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# Deformations and representations of low dimensional non-orientable hyperbolic manifolds

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By

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A thesis submitted in partial fulfillment of the requirements for the degree of Doctor in Mathematics

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Bellaterra, September 9, 2021



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Mathematics.

Bellaterra, September 9, 2021

Joan Porti Piqué

A la memoria de mi madre,

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---

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# Abstract

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In this work, we extend several known results on hyperbolic 3-manifolds and surfaces to the non-orientable case and compare them to its orientable counterpart. In particular, we focus on the deformation space of a non-orientable hyperbolic 3-manifold of finite volume, the metric completion of said deformations and the variety of representation of the Klein bottle and, more generally, of any closed non-orientable surface.

We are interested in studying the local structure of the deformation space. We approach the subject through two sides, from geometric ideal triangulations and the variety of representations. Our main result on the topic is that, for the one non-orientable cusped case, the deformation space of an ideal triangulation is homeomorphic to a half-open interval whereas deformations of representations are homeomorphic to an open interval of the real line. If we consider an orientable cusp (in a non-orientable manifold), the discrepancy is no longer observed and we obtain that its deformations are homeomorphic to an open set of  $\mathbb{C}$ .

The deformations of non-orientable cusps are related to different representations of a Klein bottle which we call type I and II. The completion of the end is either a solid Klein bottle or disc orbi-bundle for respective representations of type I and II. Furthermore, deformations of a geometric ideal triangulation can only yield representations of type I.

On the other hand, we study in depth the variety of representations of the Klein bottle and, more generally, we compute the number of connected components of the variety of representations of any closed non-orientable surfaces. For the surface of genus  $k$ , there are  $2^{k+1}$  connected components, which are distinguished by the first and second Stiefel-Whitney class of the



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associated principal bundle.

## Resumen

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En esta tesis extendemos varios resultados conocidos sobre 3-variedades hiperbólicas y superficies al caso no orientable y los comparamos con su equivalente orientable. En particular, nos centramos en el espacio de deformaciones de una 3-variedad hiperbólica no orientable de volumen finito, la completación métrica de dichas deformaciones y la variedad de representaciones de la botella de Klein y, más generalmente, de cualquier superficie cerrada no orientable.

Estamos interesados en estudiar la estructura local del espacio de deformaciones. Afrontamos el tema desde dos puntos de vista, por un lado mediante triangulaciones ideales geométricas y, por otro, por la variedad de representaciones. Nuestro principal resultado en esta cuestión es que, para el caso de una cúspide no orientable, el espacio de deformaciones de una triangulación ideal es homeomorfo a un intervalo semiabierto, mientras que las deformaciones de la representación son homeomorfas a un intervalo abierto de la recta real. Si consideramos una cúspide orientable (de una variedad no orientable), esta discrepancia ya no se observa, y obtenemos que su espacio de deformaciones es homeomorfo a un abierto de  $\mathbb{C}$ .

Las deformaciones de cúspides no orientables están relacionadas con distintas representaciones de la botella de Klein, a las que llamamos tipo I y tipo II. La completación del final es o bien una botella de Klein sólida o bien un orbi-fibrado con fibra un disco para respectivas representaciones de tipo I y II. Asimismo, deformaciones de una triangulación ideal geométrica solo pueden dar lugar a representaciones de tipo I.

Por otro lado, estudiamos en profundidad la variedad de representaciones de la botella de Klein y, más generalmente, calculamos el número

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de componentes conexas de la variedad de representaciones de cualquier superficie cerrada no orientable. Para la superficie de género  $k$ , hay  $2^{k+1}$  componentes conexas, que se distinguen por la primera y segunda clase de Stiefel-Whitney del fibrado principal asociado.

