\gamma.$$

□

7.4 Desingularization of b^m-integrable systems

In this section, we follow [8], for the definition of the desingularization of the b^m-symplectic form.

Definition 7.4.1. *The* f_ϵ***-desingularization*** ω_ϵ *form of* $\omega = \frac{dx}{x^m} \wedge \left(\sum_{i=0}^{m-1} x^i \alpha_{m-i}\right) + \beta$ *is:*

$$\omega_\epsilon = df_\epsilon \wedge \left(\sum_{i=0}^{m-1} x^i \alpha_{m-i}\right) + \beta.$$

Where in the even case, $f_\epsilon(x)$ is defined as $\epsilon^{-(2k-1)}f(x/\epsilon)$. And $f \in \mathcal{C}^\infty(\mathbb{R})$ is an odd smooth function satisfying $f'(x) > 0$ for all $x \in [-1,1]$ and satisfying outside that

$$f(x) = \begin{cases} \frac{-1}{(2k-1)x^{2k-1}} - 2 & \textit{for} \quad x < -1, \\ \frac{-1}{(2k-1)x^{2k-1}} + 2 & \textit{for} \quad x > 1. \end{cases} \tag{7.24}$$

And in the odd case, $f_\epsilon(x) = \epsilon^{-(2k)}f(x/\epsilon)$. And $f \in \mathcal{C}^\infty(\mathbb{R})$ is an even smooth positive function which satisfies: $f'(x) < 0$ if $x < 0$, $f(x) = -x^2 + 2$ for $x \in [-1,1]$, and

$$f(x) = \begin{cases} \frac{-1}{(2k+2)x^{2k+2}} - 2 & \textit{if } k > 0, x \in \mathbb{R} \setminus [-2,2] \\ \log(|x|) & \textit{if } k = 0, x \in \mathbb{R} \setminus [-2,2]. \end{cases} \tag{7.25}$$

Remark 7.4.2. *With the previous definition, we obtain smooth symplectic (in the even case) or smooth folded symplectic (in the odd case) forms that agree outside an ϵ-neighbourhood with the origial b^m-forms. Moreover, there is a convergence result in terms of m. See [26] for the details.*

To simplify notation, we introduce $F_\epsilon^{m-i}(x) = (\frac{d}{dx}f_\epsilon(x))x^i$, and hence $F_\epsilon^i(x) = (\frac{d}{dx}f_\epsilon(x))x^{m-i}$. With this notation the desingularization ω_ϵ is written:

$$\omega_\epsilon = \sum_{i=0}^{m-1} F_\epsilon^{m-i}(x)dx \wedge \alpha_{m-i} + \beta.$$

Definition 7.4.3. *The desingularization for (M, ω, μ) is the triple $(M, \omega_\varepsilon, \mu_\epsilon)$ where ω_ε is defined as above and μ_ε is:*

$$\mu \mapsto \mu_\epsilon = \left(f_{1\epsilon} = \sum_{i=1}^{m} \hat{c}_i G_\epsilon^i(x), f_2(\tilde{I}, \tilde{\phi}), \ldots, f_n(\tilde{I}, \tilde{\phi}) \right),$$

where

$$\mu = \left(f_1 = c_0 \log(x) + \sum_{i=1}^{m-1} c_i \frac{1}{x^i}, f_2(I, \phi) \ldots, f_n(I, \phi) \right)$$

$$G^i_\epsilon(x) = \int_0^x F^i_\epsilon(\tau)d\tau,$$

and $\hat{c}_1 = c_0$ *and* $\hat{c}_{i-1} = -ic_i$ *if* $i \neq 0$. *Also*

$$\begin{cases} \tilde{I} = (\tilde{I}_1, I_2, \ldots, I_n), & \tilde{I}_1 = \int_0^{I_1} \left(\frac{\sum_{i=1}^m \mathcal{K}\hat{c}_i F^i_\varepsilon(\tau)}{\sum_{i=1}^m \frac{\mathcal{K}\hat{c}_j}{\tau^j}} \right) d\tau \\ \tilde{\phi}_1 = (\tilde{\phi}_1, \phi_2, \ldots, \phi_n), & \tilde{\phi}_1 = \left(\frac{\sum_{i=1}^m \mathcal{K}\hat{c}_i F^i_\varepsilon(I_1)}{\sum_{i=1}^m \frac{\mathcal{K}\hat{c}_j}{I_1^j}} \right) \phi_1 \end{cases}$$

