PHD THESIS

WITNESSING NON-MARKOVIAN EVOLUTIONS

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Abstract

The formulation of quantum physics stands among the most revolutionary theories of the twentieth century. During the first decades of this century, many phenomena concerning the microscopic world were unexplained or had ad-hoc descriptions. The theory of quantum physics introduced a framework that allowed predicting these phenomena with unprecedented precision. While quantum mechanics offered counter-intuitive explanations for these experimental results, it predicted unexpected quantum phenomena which were considered symptoms of an ill-defined theory.

Decades passed and more and more empirical evidences sustained the existence of purely quantum effects and therefore the validity of this theory. Hence, it became a solid branch of science and physicists started to engineer scenarios where quantum effects could provide improvements if compared with classical scenarios. This approach gave birth to quantum information science, where quantum particles are manipulated to perform information tasks. Several innovative protocols, e.g. concerning state teleportation, dense coding, cryptography and integer factorization algorithms, proved that quantum physics allowed performances unattainable in classical settings.

The formulation of quantum protocols able to provide substantial speed-ups raised wide interest of the academic world and private companies. Nonetheless, the implementation of more and more complex quantum protocols became an increasingly harder task. Indeed, manipulating a large number of quantum particles with a level of noise that is small enough to obtain quantum advantages is, even nowadays, a demanding goal. The purely-quantum features essential for these speed-ups are fragile when noise influences experimental apparatus. Hence, in order to access the full potential of quantum theory, the ability to handle noisy environments is a fundamental goal.

This thesis is devoted to the study of open quantum systems (OQS), namely those where the interaction between the target quantum system and its surrounding environment is taken under consideration during the evolution. Indeed, isolated systems cannot provide realistic descriptions of dynamics. Un-

derstanding how to exploit and manipulate environments in order to obtain dynamics that are less aggressive with the information stored in our OQS is therefore an essential goal to achieve quantum advantages. There are two possible dynamical regimes for the information encoded in an OQS. We call an evolution *Markovian* when there is a one-way flow of information from the OQS to the environment. Instead, the *non-Markovian* regime is distinguished by one or more time intervals when this flow is reversed. In this case, we say that we witness information *backflows*. A characterization based on the different types of information quantifiers that can be considered in this context is fundamental to exploit these phenomena in information processing scenarios.

The main goal of this thesis is to examine the potential of correlation measures to show backflows when the OQS dynamics is non-Markovian. The first three works that we expose are devoted to this topic. First, we study how entanglement and quantum mutual information behave under non-Markovian evolutions. We follow with the formulation of a correlation measure that is able to witness almost-all non-Markovian evolutions. The last work along this topic provides the first one-to-one relation between correlation backflows and non-Markovian evolutions.

The last work in this thesis adopts a different point of view under which we can characterize OQS evolutions. We quantify non-Markovianity through the minimal amount of Markovian noise that has to be added in order to make an evolution Markovian.

Resumen

La formulación de la física cuántica se encuentra entre las teorías más revolucionadoras del siglo XX. Durante las primeras décadas de siglo, muchos fenómenos asociados al mundo microscópico yacían sin una descripción clara, o bien ésta era ad-hoc. La física cuántica introdujo un marco que permitió explicar estos fenómenos con una precisión sin precedentes. Si bien sus explicaciones eran contraintuitivas, los inesperados fenómenos cuánticos que predijo se consideraron síntomas de una teoría mal definida.

Pasaron los años y cada vez más evidencias empíricas sostuvieron la existencia de efectos puramente cuánticos, validando esta teoría. La física cuántica se convirtió en una sólida rama de la ciencia, y los físicos comenzaron a diseñar escenarios en los que sus efectos pudieran proporcionar mejoras en comparación con sus alternativas clásicas. Este enfoque dio origen al campo de la información cuántica, donde las partículas cuánticas se manipulan para realizar tareas de información. Varios innovadores protocolos, como la teletransportación de estados cuánticos, la "codificación densa", la criptografía y los algoritmos de factorización de enteros, demostraron el potencial de la física cuántica frente a estrategias clásicas.

La formulación de protocolos cuánticos capaces de proporcionar considerables mejoras despertó un gran interés en el mundo académico y en las empresas privadas. No obstante, la implementación de protocolos cuánticos cada vez más complejos se convirtió en una tarea sustancialmente más difícil. De hecho, manipular una gran cantidad de partículas cuánticas con un nivel de ruido lo suficientemente pequeño como para obtener ventajas cuánticas es, incluso a día de hoy, un objetivo exigente. Las características puramente cuánticas vitales para obtener estas mejoras son frágiles al ruido que afecta los instrumentos experimentales. Por lo tanto, para acceder a todo el potencial subyacente a la teoría cuántica, la capacidad de manejar ambientes ruidosos resulta un objetivo fundamental.

Esta tesis está dedicada al estudio de los sistemas cuánticos abiertos (SCA), es decir, aquellos en los que se tiene en cuenta la interacción entre el sis-

tema cuántico objeto y su ambiente circundante durante la evolución. De hecho, los sistemas aislados no pueden proporcionar descripciones realistas de la dinámica. Entender cómo explotar estos ambientes para obtener dinámicas menos agresivas con la información almacenada en nuestro SCA, es un objetivo primordial para conseguir ventajas cuánticas. Hay dos posibles regímenes dinámicos para la información codificada en un SCA. Decimos que una evolución es *Markoviana* cuando hay un flujo de información unidireccional desde el SCA al medio ambiente. Por contra, en el régimen *no Markoviano* se distinguen unos intervalos temporales en los que este flujo se invierte. En este caso, decimos que somos testigos de *reflujos de información*. Una caracterización basada en los diferentes tipos de cuantificadores de información que pueden considerarse en este contexto es fundamental para explotar estos fenómenos en escenarios de procesamiento de información.

El objetivo principal de esta tesis es examinar el potencial de las medidas de correlación para mostrar reflujos cuando la dinámica es no Markoviana. Los tres primeros trabajos que exponemos están dedicados a este tema. En primer lugar, estudiamos los potenciales del entanglement entrelazamiento y la información mutua cuántica. Seguidamente presentamos la formulación de una medida de correlación capaz de presenciar casi todas las evoluciones no Markovianas. Por último, proponemos la primera relación de equivalencia entre los reflujos de correlación y la no Markovianidad.

Concluimos proponiendo un punto de vista diferente bajo el cual podemos caracterizar las evoluciones de SCA. Cuantificamos la no Markovianidad a través de la mínima cantidad de ruido Markoviano que debe agregarse para tornar una evolución en Markoviana.

Contents

List of publications

Included in this thesis

- Dario De Santis, Markus Johansson, Bogna Bylicka, Nadja K. Bernardes and Antonio Acín, *Correlation measure detecting almost all non-Markovian evolutions*, Physical Review A **99**, 012303 (2019).

- Dario De Santis, Markus Johansson, Bogna Bylicka, Nadja K. Bernardes and Antonio Acín, *Witnessing non-Markovian dynamics through correlations*, Physical Review A **102**, 012214 (2020).

- Dario De Santis and Markus Johansson, *Equivalence between non-Markovian dynamics and correlation backflows*, New Journal of Physics **22**, 093034 (2020).

- Dario De Santis and Vittorio Giovannetti, *Measuring non-Markovianity via incoherent mixing with Markovian dynamics*, Physical Review A **103**, 012218 (2021).

Not included in this thesis

- Saleh Rahimi-Keshari, Mohammad Mehboudi, Dario De Santis, Daniel Cavalcanti and Antonio Acín, *Verification of joint measurability using phase-space quasiprobability distributions*, arXiv:2012.06853 (2020).

List of acronyms

OQS	Open quantum system
QMI	Quantum mutual information
QENM	Quasi-eternal non-Markovian
POVM	Positive-operator valued measure
ME−POVM	Maximally entropic POVM
GADC	Generalized amplitude damping channel
EES	Equiprobable ensemble of states

Chapter 1

Introduction

We start by introducing the basic topics that are used in this thesis, namely quantum information theory and open quantum system dynamics. We conclude the chapter with the motivations and our main results.

1.1 Quantum information theory

Quantum physics acquired more and more importance during the twentieth century and nowadays it constitutes a fundamental building block of our scientific knowledge. This theory allowed predicting outcomes and describe phenomena of the atomic-sized world that until that time were described by different inconsistent theories. The accuracy and the generality allowed by this new framework were so unparalleled that it influenced several branches of science at that time. Immediately after the first seminal works, several new phenomena were predicted and perfectly described.

Two main properties of quantum particles had a deep impact that allowed making this theory so successful. First, superposition of quantum states introduced a new paradigm under which systems could be studied. Quantum particles can be prepared in a superposition of different states that cannot be reproduced by mere statistical mixing. Secondly, the concept of non-local correlations were introduced. Rightly, entanglement is the most popular feature that people inside and outside the scientific world associate with quantum physics. This particular type of correlation between quantum particles allows experimenters to influence correlated systems even at macroscopic distances without violating the no-signaling principle.

The interplay between these features made possible to engineer quantum protocols where quantum particles were considered as carriers of information and the quantum bits, namely the *qubits*, replaced ordinary bits of information. Quantum information theory studied how to wisely apply quantum transformations, or quantum gates, on qubits. The main difference between a classical and a quantum bit is that, while the former assumes either the value 0 or 1, the latter can be considered in a coherent superposition of the two logical states. These concepts paved the way to quantum computation, where protocols were designed in a revolutionary fashion. This new framework allowed speed-ups in many computational tasks, if compared with their classical counterparts. The derivation of quantum protocols that provided improvements or even scenarios where only quantum computing could lead to a result in a feasible time (*quantum advantage*) became a central research topic.

The potential of quantum information theory was enriched by other phenomena with no classical analogue. Quantum teleportation and super-dense coding are two fundamental examples to show how the quantum realm provides many counter-intuitive and unexpected tools. Quantum cryptography is another field of quantum information theory that generated wide interest. Indeed, the whole new toolbox of techniques provided by quantum physics allowed theoretically-secure secret key distribution and communication among users that share entangled particles at large distances.

During the last decades quantum systems have been exploited more and more to fulfill information processing tasks. Research groups from the academic world and worldwide known companies such as Google, Amazon and IBM opened quantum branches in order to implement quantum technologies that aim to exploit the quantum potential in the near future. Nowadays, quantum devices that operates with larger numbers of qubits are being realized and the quantum advantage starts to be considered a reachable goal.

1.2 Open quantum systems dynamics

The easiest way to store and process quantum information starts with encoding information in qubits or higher-dimensional systems, namely *qudits*. Several quantum systems degrees of freedom can be exploited for this task, e.g. by considering photon polarization, particle spins or electron configurations of excited ions. In terms of quantum information processing tasks, the ideal framework would be given by quantum units that, once initialized, interact among themselves while being isolated from the environment. This scenario would lead to a unitary evolution of the whole system where the total information is constant in time. Unfortunately, quantum systems cannot be considered completely isolated from the surrounding environment and therefore they have to be treated

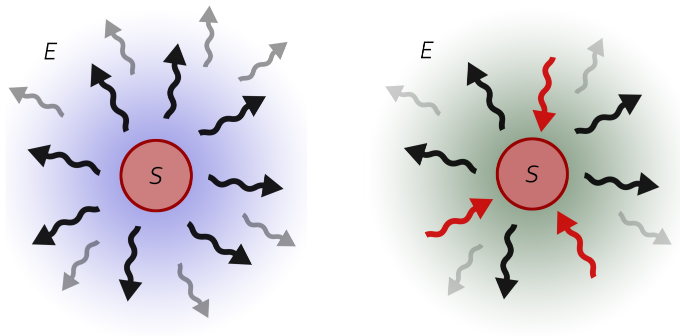


Figure 1.1: In the Markovian regime (left) the information encoded in the OQS S is lost in the environment E and never recovered during the dynamics. In this case we have a one-way flow of information from S to E. In the non-Markovian regime (right) there are one or more time intervals when this information flux is reversed from E to S. These phenomena characterize non-Markovian evolutions and are called information backflows.

as *open quantum systems* (OQS). The characterizing feature of these systems is that their evolution is not only described by their own Hamiltonian, but also by the interaction with the environment. In general this interaction leads to non-unitary transformations that make the information contained in our OQS not constant in time. As a consequence, we often obtain a very strict limiting factor in terms of coherence time, namely the time-scale that has to be considered before the quantum information becomes too poor to be processed faithfully. Therefore, an important objective is to understand how to choose and manipulate the environment that surrounds our OQS in order to make the information processing task under consideration feasible.

We can distinguish two different phenomenological regimes for the information encoded in our evolving OQS (see Fig. 1.1). In the *Markovian* regime, the interaction between the system and the environment establishes a one-way flow between the two parts: the information is monotonically lost in the environment and never recovered later in time. In these cases the environment is called *memoryless*. Indeed, no sign of the information lost by the OQS is shown by the environment during the evolution. On the contrary, we call an evolution *non-Markovian* when the information lost by the system is partially or completely recovered in one or more time intervals during the evolution, namely when a quantifier of the encoded information is non-monotonic in time. We say that a non-Markovian evolution shows a *backflow* of information whenever a temporary increase of the OQS information occurs. While these are the phenomenological differences between the Markovian and non-Markovian regimes, the mathematical framework used to make this distinction is based on a divisibility property of the maps describing OQS evolutions. According to this structure, an evolution is considered Markovian when this divisibility property is satisfied and non-Markovian when violated. The connection between the phenomenology and the mathematical description of Markovian and non-Markovian evolutions is a central topic in the study of OQS dynamics. As a general rule, we simply say "non-Markovian evolution" when the dynamics is non-Markovian according to our mathematical description, while we talk about "information backflows" when we want to emphasize the expected phenomenology.

1.3 Motivation and main contribution

The study of OQS dynamics offers several challenges. First, a complete characterization of this type of evolutions is still missing. Secondly, given a non-Markovian evolution and an initial state for the system, it is not well-understood which forms of information quantities provide backflows. Moreover, many efforts are devoted to design protocols where the exploitation of non-Markovian dynamics is crucial to obtain specific advantages that would not be possible in the Markovian regime.

The realization of quantum circuits where the corresponding environments either are as less destructive as possible or can provide backflows with useful timings is of fundamental importance for complex computations. Indeed, the study of Markovian and non-Markovian dynamics plays a central role to obtain sufficiently long coherence times needed to realize quantum protocols that include many qubits and gates.

The main target of this thesis is to study different quantifiers of information that can show backflows when finite-dimensional OQS evolve under non-Markovian evolutions. We explore the possibility to consider different initial states and ancillary systems in order to obtain the maximum non-Markovian witnessing potential of the information quantifier. To be more specific, we say that an information quantifier and an evolving OQS *witness* a non-Markovian dynamics when we obtain an information backflow that only non-Markovian evolutions can produce. Moreover, we characterize this regime of evolution in fundamentally different manners by introducing functionals, namely *measures* of non-Markovianity, that estimate the non-Markovian degree of evolutions. Point-by-point, we describe our main contributions.

1.3.1 Quantum correlations as non-Markovianity witnesses

Quantum correlations are provably one of the most important resources needed to accomplish many quantum protocols, where often highly correlated multipartite systems are indispensable. Entanglement measures and quantum mutual information (QMI) are two widely used correlation measures. For instance, the use of the former is well-known in scenarios where non-local effects generated by correlated systems are exploited to perform protocols between distant users, e.g. quantum teleportation or quantum key distribution. Instead, the latter is widely used in quantum communication tasks, for example in coding/decoding scenarios when we want to quantify the capacity of a given channel.

The use of these two correlation measures has already been considered in literature to quantify non-Markovianity [RHP10, LFS12]. Indeed, correlation measures can be used to quantify the amount of information shared between two systems and no Markovian evolution can increase these quantities if applied to one side of a bipartition. As a consequence, these quantities are perfectly suited to measure non-Markovianity. Indeed, any increase of a correlation measure is uniquely attributable to a non-Markovian dynamics. The goal of finding the initial states that are able to witness backflows for any time allowed by a non-Markovian evolution falls under the non-Markovian witnessing problem. This question is highly non-trivial and is the key to understand if a given quantifier is a good witness of non-Markovianity. Indeed, there are quantifiers of non-Markovianity that have an easy interpretation and are easy to calculate but fail to witness many evolutions. Given this scenario, we ask: *are entanglement measures and QMI able to witness all non-Markovian evolutions?*

Contribution

Our main results are the following:

- Entanglement measures cannot show backflows for a class of entanglement breaking non-Markovian evolutions and OQS-ancilla configurations;
- QMI cannot show backflows for a class of non-Markovian qubit evolutions and OQS-ancilla configurations;
- We study the interplay between the non-Markovian witnessing potential of QMI and the initial entanglement of OQS-ancilla states.

We therefore obtained several results regarding the potential of correlation measures to witness non-Markovianity. In particular, we first consider the class of *single parameter evolutions* for which we derive a non-Markovianity condition and results valid for differentiable correlation measures. Paradigmatic examples included in this class are depolarization, as well as dephasing and amplitude damping. Moreover, we consider random unitary evolutions and in particular we introduce the class of quasi-eternal non-Markovian (QENM) qubit evolutions, a generalization of the well-known eternal non-Markovian model [HCLA14, BCF17, MCPS17].

We show in which cases entanglement measures fail to witness a class of entanglement breaking non-Markovian evolutions. Moreover, we examine a QENM evolution belonging to this class. We follow by studying the more complex case of QMI. Among the various results that we derive, we explore how the QMI non-Markovian witnessing potential depends on the entanglement of initial states. Interestingly, we show cases where maximally entangled states are not the most useful choices. Moreover, we provide conditions for qubit non-Markovian evolutions under which QMI fails to show backflows when qubit ancillas are considered. Finally, we show how to build QENM evolutions satisfying these conditions.

1.3.2 A correlation measure witnessing almost-all non-Markovian evolutions

The goal of finding the explicit construction of a non-Markovian witness for any evolution, namely by proposing an information quantifier and an initial state, is of central interest. As discussed before, the absence and presence of information backflows provide the phenomenological descriptions for, respectively, Markovian and non-Markovian evolutions. Instead, the mathematical framework used to distinguish these two regimes is based on a divisibility property of the evolution maps. Hence, by proving the occurrence of backflows whenever this divisibility property is violated we confirm that the adopted mathematical definition of non-Markovianity corresponds to the expected phenomenology. On the contrary, in case an evolution that is non-Markovian according to our mathematical framework does not show any information backflow, the mathematical definition of non-Markovianity would be compromised. Secondly, this topic shades light on the possibility to exploit non-Markovianity as a resource in quantum information protocols: we must know what kind of information can be retrieved when specific evolutions are exploited and, correspondingly, which initial states have to be considered.

A constructive method that allows detecting non-Markovianity for almost-all finite-dimensional evolutions has been proposed in [BJA17]. This approach considers the distance between states defined over the evolving OQS and an ancillary system as witness of non-Markovianity. For almost-all evolutions they provide a class of pairs of initial states that are able to show an increasing distance if and only if the evolution is non-Markovian. Similarly, we ask: *can correlation measures witness almost-all non-Markovian evolutions?*

Contribution

Our main results in this direction are the following:

- We introduce a correlation measure which is able to witness almost-all non-Markovian evolutions;
- We provide a constructive method to build initial states useful for this task.

This new correlation measure quantifies the possibility to distinguish between different states obtained on one side of a bipartition if the other side is subjected to a *maximally entropic* measurement. We introduced these measurements as those such that the outcomes have the same occurrence probabilities. We construct a set of evolution-dependent initial states that suit for the non-Markovian witnessing task, namely provide a backflow of the newly introduced correlation, where the help of ancillary systems is exploited. Indeed, in order to increase the witnessing potential of initial states, we discuss how to enlarge one share of the bipartite system by introducing ancillary systems in different manners. Since this witness of non-Markovianity is highly asymmetric between the two shares, we propose a symmetrized version with the same witnessing potential. Finally, we show the details of this technique by studying an explicit example.

1.3.3 Equivalence between non-Markovianity and correlation backflows

The possibility to find an information quantifier that, given a wisely-chosen initial state that in general can be shared between the OQS and ancillary systems, allows providing a backflow for any non-Markovian evolution is of central interest. First, this result would provide a proof that the mathematical definition of non-Markovianity is indeed adherent to the phenomenological description used to describe it, namely through information backflows. Secondly, we would obtain a biunivocal connection between the particular information quantifier backflow and the non-Markovian nature of dynamics. In [BD16] the authors show how this result can be obtained by considering the guessing probability of ensembles of states defined among the evolving OQS and an ancillary system. This quantity represents the possibility to rightly guess which state has been randomly picked from an ensemble of states by performing an optimal measurement. The results in [BD16] can be used to show that also a quantity called singlet fraction [KRS09] can be exploited similarly. Therefore, we ask: *can correlation measures provide backflows for all non-Markovian evolutions?*

Contribution

In this direction, our main result is the following:

- We introduce a class of correlation measures that shows the first one-to-one relation between non-Markovian dynamics and correlation backflows.

This class of correlations is obtained through a generalization of work presented in Section 1.3.2. We propose a set of initial states shared between the OQS and ancillas and prove that for any evolution there exists a continuum of states from this set that show a backflow if and only if the evolution is non-Markovian. Finally, we build a non-Markovianity measure that collects the maximal backflow that can be provided by our correlation measure. While its computation can be in general very demanding, it is proved to be positive for any non-Markovian evolution.

1.3.4 Measuring non-Markovianity via incoherent mixing with Markovian dynamics

Non-Markovianity measures are often associated to the possibility to obtain backflows from dynamics. In this way we define a hierarchy trough the maximal amount of information that evolutions can provide for a *precise* observable. As a consequence, different non-Markovianity measures of this kind provide different properties of the evolution. In [ABCM14] it is shown that in general this order highly depends on the measure chosen. It follows that the introduction of several measures on the one hand permits to study the multi-faceted potential of non-Markovian evolutions. For instance, it may happen that an evolution provides large correlation revivals and small entropy backflows while a different evolution is characterized by a reversed behavior. On the other hand this approach may result confusing if we want to measure non-Markovianity with an objective scale. This problem suggests the introduction of a measure of non-Markovianity similar to the well-known *robustness* measure for entanglement, namely by considering the minimal distance between the given evolution and the Markovian set. The main difficulty implied by this approach is given by the non-convex geometry of the Markovian set of evolutions [WECC08]. Moreover, this feature is the main obstacle for the formulation of a resource theory of non-Markovianity. Hence, *is it possible to introduce a measure of non-Markovianity by studying convex combinations of non-Markovian and Markovian evolutions?*

Contribution

Our results are the following:

- We introduce a non-Markovianity measure through the minimal Markovian noise that has to be incoherently mixed with a non-Markovian evolution in order to make the resulting evolution Markovian;
- We show how to apply this technique for depolarizing and dephasing non-Markovian evolutions;
- We obtain analytical results for all continuous and regular-enough non-continuous depolarizing evolutions in any finite dimension.

This measure is not connected to any particular information backflow but it is strongly linked with the set of evolutions and its geometry. Therefore, given the non-convexity of the Markovian and non-Markovian sets, this approach is highly non-trivial. We conjecture that, given a non-Markovian evolution belonging to a convex set characterized by a precise symmetry, the most effective evolution needed to make it Markovian via incoherent mixing is characterized by the same symmetry.

In order to show how to use this technique, we propose an in-depth study of depolarizing evolutions in any finite dimension: for any non-Markovian depolarizing dynamics we find the optimal Markovian evolution that makes the mixture Markovian and therefore provides the value of our non-Markovianity measure. First, we derive the analytical values of this measure and the corresponding intuitive interpretations for regular-enough depolarizing evolutions. Then, we illustrate why non-continuous evolutions are highly non-trivial to measure. For this purpose, we provide a simple example that shows the intrinsic ambiguity for the choice of the optimal Markovian depolarizing evolution. Nonetheless, we design a numeric approach that singles out this solution and provides a value for the measure of non-Markovianity. Finally, in order to prove the applicability of our technique to other structured sets of evolutions, we generalize our approach to qubit dephasing evolutions.

Chapter 2

Preliminaries

In this chapter we introduce some fundamental tools of quantum information theory and open quantum systems dynamics. Moreover, we review various results concerning the characterization of Markovian and non-Markovian evolutions. In order to do so, we start in Section 2.1 by describing density operators defined over one or more parties and generalized quantum measurements. In Section 2.3 we show how closed and open quantum systems evolve, where we also discuss the definition of Markovianity adopted throughout this thesis. We follow in Section 2.4 by describing the main recent results concerning the characterization of non-Markovianity. In Section 2.5 we study several techniques that can be used to witness non-Markovianity, where we put particular emphasis on the possibility to observe information backflows for any non-Markovian evolution. Finally, in Section 2.6 we describe an exemplary class of evolutions, namely random unitary, which include several commonly studied models. Interestingly, for this class we can consider compact conditions that easily discriminate Markovian from non-Markovian evolutions.

2.1 Quantum states and measurements

We introduce those mathematical tools needed to define quantum systems, bipartitions and measurement processes. A particularly useful scenario that we describe in this section is obtained when we measure one share of a bipartite system in order to generate an ensemble of output states on the other side of the system.

2.1.1 Density operators

Consider a finite-dimensional quantum system S with d degrees of freedom. When there is a complete (statistical) certainty of the status of S, we say that our quantum system is in a *pure state*. These states are represented by vectors $|\phi\rangle_S$, often called *kets*, in a d-dimensional Hilbert space $\mathcal{H}_S$, which is isomorphic to $\mathbb{C}^d$. Given an orthonormal basis $\{|i\rangle_S\}_{i=1}^d$ for $\mathcal{H}_S$, any pure state $|\phi\rangle_S \in \mathcal{H}_S$ can be written as $|\psi\rangle_S = \sum_{i=1}^d a_i|i\rangle_S$, where we require the coordinates $a_i \in \mathbb{C}$ to be normalized $\sum_{i=1}^d |a_i|^2 = 1$. We can also define the dual space of $\mathcal{H}_S$, namely $\mathcal{H}_S^*$, where the corresponding elements are called *bra* vectors $\langle\phi|_S : \mathcal{H}_S \to \mathbb{C}$, namely linear forms from $\mathcal{H}_S$ to $\mathbb{C}$. Hence, the action of $\langle\phi|_S$ on $|\psi\rangle_S$ is written $\langle\phi|\psi\rangle_S \in \mathbb{C}$. We say that $\langle\psi|_S$ is the Hermitian conjugate of $|\psi\rangle_S$ if $\langle\psi|\psi\rangle_S = 1$ and we define the inner product between two kets $|\phi\rangle_S, |\psi\rangle_S \in \mathcal{H}_S$ as $\langle\phi|\psi\rangle_S$.

We define $B(\mathcal{H}_S)$ to be the set of linear operators $X : \mathcal{H}_S \to \mathcal{H}_S$. Moreover, the *state space* $S(\mathcal{H}_S)$ is the subset of $B(\mathcal{H}_S)$ of Hermitian, non-negative and trace-one operators ρ_S, namely such that:

$$\rho_S = \rho_S^\dagger\,, \quad \rho_S \geq 0\,, \quad \mathrm{Tr}\,[\rho_S] = 1. \tag{2.1}$$

The elements $\rho_S \in S(\mathcal{H}_S)$ are called *density operators*. In the state space $S(\mathcal{H}_S)$, pure state are represented by operators $|\phi\rangle\langle\phi|_S$. If S is in a pure state, we know that our system is in a given state $\rho_S = |\phi\rangle\langle\phi|_S$ with probability $p = 1$. Instead, in case we do not have this certainty, we can only provide probabilities $\{p_i\}_i$ that our system is in one of the pure states $\{|\phi_i\rangle\langle\phi_i|_S\}_i$ (not necessarily orthogonal), where $\{p_i\}_i$ is a probability distribution. Hence, if S is not in a pure state we say that it is in a *mixed state*, where the corresponding density operator $\rho_S \in S(\mathcal{H}_S)$ can always be written as:

$$\rho_S = \sum_i p_i|\phi_i\rangle\langle\phi_i|_S\,. \tag{2.2}$$

Hence, mixed states are statistical mixtures between different pure states of the system. Naturally, a density operator describing an intrinsically mixed state has to be characterized by an ensemble $\{p_i, |\phi_i\rangle\langle\phi_i|_S\}_i$ with at least two probabilities different from zero. A functional that quantifies the mixing degree of quantum states is the *purity*, defined as $\mathrm{Tr}[\rho_S^2] \in [1/d, 1]$, which is equal to 1 if and only if ρ_S is pure and equal to $1/d$ only for the maximally mixed state $\rho_S = \mathbb{1}_S/d$, where $\mathbb{1}_S = \sum_{i=1}^d |i\rangle\langle i|_S \in B(\mathcal{H}_S)$ is the identity operator on $\mathcal{H}_S$ and $\{|i\rangle_S\}_{i=1}^d$ is an orthonormal basis of $\mathcal{H}_S$.

Bloch representation

A geometrical approach to describe the state space is given by the Bloch representation, where each state ρ_S is associated to a real vector. In case of a qubit system, namely for $d = 2$, mixed and pure states in $S(\mathcal{H}_S)$ can be represented by three-dimensional Bloch vectors $\boldsymbol{r} = (r_x, r_y, r_z) \in \mathrm{BB}(1)$ as follows

$$\rho_S = \frac{\mathbb{1}_S + \boldsymbol{r} \cdot \boldsymbol{\sigma}}{2}, \tag{2.3}$$

where $\mathrm{BB}(R) \subset \mathbb{R}^3$ consists on the vectors inside the sphere of radius R and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector containing Pauli matrices. In order to evaluate the Bloch vector $\boldsymbol{r}$ corresponding to a state ρ_S, we simply have to evaluate the expectation values of the Pauli operators: $r_i = \mathrm{Tr}\,[\rho_S \sigma_i]$. BB_2=BB(1) is called the Bloch ball and corresponds to the qubit state space, namely there is a one-to-one relation between qubit states $\rho_S \in S(\mathcal{H}_S)$ and vectors $\boldsymbol{r} \in \mathrm{BB}_2$. We call BS_2 the Bloch sphere given by the unit vectors in BB_2, namely the border of the radius-1 sphere. These vectors represent the set of qubit pure states. Indeed, there is a one-to-one relation between qubit pure states $|\psi\rangle\langle\psi|_S \in S(\mathcal{H}_S)$ and unit vectors $\boldsymbol{r} \in \mathrm{BS}_2$. The definition that identifies mixed states as those resulting from convex combinations of pure states (see Eq. (2.2)) is particularly evident in this representation: any vector in BB_2 can be represented as a convex combinations of elements in its border BS_2. As we see in the following, this representation is very useful to visualize the effects of quantum transformations in terms of the geometry of their actions.

In the case of a qutrit, namely for $d = 3$, we have 8-dimensional vectors $\boldsymbol{r}$ representing density operators, where the Pauli operators in Eq. (2.3) are replaced by Gell-Mann matrices. Instead, given a generic d-dimensional system S, any density operator $\rho_S \in S(\mathcal{H}_S)$ can be represented by a $d^2 - 1$-dimensional real vector $\boldsymbol{r}$ as follows

$$\rho_S = \sum_{i=0}^{d^2-1} \mathrm{Tr}\,[\rho_S G_i]\, G_i = \frac{\mathbb{1}_S}{d} + \sum_{i=1}^{d^2-1} r_i G_i\,, \tag{2.4}$$

where the operators G_i (for $i = 1, \ldots, d^2 - 1$) are the traceless Hermitian generators of SU(d) such that $\mathrm{Tr}\left[G_i G_j\right] = \delta_{ij}$, $G_0 = \mathbb{1}_S / \sqrt{d}$ and $r_i = \mathrm{Tr}\,[\rho_S G_i]$ are the corresponding expectation values for $i = 1, \ldots, d^2 - 1$. The set BB_d of physical vectors $\boldsymbol{r}$, namely the subset of $\mathbb{R}^{d^2-1}$ that represents any qudit state in $S(\mathcal{H}_S)$, is defined by [Kim03]

$$\mathrm{BB}_d = \{\boldsymbol{r} \in \mathbb{R}^{d^2-1} \,|\, (-1)^j a_j(\boldsymbol{r}) \geq 0\, (j = 1, \ldots d)\}, \tag{2.5}$$

where $a_j(\boldsymbol{r})$ is the j-th coefficient of the characteristic polynomial $\det(x\mathbb{1}_S - (\mathbb{1}_S/d + \sum_{i=1}^{d^2-1} r_i G_i))$.

2.1.2 Generalized measurements

Quantum measurements are the main instruments used in quantum physics to obtain information from a quantum system. Given an experimental apparatus that measures our system, while on the one hand it provides partial information about the system, on the other hand the system is perturbed. In the following sections, we never consider a scenario where preserving the state after the measurement is needed and therefore we consider the system discarded after the output is obtained. Any measurement process on a quantum state $\rho_S \in S(\mathcal{H}_S)$ can be represented by a *positive-operator valued measure* (POVM), namely an indexed set of Hermitian and positive semi-definite operators $\{P_{S,i}\}_{i=1}^n$ of $B(\mathcal{H}_S)$ that sum up to the identity, namely such that

$$P_{S,i} = P_{S,i}^\dagger\,, \quad P_{S,i} \geq 0\,, \quad \text{for } i = 1,\ldots,n\,, \tag{2.6}$$

$$\sum_{i=1}^n P_{S,i} = \mathbb{1}_S\,, \tag{2.7}$$

where n is the number of possible measurement outcomes and $\mathbb{1}_S : B(\mathcal{H}_S) \to B(\mathcal{H}_S)$ is the identity operator in $B(\mathcal{H}_S)$. The i-th output of the measurement is represented by the POVM element $P_{S,i}$ and the Born rule states that

$$p_i = \mathrm{Tr}\left[\rho_S P_{S,i}\right]\,, \tag{2.8}$$

is the corresponding occurrence probability. In case a POVM is composed by $d = \dim(\mathcal{H}_S)$ rank-one projectors of the form $P_i = |\psi_i\rangle\langle\psi_i|_S$, where $\{|\psi_i\rangle_S\}_{i=1}^d$ is an orthonormal basis of $\mathcal{H}_S$, we say that $\{|\psi_i\rangle\langle\psi_i|_S\}_{i=1}^d$ is a *projective measurement*. Notice that, while for any set of projectors $\{|\psi_i\rangle\langle\psi_i|_S\}_{i=1}^d$ the corresponding elements are mutually orthogonal, namely $|\psi_i\rangle\langle\psi_i|_S \cdot |\psi_j\rangle\langle\psi_j|_S = \delta_{ij}|\psi_i\rangle\langle\psi_i|_S$, this is not the case for generic POVM elements.

2.2 Bipartite quantum systems

We showed how to define states and measurements for a d-dimensional system, which is considered as a single localized entity. This is the case of e.g. particles, spins or photons that are controlled by a single user. We call a system bipartite when it is shared between two parties, let say Alice and Bob. Both Alice and Bob posses a quantum particle, let say A and B, described by Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Bipartite systems $A - B$ are characterized by states that, in general, cannot be described only by the states $\rho_A \in S(\mathcal{H}_A)$ and $\rho_B \in S(\mathcal{H}_B)$ that Alice and Bob own, respectively, but by more general states. Moreover, different measurement scenarios are possible for bipartite systems: we can either measure the complete system or only one of the two shares.

2.2.1 Bipartite mixed states

The Hilbert space describing a bipartite system $A - B$ is given by the tensor product of the Hilbert spaces of its components: $\mathcal{H}_{AB} \equiv \mathcal{H}_A \otimes \mathcal{H}_B$, which implies $\dim(\mathcal{H}_{AB}) = d_A d_B$. Consider the correlated $A - B$ system such that A is Alice's system and B is Bob's system. A particularly useful characterizations of bipartite quantum states $\rho_{AB} \in S(\mathcal{H}_{AB})$ is given by the correlation between its subsystems. We say that Alice and Bob share a *product state* if there exist $\rho_A \in S(\mathcal{H}_A)$ and $\rho_B \in S(\mathcal{H}_B)$ such that ρ_{AB} can be written in the following form

$$\rho_{AB} = \rho_A \otimes \rho_B \,. \tag{2.9}$$

Product states represent uncorrelated systems: any action performed on A does not influence B and vice versa. Instead, if $A - B$ is in a probabilistic mixture of product states, we say that the system is in a *separable state*. This scenario implies the existence of the set $\{p_i, \rho_{A,i}, \rho_{B,i}\}_{i=1}^m$, where $\{p_i\}_{i=1}^m$ is a probability distribution, $\rho_{A,i} \in S(\mathcal{H}_A)$ and $\rho_{B,i} \in S(\mathcal{H}_B)$, such that

$$\rho_{AB} = \sum_{i=1}^{m} p_i \, \rho_{A,i} \otimes \rho_{B,i} \,. \tag{2.10}$$

These states are classically correlated and can be prepared by local operations (LO) on A and B assisted by classical communication (CC) between Alice and Bob. Bipartite states that cannot be written as in Eq. (2.10) are non-separable, or *entangled*. These states are correlated in a non-classical way: they cannot be prepared through LOCC and provide effects not reproducible with classical systems. A particularly useful class of non-separable states are those called *maximally entangled*. A formulation for these states that we often use in this thesis when $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = d$ is $|\phi^+\rangle_{AB} = d^{-1/2} \sum_{i=1}^d |i\rangle_A \otimes |i\rangle_B$, where $\{|i\rangle_A\}_{i=1}^d$ and $\{|i\rangle_B\}_{i=1}^d$ are orthonormal basis of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. The corresponding density matrix is:

$$\phi^+_{AB} = |\phi^+\rangle\langle\phi^+|_{AB} = \frac{1}{d} \sum_{i,j=1}^{d} |i\rangle\langle j|_A \otimes |i\rangle\langle j|_B = \frac{1}{d} \sum_{i,j=1}^{d} |ii\rangle\langle jj|_{AB} \,. \tag{2.11}$$

Finally, we note that we use the symbol "–" when we want to underline which is the bipartition under study. For instance, by writing $AB - C$ we emphasize that we are interested in studying the correlations shared between AB and C, where A and B are considered as a unique system. Indeed, given a state $\rho_{ABC} \in S(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$, the type of bipartite correlations in $AB - C$ may be completely different form those present in $A - BC$ or $AC - B$.

2.2.2 Measurements and bipartitions

A straightforward generalization of the measurement processes introduced in Section 2.1.2 leads us to define measurements for bipartite systems as those sets $\{P_{AB,i}\}_{i=1}^n$ of operators $P_{AB,i} \in B(\mathcal{H}_{AB})$ that satisfy conditions (2.6) and (2.7). Hence, by measuring the complete system described by ρ_{AB}, we obtain outputs $i = 1, \ldots, n$ with corresponding probabilities $p_i = \mathrm{Tr}\,[\rho_{AB} P_{AB,i}]$.

A different approach is given when we choose to measure only one of the two parties that constitute the bipartite system $A - B$. Consider the scenario where Alice and Bob share a mixed state $\rho_{AB} \in S(\mathcal{H}_{AB})$ and Alice applies a POVM $\{P_{A,i}\}_{i=1}^n$ on her side of ρ_{AB}. While with probability p_i Alice obtains the i-th output, Bob's share of the state is transformed into a (in general different) state $\rho_{B,i}$. In this case, we say that an *output ensemble* $\mathcal{E}(\rho_{AB}, \{P_{A,i}\}_{i=1}^n) \equiv \{p_i, \rho_{B,i}\}_{i=1}^n$ is generated on Bob's side. It is possible to see that

$$p_i = \mathrm{Tr}\,[\rho_{AB} P_{A,i} \otimes \mathbb{1}_B]\,, \quad \rho_{B,i} = \mathrm{Tr}_A\,[\rho_{AB} P_{A,i} \otimes \mathbb{1}_B]/p_i\,. \tag{2.12}$$

where $\mathrm{Tr}_A\,[\,\cdot\,]$ is the partial trace over the degrees of freedom of A. We call $\{p_i\}_{i=1}^n$ and $\{\rho_{B,i}\}_{i=1}^n$ respectively the *output probability distribution* and the *output states* of Alice's measurement $\{P_{A,i}\}_{i=1}^n$.

The transformation of Bob's system is due to the correlations in $A - B$. Indeed, if Alice and Bob share a product state $\rho_A \otimes \rho_B$, namely an uncorrelated state, we see that Bob's state is not influenced by any measurement performed by Alice. Instead, if they share the maximally entangled state, the output ensemble can be made of orthogonal states, namely perfectly distinguishable. For instance, in the two-qubit case, a maximally entangled state is given by $\rho_{AB} = |\phi^+\rangle\langle\phi^+|_{AB}$, where $|\phi^+\rangle_{AB} = (|00\rangle_{AB} + |11\rangle_{AB})/\sqrt{2}$ (see Eq. (2.11)). We see that Alice can apply the projective measurement $\{P_{A,i}\}_{i=1}^2 = \{|0\rangle\langle 0|_A, |1\rangle\langle 1|_A\}$ on her share and Bob obtains the orthogonal output states $\rho_{B,1} = |0\rangle\langle 0|_B$ and $\rho_{B,2} = |1\rangle\langle 1|_B$ with probability $p_{1,2} = 1/2$.

2.3 Evolution of open quantum systems

In this section we describe the evolution of open quantum systems (OQS), namely those having an evolution determined by the internal degrees of the system and the interaction with the surrounding environment. This approach is more general than the closed-system case, but it requires the introduction of some necessary mathematical tools. We start by describing the evolution of closed systems, then we follow by considering an evolving SE system as a composite closed system. Finally, in the same scenario, we show how to describe the evolution of the OQS S alone, while it interacts with E.

An isolated system S defined on a d-dimensional Hilbert space $\mathcal{H}_S$ has a dynamics that is completely characterized by the Hamiltonian $H_S(t)$. In general this Hermitian operator in $B(\mathcal{H}_S)$ is time-dependent. If the system S is initialized in a particular state $\rho_S(0) \in S(\mathcal{H}_S)$ at time 0, the evolution to a generic final time t, namely

$$\rho_S(t) = U_t^S \, \rho_S(0) \, (U_t^S)^\dagger \,, \tag{2.13}$$

is obtained trough the unitary transformation

$$U_t^S = T \exp\left[-i \int_0^t H_S(\tau) d\tau\right] , \tag{2.14}$$

where we fixed $\hbar = 1$ for the reduced Planck constant and T is the time-ordering operator.

Completely isolated systems cannot be considered to faithfully describe realistic scenarios. In general, in order to precisely reproduce the evolution of a quantum system, we have to include the influence of the surrounding environment. Therefore, being S the OQS that interacts with its environment E, we consider those scenarios where S is initialized in $\rho_S(0) \in S(\mathcal{H}_S)$ and is uncorrelated with the environment E. This situation is justified by considering $t = 0$ the time when S and E are put in contact. Hence, the initial state of the complete SE system is $\rho_S(0) \otimes \sigma_E(0)$ which belongs to the state space $S(\mathcal{H}_{SE}) = S(\mathcal{H}_S \otimes \mathcal{H}_E)$. The complete system SE can be considered a composite closed system and we can describe its evolution by generalizing Eq. (2.13) to this composite scenario. In order to do so, we have to consider the unitary evolution generated by the SE Hamiltonian

$$H_{SE}(t) = H_S(t) \otimes \mathbb{1}_E + \mathbb{1}_S \otimes H_E(t) + H_{SE}^{int}(t) \,, \tag{2.15}$$

where $\mathbb{1}_S$ ($\mathbb{1}_E$) is the identity operator on $\mathcal{H}_S$ ($\mathcal{H}_E$), $H_E(t) \in B(\mathcal{H}_E)$ describes the internal degrees of freedom dynamics of the environment E and the term $H_{SE}^{int}(t) \in B(\mathcal{H}_{SE})$ represents the interaction between S and E. The evolution of the interacting bipartite system $S - E$ initialized in $\rho_{SE}(0) = \rho_S(0) \otimes \sigma_E(0)$ is described by the unitary operator U_t^{SE}, where, similarly to Eqs. (2.13) and (2.14), we have

$$\rho_{SE}(t) = U_t^{SE} \left(\rho_S(0) \otimes \sigma_E(0)\right) (U_t^{SE})^\dagger \,, \tag{2.16}$$

where

$$U_t^{SE} = T \exp\left[-i \int_0^t H_{SE}(\tau) d\tau\right] . \tag{2.17}$$

2.3.1 Dynamical maps and evolutions

We implemented the environment in this scenario in order to obtain a precise description of the OQS dynamics. Indeed, the term $H_{SE}(t)$ generates a mutual influence between S and E and in general its effect on the OQS cannot be reproduced in any closed system. We underline that here we are not interested in the particular evolution of the degrees of freedom of E but only in its effective influence on S. At the same time, the calculations needed to describe the environmental dynamics may require impracticable computations. We can obtain the evolution of the OQS alone by tracing out the degrees of freedom of the environment, namely obtaining

$$\rho_S(t) = \Lambda_t(\rho_S(0)) \equiv \mathrm{Tr}_E\left[U_t^{SE} \left(\rho_S(0) \otimes \sigma_E(0)\right) (U_t^{SE})^\dagger\right] . \tag{2.18}$$

Notice that U_t^{SE} obtained through Eq. (2.17) is continuous and differentiable in time. Therefore, since the tracing operator is continuous but not invertible, the operators Λ_t are continuous and differentiable in time but may not be invertible.

The superoperator $\Lambda_t : B(\mathcal{H}_S) \to B(\mathcal{H}_S)$ is the linear operator that describes the evolution of S from the initial time to time t and is called *dynamical map*. In order to understand the properties of these superoperators, we start by noticing that, if Λ_t describes a physical evolution, it must map any initial state into a valid final state, namely

$$\rho_S(0) \in S(\mathcal{H}_S) \longrightarrow \Lambda_t(\rho_S(0)) = \rho_S(t) \in S(\mathcal{H}_S), \qquad \text{for any } t \geq 0 . \tag{2.19}$$

This condition requires Λ_t to be *positive* (P) and *trace preserving* (TP). These properties can be expressed as follows:

$$\Lambda_t \text{ is P} \longleftrightarrow \Lambda_t(X_S) \geq 0, \text{ for any } X_S \geq 0, \tag{2.20}$$

$$\Lambda_t \text{ is TP} \longleftrightarrow \mathrm{Tr}\left[\Lambda_t(X_S)\right] = \mathrm{Tr}\left[X_S\right], \tag{2.21}$$

for any $X_S \in B(\mathcal{H}_S)$.

Nonetheless, we notice that a more general scenario where Λ_t acts can occur. If S is initially correlated with an ancillary system A, which is not influenced by the evolution generated by the interaction between S and E, a physical transformation Λ_t must also preserve the physicality of these $S-A$ states. Hence, for any ancillary system with Hilbert space $\mathcal{H}_A$, we require

$$\rho_{SA}(0) \in S(\mathcal{H}_{SA}) \longrightarrow \Lambda_t \otimes I_A(\rho_{SA}(0)) = \rho_{SA}(t) \in S(\mathcal{H}_{SA}), \text{ for any } t \geq 0\,. \tag{2.22}$$

This condition generalizes Eq. (2.19) and requires Λ_t to be *completely positive* (CP) and *trace preserving*, (CPTP). The CP property can be expressed as:

$$\Lambda_t \text{ is CP} \longleftrightarrow \Lambda_t \otimes I_A(X_{SA}) \geq 0, \text{ for any } X_{SA} \geq 0\,, \tag{2.23}$$

where $X_{SA} \in B(\mathcal{H}_{SA})$ and the ancillary system A has dimension $d_A = d$, namely the same dimension as S. Indeed, it can be show that, if condition (2.23) is satisfied for $d_A = d$, then the same is true for any $d_A > d$. We conclude that the evolution of an OQS between the initial time and the final time t is described by a CPTP operator Λ_t, namely the dynamical map.

We define $\mathbf{\Lambda}$ to be the whole family of dynamical maps Λ_t for any $t \geq 0$, namely

$$\mathbf{\Lambda} \equiv \{\Lambda_t\}_{t\geq 0}. \tag{2.24}$$

Notice that from Eqs. (2.17) and (2.18) it follows that the dynamical map at the initial time corresponds to the identity map $I_S : B(\mathcal{H}_S) \to B(\mathcal{H}_S)$. As a consequence, all evolutions are characterized by

$$\Lambda_0(\cdot) = I_S(\cdot). \tag{2.25}$$

Given the physical interpretation of Eq. (2.18), it is clear that any map $\Lambda_S : B(\mathcal{H}_S) \to B(\mathcal{H}_S)$ obtained by tracing out the environment from a $S-E$ unitary evolution is a valid dynamical map and therefore CPTP. Is the opposite also true? In other words, given a CPTP map Λ_S, is it always possible to engineer an environment E interacting with S such that Λ_S is generated by Eq. (2.18)? The answer is given by the Stinespring-Kraus representation theorem [Sti55, Kra71]. It states that for any CPTP map $\Lambda_S : B(\mathcal{H}_S) \to B(\mathcal{H}_S)$, there exist: an environment E, a state $\sigma_E \in S(\mathcal{H}_E)$ and a unitary transformation U^{SE} such that Λ_S can be simulated through the application of U^{SE} on the uncorrelated $S-E$, where E is initialized in σ_E, namely

$$\Lambda_S(\rho_S) = \mathrm{Tr}_E\left[U^{SE}(\rho_S \otimes \sigma_E)(U^{SE})^\dagger\right]. \tag{2.26}$$

This result can be generalized to the case of evolutions. Indeed, given a continuous and differentiable in time family of CPTP maps $\mathbf{\Lambda} = \{\Lambda_t\}_{t\geq 0}$ where

$\Lambda_t : B(\mathcal{H}_S) \to B(\mathcal{H}_S)$ and $\Lambda_0 = I_S$, we can always find an environment E, an initial state $\sigma_E(0) \in S(\mathcal{H}_E)$ and continuous and differentiable in time unitary transformations $\{U_t^{SE}\}_{t\geq 0}$ such that the evolution to time t is given by

$$\Lambda_t(\rho_S(0)) = \mathrm{Tr}_E\left[U_t^{SE}(\rho_S(0)\otimes\sigma_E(0))(U_t^{SE})^\dagger\right]. \tag{2.27}$$

Although we said that any $S - E$ interaction leads to a continuous and differential family of CPTP maps $\mathbf{\Lambda}$, we relax this condition by allowing at most a countable set of discontinuity times. This condition is physically well motivated when considering external interventions, namely not due to the SE dynamics. Hence, having in mind this observation and the Stinespring-Kraus representation theorem, we are not interested in defining a particular physical realization, namely U_t^{SE} and $\sigma_E(0)$, for each almost-always continuous evolution $\mathbf{\Lambda}$. Therefore, we follow by simply considering S as the system on which $\mathbf{\Lambda}$ acts, while we are not interested to define the corresponding environment E.

2.3.2 Image and Kernel of evolutions

We call set of accessible states, or *image*, of Λ_t all the states of S that can be obtained by applying the map Λ_t to an initial state $\rho_S(0)$, that is:

$$\mathrm{Im}(\Lambda_t) \equiv \{\sigma_S \in S(\mathcal{H}_S)\,|\,\exists\rho_S(0)\in S(\mathcal{H}_S)\text{ s.t. }\sigma_S = \Lambda_t(\rho_S(0))\} \subseteq S(\mathcal{H}_S). \tag{2.28}$$

We underline that, while we defined Λ_t as a map over $B(\mathcal{H}_S)$, we defined $\mathrm{Im}(\Lambda_t)$ to be only the collection of states that can be obtained after an application of Λ_t. We notice that only open system dynamics can cause a shrinking of the set of accessible states during the evolution, namely $\mathrm{Im}(\Lambda_t) \subset S(\mathcal{H}_S)$. Indeed, in case of a unitary evolution (2.13), the inverse $(U_t^S)^{-1} = (U_t^S)^\dagger$ always exists. Therefore, for any $\sigma_S \in S(\mathcal{H}_S)$ there exists $\rho_S(0) = (U_t^S)^\dagger\sigma_S U_t^S \in S(\mathcal{H}_S)$ which is mapped into σ_S by Eq. (2.13). Hence, $\mathrm{Im}(U_t^S) = S(\mathcal{H}_S)$ for any unitary evolution. Instead, for generic dynamical maps Λ_t^{-1} may not be CPTP or even not exist and therefore the same reasoning cannot be used.

A particular class of evolutions is given by those such that the image at any time t is contained in the images at any earlier time:

Definition 1. *An evolution $\mathbf{\Lambda}$ is called image non-increasing if, for any $s \leq t$,*

$$Im(\Lambda_s) \supseteq Im(\Lambda_t). \tag{2.29}$$

If $\mathbf{\Lambda}$ is image non-increasing and $\rho_S \in \mathrm{Im}(\Lambda_t)$, then the same state belongs also to $\mathrm{Im}(\Lambda_s)$ for any earlier time, namely $\rho_S \in \mathrm{Im}(\Lambda_s)$ for all $s \leq t$. Hence, for any $s \in [0,t]$ there exists an initial state $\rho_S(0) \in S(\mathcal{H}_S)$ such that $\rho_S = \Lambda_s(\rho_S(0))$.

We follow by defining invertible evolutions:

Definition 2. *An evolution* $\mathbf{\Lambda}$ *is called invertible if the inverse transformation* $\Lambda_t^{-1} : B(\mathcal{H}_S) \to B(\mathcal{H}_S)$ *such that* $\Lambda_t \circ \Lambda_t^{-1} = \Lambda_t^{-1} \circ \Lambda_t = I_S$ *exists for all* $t \geq 0$.

If an evolution $\mathbf{\Lambda}$ is non-invertible, there must be a time t and at least two (different) states $\rho'_S(0)$, $\rho''_S(0) \in S(\mathcal{H}_S)$ that are mapped into the same state $\rho_S(t)$, namely $\Lambda_t(\rho'_S(0)) = \Lambda_t(\rho''_S(0)) = \rho_S(t)$. This property implies that Λ_t maps the Hermitian and traceless operator $X = \rho'_S(0) - \rho''_S(0) \in B(\mathcal{H}_S)$ into the null operator $\mathbf{0}$. This is a particularly useful example of an operator $X \in B(\mathcal{H}_S)$ belonging to the Kernel of a map. Indeed, we define the Kernel of a map $\Lambda_t : B(\mathcal{H}_S) \to B(\mathcal{H}_S)$ as

$$\mathrm{Ker}(\Lambda_t) \equiv \{Y \in B(\mathcal{H}_S) \,|\, \Lambda_t(Y) = \mathbf{0}\}\,, \tag{2.30}$$

In other words, $\mathrm{Ker}(\Lambda_t)$ is the space of operators that Λ_t maps into the null operator $\mathbf{0}$. The linearity of Λ_t implies that $\mathbf{0} \in \mathrm{Ker}(\Lambda_t)$ for any dynamical map Λ_t. Notice that, due to the trace-one property of states, TP maps, e.g. any dynamical map, cannot transform a state into the null operator. As a result, $\mathrm{Ker}(\Lambda_t) \cap S(\mathcal{H}_S) = \emptyset$ for any $\mathbf{\Lambda}$ and $t \geq 0$, where $\emptyset$ is the empty set. Nonetheless, as we saw before, when Λ_t is not invertible we can define Hermitian and traceless operators $X \in \mathrm{Ker}(\Lambda_t)$ from the states that are mapped into the same final state. On the other hand, if we find an Hermitian and traceless element $Y \in \mathrm{Ker}(\Lambda_t)$, we can always put it in the form $Y = a(\rho'_S(0) - \rho''_S(0))$ for some scalar $a \neq 0$ and states $\rho'_S(0)$, $\rho''_S(0)$ and therefore certify the non-invertibility of Λ_t.

An interesting class of evolutions is given by the following definition.

Definition 3. *An evolution* $\mathbf{\Lambda}$ *is called Kernel non-decreasing if, for any* $s \leq t$,

$$Ker(\Lambda_s) \subseteq Ker(\Lambda_t)\,. \tag{2.31}$$

We notice that that the Kernel non-decreasing property does not imply the evolution to be image non-increasing, but only [CRS18] $\dim(\mathrm{Im}(\Lambda_s)) \geq \dim(\mathrm{Im}(\Lambda_t))$ for any $s \leq t$.

2.3.3 Markovian and non-Markovian evolutions

Markovian and non-Markovian quantum evolutions have been defined in different ways and in this thesis we adopt a definition that gained large consensus

[RHP14, WECC08, HYYO11, CM14, HCLA14]. We underline that it is not trivial to formulate a unique quantum version of a concept that was originally introduced in classical physics. Classical mechanics is contained in quantum mechanics as a special case and therefore there is not a unique path to extend an idea that was originally introduced for classical systems. Our approach to define quantum Markovianity starts by considering that classical Markov processes are not influenced by past configurations of the system and therefore, once some encoded information is lost, it cannot be recovered. We focus on this feature by requiring that quantum Markovian evolutions have to show similar memoryless phenomenological properties: no information backflow from the environment E to the OQS S can occur. Hence, we discuss the framework used in this thesis to define quantum Markovianity by providing solid mathematical and phenomenological reasonings. Before doing so, we first introduce some indispensable mathematical properties of evolutions.

We start by formulating the concept of *divisibility*. An evolution, namely a family of CPTP maps $\mathbf{\Lambda} = \{\Lambda_t\}_{t\geq 0}$, whereas it describes the evolution of S for any final time t, it does not provide the operators that evolve S between two times s and t such that $0 < s < t$. Therefore, we adopt the common approach (see e.g. Ref. [CRS18]) that defines as *divisible* those evolutions for which such a linear operator can be defined:

Definition 4. *An evolution $\mathbf{\Lambda} = \{\Lambda_t\}_{t\geq 0}$ is divisible if and only if for any $0 \leq s \leq t$ there exists a linear map $V_{t,s} : B(\mathcal{H}_S) \to B(\mathcal{H}_S)$ such that*

$$\Lambda_t(\cdot) = (V_{t,s} \circ \Lambda_s)(\cdot) \equiv V_{t,s}(\Lambda_s(\cdot)) \ . \tag{2.32}$$

We also call $V_{t,s}$ the intermediate map of $\mathbf{\Lambda}$ between the times s and t.

Notice that any invertible evolution is divisible. Indeed, in this case the intermediate map always exists and is given by

$$V_{t,s} = \Lambda_t \circ \Lambda_s^{-1}. \tag{2.33}$$

Notice that the inverse is not true: an evolution could be divisible but not invertible. In Ref. [CC21] the authors discuss how to construct intermediate maps of divisible non-invertible evolutions through the use of generalized inverse operations. Interestingly, the possibility to divide an evolution is characterized by the Kernel non-decreasing property given in Definition 3:

Proposition 1 ([CRS18])**.** *An evolution $\mathbf{\Lambda}$ is divisible if and only if it satisfies the Kernel non-decreasing property (2.31).*

The CPTP nature of dynamical maps Λ_t implies that $V_{t,s}$ is TP for all the states in $S(\mathcal{H}_S)$. Nonetheless, $V_{s,t}$ may not be TP for every operator in $B(\mathcal{H}_S)$. Similarly, in order for $\Lambda_t = V_{t,s} \circ \Lambda_s$ to be CP, $V_{t,s}$ is not forced to be CP. Indeed, a legitimate intermediate map could be only P or even characterized by negative eigenvalues. When the evolution can be divided in time intervals where the intermediate maps are TP operators that are also P (CP), we say that $\mathbf{\Lambda}$ is P-divisible (CP-divisible). Hence, we adopt the following definition:

Definition 5. *An evolution $\mathbf{\Lambda} = \{\Lambda_t\}_{t\geq 0}$ is P/CP-divisible if and only if for any $0 \leq s \leq t$ there exists a PTP/CPTP linear map $V_{t,s} : B(\mathcal{H}_S) \to B(\mathcal{H}_S)$ such that*

$$\Lambda_t(\cdot) = (V_{t,s} \circ \Lambda_s)(\cdot) = V_{t,s}(\Lambda_s(\cdot)) \,. \tag{2.34}$$

We underline that a CP-divisible evolution of course is also P-divisible, while the opposite in general is not true. Hence, in order to avoid any misunderstanding, if we say that $\mathbf{\Lambda}$ is P-divisible, we take for granted that the evolution is not CP-divisible.

Finally, we introduce the evolutions satisfying the semi-group property.

Definition 6. *An evolution $\mathbf{\Lambda} = \{\Lambda_t\}_{t\geq 0}$ is a dynamical semi-group if and only if for any $t_1, t_2 \geq 0$ the following relation is verified*

$$\Lambda_{t_1+t_2}(\cdot) = \Lambda_{t_1}(\Lambda_{t_2}(\cdot)) \,. \tag{2.35}$$

It is easy to prove that any evolution satisfying the semigroup property is CP-divisible, while the inverse is not true.

Definition of Markovian evolutions

In this thesis we adopt the approach that identifies Markovian evolutions with CP-divisible evolutions. We provide two arguments that sustain this definition, where the first derives from a comparison with classical Markov processes, while the second apparently independent explanation has a purely phenomenological derivation.

We start by describing classical Markovian processes and therefore we derive a natural quantum counterpart [CRS18]. In classical information theory the topic of Markov chains is introduced as follows. Suppose X is a stochastic variable that can assume different values x_i at each time t_i. Hence, $p(x_1, t_1|x_0, 0)$ is the conditional probability that at time t_1 the random variable X assumes the value x_1, given that x_0 was its initial value at time 0. The process is called Markovian if this process satisfies [van07]

$$p(x_i, t_i|x_{i-1}, t_{i-1}, \ldots, x_0, 0) = p(x_i, t_i|x_{i-1}, t_{i-1}), \tag{2.36}$$

for each $0 \leq t_1 \leq \cdots \leq t_{i-1} \leq t_i$. A physical consequence of this condition is that the process that is generating the evolution of X does not take into account its past history, but only its last configuration. Hence, the memoryless nature of this type of processes comes out explicitly. Notice that condition (2.36) implies that, for any intermediate time s such that $0 < s < t$, we have

$$p(x, t|x_0, 0) = \sum_y p(x, t|y, s)p(y, s|x_0, 0), \tag{2.37}$$

where y can be seen as the output of a measurement process performed on X at an intermediate time s. While in the classical case a measurement of a system does not necessarily influence its state and therefore we can always assume to know the value of X at time s without influencing the dynamics, in general measurements alter quantum systems. Indeed, there is no straightforward generalization of this approach to quantum evolutions [VSL$^+$11]. Nonetheless, an approach that focuses on the operators generating the evolution is possible.

Consider a finite-dimensional classical system, where at each time t the stochastic variable X can assume one value from $\{i\}_{i=1}^d$. Then, for the starting time we have a probability vector $\boldsymbol{p}(0) = (p_1(0), p_2(0), \ldots, p_d(0))$, such that $p_i(0)$ is the initial probability that X is in the i-th configuration. The evolution to a later time t is given by a (row) *stochastic matrix* $\lambda(t, 0)$ such that $\boldsymbol{p}(t) = \lambda(t, 0)\boldsymbol{p}(0)$, where the ij-th component of $\lambda(t, 0)$ is the transition probability from i to j between the initial time and t. Markovian processes, namely satisfying Eq. (2.36), are distinguished by P-divisible stochastic matrices which can be divided into intermediate stochastic matrices $\lambda(t, s)$ [VSL$^+$11]

$$\lambda(t, 0) = \lambda(t, s)\lambda(s, 0)\,, \tag{2.38}$$

for any $s < t$. We say that these processes are stochastically P-divisible since the matrices $\lambda(t, s)$ are required to preserve the physical meaning of the evolving probability distributions, whereas no extended scenario with ancillary systems is considered. We use the term "stochastically" in order to distinguish this property from P-divisibility of quantum channels. The generalization of (2.38) to the quantum domain is given by requiring that between any two times s and t the evolution $\boldsymbol{\Lambda}$ can be divided as $\Lambda_t = V_{t,s} \circ \Lambda_s$, where $V_{t,s}$ is a valid evolving operator by its own, namely it has the same properties as Λ_t. Hence, as in the classical case this property required the matrices $\lambda(t, s)$ to be stochastic, in the quantum case we require $V_{t,s}$ to be CPTP. Hence, this analogy suggests to define quantum Markovian evolutions as those being CP-divisible.

The second reason that we consider for the identification of Markovian evolutions with CP-divisible evolutions $\boldsymbol{\Lambda}$ is the following. Consider a CP-divisible evolution $\mathbf{\Lambda}$ and its intermediate evolution between two generic times s and t. Being Λ_s and $V_{t,s}$ CPTP maps, we apply the Stinespring-Kraus representation theorem and simulate their action as in Eq. (2.26). The consecutive application of these two maps that define $\rho_S(t) = V_{t,s}(\Lambda_s(\rho_S(0)))$ can be simulated as follows. We consider a first environment E initialized in σ_E that interacts with S until time s, where the $S-E$ initial state is $\rho_{SE}(0) = \rho_S(0) \otimes \sigma_E$. Hence, between the initial time and time s, $S-E$ evolves with a unitary operator U_s^{SE} that simulates the action of Λ_s on S as in Eq. (2.26). At time s the environment E is discarded and a second environment $\overline{E}$ is initialized in $\sigma_{\overline{E}}$ and coupled with S, where $\rho_{S\overline{E}}(s) = \rho_S(s) \otimes \sigma_{\overline{E}}$ and $\rho_S(s) = \Lambda_s(\rho_S(0)) = \mathrm{Tr}_E\left[\rho_{SE}(s)\right]$. Similarly as before, between times s and t, $S-\overline{E}$ evolves with a unitary operator $U_{t,s}^{S\overline{E}}$ that simulates the action of $V_{t,s}$ on S. As shown in Fig. 2.1, the information released by S in E before time s cannot be recovered later in time when S interacts with $\overline{E}$: no information lost in the time interval $[0, s]$ can be recovered in $[s, t]$. If we extend this approach to an infinitesimal subdivision of the time axis, we obtain the connection between CP-divisibility and the expected Markovian phenomenology. Indeed, if CP-divisibility is satisfied, no information backflows can occur between any two (even infinitesimally close) times. Notice that the evolutions satisfying the semi-group property have intermediate maps $V_{t,s}$ that solely depend on the length $t-s$ of the time interval $[s, t]$ and not on the specific s and t. Indeed, due to this constancy of the information rate loss, these evolutions are often considered as the "most" memoryless.

While our definition of Markovianity is based on the idea that a Markovian evolution can be simulated through the subsequent interaction with different environments uncorrelated with the OQS, other proposals have been explored. A second approach defines as Markovian those evolutions that cannot show backflows of a given quantifier Q for the information encoded in the system. For instance, the first approach in this direction [BLP10] defined Markovianity through the decrease of distinguishability of any pair of S states, namely such that

$$\frac{d}{dt}\|\Lambda_t(\rho'_S(0) - \rho''_S(0))\|_1 \leq 0 \quad \text{for any} \quad \rho'_S(0), \rho''_S(0) \in S(\mathcal{H}_S)\,, \tag{2.39}$$

As a consequence, any violation of this condition represents an information backflow caused by the non-Markovian nature of the evolution. Nevertheless, there exist evolutions that satisfy condition (2.39) but show backflows for other quantifiers Q. Hence, this definition strongly depends on the particular Q chosen. Moreover, it is possible to prove condition (2.39) is not equivalent to CP-divisibility: there exist evolutions that contracts the trace distance between any two S states but are not CP-divisible. Actually, this condition is weaker

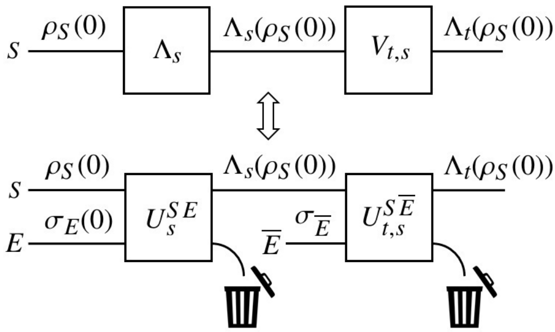


Figure 2.1: Consider an evolution $\mathbf{\Lambda}$ with a CPTP intermediate map $V_{t,s}$. The CPTP map Λ_s induces the transformation $\rho_S(0) \to \rho_S(s)$ and the CPTP map $V_{t,s}$ transforms $\rho_S(s) \to \rho_S(t)$ (above). The Stinespring-Kraus representation theorem allows simulating any CPTP map through a unitary interaction between S and an uncorrelated environment. This result can be applied both for Λ_s and $V_{t,s}$, namely through the subsequent interaction of S with E and $\overline{E}$ (below). Since E can be discarded after time s, no information lost by S in the time interval $[0, s]$ can be recovered later in $[s, t]$. By applying this result to a CP-divisible evolution, we see that no information can be retrieved between any two times.

than P-divisibility. We discuss this topic in Sections 2.4.2 and 2.5.3.

Markovianity can also be defined in a third radically different manner (see e.g. Ref. [Bud18, PRRF+18, MKPM19]). This approach considers the possible temporal memory effects arising from scenarios where the evolving OQS is discarded and replaced with a newly prepared OQS state. Hence, it may happen that the environment keeps track of the discarded state and therefore at a later time some information concerning the initial state can be retrieved. While we do not discuss the meaning and the possibility to perform such an operation, we want to underline that the main reason why we do not consider this approach is that we are interested in the study of families of CPTP maps $\mathbf{\Lambda}$ defined *solely on* S, while we do not focus on the particular $S - E$ unitary dynamics that generates the target evolution. Indeed, this approach, in order to decide whether an evolution is non-Markovian, examines also the structure of E and the particular unitary interaction U_t^{SE} that generates the evolution $\mathbf{\Lambda}$.

In many occasions we consider non-Markovian evolutions characterized by time intervals where we expect to observe information backflows, whereas in other time intervals we know that these phenomena cannot occur. For instance, this is the case when we know that the infinitesimal intermediate maps $V_{t+\epsilon,t}$ are not CPTP if and only if $t \in (t_1^{NM}, t_2^{NM})$. Indeed, for any time not in this set, the evolution behaves as Markovian. Since in the following this situation is considered several times, we use the following definition.

Definition 7. *An evolution* $\mathbf{\Lambda} = \{\Lambda_t\}_{t\geq 0}$ *is Markovian in the time interval* $[t_1, t_2]$ *if and only if for any* $t_1 \leq s \leq t \leq t_2$ *there exists a linear CPTP intermediate map* $V_{t,s} : B(\mathcal{H}_S) \to B(\mathcal{H}_S)$ *such that*

$$\Lambda_t(\cdot) = (V_{t,s} \circ \Lambda_s)(\cdot) = V_{t,s}(\Lambda_s(\cdot)) \,. \tag{2.40}$$

Similarly, we say that $\mathbf{\Lambda}$ is non-Markovian (P-divisible) in the time interval $[t_1, t_2]$ if the CP-divisibility (P-divisibility) condition is violated (satisfied) for any $t_1 \leq s \leq t \leq t_2$. We underline that, if the evolution is Markovian in a time interval $[t_1, t_2]$, it does not necessarily mean that the evolution is Markovian. Indeed, in order for $\mathbf{\Lambda}$ to represent a Markovian dynamics, $V_{t,s}$ has to be CPTP for any $0 \leq s \leq t$. On the contrary, if $\mathbf{\Lambda}$ is non-Markovian in at least one time interval $[t_1, t_2]$, then $\mathbf{\Lambda}$ is non-Markovian and moreover $[t_1, t_2]$ is a time interval when information backflows can occur.

2.3.4 The master equation

We discussed how dynamical maps can be used to express the time evolution of OQSs, where we can obtain the state at time t by applying Λ_t directly on the initial state. A second approach to express the same evolution is given by studying the dynamical differential equation for $\frac{d}{dt}\rho_S(t)$. Whereas the solution of this differential equation is the same obtained with dynamical maps, there are some insights concerning the information flows between the OQS and the environment that can be easily deduced from this differential form. Indeed, for some classes of evolutions, it is much easier to express Markovianity conditions in this framework rather than with dynamical maps.

We refer to the *master equation* of the dynamics every time it is possible to derive the generator L_t that describes the following dynamical differential equation

$$\frac{d}{dt}\rho_S(t) = L_t(\rho_S(t)) \,. \tag{2.41}$$

Gorini et. al. [GK76], Lindblad [Lin76] and Franke [Fra76] formulated the form of L_t for any evolution satisfying the semi-group property, where the generator $L_t = L$ is not time-dependent:

$$L(\rho_S(t)) \equiv i[H, \rho_S(t)] + \sum_k \gamma_k \left(G_k \rho_S(t) G_k^\dagger - \frac{1}{2} \left\{ G_k^\dagger G_k, \rho_S(t) \right\} \right) , \tag{2.42}$$

where H is a Hermitian operator representing the effective Hamiltonian of S, $\gamma_k \geq 0$ are non-negative *rates* and G_k are called Lindblad or jump operators. This equation had been generalized to CP-divisible, namely Markovian, differentiable evolutions. In these cases, the master equation is called *Lindblad equation* and the generator L_t assumes the following standard form

$$\begin{aligned} L_t(\rho_S(t)) \equiv\, & i[H(t), \rho_S(t)] \\ & + \sum_k \gamma_k(t) \left(G_k(t)\rho_S(t)G_k^\dagger(t) - \frac{1}{2}\left\{ G_k^\dagger(t)G_k(t), \rho_S(t) \right\} \right), \end{aligned} \tag{2.43}$$

where operators and rates are in general time-dependent. While $H(t)$ generates the unitary component of the dynamics, the term generated by the time-dependent operators $G_k(t)$ is called *dissipator* and characterizes the typical features of OQS evolutions, namely those given by the interaction with an environment. Indeed, master equations with no dissipator $L_t(\rho_S(t)) \equiv i[H_S(t), \rho_S(t)]$ are in the form of the Schrödinger equation for (closed) mixed states which always lead to unitary evolutions, namely Eqs. (2.13) and (2.14). We say that L_t is in the *Lindblad form* whenever it can be casted as Eq. (2.43), where $\gamma_k(t) \geq 0$ for all k. We say that L_t is in the *generalized Lindblad form* when it can be casted as Eq. (2.43) and at least one of the rates $\{\gamma_k(t)\}_k$ is negative in one or more time intervals.

An important feature of the rates $\gamma_k(t)$ is that, if they are finite, L_t gives rise to a Markovian evolution if and only if it can be written in a form where $\gamma_k(t) \geq 0$ for all k and $t \geq 0$ (see e.g. Theorem 5.1 of Ref. [RH11] for a proof). Differentiable Markovian evolutions allow time-ordered exponential representations for dynamical maps Λ_t and intermediate maps $V_{t,s}$ as follows

$$\Lambda_t = T\exp\left[\int_0^t L_\tau d\tau \right], \tag{2.44}$$

$$V_{t,s} = T\exp\left[\int_s^t L_\tau d\tau \right]. \tag{2.45}$$

Usually, an evolution $\mathbf{\Lambda}$ is considered Markovian if and only if the corresponding generator L_t is in the Lindblad form. Indeed, from this property it follows the CP-divisibility of $\mathbf{\Lambda}$ in CPTP intermediate maps (2.45). Nonetheless, this statement is true only in the case of invertible evolutions [CRS18], while there exist non-invertible CP-divisible (Markovian) evolutions with generators in the generalized Lindblad form. Moreover, while any evolution generated by a generalized Lindblad master equation where one or more $\gamma_k(t)$ is negative for some times is non-Markovian, not every non-Markovian evolution can be represented as the solution of a generalized Lindblad master equation.

Finally, we notice that a master equation in the generalized Lindblad form with non-negative rates $\gamma_k(t) \geq 0$ in a time interval $[t_1, t_2]$ generates an evolution that is Markovian in the same time interval. This result can be deduced from Eq. (2.45), where $V_{t+\epsilon,t}$ is determined solely by L_t for $t \in [t, t+\epsilon]$ and therefore it is not influenced by $\gamma_k(t)$ being negative at different times.

2.3.5 Non-convexity of Markovian and non-Markovian evolutions

We define $\mathbb{E}$ to be the collection of all the possible evolutions for a system S defined over a d-dimensional Hilbert space, where $\mathbb{E}^M$ and $\mathbb{E}^{NM}$ are the respective Markovian and non-Markovian subsets of evolutions. Here, we discuss the following statements:

- $\mathbb{E}$ is convex;
- $\mathbb{E}^M$ and $\mathbb{E}^{NM}$ are non-convex.

The first point follows directly from the convexity of the state space. Taken any pair of evolutions $\mathbf{\Lambda}^{(1,2)}$, their convex combination $\mathbf{\Lambda}^{(p)} = (1-p)\mathbf{\Lambda}^{(1)} + p\mathbf{\Lambda}^{(2)}$ is a valid evolution, namely the maps $\Lambda_t^{(p)} = (1-p)\Lambda_t^{(1)} + p\Lambda_t^{(2)}$ are CPTP for any $t \geq 0$ and $p \in [0,1]$. Indeed, $\Lambda_t^{(1,2)}$ are CPTP and for any $\rho_{SA}(0) \in S(\mathcal{H}_{SA})$ and $t \geq 0$ we have $\rho_{SA}^{(1,2)}(t) = \Lambda_t^{(1,2)} \otimes I_A(\rho_{SA}(0)) \in S(\mathcal{H}_{SA})$. Therefore, the convexity of the state space implies that $\Lambda_t^{(p)} \otimes I_A(\rho_{SA}(0)) = \rho_{SA}^{(p)}(t) = (1-p)\rho_{SA}^{(1)}(t) + p\rho_{SA}^{(2)}(t) \in S(\mathcal{H}_{SA})$ for any $t \geq 0$, $\rho_{SA}(0) \in S(\mathcal{H}_{SA})$ and $p \in [0,1]$. It follows that, for any ancilla A, the maps $\Lambda_t^{(p)}$ transforms the states of $S-A$ into valid output states and therefore $\mathbf{\Lambda}^{(p)}$ is a valid quantum evolution.

Non-convexity of the Markovian set

Intuitively, one may think that the same approach used for $\mathbb{E}$ can be used to show that also $\mathbb{E}^M$ is a convex set. Interestingly, this is not the case: Markovian evolutions define a non-convex set [WECC08]: it is possible to generate non-Markovianity from the manipulation of Markovian evolutions. Moreover, the Markovian evolutions that can be considered in this process can also be taken satisfying the semi-group property, as show in Ref. [CW15]. The explicit example that they propose is given by qubit evolutions $\mathbf{\Lambda}^{M,1}$ and $\mathbf{\Lambda}^{M,2}$ defined by Lindblad generators $L_t^{M,1}$ and $L_t^{M,2}$ of the form:

$$L_t^{M,1}(\rho_S(t)) = \gamma(\sigma_x\rho_S(t)\sigma_x - \rho_S(t)), \tag{2.46}$$

$$L_t^{M,2}(\rho_S(t)) = \gamma(\sigma_y\rho_S(t)\sigma_y - \rho_S(t)), \tag{2.47}$$

where $\gamma > 0$ and $\sigma_{x,y,z}$ are the Pauli matrices. The Lindblad generator of the evolution $\mathbf{\Lambda}^{NM} = (\mathbf{\Lambda}^{M,1} + \mathbf{\Lambda}^{M,2})/2$ assumes the form:

$$L_t^{NM}(\rho_S(t)) = \sum_{i=x,y,z} \gamma_i(t)(\sigma_i \rho_S(t)\sigma_i - \rho_S(t)), \tag{2.48}$$

where $\{\gamma_i(t)\}_{i=x,y,z} = \{\gamma, \gamma, -\gamma \tanh(2\gamma t)\}$. Notice that $\gamma_z(t) < 0$ for all $t > 0$. Indeed, these evolutions are called *eternal* non-Markovian evolutions [HCLA14] and are characterized by intermediate maps $V_{t,s}^{NM}$ that are not CPTP for any $0 < s < t$.

Under certain conditions, the evolution $\mathbf{\Lambda}^{(p)} = (1-p)\mathbf{\Lambda}^{(1)} + p\mathbf{\Lambda}^{(2)}$ is characterized by the generator obtained through the convex combination $L_t^{(p)} = (1-p)L_t^{(1)} + pL_t^{(2)}$, where $L_t^{(1)}$ ($L_t^{(2)}$) is the Lindblad generator of $\mathbf{\Lambda}^{(1)}$ ($\mathbf{\Lambda}^{(2)}$). Nonetheless, Eq. (2.48) shows that in general this is not the case: $L_t^{NM} \neq (1-p)L_t^{M,1} + pL_t^{M,2}$. Moreover, if we consider two evolutions with generic generators $L_t^{(1)}$ and $L_t^{(2)}$, the operator $L_t^{(p)} = (1-p)L_t^{(1)} + pL_t^{(2)}$ does not always generate a physical evolution. This topic is studied in e.g. Ref. [KBPLB18].

Non-convexity of the non-Markovian set

The non-convexity of $\mathbb{E}^{NM}$ is more intuitive. Without getting into the details of these cases, we briefly describe a scenario that explains the physical sense of this phenomenon. Consider two evolutions $\mathbf{\Lambda}^{NM,1}$ and $\mathbf{\Lambda}^{NM,2}$ that are non-Markovian during non-overlapping time intervals. If the environment simulated by $\mathbf{\Lambda}^{NM,2}$ is particularly dissipative when $\mathbf{\Lambda}^{NM,1}$ shows backflows and vice-versa, it is easy to imagine a Markovian evolution obtained from a convex combination of the form $(1-p)\mathbf{\Lambda}^{NM,1} + p\mathbf{\Lambda}^{NM,2}$. Moreover, the Markovian evolutions that can be obtained thorugh convex combinations of non-Markovian evolutions can also be considered satisfying the semi-group property [WC16].

2.4 Characterization of non-Markovian evolutions

In this section we want to collect those results concerning criteria that help to characterize and detect non-Markovian evolutions. We start by introducing a formalism, called k-divisibility, that helps to categorize non-Markovian evolutions thanks to a precise hierarchy. We follow by reviewing different contractivity criteria for Markovian evolutions that show how ancillary systems have to be implemented in order to fully determine the non-Markovian potential of evolutions. We end by discussing a reference approach to detect and measure non-Markovianity.

2.4.1 k-divisibility

We introduced Markovian evolutions as those families of CPTP maps that are divisible in intermediate CPTP maps for any two times $s \leq t$ (see Definition 5). In case an evolution is not Markovian, many different scenarios may occur. Indeed, consider a non-Markovian evolution $\mathbf{\Lambda}$ applied on S, which is initially correlated with an ancilla A. The dynamics of the complete system is given by $\mathbf{\Lambda} \otimes I_k = \{\Lambda_t \otimes I_k\}_{t\geq 0}$, where k is the dimension of the ancillary system. Since the evolution is non-Markovian, namely $\mathbf{\Lambda}$ is not CP-divisible, $\mathbf{\Lambda} \otimes I_d$ is not P-divisible. Nonetheless, $\mathbf{\Lambda} \otimes I_k$ may be P-divisible for some different dimension of the ancilla $k \in \{0, \dots, d-1\}$. This observation provides a straightforward hierarchy for the non-Markovianity of $\mathbf{\Lambda}$ [CM14].

We say that a map $\Phi : B(\mathcal{H}_S) \rightarrow B(\mathcal{H}_S)$ is k-positive if $\Phi \otimes I_k$ is PTP. Similarly, the evolution $\mathbf{\Lambda}$ is k-divisible if, for any $s \leq t$, it admits k-positive intermediate maps $V_{t,s}$. These definitions allow identifying a non-Markovianity degree NMD($\mathbf{\Lambda}$) as follows:

- d-divisibility corresponds to Markovianity: NMD($\mathbf{\Lambda}$) = 0;
- $(d-1)$-divisible evolutions have PTP $V_{t,s} \otimes I_{d-1}$, while some $V_{t,s} \otimes I_d$ are not PTP: NMD($\mathbf{\Lambda}$) = 1;
- 2-divisible evolutions have PTP $V_{t,s} \otimes I_2$, while some $V_{t,s} \otimes I_3$ are not PTP: NMD($\mathbf{\Lambda}$) = $d-2$;
- 1-divisible, or P-divisible, evolutions have PTP $V_{t,s}$, while some $V_{t,s} \otimes I_2$ are not PTP: NMD($\mathbf{\Lambda}$) = $d-1$;
- If $\mathbf{\Lambda}$ is not even P-divisible, some $V_{t,s}$ are not even PTP, namely it is characterized by one or more negative eigenvalues: NMD($\mathbf{\Lambda}$) = d.

We can identify as "more" non-Markovian those evolutions $\mathbf{\Lambda}$ that are k-divisible for smaller values of k: the larger is k, the larger has to be an ancilla to be used to show the non-CP-divisible nature of $\mathbf{\Lambda}$. Taken a non-Markovian $\mathbf{\Lambda}$, we say that it is: *weakly non-Markovian* if $1 \leq$NMD($\mathbf{\Lambda}$) $\leq d-1$ and *essentially non-Markovian* if NMD($\mathbf{\Lambda}$) = d.

2.4.2 Contractivity criteria

The phenomenology that we expect from Markovian evolutions corresponds to the lack of information backflows. As a consequence, if we use our system to encode information, a Markovian $\mathbf{\Lambda}$ cannot cause the increase of any information quantifier between any two times $s \leq t$. This picture explains why many of the Markovianity criteria that we show in this section require $\mathbf{\Lambda}$ to satisfy particular monotonicity relations, either by imposing conditions on Λ_t or $V_{t,s}$. We start with theorems that connect the PTP/CPTP property of maps $\Phi : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ and their ability to contract the trace-norm of Hermitian operators.

Theorem 1 ([Kos72a][Rus94]). *Any TP linear map Φ is P if and only if*

$$\|\Phi(X)\|_1 \leq \|X\|_1 \,,$$

for any Hermitian $X \in B(\mathcal{H})$.

Hence, it easily follows that

Theorem 2. *Any TP linear map Φ is CP if and only if*

$$\|\Phi \otimes I(X)\|_1 \leq \|X\|_1 \,,$$

for any Hermitian $X \in B(\mathcal{H} \otimes \mathcal{H})$, where I is the identity map on $\mathcal{H}$.

Notice that Theorems 1 and 2 can be considered to study the contractivity properties of dynamical maps $\Phi = \Lambda_t$ and intermediate maps $\Phi = V_{t,s}$.

Now, we focus on evolutions $\mathbf{\Lambda}$ and in particular on conditions that characterize their degree of non-Markovianity, namely through the k-divisibility criterion, where d-divisibility corresponds to Markovianity. We start with the following theorem:

Theorem 3 ([CM14]). *If $\mathbf{\Lambda}$ is k-divisible, then*

$$\frac{d}{dt}\|\Lambda_t \otimes I_k(X)\|_1 \leq 0 \,, \tag{2.49}$$

for all Hermitian in $X \in B(\mathcal{H}_S \otimes H_k)$, where $\dim(\mathcal{H}_k) = k$.

Since in this theorem k-divisibility is a sufficient but not necessary condition, it may happen that an evolution $\mathbf{\Lambda}$ satisfies Eq. (2.49) for any Hermitian X while not being k-divisible. In order to formulate necessary and sufficient criteria for Markovianity, we show a fundamental result for invertible evolutions. Indeed, in these cases we can provide a necessary and sufficient relation that connects a monotonicity relation with k-divisibility.