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# Introduction

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Since the work of W. Thurston, hyperbolic geometry in low dimensional topology has been a very active field of research. For the most part of it, the focus has been on orientable manifolds with little to no reference towards the non-orientable ones, despite the first example of a finite volume and complete hyperbolic 3-manifold being the Gieseking manifold, which is non-orientable (see [41], [18]). With this in mind, the aim of this PhD Thesis is to extend some of the well-known results to the non-orientable case and explore the differences between them. A good part of the results presented here have been gathered in two preprints ([5], [4]).

## The deformation space

Let us recall that a complete hyperbolic 3-manifold  $M^3$  can be described as a quotient  $\mathbb{H}^3/\Gamma$ , where  $\Gamma$  is a discrete subgroup of isometries of hyperbolic 3-space,  $\Gamma < \text{Isom}(\mathbb{H}^3)$ , and  $\text{Isom}(\mathbb{H}^3) \cong \text{PSL}(2, \mathbb{C}) \rtimes \mathbb{Z}_2$ . When  $M^3$  has finite volume and is *orientable*, it is diffeomorphic to the interior of a compact 3-manifold  $\overline{M^3}$  whose boundary is a disjoint union of tori,  $\partial\overline{M^3} = \sqcup T_i^2$ , called *peripheral tori*. More generally, if  $M^3$  has finite volume and is *non-orientable*, the boundary of the corresponding compact 3-manifold is a disjoint union of tori and Klein bottles,  $\partial\overline{M^3} = \sqcup T_i^2 \sqcup K_j^2$ .

Our main interest lies within deformations of the complete metric in a hyperbolic 3-manifold. By Mostow-Prasad Rigidity Theorem ([41, Thm. 11.8.5]), if a finite-volume, complete, hyperbolic 3-manifold  $M^3$  admits a complete structure, then the complete hyperbolic structure is unique up to isometry. Nonetheless, deformations into non-complete hyperbolic metrics

can be considered.

We can take into consideration manifolds locally isometric to  $\mathbb{H}^3$  but non-complete; in this case, the previous description as a quotient will no longer be valid. In general, when we consider a connected hyperbolic 3-manifold  $M^3$ , there exist a local diffeomorphism, called the *developing map*, from the universal cover of  $M^3$  to  $\mathbb{H}^3$ , that is,  $\text{dev} : \widetilde{M}^3 \rightarrow \mathbb{H}^3$  and a homomorphism  $\text{hol} : \pi_1(M^3) \rightarrow \text{Isom}(\mathbb{H}^3)$  called the *holonomy representation*. The developing map and the holonomy representation are often said to ‘globalize’ the coordinate charts and the coordinate changes, respectively.

We are interested in studying the local structure of the deformation space, that is, deformations of the complete metric. The results on the orientable case trace back to W. Thurston’s Notes in the seventies ([43]). To give a short account of it, it was first studied by considering geometric ideal triangulations, which topologically are tetrahedra without vertices and are realized as tetrahedra in hyperbolic 3-space with vertices at the boundary at infinity. The key fact about ideal tetrahedra is that they can be parametrized by a single complex parameter. By deforming the tetrahedra (i.e. changing continuously the parameter) and gluing them back together, a new hyperbolic structure can be obtained, as long as some *compatibility equations* are satisfied. By using this, W. Thurston showed that the deformation space of the figure eight knot exterior (the orientation covering of the Gieseking manifold) is biholomorphic to an open subset of  $\mathbb{C}$ . This result was generalized by W. Neumann and D. Zagier ([37]):

**Theorem** (W. Thurston, W. Neumann - D. Zagier). *Let  $\Delta$  be a geometric ideal triangulation of  $M^3$  and let  $\text{Def}(M^3, \Delta)$  denote the deformation space with respect to the triangulation. The deformation space  $\text{Def}(M^3, \Delta)$  is biholomorphic to an open subset of  $\mathbb{C}^l$ , where  $l$  is the number of ends of  $M^3$ .*