Remark 7.4.4. *Observe that with the last definition, when ϵ tends to 0, μ_ϵ tends to μ.*

Lemma 7.4.5. *The desingularization transforms a b^m-integrable system into an integrable system for m even on a symplectic manifld. For m odd the desingularization transforms it to a folded integrable system. The integrable systems are such that:*

$$X^\omega_{f_j} = X^{\omega_\epsilon}_{f_{j\epsilon}}.$$

Proof. Let us first check the singular part, i.e. let us check that that $X^\omega_{f_1} = X^{\omega_\epsilon}_{f_{1\epsilon}}$. Let us compute the two equations that define each one of the vector fields. We have to impose $-df_1 = \iota_{X^\omega_{f_1}}\omega$ and $-df_{1\epsilon} = \iota_{X^{\omega_\epsilon}_{f_{1\epsilon}}}\omega_\epsilon$. But observe first that we can rewrite $\omega = \sum_{i=1}^m \frac{1}{x^i} dx \wedge \alpha_i + \beta$ and $\omega_\epsilon = \sum_{i=1}^m F^i_\epsilon dx \wedge \alpha_i + \beta$. The conditions translate as:

$$-\sum_{i=1}^m \hat{c}_i \frac{1}{x^i} dx = \iota_{X^\omega_{f_1}} \left(\sum_{i=1}^m \frac{1}{x^i} dx \wedge \alpha_i + \beta \right),$$
$$-\sum_{i=0}^{m-1} \hat{c}_i F^i_\epsilon(x) dx = \iota_{X^{\omega_\epsilon}_{f_{1\epsilon}}} \left(\sum_{i=0}^{m-1} F^i_\epsilon(x) dx \wedge \alpha_i + \beta \right).$$

Since the toric action leaves ω invariant, in particular the singular set is invariant, and then $X^{\omega_\epsilon}_{f_{1\epsilon}}$ and $X^\omega_{f_1}$ are in the kernel of dx. Moreover, since β is a symplectic form in each leaf of the foliation and $X^{\omega_\epsilon}_{f_{1\epsilon}}$ and $X^\omega_{f_1}$ are transversal to this foliation, they are also in the kernel of β.

$$-\sum_{i=0}^{m-1} \hat{c}_i \frac{1}{x^i} dx = \sum_{i=0}^{m-1} \frac{1}{x^i} dx \wedge \alpha_i(X_{f_1}^{\omega}),$$
$$-\sum_{i=0}^{m-1} \hat{c}_i F_\epsilon^i(x) dx = \sum_{i=0}^{m-1} F_\epsilon^i(x) dx \wedge \alpha_i(X_{f_{1\epsilon}}^{\omega_\epsilon}).$$

Then, the conditions over $X_{f_1}^{\omega}$ and $X_{f_{1\epsilon}}^{\omega_\epsilon}$ are respectively:

$$-\hat{c}_i = \alpha_i(X_{f_1}^{\omega}),$$
$$-\hat{c}_i = \alpha_i(X_{f_{1\epsilon}}^{\omega_\epsilon}).$$

Then, the two vector fields have to be the same.

Let us now see $X_{f_j}^{\omega} = X_{f_{j\epsilon}}^{\omega_\epsilon}$ for $j > 1$. Assume now we have the b^m-symplectic form in action-angle coordinates $\omega = \sum_{i=1}^{m} \frac{\mathcal{K}\hat{c}_i}{I_1^i} dI_1 \wedge d\phi_1 + \sum_{i=1}^{n} dI_i \wedge d\phi_i$.