Theorem 4 ([CM14]). *Given an invertible evolution $\mathbf{\Lambda}$, it is k-divisible if and only if*

$$\frac{d}{dt}\|\Lambda_t \otimes I_k(X)\|_1 \leq 0 \,,$$

for all Hermitian in $X \in B(\mathcal{H}_S \otimes \mathcal{H}_k)$, where $\dim(\mathcal{H}_k) = k$.

Now we review some results concerning non-invertible evolutions. As defined in Section 4, an evolution is divisible if the linear intermediate map $V_{t,s}$ exists for any $s \leq t$. We saw that an evolution is divisible if and only if it satisfies the Kernel non-decreasing property (see Proposition 1). Nonetheless, a sufficient, but not necessary, condition for divisibility that involves a contractivity criteria is given by:

Proposition 2 ([CRS18]). *If an evolution* $\mathbf{\Lambda}$ *is such that*

$$\frac{d}{dt}\|\Lambda_t(X)\|_1 \leq 0\,, \tag{2.50}$$

for all Hermitian $X \in B(\mathcal{H})$ *and* $t \geq 0$*, then it is divisible.*

We remember that, invertibility implies the Kernel non-decreasing property (in this case $\mathrm{Ker}(\Lambda_t) = \mathrm{Ker}(\Lambda_s) = \emptyset$) and divisibility (see Eq. (2.33)). Moreover, the Kernel non-decreasing property implies $\dim(\mathrm{Im}(\Lambda_s)) \geq \dim(\mathrm{Im}(\Lambda_t))$ for any $s \leq t$. The following theorem, instead, provides a necessary condition for evolutions to be divisible with CP, but not necessarily TP, intermediate maps.

Theorem 5 ([CRS18]). *If an evolution* $\mathbf{\Lambda}$ *satisfies*

$$\frac{d}{dt}\|\Lambda_t \otimes I_d(X)\|_1 \leq 0\,, \tag{2.51}$$

for any Hermitian $X \in B(\mathcal{H}_S \otimes \mathcal{H}_A)$*, where* $\dim(\mathcal{H}_A) = d$*, then it is divisible with CP intermediate maps* $V_{t,s}$*.*

Notice that the existence of $V_{t,s}$ implies its TP property on $\mathrm{Im}(\Lambda_s)$, but not for any element in $B(\mathcal{H}_S)$. Indeed, Eq. (2.51) provides a slightly weaker condition than Markovianity, which instead requires $V_{t,s}$ to be CP and TP for any element in $B(\mathcal{H}_S)$.

If $\mathbf{\Lambda}$ satisfies the same conditions of Theorem 5 and is also image non-increasing (see Definition 1), we can certify the Markovian nature of the evolution with the following contractivity criteria:

Theorem 6 ([CRS18]). *If an image non-increasing evolution* $\mathbf{\Lambda}$ *satisfies*

$$\frac{d}{dt}\|\Lambda_t \otimes I_d(X)\|_1 \leq 0\,, \tag{2.52}$$

for any Hermitian $X \in B(\mathcal{H}_S \otimes \mathcal{H}_A)$*, where* $\dim(\mathcal{H}_A) = d$*, then it is Markovian.*

Nonetheless, notice that an evolution can be Markovian while not being image non-increasing. Indeed, while Markovian evolutions must have a decreasing volume of $\mathrm{Im}(\Lambda_t)$ (see Section 2.5.6), this is not the only condition that characterizes image non-increasing evolutions. A Markovian evolution $\mathbf{\Lambda}$ can be characterized by a decreasing volume of $\mathrm{Im}(\Lambda_t)$ while at the same time some states become accessible again after a finite time interval, and therefore leading to a violation of (2.29).

It is natural to ask whether Markovian evolutions are those satisfying the contractivity criteria required by Theorem 5, namely the evolutions that contract in trace-norm all Hermitian operators in an extended setup. Whereas this problem is still open in the general case of d-dimensional systems, it has been proved that this result is true for qubit evolutions:

Theorem 7 ([CC19]). *A qubit evolution $\mathbf{\Lambda}$ is Markovian if and only if*

$$\frac{d}{dt}\|\Lambda_t \otimes I_2(X)\|_1 \leq 0\,, \tag{2.53}$$

for any Hermitian $X \in B(\mathcal{H}_S \otimes \mathcal{H}_A)$, where $\dim(\mathcal{H}_A) = 2$.

Notice that in this case we do not require the evolution to be invertible or image non-increasing: any violation of (2.53) implies non-Markovianity and any non-Markovian qubit evolution provides a violation of (2.53).

2.5 Witnesses of non-Markovianity

We defined Markovian evolutions as those satisfying the CP-divisibility condition and we showed that, thanks to the Stinsespring-Kraus representation theorem, this property allows considering these evolutions as memoryless, namely they do not allow information backflows from the environment back to the system. Nonetheless, this picture does guarantee that all non-CP-divisible evolutions show information backflows that can be witnessed in an experimental setup, namely confirming the phenomenology that is expected. Indeed, one may argue that the CP-divisibility condition is too restrictive and that some non-CP-divisible evolutions may not able to provide information revivals for any S or $S-A$ initial setup, where A is an ancillary system. In order to get the exact correspondence between the phenomenological and the mathematical description of Markovian and non-Markovian dynamics, we look for one-to-one relations between non-Markovian dynamics and observable information backflows.

We follow by describing a generic non-Markovian witnessing scenario, where our goal is to observe characteristic phenomena through the evolution of a properly initialized system, if and only if $\mathbf{\Lambda}$ is non-Markovian (see Fig. 2.2). Whenever we say that we want to *witness* a non-Markovian evolution, we ask to define:

- *Initial condition*: an initial state $\rho_{SA}(0)$ shared between S and possibly an ancilla A which is evolved by the target evolution $\mathbf{\Lambda}$ as follows:

$$\rho_{SA}(t) = \Lambda_t \otimes I_A(\rho_{SA}(0)). \tag{2.54}$$

  The initial condition can also be an ensemble $\mathcal{E}_{SA}(0) = \{p_i, \rho_{SA,i}(0)\}_i$, where the evolved ensemble is obtained considering Eq. (2.54) for each state $\rho_{SA,i}(0)$:

$$\mathcal{E}_{SA}(t) = \{p_i, \rho_{SA,i}(t)\}_i. \tag{2.55}$$

- *Information quantifier* or *non-Markovian witness*: it is given by a functional Q that associates a real number to the evolved condition and is decreasing for any Markovian evolution. The value of $Q(\rho_{SA}(t))$ quantifies a particular type of information contained in $\rho_{SA}(t)$. We have a valid non-Markovianity witness Q if, for any $\rho_{SA} \in S(\mathcal{H}_{SA})$, it satisfies the following properties:

$$Q : S(\mathcal{H}_{SA}) \longrightarrow \mathbb{R} \qquad \text{s.t.} \tag{2.56}$$

$$Q(\Phi_S \otimes I_A(\rho_{SA})) \leq Q(\rho_{SA}) \qquad \text{for any CPTP } \Phi_S. \tag{2.57}$$

  Similar conditions can be formulated when we consider evolving state ensembles.

The last two relations corresponds to the data processing inequality: the information content of $S - A$ cannot increase through the application of local operations either on S or A. Any Q is monotonically decreasing when S is evolved by a Markovian evolution $\mathbf{\Lambda}$. Indeed, since in these cases the corresponding $V_{t,s}$ are CPTP for all $s \leq t$, from Eq. (2.57) it follows that:

$$Q(\rho_{SA}(t)) = Q(V_{t,s} \otimes I_A(\rho_{SA}(s))) \leq Q(\rho_{SA}(s)), \tag{2.58}$$

and therefore the absence of an information backflow between any two times:

$$Q(\rho_{SA}(t)) - Q(\rho_{SA}(s)) \leq 0. \tag{2.59}$$

In case of differentiable evolutions, Eq. (2.59) can be written for infinitesimally close times and the Markovian condition assumes the following differential form

$$\frac{d}{dt} Q(\rho_{SA}(t)) \leq 0. \tag{2.60}$$

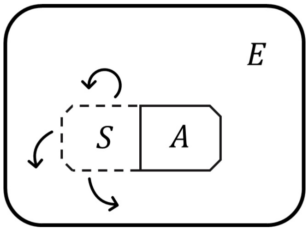


Figure 2.2: We study the information encoded in evolving states $\rho_{SA}(t)$ defined over SA, where the initial states $\rho_{SA}(0)$ evolves due to the interaction between the OQS S and the environment E, while the ancillary system A does not take part in the evolution. Notice that the information quantifiers Q that we consider are defined over the complete system SA. Our goal is to witness the non-Markovian nature of evolutions through backflows of $Q(\rho_{SA}(t))$.

We want to underline the operational role that Q plays in the characterization of non-Markovianity. Given a non-Markovian evolution $\mathbf{\Lambda}$, a non-Markovian witness Q and an initial state $\rho_{SA}(0)$ (or ensemble $\mathcal{E}_{SA}(0)$), a backflow of Q can either occur or not. Indeed, while for Markovian evolutions condition (2.59) is always satisfied, it is not obvious if it is always possible to find an initial condition and an information quantifier that show a backflow when $\mathbf{\Lambda}$ is non-Markovian. In other words, the non-trivial goal is to understand whether, for *any* non-Markovian evolution and any time interval $[s, t]$ for which there is no CPTP intermediate map $V_{t,s}$, there exist Q and $\rho_{SA}(0)$ (or $\mathcal{E}_{SA}(0)$) that show a backflow in the same time interval, namely that violate Eq. (2.59). Despite the most interesting witnessing techniques are those offering equivalencies between backflows and non-Markovian evolutions, we also discuss scenarios offering weaker conditions that however expand our knowledge of possible non-Markovian effects that can be observed and exploited.

2.5.1 Measures of non-Markovianity

A different goal connected with the characterization of non-Markovian evolutions is given by studying the degree of non-Markovianity of evolutions. Hence, *measures of non-Markovianity* provide methods to compute which non-Markovian evolutions provide larger information backflows, hence defining a hierarchy. A common method to introduce such measures is given by fixing a quantifier Q and considering the maximum total backflow that the target evolution can provide when an optimization over the initial conditions is performed.

In order to describe this procedure, we start by introducing the information flux

$$\sigma_Q(\rho_{SA}(t)) \equiv \frac{d}{dt} Q(\rho_{SA}(t)) \,. \tag{2.61}$$

It is clear that, given condition (2.57), for any t such that the infinitesimal intermediate map $V_{t+\epsilon,t}$ is CPTP, we have $\sigma_Q(\rho_{SA}(t)) \leq 0$ for any Q and $\rho_{SA}(0)$. Indeed, we say that the information Q of the system initialized in $\rho_{SA}(0)$ flows out of the system at time t when $\sigma_Q(\rho_{SA}(t)) \leq 0$. On the other hand, any violation of the differential Markovianity condition (2.60) corresponds to a positive flux $\sigma_Q(\rho_{SA}(t)) > 0$ and we say that a backflow of Q from the environment to the system is shown. These simple observations allow introducing measures of non-Markovianity $N_Q(\mathbf{\Lambda})$ that are equal to zero for all Markovian evolutions and positive only if $\mathbf{\Lambda}$ is non-Markovian

$$N_Q(\mathbf{\Lambda}) \equiv \max_{\rho_{SA}(0)} \int_{\sigma_Q(\rho_{SA}(t))>0} \sigma_Q(\rho_{SA}(t))\, dt, \tag{2.62}$$

where the states $\rho_{SA}(0)$ could be replaced by ensembles $\mathcal{E}_{SA}(0)$. Even if N_Q is not expressed in a compact form, its physical sense is very direct: we collect the information flux (2.61) whenever it flows back from the environment. The values of $N_Q(\mathbf{\Lambda})$ for different Q represent different features of the same evolution $\mathbf{\Lambda}$, namely its potential to provide backflows of Q. Nonetheless, one may think that, if $\mathbf{\Lambda}_1$ is "more" non-Markovian than $\mathbf{\Lambda}_2$ when the hierarchy provided by a precise Q is considered, namely $N_Q(\mathbf{\Lambda}_1) > N_Q(\mathbf{\Lambda}_2)$, then the same is true for any other Q. In [ABCM14] the authors propose an exhaustive comparative study to show that the ordering provided by a measure of non-Markovianity can be inverted if a different measure is considered. The measures that we introduce below in Sections 2.5.2, 2.5.3 and 2.5.7 are among those compared in [ABCM14].

Alternative approaches to quantify the non-Markovian degree of evolutions can be considered. A first strategy [PGD+16] consists in the evaluation of the maximum revival of Q that can be observed during a single time interval

$$N_Q^{max}(\mathbf{\Lambda}) \equiv \max_{s\leq t,\, \rho_{SA}(0)} Q(\rho_{SA}(t)) - Q(\rho_{SA}(s)) \,. \tag{2.63}$$

A second alternative [PGD+16] can be defined through the maximum difference between $Q(\rho_{SA}(t))$ and the time-average of the same quantity, namely $\langle Q(\rho_{SA}(t))\rangle$, during the previous times

$$N_Q^{\langle\cdot\rangle}(\mathbf{\Lambda}) \equiv \max_{\rho_{SA}(0)} \left\{0, \max_t Q(\rho_{SA}(t)) - \langle Q(\rho_{SA}(t))\rangle\right\} , \tag{2.64}$$

where $\langle Q(\rho_{SA}(t))\rangle = \left(\int_0^t Q(\rho_{SA}(t'))\,dt'\right)/t$ is the time average of Q in the time interval $[0, t]$ when the system is initialized in $\rho_{SA}(0)$. It is easy to prove that, for any Q and $\mathbf{\Lambda}$

$$N_Q^{\langle\cdot\rangle}(\mathbf{\Lambda}) \leq N_Q^{max}(\mathbf{\Lambda}) \leq N_Q(\mathbf{\Lambda})\,. \tag{2.65}$$

Moreover, while

$$N_Q^{max}(\mathbf{\Lambda}) > 0 \quad \longleftrightarrow \quad N_Q(\mathbf{\Lambda}) > 0\,, \tag{2.66}$$

a similar relation does not hold for $N_Q^{\langle\cdot\rangle}(\mathbf{\Lambda})$, namely $N_Q(\mathbf{\Lambda}) > 0$ do not imply $N_Q^{\langle\cdot\rangle}(\mathbf{\Lambda}) > 0$. Nonetheless, it can be shown [PGD+16] that $N_Q^{\langle\cdot\rangle}(\mathbf{\Lambda})$ has a very specific operational meaning connected with the probability to store and faithfully retrieve information, as measured by Q, by state preparation and measurement, where an attack performed by an eavesdropper may occur. Larger values of $N_Q^{\langle\cdot\rangle}(\mathbf{\Lambda})$ implies higher probabilities to succeed in this task. Hence, even if $N_Q^{\langle\cdot\rangle}(\mathbf{\Lambda})$ is not as accurate as $N_Q(\mathbf{\Lambda})$, namely it may be equal to zero even if $\mathbf{\Lambda}$ is non-Markovian, its value has a precise operational meaning.

This discussion suggests an interesting starting point to explore the advantages that only non-Markovian evolutions can provide in quantum protocols, where the occurrence of information backflows are exploited to obtain performance improvements. Whereas this topic is particularly interesting and it is connected with the possibility to formulate a resource theory of non-Markovian evolutions [RHP14], it goes beyond the purposes of this thesis.

2.5.2 Rivas-Huelga-Plenio measure of non-Markovianity

We gave three formulations of non-Markovianity measures, namely N_Q, N_Q^{max} and $N_Q^{\langle\cdot\rangle}$, based on the possibility to obtain backflows of an information quantifier Q. Nonetheless, the procedure given in Ref. [RHP10] by Rivas, Huelga and Plenio was one of the first methods able to detect all non-Markovian evolutions but it cannot be expressed through backflows of a specific Q. The idea behind their method exploits the Choi-Jamiołkowski isomorphism [Cho75, Jam72], which states that a map Φ_S defined for an arbitrary finite-dimensional system S is CPTP if and only if $\Phi_S \otimes I_A$ maps the maximally entangled state $\phi_{SA}^+ = d^{-1}\sum_{ij=1}^d |i\rangle\langle j|_S \otimes |i\rangle\langle j|_A$ into a physical state, where S and A have the same dimension, namely $\dim(\mathcal{H}_A) = d = \dim(\mathcal{H}_S)$. As a consequence, if Φ_S is TP, then it is CP if and only if $\Phi_S \otimes I_A(\phi_{SA}^+) \geq 0$, namely

$$\Phi_S \text{ is CP } \leftrightarrow \|\Phi_S \otimes I_A(\phi_{SA}^+)\|_1 = 1\,. \tag{2.67}$$

Consider a differentiable quantum evolution $\mathbf{\Lambda}$ with intermediate maps $V_{t+\epsilon,t}$ for the infinitesimal time intervals $[t, t+\epsilon]$. We can define the functional $f_{\mathbf{\Lambda}}(t+$

$\epsilon, t) \equiv \|V_{t+\epsilon,t} \otimes I_A(\phi^+_{SA})\|_1$, which is equal to 1 if and only if $V_{t+\epsilon,t}$ is CP, otherwise $f_{\mathbf{\Lambda}}(t+\epsilon,t) > 1$. Therefore, by defining

$$g_{\mathbf{\Lambda}}(t) \equiv \lim_{\epsilon \to 0^+} \frac{f_{\mathbf{\Lambda}}(t+\epsilon,t) - 1}{\epsilon}, \tag{2.68}$$

we can measure the non-Markovian degree of $\mathbf{\Lambda}$ with

$$N_{RHP}(\mathbf{\Lambda}) = \int g_{\mathbf{\Lambda}}(t)\, dt\,. \tag{2.69}$$

This measure of non-Markovianity is equal to zero if and only if $\mathbf{\Lambda}$ is divisible in CP intermediate maps, namely is Markovian. This approach can be problematic for non-divisible evolutions that do not allow the evaluation of $f_{\mathbf{\Lambda}}(t+\epsilon,t)$ for some t.

Finally, we want to discuss another subtle detail of this technique. The straightforward operational scenario suggested by the definition of $f_{\mathbf{\Lambda}}(t+\epsilon,t)$ for its evaluation in a laboratory would be given by applying $V_{t+\epsilon,t}$ on ϕ^+_{SA}, which is a maximally entangled (pure) state. Therefore, to do so, one may consider to find an[1] initial state $\phi_{SA}(0)$ such that $\Lambda_t \otimes I_A(\phi_{SA}(0)) = \phi^+_{SA}$. Indeed, in this way the following infinitesimal intermediate map $V_{t+\epsilon,t}$ would be applied on ϕ^+_{SA}. In general, this approach cannot be followed for two reasons: (i) ϕ^+_{SA} may not be inside $\mathrm{Im}(\Lambda_t)$ and, most importantly, (ii) even if this first condition is verified, we would obtain that $\phi_{SA}(t+\epsilon) = V_{t+\epsilon,t} \otimes I_A(\phi^+_{SA})$ would not be physical because $\|\phi_{SA}(t+\epsilon)\|_1 = f_{\mathbf{\Lambda}}(t+\epsilon,t) > 1$. This result is indeed not acceptable because evolutions (Markovian and non-Markovian) map initial (physical) states into (physical) states. Hence, if $V_{t,s}$ is not CPTP, $\phi^+_{SA} \notin \mathrm{Im}(\Lambda_s \otimes I_A)$. Similarly, if $\phi^+_{SA} \in \mathrm{Im}(\Lambda_s \otimes I_A)$, then $V_{t,s}$ must be CPTP. Therefore, even if $N_{RHP}(\mathbf{\Lambda}) > 0$ if and only if $\mathbf{\Lambda}$ is non-Markovian, this principle does not prove if it is possible to *witness* the non-Markovian nature of $\mathbf{\Lambda}$ by evolving an initial state which is later in time measured. Nonetheless, this measure of non-Markovianity is easily evaluable in many instances and quantifies non-Markovianity without considering the observation of a particular observable. This property gives to N_{RHP} an absolute meaning and is often considered as a reference measure.

2.5.3 Distinguishability of states

A measure of non-Markovianity is given by considering as witness Q the distance between pairs of states of S [BLP10], where no ancilla is exploited. Consider the measure Q_{BLP} defined as the trace distance between two evolving

[1] There may exist more than one initial state if the evolution is not bijective.

states of the system

$$Q_{BLP}(\{\rho'_S(t), \rho''_S(t)\}) = \frac{1}{2}\left(1 + \frac{\|\rho'_S(t) - \rho''_S(t)\|_1}{2}\right). \tag{2.70}$$

The value of $Q_{BLP}(\{\rho'_S(t), \rho''_S(t)\})$ represents the *distinguishability* between $\rho'_S(t)$ and $\rho''_S(t)$: its maximum value 1 is obtained when the states are orthogonal, namely perfectly distinguishable with a quantum measurement, and the minimal value $1/2$ is obtained when the states are identical and the best strategy to distinguish them is to randomly guess. In other words, it is the probability of distinguishing $\rho'_S(t)$ from $\rho''_S(t)$ in an optimal measurement scenario. Indeed, this quantity can be evaluated as

$$Q_{BLP}(\{\rho'_S(t), \rho''_S(t)\}) = \max_{\{P'_S, P''_S\}} \frac{1}{2}\left(\mathrm{Tr}\left[\rho'_S(t)P'_S\right] + \mathrm{Tr}\left[\rho''_S(t)P''_S\right]\right), \tag{2.71}$$

where the maximization is performed over POVMs $\{P'_S, P''_S\}$ on S. We define the *flux of information*

$$\sigma_{BLP}(\{\rho'_S(t), \rho''_S(t)\}) = \frac{d}{dt} Q_{BLP}(\{\rho'_S(t), \rho''_S(t)\}), \tag{2.72}$$

where we are assuming the evolution to be differentiable. In case of a Markovian evolution

$$\sigma_{BLP}(\{\rho'_S(t), \rho''_S(t)\}) \leq 0 \tag{2.73}$$

and we can interpret this as a flux of information going from S to E. On the other hand, $\sigma_{SA}(\{\rho'_S(t), \rho''_S(t)\}) > 0$ represents a flux from E back to S, namely a backflow. The measure of non-Markovianity for evolutions based on Q_{BLP} is given by collecting the maximum backflow that $\mathbf{\Lambda}$ can provide when we maximize over the possible initial pairs, namely

$$N_{BLP}(\mathbf{\Lambda}) = \max_{\{\rho'_S(0), \rho''_S(0)\}} \int_{\sigma_{BLP}(\{\rho'_S(t), \rho''_S(t)\})>0} \sigma_{BLP}(\{\rho'_S(t), \rho''_S(t)\})\, dt. \tag{2.74}$$

We have a positive value $N_{BLP}(\mathbf{\Lambda}) > 0$ only if $\mathbf{\Lambda}$ is non-Markovian, but the converse is not true. We remember that, while any PTP $V_{t,s}$ maps $S(\mathcal{H}_S)$ into itself, this is no longer true if $V_{t,s} \otimes I_A$ is applied on states in $S(\mathcal{H}_{SA})$ for a generic ancilla (see Section 2.3). If $V_{t,s}$ is PTP, the distinguishability between any two states cannot increase during $[s, t]$, namely $\|\rho'_{SA}(t) - \rho''_{SA}(t)\|_1 \leq \|\rho'_{SA}(s) - \rho''_{SA}(s)\|_1$. Indeed, as we explained in detail in Section 2.4.2, a PTP map applied on an Hermitian operator, e.g. $X = \rho'_{SA}(s) - \rho''_{SA}(s) \in B(\mathcal{H}_S)$, cannot increase its trace-one norm. It follows that we cannot observe backflows of Q_{BLP}

in a given time interval if the corresponding intermediate map is PTP. Hence, P-divisible (non-Markovian) evolutions are characterized by $N_{BLP}(\mathbf{\Lambda}) = 0$ and therefore this measure cannot detect this class of non-Markovian evolutions. In the following section we see how this picture drastically changes when initial states defined over S and supplementary ancillary systems are considered.

2.5.4 Distinguishability of states assisted by an ancilla

The results shown in Section 2.4.2 can be translated from the ability of Λ_t to contract the trace-norm of Hermitian operators to its potential to decrease the distinguishability of mixed states. We start by noticing a useful property of Hermitian matrices. Any Hermitian $X \in S(\mathcal{H}_S \otimes \mathcal{H}_A)$, up to a normalizing factor, can be written as [Hel76] $X = p\rho'_{SA} - (1-p)\rho''_{SA}$ for some $p \in [0,1]$ and $\rho'_{SA}, \rho''_{SA} \in S(\mathcal{H}_{SA})$. An Hermitian matrix written in this form is often called Helstrom matrix. The quantity $\|X\|_1 = \|p\rho'_{SA} - (1-p)\rho''_{SA}\|_1$ describes the possibility to distinguish ρ'_{SA} from ρ''_{SA} when they are prepared with *a-priori* probabilities p and $1-p$, respectively. Indeed, the probability to success in this task when an optimal measurement is performed is given by

$$P_g^{(p)}(\rho'_{SA}, \rho''_{SA}) = \frac{1}{2}(1 + \|p\rho'_{SA} - (1-p)\rho''_{SA}\|_1)\,, \tag{2.75}$$

which is indeed maximal if the states are orthogonal and minimal if they are identical. We call this quantity p-distinguishability and, if $p = 1/2$, we simply call it distinguishability. Indeed, for $p = 1/2$ the two states are prepared with the same probability and $P_g^{(1/2)}$ corresponds to Q_{BLP} introduced in Eq. (2.70). Similarly to Eq. (2.71), the operational meaning of $P_g^{(p)}$ is made explicit by the following formulation:

$$P_g^{(p)}(\rho'_{SA}, \rho''_{SA}) = \max_{\{P'_{SA}, P''_{SA}\}} \left(p\mathrm{Tr}\left[P'_{SA}\rho'_{SA}\right] + (1-p)\mathrm{Tr}\left[P''_{SA}\rho''_{SA}\right]\right)\,, \tag{2.76}$$

where the maximization is performed over 2-output POVMs $\{P'_{SA}, P''_{SA}\}$ on $S-A$.

We start by considering Theorem 2 and we show that it can be exploited to connect increases in p-distinguishability with the presence of non-CPTP intermediate maps during an evolution. Hence, instead of a generic TP map Φ_S, we consider the non-CPTP $V_{t,s}$. The problem that we encounter in adopting Theorem 2 to operationally witness non-Markovianity with p-distinguishability is that we should be certain that ρ'_{SA} and ρ''_{SA} belong to the image of the preceding evolution $\mathrm{Im}(\Lambda_s)$. In [BJA17] the authors show that, if Λ_s is invertible, then the existence of at least one Helstrom matrix $\overline{X} = p\overline{\rho}'_{SA} - (1-p)\overline{\rho}''_{SA}$ with a decreasing trace-norm in the time interval $[s,t]$ implies the existence of a second Helstrom matrix $X = p\rho'_{SA} - (1-p)\rho''_{SA}$ where ρ'_{SA} and ρ''_{SA} belong to $I(\Lambda_s \otimes I_A)$. Hence, if Λ_s^{-1} exists, there exist initial states $\rho'_{SA}(0)$ and $\rho''_{SA}(0)$ such that their p-distinguishability increases in the time interval $[s,t]$ if and only if $V_{t,s}$ is not CPTP.

This approach can be considered also to study k-divisibility: an evolution $\mathbf{\Lambda}$ is k-divisible if $\mathbf{\Lambda} \otimes I_k$ $(\Phi \otimes I_k)$ is P-divisible, where I_k is the identity operator on a k-dimensional ancillary system (see Section 2.4.1).

Theorem 8 ([BJA17]). *Given an invertible evolution* $\mathbf{\Lambda}$*, the intermediate map* $V_{t,s}$ *is* k*-positive if and only if*

$$\|V_{t,s} \otimes I_k \left(p\rho'_{SA}(s) - (1-p)\rho''_{SA}(s)\right)\|_1 \leq \|p\rho'_{SA}(s) - (1-p)\rho''_{SA}(s)\|_1 \,, \quad (2.77)$$

for any $p \in (0,1)$ *and pair of states* $\{\rho'_{SA}(0), \rho''_{SA}(0)\}$*, where* $\dim(\mathcal{H}_A) = k$.

We remember that Markovianity, namely CP-divisibility, corresponds to d-divisibility of the evolution, namely the d-positivity of the corresponding intermediate maps. We notice that, in case of differentiable evolutions, Theorem 4 can be casted as follows

Theorem 9 ([CM14, CKR11]). *Given an invertible evolution* $\mathbf{\Lambda}$*, it is* k*-divisible if and only if*

$$\frac{d}{dt}\|\Lambda_t \otimes I_k(p\rho'_{SA}(0) - (1-p)\rho''_{SA}(0))\|_1 \leq 0 \,, \quad (2.78)$$

for any $p \in (0,1)$ *and pair of states* $\{\rho'_{SA}(0), \rho''_{SA}(0)\}$*, where* $\dim(\mathcal{H}_A) = k$.

Therefore, given any invertible non-Markovian evolution, there exist two initial $S-A$ states and a $p \in (0,1)$ such that their p-distinguishability increases during at least one time interval. Similarly, also the results concerning non-invertible evolutions, namely Theorems 5, 6 and 7, can be formulated in terms of p-distinguishabilities. In particular, the latter can be written as

Theorem 10 ([CC19]). *A qubit evolution* $\mathbf{\Lambda}$ *is Markovian if and only if*

$$\frac{d}{dt}\|\Lambda_t \otimes I_2(p\rho'_{SA}(0) - (1-p)\rho''_{SA}(0))\|_1 \leq 0 \,, \quad (2.79)$$

for any $p \in (0,1)$ *and pair of two-qubit states* $\{\rho'_{SA}(0), \rho''_{SA}(0)\}$.

The price we have to pay in order to be able to detect any non-Markovian evolution without the need to check all p-distinguishabilities, but only the ordinary ($p = 1/2$) distinguishability, is to increase the dimension of the ancillary system from d to $d + 1$ [BJA17]. Indeed, the contractivity criteria of the type $\frac{d}{dt}\|\Lambda_t(X)\|_1 \leq 0$ for Hermitian $X \in S(\mathcal{H}_S \otimes \mathcal{H}_A)$ with a d-dimensional ancilla can be replaced by $\frac{d}{dt}\|\Lambda_t \otimes I_{d+1}(\rho'_{SA}(0) - \rho''_{SA}(0))\|_1 \leq 0$ for any pair of $S - A$ states, where $\dim(\mathcal{H}_A) = d + 1$.

This result can be casted in differential or non-differential forms:

Theorem 11 ([BJA17])**.** *Given an invertible or point-wise non-invertible evolution* $\mathbf{\Lambda}$ *such that* Λ_s^{-1} *exists,* $V_{t,s}$ *is CPTP if and only if*

$$\|V_{t,s} \otimes I_{d+1}\left(\rho'_{SA}(s) - \rho''_{SA}(s)\right)\|_1 \leq \|\rho'_{SA}(s) - \rho''_{SA}(s)\|_1\,, \tag{2.80}$$

for any pair of states $\{\rho'_{SA}(0), \rho''_{SA}(0)\}$*, where* $\dim(\mathcal{H}_A) = d + 1$.

Theorem 12 ([BJA17])**.** *Given an invertible or point-wise non-invertible evolution* $\mathbf{\Lambda}$*, it is Markovian if and only if*

$$\frac{d}{dt}\|\Lambda_t \otimes I_{d+1}(\rho'_{SA}(0) - \rho''_{SA}(0))\|_1 \leq 0\,, \tag{2.81}$$

for any pair of states $\{\rho'_{SA}(0), \rho''_{SA}(0)\}$*, where* $\dim(\mathcal{H}_A) = d + 1$.

A constructive method for the initial witnessing pair of states

We saw several criteria that connects non-Markovianity with the increase of the evolving p-distinguishability of $\rho'_{SA}(t)$ and $\rho''_{SA}(t)$. Nonetheless, only [BJA17] proposes a method to construct the initial states $\rho'_{SA}(0)$ and $\rho''_{SA}(0)$ needed for this task. In particular, they study Theorem 11 and provide initial states to witness any invertible or point-wise non-bijective evolutions characterized by a non-CPTP intermediate map $V_{t,s}$. The states provided by this method depends solely on the initial time s. These states can also be used for Theorem 12, where the same states would violate Eq. (2.81) during at least one time interval contained in $[s, t]$ if and only if $V_{t,s}$ is not CPTP.

Any p-distinguishability is a functional satisfying the non-Markovian witnesses conditions (2.56) and (2.57) for ensembles. Hence, we simply define

$$Q_{BJA}(\{\rho'_{SA}(t), \rho''_{SA}(t)\}) \equiv P_g^{(1/2)}(\rho'_{SA}(t), \rho''_{SA}(t))\,, \tag{2.82}$$

where $d_A = d + 1$ and the corresponding initial condition is given by the pair of $S - A$ states $\rho'_{SA}(0)$ and $\rho''_{SA}(0)$. Now, we show the details of the constructive method that provides the initial witnessing pair of states. Given a non-Markovian evolution, we can always individuate a time interval when there is no CPTP intermediate map $V_{t,s}$. Now, we consider those states that at time s assume the form

$$\rho'_{SA}(s) = (1-p)\sigma_{SA} + p\phi^+_{SA}\,, \tag{2.83}$$

$$\rho''_{SA}(s) = (1-p)\sigma_{SA} + p\rho_S \otimes |d+1\rangle\langle d+1|_A\,, \tag{2.84}$$

where $\phi^+_{SA} = d^{-1}\sum_{ij=1}^{d} |i\rangle\langle j|_S \otimes |i\rangle\langle j|_A$ is the maximally entangled state between S and the first d degrees of freedom of A, σ_{SA} is an arbitrary state in the interior of $\mathrm{Im}(\Lambda_s \otimes I_A)$, namely not in its border, and ρ_S is an arbitary state in $S(\mathcal{H}_S)$. It is straightforward to see that, for small-enough values of $p > 0$, we have $\rho'_{SA}(s), \rho''_{SA}(s) \in \mathrm{Im}(\Lambda_s \otimes I_A)$. Notice that $\rho'_{SA}(s) - \rho''_{SA}(s) = p(\phi^+_{SA} - \rho_S \otimes |d+1\rangle\langle d+1|_A)$. Since ϕ^+_{SA} has no support on the $d+1-$th degree of freedom of A, this state is orthogonal to $\rho_S \otimes |d+1\rangle\langle d+1|_A$. Hence, $\|\rho'_{SA}(s) - \rho''_{SA}(s)\|_1 = p\|\phi^+_{SA}\|_1 + p\|\rho_S \otimes |d+1\rangle\langle d+1|_A\|_1 = 2p$. The orthogonality between these components is preserved after the action of the intermediate map $V_{t,s} \otimes I_A$ and at time t we obtain $\|\rho'_{SA}(t) - \rho''_{SA}(t)\|_1 = p\|V_{t,s} \otimes I_A(\phi^+_{SA})\|_1 + p\|V_{t,s}(\rho_S) \otimes |d+1\rangle\langle d+1|_A\|_1$. While $p\|V_{t,s}(\rho_S) \otimes |d+1\rangle\langle d+1|_A\|_1 \geq 1$, we focus on $\|V_{t,s} \otimes I_A(\phi^+_{SA})\|_1$. This quantity, due to the Choi-Jamiołkowski isomorphism [Cho75, Jam72], is greater than 1 if and only if $V_{t,s}$ is not CPTP. Therefore, a non-CPTP intermediate map $V_{t,s}$ causes an increase of the distinguishability between these two states during the time interval $[s,t]$. Finally, notice that the initial states $\rho'_{SA}(0)$ and $\rho''_{SA}(0)$ can be obtained by applying $\Lambda_s^{-1} \otimes I_A$ on Eqs. (2.83) and (2.84). Moreover, σ_{SA}, ρ_S and p can be chosen such that $\rho'_{SA}(0)$ and $\rho''_{SA}(0)$ are arbitrary close and/or separable.

In summary, this method provides a pair of initial states $\rho'_{SA}(0)$ and $\rho''_{SA}(0)$ such that, if the evolution in $[s,t]$ is described by a non-CPTP intermediate map $V_{t,s}$, then the witness Q_{BJA} provide a backflow

$$Q_{BJA}(\{\rho'_{SA}(t), \rho''_{SA}(t)\}) - Q_{BJA}(\{\rho'_{SA}(s), \rho''_{SA}(s)\}) > 0\,. \tag{2.85}$$

Notice that the authors in [BJA17] show that also point-wise non-invertible evolutions can be considered. These evolutions are those $\mathbf{\Lambda}$ for which the inverse map Λ_t^{-1} does not exist only for a discrete set of times $\{t_i\}_i$. In case s is a time when $\mathbf{\Lambda}$ is non-invertible, in order to recover Eq.(2.85) it is enough to find a pair of states $\rho'_{SA}(0) \neq \rho''_{SA}(0)$ that are mapped by Λ_s into the same state $\rho'_{SA}(s) = \rho''_{SA}(s)$. Indeed, this condition implies that $Q_{BJA}(\{\rho'_{SA}(s), \rho''_{SA}(s)\}) = 1/2$ and, when the invertibility is recovered for some later time t, these two states $\rho'_{SA}(t) \neq \rho''_{SA}(t)$ become distinguishable again, $Q_{BJA}(\{\rho'_{SA}(t), \rho''_{SA}(t)\}) > 1/2$ and the backflow (2.85) is recovered. Finally, they notice that evolutions that are not invertible or point-wise non-bijective are contained in a zero-measure set in the space of quantum evolutions. Hence, any random infinitesimal perturbation of any evolution is either invertible or point-wise non-invertible.

2.5.5 Guessing probability of ensembles

The approach followed by Buscemi and Datta in [BD16] is slightly different and requires the introduction of the guessing probability of ensembles, a generalization of $P_g^{(p)}$ to ensembles of any number of states. Consider the task of identifying a state that we randomly choose from a known ensemble $\mathcal{E} = \{p_i, \rho_i\}_{i=1}^n$ of states of $S(\mathcal{H})$. The guessing probability $P_g(\mathcal{E})$ is the average probability to successfully identify the extracted state with an optimal measurement, that is

$$P_g(\mathcal{E}) \equiv \max_{\{P_i\}_{i=1}^n} \sum_{i=1}^n p_i \operatorname{Tr}[\rho_i P_i] \,, \tag{2.86}$$

where the maximization is performed over the n-output POVMs of $B(\mathcal{H})$ (compare it with Eq. (2.76)). We say that the larger is $P_g(\mathcal{E})$, the more $\mathcal{E}$ is *distinguishable*. Notice that the maximum value $P_g(\mathcal{E}) = 1$ is obtained for orthogonal state ensembles. Moreover, $P_g(\mathcal{E})$ can be used to define witnesses of non-Markovianity: under the action of any CPTP map $\Phi : B(\mathcal{H}_S) \to B(\mathcal{H})$ on the states of $\mathcal{E} = \{p_i, \rho_i\}_i$, the guessing probability $P_g(\mathcal{E})$ is non-increasing: $P_g(\{p_i, \rho_i\}_i) \geq P_g(\{p_i, \Phi(\rho_i)\}_i)$.

Now we explain how we can use the guessing probability to witness *any* non-Markovian dynamics. We consider a finite-dimensional system $S - A$, where the d-dimensional system S is evolved by a generic evolution $\mathbf{\Lambda}$ and A is an ancillary system. Given an initial ensemble $\mathcal{E}_{SA}(0) = \{p_i, \rho_{SA,i}(0)\}_i$, we consider its evolution $\mathcal{E}_{SA}(t) = \{p_i, \Lambda_t \otimes I_A(\rho_{SA,i}(0))\}_i$. Therefore, if $\mathbf{\Lambda}$ is Markovian,

$$P_g(\mathcal{E}_{SA}(t)) - P_g(\mathcal{E}_{SA}(s)) \leq 0 \,, \tag{2.87}$$

for any time interval $[s, t]$. Hence, the quantifier of information considered in this scenario is

$$Q_{BD}(\mathcal{E}_{SA}(t)) = P_g(\mathcal{E}_{SA}(t)) \,,$$

where the initial condition is given by $\mathcal{E}_{SA}(0)$. The authors of [BD16] show that, for any evolution $\mathbf{\Lambda}$ and time interval $[s, t]$, there exist an ancillary system A and an initial ensemble $\overline{\mathcal{E}}_{SA}(0)$ of separable states of $S(\mathcal{H}_{SA})$

$$\overline{\mathcal{E}}_{SA}(0) \equiv \{\overline{p}_i, \overline{\rho}_{SA,i}\}_{i=1}^{\overline{n}} \,, \tag{2.88}$$

such that we have a backflow

$$P_g(\overline{\mathcal{E}}_{SA}(t)) - P_g(\overline{\mathcal{E}}_{SA}(s)) > 0 \,, \tag{2.89}$$

if and only if there exists no CPTP intermediate map $V_{t,s}$, and therefore a violation of the Markovian condition (2.59). Moreover, the probability distribution

$\overline{\mathcal{P}} \equiv \{\overline{p}_i\}_{i=1}^{\overline{n}}$ has a finite size of $\overline{n} \leq d^4$ elements and $\dim(\mathcal{H}_A) \leq d$. Notice that, even if we do not make it explicit, $\overline{\mathcal{E}}_{SA}(0)$ depends on $\mathbf{\Lambda}$ and $[s, t]$. This result is completely general, applies to any finite-dimensional evolution and, while it is not able to provide the explicit states needed to define $\overline{\mathcal{E}}_{SA}(0)$, it proves the first one-to-one relation between information backflows and non-Markovianity.

2.5.6 Volume of accessible states

A way to characterize and study non-Markovianity is given by the study of $\mathrm{Im}(\Lambda_t)$ and the temporal evolution of its volume $V(t)$ [LPP13]. Indeed, we can see that it is contractive under CPTP maps and therefore it can be used to study non-Markovianity. In order to discuss this technique, we briefly introduce the technique used. Given a d-dimensional system S, any density operator $\rho_S \in S(\mathcal{H}_S)$ can be represented by a $d^2 - 1$ dimensional real vector $\boldsymbol{r}$ in the Bloch representation (see Section 2.1.1). Therefore, any state $\rho_S(t)$ evolving under the dynamics defined by $\mathbf{\Lambda}$ can be represented by a time dependent vector $\boldsymbol{r}(t)$. In this formalism, the action of the dynamical map Λ_t induces the following affine transformation

$$\boldsymbol{r}(0) \to \boldsymbol{r}(t) = A(t)\boldsymbol{r}(t) + \boldsymbol{q}(t)/\sqrt{d}\,, \tag{2.90}$$

where $A(t)$ is a $(d^2 - 1) \times (d^2 - 1)$ real matrix. This matrix can be decomposed as $A(t) = O^{(1)}(t)D(t)O^{(2)}(t)$, where $O^{(i)}(t)$ are orthogonal matrices and $D(t)$ is positive semi-definite and diagonal. It follows that the action of Λ_t on the space of state vectors $\boldsymbol{r}$ corresponds to a first rotation $O^{(1)}(t)$ (possibly composed with an inversion), then a shrink of the vectors $D(t)$ followed by a second rotation $O^{(2)}(t)$ and a translation $\boldsymbol{q}(t)/\sqrt{d}$. The contraction factor of the available state vectors is given by $\det A(t) = \det D(t)$. Indeed, it can be shown that $V(t) = |\det A(t)|V(0)$. Moreover, $|\det A(t)|$ is monotonically decreasing for P-divisible evolutions [WC08] and therefore, if $\mathbf{\Lambda}$ is P-divisible,

$$\frac{dV(t)}{dt} = \frac{d|\det A(t)|}{dt} \leq 0\,. \tag{2.91}$$

This result implies that whenever an increase of $V(t)$ occurs in a time interval $[t_1, t_2]$, we can infer that V_{t_2,t_1} is not even P and $\mathbf{\Lambda}$ is essentially non-Markovian, namely not even P-divisible. The following measure can be considered

$$N_{LPP}(\mathbf{\Lambda}) = \frac{1}{V(0)} \int_{dV(t)/dt>0} \frac{dV(t)}{dt} dt = \int_{d|\det A(t)|/dt>0} \frac{d|\det A(t)|}{dt} dt\,.$$

We have that $N_{LPP}(\mathbf{\Lambda}) > 0$ only if $\mathbf{\Lambda}$ is essentially non-Markovian.

The authors of [LPP13] show that this measure of non-Markovianity is also connected to the amount of classical information that can be retrieved from the environment. Indeed, if we perform an encoding of classical information by preparing quantum states according to a probability distribution $p_{\boldsymbol{r}}$ and we evolve such states with the dynamical map Λ_t, we can see that change in entropy of the probability distribution is

$$S_C(p_{\boldsymbol{r}(t)}) - S_C(p_{\boldsymbol{r}(0)}) = \log_2 |\det A(t)|\,,$$

where $S_C(p_{\boldsymbol{r}}) = -\int p_{\boldsymbol{r}} \log(p_{\boldsymbol{r}}) d\boldsymbol{r}$. It follows that a contraction of the volume of accessible states corresponds to a loss of classical information.

In [CMM17] this vectorial approach is applied to the Breuer-Laine-Piilo non-Markovianity measure [BLP10](see Section 2.5.3), where $\Lambda_t(\rho'_S(0)-\rho''_S(0))$ is studied to witness non-Markovianity. This quantity, in the representation introduced here, assumes the form $A(t)(\boldsymbol{r}'(0) - \boldsymbol{r}''(0))$, where $\boldsymbol{r}'(0)$ ($\boldsymbol{r}''(0)$) is the vectorial representation of $\rho'_S(0)$ ($\rho''_S(0)$) obtained through Eq. (2.4). Therefore, the BLP condition depends on $A(t)$ but not on $\boldsymbol{q}(t)$. The P-divisibility of $\boldsymbol{\Lambda}$ and therefore of the evolution induced by Eq. (2.90), relies on both $A(t)$ and $\boldsymbol{q}(t)$ and therefore both the BLP condition (2.73) and the condition (2.91) are weaker than P-divisibility. Notice this is not true for unital evolutions, namely such that $\Lambda_t(\mathbb{1}_S) = \mathbb{1}_S$ for any $t \geq 0$, where $\boldsymbol{q}(t) = 0$.

It may seem unfeasible to perform an experiment that estimates $V(t)$ for a generic evolution of a d-dimensional systems. Nonetheless, the authors of [LPP13] show that, in order to evaluate $V(t)$, it is enough to perform state tomography on $d^2 - 1$ states that initially correspond to an orthogonal basis of Bloch vectors in $\mathbb{R}^{d^2-1}$ and the maximally mixed state. Hence, in the context introduced at the beginning of this section, we can consider the witness $Q_{LPP}(t) = V(t)$, where the initial condition is given by an orthogonal set $\{\rho_{S,i}(0)\}_{i=1}^{d^2-1}$.

Additional results concerning commutative evolutions, namely such that $\Lambda_t \circ \Lambda_s = \Lambda_s \circ \Lambda_t$ for any $s \leq t$, normal commutative evolutions, namely commutative evolutions such that $\Lambda_t \circ \Lambda_t^* = \Lambda_t^* \circ \Lambda_t$ for any $t \geq 0$ with Λ_t^* being the dual of Λ_t, and Hermitian commutative evolutions, namely commutative evolutions such that $\Lambda_t = \Lambda_t^*$ for any $t \geq 0$, are presented together with examples in [CMM17].

2.5.7 Correlation measures

The higher performance that quantum protocols can achieve, if compared with the corresponding classical counterparts, are often due to quantum correlated multipartite system. Indeed, quantum correlations are among the most representative type of information that can be considered for quantum systems. Hence, scenarios that admit backflows of these quantities are of great interest.

In Section 2.2 we described three types of bipartite $A - B$ states, namely product states (no correlations), separable states (classical correlations) and entangled states (quantum correlations), as those having increasing degrees of shared correlation. In this section we give precise rules to define functionals to measure correlations over a bipartition. Given a bipartite system $A-B$, a *correlation measure* M quantifies correlations shared between the subsystems A and B. Several different measures have been formulated, where each one studies qualitatively different types of correlations. In order for $M : S(\mathcal{H}_{AB}) \to \mathbb{R}$ to be considered an operationally meaningful correlation measure, we require it to satisfy the following properties:

- $M(\rho_{AB})$ is non-increasing under local operations on A and B;
- $M(\rho_{AB}) \geq 0$ for any state ρ_{AB};
- $M(\rho_{AB}) = 0$ if ρ_{AB} is a product state $\rho_A \otimes \rho_B$.

The first condition encapsulates the natural requirement that correlations cannot be created by local operations. Notice that, while for generic Q we imposed contractivity under local operations only for one subsystem (see condition (2.57)), correlation measures require contractivity under local operations for both subsystems. This property implies that in general M decreases when we apply CPTP maps on A and/or B, while it has to be invariant under unitary local transformations. Indeed, for any bipartite system state $\rho_{AB} \in S(\mathcal{H}_{AB})$ and local unitaries $U_A \in B(\mathcal{H}_A)$ and $U_B \in B(\mathcal{H}_B)$, we have $M(\rho_{AB}) \geq M((U_A \otimes U_B)\rho_{AB}(U_A \otimes U_B)^\dagger) = M(\rho'_{AB}) \geq M((U_A \otimes U_B)^\dagger \rho_{AB}(U_A \otimes U_B)) = M(\rho_{AB})$, where $\rho'_{AB} = (U_A \otimes U_B)\rho_{AB}(U_A \otimes U_B)^\dagger$. Notice that since any product state can be prepared by local operations, all these states should give the same value of M, which also corresponds to the minimum of M over all quantum states. We obtain the second and third conditions if, without loss of generality, we impose this minimal value to be equal to zero. Indeed, the first of these three conditions is the central property that characterizes correlation measures and distinguishes them from other functionals.

Non-Markovian evolutions and correlation measures

Notice that the monotonicity of correlation measures under CPTP local maps implies that any Markovian evolution on S monotonically decrease correlations

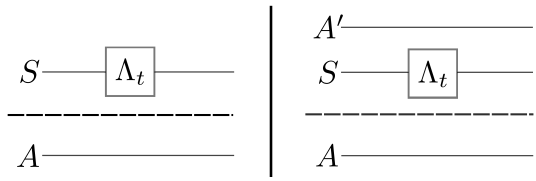


Figure 2.3: Left: in the first setting, an initial state between system S and ancilla A is used. An increase of correlations between these two parts witnesses the presence of non-Markovian effects. Right: in our second extended setting, the whole setup consists of three parts, the systems S and A as before, plus an extra ancilla A'. An increase of the correlations over the bipartition SA' versus A can be used to witness non-Markovian evolutions. By taking an initial product state along the bipartition $A' - AS$, we recover the first scenario of $S - A$ states.

shared in $S-A$ systems. In fact, consider the scenario of Fig. 2.3 left, where S is evolved by $\mathbf{\Lambda}$ and is correlated with an ancillary system A, namely $\rho_{SA}(t) = \Lambda_t \otimes I_A(\rho_{SA}(0))$. If there is a correlation backflow between S and A in a time interval $[s, t]$, the evolution $\mathbf{\Lambda}$ is non-Markovian and more precisely the corresponding intermediate map $V_{t,s}$ cannot be CPTP.

Given the definition of correlation functionals, these quantities automatically satisfy the conditions for non-Markovian witnesses Q. Indeed, different correlation measures have been proposed to witness and quantify non-Markovian effects, e.g. quantum entanglement [RHP10] and quantum mutual information (QMI) [LFS12].

From an OQS perspective, a decrease in correlations between S and A during the dynamics may be caused by a non-recoverable loss of $S-A$ correlations but it may also be that these correlations have been transformed into potentially recoverable correlations of the environment-OQS-ancilla. A revival can therefore be seen as a flow of correlations lost during the previous evolution back to $S - A$.

In what follows we also consider a slightly more complex setting with two ancillas A and A', where the evolution is again applied only on S but we are interested in the correlations shared between SA' and A (see Fig.2.3 right). It is straightforward to see that any correlation measure of this kind cannot increase under Markovian dynamics. Notice that the previous setting can be recovered by taking an initial state which is product along the bipartition $A' - AS$.

Entanglement measures

We start by describing an exemplary correlation measure widely used in quantum information theory: entanglement. It is exploited in different areas, e.g. as a fundamental tool in quantum protocols and resource theories [HHHH09, PV07, CG19]. Entanglement M_E, being typically a non-local correlation, was originally constructed to capture only non-classical correlations. The idea of LOCC was originally introduced to describe the effects that distant experimenters can induce on a correlated states by using local operations. It was later understood that entanglement can be defined to be that correlation that is non-increasing under any LOCC applied by the parties that share the state. This implies that $M_E(\rho_{SA}) = 0$ if ρ_{SA} is a separable state. A widely used measure of entanglement is *negativity*:

$$\mathrm{NEG}(\rho_{SA}) = \frac{\|\rho_{SA}^{T_A}\|_1 - 1}{2} \geq 0\,, \tag{2.92}$$

where T_A, a PTP transformation, represents the partial transposition of A. We have that $\mathrm{NEG}(\rho_{SA}) = 0$ for any separable states, while $\mathrm{NEG}(\rho_{SA}) > 0$ implies that S and A share quantum correlations. This measure derives from the positivity of the partial transpose (PPT) condition [Per96], which states that $\rho_{SA}^{T_A} \geq 0$ is a necessary condition for the separability of ρ_{SA}. The PPT criterion is also a sufficient condition only for qubit-qubit and qubit-qutrit bipartite systems [HHH96]. Hence, for these systems, $\mathrm{NEG}(\rho_{SA}) > 0$ if and only if ρ_{SA} is entangled. Instead, this is no longer true for larger subsystems [HHH98], where there exist entangled states satisfying the PPT condition and therefore $\mathrm{NEG}(\rho_{SA}) = 0$.

Rivas-Huelga-Plenio, in Ref. [RHP10], first introduced the idea of using entanglement measures to witness non-Markovianity as follows. The authors considered a $S - A$ bipartite system prepared in the maximally entangled state $|\phi^+\rangle\langle\phi^+|_{SA} = d^{-1}\sum_{ij=1}^{d} |i\rangle\langle j|_S \otimes |i\rangle\langle j|_A$, where S is evolved with the target evolution $\mathbf{\Lambda}$. In the context introduced in this section, our non-Markovianity witness can be any entanglement measure M_E, e.g. negativity, and the initial condition is given by $\phi_{SA}(0) = |\phi^+\rangle\langle\phi^+|_{SA}$. The evolution induces the transformation $\phi_{SA}(t) = \Lambda_t \otimes I_A(|\phi^+\rangle\langle\phi^+|_{SA})$ and any increase of $M_E(\phi_{SA}(t))$ has to be attributed to a non-CPTP intermediate map and therefore to a non-Markovian evolution. Finally, a measure of non-Markovianity $N_E(\mathbf{\Lambda})$ similar to Eq. (2.74) can be formulated once we consider the flux of entanglement $\sigma_E(t) = \frac{d}{dt}M_E(\phi_{SA}(t))$. Notice that in this case no maximization over initial states is required and we obtain

$$N_E(\mathbf{\Lambda}) = \int \sigma_E(t)dt\,.$$

Quantum mutual information

Classical mutual information quantifies the amount of mutual dependence between two random variables. It evaluates the bits of information we gain about one variable if we observe the other variable and vice versa. Its formulation is inherently connected with the entropy of a random variable, where the larger is the *Shannon entropy* of X, namely $S_C(X)$, the less we know about the random variable X. Given X with possible outcomes x_i and occurrence probabilities p_i, the Shannon entropy of X is defined as

$$S_C(X) \equiv -\sum_i p_i \log p_i \,. \tag{2.93}$$

The mutual information shared between two random variables X and Y is therefore given by

$$I_C(X;Y) \equiv S_C(X) + S_C(Y) - S_C(X,Y)\,, \tag{2.94}$$

where $S(X,Y)$ is calculated with the joint probability distributions for the outcomes of X and Y. The quantum generalization of Eq. (2.93) is given by the von Neumann entropy

$$S(\rho) \equiv -\mathrm{Tr}\left[\rho \log \rho\right]\,, \tag{2.95}$$

where, in this context, log denotes the matrix logarithm. Similarly, we can define $I(\rho_{SA})$ as the information shared between the quantum systems S and A

$$I(\rho_{SA}) \equiv S(\rho_S) + S(\rho_A) - S(\rho_{SA})\,, \tag{2.96}$$

where ρ_{SA} is the state of the bipartite system $S-A$ and $\rho_S = \mathrm{Tr}_A\left[\rho_{SA}\right]$ ($\rho_A = \mathrm{Tr}_S\left[\rho_{SA}\right]$) is the corresponding reduced state of S (A). The QMI is a continuous function on the set of states and is analytic on the interior of the set of states, namely it is infinitely differentiable and equals its Taylor series in a neighborhood of any point. Moreover, $I(\rho_{SA})$ measures both classical and non-classical correlations. Indeed, this measure in general is not null for separable states. In the following we describe how this feature distinguishes the non-Markovian witnessing potential of entanglement and QMI.

In Ref. [LFS12] the authors used QMI to witness non-Markovianity and constructed a corresponding measure obtained by the scheme given in Section 2.5, where a maximization over ancillary systems A and initial states $\rho_{SA}(0)$ is performed

$$N_{LFS}(\mathbf{\Lambda}) \equiv \sup_{A,\rho_{SA}(0)} \int_{\sigma_{LFS}(\rho_{SA}(t))>0} \sigma_{LFS}(\rho_{SA}(t))\, dt\,, \tag{2.97}$$

where $\rho_{SA}(t) = \Lambda_t \otimes I_A(\rho_{SA}(0))$ and

$$\sigma_{LFS}(\rho_{SA}(t)) \equiv \frac{d}{dt} I(\rho_{SA}(t))\,. \tag{2.98}$$

2.5.8 Entropic quantities

Several entropic quantities have been taken in consideration in the context of witnessing non-Markovianity. See Ref. [ABC18] for a review on this topic. The monotonicity of the relative entropy evaluated between two evolving S states can be connected with a divisibility property of $\mathbf{\Lambda}$ [Uhl77, OP04]. Being the relative entropy between two density operators in $S(\mathcal{H})$ defined as $S(\rho\|\sigma) \equiv \mathrm{Tr}\,[\rho(\log\rho - \log\sigma)]$, we have:

Proposition 3 ([MHR17])**.** *If $\mathbf{\Lambda}$ is k-divisible, then*

$$\frac{d}{dt} S(\rho'_{SA}(t)\|\rho''_{SA}(t)) \leq 0\,, \tag{2.99}$$

for any pair of states $\rho'_{SA}(0)$ and $\rho'_{SA}(0)$ in $S(\mathcal{H}_{SA})$, where $\dim(\mathcal{H}_A) = k$ and $S - A$ states are evolved by $\mathbf{\Lambda} \otimes I_k$,

Hence, a violation of Eq. (2.99) implies that the target evolution is either P-divisible or essentially non-Markovian. Results similar to Proposition 3 can be obtained with other entropic quantities, such as: the Rényi-α divergence [OP04, HMPB11, MHR17], the sandwiched Rényi divergences [MLDS$^+$13, FL13, MO14, MHR17] and the conditional Rényi entropy [Tom15].

Finally, by defining the entropic quantity called min-entropy as follows

$$H_{min}(\rho_{SA}) = \min_{\sigma_S \in S(\mathcal{H}_S)} - \log \|(\sigma_S^{-1/2} \otimes \mathbb{1}_A)\rho_{SA}(\sigma_S^{-1/2} \otimes \mathbb{1}_A)\|_\infty\,, \tag{2.100}$$

we can also consider the *quantum correlation* $q_{corr}(\rho_{SA}) = 2^{-H_{min}(\rho_{SA})}$ [KRS09, ABC18]. This quantity can be related to the singlet fraction of ρ_{SA}, namely:

$$q_{corr}(\rho_{SA}) = d_A \max_{\Phi_S} \langle\phi^+|(\Phi_S \otimes \mathbb{1}_A)(\rho_{SA})|\phi^+\rangle^2_{SA}, \tag{2.101}$$

that is the maximum fidelity with a maximally entangled state $|\phi^+\rangle_{SA}$ optimized over local operations Φ_S. It is easy to see that this measure cannot increase by local operations on the first system, as any further local processing can always be adsorbed in the optimization in (2.101).

For quantum-classical states $\rho_{SA}(t) = \sum_i p_i \rho_{S,i}(t) \otimes |i\rangle\langle i|_A$, where $|i\rangle_A \in \mathcal{H}_A$ are orthogonal states, $q_{corr}(\rho_{SA}(t)) = P_g(\{p_i, \rho_{S,i}(t)\})$ and the results of Section 2.5.5 can be recovered. Notice that, while the name of this quantity contains the word "correlation", it is not monotonic under CPTP maps on the ancillary system and therefore we do not consider q_{corr} a proper correlation measure.

Each one of the entropic quantities introduced in this section can be used to define a witness Q and therefore a corresponding flux σ_Q and a measure of non-Markovianity N_Q, e.g. as in Section 2.5.3. Notice that these procedures always imply maximizations over the possible initial states or pair of states.

2.5.9 Channel discrimination

Consider the task of distinguishing a channel between different possible CPTP maps $\Phi_{S,i} : B(\mathcal{H}_S) \to B(\mathcal{H}_S)$, where each channel on S is given with *a-priori* probability p_i. In this scenario, we have the freedom to choose on which state ρ_{SA} we apply the unknown channel, where S is transformed by one channel from $\{\Phi_{S,i}\}_{i=1}^n$ while the k-dimensional ancilla A is left untouched. The i-th channel transforms ρ_{SA} in $\rho_{SA,i} = \Phi_{S,i} \otimes I_A(\rho_{SA})$ and therefore the task of distinguishing the channel ensemble $\{p_i, \Phi_{S,i}\}_i$ can be translated into the evaluation of $P_g(\{p_i, \rho_{SA,i}\}_i)$, namely the distinguishability of the state ensemble $\{p_i, \rho_{SA,i}\}_i$. Indeed, if we use a k-dimensional ancilla A, we define the distinguishability of the channel ensemble $\{p_i, \Phi_{S,i}\}_i$ as follows

$$P_g^{(k)}(\{p_i, \Phi_{S,i}\}_i) \equiv \max_{\rho_{SA} \in S(\mathcal{H}_{SA})} P_g(\{p_i, \rho_{SA,i}\}_i) . \tag{2.102}$$

The presence of an ancillary system used to define initial correlations among S and A helps in this witnessing task. Indeed, one finds that:

$$P_g^{(2)}(\{p_i, \Phi_{S,i}\}_i) \leq P_g^{(3)}(\{p_i, \Phi_{S,i}\}_i) \leq \cdots \leq P_g^{(d)}(\{p_i, \Phi_{S,i}\}_i) . \tag{2.103}$$

Now, consider the scenario where, after the application of a channel from $\{p_i, \Phi_{S,i}\}_i$ on the initial state $\rho_{SA}(0)$, we evolve the system to time t with an evolution $\boldsymbol{\Lambda}$. Hence, in this case the initial $\rho_{SA}(0)$ is transformed with probability p_i into $\rho_{SA,i}(t) = \Lambda_t \circ \Phi_{S,i} \otimes I_A(\rho_{SA}(0))$. As expected, distinguishing the ensemble $\{p_i, \rho_{SA,i}(0)\}_i$ is easier than for $\{p_i, \rho_{SA,i}(t)\}_i$ (see Section 2.5.5). Therefore, the same can be said comparing $\{p_i, \Phi_{S,i}\}_i$ and $\{p_i, \Lambda_t \circ \Phi_{S,i}\}_i$. Indeed, for any $k \geq 2$ and CPTP map Λ_S:

$$P_g^{(k)}(\{p_i, \Phi_{S,i}\}_i) \geq P_g^{(k)}(\{p_i, \Lambda_S \circ \Phi_{S,i}\}_i) .$$

Actually, it can be proven that:

Proposition 4 ([BC16]). *If $\boldsymbol{\Lambda}$ is k-divisible, then*

$$\frac{d}{dt} P_g^{(l)}(\{p_i, \Lambda_t \circ \Phi_{S,i}\}_i) \leq 0 , \tag{2.104}$$

for any $l \leq k$.

Consider the case of two channels $\{\Phi_{S,1}, \Phi_{S,2}\}$ which are respectively applied with *a-priori* probabilities p and $1-p$ on states $\rho_{SA} \in S(\mathcal{H}_{SA})$, where A is a k-dimensional ancilla. We can define their distinguishability as

$$P_g^{(k,p)}(\Phi_{S,1}, \Phi_{S,2}) \equiv \max_{\rho_{SA}} \frac{1}{2} \left(1 + \|(p\Phi_{S,1} - (1-p)\Phi_{S,2}) \otimes I_A(\rho_{SA})\|_1\right), \tag{2.105}$$

where A is a k-dimensional ancilla. Notice that, for fixed values of p, $P^{(k,p)}$ satisfies Eq. (2.103) when different dimensions of the ancilla are considered.

The following necessary and sufficient relation connects invertible Markovian evolutions and the monotonic decrease of channels dinstinguishability:

Proposition 5 ([BC16]). *If $\mathbf{\Lambda}$ is invertible, then it is k-divisible if and only if*

$$\frac{d}{dt} P_g^{(k,p)}(\Lambda_t \circ \Phi_{S,1}, \Lambda_t \circ \Phi_{S,2}) \leq 0\,, \tag{2.106}$$

for any $p \in (0,1)$ and pair of CPTP maps $\{\Phi_{S,1}, \Phi_{S,2}\}$.

In practice, detecting a non-Markovian evolution with the procedure suggested here can be demanding in the general case. Indeed, Theorem 5 requires to consider every pair of CPTP maps $\{\Phi_{S,1}, \Phi_{S,2}\}$, $p \in (0,1)$ and a maximization over the set of initial states $\rho_{SA}(0)$ (see Eq. (2.105)).

2.6 Random unitary evolutions

In this section we introduce an exemplary set of quantum evolutions that are considered throughout this thesis. The wide interest that it attracted is due to its simple formulation and the possibility to easily characterize some properties, e.g. P-divisibility, by checking simple conditions on the parameters that define this set.

We start by defining *unital* evolutions as those that preserve the identity operator during the dynamics, namely $\mathbf{\Lambda}$ is unital if and only if $\Lambda_t(\mathbb{1}_S) = \mathbb{1}_S$ for any $t \geq 0$. In other words, unital evolutions are those having the maximally mixed state $\mathbb{1}_S/d$ as steady state, where d is the dimension of S. Notice that this property has not to be true *only* for $\mathbb{1}_S/d$, other steady states different from $\mathbb{1}_S/d$ may exist.

We follow by defining as random unitary those maps $\Phi_S : S(\mathcal{H}_S) \to S(\mathcal{H}_S)$ that can be written as

$$\Phi(\rho_S) = \sum_k p_k U_k \rho_S U_k^\dagger\,, \tag{2.107}$$

where $\{p_k\}_k$ is a probability distribution and $\{U_k\}_k$ a set of unitary transformations [CW15]. Therefore, $\mathbf{\Lambda}$ is a random unitary evolution if the corresponding Λ_t can be written in this form for any $t \geq 0$. If $d = 2$, that is for qubit evolutions, any unital channel is random unitary [LS93]. However, when $d \geq 3$, the set of random unitary channels is strictly included in the set of unital channels.

A particularly handy subset of random unitary evolutions is given by Pauli evolutions, which are defined as those where the dynamical maps assume the form

$$\Lambda_t(\rho_S(0)) = \sum_{k=0}^{d^2-1} p_k(t)\, \sigma_k \rho_S(0) \sigma_k^\dagger\,, \tag{2.108}$$

where the unitary operators σ_k, for $k = 1, \ldots, d^2 - 1$, are such that $\sigma_0 = \mathbb{1}_S$ and $\mathrm{Tr}\left[\sigma_i \sigma_j^\dagger\right] = d\delta_{ij}$ for $i, j = 0, 1, \ldots, d^2 - 1$. This class is studied in Ref. [CW15], where the operators σ_k are given by the set of unitary generalized spin Weyl operators. Since $\{p_k(t)\}_k$ is a probability distribution, we have that $\sum_{k=0}^{d^2-1} p_k(t) = 1$ and $p_k(t) \geq 0$ for any k. Notice that from the initial condition $\Lambda_0 = I_S$ we get $p_0(0) = 1$.

2.6.1 Qubit Pauli evolutions

In the case that S is a qubit, namely for $d = 2$, the Pauli dynamical maps (2.108) assume the form

$$\Lambda_t(\rho_S(0)) = p_0(t)\rho_S(0) + \sum_{k=x,y,z} p_k(t)\, \sigma_k \rho_S(0) \sigma_k\,, \tag{2.109}$$

where the operators σ_x, σ_y and σ_z are the Pauli operators. Now we show how to connect Pauli evolutions with the dynamics generated by the following master equation in the generalized Lindblad form (2.43)

$$L_t(\rho_S(t)) = \sum_{k=x,y,z} \gamma_k(t)(\sigma_k \rho_S(t) \sigma_k - \rho_S(t))\,, \tag{2.110}$$

where $\gamma_{x,y,z}(t)$ are real valued time-dependent functions. We consider Eq. (2.44) to construct Λ_t and we study its action on the Pauli operators.

$$\begin{aligned}
\Lambda_t(\sigma_x) &= \exp\left[-\int_0^t (\gamma_z(\tau) + \gamma_y(\tau))d\tau\right]\sigma_x, \\
\Lambda_t(\sigma_y) &= \exp\left[-\int_0^t (\gamma_z(\tau) + \gamma_x(\tau))d\tau\right]\sigma_y, \\
\Lambda_t(\sigma_z) &= \exp\left[-\int_0^t (\gamma_x(\tau) + \gamma_y(\tau))d\tau\right]\sigma_z, \\
\Lambda_t(\mathbb{1}_S) &= \mathbb{1}_S\,,
\end{aligned} \tag{2.111}$$

Remember that, as we saw in Eq. (2.3), any qubit state can be written in terms of components proportional to $\sigma_{x,y,z}$. Note that if we allow the rates to take values in the extended reals $\mathbb{R} \cup \{-\infty, +\infty\}$ we can have non-bijective dynamical maps Λ_t since $\exp[-\infty] = 0$. Therefore, any qubit random unitary dynamical map generated by finite rates is bijective. In these cases, Λ_t^{-1} exists for all $t \geq 0$ (the exponential function is always non-zero) and the intermediate maps are given by considering Eqs. (2.111) in Eq. (2.33), namely:

$$
\begin{aligned}
V_{t,s}(\sigma_x) &= \exp\left[-\int_s^t (\gamma_z(\tau) + \gamma_y(\tau))d\tau\right]\sigma_x, \\
V_{t,s}(\sigma_y) &= \exp\left[-\int_s^t (\gamma_z(\tau) + \gamma_x(\tau))d\tau\right]\sigma_y, \\
V_{t,s}(\sigma_z) &= \exp\left[-\int_s^t (\gamma_x(\tau) + \gamma_y(\tau))d\tau\right]\sigma_z, \\
V_{t,s}(\mathbb{1}_S) &= \mathbb{1}_S.
\end{aligned}
\tag{2.112}
$$

Any Pauli evolution can be defined either by the time-dependent probability distribution $p_{0,x,y,z}(t)$ (see Eq. (2.109)) or by the time-dependent set of rates $\gamma_{x,y,z}(t)$ that defines the generator L_t and the dynamical map Λ_t through Eqs. (2.111) and (2.110). In order to derive Λ_t from L_t and vice-versa, the following equations can be considered [CW13]

$$
\begin{aligned}
p_0(t) &= \frac{1}{4}\left(1 + A_{xy}(t) + A_{xz}(t) + A_{yz}(t)\right), \\
p_x(t) &= \frac{1}{4}\left(1 - A_{xy}(t) - A_{xz}(t) + A_{yz}(t)\right), \\
p_y(t) &= \frac{1}{4}\left(1 - A_{xy}(t) + A_{xz}(t) - A_{yz}(t)\right), \\
p_z(t) &= \frac{1}{4}\left(1 + A_{xy}(t) - A_{xz}(t) - A_{yz}(t)\right).
\end{aligned}
\tag{2.113}
$$

We end this section by showing some simple conditions on $\gamma_{x,y,z}(t)$ that allows understanding whether the generated evolution is physical, P-divisible or Markovian. We start by defining:

$$
A_{ij}(t) \equiv \exp\left[-2\int_0^t (\gamma_i(\tau) + \gamma_j(\tau))\, d\tau\right] \geq 0\,, \tag{2.114}
$$

for $i \neq j$ and $i, j \in x, y, z$. From Eqs. (2.111) and (2.114), we obtain

$$
\Lambda_t(\sigma_i) = \lambda_i(t)\sigma_i\ , \quad \text{where } \lambda_i(t) = \sqrt{A_{jk}(t)}\,, \tag{2.115}
$$

where $(i, j, k) = (x, y, z), (y, z, x), (z, x, y)$. Notice that the i-th eigenoperator of Λ_t is σ_i, where λ_i is the corresponding eigenvalue, while $\mathbb{1}_S$ is an eigenoperator with eigenvalue 1 at any time. In case $\gamma_{x,y,z}(t) \geq 0$ is finite for any $t \geq 0$ we know from Section 2.3.4 that Eq. (2.110) defines a Markovian evolution.

Instead, if one or more of these rates assume negative values, we use the following conditions in order to understand whether the given set of rates defines an evolution that is physical (Λ_t is CPTP at any time) or P-divisible:

- Λ_t is CPTP if and only if

$$B_{ijk}(t) \equiv 1 + A_{ij}(t) - A_{jk}(t) - A_{ki}(t) \geq 0\,, \tag{2.116}$$

  for $(i, j, k) = (x, y, z), (y, z, x), (z, x, y)$ [HCLA14];

- $\mathbf{\Lambda}$ is P-divisible if and only if, for any $t \geq 0$,

$$\begin{aligned} \gamma_x(t) + \gamma_y(t) &\geq 0, \\ \gamma_y(t) + \gamma_z(t) &\geq 0, \\ \gamma_z(t) + \gamma_x(t) &\geq 0, \end{aligned} \tag{2.117}$$

  since the intermediate maps are then contractive in trace norm [Kos72b, Kos72a, Rus94, CW13];

- $\mathbf{\Lambda}$ is Markovian if and only if $\gamma_{x,y,z}(t) \geq 0$ for any $t \geq 0$.

2.6.2 Depolarizing evolutions

We gave conditions to distinguish Markovian from non-Markovian Pauli evolutions based on the behavior of the rates $\gamma_{x,y,z}(t)$. In this section we introduce a widely studied subclass of Pauli evolutions called depolarizing evolutions. An evolution $\mathbf{D} = \{D_t\}_t$ defined on a d-dimensional $\mathcal{H}_S$ is depolarizing if and only if at any time $t \geq 0$ the corresponding dynamical map D_t can be written as a linear combination of the identity transformation I_S and the map that sends every input state into the completely mixed state. Specifically, we have

$$D_t(\cdot) = f(t)\, I_S(\cdot) + (1 - f(t))\mathrm{Tr}\,[\cdot]\, \frac{\mathbb{1}_S}{d}\,, \tag{2.118}$$

with the *characteristic function* $f(t)$ being a real quantity belonging to the interval

$$J_{\mathbb{D}} \equiv \left[-\frac{1}{d^2 - 1}, 1 \right], \tag{2.119}$$

where this last property is necessary and sufficient for D_t to be CPTP [Kin03]. We define $\mathbb{D}$ to be the set of depolarizing evolutions, where the specific dimension of the corresponding Hilbert space is left unspecified. It is easy to verify that $\mathbb{D}$ is a closed set [Hol12, Wil13, Kin03]. Indeed, given any two depolarizing evolutions $\mathbf{D}^{(1)} = \{D_t^{(1)}\}_t$ and $\mathbf{D}^{(2)} = \{D_t^{(2)}\}_t$, we have that $D_t^{(p)} = (1-p)D_t^{(1)} +$

$pD_t^{(2)}$ assumes the form (2.118) with characteristic function $f^{(p)}(t) \in J_{\mathbb{D}}$. It is possible to prove that, for any d, the depolarizing evolutions are random unitary. From Eq. (2.118) it is clear that we can use the function $f(t)$ to uniquely characterize the elements of $\mathbb{D}$.

Depolarizing evolutions for qubits

We show how to connect the characteristic function $f(t)$ to the parameters used in the previous sections to describe evolutions, namely the probabilities $p_{x,y,z}(t)$ given in Eq. (2.109) and the rates $\gamma_{x,y,z}(t)$ given in Eq. (2.110). The dynamical map D_t is in the Pauli form. Indeed, the channel (2.109) is equal to the depolarizing map (2.118) when

$$p_0(t) = \frac{3f(t)+1}{4} \quad \text{and} \quad p_x(t) = p_y(t) = p_z(t) = \frac{1-p_0(t)}{3} . \tag{2.120}$$

Notice that, if $f(t)$ assumes its maximum value 1, namely when D_t acts as the identity map, $p_0(t) = 1$. On the contrary, if $f(t)$ assumes its minimal value $-1/3$, we obtain $p_0(t) = 0$. Now we show the connection between **D** and the corresponding master equation L_t that generates the same evolution.

Given the symmetric action of D_t, it is straightforward to see that the three rates $\gamma_{x,y,z}(t)$ are identical: $\gamma_{x,y,z}(t) = \gamma(t)$. Therefore, if we compare Eq. (2.110), which now assumes the form

$$\frac{d}{dt}\rho_S(t) = \gamma(t) \sum_{i=x,y,z} \left(\sigma_i \rho_S(t)\sigma_i - \rho_S(t)\right) . \tag{2.121}$$

with the time derivative of Eq. (2.118), namely

$$\frac{d}{dt} D_t(\rho_S(0)) = \dot{f}(t)\left(\rho_S(0) - \frac{\mathbb{1}_S}{2}\right) , \tag{2.122}$$

we obtain

$$\gamma(t) = -\frac{\dot{f}(t)}{4f(t)} . \tag{2.123}$$

Hence, Eqs. (2.120) provides the connection between $p_{0,x,y,z}(t)$ and $f(t)$ and (2.123) provides the connection between $\gamma(t)$ and $f(t)$. In the following we characterize Markovian and non-Markovian depolarizing evolutions in terms of the behavior of $f(t)$. Nonetheless, we can notice that

- A qubit depolarizing evolution can be either Markovian or non-Markovian, but not P-divisible. Indeed, if $\gamma_{x,y,z}(t) = \gamma(t) < 0$ for some time, the P-divisibility conditions (2.117) cannot be satisfied.

- The times for which the infinitesimal intermediate maps $V_{t+\epsilon,t}$ are CPTP are characterized by $\gamma(t) \geq 0$. For the same times, the characteristic function $f(t)$ is either positive and non-increasing or negative and non-decreasing. If $f(t)$ behaves differently, the evolution is non-Markovian, $\gamma(t) < 0$ and $V_{t+\epsilon,t}$ is not CPTP.

2.6.3 Eternal non-Markovian model

This particular non-Markovian model [HCLA14, BCF17] is a Pauli evolution with a generator (2.110) defined by the following rates

$$\left\{\gamma_x(t), \gamma_y(t), \gamma_z(t)\right\} = \frac{\alpha}{2}\{1, 1, -\tanh t\}, \quad \text{for } \alpha \geq 1 . \tag{2.124}$$

It is straightforward to see these evolutions are P-divisible for any $\alpha > 0$ (see Eqs. (2.117)). Nonetheless, the physicality condition (2.116) is violated for $\alpha \in (0, 1)$. Indeed, in these cases, the corresponding dynamical map Λ_t would not be CPTP for all $t \geq 0$. Hence, we restrict to those P-divisible evolutions defined by $\alpha \geq 1$.

The notable feature of these evolutions is given by the behavior of the rate $\gamma_z(t)$, which is negative at any $t > 0$. As a consequence, any intermediate map V_{t_2,t_1} with $t_1 > 0$ is P but not CP. Hence, for any time interval $[t_1, t_2]$ with $t_1 > 0$, we expect to be able to witness an information backflow. A counterintuitive consequence is given by the possibility to obtain a backflow even after the evolution started from an infinitesimal time. Indeed, one could expect that any S requires a minimal amount time to lose information before that it can be recovered from the environment. Hence, consider $V_{t,s}$ defined for a s very close to zero. The data processing inequality applied on Λ_s and Λ_t implies that $Q(\rho_{SA}(s)) \leq Q(\rho_{SA}(0))$ and $Q(\rho_{SA}(t)) \leq Q(\rho_{SA}(0))$ for any information quantifier Q monotonic under CPTP maps. Now, if Q is continuous and it increases in the time interval (s, t), there is a narrow time-window for a backflow: $Q(\rho_{SA}(s)) < Q(\rho_{SA}(t)) \leq Q(\rho_{SA}(0))$, where $Q(\rho_{SA}(s))$ and $Q(\rho_{SA}(0))$ are infinitesimally close. Hence, by fixing t, as s approaches 0, $Q(\rho_{SA}(s))$ approaches $Q(\rho_{SA}(0))$ and the possible backflows $Q(\rho_{SA}(t)) - Q(\rho_{SA}(s)) > 0$ become increasingly smaller. Therefore, we both have the intuitive scenario where large backflows are possible only after a minimum elapsed time from the beginning of the evolution and the possibility to obtain (infinitesimal) backflows for time intervals with starting times infinitesimally close to zero.

Finally, we mention that in Ref. [MCPS17] the authors show how to obtain non-Markovian evolutions with rates similar to Eq. (2.124) through the convex combination of Markovian Pauli evolutions with non-negative and constant rates, namely satisfying the semi-group property. Therefore, this work studies the non-convexity of the Markovian set by showing that even the peculiar eternal non-Markovian model can be obtained by mixing evolutions satisfying the semi-group property (2.35).

Chapter 3

Witnessing non-Markovianity through correlations

Non-Markovian effects in an open-system dynamics are usually associated to information backflows from the environment to the system. However, the way these backflows manifest and how to detect them is unclear. A natural approach is to study the backflow in terms of the correlations the evolving system displays with another unperturbed system during the dynamics. In this chapter, we study the power of this approach to witness non-Markovian dynamics using different correlation measures. We identify simple dynamics where the failure of CP-divisibility is in one-to-one correspondence with a correlation backflow. We then focus on specific correlation measures, such as those based on entanglement and the mutual information, and identify their strengths and limitations. The results exposed in this chapter are contained in the original work [DJB$^+$20].

3.1 Introduction

The dynamics of open quantum systems [BP07, Wei00, RH11] has been investigated extensively in recent years for both fundamental and applicative reasons. In particular the problem of understanding and characterizing memoryless dynamics, the so-called Markovian regime, and dynamics exhibiting memory effects, the non-Markovian regime, have been considered in a wide range of different ways (for extended reviews see [RHP14, BLPV16]). Intuitively, one expects that non-Markovian effects are associated to a backflow of information from the environment to the system. Several approaches have been pursued to put this intuition in rigorous terms. A standard procedure, see for example [BLP10, LFS12, BD16, BJA17], consists of considering operational quantities Q that are monotonically non-increasing under CP maps (see Section 2.5). An increase of any such quantities implies that the evolution is non-Markovian, although the converse is not true in general. One of the quantities considered in the context of non-Markovian characterization is correlations, a fundamental concept for our understanding of quantum theory and also a resource for many quantum information protocols. The general idea of this approach consists of monitoring the evolution of the correlations between a system subjected to a dynamics and an additional system that does not take part in the evolution. If at some point a correlation backflow, that is, an increase in the correlations quantified by a given correlation measure, is observed, then the dynamics must be non-Markovian.

Non-Markovian evolutions show advantages in different quantum information processing protocols; for example in quantum metrology [CHP12], quantum teleportation [LBP14], entanglement generation [HRP12], and quantum communication [BCM14]. An increase of correlations such as entanglement and quantum mutual information (QMI) between system and an ancilla maybe in many of these examples responsible for these achievements. Therefore, understanding for which measures and channels a correlation backflow appears is not only a fundamental question but may also clarify how non-Markovianity can be helpful for quantum information processing protocols.

In this chapter we focus on the first side of this problem and our main goal is to understand the power of correlations to witness non-Markovian evolutions. In particular, we study the strengths and weaknesses of several well-known correlation measures for this task and we derive several results that improve our understanding of this question. First, we introduce the quasi-eternal non-Markovian model and we study how to tune its parameters in order to delay the appearance of its non-Markovian effects. Then, it is shown how for a class of differentiable evolutions termed single parameter, which includes relevant examples such as depolarization, dephasing, and amplitude damping, any continuously differentiable correlation measure increases during non-Markovian dynamics unless it is time independent on the whole image of the preceding evolution. Secondly, we focus on two fundamental quantum correlation measures: entanglement measures and QMI. For the first, we provide a simple argument explaining how it fails to witness non-Markovianity in many situations. For the second, we study its behavior in different scenarios. We first show that QMI witnesses the non-Markovianity of any bijective unital and non-P-divisible dynamics on a qubit. We then provide several examples of non-Markovian dynamics where no QMI backflow is observed when using maximally entangled states. For some of these examples, we demonstrate that a backflow in the QMI does appear when using non-maximally entangled states, in some cases even arbitrarily weakly entangled pure states. This highlights how a high degree of initial correlations is not necessarily beneficial for the detection of non-Markovianity when using the QMI as a witness. Lastly, we discuss conditions under which the QMI shared between an evolving system and an ancilla cannot show backflows. Moreover, we show how to construct examples of quasi-eternal non-Markovian dynamics that do not display any these QMI backflows.

3.2 Non-Markovian dynamics

In Section 2.3.4 we saw that, for a differentiable evolution $\mathbf{\Lambda}$, any dynamical map Λ_t and any intermediate map $V_{t,s}$ can be expressed as time-ordered exponentials

$$\Lambda_t(\rho) = Te^{\int_0^t L_\tau d\tau}, \qquad V_{t,s} = Te^{\int_s^t L_\tau d\tau}, \tag{3.1}$$

where L_t is the generator of the evolution. We say that L_t is casted in the generalized Lindblad form when it can be written as

$$\begin{aligned} L_t(\rho_S(t)) \equiv\ & i[H(t), \rho_S(t)] \\ & + \sum_k \gamma_k(t) \left(G_k(t)\rho_S(t)G_k^\dagger(t) - \frac{1}{2}\left\{ G_k^\dagger(t)G_k(t), \rho_S(t) \right\} \right), \end{aligned} \tag{3.2}$$

where $\gamma_k(t)$ are real time-dependent functions, $G_k(t)$ are time-dependent operators and $H(t)$ a Hermitian possibly time-dependent operator. The Hamiltonian term of the generator describes the unitary part of the dynamics generated by $H(t)$ and the second term describes the dissipative part of the dynamics generated by the operators $G_k(t)$.