The biholomorphism is achieved by fixing longitude and meridian pairs  $(l_i, m_i)$  in each peripheral tori. Then, the map from some deformation to  $(\log \text{hol } l_i) \in \mathbb{C}^l$  is a biholomorphism (the same happens if we choose the meridians instead). Moreover, we can solve for  $(p_i, q_i) \in \mathbb{R}^2 \cup \{\infty\}$  in

Thurston's equation

$$p_i \log \operatorname{hol} l_i + q_i \log \operatorname{hol} m_i = 2\pi i.$$

The terms  $(p_i, q_i)$  are called the *generalized Dehn filling coefficients* and the map from a deformation to its coefficients is a homeomorphism between the deformation space and a neighbourhood of  $(\infty, \dots, \infty)$ , so we can use them to parametrize deformations too. In terms of generalized Dehn filling coefficients, the hyperbolic structures corresponding to  $(p, q)$  and  $(-p, -q)$  are isometric.

The upside to the approach through ideal triangulations is that it allows us to reduce the problem of computing metrics to a combinatorial one, however it does not come without disadvantages. The main issue is its generality; where the following question is still an open problem:

**Question.** Does every complete hyperbolic 3-manifold of finite volume admit a geometric ideal triangulation?

Epstein-Penner Theorem ([13]) states that these manifolds can be obtained by gluing faces of an ideal polyhedron, so the first idea that comes to mind is to try to obtain a subdivision of the polyhedron into ideal tetrahedra, however so far the efforts of obtaining such subdivision have yielded no result. On a positive note, in 2008, F. Luo, S. Schleimer and S. Tillmann proved ([32]) that virtually any such manifold admits a geometric ideal triangulation, that is, for any manifold there exist some finite-sheeted covering having ideal triangulations. Moreover, recently D. Futer, E. Hamilton and N. Hoffman proved ([16]) based upon the work of Luo, Schleimer and Tillman that, in fact, any manifold under our hypothesis has a finite-sheeted covering admitting infinitely many geometric ideal triangulations.

On the other hand, we have the algebraic approach. The deformation space of a manifold can be identified with a neighbourhood of the representation variety of the fundamental group (quotiented by the inner automorphisms) ([10]). To be more precise, let  $\Gamma = \pi_1(M^3)$  be the fundamental group of  $M^3$ , let  $G = \operatorname{Isom}(\mathbb{H}^3)$  be the group of isometries of hyperbolic



space, the representation variety is the set of homomorphisms  $\text{hom}(\Gamma, G)$ ; however, two representations in the same conjugacy class by the action of  $G$  correspond to the same structure, so we have to consider the quotient

$$\mathcal{R}(\Gamma, G) := \text{hom}(\Gamma, G)/G.$$

Now the question lies in how to parametrize this space. Based on work of M. Kapovich [31], M. Boileau and J. Porti show ([8, Thm. B.1.2]) that it is enough to fix in each peripheral tori a nontrivial loop and consider the trace of the image of the loop by the representation. This map is a bi-analytic isomorphism.

Both the combinatorial and algebraic approach are practically equivalent, a neighbourhood of hyperbolic structures is in one to one correspondence with a neighbourhood of  $\mathcal{R}(\Gamma, G)$ , whereas there is a two to one ramified covering from a neighbourhood of generalized Dehn filling coefficients to hyperbolic structures. As we stated before, deformations with generalized Dehn filling coefficients  $(p, q)$  and  $-(p, q)$  in some end correspond to the same structure. As we will see in the following result, in the non-orientable case, an interesting phenomenon happens regarding this equivalence.

**Theorem A.** Let  $M^3$  be a complete, non-orientable, hyperbolic 3-manifold of finite volume with  $l$  orientable ends and  $k$  non-orientable ones.