The differential of the functions are

$$\begin{aligned} df_i^\varepsilon &= \tfrac{\partial f_i^\varepsilon}{\partial I_1} dI_1 + \tfrac{\partial f_i^\varepsilon}{\partial \phi_1} d\phi_1 + \textstyle\sum_{j=2}^{n} \left(\tfrac{\partial f_i^\varepsilon}{\partial I_j} dI_j + \tfrac{\partial f_i^\varepsilon}{\partial \phi_j} d\phi_j \right) \\ &= \tfrac{\partial f_i}{\partial I_1} \left(\frac{\sum_{i=1}^{m} \mathcal{K}\hat{c}_i F_\varepsilon^i(\tau)}{\sum_{i=1}^{m} \frac{\mathcal{K}\hat{c}_j}{\tau^j}} \right) dI_1 + \tfrac{\partial f_i}{\partial \phi_1} \left(\frac{\sum_{i=1}^{m} \mathcal{K}\hat{c}_i F_\varepsilon^i(\tau)}{\sum_{i=1}^{m} \frac{\mathcal{K}\hat{c}_j}{\tau^j}} \right) d\phi_1 \\ &\quad + \textstyle\sum_{j=2}^{n} \left(\tfrac{\partial f_i^\varepsilon}{\partial I_j} dI_j + \tfrac{\partial f_i^\varepsilon}{\partial \phi_j} d\phi_j \right). \end{aligned}$$

On the other hand, the desingularized form is:

$$\omega^\varepsilon = \sum_{j=1}^{m} \mathcal{K}\hat{c}_i F_\varepsilon^j(I_1) dI_1 \wedge d\phi_1 + \sum_{j=2}^{m} dI_j \wedge d\phi_j.$$

Hence, one can see that the expression for both $X_{f_j}^{\omega}$ and $X_{f_{j\epsilon}}^{\omega_\epsilon}$ is

$$X_{f_j}^{\omega} = X_{f_{j\epsilon}}^{\omega_\epsilon} = \frac{\frac{\partial f_i}{\partial I_1}}{\sum_{i=1}^{m} \frac{\mathcal{K}\hat{c}_i}{I_1^i}} \frac{\partial}{\partial \phi_1} - \frac{\frac{\partial f_i}{\partial \phi_1}}{\sum_{i=1}^{m} \frac{\mathcal{K}\hat{c}_i}{I_1^i}} \frac{\partial}{\partial I_1} + \sum_{j=2}^{n} \left(\frac{\partial f_i^\varepsilon}{\partial I_j} dI_j + \frac{\partial f_i^\varepsilon}{\partial \phi_j} d\phi_j \right)$$

□

Remark 7.4.6. *The previous lemma tells us that the dynamics of the desingularized system are identical to the dynamics of the original b^m-integrable system in the b^m-symplectic manifold.*

Hence the desingularized b^m-form goes to folded in the case $m = 2k+1$ and to symplectic for $m = 2k$. And the b^m-integrable system goes to a folded integrable system (see [35]) in the case $m = 2k+1$ and to a standard integrable system for $n = 2k$.

7.5 Desingularization of the KAM theorem on b^m-symplectic manifolds

The idea of this section is to recover some version of the classical KAM theorem by "desingularizing the b^m-KAM theorem", as well as a new version of a KAM theorem that works for folded symplectic forms. Observe that no KAM theorem is known for folded symplectic forms. The best that is known is a KAM theorem for presymplectic structures that was done in [36]. Desingularizing the KAM means applying the b^m-KAM in the b^m-manifold and then translate the result to the desingularized setting.

To be able to obtain proper desingularized theorems we need to identify which integrable systems that can be obtained as a desingularization of a b^m-integrable system. To simplify computations we are going to use a particular case of b^m-integrable systems, where $f_1 = \frac{1}{I_1^{m-1}}$. We call these systems *simple*. Observe that by taking a particular case of b^m-integrable systems we will not get all the systems that can be obtained by desingularizing a b^m-integrable system, but some of them.

1. **Even case** $m = 2k$.