The generator L_t can be defined in terms of the intermediate map as $L_t = \frac{dV_{t,s}}{ds}\big|_{s=t}$. Often, it is convenient to describe the intermediate map $V_{t,s} \otimes I_A$ and the generator L_t by how they act on a basis of $B(\mathcal{H}_{SA})$. Such a basis can be constructed from operators of the form $\chi_S \otimes \chi_A$ where χ_S is an operator on $\mathcal{H}_S$ and χ_A is an operator on $\mathcal{H}_A$. Let the dimension of $\mathcal{H}_S$ be d and let χ_{Sk} be d^2-1 traceless Hermitian operators such that $\mathrm{Tr}[\chi_{Sk}\chi_{Sl}] = \delta_{kl}d$. Likewise let d_A be the dimension of $\mathcal{H}_A$ and let χ_{Ak} be $d_A^2 - 1$ traceless Hermitian operators such that $\mathrm{Tr}[\chi_{Ak}\chi_{Al}] = \delta_{kl}d_A$. Then one can choose an orthonormal basis $\{e_i\}_{i=0}^{d_A^2 d^2-1}$ for $B(\mathcal{H}_{SA})$ by constructing the basis vectors e_i as the tensor products $\chi_{Sj}\otimes\chi_{Ai}$, $\mathbb{1}_S \otimes \chi_{Ai}$, $\chi_{Sj} \otimes \mathbb{1}_A$, and $\mathbb{1}_S \otimes \mathbb{1}_A$ for all i, j. Since there are $d^2 - 1$ traceless χ_{Si} and $d_A^2 - 1$ traceless χ_{Ai}, one obtains $(d^2 - 1)(d_A^2 - 1)$ traceless elements of the form $\chi_{Sj} \otimes \chi_{Ai}$, $(d^2 - 1)$ traceless elements of the form $\mathbb{1}_A \otimes \chi_{Sj}$, $(d_A^2 - 1)$

traceless elements of the form $\chi_{Ai} \otimes \mathbb{1}_S$ and a single element with trace $\mathbb{1}_S \otimes \mathbb{1}_A$. We denote $e_0 \equiv \mathbb{1}_S \otimes \mathbb{1}_A$. A state $\rho_{SA} \in S(\mathcal{H}_{SA})$ can be represented in this basis by coordinates given by the real numbers $a_i = \frac{1}{dd_A} \operatorname{Tr}(\rho_{SA} e_i)$, namely

$$\rho_{SA} = \frac{1}{dd_A} \mathbb{1}_S \otimes \mathbb{1}_A + \sum_{i=1}^{d^2 d_A^2 - 1} a_i e_i. \tag{3.3}$$

We can now describe the intermediate map $V_{t,s} \otimes I_A$ by how it acts on each basis elements e_i. For this purpose we define

$$\mathcal{V}_{ij}(t,s) \equiv \operatorname{Tr}\left[e_i V_{t,s} \otimes I_A(e_j)\right]. \tag{3.4}$$

Note that $\mathcal{V}_{ij}(t,s)$ is real for all i, j since $V_{t,s}$ is hermiticity preserving. Moreover, since the map $V_{t,s}$ is trace preserving it follows that $\mathcal{V}_{00}(t,s) = 1$ and $\mathcal{V}_{0j}(t,s) = 0$ for $j \neq 0$. Let the coordinates $\bar{a} \equiv \{a_i\}$, where $a_0 = 1/d_A d$, describe a state at time t. This state is mapped by $V_{t,s} \otimes I_A$ to coordinates $\bar{a}(s) = \{a_i(s) \equiv \sum_j \mathcal{V}_{ij}(t,s) a_j\}$. Analogously to Eq. (3.4), we define the time derivatives of the components $\mathcal{V}_{ij}(t,s)$ as

$$\left.\frac{d\mathcal{V}_{ij}(t,s)}{ds}\right|_{s=t} \equiv \operatorname{Tr}\left[e_i \frac{dV_{t,s}}{ds} \otimes I_A(e_j)\right]\Bigg|_{s=t}. \tag{3.5}$$

3.2.1 The quasi-eternal non-Markovian model

We present a class of non-Markovian Pauli evolutions that generalizes the eternal non-Markovian model shown in Section 2.6.3. We analyze the evolutions $\mathbf{\Lambda}^{(t^{NM},\alpha)}$, defined by Eqs. (2.111), with parameterized time-dependent rates

$$\left\{\gamma_x(t), \gamma_y(t), \gamma_z(t)\right\} = \frac{\alpha}{2}\left\{1, 1, -\tanh(t - t^{NM})\right\}, \tag{3.6}$$

where $\alpha > 0$ and $t^{NM} \geq 0$. Whether $\Lambda_t^{(t^{NM},\alpha)}$ generated by Eqs. (2.111) define a physical evolution, namely they are CPTP maps for every $t \geq 0$, depends on the values of α and t^{NM}. We saw that, if $t^{NM} = 0$, the maps $\Lambda_t^{(0,\alpha)}$ are CPTP if and only if $\alpha \geq 1$. Otherwise, if $\alpha \in (0,1)$, the map $\Lambda_t^{(0,\alpha)}$ is not CP for every $t \geq 0$.

We define as *quasi-eternal non-Markovian evolution* any physical evolution generated by Eq. (3.6) where $t^{NM} > 0$. First, $\mathbf{\Lambda}^{(t^{NM},\alpha)}$ represents an evolution for all $\alpha \geq 1$. Instead, if $0 < \alpha < 1$, in order for $\Lambda_t^{(t^{NM},\alpha)}$ to be CPTP for every $t \geq 0$, we have to consider wisely-chosen values of t^{NM}. In order to satisfy the physicality condition (2.116), we calculate the quantities $A_{ij}^{(t^{NM},\alpha)}(t)$ that define $\Lambda_t^{(t^{NM},\alpha)}$ through Eq. (2.115) and we obtain

$$A_{xy}^{(t^{NM},\alpha)}(t) = e^{-2\alpha t}\,, \tag{3.7}$$

$$A_{yz}^{(t^{NM},\alpha)}(t) = A_{zx}^{(t^{NM},\alpha)}(t) = \left(e^{-t}\frac{\cosh(t-t^{NM})}{\cosh(t^{NM})}\right)^{\alpha}\,. \tag{3.8}$$

Now, we derive the physicality conditions for α and t^{NM}. In order to satisfy the CPTP conditions given in Eq. (2.116), we notice that $B_{yzx}^{(t^{NM},\alpha)}(t) = B_{zxy}^{(t^{NM},\alpha)}(t) = 1 - A_{xy}^{(t^{NM},\alpha)}(t) = 1 - e^{-2\alpha t} \geq 0$ for any $\alpha > 0$ and $t \geq 0$. It is straightforward to verify that the last condition $B_{xyz}^{(t^{NM},\alpha)}(t) = 1 + A_{xy}^{(t^{NM},\alpha)}(t) - 2A_{yz}^{(t^{NM},\alpha)}(t) \geq 0$ is satisfied for any $t \geq 0$ if and only if $\lim_{t\to\infty} B_{xyz}^{(t^{NM},\alpha)}(t) \geq 0$. Therefore, the relation between $\alpha > 0$ and $t^{NM} \geq 0$ that implies $\Lambda_t^{(t^{NM},\alpha)}$ to be CPTP for any $t \geq 0$ is $1 - 2(e^{2t^{NM}} + 1)^{-\alpha} \geq 0$, namely

$$t^{NM} \geq \bar{t}^{NM}(\alpha) \equiv \frac{1}{2}\log\left(2^{1/\alpha} - 1\right)\,, \tag{3.9}$$

or equivalently

$$\alpha \geq \overline{\alpha}(t^{NM}) \equiv \frac{1}{\log_2(e^{2t^{NM}} + 1)}\,. \tag{3.10}$$

After having described how to construct quasi-eternal evolutions $\mathbf{\Lambda}^{(t^{NM},\alpha)}$, we discuss their properties. We start by noticing that, while $\gamma_x(t)$ and $\gamma_y(t)$ are positive and constant, $\gamma_z(t) < 0$ for any $t > t^{NM}$. Hence, the negativity of $\gamma_z(t)$ implies the non-Markovianity of these evolutions. It is easy to show that the P-divisibility conditions given by Eqs. (2.117) are satisfied for any $\alpha > 0$, $t^{NM} \geq 0$ and $t \geq 0$. It follows that the evolutions of this class are non-Markovian and P-divisible. Indeed, the intermediate maps $V_{t,s}^{(t^{NM},\alpha)}$ are P (but not CP) for any time interval (t, s) such that $t^{NM} < s < t$. Notice that the non-Markovian evolutions $\mathbf{\Lambda}^{(t^{NM},\alpha)}$ behave as a Markovian evolution for $t \in [0, t^{NM}]$, namely during those times when $\gamma_{x,y,z}(t)$ are non-negative.

3.2.2 Tuning of the quasi-eternal non-Markovian model

We show how we can use the quasi-eternal model to tune when non-Markovian effects take place. We fix $\alpha = 2/5$ and we consider the evolutions $\mathbf{\Lambda}^{(t^{NM})}$ defined by different values of t^{NM}, namely generated by the rates

$$\left\{\gamma_x(t), \gamma_y(t), \gamma_z^{(t^{NM})}(t)\right\} = \frac{1}{5}\left\{1, 1, -\tanh(t - t^{NM})\right\}\,. \tag{3.11}$$

In this section we show that, by increasing the value of t^{NM}, we can arbitrarily delay the time when the evolution starts to be not CP-divisible. Let $\Lambda_t^{(t^{NM})}$ be the qubit dynamical map at time t of the evolution $\mathbf{\Lambda}^{(t^{NM})}$ and $V_{t'',t'}^{(t^{NM})}$ be corresponding intermediate map for the time interval $[t', t'']$. The parameters $t^{NM} = 1$ and $\alpha = 2/5$ define a physical evolution: from Eq. (3.9), $\Lambda_t^{(t^{NM})}$ is a CPTP map for any $t \geq 0$ and $t^{NM} \geq \bar{t}^{NM}(2/5) \simeq 0.769$. Moreover, the same is true for any larger value of $t^{NM} > 1$.

Before going through the technical details of these evolution, we explain the structure of this section. We start by picking a first value of t^{NM}, namely t_1^{NM}, to define the quasi-eternal evolution $\mathbf{\Lambda}^{(t_1^{NM})}$. Then we consider a P (but not CP) intermediate map $\overline{V}$ that occurs in the time interval (t_1', t_1''), where $t_1^{NM} < t_1' < t_1''$, namely $\overline{V} = V_{t_1'',t_1'}^{(t_1^{NM})}$. Therefore, we consider a second evolution defined by a different t_2^{NM}, namely $\mathbf{\Lambda}^{(t_2^{NM})}$, characterized by $t_1^{NM} < t_2^{NM}$. We show that the same intermediate map $\overline{V}$ considered for $\mathbf{\Lambda}^{(t_1^{NM})}$ occurs for $\mathbf{\Lambda}^{(t_2^{NM})}$ in a different time interval $(t_2', t_2'') = (t_1' + \Delta t^{NM}, t_1'' + \Delta t^{NM})$, where $\Delta t^{NM} = t_2^{NM} - t_1^{NM} > 0$. In other words $V_{t_2'',t_2'}^{(t_2^{NM})} = V_{t_1'',t_1'}^{(t_1^{NM})} = \overline{V}$. Therefore, both the evolutions $\mathbf{\Lambda}^{(t_1^{NM})}$ and $\mathbf{\Lambda}^{(t_2^{NM})}$ have $\overline{V}$ as intermediate map during their evolutions but, while for the first evolution $\overline{V}$ occurs in the time interval (t_1', t_1''), the second evolution has $\overline{V}$ as intermediate map in the later time interval (t_2', t_2''), namely with a delay $\Delta t^{NM} = t_2^{NM} - t_1^{NM} > 0$. Hence, by considering increasing values of t^{NM} we obtain evolution $\mathbf{\Lambda}^{(t^{NM})}$ for which the same intermediate map $\overline{V}$ occurs later and later in time. We follow by studying the image of the dynamics at the time t', namely the time when $\overline{V}$ occurs. Notice that t' increases together with t^{NM}. The final purpose is to show that, by increasing enough the value of t^{NM}, $\mathrm{Im}(\Lambda_{t'}^{(t^{NM})})$ is contained in an arbitrarily small neighbor of the stationary state $\rho_S = \mathbb{1}_S/2$.

Following the steps we have described, we start by considering the evolutions $\mathbf{\Lambda}^{(t_1^{NM})}$ and $\mathbf{\Lambda}^{(t_2^{NM})}$, where $1 \leq t_1^{NM} < t_2^{NM}$ and $\Delta t^{NM} \equiv t_2^{NM} - t_1^{NM} > 0$. From Eqs. (2.111) and (2.112) we see that the rates that defines the two evolutions differ by a simple time-shift (see Eq. (3.11)). Hence, for $t \geq \Delta t^{NM}$, we can express the intermediate maps of $\mathbf{\Lambda}^{(t_2^{NM})}$ starting at time Δt^{NM} in terms of the dynamical maps of $\mathbf{\Lambda}^{(t_1^{NM})}$. Indeed, for $t > \Delta t^{NM}$

$$V_{t,\Delta t^{NM}}^{(t_2^{NM})} = \Lambda_{t-\Delta t^{NM}}^{(t_1^{NM})} . \tag{3.12}$$

As a consequence, if $t > \Delta t^{NM}$, the dynamical map $\Lambda_t^{(t_2^{NM})}$ itself can be ex-

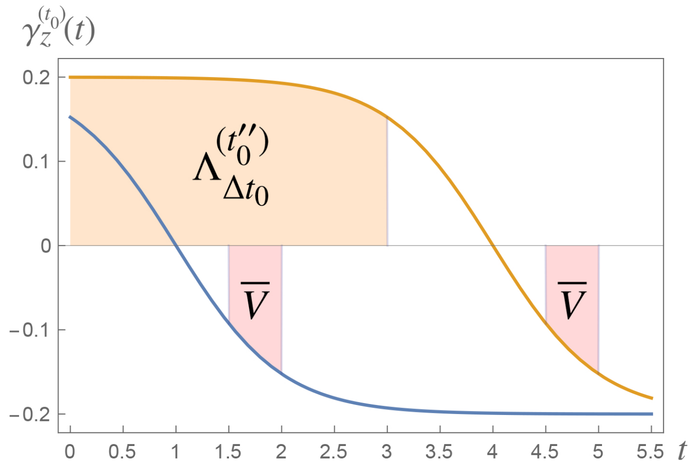


Figure 3.1: Plots of the quasi-eternal non-Markovian model rate $\gamma_z^{(t_{1,2}^{NM})}(t) = -\tanh(t - t_{1,2}^{NM})/5$ for $t_1^{NM} = 1$ (blue) and $t_2^{NM} = 4$ (orange). The differences between $\mathbf{\Lambda}^{(1)}$ and $\mathbf{\Lambda}^{(4)}$ are given by the different values of the integral of $\gamma_z^{(1)}(t)$ and $\gamma_z^{(4)}(t)$ (see Eqs. (2.111) and (3.11)). Let $\overline{V}$ be the P intermediate map of $\mathbf{\Lambda}^{(1)}$ in the time interval $(t_1', t_1'') = (2, 3)$ (left pink region). $\overline{V}$ is also the intermediate map of $\mathbf{\Lambda}^{(4)}$ that occurs in a time interval shifted by $\Delta t^{NM} = t_2^{NM} - t_1^{NM} = 3$, namely in $(t_2', t_2'') = (5, 6)$ (right pink region). The difference between the images of the two maps before the action of $\overline{V}$ is given by the contractive action of $\Lambda_{\Delta t^{NM}}^{(4)}$ (orange region). This result follows from $\gamma_z^{(1)}(t) = \gamma_z^{(4)}(t + \Delta t^{NM})$, Eqs. (2.111) and (3.11).

pressed as the composition of $\Lambda_{\Delta t^{NM}}^{(t_2^{NM})}$ and $\Lambda_{t-\Delta t^{NM}}^{(t_1^{NM})}$

$$\Lambda_t^{(t_2^{NM})} = \Lambda_{t-\Delta t^{NM}}^{(t_1^{NM})} \Lambda_{\Delta t^{NM}}^{(t_2^{NM})} . \tag{3.13}$$

Indeed, by composing Eqs. (3.12) and (3.13), $\Lambda_t^{(t_2^{NM})} = V_{t,\Delta t^{NM}}^{(t_2^{NM})} \Lambda_{\Delta t^{NM}}^{(t_2^{NM})}$. From these equations we should convince ourselves that, for $t \geq t^{NM}$, the dynamical maps of $\mathbf{\Lambda}^{(t_2^{NM})}$ are exactly the ones given by $\mathbf{\Lambda}^{(t_1^{NM})}$ shifted by $-t^{NM}$ and composed with the time *independent-map* $\Lambda_{\Delta t^{NM}}^{(t_2^{NM})}$.

Now, we consider a time interval $[t_1', t_1'']$ such that $t_1^{NM} < t_1' < t_1''$. In this time interval, $\gamma_z^{(t_1^{NM})}(t) < 0$ and the intermediate map $\overline{V} \equiv V_{t_1'',t_1'}^{(t_1^{NM})}$ is P but not CP (see Section 3.2.1). From the results obtained above, it is clear that the action of $\overline{V}$ can also be obtained as the intermediate map of $\mathbf{\Lambda}^{(t_2^{NM})}$ that occurs in the time interval $(t_2', t_2'') \equiv (t_1' + \Delta t^{NM}, t_1'' + \Delta t^{NM})$. Hence, we have the identity

$$\overline{V} = V^{(t_1^{NM})}_{t_1'',t_1'} = V^{(t_2^{NM})}_{t_2'',t_2'} . \tag{3.14}$$

Therefore, in order to witness the non-Markovian effect of $\overline{V}$ while considering $\mathbf{\Lambda}^{(t_1^{NM})}$, its action starts at time t_1'. Instead, by considering $\mathbf{\Lambda}^{(t_2^{NM})}$, $\overline{V}$ occurs from the delayed time $t_2' = t_1' + \Delta t^{NM}$. In summary, for both $\Lambda^{(t_1^{NM})}_{t_1'}$ and $\Lambda^{(t_2^{NM})}_{t_2'}$, $\overline{V}$ is the intermediate map of the evolution that follows for a time interval that lasts $t_1' - t_1'' = t_2' - t_2''$ (see Fig. 3.1).

We proceed by checking the images of the respective preceding evolutions, namely $\mathrm{Im}(\Lambda^{(t_1^{NM})}_{t_1'})$ and $\mathrm{Im}(\Lambda^{(t_2^{NM})}_{t_2'})$. The difference between the two images is given by the CP map $\Lambda^{(t_2^{NM})}_{\Delta t^{NM}}$ (see Eq. (3.13)). In order to understand the action of this map, since $\Delta t^{NM} < t_2^{NM}$, the rates that define $\Lambda^{(t_2^{NM})}_{\Delta t^{NM}}$ through Eqs. (2.111) are strictly positive in the time interval $[0, \Delta t^{NM}]$. Therefore, the action of $\Lambda^{(t_2^{NM})}_{\Delta t^{NM}}$ is CPTP and behaves as a "global" contraction, namely it contracts the state space in every direction. Indeed, by using Eq. (2.115) for $\Lambda^{(t_2^{NM})}_{\Delta t^{NM}}$,

$$\Lambda^{(t_2^{NM})}_{\Delta t^{NM}}(\sigma_i) = \lambda_i^{(t_2^{NM})}(\Delta t^{NM})\sigma_i \, , \tag{3.15}$$

where $\lambda_i^{(t_2^{NM})}(\Delta t^{NM}) < 1$ for $i = x, y, z$. Moreover, since $\Lambda^{(t_1^{NM})}_{t_1'}$ is CPTP, we can write

$$\Lambda^{(t_1^{NM})}_{t_1'}(\sigma_i) = \lambda_i^{(t_1^{NM})}(t_1')\sigma_i \, , \tag{3.16}$$

where $\lambda_i^{(t_1^{NM})}(t_1') < 1$ for $i = x, y, z$. Considering Eqs. (3.15) and (3.16) in Eq. (3.13), we obtain

$$\Lambda^{(t_2^{NM})}_{t_2'}(\sigma_i) = \lambda_i^{(t_2^{NM})}(t_2')\sigma_i = \lambda_i^{(t_1^{NM})}(t_1')\lambda_i^{(t_2^{NM})}(\Delta t^{NM})\sigma_i \, , \tag{3.17}$$

where we remember that $t_2' = t_1' + \Delta t^{NM}$. Since $\lambda_i^{(t_1^{NM})}(t_1') < 1$ and $\lambda_i^{(t_2^{NM})}(\Delta t^{NM}) < 1$, we get $\lambda_i^{(t_2^{NM})}(t_2') < \min\{\lambda_i^{(t_1^{NM})}(t_1'), \lambda_i^{(t_2^{NM})}(\Delta t^{NM})\}$ and we conclude that

$$\mathrm{Im}(\Lambda^{(t_2^{NM})}_{t_2'}) = \mathrm{Im}(\Lambda^{(t_1^{NM})}_{t_1'}\Lambda^{(t_2^{NM})}_{\Delta t^{NM}}) \subset \mathrm{Im}(\Lambda^{(t_1^{NM})}_{t_1'}) \, . \tag{3.18}$$

Therefore, both $\mathbf{\Lambda}^{(t_1^{NM})}$ and $\mathbf{\Lambda}^{(t_2^{NM})}$ have $\overline{V}$ as intermediate map starting, respectively, from time t_1' and t_2', but in the latter case the space of accessible states $\mathrm{Im}(\Lambda^{(t_2^{NM})}_{t_2'})$ is strictly included in $\mathrm{Im}(\Lambda^{(t_1^{NM})}_{t_1'})$ obtained with the first map.

The next step is to fix $t_1^{NM} = 1$ and increase t_2^{NM}. Notice that, as a consequence, $\Delta t^{NM} = t_2^{NM} - t_1^{NM}$ increases. In this way, at the same time, we delay the occurrence of $\overline{V}$ for $\mathbf{\Lambda}^{(t_2^{NM})}$ and show that we can make the evolution that precedes its action more and more contractive, in the sense that $\mathrm{Im}(\Lambda_{t_2'}^{t_2^{NM}})$ becomes smaller and smaller (see Fig. 3.2). From Eq. (3.18) we notice that $\mathrm{Im}(\Lambda_{t_2'}^{(t_2^{NM})})$ is obtained by contracting $\mathrm{Im}(\Lambda_{t_1'}^{(t_1^{NM})})$ with $\Lambda_{\Delta t^{NM}}^{(t_2^{NM})}$, where the action of this map is more and more contractive as Δt^{NM}, or equivalently t_2^{NM}, increases. Indeed, now we prove that for any $\epsilon > 0$ there exists a value of t_2^{NM} such that $\lambda_i^{(t_2^{NM})}(t_2') < \epsilon$ for each $i = x, y, z$. From Eqs. (2.111) and (3.11) it is easy to check that $\lambda_x^{(t_2^{NM})}(t) = \lambda_y^{(t_2^{NM})}(t) > \lambda_z^{(t_2^{NM})}(t)$ for any $t > 0$. Indeed,

$$\lambda_x^{(t_2^{NM})}(t_2') = \left(e^{-t_2'} \frac{\cosh(t_2' - t_2^{NM})}{\cosh(t_2^{NM})} \right)^{1/5} < \left(2e^{-2t_2^{NM}} \right)^{1/5}, \tag{3.19}$$

$$\lambda_z^{(t_2^{NM})}(t_2') = \left(2e^{-2t_2^{NM}} \right)^{1/5} \left(\frac{e^{-2(t_1'-1)}}{2} \right)^{1/5}. \tag{3.20}$$

Therefore, for any $\epsilon > 0$, if the following condition is satisfied

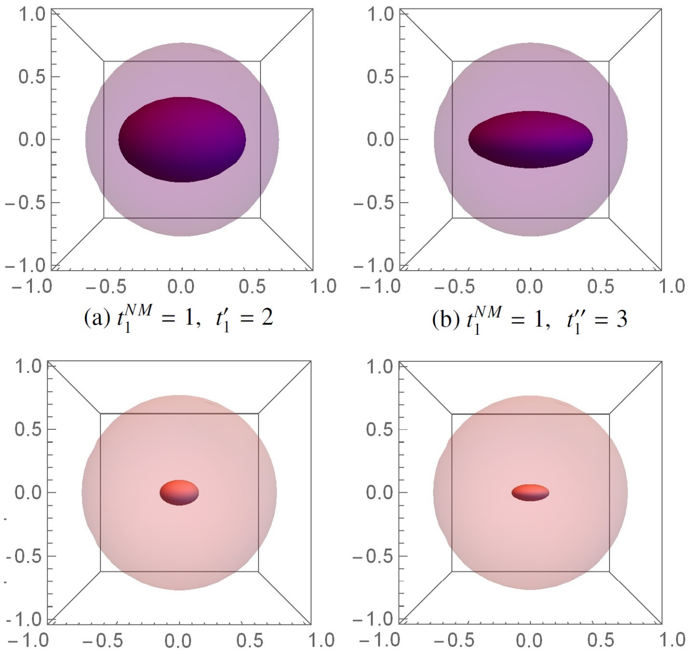
$$t_2^{NM} > \log \sqrt{2/\epsilon^5}\,, \tag{3.21}$$

we have $\lambda_z^{(t_2^{NM})}(t_2') < \lambda_x^{(t_2^{NM})}(t_2') = \lambda_y^{(t_2^{NM})}(t_2') < \epsilon$.

We want to understand the effects on the set of accessible states of $\Lambda_{t_2'}^{(t_2^{NM})}$ that we obtain when t_2^{NM} is increased over the bound given by Eq. (3.21). Therefore, we consider a generic initial state $\rho_S(0) = (\mathbb{1}_S + \mathbf{v}(0) \cdot \boldsymbol{\sigma})/2$, represented by the Bloch vector $\mathbf{v}(0) = (v_x(0), v_y(0), v_z(0))$, where in the vector $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ we collect the Pauli operators. We evolve this qubit state with the dynamical map $\Lambda_t^{(t_2^{NM})}$, where the condition of Eq. (3.21) is satisfied for some $\epsilon > 0$. At time $t = t_2'$ the Bloch vector is evolved to $\mathbf{v}(t_2') = (v_x(t_2'), v_y(t_2'), v_z(t_2'))$. From Eq. (3.19) and (3.20), it is straightforward to show that $\max_i v_i(t_2') < \epsilon \max v_i(0)$. In particular

$$\|\rho_S(t_2') - \mathbb{1}_S/2\|_1 = \frac{1}{2} \sqrt{\sum_i v_i^2(t_2')} < \frac{\epsilon}{2}\,. \tag{3.22}$$

In other words, if the condition of Eq. (3.21) is satisfied, $\mathrm{Im}(\Lambda_{t_2'}^{(t_2^{NM})})$ is inside a neighbor of radius ϵ centered in the maximally mixed state $\mathbb{1}_S/2$, namely the



(a) $t_1^{NM} = 1, \; t_1' = 2$ (b) $t_1^{NM} = 1, \; t_1'' = 3$

(c) $t_2^{NM} = 4, \; t_2' = 5$ (d) $t_2^{NM} = 4, \; t_2'' = 6$

Figure 3.2: Plots of $\mathrm{Im}(\Lambda_t^{(t^{NM})})$ for different t^{NM} and t as subsets of the Bloch sphere (the qubit state space). First, we set (purple) $t^{NM} = t_1^{NM} = 1$, (a) $t = t_1' = 2$ and (b) $t = t_1'' = 3$. The map that transforms $\mathrm{Im}(\Lambda_2^{(1)})$ into $\mathrm{Im}(\Lambda_3^{(1)})$ is the P intermediate map $V_{3,2}^{(1)} = \overline{V}$. By considering (pink) $t^{NM} = t_2^{NM} = 4$, (c) $t = t_2' = 5$ and (d) $t = t_2'' = 6$, the map that transforms $\mathrm{Im}(\Lambda_5^{(4)})$ into $\mathrm{Im}(\Lambda_6^{(4)})$ is again $\overline{V}$, namely $V_{6,5}^{(4)} = V_{3,2}^{(1)} = \overline{V}$ (see Eq. (3.14)). While the map that transforms (a) into (b) and (c) into (d) is $\overline{V}$ in both cases, in the first case (a) the starting set where $\overline{V}$ acts is larger than in the second case (c). The more we increase t^{NM}, the smaller is $\mathrm{Im}(\Lambda_{t'}^{(t^{NM})})$ at the time t' when $\overline{V}$ starts to act.

stationary state of these evolutions. Hence, we can make $\mathrm{Im}(\Lambda_{t_2'}^{(t_2^{NM})})$ fit in an arbitrary small sphere of radius ϵ by increasing t_2^{NM}.

3.2.3 Single parameter evolutions

A simple family of open-system dynamics is given by those differentiable evolutions such that, for $t \geq 0$, the generator L_t can be expressed as

$$L_t(\rho_S(t)) = i[H(t), \rho_S(t)] + \gamma(t) \sum_k \left(G_k \rho_S(t) G_k^\dagger - \frac{1}{2} \left\{ G_k^\dagger G_k, \rho_S(t) \right\} \right), \quad (3.23)$$

where G_k are time-independent operators and $\gamma(t)$ is a continuous function of time. We call those evolutions with this property *single parameter* since $\gamma(t)$ alone describes the time dependence of the dissipative part. Paradigmatic examples of single parameter evolutions are depolarization, as well as dephasing and amplitude damping in a time independent basis. Using the representation defined in Eq. (3.5) we can state the single parameter property as

$$\left. \frac{d\mathcal{V}_{ij}(t,s)}{ds} \right|_{s=t} = g_{ij}\gamma(t) + h_{ij}(t), \quad (3.24)$$

where $h_{lj}(t) \equiv \mathrm{Tr}\left[i[H(t) \otimes I_A, e_j] e_l \right]$ and the time independent

$$g_{ij} \equiv \mathrm{Tr}\left[e_i \sum_k \left(G_k \otimes I_A e_j G_k^\dagger \otimes I_A - \frac{1}{2} \left\{ G_k^\dagger G_k \otimes I_A, e_j \right\} \right) \right].$$

An important property of single-parameter evolutions is that they can be divided into CP-divisible and not P-divisible time intervals, that is, for sufficiently small time intervals they never present intermediate maps that are not CP but P.

Proposition 6. *Let* Λ *be a single parameter evolution. Then the corresponding intermediate map* $V_{t+\epsilon,t}$ *is either CP or not P for any* t *and sufficiently small* $\epsilon > 0$.

Proof. An intermediate map $V_{t,s}$ is CP if the Choi matrix $C_{V_{t,s}}$ has non-negative eigenvalues [Cho75, Jam72]. Moreover, $V_{t,s}$ is P if $V_{t,s}(\rho_S)$ is positive semidefinite for all $\rho_S \in S(\mathcal{H}_S)$. In particular, if $V_{t,s}$ is P, $V_{t,s}(\rho_S)$ is positive semidefinite for any positive semidefinite rank one ρ_S.

The eigenvalues of $V_{t,s}(\rho_S)$ for a rank-one pure state ρ_S and the eigenvalues of $C_{V_{t,s}}$ are functions of the parameters $\bar{a}$ and $\mathcal{V}_{ij}(t,s)$. Since we assumed that

$\gamma(t)$ is continuous it follows that $V_{t,s}(\rho_S)$ and $C_{V_{t,s}}$ are both continuously differentiable. This in turn implies that the eigenvalues of $V_{t,s}(\rho_S)$ and $C_{V_{t,s}}$ can be described by continuously differentiable functions [Rel54].

If $\gamma(t) \neq 0$, so that $V_{t,s}$ is not unitary, there always exists at least one eigenvalue $\lambda_C(V_{t,s})$ of the Choi matrix that is zero and has a nonzero time derivative at $s = t$. There also exists at least one rank-one state η_S with eigenvalue $\lambda(\eta)$ that is zero and its evolution $\lambda(\eta_S(t-s))$, defined by $\eta_S(t-s) \equiv V_{t,s} \otimes I_A(\eta_S)$, has a nonzero time derivative at $s = t$ (see Appendix A.1).

Consider the temporal derivatives $\frac{d\lambda_C(V_{t,s})}{ds}\Big|_{s=t} = \sum_{ij} \frac{\partial \lambda_C(V_{t,s})}{\partial \mathcal{V}_{ij}(t,s)}\Big|_{s=t} g_{ij}\gamma(t)$ and $\frac{d\lambda(\eta(t-s))}{ds}\Big|_{s=t} = \sum_{ij} \frac{\partial \lambda(\eta(t-s))}{\partial \mathcal{V}_{ij}(t,s)}\Big|_{s=t} g_{ij}\gamma(t)$. Here we used the invariance of eigenvalues under continuous unitary evolution. Therefore, $\sum_{ij} \frac{\partial \lambda_C(V_{t,s})}{\partial \mathcal{V}_{ij}(t,s)}\Big|_{s=t} h_{ij}(t) = 0$ and $\sum_{ij} \frac{\partial \lambda(\eta(t-s))}{\partial \mathcal{V}_{ij}(t,s)}\Big|_{s=t} h_{ij}(t) = 0$. Since the time derivatives are non-zero, they are proportional to $\gamma(t)$. If $V_{t+\epsilon,t}$ is CP for any sufficiently small ϵ it follows that $\lambda_C(V_{t+\epsilon,t})$ and $\lambda(V_{t+\epsilon,t}(\eta))$ are positive and that $\frac{d\lambda_C(V_{t,s})}{ds}\Big|_{s=t} > 0$ and $\frac{d\lambda(\eta(t-s))}{ds}\Big|_{s=t} > 0$. Then if a t' exists such that $\text{sign}[\gamma(t')] = -\text{sign}[\gamma(t)]$, there exists an ϵ' such that $V_{t'+\epsilon',t'}$ is neither CP or P since $\frac{d\lambda_C(V_{s',t'})}{ds'}\Big|_{s'=t'} < 0$ and $\frac{d\lambda(\eta(s'-t'))}{ds'}\Big|_{s'=t'} < 0$. From this follows that it is impossible for $V_{t'+\epsilon',t'}$ to be P but not CP for sufficiently small ϵ'.

□

Thus, if the assumptions of Proposition 6 hold, we can conclude that the evolution can be divided into closed time intervals where $\gamma(t) \geq 0$ and open intervals for which $\gamma(t) < 0$. In the closed intervals where $\gamma(t) \geq 0$ the dynamics is CP-divisible and in the open intervals where $\gamma(t) < 0$ the dynamics is not P-divisible.

3.3 Correlations as witnesses of non-Markovianity

In this section we see how correlations can be used for the detection of non-Markovian dynamics. In Section 2.5.7 we defined the minimal requirements for the functionals of bipartite systems that we call correlations. In this chapter we show several results, where some are focused on particular correlation measures or evolutions.

We start by noticing the following result valid for generic correlation measures $M(\rho_{SA})$ that are continuously differentiable on the state space $S(\mathcal{H}_{SA})$ of bipartite systems $S-A$. If the intermediate map $V_{t,s} \otimes I_A$ is differentiable with respect to t, the time derivative $\frac{d}{ds}M(\eta_{SA}(t-s))|_{s=t} \equiv \frac{d}{ds}M(V_{t,s} \otimes I_A(\eta_{SA}))|_{s=t}$

for any $\eta_{SA} \in S(\mathcal{H}_{SA})$ can be expressed as

$$\sum_i \frac{\partial M(\eta_{SA}(t-s))}{\partial a_i(s)} \frac{da_i(s)}{ds}\Big|_{s=t} = \sum_{ij} \frac{\partial M(\eta_{SA})}{\partial a_i} a_j \frac{d\mathcal{V}_{ij}(t,s)}{ds}\Big|_{s=t}, \tag{3.25}$$

where a_i ($a_i(s)$) are the coordinates of η_{SA} ($\eta_{SA}(t-s)$). We see that $\frac{d}{ds}M(\eta_{SA}(t-s))|_{s=t}$ depends only on a_i and the time derivatives of the components $\mathcal{V}_{ij}(t,s)$ of the intermediate map. As long as $\frac{\partial M(\rho_{SA})}{\partial a_i}a_j \neq 0$ for some i, j and ρ_{SA}, there exists some intermediate map that will induce either a decrease or increase of $M(\rho_{SA})$. However, it is not always the case that a given measure M satisfies $\frac{\partial M(\rho_{SA})}{\partial a_i}a_j \neq 0$ for a ρ_{SA} in the image of the dynamical map Λ_t. In Section 3.4 we show that entanglement measures and entanglement breaking evolutions [HSR03] can provide examples of this situation.

3.3.1 Single parameter evolutions

Consider a generic single-parameter evolution and any correlation measure that is continuously differentiable. For these evolutions, the sign of any non-zero time derivative of a continuously differentiable correlation measure is determined by the sign of $\gamma(t)$. Therefore any non-Markovian effect leads to a correlation backflow, no matter which measure of this kind is used for the quantification as long as it is not time independent on the whole $\mathrm{Im}(\Lambda_t)$. That is, it witnesses non-Markovianity as long as it is capable of witnessing any change in correlations at all.

Proposition 7. *Let M be a continuously differentiable correlation measure and let $\mathbf{\Lambda}$ be a single parameter evolution. Then,*

$$sign\left[\frac{d}{ds}M(V_{t,s}\otimes I_A(\eta_{SA}))\Big|_{s=t}\right] = -sign[\gamma(t)]\,, \tag{3.26}$$

for all $\eta_{SA} \in S(\mathcal{H}_{SA})$ such that $\frac{d}{ds}M(V_{t,s}\otimes I_A(\eta_{SA}))|_{s=t} \neq 0$.

Proof. For a continuously differentiable correlation measure M the time derivative $\frac{d}{ds}M(V_{t,s}\otimes I_A(\eta_{SA}))|_{s=t}$ under an evolution of this type can be expressed, using Eqs. (3.24) and (3.25), as $F(\eta_{SA})\gamma(t)$, where $F(\eta_{SA}) \equiv \sum_{ij}\frac{\partial M(\eta_{SA})}{\partial a_i}a_j g_{ij}$ is a time independent function. Here we used that $\sum_{ij}\frac{\partial M(\eta_{SA})}{\partial a_i}a_j h_{ij}(t) = 0$ since M is invariant under unitary evolution. Thus, $\frac{d}{ds}M(V_{t,s}\otimes I_A(\eta_{SA}))|_{s=t}$ is proportional to $\gamma(t)$.

If $\gamma(t) > 0$, so that $V_{\tau,t}$ is a non-unitary CP map for sufficiently small $\tau - t$, the time derivative $\frac{d}{d\tau}M(V_{\tau,t}\otimes I_A(\eta_{SA}))|_{\tau=t}$ is non-positive for all η_{SA}.

This implies that $F(\eta_{SA})$ is non-positive for all η_{SA}. Assume that $F(\zeta_{SA}) \neq 0$ for some $\zeta_{SA} \in S(\mathcal{H}_{SA})$. Then it follows that $\text{sign}[\frac{d}{d\tau} M(V_{\tau,t} \otimes I_A(\zeta_{SA}))|_{\tau=t}] = -\text{sign}[\gamma(t)]$. □

It follows from Proposition 7 that a continuously differentiable correlation measure M shows an increase in the time intervals where $\mathbf{\Lambda}$ is not P-divisible as long as $\frac{d}{dt} M(\rho_{SA}(t)) \neq 0$ for some $\rho_{SA}(t) \in \text{Im}(\Lambda_t)$.

Example: Dephasing evolution

Dephasing in a fixed basis is an example of a random unitary and single-parameter evolution that satisfies the conditions in Propositions 6 and 7. The pure qubit dephasing dynamics is described by the dynamical maps

$$\begin{aligned} \Lambda_t(\sigma_x) &= e^{-\int_0^t \gamma(\tau) d\tau} \sigma_x, \\ \Lambda_t(\sigma_y) &= e^{-\int_0^t \gamma(\tau) d\tau} \sigma_y, \\ \Lambda_t(\sigma_z) &= \sigma_z, \\ \Lambda_t(\mathbb{1}_S) &= \mathbb{1}_S. \end{aligned} \tag{3.27}$$

The dynamical maps are bijective for all times and the intermediate maps $V_{t,s}$ are therefore given by $V_{t,s} = \Lambda_t \Lambda_s^{-1}$. Explicitly $V_{t,s}$ is given by

$$\begin{aligned} V_{t,s}(\sigma_x) &= e^{-\int_s^t \gamma(\tau) d\tau} \sigma_x, \\ V_{t,s}(\sigma_y) &= e^{-\int_s^t \gamma(\tau) d\tau} \sigma_y, \\ V_{t,s}(\sigma_z) &= \sigma_z, \\ V_{t,s}(\mathbb{1}_S) &= \mathbb{1}_S. \end{aligned} \tag{3.28}$$

The generator L_t is given by

$$L_t(\rho_S(t)) = \gamma(t)(\sigma_z \rho_S(t) \sigma_z - \rho_S(t)), \tag{3.29}$$

where $\gamma(t)$ is the time dependent dephasing rate. The dynamics is Markovian if and only if the dephasing rate $\gamma(t) \geq 0$. Furthermore, the dephasing dynamics is not P-divisible when $\gamma(t) < 0$ (see Proposition 6).

We can see from Eq. (3.28) that for all i, j such that $\mathcal{V}_{ij}(t, s)$ has non-zero time derivatives it holds that $\frac{d\mathcal{V}_{ij}(t,s)}{ds}\Big|_{s=t} = -\gamma(t)$. Thus for any continuously differentiable correlation measure $M(\rho_{SA}(t))$ the dephasing rate $\gamma(t)$ determines the sign of $\frac{d}{dt} M(\rho_{SA}(t))$ and $\frac{d}{dt} M(\rho_{SA}(t)) \geq 0$ for $\gamma(t) \leq 0$.

Example: Generalized amplitude damping evolution

Generalized amplitude damping evolution in a fixed basis is a second example that is single-parameter and satisfies the conditions in Propositions 6 and 7 except where it is non-differentiable. The dynamics of generalized amplitude damping on a qubit is described by the dynamical maps

$$\begin{aligned}\Lambda_t(\sigma_x) &= G(t)\sigma_x,\\ \Lambda_t(\sigma_y) &= G(t)\sigma_y,\\ \Lambda_t(\sigma_z) &= G^2(t)\sigma_z,\\ \Lambda_t(\mathbb{1}_S) &= \mathbb{1}_S + (2p-1)(1-G^2(t))\sigma_z,\end{aligned} \tag{3.30}$$

where $0 \le G(t) \le 1$ and $0 \le p \le 1$. For $G(t) > 0$ the dynamical maps are bijective and the intermediate maps are given by

$$\begin{aligned}V_{t,s}(\sigma_x) &= \frac{G(t)}{G(s)}\sigma_x,\\ V_{t,s}(\sigma_y) &= \frac{G(t)}{G(s)}\sigma_y,\\ V_{t,s}(\sigma_z) &= \left(\frac{G(t)}{G(s)}\right)^2\sigma_z,\\ V_{t,s}(\mathbb{1}_S) &= \mathbb{1}_S + (2p-1)\left(1-\left(\frac{G(t)}{G(s)}\right)^2\right)\sigma_z.\end{aligned} \tag{3.31}$$

For s and t such that $G(s) = 0$ the intermediate map only exists if $G(t) = 0$ and can be defined as the identity map. If $G(s) = 0$ and $G(t) \neq 0$ the intermediate map does not exist since the evolution is many-to-one. For t where L_t is well defined, it is given by

$$\begin{aligned}L_t(\rho_S(t)) \;=\;& p\gamma(t)(\sigma_-\rho_S(t)\sigma_+ - 1/2\{\sigma_+\sigma_-, \rho_S(t)\})\\ &+(1-p)\gamma(t)(\sigma_+\rho_S(t)\sigma_- - 1/2\{\sigma_-\sigma_+, \rho_S(t)\}),\end{aligned} \tag{3.32}$$

where $\sigma_\pm = 1/2(\sigma_x \pm i\sigma_y)$ and $\gamma(t)$ is given by

$$\gamma(t) = -2\frac{d}{ds}\frac{G(t)}{G(s)}\bigg|_{s=t} = -\frac{2}{G(t)}\frac{d}{dt}G(t), \tag{3.33}$$

whenever $G(t) > 0$ and differentiable. The dynamics is Markovian in a generic time interval $[t_1, t_2]$ when $\gamma(t) \geq 0$ for any $t \in [t_1, t_2]$. Furthermore, the amplitude damping dynamics is not P-divisible in $[t_1, t_2]$ when $\gamma(t) < 0$ for any $t \in [t_1, t_2]$.

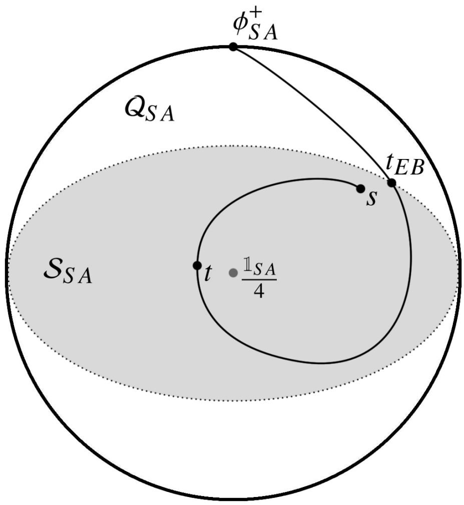


Figure 3.3: Depiction of the trajectory of the evolution of a maximally entangled state ϕ^+_{SA} , where the system S evolves under an entanglement breaking Λ_t^{EB}. Therefore, if $t > t_{EB}$, any initial state $\rho_{SA}(0) \in Q_{SA} = S(\mathcal{H}_{SA})$ is evolved into a separable state $\rho_{SA}(t) \in S_{SA}$. Suppose that Λ_t^{EB} is non-Markovian but CP-divisible in $[0, t_{EB}]$ and with a non-CP intermediate map $V_{t,s}^{EB}$ for $s > t > t_{EB}$. In this case, it is not possible to witness backflows of any entanglement measure. Indeed, entanglement is zero in the set of separable states S_{SA}.

We can easily see that when $G(t) > 0$ and differentiable, all non-zero time derivatives of the components of the intermediate map are proportional to $\gamma(t)$. Thus $\gamma(t)$ determines the sign of $\frac{d}{dt}M(\rho_{SA}(t))$ for any continuously differentiable correlation measure M.

3.4 Entanglement measures

It is relatively straightforward to see that entanglement measures M_E cannot detect all non-Markovian dynamics when we consider $S-A$ correlated systems. In fact, consider the situation in which a dynamics Λ_t^{EB} becomes and remains entanglement breaking (EB) after a given time t_{EB}, see Fig. 3.3. Any entanglement measure evaluated over $S-A$ remains equal to zero for $t \geq t_{EB}$ and will therefore be unable to detect any non-Markovian effect taking place for $t \geq t_{EB}$. This is for instance the case of non-Markovian P-divisible evolutions that are EB for any $t \geq t_{EB}$. Indeed, in these cases, all the states in $\mathrm{Im}(\Lambda_{t_{EB}} \otimes I_A)$ are separable, and remain separable after t_{EB} since local P maps do not create any entanglement when acting on separable states. Note that, from Prop. 7, it follows that a continuously differentiable entanglement measure can detect non-Markovian single-parameter evolutions unless it is time independent on the whole image of the evolution at the time where non-Markovianity occurs. However, no single-parameter evolution is P-divisible unless CP-divisible since the intermediate maps of single-parameter evolutions are either CP or not P for sufficiently small time intervals.

For the sake of clarity, in what follows we provide an example of a non-Markovian qubit dynamics Λ_t^{EB} which is CP-divisible in $[0, t^{NM}]$, while it is only P-divisible for $t > t^{NM}$. We show that the time $t_{EB}(\Lambda_t^{EB})$ when the dynamics starts to be EB precedes t^{NM} and therefore we prove that this non-Markovian dynamics does not display any backflow of the entanglement shared between S and A.

3.4.1 Example: the quasi-eternal non-Markovian model

We consider a bipartite system, where A and S are qubits. We consider the evolution $\mathbf{\Lambda}^{(2/5,2)}$ that belongs to the class of quasi-eternal non-Markovian P-divisible evolutions introduced in Section 3.2.1. The corresponding rates are

$$\left\{\gamma_x(t), \gamma_y(t), \gamma_z(t)\right\} = \frac{1}{5}\{1, 1, -\tanh(t-2)\}\ , \tag{3.34}$$

where we fixed $\alpha = 2/5$, $t^{NM} = 2$ and the condition of physicality (3.9) is satisfied. This evolution is not single parameter (see Eq. (3.24)) and the results of Prop. 7 do not apply. As explained in Section 2.6.1, $\gamma_z(t)$ implies that the evolution is non-Markovian for any $t \geq t^{NM} = 2$ and moreover the intermediate maps $V_{t,s}^{(2/5,2)}$ of this evolution are P but not CP for any $t^{NM} < s < t$.

This temporal evolution becomes entanglement breaking. Indeed, consider $\phi^+_{SA}(t) = \Lambda_t^{(2/5,2)} \otimes I_A(\phi^+_{SA})$, namely the temporal evolution of the maximally entangled state $\phi^+_{SA} = |\phi^+\rangle\langle\phi^+|_{SA}$, where $|\phi^+\rangle_{SA} = (|00\rangle_{SA} + |11\rangle_{SA})/\sqrt{2}$. We obtain the separability of $\phi^+_{SA}(t)$, namely $\mathrm{NEG}(\phi^+_{SA}(t)) = 0$, for any $t \geq t_{EB}(\Lambda_t^{(2/5,2)}) \simeq 1.47$, where $\mathrm{NEG}(\cdot)$ is negativity, an entanglement measure introduced in Section 2.5.7. Therefore, since $t_{EB}(\Lambda_t^{(2/5,2)}) < t^{NM}$, we conclude that it is not possible to observe a non-Markovian backflow of negativity, and more generally of any entanglement measure, for any initial state $\rho_{SA}(0)$.

3.5 Quantum mutual information

A commonly used correlation measure is QMI: the entropic measures based on the von Neumann entropy introduced in Section 2.5.7. We recall that, given a bipartite quantum system $S-A$, it assumes the form

$$I(\rho_{SA}) = S(\rho_S) + S(\rho_A) - S(\rho_{SA}). \tag{3.35}$$

In the following, we present many different scenarios where the QMI witnesses non-Markovian effects, but we conclude by identifying a situation where it fails in this task.

- In Section 3.5.1 we show that an increase in the QMI can be witnessed for any qubit random unitary non-Markovian dynamics that is not P-divisible.
- In Section 3.5.2 we continue by considering non-Markovian random unitary dynamics that are P-divisible. In particular, we provide examples that prove that maximally entangled states are not always optimal to detect backflows of QMI.
- In Section 3.5.3 we turn our attention to an evolution that is not random unitary, namely a generalized amplitude damping channel. We provide a class of initial states that efficiently witness the non-Markovian nature of this evolution. Similarly to Section 3.5.2, we show that maximally entangled initial states are not optimal to witness backflows of QMI.
- In Sections 3.5.4 and 3.5.5, we study non-Markovian evolutions for which the QMI does not provide backflows for any initial state, where we exploit the results of Section 3.2.2.

3.5.1 Non-Markovian non-P-divisible random unitary qubit dynamics

Several commonly studied dynamics, including dephasing and random unitary dynamics, are unital, namely they preserve the identity through the evolution. We now show that for bijective unital dynamical maps acting on a qubit an increase of the QMI can be observed for any non-P intermediate map $V_{t,s}$.

Theorem 13. *Let $\mathbf{\Lambda}$ be a unital bijective qubit evolution. Furthermore, assume that the intermediate map $V_{t,s}$ is analytic and non-P for some t. Then there exist states $\rho_{SA} \in B(\mathcal{H}_{SA})$ in the image of Λ_t for which $I(V_{t,s} \otimes I_A(\rho_{SA})) > I(\rho_{SA})$.*

Proof. Assume that $V_{t,s}$ is not P. Then there exists a pure state $|\phi\rangle_S \in \mathcal{H}_S$ such that $V_{t,s}(|\phi\rangle\langle\phi|_S)$ has a negative eigenvalue. Let the eigenvalues of $V_{t,s}(|\phi\rangle\langle\phi|_S)$ be $1+\epsilon(s)$ and $-\epsilon(s)$, where $\epsilon(t)=0$ and $\epsilon(s)\geq 0$. Consider the state at time s

$$\rho_{SA}(s) \equiv \frac{1}{2}\left(p|\phi\rangle\langle\phi|_S + (1-p)\frac{\mathbb{1}_S}{2}\right)\otimes|0\rangle\langle 0|_A + \frac{1}{2}\left(p|\phi^\perp\rangle\langle\phi^\perp|_S + (1-p)\frac{\mathbb{1}_S}{2}\right)\otimes|1\rangle\langle 1|_A\,, \tag{3.36}$$

where $|\phi^\perp\rangle_S$ is the pure state orthogonal to $|\phi\rangle_S$, and $|0\rangle\langle 0|_A$ and $|1\rangle\langle 1|_A$ are orthogonal states in $S(\mathcal{H}_A)$. Notice that, since Λ_s is bijective and unital: (i) there always exists a sufficiently small p such that $\rho_{SA}(s)$ is in the image of Λ_s and (ii) the eigenvalues of $V_{t,s}(|\phi^\perp\rangle\langle\phi^\perp|_S)$ are $1+\epsilon(s)$ and $-\epsilon(s)$. Note also that the reduced density matrices of both the system and the ancilla are maximally mixed. Therefore, the reduced states are unchanged by a unital map $V_{t,s}$. The difference in QMI between time s and time t for a unital map is thus

$$I(\rho_{SA}(t)) - I(\rho_{SA}(s)) = -S(V_{t,s}\otimes I_A(\rho_{SA}(s))) + S(\rho_{SA}(s))\,, \tag{3.37}$$

where $I(\rho_{SA}(t)) = I(V_{t,s}\otimes I_A(\rho_{SA}))$ reads

$$I(\rho_{SA}(t)) = \left(\frac{1+p}{2} + p\epsilon(s)\right)\ln\left(\frac{1+p}{2} + p\epsilon(s)\right) + \left(\frac{1-p}{2} - p\epsilon(s)\right)\ln\left(\frac{1-p}{2} - p\epsilon(s)\right) + \ln 2. \tag{3.38}$$

Its time derivative at $s=t$ is

$$-\frac{d}{ds}S(V_{t,s}\otimes I_A(\rho_{SA}(s)))|_{s=t} = \frac{d\epsilon(t)}{dt}p\,(\ln(1+p) - \ln(1-p))\,. \tag{3.39}$$

Note that $p(\ln(1+p) - \ln(1-p)) > 0$ for $0 < p < 1$. Therefore, for $\frac{d\epsilon(t)}{dt} > 0$ the time derivative of the QMI is positive for $\rho_{SA}(t)$. Moreover, since $\epsilon(t)$ is assumed to be analytic $\frac{d\epsilon(t)}{dt} > 0$ implies that the map $V_{t,t-\delta t}$ is non-P for a sufficiently small δ. □

3.5.2 Non-Markovian random unitary dynamics and maximally entangled states

We provide a condition for the QMI not to show backflows when a maximally entangled state is evolved by a random unitary dynamical map. Thereafter, we formulate a version of this condition that applies to qubits, where the P-divisibility of the dynamical map is implied.

We consider the bipartite scenario where the systems S and A are qudits ($d_S = d_A = d$), S is evolved by a random unitary evolution $\boldsymbol{\Lambda}$ and the ancillary system A is left untouched. We study the evolution of a maximally entangled state $\phi_{SA} = |\phi\rangle\langle\phi|_{SA}$, where

$$|\phi\rangle_{SA} = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |s_i\rangle_S \otimes |a_i\rangle_A \,, \tag{3.40}$$

and $\{|s_i\rangle_S\}_i$ ($\{|a_i\rangle_A\}_i$) is an orthonormal basis of $\mathcal{H}_S$ ($\mathcal{H}_A$). First, we show that the evolved state is diagonal in a Bell basis with eigenvalues given by the same probability distribution $\{p_k(t)\}_k$ that defines Λ_t. Indeed

$$\phi_{SA}(t) = \sum_{k=1}^{N} p_k(t)(\mathbb{1}_A \otimes \sigma_k)|\phi\rangle\langle\phi|_{SA}(\mathbb{1}_A \otimes \sigma_k) \equiv \sum_{k=0}^{N} p_k(t)|\phi_k\rangle\langle\phi_k|_{SA}, \tag{3.41}$$

where $N = d^2 - 1$. The set of states $\{|\phi_k\rangle_{SA}\}_k \equiv \{(\sigma_k \otimes \mathbb{1}_A)|\phi\rangle_{SA}\}_k$ define a Bell basis and are orthonormal since: $\langle\phi_i|\phi_j\rangle_{SA} = \mathrm{Tr}\left[\phi_{SA}(\sigma_i\sigma_j \otimes \mathbb{1}_A)\right] = \mathrm{Tr}_S\left[\mathrm{Tr}_A\left[\phi_{SA}\right]\sigma_i\sigma_j\right] = \frac{1}{d}\mathrm{Tr}_S\left[\sigma_i\sigma_j\right] = \delta_{ij}$. It follows that the von Neumann entropy of $\phi_{SA}(t)$ is defined by the distribution $p_k(t)$

$$S(\phi_{SA}(t)) = -\sum_{k=0}^{N} p_k(t) \ln p_k(t) \,. \tag{3.42}$$

The reduced states of $\phi_{SA}(t)$ to the subsystems S and A are maximally mixed: $\rho_S(t) = \mathrm{Tr}_A\left[\phi_{SA}(t)\right] = \mathbb{1}_S/d$ and $\rho_A(t) = \mathrm{Tr}_S\left[\phi_{SA}(t)\right] = \mathbb{1}_A/d$. Thus, the only evolving component of the QMI of $\phi_{SA}(t)$ is given by (3.42): $I(\phi_{SA}(t)) = 2\log_2 d - S(\phi_{SA}(t))$. The time derivative of this quantity is

$$\begin{aligned}
\frac{d}{dt}I(\phi_{SA}(t)) &= \sum_{k=0}^{N} \frac{dp_k(t)}{dt}(\ln p_k(t) + 1) \\
&= \frac{dp_0(t)}{dt}(\ln p_0(t) + 1) + \sum_{k=1}^{N} \frac{dp_k(t)}{dt}(\ln p_k(t) + 1) \\
&= \sum_{k=1}^{N} \frac{dp_k(t)}{dt} \ln \frac{p_k(t)}{p_0(t)} \,.
\end{aligned} \tag{3.43}$$

It follows that, the conditions $\frac{d}{dt}p_k(t) \geq 0$ and $p_0(t) \geq p_k(t)$ for $k = 1, 2, \ldots, N$ implies $\frac{d}{dt}I(\phi_{SA}(t)) \leq 0$, namely we cannot witness any backflow of QMI with a maximally entangled state.

Finally, we consider the qubit case, namely when $d = 2$. From Eqs. (2.114) and (2.113) it follows that the conditions $p_0(t) \geq p_k(t)$, for $k = x, y, z$ and $t \geq 0$, are always satisfied. We conclude that, when S and A are qubits, if $\frac{d}{dt} p_k(t) \geq 0$ for any $k = x, y, z$ and $t \geq 0$, we cannot obtain any backflow of QMI if the initial state is maximally entangled.

We notice that in a time interval where $\frac{d}{dt} p_k(t) \geq 0$ the dynamics is P-divisible, but not necessarily CP-divisible. Thus, there are some cases of non-Markovian P-divisible qubit dynamics that cannot be witnessed by the QMI of an evolved maximally entangled state. In order to prove this result, we write the time derivative of $p_x(t)$

$$\begin{aligned}\frac{d}{dt} p_x(t) = \; & \tfrac{1}{2}\Big((\gamma_x(t) + \gamma_y(t))A_{xy}(t) + (\gamma_z(t) + \gamma_x(t))A_{zx}(t) \\ & - (\gamma_y(t) + \gamma_z(t))A_{yz}(t)\Big). \end{aligned} \tag{3.44}$$

Similarly, we can write $\frac{d}{dt} p_y(t)$ and $\frac{d}{dt} p_z(t)$. We notice that $\frac{d}{dt} p_x(t) + \frac{d}{dt} p_y(t) = (\gamma_x(t) + \gamma_y(t))A_{xy}(t)$. Therefore, given the positivity of $A_{xy}(t)$, $A_{yz}(t)$ and $A_{zx}(t)$ (see Eq. (2.114)), if $\frac{d}{dt} p_k(t) \geq 0$ for $k = x, y, z$, the dynamics is P-divisible (in this case the conditions given in Eqs. (2.117) are automatically satisfied). However, in general the converse is not true. Indeed, in Section 3.5.2 we study two similar P-divisible evolutions for qubits where in the first the conditions $\frac{d}{dt} p_k(t) \geq 0$ for $k = x, y, z$ are not satisfied, while in the second they are.

In order to obtain an intuitive meaning of the conditions presented in this section, we look at the definition given in Eq. (2.108) for random unitary evolutions. We notice that $p_0(t)$ represents the fraction of Λ_t that acts as the identity map on $\rho_S(0)$. Therefore, if the value of $p_0(t)$ is increasing for some t, namely $\frac{d}{dt} p_0(t) > 0$, it is reasonable to expect that at time t the system S is getting closer to its initial configuration $\rho_S(0)$ and therefore evolving under a non-Markovian evolution that can be witnessed. Conversely, since $\sum_{k=0}^{N} p_k(t) = 1$, if $\frac{d}{dt} p_k(t) \geq 0$ for any $k \neq 0$, it follows that $\frac{d}{dt} p_0(t) \leq 0$. We expect that in this situation, where the overlap of $\rho_S(t)$ with its initial configuration decreases, S undergoes an evolution that cannot be distinguished from a Markovian one.

Example: the quasi-eternal non-Markovian model

In the previous section we gave a set of conditions for random unitary dynamics such that, if satisfied, the QMI is never increasing when evolving maximally entangled states. In the case of qubits, these conditions are given in terms of the time derivative of the probability distribution $\{p_k(t)\}_k$ that defines Eq. (2.108). In this section we consider two examples. First, we consider a non-Markovian model that does not satisfy the conditions given in Section 3.5.2 and we check if, apart from the maximally entangled states, the evolution of random pure states can provide any backflow. Secondly, we consider another non-Markovian model that satisfies these conditions, namely cannot be witnessed by any initially maximally entangled state, and we perform numerical tests showing that the evolution of random pure initial states does not provide any backflow of QMI either.

First of all, we consider the qubit random unitary quasi-eternal evolution defined by $\alpha = 2/5$ and $t^{NM} = 1$ (see Section 3.2.1), which is non-Markovian for all $t > t^{NM} = 1$. This example satisfies the condition of physicality given by Eq. (3.9), as $\lim_{t\to\infty} B_{xyz}^{(2/5,1)}(t) \simeq 0.146 > 0$. The time-dependent distribution $\{p_k(t)\}_k$, that defines the corresponding random unitary evolution is

$$p_x(t) = p_y(t) = \frac{1}{4}\left(1 - e^{-4t/5}\right),$$

$$p_z(t) = \frac{1}{4}\left(1 + e^{-4t/5} - 2e^{-2t/5}\left(\frac{\cosh(t-1)}{\cosh(1)}\right)^{2/5}\right),$$

where $p_0(t) = 1 - p_x(t) - p_y(t) - p_z(t) \geq 0$ and $p_0(0) = 1$.

We define $t^{MI}(\rho_{SA}(0))$, the time when the backflow of QMI starts if the initial state considered is $\rho_{SA}(0)$. We evaluated $t^{MI}(\rho_{SA}(0))$ for $2 \cdot 10^4$ pure random states $\rho_{SA}(0) = |\psi\rangle\langle\psi|_{SA}$ of the form

$$|\psi\rangle_{SA} = a_1|00\rangle_{SA} + a_2|01\rangle_{SA} + a_3|10\rangle_{SA} + a_4|11\rangle_{SA},$$

where the parameters a_i are normalized complex random numbers. The minimum value of $t^{MI}(\rho_{SA}(0))$ obtained has been $t^{MI}(\overline{\rho}_{SA}(0)) \simeq 2.404$, where the values of the parameters that generates $\overline{\rho}_{SA}(0)$ are characterized by: $|\overline{a}_1| \simeq |\overline{a}_3| \simeq |\overline{a}_4| \simeq 0$ and $|\overline{a}_2| \simeq 1$ up to local unitary operations on A. Our numerical analysis does not give any insight about the possible existence of a class of initial states for which the corresponding $t^{MI}(\rho_{SA}(0))$ is arbitrarily close to $t^{NM} = 1$, namely the earliest time for which the intermediate map $V_{t+\epsilon,t}$ is P but not CP. If there exist pure states with t^{MI} closer to $t^{NM} = 1$, they must belong to a small subset that we did not sample. We point out that for this model, while $\frac{d}{dt}p_x(t) \geq 0$ and $\frac{d}{dt}p_y(t) \geq 0$ for any $t \geq 0$, $\frac{d}{dt}p_z(t) < 0$ for $t > 1.3254$. Indeed, the evolution of a maximally entangled state $|\phi^+\rangle_{SA} = (|00\rangle_{SA} + |11\rangle_{SA})/\sqrt{2}$ shows a backflow of QMI with $t^{MI}(|\phi^+\rangle\langle\phi^+|_{SA}) \simeq 2.741$, larger than what obtained for some initial non-maximally entangled states.

As a second example, we study the standard eternal non-Markovian model given by $\alpha = 1$ and $t^{NM} = 0$. Interestingly, in this case $\frac{d}{dt}p_x(t) \geq 0$, $\frac{d}{dt}p_y(t) \geq 0$

and $\frac{d}{dt}p_z(t) \geq 0$ for any $t \geq 0$. Therefore, this model satisfies the conditions given in Section 3.5.2, namely maximally entangled states that are evolved by this dynamics never show backflow of QMI. Then, we study the evolution of the QMI for 10^3 different random pure initial states that are not maximally entangled. Interestingly, also for these initial states no backflow of QMI is observed.

In summary, for a random unitary evolution that can be witnessed with maximally entangled states ($\alpha = 2/5$ and $t^{NM} = 1$), we have been able to find different initial states that provide backflow of QMI at earlier times than maximally entangled states, namely such that $t^{MI}(\rho_{SA}(0)) < t^{MI}(|\phi^+\rangle\langle\phi^+|_{SA})$. Instead, for a non-Markovian dynamics for which maximally entangled states do not show any backflow of QMI ($\alpha = 1$ and $t^{NM} = 0$), we could not find any other state able to do so either.

3.5.3 Non-maximally entangled states improve the detection precision of non-Markovianity

The purpose of this section is to examine, through a concrete example, how the use of non-maximally entangled states can be highly beneficial for the detection of non-Markovian effects. In fact, we see that in some situations, to detect a backflow in the QMI, one has to use initial states with an arbitrarily small amount of pure-state entanglement. This example also serves as an illustration of the witnessing potential of the QMI for non-Markovian dynamics that are neither random unitary nor P-divisible.

The model $\mathbf{\Lambda}$ that we consider here is a generalized amplitude damping channel (GADC) with two time dependent parameters, defined by the following set of Kraus operators

$$
\begin{aligned}
K_1(t) &= \sqrt{s(t)}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{r(t)} \end{pmatrix}, \\
K_2(t) &= \sqrt{s(t)}\begin{pmatrix} 0 & \sqrt{1-r(t)} \\ 0 & 0 \end{pmatrix}, \\
K_3(t) &= \sqrt{1-s(t)}\begin{pmatrix} \sqrt{r(t)} & 0 \\ 0 & 1 \end{pmatrix}, \\
K_4(t) &= \sqrt{1-s(t)}\begin{pmatrix} 0 & 0 \\ \sqrt{1-r(t)} & 0 \end{pmatrix},
\end{aligned} \tag{3.45}
$$

where $s(t) = \cos^2(5t)$ and $r(t) = e^{-t}$. The evolution induced by these operators is equivalent to that described by a generator L_t of the form given in Eq.(2.43) with $G_- = \sigma_-$ and $G_+ = \sigma_+$ and the respective time-dependent rates

$$\gamma_-(t) = \cos^2(5t) - 5(1 - e^{-t})\sin(10t)\,, \tag{3.46}$$
$$\gamma_+(t) = \sin^2(5t) + 5(1 - e^{-t})\sin(10t)\,, \tag{3.47}$$

for which the following equality holds

$$\gamma_-(t) + \gamma_+(t) = 1\,. \tag{3.48}$$

In order to understand the non-Markovian behavior of this model, it is possible to calculate the function $g_{\mathbf{\Lambda}}(t)$ introduced in Section 2.5.2

$$g_{\mathbf{\Lambda}}(t) = \lim_{\epsilon\to 0^+} \frac{\|(V_{t+\epsilon,t} \otimes I_A)(\phi_{AS})\|_1 - 1}{\epsilon}\,,$$

where ϕ_{AS} is the maximally entangled state (3.40). Through the value of this quantity we can understand if the intermediate map $V_{t+\epsilon,t}$ is CP or not and therefore if the evolution is Markovian or non-Markovian. Indeed, a CPTP intermediate map $V_{t+\epsilon,t}$ implies that $g(t) = 0$, while $g(t) > 0$ if and only if $V_{t+\epsilon,t}$ is not CPTP. In our case $g(t) > 0$ if and only if either $\gamma_-(t)$ or $\gamma_+(t)$ is negative

$$g(t) = \frac{1}{2}\sum_{i=\pm} |\gamma_i(t)| - \gamma_i(t) = \begin{cases} -\gamma_-(t) & t \in T^- \\ -\gamma_+(t) & t \in T^+ \\ 0 & \text{otherwise} \end{cases}\,,$$

where $T^\pm \equiv \{t : \gamma_\pm(t) < 0\}$ are two non-overlapping sets of time intervals. Indeed, we define T^- as the union of the time intervals $T_k^- \equiv (t^-_{in,k}, t^-_{fin,k})$ when the rate $\gamma_-(t)$ is negative

$$T^- \equiv \bigcup_{k=1}^{\infty} T_k^- \equiv \bigcup_{k=1}^{\infty} \left(t^-_{in,k}, t^-_{fin,k}\right)\,.$$

Similarly, we define the time intervals T_k^+ and the collection T^+. In Ref. [HKS$^+$14] the authors compare the ability of QMI and entanglement of formation, namely an entanglement measure, to witness non-Markovianity when a maximally entangled state is shared between S and A for the considered dynamics. They note that the QMI does not show any backflow during the first time interval where the dynamics is not CP-divisible, namely for $t \in T_1^-$, while the entanglement of formation shows a backflow in a time interval that is a proper subset of T_1^-. However, in order to fairly compare the witnessing potential of two different correlation measure, we must consider any possible initial state.

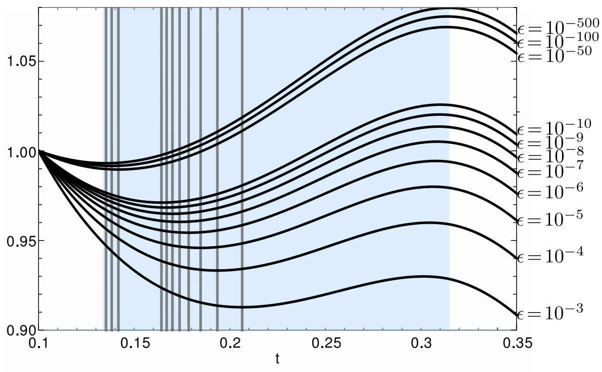


Figure 3.4: The QMI relative to its value at $t = 0.1$, $I(\rho^-_{SA}(t,\epsilon))/I(\rho^-_{SA}(0.1,\epsilon))$ as a function of t for values of ϵ between 10^{-3} and 10^{-500} (black curves). With successively smaller ϵ the QMI increases in a larger part of the interval $0.13437 \lesssim t \lesssim 0.31416$ where the dynamics is non CP-divisible (light blue area), and the t where the QMI begins to increase approaches the beginning of the interval (grey vertical lines). For $\epsilon = 10^{-500}$ the increase in QMI begins at $t \simeq 0.1352$.

From now on we focus on detecting backflows of QMI during the first time interval when $\gamma_-(t) < 0$, i.e., for $t \in T_1^- \simeq (0.13437, 0.31416)$. We give numerical results indicating that the QMI, depending on the chosen initial state, can provide backflows for any $t \in T_1^-$. In fact, we consider the following initial state

$$|\psi^-(\epsilon)\rangle_{SA} \equiv \sqrt{1-\epsilon^2}|00\rangle_{SA} + \epsilon|11\rangle_{SA}\,, \tag{3.49}$$

and provide strong evidence that these state provides backflows of QMI for any time $t \in T_1^-$ for $\epsilon \to 0^+$. More precisely, we have observed backflows in the QMI for $t \in \widetilde{T}_1^- \subset T_1^-$, where $\widetilde{T}_1^- \equiv (t^-_{in,1} + \delta\tau, t^-_{fin,1} - \delta\tau)$ and $\delta\tau = 10^{-10}$. That is, we have strong evidence that by taking initial pure states with an arbitrarily small amount of entanglement a backflow of QMI is observed in the whole range where the evolution is not CP-divisible.

We consider $\rho^-_{SA}(0,\epsilon) = |\psi^-(\epsilon)\rangle\langle\psi^-(\epsilon)|_{SA}$ as the initial state of our complete system and we study its evolution $\rho^-_{SA}(t,\epsilon) = \Lambda_t \otimes I_A(\rho^-_{SA}(0,\epsilon))$, where Λ_t represents the GADC described above. In Fig. 3.4 we show the behavior of $I(\rho^-_{SA}(t,\epsilon))$ for several values of ϵ. We notice that as ϵ approaches zero, the time interval where $I(\rho^-_{SA}(t,\epsilon))$ is increasing widens and approaches T_1^-, while the amplitude of the QMI decreases. The latter effect, and the increasingly small values of ϵ, makes it difficult to numerically verify the possibility to witness a

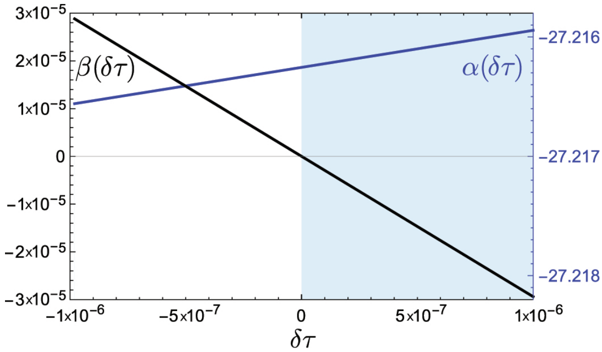


Figure 3.5: The coefficients $\alpha(\delta\tau)$ (dark blue curve) and $\beta(\delta\tau)$ (black curve) of the leading order of the series expansion of $\frac{d}{dt}I(\rho^-_{SA}(t,\epsilon))$ in ϵ as functions of $\delta\tau$ for $-10^{-6} \leq \delta\tau \leq 10^{-6}$. The dynamics is non-Markovian for $\delta\tau > 0$ (light blue area). For $\delta\tau \leq 0$ the coefficient $\beta(\delta\tau)$ is non-negative while $\alpha(\delta\tau)$ is negative and therefore the leading order term of the expansion is negative for any ϵ. For $\delta\tau > 0$ both $\beta(\delta\tau)$ and $\alpha(\delta\tau)$ are negative and therefore, for sufficiently small ϵ the leading order term of the expansion is positive. For $\delta\tau = 0$ the value of $\beta(\delta\tau)$ is zero to within numerical precision.

backflow of QMI for $t \in T_1^-$ arbitrarily close to $t^-_{in,1}$ and $t^-_{fin,1}$.

To better understand the behavior of $I(\rho^-_{SA}(t,\epsilon))$ when $t \simeq t^-_{in,1}$, we consider a series expansion in ϵ of the time derivative of this quantity for times close to the beginning of T_1^-, namely for $|\delta\tau| \equiv |t - t^-_{in,1}| \leq 10^{-6}$. We find

$$\frac{d}{dt}I(\rho^-_{SA}(t^-_{in,1} + \delta\tau, \epsilon)) = (\alpha(\delta\tau) + \beta(\delta\tau)\ln(\epsilon))\epsilon^2 + O(\epsilon^3)\,.$$

This expansion, see Fig. 3.5, is characterized by $\alpha(\delta\tau) < 0$ and the relation $\mathrm{sign}(\beta(\delta\tau)) = -\,\mathrm{sign}(\delta\tau)$, which has been verified up to $\delta\tau = \pm 10^{-10}$. An analogous result is obtained when $t \simeq t^-_{fin,1}$. For this case $\alpha(\delta\tau)$ is negative and $\mathrm{sign}(\beta(\delta\tau)) = \mathrm{sign}(\delta\tau)$. In summary, the numerical analysis indicates that for any value of $t \in T_1^-$ there exists a positive number ϵ_t such that, if $\rho_{SA}(0) = \rho^-_{SA}(0,\epsilon)$, we have a backflow of QMI at time t, namely $\frac{d}{dt}I(\rho^-_{SA}(t,\epsilon)) > 0$, for any $0 < \epsilon < \epsilon_t$.

We focused just on the first time interval of non-Markovianity, namely T_1^-, because in this case QMI does not show any backflow for the maximally entangled state. Indeed, given a value of $\epsilon > 0$, the fraction of the time interval of T_1^- for which $\frac{d}{dt}I(\rho^-_{SA}(t,\epsilon)) > 0$ is smaller than the one that we have for T_k^- when $k \geq 2$. Similarly, to observe backflows of QMI in the time intervals T_k^+, we can consider a different class of initial states $\rho^{+}_{SA}(0,\epsilon) \equiv |\psi^{+}(\epsilon)\rangle\langle\psi^{+}(\epsilon)|_{SA}$, where $\epsilon > 0$ approaches zero and $|\psi^{+}(\epsilon)\rangle_{SA} \equiv \epsilon|00\rangle_{SA} + \sqrt{1-\epsilon^2}|11\rangle_{SA}$. We underline that we cannot perform this numerical analysis for each time interval $T^{\pm}_{k}$ since there is an infinite number of such intervals.

The results in this section demonstrate that it appears difficult to fully determine when a given non-Markovian dynamics experiences correlation backflow in terms of the QMI, as one needs to consider all possible initial states. In fact, to our knowledge, it is not even clear if one can restrict the study to initial pure states. Despite all these difficulties, in the next sections we construct examples of non-Markovian dynamics where it can be proven that no backflow in the QMI takes place.

3.5.4 Taylor expansion of the quantum mutual information time derivative

To study the time dependence of the QMI perturbatively, we here outline how the time derivative of the QMI $\frac{d}{dt}I(\rho_{SA}(t)) \equiv \frac{d}{ds}I(\bar{a}, V_{t,s})\big|_{s=t}$ in a neighborhood of a state that a time t is $\rho_{SA}(t)$ can be described by a Taylor expansion in the coordinates a_i. In particular we consider Taylor expansions as a tool to investigate the neighborhoods of stationary states.

The QMI as a function on the set of states $S(\mathcal{H}_{SA})$ is analytic for all states ρ_{SA} of full rank, namely everywhere in the interior of the set of states, here denoted int$[S(\mathcal{H}_{SA})]$. Thus, in any open neighborhood $U \subset \text{int}[S(\mathcal{H}_{SA})]$ of a state ρ_{SA} the QMI equals its Taylor series and we can use Taylor expansions to analyze it perturbatively around ρ_{SA}. Moreover, if the dynamics is differentiable the time derivative of any analytic correlation measure is analytic as well.

Proposition 8. *Let M be a correlation measure that is analytic at $\bar{a}$. If $\frac{dV_{t,s}}{ds}\big|_{s=t}$ is well defined it follows that $\frac{d}{dt}M(\bar{a},t) \equiv \frac{d}{ds}M[\bar{a}, V_{t,s}]\big|_{s=t}$ is analytic at $\bar{a}$.*

Proof. We can write the time derivative $\frac{d}{dt}M(\bar{a},t) \equiv \frac{d}{ds}M[\bar{a}, \mathcal{V}_{ij}(t,s)]\big|_{s=t}$ as $\frac{d}{dt}M(\bar{a},t) = \sum_{i,j} a_j \frac{d\mathcal{V}_{ij}(t,s)}{ds}\big|_{s=t}\frac{\partial}{\partial a_i}M(\bar{a},t)$. Assume that $\frac{d\mathcal{V}_{ij}(t,s)}{ds}\big|_{s=t}$ is well defined for each i, j. Then, since products, linear combinations, and derivatives of analytic functions are analytic it follows that $\frac{d}{dt}M(\bar{a},t)$ is analytic as a function of $\bar{a}$. □

Thus, in particular, if $V_{t,s}$ is differentiable, $\frac{d}{dt}I(\rho_{SA}(t))$ can be described perturbatively in any open neighborhood of a state in int$[S(\mathcal{H}_{SA})]$ by a Taylor expansion. On the other hand, for states of less than full rank, namely states on the boundary of $S(\mathcal{H}_{SA})$, the partial derivatives in the coordinates a_i need not even be well defined to all orders.

Now consider a full-rank stationary state ρ^0_{SA} of a divisible evolution $\mathbf{\Lambda}$. It follows that $\rho^0_{SA} \in \mathrm{int}[S(\mathcal{H}_{SA})]$. Since $\frac{d}{dt}I(\rho^0_{SA}(t)) = 0$ the sign of $\frac{d}{dt}I(\rho_{SA}(t))$ for a $\rho_{SA}(t)$ in a neighborhood of ρ^0_{SA} is determined by the terms of order greater than zero in the Taylor expansion of $\frac{d}{dt}I(\rho_{SA}(t))$.

In general $\frac{d}{dt}I(\rho_{SA}(t))$ may take both positive and negative values for $\rho_{SA}(t)$ in a neighborhood of ρ^0_{SA}. If a neighborhood of ρ^0_{SA} exists where $\frac{d}{dt}I(\rho_{SA}(t))$ is everywhere non-negative, or alternatively everywhere non-positive, depends only on ρ^0_{SA} and $\frac{d}{ds}V_{t,s}\big|_{s=t}$. In particular, the properties of the neighborhood is independent of the previous dynamic Λ_t and the properties of $\mathrm{Im}(\Lambda_t)$ since we assumed linear divisibility of the dynamics.

This last observation allows us to make the following two statements about the change of the QMI. If there is a neighborhood of ρ^0_{SA} such that $\frac{d}{dt}I(\rho_{SA}(t))$ is somewhere positive, and this neighborhood is contained in $\mathrm{Im}(\Lambda_t)$, we can observe an increase of the QMI. If there is some neighborhood of ρ^0_{SA} where $\frac{d}{dt}I(\rho_{SA}(t))$ is non-positive, and $\mathrm{Im}(\Lambda_t)$ is contained in this neighborhood, we cannot observe any increase of QMI.

Neighborhoods of critical points

In the case that one or more first derivatives are zero one must consider higher order terms of the Taylor expansion to study how the sign of $\frac{d}{dt}I(\rho_{SA}(t))$ behaves in a neighborhood of a stationary state ρ^0_{SA}. In particular this is true if all first derivatives with respect to the a_i are zero, namely if ρ^0_{SA} is a critical point of $\frac{d}{dt}I(\rho_{SA}(t))$. The relevance of considering critical points in relation to stationary states can be understood from the following two observations. For any continuously differentiable evolution, a product state in the interior of the set of states is a critical point of $\frac{d}{dt}M$, if M is analytic.

Proposition 9. *Let M be a correlation measure that is analytic at a state ρ_{SA}. Assume that $V_{t,s}$ is continuously differentiable. Then if $\rho_{SA} \in \mathrm{int}[S(\mathcal{H}_{SA}]$ and is a product state it is a critical point of $\frac{dM}{dt}$.*

Proof. See Appendix A.2. □

Thus, in particular, all product states in $\mathrm{int}[S(\mathcal{H}_{SA})]$ are critical points of $\frac{d}{dt}I(\rho_{SA}(t))$. Note the product states at the boundary of the set of states, namely product states of less than full rank, are not necessarily critical points of $\frac{d}{dt}M$ because M is not necessarily constrained to be non-negative outside the set of states.

For qubit evolutions a stationary state in the interior of the set of states is a critical point of $\frac{d}{dt}M$, if M is analytic.

Proposition 10. *Let M be a correlation measure that is analytic at ρ_{SA} and let $\mathbf{\Lambda}$ be a continuously differentiable dynamical qubit evolution. If $\rho_{SA} \in$ $\mathrm{int}[S(\mathcal{H}_{SA})]$ and is a stationary state of $\mathbf{\Lambda}$, it is a critical point of $\frac{d}{dt}M$.*

Proof. See Appendix A.3. □

Thus, for qubit evolutions all stationary states in $\mathrm{int}[S(\mathcal{H}_{SA})]$ are critical points of $\frac{d}{dt}I(\rho_{SA}(t))$.

The nature of a critical point ρ_{SA}^0 can be analyzed by obtaining the eigenvalues of the Hessian matrix, namely the matrix of second derivatives $\mathbf{H}_{i,j} = \frac{\partial^2}{\partial a_i \partial a_j}\frac{d}{dt}I(\rho_{SA}(t))$. However, for any stationary state, the Hessian $\mathbf{H}_{i,j}$ is of less than full rank since $\frac{d}{dt}I(\rho_{SA}(t)) = 0$ on the set of stationary states of $V_{t,s}$, denoted S_s, and on the set of product states, denoted S_p. From this follows that any eigenvector of the Hessian that is tangent to $S_s \cup S_p$ corresponds to a zero eigenvalue. Thus, the sign of $\frac{d}{dt}I(\rho_{SA}(t))$ in the part of the neighborhood of ρ_{SA}^0 that coincides with the zero-eigenspace E_0 of $\mathbf{H}_{i,j}$ cannot be determined from the Hessian matrix alone since it depends on higher order derivatives.

On the overlap of the neighborhood of ρ_{SA}^0 with the complement of E_0, namely with $E_0^C \equiv B(\mathcal{H}_{SA})\backslash E_0$, the Hessian can be used to determine the sign of $\frac{d}{dt}I(\rho_{SA}(t))$, if the neighborhood is sufficiently small. In particular, if all non-zero eigenvalues of the Hessian, which correspond to eigenvectors tangent to E_0^C, are negative there exists a neighborhood $U^-_{\rho_{SA}^0}$ of ρ_{SA}^0 where $\frac{d}{dt}I(\rho_{SA}(t))$ is negative in $U^-_{\rho_{SA}^0} \cap E_0^C$. If all the non-zero eigenvalues of the Hessian are positive there exists a neighborhood $U^+_{\rho_{SA}^0}$ of ρ_{SA}^0 where $\frac{d}{dt}I(\rho_{SA}(t))$ is positive in $U^+_{\rho_{SA}^0} \cap E_0^C$.

Since the stationary states of a particular evolution $\mathbf{\Lambda}$ are always in the image of Λ_t for any t, the behavior of a correlation measure or other witness of non-Markovianity in a neighborhood of such a state may be more relevant than its ability to witness correlation backflows for states that are outside the image during the non-Markovian part of the dynamics. In particular this is true for evolutions where the image shrinks to a small neighborhood of the stationary states. For the case of correlation measures M that are locally analytic functions the stationary states are in many cases critical points of $\frac{d}{dt}M$. Thus the behavior of $\frac{d}{dt}M$ in a neighborhood of the stationary states can be investigated by calculating second order or higher partial derivatives at the stationary states.