- (a) If  $M^3$  admits an ideal triangulation  $\Delta$ , then,  $\text{Def}(M^3, \Delta) \cong (-1, 1)^k \times B(1)^l$ , where  $B(1) \subset \mathbb{C}$  denotes the unit ball centered at 0, and where the parameters  $(\pm t_1, \dots, \pm t_{k+l}) \in (-1, 1)^k \times B(1)^l$  correspond to the same structure.
- (b) A neighborhood of the holonomy in  $\mathcal{R}(\pi_1(M^3), \text{Isom}(\mathbb{H}^3))$  is homeomorphic to  $(-1, 1)^k \times B(1)^l$ .

Furthermore, the holonomy map  $\text{Def}(M^3, \Delta) \rightarrow \mathcal{R}(\pi_1(M^3), \text{Isom}(\mathbb{H}^3))$  is a  $2^{k+l}$  branched covering on the image and folds each interval  $(-1, 1)$  at 0. Its image is the product of half-open intervals and open balls  $[0, 1)^k \times B(1)^l$ , where  $(0, \dots, 0)$  corresponds to the complete structure.

**Remark 1.** The previous branched covering is not exhaustive. In particular, there are deformations of the holonomy that cannot be achieved through deformations of the geometric ideal triangulation.

When we tackle the problem for the non-orientable case, we have two choices, either try to extend the ideas to some non-orientable manifold  $N^3$ , or consider the orientation covering  $N_+^3$ , where the known results apply, and also consider the group of deck transformations  $\mathbb{Z}_2$ . Then, hopefully we will be able to identify deformations of  $N^3$  with deformations of  $N_+^3$  fixed by the action of  $\mathbb{Z}_2$ . We opt for the latter one.

In the first place, when we take the approach through ideal triangulations, it is mainly a matter of finding the action of the covering transformation generating the cyclic group on the deformation space. The generalization of the compatibility equations is pretty straightforward and, from here, the identification between both the deformation space of  $N^3$  and the deformations of the orientation covering  $N_+^3$  fixed by the action of the covering transformation comes naturally. In order to find its local structure, we translate the action to generalized Dehn filling coefficients. For the one-cusped case with a peripheral Klein bottle, the fixed coefficients by the action are the ones of type  $(0, q)$ , which happens to be homeomorphic to a real line.

If we take into account the equivalence between deformations with parameters  $(p, q)$  and  $-(p, q)$  what we have is that these deformations coming from an ideal triangulation are homeomorphic to a half-open interval. Nonetheless, it is known that the deformation space of an orientable hyperbolic 3-manifold is smooth at the complete structure, and it can be proved that the same holds for a non-orientable one. Therefore, some structures cannot be achieved through deformations of an ideal triangulation. This leads us to also compute deformations by means of the representation variety.

The strategy we take in the representation variety is analogous to the one of the combinatorial method; there is an induced action of the covering transformation into the representation variety  $\mathcal{R}(\Gamma_+, G)$  where  $\Gamma_+ < \Gamma$  is

the fundamental group of  $N_+^3$  and this allows us to identify the representation variety of our interest  $\mathcal{R}(\Gamma, G)$  with the fixed points of  $\mathcal{R}(\Gamma_+, G)$ . Actually, as a technical step we have to consider  $\mathcal{R}(\Gamma_+, G_+)$ , where  $G_+ = \text{Isom}^+(\mathbb{H}^3)$ , the subgroup of orientation preserving isometries. The action can be translated through the trace map and we can identify a neighbourhood of  $\mathcal{R}(\Gamma, G)$  with the representations with real trace (when restricted to non-orientable ends). Thus, by means of deformations of the holonomy we can attain any possible deformation and we can see that the deformation space is indeed smooth.

The discussion of the previous paragraphs and the statement of Theorem A is illustrated in the Gieseking manifold, discovered by H. Gieseking in his doctoral Thesis ([18]) under the supervision of M. Dehn. The Gieseking manifold admits an ideal triangulation by one ideal tetrahedron, hence computing the deformation space with respect to the triangulation is quite straightforward. On the other hand, it also admits a fiber bundle structure, which in turn facilitates the computation of its variety of representations.