   $F = (f_1 = \frac{1}{I_1^{2k-1}}, f_2, \ldots, f_n)$, $\omega = \frac{1}{I_1^m} dI_1 \wedge d\phi_1 + \sum_{j=1}^n dI_j \wedge d\phi_j$. Observe that close to Z in the even case we can assume $f(I_1) = cI_1$ for some $c > (2 - \frac{1}{2^{2k-1}})$. Then $f_\varepsilon(I_1) = \frac{1}{\varepsilon^{(2k-1)}} \frac{cI_1}{\varepsilon} = c'I$, hence $\omega_\varepsilon = c' dI_1 \wedge d\phi_1 + \sum_{j=1}^n dI_j \wedge d\phi_j$. Also $F_\varepsilon^m(I_1) = c'$, $G_\varepsilon^m(I_1) = c'I_1$. Then,

$$\begin{cases} \tilde{I}_1 &= \int_0^{I_1} \frac{c'}{1/\tau^m} d\tau = \int_0^{I_1} c'\tau^m d\tau = c' \frac{I_1^{m+1}}{m+1}, \\ \tilde{\phi}_1 &= \frac{c'}{1/I_1^m} \phi_1 = c' I_1^m \phi_1 \end{cases}$$

$$F^\varepsilon = ((m-1)c_{m-1}c' I_1, f_2(\tilde{I}, \tilde{\phi}), \dots f_n(\tilde{I}, \tilde{\phi})). \tag{7.26}$$

Hence, the systems in this form can be viewed as a desingularization of a b^m-integrable system.

Theorem 7.5.1 (Desingularized KAM for symplectic manifolds)**.** *Consider a neighborhood of a Liouville torus of an integrable system F_ε as in 7.26 of a symplectic manifold (M, ω_ε) semilocally endowed with coordinates (I, ϕ), where ϕ are the angular coordinates of the torus, with $\omega_\varepsilon = c' dI_1 \wedge d\phi_i + \sum_{j=1}^n dI_j \wedge d\phi_j$. Let $H = (m-1)c_{m-1}c' I_1 + h(\tilde{I}) + R(\tilde{I}, \tilde{\phi})$ be a nearly integrable system where*

$$\begin{cases} \tilde{I}_1 &= c' \frac{I_1^{m+1}}{m+1}, \\ \tilde{\phi}_1 &= c' I_1^m \phi_1, \end{cases}$$

and

$$\begin{cases} \tilde{I} &= (\tilde{I}_1, I_2, \dots, I_n), \\ \tilde{\phi} &= (\tilde{\phi}_1, \phi_2, \dots, \phi_n). \end{cases}$$

Then the results for the b^m-KAM theorem 7.3.1 applied to $H_{sing} = \frac{1}{I_1^{2k-1}} + h(I) + R(I, \phi)$ hold for this desingularized system.

Remark 7.5.2. *This theorem is not as general as the standard KAM, but we also know extra information of the dynamics. For instance the perturbed of trajectories in tori inside of Z will be trajectories lying inside of Z. In this sense the theorem is new because it leaves invariant an hypersurface of the manifold.*

2. **Odd case** $m = 2k+1$.

   $F = (f_1 = \frac{1}{I_1^{2k}}, f_2, \dots, f_n)$ and $\omega = \frac{1}{I_1^{2k+1}} dI_1 \wedge d\phi_1 + \sum_{j=1}^n dI_j \wedge d\phi_j$. Before continuing we need the following notions defined in [35].

Definition 7.5.3. *A function $f : M \to \mathbb{R}$ in a folded symplectic manifold (M, ω) is folded if $df|_Z(v) = 0$ for all $v \in V = \ker\omega|_Z$.*

Definition 7.5.4. *An integrable system in a folded symplectic manifold (M, ω) with critical surface Z is a set of functions $F = (f_1, \dots, f_n)$ such that they define Hamiltonian vector fields which are independent ($df_1 \wedge \dots \wedge df_n \neq 0$ in the folded cotangent bundle) on a dense subset of Z and M, and commute with respect to ω.*

Note that we need to prove that the desingularized functions in this case are folded.