Calculating partial derivatives

Directly calculating the derivatives of $\frac{d}{dt}I(\rho_{SA}(t))$ with respect to the coordinates a_i can be demanding since the eigenvalues of ρ are the roots of a polynomial with degree equal to $\dim(\mathcal{H}_{SA})$. However, even if the general expression of ρ_{SA} as a function of the coordinates a_i is difficult to diagonalize, the eigenvalues and eigenvectors may be known in the point where the derivatives have to be calculated. In this case one may circumvent the difficulty of diagonalizing the general expression and instead calculate the derivatives and second derivatives at a state ρ_{SA} using a method adapted from Ref. [TFV94]. The method described in this reference is valid for real symmetric matrices but generalizing it to Hermitian complex matrices is very straightforward. We present the version of this method that works for Hermitian matrices in the following paragraphs.

Let f be a spectral function defined on a set of $n \times n$ Hermitian matrices A which are parameterized by real numbers a_i. By spectral function we refer to a function that only depends on the eigenvalues $\{\lambda_k\}_{k=1}^n$ of A but not on the ordering of the eigenvalues. Furthermore, assume that f is analytic in a given point $\bar{a}$ and let $\lambda_k(\bar{a})$ be the eigenvalue of $A(\bar{a})$ with corresponding normalized eigenvector $u_k(\bar{a})$. Then the first and second order partial derivatives of f with respect to the parameters a_i in the point $\bar{a}$ can be expressed as

$$\frac{\partial f(\bar{a})}{\partial a_i} = \sum_k \frac{\partial f[\lambda(\bar{a})]}{\partial \lambda_k} h_i^k(\bar{a}) , \tag{3.50}$$

and

$$\frac{\partial^2 f(\bar{a})}{\partial a_i \partial a_j} = \sum_{k,l} \frac{\partial^2 f[\lambda(\bar{a})]}{\partial \lambda_k \partial \lambda_l} h_i^k(\bar{a}) h_j^l(\bar{a}) + \sum_k \frac{\partial f[\lambda(\bar{a})]}{\partial \lambda_k} h_{ij}^k(\bar{a}) + \eta_{ij}(\bar{a}) , \tag{3.51}$$

respectively, where

$$
\begin{aligned}
h_i^k(\bar{a}) &= u_k^\dagger \frac{\partial A(\bar{a})}{\partial a_i} u_k, \\
h_{ij}^k(\bar{a}) &= u_k^\dagger \frac{\partial^2 A(\bar{a})}{\partial a_i \partial a_j} u_k + \sum_{l|\lambda_k \neq \lambda_l} \frac{\alpha_{ij}^{kl}(\bar{a})}{\lambda_k(\bar{a}) - \lambda_l(\bar{a})}, \\
\alpha_{ij}^{kl}(\bar{a}) &= \left(u_k^\dagger(\bar{a}) \frac{\partial A(\bar{a})}{\partial a_i} u_l(\bar{a}) \right) \left(u_l^\dagger(\bar{a}) \frac{\partial A(\bar{a})}{\partial a_j} u_k(\bar{a}) \right) \\
&\quad + \left(u_k^\dagger(\bar{a}) \frac{\partial A(\bar{a})}{\partial a_j} u_l(\bar{a}) \right) \left(u_l^\dagger(\bar{a}) \frac{\partial A(\bar{a})}{\partial a_i} u_k(\bar{a}) \right), \\
\eta_{ij}(\bar{a}) &= \sum_{k,l|\lambda_k = \lambda_l, k<l} \alpha_{ij}^{kl}(\bar{a}) \frac{\partial^2 f[\lambda(\bar{a})]}{\partial^2 \lambda_k}.
\end{aligned}
\tag{3.52}
$$

When two or more eigenvalues coincide, the corresponding eigenvectors cannot be uniquely defined. Nevertheless, the method here can still be used since, while the expressions given in Eq. 3.52, namely h_i^k, may depend on the choice of eigenvectors, the partial derivatives themselves are independent and can be evaluated using any such choice.

If the diagonal form of A in the point $\bar{a}$ and the corresponding eigenvectors $u_k(\bar{a})$ are known, the method described here can greatly simplify the computation of the partial derivatives.

3.5.5 Non-Markovian dynamics that the quantum mutual information cannot witness

We analyzed several situations where correlation backflows as measured by the QMI detect non-Markovianity, including explicit examples of non-P-divisible, P-divisible, unital and non-unital non-Markovian evolutions. We now show that the QMI is non-increasing for some cases of random unitary qubit dynamics that are P-divisible but not CP-divisible by studying a neighborhood of the stationary states using the methods described in Section 3.5.4.

We consider an ancilla that is also a qubit and explicitly introduce coordinates a_i for $B(\mathcal{H}_{SA})$ with respect to an orthonormal basis $\{e_i\}_{i=0}^{15}$ defined by

$$\begin{aligned} e_0 &= \mathbb{1}_S \otimes \mathbb{1}_A, & e_8 &= \sigma_y \otimes \mathbb{1}_A, \\ e_1 &= \mathbb{1}_S \otimes \sigma_x, & e_9 &= \sigma_y \otimes \sigma_x, \\ e_2 &= \mathbb{1}_S \otimes \sigma_y, & e_{10} &= \sigma_y \otimes \sigma_y, \\ e_3 &= \mathbb{1}_S \otimes \sigma_z, & e_{11} &= \sigma_y \otimes \sigma_z, \\ e_4 &= \sigma_x \otimes \mathbb{1}_A, & e_{12} &= \sigma_z \otimes \mathbb{1}_A, \\ e_5 &= \sigma_x \otimes \sigma_x, & e_{13} &= \sigma_z \otimes \sigma_x, \\ e_6 &= \sigma_x \otimes \sigma_y, & e_{14} &= \sigma_z \otimes \sigma_y, \\ e_7 &= \sigma_x \otimes \sigma_z, & e_{15} &= \sigma_z \otimes \sigma_z, \end{aligned} \tag{3.53}$$

where all operators are of the form $\chi_S \otimes \chi_A$ for $\chi_S \in B(\mathcal{H}_S)$ and $\chi_A \in B(\mathcal{H}_A)$. A state ρ_{SA} is represented as

$$\rho_{SA} = \frac{1}{4}\mathbb{1}_S \otimes \mathbb{1}_A + \sum_{i=1}^{15} a_i e_i\,, \tag{3.54}$$

where $a_i = \frac{1}{4}\,\mathrm{Tr}(\rho_{SA} e_i)$.

The analysis of $\frac{d}{dt}I(\rho_{SA}(t))$ in the neighborhood of the stationary states is done by first considering the states of full rank, namely the states in $\mathrm{int}[S(\mathcal{H}_{SA})]$, where $\frac{d}{dt}I(\rho_{SA}(t))$ is analytic. There we calculate the second derivatives of $\frac{d}{dt}I(\rho_{SA}(t))$ at the stationary states and find the eigenvalues of the Hessian matrix. On the subset of states that fall in the zero eigenspace of the Hessian we then directly evaluate $\frac{d}{dt}I(\rho_{SA}(t))$. Finally, we consider the states of less than full rank and describe the neighborhood of the intersection of the stationary states with the boundary of the set of states.

The stationary states are of the form $\mathbb{1}_S/2 \otimes \rho_A$ for arbitrary ρ_A. Since the stationary states in $\mathrm{int}[S(\mathcal{H}_{SA})]$ are critical by Propositions 9 and 10 and such that $\frac{d}{dt}I(\rho_{SA}(t)) = 0$, there exists some sufficiently small neighborhood of the set of stationary states where the second order terms of the Taylor expansion in the a_i determine the sign of $\frac{d}{dt}I(\rho_{SA}(t))$, in all directions where the second derivative is non-zero. To simplify the calculation of these derivatives we note that unitary transformations on the ancilla do not change the QMI and it is therefore sufficient to consider diagonal ρ_A. In other words, the purity of the state of the ancilla is the only parameter that is relevant for our analysis. The diagonal stationary states are of the form $\frac{1}{4}\mathbb{1}_S \otimes \mathbb{1}_A + a_{12}\mathbb{1}_S \otimes \sigma_z$ for $-1/4 \leq a_{12} \leq 1/4$. The states for which $-1/4 < a_{12} < 1/4$ are in $\mathrm{int}[S(\mathcal{H}_{SA})]$ and the states with coordinates $a_{12} = \pm 1/4$ are at the boundary of the set of states.

The second derivatives at the diagonal stationary states of full rank were calculated using the method described in Section 3.5.4 and the eigenvalues of the Hessian matrix were obtained.The Hessian has six eigenvalues that are zero for all stationary states in $\mathrm{int}[S(\mathcal{H}_{SA})]$, and for all values of the parameters $\gamma_k(t)$. The remaining nine eigenvalues are functions of the parameters $\gamma_k(t)$ and of a_{12}, and are given by

$$\begin{aligned}
&32[\gamma_y(t)+\gamma_z(t)]\left(\frac{16a_{12}^2+1}{16a_{12}^2-1}\right),\ -8[\gamma_y(t)+\gamma_z(t)]\frac{\mathrm{atanh}(4a_{12})}{a_{12}},\\
&32[\gamma_x(t)+\gamma_z(t)]\left(\frac{16a_{12}^2+1}{16a_{12}^2-1}\right),\ -8[\gamma_y(t)+\gamma_z(t)]\frac{\mathrm{atanh}(4a_{12})}{a_{12}},\\
&32[\gamma_x(t)+\gamma_y(t)]\left(\frac{16a_{12}^2+1}{16a_{12}^2-1}\right),\ -8[\gamma_x(t)+\gamma_z(t)]\frac{\mathrm{atanh}(4a_{12})}{a_{12}},\\
&-8[\gamma_x(t)+\gamma_z(t)]\frac{\mathrm{atanh}(4a_{12})}{a_{12}},\ -8[\gamma_x(t)+\gamma_y(t)]\frac{\mathrm{atanh}(4a_{12})}{a_{12}},\\
&-8[\gamma_x(t)+\gamma_y(t)]\frac{\mathrm{atanh}(4a_{12})}{a_{12}}.
\end{aligned} \tag{3.55}$$

These nine eigenvalues are all non-positive if and only if the conditions in Eq. (2.117) are satisfied, namely if and only if the dynamics is P-divisible. In particular, they are all strictly negative if $\gamma_i(t)+\gamma_j(t) > 0$ for all i, j. In this case there thus exists a sufficiently small neighborhood of the stationary states where $\frac{d}{dt}I(\rho_{SA}(t))$ is negative in the intersection of the neighborhood with the complement of the zero eigenspace of the Hessian.

Next, we need to investigate $\frac{d}{dt}I(\rho_{SA}(t))$ on the intersection of a neighborhood around a stationary state with the zero eigenspace of the Hessian. Here we would need higher order terms in the Taylor expansion to determine the sign of $\frac{d}{dt}I(\rho_{SA}(t))$, however on this eigenspace we can evaluate it directly. The zero eigenspace $E_0(a_{12})$ as a function of a_{12}, is spanned by the six eigenvectors $\sigma_i \otimes (\mathbb{1}_A + 4a_{12}\sigma_z)$ and $\mathbb{1}_S \otimes \sigma_i$ for $i = x, y, z$. These eigenvectors are tangent to the set of product states for all a_{12}, but the plane they span, namely $E_0(a_{12})$, also contains correlated states. For a given stationary state $\rho^0_{SA} = \frac{1}{4}\mathbb{1}_S \otimes \mathbb{1}_A + a_0\mathbb{1}_S \otimes \sigma_z$ we can parameterize $E_0(a_0)$. The states in the $E_0(a_0)$ are of the form

$$\begin{aligned}
\frac{1}{4}\ &\mathbb{1}_S \otimes \mathbb{1}_A + (a_1\sigma_x + a_2\sigma_y + a_3\sigma_z) \otimes (\mathbb{1}_A + 4a_0\sigma_z)\\
&+\mathbb{1}_S \otimes (a_4\sigma_x + a_8\sigma_y + a_{12}\sigma_z).
\end{aligned} \tag{3.56}$$

Note that $E_0(a_0)$ is an invariant subspace of $V_{t,s}$ for all a_0 since the evolution is unital. Thus any state in $E_0(a_0)$ is mapped into a state also belonging to

$E_0(a_0)$. Furthermore, a_4, a_8 and a_{12} are time independent. Therefore, the time derivative of the QMI I as a function on $E_0(a_0)$ depends only on the coordinates a_1, a_2 and a_3. Since the QMI is independent of unitary transformations on the system we can diagonalize $a_1\sigma_x + a_2\sigma_y + a_3\sigma_z$ without changing its value. Let $\pm\lambda(s) = \pm\sqrt{a_1^2(s) + a_2^2(s) + a_3^2(s)}$ be the corresponding eigenvalues as functions of time where

$$\begin{aligned} a_1(s) &= a_1 \exp\left[-\int_t^s (\gamma_z(\tau) + \gamma_y(\tau))d\tau\right], \\ a_2(s) &= a_2 \exp\left[-\int_t^s (\gamma_z(\tau) + \gamma_x(\tau))d\tau\right], \\ a_3(s) &= a_3 \exp\left[-\int_t^s (\gamma_x(\tau) + \gamma_y(\tau))d\tau\right]. \end{aligned} \tag{3.57}$$

Since the only time dependence of I is its dependence on $\lambda(t)$ we can express the time derivative of the QMI as

$$\frac{dI[\rho_{SA}(t))]}{dt} = \frac{dI[\rho_{SA}(s)]}{d\lambda(s)} \frac{d\lambda(s)}{ds}\Big|_{s=t}$$

for any $\rho_{SA}(t) \in E_0(a_0)$, where $\frac{d\lambda(s)}{ds}\Big|_{s=t}$ is given by

$$-\frac{a_1^2[\gamma_z(t) + \gamma_y(t)] + a_2^2[\gamma_x(t) + \gamma_z(t)] + a_3^2[\gamma_x(t) + \gamma_y(t)]}{\sqrt{a_1^2 + a_2^2 + a_3^2}}.$$

When the conditions in Eq. (2.117) are satisfied, namely when the evolution is P-divisible, $\frac{d\lambda(s)}{ds}\Big|_{s=t}$ is non-positive for all a_1, a_2 and a_3. This is equivalent to stating that the length of the Bloch vector of the reduced state of the system does not increase when the dynamics is P-divisible. Moreover, we see that for all states in $E_0(a_0)$ except those of the form $\mathbb{1}_S/2\otimes\rho_A$, for which $a_1 = a_2 = a_3 = 0$, there exists some CP-divisible random unitary dynamics such that $\frac{d\lambda(s)}{ds}\Big|_{s=t} < 0$. Since we know that $\frac{dI[\rho_{SA}(t)]}{dt} \leq 0$ for all $\rho_{SA}(t) \in E_0(a_0)$ when the evolution is CP-divisible it follows that $\frac{dI[\rho_{SA}(s)]}{d\lambda(s)}\Big|_{s=t}$ is non-negative for all $\rho_{SA}(s) \in E_0(a_0)$ not of the form $\mathbb{1}_S/2 \otimes \rho_A$. Therefore we can conclude that $\frac{dI[\rho_{SA}(t)]}{dt} \leq 0$ for all $\rho_{SA}(t) \in E_0(a_0)$ when $V_{t,s}$ is P-divisible.

In the above analysis we have seen that there exists random unitary non-Markovian P-divisible evolutions for which no increase of the QMI occurs in a sufficiently small neighborhood of the stationary states in int$[S(\mathcal{H}_{SA})]$. It remains to consider the neighborhood of the stationary states of less than full rank, namely of stationary states in the boundary of the set of states. For these stationary states $a_{12} = \pm 1/4$ and they are thus of the form $\mathbb{1}_S \otimes (\mathbb{1}_A + \sigma_z)/4$ and $\mathbb{1}_S \otimes (\mathbb{1}_A - \sigma_z)/4$. It is sufficient to consider restricted neighborhoods of these states where the coordinate a_{12} is held fixed at $1/4$ or $-1/4$ respectively. Any other point in their neighborhoods either belongs to a neighborhood of a stationary state in $\mathrm{int}[S(\mathcal{H}_{SA})]$, or is unphysical. The physical states in these restricted neighborhoods for which $a_{12} = \pm 1/4$ are product states of the form $\rho_S \otimes (\mathbb{1}_A \pm \sigma_z)/4$, where $\rho_S \in B(\mathcal{H}_S)$. This can be seen by noting that if $a_{12} = \pm 1/4$, one must choose $a_4 = a_8 = 0$ to ensure non-negative eigenvalues of the reduced state on $\mathcal{H}_A$. Therefore, for all physical states in the restricted neighborhoods the reduced state of the ancilla is pure and of the form $(\mathbb{1}_A \pm \sigma_z)/4$, which implies that all such states are product states. Since any product state has zero QMI and remains a product state during the evolution it follows that $\frac{d}{dt} I(\rho_{SA}(t))$ is zero for all states in any neighborhood of $\mathbb{1}_S \otimes (\mathbb{1}_A \pm \sigma_z)/4$ where $a_{12} = \pm 1/4$.

Finally we can conclude that for random unitary evolutions where the non-Markovian dynamics is P-divisible and all initial states have been mapped to a sufficiently small neighborhood of the stationary states by the preceding Markovian evolution no increase in the QMI occurs. Moreover, the neighborhood where no increase of the QMI occurs only depends on $V_{t,s}$ and is independent of the preceding dynamics. Therefore, for any random unitary P-divisible evolution subsequent to time t, it is always possible to find a random unitary evolution $\mathbf{\Lambda}$ that is CP-divisible in $[0, t]$ such that $\mathrm{Im}(\Lambda_t \otimes I_A)$ is contained in this neighborhood by appropriately choosing the rates $\gamma_k(\tau) > 0$ for $0 \leq \tau \leq t$. Now we discuss how to obtain quasi-eternal non-Markovian evolutions that cannot show backflows of the QMI shared between S and A.

Example: the quasi-eternal non-Markovian model

In Section 3.2.2 we studied the dependence of quasi-eternal evolutions $\mathbf{\Lambda}^{(t_2^{NM})}$ from the parameter t_2^{NM} when $\alpha = 2/5$. We showed that we can make the image of the evolution at the time when it starts to be non-Markovian, namely $t = t_2^{NM}$, as small as we want, or, in other words, as close as desired to the stationary state of the evolution. Now we show how to obtain non-Markovian evolutions that cannot show backflows of QMI.

Consider two qubits S and A, where S is evolved by $\mathbf{\Lambda}^{(t_2^{NM})}$ (see Section 3.2.2) and A is an ancilla. In this scenario, we want to witness a backflow of the QMI I shared between S and A in the time interval $[t_2', t_2'']$, namely when the P intermediate map $\overline{V}$ evolves S. Hence, we increase the value of t_2^{NM} until the image $\mathrm{Im}(\Lambda_{t_2'}^{(t_2^{NM})} \otimes I_A)$ of the evolution that precedes the action of $\overline{V} \otimes I_A$ is in a neighbor of radius ϵ of the stationary states of the dynamics, namely $\rho_A \otimes \mathbb{1}_S/2$, where ρ_A is any state of $S(\mathcal{H}_A)$. Therefore, at time t_2' we have that

- The states in $\mathrm{Im}(\Lambda_{t_2'}^{(t_2^{NM})} \otimes I_A)$ are ϵ-close to the stationary state of the evolution;

- The evolution of $S - A$ in the following time interval $[t_2', t_2'']$ is described by the intermediate map $\overline{V} \otimes I_A$;

where these properties are valid for any $t_2'' - t_2' > 0$.

Hence, this method provide examples of random unitary P-divisible evolutions in which non-Markovian effects take place only for states arbitrarily close to the stationary states. For these cases, we can apply the perturbative analysis of Sections 3.5.4 and 3.5.5 and conclude that these non-Markovian evolutions do not allow QMI revivals for any initial state in $S(\mathcal{H}_{SA})$.

3.6 Discussion

Understanding the operational consequences of non-Markovian effects in terms of information backflow is a fundamental question. In this chapter, we focused on correlations and studied how they can be used to detect the failure of CP-divisibility. We identified strengths and weaknesses of several known correlation measures. In particular, we have shown that:

- Non-Markovian effects in single-parameter dynamics, such as depolarization, dephasing or amplitude damping, always lead to correlation backflows for any continuously differentiable measure that is time-dependent on the image of the preceding evolution;

- Measures of entanglement between S and an ancilla A cannot provide any backflow in those cases where the non-Markovian dynamics is P-divisible and appear only after the dynamics has become entanglement breaking;

- It is possible to detect backflows in the QMI for any qubit unital non-P-divisible dynamics;

- Maximally entangled states are not necessarily optimal for observing backflows in the QMI;

- There exist quasi-eternal non-Markovian dynamics with no backflow in the QMI evaluated between S and an ancilla A.

Our results clarify many issues but also point to several open questions. The most obvious one is to construct a correlation measures able to detect all non-Markovian evolutions, either by adapting the results in [BD16] to our approach, or by considering a novel approach. A second open question is to understand if the use of the second additional system can be of use for existing correlation measures, such as those based on entanglement or QMI.

Chapter 4

Correlation measure witnessing almost-all non-Markovian evolutions

We study the ability of correlation measures to witness non-Markovian open quantum system dynamics. A correlation measure is introduced, and it is proven that, in an enlarged setting with two ancillary systems, this measure detects almost all non-Markovian dynamics, except possibly a zero-measure set of dynamics that is non-bijective in finite time-intervals. Our proof is constructive and provides different initial states detecting the non-Markovian evolutions. These states are all separable and some are arbitrarily close to a product state. The results exposed in this chapter are contained in the original works [DJB$^+$19] and [DJB$^+$20].

4.1 Introduction

The dynamics of open quantum systems [BP07, Wei00, RH11] has been investigated extensively in recent years for both fundamental and applicative reasons. In particular the phenomenon of reservoir memory effects has been studied since such effects can induce a recovery of correlations or coherence and are therefore viewed as a potential resource for the performance of quantum technologies. The problem of characterizing memoryless dynamics, the so-called Markovian regime, and dynamics exhibiting memory effects, the non-Markovian regime, has been considered in a wide range of different ways (for extended reviews see [RHP14, BLPV16]). Markovianity is frequently identified with the property of CP-divisibility (see Section 2.3.3): an evolution $\mathbf{\Lambda}$ is

CP-divisible if between any two times it can be described by a CPTP-map. This idea generalizes the concept of classical Markovian processes [GK76, Lin76].

A complementary way of addressing memory effects consists of identifying operational quantities that can detect the information backflow expected in non-Markovian evolutions [RHP10, LFS12, BD16, BJA17, CKR11]. A common approach is to study functions that are monotonically non-increasing under local CP-maps, namely the witnesses of non-Markovianity Q introduced in Section 2.5. An increase of such a quantity implies that the evolution is not CP-divisible, hence non-Markovian, although the converse may not be true in general. Investigating under what conditions a non-increase of these quantities is in one-to-one correspondence with CP-divisibility is relevant for evaluating current methods for non-Markovianity detection, finding new ones, and understand the operational consequences of non-Markovianity. It is also relevant to understand how these different detection methods are related, and to what extent they are equivalent. In particular, it has been shown that the guessing probability of minimum error state discrimination can be used to witness any non-Markovian dynamics [BD16]. However, no method for constructing state ensembles required for this is known. A constructive method to witness any bijective non-Markovian dynamics using an ensemble of two equiprobable states has subsequently been proposed [BJA17].

In this chapter we investigate the relation between non-Markovianity and correlations. As we saw in Chapter 3, the quantum mutual information (QMI) between system and ancilla as well as any entanglement measure are unable to witness all non-Markovian dynamics. The next natural question is to understand whether there exists a correlation measures that is able to provide backflows for any non-Markovian evolution. To investigate this, we first introduce a bipartite correlation measure based on the distinguishability of an ensemble of remotely prepared states. We then use this measure in an extended setting consisting of the system and two ancillary systems and prove that the non-increase of this measure is in one-to-one correspondence with CP-divisibility for almost all evolutions. More precisely, we show how to detect a correlation backflow for all non CP-divisible evolutions that are bijective or at most point-wise non-bijective. Our method is constructive and provides a family of initial states able to detect the correlation backflow. The states in this family are all separable and include states that are arbitrarily close to uncorrelated.

4.2 Introduction of a correlation measure

In Chapter 3 we provided examples of non-Markovian dynamics that could not be detected with ordinarily used correlation measures. Now, one may wonder if this limitation applies to any correlation measure or, on the contrary, if there exists a correlation measure that witnesses all non-Markovian dynamics. Motivated by this question we introduce a correlation measure based on the distinguishability of the ensembles one party prepares for the other party by performing local measurements on half of a bipartite state. For this, we first need to discuss several concepts related to the distinguishability of quantum states.

4.2.1 Maximally entropic measurements

In Section 2.1.2 we showed that an n-outcome measurement on a quantum system $\rho \in S(\mathcal{H})$ is represented by a positive-operator valued measure (POVM), namely a collection of positive semi-definite operators $\{P_i\}_{i=1}^n$ in $B(\mathcal{H})$ such that $\sum_{i=1}^n P_i = \mathbb{1}$. Each P_i represents a possible outcome with the probability of occurrence $p_i = \mathrm{Tr}[\rho P_i]$.

We say that an n-output POVM is *maximally entropic* (ME-POVM) *for* ρ if, when applied on ρ, each outcome has the same probability of occurrence: $p_i = 1/n$. Indeed, if $S_C(\{p_i\}_i) = -\sum_i p_i \log_n p_i$ is the Shannon entropy of the resulting n-outcome probability distribution, where we take as the basis of the logarithm in the entropy the number of outputs, $S_C(\{p_i\}_i) = 1$ if and only if $p_i = 1/n$. We define the set of n-output ME-POVMs for ρ as

Definition 8. *The set of n-output ME-POVMs on $\mathcal{H}$ for $\rho \in S(\mathcal{H})$ is defined as*

$$\Pi_n(\rho) \equiv \left\{ \{P_i\}_{i=1}^n : Tr[\rho P_i] = \frac{1}{n} \right\}, \tag{4.1}$$

where $\{P_i\}_{i=1}^n$ is a generic n-output POVM.

Moreover, we can define the whole set ME-POVMs as follows

Definition 9. *The set of ME-POVMs on $\mathcal{H}$ for $\rho \in S(\mathcal{H})$ is defined as*

$$\Pi(\rho) \equiv \bigcup_{n\geq 2} \Pi_n(\rho). \tag{4.2}$$

For any state ρ, this collection is non-empty and contains measurements with any number of outputs (see Appendix B.1).

Notice that each state in $\rho \in S(\mathcal{H})$ defines a different set of ME-POVMs $\Pi(\rho)$. Indeed, if $\{P_i\}_i$ is a ME-POVM for ρ, namely $\{P_i\}_i \in \Pi(\rho)$, in general the same is not true for a state ρ' different from ρ, namely $\{P_i\}_i \notin \Pi(\rho')$. The only POVMs that belong to $\Pi(\rho)$ for any ρ are the trivial measurements $\{\mathbb{1}/n\}_{i=1}^n$ for $n \geq 2$, which provide equiprobable outcomes for any ρ. Indeed, $p_i = \mathrm{Tr}\left[\rho \mathbb{1}/n\right] = 1/n$ for any ρ and $n \geq 2$.

Consider a bipartite state ρ_{AB} defined on a finite dimensional state space of a composed system $S(\mathcal{H}_A \otimes \mathcal{H}_B)$. A measurement with an arbitrary number of outcomes $\{P_{A,i}\}_i$ performed on system A prepares on B the output ensemble $\mathcal{E}(\rho_{AB}, \{P_{A,i}\}_i) \equiv \{p_i, \rho_{B,i}\}_i$ defined by (see Section 2.1.2)

$$p_i = \mathrm{Tr}\left[\rho_{AB}\, P_{A,i}\right], \quad \rho_{B,i} = \frac{\mathrm{Tr}_A\left[\rho_{AB}\, P_{A,i} \otimes \mathbb{1}_B\right]}{p_i}, \tag{4.3}$$

where $\rho_A = \mathrm{Tr}_B\left[\rho_{AB}\right]$ is the reduced state on A. From Eq. (4.3) it follows that a ME-POVM for ρ_{AB} of the form $\{P_{A,i} \otimes \mathbb{1}_B\}_{i=1}^n$ implies that $\{P_{A,i}\}_{i=1}^n$ is a ME-POVM for $\rho_A = \mathrm{Tr}_B\left[\rho_{AB}\right]$. Indeed, it is easy to show that $\{\{P_{A,i}\}_i : \{P_{A,i} \otimes \mathbb{1}_B\}_i \in \Pi(\rho_{AB})\} = \Pi(\rho_A)$.

Now we restrict the previous analysis to ME-POVMs for the reduced state on system A, namely for $\rho_A = \mathrm{Tr}_B\left[\rho_{AB}\right]$. If the n-output $\{P_{A,i}\}_{i=1}^n$ is a ME-POVM for ρ_A, from Eqs. (4.2) and (4.3) it follows that,

$$\mathcal{E}(\rho_{AB}, \{P_{A,i}\}_i) = \left\{ p_i = \frac{1}{n}, \;\; \rho_{B,i} = n\, \mathrm{Tr}_A\left[\rho_{AB} P_{A,i} \otimes \mathbb{1}_B\right] \right\}_{i=1}^n . \tag{4.4}$$

Therefore, Alice measures ρ_{AB} with $\{P_{A,i}\}_{i=1}^n \in \Pi(\rho_A)$ and Bob obtains an *equiprobable ensemble of states* (EES), namely an n-state output ensemble where the probability distribution of occurrence of each state $\rho_{B,i}$ is uniform (see Fig. 4.1).

As we saw in Section 2.5.5, the average probability to correctly identify a state extracted from an ensemble $\mathcal{E} = \{p_i, \rho_i\}_{i=1}^n$ when we maximize over all possible measurements, is the guessing probability of the ensemble

$$P_g(\mathcal{E}) \equiv \max_{\{P_i\}_i} \sum_{i=1}^n p_i \mathrm{Tr}\left[\rho_i\, P_i\right] , \tag{4.5}$$

where the maximization is performed over the space of the n-output POVMs. Using the definition $\overline{p} \equiv \max_i \{p_i\}_{i=1}^n \geq 1/n$, it follows that $P_g(\mathcal{E}) \geq \overline{p}$, where the equality holds if $\mathcal{E}$ is made of identical states. Hence, $P_g(\{p_i = 1/n, \rho_i = \rho\}_{i=1}^n) = 1/n$. Note that when the target ensemble is an EES of two states, namely for $\mathcal{E}^{eq} = \{\{p_{1,2} = 1/2\}, \{\rho_1, \rho_2\}\}$, the quantity $P_g(\mathcal{E}^{eq})$ can be expressed

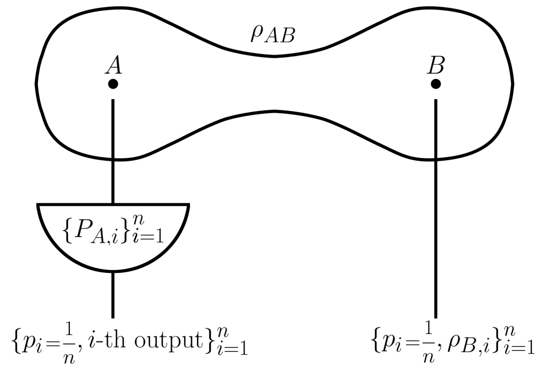


Figure 4.1: The measurement scenario where Alice, measuring her side of ρ_{AB} with an n-ouput ME-POVM $\{P_{A,i}\}_{i=1}^{n}$, produces on Bob's side the EES given by Eq. (4.4).

in terms of the distinguishability $Q_{BLP}(\rho_1, \rho_2) = \frac{1}{4}(2 + \|\rho_1 - \rho_2\|_1)$ between ρ_1 and ρ_2 given in Eq. (2.70) and we write

$$P_g(\mathcal{E}^{eq}) = \frac{1}{4}\left(2 + \|\rho_1 - \rho_2\|_1\right) , \tag{4.6}$$

4.2.2 Definition of the correlation measure

We now have all the ingredients needed to define our correlation measure. A correlation measure $C_A^{(2)} : S(\mathcal{H}_{AB}) \to \mathbb{R}^+$ that satisfies the properties mentioned in Section 2.5.7 is obtained by maximizing the guessing probability of these ensembles of B over the 2-output ME-POVMs on A, namely

$$C_A^{(2)}(\rho_{AB}) \equiv \max_{\{P_{A,1}, P_{A,2}\} \in \Pi_2(\rho_A)} P_g\left(\mathcal{E}\left(\rho_{AB}, \{P_{A,1}, P_{A,2}\}\right)\right) - \frac{1}{2} , \tag{4.7}$$

where $\rho_A = \mathrm{Tr}_B\left[\rho_{AB}\right]$ is the reduced state on A. The scenario that reproduces the value of $C_A^{(2)}(\rho_{AB})$ is described in Fig. 4.1, where Alice chooses a 2-output ME-POVM that maximizes the guessing probability of the output ensemble generated on Bob's side. Moreover, we can use Eq. (4.6) to rewrite $C_A^{(2)}(\rho_{AB})$ in the following way

$$C_A^{(2)}(\rho_{AB}) = \max_{\{P_{A,1}, P_{A,2}\} \in \Pi_2(\rho_A)} \frac{\|\rho_{B,1} - \rho_{B,2}\|_1}{4} , \tag{4.8}$$

where $\rho_{B,1}$ and $\rho_{B,2}$ are the two output states obtained when Alice applies the ME-POVM $\{P_{A,1}, P_{A,2}\}$ (see Eq. (4.4)).

Alternatively, we can perform 2-output ME-POVMs on the system B and obtain another measure

$$C_B^{(2)}(\rho_{AB}) \equiv \max_{\{P_{B,1},P_{B,2}\}\in\Pi_2(\rho_B)} P_g\left(\mathcal{E}\left(\rho_{AB},\{P_{B,1},P_{B,2}\}\right)\right) - \frac{1}{2}\,, \tag{4.9}$$

where $\rho_B = \mathrm{Tr}_A\left[\rho_{AB}\right]$ is the reduced state on B. We underline that the guessing probabilities that appear in Eq. (4.7) and (4.9) can be evaluated using Eq. (4.6). A natural way to construct a symmetric measure with respect to A and B is the following

$$C^{(2)}(\rho_{AB}) \equiv \max\left\{C_A^{(2)}(\rho_{AB}),\ C_B^{(2)}(\rho_{AB})\right\}\,. \tag{4.10}$$

Operationally, $C_A^{(2)}(\rho_{AB})$ ($C_B^{(2)}(\rho_{AB})$) corresponds to the largest distinguishability between the pairs of equiprobable states of B (A) that we can obtain from ρ_{AB} by performing measurements on A (B).

Similar correlation measures $C^{(n)}$ can be obtained by fixing the number of outputs of the ME-POVMs to any integer $n \geq 3$ and replacing the term $1/2$ in Eqs. (4.7) and (4.9) by $1/n$. We define $C_A(\rho_{AB})$ ($C_B(\rho_{AB})$) as the correlation measure obtained without fixing the number of outputs of the ME-POVMs

$$C_A(\rho_{AB}) \equiv \max_{\{P_{A,i}\}_i\in\Pi(\rho_A)} P_g\left(\mathcal{E}\left(\rho_{AB},\{P_{A,i}\}_i\right)\right) - \frac{1}{2}\,. \tag{4.11}$$

We define $C_B(\rho_{AB})$ similarly. Finally, we define

$$C(\rho_{AB}) \equiv \max\{C_A(\rho_{AB}),\ C_B(\rho_{AB})\}, \tag{4.12}$$

which represents the maximum distinguishability of the output ensembles that can be generated by measuring either A or B with ME-POVMs.

We give two examples that provide an intuitive idea of the meaning of the correlation measures $C_A^{(2)}$, $C_B^{(2)}$ and $C^{(2)}$. First, we consider a generic product state $\rho_A \otimes \rho_B$, which is a completely uncorrelated state (classically and quantumly). In this case, if Alice measures her side of ρ_{AB} with a 2-output ME-POVMs $\{P_{A,i}\}_{i=1}^2$, the ensemble generated on Bob's side consists of the two states: $\rho_{B,i} = 2\,\mathrm{Tr}_A\left[\rho_A \otimes \rho_B \cdot P_{A,i} \otimes \mathbb{1}_B\right] = \rho_B$, which are identical and equal to ρ_B. The corresponding guessing probability is $P_g = 1/2$ and therefore $C_A^{(2)}(\rho_A \otimes \rho_B) = 0$. In fact, the $1/2$ factor is chosen just to make the value of the correlation measure equals to zero for product states, which are therefore uncorrelated also with respect to $C_A^{(2)}$. It is straightforward to show $\rho_A \otimes \rho_B$ is uncorrelated also respect to $C_B^{(2)}$ and C.

The second example is given by the two-qubit maximally entangled state $\phi_{AB}^+ = |\phi^+\rangle\langle\phi^+|_{AB}$, where $|\phi^+\rangle_{AB} = (|00\rangle_{AB} + |11\rangle_{AB})/\sqrt{2}$. In order to evaluate $C_A^{(2)}(\phi_{AB}^+)$, it is easy to realize that the projective measurement $\{P_{A,i}^{(\mathrm{proj})}\}_{i=1}^2 =$

$\{|0\rangle\langle 0|_A, |1\rangle\langle 1|_A\}$ is the ME-POVM obtained by the maximization of Eq. (4.7). Indeed, in this case, Alice generates on Bob's side an orthogonal ensemble of two states: $\rho_{B,i} = 2\,\mathrm{Tr}_A\left[\phi^+_{AB} \cdot |i\rangle\langle i|_A \otimes \mathbb{1}_B\right] = |i\rangle\langle i|_B$, which is perfectly distinguishable: $P_g(\{p_i = 1/2, |i\rangle\langle i|_B\}_{i=1}^2) = 1$. It follows that, since the guessing probability of an ensemble cannot be greater than 1, maximally entangled states are maximally correlated states with respect to $C_A^{(2)}$ and, as it is straightforward to prove, also to $C_B^{(2)}$ and $C^{(2)}$. Note however that the same maximum value can be obtained by a maximally correlated classical bit, defined by the equal mixture of states $|00\rangle$ and $|11\rangle$.

We remember that, in order to show that these functionals are proper correlation measures, we must prove that they are (i) monotone under local operations (see Section 2.5.7). In Appendix B.2 we prove that this fundamental monotonicity property holds for C and $C^{(n)}$, for any $n \geq 2$. Moreover, we can prove that these correlations are (ii) zero-valued for product states and (iii) non-negative. First, we prove that property (ii) holds for $C_A^{(2)}$. For any product state $\rho_{AB} = \rho_A \otimes \rho_B$ and ME-POVM on A, the equiprobable output states $\rho_{B,1}$ and $\rho_{B,2}$ are identical. Hence, $\rho_{B,1} = \rho_{B,2} = \rho_B$ and $C_A^{(2)}(\rho_A \otimes \rho_B) = \|\rho_{B,1} - \rho_{B,2}\|/4 = 0$. The generalizations to prove that (ii) is valid also for $C^{(n)}$ for any $n \geq 2$ and C are obvious. Consequently, property (iii) is trivial.

While we have defined a whole class of correlation measures, in the following we focus on the potential of $C^{(2)}$ to witness non-Markovian dynamics. Therefore, unless otherwise specified the correlation measure referred to is $C^{(2)}$.

4.3 Witnessing non-Markovian dynamics

We now show how to use the correlation measures introduced above to detect non-Markovian evolutions. We prove that for any evolution that is at most point-wise non-bijective, we can find an initial state $\rho_{AB}^{(\tau)}(0)$ such that $C^{(2)}(\rho_{AB}^{(\tau)}(t))$ increases between time $t = \tau$ and $t = \tau + \Delta t$ if and only if there is no CP intermediate map $V_{\tau+\Delta t,\tau}$. By "at most point-wise non-bijective" evolutions we refer to evolutions where multiple initial states are mapped to the same state by Λ_t for at most a discrete set of times $t \in \{t_i\}_i$. Although our method applies to any bijective or pointwise non-bijective evolution, at the moment we are unable to extend the proof to non-Markovian evolutions that are non-bijective in finite time intervals. Note however that the set of non-Markovian evolutions not covered by our result has zero measure in the space of evolutions. More precisely, if we take an evolution that is non-bijective in a finite time interval and add a perturbation chosen at random with respect to a Borel measure, this yields an at most point-wise non-bijective evolution with probability one [OY05].

To take full advantage of this measure, we extend the standard setting and consider a scenario where A is an ancillary qubit and $B = SA$ is composed of the system S undergoing evolution and a suitably chosen ancilla A', see Fig.2.3. Hence, following the scheme introduce in Section 2.5, in this chapter we focus on the non-Markovian witness $Q = C^{(2)}$ and its potential to show backflows once that the bipartite system $A-B$ is initialized in a precise state $\rho_{AB}^{(\tau)}(0)$. First, we show how to construct the state $\rho_{AB}^{(\tau)}(0)$ to be used as a probe, namely the initial condition. Second, we show that for the class of non-Markovian dynamics specified above, $C^{(2)}(\rho_{AB}^{(\tau)}(t))$ provides a correlation backflow every time an at most pointwise non-bijective non-Markovian $\boldsymbol{\Lambda}$ evolves $\rho_{AB}^{(\tau)}(0)$.

4.3.1 The probe

Let $\boldsymbol{\Lambda}$ represent a bijective or pointwise non-bijective non-Markovian evolution that acts on the system S and introduce an ancillary system A'. Hence, we call B the complete $A'S$ system, where the corresponding Hilbert space is $\mathcal{H}_B = \mathcal{H}_{A'} \otimes \mathcal{H}_S$. As we discussed in Section 2.5.4, for any of these dynamics we can construct a class of pairs of initial states in B, namely $\{\rho_B'^{(\tau)}(0), \rho_B''^{(\tau)}(0)\} \in S(\mathcal{H}_B) = S(\mathcal{H}_{A'} \otimes \mathcal{H}_S)$, that show an increase in distinguishability between time $t = \tau$ and $t = \tau + \Delta t$

$$\left\|\rho_B'^{(\tau)}(\tau + \Delta t) - \rho_B''^{(\tau)}(\tau + \Delta t)\right\|_1 > \left\|\rho_B'^{(\tau)}(\tau) - \rho_B''^{(\tau)}(\tau)\right\|_1 , \tag{4.13}$$

if and only if there is no CP intermediate map $V_{\tau+\Delta t,\tau}$, where the evolution of the system B is given by the $I_{A'} \otimes \boldsymbol{\Lambda} = \{I_{A'} \otimes \Lambda_t\}_{t\geq 0}$ and $I_{A'}$ is the identity map on A'. We underline that the parameter τ that appears in the definition of $\rho_B'^{(\tau)}(0)$ and $\rho_B'^{(\tau)}(0)$ corresponds to the time for which its proved to witness non-Markovianity through Eq. (4.13), namely in the time interval $[\tau, \tau + \Delta\tau]$. Indeed, a different construction of these states is needed for each τ.

The particular initial bipartite separable states $\rho_{AB}^{(\tau)}(0)$ for which we examine the correlation $C^{(2)}$ are classical-quantum states that belong to $S(\mathcal{H}_A \otimes \mathcal{H}_B)$, where A is an ancillary qubit. Indeed, we define our initial probe state with the following "flagged" structure

$$\rho_{AB}^{(\tau)}(0) \equiv \frac{1}{2}\left(|0\rangle\langle 0|_A \otimes \rho_B'^{(\tau)}(0) + |1\rangle\langle 1|_A \otimes \rho_B''^{(\tau)}(0)\right) , \tag{4.14}$$

where $\mathcal{B}_A \equiv \{|0\rangle_A, |1\rangle_A\}$ is an orthonormal basis for the Hilbert space $\mathcal{H}_A$ of the qubit A and $\rho_B'^{(\tau)}(0)$ and $\rho_B''^{(\tau)}(0)$ are the corresponding initial states of those appearing in Eq. (4.13). Notice that, since we are considering bijective or pointwise non-bijective evolutions, we can always obtain those states through the application of $I_{A'} \otimes \Lambda_\tau^{-1}$ (see Section 2.5.4 for a discussion about the cases where τ is a time of non-invertibility).

The system B is the only component involved in the evolution. Therefore, $\rho_{AB}^{(\tau)}(t)$ assumes the same flagged structure of $\rho_{AB}^{(\tau)}(0)$ at any $t \geq 0$:

$$\rho_{AB}^{(\tau)}(t) \equiv \frac{1}{2}\left(|0\rangle\langle 0|_A \otimes \rho_B'^{(\tau)}(t) + |1\rangle\langle 1|_A \otimes \rho_B''^{(\tau)}(t)\right), \tag{4.15}$$

where $\rho_{AB}^{(\tau)}(t) = I_A \otimes I_{A'} \otimes \Lambda_t(\rho_{AB}^{(\tau)}(0))$. Note that from Eq. (4.15) it follows that $\rho_{AB}^{(\tau)}(t)$ does not contain any entanglement for all $t \geq 0$. Moreover, the state can be chosen arbitrarily close to an uncorrelated state since, as shown in [BJA17], one can always choose states $\rho_B'^{(\tau)}(0)$ and $\rho_B''^{(\tau)}(0)$ to be arbitrarily close to each other.

4.3.2 Detecting the correlation backflow

We now show how the correlation measure $C_A^{(2)}(\rho_{AB}^{(\tau)}(t))$ witnesses bijective or pointwise non-bijective non-Markovian dynamics. Moreover, we show that the same result can be obtained also for $C^{(2)}(\rho_{AB}^{(\tau)}(t))$.

To evaluate $C_A^{(2)}(\rho_{AB}^{(\tau)}(t))$, we have to find a ME-POVM $\{P_{A,1}, P_{A,2}\}$ that, once applied on $\rho_{AB}^{(\tau)}(t)$, generates an output ensemble of states $\{\{p_{1,2} = 1/2\}, \{\rho_{B,1}(t), \rho_{B,2}(t)\}\}$ with the largest value of $\|\rho_{B,1}(t) - \rho_{B,2}(t)\|_1$. Let $\lambda \in [0,1]$ and $\eta \in [0,1]$ be the diagonal elements of $P_{A,1}$ in the basis $\mathcal{B}_A$. It is easy to show that $\lambda + \eta = 1$ is a necessary condition for ME-POVMs. The corresponding output states parameterized by λ and η are

$$\rho_{B,1}(t) = \lambda\rho_B'^{(\tau)}(t) + \eta\rho_B''^{(\tau)}(t), \tag{4.16}$$

$$\rho_{B,2}(t) = (1-\lambda)\rho_B'^{(\tau)}(t) + (1-\eta)\rho_B''^{(\tau)}(t). \tag{4.17}$$

It follows that

$$\|\rho_{B,1}(t) - \rho_{B,2}(t)\|_1 = |\lambda - \eta|\, \|\rho_B'^{(\tau)}(t) - \rho_B''^{(\tau)}(t)\|_1 . \tag{4.18}$$

Since $0 \leq |\lambda - \eta| \leq 1$, the maximum is obtained when either λ or η is equal to 1. In both cases the equiprobable output states are exactly the states $\rho_B'^{(\tau)}(t)$ and $\rho_B''^{(\tau)}(t)$ and we can consider Eq. (4.8) in order to write $C_A^{(2)}(\rho_{AB}^{(\tau)}(t))$ in following simple form

$$C_A^{(2)}(\rho_{AB}^{(\tau)}(t)) = \frac{\|\rho_B'^{(\tau)}(t) - \rho_B''^{(\tau)}(t)\|_1}{4} . \tag{4.19}$$

In Appendices B.3 and B.5 we prove the following relation $C^{(2)}(\rho_{AB}^{(\tau)}(t)) = C_A^{(2)}(\rho_{AB}^{(\tau)}(t)) \geq C_B^{(2)}(\rho_{AB}^{(\tau)}(t))$. Therefore, using Eqs. (4.13) and (4.19), we obtain the main result of this chapter, namely a correlation backflow

$$C^{(2)}\left(\rho_{AB}^{(\tau)}(\tau + \Delta t)\right) > C^{(2)}\left(\rho_{AB}^{(\tau)}(\tau)\right), \tag{4.20}$$

if and only if there is no CPTP intermediate map $V_{\tau+\Delta t,\tau}$.

In Appendices B.4 and B.5 we prove that $C_B(\rho_{AB}^{(\tau)}(t)) = C_B^{(2)}(\rho_{AB}^{(\tau)}(t))$ and $C_A(\rho_{AB}^{(\tau)}(t)) = C_A^{(2)}(\rho_{AB}^{(\tau)}(t))$. From these additional results it follows that for this initial probe state $C^{(2)}(\rho_{AB}^{(\tau)}(t)) = C(\rho_{AB}^{(\tau)}(t))$ at any time $t \geq 0$. Hence, once the system is initialized in $\rho_{AB}^{(\tau)}(0)$, the correlation $C(\rho_{AB}^{(\tau)}(t))$ witnesses non-Markovianity with the same efficiency as $C^{(2)}(\rho_{AB}^{(\tau))}(t)$.

Example: the quasi-eternal non-Markovian model

For the sake of clarity, we illustrate the previous general results through a specific evolution. Let us consider the example introduced in Section 3.2.1, where the corresponding dynamical maps $\Lambda_t^{(t^{NM},\alpha)}$ are characterized by some $\alpha > 0$ and $t^{NM} \geq 0$ that satisfty the relations (3.9) and (3.10). Moreover, we recall that in Section 3.5.5 we showed that the QMI fails to detect some non-Markovian evolutions belonging to this class.

The parameter t^{NM} represents the time when the evolution $\mathbf{\Lambda}^{(t^{NM},\alpha)}$ starts to be non-Markovian, namely such that the intermediate maps $V_{\tau+\Delta t,\tau}$ are not CPTP for any $t^{NM} < \tau < \tau + \Delta t$. Hence, we focus on the construction of the initial states $\rho_B^{\prime(\tau)}(0)$ and $\rho_B^{\prime\prime(\tau)}(0)$ that appear in Eqs. (4.13) when the evolution is given by $\mathbf{\Lambda}^{(t^{NM},\alpha)}$. We pick $\tau > t^{NM}$ so that $\rho_{AB}^{(\tau)}(0)$ given by Eq. (4.14) is able to witness non-Markovian phenomena for time intervals $[\tau, \tau + \Delta t]$ when the evolution has no CPTP intermediate maps. By following the constructive method given in [BJA17], we have to consider, together with the qubit S evolved by $\mathbf{\Lambda}^{(t^{NM},\alpha)}$, an ancillary qutrit A': $\mathcal{S}(\mathcal{H}_B) = \mathcal{S}(\mathcal{H}_{A'} \otimes \mathcal{H}_S)$. Now, being $\{|0\rangle_{A'}, |1\rangle_{A'}, |2\rangle_{A'}\}$ and $\{|0\rangle_S, |1\rangle_S\}$ orthonormal basis respectively for $\mathcal{H}_{A'}$ and $\mathcal{H}_S$, we have:

$$\rho_{A'S}^{\prime(\tau)}(\tau) = (1-p)\sigma_{A'S} + p\phi_{A'S}^{+}, \tag{4.21}$$

$$\rho_{A'S}^{\prime\prime(\tau)}(\tau) = (1-p)\sigma_{A'S} + p|2\rangle\langle 2|_{A'} \otimes \rho_S, \tag{4.22}$$

where $\phi_{A'S}^{+} \equiv |\phi^+\rangle\langle\phi^+|_{A'S}$ is the maximally entangled state, $|\phi^+\rangle_{A'S} \equiv (|00\rangle_{A'S} + |11\rangle_{A'S})/\sqrt{2}$ and $\sigma_{A'S}$ is a state in the interior of $\mathrm{Im}(\Lambda_\tau^{(t^{NM},\alpha)})$.

In order to define completely $\rho'^{(\tau)}_{A'S}(\tau)$ and $\rho''^{(\tau)}_{A'S}(\tau)$, we fix their free components: $\sigma_{A'S} \equiv (|0\rangle\langle 0|_{A'} + |1\rangle\langle 1|_{A'})/2 \otimes \mathbb{1}_S/2$ and $\rho_S \equiv \mathbb{1}_S/2$ and we get:

$$\rho'^{(\tau)}_{A'S}(\tau) = \frac{(|0\rangle\langle 0|_{A'} + |1\rangle\langle 1|_{A'}) \otimes \mathbb{1}_S}{4} + p\,\frac{\sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z}{4}\,, \quad (4.23)$$

$$\rho''^{(\tau)}_{A'S}(\tau) = \left((1-p)\frac{|0\rangle\langle 0|_{A'} + |1\rangle\langle 1|_{A'}}{2} + p|2\rangle\langle 2|_{A'}\right) \otimes \frac{\mathbb{1}_S}{2}\,. \quad (4.24)$$

Considering the rates given in Eq. (3.11), the evolution induced by the dynamical map $\Lambda^{(t^{NM},\alpha)}_{\tau}$ (see Eqs. (2.111)) that precedes the action of $V^{(t^{NM},\alpha)}_{\tau+\Delta t,\tau}$, can be written as:

$$\begin{aligned}
\Lambda^{(t^{NM},\alpha)}_{\tau}(\sigma_x) &= \left(e^{-\tau}\frac{\cosh(\tau - t^{NM})}{\cosh(t^{NM})}\right)^{\alpha/2} \sigma_x \equiv \lambda^{(t^{NM},\alpha)}_{xy}(\tau)\,\sigma_x\,,\\
\Lambda^{(t^{NM},\alpha)}_{\tau}(\sigma_y) &= \left(e^{-\tau}\frac{\cosh(\tau - t^{NM})}{\cosh(t^{NM})}\right)^{\alpha/2} \sigma_y \equiv \lambda^{(t^{NM},\alpha)}_{xy}(\tau)\,\sigma_y\,,\\
\Lambda^{(t^{NM},\alpha)}_{\tau}(\sigma_z) &= e^{-\alpha\tau}\sigma_z \equiv \lambda^{(\alpha)}_{z}(\tau)\,\sigma_z\,,\\
\Lambda^{(t^{NM},\alpha)}_{\tau}(\mathbb{1}_S) &= \mathbb{1}_S\,,
\end{aligned} \quad (4.25)$$

where, for $\tau > t^{NM}$, we have $\lambda^{(t^{NM},\alpha)}_{xy}(\tau) > \lambda^{(\alpha)}_{z}(\tau)$. The state $\rho''^{(\tau)}_{A'S}(\tau)$ assumes the form $\rho_{A'} \otimes \mathbb{1}_S/2$ and therefore, since the evolution is random unitary, it is stationary for $I_{A'} \otimes \mathbf{\Lambda}^{(t^{NM},\alpha)}$. Therefore, $\rho''^{(\tau)}_{A'S}(0) = (I_{A'} \otimes \Lambda^{(t^{NM},\alpha)}_{t})^{-1}(\rho''^{(\tau)}_{A'S}(\tau)) = \rho''^{(\tau)}_{A'S}(\tau)$. Conversely, $\rho'^{(\tau)}_{A'S}(\tau)$ is not stationary and $(I_{A'} \otimes \Lambda^{(t^{NM},\alpha)}_{\tau})^{-1}(\rho'^{(\tau)}_{A'S}(\tau))$ is not physical for every $p \in [0,1]$. Indeed, we can write the operator $\rho'^{(\tau)}_{A'S}(0)$ obtained by $(\Lambda^{(t^{NM},\alpha)}_{\tau})^{-1}(\rho'^{(\tau)}_{A'S}(\tau))$ as follows

$$\begin{aligned}
\rho'^{(\tau)}_{A'S}(0) \;=\; & \frac{(|0\rangle\langle 0|_{A'} + |1\rangle\langle 1|_{A'}) \otimes \mathbb{1}_S}{4} + \frac{p}{\lambda^{(t^{NM},\alpha)}_{xy}(\tau)}\frac{\sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y}{4}\\
+\; & \frac{p}{\lambda^{(\alpha)}_{z}(\tau)}\frac{\sigma_z \otimes \sigma_z}{4}\,,
\end{aligned} \quad (4.26)$$

which represents physical state for $p/\lambda^{(t^{NM},\alpha)}_{xy}(\tau) < p/\lambda^{(\alpha)}_{z}(\tau) \leq 1$ (see Section 3.5.5). Therefore, if p satisfies the condition $p < \lambda^{(\alpha)}_{z}(\tau)$, the operator given by Eq. (4.26) represents a physical initial state $\rho'^{(\tau)}_{A'S}(0) \in S(\mathcal{H}_{A'} \otimes \mathcal{H}_S)$ which at time τ is evolved to the state given in Eq. (4.23), and therefore, together with $\rho''^{(\tau)}_{A'S}(0)$, fulfills the requirements of the constructive method given in [BJA17].

The construction of the state $\rho_{AB}^{(\tau)}(0)$ through Eq. (4.14) is now straightforward. Being A an ancillary qubit for which we adopt the orthonormal basis $\{|0\rangle\langle 0|_A, |1\rangle\langle 1|_A\}$, we have

$$\rho_{AB}^{(\tau)}(0) = \frac{1}{2}\left(|0\rangle\langle 0|_A \otimes \rho_B'^{(\tau)}(0) + |1\rangle\langle 1|_A \otimes \rho_B''^{(\tau)}(0)\right), \tag{4.27}$$

where $\rho_B'^{(\tau)}(0)$ is given in Eq. (4.26) and $\rho_B''^{(\tau)}(0)$ in Eq. (4.24).

4.3.3 Quasi-correlation measures

We would like to conclude our study by discussing the use of what we called quasi-correlation measures in the context of non-Markovian detection (see Section 2.5.8). Note that while for a correlation measure it is demanded that it does not increase by local operations, for the detection of non-Markovianity it suffices to consider functions that do not increase under the action of operations by one of the parties (the one evolving through the dynamics). An increase on the value of these measures is enough to detect non-Markovian evolutions. We name quasi-correlation measures those functions of a bipartite state that do not increase when applying operations on only one share of the state.

An example of such measures is the quantum correlation (or singlet fraction) $q_{corr}(\rho_{SA})$ [KRS09]. As we saw in Section 2.5.8, given a bipartite state ρ_{SA}, it is defined as

$$q_{corr}(\rho_{SA}) = d_A \max_{\Phi_S} \langle \phi^+ | \Phi_S \otimes \mathbb{1}_A(\rho_{SA}) | \phi^+ \rangle_{SA}^2. \tag{4.28}$$

This quantity detects all non-Markovian dynamics. Indeed, in [KRS09] the authors showed that for classical-quantum correlated states of the form

$$\rho_{SA} = \sum_i p_i \, \rho_{S,i} \otimes |i\rangle\langle i|_A, \tag{4.29}$$

q_{corr} is equal to the guessing probability of the ensemble $\mathcal{E}_S = \{p_i, \rho_{S,i}\}_i$. Therefore, we can combine this with the results in Ref. [BD16] (see Section 2.5.5), proving the existence of an ensemble with increasing guessing probability for any non-Markovian dynamics to conclude that this version of the singlet fidelity also detects all such dynamics (see also Ref. [Bus17]).

4.4 Discussion

The main motivation of this chapter is to understand the power of correlations to witness non-Markovian evolutions. We introduced a correlation measure and showed that, in an extended setting with a second ancilla, it displays backflow for almost all non-Markovian evolutions. More precisely, it displays backflows for all non-Markovian evolutions that are bijective or at most point-wise non-bijective. For a given evolution we described how probe states that exhibit such an increase in correlations can be constructed. These states have no entanglement across the given bipartition and can be chosen to be arbitrarily close to an uncorrelated state. We showed how to apply our method to a set of evolutions, namely eternal and quasi-eternal non-Markovian evolutions, by explicitly constructing all the components of the probe states. Finally, we reviewed quasi-correlation measures that can be used for non-Markovianity detection and always show a backflow.

The question if there exists a measure of correlation with the property of being non-increasing if and only if the dynamics is CP-divisible, without any restrictions on the dynamics, is still open, both in the case of system-ancilla correlations and in the extended setting with a second ancilla. A possible avenue consists of understanding how to adapt the results in [BD16], valid for any non-Markovian evolution, to our correlation measure.

Chapter 5

Equivalence between non-Markovianity and correlation backflows

The information encoded into an open quantum system evolving with a Markovian dynamics is always monotonically non-increasing. Nonetheless, for a given quantifier of the information contained in the system, it is in general not clear if for all non-Markovian dynamics it is possible to observe a non-monotonic evolution of this quantity, namely a backflow. We address this problem by considering correlations of finite-dimensional bipartite systems. For this purpose, we consider a class of correlation measures and prove that if the dynamics is non-Markovian there exists at least one element from this class that provides a correlation backflow. Moreover, we provide a set of initial probe states that accomplish this witnessing task. This result provides the first one-to-one relation between non-Markovian quantum dynamics and correlation backflows. Finally, we introduce a new measure of non-Markovianity. The results exposed in this chapter are contained in the original works [DJB$^+$20] and [DJ20].

5.1 Introduction

The study of open quantum systems dynamics [BP07, RH11] is of central interest in quantum mechanics. Since there are no experimental scenarios where a quantum system can be considered completely isolated, this approach provides a more realistic description of quantum evolutions.

The interaction between an open quantum system S and its environment E

leads to two possible regimes of evolution. The phenomena associated with the *Markovian* regime are characterized by the monotonic non-increase of the information contained in the open system. Instead, in the *non-Markovian* regime, part of the information lost is recovered in one or more subsequent time intervals. This phenomenon is called *backflow* of information. For some detailed reviews on non-Markovian evolutions, see Refs. [BLPV16, dVA17, RHP14, LHW18].

It is nonobvious what mathematical structure is better suited to reproduce the backflow phenomenology. A framework based on a notion of *divisibility* of dynamical maps, namely the operators describing the dynamical evolution of the system, has achieved a promising consensus [RH11, TLSM18, ABCM14, CKR11, RHP10, BLP10, BD16, BJA17, LFS12]. A characterizations of non-Markovian evolutions based on divisibility is proposed in Ref. [CM14], where the authors introduce a degree of non-Markovianity to classify evolutions. Many efforts are presently directed towards testing this mathematical definition by studying the characteristic backflows that different physical quantities show when the evolution is non-Markovian. Once we consider a quantity that is non-increasing under Markovian evolutions, we can study its potential to show a backflow when the dynamics is non-Markovian. Distinguishability between states [BLP10, BD16, BJA17], correlation measures [LFS12, DJB$^+$19, DJB$^+$20, KRS20], channel capacities [BCM14] and the volume of accessible states [LPP13] and quantum Fisher information [LWS10] are some quantities that have been studied in this scenario. The non-trivial point that has to be analyzed is if it is possible to obtain one-to-one relations between backflows of these quantities and non-Markovian evolutions. Indeed, this result would imply a correspondence between the phenomenological and the mathematical description of non-Markovianity that we have presented. In Ref. [CKR11] it was suggested that for bijective evolutions there is a one-to-one correspondence between backflow of the distinguishability of two-state ensembles and non-Markovianity. This correspondence follows from the results of [Kos72a, Rus94] together with an addendum given in Ref. [BJA17]. Later it was shown in Ref. [BD16] that for general evolutions a one-to-one correspondence exists between non-Markovianity and backflow of the guessing probability for some ensemble of states. Furthermore, it was shown in Ref. [BJA17] that for evolutions that are non-bijective for at most a discrete set of times there is a one-to-one correspondence between backflows of the distinguishability of an equiprobable two-state ensemble and non-Markovianity.

In this chapter we focus on the connection between revivals of bipartite correlation measures and non-Markovian evolutions on S. Several measures have been considered in this scenario, e.g. quantum mutual information [LFS12, DJB$^+$20] and entanglement measures [RHP10, KRS20]. In the previous chapter we showed how to introduce a correlation that witnesses almost all non-Markovian dynamics. However, it is unknown if any of these correlations can witness all non-Markovian dynamics [DJB$^+$20].

The main result of this chapter is the first one-to-one relation between correlation backflows and non-Markovian dynamics. We consider a class of bipartite correlation measures that provides backflows if and only if the dynamics is not Markovian. For this purpose, we exploit supplementary ancillary systems to define initial probe states that succeed in this witnessing task. Finally, by considering the maximum backflow that these correlation measures can show when bipartite states evolve, we introduce a class of non-Markovianity measures. We prove that for any non-Markovian evolution there exists at least one measure from this class that is positive.

5.2 Measurements having fixed output probability distributions

As we discussed in Section 2.1.2, any measurement process on a quantum state $\rho \in S(\mathcal{H})$ is defined by a *positive-operator valued measure* (POVM), namely an indexed set of Hermitian and positive semi-definite operators $\{P_i\}_{i=1}^n$ of $B(\mathcal{H})$ such that $\sum_{i=1}^n P_i = \mathbb{1}$, where $\mathbb{1} \in B(\mathcal{H})$ is the identity operator on $\mathcal{H}$ and n is the number of possible outcomes. The i-th output of the measurement is represented by P_i, where $p_i = \mathrm{Tr}[\rho P_i]$ is the corresponding occurrence probability.

We consider (normalized) finite probability distributions $\mathcal{P} = \{p_i\}_{i=1}^n$, where $\sum_{i=1}^n p_i = 1$ and define the set of n-output POVMs that, if applied on $\rho \in S(\mathcal{H})$, provide $\mathcal{P}$-distributed outcomes.

Definition 10. *Given the finite probability distribution $\mathcal{P} = \{p_i\}_{i=1}^n$, the set of $\mathcal{P}$-POVMs $\{P_i\}_{i=1}^n$ on $\mathcal{H}$ for $\rho \in S(\mathcal{H})$ is defined as*

$$\Pi^{\mathcal{P}}(\rho) \equiv \{\{P_i\}_{i=1}^n : Tr[\rho P_i] = p_i, \forall i = 1, \dots, n\}.$$

These sets of POVMs generalize the idea of ME-POVMs presented in Chapter 4, where only uniform distributions $\mathcal{P} = \{p_i = 1/n\}_{i=1}^n$ were considered. We prove that $\Pi^{\mathcal{P}}(\rho) \neq \emptyset$ for all $\mathcal{P}$ and ρ in Appendix C.1.

Now we consider a bipartite scenario where Alice and Bob share a state $\rho_{AB} \in S(\mathcal{H}_A \otimes \mathcal{H}_B)$. If Alice applies a POVM $\{P_{A,i}\}_{i=1}^n$ on her side of ρ_{AB},

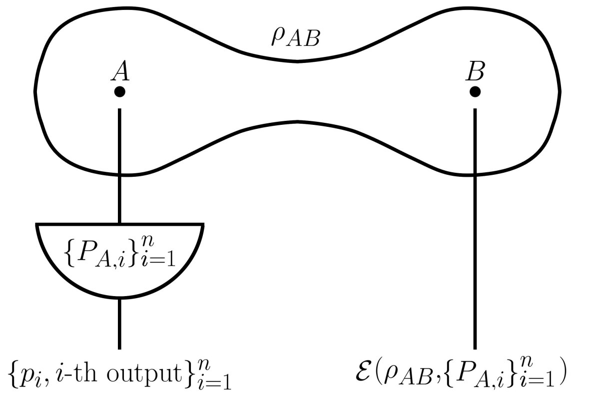


Figure 5.1: Given a probability distribution $\mathcal{P} = \{p_i\}_{i=1}^n$, $\{P_{A,i}\}_{i=1}^n$ is a $\mathcal{P}$-POVM for ρ_{AB} if and only if the output probability distribution of this measurement is $\mathcal{P}$. The correlation $C_A^{\mathcal{P}}(\rho_{AB})$ considers the scenario where ρ_{AB} is measured with a $\mathcal{P}$-POVM $\{P_{A,i}\}_{i=1}^n$ that generates the most distinguishable $\mathcal{P}$-distributed ensemble on B.

an *output ensemble* $\mathcal{E}(\rho_{AB}, \{P_{A,i}\}_{i=1}^n) \equiv \{p_i, \rho_{B,i}\}_{i=1}^n$ is generated on Bob's side, where each state $\rho_{B,i} \in S(\mathcal{H}_B)$ occurs with probability p_i as follows

$$\mathcal{E}\left(\rho_{AB}, \{P_{A,i}\}_{i=1}^n\right) = \left\{ p_i = \mathrm{Tr}\left[\rho_{AB} P_{A,i} \otimes \mathbb{1}_B\right], \rho_{B,i} = \frac{\mathrm{Tr}_A\left[\rho_{AB} P_{A,i} \otimes \mathbb{1}_B\right]}{p_i} \right\}. \tag{5.1}$$

In particular, with probability p_i, Alice obtains the i-th outcome of the measurement and Bob's side of the shared state is transformed into $\rho_{B,i}$. We call $\{p_i\}_{i=1}^n$ and $\{\rho_{B,i}\}_{i=1}^n$ respectively the output probability distribution and the output states of the measurement.

Similarly to $\Pi^{\mathcal{P}}(\rho)$, we define the measurements that Alice can perform on ρ_{AB} to generate $\mathcal{P}$-distributed output ensembles on Bob's side (see Fig. 5.1).

Definition 11. *Given the finite probability distribution $\mathcal{P} = \{p_i\}_{i=1}^n$, the set of $\mathcal{P}$-POVMs $\{P_{A,i}\}_{i=1}^n$ on $\mathcal{H}_A$ for $\rho_{AB} \in S(\mathcal{H}_{AB})$ is defined as*

$$\Pi_A^{\mathcal{P}}(\rho_{AB}) \equiv \{\{P_{A,i}\}_{i=1}^n : Tr\left[\rho_{AB} P_{A,i} \otimes \mathbb{1}_B\right] = p_i, \forall i = 1, \ldots, n\}.$$

Analogously, we can define $\Pi_B^{\mathcal{P}}(\rho_{AB})$. We notice that $\Pi_A^{\mathcal{P}}(\rho_{AB}) = \Pi^{\mathcal{P}}(\rho_A)$ for any $\mathcal{P}$ and ρ_{AB}, where $\rho_A = \mathrm{Tr}_B\left[\rho_{AB}\right]$. Moreover, $\Pi^{\mathcal{P}}(\rho)$ $(\Pi_A^{\mathcal{P}}(\rho_{AB}))$ is a non-empty convex set for any ρ (ρ_{AB}) and $\mathcal{P}$.

5.3 Non-Markovianity and the guessing probability

We consider the task of identifying a state that we randomly extract from a known ensemble $\mathcal{E} = \{p_i, \rho_i\}_{i=1}^n$ of states of $S(\mathcal{H})$. The guessing probability $P_g(\mathcal{E})$ is the average probability to successfully identify the extracted state with an optimal measurement, that is

$$P_g(\mathcal{E}) \equiv \max_{\{P_i\}_{i=1}^n} \sum_{i=1}^{n} p_i \operatorname{Tr}[\rho_i P_i] \ , \tag{5.2}$$

where the maximization is performed over the n-output POVMs of $B(\mathcal{H})$. Notice that $P_g(\mathcal{E}) \geq p_{max} \equiv \max_i p_i$. Indeed, the best strategy that can be adopted when no measurement is performed corresponds to guess on the most probable state. Therefore, when we collect information from a measurement, we can only improve our knowledge about the extracted state. We discussed this quantity and its potential to witness non-Markovianity in Section 2.5.5, while here we remind some aspects that are needed in this chapter. Note that $P_g(\mathcal{E})$ can be used to define a witness of non-Markovianity. Indeed, it is non-increasing under the action of any CPTP map Φ acting on the states ρ_i: $P_g(\{p_i, \rho_i\}_i) \geq P_g(\{p_i, \Phi(\rho_i)\}_i)$.

Now we explain how we can use the guessing probability to witness *any* non-Markovian dynamics [BD16]. We consider a finite-dimensional system $S - A$, where the d-dimensional system S is evolved by a generic evolution $\mathbf{\Lambda}$ and A is an ancillary system. By evolving an initial ensemble $\mathcal{E}_{SA}(0) = \{p_i, \rho_{SA,i}(0)\}_i$ of states $\rho_{SA,i} \in S(\mathcal{H}_{SA})$, we obtain $\mathcal{E}_{SA}(t) = \{p_i, \Lambda_t \otimes I_A(\rho_{SA,i}(0))\}_i$. Therefore, if the evolution $\mathbf{\Lambda}$ is Markovian, $P_g(\mathcal{E}_{SA}(t)) - P_g(\mathcal{E}_{SA}(s)) \leq 0$, for any time interval $[s, t]$.

The authors of [BD16] show that, for any evolution $\mathbf{\Lambda}$ and time interval $[s, t]$, there exist an ancillary system A and an initial ensemble $\overline{\mathcal{E}}_{SA}(0)$ of separable states of $S(\mathcal{H}_{SA})$

$$\overline{\mathcal{E}}_{SA}(0) \equiv \{\overline{p}_i, \overline{\rho}_{SA,i}\}_{i=1}^{\overline{n}} \,, \tag{5.3}$$

such that we have a backflow

$$P_g(\overline{\mathcal{E}}_{SA}(t)) - P_g(\overline{\mathcal{E}}_{SA}(s)) > 0 \,, \tag{5.4}$$

if and only if there exists no CPTP intermediate map $V_{t,s}$ for the time interval $[s, t]$. Notice that Eq. (5.4) corresponds to a violation of the Markovian condition (2.59). Regarding the details of $\overline{\mathcal{E}}_{SA}(0)$, the probability distribution $\overline{\mathcal{P}} \equiv \{\overline{p}_i\}_{i=1}^{\overline{n}}$ has a finite size of $\overline{n} \leq d^4$ elements and $\dim(\mathcal{H}_A) \leq d$. Notice that, even if we do not make it explicit, $\overline{\mathcal{E}}_{SA}(0)$ depends on $\mathbf{\Lambda}$ and $[s,t]$. This result is completely general, it applies to any finite-dimensional evolution and, while it does not provide the explicit states and probabilities needed to define $\overline{\mathcal{E}}_{SA}(0)$, it proves the first one-to-one relation between information backflows and non-Markovianity.

5.4 A class of correlation measures

In this section we define the correlation measures that we use to prove that correlation backflows occur for all non-Markovian evolutions. Let $\mathcal{P} \equiv \{p_i\}_i$ be a generic probability distribution and $\rho_{AB} \in S(\mathcal{H}_A \otimes \mathcal{H}_B)$ a generic bipartite system state. We consider the correlation measure

$$C_A^{\mathcal{P}}(\rho_{AB}) \equiv \max_{\{P_{A,i}\}_i \in \Pi_A^{\mathcal{P}}(\rho_{AB})} P_g\left(\rho_{AB}, \{P_{A,i}\}_i\right) - p_{max}\,, \tag{5.5}$$

where the maximization is performed over the $\mathcal{P}$-POVMs on A for ρ_{AB} and we defined $P_g(\rho_{AB}, \{P_{A,i}\}_i) \equiv P_g(\mathcal{E}(\rho_{AB}, \{P_{A,i}\}_i))$ and $p_{max} = \max_i p_i$ (see Fig. 5.1). Therefore, we can consider the class of correlation measures where each element is defined by a different distribution $\mathcal{P}$. Notice that, if $\mathcal{P}_1$ can be transformed into $\mathcal{P}_2$ by a permutation and an addition or removal of one or more zero-valued probabilities, $C_A^{\mathcal{P}_1}(\rho_{AB}) = C_A^{\mathcal{P}_2}(\rho_{AB})$ for any ρ_{AB}.

The operational meaning of this correlation measure for a given $\mathcal{P}$ is the following. The value of $C_A^{\mathcal{P}}(\rho_{AB})$ corresponds to the guessing probability of the most distinguishable $\mathcal{P}$-distributed state ensembles of B that Alice can generate measuring her side of ρ_{AB}. We apply the $-p_{max}$ correction in order to remove the contribution coming from the no-measurement strategy discussed above. Then, $C_A^{\mathcal{P}}(\rho_{AB}^{(1)}) > C_A^{\mathcal{P}}(\rho_{AB}^{(2)})$ implies that the largest distinguishability of the $\mathcal{P}$-distributed output ensembles of B that Alice can generate measuring $\rho_{AB}^{(1)}$ is greater than the largest distinguishability of the $\mathcal{P}$-distributed output ensembles of B that Alice can generate measuring $\rho_{AB}^{(2)}$.

As pointed out in the previous chapter and in Section 2.5.7, in order to consider $C_A^{\mathcal{P}}$ a proper correlation measure, in Appendix C.2 we prove that (i) it is non-increasing under local operations. Moreover, we can also prove that it is (ii) zero-valued for product states and (iii) non-negative. In order to prove the former property, given a generic product state $\rho_{AB} = \rho_A \otimes \rho_B$, the output ensemble $\mathcal{E}(\rho_A \otimes \rho_B, \{P_{A,i}\}_i) = \{p_i, \rho_B\}_i$ is made of identical states for any POVM $\{P_{A,i}\}_i$ and $P_g(\{p_i, \rho_B\}_i) = p_{max}$. Therefore, $C_A^{\mathcal{P}}(\rho_{AB}) \geq 0$ is now trivial.

Similarly to $C_A^{\mathcal{P}}$, we define

$$C_B^{\mathcal{P}}(\rho_{AB}) \equiv \max_{\{P_{B,i}\}_i \in \Pi_B^{\mathcal{P}}(\rho_{AB})} P_g\left(\rho_{AB}, \{P_{B,i}\}_{i=1}^n\right) - p_{max}\,. \tag{5.6}$$

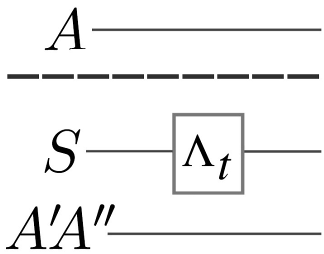


Figure 5.2: The initial probe state $\rho_{AB}^{(\lambda)}(0)$ belongs to the bipartite system $S(\mathcal{H}_A \otimes \mathcal{H}_B)$, where $\mathcal{H}_B = \mathcal{H}_S \otimes \mathcal{H}_{A'} \otimes \mathcal{H}_{A''}$. We consider the correlation $C_A^{\overline{\mathcal{P}}}(\rho_{AB}^{(\lambda)}(t))$ given by the bipartition between the subsystems A and B, where S is evolved by $\mathbf{\Lambda}$ and A, A' and A'' are ancillary systems.

Since in general $C_A^{\mathcal{P}}(\rho_{AB}) \neq C_B^{\mathcal{P}}(\rho_{AB})$, we can consider the symmetric class of measures

$$C_{AB}^{\mathcal{P}}(\rho_{AB}) \equiv \max\left\{C_A^{\mathcal{P}}(\rho_{AB}), C_B^{\mathcal{P}}(\rho_{AB})\right\} . \tag{5.7}$$

Finally, we notice that Eqs. (5.5), (5.6) and (5.7) are generalizations of the correlation measures introduced in Chapter 4. Moreover, we proved that by considering $C_{AB}^{\mathcal{P}}$ with $\mathcal{P} = \{1/2, 1/2\}$ it is possible to witness any bijective or pointwise non-bijective evolution, while a proof for generic non-Markovian evolutions was not provided.

5.4.1 The probe states

The goal of this chapter is to prove a one-to-one correspondence between non-Markovian evolutions and correlation backflows. Therefore, we consider the most general evolution $\mathbf{\Lambda}$ and we focus on a generic time interval $[\tau, \tau+\Delta\tau]$. We provide an *initial probe state* shared between Alice and Bob and a distribution $\mathcal{P}$ for which the correlation measure $C_A^{\mathcal{P}}$ shows a backflow between τ and $\tau+\Delta\tau$ if and only if there is no CPTP intermediate map $V_{\tau+\Delta\tau,\tau}$.

First, we introduce the bipartition and the state space needed to consider $C_A^{\mathcal{P}}$ and the initial probe state. We define the bipartite system $S(\mathcal{H}_A \otimes \mathcal{H}_B)$ such that $\dim(\mathcal{H}_A) = \overline{n}$ and $\mathcal{H}_B \equiv \mathcal{H}_S \otimes \mathcal{H}_{A'} \otimes \mathcal{H}_{A''}$, where $\dim(\mathcal{H}_S) = \dim(\mathcal{H}_{A'}) = d_S$ and $\dim(\mathcal{H}_{A''}) = \overline{n}+1$. We fix the following orthonormal basis for $\mathcal{H}_A$ and $\mathcal{H}_{A''}$: $\mathcal{M}_A \equiv \{|i\rangle_A\}_{i=1}^{\overline{n}} = \{|1\rangle_A, |2\rangle_A, \dots, |\overline{n}\rangle_A\}$ and $\mathcal{M}_{A''} \equiv \{|i\rangle_{A''}\}_{i=1}^{\overline{n}+1} = \{|1\rangle_{A''}, |2\rangle_{A''}, \dots, |\overline{n}+1\rangle_{A''}\}$. Notice that the ancillas A' and A'' can be considered as a single ancilla with Hilbert space $\mathcal{H}_{A'} \otimes \mathcal{H}_{A''}$ (see Fig. 5.2).

We define $\overline{\rho}_{B,i} \equiv \overline{\rho}_{SA',i} \otimes |\overline{n}+1\rangle\langle\overline{n}+1|_{A''} \in S(\mathcal{H}_B)$, for $i = 1, \dots, \overline{n}$, where we made use of the elements of $\overline{\mathcal{E}}_{SA'}(0) = \{\overline{p}_i, \overline{\rho}_{SA',i}\}_{i=1}^{\overline{n}}$ (see Eq. (5.3)). We introduce a class of initial probe states $\rho_{AB}^{(\lambda)}(0) \in S(\mathcal{H}_A \otimes \mathcal{H}_B)$ parameterized by $\lambda \in [0,1)$

$$\rho_{AB}^{(\lambda)}(0) \equiv \sum_{i=1}^{\overline{n}} \overline{p}_i \, |i\rangle\langle i|_A \otimes \left(\lambda \, \sigma_{SA'} \otimes |i\rangle\langle i|_{A''} + (1-\lambda)\, \overline{\rho}_{B,i}\right), \tag{5.8}$$

where $\sigma_{SA'}$ is a generic state of $S(\mathcal{H}_S \otimes \mathcal{H}_{A'})$. Notice that the index i runs from 1 to $\overline{n}$, while $\dim(\mathcal{H}_{A''}) = \overline{n}+1$. Since the ancillary systems do not evolve, the action of the dynamical map of the evolution on the probe state preserves the initial classical-quantum separable structure for any $t \geq 0$

$$\rho_{AB}^{(\lambda)}(t) = \sum_{i=1}^{\overline{n}} \overline{p}_i \, |i\rangle\langle i|_A \otimes \left(\lambda \, \sigma_{SA'}(t) \otimes |i\rangle\langle i|_{A''} + (1-\lambda)\, \overline{\rho}_{B,i}(t)\right), \tag{5.9}$$

where $\overline{\rho}_{B,i}(t) = \Lambda_t \otimes I_{A'A''}(\overline{\rho}_{B,i})$ and $\sigma_{SA'}(t) = \Lambda_t \otimes I_{A'}(\sigma_{SA'})$. Finally, since $\mathrm{Tr}_B[\rho_{AB}^{(\lambda)}(t)] = \sum_{i=1}^{\overline{n}} \overline{p}_i |i\rangle\langle i|_A$, the set of $\mathcal{P}$-POVMs $\Pi_A^{\mathcal{P}}(\rho_{AB}^{(\lambda)}(t)) = \Pi^{\mathcal{P}}(\mathrm{Tr}_B[\rho_{AB}^{(\lambda)}(t)])$ does not depend on t and λ.

5.4.2 Witnessing non-Markovianity with correlations

In the case of bijective or pointwise non-bijective $\mathbf{\Lambda}$, we can witness correlation backflows with the technique described in Section 4. Moreover, in Ref. [KRS20] it is proved that negativity, the entanglement measure given in Eq. (2.92), witnesses any non-Markovian qubit evolution. Now, we provide a proof for the possibility to witness any non-Markovian dynamics with a correlation backflow. If we consider the formalism introduced in Section 2.5, we can define the class of witnesses of non-Markovianity $Q_{A,\mathcal{P}}$ with the correlations $C_A^{\mathcal{P}}$, where the corresponding initial conditions are provided by the probe states $\rho_{AB}^{(\lambda)}(0)$. Later, we also introduce an associated measure of non-Markovianity $N_{\mathcal{P}}$, which we prove to be positive for any non-Markovian evolution.

In order to witness non-Markovianity through revivals of $C_A^{\overline{\mathcal{P}}}$, the evolution of the initial state $\rho_{ASA'} = \sum_{i=1}^{\overline{n}} \overline{p}_i |i\rangle\langle i|_A \otimes \overline{\rho}_{SA',i}$ is an intuitive choice. Indeed, we have that $\{|i\rangle\langle i|_A\}_{i=1}^{\overline{n}} \in \Pi_A^{\overline{\mathcal{P}}}(\rho_{ASA'}(t))$ for all $t \geq 0$ and $P_g(\rho_{ASA'}(t), \{|i\rangle\langle i|_A\}_{i=1}^{\overline{n}}) = P_g(\overline{\mathcal{E}}_{SA'}(t))$ (see Eq. (5.4)). Nonetheless, in general $\{|i\rangle\langle i|_A\}_{i=1}^{\overline{n}}$ is not selected by the maximization that defines $C_A^{\overline{\mathcal{P}}}(\rho_{ASA'}(t))$. In Appendix C.8 we study an explicit example where this situation is encountered.

We are now able to present the main result of this chapter.

Theorem 14. *For any evolution* $\boldsymbol{\Lambda}$ *defined on a finite-dimensional system* S *and time interval* $[\tau, \tau + \Delta\tau]$ *there exist at least one ancillary system* $\mathcal{H}$*, one bipartite system* $\mathcal{H}_A \otimes \mathcal{H}_B$*, where* $\mathcal{H}_B = \mathcal{H}_S \otimes \mathcal{H}$*, a correlation measure for bipartite systems* C_{AB} *and an initial state* $\rho_{AB}(0) \in S(\mathcal{H}_A \otimes \mathcal{H}_B)$ *such that a backflow*

$$C_{AB}\left(\rho_{AB}(\tau + \Delta\tau)\right) - C_{AB}\left(\rho_{AB}(\tau)\right) > 0\,,$$

occurs if and only if there is no CPTP intermediate map $V_{\tau+\Delta\tau,\tau}$*, where* S *is the only system that evolves during the evolution.*

Proof. We consider the ancillary system $\mathcal{H} = \mathcal{H}_{A'} \otimes \mathcal{H}_{A''}$, the correlation measure $C_{AB} = C_A^{\overline{\mathcal{P}}}$ and the initial probe states $\rho_{AB}^{(\lambda)}(0)$. We prove that, for wisely chosen values of λ, we obtain the backflow

$$\Delta C_A^{\overline{\mathcal{P}}} \equiv C_A^{\overline{\mathcal{P}}}\left(\rho_{AB}^{(\lambda)}(\tau + \Delta\tau)\right) - C_A^{\overline{\mathcal{P}}}\left(\rho_{AB}^{(\lambda)}(\tau)\right) > 0\,, \tag{5.10}$$

if and only if there is no CPTP intermediate map $V_{\tau+\Delta\tau,\tau}$. As stated before, in Appendix C.2 we prove that $C_A^{\overline{\mathcal{P}}}$ is monotonically decreasing under local operations. It follows that $C_A^{\overline{\mathcal{P}}}(\rho_{AB}^{(\lambda)}(\tau))$ cannot increase in case of CPTP intermediate maps $V_{\tau+\Delta\tau,\tau}$ acting on $\rho_{AB}^{(\lambda)}(\tau)$. Moreover, any correlation measure C_{AB} is by definition monotonically decreasing under local operations. Therefore, in order to prove Theorem 14, we follow by studying the cases where there is no CPTP intermediate map $V_{\tau+\Delta\tau,\tau}$.