## The metric completion

A natural question when obtaining deformations is to consider its *completion*. For the orientable case, this can be found again in Thurston's Notes, and the resulting manifolds are the core of the proof of Thurston's hyperbolic Dehn surgery Theorem. In order to construct the completion on the non-orientable case, we have to understand first how the corresponding holonomy representation restrict to the peripheral Klein bottles related to the non-orientable ends. This is achieved by taking into account the square map

$$[A] \in \text{Isom}^-(\mathbb{H}^3) \mapsto A^2 \in \text{SL}(2, \mathbb{C}),$$

and the fact that the variety of representations of the torus is well-known. A neighbourhood of *orientation type preserving representations* of the Klein bottle, that is, representations mapping a loop to an element of  $\text{Isom}^+(\mathbb{H}^3)$  iff the loop is orientable, consists of three types of representations: parabolic,

type I and type II. In order to describe them, it is better to fix generators of the fundamental group of the Klein bottle,  $\pi_1(K^2)$ . The fundamental group  $\pi_1(K^2)$  admits a presentation

$$\pi_1(K^2) = \langle a, b \mid aba^{-1} = b^{-1} \rangle.$$

In terms of the generators  $a$  and  $b$  and writing the isometries as linear fractional transformations, we define  $\rho$  to be a *non-degenerate parabolic transformation* if, up to conjugation,  $\rho(a)(z) = \bar{z} + 1$ ,  $\rho(b)(z) = z + \tau i$ , for  $\tau \in \mathbb{R} \setminus \{0\}$ . A neighbourhood of a non-degenerate parabolic transformation consists of three types of representations:

**Proposition B.** Let  $\rho_0 \in \text{hom}(\pi_1(K^2), G)$  preserve the orientation type and be non-degenerate parabolic. Let  $\rho$  be in a small neighbourhood of  $\rho_0$ . Then, up to conjugation, one of the following holds:

- a)  $\rho(a)(z) = \bar{z} + 1$ ,  $\rho(b)(z) = z + \tau i$ , with  $\tau \in \mathbb{R}_{>0}$ .
- b)  $\rho(a)(z) = e^l \bar{z}$ ,  $\rho(b)(z) = e^{i\theta} z$ , with  $l \in \mathbb{R}_{\geq 0}$ ,  $\theta \in (0, \pi]$ .
- c)  $\rho(a)(z) = e^{i\theta} / \bar{z}$ ,  $\rho(b)(z) = e^l z$ , with  $l \in \mathbb{R}_{>0}$ ,  $\theta \in [0, \pi]$ .

A representation in case a) is called *parabolic*, in case b), *type I*, and in case c), *type II*.

The list is not exhaustive, however in a neighbourhood of a non-degenerate parabolic representation it is so. From a geometric point of view, both type I and II leave invariant a geodesic of  $\mathbb{H}^3$ , but in one of them, the reflection coming from a non-orientable element is with respect to a plane containing said invariant geodesic (type I), whereas in the other, the plane is orthogonal to the geodesic (type II). This dicotomy is at the core of the discrepancy between the algebraic and combinatorial approach when computing deformations: the gluing of ideal tetrahedra can only yield representations of type I. The proof that any representation in a neighbourhood falls in one of the three previous types comes from restricting representations to the

orientation covering and from here, compute the preimage through the restriction map (this is where the aforementioned square map plays a role).

Once we know how the deformations restrict to the peripheral Klein bottles, we can inspect the completion. There are some nuances when identifying holonomy representations and structures. For instance, we can consider an open subset of a non-complete manifold such that both have the same holonomy. This is circumvented by constructing a maximal structure which we call the *radial thickening*. The radial thickening happens to identify with the canonical structure for deformations coming from an ideal triangulation. According to the type of representations we have 3 possibilities.