Observe that close to Z in the odd case we can assume $f(I_1) = -I_1^2 + 2$. Then $f_\varepsilon(I_1) = \varepsilon^{-(2k)} f(\frac{I_1}{\varepsilon}) = \frac{1}{\varepsilon^{2k}}(-(\frac{I_1}{\varepsilon})^2 + 2) = cI_1^2 + \frac{2}{\varepsilon^{2k}}$. Then

$$\omega_\varepsilon = 2cI_1 dI_1 \wedge d\phi_1 + \sum_{j=1}^{n} dI_j \wedge d\phi_j.$$

Also $F_\varepsilon^m(I_1) = 2cI_1$, $G_\varepsilon^m(I_1) = cI_1^2$. Then,

$$\left\{ \begin{array}{lcl} \tilde{I}_1 & = & \int_0^{I_1} \frac{2c\tau}{1/\tau^m} d\tau = 2c\frac{I_1^{(m+2)}}{(m+2)}, \\ \tilde{\phi}_1 & = & 2cI_1^{m+1}\phi_1 \end{array} \right.$$

Then the desingularized moment map becomes

$$F^\varepsilon = ((m-1)c_{m-1}cI_1^2, f_2(\tilde{I}, \tilde{\phi}), \dots f_n(\tilde{I}, \tilde{\phi})). \tag{7.27}$$

It is a simple computation to check that these functions are actually folded and hence they form a folded integrable system. Note that the systems of the form 7.27 can be viewed as a desingularization of a b^m-integrable system. Then, like we proceeded in the even case:

Theorem 7.5.5 (Desingularized KAM for folded symplectic manifolds)**.** *Consider a neighborhood of a Liouville torus of an integrable system F_ε as in 7.27 of a folded symplectic manifold (M, ω_ε) semilocally endowed with coordinates (I, ϕ), where ϕ are the angular coordinates of the Torus, with $\omega_\varepsilon = 2cI_1 dI_1 \wedge d\phi_1 + \sum_{j=2}^m dI_j \wedge d\phi_j$. Let $H = (m-1)c_{m-1}cI_1^2 + h(\tilde{I}) + R(\tilde{I}, \tilde{\phi})$ a nearly integrable system with*

$$\begin{cases} \tilde{I}_1 &= 2c\frac{I_1^{m+2}}{m+2}, \\ \tilde{\phi}_1 &= 2cI_1^{m+1}\phi_1, \end{cases}$$

and

$$\begin{cases} \tilde{I} &= (\tilde{I}_1, I_2, \dots, I_n), \\ \tilde{\phi} &= (\tilde{\phi}_1, \phi_2, \dots, \phi_n). \end{cases}$$

Then the results for the b^m-KAM theorem 7.3.1 applied to $H_{sing} = \frac{1}{I_1^{2k}} + h(I) + R(I, \phi)$ hold for this desingularized system.

Remark 7.5.6. *The last two theorems can be improved if considering b^m-integrable systems non necessarily* simple.

7.6 Applications to Celestial mechanics

In this last section of the thesis we catch up with the circular planar restricted three body problem which we discussed in Chapter 2. Given an autonomous Hamiltonian system of a symplectic manifold of dimension $2n$, the level sets of the Hamiltonian function are often endowed with a contact structure (a contact structure is given by a one form α satisfying a condition of type $\alpha \wedge (d\alpha)^{n-1} \neq 0$).

In [37, 38] the authors discuss applications of the b-apparatus in this context. In particular the notion of b^m-contact structures is introduced by translating the condition above for b^m-forms. The classical notions in the contact realm such as Reeb vector fields can also be introduced in this set-up.

In particular by considering the Mc Gehee change as we did in the contact context, the authors of [38] prove:

Theorem 7.6.1. *After the McGehee change, the Liouville vector field* $Y = p\frac{\partial}{\partial p}$ *is a* b^3*-vector field that is everywhere transverse to the level sets of the Hamiltonian* Σ_c *for* $c > 0$ *and the level-sets* $(\Sigma_c, \iota_Y\omega)$ *for* $c > 0$ *are* b^3*-contact manifolds. Topologically, the critical set of this contact manifold is a cylinder (which can be interpreted as a subset of the line at infinity) and the Reeb vector field admits infinitely many non-trivial periodic orbits on the critical set.*

One of the possible applications of our KAM theorem would be to find new periodic orbits of the restricted three body problem close to infinity by perturbing the periodic orbits described above.

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