First, $\Pi_A^{\overline{\mathcal{P}}} \equiv \Pi_A^{\overline{\mathcal{P}}}(\rho_{AB}^{(\lambda)}(t))$ does not depend on λ or t, and $\{|i\rangle\langle i|_A\}_{i=1}^{\overline{n}} \in \Pi_A^{\overline{\mathcal{P}}}$. In the following, if not specified otherwise, the index i runs from 1 to $\overline{n}$. Moreover, we omit the dependence on τ of some quantities to increase readability.

By applying $\{|i\rangle\langle i|_A\}_i$ on $\rho_{AB}^{(\lambda)}(t)$, we get the output ensemble

$$\mathcal{E}(\rho_{AB}^{(\lambda)}(t), \{|i\rangle\langle i|_A\}_i) = \{\overline{p}_i, \lambda\sigma_{SA'}(t) \otimes |i\rangle\langle i|_{A''} + (1-\lambda)\overline{\rho}_{B,i}(t)\}_i,$$

and (see Appendix C.3)

$$P_g\left(\rho_{AB}^{(\lambda)}(t), \{|i\rangle\langle i|_A\}_i\right) = \lambda + (1-\lambda)\, P_g\left(\overline{\mathcal{E}}_{SA'}(t)\right). \tag{5.11}$$

For a $\{P_{A,i}\}_i \in \Pi_A^{\overline{\mathcal{P}}}$ different from the projective measurement $\{|i\rangle\langle i|_A\}_i$ we obtain (see Appendix C.3): $\mathcal{E}(\rho_{AB}^{(\lambda)}(t), \{P_{A,i}\}_i) = \{\overline{p}_i, \lambda\sigma_{B,i}^{\perp}(t) + (1-\lambda)\sigma_{B,i}^{||}(t)\}_i$. Here $\sigma_{B,i}^{\perp}(t) \equiv \sigma_{SA'}(t) \otimes \rho_{A'',i}^{\perp}$, where $\rho_{A'',i}^{\perp}$ is a convex combination of $\{|k\rangle\langle k|_{A''}\}_{k=1}^{\overline{n}}$. Analogously, $\sigma_{B,i}^{||}(t) \equiv \rho_{SA',i}^{||}(t) \otimes |\overline{n}+1\rangle\langle\overline{n}+1|_{A''}$, where $\rho_{SA',i}^{||}(t)$ is a convex combination of $\{\overline{\rho}_{SA',k}(t)\}_{k=1}^{\overline{n}}$ (see Appendix C.3). The corresponding guessing probability is

$$P_g\left(\rho_{AB}^{(\lambda)}(t), \{P_{A,i}\}_i\right) = \lambda P_g\left(\{\overline{p}_i, \rho_{A'',i}^{\perp}\}_i\right) + (1-\lambda) P_g\left(\{\overline{p}_i, \rho_{SA',i}^{||}(t)\}_i\right). \tag{5.12}$$

Next consider the following lower bound for Eq. (5.10)

$$\Delta C_A^{\overline{\mathcal{P}}} \geq P_g\left(\rho_{AB}^{(\lambda)}(\tau+\Delta\tau), \{|i\rangle\langle i|_A\}_i\right) - P_g\left(\rho_{AB}^{(\lambda)}(\tau), \{P_{A,i}^{(\lambda)}\}_i\right), \tag{5.13}$$

where we define $\{P_{A,i}^{(\lambda)}\}_i$ to be one of the *optimal* $\overline{\mathcal{P}}$-POVMs for the maximization that defines $C_A^{\overline{\mathcal{P}}}(\rho_{AB}^{(\lambda)}(t))$ at $t=\tau$, namely

$$C_A^{\overline{\mathcal{P}}}(\rho_{AB}^{(\lambda)}(\tau)) = P_g(\rho_{AB}^{(\lambda)}(\tau), \{P_{A,i}^{(\lambda)}\}_i) - \overline{p}_{max}.$$

Consider Eq. (5.12) for $\{P_{A,i}^{(\lambda)}\}_i$ and define $\mathcal{E}^{\perp}(\{P_{A,i}^{(\lambda)}\}_i)$ and $\mathcal{E}^{||}(\{P_{A,i}^{(\lambda)}\}_i)$, such that

$$P_g\left(\rho_{AB}^{(\lambda)}(\tau), \{P_{A,i}^{(\lambda)}\}_i\right) = \lambda P_g(\mathcal{E}^{\perp}(\{P_{A,i}^{(\lambda)}\}_i)) + (1-\lambda)P_g(\mathcal{E}^{||}(\{P_{A,i}^{(\lambda)}\}_i)). \tag{5.14}$$

In order to evaluate $\Delta C_A^{\overline{\mathcal{P}}}$, we analyze the two possible scenarios for the quantity $P_g(\rho_{AB}^{(\lambda)}(\tau), \{P_{A,i}^{(\lambda)}\}_i)$:

(A): $\{|i\rangle\langle i|_A\}_i$ is an optimal $\overline{\mathcal{P}}$-POVMs for some $\lambda \in [0,1)$;

(B): $\{|i\rangle\langle i|_A\}_i$ is not an optimal $\overline{\mathcal{P}}$-POVMs for any $\lambda \in [0,1)$.

We start by studying case (A): if $\{|i\rangle\langle i|_A\}_i$ is an optimal $\overline{\mathcal{P}}$-POVM for some λ^*, then the same is true for any $\lambda \in (\lambda^*, 1)$ (see Appendix C.4). From Eqs. (5.4), (5.11) and (5.13), for $\lambda \in (\lambda^*, 1)$

$$\Delta C_A^{\overline{\mathcal{P}}} \geq (1-\lambda)\left(P_g(\overline{\mathcal{E}}_{SA'}(\tau+\Delta\tau)) - P_g(\overline{\mathcal{E}}_{SA'}(\tau))\right) > 0, \tag{5.15}$$

if and only if there is no CPTP intermediate map $V_{\tau+\Delta\tau,\tau}$.

Case (B): we start by noting that $P_g(\rho_{AB}^{(\lambda)}(\tau), \{P_{A,i}\}_i)$ is Lipschitz continuous in λ, $P_g(\rho_{AB}, \{P_{A,i}\}_i)$ is Lipschitz continuous in $\{P_{A,i}\}_i$ and the unique optimal $\overline{\mathcal{P}}$-POVM for $\lambda=1$ is $\{|i\rangle\langle i|_A\}_i$ (see Appendix C.5). Therefore, the set of optimal $\overline{\mathcal{P}}$-POVMs is contained in a neighborhood of $\{|i\rangle\langle i|_A\}_i$ with size decreasing towards zero as $\lambda \to 1$. This in turn implies that the values of $P_g(\mathcal{E}^{||}(\{P_{A,i}^{(\lambda)}\}_i))$ for different $\{P_{A,i}^{(\lambda)}\}_i$ are inside an interval that converges on $P_g(\overline{\mathcal{E}}_{SA'}(\tau))$ (see Appendix C.5). If we define $P_g^{||(\lambda)} \equiv \max_{\{P_{A,i}^{(\lambda)}\}_i} P_g(\mathcal{E}^{||}(\{P_{A,i}^{(\lambda)}\}_i))$ and $P_g^{\perp(\lambda)} \equiv \max_{\{P_{A,i}^{(\lambda)}\}_i} P_g(\mathcal{E}^{\perp}(\{P_{A,i}^{(\lambda)}\}_i))$, it holds that

$$\forall\delta > 0,\ \exists\lambda_\delta > 0:\ P_g^{||(\lambda)} - P_g(\overline{\mathcal{E}}_{SA'}(\tau)) < \delta\ ,\ \forall\lambda \in (\lambda_\delta, 1)\,. \tag{5.16}$$

Hence, for $\overline{\delta} \equiv P_g(\overline{\mathcal{E}}_{SA'}(\tau+\Delta\tau)) - P_g(\overline{\mathcal{E}}_{SA'}(\tau)) > 0$, there exists $\overline{\lambda} \in [0,1)$ such that $P_g^{||(\lambda)} - P_g(\overline{\mathcal{E}}_{SA'}(\tau)) < P_g(\overline{\mathcal{E}}_{SA'}(\tau+\Delta\tau)) - P_g(\overline{\mathcal{E}}_{SA'}(\tau))$ for any $\lambda \in (\overline{\lambda}, 1)$, namely

$$P_g(\overline{\mathcal{E}}_{SA'}(\tau+\Delta\tau)) - P_g^{||(\lambda)} > 0,\ \ \forall\lambda \in (\overline{\lambda}, 1)\,. \tag{5.17}$$

We consider inequalities (5.13) and (5.17) for $\lambda \in (\overline{\lambda}, 1)$ and obtain a backflow

$$\Delta C_A^{\overline{\mathcal{P}}} \geq \lambda \left(1 - P_g^{\perp(\lambda)}\right) + (1-\lambda)\left(P_g(\overline{\mathcal{E}}_{SA'}(\tau+\Delta\tau)) - P_g^{\|(\lambda)}\right) > 0, \tag{5.18}$$

if and only if there is no CPTP intermediate map $V_{\tau+\Delta\tau,\tau}$. □

We proved that any non-Markovian evolution can be witnessed with backflows of $C_A^{\overline{\mathcal{P}}}$ by considering the probe states $\rho_{AB}^{(\lambda)}(0)$. These backflows are robust, namely, if we add sufficiently small perturbations to $\rho_{AB}^{(\lambda)}(0)$ and the optimal $\overline{\mathcal{P}}$-POVMs obtained by the maximization in Eq. (5.5) for any given non-Markovian dynamics, we still obtain a backflow of $C_A^{\overline{\mathcal{P}}}$. Hence, there exists a set of initial states with the same dimension as $S(\mathcal{H}_A \otimes \mathcal{H}_B)$ that provide a backflow of $C_A^{\overline{\mathcal{P}}}$ in the scenario described above (see Appendix C.6 for more details).

We did not take advantage or made assumptions about any particular structure of the components of $\overline{\mathcal{E}}_{SA'}(0)$. As a consequence, it is straightforward to adapt our technique to any other ensemble. In particular, if the evolution of an initial ensemble $\{p_i, \phi_{SA',i}\}_{i=1}^n$ provides a backflow of $P_g(\{p_i, \phi_{SA',i}(t)\}_{i=1}^n)$ in a time interval $[\tau, \tau+\Delta\tau]$, we can consider $C_A^{\mathcal{P}}(\psi_{AB}^{(\lambda)}(0))$, where $\mathcal{P} = \{p_i\}_{i=1}^n$, and

$$\psi_{AB}^{(\lambda)}(0) = \sum_{i=1}^n p_i \, |i\rangle\langle i|_A \otimes (\lambda \sigma_{SA'} \otimes |i\rangle\langle i|_{A''} + (1-\lambda)\, \phi_{SA',i} \otimes |n+1\rangle\langle n+1|_{A''}),$$

which provide a backflow of $C_A^{\mathcal{P}}(\psi_{AB}^{(\lambda)}(t))$ in $[\tau, \tau+\Delta\tau]$. We make some examples of ensembles (different from $\overline{\mathcal{E}}_{SA'}(0)$) that can be considered to witness particular classes of non-Markovian evolutions. A constructive method that provides ensembles of two equiprobable states that witness any bijective or pointwise non-bijective non-Markovian dynamics is given in Ref. [BJA17], namely the technique described in Section 2.5.4. The existence of two-state ensembles that detect any image non-increasing evolution, namely such that $\mathrm{Im}(\Lambda_t) \subseteq \mathrm{Im}(\Lambda_s)$ for any $s < t$, is proven in Ref. [CRS18] (see Theorem 6). Finally, two-state ensembles suffices to witness any non-Markovian qubit evolution [CC19] (see Theorem 7).

5.4.3 A measure of non-Markovianity

Similarly to prior measures of non-Markovianity N_Q described in Sections 2.4 and 2.5 for different information quantifiers Q, we can introduce a measure associated to the non-Markovian witness $Q_{A,\mathcal{P}}$ defined by $C_A^{\mathcal{P}}(\rho_{AB})$ as follows

$$N_{\mathcal{P}}(\mathbf{\Lambda}) \equiv \sup_{\rho_{ASA'}(0)} \int_{\frac{d}{dt} C_A^{\mathcal{P}}(\rho_{ASA'}(t))>0} \frac{d}{dt} C_A^{\mathcal{P}}(\rho_{ASA'}(t)) dt\,, \tag{5.19}$$

where the sup is over the possible ancillary systems (A and A') and the initial states $\rho_{ASA'}(0) \in S(\mathcal{H}_A \otimes \mathcal{H}_S \otimes \mathcal{H}_{A'})$, where we impose $\dim(\mathcal{H}_A) \leq d^4$ and $\dim(\mathcal{H}_{A'}) \leq d$. As a consequence of Theorem 14, if $C_A^{\overline{\mathcal{P}}}(\rho_{ASA'}(t))$ is differentiable, $N_{\overline{\mathcal{P}}}(\mathbf{\Lambda}) > 0$ if and only if the evolution is non-Markovian (see Appendix C.7 for details and a discussion of the non-differentiable case). Indeed, for any time interval where the evolution cannot be described by a CPTP intermediate map, we proved the existence of a set of initial states that show an increase of $C_A^{\overline{\mathcal{P}}}$ in the same time interval. We notice that, if we fix $\mathcal{P} = \{1/2, 1/2\}$, $N_{\mathcal{P}}(\mathbf{\Lambda}) > 0$ for any bijective or pointwise non-bijective non-Markovian evolution $\mathbf{\Lambda}$. Indeed, in Chapter 4 we saw that two-ouput ME-POVMs are sufficient to provide backflows of $C_A^{(2)}$ for these evolutions.

The measure of non-Markovianity N_{RHP} described in Section 2.5.2 and the class $N_{\mathcal{P}}$ are the only measures that are proved to be positive for any non-Markovian evolution. Notice that the value of $N_{\mathcal{P}}(\mathbf{\Lambda})$, differently from $N_{RHP}(\mathbf{\Lambda})$, represents the backflow of a physical quantity, namely $C_A^{\mathcal{P}}$, shown by a state that undergoes the target evolution. Nonetheless, while N_{RHP} is easy to compute in many different cases, the computation of the class $N_{\mathcal{P}}$ may be difficult in the general case, since it involves a supremum over initial states.

5.5 Discussion

In this chapter we showed that any non-Markovian dynamics can be witnessed through backflows of the correlation measure $C_A^{\overline{\mathcal{P}}}$. For this purpose, we introduced a class of initial probe states $\rho_{AB}^{(\lambda)}(0)$ that allows us to accomplish this task. Hence, we proved the first one-to-one correspondence between CP-divisibility of evolutions, namely Markovianity, and the absence of correlation backflows.

It would be useful to obtain a constructive method that provides the elements of $\overline{\mathcal{E}}_{SA'}(0)$ that we used to define the initial probe state. Moreover, since the class of bipartite correlations that we studied does not consider the subsystems A and B symmetrically, an open question is to understand if also $C_{AB}^{\overline{\mathcal{P}}}$ (see Eq. (5.7)) is able to witness any non-Markovian evolution.

Different approaches that manipulate and evolve two-state ensembles defined over S and particular ancillary systems are proved to witness any bijective or alternatively at most point-wise non-bijective non-Markovian evolution [BJA17, KRS20, DJB$^+$19]. On the other hand, methods that allow witnessing any non-Markovian evolution, e.g. [BD16] and the one presented in this chapter, make use of ensembles that in general are made by more than two states. We therefore find it interesting to know if the use of larger ensembles in [BD16] and in this chapter is necessary to witness any non-Markovian evolution. Finding an example of a non-Markovian evolution that two-state ensembles cannot witness would prove this in the positive. Such an example, if it exists, could perhaps also help elucidate how to explicitly construct the elements of $\overline{\mathcal{E}}_{SA'}(0)$.

We consider interesting the possibility to formulate simplified versions of the non-Markovianity measures $N_{\mathcal{P}}$ that permit simplified computations while still being positive for any non-Markovian evolution.

Chapter 6

Non-Markovianity measure via mixing with Markovian dynamics

We introduce a non-Markovianity measure based on the minimal amount of extra Markovian noise we have to add to the process via incoherent mixing, in order to make the resulting transformation Markovian at all times. We show how to evaluate this measure by considering the set of depolarizing evolutions in arbitrary dimension and the set of dephasing evolutions for qubits. The results exposed in this chapter are contained in the original work [DG21].

6.1 Introduction

In open quantum system dynamics [BP07] Markovian evolutions are characterized by the existence of a one-way flow of information from the system to its environment. While approximately valid in many contexts of physical relevance (in particular for system-environment weak-coupling conditions), in the vast majority of settings the Markovianity of the dynamical evolution is lost and one witnesses *backflows* of information from the environment to the system [Bre12, RHP14, BLPV16, LGP19]. The study of these non-Markovian effects is a central topic of quantum information theory both because they arise almost everywhere, but also because, when properly exploited, they may show advantages in different quantum information processing tasks, such as quantum metrology [CHP12], quantum key distribution [VOPM11], quantum teleportation [LBP14], entanglement generation [HRP12], quantum communication [BCM14] and quantum thermodynamics [WGE16, LWEG18, PLWR+18, AG19].

The standard procedure to characterize and measure the non-Markovianity of a given evolution is to target functionals that are guaranteed to be monotonic under arbitrary Markovian evolutions and to check for violations of such behavior. Many quantities have been studied in this framework: the distance between pair of states [BLP10, LPB10], channel capacities [BCM14], the guessing probability of evolving ensembles of states [BD16], the volume of the accessible states [LPP13] and correlation measures [LFS12, DJB$^+$19]. In this chapter we introduce a conceptually different approach to the problem which tries to quantify non-Markovian character of a dynamical evolution by computing the minimal amount of extra noise that one has to *inject* into the system dynamics in order to stop the information backflow at all times. Specifically we consider the minimum value of the probability needed to introduce Markovianity for the entire temporal evolution of the system by incoherently mixing it with an arbitrary extra process which is already Markovian. Our measure has a clear operational meaning due to the fact that creating stochastic convolutions of processes is a well defined physical procedure.

We remark however that since neither the set of Markovian evolutions, nor its complementary counterpart, are convex [WECC08] the explicit evaluation of the proposed measure is typically hard to comply. At variance with the approaches presented in Refs. [BBM20, AB19] which discuss similar ideas focusing on infinitesimal Markovian evolutions [Kos72b, Lin76, GK76], non-convexity also prevents us from framing our proposal in the context of a conventional (convex) resource theory of evolutions where Markovian dynamics constitute the resource-free set [RBTL20, BG15].

After introducing the procedure in the general case of arbitrary open quantum evolutions we focus on the special subset of depolarizing transformations of arbitrary dimension and for qubit dephasing channels [Hol12, Wil13, Kin03] which, thanks to their highly symmetric character, allow for an explicit analytical treatment. Depolarizing channels represent an important error model in quantum information theory. Indeed by pre- and post- processing and classical communication via twirling [HHH99], any other open quantum dynamics can be mapped into a depolarizing channel whose efficiency in protecting the information stored into the system is lower than or equal to the corresponding one of the original process. Accordingly the study of the non-Markovian character of this special set of open quantum evolutions is an important task in its own.

The chapter is organized as follows. In Section 6.2 we introduce the depolarizing evolutions set. In addition, we describe its Markovian and non-Markovian subsets (Section 6.2.1), we discuss some geometrical properties of these subsets (Section 6.2.2) and we characterize continuous depolarizing evolutions (Section 6.2.3). In Section 6.3 we present the measure of non-Markovianity that we study throughout this chapter and we describe how to apply it to non-Markovian depolarizing evolutions (Section 6.3.1). We follow in Section 6.4 by evaluating this measure of non-Markovianity for continuous depolarizing evolutions. Section 6.5 is dedicated to show that, considering the task of making continuous depolarizing evolutions Markovian by mixing them with Markovian evolutions, non-continuous Markovian evolutions are less efficient than continuous Markovian evolutions. From Section 6.6 we start to study non-continuous non-Markovian depolarizing evolutions. In particular, we show that in some particular cases the approaches considered for continuous non-Markovian evolutions are still valid to evaluate the degree of non-Markovianity of these evolutions. In Section 6.7 we consider our measure of non-Markovianity applied to generic non-continuous non-Markovian depolarizing evolutions. We start by noticing some features of these evolutions that imply an ambiguity for the identification of the optimal Markovian evolution that makes a generic non-Markovian depolarizing evolution Markovian (Section 6.7.1). Hence, in Section 6.7.2, we propose a strategy to calculate our measure of non-Markovianity for any non-continuous depolarizing evolutions. Finally, in Section 6.8 we extend the analysis to the case of dephasing channels for qubits. The chapter ends in Section 6.9 with the conclusions. Technical material is presented in the appendices.

6.2 Depolarizing evolutions

Let $S(\mathcal{H}_S)$ be the set of density matrices on a d-dimensional Hilbert space $\mathcal{H}_S$. As we defined in Chapter 2, an evolution on $S(\mathcal{H}_S)$ is a one-parameter family $\mathbf{\Lambda} = \{\Lambda_t\}_{t\geq 0}$ of CPTP maps $\Lambda_t : B(\mathcal{H}_S) \to B(\mathcal{H}_S)$, namely dynamical maps. Moreover, we imposed that for $t = 0$ the dynamical map should correspond to the identity map, namely

$$\Lambda_0(\cdot) = I_S(\cdot) , \tag{6.1}$$

and require the family $\mathbf{\Lambda}$ to be continuous and differentiable almost always, allowing at most a countable set of discontinuity times. In Section 2.3.1 we showed why these assumptions are well motivated. We hence defined $\mathbb{E} \equiv \{\mathbf{\Lambda}\}$ to be the set of all the evolutions on $S(\mathcal{H}_S)$ that obey the above constraints. One can easily verify that such set is closed under convex combination. As discussed in Section 2.3.3, we identify Markovian evolutions with those being CP-divisible. Therefore, we identify the Markovian and non-Markovian subsets

$\mathbb{E}^M$ and $\mathbb{E}^{NM}$ of $\mathbb{E}$. As already mentioned in the Section 2.3.5, neither $\mathbb{E}^M$ nor $\mathbb{E}^{NM}$ are closed under convex combinations.

In Section 6.2 we saw that depolarizing evolutions $\mathbb{D}$ form a closed convex subset of $\mathbb{E}$ [Hol12, Wil13, Kin03]. We remember that an evolution $\boldsymbol{D} = \{D_t\}_{t\geq 0}$ belongs to $\mathbb{D}$ if and only if at any time $t \geq 0$ the corresponding dynamical map D_t can be written as

$$D_t(\cdot) = f(t)\, I_S(\cdot) + (1 - f(t))\mathrm{Tr}\,[\cdot]\, \frac{\mathbb{1}_S}{d}\,, \tag{6.2}$$

where $f(t)$ is a real quantity belonging to the interval

$$J_{\mathbb{D}} \equiv \left[-\frac{1}{d^2-1}, 1\right]. \tag{6.3}$$

From Eq. (6.2) it is clear that we can use the function $f(t)$ to uniquely characterize the elements of $\mathbb{D}$. In order to comply with the structural requirements we imposed on $\mathbb{E}$ in the previous section, we focus on the collection of functions $f(t) : \mathbb{R}^+ \to J_{\mathbb{D}}$ that

1. are continuous for almost-all t;
2. admit right and left time derivatives ($\dot{f}(t^{\pm}) \equiv \lim_{\epsilon\to 0^{\pm}} \frac{f(t+\epsilon)-f(t)}{\epsilon}$);
3. satisfy $f(0) = 1$;

the last property being introduced to enforce Eq. (6.1). We define $\mathfrak{F}$ to be the set of *characteristic functions* $f(t)$ that satisfy the above conditions and use Eq. (6.2) to establishing a one-to-one relation between such set and $\mathbb{D}$. We also introduce the special subset of continuous depolarizing evolutions $\mathbb{D}_C$ as the collection of depoloring evolutions (6.2) whose functions $f(t)$ belong to the subset $\mathfrak{F}_C \subset \mathfrak{F}$ formed by continuous characteristic functions.

To fix the notation, if $\{t_i\}_i$ is the discrete collection of times when $f(t)$ is discontinuous, we have that $f(t_i^+) \equiv \lim_{\epsilon\to 0^+} f(t_i + \epsilon)$ is different from $f(t_i^-) \equiv \lim_{\epsilon\to 0^+} f(t_i - \epsilon)$. To describe the discontinuous behavior of $f(t)$ we hence introduce the quantity

$$\xi(f(t)) \equiv \frac{f(t^+)}{f(t^-)}\,, \tag{6.4}$$

which assumes values in $[-\infty, +\infty]$, where we fix $\xi(f(t)) = \pm\infty$ when we have $\mathrm{sign}(f(t^+)) = \pm 1$ and $f(t^-) = 0$. Moreover, when $f(t^+) = f(t^-) = 0$ we define $\xi(f(t)) = 1$. From Eq. (6.4) it follows that $f(t)$ is continuous at time t if $\xi(f(t)) = 1$ and that $f(t) \in \mathfrak{F}_C$ if and only if $\xi(f(t)) = 1$ for any $t \geq 0$. On the contrary, from Eq. (6.4) it also follows that

- $\xi(f(t)) > 1$: the discontinuity distances $f(t)$ from zero, namely depending on the sign of $f(t^-)$ we either have $0 < f(t^-) < f(t^+)$ or $0 > f(t^-) > f(t^+)$;
- $\xi(f(t)) < 0$: $f(t)$ changes sign;
- $\xi(f(t)) = 0$: $f(t^+) = 0$ and $f(t^-) \neq 0$.

6.2.1 Markovian and non-Markovian depolarizing evolutions

In view of the one-to-one correspondence between $\mathbb{D}$ and $\mathfrak{F}$, we define the Markovian and non-Markovian depolarizing subsets $\mathbb{D}^M \equiv \mathbb{D} \cap \mathbb{E}^M$ and $\mathbb{D}^{NM} \equiv \mathbb{D} \cap \mathbb{E}^{NM} = \mathbb{D} \setminus \mathbb{D}^M$ by assigning the corresponding sets of the associated characteristic functions $\mathfrak{F}^M$ and $\mathfrak{F}^{NM}$.

We start by observing that, if the characteristic function of an element $\boldsymbol{D}$ of $\mathbb{D}$ assumes a zero value at s (namely $f(s) = 0$), then D_s becomes the complete depolarizing channel $\mathrm{Tr}\,[\cdot]\,\frac{\mathbb{1}_S}{d}$, loosing memory of the input state of the system. Accordingly the only possibility we have to fulfill the CP-divisibility given in Definition 5 for Markovianity is that D_t correspond to $\mathrm{Tr}\,[\cdot]\,\frac{\mathbb{1}_S}{d}$ too, namely

$$f(s) = 0 \quad \Longrightarrow \quad f(t) = 0\,, \qquad \forall t \geq s\,. \tag{6.5}$$

On the contrary, if $f(s) \neq 0$, Markovianity can be enforced by observing that the intermediate map $V_{t,s}$ assumes the same form of Eq. (6.2), namely

$$V_{t,s}(\cdot) = \frac{f(t)}{f(s)}\, I_S(\cdot) + \left(1 - \frac{f(t)}{f(s)}\right) \mathrm{Tr}\,[\cdot]\, \frac{\mathbb{1}_S}{d}\ , \tag{6.6}$$

which is CPTP if and only if

$$\frac{f(t)}{f(s)} \in J_{\mathbb{D}}\ , \tag{6.7}$$

with $J_{\mathbb{D}}$ the interval defined in Eq. (6.3). This includes also the case (6.5) by noticing that only with $f(t) = 0$ we prevent $f(t)/f(s)$ from diverging when $f(s) = 0$. As shown in Appendix D.1, Eq. (6.7) can be conveniently casted in the following inequality that in some case is easier to handle, namely

$$C(t,s) \equiv \left|2(d^2-1)f(t) - (d^2-2)f(s)\right| - d^2|f(s)| \leq 0\,. \tag{6.8}$$

From Definition 5 we have hence that $\boldsymbol{D} \in \mathbb{D}^M$ if and only if its characteristic function $f(t)$ is such that (6.7) (or equivalently (6.8)) holds true for any $t \geq s \geq 0$, namely

$$\mathfrak{F}^M \equiv \{f(t) \in \mathfrak{F}\,|\, C(t,s) \leq 0\ ,\ \forall t \geq s \geq 0\}\ . \tag{6.9}$$

Considering the property (6.5) and that for $f(t) \in \mathfrak{F}$ we must have $f(0) = 1$, it is easy to verify that all continuous elements of $\mathfrak{F}^M$ are non-negative and non-increasing (more on this in Section 6.2.3). Markovian characteristic functions can however change their sign through discontinuities. Indeed according to (6.7) a non continuous element $f(t)$ of $\mathfrak{F}^M$ can jump either to a value $f(t^+)$ with the same sign and $|f(t^+)| < |f(t^-)|$, namely $\xi(f(t)) \in [0, 1)$, or to a value with opposite sign and $|f(t^+)| \leq |f(t^-)|/(d^2 - 1)$, namely $\xi(f(t)) \in [-1/(d^2 - 1), 0]$. These facts can be formalized by saying that a generic $f(t) \in \mathfrak{F}$ exhibits a *Markovian behavior* at time $\tau \geq 0$ if one of the two conditions applies

$$\begin{array}{ll} \mathbf{CM_1}(\tau): & \xi(f(\tau)) = 1 \text{ and } \frac{d}{d\tau}|f(\tau)| \leq 0; \\ \mathbf{CM_2}(\tau): & \xi(f(\tau)) \in J_{\mathbb{D}} \setminus 1; \end{array} \tag{6.10}$$

where $\mathbf{CM_1}(\tau)$ has to be replaced by $\dot{f}(\tau^{\pm})f(\tau) \leq 0$ when $\dot{f}(\tau)$ is non-continuous, namely $\dot{f}(\tau^-) \neq \dot{f}(\tau^+)$. Notice that the conditions given in Eq. (6.10) do not explicitly exclude the cases for which $\dot{f}(t) \neq 0$ and $f(t) = 0$. Nonetheless, the properties of $\mathfrak{F}$ would imply that $\exists \delta > 0$ such that $\dot{f}(t + \delta)f(t + \delta) > 0$, which would exclude $f(t)$ from $\mathfrak{F}^M$. It is worth stressing that imposing (6.10) for all $\tau \geq 0$ is equivalent to enforce (6.7) (or (6.8)) for all couples $0 \leq s \leq t$. Hence, Eq. (6.9) can be casted in the form

$$\mathfrak{F}^M = \{f(t) \in \mathfrak{F} \,|\, \mathbf{CM_1}(\tau) \text{ or } \mathbf{CM_2}(\tau) = \text{TRUE}, \forall \tau \geq 0\} \ , \tag{6.11}$$

which involves only local properties of $f(t)$. By construction any $f(t) \in \mathfrak{F}$ that fails to fulfill both the constraints of Eq. (6.10) at least for one τ, or the inequality (6.8) for some couple s and t, defines an element of the non-Markovian characteristic function set $\mathfrak{F}^{NM} \equiv \mathfrak{F} \setminus \mathfrak{F}^M$ which describes the non-Markovian depolarizing evolutions $\mathbb{D}^{NM}$. At variance with the elements of $\mathfrak{F}^M$ a characteristic function $f(t)$ which is non-Markovian can show any increasing or decreasing continuous behavior and discontinuities with $\xi(f(t)) \in [-\infty, +\infty]$. In Fig. 6.1 we show the typical behavior of characteristic functions in $\mathfrak{F}^M$ and $\mathfrak{F}^{NM}$.

We notice that any element of $\mathfrak{F}^{NM}$ can still obey the constraints (6.10) on some part of the real axis. In particular we say that $f(t) \in \mathfrak{F}^{NM}$ has a Markovian behavior in (t_1, t_2) if the function satisfies at least one of the conditions of Eq. (6.10) for any $\tau \in (t_1, t_2)$. Finally, we say that τ is a time when $f(t) \in \mathfrak{F}$ shows a *Markovian discontinuity* if $\xi(f(\tau)) \in J_{\mathbb{D}} \setminus 1$. Instead, if $\xi(f(\tau)) \notin J_{\mathbb{D}}$, we say that τ is a time when $f(t)$ shows a *non-Markovian discontinuity*.

6.2.2 Border and geometry of the depolarizing evolutions

It is possible to show that the following properties hold:

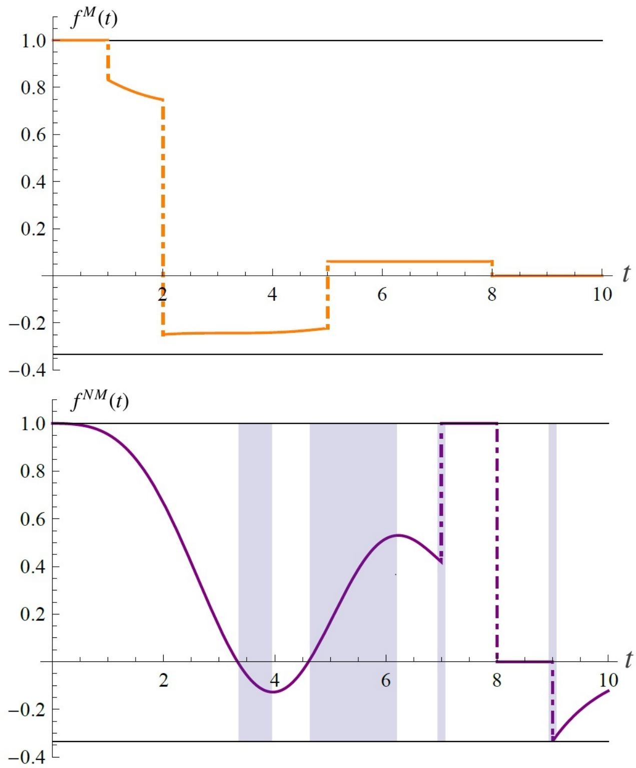


Figure 6.1: Example of a non-continuous Markovian characteristic function (above) $f^M(t) \in \mathfrak{F}^M$ and a non-continuous non-Markovian characteristic function (below) $f^{NM}(t) \in \mathfrak{F}^{NM}$ for $d = 2$. Given Eq. (6.3), any characteristic function has to assume values in $J_{\mathbb{D}} = [-1/3, 1]$. Discontinuities are underlined by dotted dashed lines. $f^M(t)$, when continuous, satisfies $\mathbf{CM_1}(\tau)$, namely it does not increase its distance from zero. When $f^M(t)$ is not continuous it satisfies $\mathbf{CM_2}(\tau)$: for the times $\tau = 1, 2, 5$ and 8, we have $\xi(f^M(1)) = 0.83$, $\xi(f^M(2)) = -0.33$, $\xi(f^M(5)) = -0.27$ and $\xi(f^M(8)) = 0$. Since $f^M(8^+) = 0$, $f^M(t)$ has to be equal to 0 for any $t > 8$. The times when $f_C^{NM}(t)$ has a non-Markovian behavior are colored in purple. This characteristic function shows both time intervals and times of discontinuity when, respectively, $\mathbf{CM_1}(\tau)$ and $\mathbf{CM_2}(\tau)$ are violated. Indeed, for $\tau = 7$ and 9 we have non-Markovian discontinuities $\xi(f^{NM}(7)) = 2.39$ and $\xi(f^{NM}(t)) = -\infty$, while at $\tau = 8$ we have $\xi(f^{NM}(8)) = 0$, namely a Markovian discontinuity. The temporal parameter t in the plots is expressed in arbitrary units.

- $\mathbb{D}$ is convex,
- $\mathbb{D}^M$ is closed, non-convex, and **border**$(\mathbb{D}^M) = \mathbb{D}^M$,
- $\mathbb{D}^{NM}$ is open, non-convex, and dense.

The non convexity of $\mathbb{D}^M$ and $\mathbb{D}^{NM}$ (and hence $\mathfrak{F}^M$ and $\mathfrak{F}^{NM}$) can be easily proven by presenting some explicit counter-examples (see Appendix D.2). In order to prove the non-convexity of $\mathbb{D}^{NM}$, we mix two depolarizing evolutions that show non-Markovian features in non-overlapping time intervals (see Appendix D.2.1). The example that we study for the non-convexity of $\mathbb{D}^M$ underlines the important role played by discontinuities: we mix an evolution defined by $f^{M,1}(t) = 1$ for all $t \geq 0$ and a second Markovian characteristic function $f^{M,2}(t)$ with two Markovian discontinuities. We show that for all $p \in (0, 1)$ the resulting $f^{(p)}(t)$ shows a non-Markovian discontinuity (see Appendix D.2.2).

To show that $\mathbb{D}^M$ coincides with its border we proceed as follows: given a generic Markovian depolarizing evolution $\boldsymbol{D}^M \in \mathbb{D}^M$, consider a time $s > 0$ where the associated characteristic function $f^M(t)$ is continuous, namely $\xi(f^M(s)) = 1$ (of course such s can always be found since the set of discontinuity points for a generic element of $\mathfrak{F}$ is at most countable). Take then a non-Markovian depolarizing evolution $\boldsymbol{D}^{NM} \in \mathbb{D}^{NM}$ with characteristic function $f^{NM}(t)$ which instead has $\xi(f^{NM}(s)) > 1$ and $\text{sign}(f^{NM}(s^-)) = \text{sign}(f^M(s^+))$ (such an element can always be identified). It is then straightforward to verify that the whole family of elements of $\mathbb{D}$ defined as $\boldsymbol{D}^{(p)} = (1-p)\boldsymbol{D}^{NM} + p\boldsymbol{D}^M$ for $p \in [0, 1)$ is non-Markovian: indeed for all such values, at $t = s$ the characteristic function

$$f^{(p)}(t) = (1-p)f^{NM}(t) + pf^M(t) \, , \tag{6.12}$$

of $\boldsymbol{D}^{(p)}$ has a non-Markovian discontinuity ($\xi(f^{(p)}(s)) > 1$). Notice also that as $p \to 1$, $\boldsymbol{D}^{(p)}$ gets arbitrarily close to $\boldsymbol{D}^M$ in any conceivable norm one can introduce on $\mathbb{E}$ or $\mathbb{D}$ (indeed $\|\boldsymbol{D}^{(p)} - \boldsymbol{D}^M\| = (1-p)\|\boldsymbol{D}^{NM} - \boldsymbol{D}^M\|$). The above argument shows that any neighbor of a Markovian depolarizing trajectory contains non-Markovian processes, namely that $\mathbb{D}^M$ is a set of measure zero, or equivalently, that almost-all depolarizing evolutions are non-Markovian. On the contrary, for any non-Markovian depolarizing evolution $\boldsymbol{D}^{NM}$ one can show that there exists no Markovian $\boldsymbol{D}^M$ such that the convex combination $\boldsymbol{D}^{(p)} = (1-p)\boldsymbol{D}^{NM} + p\boldsymbol{D}^M$ is Markovian for any $p \in (0, 1]$. More precisely, it is possible to identify a probability value $p^*(\boldsymbol{D}^{NM}) \in (0, 1]$ such that, independently from the choice of $\boldsymbol{D}^M$, we have

$$\boldsymbol{D}^{(p)} \in \mathbb{D}^{NM} \qquad \forall p < p^*(\boldsymbol{D}^{NM}) \, . \tag{6.13}$$

Indeed, since $\boldsymbol{D}^{NM}$ is explicitly non-Markovian, there must exist times $t \geq s \geq 0$ such that its the characteristic function violate the constraint (6.8) which we rewrite here as

$$A^{NM}(t,s) \equiv \left|2(d^2-1)f^{NM}(t) - (d^2-2)f^{NM}(s)\right| > d^2|f^{NM}(s)| \ . \tag{6.14}$$

On the contrary, if $\boldsymbol{D}^{(p)}$ is Markovian, its characteristic function must fulfill (6.8), namely

$$\left|2(d^2-1)f^{(p)}(t) - (d^2-2)f^{(p)}(s)\right| \leq d^2|f^{(p)}(s)| \ . \tag{6.15}$$

Using (6.12) we notice however that the left-hand-side of the above expression can be lower bounded as follows

$$\begin{aligned}
&\left|2(d^2-1)f^{(p)}(t) - (d^2-2)f^{(p)}(s)\right| \\
&\quad \geq (1-p)A^{NM}(t,s) - p\left|2(d^2-1)f^{M}(t) - (d^2-2)f^{M}(s)\right| \\
&\quad \geq (1-p)A^{NM}(t,s) - p(3d^2-4) \ ,
\end{aligned} \tag{6.16}$$

where in the last inequality we exploit the fact that all characteristic functions must have modulus smaller or equal to 1. Similarly the right-hand-side of (6.15) can be upper bounded as

$$\left|f^{(p)}(s)\right| \leq (1-p)\left|f^{NM}(s)\right| + p\left|f^{M}(s)\right| \leq (1-p)\left|f^{NM}(s)\right| + p \ . \tag{6.17}$$

Hence a necessary condition for (6.15) is to have

$$4p(d^2-1) \geq (1-p)C^{NM}(t,s) \ , \tag{6.18}$$

where $C^{NM}(t,s) \equiv A^{NM}(t,s) - d^2|f^{NM}(s)|$. Due to the strict positivity of the rightmost term of Eq. (6.18) (see (6.14)), it cannot be fulfilled for all $p \in (0,1]$. Eq. (6.13) finally follows from (6.18), e.g. by setting

$$p^*(\boldsymbol{D}^{NM}) = \frac{C^{NM}(t,s)}{C^{NM}(t,s) + 4(d^2-1)} \ . \tag{6.19}$$

It is easy to show that this value of $p^*(\boldsymbol{D}^{NM})$ belongs to $(0,1]$ if and only if $C^{NM}(t,s)$ violates Eq. (6.8).

6.2.3 Continuous depolarizing evolutions

Important subsets of $\mathbb{D}^M$ and $\mathbb{D}^{NM}$ are obtained by considering their intersections with the continuous subset $\mathbb{D}_C$ of $\mathbb{D}$, namely

$$\mathbb{D}_C^M \equiv \mathbb{D}_C \cap \mathbb{D}^M \ , \qquad \mathbb{D}_C^{NM} \equiv \mathbb{D}_C \cap \mathbb{D}^{NM} \ . \tag{6.20}$$

By construction $\mathbb{D}_C^M$ and $\mathbb{D}_C^{NM}$ are composed by depolarizing process whose associated characteristic functions $f(t)$ belong respectively to the intersections $\mathfrak{F}_C^M \equiv \mathfrak{F}_C \cap \mathfrak{F}^M$ and $\mathfrak{F}_C^{NM} \equiv \mathfrak{F}_C \cap \mathfrak{F}^{MN}$. From Eq. (6.10) we deduce that the elements of $\mathfrak{F}_C^M$ are *monotonically non increasing*, continuous functions $f_C^M(t) \in [0, 1]$. In particular, since any convex combination of two continuous functions in $\mathfrak{F}_C^M$ belongs to $\mathfrak{F}_C^M$, we have

- $\mathbb{D}_C$ is convex,
- $\mathbb{D}_C^M$ is closed and convex,
- $\mathbb{D}_C^{NM}$ is open and non-convex.

Furthermore, if $f_C^M(t') = 0$ for some time t', the time derivative of $f_C^M(t)$ cannot be different from zero for any $t > t'$ without violating the first condition of Eq. (6.10). Instead the elements of $\mathfrak{F}_C^{NM}$ are continuous functions $f_C^{NM}(t)$ that can assume any value in $J_{\mathbb{D}}$ such that $f_C^{NM}(0) = 1$. In Fig. 6.2 we show the typical behavior of continuous characteristic functions in $\mathfrak{F}_C^M$ and $\mathfrak{F}_C^{NM}$.

In Appendix D.3 we introduce another convex subset of $\mathbb{D}$ given by the positive depolarizing evolutions, namely defined by, in general non-continuous, positive characteristic functions. The Markovian subset of these evolutions is convex and, as we show, it contains the set of continuous Markovian evolutions.

6.3 A measure of non-Markovianity by noise addition

In this section we introduce our measure of non-Markovianity. Given $\mathbf{\Lambda} \in \mathbb{E}$ the quantum process we are interested in, consider the quantum trajectories $\mathbf{\Lambda}^{(p)} \in \mathbb{E}$ defined by the convex sums

$$\mathbf{\Lambda}^{(p)} = (1 - p)\mathbf{\Lambda} + p\mathbf{\Lambda}^M, \qquad p \in [0, 1], \tag{6.21}$$

one gets by incoherently mixing the original evolution with an element $\mathbf{\Lambda}^M$ of the Markovian subset $\mathbb{E}^M$ with time-independent weights $1 - p$ and p. It is worth stressing that the dynamical evolution (6.21) can be physically implemented, at least in principle, by a simple random event taking place at time $t = 0$ which decides whether to transform the state of the system under the action of $\mathbf{\Lambda}$ or under the action of $\mathbf{\Lambda}^M$. Implementations of this kind of dynamical evolutions has been theoretically proposed in Ref. [FPMZ17] within the collisional model setting, and in Ref. [UWS+20] using a photonic platform in which different optical paths that simulates the alternative evolutions of the system are incoherently recombined at the output of the setup.

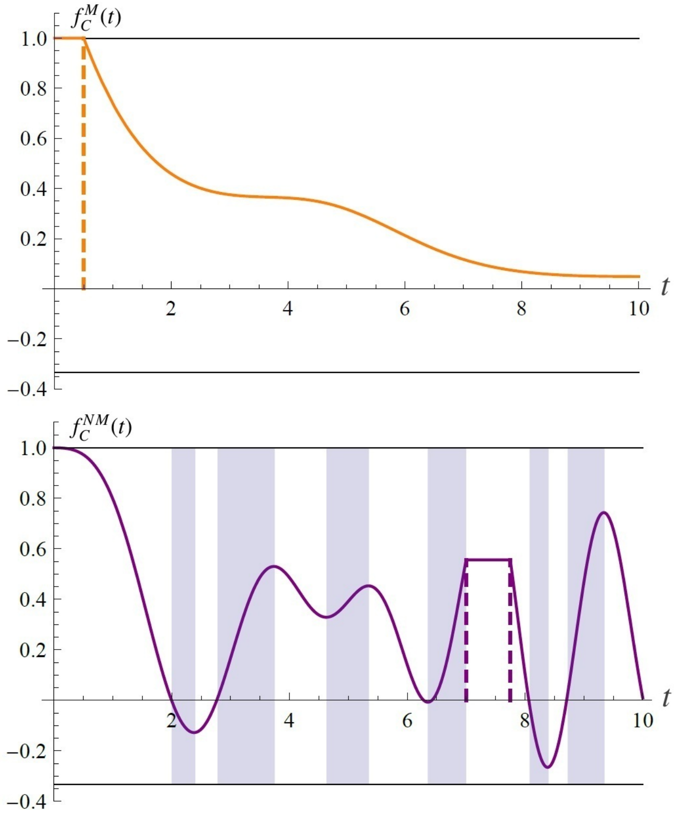


Figure 6.2: Example of a continuous Markovian characteristic function (above) $f_C^M(t) \in \mathfrak{F}_C^M$ and a continuous non-Markovian characteristic function (below) $f_C^{NM}(t) \in \mathfrak{F}_C^{NM}$ for $d = 2$. Given Eq. (6.3), any characteristic function has to assume values in $J_{\mathbb{D}} = [-1/3, 1]$. $f_C^M(t)$ is non-increasing and assumes values in $[0, 1]$. $f_C^{NM}(t)$ assumes values in $J_{\mathbb{D}} = [-1/3, 1]$ (horizontal lines) and violates the Markovian condition $\mathbf{CM_1}(\tau)$ in the time intervals colored in purple, namely when it increases its distance from zero. Dahsed lines underline the times when the respective time derivatives are non-continuous. The temporal parameter t in the plots is expressed in arbitrary units.

We introduce a measure of non-Markovianity $p(\mathbf{\Lambda})$ by considering the smallest p that enables us to make $\mathbf{\Lambda}^{(p)}$ Markovian, namely CP-divisible, for some $\mathbf{\Lambda}^M$, namely

$$p(\mathbf{\Lambda}) \equiv \min_p \{p \,|\, \exists \mathbf{\Lambda}^M \in \mathbb{E}^M \text{ s.t. } \mathbf{\Lambda}^{(p)} \in \mathbb{E}^M\}, \tag{6.22}$$

and call *optimal* a Markovian evolution $\mathbf{\Lambda}^M$ that allows us to attain such value. In other contexts, e.g. resource theories [NBC+16, TRB+19], the measure of non-Markovianity $p(\mathbf{\Lambda})$ is after referred to as a *robustness* measure. The value $p(\mathbf{\Lambda})$ is always well defined since the set of p entering the optimization contains at least the point 1. âĂćThe rational of this choice is that, the greater is p, the stronger is the perturbation we add into the system by the mixing operation (6.21): indeed, for fixed $\mathbf{\Lambda}^M$, the distance between $\mathbf{\Lambda}^{(p)}$ and the original trajectory $\mathbf{\Lambda}$ is always proportional to p. For instance, at any given time t we can write $\|\Lambda_t^{(p)} - \Lambda_t\| = p\|\Lambda_t^M - \Lambda_t\|$ where $\|\cdot\|$ stands for (say) the diamond norm for super-operators [KSV02]. As a consequence, $p(\mathbf{\Lambda})$ is the minimum perturbation one needs to introduce via the mixing procedure (6.21) to enforce Markovianity into the system evolution. The maximum value of this quantity has a precise meaning: $p(\mathbf{\Lambda}) = 1$ implies that $\mathbf{\Lambda}$ cannot be made Markovian by any non-trivial mixture (6.21). On the contrary, since $p(\mathbf{\Lambda}) = 0$ if and only if $\mathbf{\Lambda} \in \mathbb{E}^M$, it is clear that (6.22) is a faithful measure of non-Markovianity.

We can consider the case where in Eq. (6.21) $\mathbf{\Lambda}^{(p)}$ is asked to belong to a specific Markovian target subset $\mathbb{T}^M$ of $\mathbb{E}^M$, while at same time $\mathbf{\Lambda}^M$ belongs to a particular set $\mathbb{A}^M$ of $\mathbb{T}^M$ (namely $\mathbb{A}^M \subseteq \mathbb{T}^M \subseteq \mathbb{E}^M$). This leads to the functional

$$p(\mathbf{\Lambda}|\, \mathbb{A}^M, \mathbb{T}^M) \;\; \equiv \;\; \min_p \{p \,|\, \exists \mathbf{\Lambda}^M \in \mathbb{A}^M \text{ s.t. } \mathbf{\Lambda}^{(p)} \in \mathbb{T}^M\}, \tag{6.23}$$

which by construction provides a bound for (6.22)

$$p(\mathbf{\Lambda}|\, \mathbb{A}^M, \mathbb{T}^M) \geq p(\mathbf{\Lambda}|\, \mathbb{A}^M, \mathbb{E}^M) \geq p(\mathbf{\Lambda})\,, \tag{6.24}$$

A typical situation where $p(\mathbf{\Lambda}|\, \mathbb{A}^M, \mathbb{T}^M)$ can be considered is given when $\mathbb{A}^M$ represents the accessible Markovian evolutions that we are able to reproduce in our laboratory and mix with $\mathbf{\Lambda}$, while $\mathbb{T}^M$ represents a particular subset of $\mathbb{E}^M$ for which Markovianity is easy to certify, or which possesses some additional features that we demand. From this perspective Eq. (6.24), besides being an upper bound for Eq. (6.22) can also be seen as a different approach to quantify the degree of non-Markovianity of the process $\mathbf{\Lambda}$. A case of special interest is provided by the scenario where the subsets $\mathbb{A}^M$ and $\mathbb{T}^M$ entering (6.23) coincide and correspond to the Markovian part of a convex subset of the system evolutions $\mathbb{B} \subset \mathbb{E}$, namely $\mathbb{A}^M = \mathbb{T}^M = \mathbb{B}^M \equiv \mathbb{B} \cap \mathbb{E}^M$. Under these conditions from (6.21) it follows that we can write

$$p(\mathbf{\Lambda}|\mathbb{B}^M) \equiv p(\mathbf{\Lambda}|\mathbb{B}^M, \mathbb{B}^M) = p(\mathbf{\Lambda}|\mathbb{B}^M, \mathbb{E}^M) , \qquad \forall \mathbf{\Lambda} \in \mathbb{B} , \tag{6.25}$$

showing that for the elements of $\mathbb{B}$, at least the first of the inequalities in (6.24) closes (of course this does not necessarily hold if $\mathbb{B}$ is not convex, as in this case there could be maps $\mathbf{\Lambda}^{(p)}$ in $\mathbb{E}^M$ which are not necessarily in $\mathbb{B}^M$).

Furthermore, while we have no explicit evidence in support of this claim, if $\mathbb{B}$ is a sufficiently "structured" set as in the case of the depolarizing evolutions addressed in the following subsection, it is also tempting to conjecture that the second gap in (6.24) should collapse too, implying that in this case $p(\mathbf{\Lambda}|\mathbb{B})$ should coincide with $p(\mathbf{\Lambda})$ for all $\mathbf{\Lambda} \in \mathbb{B}$, or equivalently that

$$\textbf{(CONJECTURE)} \quad p(\mathbf{\Lambda}) = p(\mathbf{\Lambda}|\mathbb{B}^M) , \qquad \forall \mathbf{\Lambda} \in \mathbb{B} . \tag{6.26}$$

In order to discuss this conjecture, we notice that the dynamical maps and the intermediate maps (see respectively Eqs. (6.2) and (6.6)) that define depolarizing evolutions transform the state space with the same spherical symmetry. Therefore, non-Markovian effects arising from these evolutions are characterized by the same property. As a result, it is reasonable to conjecture that, in order to contrast the non-Markovian effects of a generic $\boldsymbol{D}^{NM}$ in the most efficient manner, namely by finding a Markovian evolution $\mathbf{\Lambda}^M$ that allows the minimum value of p in Eq. (6.22), it is enough to consider only Markovian depolarizing evolutions $\mathbb{D}^M$. We expect that the same argument can be applied to other convex sets of evolutions with analogous symmetries, e.g. dephasing evolutions (see Section 6.8).

6.3.1 Measuring the non-Markovianity of depolarizing evolutions

To study the non-Markovian behavior of depolarizing evolutions $\boldsymbol{D} \in \mathbb{D}$ we shall focus on the case where the set $\mathbb{B}$ entering in Eq. (6.25) corresponds to $\mathbb{D}$ itself, namely the quantity $p(\boldsymbol{D}|\mathbb{D}^M)$. While for elements of the Markovian subset $p(\boldsymbol{D}|\mathbb{D}^M)$ is clearly equal to 0, in the case $\boldsymbol{D}^{NM} \in \mathbb{D}^{NM}$ we can invoke (6.13) to claim the following lower bound

$$p(\boldsymbol{D}^{NM}|\mathbb{D}^M) \geq p^*(\boldsymbol{D}^{NM}) , \tag{6.27}$$

which is non trivial due to the fact that $p^*(\boldsymbol{D}^{NM})$ is strictly larger than 0. Since $\mathbb{D}_C$ is a proper subset of $\mathbb{D}$, it is also clear that in general the following ordering holds

$$p(\boldsymbol{D}|\mathbb{D}_C^M, \mathbb{D}^M) \geq p(\boldsymbol{D}|\mathbb{D}^M) , \qquad \forall \boldsymbol{D} \in \mathbb{D} . \tag{6.28}$$

In particular if the channel we test is an element of the continuous subset of $\mathbb{D}$, the inequality in Eq. (6.28) closes, leading to

$$p(\boldsymbol{D}_C|\mathbb{D}_C^M) = p(\boldsymbol{D}_C|\mathbb{D}^M)\,, \qquad \forall \boldsymbol{D}_C \in \mathbb{D}_C\,. \tag{6.29}$$

Notice that we used the fact that, due to the convexity of $\mathbb{D}_C$, one has that $p(\boldsymbol{D}_C|\mathbb{D}_C^M, \mathbb{D}^M)$ corresponds to $p(\boldsymbol{D}_C|\mathbb{D}_C^M) \equiv p(\boldsymbol{D}_C|\mathbb{D}_C^M, \mathbb{D}_C^M)$ when evaluated on $\boldsymbol{D}_C \in \mathbb{D}_C$. The proof of Eq. (6.29) is rather cumbersome and we postpone it to Section 6.5, focusing first on the explicit computation of $p(\boldsymbol{D}_C|\mathbb{D}_C^M)$, which we present in Section 6.4.

6.4 Non-Markovianity measure for continuous depolarizing evolutions

In this section we evaluate our measure of non-Markovianity

$$p(\boldsymbol{D}_C|\mathbb{D}_C^M)\,, \tag{6.30}$$

for the cases where $\boldsymbol{D}_C$ is an arbitrary element of the continuous subset $\mathbb{D}_C$ of the depolarizing evolutions, under the assumption that also the transformations $\boldsymbol{D}^M$ of (6.31) are elements of $\mathbb{D}_C$. Before entering into the details of the analysis it is worth clarifying that in computing $p(\boldsymbol{D}_C|\mathbb{D}_C^M)$ the map $\boldsymbol{\Lambda}^{(p)}$ of Eq. (6.21) has the form

$$\boldsymbol{D}_C^{(p)} = (1-p)\boldsymbol{D}_C + p\boldsymbol{D}_C^M, \tag{6.31}$$

where $\boldsymbol{D}_C^M \in \mathbb{D}_C^M$ and $\boldsymbol{D}_C \in \mathbb{D}_C$. Thus, since $\mathbb{D}_C$ is convex, for any p, $\boldsymbol{D}_C$ and $\boldsymbol{D}_C^M$, we have that $\boldsymbol{D}_C^{(p)} \in \mathbb{D}_C$ with characteristic function $f_C^{(p)}(t) \in \mathfrak{F}_C$ given by the convex sum of the characteristic functions $f_C(t)$ and $f_C^M(t)$ associated with $\boldsymbol{D}_C$ and $\boldsymbol{D}_C^M$ respectively, namely

$$f_C^{(p)}(t) = (1-p)f_C(t) + pf_C^M(t)\,. \tag{6.32}$$

In order to evaluate $p(\boldsymbol{D}_C|\mathbb{D}^M)$ our goal is hence to obtain the optimal choice of $f_C^M(t) \in \mathfrak{F}_C^M$ that allows the minimum value of p such that $f_C^{(p)}(t) \in \mathfrak{F}_C^M$.

As notice before, if $\boldsymbol{D}_C$ is an element of $\mathbb{D}_C^M$ then we can simply take $p = 0$, namely $p(\boldsymbol{D}_C^M|\mathbb{D}_C^M) = 0$. For the depolarizing evolutions which instead have a continuous characteristic function $f_C^{NM}(t)$ that possesses some degree of non-Markovianity, the computation of (6.30) requires instead some non trivial work. In this case Eq. (6.32) becomes

$$f_C^{(p)}(t) = (1-p)f_C^{NM}(t) + pf_C^M(t)\,. \tag{6.33}$$

While the continuity of $f_C^{(p)}(t)$ is automatically ensured by construction, finding the minimum p that forces this function into $\mathfrak{F}_C^M$ (namely that allows it to be also positive and non-increasing) is not a simple task. In order to tackle this problem we start by illustrating the relatively simple case of non-Markovian depolarizing evolutions with positive $f_C^{NM}(t) \in \mathfrak{F}_C^{NM}$ (see Section 6.4.1). Next we discuss the slightly more complex scenario of $f_C^{NM}(t) \in \mathfrak{F}_C^{NM}$ having a non definite sign, but which exhibit their non-Markovian character exclusively on the time intervals where they are negative (Section 6.4.2). Finally we conclude by addressing the general case of a non-Markovian continuous characteristic functions $f_C^{NM}(t) \in \mathfrak{F}_C^{NM}$ in Section 6.4.3.

6.4.1 Positive non-Markovian continuous characteristic functions

In this section we study depolarizing processes $\boldsymbol{D}_C^{NM}$ characterized by $f_C^{NM}(t) \in \mathfrak{F}_C^{NM}$ which are positive and which have a number $L > 0$ of intervals $T_k^+ \equiv (t_k^{(in)}, t_k^{(fin)})$ of non-Markovianity where $\dot{f}^{NM}(t^\pm) > 0$, namely

$$\begin{cases} f_C^{NM}(t) \geq 0, \dot{f}_C^{NM}(t^\pm) \leq 0, \xi(f_C^{NM}(t)) = 1 \quad t \notin T^{NM} \;, \\ \\ f_C^{NM}(t) \geq 0, \dot{f}_C^{NM}(t^\pm) > 0, \xi(f_C^{NM}(t)) = 1 \quad t \in T^{NM} \;, \end{cases} \tag{6.34}$$

with $T^{NM} \equiv \bigcup_{k=1}^L T_k^+$ being the collection of the intervals T_k^+. As we shall see, in this case the quantity (6.30) is a monotonically increasing function of the gaps

$$\Delta_k^{NM} \equiv f_C^{NM}(t_k^{(fin)}) - f_C^{NM}(t_k^{(in)}) > 0 \,, \tag{6.35}$$

which certify the non-Markovian character of $f_C^{NM}(t)$ on the intervals T_k^+. Specifically, given

$$\Delta^{NM} \equiv \sum_{k=1}^{L} \Delta_k^{NM} \;, \tag{6.36}$$

we have

$$p(\boldsymbol{D}_C^{NM} | \mathbb{D}_C^M) = \frac{\Delta^{NM}}{1 + \Delta^{NM}} \;, \tag{6.37}$$

which saturates to its upper bound 1 in the case where Δ^{NM} diverges, e.g. when $f_C^{NM}(t)$ exhibit infinite, not properly dumped, oscillations. In order to derive (6.37) we first address the simple case of a single non-Markovian interval ($L = 1$), and then generalize it to the case of arbitrary (possibly infinite) L, where this last case is studied in Appendix D.4.

One time interval of non-Markovianity for positive characteristic functions

Let $\boldsymbol{D}_C^{NM}$ be an element of $\mathbb{D}_C^{NM}$ with characteristic function $f_C^{NM}(t) \in \mathfrak{F}_C^{NM}$ that is always positive and which has positive derivative (hence non-Markovian character) in a single time interval $T_1^+ = (t_1^{(in)}, t_1^{(fin)})$ ($t_1^{(fin)}$ being possibly infinite), i.e,

$$\begin{cases} f_C^{NM}(t) \geq 0, \dot{f}_C^{NM}(t^\pm) \leq 0, \xi(f_C^{NM}(t)) = 1 & t \notin T_1^+ \,, \\ f_C^{NM}(t) \geq 0, \dot{f}_C^{NM}(t^\pm) > 0, \xi(f_C^{NM}(t)) = 1 & t \in T_1^+ \,. \end{cases} \tag{6.38}$$

Our goal is to determine the minimum value of p which allows $f_C^{(p)}(t)$ of (6.33) to be an element of $\mathfrak{F}_C^M$, namely to obey to the first of the constraints (6.10) – the function being already continuous by construction. Since both $f_C^{NM}(t)$ and $f_C^M(t)$ are non-negative, this is equivalent to impose

$$\dot{f}_C^{(p)}(t^\pm) = (1-p)\dot{f}_C^{NM}(t^\pm) + p\dot{f}_C^M(t^\pm) \leq 0 \,, \tag{6.39}$$

which is automatically verified for $t \notin T_1^+$. A necessary condition for (6.39) can then be obtained by imposing that $f_C^{(p)}(t)$ experiences a negative gap at the extreme points of T_1^+, namely

$$\Delta_1^{(p)} \equiv f_C^{(p)}(t_1^{(fin)}) - f_C^{(p)}(t_1^{(in)}) \leq 0 \,. \tag{6.40}$$

From (6.33) we can cast this into the condition

$$\Delta_1^{(p)} = (1-p)\Delta_1^{NM} + p\Delta_1^M \leq 0 \,, \tag{6.41}$$

where Δ_1^{NM} is the positive gap defined as in Eq. (6.35) and

$$\Delta_1^M \equiv f_C^M(t_1^{(fin)}) - f_C^M(t_1^{(in)}) \,, \tag{6.42}$$

is the associated gap of $f_C^M(t)$. Notice that from the properties of $f_C^M(t)$ it follows that the latter quantity is non-negative and larger than -1 (which is the minimum allowed gap for an element of $\mathfrak{F}_C^M$), namely

$$\Delta_1^M \in [-1, 0] \quad \Longrightarrow \quad |\Delta_1^M| \leq 1 \,. \tag{6.43}$$

From Eq. (6.41) it follows that a necessary condition for p is

$$p \geq \frac{\Delta_1^{NM}}{|\Delta_1^M| + \Delta_1^{NM}} \geq \frac{\Delta_1^{NM}}{1 + \Delta_1^{NM}} \equiv p_1 \,, \tag{6.44}$$

where the last inequality follows from (6.43).

To show that (6.44) is also a sufficient condition for (6.39), we provide a particular example of $f_C^M(t)$ such that $\dot{f}_C^{(p)}(t) \leq 0$ for $p \geq p_1$. For this purpose consider $g_C^M(t) \in \mathfrak{F}_C^M$ such that

$$g_C^M(t) = \begin{cases} 1 & t \leq t_1^{(in)} \ , \\ 1 - \left(f^{NM}(t) - f^{NM}(t_1^{(in)})\right)/\Delta_1^{NM} & t \in T_1^+ \ , \\ 0 & t \geq t_1^{(fin)} \ . \end{cases} \tag{6.45}$$

This function, for $t \in T_1^+$, is a linear manipulation of $f_C^{NM}(t)$, where its slope is stretched and inverted. Moreover, in this case $\Delta_1^M = -1$ and $\Delta_1^{(p)} \leq 0$ for $p \geq p_1$. Finally, if we consider $g_C^M(t)$ in $f^{(p)}(t)$, for $p = p_1$, we obtain

$$f^{(p_1)}(t) = \frac{f^{NM}(t_1^{(fin)})}{1 + \Delta_1^{NM}} \ , \qquad \text{for } t \in T_1^+ \ , \tag{6.46}$$

which is a constant. Hence, in this case $\dot{f}^{(p_1)}(t) \leq 0$ for any $t \geq 0$. Putting all together we can hence claim that

$$p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M) = p_1 = \frac{\Delta_1^{NM}}{1 + \Delta_1^{NM}} \ , \tag{6.47}$$

which proves the validity of (6.37) at least for the functions we are considering here, namely when $L = 1$. In Appendix D.4 we show how to extend these results to any $L > 1$.

6.4.2 Characteristic functions that exhibit non-Markovianity only when negative

Here we consider elements of $\mathbb{D}_C^{NM}$ with $f_C^{NM}(t)$ such that their non-Markovian nature is shown only in a number $m > 0$ of time intervals $T_j^- \equiv (t_j^{(in)}, t_j^{(fin)})$ where it assumes negative values while being strictly decreasing, namely violating $\mathbf{CM_1}(\tau)$ while being negative, as notified by the following negative gaps

$$\Theta_j^{NM} \equiv f_C^{NM}(t_j^{(fin)}) - f_C^{NM}(t_j^{(in)}) < 0 \ . \tag{6.48}$$

It is worth observing that under the above assumption $f_C^{NM}(t)$ cannot be positive after that it becomes negative for the first time. Otherwise, for some time we would have $f_C^{NM}(t) \geq 0$ and $\dot{f}_C^{NM}(t^+) > 0$, which contradicts our premise. Therefore, we have that

$$f_C^{NM}(t) \leq 0 \ , \qquad \forall t \geq t_1^{(in)} \ . \tag{6.49}$$

We shall see that in this scenario the measure of non-Markovianity (6.30) reduces to

$$p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M) = \frac{|\Theta^{NM}|}{1+|\Theta^{NM}|}\,, \tag{6.50}$$

with

$$\Theta^{NM} \equiv \sum_{j=1}^{m} \Theta_j^{NM}\ . \tag{6.51}$$

As in the previous section, to derive the above identity first we obtain a necessary condition for $f^{(p)}(t)$ to belong to $\mathfrak{F}_C^M$ and then we provide an explicit example that saturates this value. In this case however we find it useful to treat separately the case of finite m from those where m is unbounded which introduce some technicalities which have to be dealt carefully. The study for unbounded values of m is given in Appendix D.5.

The m finite case

If m is finite the function $f_C^{NM}(t)$ cannot exhibit infinite oscillations. Therefore,

$$\lim_{t\to\infty} f_C^{NM}(t) = f_C^{NM}(\infty) \leq 0\ . \tag{6.52}$$

Define now $\overline{T}_j = (\bar{t}_j^{(in)}, \bar{t}_j^{(fin)})$ to be the time intervals when $f_C^{NM}(t) \leq 0$ and $\dot{f}_C^{NM}(t^\pm) \geq 0$, namely the times when the Markovian condition $\mathbf{CM_1}(\tau)$ is satisfied while $f_C^{NM}(t)$ is negative. We notice that, since $f_C^{NM}(t)$ is continuous, for any T_j^- there exists a $\overline{T}_j$ such that $t_j^{(fin)} = \bar{t}_j^{(in)}$, the only case when it does not happen is for $t_j^{(fin)} = \infty$: accordingly the total number $\overline{m}$ of the intervals $\overline{T}_j$ is either equal to m or to $m-1$ and is hence also finite by assumption.

We consider now the associated gaps of the functions $f_C^{NM}(t)$, $f_C^M(t)$, and $f_C^{(p)}(t)$, namely the quantities

$$\delta_j^{NM} \equiv f_C^{NM}(\bar{t}_j^{(fin)}) - f_C^{NM}(\bar{t}_j^{(in)})\,, \tag{6.53}$$

$$\delta_j^{M} \equiv f_C^{M}(\bar{t}_j^{(fin)}) - f_C^{M}(\bar{t}_j^{(in)})\,, \tag{6.54}$$

$$\delta_j^{(p)} \equiv f_C^{(p)}(\bar{t}_j^{(fin)}) - f_C^{(p)}(\bar{t}_j^{(in)}) = (1-p)\delta_j^{NM} + p\delta_j^M\ . \tag{6.55}$$

By definition we have that the δ_j^{NM} must be non-negative, while the δ_j^M must be non-positive, namely

$$\delta_j^{NM} \geq 0\ , \qquad \delta_j^M \leq 0\ , \qquad \forall j. \tag{6.56}$$

If $f_C^{(p)}(t)$ is Markovian it has to be positive and non-increasing. Therefore, we should also have

$$\delta_j^{(p)} \leq 0 \ , \qquad \forall j. \tag{6.57}$$

Therefore a necessary condition for the Markovianity of $f_C^{(p)}(t)$ is given by the following inequality

$$\delta^{(p)} \equiv \sum_{j=1}^{\overline{m}} \delta_j^{(p)} = (1-p)\delta^{NM} - p|\delta^M| \leq 0 \, , \tag{6.58}$$

where $\delta^M \equiv \sum_{j=1}^{\overline{m}} \delta_j^M \leq 0$ and $\delta^{NM} \equiv \sum_{j=1}^{\overline{m}} \delta_j^{NM} \geq 0$.

Observe that since $f_C^M(t)$ and $f_C^{(p)}(t)$ are both elements of $\mathfrak{F}_C^M$ their limiting values for $t \to \infty$ exist and fulfill the following constraints

$$f_C^M(t) \geq f_C^M(\infty) \geq 0 \, , \qquad f_C^{(p)}(t) \geq f_C^{(p)}(\infty) \geq 0 \, , \tag{6.59}$$

for all $t \geq 0$. Notice finally that since $f_C^M(t)$ is non increasing and upper bounded by 1, its limiting value must fulfill the constraint

$$1 \geq f_C^M(\infty) + |\delta^M| \, . \tag{6.60}$$

Accordingly from (6.52) we can write

$$f_C^{(p)}(\infty) = (1-p) f_C^{NM}(\infty) + p f_C^M(\infty) \geq 0 \, , \tag{6.61}$$

or equivalently

$$-(1-p)(\delta^{NM} + \Theta^{NM}) - p f_C^M(\infty) \leq 0 \, , \tag{6.62}$$

where we used

$$f_C^{NM}(\infty) = \delta^{NM} + \Theta^{NM} \ , \tag{6.63}$$

with Θ^{NM} as in Eq. (6.51). Summing up (6.62) with (6.58) term by term, the following necessary constraint for p can finally be obtained

$$-(1-p)\Theta^{NM} - p(f_C^M(\infty) + |\delta^M|) \leq 0 \, , \tag{6.64}$$

which implies

$$p \geq \frac{|\Theta^{NM}|}{f_C^M(\infty) + |\delta^M| + |\Theta^{NM}|} \geq \frac{|\Theta^{NM}|}{1 + |\Theta^{NM}|} \equiv p_m \ , \tag{6.65}$$

where in the last passage we used the inequality (6.60). Accordingly we can conclude that the quantity p_m is lower bound for the value $p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)$ associated with the evolutions $\boldsymbol{D}_C^{NM}$ we are considering here.

In order to show that p_m does indeed correspond to $p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)$ we now present a example of $f_C^M(t)$ which makes $f_C^{(p)}(t)$ an element of $\mathfrak{F}_C^M$ for $p = p_m$. To do so we define $\overline{g}_C^M(t) \in \mathfrak{F}_C^M$ to be equal to

$$\begin{cases} 1 & t \le t_1^{(fin)} \\ 1 - \left(f^{NM}(t) - f^{NM}(\bar{t}_1^{(in)})\right)/|\Theta^{NM}| & t \in \overline{T}_1 \\ 1 - \delta_1^{NM}/|\Theta^{NM}| & t \in T_2^- \\ (1 - \delta^{NM}/|\Theta^{NM}|) - \left(f^{NM}(t) - f^{NM}(\bar{t}_2^{(in)})\right)/|\Theta^{NM}| & t \in \overline{T}_2 \\ \dots & \\ \overline{g}_C^M(\bar{t}_{j-1}^{(fin)}) - \left(f^{NM}(t) - f^{NM}(\bar{t}_j^{(in)})\right)/|\Theta^{NM}| & t \in \overline{T}_j \\ 1 - \sum_{i=1}^j \delta_i^{NM}/|\Theta^{NM}| & t \in T_{j+1}^- \\ \dots & \\ 1 - \delta^{NM}/|\Theta^{NM}| & t \to \infty \end{cases} . \tag{6.66}$$

The temporal derivative of $\overline{g}_C^M(t)$ assumes the simple form

$$\dot{\overline{g}}_C^M(t^\pm) = \begin{cases} -\dot{f}_C^{NM}(t^\pm)/|\Theta^{NM}| & t \in \overline{T}_j \\ 0 & \text{otherwise} \end{cases} . \tag{6.67}$$

It is easy to show that $f_C^{(p)}(t) = (1-p)f_C^{NM}(t) + p\overline{g}_C^M(t)$ Markovian for $p \ge p_m$. Therefore, for any $f_C^{NM}(t)$ that shows a non-Markovian behavior while being negative, we have that

$$p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M) = p_m = \frac{|\Theta^{NM}|}{1 + |\Theta^{NM}|}, \tag{6.68}$$

which proves (6.50). We study the cases where we have unbounded values of m in Appendix D.5.

6.4.3 Multiple time intervals of non-Markovianity for continuous characteristic functions: the general case

Building up from the previous sections here we compute $p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)$ for the general case of a non-Markovian depolarizing processes with continuous characteristic function $f_C^{NM}(t)$. At variance with the examples discussed before, now $f_C^{NM}(t)$ may possess both a collection of time intervals $T_k^+ \equiv (t_k^{(in)}, t_k^{(fin)})$ where it is positive and increasing, and also time intervals $T_j^- \equiv (t_j^{(in)}, t_j^{(fin)})$

where instead it is negative and decreasing (namely it may exhibit all the non-Markovian features detailed separately in Section 6.4.1 and Section 6.4.2).

In this case we can show that Eqs. (6.37) and (6.50) get replaced by the more general formula

$$p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M) = \frac{\Gamma^{NM}}{1+\Gamma^{NM}} , \tag{6.69}$$

with Γ^{NM} being given by the expression

$$\Gamma^{NM} \equiv \Delta^{NM} + |\Theta^{NM}| , \tag{6.70}$$

where Δ^{NM} and Θ^{NM}, defined as in Eqs. (6.36) and (6.51), are the sums of the non-Markovian increments the function $f_C^{NM}(t)$ experiences on the intervals T_k^+ and T_j^-, respectively.

Since $f_C^{NM}(t)$ may not admit a limiting value for $t \to \infty$, to prove (6.69) we shall proceed as in Appendix D.5, determining first the conditions under which the associated $f_C^{(p)}(t)$ is guaranteed to be Markovian at least in the time interval $[0,T]$ with T finite. Under this condition the numbers $L(T)$ and $m(T)$ of intervals T_k^+ and T_j^- of $f_C^{NM}(t)$ that fit on the considered domain, are both finite. We introduce also the time intervals $\overline{T}_j \equiv (\bar{t}_j^{(in)}, \bar{t}_j^{(fin)})$ of $[0,T]$ where $f_C^{NM}(t)$ is negative and non decreasing (their number $\overline{m}(T)$ being finite too), and define the gaps $\Delta_k^{NM}(T)$, $\Delta_k^M(T)$, $\Delta_k^{(p)}(T)$, $\Theta_j^{NM}(T)$, $\delta_j^{NM}(T)$, $\delta_j^M(T)$ and $\delta_j^{(p)}(T)$ as in Eqs. (6.35), (D.21), (D.22), (6.48), (6.53), (6.54), and (6.55). By construction we have the following conditions

$$\begin{aligned}
&\Delta_k^{NM}(T) > 0 , \qquad \Theta_j^{NM}(T) < 0 , \qquad \delta_j^{NM}(T) \geq 0 , \\
&\Delta_k^{M}(T) \leq 0 , \qquad \delta_j^{M}(T) \leq 0 , \\
&\Delta_k^{(p)}(T) = (1-p)\Delta_k^{NM}(T) + p\Delta_k^{M}(T) , \qquad (6.71)\\
&\delta_j^{(p)}(T) = (1-p)\delta_j^{NM}(T) + p\delta_j^{M}(T) , \qquad (6.72)
\end{aligned}$$

for all k and j. A necessary condition for $f^{(p)}(t)$ being Markovian on the considered domain is that all its gaps $\Delta_k^{(p)}(T)$ and $\delta_j^{(p)}(T)$ are non-positive, namely

$$(1-p)\Delta_k^{NM}(T) + p\Delta_k^{M}(T) \leq 0 , \tag{6.73}$$
$$(1-p)\delta_j^{NM}(T) + p\delta_j^{M}(T) \leq 0 . \tag{6.74}$$

By summing up term by term, all contributions from (6.73) and (6.74) we get

$$(1-p)(\Delta^{NM}(T) + \delta^{NM}(T)) - p(|\Delta^{M}(T)| + |\delta^{M}(T)|) \leq 0 , \tag{6.75}$$

where

$$\Delta^{NM}(T) \equiv \sum_{k=1}^{L(T)} \Delta_k^{NM}(T) > 0 \; , \qquad \Delta^{M}(T) \equiv \sum_{k=1}^{L(T)} \Delta_k^{M}(T) \leq 0 \; ,$$
$$\delta^{NM}(T) \equiv \sum_{k=1}^{\overline{m}(T)} \Delta_k^{NM}(T) > 0 \; , \qquad \delta^{M}(T) \equiv \sum_{k=1}^{\overline{m}(T)} \Delta_k^{M}(T) \leq 0 \; .$$

Suppose now that $f_C^{NM}(T)$ is a non-negative quantity, namely $f_C^{NM}(T) \geq 0$. Under this condition it is easy to verify that the sum of gaps this function experiences on the interval where it is negative must nullify, namely

$$\delta^{NM}(T) = |\Theta^{NM}(T)| \; , \tag{6.76}$$

with

$$\Theta^{NM}(T) \equiv \sum_{j=1}^{m(T)} \Theta_j^{NM}(T) < 0 \; . \tag{6.77}$$

Replacing this into (6.75) we hence get the condition

$$\begin{aligned} p \;\; &\geq \;\; \frac{\Delta^{NM}(T) + |\Theta^{NM}(T)|}{|\Delta^{M}(T)| + |\delta^{M}(T)| + \Delta^{NM}(T) + |\Theta^{NM}(T)|} \\ &\geq \;\; \frac{\Delta^{NM}(T) + |\Theta^{NM}(T)|}{1 + \Delta^{NM}(T) + |\Theta^{NM}(T)|} \; , \end{aligned} \tag{6.78}$$

where in the second line we used the fact that the sum over the gaps of a continuous Markovian function cannot cannot be larger than 1, namely $|\Delta^{M}(T)| + |\delta^{M}(T)| \leq 1$.

If $f_C^{NM}(T)$ is negative, namely $f_C^{NM}(T) < 0$, we can still show that (6.78) holds, but we need to change the derivation. In this case we observe that Eq. (6.76) is substituted by the constraint

$$f_C^{NM}(T) \;\; = \;\; \delta^{NM}(T) + \Theta^{NM}(T) \; , \tag{6.79}$$

which allows us to rewrite positivity of $f_M^{(p)}(t)$ for $t = T$ (a necessary condition for $f^{(p)}(t)$ to be Markovian on $[0, T]$) as

$$(1-p)(\delta^{NM}(T) + \Theta^{NM}(T)) + p f_C^{M}(T) \;\; \geq \;\; 0 \, . \tag{6.80}$$

Together with (6.75) the above expression finally leads to

$$(1-p)(\Delta^{NM}(T) - \Theta^{NM}(T)) \leq p(|\Delta^{M}(T)| + |\delta^{M}(T)| + f_C^{M}(T)) \leq p \; , \tag{6.81}$$

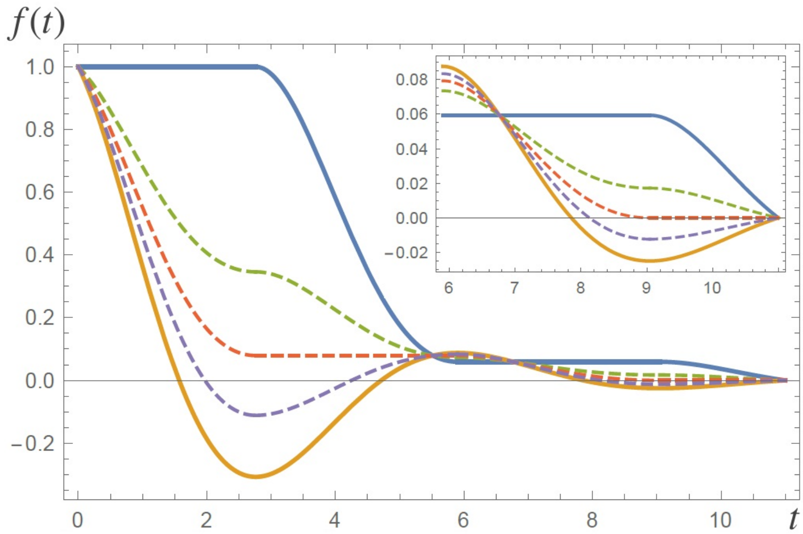


Figure 6.3: Plots of $f_C^{NM}(t) = e^{-2t/5}\cos(t)$ (yellow), the corresponding optimal Markovian characteristic function $h_C^M(t)$ (blue) and $f^{(p)}(t)$ for different values of p (dashed lines) in the time interval $t \in [0, 7\pi/2]$. The inset shows their behavior for $t \geq 5.90$. In this example $T_1^- \simeq (\pi/2, 2.76)$, $T_1^+ \simeq (3\pi/2, 5.90)$ and $T_2^- \simeq (5\pi/2, 9.04)$ are the time intervals of non-Markovianity of $f^{NM}(t)$ and $\Theta_1^{NM} \simeq -0.31$, $\Delta_1^{NM} \simeq 0.09$ and $\Theta_2^{NM} \simeq -0.02$ are the corresponding non-Markovian gaps. The value of the measure of non-Markovianity is $p(\boldsymbol{D}_C^{NM}|\mathbb{D}^M) \simeq 0.30$. If $p = 0.5 > p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)$, $f^{(p)}(t) \in \mathfrak{F}_C^M$ is monotonically decreasing (green dashed line). If $p = p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M) \simeq 0.30$, $f^{(p)}(t) \in \mathfrak{F}_C^M$ is monotonically decreasing and constant when $\dot{f}^{NM}(t) > 0$ (red dashed line). If $p = 0.15 < p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)$, $f^{(p)}(t) \in \mathfrak{F}_C^{NM}$ is not monotonic nor positive in more than one time interval (purple dashed line).

where in the last passage we used the fact that continuous Markovian characteristic function cannot have drops larger than 1, namely $|\Delta^M(T)| + |\delta^M(T)| + f_C^M(T) \leq 1$. Eq. (6.81) coincides with (6.78) which hence holds true independently from the sign of $f_C^{NM}(T)$. Taking the limit $T \to \infty$ we can finally conclude that a necessary condition for $f_C^{(p)}(t)$ to be Markovian is

$$p \geq \frac{\Gamma^{NM}}{1 + \Gamma^{NM}} , \tag{6.82}$$

with Γ^{NM} as in (6.70) with Δ^{NM} and Θ^{NM} formally given by

$$\Delta^{NM} = \lim_{T\to\infty} \Delta^{NM}(T) , \qquad \Theta^{NM} = \lim_{T\to\infty} \Theta^{NM}(T) . \tag{6.83}$$

To show that the inequality (6.82) is also a sufficient condition for the Markovianity of $f_C^{(p)}(t)$ we now provide an explicit example that saturates it – in Appendix D.6 we also prove that the solution we present here is unique.

It is intuitive to understand that the function $h_C^M(t) \in \mathfrak{F}_C^M$ that we are looking for must be a combination of $g_C^M(t)$ (see Eq. (D.26)) and $\overline{g}_C^M(t)$ (see Eq. (6.66)). In order to simplify its complicated formulation, we express $h_C^M(t)$ only through its temporal derivative

$$\dot{h}_C^M(t^\pm) = \begin{cases} -\dot{f}_C^{NM}(t^\pm)/\Gamma^{NM} & t \in T_k^+ \\ -\dot{f}_C^{NM}(t^\pm)/\Gamma^{NM} & t \in \overline{T}_j \\ 0 & \text{otherwise} \end{cases} , \tag{6.84}$$

which can be rewritten in a particularly simple form

$$\dot{h}_C^M(t^\pm) = \begin{cases} -\dot{f}_C^{NM}(t^\pm)/\Gamma^{NM} & \text{if } \dot{f}_C^{NM}(t) > 0 \\ 0 & \text{otherwise} \end{cases} , \tag{6.85}$$

(see Figure 6.3 for an example). After a long but straightforward calculation, it is possible to show that $f^{(p)}(t) = (1-p)f_C^{NM}(t) + ph_C^M(t)$ belongs to the Markovian set for all p fulfilling (6.82). Therefore, this proves that

$$p(\boldsymbol{D}_C^{NM}|\mathbb{D}^M) = \frac{\Gamma^{NM}}{1 + \Gamma^{NM}} , \tag{6.86}$$

and therefore (6.69).