**Theorem C.** For a deformation of the holonomy of  $M^3$ , the corresponding deformation of the metric can be chosen so that on a non-orientable end one of the following holds:

- It is a cusp, a metrically complete end, if the peripheral holonomy is parabolic.
- The metric completion is a solid Klein bottle with singular soul if the peripheral holonomy is of type I.
- The metric completion is a disc orbi-bundle with singular soul if the peripheral holonomy is of type II.

Regarding the third case, it is a pseudomanifold with two singular points: the endpoints of the singular soul. More precisely, a neighbourhood of each singular point is isometric to the metric cone on the projective plane  $P^2$ .

## Representation varieties of surfaces

The main motivation to study the variety of representations of the Klein bottle is to obtain the previous result on the completion of the deformed structures. Nonetheless, it is also an interesting object by itself. For this reason, several questions about the variety of representations are addressed. We should note first that the study of the square map leads to a classification

of the non-orientable isometries of  $\mathbb{H}^3$  up to conjugation. We are interested in the properties of the restriction map to the orientation covering. We consider the variety of orientation type preserving representation, which we denote by  $\mathcal{R}_+(\pi_1(K^2), G)$ , and prove that the restriction map

$$\text{res} : \mathcal{R}_+(\pi_1(K^2), G) \mapsto \mathcal{R}(\pi_1(T^2), G_+),$$

although not an homeomorphism (it is not even injective), is locally an homeomorphism around a parabolic representation. We also compute the homology and cohomology groups of the Klein bottle twisted by  $\mathfrak{sl}(2, \mathbb{C})$ , which is related to the smoothness of the variety of representations and its dimension. We prove that it is indeed smooth and it has real dimension 7. Finally, the connected components of the variety of representations are computed in a very straightforward way (we compute every possible representation).

**Proposition D.** The variety of representations  $\text{hom}(\pi_1(K^2), \text{Isom}(\mathbb{H}^3))$  has 8 connected components.

The connected components are distinguished according to the orientability behaviour of the image of the generators and whether the representation can be lifted to the universal cover or not.

Finally, we generalize the previous proposition on connected components for every closed non-orientable surface. This in turn is the non-orientable version of a result of W. Goldman ([20]):

**Theorem** (W. Goldman). *Let  $\Sigma_g$  be the orientable closed surface of genus  $g$ . Then, the variety of representations  $\text{hom}(\pi_1(\Sigma_g), \text{Isom}^+(\mathbb{H}^3))$  has exactly two connected components.*

In the same paper, W. Goldman consider the  $\text{PSL}(2, \mathbb{R})$  and  $\text{PSL}(2, \mathbb{C})$ -cases for orientable surfaces. Similar results have been obtained for other groups and orientable surfaces. However, the non-orientable case has not been inspected that much. Ho and Liu ([26], [27]) proved very general results for non-orientable surfaces but for connected groups. E. Xia ([46])

considered the group  $\mathrm{PGL}(2, \mathbb{R})$ , although for orientable surfaces. More closely related we find Palesi's papers, [39] and [40], where the  $\mathrm{PSL}(2, \mathbb{R})$  and  $\mathrm{PGL}(2, \mathbb{R})$ -cases are inspected for non-orientable surfaces. We obtain the following theorem in the  $\mathrm{Isom}(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C}) \rtimes \mathbb{Z}_2$ -case:

**Theorem E.** Let  $N_k$  be the closed non-orientable surface of genus  $k$ . The variety of representations  $\mathrm{hom}(\pi_1(N_k), \mathrm{Isom}(\mathbb{H}^3))$  has  $2^{k+1}$  connected components. In particular, the connected components are classified by the first and second Stiefel-Whitney class of the associated bundle.