6.5 Continuity of the optimal characteristic functions for continuous non-Markovian evolutions

In this section we prove the identities (6.29) showing that in the case of continuous characteristic functions $f_C(t)$, non-continuous Markovian characteristic functions $f^M(t) \notin \mathfrak{F}_C^M$ cannot make their convex combination $f^{(p)}(t)$ Markovian for values of p smaller than $p(\boldsymbol{D}_C|\mathbb{D}_C^M)$. This is trivial if $f_C(t)$ is already Markovian as in this case $p(\boldsymbol{D}_C|\mathbb{D}_C^M)$ saturates to the minimum allowed value 0. For characteristic functions which are explicitly non-Markovian in Section 6.4.1 we analyze the simple scenario of positive functions which exhibit non-Markovianity only in a single interval. Then in Section 6.5.2 we discuss the case of functions that have non-Markovian behavior when negative, and conclude in Section 6.5.3 with the general case.

6.5.1 Single non-Markovian time interval with $f_C^{NM}(t) \geq 0$

We start by studying the cases discussed in Section 6.4.1, where $f_C^{NM}(t)$ has a single time interval (t_1, t_2) of non-Markovianity when $f_C^{NM}(t) \geq 0$ and $\dot{f}_C^{NM}(t) > 0$. In this case the optimal continuous Markovian function $g_C^M(t)$ which makes the corresponding $f^{(p)}(t)$ Markovian for the smallest p is given in Eq. (6.45) and leads to

$$p \geq p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M) = \frac{\Delta^{NM}}{1+\Delta^{NM}} , \tag{6.87}$$

where $\Delta^{NM} = f_C^{NM}(t_2) - f_C^{NM}(t_1) > 0$. To show that Eq. (6.87) cannot be improved by allowing $f^M(t)$ being non continuous, we start noticing that in this scenario also $f^{(p)}(t)$ will be non-continuous. We distinguish then six possible cases:

(i) $f^M(t_1) > 0$ and $f^M(t_2) \geq 0$ with a discontinuity at $T \in (t_1, t_2)$;

(ii) $f^M(t_1) \geq 0$ and $f^M(t_2) < 0$ with a discontinuity at $T \in (t_1, t_2)$;

(iii) $f^M(t_1) < 0$ and $f^M(t_2) \leq 0$ with $f^M(t)$ continuous in (t_1, t_2);

(iv) $f^M(t_1) < 0$ and $f^M(t_2) \leq 0$ with a discontinuity at $T \in (t_1, t_2)$;

(v) $f^M(t_1) < 0$ and $f^M(t_2) > 0$ with a discontinuity at $T \in (t_1, t_2)$;

(vi) $f^M(t_1) > 0$ and $f^M(t_2) \geq 0$ with $f^M(t)$ showing discontinuities before t_1.

Notice that in the cases (iii) and (v) where $f^M(t_1) < 0$ implicitly imply a discontinuity $\xi(f^M(T_0)) \in [-1/(d^2-1), 0)$ at some $T_0 < t_1$.

In case (i) we have that at time $T \in (t_1, t_2)$ a discontinuity is shown such that $f^M(T^+) - f^M(T^-) = -\epsilon < 0$, where $\epsilon \in (0, 1)$. Notice that $\epsilon = 1$ implies that $f^M(T^-) = 1$ and $f^M(T^+) = 0$, and therefore this choice does not make sense if our purpose is to make $f^{(p)}(t)$ Markovian. Fixed this ϵ-jump for $f^M(t)$, we build the optimal behavior that makes $f^{(p)}(t)$ Markovian for the smallest p possible. Using the same technique used to obtain Eq. (6.85), we see that this function is characterized by $f^M(t_1) = 1$ and $\dot{f}^M(t) = -\dot{f}^{NM}(t)(1-\overline{p})/\overline{p}$ for $t \in (t_1, t_2)$ and the smallest value of $\overline{p}$ for which $f^M(t)$ is Markovian in (t_1, t_2). Indeed, with this structure $f^{(p)}(t)$ is non-increasing for any $p \geq \overline{p}$ and $\dot{f}^{(\overline{p})}(t) = 0$ for $t \in (t_1, t_2)$. By studying the condition of Markovianity $f^M(t_2) \geq 0$, we obtain

$$\overline{p} \geq \frac{\Delta^{NM}/(1-\epsilon)}{1+\Delta^{NM}/(1-\epsilon)} > p(\boldsymbol{D}_C^M|\mathbb{D}_C)\,,$$

where the last inequality holds for any $\epsilon \in (0,1)$, namely for any discontinuity of this type.

Cases (ii), (iii) and (iv) can be proven to be inefficient to make $f^{(p)}(t)$ Markovian thanks to the following argument. Since $\dot{f}^{NM}(t) > 0$ for $t \in (t_1, t_2)$, in order to make $f^{(p)}(t)$ Markovian, we have to require that $f^{(p)}(t_2) \leq 0$, namely it has to assume the same sign of $f^M(t_2)$. It implies that

$$\begin{aligned} p \;\; &\geq \;\; \frac{f^M(t_2)/|f^M(t_2)|}{1+f^M(t_2)/|f^M(t_2)|} \geq \frac{\Delta^{NM}/|f^M(t_2)|}{1+\Delta^{NM}/|f^M(t_2)|} \\ &\geq \;\; \frac{(d^2-1)\Delta^{NM}}{1+(d^2-1)\Delta^{NM}} > p(\boldsymbol{D}_C|\mathbb{D}_C^M)\,, \end{aligned} \tag{6.88}$$

where we used $f^{NM}(t_2) \geq \Delta^{NM}$ and $|f^M(t_2)| \leq 1/(d^2-1)$.

For case (v) we start by noticing that the discontinuity at time T may lead to a non-Markovian discontinuity for $f^{(p)}(t)$. Therefore, we parameterize the discontinuity of $f^M(t)$ in the following way: $f^M(T^+) = |f^M(T^-)|\,\lambda/(d^2-1)$, where $\lambda \in [0,1]$. Moreover, in order for $f^M(t)$ to make $f^{(p)}(t)$ Markovian, $f^{(p)}(T^-) < 0$. Hence, $f^{(p)}(t)$ shows a Markovian discontinuity at time $t = T$ if and only if $\xi(f^{(p)}(T)) \geq -1/(d^2-1)$. This condition can be written as

$$\lambda \leq 1 - \frac{(1-p)d^2}{p}\frac{f_C^{NM}(T)}{|f^M(T^-)|}\,. \tag{6.89}$$

If we consider this bound for $p = p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)$, we have that the difference $h_C^M(T) - f^M(T^+)$ becomes

$$h_C^M(T) - f^M(T^+) \geq \frac{1}{\Delta^{NM}}\left(\frac{f_C^{NM}(T)}{d^2-1} + f_C^{NM}(t_1)\right) > 0\,, \tag{6.90}$$

where $h_C^M(T) = 1 - (f^{NM}(T) - f^{NM}(t_1))/\Delta^{NM}$ (see Eq. (6.45)) and we used that in the optimal case $f^M(T^-) = -1/(d^2-1)$. By considering the Markovianity of $f^{(p)}(t)$ in the time interval (T, t_2), the optimal strategy imposes that $\dot{f}^M(t) = -\dot{f}_C^{NM}(t)(1-\overline{p})/\overline{p}$ for $t \in (T, t_2)$ and some $\overline{p} < 1$. In analogy to what we found in case (i), Eq. (6.90) implies that $f^M(t)$ cannot make $f^{(p)}(t)$ Markovian for $p = p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)$.

The last case we need to check is (vi), where $f^M(t)$ is continuous (hence non increasing) in (t_1, t_2) but exhibits some discontinuities before t_1. Since by construction $f^{(p)}(t)$ is continuous in (t_1, t_2), it can be Markovian only if it is non increasing in this interval, which in particular implies

$$\begin{aligned} 0 &\geq f^{(p)}(t_2^-) - f^{(p)}(t_1^+) \\ &= (1-p)(f^{NM}(t_2) - f^{NM}(t_1)) - p(f^M(t_1^+) - f^M(t_2^-)) \\ &= (1-p)\Delta^{NM} - p(f^M(t_1^+) - f^M(t_2^-)) \,, \end{aligned} \tag{6.91}$$

that leads to

$$p \geq \frac{\Delta^{NM}}{f^M(t_1^+) - f^M(t_2^-) + \Delta^{NM}} > p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M) \,, \tag{6.92}$$

where in the last passage we used the fact that $f^M(t)$ is positive, continuous in (t_1, t_2) and, since it shows discontinuities before t_1, $f^M(t_1^+) < 1$ and therefore $f^M(t_1^+) - f^M(t_2^-) \in [0, 1)$.

6.5.2 Single non-Markovian time interval with $f^{NM}(t) < 0$

Let us consider a non-Markovian $f_C^{NM}(t)$ such that it has a single time interval of non-Markovianity (t_1, t_2) when $f_C^{NM}(t) < 0$ and $\dot{f}_C^{NM}(t) < 0$. An important difference from discontinuous non-Markovian characteristic functions is that $f_C^{NM}(t)$ can become negative if and only if it shows a time interval of non-Markovianity of this type. Indeed, $f_C^{NM}(t_1) = 0$. Notice that in the non-continuous case a characteristic function can change its sign without being non-Markovian.

The optimal continuous Markovian characteristic function $h_C^M(t)$ is constant and equal to 1 for any $t \in [0, t_2]$ and it decreases depending on the behavior of $f_C^{NM}(t)$ (see Eq. (6.66) or (6.85)) for $t \geq t_2$. It can make the corresponding $f^{(p)}(t)$ Markovian for $p \geq p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M) = |\Theta^{NM}|/(1 + |\Theta^{NM}|)$, where $\Theta^{NM} = f_C^{NM}(t_2) - f_C^{NM}(t_2) < 0$.

Now we consider non-continuous Markovian characteristic functions $f^M(t)$ and we study which scenarios could potentially make $f^{(p)}(t)$ Markovian for some $p < p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)$. We have to study the following scenarios:

(i) $f^M(t_2) \in (0, 1)$;

(ii) $f^M(t)$ jumps at time $T \leq t_1$ to some negative value and $f^M(t_2) < 0$;

(iii) $f^M(t)$ jumps at time $T \in (t_1, t_2)$ to negative values and $f^M(t_2) < 0$.

In case (i) we include all those situations where $f^M(t)$ shows discontinuities with or without changes of sign for one or more times prior to t_2 and such that $f^M(t_2) > 0$. A necessary condition for $f^M(t)$ to make $f^{(p)}(t)$ Markovian is $f^{(p)}(t_2) \geq 0$. The non-negativity of $f^{(p)}(t_2)$ holds if and only if

$$p \geq \frac{|\Theta^{NM}|/f^M(t_2)}{1 + |\Theta^{NM}|/f^M(t_2)}\,.$$

Since $f^M(t_2) = 1$ if and only if $f^M(t) = 1$ for any $t \in [0, t_2]$ we have that all the $f^M(t)$ with discontinuities of this type cannot perform better than $h^M_C(t)$ in making $f^{(p)}(t)$ Markovian.

Considering case (ii), we start by noticing that, if $f^M(t_1) < 0$ and $f^M(t)$ is continuous for any $t \in (t_1, t_2)$, the optimal $f^M(t)$ of this type can make $f^{(p)}(t)$ Markovian for

$$p \geq \frac{|\Theta^{NM}|/f^M(t_1)}{1 + |\Theta^{NM}|/f^M(t_1)} \geq \frac{(d^2-1)|\Theta^{NM}|}{1 + (d^2-1)|\Theta^{NM}|} > p(\boldsymbol{D}^{NM}_C|\mathbb{D}^M_C)\,,$$

where $p(\boldsymbol{D}^{NM}_C|\mathbb{D}^M_C) = |\Theta^{NM}|/(1+|\Theta^{NM}|)$. In the case of a discontinuity of $f^M(t)$ (without change of sign) during the time interval (t_1, t_2), in analogy with case (i) of the previous section, we conclude that $f^M(t)$ cannot make $f^{(p)}(t)$ Markovian for $p < p(\boldsymbol{D}^{NM}_C|\mathbb{D}^M_C)$ also in this scenario.

In case (iii) $f^M(T^-) > 0$ and $f^M(T^+) < 0$ for some $T \in (t_1, t_2)$. We have to make $f^{(p)}(t)$ Markovian in (t_1, t_2) and in order to obtain this result we need that $f^{(p)}(t)$ and $f^M(t)$ have the same sign. As a consequence, $f^{(p)}(t)$ shows a discontinuity at time T such that $\xi(f^{(p)}(T)) < 0$. If we study the condition of Markovianity $\xi(f^{(p)}(T)) \geq -1/(d^2-1)$, we obtain

$$\xi(f^{(p)}(T)) = \frac{(1-p)f^{NM}(T) + pf^M(T^+)}{(1-p)f^{NM}(T) + pf^M(T^-)}$$

$$= \frac{-(1-p)|f^{NM}(T)| - p\lambda f^M(T^-)/(d^2-1)}{-(1-p)|f^{NM}(T)| + pf^M(T^-)} \geq \frac{-1}{d^2-1}\,, \qquad (6.93)$$

where we used $f^{NM}(T) = -|f^{NM}(T)|$ and $|f^M(T^+)| = f^M(T^-)\lambda/(d^2-1)$, where $\lambda \in (0, 1)$. We can use Eq. (6.93) to find a p-dependent bound for the values of λ that make $\xi(f^{(p)}(T)) \geq -1/(d^2-1)$. By doing so we obtain $\lambda \leq 1 - (1-p)d^2|f^{NM}(T)|/(pf^M(T^-))$. Now we check if the $f^M(t)$ of this case can make

$f^{(p)}(t)$ Markovian for $p = p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M) = |\Theta^{NM}|/(1+|\Theta^{NM}|)$. The optimal scenario is obtained when $f^M(T^-) = 1$ and therefore we get

$$f^M(T^+) = \frac{-\lambda}{d^2-1} \geq \frac{-1}{d^2-1} + \frac{d^2|f^{NM}(T)|}{(d^2-1)|\Theta^{NM}|},$$

where we used $(1 - p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M))/p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M) = 1/|\Theta^{NM}|$. The optimal behavior of $f^M(t)$ that makes the derivative $\dot{f}^{(p)}(t) \geq 0$ for the smallest increase of $f^M(t)$ in (T, t_2) is achieved by considering $\dot{f}^M(t) = -\dot{f}^{NM}(t)(1-\overline{p})/\overline{p}$, for the smallest $\overline{p}$ that allows a Markovian $f^M(t)$. Therefore, for $\overline{p} = p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M) = |\Theta^{NM}|/(1+|\Theta^{NM}|)$, we get $\dot{f}^M(t) = -\dot{f}^{NM}(t)/|\Theta^{NM}|$. This implies that at time t_2 we have

$$f^M(t_2) \geq \left(\frac{d^2|f^{NM}(T)|}{(d^2-1)|\Theta^{NM}|} - \frac{1}{d^2-1}\right) + \frac{|f^{NM}(t_2)| - |f^{NM}(T)|}{|\Theta^{NM}|}$$

$$= |f^{NM}(T)|\left(\frac{d^2}{(d^2-1)|\Theta^{NM}|} - \frac{1}{|\Theta^{NM}|}\right) + 1 - \frac{1}{d^2-1} > 0\,, \tag{6.94}$$

where we used $f^{NM}(t_2) = \Theta^{NM} < 0$. In summary, we proved that a $f^M(t)$ that jumps at $T \in (t_1, t_2)$ to some negative value such that $f^{(p)}(t)$ does not show a non-Markovian jump at time $t = T$, cannot make $f^{(p)}(t)$ Markovian in the time interval (T, t_2) for $p = p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)$. Indeed, the Markovianity of $f^{p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)}(t)$ in this time interval implies that $f^M(t_2) > 0$, namely $f^M(t)$ should change sign while being continuous (this behavior is not allowed for Markovian characteristic functions). We underline that Markovian functions of case (iii) can make $f^{(p)}(t)$ Markovian but only for values of p larger than $p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)$, namely by imposing $\dot{f}^M(t) = -\dot{f}^{NM}(t)(1-\overline{p})/\overline{p}$ in (T, t_2) with some $\overline{p} > p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)$ that allows $f^M(t_2) \leq 0$.

From the results obtained in this section it is clear that, if we add to cases (i), (ii) and (iii) any additional discontinuity in (t_1, t_2), we cannot reduce the value of p for which $f^{(p)}(t)$ can be made Markovian with a discontinuous $f^M(t) \in \mathfrak{F}^M$.

6.5.3 General case

In order to prove (6.29) for any $\boldsymbol{D}_C \in \mathbb{D}_C^{NM}$ represented by a $f_C^{NM}(t) \in \mathfrak{F}_C^{NM}$, we notice that the same technique that we used to derive the optimal continuous solution $h_C^M(t)$ given in Eq. (6.85) can be generalized to the case where we fix the discontinuities that the Markovian characteristic function has to show. Indeed, the rules given in Eq. (6.85) can be generalized to the cases where $f^M(t)$

jumps with or without a change of sign and we obtain

$$h_{NC}^{M}(t) = \begin{cases} -\dot{f}_C^{NM}(t)/\Gamma' & \text{if } \dot{f}_C^{NM}(t) > 0 \text{ and } h_{NC}^{M}(t) > 0 \\ 0 & \text{if } \dot{f}_C^{NM}(t) \leq 0 \text{ and } h_{NC}^{M}(t) > 0 \\ -\dot{f}_C^{NM}(t)/\Gamma' & \text{if } \dot{f}_C^{NM}(t) < 0 \text{ and } h_{NC}^{M}(t) < 0 \\ 0 & \text{if } \dot{f}_C^{NM}(t) \geq 0 \text{ and } h_{NC}^{M}(t) < 0 \end{cases}, \tag{6.95}$$

where the sign of $h_{NC}^{M}(t)$ depends on the discontinuities $\xi(h_{NC}^{M}(t)) \in J_{\mathbb{D}}$ that we impose and $\Gamma' > 0$ has to be chosen such that $h_{NC}^{M}(t)$ is Markovian and $f^{(p)}(t)$ is made Markovian for the smallest possible p.

The main difference between $h_C^M(t)$ and $h_{NC}^{M}(t)$ is that Γ^{NM} is replaced by Γ', which in general depends on the particular jumps that $h_{NC}^{M}(t)$ has to show. Notice that in the previous two sections we used $\Gamma' = \overline{p}/(1-\overline{p})$. Our goal is to prove that in every scenario $\Gamma' > \Gamma^{NM}$. Indeed, $h_{NC}^{M}(t)$ makes $f^{(p)}(t)$ Markovian for $p \geq \Gamma'/(1+\Gamma') = \overline{p}$ and $\Gamma' > \Gamma^{NM}$ implies that $\overline{p} > p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^{M}) = \Gamma^{NM}/(1+\Gamma^{NM})$.

We consider those cases where the discontinuities of $h_{NC}^{M}(t)$ do not take place during time intervals of non-Markovianity of $f_C^{NM}(t)$. We show that, even if we ignore possible non-Markovian discontinuities of $f^{(p)}(t)$ caused by the discontinuities of $h_{NC}^{M}(t)$ (which may increase the minimum p for which $f^{(p)}(t)$ can be made Markovian by $h_{NC}^{M}(t)$), $\Gamma' > \Gamma^{NM}$. We use the following notation for the intervals of non-Markovianity of $f_C^{NM}(t)$: the i-th interval $(t_i^{(in)}, t_i^{(fin)})$ can either be a time interval where $f_C^{NM}(t)$ shows a non-Markovian behavior while being positive or negative. The i-th gap $\Gamma_i^{NM} \equiv |f_C^{NM}(t_i^{(fin)}) - f_C^{NM}(t_i^{(in)})| > 0$ is therefore the non-Markovian gap shown in the time interval $(t_i^{(in)}, t_i^{(fin)})$. Notice that $\Gamma^{NM} = \sum_i \Gamma_i^{NM}$ (see Eq. (6.70)). Let start with the case of a $h_{NC}^{M}(t)$ that shows a single discontinuity at time $T_1 < t_1^{(in)}$, where $\xi_1 = \xi(h_{NC}^{M}(T_1)) \in \{J_{\mathbb{D}} \setminus 1\}$. It is easy to prove that the minimum probability $\overline{p}$ for which $h_{NC}^{M}(t)$ can make $f^{(p)}(t)$ Markovian satisfies the following lower bound $\overline{p} \geq (\Gamma^{NM}/|\xi_1|)/(1+\Gamma^{NM}/|\xi_1|)$. Therefore, in these cases

$$\Gamma' = \Gamma^{NM}/|\xi_1| > \Gamma^{NM}. \tag{6.96}$$

Now, suppose that a discontinuity characterized by $\xi_1 = \xi(h_{NC}^{M}(T_1)) \in \{J_{\mathbb{D}} \setminus 1\}$ is verified for $t_{k_1}^{(fin)} \leq T_1 \leq t_{k_1+1}^{(in)}$, namely between the k_1-th and the k_1+1-th non-Markovian time interval. It is easy to show that in this case

$$\Gamma' = \sum_{i=1}^{k_1} \Gamma_i^{NM} + \frac{\sum_{i=k_1+1}^{N} \Gamma_i^{NM}}{|\xi_1|} > \Gamma^{NM}, \tag{6.97}$$

where N (which may be infinite) is the number of non-Markovianity intervals of $f_C^{NM}(t)$. In the case of an additional discontinuity $\xi_2 = \xi(h_{NC}^M(T_2)) \in \{J_{\mathbb{D}} \setminus 1\}$ that is shown at time $t_{k_2}^{(fin)} \leq T_2 \leq t_{k_2+1}^{(in)}$, we have

$$\Gamma' = \sum_{i=1}^{k_1} \Gamma_i^{NM} + \frac{\sum_{i=k_1+1}^{k_2} \Gamma_i^{NM}}{|\xi_1|} + \frac{\sum_{i=k_2+1}^{N} \Gamma_i^{NM}}{|\xi_1\, \xi_2|} > \Gamma^{NM} . \tag{6.98}$$

We notice that, the presence of two Markovian discontinuities for $h_{NC}^M(t)$ provides a value of Γ' that is strictly larger than the value obtained with only the first or the second discontinuity (see Eq. (6.97)). The generalization of Eq. (6.98) to any number of this type of discontinuities is trivial. We conclude that the $h_{NC}^M(t)$ obtained by any number of discontinuities $\{\xi_j\}_j$ of this type are always characterized by $\Gamma' > \Gamma^{NM}$.

In the previous sections we proved that the presence of any discontinuity that takes place during a single time interval of non-Markovianity (t_1, t_2) does not allow making $f^{(p)}(t)$ Markovian for $p \leq p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)$. It is clear that Eq. (6.95) provides an optimal non-continuous Markovian solution for any set of discontinuities that takes place inside or outside the time intervals $(t_i^{(in)}, t_i^{(fin)})$. Moreover, combining the previous results together we obtain that in every scenario $\Gamma' = \overline{p}/(1-\overline{p})$ is larger than $\Gamma^{NM} = p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)/(1 + p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M))$ hence proving Eq. (6.29).

6.6 A special subset of non-continuous depolarizing dynamics

As we shall see in details in the next section, computing our measure of non-Markovianity for depolarizing trajectories which are explicitly non continuous is rather demanding. For this reason we find it useful to remark that the construction presented in Section 6.4 can however be shown to generalize beyond the domain $\mathbb{D}_C^{NM}$ allowing us to compute $p(\boldsymbol{D}^{NM}|\mathbb{D}_C^M, \mathbb{D}^M)$ at least for some non continuous elements $\boldsymbol{D}^{NM}$. In particular, following the same approach we used in Section 6.4.1, the function $g_C^M(t)$ of Eq. (6.45) can be shown to provide the optimal choice for the computation of $p(\boldsymbol{D}^{NM}|\mathbb{D}_C^M, \mathbb{D}^M)$ for the whole set of non-Markovian evolutions $\boldsymbol{D}^{NM} \in \mathbb{D}^{NM}$ with characteristic functions of the form

$$\begin{cases} f^{NM}(t) \geq 0, \dot{f}^{NM}(t^{\pm}) \leq 0, \xi(f^{NM}(t)) \in [0,1] & t < t_1^{(in)} \\ f^{NM}(t) \geq 0,\ \dot{f}^{NM}(t^{\pm}) > 0, \xi(f^{NM}(t)) = 1 & t \in T_1^+ \\ \text{“Markovian”} & t > t_1^{(fin)}\ . \end{cases} \tag{6.99}$$

Notice that differently from the case addressed in Eq. (6.38) this new set of functions (i) can show Markovian discontinuities without changing their sign for any $t < t_1^{(in)}$, and (ii) can follow any behavior allowed by the Markovian conditions (see Eq. (6.10)), even changing sign, for $t > t_1^{(fin)}$. Since $\mathbb{D}^M$ is non-convex (see Appendix D.2.2), the mixture between $f^{NM}(t)$ and $g_C^M(t)$ may in principle make $f^{(p)}(t)$ non-Markovian for one or more times when $f^{NM}(t)$ behaves as a Markovian characteristic function. Nonetheless, this is not the case. Indeed, for $t > t_1^{(fin)}$, we have $g_C^M(t) = 0$ and therefore $f^{(p)}(t) = (1 - p)f^{NM}(t)$ is always Markovian. Instead, for $t < t_1^{(in)}$, since $g_C^M(t)$ and $f^{NM}(t)$ are positive, $f^{(p)}(t)$ cannot behave as a non-Markovian characteristic function. As a result of this observation one has that for the functions of the form (6.99) we have

$$p(\boldsymbol{D}^{NM}|\mathbb{D}_C^M, \mathbb{D}^M) = \frac{\Delta_1^{NM}}{1 + \Delta_1^{NM}}\ , \tag{6.100}$$

with Δ_1^{NM} being the gap associated with the non-Markovian character of the function on T_1^+.

Analogously, the function $g_C^M(t)$ given in Eq. (D.26) can be shown to provide the value of $p(\boldsymbol{D}^{NM}|\mathbb{D}_C^M, \mathbb{D}^M)$ also for the following class of not necessarily continuous, non-Markovian characteristic functions $f^{NM}(t)$ of the form

$$\begin{cases} f^{NM}(t) \geq 0, \dot{f}^{NM}(t) \leq 0, \xi(f^{NM}(t)) \in [0,1] & t \notin T^{NM} \\ f^{NM}(t) \geq 0,\ \dot{f}^{NM}(t) > 0, \xi(f^{NM}(t)) = 1 & t \in T^{NM} \\ \text{“Markovian”} & t > t_N^{(fin)}\ , \end{cases} \tag{6.101}$$

where, if $t_N^{(fin)} < \tau$ for some $\tau > 0$, the latter of Eq. (6.101) is the condition that we consider for $t > t_N^{(fin)}$. Therefore, also for the depolarizing evolutions $\boldsymbol{D}^{NM}$ defined by Eq. (6.101), we have

$$p(\boldsymbol{D}^{NM}|\mathbb{D}_C^M, \mathbb{D}^M) = \frac{\Delta^{NM}}{1 + \Delta^{NM}}\ . \tag{6.102}$$

By the same token one can show that $h_C^M(t)$ of Eq. (6.84) yields the measure of non-Markovianity $p(\boldsymbol{D}^{NM}|\mathbb{D}_C^M, \mathbb{D}^M)$ also for the class of characteristic functions of the form

$$\begin{cases} f^{NM}(t) \geq 0, \dot{f}^{NM}(t) \leq 0, \xi(f^{NM}(t)) \in [0,1] & t \notin T^{NM} \\ f^{NM}(t) \leq 0, \dot{f}^{NM}(t) \geq 0, \xi(f^{NM}(t)) = 1 & t \notin T^{NM} \\ f^{NM}(t) \geq 0, \ \dot{f}^{NM}(t) > 0, \xi(f^{NM}(t)) = 1 & t \in T^{NM} \\ f^{NM}(t) \leq 0, \ \dot{f}^{NM}(t) < 0, \xi(f^{NM}(t)) = 1 & t \in T^{NM} \\ \text{“Markovian”} & t > t^{(fin)} \,, \end{cases} \tag{6.103}$$

with $T^{NM} = (\cup_k T_k^+) \cup (\cup_j T_j^-)$ being the same intervals defined in Section 6.4.3 and where, if there exits a time $t^{(fin)}$ such that $f_C^{NM}(t)$ does not show any non-Markovian behavior for $t \geq t^{(fin)}$, the last condition replaces the first two for $t \geq t^{(fin)}$. In this case we get

$$p(\boldsymbol{D}^{NM}|\mathbb{D}_C^M, \mathbb{D}^M) = \frac{\Gamma^{NM}}{1 + \Gamma^{NM}} \,, \tag{6.104}$$

where again Γ^{NM} is defined as in (6.70).

6.7 Non-continuous depolarizing evolutions

It is rather complex to extend the results of the previous sections to the general case of non-Markovian depolarizing evolutions $\boldsymbol{D}^{NM}$ which are not necessarily continuous. This has to due with the fact that in computing $p(\boldsymbol{D}^{NM}|\mathbb{D}^M)$ we have to perform an optimization with respect to all the elements of $\mathbb{D}^M$, which as discussed in Section 6.2.2 is not convex. As we shall see in Section 6.7.1 this introduces an ambiguity in the definition of the optimal Markovian element which is hard to handle. Nonetheless, in Section 6.7.2 we propose a solution to the problem which, even though it does not allow deriving a closed formula for $p(\boldsymbol{D}^{NM}|\mathbb{D}^M)$, it leads in principle to the exact results for any assigned non-Markovian depolarizing evolution $\boldsymbol{D}^{NM}$.

Before entering into the details of the analysis we define two sets of times: W_C is the set of times when $f^{NM}(t)$ is continuous, namely $\xi(f^{NM}(t)) = 1$ if and only if $t \in W_C$ and $W_{NC} \equiv \{t_{NC,i}\}_i = \mathbb{R}^+ \setminus W_C$ is the discrete set of times when $f^{NM}(t)$ is discontinuous, namely $\xi(f^{NM}(t)) \neq 1$ if and only if $t \in W_{NC}$. Moreover, we divide W_{NC} in W_{NC}^M and W_{NC}^{NM}, namely the times when $f^{NM}(t)$

shows Markovian ($\xi(f^{NM}(t^{M}_{NC,i})) \in J_{\mathbb{D}}$) and non-Markovian ($\xi(f^{NM}(t^{NM}_{NC,i})) \notin J_{\mathbb{D}}$) discontinuities, respectively.

6.7.1 Ambiguity for the choice of the optimal Markovian evolution

In Section 6.4, while evaluating $p(\boldsymbol{D}^{NM}_{C}|\mathbb{D}^{M})$ for continuous evolutions, we never assumed any particular shape for $f^{NM}_{C}(t)$ and $\dot{f}^{NM}_{C}(t)$ in order to provide the optimal $f^{M}_{C}(t)$ needed to calculate this measure. In the following example, instead, we show that for non-continuous evolutions there is an ambiguity for the choice of the times when the optimal $f^{M}(t)$ shows discontinuities. This ambiguity is solved only if we know exactly the shape of $f^{NM}(t)$. Moreover, in these cases the value of the measure of non-Markovianity does not depend solely from Γ^{NM}.

We consider the non-Markovian characteristic function for qubits $f^{NM}_{\Theta}(t) \in \mathfrak{F}^{NM}$ with a single Markovian discontinuity at time t_{NC}, namely $W^{M}_{NC} = \{t_{NC}\}$, and a single time interval of non-Markovianity $T^{-} = (t^{(in)}, t^{(fin)})$ when the characteristic function and its time derivative are negative. More in details

$$f^{NM}_{\Theta}(t) = \begin{cases} 1 & t \in [0, t_{NC}] \\ -1/3 & t \to t^{+}_{NC} \\ f^{NM}_{\Theta}(t) \leq 0, \dot{f}^{NM}_{\Theta}(t) \geq 0 & t \in [t_{NC}, t^{(in)}] \\ \Theta - 1/3 & t = t^{(in)} \\ f^{NM}_{\Theta}(t) \leq 0, \dot{f}^{NM}_{\Theta}(t) < 0 & t \in (t^{(in)}, t^{(fin)}) \\ -1/3 & t \geq t^{(fin)} \end{cases}, \tag{6.105}$$

where $\Theta \in (0, 1/3]$. It is clear that this function is characterized by a null positive non-Markovian gap $\Delta^{NM} = 0$ and a negative non-Markovian gap $\Theta^{NM} = -\Theta$ that is shown in the time interval $T^{-} = (t^{(in)}, t^{(fin)})$. This example can be easily generalized to the qudit case: if we have a d-dimensional system, we have to replace the following conditions $f^{NM}_{\Theta}(t^{+}_{NC}) = -1/(d^2-1)$, $f^{NM}_{\Theta}(t) = -1/(d^2-1)$ for any $t \geq t^{(fin)}$, $f^{NM}_{\Theta}(t^{(in)}) = \Theta - 1/(d^2-1)$ and $\Theta \in (0, 1/(d^2-1)]$.

We can adopt two non-equivalent $f^{M,1}(t)$ and $f^{M,2}(t)$ in order to make $f^{(p)}(t) = (1-p)f^{NM}_{\Theta}(t) + pf^{M}(t)$ Markovian. We show that the form of the optimal Markovian characteristic function needed for the evaluation of $p(\boldsymbol{D}^{NM}_{\Theta}|\mathbb{D}^{M})$ depends on the particular value of Θ. Indeed, consider

$$f^{M,1}(t) = \begin{cases} 1 & t \in [0, t_{NC}] \\ -1/3 & t \in (t_{NC}, t^{(in)}] \\ f^{M,2}(t) \leq 0, \dot{f}^{M,2}(t) > 0 & t \in (t^{(in)}, t^{(fin)}] \\ 0 & t \geq t^{(fin)} \end{cases}, \tag{6.106}$$

or

$$f^{M,2}(t) = \begin{cases} 1 & t \in [0, t^{(in)}] \\ f^{M,1}(t) > 0, \dot{f}^{M,1}(t) < 0 & t \in (t^{(in)}, t^{(fin)}] \\ f^{M,1}(t) > 0, \dot{f}^{M,1}(t) = 0 & t \geq t^{(fin)} \end{cases} , \tag{6.107}$$

where, when the time derivative of the characteristic function is different from zero, we impose it to be equal to $-f_{\Theta}^{NM}(t)/\Delta_1^{eff}$ and $-f_{\Theta}^{NM}(t)/\Delta_2^{eff}$, respectively. In Fig. 6.4 and 6.5 we provide an example of this situation. We find that $f^{(p)}(t)$ can be made Markovian for

- $p \geq \frac{3\Theta}{1+3\Theta}$, if we consider $f^{M,1}(t)$ with $\Delta_1^{eff} = 3\Theta$;
- $p \geq \frac{1/3+\Theta}{4/3+\Theta}$, if we consider $f^{M,2}(t)$ with $\Delta_2^{eff} = \Theta + \frac{1}{3}$.

It follows that, depending on the value of $\Theta \in (0, 1/3]$, the optimal Markovian characteristic function needed to evaluate the measure of non-Markovianity is different, namely it is $f^{M,1}(t)$, if $\Theta \in (0, 1/6]$ and $f^{M,2}(t)$, if $\Theta \in [1/6, 1/3]$. As a consequence

$$p(\boldsymbol{D}_{\Theta}^{NM}|\mathbb{D}) = \begin{cases} \frac{3\Theta}{1+3\Theta} & \Theta \in (0, \frac{1}{6}] \\ \frac{1/3+\Theta}{4/3+\Theta} & \Theta \in [\frac{1}{6}, \frac{1}{3}] \end{cases} . \tag{6.108}$$

We notice that, differently from the continuous case, given the signs of $f^{NM}(t)$ and $\dot{f}^{NM}(t)$, it is not possible to know a-priori which are the signs of the optimal $f^{M}(t)$ and $\dot{f}^{M}(t)$ that make $f^{(p)}$ Markovian for the smallest value of p. Indeed, we have to consider all the possible alternatives for the optimal $f^{M}(t)$ and evaluate the minimum p for which each one make the corresponding $f^{(p)}(t)$ Markovian. This ambiguity is generated by the sign that we decide to assign to $f^{M}(t)$ during its evolution. Notice that in the continuous case $f_{C}^{M}(t)$ could not change its sign and we had no ambiguity in the definition of the optimal Markovian characteristic function. For instance, as we concluded studying $f_{\Theta}^{NM}(t)$, the difference between $f^{M,1}(t)$ and $f^{M,2}(t)$ is obtained solely by the choice of making the Markovian characteristic function change its sign at time t_{NC} with a discontinuity or not. The remaining part of their definitions are analogous to the optimal solution obtained for continuous evolutions (see Eq. (6.95))

In the following, we describe how to evaluate the non-Markovianity measure for generic non-Markovian depolarizing evolutions, where we pay particular attention to all the possible choices for the signs of the Markovian characteristic function during its evolution.

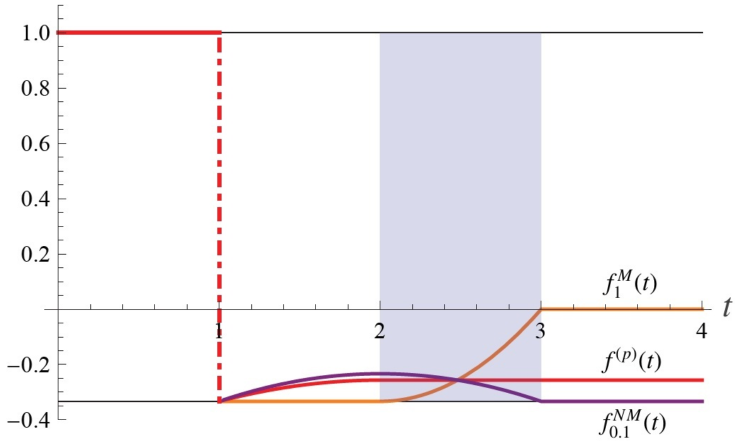


Figure 6.4: Plots of $f_1^M(t)$, $f_\Theta^{NM}(t)$ and $f^{(p)}(t)$ for a non-Markovian gap $\Theta = -\Theta^{NM} = 0.1$ and $p = p(\boldsymbol{D}_\Theta^{NM}|\mathbb{D}) \simeq 0.77$. The time interval of non-Markovianity $T^- = (2, 3)$ of $f_{0.1}^{NM}(t)$ is colored in purple. Since $\Theta < 1/6$, the optimal Markovian characteristic function is $f_1^M(t)$.

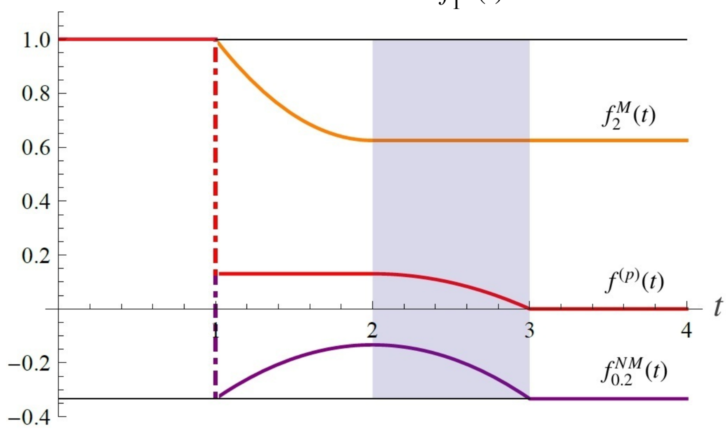


Figure 6.5: Plots of $f_2^M(t)$, $f_\Theta^{NM}(t)$ and $f^{(p)}(t)$ for a non-Markovian gap $\Theta = -\Theta^{NM} = 0.2$ and $p = p(\boldsymbol{D}_\Theta^{NM}|\mathbb{D}) \simeq 0.65$. The time interval of non-Markovianity $T^- = (2, 3)$ of $f_{0.2}^{NM}(t)$ is colored in purple. Since $\Theta > 1/6$, the optimal Markovian characteristic function is $f_2^M(t)$.

6.7.2 Measure of non-Markovianity for non-continuous depolarizing evolutions

In this section we propose a technique to evaluate the non-Markovianity measure for any non-Markovian depolarizing channel. For this purpose, we collect the results of the previous sections in order to find a strategy that singles out the optimal $\boldsymbol{D}^M$ needed to evaluate this measure.

Given the previous results, we consider two rules

- If $t' \in W_C$, the $f^M(t)$ that are discontinuous at $t = t'$ do not provide larger values of p (if compared with the $f^M(t)$ that are continuous for $t = t'$);
- If $t' \in W_{NC}$, the $f^M(t)$ that are discontinuous at $t = t'$ may provide larger values of p.

Therefore, the optimal Markovian evolution needed to evaluate $p(\boldsymbol{D}^{NM}|\mathbb{D}^M)$ is continuous at least for any $t \in W_C$.

Vector of signs

We define $T_{C,i} = (t_{NC,i-1}, t_{NC,i})$ to be the time intervals defined between the times in $W_{NC} = \{t_{NC,i}\}_{i=1}^N$, where we fix $t_{NC,0} = 0$ and, if N is finite, $t_{NC,N+1} = \infty$. With this procedure we define $N + 1$ time intervals such that $\cup_i T_{C,i} = W_C$.

We consider a dichotomic variable $\sigma_i \in \{-1, 1\}$ that we attach to each time interval $T_{C,i}$. Therefore, we obtain a vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots)$ of values equal to +1 or -1. We have a countable number of combinations for this vector. We label each combination $\boldsymbol{\sigma}_a = (\sigma_{a,1}, \sigma_{a,2}, \dots)$ with a different value of an integer number $a = 1, 2, \dots$. We impose $\sigma_{a,0} = +1$ for each combination and we fix a labeling scheme, for instance

$$
\begin{aligned}
\boldsymbol{\sigma}_1 &= (+1,+1,+1,+1,\dots), & \boldsymbol{\sigma}_5 &= (+1,+1,+1,-1,\dots),\\
\boldsymbol{\sigma}_2 &= (+1,-1,+1,+1,\dots), & \boldsymbol{\sigma}_6 &= (+1,-1,+1,-1,\dots),\\
\boldsymbol{\sigma}_3 &= (+1,+1,-1,+1,\dots), & \boldsymbol{\sigma}_7 &= (+1,+1,-1,-1,\dots),\\
\boldsymbol{\sigma}_4 &= (+1,-1,-1,+1,\dots), & \boldsymbol{\sigma}_8 &= (+1,-1,-1,-1,\dots), \quad \dots
\end{aligned}
$$

We call each $\boldsymbol{\sigma}_a$ a *vector of signs* for the following reason. We call $f_a^M(t)$ the Markovian characteristic functions such that their sign is defined by $\boldsymbol{\sigma}_a$ as follows

$$
\mathrm{sign}(f_a^M(t)) = \begin{cases} \sigma_{a,1} = +1 & t \in [0, t_{NC,1}] \\ \sigma_{a,2} & t \in (t_{NC,1}, t_{NC,2}] \\ \sigma_{a,3} & t \in (t_{NC,2}, t_{NC,3}] \\ \dots & \dots \end{cases} . \tag{6.109}
$$

We underline that, as noticed in Section 6.2.1, a Markovian characteristic function can change its sign only with discontinuities such that $\xi(f^M(t)) \in [-1/(d^2-1), 0)$. Indeed, we imposed that $f_a^M(t)$ is continuous at least for any $t \in W_C$. Indeed, $f_a^M(t)$ can show a discontinuity only when $f^{NM}(t)$ shows a discontinuity. Therefore,

- $\sigma_{a,i} = \sigma_{a,i+1}$: $f_a^M(t)$ can either be continuous or show a discontinuity at $t = t_{NC,i}$;
- $\sigma_{a,i} = -\sigma_{a,i+1}$: $f_a^M(t)$ must show a discontinuity $\xi(f^M(t_{NC,i})) \in [-1/(d^2-1), 0)$ while it changes sign.

The Markovian characteristic functions with these features define the set $\mathfrak{F}_a^M$.

Consider the convex sum $f^{(p)}(t) = (1-p)f^{NM}(t) + pf_a^M(t)$. First, it is continuous for any $t \in W_C$. Second, if it is Markovian for some p and $f_a^M(t)$, it also has to belong to $\mathfrak{F}_b^M$ for some vector of signs $\boldsymbol{\sigma}_b$, namely such that $\text{sign}(f^{(p)}(t)) = \sigma_{b,i}$ for any $t \in T_{C,i}$. Notice that $\boldsymbol{\sigma}_b$ may be different from $\boldsymbol{\sigma}_a$. Therefore, in order to obtain $p(\boldsymbol{D}^{NM}|\mathbb{D}^M)$ we proceed as follows. We fix a vector $\boldsymbol{\sigma}_a$ for $f_a^M(t)$ and we make $f^{(p)}(t) \in \mathfrak{F}_b^M$ for the smallest p

$$p_{a,b} \equiv \min\{p \,|\, \exists f_a^M(t) \in \mathfrak{F}_a^M \text{ s.t. } f^{(p)}(t) \in \mathfrak{F}_b^M\}, \tag{6.110}$$

Therefore, we get

$$p(\boldsymbol{D}^{NM}|\mathbb{D}^M) = \min_{a,b} p_{a,b}\,. \tag{6.111}$$

The procedure to evaluate $p_{a,a}$ is given in Section 6.7.2, while the evaluation of $p_{a,b}$ for $a \neq b$ is given in Appendix D.7. In both cases, we simplify the minimization over a functional space given in Eq. (6.110) with a minimization over a discrete set of real parameters.

Optimal Markovian function for a generic vector of signs

In this section we evaluate $p_{a,a}$. We fix a vector of signs $\boldsymbol{\sigma}_a$ that describes the signs of $f_a^M(t)$ and $f^{(p)}(t)$, namely $\sigma_{a,i} = \text{sign}(f_a^M(t)) = \text{sign}(f^{(p)}(t))$ for any $t \in T_{C,i}$. A generic $f^{NM}(t) \in \mathfrak{F}^{NM}$ is characterized by:

- Time intervals $T_{C,i} = (t_{NC,i-1}, t_{NC,i})$ when $f^{NM}(t)$ is continuous, namely $\cup_i T_{C,i} = W_C$.
- Discrete set of times $W_{NC}^M = \{t_{NC,i}^M\}_i$ when $f^{NM}(t)$ shows Markovian discontinuities $\xi(f^{NM}(t)) \in J_{\mathbb{D}}$ for any $t \in W_{NC}^M$. We define $W_{NC} = W_{NC}^{NM} \cup W_{NC}^M$.

- Discrete set of times $W_{NC}^{NM} = \{t_{NC,i}^{NM}\}_i$ when $f^{NM}(t)$ shows non-Markovian discontinuities $\xi(f^{NM}(t)) \notin J_{\mathbb{D}}$ for any $t \in W_{NC}^{NM}$.

Our goal is not only to make $f^{(p)}(t)$ Markovian during the times when $f^{NM}(t)$ behaves as a non-Markovian characteristic function, but we also have to take care of the possible non-Markovianity generated from the convex sum of two characteristic functions, namely $f^{NM}(t)$ and $f_a^M(t)$, that for for some times behave as Markovian functions (see the example in Appendix D.2.2).

We adopt the following strategy. First, we generalize the technique introduced in Section 6.4 in order to make $f^{(p)}(t)$ behave as a Markovian characteristic function for any $t \in W_C$ (Section 6.7.2). Second, we make sure not to generate non-Markovianity for those times $t \in W_{NC}^{M}$ when $f^{NM}(t)$ shows Markovian discontinuities (Section 6.7.2). Finally, we study the cases of those times $t \in W_{NC}^{NM}$ when $f^{NM}(t)$ shows non-Markovian discontinuities (Section 6.7.2).

Times of continuity

Consider those times $t \in W_C$ when $f^{NM}(t)$ is continuous. Following what we saw in Section 6.4.3, it is straightforward to obtain the behavior of the optimal $f_a^M(t)$ that allows obtaining $p_{a,a}$. The definition of $f_a^M(t)$ has to change depending on (i) the Markovian/non-Markovian behavior of $f^{NM}(t)$ at time t, (ii) the sign of $f^{NM}(t)$ at time t and (iii) the sign of $f_a^M(t)$ at time t. Therefore, we focus on a generic $T_{C,i} = (t_{NC,i-1}, t_{NC,i})$ when $\text{sign}(f_a^M) = \sigma_{a,i}$. Then, the definition of the time derivative of $f_a^M(t)$ is given in Table 6.1. The adopted strategy has the following purpose. We have $\dot{f}_a^M(t) = 0$ for all those times when a non-zero derivative is not needed to make $f^{(p)}(t)$ Markovian. This strategy cannot be used when the sign of the time derivative of $f^{NM}(t)$ is such that $\text{sign}(\dot{f}^{NM}(t))\text{sign}(f_a^{(p)}(t)) = +1$. Indeed, if we have $\dot{f}_a^M(t) = 0$, then $\text{sign}(\dot{f}^{(p)}(t))\text{sign}(f_a^{(p)}(t)) = +1$ and $f^{(p)}(t)$ would not satisfy the first Markovian condition (6.10). The condition $\dot{f}^{(p)}(t) = 0$ is given in analogy to the continuous case. In order to apply it, we introduce a parameter $\Delta > 0$ as follows: $\dot{f}_a^M(t) = -\dot{f}^{NM}(t)/\Delta$[1], which indeed makes $f^{(p)}(t)$ Markovian in these time intervals for $p \geq \Delta/(1+\Delta)$. We notice that not all values of $\Delta > 0$ are allowed. Indeed, if Δ is not large enough, $f_a^M(t)$ could violate the Markovian conditions of Eq. (6.10). The introduction of this parameter imposes to consider $f_a^M(t)$ as a function of t and Δ:

$$f_a^M(t) = f_a^M(t, \Delta) \,. \tag{6.112}$$

If not necessary, we omit this dependence on Δ.

[1] We introduce this parameter in analogy with Eq. (6.84). If $f^{NM}(t)$ does not show any discontinuity, $\Delta = \Gamma^N M$.

$t \in T_{C,i}$	$t \in T^M$	$t \in T^{NM}$
$\text{sign}(f^{NM}(t)) = \sigma_{a,i}$	$\dot{f}_a^M(t) = 0$	$\dot{f}^{(p)}(t) = 0$
$\text{sign}(f^{NM}(t)) = -\sigma_{a,i}$	$\dot{f}^{(p)}(t) = 0$	$\dot{f}_a^M(t) = 0$

Table 6.1: The conditions for time derivative of the optimal $f_a^M(t)$ for $t \in T_{C,i}$ depends on $\sigma_{a,i}$, $f^{NM}(t)$ and $\dot{f}^{NM}(t)$. T^M (T^{NM}) is the set of times when $f^{NM}(t)$ behaves as a Markovian (non-Markovian) characteristic function.

$t \in W_{NC}^M$	$\sigma_{a,i} = +1$ $\sigma_{a,i+1} = +1$	$\sigma_{a,i} = +1$ $\sigma_{a,i+1} = -1$
$\text{sign}(f^{NM}(t^-_{NC,i})) = +1$ $\text{sign}(f^{NM}(t^+_{NC,i})) = +1$	(a)	(b)
$\text{sign}(f^{NM}(t^-_{NC,i})) = +1$ $\text{sign}(f^{NM}(t^+_{NC,i})) = -1$	(a)	(b)
$\text{sign}(f^{NM}(t^-_{NC,i})) = -1$ $\text{sign}(f^{NM}(t^+_{NC,i})) = +1$	(c)	(d)
$\text{sign}(f^{NM}(t^-_{NC,i})) = -1$ $\text{sign}(f^{NM}(t^+_{NC,i})) = -1$	(c)	(d)

Table 6.2: Discontinuities of $f_a^M(t)$ depending of $\sigma_{a,i}$, $\sigma_{a,i+1}$, $\text{sign}(f^{NM}(t^-_{NC,i}))$ and $\text{sign}(f^{NM}(t^+_{NC,i}))$ in the case that $t_{NC,i}$ is a Markovian discontinuity for $f^{NM}(t)$. The remaining combinations are obtained by flipping all the signs of this table, where the optimal strategies are the same.

Markovian discontinuities

In this section we define the behavior of the optimal $f_a^M(t)$ for those times when $f^{NM}(t)$ shows Markovian discontinuities, namely we consider times $t_{NC,i} \in W_{NC}^M$ such that $\xi(f^{NM}(t_{NC,i})) \in J_{\mathbb{D}}$. Having fixed the vector of signs $\boldsymbol{\sigma}_a = (\sigma_{a,1}, \dots, \sigma_{a,i}, \sigma_{a,i+1}, \dots)$, we know the sign of $f_a^M(t)$ and $f_a^{(p)}(t)$ before and after $t_{NC,i}$. Moreover, we need to decide what value has to assume $f_a^M(t^+_{NC,i})$, while we consider $f_a^M(t^-_{NC,i})$ fixed by its behavior in the time interval $T_{C,i} = (t_{NC,i-1}, t_{NC,i})$.

If $\sigma_{a,i} = \sigma_{a,i+1}$, for $f_a^M(t)$ the time $t = t_{NC,i}$ can be either (i) a time of continuity $\xi(f_a^M(t)) = 1$ or (ii) a time of discontinuity when it does not change its sign, namely $\xi(f_a^M(t_{NC,i})) \in [0, 1)$[2]. Instead, if $\sigma_{a,i} = -\sigma_{a,i+1}$, for $f_a^M(t)$ the

[2]We remember that $\xi(f_a^M(t_{NC,i}))$ if and only if $f_a^M(t^+_{NC,i}) = 0$ and $f_a^M(t) = 0$ for any $t \geq t_{NC,i}$. Therefore, we can pick this value if and only if $f^{NM}(t)$ does not show any non-Markovian behavior for $t \geq t_{NC,i}$.

time $t = t_{NC,i}$ is a time of (Markovian) discontinuity $\xi(f^{NM}(t)) \in [-1/(d^2-1), 0)$ when its sign changes.

Straightforward calculations show that, if the starting sign of $f^{NM}(t)$ and $f_a^M(t)$ are the same and they are both showing a Markovian discontinuity, $f^{(p)}(t)$ shows a Markovian discontinuity independently from their final signs. In order to illustrate the discontinuities that $f_a^M(t)$ has to show for any combination of $\sigma_{a,i}$, $\sigma_{a,i+1}$, $\mathrm{sign}(f^{NM}(t^-_{NC,i}))$ and $\mathrm{sign}(f^{NM}(t^+_{NC,i}))$, we follow the scheme of Table 6.2.

(a) $f_a^M(t)$ preserves its sign and, independently from the final value and sign of $f^{NM}(t^+_{NC,i})$, the time $t_{NC,i}$ is not a non-Markovian discontinuity for $f^{(p)}(t)$. Therefore, the best strategy is to consider $f_a^M(t_{NC,i})$ continuous: $\xi(f_a^M(t_{NC,i})) = 1$.

(b) Similarly to (a), $t_{NC,i}$ is never a non-Markovian discontinuity for $f^{(p)}(t)$. Since $f_a^M(t)$ has to change sign, the best strategy is to maximize the final distance from zero. Therefore, we impose $\xi(f_a^M(t_{NC,i})) = -1/(d^2-1)$.

(c) From $\xi(f_a^M(t_{NC,i})) = 1$ it follows a non-Markovian discontinuity for $f^{(p)}(t_{NC,i})$ for any $p < 1$. Since $\xi(f_a^M(t_{NC,i})) < 1$ makes $f_a^M(t_{NC,i})$ and $f^{(p)}(t_{NC,i})$ closer to zero, we need the minimal intervention to make $f^{(p)}(t)$ Markovian and positive. Due to this ambiguity, we introduce the parameter $\Xi_i = \xi^M(f_a^M(t_{NC,i})) \in [0,1)$[3].

(d) $\xi(f_a^M(t)) = -1/(d^2-1)$ implies $\xi(f^{(p)}(t_{NC,i})) < -1/(d^2-1)$ for any $p < 1$. In this case, we define the parameter $\Xi_i = \xi^M(f_a^M(t_{NC,i})) \in (-1/(d^2-1), 0]$.

Therefore, these conditions fix the behavior of $f_a^M(t)$ when $f^{NM}(t)$ shows a Markovian discontinuity.

Non-Markovian discontinuities

In this section we define the behavior of the optimal $f_a^M(t)$ for those times when $f^{NM}(t)$ shows non-Markovian discontinuities, namely we consider times $t_{NC,i} \in W_{NC}^{NM}$ when $\xi(f^{NM}(t_{NC,i})) \notin J_{\mathbb{D}}$. Having fixed $\boldsymbol{\sigma}_a = (\sigma_{a,1}, \ldots, \sigma_{a,i}, \sigma_{a,i+1}, \ldots)$,

[3]For each value of Ξ_i we have a different interval of p such that $f^{(p)}(t)$ is Markovian and with the same sign of $f_a^M(t)$. If $\xi(f^{NM}(t_{NC,i})) > 0$, the choice $\Xi_i = \xi(f^{NM}(t_{NC,i}))$ allows the largest value of p for which we can make $f^{(p)}(t)$ Markovian and with the same sign of $f_a^M(t)$, but it implies that $f^{(p)}(t^+_{NC,i}) = 0$ and it denies any further possibility to make $f^{(p)}(t)$ Markovian for $t > t^+_{NC,i}$. Therefore, chosen a value of Ξ_i, we obtain some conditions $p \leq p(\Xi_i)$ for which $f^{(p)}(t)$ is Markovian and with the same sign of $f_a^M(t)$.

$t \in W_{NC}^{NM}$	$\sigma_{a,i} = +1$ $\sigma_{a,i+1} = +1$	$\sigma_{a,i} = +1$ $\sigma_{a,i+1} = -1$
$\text{sign}(f^{NM}(t_{NC,i}^-)) = +1$ $\text{sign}(f^{NM}(t_{NC,i}^+)) = +1$	(e)	(g)
$\text{sign}(f^{NM}(t_{NC,i}^-)) = +1$ $\text{sign}(f^{NM}(t_{NC,i}^+)) = -1$	(f)	(h)
$\text{sign}(f^{NM}(t_{NC,i}^-)) = -1$ $\text{sign}(f^{NM}(t_{NC,i}^+)) = +1$	(e)	(g)
$\text{sign}(f^{NM}(t_{NC,i}^-)) = -1$ $\text{sign}(f^{NM}(t_{NC,i}^+)) = -1$	(f)	(h)

Table 6.3: Discontinuities of $f_a^M(t)$ depending of $\sigma_{a,i}$, $\sigma_{a,i+1}$, $\text{sign}(f^{NM}(t_{NC,i}^-))$ and $\text{sign}(f^{NM}(t_{NC,i}^+))$ in the case that $t_{NC,i}$ is a non-Markovian discontinuity for $f^{NM}(t)$. The remaining combinations are obtained by flipping all the signs of this table, where the optimal strategies are the same.

we know the sign of $f_a^M(t)$ and $f_a^{(p)}(t)$ before and after $t_{NC,i}$. Moreover, we need to decide what value has to assume $f_a^M(t_{NC,i}^+)$.

In order to illustrate the discontinuities that $f_a^M(t)$ has to show for any combination of $\sigma_{a,i}$, $\sigma_{a,i+1}$, $\text{sign}(f^{NM}(t_{NC,i}^-))$ and $\text{sign}(f^{NM}(t_{NC,i}^+))$, we follow the scheme of Table 6.3.

(e) Similarly to case (c), we define the parameter $\Xi_i = \xi(f_a^M(t_{NC,i})) \in [0, 1)$.

(f) Calculations show that the optimal $f_a^M(t)$ is obtained when $f_a^M(t)$ is continuous at time $t = t_{NC,i}$, namely by imposing $\xi(f_a^M(t_{NC,i})) = 1$.

(g) Calculations show that the optimal $f_a^M(t)$ is obtained for $\xi(f_a^M(t_{NC,i})) = -1/(d^2 - 1)$.

(h) Similarly to case (d), we define parameters $\Xi_i = \xi(f_a^M(t_{NC,i})) \in (-1/(d^2 - 1), 0]$.

Therefore, these conditions fix the behavior of $f_a^M(t)$ when $f^{NM}(t)$ shows a non-Markovian discontinuity.

Evaluation of $p_{a,a}$

We show the procedure to define the optimal $f_a^M(t)$ until $t = t_{NC,2}$.

- First interval of continuity $[0, t_{NC,1})$: we start by imposing the condition $f_a^M(0) = 1$. We have $\text{sign}(f_a^M(t)) = \text{sign}(f^{(p)}(t)) = +1$. The evolution of $f_a^M(t)$ for $t \in T_{C,1} = (0, t_{NC,1})$ is given in Table 6.1;

- First time of discontinuity $t_{NC,1}$: the behavior of $f_a^M(t)$ for $t = t_{NC,1}$ is given by Table 6.2 if $t_{NC,1}$ is a Markovian discontinuity for $f^{NM}(t)$ and by Table 6.3 if $t_{NC,1}$ is a non-Markovian discontinuity for $f^{NM}(t)$;

- Second interval of continuity $T_{C,2} = (t_{NC,1}, t_{NC,2})$: we have $\text{sign}(f_a^M(t)) = \text{sign}(f^{(p)}(t)) = \sigma_{a,2}$. The evolution of $f_a^M(t)$ is given in Table 6.1.

The definition of this characteristic function for any $t \geq t_{NC,2}$ is now obvious.

We saw that in order to define $f_a^M(t)$ for $t \in (t_{NC,i-1}, t_{NC,i})$ it may be necessary to introduce a parameter $\Delta > 0$ that allows making $\dot{f}^{(p)}(t) = 0$ when the cross-diagonal conditions of Table 6.1 occur (see Eq. (6.112)). Moreover, for each time of discontinuity $t_{NC,i} \in W_{NC}$ we have to define $\xi(f_a^M(t_{NC,i}))$. For each discontinuity of type (a) or (f), we impose $\xi(f_a^M(t_{NC,i})) = 1$. For each discontinuity of type (b) or (g), we impose $\xi(f_a^M(t_{NC,i})) = -1/(d^2 - 1)$. For each discontinuity of type (e) or (c), we introduce a parameter $\Xi_i \in [0, 1)$. For each discontinuity of type (d) or (h), we introduce a parameter $\Xi_i = \xi(f_a^M(t_{NC,i})) \in (-1/(d^2 - 1), 0]$. Therefore, in general, we introduce a set of parameters that defines $f_a^M(t)$:

$$f_a^M(t) = f_a^M(t, \Delta, \{\Xi_i\}_i)\,. \tag{6.113}$$

We seek a combination of Δ and $\{\Xi_i\}_i$ that minimizes the value of p for which $f^{(p)}(t) \in \mathfrak{F}_a^M$. Eq. (6.110) becomes

$$p_{a,a} = \min_{\Delta, \{\Xi_i\}_i} \{p \,|\, f_a^M(t, \Delta, \{\Xi_a\}_i) \text{ and } f^{(p)}(t) \in \mathfrak{F}^M\}\,. \tag{6.114}$$

This relation provides a drastic simplification of the minimization required in Eq. (6.110). Indeed, to calculate $p_{a,b}$, we formally need to perform a minimization over the elements of $\mathfrak{F}_a^M$, which have infinite degrees of freedom. Instead, thanks to this procedure, we only need to perform a minimization over Δ and $\{\Xi_i\}_i$. Notice that, if the discontinuities of type (c), (d), (e) and (h) are finite, the total number of parameters over which we need to optimize $p_{a,a}$ is finite.

6.8 Dephasing evolutions

In this section we show that the class of dephasing evolutions for qubits $\mathbb{Z}$ requires a method to evaluate the corresponding non-Markovianity measure $p(\boldsymbol{Z}^{NM}|\mathbb{Z}^M)$ similar to the depolarizing case. A dephasing evolution $\boldsymbol{Z} =$

$\{Z_t\}_{t\geq 0} \in \mathbb{Z}$ corresponds to a family of dynamical maps Z_t that at any time $t \geq 0$ assumes the form

$$Z_t(\cdot) = \phi(t)\, I_S(\cdot) + (1 - \phi(t))\sigma_z \cdot \sigma_z\,, \tag{6.115}$$

with $\sigma_z = \mathrm{diag}(1, -1)$ being the diagonal z-Pauli matrix . We have that $\phi(t) \in [0, 1]$ is a necessary and sufficient condition to ensure Z_t to be CPTP. We rewrite Eq. (6.115) making use of $\varphi(t) \equiv 2\phi(t) - 1$, namely considering

$$Z_t(\cdot) = \frac{1 + \varphi(t)}{2}\, I_S(\cdot) + \frac{1 - \varphi(t)}{2}\sigma_z \cdot \sigma_z\,, \tag{6.116}$$

where $\varphi(t)$ belonging to

$$J_{\mathbb{Z}} \equiv [-1, 1]\,, \tag{6.117}$$

is the necessary and sufficient condition to ensure Z_t to be CPTP.

In order to characterize Markovian dephasing evolutions, similarly to the case of depolarizing channels, if $\varphi(s) = 0$ for some $s > 0$, then the intermediate map $Z_{t,s}$ from s to $t \geq s$ of a dephasing channel can be CPTP if and only if $\varphi(t) = 0$ for any $t \geq s$, namely $Z_{t,s}(\cdot) = I_S(\cdot)$ for any $t \geq s$. In the case of a non-zero value of $\phi(s)$, the parameterization given in Eq. (6.118) allows us to write the intermediate map $Z_{t,s}$ for $t \geq s$ in the following convenient form

$$Z_{t,s}(\cdot) = \frac{1 + \varphi(t)/\varphi(s)}{2}\, I_S(\cdot) + \frac{1 - \varphi(t)/\varphi(s)}{2}\sigma_z \cdot \sigma_z\,, \tag{6.118}$$

which is a dephasing channel characterized by the value of $\varphi(t)/\varphi(s)$. As a consequence, $Z_{t,s}$ is CPTP if and only if $\varphi(t)/\varphi(s) \in J_{\mathbb{Z}}$.

From Eq. (6.118) it is clear that we can use $\varphi(t)$ to uniquely characterize $\boldsymbol{Z}$. We define the set of dephasing characteristic functions $\mathfrak{S}$ by requiring the same conditions of regularity considered in Section 6.2 for depolarizing evolutions. As a result, we have a one-to-one correspondence between dephasing evolutions $\boldsymbol{Z} \in \mathbb{Z}$ and "regular" (in general non-continuous) characteristic functions that take values in $J_{\mathbb{Z}}$, namely $\varphi(t) \in \mathfrak{S}$.

In analogy to Eq. (6.4), the non-continuous behavior of $\varphi(t)$ can be studied by considering the quantity

$$\xi(\varphi(t)) = \frac{\varphi(t^+)}{\varphi(t^-)}\,. \tag{6.119}$$

Similarly to the depolarizing case, we have a Markovian discontinuity when $\xi(\varphi(t)) \in J_{\mathbb{Z}} \setminus 1$, a non-Markovian discontinuity when $\xi(\varphi(t)) \notin J_{\mathbb{Z}}$ and a time of continuity when $\xi(\varphi(t)) = 1$.

The similarities between the CPTP conditions for dephasing and depolarizing channels and the role of the corresponding characteristic functions allow us to conclude that a dephasing evolution $\boldsymbol{Z}$ with characteristic function $\varphi(t)$ exhibits a *Markovian behavior* at time $\tau \geq 0$ if one of the two conditions applies

$$\begin{aligned} &\mathbf{CM_1}(\tau): && \xi(\varphi(\tau)) = 1 \text{ and } \tfrac{d}{d\tau}|\varphi(\tau)| \leq 0; \\ &\mathbf{CM_2}(\tau): && \xi(\varphi(\tau)) \in J_{\mathbb{Z}} \setminus 1; \end{aligned} \tag{6.120}$$

where $\mathbf{CM_1}(\tau)$ has to be replaced by $\dot{\varphi}(\tau^{\pm})\varphi(\tau) \leq 0$ when $\dot{\varphi}(\tau)$ is non-continuous, namely $\dot{\varphi}(\tau^-) \neq \dot{\varphi}(\tau^+)$. We define the set of Markovian dephasing characteristic functions as

$$\mathfrak{S}^M = \{\varphi(t) \in \mathfrak{S} \,|\, \mathbf{CM_1}(\tau) \text{ or } \mathbf{CM_2}(\tau) = \text{TRUE}, \forall \tau \geq 0\} \ , \tag{6.121}$$

which involves only local properties of $\varphi(t)$. Consequently, we can define $\mathfrak{S}^{NM} \equiv \mathfrak{S} \setminus \mathfrak{S}^M$, $\mathbb{Z}^M$ and $\mathbb{Z}^{NM}$.

We can summarize the behavior of Markovian dephasing functions as follows. $\varphi^M(t) \in \mathfrak{S}^M$, when continuous ($\xi(\varphi(t)) = 1$), does not increase its distance from zero, namely its modulus is non-increasing. Therefore, in the time intervals where it is positive (negative) and it is continuous, it is monotonically non-increasing (non-decreasing). As a consequence, $\varphi^M(t)$ cannot change sign while being continuous, namely if $\varphi^M(s) = 0$ for some $s \geq 0$, then $\varphi^M(t) = 0$ for any $t \geq s$. Discontinuities of Markovian characteristic functions cannot make $\varphi^M(t)$ increase its modulus. Therefore, $\varphi^M(t)$ can change its sign at a generic time τ (only) with a discontinuity, where $|\varphi^M(\tau^+)| \leq |\varphi^M(\tau^-)|$. Non-Markovian characteristic functions $\varphi^{NM}(t) \in \mathfrak{S}^{NM}$, instead, can show any discontinuity and non-monotonic behavior, with the only constraint of assuming values in $J_{\mathbb{Z}} = [-1, 1]$ at any time.

We notice that the characterizations of Markovian dephasing evolutions and depolarizing evolutions are analogous. Given the similarities between the Markovian conditions (6.10) and (6.120) and the dependence of the intermediate maps (6.6) and (6.119) from the respective characteristic functions $f(t)$ and $\varphi(t)$, we obtain a very similar procedure needed to evaluate the measure of non-Markovianity $p(\boldsymbol{Z}^{NM}|\mathbb{Z}^M)$. Indeed, in this case we need to find a $\boldsymbol{Z}^M \in \mathbb{Z}^M$ that allows making $\boldsymbol{Z}^{(p)} = (1-p)\boldsymbol{Z}^{NM} + p\boldsymbol{Z}^M$ Markovian for the smallest value of $p \in [0, 1]$, where the Markovian condition for $\boldsymbol{Z}^{(p)}$ can be studied by imposing $\varphi^{(p)} = (1-p)\varphi^{NM}(t) + p\varphi^M(t)$ to satisfy the Markovian conditions (6.120). The main difference between the evaluations of $p(\boldsymbol{Z}^{NM}|\mathbb{Z}^M)$ and $p(\boldsymbol{D}^{NM}|\mathbb{D}^M)$ for generic $\boldsymbol{Z}^{NM} \in \mathbb{Z}^{NM}$ and $\boldsymbol{D}^{NM} \in \mathbb{D}^{NM}$ is given by the fact that $J_{\mathbb{D}} \neq J_{\mathbb{Z}}$, which in particular implies that Markovian and non-Markovian characteristic functions of dephasing and depolarizing evolutions have different freedoms to assume values and show discontinuities (compare Eqs. (6.3) and (6.117) for the values of physicality of characteristic functions and $\mathbf{CM_2}(\tau)$ of Eqs. (6.10) and (6.120) for the definition of Markovian discontinuities). Nonetheless, the evaluation of $p(\boldsymbol{Z}^{NM}|\mathbb{Z}^M)$ does not require any particular additional technique compared to the depolarizing case.

Generalizing this approach to convex set of dynamics of similar forms is straightforward. Some examples are (i) $\mathcal{X}$ and $\mathcal{Y}$ obtained by replacing in Eq. (6.115) σ_z with the Pauli matrix, respectively, σ_x and σ_y and, more in general, (ii) $\mathcal{N}$ obtained by replacing in Eq. (6.115) σ_z with any $\sigma_n = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$ where (n_x, n_y, n_z) is a unit real vector.

Finally, we discuss the option of evaluating $p(\boldsymbol{Z}^{NM}|\mathbb{D}^M, \mathbb{E}^M)$, namely the minimum value of p in Eq. (6.23) for which it is possible to make a non-Markovian dephasing evolution a Markovian evolution through the incoherent mixing with a Markovian depolarizing evolution. We notice that while $\mathbb{Z}$ and $\mathbb{D}$ are convex sets, this is not the case for $\mathbb{Z} \cup \mathbb{D}$. Indeed, it is easy to check that $(1-p)\boldsymbol{Z}^{NM} + p\boldsymbol{D}^M$ is neither a dephasing nor a depolarizing evolution. Studying the Markovian conditions that apply to the evolutions coming from convex sums of this type, namely the evolutions in the convex hull of $\mathbb{Z} \cup \mathbb{D}$, is beyond the scope of this chapter.

6.9 Discussion

We introduced a non-Markovianity measure inspired by the intuitive concept for which, in order to consider an evolution highly non-Markovian, it has to be difficult to make it Markovian via incoherent mixing with Markovian dynamics. We showed how to evaluate this measure in the case of depolarizing evolutions in arbitrary dimensions and we discussed the case of dephasing evolutions for qubits.

Analytical results are derived for evolutions that satisfy precise continuity and regularity criteria, while we proposed a numerical approach for generic depolarizing evolutions. In particular, in case of a continuous non-Markovian depolarizing evolution $\boldsymbol{D}_C^{NM}$ with characteristic function $f_C^{NM}(t)$:

- The measure of non-Markovianity $p(\boldsymbol{D}_C^{NM}|\mathbb{D}^M)$ can be obtained by only considering continuous Markovian depolarizing evolutions, namely

$$p(\boldsymbol{D}_C^{NM}|\mathbb{D}^M) = p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)\,;$$

- There is an analytical relation between $p(\boldsymbol{D}_C^{NM}|\mathbb{D}^M)$ and $\Gamma^{NM} = \Delta^{NM} + |\Theta^{NM}|$, where Δ^{NM} and Θ^{NM} are the sums of the non-Markovian gaps that $f_C^{NM}(t)$ shows while being, respectively, positive and negative. We showed that

$$p(\boldsymbol{D}_C^{NM}|\mathbb{D}^M) = \frac{\Gamma^{NM}}{1+\Gamma^{NM}} ;$$

- We derived the form of the optimal Markovian characteristic function $h_C^M(t)$ that makes the incoherent mix $f^{(p)}(t) = (1-p)f_C^{NM}(t) + p\,h_C^M(t)$ Markovian for $p = p(\boldsymbol{D}_C^{NM}|\mathbb{D}^M)$ (see Eqs. (6.84) and (6.85)).

In case of non-continuous depolarizing evolutions $\boldsymbol{D}^{NM}$:

- We identified a class of non-continuous depolarizing evolutions for which $p(\boldsymbol{D}^{NM}|\mathbb{D}^M)$ is given by the same relation derived in the continuous case (see Section 6.6);

- We provided a numerical procedure that allows to evaluate $p(\boldsymbol{D}^{NM}|\mathbb{D}^M)$ for any non-continuous depolarizing evolution $\boldsymbol{D}^{NM}$ (see Section 6.7.2).

Finally, we studied dephasing evolutions for qubits and showed that the evaluation of our non-Markovianity measure is similar to the depolarizing case (see Section 6.8).

It would be interesting to generalize this analysis to other (even non-convex) classes of evolutions with particular symmetries, e.g. generalized amplitude damping channels, higher-dimensional dephasing evolutions and non-unital evolutions. Moreover, a proof for conjecture (6.26) is missing. In this direction, it would be interesting to study the value of $p(\boldsymbol{Z}^{NM}|\mathbb{D}^{NM}, \mathbb{E}^M)$ for generic dephasing non-Markovian evolutions and compare it with $p(\boldsymbol{Z}^{NM}|\mathbb{Z}^M)$.

Chapter 7

Conclusions and outlook

A realistic approach to the description of evolving quantum systems has to include the interaction with the corresponding surrounding environment. We saw that there exist two dynamical regimes for open quantum systems (OQS), namely Markovian and non-Markovian. Differently from the Markovian regime, non-Markovian evolutions allow obtaining a variety of information backflows, where we put particular emphasis on the possibility to witness bipartite correlation backflows. The study of these phenomena is fundamental for different reasons. From a theoretical point of view, it is needed to understand the precise relation between observables and initializations, e.g. particular initial states or ancillary systems, that have to be considered in order to obtain backflows whenever a generic class of non-Markovian evolutions is studied. Moreover, also the advantages that can be obtained in experimental setups are relevant. Indeed, non-Markovian effects can be particularly useful in many branches of quantum information technologies, from the possibility to achieve longer coherence times to the formulation of security protocols. Hence, the possibility to precisely engineer environments and the corresponding interactions with our quantum systems is a major challenge that needs to be tackled.

We analyzed a technique to quantify the potential of non-Markovian evolutions to provide different information backflows, namely through the introduction of non-Markovianity measures. As we saw, this technique can be also used to study other features, such as the non-convex geometries of Markovian and non-Markovian evolutions.

While in this thesis we contributed to the study of non-Markovian evolutions defined over finite-dimensional quantum systems, we did not consider infinite-dimensional cases. The study of generic non-Markovian evolutions for continuous variable systems results particularly difficult to approach. Nevertheless, the relevant subset of Gaussian evolutions recently proved to be a prominsing and prolific starting point to tackle this topic.

In this chapter we review our main contributions to these topics and we individuate possible future research lines.

Witnessing non-Markovianity through correlations

We addressed the characterization of non-Markovian backflows from the point of view of correlation measures revivals in bipartite systems. The bipartition considered throughout this analysis consisted in the evolving OQS and an ancillary system. We approached this topic by deriving properties common to a vast class of measures. We showed that non-Markovian effects in single-parameter evolutions, e.g. depolarization, dephasing or amplitude damping, always provide backflows for continuously differentiable correlations that are not time-independent on the image of the preceding evolution.

We followed by focusing on two of the mostly used quantum correlations: entanglement and quantum mutual information (QMI). We started by showing that a class of entanglement breaking evolutions do not allow entanglement backflows and we provided a corresponding dynamical example. For what concerns QMI, we showed that we can always obtain QMI backflows when the qubit evolution is essentially non-Markovian, namely not even P-divisible. Then, we followed by studying the relation between entanglement in the initial bipartition and the potential of QMI to provide backflows and we proved that maximally entangled states are not always optimal. Finally, we showed in which cases non-Markovian evolutions cannot be witnessed through backflows of QMI and we gave an explicit example of such an evolution. Among the different evolutions studied, we made use of the newly introduced quasi-eternal non-Markovian evolutions, which generalize the well-known eternal non-Markovian model.

There are many possible paths that could lead to interesting extensions of our results. A first interesting topic consists in studying the witnessing potential of correlations when the ancilla has a dimension larger than the OQS. While this approach cannot lead to any improvement when entanglement measures are studied, QMI may provide backflows for a wider class of non-Markovian evolutions.

A final goal is to exploit these correlation backflows in computational and/or communication quantum protocols and quantify the advantages over the Markovian strategies. Indeed, one of the major recent lines of research is given by the formulation of quantum protocols where backflows are exploited to obtain performance advantages.

A correlation measure witnessing almost-all non-Markovian evolutions

We introduced a new correlation measure that provides backflows for almost-all non-Markovian evolutions. The definition of this measure enforces the intuitive idea that, if Alice an Bob share a poorly correlated state, the former can only induce scarcely distinguishable effects on the latter system and vice-versa. We show that maximally entangled states are maximally correlated also with respect to this measure and at the same time classically correlated systems do not necessarily show minimal values. The actions allowed by one party to influence the second share of the bipartition are the newly introduced maximally entropic measurements, where each outcome has the same occurrence probability. We showed that this measure is able to provide backflows for almost-all non-Markovian evolutions, where Alice owns an ancillary qubit and Bob's system consists in the evolving OQS and an ancilla. In order to do so, we also showed how to construct the initial probe states that have to be considered in this witnessing process. Interestingly, these initial states are separable and as close an needed to uncorrelated states. Finally, we showed how to apply our technique to a quasi-eternal non-Markovian evolution.

A major challenge is to explore the non-Markovian witnessing potential of other correlation measures when two ancillas are deployed together with the OQS, namely as we did in this work. Recently, it has been shown that this setting allows entanglement to witness almost-all non-Markovian evolutions and all qubit non-Markovian evolutions [KRS20]. Since QMI quantifies both classical and quantum correlations, we expect that similar results can be obtained also with this measure. Moreover, whether these correlations are able to witness all non-Markovian evolutions is still unknown.

In the case we choose our probe states as initializations of the bipartite system, the computation of our correlation measure is straightforward. Hence, interesting goals consist in finding: (i) other classes of states for which our measure is easily computable and (ii) an efficient algorithm for generic states.

A second interesting question is whether this correlation measure can be considered as a figure of merit in an information protocol. Our interest in its usability come from the similarities between our correlation measure and a form of quantum correlations called steering.

Equivalence between non-Markovianity and correlation backflows

We presented the first one-to-one relation between backflows of correlations and non-Markovian evolutions. In other words, for every time interval where the evolution cannot be formulated as the action of a CPTP map, there exist initial states and at least one correlation measure for which it is possible to observe a correlation revival in the same time interval. In many cases this result can be obtained by considering correlations such as QMI, entanglement or our previously introduced measure. Nonetheless, in case of generic non-bijective evolutions, there is no proof that these measures are able to provide backflows. Hence, we introduced a new set of correlations that succeed also in the non-bijective case. In order to prove their potential to witness all non-Markovian evolutions, we formulate a class of initial states that can be used in this procedure. We exploited a bipartition where the first share consists in the evolving OQS and an ancilla and the second share is another ancilla.

We proved the existence of initial states that, together with our correlation measures, are able to show backflows for any non-Markovian evolutions. Nevertheless, we could not provide a constructive procedure to prepare these states. Once we attain their form, we could understand how hard is to compute these backflows and how we can experimentally implement this technique.

A future goal could be to extend this approach to other information quantifiers. The observables that should receive the main attention are those easy to compute and with intuitive and feasible physical realizations. An interesting example that goes in this direction is the quantum Fisher information.

Measuring non-Markovianity via incoherent mixing with Markovian dynamics

In this work we showed how to measure non-Markovianity through the minimal amount of Markovian noise that needs to be incoherently mixed with an evolution in order to make it Markovian. While this approach mimics the concept of robustness used to measure entanglement, in this case the non-convexity of the Markovian set makes this approach more intricate. Indeed, whenever we add some Markovian noise to an evolution that is Markovian in a particular time interval, we have to take care not to generate new non-Markovian features in the same time interval. Notice that the non-convexity of this set is also the reason why a resource theory of non-Markovian evolutions cannot be formulated as for other resources, e.g. entanglement.

We focused on the study of depolarizing evolutions and we showed how to evaluate our measure by making the assumption that the Markovian depolarizing evolutions are those that can make Markovian a non-Markovian depolarizing evolution with the highest efficiency. We obtained analytical results for all continuous (or with discontinuities of a certain class) depolarizing evolutions, where the value of the measure assumes an intuitive meaning. We tackled the generic case of non-continuous depolarizing evolutions and we provided a computational procedure that reduces a minimization problem defined over an, in general, infinite dimensional set into a minimization over a finite number of parameters. Finally, we discussed the dephasing case in order to show how to generalize our approach to other classes of evolutions.

In this work we studied how to make a non-Markovian evolution belonging to a well-structured set Markovian via incoherent mixing with Markovian evolutions. We conjectured that in different instances the Markovian evolution that accomplishes this task with the maximum efficiency belongs to the same structured set. A first interesting goal would be to prove our conjecture, namely understand for which classes of evolutions it can be considered valid. It is intuitive that non-convex sets of evolutions, such as amplitude damping, cannot satisfy this conjecture. Moreover, it is also plausible that the convexity of a given set of evolutions cannot be enough to consider this conjecture true. For instance, by considering a convex subset inside the depolarizing evolutions it seems intuitive that our conjecture would not hold true. Hence, it would be interesting to find minimal conditions for convex sets of evolutions for which this conjecture can be confirmed.

The measure of non-Markovianity that we introduced is not a proper distance in the set of evolutions. Indeed, instead of finding the minimal distance between our non-Markovian evolution and Markovian evolutions, we look for the Markovian evolution that has the largest distance from our non-Markovian evolution which at the same time makes it Markovian through a minimal mixing. Hence, an interesting approach would be to formulate a non-Markovianity measure that is purely geometrical in the set of evolutions.

Appendix A

Appendix of Chapter 3

A.1 Nonzero time derivatives of initially zero eigenvalues of rank one matrices for non-unitary single parameter maps

Let $\rho(t)$ be a positive semidefinite Hermitian matrix. Consider an eigenvalue $\lambda_k(t)$ of $\rho(t)$ and its corresponding normalized eigenvector $u_k(t)$ in a scenario where $\rho(t)$ evolves in time. In order to make the notation lighter, the time-dependence of these quantities will not be made explicit in the following equations. By definition it holds that $\lambda_k = u_k^\dagger \rho u_k$. The time derivative of λ_k is

$$\frac{d\lambda_k}{dt} = \frac{du_k^\dagger}{dt}\rho u_k + u_k^\dagger \rho \frac{du_k}{dt} + u_k^\dagger \frac{d\rho}{dt} u_k = \lambda_k \left(\frac{du_k^\dagger}{dt} u_k + u_k^\dagger \frac{du_k}{dt} \right) + u_k^\dagger \frac{d\rho}{dt} u_k. \tag{A.1}$$

Since u_k is normalized, namely $u_k^\dagger u_k = 1$ for all t, it follows that $\frac{du_k^\dagger}{dt} u_k + u_k^\dagger \frac{du_k}{dt} = 0$. If the evolution of ρ is described by a continuously differentiable family of dynamical maps so that $\frac{d\rho}{dt} = \frac{d}{ds} V_{t,s}(\rho)|_{s=t} = L_t(\rho)$ it follows that $\frac{d\lambda_k}{dt} = u_k^\dagger L_t(\rho) u_k$.

Next consider the special case where ρ is a rank one positive semidefinite trace one $n \times n$ matrix and consider its block diagonal form.

$$\rho = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{A.2}$$

where 1 represents the 1×1 block corresponding to the nonzero eigenvalue 1 and **0** represents the $n \times 1$, $1 \times n$, and $n \times n$ zero blocks. We want to investigate $\frac{d}{ds} V_{t,s}(\rho)|_{s=t} = L_t(\rho)$ for the case of single parameter evolution. In particular we want to study the projection of $L_t(\rho)$ onto the zero-eigenspace of ρ.

First we consider the unitary part of L_t. Let P_0 be the projector onto the zero eigenspace of ρ. One easily finds that $P_0[H, \rho]P_0 = 0$ for any H.

Then we consider $(G_k \rho G_k^\dagger - \frac{1}{2}\{G_k^\dagger G_k, \rho\})$ and express the matrix G_k on the same block form as ρ, namely

$$G_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}, \tag{A.3}$$

where A_k is the 1×1 block. The projection of $(G_k \rho G_k^\dagger - \frac{1}{2}\{G_k^\dagger G_k, \rho\})$ onto the zero eigenspace of ρ is then

$$P_0 \left(G_k \rho G_k^\dagger - \frac{1}{2} \left\{ G_k^\dagger G_k, \rho \right\} \right) P_0 = C_k C_k^\dagger. \tag{A.4}$$

The matrix $C_k C_k^\dagger$ is clearly Hermitian and positive semidefinite. It follows that

$$P_0 L_t P_0 = \gamma(t) \sum_k C_k C_k^\dagger, \tag{A.5}$$

is also a positive semidefinite matrix if $\gamma(t) > 0$ and negative semidefinite if $\gamma(t) < 0$. Moreover, $P_0 L_t P_0$ is zero if and only if C_k is zero for every k. Hence, if and only if for each k the lower off-diagonal $n \times 1$ block of G_k is zero in every basis will there be no rank one ρ such that $P_0 L_t P_0$ is nonzero. In this case G_k is proportional to the identity which implies $(G_k \rho G_k^\dagger - \frac{1}{2}\{G_k^\dagger G_k, \rho\}) = 0$ for every ρ. Thus, for any L_t with non-zero dissipative part there exist at least one rank one ρ such that the time derivative of the initial zero-eigenspace is nonzero.

To analyze the special case when $\rho_{AB} = \phi_{AB}^+$ where ϕ_{AB}^+ is the maximally entangled state on $\mathcal{H}_A \otimes \mathcal{H}_B$ and $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = n$ we note that the condition $C_k = 0$ can be formulated as $G_k \rho_{AB} = \rho_{AB} G_k \rho_{AB}$. In the following, in order to make the notation lighter, we simply write ρ and ϕ^+. We write $\phi^+ = (1/n) \sum_{ij} E_{ij} \otimes E_{ij}$ where E_{ij} is the matrix with the ij-th element equal to 1 and all other elements equal to zero. We write $G_k = \mathbb{1} \otimes F_k$ where F_k is any matrix. Then

$$\begin{aligned} G_k \rho &= \frac{1}{n} \sum_{ij} E_{ij} \otimes F_k E_{ij} \\ \rho G_k \rho &= \frac{1}{n^2} \sum_{ijl} E_{li} E_{ij} \otimes E_{li} F_k E_{ij} = \frac{\mathrm{Tr}(F_k)}{n^2} \sum_{jl} E_{lj} \otimes E_{lj}. \end{aligned} \tag{A.6}$$

These two expressions are equal if and only if $F_k E_{ij} = \mathrm{Tr}(F_k)/nE_{ij}$ for each ij. Since the matrices E_{ij} form a basis for the matrix space it follows that this relation is satisfied for all E_{ij} if and only if $F_k \propto \mathbb{1}$. Thus, $G_k\rho = \rho G_k\rho$ if and only if $G_k \propto \mathbb{1} \otimes \mathbb{1}$. As noted before this implies $(G_k\rho G_k^\dagger - \frac{1}{2}\{G_k^\dagger G_k, \rho\}) = 0$ for every ρ. Considering $V_{t,s}(\phi_n^+)$ we can now conclude that for any L_t with non-zero dissipative part there is an eigenvalue of $V_{t,s}(\phi^+)$ that is zero for $s = t$ but has a non-zero time derivative.

A.2 Proof of Proposition 9

Let $M(\bar{a})$ be a correlation measure that is an analytic function of the coordinates a_i in a point $\bar{a}$ corresponding to a state that at time t is in a product state ρ_{SA} and let $V_{t,s}$ be a continuously differentiable intermediate map for all $s \leq t$. In order to make the notation lighter, we simply write ρ. Consider a family of states $\rho_\epsilon = \rho + \epsilon\chi$ where χ is Hermitian. The Taylor expansion of $M(\rho_\epsilon)$ in ϵ around $\epsilon = 0$ is

$$M(\rho_\epsilon) = \left.\frac{\partial M(\rho_\epsilon)}{\partial \epsilon}\right|_{\epsilon=0} \epsilon + \left.\frac{\partial^2 M(\rho_\epsilon)}{2\partial \epsilon^2}\right|_{\epsilon=0} \epsilon^2 + \ldots, \tag{A.7}$$

where we have used that $M(\rho_\epsilon)|_{\epsilon=0} = 0$. Since $M \geq 0$ on $S(\mathcal{H}_{SA})$ it follows that the first order term of the expansion must be zero if $\rho \in \mathrm{int}[S(\mathcal{H}_{SA})]$. Otherwise there would be a sufficiently small ϵ for which both $\rho_\epsilon, \rho_{-\epsilon} \in \mathrm{int}[S(\mathcal{H}_{SA})]$ and either $M(\rho_\epsilon)$ or $M(\rho_{-\epsilon})$ was negative. Note that if ρ is not in $\mathrm{int}[S(\mathcal{H}_{SA})]$ this argument cannot be made since M could be negative outside $S(\mathcal{H}_{SA})$.

Assume that $\rho \in \mathrm{int}[S(\mathcal{H}_{SA})]$ and consider the Taylor expansion of $M[V_{t,s} \otimes I(\rho_\epsilon)]$ in ϵ around $\epsilon = 0$

$$M[V_{t,s} \otimes I(\rho_\epsilon)] = \left.\frac{\partial M[V_{t,s} \otimes I(\rho_\epsilon)]}{\partial \epsilon}\right|_{\epsilon=0} \epsilon + \left.\frac{\partial^2 M[V_{t,s} \otimes I(\rho_\epsilon)]}{2\partial \epsilon^2}\right|_{\epsilon=0} \epsilon^2 + \ldots, \tag{A.8}$$

where we have used that $M(V_{t,s} \otimes I(\rho_\epsilon))|_{\epsilon=0} = 0$. Since $M \geq 0$ it follows again that the first order term of the expansion must be zero. Thus $\left.\frac{\partial M[(V_{t,s} \otimes I(\rho_\epsilon)]}{\partial \epsilon}\right|_{\epsilon=0} = 0$ for all s. Next, consider the Taylor expansion of the derivative $\frac{d}{ds} M[V_{t,s} \otimes$

$I(\rho_\epsilon)]|_{s=t}$ in ϵ

$$\frac{d}{ds}M[V_{t,s}\otimes I(\rho_\epsilon)]\bigg|_{s=t} = \epsilon\frac{\partial}{\partial\epsilon}\frac{d}{ds}M[V_{t,s}\otimes I(\rho_\epsilon)]\bigg|_{s=t}\bigg|_{\epsilon=0} + \epsilon^2\frac{\partial^2}{2\partial\epsilon^2}\frac{d}{ds}M[V_{t,s}\otimes I(\rho_\epsilon)]\bigg|_{s=t}\bigg|_{\epsilon=0} + \dots, \quad \text{(A.9)}$$

where we have used that $\frac{d}{ds}M(V_{t,s}\otimes I(\rho_\epsilon))\big|_{\epsilon=0} = 0$.

From the analyticity of M and the continuous differentiability of Λ_t it follows that $\frac{d}{dt}M$, $\frac{d}{d\epsilon}M$ and $\frac{d}{dt}\frac{d}{d\epsilon}M$ exist and that $\frac{d}{dt}\frac{d}{d\epsilon}M$ is continuous as a function of ϵ and t. Therefore, it holds that $\frac{d}{d\epsilon}\frac{d}{dt}M$ exist and $\frac{d}{d\epsilon}\frac{d}{dt}M = \frac{d}{dt}\frac{d}{d\epsilon}M$ [Rud76]. It follows that the first order term in the Taylor expansion is zero. Since this holds for every χ, it follows that every product state in int$[S(\mathcal{H}_{SA})]$ is a critical point of M.

A.3 Proof of Proposition 10

We begin by considering the following two propositions.

Proposition 11. *Let $\mathbf{\Lambda}$ be a qubit evolution. If the set of stationary states in $S(\mathcal{H}_S)$ is of non-zero dimension, $\mathbf{\Lambda}$ is unital.*

Proof. Assume that $\Lambda_t(\mathbb{1}_S) = \mathbb{1}_S + \theta$ and $\Lambda_t(\rho_1) = \rho_1$, $\Lambda_t(\rho_2) = \rho_2$ where $\rho_1 \neq \rho_2$ belong to $S(\mathcal{H}_S)$. It follows that $\Lambda_t(\mathbb{1}_S + x(\rho_1-\rho_2)) = \mathbb{1}_S + x(\rho_1-\rho_2)+\theta$. Note that x can be chosen such that $\mathbb{1}_S \pm x(\rho_1 - \rho_2)$ are rank one. Since these rank one qubit states are antipodal points on the Bloch ball $S(\mathcal{H}_S)$ it follows that unless $\theta = 0$ at least one of $\mathbb{1}_S + x(\rho_1-\rho_2)+\theta$ and $\mathbb{1}_S - x(\rho_1-\rho_2)+\theta$ is not positive semidefinite. Thus if the set of stationary states has dimension greater than zero, it follows that Λ_t is unital. □

Proposition 12. *The set of stationary states of any qubit evolution $\mathbf{\Lambda}$ has a dimension different from 2.*

Proof. Assume that the dimension of the set of stationary states in $S(\mathcal{H}_S)$ is 2. From Prop. 11 follows that Λ_t is unital. Without losing generality we assume that $\Lambda_t(\sigma_z) = \sigma_z$ and $\Lambda_t(\sigma_y) = \sigma_y$ and $\Lambda_t(\sigma_x) = a\sigma_x + b\sigma_y + c\sigma_z$. The Choi matrix of Λ_t has eigenvalues $\pm\sqrt{1-2a+a^2+b^2+c^2}$ and $2 \pm \sqrt{1+2a+a^2+b^2+c^2}$. Therefore, Λ_t is CP if and only if $a = 1$ and $b = c = 0$, namely if an only if $\Lambda_t = I_S$. In this case the set of stationary states has dimension 3, contradicting the assumption. □

Next, consider a family of continuously differentiable dynamical qubit maps $\mathbf{\Lambda}$ and a correlation measure M. If the set of stationary states in $S(\mathcal{H}_S)$ is non-empty its dimension is either zero or non-zero. If the dimension is zero, the set of stationary states in $S(\mathcal{H}_{SA})$ is a set of product states. Then it follows from Prop. 9 that such a stationary state is a critical point if M is analytical at the state and the state is in the interior of $S(\mathcal{H}_{SA})$. If the dimension is 3 all states are stationary points and $\frac{dM}{dt} = 0$ on all of $S(\mathcal{H}_{SA})$. Thus all states are critical points. Moreover, by Prop. 12 the dimension is never 2. The remaining case is a one-dimensional set of stationary states. Without loss of generality, we can express any state in this set as $\rho_{SA} = \mathbb{1}_S \otimes \rho_A + \sigma_z \otimes \chi_A$ for some ρ_A, $\chi_A \in B(\mathcal{H}_A)$.

Now, assume that ρ_{SA} is in the interior of $S(\mathcal{H}_{SA})$ and that the correlation measure M is an analytic function at ρ_{SA}. Then consider the family of states $\rho_\epsilon = \rho_{SA} + \epsilon \chi_A \otimes \chi_S$ parameterized by ϵ, where χ_S,χ_A are Hermitian and $\mathrm{Tr}(\chi_S \otimes \chi_A) = 0$. If $\chi_S = \sigma_z$ or $\chi_S = \mathbb{1}_S$, it follows that ρ_ϵ is also a stationary state and thus $\frac{\partial}{\partial \epsilon} \frac{d}{dt} M(\rho_\epsilon, t)\big|_{\epsilon=0} = 0$. If $\chi_S = \sigma_x$ or $\chi_S = \sigma_y$ there exists a local unitary operation, $\sigma_z \otimes \mathbb{1}_A$, that commutes with ρ_{SA} but anticommutes with $\chi_S \otimes \chi_A$. Since $M(\bar{a}, t)$ is invariant under local unitary operations it follows that $M(\rho_\epsilon) = M(\rho_{-\epsilon})$. Thus $M(\rho_\epsilon)$ is an even analytic function in ϵ and it follows that $\frac{\partial}{\partial \epsilon} M(\rho_\epsilon, t)\big|_{\epsilon=0} = 0$. Since the $\sigma_x, \sigma_y, \sigma_z$ and $\mathbb{1}_S$ span $B(\mathcal{H}_S)$ we can conclude that $\frac{\partial}{\partial \epsilon} M(\rho_\epsilon, t)\big|_{\epsilon=0} = 0$ for every $\chi_S \otimes \chi_A$. Moreover, $\frac{\partial}{\partial \epsilon} M(\rho_\epsilon, t)\big|_{\epsilon=0} = 0$ holds for any t. Therefore, we can conclude that $\frac{d}{dt} \frac{\partial}{\partial \epsilon} M(\rho_\epsilon, t)\big|_{\epsilon=0} = 0$. By the analyticity of M and the continuous differentiability of Λ_t, it follows that $\frac{d}{dt} M$, $\frac{d}{d\epsilon} M$ and $\frac{d}{dt} \frac{d}{d\epsilon} M$ exist and that $\frac{d}{dt} \frac{d}{d\epsilon} M$ is continuous as a function of ϵ and t. Therefore, it follows that $\frac{d}{d\epsilon} \frac{d}{dt} M$ exist and $\frac{d}{d\epsilon} \frac{d}{dt} M = \frac{d}{dt} \frac{d}{d\epsilon} M$ [Rud76]. We can conclude that all first derivatives of $\frac{d}{dt} M(\bar{a}, t)$ with respect to $\bar{a}$ equal zero for states in the interior of $S(\mathcal{H}_{SA})$ that are stationary under a continuously differentiable $\mathbf{\Lambda}$.

Appendix B

Appendix of Chapter 4

B.1 The set of maximally entropic measurements is non-empty

We explicitly construct an element $\{P_i\}_i$ of $\Pi(\rho)$ for an arbitrary state ρ. The method that we use should convince the reader that there are innumerable other ways to construct a ME-POVM with any number of outputs.

By definition $\{P_i\}_{i=1}^n \in \Pi(\rho)$ if the output ensemble $\mathcal{E}(\rho, \{P_i\}_i) = \{p_i, \rho_i\}_i$ is characterized by $p_i = 1/n$. In general, we have that $p_i = \mathrm{Tr}[\rho P_i]$. Using an orthogonal decomposition of ρ, we can always write it as $\rho = \sum_{i=1}^d \pi_i |i\rangle\langle i|$, where $\{|i\rangle\}_i$ is an orthonormal basis of the Hilbert space $\mathcal{H}$. The condition $\sum_{i=1}^d \pi_i = 1$ implies that there exist an $\bar{i}$, such that $S(\bar{i}) \equiv \sum_{i=1}^{\bar{i}} \pi_i > 1/2$ and $S(\bar{i}-1) \equiv \sum_{i=1}^{\bar{i}-1} \pi_i \leq 1/2$. We consider the following class of 2-output POVM that depends on a real parameter $\omega \in [0,1]$: $P_1(\omega) = \sum_{i=1}^{\bar{i}-1} |i\rangle\langle i| + \omega |\bar{i}\rangle\langle\bar{i}|$, $P_2(\omega) = (1-\omega)|\bar{i}\rangle\langle\bar{i}| + \sum_{i=\bar{i}+1}^{d} |i\rangle\langle i|$. We evaluate p_1 for a general value of ω and we obtain: $p_1(\omega) = \sum_{i=1}^{\bar{i}-1} \pi_i + \omega \pi_{\bar{i}} = S(\bar{i}-1) + \omega \pi_{\bar{i}}$. It is clear that, since $p_1(0) = S(\bar{i}-1) \leq 1/2$ and $p_1(1) = S(\bar{i}) > 1/2$, the value $\omega = \overline{\omega} \equiv (1/2 - S(\bar{i}-1))/\pi_{\bar{i}}$, gives the uniform distribution $p_{1,2}(\overline{\omega}) = 1/2$ and consequently $\{P_i(\overline{\omega})\}_i \in \Pi(\rho)$, i.e,, is a ME-POVM for ρ.

Finally, we point out that in Appendix C.1 we tackle this problem by considering a different approach that allows to obtain the same result. In particular, we show that for any ρ and n there always exist a mapping that allows to obtain a n-output ME-POVM from a generic n-output POVM.

B.2 Monotonic behavior of C and $C^{(n)}$ under local operations

Firstly, we prove that C_A is monotone under local operations of the form $\Lambda_A \otimes \mathcal{I}_B$, and secondly we consider the case where the local operation is $\mathcal{I}_A \otimes \Lambda_B$, where Λ_A (Λ_B) is a CPTP map on A (B) and $\mathcal{I}_A$ ($\mathcal{I}_B$) is the identity map on A (B). The proof for C_A easily generalizes to C_B and C. Finally, we prove that the same monotonicity property holds for $C^{(n)}$ for any $n \geq 2$. We denote the set of ME-POVMs acting on A for the state ρ_{AB} by $\Pi_A(\rho_{AB})$ and similarly for B.

In order to show the effect of the application of a local operation of the form $\Lambda_A \otimes \mathcal{I}_B$ on $C_A(\rho_{AB})$, we look at $\Pi_A(\rho_{AB})$ in a different way. Each element of this collection is a ME-POVM for ρ_{AB}, namely they generate sets of *equiprobable ensembles of states* (EES) from ρ_{AB}. In fact

$$C_A(\rho_{AB}) \equiv \max_{\{P_{A,i}\}_i \in \Pi_A(\rho_A)} P_g\left(\mathcal{E}\left(\rho_{AB}, \{P_{A,i}\}_i\right)\right) - \frac{1}{2}\,. \tag{B.1}$$

is a maximization over all the possible EES that we can generate from ρ_{AB} with a measurement procedure on A.

The effect of the first local operation that we consider is: $\tilde{\rho}_{AB} = \Lambda_A \otimes \mathcal{I}_B(\rho_{AB}) = \sum_k (E_k \otimes \mathbb{1}_B) \cdot \rho_{AB} \cdot (E_k \otimes \mathbb{1}_B)^\dagger$, where $\{E_k\}_k$ is the set of the Kraus operators that defines Λ_A. What is the relation between $\Pi_A(\rho_{AB})$ and $\Pi_A(\tilde{\rho}_{AB})$? Given an n-output ME-POVM for $\tilde{\rho}_{AB}$, namely $\{P_{A,i}\}_i \in \Pi_A(\tilde{\rho}_{AB})$, the probabilities and the states of the output ensemble $\mathcal{E}(\tilde{\rho}_{AB}, \{P_{A,i}\}_i)$ are $\tilde{p}_i = \mathrm{Tr}\left[\tilde{\rho}_{AB} \cdot P_{A,i}\right] = 1/n$ and $\tilde{\rho}_{B,i} = \mathrm{Tr}_A\left[\tilde{\rho}_{AB} \cdot P_{A,i}\right]/\tilde{p}_i$. Now we look at the term

$$\mathrm{Tr}_A\left[\tilde{\rho}_{AB} P_{A,i}\right] = \mathrm{Tr}\left[\Lambda_A \otimes \mathcal{I}_B(\rho_{AB}) P_{A,i}\right] = \mathrm{Tr}_A\left[\sum_k (E_k \otimes \mathbb{1}_B)\rho_{AB}(E_k^\dagger \otimes \mathbb{1}_B) P_{A,i}\right]$$

$$= \mathrm{Tr}_A\left[\rho_{AB} \sum_k (E_k^\dagger \otimes \mathbb{1}_B) P_{A,i} (E_k \otimes \mathbb{1}_B)\right] = \mathrm{Tr}_A\left[\rho_{AB} \Lambda_A^*(P_{A,i})\right] = \mathrm{Tr}_A\left[\rho_{AB} \tilde{P}_{A,i}\right],$$

and we rewrite the output ensemble elements as: $\tilde{p}_i = \mathrm{Tr}[\rho_{AB}\tilde{P}_{A,i}] = 1/n$ and $\rho_{B,i} = \mathrm{Tr}_A[\rho_{AB}\tilde{P}_{A,i}]/\tilde{p}_i$. This ensemble is an EES. Next we show that: $\{\tilde{P}_{A,i}\}_i = \{\Lambda_A^*(P_{A,i})\}_i = \{\sum_k E_k^\dagger P_{A,i} E_k\}_i$ is a POVM. The elements of $\{\tilde{P}_{A,i}\}_i$ sum up to the identity: $\sum_i \tilde{P}_{A,i} = \sum_{k,i} E_k^\dagger P_{A,i} E_k = \sum_k E_k^\dagger \left(\sum_i P_{A,i}\right) E_k = \sum_k E_k^\dagger E_k = \mathbb{1}_B$, and they are positive operators: $\tilde{P}_{A,i} = \sum_k E_k^\dagger P_{A,i} E_k = \sum_k E_k^\dagger M_{A,i}^\dagger M_{A,i} E_k = \tilde{M}_{A,i}^\dagger \tilde{M}_{A,i}$, where the decomposition $P_{A,i} = M_{A,i}^\dagger M_{A,i}$ exists since $P_{A,i}$ is positive-semidefinite and $\tilde{M}_{A,i} = \sum_k M_{A,i} E_k$. It follows that, $\{\tilde{P}_{A,i}\}_i$ is a ME-POVM for ρ_{AB}, namely $\{\tilde{P}_{A,i}\}_i \in \Pi_A(\rho_{AB})$. Thus, for every ME-POVM $\{P_{A,i}\}_i \in \Pi_A(\tilde{\rho}_{AB})$ for $\tilde{\rho}_{AB}$, there is a ME-POVM $\{\tilde{P}_{A,i}\}_i \in \Pi_A(\rho_{AB})$ for ρ_{AB}, such that the output ensembles are identical: $\mathcal{E}(\tilde{\rho}_{AB}, \{P_{A,i}\}_i) = \mathcal{E}(\rho_{AB}, \{\tilde{P}_{A,i}\}_i)$. Thus, any EES that can be generated from $\tilde{\rho}_{AB}$, is obtainable from ρ_{AB} as well

$$\bigcup_{\{P_{A,i}\}_i \in \Pi_A(\tilde{\rho}_{AB})} \mathcal{E}\left(\tilde{\rho}_{AB}, \{P_{A,i}\}_i\right) \subseteq \bigcup_{\{P_{A,i}\}_i \in \Pi_A(\rho_{AB})} \mathcal{E}\left(\rho_{AB}, \{P_{A,i}\}_i\right) . \tag{B.2}$$

Finally, because $C_A(\rho_{AB})$ could be thought as the maximum guessing probability of the EESs that can be generated from ρ_{AB} (see Eq. (B.1)), we conclude that

$$C_A\left(\rho_{AB}\right) \geq C_A\left(\Lambda_A \otimes \mathcal{I}_B\left(\rho_{AB}\right)\right) , \tag{B.3}$$

for any state ρ_{AB} and CPTP map Λ_A.

Fixing the number n of outputs of the ME-POVMs considered in (B.1), Eq. (C.5) becomes:

$$\bigcup_{\{P_{A,i}\}_{i=1}^n \in \Pi_A(\tilde{\rho}_{AB})} \mathcal{E}\left(\tilde{\rho}_{AB}, \{P_{A,i}\}_i\right) \subseteq \bigcup_{\{P_{A,i}\}_{i=1}^n \in \Pi_A(\rho_{AB})} \mathcal{E}\left(\rho_{AB}, \{P_{A,i}\}_i\right) . \tag{B.4}$$

Therefore, it follows that:

$$C_A^{(n)}\left(\rho_{AB}\right) \geq C_A^{(n)}\left(\Lambda_A \otimes \mathcal{I}_B\left(\rho_{AB}\right)\right) , \tag{B.5}$$

for any integer $n \geq 2$, state ρ_{AB} and CPTP map Λ_A.

Next we show the property of monotonicity of $C_A(\rho_{AB})$ under the action of local operations of the form $\mathcal{I}_A \otimes \Lambda_B$. We find that the collection of the ME-POVMs for $\tilde{\rho}_{AB} = \mathcal{I}_A \otimes \Lambda_B\left(\rho_{AB}\right)$, namely $\Pi_A(\tilde{\rho}_{AB})$, coincides with $\Pi_A(\rho_{AB})$.

In order to prove this, we apply a general POVM $\{P_{A,i}\}_i$ on both ρ_{AB} and $\tilde{\rho}_{AB}$ and we show that the respective output ensembles are defined by the same probabilities. We can write $p_i = \mathrm{Tr}\left[\rho_{AB} P_{A,i}\right]$ and $\tilde{p}_i = \mathrm{Tr}\left[\mathcal{I}_A \otimes \Lambda_B\left(\rho_{AB}\right) P_{A,i}\right] = \mathrm{Tr}\left[\rho_{AB} P_{A,i}\right]$, where the last step uses the trace-preserving property of the superoperator $\mathcal{I}_A \otimes \Lambda_B$. Consequently, $p_i = 1/n$ if and only if $\tilde{p}_i = 1/n$ and $\{P_{A,i}\}_i \in \Pi_A(\rho_{AB})$ if and only if $\{P_{A,i}\}_i \in \Pi_A(\tilde{\rho}_{AB})$

$$\Pi_A(\rho_{AB}) = \Pi_A(\tilde{\rho}_{AB}) . \tag{B.6}$$

Given a ME-POVM for both ρ_{AB} and $\tilde{\rho}_{AB}$, we relate the output states

$$\tilde{\rho}_{B,i} = \Lambda_B \mathrm{Tr}_A\left[\rho_{AB} P_{A,i}\right] / p_i = \Lambda_B(\rho_{B,i}) . \tag{B.7}$$

From Eq. (C.8) and the definition of the guessing probability, it follows that

$$P_g\left(\{p_i, \rho_{B,i}\}_i\right) \geq P_g\left(\{p_i, \Lambda_B(\rho_{B,i})\}_i\right) , \tag{B.8}$$

and, considering Eq. (C.7), Eq. (C.8) and Eq. (C.9)

$$C_A(\rho_{AB}) \geq C_A(\mathcal{I}_A \otimes \Lambda_B(\rho_{AB})) \,, \tag{B.9}$$

that is true for any state ρ_{AB} and CPTP map Λ_B.

From Eq. (C.7) it follows the collection of the n-output ME-POVMs does not change if we apply a CPTP map Λ_B on ρ_{AB}. Therefore, since Eq. (C.9) is true for any number of outputs:

$$C_A^{(n)}(\rho_{AB}) \geq C_A^{(n)}(\mathcal{I}_A \otimes \Lambda_B(\rho_{AB})) \,, \tag{B.10}$$

for any integer $n \geq 2$, state ρ_{AB} and CPTP map Λ_B.

We underline that from this proof we automatically obtain the invariance under local unitary transformations of C and $C^{(n)}$ for any $n \geq 2$.

B.3 Proof that $C_A(\rho_{AB}^{(\tau)}) \geq C_B^{(2)}(\rho_{AB}^{(\tau)})$

In this appendix (where from now on we omit the time dependence of $\rho_{AB}^{(\tau)}(t)$, $\rho_B'^{(\tau)}(t)$ and $\rho_B''^{(\tau)}(t)$) we show that $C_A(\rho_{AB}^{(\tau)}) \geq C_B^{(2)}(\rho_{AB}^{(\tau)})$, where $C_B^{(2)}(\rho_{AB}^{(\tau)})$ is defined by

$$C_B^{(2)}(\rho_{AB}^{(\tau)}) = \max_{\{P_{B,i}\}_i \in \Pi_B^{(2)}\left(\rho_{AB}^{(\tau)}\right)} P_g\left(\mathcal{E}\left(\rho_{AB}^{(\tau)}, \{P_{B,i}\}_i\right)\right) - \frac{1}{2} \,,$$

where $\Pi_B^{(2)}(\rho_{AB}^{(\tau)})$ is the set of the 2-output ME-POVMs acting on B. In Appendix B.4 we show that $C_B^{(2)}(\rho_{AB}^{(\tau)}) = C_B(\rho_{AB}^{(\tau)})$ and this completes the proof that $C_A(\rho_{AB}^{(\tau)}) \geq C_B(\rho_{AB}^{(\tau)})$.

We apply a general but fixed 2-output ME-POVM for $\rho_{AB}^{(\tau)}$, where now the measured system is B: $\{P_{B,i}^{(2)}\}_i = \{P_B, \overline{P}_B\} \in \Pi_B(\rho_{AB}^{(\tau)})$, where $\overline{P}_B = \mathbb{1}_B - P_B$. The output ensemble $\mathcal{E}(\rho_{AB}^{(\tau)}, \{P_{B,i}^{(2)}\}_i) = \{p_{A,i}, \rho_{A,i}\}_i$ is composed by an uniform distribution (by definition of ME-POVM) and states in the following form

$$p_{A,1} = \frac{1}{2}\mathrm{Tr}_B\left[\left(\rho_B'^{(\tau)} + \rho_B''^{(\tau)}\right)P_B\right] = \frac{1}{2} \,, \tag{B.11}$$

$$p_{A,2} = \frac{1}{2}\mathrm{Tr}_B\left[\left(\rho_B'^{(\tau)} + \rho_B''^{(\tau)}\right)\overline{P}_B\right] = \frac{1}{2} \,, \tag{B.12}$$

$$\rho_{A,1} = |0\rangle\langle 0|_A \mathrm{Tr}_B\left[\rho_B'^{(\tau)} P_B\right] + |1\rangle\langle 1|_A \mathrm{Tr}_B\left[\rho_B''^{(\tau)} P_B\right] \,, \tag{B.13}$$

$$\rho_{A,2} = |0\rangle\langle 0|_A \mathrm{Tr}_B\left[\rho_B'^{(\tau)} \overline{P}_B\right] + |1\rangle\langle 1|_A \mathrm{Tr}_B\left[\rho_B''^{(\tau)} \overline{P}_B\right] \,. \tag{B.14}$$

Since $\mathcal{E}(\rho_{AB}^{(\tau)}, \{P_{B,i}^{(2)}\}_i)$ is an equiprobable ensemble of two states, we obtain that $P_g(\mathcal{E}(\rho_{AB}^{(\tau)}, \{P_{B,i}^{(2)}\}_i)) = (2 + \|\rho_{A,1} - \rho_{A,2}\|_1)/4$. Hence, with Eqs. (B.11)-(B.14), we can write it as

$$\left\| |0\rangle\langle 0|_A \mathrm{Tr}_B\left[\rho_B'^{(\tau)} \Delta P_B\right] + |1\rangle\langle 1|_A \mathrm{Tr}_B\left[\rho_B''^{(\tau)} \Delta P_B\right] \right\|_1$$

$$= \left|\mathrm{Tr}_B\left[\rho_B'^{(\tau)} \Delta P_B\right]\right| + \left|\mathrm{Tr}_B\left[\rho_B''^{(\tau)} \Delta P_B\right]\right| ,$$

where $\Delta P_B = P_B - \overline{P}_B$. Hence

$$\|\rho_{A,1} - \rho_{A,2}\|_1 = \max_{\pm} \left|\mathrm{Tr}_B\left[(\rho_B'^{(\tau)} \pm \rho_B''^{(\tau)}) \Delta P_B\right]\right| .$$

Using Eq. (B.11) and Eq. (B.12) we see that $|\mathrm{Tr}_B\left[(\rho_B'^{(\tau)} + \rho_B''^{(\tau)}) \Delta P_B\right]| = |\mathrm{Tr}_B\left[(\rho_B'^{(\tau)} + \rho_B''^{(\tau)}) P_B\right] - \mathrm{Tr}_B\left[(\rho_B' + \rho_B''^{(\tau)}) \overline{P}_B\right]| = 2|p_{A,1} - p_{A,2}| = 0$. Hence, we have that $\|\rho_{A,1} - \rho_{A,2}\|_1$ is equal to

$$\left|\mathrm{Tr}_B\left[(\rho_B'^{(\tau)} - \rho_B''^{(\tau)})(2P_B - \mathbb{1}_B)\right]\right| = 2\left|\mathrm{Tr}_B\left[(\rho_B'^{(\tau)} - \rho_B''^{(\tau)}) P_B\right]\right| ,$$

from which follows that

$$C_B^{(2)}(\rho_{AB}^{(\tau)}) = \max_{\{P_{B,i}^{(2)}\}_i \in \Pi_B(\rho_{AB}^{(\tau)})} \frac{\left|\mathrm{Tr}_B\left[(\rho_B'^{(\tau)} - \rho_B''^{(\tau)}) P_B\right]\right|}{2}. \tag{B.15}$$

To compare $C_B^{(2)}(\rho_{AB}^{(\tau)})$ with $C_A(\rho_{AB}^{(\tau)})$, we write

$$C_A(\rho_{AB}^{(\tau)}) = P_g(\{\{p_{A,1,2} = 1/2\}_i, \{\rho_B'^{(\tau)}, \rho_B''^{(\tau)}\}\}) - \frac{1}{2}$$

$$= \max_{\{P_{B,i}\}_i} \frac{\mathrm{Tr}_B\left[\rho_B'^{(\tau)} P_B + \rho_B''^{(\tau)} \overline{P}_B\right]}{2} - \frac{1}{2} = \max_{\{P_{B,i}\}_i} \frac{\mathrm{Tr}_B\left[(\rho_B'^{(\tau)} - \rho_B''^{(\tau)}) P_B\right]}{2}$$

$$= \max_{\{P_{B,i}\}_i} \frac{\left|\mathrm{Tr}_B\left[(\rho_B'^{(\tau)} - \rho_B''^{(\tau)}) P_B\right]\right|}{2} .$$

The only difference between $C_B^{(2)}(\rho_{AB}^{(\tau)})$ and $C_A(\rho_{AB}^{(\tau)})$ is in the maximization procedure: in the former we maximize only over the 2-output ME-POVMs $\Pi_B(\rho_{AB}^{(\tau)})$, while in the latter we can pick any 2-output POVM: $C_A(\rho_{AB}^{(\tau)}(t)) \geq C_B^{(2)}(\rho_{AB}^{(\tau)}(t))$ follows as a natural consequence.

B.4 Proof that $C_B(\rho_{AB}^{(\tau)}) = C_B^{(2)}(\rho_{AB}^{(\tau)})$

In this Appendix, in contrast to Appendix B.3, we consider the action of any ME-POVM over B for $\rho_{AB}^{(\tau)}$. We want to show that for each ME-POVM $\{P_{B,i}^{(n)}\}_i$ that we can consider in $C_B(\rho_{AB}^{(\tau)})$, where i runs from 1 to $n > 2$, we can always find at least one 2-output ME-POVM acting on B, namely $\{P_{B,1}, P_{B,2}\} \in \Pi_B(\rho_{AB}^{(\tau)})$, that provides an ensemble with a higher value of $P_g(\cdot)$. We recall that, if $\mathcal{E} = \{p_i, \rho_i\}_i$ is a generic ensemble of n states defined on $S(\mathcal{H})$, where $\mathcal{H}$ is a generic finite dimensional Hilbert space, the guessing probability of $\mathcal{E}$ is

$$P_g(\mathcal{E}) \equiv \max_{\{P_i\}_i} \sum_{i=1}^{n} p_i \mathrm{Tr}\left[\rho_i P_i\right] , \tag{B.16}$$

where the maximization is performed over the space of the n-output POVMs $\{P_i\}_i$ on $S(\mathcal{H})$. Starting from a general n-output ME-POVM $\{P_{B,i}^{(n)}\}_i$, we construct the corresponding 2-output ME-POVM $\{P_{B,1}, P_{B,2}\} \in \Pi_B(\rho_{AB}^{(\tau)})$ that accomplishes this task.

For every given n-output ME-POVM $\{P_{B,i}^{(n)}\}_i$ for $\rho_{AB}^{(\tau)}$, we can generate an equiprobable ensemble of states (EES) of the form $\mathcal{E}(\rho_{AB}^{(\tau)}, \{P_{B,i}^{(n)}\}_i) = \{\{p_i = 1/n\}, \{\rho_{A,i}\}\}_i$. The guessing probability of this ensemble, which we denote by $P_g^{(n)} = P_g(\mathcal{E}(\rho_{AB}^{(\tau)}, \{P_{B,i}^{(n)}\}_i))$, is

$$P_g^{(n)} = \mathrm{Tr}\left[\rho_{AB}^{(\tau)}\left(\sum_{i=1}^{n} \overline{P}_{A,i}^{(n)} \otimes P_{B,i}^{(n)}\right)\right] , \tag{B.17}$$

where $\{\overline{P}_{A,i}^{(n)}\}_i$ is a POVM that provides the maximum in Eq. (B.16). If n is even we consider the following 2-output POVM

$$P_{B,1}^{(2)} = \sum_{i\in E_1} P_{B,i}^{(n)}, \quad P_{B,2}^{(2)} = \sum_{i\in E_2} P_{B,i}^{(n)}, \tag{B.18}$$

where E_1 and E_2 are any two sets of $n/2$ indices such that $E_1 \cup E_2 = \{1, 2, \ldots, n\}$. This structure guarantees that Eq. (B.18) is a 2-output ME-POVM for $\rho_{AB}^{(\tau)}$. We compare Eq. (B.17) with the guessing probability of the output ensemble that we obtain applying Eq. (B.18) on $\rho_{AB}^{(\tau)}$

$$P_g^{(2)} = \max_{\{P_{A,i}\}_{i=1,2}} \mathrm{Tr}\left[\rho_{AB}^{(\tau)}\left(\sum_{i=1}^{2} P_{A,i} \otimes P_{B,i}^{(2)}\right)\right] \geq \mathrm{Tr}\left[\rho_{AB}^{(\tau)}\left(\sum_{i=1}^{2} P_{A,i}^{(2)} \otimes P_{B,i}^{(2)}\right)\right] , \tag{B.19}$$

where the POVM $\{P_{A,i}^{(2)}\}_i$ is defined by

$$P_{A,1}^{(2)} = \sum_{i\in E_1} \overline{P}_{A,i}^{(n)}, \quad P_{A,2}^{(2)} = \sum_{i\in E_2} \overline{P}_{A,i}^{(n)}. \tag{B.20}$$

$$P_g^{(2)} \geq \mathrm{Tr}\left[\rho_{AB}^{(\tau)} \left(P_{A,1}^{(2)} \otimes P_{B,1}^{(2)} + P_{A,2}^{(2)} \otimes P_{B,2}^{(2)}\right)\right] = \mathrm{Tr}\left[\rho_{AB}^{(\tau)} \left(\sum_{i=1}^{n} \overline{P}_{A,i}^{(n)} \otimes P_{B,i}^{(n)} + P_{AB}^{mix}\right)\right]$$

$$= P_g^{(n)} + \mathrm{Tr}\left[\rho_{AB}^{(\tau)} P_{AB}^{mix}\right] \geq P_g^{(n)}, \tag{B.21}$$

where P_{AB}^{mix} is a sum of mixed terms of the form $\overline{P}_{A,i}^{(n)} \otimes P_{B,j}^{(n)}$ with $i \neq j$, and it provides a non-negative contribution.

On the other hand, if n is odd, we define

$$P_{B,k}^{(2)} = \frac{1}{2} P_{B,x}^{(n)} + \sum_{i\in O_k^x} P_{B,i}^{(n)} \qquad (k = 1, 2) \tag{B.22}$$

$$P_{A,k}^{(2)} = \frac{1}{2} \overline{P}_{A,x}^{(n)} + \sum_{i\in O_k^x} \overline{P}_{A,i}^{(n)} \qquad (k = 1, 2) \tag{B.23}$$

where O_1^x and O_2^x are any two sets of $(n-1)/2$ indices such that $O_1^x \cup O_2^x = \{1, 2, \ldots, n\}\backslash x$ (the value of x will be fixed later). We consider again Eq. (B.19), where $\{P_{B,i}^{(2)}\}_i$ is now given by Eq. (B.22) and and $P_{A,i}^{(2)}$ is now given by Eq. (B.23). Since $P_{A,i}^{(2)}$ is not necessarily a POVM that maximizes Eq. (B.16) we have the following inequality for $P_g^{(2)}$

$$P_g^{(2)} \geq \mathrm{Tr}\left[\rho_{AB}^{(\tau)} \left(\sum_{i\neq x} \overline{P}_{A,i}^{(n)} \otimes P_{B,i}^{(n)} + \frac{1}{2}\overline{P}_{A,x}^{(n)} \otimes P_{B,x}^{(n)} + \frac{1}{2}\left(\sum_{i\neq x} \overline{P}_{A,i}^{(n)}\right) \otimes P_{B,x}^{(n)} + P_{AB}^{mix}\right)\right] \geq$$

$$\geq \mathrm{Tr}\left[\rho_{AB}^{(\tau)} \left(\sum_{i=1}^{n} P_{A,i}^{(n)} \otimes P_{B,i}^{(n)} - \frac{1}{2}\overline{P}_{A,x}^{(n)} \otimes P_{B,x}^{(n)} + \frac{1}{2}\left(\sum_{i\neq x} P_{A,i}^{(n)}\right) \otimes P_{B,x}^{(n)}\right)\right] =$$

$$= P_g^{(n)} + \mathrm{Tr}\left[\rho_{AB}^{(\tau)} \left(\frac{-\overline{P}_{A,x}^{(n)}}{2} \otimes P_{B,x}^{(n)} + \frac{\sum_{i\neq x} P_{A,i}^{(n)}}{2} \otimes P_{B,x}^{(n)}\right)\right]$$

$$= P_g^{(n)} + \mathrm{Tr}\left[\rho_{AB}^{(\tau)} \frac{\mathbb{1}_A - 2\overline{P}_{A,x}^{(n)}}{2} \otimes P_{B,x}^{(n)}\right],$$

where P_{AB}^{mix} represents terms that provide positive contributions to $P_g^{(2)}$. We have to find a value of x that makes the second term of the last relation positive. Let a_x and b_x be the diagonal elements of $\overline{P}_{A,x}^{(n)}$ in the orthonormal basis $\{|0\rangle_A, |1\rangle_A\}$. We recall that $\rho_{AB}^{(\tau)} = (|0\rangle\langle 0|_A \otimes \rho_B'^{(\tau)} + |1\rangle\langle 1|_A \otimes \rho_B''^{(\tau)})/2$ and we obtain

$$P_g^{(2)} \geq P_g^{(n)} + \mathrm{Tr}_B\left[\left(\frac{1-2a_x}{4}\rho_B'^{(\tau)} + \frac{1-2b_x}{4}\rho_B''^{(\tau)}\right) P_{B,x}^{(n)}\right], \tag{B.24}$$

where the second term on the right-hand side of the inequality is definitely positive when $a_x, b_x \leq 1/2$. From $\sum_i \overline{P}_{A,i}^{(n)} = \mathbb{1}_A$ follows that $\sum_{i=1}^n a_i = 1$ and $\sum_{i=1}^n b_i = 1$. Therefore, if $a_x > 1/2$ ($b_x > 1/2$), then $a_y \leq 1/2$ ($b_y \leq 1/2$) for any $y \neq x$. In order to fix the value of x, we must consider that a_x and b_x could be bigger than $1/2$ for two different values of x: let's say x_a and x_b. Even in this "worst-case" scenario we still have $n-2$ other possible choices for x such that $(1-2a_x), (1-2b_x) \geq 0$. We pick one of these values, and we call it $\overline{x} \in \{1, \ldots, n\} \setminus \{x_a, x_b\}$. Finally, if we use $\overline{x}$ in the definition of the POVMs $\{P_{A,i}^{(2)}\}_i$ and $\{P_{B,i}^{(2)}\}_i$, from Eq. (B.24) we obtain

$$P_g^{(2)} \geq P_g^{(n)}. \tag{B.25}$$

Eqs. (B.21) and (B.25) show that, when we evaluate $C_B(\rho_{AB}^{(\tau)})$, the guessing probability of the ensembles generated by the n-output ME-POVMs is never bigger than the one that we obtain if we only consider the 2-output ME-POVMs: $C_B^{(2)}(\rho_{AB}^{(\tau)}) = C_B(\rho_{AB}^{(\tau)})$. Thanks to this result we can finally say that $C_A(\rho_{AB}^{(\tau)}(t)) \geq C_B(\rho_{AB}^{(\tau)}(t))$ and $C(\rho_{AB}^{(\tau)}(t)) = C_A(\rho_{AB}^{(\tau)}(t))$. This result is valid if we consider $\rho_{AB}^{(\tau)}$, but in general it is not true.

B.5 Proof that $C_A(\rho_{AB}^{(\tau)}) = C_A^{(2)}(\rho_{AB}^{(\tau)})$

When we considered $C_A(\rho_{AB}^{(\tau)})$, we have seen that if the maximization over the ME-POVMs is considered only over the 2-output ones, the maximum is obtained for $\{P_{A,i}^{proj}\}_i = \{|0\rangle\langle 0|_A, |1\rangle\langle 1|_A\}$. In order to complete the proof, we need to show that even if we consider general n-output ME-POVMs (as in the definition (B.1)), we don't get higher guessing probabilities of the corresponding output ensembles. In other words, if we use the definition

$$C_A^{(2)}(\rho_{AB}^{(\tau)}) = \max_{\{P_{A,i}\}_i \in \Pi_A^{(2)}(\rho_{AB}^{(\tau)})} P_g\left(\mathcal{E}\left(\rho_{AB}^{(\tau)}, \{P_{A,i}\}_i\right)\right) - \frac{1}{2},$$

where $\Pi_A^{(2)}(\rho_{AB}^{(\tau)})$ contains only the 2-output ME-POVMs of $\rho_{AB}^{(\tau)}$, then $C_A(\rho_{AB}^{(\tau)}) = C_A^{(2)}(\rho_{AB}^{(\tau)})$.

To see this we can make the same analysis as done in Appendix B.4 for $C_B(\rho_{AB}^{(\tau)})$ but we switch the role of A and B in Eq. (B.18) and Eq. (B.20) when n is even and Eq. (B.22) and Eq. (B.23) when n is odd. The definitions for $P_g^{(n)}$, $P_g^{(2)}$, $E_{1,2}$ and $O_{1,2}^x$ are preserved.

The guessing probability of an EES generated by a ME-POVM $\{P_{A,i}^{(n)}\}_i$ with an even number of outputs is

$$P_g^{(n)} = \mathrm{Tr}\left[\rho_{AB}^{(\tau)}\left(\sum_{i=1}^{n} P_{A,i}^{(n)} \otimes \overline{P}_{B,i}^{(n)}\right)\right],$$

where $\{\overline{P}_{B,i}^{(n)}\}_i$ is a POVM that maximizes the guessing probability in Eq. (B.16). The 2-output ME-POVM that provides a higher guessing probability is

$$P_{A,1}^{(2)} = \sum_{i\in E_1} P_{A,i}^{(n)}, \quad P_{A,2}^{(2)} = \sum_{i\in E_2} P_{A,i}^{(n)}. \tag{B.26}$$

We define the following POVM on the system B

$$P_{B,1}^{(2)} = \sum_{i\in E_1} \overline{P}_{B,i}^{(n)}, \quad P_{B,2}^{(2)} = \sum_{i\in E_2} \overline{P}_{B,i}^{(n)}. \tag{B.27}$$

Consequently, we consider the following inequality

$$P_g^{(2)} \geq \mathrm{Tr}\left[\rho_{AB}^{(\tau)} \sum_{i=1,2} P_{A,i}^{(2)} \otimes P_{B,i}^{(2)}\right] = P_g^{(n)} + \mathrm{Tr}\left[\rho_{AB}^{(\tau)} \sum_{k=1}^{2} \sum_{i\neq j}^{i,j\in E_k} P_{A,i}^{(n)} \otimes \overline{P}_{B,j}^{(n)}\right],$$

which shows that $P_g^{(2)} \geq P_g^{(n)}$. If n is odd, we use again the technique from Appendix B.4, where we switch the role of A and B, to obtain the inequality

$$P_g^{(2)} \geq P_g^{(n)} + \mathrm{Tr}\left[\rho_{AB}^{(\tau)} \frac{\mathbb{1}_A - 2P_{A,x}^{(n)}}{2} \otimes \overline{P}_{B,x}^{(n)}\right],$$

where the right-hand side is greater than $P_g^{(n)}$ if x is suitably chosen. We underline that the results given in this section and Appendix B.3 suffice to state that $C^{(2)}(\rho_{AB}^{(\tau)}) = C_A^{(2)}(\rho_{AB}^{(\tau)}) \geq C_B^{(2)}(\rho_{AB}^{(\tau)})$.

Appendix C

Appendix of Chapter 5

C.1 The set of $\mathcal{P}$-POVMs is non-empty

In this section we prove that $\Pi_{\mathcal{P}}(\rho) \neq \emptyset$ for all states ρ and distributions $\mathcal{P}$. First, we notice that the POVM $\{\pi_1 \mathbb{1}, \pi_2 \mathbb{1}, \dots, \pi_n \mathbb{1}\}$ is a $\mathcal{P}$-POVM for all ρ. Nonetheless, we can prove this result also by realizing a mapping that provides a $\mathcal{P}$-POVM from any given POVM $\{P_i\}_{i=1}^n$ outside $\Pi_{\mathcal{P}}(\rho)$, where in general the outcome is different from $\{\pi_1 \mathbb{1}, \pi_2 \mathbb{1}, \dots, \pi_n \mathbb{1}\}$.

We start by fixing the generic distribution studied as $\mathcal{P} = \{\pi_i\}_{i=1}^n$. Hence, suppose that $p_i = \mathrm{Tr}\left[\rho P_i\right] \neq \pi_i$ for two or more values of $i = 1, \dots n$. We can always consider a $n \times n$ left-stochastic matrix M_{ij} that maps the probability distribution $\{p_i\}_{i=1}^n$ into $\mathcal{P} = \{\pi_i\}_{i=1}^n$ through the relation

$$\pi_i = \sum_{j=1}^{n} M_{ij} p_j \quad \text{for } i = 1, \dots, n \, . \tag{C.1}$$

Since M_{ij} is a left-stochastic matrix, $M_{ij} \in [0, 1]$ for all $i, j = 1, \dots, n$ and $\sum_i M_{ij} = 1$ for all $j = 1, \dots, n$, where the latter implies that M_{ij} has columns that sum up to 1. Hence, if we use M_{ij} to map the POVM $\{P_i\}_{i=1}^n$ into the set of operators $\{\tilde{P}_i\}_{i=1}^n$ through the relation

$$\tilde{P}_i = \sum_{j=1}^{n} M_{ij} P_j \quad \text{for } i = 1, \dots, n \, , \tag{C.2}$$

we can verify that it is a POVM. Indeed, from the property $\sum_{i=1}^n M_{ij} = 1$ for all $j = 1, \dots, n$, it follows that $\sum_{i=1}^n \tilde{P}_i = \sum_{i=1}^n P_i = \mathbb{1}$. Moreover, form $M_{ij} \in [0, 1]$ it follows that the operators $\tilde{P}_i$ are positive semi-definite. Now, if the left-stochastic matrix used in Eq. (C.2) is the same that appears in Eq. (C.1), we can also prove that $\{\tilde{P}_i\}_{i=1}^n \in \Pi_{\mathcal{P}}(\rho)$, namely that it is a $\mathcal{P}$-POVM. We can show this result by noticing that

$$\mathrm{Tr}\left[\rho\tilde{P}_i\right] = \mathrm{Tr}\left[\rho\sum_{j=1}^n M_{ij}P_j\right] = \sum_{j=1}^n M_{ij}\mathrm{Tr}\left[\rho P_j\right] = \sum_{j=1}^n M_{ij}p_j = \pi_i\,. \tag{C.3}$$

Hence, we proved that the POVM $\{\tilde{P}_i\}_{i=1}^n$ obtained from a generic POVM $\{P_i\}_{i=1}^n$ through the combinations given by the left-stochastic matrix M_{ij} satisfying Eq. (C.1) is a $\mathcal{P}$-POVM.

Notice that the approach shown here can be used to generate ME-POVMs (see Chapter 4) just by considering $\{\pi_i = 1/n\}_{i=1}^n$ for some $n \geq 2$. Therefore, this section generalizes the results of Appendix B.1.

In general, this procedure allows to obtain $\mathcal{P}$-POVMs different from the trivial measurement $\{\pi_1\mathbb{1}, \pi_2\mathbb{1}, \ldots, \pi_n\mathbb{1}\}$. Notice that the transformation that maps any n-output POVM into the $\mathcal{P}$-POVM $\{\pi_1\mathbb{1}, \pi_2\mathbb{1}, \ldots, \pi_n\mathbb{1}\}$ through Eq. (C.2) is given by the M_{ij} having columns equal to $\{\pi_i\}_{i=1}^n$, namely

$$M = \begin{pmatrix} \pi_1 & \pi_1 & \ldots & \pi_1 \\ \pi_2 & \pi_2 & \ldots & \pi_2 \\ \vdots & \vdots & \ddots & \vdots \\ \pi_n & \pi_n & \ldots & \pi_n \end{pmatrix}.$$

Example We apply this technique to the following example. We consider the qubit state $\rho \in S(\mathcal{H})$ that with respect to the basis $\{|0\rangle, |1\rangle\}$ of $\mathcal{H}$ assumes the following matrix form

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}.$$

Consider the POVM $\{P_i\}_{i=1}^3$ which, in the same basis, is given by

$$P_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/4 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 \\ 0 & 3/4 \end{pmatrix}.$$

The output probabilities $p_i = \mathrm{Tr}\left[\rho P_i\right]$ that we obtain by measuring ρ with $\{P_i\}_{i=1}^3$ are $\{p_i\}_{i=1}^3 = \{1/4, 3/8, 3/8\}$. Now, imagine that our target probability distribution is $\{\pi_i\}_{i=1}^3 = \{1/\sqrt{5}, 1/\sqrt{5}, 1-2/\sqrt{5}\}$. A 3×3 left-stochastic matrix M_{ij} mapping $\{p_i\}_{i=1}^3$ into $\{\pi_i\}_{i=1}^3$ through Eq. (C.1) is given by

$$M = \begin{pmatrix} a & b & 0 \\ 0 & 1-b & c \\ 0 & 0 & 1-c \end{pmatrix},$$

where $a = 1$, $b = (8\sqrt{5} - 10)/15 \simeq 0.52590$ and $c = (16 - 5\sqrt{5})/(3\sqrt{5}) \simeq 0.71847$ (the property of left-stochastic matrices for which columns sum up to 1 is particularly evident in this case). If we use the matrix M_{ij} and the POVM $\{P_i\}_{i=1}^3$ of this example in Eq. (C.2) we obtain the $\mathcal{P}$-POVM $\{\tilde{P}_i\}_{i=1}^3$ given by

$$\tilde{P}_1 = P_1 + bP_2, \quad \tilde{P}_2 = (1-b)P_2 + cP_3, \quad \tilde{P}_3 = (1-c)P_3 .$$

We can verify that $\mathrm{Tr}\left[\rho\tilde{P}_i\right] = \pi_i$ and that $\{\tilde{P}_i\}_{i=1}^3$ is different from the trivial $\mathcal{P}$-POVM $\{\pi_1\mathbb{1}, \pi_2\mathbb{1}, \pi_3\mathbb{1}\}$. For instance, we have that $\tilde{P}_1 = \mathrm{diag}((1+b)/2, b/4) \simeq \mathrm{diag}(0.76295, 0.13148) \neq \pi_1\mathrm{diag}(1,1)$, where $\pi_1 = 1/\sqrt{5} = 0.44721$. The technique used to construct the matrix M of this example is inspired by the results explained in Appendix B.1 for ME-POVMs.

C.2 Monotonic behavior of $C_A^{\mathcal{P}}$ under local operations

We consider a general bipartite finite-dimensional quantum system with Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. Therefore, the states that we consider are $\rho_{AB} \in S(\mathcal{H}_{AB})$. We consider a generic finite probability distribution $\mathcal{P} = \{p_i\}_{i=1}^n$ and we prove that $C_A^{\mathcal{P}}$ is monotone under local operations of the form $\Lambda_A \otimes I_B$ and $I_A \otimes \Lambda_B$ on ρ_{AB}, where Λ_A (Λ_B) is a CPTP map on A (B) and I_A (I_B) is the identity map on $S(\mathcal{H}_A)$ ($S(\mathcal{H}_B)$).

In order to show the effect of the application of a local operation of the form $\Lambda_A \otimes I_B$ on $C_A^{\mathcal{P}}(\rho_{AB})$, we look at $\Pi_A^{\mathcal{P}}(\rho_{AB})$ in a different way. Each element of this collection is a $\mathcal{P}$-POVM for ρ_{AB}, namely they generate output ensembles where the output probability distribution is $\mathcal{P} = \{p_i\}_i$. In fact, we can consider $C_A^{\mathcal{P}}(\rho_{AB})$ as the maximization over all the possible output ensembles with output probability distribution $\mathcal{P}$ that we can generate measuring the subsystem A of ρ_{AB}.

The effect of the first local operation that we consider is: $\tilde{\rho}_{AB} = \Lambda_A \otimes I_B(\rho_{AB}) = \sum_k (E_k \otimes \mathbb{1}_B)\rho_{AB}(E_k \otimes \mathbb{1}_B)^\dagger$, where $\{E_k\}_k$ is a set of Kraus operators that corresponds to Λ_A. Now we analyze the relation between $\Pi_A^{\mathcal{P}}(\rho_{AB})$ and $\Pi_A^{\mathcal{P}}(\tilde{\rho}_{AB})$. Given a $\mathcal{P}$-POVM for $\tilde{\rho}_{AB}$, namely $\{P_{A,i}\}_i \in \Pi_A^{\mathcal{P}}(\tilde{\rho}_{AB})$, the output ensemble $\mathcal{E}(\tilde{\rho}_{AB}, \{P_{A,i}\}_i)$ is defined by $\mathrm{Tr}\left[\tilde{\rho}_{AB}P_{A,i} \otimes \mathbb{1}_B\right] = p_i$ and $\tilde{\rho}_{B,i} = \mathrm{Tr}_A\left[\tilde{\rho}_{AB}P_{A,i} \otimes \mathbb{1}_B\right]/p_i$. Now we write the i-th element of the output probability distribution that we obtain applying $\{P_{A,i}\}_i$ on $\tilde{\rho}_{AB}$, namely the probabilities

$p_i = \mathrm{Tr}\left[\Lambda_A \otimes I_B\left(\rho_{AB}\right) P_{A,i} \otimes \mathbb{1}_B\right]$, as follows

$$\begin{aligned} p_i &= \mathrm{Tr}\left[\sum_k (E_k \otimes \mathbb{1}_B)\rho_{AB}(E_k^\dagger \otimes \mathbb{1}_B) P_{A,i} \otimes \mathbb{1}_B\right] \\ &= \mathrm{Tr}\left[\rho_{AB}\sum_k (E_k^\dagger \otimes \mathbb{1}_B) P_{A,i}(E_k \otimes \mathbb{1}_B)\right] = \mathrm{Tr}\left[\rho_{AB}\Lambda_A^*(P_{A,i}) \otimes \mathbb{1}_B\right] \\ &= \mathrm{Tr}\left[\rho_{AB}\tilde{P}_{A,i} \otimes \mathbb{1}_B\right], \end{aligned} \tag{C.4}$$

where we have defined $\tilde{P}_{A,i} \equiv \Lambda_A^*(P_{A,i}) = \sum_k (E_k^\dagger \otimes \mathbb{1}_B) P_{A,i}(E_k \otimes \mathbb{1}_B)$. Similarly, we can write $\tilde{\rho}_{B,i} = \mathrm{Tr}_A[\tilde{\rho}_{AB}P_{A,i}]/p_i = \mathrm{Tr}_A[\rho_{AB}\tilde{P}_{A,i}]/p_i$. Therefore, since $p_i = \mathrm{Tr}[\rho_{AB}\tilde{P}_{A,i}]$ and $\tilde{\rho}_{B,i} = \mathrm{Tr}_A[\rho_{AB}\tilde{P}_{A,i}]/p_i$, if we apply $\{\tilde{P}_{A,i}\}_i$ on ρ_{AB} we obtain the same $\mathcal{P}$-distributed output ensemble $\{p_i, \tilde{\rho}_{B,i}\}_i$ that we obtain applying $\{P_{A,i}\}_i$ on $\tilde{\rho}_{AB}$. Next we show that: $\{\tilde{P}_{A,i}\}_i = \left\{\Lambda_A^*\left(P_{A,i}\right)\right\}_i = \{\sum_k E_k^\dagger P_{A,i} E_k\}_i$, is a proper n-output POVM. First, the elements of $\{\tilde{P}_{A,i}\}_i$ sum up to the identity: $\sum_i \tilde{P}_{A,i} = \sum_{k,i} E_k^\dagger\, P_{A,i}\, E_k = \sum_k E_k^\dagger\, \left(\sum_i P_{A,i}\right)\, E_k = \sum_k E_k^\dagger\, E_k = \mathbb{1}_B$. Moreover, we show that they are positive semi-definite operators. Indeed, for any $|\psi\rangle_A \in \mathcal{H}_A$, we have

$$\langle\psi|_A\tilde{P}_{A,i}|\psi\rangle_A = \sum_k (\langle\psi|_A E_k^\dagger)\, P_{A,i}\, (E_k|\psi\rangle_A) = \sum_k \langle\psi_k|_A P_{A,i}|\psi_k\rangle_A \geq 0,$$

where each element of the last sum is non-negative because $P_{A,i}$ is positive semi-definite. It follows that $\{\tilde{P}_{A,i}\}_i$ is a POVM and in particular a $\mathcal{P}$-POVM for ρ_{AB}, namely $\{\tilde{P}_{A,i}\}_i \in \Pi_A^{\mathcal{P}}(\rho_{AB})$. Thus, for every $\mathcal{P}$-POVM $\{P_{A,i}\}_i \in \Pi_A^{\mathcal{P}}(\tilde{\rho}_{AB})$ for $\tilde{\rho}_{AB}$, there is a $\mathcal{P}$-POVM $\{\tilde{P}_{A,i}\}_i \in \Pi_A^{\mathcal{P}}(\rho_{AB})$ for ρ_{AB}, such that the output ensembles are identical: $\mathcal{E}(\tilde{\rho}_{AB}, \{P_{A,i}\}_i) = \mathcal{E}(\rho_{AB}, \{\tilde{P}_{A,i}\}_i)$. Hence, any $\mathcal{P}$-distributed ensemble of B that can be generated from $\tilde{\rho}_{AB}$ can also be obtained from ρ_{AB}. Therefore, we obtain the following inclusion

$$\bigcup_{\{P_{A,i}\}_i \in \Pi_A^{\mathcal{P}}(\tilde{\rho}_{AB})} \mathcal{E}\left(\tilde{\rho}_{AB},\, \{P_{A,i}\}_i\right) \subseteq \bigcup_{\{P_{A,i}\}_i \in \Pi_A^{\mathcal{P}}(\rho_{AB})} \mathcal{E}\left(\rho_{AB},\, \{P_{A,i}\}_i\right). \tag{C.5}$$

Finally, since as we said above $C_A^{\mathcal{P}}(\rho_{AB})$ is the maximum guessing probability of the $\mathcal{P}$-distributed output ensembles that can be generated from ρ_{AB}, from Eq. (C.5) we conclude that $C_A^{\mathcal{P}}(\rho_{AB})$ is defined as a maximization over a set that includes the set over which maximization defines $C_A^{\mathcal{P}}(\tilde{\rho}_{AB})$. Hence, for any state ρ_{AB} and CPTP map Λ_A, we obtain

$$C_A^{\mathcal{P}}\left(\rho_{AB}\right) \geq C_A^{\mathcal{P}}\left(\Lambda_A \otimes I_B\left(\rho_{AB}\right)\right). \tag{C.6}$$

Next we show that $C_A^{\mathcal{P}}(\rho_{AB})$ is monotonic under local operations of the form $I_A \otimes \Lambda_B$. We find that the collection of the $\mathcal{P}$-POVMs for $\tilde{\rho}_{AB} = I_A \otimes \Lambda_B(\rho_{AB})$, namely $\Pi_A^{\mathcal{P}}(\tilde{\rho}_{AB})$, coincides with $\Pi_A^{\mathcal{P}}(\rho_{AB})$. In order to prove this, we apply a general POVM $\{P_{A,i}\}_i$ both on ρ_{AB} and $\tilde{\rho}_{AB}$ and we show that the respective output ensembles are defined by the same probability distribution. Indeed, being $\mathrm{Tr}[\rho_{AB} P_{A,i}]$ ($\mathrm{Tr}[I_A \otimes \Lambda_B(\rho_{AB}) P_{A,i}]$) the probability for the i-th output of the POVM considered when it is applied on ρ_{AB} ($\tilde{\rho}_{AB}$), we have $\mathrm{Tr}[I_A \otimes \Lambda_B(\rho_{AB}) P_{A,i}] = \mathrm{Tr}[\rho_{AB} P_{A,i}]$, where this identity uses the trace-preserving property of the superoperator $I_A \otimes \Lambda_B$. Consequently, if $\{P_{A,i}\}_i$ is a $\mathcal{P}$-POVM for ρ_{AB}, which means that $\mathrm{Tr}[\rho_{AB} P_{A,i}] = p_i$, in the same way $\mathrm{Tr}[I_A \otimes \Lambda_B(\rho_{AB}) P_{A,i}] = p_i$. Hence, $\{P_{A,i}\}_i \in \Pi_A^{\mathcal{P}}(\rho_{AB})$ if and only if $\{P_{A,i}\}_i \in \Pi_A^{\mathcal{P}}(\tilde{\rho}_{AB})$, namely

$$\Pi_A^{\mathcal{P}}(\rho_{AB}) = \Pi_A^{\mathcal{P}}(\tilde{\rho}_{AB})\,. \tag{C.7}$$

Given a $\mathcal{P}$-POVM $\{P_{A,i}\}_i$ both for ρ_{AB} and $\tilde{\rho}_{AB}$, we compare the corresponding output states

$$\tilde{\rho}_{B,i} = \Lambda_B\left(\mathrm{Tr}_A\left[\rho_{AB} P_{A,i} \otimes \mathbb{1}_B\right]/p_i\right) = \Lambda_B(\rho_{B,i})\,. \tag{C.8}$$

From Eq. (C.8) and the definition of the guessing probability, it follows that

$$P_g\left(\{p_i,\, \rho_{B,i}\}_i\right) \geq P_g\left(\{p_i,\, \Lambda_B(\rho_{B,i})\}_i\right)\,. \tag{C.9}$$

The consequence of the last relation is that for any $\mathcal{P}$-distributed output ensemble ensemble that we can generate from $\tilde{\rho}_{AB}$ there exists at least one $\mathcal{P}$-distributed output ensemble that we can generate from ρ_{AB} for which the guessing probability is equal or greater. Hence, considering the definition of $C_A^{\mathcal{P}}$, Eqs. (C.7) and (C.9), we conclude that

$$C_A^{\mathcal{P}}(\rho_{AB}) \geq C_A^{\mathcal{P}}(I_A \otimes \Lambda_B(\rho_{AB}))\,, \tag{C.10}$$

for any state ρ_{AB} and CPTP map Λ_B.

C.3 Performing $\overline{\mathcal{P}}$-POVMs on the probe state

In this section we prove that, if we apply the projective $\overline{\mathcal{P}}$-POVM $\{|i\rangle\langle i|_A\}_i$ on A for $\rho_{AB}^{(\lambda)}(t)$, we obtain

$$P_g\left(\rho_{AB}^{(\lambda)}(t), \{|i\rangle\langle i|_A\}_i\right) = \lambda + (1-\lambda)\, P_g\left(\overline{\mathcal{E}}_{SA'}(t)\right). \tag{C.11}$$

Moreover, for a general $\overline{\mathcal{P}}$-POVM on A for $\rho_{AB}^{(\lambda)}(t)$ different from $\{|i\rangle\langle i|_A\}_i$, we have

$$P_g\left(\rho_{AB}^{(\lambda)}(t), \{P_{A,i}\}_i\right) = \lambda P_g\left(\{\overline{p}_i, \rho_{A'',i}^{\perp}\}_i\right) + (1-\lambda) P_g\left(\{\overline{p}_i, \rho_{SA',i}^{\parallel}(t)\}_i\right), \tag{C.12}$$

for some $\{\rho^{\perp}_{A'',i}\}_i$ and $\{\rho^{||}_{SA',i}(t)\}_i$ that we define. First, we notice that the projective measurement $\{|i\rangle\langle i|_A\}_{i=1}^{\overline{n}}$ is a $\overline{\mathcal{P}}$-POVM on A for $\rho^{(\lambda)}_{AB}(t)$ for any t and λ. We consider $\mathcal{E}(\rho^{(\lambda)}_{AB}(t), \{|i\rangle\langle i|_A\}_i)$, namely the ensemble of B that we obtain measuring $\rho^{(\lambda)}_{AB}(t)$ with $\{|i\rangle\langle i|_A\}_i$:

$$\mathcal{E}(\rho^{(\lambda)}_{AB}(t), \{|i\rangle\langle i|_A\}_i) = \left\{\overline{p}_i, \lambda\, \sigma_{SA'}(t) \otimes |i\rangle\langle i|_{A''} + (1-\lambda)\overline{\rho}_{B,i}(t)\right\}_{i=1}^{\overline{n}}, \qquad \text{(C.13)}$$

where $\overline{\rho}_{B,i} = \overline{\rho}_{SA',i} \otimes |\overline{n}+1\rangle\langle\overline{n}+1|_{A''}$. We evaluate the guessing probability of this ensemble and we obtain

$$\begin{aligned} P_g(\rho^{(\lambda)}_{AB}(t), \{|i\rangle\langle i|_A\}_i) = & \max_{\{P_{B,i}\}_i} \sum_{i=1}^{\overline{n}} \overline{p}_i \operatorname{Tr}_B\Big[(\lambda\sigma_{SA'}(t) \otimes |i\rangle\langle i|_{A''} \\ & +(1-\lambda)\overline{\rho}_{SA',i}(t) \otimes |\overline{n}+1\rangle\langle\overline{n}+1|_{A''}\Big) P_{B,i}\Big] \\ = & \max_{\{P_{B,i}\}_i} \sum_{i=1}^{\overline{n}} \overline{p}_i \Big(\lambda \operatorname{Tr}_B\Big[\sigma_{SA'}(t) \otimes |i\rangle\langle i|_{A''}\, P_{B,i}\Big] \\ & +(1-\lambda)\operatorname{Tr}_B\Big[\overline{\rho}_{SA',i}(t) \otimes |\overline{n}+1\rangle\langle\overline{n}+1|_{A''}\, P_{B,i}\Big]\Big). \qquad \text{(C.14)} \end{aligned}$$

We notice that, for any $i = 1, \ldots, \overline{n}$, every state that belongs to the set $\{\sigma_{SA'}(t) \otimes |i\rangle\langle i|_{A''}\}_i$ is orthogonal to every state of the set $\{\overline{\rho}_{SA',i}(t) \otimes |\overline{n}+1\rangle\langle\overline{n}+1|_{A''}\}_i$. It follows that, for any $i = 1, \ldots, \overline{n}$, $\operatorname{Tr}_B\left[\sigma_{SA'}(t) \otimes |i\rangle\langle i|_{A''}\, P_{B,i}\right]$ depends only on the components of $P_{B,i}$ that belong to $\operatorname{span}(\{|i\rangle\langle j|_B\}_{ij})$, where $|i\rangle_B$ and $|j\rangle_B$ belong to the tensor product between the elements of $\mathcal{M}_{SA'}$, namely an orthonormal basis of $\mathcal{H}_S \otimes \mathcal{H}_{A'}$, and $\{|k\rangle_{A''}\}_{k=1}^{\overline{n}}$ (notice that $\dim(\mathcal{H}_{A''}) = \overline{n}+1$). Similarly, for any $i = 1, \ldots, \overline{n}$, the value of $\operatorname{Tr}_B\left[\overline{\rho}_{SA',i}(t) \otimes |\overline{n}+1\rangle\langle\overline{n}+1|_{A''}\, P_{B,i}\right]$ depends only on the components of $P_{B,i}$ that belong to $\operatorname{span}(\{|i'\rangle\langle j'|_B\}_{i'j'})$, where $|i'\rangle_B$ and $|j'\rangle_B$ belong to the tensor product between the elements of $\mathcal{M}_{SA'}$ and the vector $|\overline{n}+1\rangle_{A''}$. We further note that no operator defined on $\operatorname{span}(\{|i\rangle\langle j|_B\}_{ij}) \oplus \operatorname{span}(\{|i'\rangle\langle j'|_B\}_{i'j'})$ that is not positive semidefinite can be made positive semidefinite by adding something outside $\operatorname{span}(\{|i\rangle\langle j|_B\}_{ij}) \oplus \operatorname{span}(\{|i'\rangle\langle j'|_B\}_{i'j'})$. Therefore, we can limit the maximization in Eq. (C.14) to be over POVMs $P_{B,i}$ that are defined on $\operatorname{span}(\{|i\rangle\langle j|_B\}_{ij}) \oplus \operatorname{span}(\{|i'\rangle\langle j'|_B\}_{i'j'})$, without affecting the optimal value. Since $\operatorname{span}(\{|i\rangle\langle j|_B\}_{ij})$ is orthogonal to $\operatorname{span}(\{|i'\rangle\langle j'|_B\}_{i'j'})$, the maximization in Eq. (C.14) can be divided in two independent maximizations

$$
\begin{aligned}
P_g(\rho_{AB}^{(\lambda)}(t),\{|i\rangle\langle i|_A\}_i) &= \lambda \max_{\{P_{B,i}\}_i} \sum_{i=1}^{\overline{n}} \overline{p}_i \mathrm{Tr}_B\left[\sigma_{SA'}(t)\otimes|i\rangle\langle i|_{A''}P_{B,i}\right] \\
&\quad +(1-\lambda)\max_{\{P_{B,i}\}_i} \sum_{i=1}^{\overline{n}} \overline{p}_i \mathrm{Tr}_B\left[\overline{\rho}_{SA',i}(t)\otimes|\overline{n}+1\rangle\langle\overline{n}+1|_{A''}P_{B,i}\right] \\
&= \lambda P_g(\{\overline{p}_i,\sigma_{SA'}(t)\otimes|i\rangle\langle i|_{A''}\}_i) \\
&\quad +(1-\lambda)P_g(\{\overline{p}_i,\overline{\rho}_{SA',i}(t)\otimes|\overline{n}+1\rangle\langle\overline{n}+1|_{A''}\}_i) \\
&= \lambda P_g(\{\overline{p}_i,|i\rangle\langle i|_{A''}\}_i)+(1-\lambda)P_g(\{\overline{p}_i,\overline{\rho}_{SA',i}(t)\}_i) \\
&= \lambda+(1-\lambda)P_g(\overline{\mathcal{E}}_{SA'}(t))\,, \qquad \text{(C.15)}
\end{aligned}
$$

where we have used $P_g(\{\overline{p}_i,|i\rangle\langle i|_{A''}\}_i)=1$, namely the possibility to perfectly distinguish orthonormal states, $P_g(\{\overline{p}_i,\sigma_{SA'}(t)\otimes|i\rangle\langle i|_{A''}\}_i)=P_g(\{\overline{p}_i,|i\rangle\langle i|_{A''}\}_i)$ and $P_g(\{\overline{p}_i,\overline{\rho}_{SA',i}(t)\otimes|\overline{n}+1\rangle\langle\overline{n}+1|_{A''}\}_i)=P_g(\{\overline{p}_i,\overline{\rho}_{SA',i}(t)\}_i)=P_g(\overline{\mathcal{E}}_{SA'}(t))$.

The output ensemble that we obtain applying a generic $\overline{\mathcal{P}}$-POVM $\{P_{A,i}\}_i$ on A for $\rho_{AB}^{(\lambda)}(t)$ different from $\{|i\rangle\langle i|_A\}_i$ is $\mathcal{E}(\rho_{AB}^{(\lambda)}(t),\{P_{A,i}\}_i)$. The k-th state of this ensemble is

$$
\begin{aligned}
\rho_{B,k}^{(\lambda)}(t) &= \frac{\mathrm{Tr}_A\left[\rho_{AB}^{(\lambda)}(t)P_{A,k}\otimes\mathbb{1}_B\right]}{\overline{p}_k} \\
&= \sum_{i=1}^{\overline{n}}\frac{\overline{p}_i}{\overline{p}_k}\mathrm{Tr}_A\left[|i\rangle\langle i|_A P_{A,k}\right]\left(\lambda\sigma_{SA'}(t)\otimes|i\rangle\langle i|_{A''}+(1-\lambda)\overline{\rho}_{B,i}(t)\right) \\
&= \sum_{i=1}^{\overline{n}}\frac{\overline{p}_i\,(P_{A,k})_{ii}}{\overline{p}_k}\left(\lambda\sigma_{SA'}(t)\otimes|i\rangle\langle i|_{A''}+(1-\lambda)\overline{\rho}_{B,i}(t)\right)\,, \qquad \text{(C.16)}
\end{aligned}
$$

where $(P_{A,k})_{ii}=\langle i|_A P_{A,k}|i\rangle_A\geq 0$ is the i-th diagonal element of $P_{A,k}$ in the basis $\mathcal{M}_A=\{|i\rangle_A\}_{i=1}^{\overline{n}}$. Keeping in mind that $\overline{\mathcal{P}}$ is a finite probability distribution and $\overline{p}_k>0$ for any k, we define the parameters $e_{ik}\equiv(P_{A,k})_{ii}\overline{p}_i/\overline{p}_k\geq 0$. Since $\rho_{B,k}^{(\lambda)}(t)$ and the states $\lambda\sigma_{SA'}(t)\otimes|i\rangle\langle i|_{A''}+(1-\lambda)\overline{\rho}_{B,i}(t)$ are trace one operators for any $i=1,\ldots,\overline{n}$, we conclude that $\sum_i e_{ik}=1$ for any $k=1,\ldots,\overline{n}$. Therefore, $\{e_{ik}\}_{i=1}^{\overline{n}}$ is an $\overline{n}$-element probability distribution for any value of $k=1,...,\overline{n}$. We write:

$$\begin{aligned}\rho_{B,k}^{(\lambda)}(t) &= \sum_{i=1}^{\bar{n}} e_{ik}\left(\lambda\sigma_{SA'}(t)\otimes|i\rangle\langle i|_{A''}+(1-\lambda)\overline{\rho}_{B,i}(t)\right)\\ &= \lambda\sum_{i=1}^{\bar{n}} e_{ik}\sigma_{SA'}(t)\otimes|i\rangle\langle i|_{A''}+(1-\lambda)\sum_{i=1}^{\bar{n}} e_{ik}\overline{\rho}_{B,i}(t)\\ &= \lambda\sigma_{B,k}^{\perp}(t)+(1-\lambda)\sigma_{B,k}^{||}(t), \end{aligned}\tag{C.17}$$

where we have used the definitions

$$\sigma_{B,k}^{\perp}(t)\equiv\sigma_{SA'}(t)\otimes\left(\sum_{i=1}^{\bar{n}} e_{ik}|i\rangle\langle i|_{A''}\right)\equiv\sigma_{SA'}(t)\otimes\rho_{A'',k}^{\perp}\,,\tag{C.18}$$

$$\sigma_{B,k}^{||}(t)\equiv\left(\sum_{i=1}^{\bar{n}} e_{ik}\,\overline{\rho}_{SA',i}(t)\right)\otimes|\bar{n}+1\rangle\langle\bar{n}+1|_{A''}\equiv\rho_{SA',k}^{||}(t)\otimes|\bar{n}+1\rangle\langle\bar{n}+1|_{A''}.\tag{C.19}$$

Each state $\rho_{A'',k}^{\perp}$ ($\rho_{SA',k}^{||}(t)$) is a convex combination of the states $\{|i\rangle\langle i|_{A''}\}_{i=1}^{\bar{n}}$ ($\{\overline{\rho}_{SA',i}(t)\}_{i=1}^{\bar{n}}$) that does not depend on λ but depends on the $\overline{\mathcal{P}}$-POVM $\{P_{A,i}\}_i$ chosen. From Eq. (C.17) it follows that, if we consider a generic $\overline{\mathcal{P}}$-POVM $\{P_{A,i}\}_i$ for $\rho_{AB}^{(\lambda)}(t)$, we obtain

$$\mathcal{E}(\rho_{AB}^{(\lambda)}(t),\{P_{A,i}\}_i)=\{\overline{p}_i,\lambda\sigma_{B,i}^{\perp}(t)+(1-\lambda)\sigma_{B,i}^{||}(t)\}_i\,,\tag{C.20}$$

and therefore, similarly to Eq. (C.15), now we can write

$$P_g(\rho_{AB}^{(\lambda)}(t),\{P_{A,i}\}_i)=\lambda P_g(\{\overline{p}_i,\rho_{A'',i}^{\perp}\}_i)+(1-\lambda)P_g(\{\overline{p}_i,\rho_{SA',i}^{||}(t)\}_i)\,.\tag{C.21}$$

C.4 Analysis of case (A)

Let assume that for some $\alpha\in[0,1)$ we have that $\{P_{A,i}^{(\alpha)}\}_i=\{|i\rangle\langle i|_A\}_i$, namely this projective measurement is one of the optimal $\overline{\mathcal{P}}$-POVM that accomplishes the maximization for $C_A^{\overline{\mathcal{P}}}(\rho_{AB}^{(\alpha)}(\tau))$, and that for some $\beta>\alpha$ instead we have that $\{|i\rangle\langle i|_A\}_i$ is not optimal. In this section we show that these two assumptions are incompatible and lead to a contradiction. The first condition implies that, when $\lambda=\alpha$ the optimal $\overline{\mathcal{P}}$-POVM that provides the greatest value of $P_g(\rho_{AB}^{(\alpha)}(\tau),\{P_{A,i}\}_i)$ is $\{P_{A,i}^{(\alpha)}\}_i=\{|i\rangle\langle i|_A\}_i$ and therefore

$$\alpha+(1-\alpha)P_g(\overline{\mathcal{E}}_{SA'}(\tau))\geq\alpha P_g(\mathcal{E}^{\perp(\beta)})+(1-\alpha)P_g(\mathcal{E}^{||(\beta)})\,,$$

$$\alpha\left(1 - P_g(\mathcal{E}^{\perp(\beta)})\right) + (1-\alpha)\left(P_g(\overline{\mathcal{E}}_{SA'}(\tau)) - P_g(\mathcal{E}^{\|(\beta)})\right) \geq 0\,, \tag{C.22}$$

where we also considered the cases where $\{P^{(\beta)}_{A,i}\}_i$ is optimal both for $\lambda = \alpha$ and $\lambda = \beta$. On the other hand, for $\lambda = \beta > \alpha$ we have that $\{|i\rangle\langle i|_A\}_i$ is not an optimal $\overline{\mathcal{P}}$-POVM for the maximization needed for $C^{\overline{\mathcal{P}}}_A(\rho^{(\beta)}_{AB}(\tau))$ and

$$\beta P_g(\mathcal{E}^{\perp(\beta)}) + (1-\beta)P_g(\mathcal{E}^{\|(\beta)}) > \beta + (1-\beta)P_g(\overline{\mathcal{E}}_{SA'}(\tau))\,, \tag{C.23}$$

which can be written as

$$\beta\left(P_g(\mathcal{E}^{\perp(\beta)}) - 1 + P_g(\overline{\mathcal{E}}_{SA'}(\tau)) - P_g(\mathcal{E}^{\|(\beta)})\right) > P_g(\overline{\mathcal{E}}_{SA'}(\tau)) - P_g(\mathcal{E}^{\|(\beta)})\,, \tag{C.24}$$

and therefore, by subtracting $\alpha\left(P_g(\mathcal{E}^{\perp(\beta)}) - 1 + P_g(\overline{\mathcal{E}}_{SA'}(\tau)) - P_g(\mathcal{E}^{\|(\beta)})\right)$ from each side of inequality (C.24), we obtain

$$\begin{aligned}(\beta-\alpha)&\left(P_g(\mathcal{E}^{\perp(\beta)}) - 1 + P_g(\overline{\mathcal{E}}_{SA'}(\tau)) - P_g(\mathcal{E}^{\|(\beta)})\right)\\ &> \alpha\left(1 - P_g(\mathcal{E}^{\perp(\beta)})\right) + (1-\alpha)\left(P_g(\overline{\mathcal{E}}_{SA'}(\tau)) - P_g(\mathcal{E}^{\|(\beta)})\right).\end{aligned} \tag{C.25}$$

If inequality (C.23) holds, then $P_g(\mathcal{E}^{\|(\beta)}) > P_g(\overline{\mathcal{E}}_{SA'}(\tau))$. Therefore, we have $P_g(\overline{\mathcal{E}}_{SA'}(\tau)) - P_g(\mathcal{E}^{\|(\beta)}) < 0$ and we conclude that the left-hand side of inequality (C.25) is negative. The right-hand side of the same inequality is instead non-negative for inequality (C.22). This contradiction shows that if for some value of the parameter λ the orthogonal measurement $\{|i\rangle\langle i|_A\}_i$ maximizes $P_g(\rho^{(\lambda)}_{AB}(\tau), \{P_{A,i}\}_i)$, then it is also the case for any greater value of λ. In conclusion, if one of the optimal measurement is $\{|i\rangle\langle i|_A\}_i$ for $\lambda = \alpha$, the same is true for any $\beta \in [\alpha, 1)$.

C.5 Study of the limit $\lambda \to 1$ in case (B)

First, we notice that the set of $\overline{\mathcal{P}}$-POVMs on A for $\rho^{(\lambda)}_{AB}(t)$ is a set that does not depend on λ and t. Indeed, we use the notation $\Pi^{\overline{\mathcal{P}}}_A = \Pi^{\overline{\mathcal{P}}}_A(\rho^{(\lambda)}_{AB}(\tau))$. Now we prove that the only optimal $\overline{\mathcal{P}}$-POVM for $C^{\overline{\mathcal{P}}}_A(\rho^{(1)}_{AB}(\tau))$ is the projective measurement $\{|i\rangle\langle i|_A\}_i$. In the case of an optimal $\{P_{A,i}\}_i \in \Pi^{\overline{\mathcal{P}}}_A$ for $\rho^{(1)}_{AB}(\tau)$ we obtain the output ensemble (see Eq. (C.18))

$$\mathcal{E}(\rho^{(1)}_{AB}(\tau), \{P_{A,i}\}_i) = \{\overline{p}_i, \sigma_{SA'}(\tau) \otimes \sum_j e_{ji}|j\rangle\langle j|_{A''}\}_i\,, \tag{C.26}$$

where $\sum_j e_{ji} = 1$ for any $i = 1, \ldots, \overline{n}$. Since $P_g(\rho^{(1)}_{AB}(\tau), \{|i\rangle\langle i|_A\}_i) = 1$, an optimal $\overline{\mathcal{P}}$-POVM different from $\{|i\rangle\langle i|_A\}_i$ must provide an output ensemble of orthogonal states $\mathcal{E}(\rho_{AB}^{(1)}(\tau), \{P_{A,i}\}_i)$. Given the identity $P_g(\mathcal{E}(\rho_{AB}^{(1)}(\tau), \{P_{A,i}\}_i) = P_g(\{\overline{p}_i, \sum_j e_{ji}|j\rangle\langle j|_{A''}\}_i)$, we have to check if, for some e_{ij}, the following ensemble $\{\overline{p}_i, \sum_j e_{ji}|j\rangle\langle j|_{A''}\}_i$ can be an orthogonal ensemble of states different from $\{\overline{p}_i\, |i\rangle\langle i|_{A''}\}_i$. Each state $\rho_{A'',i}^{\perp} = \sum_j e_{ji}|j\rangle\langle j|_{A''}$ is defined as a convex combination of the states $\{|i\rangle\langle i|_{A''}\}_i$. Two such states are orthogonal only if the respective convex combinations do not have any element $|i\rangle\langle i|_{A''}$ in common. Therefore, the only way to have $\bar{n}$ orthogonal output states is if for each i the state is of the form $\rho_{A'',i}^{\perp} = |j\rangle\langle j|_{A''}$ for some $j = j(i)$ exclusively assigned to i. Thus, each $P_{A,i}$ has only one nonzero diagonal element $(P_{A,i})_{jj} = \langle j|_A P_{A,i}|j\rangle_A$. Since $\sum_i P_{A,i} = \mathbb{1}_A$ this is only possible if $\{P_{A,i}\}_i = \{|i\rangle\langle i|_A\}_i$.

We proved that $\{|i\rangle\langle i|_A\}_i \in \Pi_A^{\overline{\mathcal{P}}}$ is the only optimal $\overline{\mathcal{P}}$-POVM for the evaluation of $C_A^{\overline{\mathcal{P}}}(\rho_{AB}^{(1)}(\tau))$. Therefore, for any $\overline{\mathcal{P}}$-POVM $\{P_{A,i}\}_i \neq \{|i\rangle\langle i|_A\}_i$ we have that $P_g(\rho_{AB}^{(1)}(\tau), \{P_{A,i}\}_i) < 1$. We notice that the set $\Pi_A^{\overline{\mathcal{P}}}$ is closed and bounded, namely it is compact. Indeed, it is a subset of $\mathcal{B}(\mathcal{H}_A)$ that is defined through linear constraints involving identities and relations of semi-positivity. The guessing probability $P_g(\rho_{AB}^{(1)}(\tau), \{P_{A,i}\}_i)$ is a continuous function on this compact set of $\overline{\mathcal{P}}$-POVMs.

We now show that $P_g(\rho_{AB}^{(\lambda)}(\tau), \{P_{A,i}\}_i)$ is Lipschitz continuous in λ. In other words we construct a bound on the change of the guessing probability for a given change in λ. To do so we first show that $P_g(\rho_{AB}, \{P_{A,i}\}_i)$ is Lipschitz continuous on the set of states. Consider $P_g(\rho_{AB}, \{P_{A,i}\}_i)$ as a function of ρ_{AB}. We consider a pair ρ_{AB}^1, ρ_{AB}^2 and observe that

$$\begin{aligned}\max_{\{P_{B,i}\}_i} \sum_i \mathrm{Tr}[P_{A,i} \otimes P_{B,i}\rho_{AB}^1] &= \max_{\{P_{B,i}\}_i} \sum_i \mathrm{Tr}[P_{A,i} \otimes P_{B,i}(\rho_{AB}^2 + (\rho_{AB}^1 - \rho_{AB}^2)] \\ &\leq \max_{\{P_{B,i}\}_i} \sum_i \mathrm{Tr}[P_{A,i} \otimes P_{B,i}\rho_{AB}^2] + \max_{\{P_{B,i}\}_i} \sum_i \mathrm{Tr}[P_{A,i} \otimes P_{B,i}(\rho_{AB}^1 - \rho_{AB}^2)].\end{aligned} \tag{C.27}$$

Let Δ be a diagonal matrix such that $\Delta = U(\rho_{AB}^1 - \rho_{AB}^2)U^\dagger$ for a unitary U. Let Δ_+ and Δ_- be the two diagonal positive semidefinite matrices such that $\Delta = \Delta_+ - \Delta_-$. Note that $U^\dagger\Delta_+U$ and $U^\dagger\Delta_-U$ are positive semidefinite. This implies

$$\begin{aligned}&\max_{\{P_{B,i}\}_i} \sum_i \mathrm{Tr}[P_{A,i} \otimes P_{B,i}(\rho_{AB}^1 - \rho_{AB}^2)] = \max_{\{P_{B,i}\}_i} \sum_i \mathrm{Tr}[P_{A,i} \otimes P_{B,i}U^\dagger(\Delta_+ - \Delta_-)U] \\ &\leq \max_{\{P_{B,i}\}_i} \sum_i \mathrm{Tr}[P_{A,i} \otimes P_{B,i}(U^\dagger\Delta_+U)] + \max_{\{P_{B,i}\}_i} \sum_i \mathrm{Tr}[P_{A,i} \otimes P_{B,i}(U^\dagger\Delta_-U)].\end{aligned} \tag{C.28}$$

Since POVM elements are positive semidefinite $\mathrm{Tr}[P_{A,i} \otimes P_{B,j}(U^\dagger \Delta_+ U)]$ is positive for each pair $P_{A,i}$, $P_{B,j}$. Therefore $\mathrm{Tr}[\sum_i P_{A,i} \otimes P_{B,i}(U^\dagger \Delta_+ U)] \le \mathrm{Tr}[\sum_i P_{A,i} \otimes \sum_j P_{B,j}(U^\dagger \Delta_+ U)] = \mathrm{Tr}[U^\dagger \Delta_+ U] = \mathrm{Tr}[\Delta_+]$. Likewise, we have $\sum_i \mathrm{Tr}[P_{A,i} \otimes P_{B,i}(U^\dagger \Delta_- U)] \le \mathrm{Tr}[\Delta_-]$. Thus,

$$\max_{\{P_{B,i}\}_i} \sum_i \mathrm{Tr}[P_{A,i} \otimes P_{B,i}(\rho^1_{AB} - \rho^2_{AB})] \le \mathrm{Tr}[\Delta_+ + \Delta_-] = \|\rho^1_{AB} - \rho^2_{AB}\|_1. \quad \text{(C.29)}$$

Considering Eqs. (C.27) and (C.29) we can now conclude that

$$P_g(\rho^1_{AB}, \{P_{A,i}\}_i) - P_g(\rho^2_{AB}, \{P_{A,i}\}_i) \le \|\rho^1_{AB} - \rho^2_{AB}\|_1. \quad \text{(C.30)}$$

By exchanging the 1 and 2 in the above derivation we obtain

$$P_g(\rho^2_{AB}, \{P_{A,i}\}_i) - P_g(\rho^1_{AB}, \{P_{A,i}\}_i) \le \|\rho^1_{AB} - \rho^2_{AB}\|_1. \quad \text{(C.31)}$$

Thus,

$$|P_g(\rho^1_{AB}, \{P_{A,i}\}_i) - P_g(\rho^2_{AB}, \{P_{A,i}\}_i)| \le \|\rho^1_{AB} - \rho^2_{AB}\|_1. \quad \text{(C.32)}$$

Note that this bound is independent of $\{P_{A,i}\}_i$. Thus we see that $P_g(\rho_{AB}, \{P_{A,i}\}_i)$ is Lipschitz continuous on the set of states. Next we consider the pair of states $\rho^{(\lambda_1)}_{AB}(\tau), \rho^{(\lambda_2)}_{AB}(\tau)$ and note that the trace norm $\|\rho^{(\lambda_1)}_{AB}(\tau) - \rho^{(\lambda_2)}_{AB}(\tau)\|_1 = 2|\lambda_1 - \lambda_2|$. Therefore,

$$|P_g(\rho^{(\lambda_1)}_{AB}(\tau), \{P_{A,i}\}_i) - P_g(\rho^{(\lambda_2)}_{AB}(\tau), \{P_{A,i}\}_i)| \le 2|\lambda_1 - \lambda_2|. \quad \text{(C.33)}$$

Thus we see that $P_g(\rho^{(\lambda)}_{AB}(\tau), \{P_{A,i}\}_i)$ is Lipschitz continuous in λ.

We next consider how the set of optimal $\overline{\mathcal{P}}$-POVMs converges to $\{|i\rangle\langle i|_A\}_i$ as $\lambda \to 1$ using the bound in Eq. (C.33). Consider a semi-open neighborhood O_1 of the projective $\overline{\mathcal{P}}$-POVM $\{|i\rangle\langle i|_A\}_i$ such that the set $S_1 \equiv \Pi^{\overline{\mathcal{P}}}_A - O_1$ of $\overline{\mathcal{P}}$-POVMs not in O_1 is closed. Since the set S_1 is closed and bounded and $P_g(\rho^{(1)}_{AB}(\tau), \{P_{A,i}\}_i)$ is a continuous function on $\Pi^{\overline{\mathcal{P}}}_A$ there exists a maximum value $m_1 < 1$ of $P_g(\rho^{(1)}_{AB}(\tau), \{P_{A,i}\}_i)$ on S_1, namely

$$m_1 \equiv \max_{\{P_{A,i}\}_i \in S_1} P_g(\rho^{(1)}_{AB}(\tau), \{P_{A,i}\}_i) < 1.$$

Then, due to Eq. (C.33), for $\epsilon > 0$ and $\lambda = 1 - \epsilon$ we get $P_g(\rho^{(1-\epsilon)}_{AB}(\tau), \{P_{A,i}\}_i) \le m_1 + 2\epsilon$ on S_1 and the maximum value of $P_g(\rho^{(1-\epsilon)}_{AB}(\tau), \{P_{A,i}\}_i)$ on O_1 is larger or equal to $1 - 2\epsilon$. There exists a sufficiently small $\epsilon_1 > 0$ such that $1 - 2\epsilon_1 = m_1 + 2\epsilon_1$. For all $\epsilon < \epsilon_1$ the set of optimal $\overline{\mathcal{P}}$-POVMs belongs to O_1.

We next consider a sequence of semi-open sets O_i which all contain $\{|i\rangle\langle i|_A\}_i$ and are such that $O_{i+1} \subset O_i$. There is a corresponding sequence of closed sets

$S_i \equiv \Pi_A^{\overline{\mathcal{P}}} - O_i$ and non-decreasing sequence of maximal values $m_i < 1$ of $P_g(\rho_{AB}^{(1)}(\tau), \{P_{A,i}\}_i)$ on S_i. For each m_i there is an ϵ_i such that for all $\epsilon < \epsilon_i$ the optimal $\overline{\mathcal{P}}$-POVMs, namely the $\overline{\mathcal{P}}$-POVMs that maximize $P_g(\rho_{AB}^{(1-\epsilon)}(\tau), \{P_{A,i}\}_i)$, belong to O_i. The sequence of ϵ_i is non-increasing since the sequence of m_i is non-decreasing.

Let us consider a distance measure $d(\cdot,\cdot)$ on $\mathcal{B}(\mathcal{H}_A)$ and define a sequence $O(\delta_i)$ of semi-open sets as the $\overline{\mathcal{P}}$-POVMs $\{P_{A,i}\}_i$ such that we get $d(P_{A,i}, |i\rangle\langle i|_A) < \delta_i$ for any $i = 1, ..., \overline{n}$, for a strictly decreasing sequence $\delta_{i+1} < \delta_i$ where $\delta_i \to 0$ as $i \to \infty$.

Then from the above argument we can conclude that, for any $\delta > 0$ there exists a value $\lambda_\delta \in (0, 1)$ such that, if $\lambda \in (\lambda_\delta, 1)$, any optimal $\overline{\mathcal{P}}$-POVM $\{P_{A,i}^{(\lambda)}\}_i$ for this λ is such that $d(P_{A,i}^{(\lambda)}, |i\rangle\langle i|_A) < \delta$ for any $i = 1, ..., \overline{n}$.

Next we show that $P_g(\rho_{AB}, \{P_{A,i}\}_i)$ is Lipschitz continuous as a function of $\{P_{A,i}\}_i$. In other words, we construct a bound on the change of the guessing probability proportional to a distance measure quantifying the change of the POVM $\{P_{A,i}\}_i$, valid for any $\rho_{AB} \in S(\mathcal{H}_A \otimes \mathcal{H}_B)$. We select a pair $\{P_{A,i}^1\}_i$, $\{P_{A,i}^2\}_i$ and observe that

$$
\begin{aligned}
&\max_{\{P_{B,i}\}_i} \sum_i \mathrm{Tr}[P_{A,i}^1 \otimes P_{B,i}\rho_{AB}] \\
&= \max_{\{P_{B,i}\}_i} \sum_i \mathrm{Tr}[P_{A,i}^2 \otimes P_{B,i}\rho_{AB} + (P_{A,i}^1 - P_{A,i}^2) \otimes P_{B,i}\rho_{AB}] \\
&\leq \max_{\{P_{B,i}\}_i} \sum_i \mathrm{Tr}[P_{A,i}^2 \otimes P_{B,i}\rho_{AB}] + \max_{\{P_{B,i}\}_i} \sum_i \mathrm{Tr}[(P_{A,i}^1 - P_{A,i}^2) \otimes P_{B,i}\rho_{AB}]. \qquad (C.34)
\end{aligned}
$$

Let Δ_i be a diagonal matrix such that $\Delta_i = U_i(P_{A,i}^1 - P_{A,i}^2)U_i^\dagger$ for a unitary U_i. Let Δ_{i+} and Δ_{i-} be the two diagonal positive semidefinite matrices such that $\Delta_i = \Delta_{i+} - \Delta_{i-}$. Note that $U_i^\dagger \Delta_{i+} U_i$ and $U_i^\dagger \Delta_{i-} U_i$ are positive semidefinite. This implies

$$
\begin{aligned}
&\max_{\{P_{B,i}\}_i} \sum_i \mathrm{Tr}[(P_{A,i}^1 - P_{A,i}^2) \otimes P_{B,i}\rho_{AB}] = \max_{\{P_{B,i}\}_i} \sum_i \mathrm{Tr}[U_i^\dagger(\Delta_{i+} - \Delta_{i-})U_i \otimes P_{B,i}\rho_{AB}] \\
&\leq \max_{\{P_{B,i}\}_i} \sum_i \mathrm{Tr}[U_i^\dagger(\Delta_{i+})U_i \otimes P_{B,i}\rho_{AB}] + \max_{\{P_{B,i}\}_i} \sum_i \mathrm{Tr}[U_i^\dagger(\Delta_{i-})U_i \otimes P_{B,i}\rho_{AB}]. \quad (C.35)
\end{aligned}
$$

Since POVM elements are positive semidefinite $\mathrm{Tr}[U_i^\dagger(\Delta_{i+})U_i \otimes P_{B,i}\rho_{AB}]$ is positive for each $P_{B,j}$. Therefore, $\mathrm{Tr}[U_i^\dagger(\Delta_{i+})U_i \otimes P_{B,i}\rho_{AB}] \leq \mathrm{Tr}[U_i^\dagger(\Delta_{i+})U_i \otimes$

$\sum_j P_{B,j}\rho_{AB}] = \mathrm{Tr}[U_i^\dagger(\Delta_{i+})U_i \otimes \mathbb{1}_B\rho_{AB}]$. Likewise, we can write $\mathrm{Tr}[U_i^\dagger(\Delta_{i-})U_i \otimes P_{B,i}\rho_{AB}] \leq \mathrm{Tr}[U_i^\dagger(\Delta_{i-})U_i \otimes \mathbb{1}_B\rho_{AB}]$. Using this we find that

$$\begin{aligned}\max_{\{P_{B,i}\}_i} &\sum_i \mathrm{Tr}[(P^1_{A,i} - P^2_{A,i}) \otimes P_{B,i}\rho_{AB}] \leq \\ &\leq \sum_i \mathrm{Tr}[U_i^\dagger(\Delta_{i+} + \Delta_{i-})U_i \otimes \mathbb{1}_B\rho_{AB}] \leq \sum_i \mathrm{Tr}[U_i^\dagger(\Delta_{i+} + \Delta_{i-})U_i \otimes \mathbb{1}_B] \\ &= (\overline{n}+1)d_S^2 \sum_i \mathrm{Tr}[\Delta_{i+} + \Delta_{i-}] = (\overline{n}+1)d_S^2 \sum_i \|P^1_{A,i} - P^2_{A,i}\|_1 . \end{aligned} \tag{C.36}$$

where we used that $\mathrm{Tr}[\mathbb{1}_B] = (\overline{n}+1)d_S^2$ and for the second inequality we have used von Neumann's trace inequality and that the largest eigenvalue of ρ_{AB} is smaller or equal to 1. By combining Eq. (C.34) and Eq. (C.36) we can now conclude that

$$P_g(\rho_{AB}, \{P^1_{A,i}\}_i) - P_g(\rho_{AB}, \{P^2_{A,i}\}_i) \leq (\overline{n}+1)d_S^2 \sum_i \|P^1_{A,i} - P^2_{A,i}\|_1 . \tag{C.37}$$

By exchanging the $\{P^1_{A,i}\}_i$ and $\{P^2_{A,i}\}_i$ in the above derivation we obtain

$$P_g(\rho_{AB}, \{P^2_{A,i}\}_i) - P_g(\rho_{AB}, \{P^1_{A,i}\}_i) \leq (\overline{n}+1)d_S^2 \sum_i \|P^1_{A,i} - P^2_{A,i}\|_1 . \tag{C.38}$$

Therefore

$$|P_g(\rho_{AB}, \{P^1_{A,i}\}_i) - P_g(\rho_{AB}, \{P^2_{A,i}\}_i)| \leq (\overline{n}+1)d_S^2 \sum_i \|P^1_{A,i} - P^2_{A,i}\|_1 . \tag{C.39}$$

Thus we have shown that $P_g(\rho_{AB}, \{P_{A,i}\}_i)$ is Lipschitz continuous as a function of $\{P_{A,i}\}_i$ for any $\rho_{AB} \in S(\mathcal{H}_A \otimes \mathcal{H}_B)$.

We now study the guessing probability of the ensemble that we obtain applying $\{P_{A,i}\}_i \in \Pi_A^{\overline{\mathcal{P}}}$ on $\rho^{(\lambda)}_{AB}(t)$ given by

$$P_g\left(\rho^{(\lambda)}_{AB}(t), \{P_{A,i}\}_i\right) = \lambda P_g\left(\{\overline{p}_i, \rho^{\perp}_{A'',i}\}_i\right) + (1-\lambda)P_g\left(\{\overline{p}_i, \rho^{\|}_{SA',i}(t)\}_i\right). \tag{C.40}$$

We consider Eq. (C.40) when an optimal $\{P^{(\lambda)}_{A,i}\}_i$ is chosen. We define the corresponding ensembles that appear in this expression $\mathcal{E}^{\perp}(\{P^{(\lambda)}_{A,i}\}_i) \equiv \{\overline{p}_i, \rho^{\perp}_{A'',i}\}_i$ and $\mathcal{E}^{\|}(\{P^{(\lambda)}_{A,i}\}_i) \equiv \{\overline{p}_i, \rho^{\|}_{SA',i}(t)\}_i$, so that

$$P_g\left(\rho^{(\lambda)}_{AB}(\tau), \{P^{(\lambda)}_{A,i}\}_i\right) = \lambda P_g(\mathcal{E}^{\perp}(\{P^{(\lambda)}_{A,i}\}_i)) + (1-\lambda)P_g(\mathcal{E}^{\|}(\{P^{(\lambda)}_{A,i}\}_i)) . \tag{C.41}$$

The ensembles $\mathcal{E}^{\perp}(\{P_{A,i}^{(\lambda)}\}_i)$ and $\mathcal{E}^{\parallel}(\{P_{A,i}^{(\lambda)}\}_i)$ are functions on the set of optimal $\overline{\mathcal{P}}$-POVMs $\{P_{A,i}^{(\lambda)}\}_i$ for a given λ. Thus the image of the function $P_g(\mathcal{E}^{\perp}(\{P_{A,i}^{(\lambda)}\}_i))$ over the set of optimal $\overline{\mathcal{P}}$-POVMs $\{P_{A,i}^{(\lambda)}\}_i$ for a given λ, denoted $\mathrm{Im}(P_g^{(\lambda)}(\mathcal{E}^{\perp})) \equiv \{P_g(\mathcal{E}^{\perp}(\{P_{A,i}^{(\lambda)}\}_i)) : \{P_{A,i}^{(\lambda)}\}_i$ is optimal$\}$, is a subset of the interval $[0,1]$, namely $\mathrm{Im}(P_g^{(\lambda)}(\mathcal{E}^{\perp})) \subset [0,1]$. Likewise, the function $P_g(\mathcal{E}^{\parallel}(\{P_{A,i}^{(\lambda)}\}_i))$ takes values in a set $\mathrm{Im}(P_g^{(\lambda)}(\mathcal{E}^{\parallel})) \subset [0,1]$ for a given λ.

Using Eq. (C.39) we can construct bounds on $\mathrm{Im}(P_g^{(\lambda)}(\mathcal{E}^{\perp}))$ and $\mathrm{Im}(P_g^{(\lambda)}(\mathcal{E}^{\parallel}))$ for a given λ. First, based on the above argument we make the following observation: for any $\eta > 0$ there exists a value $\lambda_\eta \in (0,1)$ such that, if $\lambda \in (\lambda_\eta, 1)$, any optimal $\overline{\mathcal{P}}$-POVM $\{P_{A,i}^{(\lambda)}\}_i$ for this λ is such that $\|P_{A,i}^{(\lambda)} - |i\rangle\langle i|_A\|_1 < \eta$ for any $i = 1, ..., \overline{n}$. Thus, by Eq. (C.39) the values in the image of $P_g(\mathcal{E}^{\perp}(\{P_{A,i}^{(\lambda)}\}_i))$ for $\lambda \in (\lambda_\eta, 1)$ differ from $P_g(\mathcal{E}^{\perp}(\{|i\rangle\langle i|_A\}_i)) = 1$ by less than $\overline{n}(\overline{n}+1)d_S^2\eta$, namely $|P_g(\mathcal{E}^{\perp}(\{P_{A,i}^{(\lambda)}\}_i)) - 1| < \overline{n}(\overline{n}+1)d_S^2\eta$ for all optimal $\{P_{A,i}^{(\lambda)}\}_i : \lambda \in (\lambda_\eta, 1)$. Likewise, the values in the range of $P_g(\mathcal{E}^{\parallel}(\{P_{A,i}^{(\lambda)}\}_i))$ for $\lambda \in (\lambda_\eta, 1)$ differ from $P_g(\mathcal{E}^{\parallel}(\{|i\rangle\langle i|_A\}_i)) = P_g(\overline{\mathcal{E}}_{SA'}(\tau))$ by less than the quantity $\overline{n}(\overline{n}+1)d_S^2\eta$, namely $|P_g(\mathcal{E}^{\parallel}(\{P_{A,i}^{(\lambda)}\}_i)) - P_g(\overline{\mathcal{E}}_{SA'}(\tau))| < \overline{n}(\overline{n}+1)d_S^2\eta$ for all optimal $\{P_{A,i}^{(\lambda)}\}_i : \lambda \in (\lambda_\eta, 1)$. Using this we can state the following

$$\forall\delta>0,\ \exists\lambda_\delta>0:\ P_g(\mathcal{E}^{\parallel}(\{P_{A,i}^{(\lambda)}\}_i)) - P_g(\overline{\mathcal{E}}_{SA'}(\tau))<\delta, \forall\{P_{A,i}^{(\lambda)}\}_i : \lambda \in (\lambda_\delta, 1)\,. \tag{C.42}$$

C.6 Lipschitz continuity of $C_A^{\mathcal{P}}$ on the set of states

Consider a POVM $\{P_{A,i}\}_i$ and two states ρ_{AB}, $\tilde{\rho}_{AB}$. Let $p_i = \mathrm{Tr}[P_{A,i}\rho_{AB}]$ and $\tilde{p}_i = \mathrm{Tr}[P_{A,i}\tilde{\rho}_{AB}]$. Let Δ be a diagonal matrix such that $\Delta = U(\tilde{\rho}_{AB} - \rho_{AB})U^\dagger$ for a unitary U. Let Δ_+ and Δ_- be the two diagonal positive semidefinite matrices such that $\Delta = \Delta_+ - \Delta_-$. Note that $U^\dagger\Delta_+U$ and $U^\dagger\Delta_-U$ are positive semidefinite. Then

$$\begin{aligned}\tilde{p}_i - p_i = \mathrm{Tr}[P_{A,i}(\tilde{\rho}_{AB} - \rho_{AB})] &= \mathrm{Tr}[P_{A,i}(U^\dagger\Delta_+U - U^\dagger\Delta_-U)] \\ &\leq \mathrm{Tr}[P_{A,i}U^\dagger\Delta_+U] + \mathrm{Tr}[P_{A,i}U^\dagger\Delta_-U].\end{aligned} \tag{C.43}$$

Since POVM elements are positive semidefinite $\mathrm{Tr}[P_{A,j}U^\dagger\Delta_+U]$ is positive for each $P_{A,j}$. Therefore $\mathrm{Tr}[P_{A,i}U^\dagger\Delta_+U] \leq \mathrm{Tr}[\sum_j P_{A,j}U^\dagger\Delta_+U] = \mathrm{Tr}[U^\dagger\Delta_+U] = \mathrm{Tr}[\Delta_+]$. Likewise $\mathrm{Tr}[P_{A,i}(U^\dagger\Delta_-U)] \leq \mathrm{Tr}[\Delta_-]$. Thus,

$$\mathrm{Tr}[P_{A,i}U^\dagger\Delta_+U]+\mathrm{Tr}[P_{A,i}U^\dagger\Delta_-U]\leq\mathrm{Tr}[\Delta_+ + \Delta_-]=\|\tilde{\rho}_{AB} - \rho_{AB}\|_1. \tag{C.44}$$

It follows that

$$\tilde{p}_i - p_i \leq \|\tilde{\rho}_{AB} - \rho_{AB}\|_1. \tag{C.45}$$

By exchanging p_i and $\tilde{p}_i$ in the above derivation we obtain

$$p_i - \tilde{p}_i \leq \|\tilde{\rho}_{AB} - \rho_{AB}\|_1. \tag{C.46}$$

From this we can conclude that

$$|\tilde{p}_i - p_i| \leq \|\tilde{\rho}_{AB} - \rho_{AB}\|_1. \tag{C.47}$$

Assume now that $\{P_{A,i}\}_i$ is a $\mathcal{P}$-POVM for ρ_{AB} but not necessarily for $\tilde{\rho}_{AB}$. We can create a $\mathcal{P}$-POVM for $\tilde{\rho}_{AB}$ from $\{P_{A,i}\}_i$ in the following way. If $\tilde{p}_i - p_i > 0$ we subtract $(1 - p_i/\tilde{p}_i)P_{A,i}$ from $P_{A,i}$ to create a new element $\tilde{P}_{A,i} \equiv p_i/\tilde{p}_i P_{A,i}$. Let $P_r \equiv \sum_{i\in\{i+\}}(1 - p_i/\tilde{p}_i)P_{A,i}$ where the $\{i+\}$ is the set of all i such that $\tilde{p}_i - p_i > 0$ and let $p_r \equiv \mathrm{Tr}[P_r\tilde{\rho}_{AB}] = \sum_{i\in\{i+\}} \tilde{p}_i - p_i$. If $\tilde{p}_i - p_i < 0$ we add $(p_i - \tilde{p}_i)/(p_r)P_r$ to $P_{A,i}$ to create a new element $\tilde{P}_{A,i} \equiv P_{A,i} + (p_i - \tilde{p}_i)/(p_r)P_r$.

Next consider the trace distance between $\{\tilde{P}_{A,i}\}_i$ and $\{P_{A,i}\}_i$.

$$\begin{aligned}\sum_i \|\tilde{P}_{A,i} - P_{A,i}\|_1 &= \sum_{i\in\{i+\}} \left|\frac{\tilde{p}_i - p_i}{\tilde{p}_i}\right| \|P_{A,i}\|_1 + \sum_{i\notin\{i+\}} \left|\frac{p_i - \tilde{p}_i}{p_r}\right| \|P_r\|_1 \\ &= \sum_{i\in\{i+\}} \left|\frac{\tilde{p}_i - p_i}{\tilde{p}_i}\right| \|P_{A,i}\|_1 + \|P_r\|_1,\end{aligned} \tag{C.48}$$

where we used that $\sum_{i\notin\{i+\}} p_i - \tilde{p}_i = p_r$. Since each $P_{A,i}$ is positive semidefinite with all eigenvalues less or equal to 1 it follows that $\|P_{A,i}\|_1 \leq n_A$ where $n_A \equiv \dim(\mathcal{H}_A)$. Moreover, $\|P_r\|_1 = \|\sum_{i\in\{i+\}}(1 - p_i/\tilde{p}_i)P_{A,i}\|_1 \leq \sum_{i\in\{i+\}} |1 - p_i/\tilde{p}_i| \|P_{A,i}\|_1$. Therefore,

$$\sum_i \|\tilde{P}_{A,i} - P_{A,i}\|_1 \leq 2 \sum_{i\in\{i+\}} \left|\frac{\tilde{p}_i - p_i}{\tilde{p}_i}\right| \|P_{A,i}\|_1 \leq 2n_A \sum_{i\in\{i+\}} \left|\frac{\tilde{p}_i - p_i}{\tilde{p}_i}\right|. \tag{C.49}$$

We further note that $\tilde{p}_i > p_i$ for $i \in \{i+\}$ and thus if $p_{min} \equiv \min_i p_i$ we have that $\tilde{p}_i > p_{min}$ for $i \in \{i+\}$. It follows that $|(\tilde{p}_i - p_i)/\tilde{p}_i| < |(\tilde{p}_i - p_i)/p_{min}|$ for $i \in \{i+\}$. Hence,

$$\begin{aligned}\sum_i \|\tilde{P}_{A,i} - P_{A,i}\|_1 &< \frac{2n_A}{p_{min}} \sum_{i\in\{i+\}} |\tilde{p}_i - p_i| \leq \frac{2n_A}{p_{min}} \sum_{i\in\{i+\}} \|\tilde{\rho}_{AB} - \rho_{AB}\|_1 \\ &< \frac{2n_A|\mathcal{P}|}{p_{min}} \|\tilde{\rho}_{AB} - \rho_{AB}\|_1,\end{aligned} \tag{C.50}$$

where $|\mathcal{P}|$ is the number of elements of $\mathcal{P}$ and we have used Eq. (C.47). Thus if $\{P_{A,i}\}_i$ is a $\mathcal{P}$-POVM for ρ_{AB} the minimum trace distance between $\{P_{A,i}\}_i$ and a $\mathcal{P}$-POVM for $\tilde{\rho}_{AB}$ is upper bounded by $2n_A|\mathcal{P}|\|\tilde{\rho}_{AB} - \rho_{AB}\|_1/p_{min}$. By an analogous argument if $\{\tilde{P}_{A,i}\}_i$ is a $\mathcal{P}$-POVM for $\tilde{\rho}_{AB}$ the minimum trace distance between $\{\tilde{P}_{A,i}\}_i$ and a $\mathcal{P}$-POVM for ρ_{AB} is upper bounded by $2n_A|\mathcal{P}|\|\tilde{\rho}_{AB} - \rho_{AB}\|_1/p_{min}$

We now recall Eq. (C.32) and Eq. (C.39) from C.5 showing that the guessing probability $P_g(\rho_{AB}, \{P_{A,i}\}_i)$ is Lipschitz continuous on the set of states for a fixed $\{P_{A,i}\}_i$

$$|P_g(\tilde{\rho}_{AB}, \{P_{A,i}\}_i) - P_g(\rho_{AB}\{P_{A,i}\}_i)| \leq \|\tilde{\rho}_{AB} - \rho_{AB}\|_1, \tag{C.51}$$

and Lipschitz continuous on the set of POVMs for a fixed ρ_{AB}

$$|P_g(\rho_{AB}, \{\tilde{P}_{A,i}\}_i) - P_g(\rho_{AB}, \{P_{A,i}\}_i)| \leq n_B \sum_i \|\tilde{P}_{A,i} - P_{A,i}\|_1, \tag{C.52}$$

where $n_B \equiv \dim(\mathcal{H}_B)$.

We are now ready to show Lipschitz continuity of $C_A^{\mathcal{P}}$ on the set of states. When ρ_{AB} changes to $\tilde{\rho}_{AB}$ the minimum trace distance between any $\mathcal{P}$-POVM for $\tilde{\rho}_{AB}$ and a $\mathcal{P}$-POVM for ρ_{AB} is upper bounded by $2n_A|\mathcal{P}|\|\tilde{\rho}_{AB} - \rho_{AB}\|_1/p_{min}$. From this and Eq. (C.52) follows that the difference between the maximum of $P_g(\rho_{AB}, \{P_{A,i}\}_i)$ evaluated on the set $\Pi_A^{\mathcal{P}}(\tilde{\rho}_{AB})$ of $\mathcal{P}$-POVMs for $\tilde{\rho}_{AB}$ and the maximum of $P_g(\rho_{AB}, \{P_{A,i}\}_i)$ evaluated on the set $\Pi_A^{\mathcal{P}}(\rho_{AB})$ of $\mathcal{P}$-POVMs for ρ_{AB} is upper bounded by $2n_A n_B|\mathcal{P}|\|\tilde{\rho}_{AB} - \rho_{AB}\|_1/p_{min}$. Moreover, by Eq. (C.51) the difference between $P_g(\rho_{AB}, \{P_{A,i}\}_i)$ and $P_g(\tilde{\rho}_{AB}, \{P_{A,i}\}_i)$ for any given $\{P_{A,i}\}_i$ in the union $\Pi_A^{\mathcal{P}}(\rho_{AB}) \cup \Pi_A^{\mathcal{P}}(\tilde{\rho}_{AB})$ of the set of $\mathcal{P}$-POVMs for $\tilde{\rho}_{AB}$ and the set of $\mathcal{P}$-POVMs for ρ_{AB} is upper bounded by $\|\tilde{\rho}_{AB} - \rho_{AB}\|_1$. In conclusion the change of $C_A^{\mathcal{P}}$ when ρ_{AB} changes to $\tilde{\rho}_{AB}$ is upper bounded by $(1 + 2n_A n_B|\mathcal{P}|/p_{min})\|\tilde{\rho}_{AB} - \rho_{AB}\|_1$, namely

$$|C_A^{\mathcal{P}}(\tilde{\rho}_{AB}) - C_A^{\mathcal{P}}(\rho_{AB})| < \left(1 + \frac{2n_A n_B|\mathcal{P}|}{p_{min}}\right)\|\tilde{\rho}_{AB} - \rho_{AB}\|_1, \tag{C.53}$$

Thus $C_A^{\mathcal{P}}$ is Lipschitz continuous on the set of states.

Using Eq. (C.53) we can make some observations about the robustness of correlation backflows. If we have a backflow in the interval $[\tau, \tau+\Delta\tau]$ for an initial state $\rho_{AB}(0)$, namely $C_A^{\mathcal{P}}(\rho_{AB}(\tau+\Delta\tau)) - C_A^{\mathcal{P}}(\rho_{AB}(\tau)) > 0$, any state ρ'_{AB} such that the quantity $\|\rho'_{AB} - \rho_{AB}(\tau+\Delta\tau)\|_1 < p_{min}/(p_{min} + 2n_A n_B|\mathcal{P}|)|C_A^{\mathcal{P}}(\rho_{AB}(\tau + \Delta\tau)) - C_A^{\mathcal{P}}(\rho_{AB}(\tau))|$ satisfies $C_A^{\mathcal{P}}(\rho'_{AB}) - C_A^{\mathcal{P}}(\rho_{AB}(\tau)) > 0$. Likewise, if we have $C_A^{\mathcal{P}}(\rho_{AB}(\tau+\Delta\tau)) - C_A^{\mathcal{P}}(\rho_{AB}(\tau)) > 0$ any state ρ''_{AB} such that the quantity $\|\rho''_{AB} - \rho_{AB}(\tau)\|_1 < p_{min}/(p_{min} + 2n_A n_B|\mathcal{P}|)|C_A^{\mathcal{P}}(\rho_{AB}(\tau + \Delta\tau)) - C_A^{\mathcal{P}}(\rho_{AB}(\tau))|$ satisfies

$C_A^{\mathcal{P}}(\rho_{AB}(\tau+\Delta\tau)) - C_A^{\mathcal{P}}(\rho''_{AB}) > 0$. Moreover, if $C_A^{\mathcal{P}}(\rho_{AB}(\tau+\Delta\tau)) - C_A^{\mathcal{P}}(\rho_{AB}(\tau)) > 0$ any pair of states ρ'_{AB} and ρ''_{AB} such that $\|\rho'_{AB} - \rho_{AB}(\tau+\Delta\tau)\|_1 + \|\rho''_{AB} - \rho_{AB}(\tau)\|_1 < p_{min}/(p_{min} + 2n_A n_B|\mathcal{P}|)|C_A^{\mathcal{P}}(\rho_{AB}(\tau+\Delta\tau)) - C_A^{\mathcal{P}}(\rho_{AB}(\tau))|$ satisfies $C_A^{\mathcal{P}}(\rho'_{AB}) - C_A^{\mathcal{P}}(\rho''_{AB}) > 0$.

Thus a backflow can be seen also for evolution of a perturbed initial state $\rho_{AB}(0) + \chi$ where χ is traceless Hermitian if $\|\Lambda(\tau+\Delta\tau, 0) \otimes \mathbb{1}_B(\chi)\|_1 + \|\Lambda(\tau, 0) \otimes \mathbb{1}_B(\chi)\|_1 < p_{min}/(p_{min} + 2n_A n_B|\mathcal{P}|)|C_A^{\mathcal{P}}(\rho_{AB}(\tau+\Delta\tau)) - C_A^{\mathcal{P}}(\rho_{AB}(\tau))|$. Since $\Lambda(t, 0)$ is CPTP for every t it holds that $\|\Lambda(\tau+\Delta\tau, 0) \otimes \mathbb{1}_B(\chi)\| \leq \|\chi\|$ and $\|\Lambda(\tau, 0) \otimes \mathbb{1}_B(\chi)\| \leq \|\chi\|$. Thus there is a neighborhood of $\rho_{AB}(0)$ such that all states in this neighborhood show a backflow in the interval $[\tau, \tau+\Delta\tau]$ and it includes all states $\rho_{AB}(0) + \chi$ such that $2\|\chi\|_1 < p_{min}/(p_{min} + 2n_A n_B|\mathcal{P}|)|C_A^{\mathcal{P}}(\rho_{AB}(\tau+\Delta\tau)) - C_A^{\mathcal{P}}(\rho_{AB}(\tau))|$. Hence, this neighborhood has the same dimension as $S(\mathcal{H}_A \otimes \mathcal{H}_B)$.

C.7 The non-differentiable case

Here we discuss the non-Markovianity measure

$$N_{\mathcal{P}}(\mathbf{\Lambda}) \equiv \sup_{\rho_{ASA'}(0)} \int_{\frac{d}{dt}C_A^{\mathcal{P}}(\rho_{ASA'}(t))>0} \frac{d}{dt} C_A^{\mathcal{P}}(\rho_{ASA'}(t)) dt \,, \tag{C.54}$$

and how it can be extended to work for almost-everywhere differentiable functions $C_A^{\mathcal{P}}(\rho_{ASA'}(t))$. We also comment on how one may construct measures of non-Markovianity based on $C_A^{\mathcal{P}}(\rho_{ASA'}(t))$ using finite differences.

First we consider the case where $C_A^{\mathcal{P}}(\rho_{ASA'}(t))$ is differentiable. Consider the non-Markovianity measure introduced in Eq. (C.54) and let $[t_1, t_2]$ be a closed time interval for which it holds that $\frac{d}{dt}C_A^{\mathcal{P}}(\rho_{ASA'}(t)) > 0$. In Eq. (C.54) the type of integration used is not specified, but if the Henstock-Kurzweil integral is used it holds that

$$\int_{t_1}^{t_2} \frac{d}{dt} C_A^{\mathcal{P}}(\rho_{ASA'}(t)) dt = C_A^{\mathcal{P}}(\rho_{ASA'}(t_2)) - C_A^{\mathcal{P}}(\rho_{ASA'}(t_1)) \,, \tag{C.55}$$

if $C_A^{\mathcal{P}}(\rho_{ASA'}(t))$ is differentiable in $[t_1, t_2]$. If the Riemann or Lebesgue integral is used there would be the additional request that $\frac{d}{dt}C_A^{\mathcal{P}}(\rho_{ASA'}(t))$ is Riemann or Lebesgue integrable, respectively.

Next we consider the case where $C_A^{\mathcal{P}}(\rho_{ASA'}(t))$ is almost everywhere differentiable, namely $C_A^{\mathcal{P}}(\rho_{ASA'}(t))$ is non-differentiable for at most a countable set of times t_i. At the times where $C_A^{\mathcal{P}}(\rho_{ASA'}(t))$ fails to be differentiable, it is either non-differentiable but continuous or has a discontinuity. Since $C_A^{\mathcal{P}}(\rho_{ASA'}(t))$ is a continuous function on the set of states it has a discontinuity only if the evolution of $\rho_{ASA'}(t)$ is discontinuous. To deal with these non-differentiability points, we can define a function $\frac{d}{dt}C_A^{\mathcal{P}}(\rho_{ASA'}(t))^*$ that is equal to $\frac{d}{dt}C_A^{\mathcal{P}}(\rho_{ASA'}(t))$ for all t for which $C_A^{\mathcal{P}}(\rho_{ASA'}(t))$ is differentiable, and is equal to zero otherwise. If we use the Henstock-Kurzweil integral in the definition of the measure $N_{\mathcal{P}}(\mathbf{\Lambda})$ it is insensitive to how we define $\frac{d}{dt}C_A^{\mathcal{P}}(\rho_{ASA'}(t))^*$ in the countable set of t_i where $C_A^{\mathcal{P}}(\rho_{ASA'}(t))$ is not differentiable. Thus, we can define the measure

$$N_{\mathcal{P}}(\mathbf{\Lambda}) \equiv \sup_{\rho_{ASA'}(0)} \int_{\frac{d}{dt}C_A^{\mathcal{P}}(\rho_{ASA'}(t))^*>0} \frac{d}{dt}C_A^{\mathcal{P}}(\rho_{ASA'}(t))^* dt + \sum_{t_i} \Delta_+(t_i)\,, \tag{C.56}$$

where $\Delta_+(t_i)$ is the value of a discontinuous increase of $C_A^{\mathcal{P}}(\rho_{ASA'}(t))$ at a time t_i. This definition reduces to that of Eq. (C.54) when $C_A^{\mathcal{P}}(\rho_{ASA'}(t))$ is differentiable.

For the case when $C_A^{\mathcal{P}}(\rho_{ASA'}(t))$ is not almost everywhere differentiable the measure in Eq. (C.56) is not well defined. In this case one can resort to finite difference methods to estimate the amount of non-Markovianity in a given interval. A simple measure of this kind is

$$N_{\mathcal{P},finite}(\mathbf{\Lambda}) \equiv \sup_{\rho_{ASA'}(0),t_i<t_f} \{0, C_A^{\mathcal{P}}(\rho_{ASA'}(t_f)) - C_A^{\mathcal{P}}(\rho_{ASA'}(t_i))\}, \tag{C.57}$$

where t_i and t_f and belong to the interval of interest. We know that if the evolution is non-Markovian there always exists at least one $\mathcal{P}$, some ancillas A and A', an initial state $\rho_{ASA'}(0)$ and a pair of times t_i and t_f such that $C_A^{\mathcal{P}}(\rho_{ASA'}(t_f)) - C_A^{\mathcal{P}}(\rho_{ASA'}(t_i)) > 0$ (see Theorem 14). Therefore, $N_{\mathcal{P},finite}(\mathbf{\Lambda}) > 0$ if and only if the evolution $\mathbf{\Lambda}$ is non-Markovian.

C.8 Problems to witness any non-Markovian dynamics

In Section 2.5.5 we showed that for any non CP-divisible intermediate map $V_{t,s}$ an ensemble of states $\overline{\mathcal{E}}_{SA} = \{p_i, \rho_{SA,i}\}$ exists such that the guessing probability $P_g(\overline{\mathcal{E}}_{SA})$ increases in the time interval $[s,t]$. A natural question is therefore if a correlation measure of the type defined in Eq. (5.5), namely $C_A^{\mathcal{P}}$, can show revivals when $P_g(\overline{\mathcal{E}}_{SA})$ increases.

A first attempt in this direction could go as follows. The idea would be to mimic the previous results for $C_A^{(2)}$ by using as initial probe a state consisting of the ensemble states $\overline{\mathcal{E}}_{SA} = \{p_i, \rho_{SA,i}\}_i$ for systems S and A' correlated with orthogonal states on A according to the probabilities in the ensemble. Unfortunately, this approach does not work. In fact, we show that for a classical-quantum state ρ_{AB} of the type

$$\rho_{AB} = \sum_i p_i |i\rangle\langle i|_A \otimes \rho_{SA',i}. \tag{C.58}$$

the ensemble that maximizes $P_g(\rho_{AB}, \{P_{A,i}\}_i)$ is not defined by the projection $\{|i\rangle\langle i|_A\}_i$ in system A. We show this by constructing a counterexample. Consider the state

$$\rho_{AB} = p_1|1\rangle\langle 1| \otimes \rho_1 + p_2|2\rangle\langle 2| \otimes \rho_2 + p_3|3\rangle\langle 3| \otimes \rho_3, \tag{C.59}$$

where $2p_3 > p_1 > 2p_2$ and

$$\rho_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \; \rho_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; \rho_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{C.60}$$

For this example we can directly find P_g. To do so, we use that P_g is the solution to a convex optimization problem where strong duality holds, namely P_g is the solution to the dual optimization problem [YKL75, EMV03]. The dual formulation of P_g is

$$P_g = \min_K \operatorname{Tr} K \;\; \text{s.t.} \;\; K \geq p_i\rho_i \;\; \forall i. \tag{C.61}$$

For the output ensemble $\mathcal{E} = \{\{p_1, p_2, p_3\}, \{\rho_1, \rho_2, \rho_3\}\}$, achieved by the $\mathcal{P}$-POVM $\{|1\rangle\langle 1|, |2\rangle\langle 2|, |3\rangle\langle 3|\}$, it can be seen that $P_g = p_1/2 + p_3$ by directly constructing the optimal K. However, for the output ensemble $\{\{p_1, p_2, p_3\}, \{(1 - p_2/p_1)\rho_1 + p_2/p_1\rho_2, \rho_1, \rho_3\}\}$ achieved by a non-projective $\mathcal{P}$-POVM of the form $\{(1 - p_2/p_1)|1\rangle\langle 1| + |2\rangle\langle 2|, p_2/p_1|1\rangle\langle 1|, |3\rangle\langle 3|\}$ the construction of the optimal K gives $P_g = p_1/2 + p_2/2 + p_3$. Thus, in general the maximization of P_g over $\Pi_A^p(\rho_A)$ does not produce the desired ensemble for systems S and A'.

Appendix D

Appendix of Chapter 6

D.1 Derivation of Eq. (6.8)

To cast inequality (6.7) into the equivalent form (6.8) let us first consider the case where $f(s) > 0$. Under this condition (6.7) forces $f(t)$ to belong to the interval $[-\frac{f(s)}{d^2-1}, f(s)]$ which is centred on the point

$$f_M \equiv \frac{1}{2}\left(f(s) - \frac{f(s)}{d^2-1}\right) = \frac{d^2-2}{2(d^2-1)} f(s) \;, \tag{D.1}$$

and has width

$$W \equiv f(s) + \frac{f(s)}{d^2-1} = \frac{d^2}{d^2-1} f(s) \;. \tag{D.2}$$

Accordingly imposing $f(t) \in [-\frac{f(s)}{d^2-1}, f(s)]$ is equivalent to require

$$|f(t) - f_M| \leq W/2 \;, \tag{D.3}$$

that is

$$|2(d^2-1)f(t) - (d^2-2)f(s)| \leq d^2 f(s) \;, \tag{D.4}$$

which corresponds to (6.8). Similarly if $f(s) \leq 0$, Eq. (6.7) forces $f(t)$ to belong to the interval $[f(s), -\frac{f(s)}{d^2-1}]$ which can still be expressed as in (D.3) by observing that f_M is still as in (D.1) while W becomes

$$W \equiv -\frac{f(s)}{d^2-1} - f(s) = -\frac{d^2}{d^2-1} f(s) \;. \tag{D.5}$$

In this case hence we get

$$|2(d^2-1)f(t) - (d^2-2)f(s)| \leq -d^2 f(s) \;, \tag{D.6}$$

which corresponds to (6.8) for nonpositive values of $f(s)$.

D.2 Non-convex geometries of the Markovian and non-Markovian sets

From the results of Ref. [WECC08] it follows that neither the Markovian subset $\mathbb{E}^M$ nor its complement $\mathbb{E}^{NM}$ are convex (or equivalently that $\mathbb{E}^M$ is neither convex nor concave). In Sections D.2.1 and D.2.2 we show that the same property holds also for the Markovian and non-Markovian parts of the depolarizing trajectories $\mathbb{D}$.

D.2.1 Non-convexity of $\mathbb{D}^{NM}$

Consider the pair of non-Markovian depolarizing evolutions $\boldsymbol{D}^{NM,1}$ and $\boldsymbol{D}^{NM,2}$ with characteristic functions

$$f^{NM,1}(t) \equiv \theta_H(1-t) + \theta_H(t-1)\cos^2(t-1)\,, \tag{D.7}$$

$$f^{NM,2}(t) \equiv \theta_H(1-t) + \theta_H(t-1)\sin^2(t-1)\,, \tag{D.8}$$

where $\theta_H(\tau) = 1$ for $\tau \geq 0$ and $\theta_H(\tau) = 0$ for $\tau < 0$. The characteristic functions $f^{NM,1}(t)$ and $f^{NM,2}(t)$ belong to $\mathfrak{F}$ but fail to fulfill the conditions (6.10) for all t, hence they are elements of $\mathfrak{F}^{NM}$. Interestingly these two evolutions are maximally non-Markovian. Indeed, they show infinitely many non-Markovian gaps $\Delta_k^{NM} = 1$ while being positive and continuous. $f^{NM,1}(t)$ is continuous at any time and $f^{NM,2}(t)$, even if it is not continuous at $t = 1$, belongs to the family described in Eq. (6.101). Hence, since for both of them we have $\Delta^{NM} = \sum_k \Delta_k^{NM} = +\infty$, they assume the maximal value for the measure of non-Markovianity $p(\boldsymbol{D}^{NM,1}|\mathbb{D}^M) = p(\boldsymbol{D}^{NM,2}|\mathbb{D}^M) = 1$ (see Eq. (D.28)). Nonetheless, the convex combination $f^{(p)}(t) = (1-p)f^{NM,1}(t) + pf^{NM,2}(t)$ is Markovian for $p = 1/2$. Indeed, we have

$$f^{(1/2)}(t) = \theta_H(1-t) + \frac{\theta_H(t-1)}{2} = \begin{cases} 1 & t \in [0,1]\,, \\ \frac{1}{2} & t > 1\,, \end{cases} \tag{D.9}$$

which is an element of $\mathfrak{F}^M$ with a Markovian discontinuity at $t = 1$ (indeed $\xi(f^{(1/2)}(1)) = 1/2 \in J_{\mathbb{D}}$). Accordingly the process $(\boldsymbol{D}^{NM,1} + \boldsymbol{D}^{NM,2})/2$ is an element of $\mathbb{D}^M$ proving that $\mathbb{D}^{NM}$ is not closed under convex combination.

D.2.2 Non-convexity of $\mathbb{D}^M$

Focusing on the qubit case $d = 2$, we show an example where any non-trivial convex combination of two Markovian depolarizing evolutions provide a non-

Markovian depolarizing evolution, with the generalization for $d > 2$ being trivial. Therefore, this proves that the Markovian set of depolarizing channels is non-convex and that the two Markovian evolutions used in this example belong to the border of the Markovian set $\mathbb{E}^M$.

Consider two Markovian qubit evolutions $\boldsymbol{D}^{M,1}$ and $\boldsymbol{D}^{M,2}$ defined by the characteristic functions $f^{M,1}(t)$ and $f^{M,2}(t)$, respectively. First, we fix $f^{M,1}(t) = 1$ for all $t \geq 0$. Notice that $\boldsymbol{D}_t^{M,1}(\cdot) = I_S(\cdot)$ is the identical map for any $t \geq 0$. Secondly we take

$$f^{M,2}(t) \equiv \begin{cases} 1 & t \leq t_{NC,1} \\ -1/3 & t \in (t_{NC,1}, t_{NC,2}] \\ 1/9 & t > t_{NC,2} \end{cases} , \tag{D.10}$$

which exhibits Markovian discontinuities

$$\xi(f^{M,2}(t_{NC,1})) = \xi(f^{M,2}(t_{NC,2})) = -\frac{1}{d^2 - 1} = -\frac{1}{3} . \tag{D.11}$$

The convex combination $\boldsymbol{D}^{(p)} = (1 - p)\boldsymbol{D}^{M,1} + p\boldsymbol{D}^{M,2}$ is characterized by $f^{(p)}(t) = (1 - p)f^{M,1}(t) + pf^{M,2}(t)$. While the discontinuity that $f^{(p)}(t)$ shows at $t = t_{NC,1}$ is always Markovian, at $t = t_{NC,2}$ we have

$$\xi(f^{(p)}(t_{NC,2})) = \frac{9 - 8p}{9 - 12p} \notin J_{\mathbb{D}} , \;\; \forall p \in (0, 1) . \tag{D.12}$$

Indeed, $\xi(f^{(p)}(t_{NC,2})) > 1$ for any $p \in (0, 3/4)$, $\xi(f^{(p)}(t_{NC,2})) < -1/3$ for any $p \in (3/4, 1)$ and it diverges for $p = 3/4$: $\lim_{p \to 3/4^{\mp}} \xi(f^{(p)}(t_{NC,2})) = \pm\infty$ (see Fig. D.1). Therefore, *any* depolarizing evolution $\boldsymbol{D}^{(p)}$ obtained by the non-trivial convex combination of the Markovian depolarizing evolutions $\boldsymbol{D}^{M,1}$ and $\boldsymbol{D}^{M,2}$ is non-Markovian.

D.3 Markovian and non-Markovian positive depolarizing evolutions

We define $\mathbb{D}_+ \subset \mathbb{D}$ to be the class of positive depolarizing evolutions which is defined by non-negative characteristic functions, namely the set $\mathfrak{F}_+ \subset \mathfrak{F}$ made by the elements of $\mathfrak{F}$ that are non-negative for any $t \geq 0$. Given the defining feature of the elements of the $\mathfrak{F}_+$, it is clear that the positive depolarizing evolutions form a convex set. We define $\mathbb{D}_+^M$ to be the Markovian subset of $\mathbb{D}_+$ which is in one-to-one correspondence with the set of characteristic functions $\mathfrak{F}_+^M \subset \mathfrak{F}^M$. Similarly, we define $\mathbb{D}_+^{NM}$ and $\mathfrak{F}_+^{NM}$.

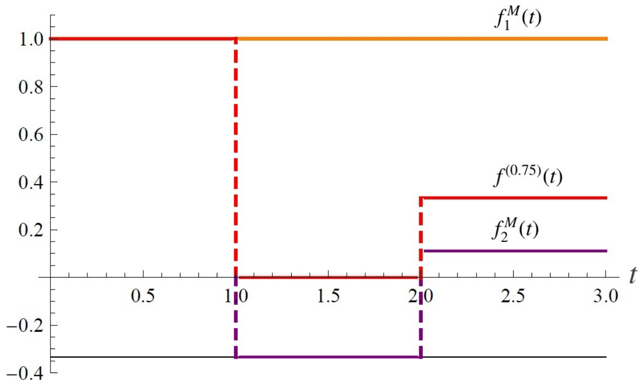


Figure D.1: Plots of $f_1^M(t) = 1$ (orange), $f_2^M(t)$ (purple) and $f^{(p)}(t) = pf_1^M(t) + (1-p)f_2^M(t)$ for $p = 0.75$ (red), where $t_{NC,1} = 1$ and $t_{NC,2} = 2$. While $f_{1,2}^M(t)$ satisfy the conditions (6.10) at any time, this is not the case for $f^{(0.75)}(t)$: it shows a non-Markovian discontinuity at $t_{NC,2} = 2$ when $\mathbf{CM_2}(2)$ is violated. Indeed, $f^{(0.75)}(2^-) = 0$, $f^{(0.75)}(2^+) > 0$ and $\xi(f^{(0.75)}(2)) = +\infty$.

The characteristic functions of $\mathfrak{F}_+$ are in general non-continuous. Indeed, we require that $\xi(f^M(t)) \in [0,1]$ for any $f_+^M(t) \in \mathfrak{F}_+^M$ and $t \geq 0$. Analogously, $\xi(f^{NM}(t)) \in [0,+\infty]$ for any $f_+^{NM}(t) \in \mathfrak{F}_+^{NM}$ and $t \geq 0$. A negative value of $\xi(f(t))$ implies that $f(t)$ changes sign at time t and this circumstance cannot occur for positive $f(t)$. A straightforward calculation shows that $f^{(p)}(t) = (1-p)f_+^{M,1}(t) + pf_+^{M,2}(t)$ cannot show non-Markovian discontinuities and more in general cannot be non-Markovian. Hence,

- $\mathbb{D}_+$ is convex,
- $\mathbb{D}_+^M$ is closed and convex,
- $\mathbb{D}_+^{NM}$ is open and non-convex.

We remember that the set of continuous depolarizing evolutions $\mathbb{D}_C$ has a convex Markovian subset that we called $\mathbb{D}_C^M$. The set of characteristic functions that corresponds to $\mathbb{D}_C^M$ is $\mathfrak{F}_C^M$, which is the collection of non-increasing continuous non-negative functions $f_C^M(t)$. Therefore, we can conclude that

$$\mathbb{D}_C^M \subset \mathbb{D}_+^M \,, \tag{D.13}$$

where $\mathbb{D}_+^M \setminus \mathbb{D}_C^M$ is given by those evolutions of $\mathbb{D}_+^M$ that show at least one (Markovian) discontinuity. Moreover, since no $f_+^{NM}(t) \in \mathfrak{F}_+^{NM}$ can assume negative values, it is easy to see that $\mathbb{D}_C^{NM} \not\subset \mathbb{D}_+^{NM}$, $\mathbb{D}_+^{NM} \not\subset \mathbb{D}_C^{NM}$ and $\mathbb{D}_+^{NM} \cap \mathbb{D}_C^{NM} \neq \emptyset$, namely the intersection is not empty.

D.3.1 Positive non-Markovian characteristic functions

We study the value of $p(\boldsymbol{D}_+^{NM}|\mathbb{D}_+^M)$ when $\boldsymbol{D}_+^{NM} \in \mathbb{D}_+^M$, namely a non-Markovian depolarizing evolution with a positive characteristic function $f_+^{NM}(t) \in \mathfrak{F}_+^{NM}$. Therefore we have to consider the convex combination $f^{(p)}(t) = (1-p)f_+^{NM}(t) + pf_+^M(t)$ and evaluate the smallest p for which there exists a $f_+^M(t) \in \mathfrak{F}_+^M$ that makes $f^{(p)}(t)$ Markovian, more precisely an element of $\mathfrak{F}_+^M$.

Similarly to the previous sections, we define T_+^{NM} and Δ^{NM} exactly as in the continuous case, namely the collection of the time intervals $T_k^+ = (t_k^{(in)}, t_k^{(fin)})$ where a non-Markovian gap $\Delta_k^{NM} > 0$ is shown while the non-negative $f_+^{NM}(t)$ is continuous. Analogously to Appendix D.4, we introduce

$$\Delta_k^M \in [-1, 0] \ , \qquad \Delta^M \equiv \sum_{k=1}^{L} \Delta_k^M \in [-1, 0] \, , \tag{D.14}$$

where $\Delta_k^M = f_+^M(t_k^{(fin)}) - f_+^M(t_k^{(in)}) \leq 0$ is the gap that $f_+^M(t)$ describes when $f_+^{NM}(t)$ is increasing.

Moreover, we introduce $W_+^{NM} \equiv \{\tau_i\}_i$ as the discrete set of times when $f_+^{NM}(t)$ shows a non-Markovian discontinuity, namely such that $\xi(f_+^{NM}(\tau_i)) \in (1, \infty]$ (remember that $\xi(f_+^{NM}(t))$ and $f_+^{NM}(t)$ itself cannot be negative). Analogously to Δ_k^{NM}, we introduce the quantities

$$\pi_i^{NM} \equiv f_+^{NM}(\tau_i^+) - f_+^{NM}(\tau_i^-) > 0 \, , \tag{D.15}$$

$$\pi_i^{M} \equiv f_+^{M}(\tau_i^+) - f_+^{M}(\tau_i^-) < 0 \, , \tag{D.16}$$

respectively the non-Markovian gaps shown by $f_+^{NM}(t)$ and the Markovian gaps shown by $f_+^M(t)$ at the times when $f_+^{NM}(t)$ has non-Markovian discontinuities. Moreover,

$$\pi^{NM} = \sum_i \pi_i^{NM} > 0 \, , \tag{D.17}$$

$$\pi^{M} = \sum_i \pi_i^{M} \in [-1, 0] \, , \tag{D.18}$$

are respectively the sum of all the non-Markovian jumps shown by $f_+^{NM}(t)$ and all the Markovian jumps shown by $f_+^M(t)$. Notice that, since $f_+^M(t)$ is non-increasing, $\Delta^M + \pi^M \in [-1, 0]$[1]. Indeed, it is easy to show that, in order to

[1] This quantity can be equal to -1 if and only if $f_+^M(t)$ is constant for those times when $f_+^{NM}(t)$ behaves as a Markovian characteristic function.

calculate $p(\boldsymbol{D}_+^{NM}|\mathbb{D}_+^M)$, by considering $f_+^M(t)$ with (Markovian) discontinuities for some $t \notin W_+^{NM}$ we do not obtain an advantage. More precisely, we have to consider $f_+^M(t)$ that show Markovian discontinuities if and only if $t \in W_+^{NM}$.

A necessary condition to make $f^{(p)}(t)$ Markovian is that $p \geq p_+ \equiv (\Delta^{NM} + \pi^{NM})/(1 + \Delta^{NM} + \pi^{NM})$. This relation is obtained as Eq.(D.25), where we also require that $\pi_i^{(p)} = (1-p)\pi_i^{NM} + p\pi_i^M \leq 0$, namely that the discontinuities of $f^{(p)}(t)$ are Markovian.

In order to evaluate the measure of non-Markovianity of $f_+^{NM}(t)$, we adapt the tools introduced in Appendix D.4 (where we studied non-negative continuous non-Markovian characteristic functions) to implement the cases where $f_+^{NM}(t)$ shows non-Markovian discontinuities. We define $g_+^M(t)$ as the following function

$$\begin{cases} 1 & t \leq t_1 \,, \\ \cdots & \\ g_+^M(t_{k-1}^{(fin)}) - \left(f^{NM}(t) - f^{NM}(t_k^{(in)})\right)/(\Delta^{NM} + \pi^{NM}) & t \in T_k^+ \,, \\ \cdots & \\ g_+^M(\tau_i^-) - \pi_i^{NM}/(\Delta^{NM} + \pi^{NM}) & t = \tau_i \,. \end{cases} \tag{D.19}$$

where $t_1 \equiv \min\{\tau_1, t_1^{(in)}\}$. It is easy to see that Eq. (D.19) is obtained from Eq. (D.26) by replacing Δ^{NM} with $\Delta^{NM}+\pi^{NM}$ and by implementing the Markovian gaps $\pi_i^M \equiv -\pi_i^{NM}/(\Delta^{NM} + \pi^{NM})$ that $g_+^M(t)$ shows when $f_+^{NM}(t)$ shows a non-Markovian discontinuity. Moreover, we notice that by considering $g_+^M(t)$ we have $\Delta^M + \pi^M = -1$. The function $f^{(p)}(t) = (1-p)f_+^{NM}(t) + pg_+^M(t)$ belongs to $\mathfrak{F}_+^M$ for any $p \geq p_+$, where $f^{(p_+)}(t)$ is constant for any $t \in T^{NM}$ and continuous for any $t \in W_+^{NM}$. Finally, we can state that

$$p(\boldsymbol{D}_+^{NM}|\mathbb{D}_+^M) = \frac{\Delta^{NM} + \pi^{NM}}{1 + \Delta^{NM} + \pi^{NM}} \,, \tag{D.20}$$

and therefore this measure of non-Markovianity depends on the non-Markovian gaps shown by the non-Markovian characteristic function (in this case $\Delta^{NM} + \pi^{NM}$) as in the continuous case.

We notice that, while for continuous depolarizing evolutions $p(\boldsymbol{D}_C^{NM}|\mathbb{D}^M) = p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)$ (see Section 6.5), in the case of positive depolarizing evolutions we have $p(\boldsymbol{D}_+^{NM}|\mathbb{D}^M) \leq p(\boldsymbol{D}_+^{NM}|\mathbb{D}_+^M)$. We present a simple example that shows this feature. Consider $f_+^{NM}(t)$ with (i) a single non-Markovian discontinuity $W_+^{NM} = \{\tau\}$, (ii) no non-Markovian intervals of non-Markovianity T_k^{NM} and (iii) such that $\dot{f}_+^{NM}(t) = 0$ for any $t \geq \tau$. In this case we have that $g_+^M(t) = 1$ for any $t \leq \tau$ and $g_+^M(t) = 0$ for any $t > \tau$. Therefore, we obtain

$p(\boldsymbol{D}_{+}^{NM}|\mathbb{D}_{+}^{M}) = \pi^{NM}/(1+\pi^{NM})$, where $\pi^{NM} = f_{+}^{NM}(\tau^{+}) - f_{+}^{NM}(\tau^{-})$. By considering the non-positive Markovian characteristic function $g^{M}(t) = 1$ for $t \leq \tau$ and $g^{M}(t) = -1/(d^2-1)$ for $t \geq \tau$ we have[2]

$$p(\boldsymbol{D}_{+}^{NM}|\mathbb{D}^{M}) = \frac{\pi^{NM}}{1+\frac{1}{d^2-1}+\pi^{NM}} < p(\boldsymbol{D}_{+}^{NM}|\mathbb{D}_{+}^{M})\,.$$

D.4 Multiple non-Markovianity time intervals for positive characteristic functions

Here we extend the construction of Section 6.4.1 to address the general case of continuous non-Markovian characteristic functions of the form (6.38), namely which are positive and which have an arbitrary (possibly infinite) number $L > 0$ of intervals $T_k^{+} \equiv (t_k^{(in)}, t_k^{(fin)})$ of non-Markovianity. As in the previous section for each of the intervals T_k^{+} we introduce the gaps

$$\Delta_k^{M} \equiv f_C^{M}(t_k^{(fin)}) - f_C^{M}(t_k^{(in)})\,, \tag{D.21}$$

$$\Delta_k^{(p)} \equiv f_C^{(p)}(t_k^{(fin)}) - f_C^{(p)}(t_k^{(in)}) = (1-p)\Delta_k^{NM} + p\Delta_k^{M}\,, \tag{D.22}$$

with Δ_k^{NM} the positive quantities defined in (6.35). Observe that, due to the fact that $f_C^{M}(t)$ is in $\mathfrak{F}_C^{M}$, the Δ_k^{M} are all non-positive while their global sum is larger than -1, namely

$$\Delta_k^{M} \in [-1, 0]\ , \qquad \Delta^{M} \equiv \sum_{k=1}^{L} \Delta_k^{M} \in [-1, 0]\ . \tag{D.23}$$

This is just a consequence of the fact that the maximum gap of a continuous Markovian characteristic function is at most equal to -1. A necessary condition for the Markovianity of $f_M^{(p)}(t)$ can then be obtained by imposing that $\Delta_k^{(p)} \leq 0$ for all k, which in turn implies

$$0 \geq \sum_{k=1}^{L} \Delta_k^{(p)} = (1-p)\Delta^{M} + p\Delta^{NM} \tag{D.24}$$

$$\Longrightarrow p \geq \frac{\Delta^{NM}}{|\Delta^{M}| + \Delta^{NM}} \geq \frac{\Delta^{NM}}{1+\Delta^{NM}} \equiv p_L\,, \tag{D.25}$$

[2] We think that it is not necessary to prove that there are no $f^{M}(t) \in \mathfrak{F}^{M}$ that are able to make $f^{(p)}(t)$ Markovian for smaller values of p.

where D.25 we used (6.36) and (D.23).

Now we show that a $g_C^M(t) \in \mathfrak{F}_C^M$ that makes $f_C^{(p)}(t)$ Markovian for any $p \geq p_L$ exists. We consider the following monotonically decreasing function

$$g_C^M(t) = \begin{cases} 1 & t \leq t_1^{(in)} \\ 1 - \left(f^{NM}(t) - f^{NM}(t_1^{(in)})\right)/\Delta^{NM} & t \in T_1^+ \\ g_C^M(t_1^{(fin)}) - \left(f^{NM}(t) - f^{NM}(t_2^{(in)})\right)/\Delta^{NM} & t \in T_2^+ \\ \cdots & \\ g_C^M(t_{k-1}^{(fin)}) - \left(f^{NM}(t) - f^{NM}(t_k^{(in)})\right)/\Delta^{NM} & t \in T_k^+ \\ \cdots & \end{cases}, \tag{D.26}$$

that we define constant and equal to $g_C^M(t_{k-1}^{(fin)})$ in the intervals $[t_{k-1}^{(fin)}, t_k^{(in)}]$, for $k = 1, \ldots, L$. Therefore, the temporal derivative of $g_C^M(t)$ is particularly simple

$$\dot{g}_C^M(t^\pm) = \begin{cases} -\dot{f}_C^{NM}(t^\pm)/|\Delta^{NM}| & t \in T_k^+ \, , \\ \\ 0 & \text{otherwise.} \end{cases} \tag{D.27}$$

As a consequence, for $t \in T_k^+$, the function $g_C^M(t)$ decreases by a factor proportional to the increase of $f^{NM}(t)$ in the same time interval, namely $\Delta_k^M = -\Delta_k^{NM}/\Delta^{NM} < 0$. An intuitive explanation for the form of $g_C^M(t)$ is the following. The "resource" of a continuous Markovian characteristic function to contrast the non-Markovianity of $f_C^{NM}(t)$ is its distance from zero. Once that $f_C^{NM}(t)$ decreases, it cannot increase again. Therefore, to efficiently use the maximum available gap allowed for Markovian characteristic functions, namely $\Delta^M = -1$, $g_C^M(t)$ is constant whenever $f_C^{NM}(t)$ behaves as a Markovian characteristic function. Instead, when this behavior is non-Markovian, $g_C^{NM}(t)$ decreases accordingly to the increase of $f_C^{NM}(t)$ in order to make their convex sum $f_C^{(p)}(t) = (1-p)f_C^{NM}(t) + pg_C^M(t)$ constant for the smallest value of p. This proves that, for the continuous depolarizing evolutions defined as in Eq. (6.38), $p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M) = p_L$. Therefore, the corresponding non-Markovianity measure (6.30) is equal to

$$p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M) = p_L = \frac{\Delta^{NM}}{1+\Delta^{NM}}\, , \tag{D.28}$$

which corresponds to Eq. (6.37).

D.5 Removing the finite *m* constraint

In Section 6.4.2 we have assumed m to be explicitly finite, a useful hypothesis which allowed us to assume the existence of (6.52) and to express its value as in (6.63). It turns out however that this assumption is not fundamental and that Eq. (6.50) holds true also if we drop it. In order to show this, instead of studying the Markovian character of $f_C^{(p)}(t)$ for all $t \geq 0$, we limit the analysis for just all $t \leq T$ with T being finite quantity. Observe then that the number $m(T)$ of time intervals $T_j^- = (t_j^{(in)}, t_j^{(fin)})$ contained into domain $[0, T]$, where the characteristic function $f_C^{NM}(t)$ is negative and decreasing, is by construction finite. Same considerations holds for the total number $\overline{m}(T)$ of the time intervals $\overline{T}_j = (\bar{t}_j^{(in)}, \bar{t}_j^{(fin)})$ when $f_C^{NM}(t) \leq 0$ and $\dot{f}_C^{NM}(t^\pm) \geq 0$ and which fit on $[0, T]$. Following the same reasoning we adopted in the previous section, the following relations can then be derived

$$
\begin{aligned}
f_C^{NM}(T) &= \delta^{NM}(T) + \Theta^{NM}(T) \,, && \text{(D.29)}\\
1 &\geq f_C^M(T) + |\delta^M(T)| \,, && \text{(D.30)}
\end{aligned}
$$

with

$$
\begin{gathered}
\delta^M(T) \equiv \textstyle\sum_{j=1}^{\overline{m}(T)} \delta_j^M \leq 0 \,, \qquad \delta^{NM}(T) \equiv \textstyle\sum_{j=1}^{\overline{m}(T)} \delta_j^{NM} \geq 0 \,, \\
\Theta^{NM}(T) \equiv \textstyle\sum_{j=1}^{m(T)} \Theta_j^{NM} < 0 \,. \qquad \text{(D.31)}
\end{gathered}
$$

Furthermore Eqs. (6.58) and (6.62) get replaced by

$$
\begin{aligned}
(1-p)\delta^{NM}(T) - p|\delta^M(T)| &\leq 0 \,, && \text{(D.32)}\\
-(1-p)(\delta^{NM}(T) + \Theta^{NM}(T)) - p f_C^M(T) &\leq 0 \,, && \text{(D.33)}
\end{aligned}
$$

that summed up term by term lead to

$$
p \geq \frac{|\Theta^{NM}(T)|}{1 + |\Theta^{NM}(T)|} \,, \qquad \text{(D.34)}
$$

which is a necessary condition to have $f_C^{(p)}(t)$ Markovian at least on $[0, T]$. Following then a construction which is analogous to the one given in (6.66) we can also show that indeed the right-hand-side term of (D.34) is the minimum value for p to ensure the Markovianity of $f_C^{(p)}(t)$ on $[0, T]$. The final result thus can be derived by taking the limit $T \to \infty$ which leads to (6.50) where now Θ^{NM} is properly computed as $\Theta^{NM} = \lim_{T\to\infty} \Theta^{NM}(T)$. Notice in particular that having extend (6.50) to the case of infinite m it is now possible that $|\Theta^{NM}|$ will diverge (a case that for instance happen whenever $f_C^{NM}(t)$ has infinitely many – not properly dumped – oscillations) leading to the maximum value for the measure of non-Markovianity, namely $p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M) = 1$.

D.6 Uniqueness of the optimal continuous Markovian characteristic function

We consider the evaluation of $p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)$ when $\boldsymbol{D}_C^{NM} \in \mathbb{D}_C^{NM}$ and $f_C^{NM}(t)$ is the corresponding continuous characteristic function. For the purpose of evaluating this quantity, in Section 6.4.3 we saw that its value is given by $\Gamma^{NM}/(1+\Gamma^{NM})$ and a continuous characteristic function that makes the corresponding $f^{(p)}(t)$ Markovian for $p = p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)$ is $h_C^M(t)$ (see Eq. (6.85)). In this section we show that $h_C^M(t)$ is the *only* continuous Markovian characteristic function that makes $f^{(p)}(t)$ Markovian for any $p \geq p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)$.

Any continuous Markovian characteristic function $f_C^M(t)$ assumes values in $[0,1]$ and is non-increasing. Let start noticing that, if $f_C^M(t)$ decreases while $h_C^M(t)$ is constant, given what we discussed in Section 6.4 we conclude that the former has no chance to perform better than the latter. Therefore, consider a time interval (t_1,t_2) of non-Markovianity where $\dot{h}_C^M(t) < 0$ and $\dot{f}^{NM}(t) > 0$. If for some $t \in (t_1,t_2)$ we have $\dot{h}_C^M(t) < \dot{f}_C^M(t) \leq 0$, the $f^{(p)}(t)$ obtained with $f_C^M(t)$ has a time derivative that can be made non-positive for larger values of p if compared with the $f^{(p)}(t)$ obtained with $h_C^M(t)$. Therefore, in this situation $f_C^M(t)$ is less efficient than $h_C^M(t)$ to make $f^{(p)}(t)$ Markovian.

Consider a time interval of non-Markovianity $(t_k^{(in)}, t_k^{(fin)})$ where we have $\dot{f}_C^M(t) < \dot{h}_C^M(t) < 0$ for some $t \in (t_k^{(in)}, t_k^{(fin)})$ and $\dot{f}_C^M(t) \leq \dot{h}_C^M(t) < 0$ for every $t \in (t_k^{(in)}, t_k^{(fin)})$. Assume that $f^{NM}(t) \geq 0$ and therefore $\lim_{t\to\infty} h_C^M(t) = 0$. Using the notation introduced in Eqs. (6.35), (D.21) and (D.22), we see that by using $h_C^M(t)$ all the $\Delta_k^{(p)}$ are non-positive for $p \geq p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)$, while they are all positive for $p < p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)$. In the case of the $f_C^M(t)$ described above, we may have that some$\Delta_k^{(p)}(t)$ can be made non-positive for some $p < p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C^M)$. Since $\sum_k \Delta_k^M \in [-1,0]$ and by considering that with $h_C^M(t)$ we have $\sum_k \Delta_k^M = -1$, there must be a $k' \neq k$ such that the value of $|\Delta_{k'}^M|$ obtained with $f_C^M(t)$ is smaller than the one obtained with $h_C^M(t)$. Hence, while $h_C^M(t)$ can make $f^{(p)}(t)$ Markovian for $p = p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C)$, $f_C^M(t)$ cannot do the same. A similar argument can be used for $f^{NM}(t)$ that assume positive and negative values.

Finally, since any characteristic function $f_C^M(t)$ that makes $f^{(p)}(t)$ Markovian for $p = p(\boldsymbol{D}^{NM}|\mathbb{D}_C^M)$ cannot have a time derivative different from $\dot{h}_C^M(t)$, $h_C^M(t)$ is the only continuous Markovian characteristic function that is optimal to make $f^{(p)}(t)$ Markovian, namely $f^{(p)}(t)$ can be made Markovian for $p = p(\boldsymbol{D}_C^{NM}|\mathbb{D}_C)$ with a continuous characteristic function if and only if we consider $h_C^M(t)$. In particular, given the results of Section 6.5, we can state that the optimal Markovian evolution needed to evaluate $p(\boldsymbol{D}_C^{NM}|\mathbb{D}^M)$ is unique and defined by $h_C^M(t)$.

D.7 Different vectors of signs for $f_a^M(t)$ and $f^{(p)}(t)$

We consider $f^{(p)}(t) \in \mathfrak{F}_b^M$ for some $\boldsymbol{\sigma}_b$, where $f_a^M(t) \in \mathfrak{F}_a^M$ for some $\boldsymbol{\sigma}_a \neq \boldsymbol{\sigma}_b$. In the following, b is always the index attached to $f^{(p)}(t)$ and a is always the index attached to $f_a^M(t)$.

First, we make the following consideration. In the case that $f^{NM}(t)$ violates the conditions of Markovianity given in Eq. (6.10) in a time interval in $T_{C,i}$, then we must consider $\sigma_{a,i} = \sigma_{b,i}$. Indeed, if $\text{sign}(f^M(t)) = -\text{sign}(f^{NM}(t))$ and $f^{NM}(t)$ shows a non-Markovian behavior while being continuous at time t, $f^{(p)}(t)$ can be made Markovian at time t if and only if $\text{sign}(f^{(p)}(t)) = \text{sign}(f^M(t))$. Therefore:

(A) If $T_{C,i}$ is a time interval when $f^{NM}(t)$ behaves as a non-Markovian characteristic function, then $\sigma_{a,i} = \sigma_{b,i}$.

Therefore, if $\boldsymbol{\sigma}_a$ and $\boldsymbol{\sigma}_b$ do not satisfy (A) for at least one time interval $T_{C,i}$ we set $p_{a,b} = 1$ because $f_a^M(t)$ cannot make $f^{(p)}(t) \in \mathfrak{F}_b^M$.

D.7.1 Times of continuity

Let us consider $\boldsymbol{\sigma}_a$ and $\boldsymbol{\sigma}_b$ that satisfy (A) for each $T_{C,i}$ and define the optimal $f_a^M(t)$ for a generic time interval $T_{C,i}$ when $\sigma_{a,i} \neq \sigma_{b,i}$. During this time interval $f^{NM}(t)$ behaves as a continuous Markovian characteristic function and therefore cannot change its sign. As a consequence, we must be in a situation where $\sigma_{a,i} = -\sigma_{b,i}$ while $\text{sign}(f_a^M(t)) = \sigma_{a,i}$ and $\text{sign}(f^{NM}(t)) = \text{sign}(f^{(p)}(t)) = \sigma_{b,i}$. Therefore, with opposite signs, $f_a^M(t)$ and $f^{NM}(t)$ are approaching continuously zero and we need to make their convex combination be of the same sign of $f^{NM}(t)$.

We study the scenario where $\sigma_{a,i} = +1$ and $\sigma_{b,i} = -1$, where $\text{sign}(f_a^M(t)) = +1$ and $\text{sign}(f^{NM}(t)) = \text{sign}(f^{(p)}(t)) = -1$ for any $t \in [t_{NC,i-1}, t_{NC}]$ (the same results can be derived for $\text{sign}(f_a^M(t)) = -1$ and $\text{sign}(f^{NM}(t)) = \text{sign}(f^{(p)}(t)) = +1$). We write $f_a^M(t_{NC,i}) = |f_a^M(t_{NC,i})| = |f_a^M(t_{NC,i-1})| - \delta_{M,i}$ and moreover we have $f^{NM}(t_{NC,i}) = -|f^{NM}(t_{NC,i})| = -|f^{NM}(t_{NC,i-1})| + \delta_i^{NM}$, where $\delta_i^M, \delta_i^{NM} \geq 0$. Indeed, between $t_{NC,i-1}$ and $t_{NC,i}$, $f_a^M(t) \geq 0$ decreases and $f^{NM}(t) \leq 0$ increases. Therefore, if we consider the value of $f_a^M(t_{NC,i-1})$ fixed by the study of the time interval $[t_{NC,i-2}, t_{NC,i-1}]$, we have to study for which values of p the function $f^{(p)}(t)$ in negative and non-decreasing in $[t_{NC,i-1}, t_{NC,i}]$, when δ_i^M varies. These two conditions can be respectively written as:

$$p \leq \frac{|f^{NM}(t_{NC,i})|}{|f^{NM}(t_{NC,i})| + |f^{NM}(t_{NC,i})| - \delta_i^M} \leq 1 \,, \tag{D.35}$$

$$p \leq \frac{\delta_i^{NM}}{\delta_i^{NM} + \delta_i^M} \leq 1\,, \tag{D.36}$$

which are upper bounds for p. This is the first time that we obtain upper bounds on p rather than lower bounds. The reason of this new situation is given by the fact that we impose $f^{(p)}(t)$ to have the same sign of $f^{NM}(t)$ and the opposite sign of $f_a^M(t)$. Hence, this condition cannot be satisfied if p is too large and it is surely verified when p is small enough. Notice that (D.35) provides the largest interval of validity when δ_i^M is as large as possible, namely $|f^{NM}(t_{NC,i})| - \delta_i^M = 0$, while for (D.36) we have the opposite situation. Since they are both upper bounds, $\delta_i^M = 0$ may seem the best choice. Nonetheless, we have to consider that (D.35) have to be consistent with the lower bounds on p that we obtain when we impose Markovianity for $f^{(p)}(t)$ in the other time intervals and times of discontinuity. As a consequence, the choice of δ_i^M is not obvious, and we have to implement a variable δ_i^M that we fix when we calculate $p_{a,b}$. Therefore, for each time interval where $\sigma_{a,i} \neq \sigma_{b,i}$ we introduce a parameter δ_i^M and $f_a^M(t)$ has to be parameterized by this set, namely $f_a^M(t) = f_a^M(t, \{\delta_i^M\}_i)$.

Notice that, for the time intervals when instead we have $\sigma_{a,i} = \sigma_{b,i}$, we use the conditions introduced in Table 6.1. Therefore, we defined the behavior of $f_a^M(t)$ for all the times $t \in W_C$.

D.7.2 Discontinuities

Let us consider those discontinuities that cannot be described by Tables 6.2 and 6.3, namely those times $t_{NC,i}$ such that:

$$\begin{cases} \sigma_{a,i-1} = -\sigma_{b,i-1} & \text{and} \quad \sigma_{a,i} = \sigma_{b,i} \\ \sigma_{a,i-1} = \sigma_{b,i-1} & \text{and} \quad \sigma_{a,i} = -\sigma_{b,i} \\ \sigma_{a,i-1} = -\sigma_{b,i-1} & \text{and} \quad \sigma_{a,i} = -\sigma_{b,i} \end{cases} . \tag{D.37}$$

These discontinuities, analogously to (D.35) and (D.36), often provide upper-bounds for p. Consider that, if there is just one time interval where the cross-diagonal conditions of Table 6.1 occur, we obtain a lower-bound $p \geq p_{up} = \Delta/(1+\Delta)$. Moreover, lower-bound conditions are obtained when we consider Tables 6.2 and 6.3 (while (D.37) does not apply). Moreover we notice that we must have at least one lower-bound condition, otherwise $p = 0$ would be consistent with $f^{(p)}(t)$ being Markovian, which is a contradiction. Therefore, we may be interested to maximize p_{lim} as a function of $\xi(f_a^M(t))$ in order to make it compatible with one or more lower-bound conditions. In several cases given by Eq. (D.37) this result is obtained for $\xi(f_a^M(t)) = 0$. Therefore, there is a trade-off between p_{lim} and the ability of $f_a^M(t)$ to make $f^{(p)}(t)$ for later times. We conclude that, for each Markovian and non-Markovian discontinuity of type (D.37), we introduce a parameter $\overline{\Xi}_i$ that defines the value of $\xi(f_a^M(t_{NC,i}))$.

D.7.3 Evaluation of p_{ab}

Therefore, having $\boldsymbol{\sigma}_a \neq \boldsymbol{\sigma}_b$ such that condition (A) is satisfied, in general we need to consider an $f_a^M(t)$ that depends on the parameters Δ (see Section 6.7.2), $\{\Xi_i\}_i$ (see Sections 6.7.2 and 6.7.2), δ_i^M (see Section D.7.1) and $\{\overline{\Xi}_i\}_i$ (see Section D.7.2) and $p_{a,b}$ is obtained by the optimization

$$p_{a,b} = \min_{\Delta,\{\delta_i^M\}_i,\{\Xi_i\}_i,\{\overline{\Xi}_i\}_i} \{p \,|\, f_a^M(\Delta,\{\delta_i^M\}_i,\{\Xi_i\}_i,\{\overline{\Xi}_i\}_i) \in \mathfrak{F}_a^M, f^{(p)}(t) \in \mathfrak{F}_b^M\}. \quad \text{(D.38)}$$

It is plausible that, even if condition (A) holds, the maximization required for $p_{a,b}$ has no solution. Indeed, the upper-bound and lower-bound conditions discussed above may not be made compatible for any $f_a^M(t) \in \mathfrak{F}_a^M$.

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