We take an approach which follows closely W. Goldman's paper for the  $\mathrm{PSL}(2, \mathbb{C})$ -case. The main idea is that given a representation in a Lie group  $G$ , there is an associated flat  $G$ -bundle over the surface. The Stiefel-Whitney classes of said bundle are invariant over the connected components and, in fact, it is all that is needed to distinguish them. Moreover, they can be computed directly from the representation of the generators: the first Stiefel-Whitney class corresponds to the image of the generators being orientation preserving or not, whereas the second Stiefel-Whitney class corresponds to the lift of the relation between the generators to the universal cover  $\mathrm{SL}(2, \mathbb{C})$ . Then, the case of non-orientable genus 3 is computed by cutting along the Möbius strips in order to obtain a representation of a 3-punctured sphere and making use of Fricke-Klein Theorem. The general case is obtained by induction: the surface of genus  $k$  is the connected sum of the surface of genus  $k-2$  and the one of genus 2. The proofs of these results need several technical results on the possibility of lifting paths through the square map.

## Organization of the thesis

The thesis is organized as follows. It is divided in 5 chapters, all of them mostly self-contained, which, although leading to some repetition, we find to be a strong point of the structure. The first chapter is devoted to ideal triangulations and the Deformation Space with respect to an ideal triangulation. The second chapter, on the other hand, covers the approach through

the representation variety  $\mathcal{R}(N^3, G)$ . At the end of each of those chapters, the Gieseking manifold is used as an example of the results. On the third chapter all the topics we have already discussed on the representation variety of the Klein bottle are inspected, in particular, we obtain here the distinction in parabolic, type I and type II representations. Later on, in the fourth chapter, the results of the third chapter are applied in order to obtain the completion of the deformed ends. Finally, the fifth chapter is devoted to computing connected components of representations for closed surfaces.





# 1

## Deformation space from ideal triangulations

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Before discussing non-orientable manifolds, we recall first the orientable case. The computation of the deformation space from an ideal triangulation was first exemplified by Thurston in his notes [43] for the figure eight knot exterior, and the general case was constructed by Neumann and Zagier in [37].

From the point of view of a triangulation, the deformation of the hyperbolic structure on a manifold with a given *geometric ideal* triangulation is the space of parameters of ideal tetrahedra, subject to compatibility equations.

A *geometric ideal* tetrahedron is a geodesic tetrahedron of  $\mathbb{H}^3$  with all of its vertices in the ideal sphere,  $\partial_\infty\mathbb{H}^3$ . We say that a hyperbolic 3-manifold *admits a geometric ideal triangulation* if it is the union of such tetrahedra, glued along the geodesic faces. Though it has been established in many cases and for some time it seemed it could be derived from Epstein-Penner theorem ([13]), it is still an open problem to decide whether every orientable hyperbolic three-manifold of finite volume admits a geometric ideal triangulation. Some examples of manifolds admitting geometric ideal triangulations are two-bridge link exteriors and once-punctured torus bundles over the circle ([23], [42], [14]).

Given an ideal tetrahedron in  $\mathbb{H}^3$ , up to isometry we may assume that its ideal vertices in  $\partial_\infty\mathbb{H}^3 \cong \mathbb{C} \cup \{\infty\}$  are  $0$ ,  $1$ ,  $\infty$  and  $z \in \mathbb{C}$ , where  $\text{Im}(z) > 0$ . The idea of Thurston is to equip the (unoriented) edge between

0 and  $\infty$  with the complex number  $z$ , called *edge invariant*. The edge invariant determines the isometry class of the tetrahedron, and for different edges the corresponding invariants satisfy some relations, called *tetrahedron relations*:

- Opposite edges have the same invariant.
- Given 3 edges with a common end-point and invariants  $z_1, z_2, z_3$ , indexed following the right hand rule towards the common ideal vertex, they are related to  $z_1$  by  $z_2 = \frac{1}{1-z_1}$  and  $z_3 = \frac{z_1-1}{z_1}$ .

The edge invariant may also be introduced by means of the *link* of a vertex. The link at a vertex  $v$ , denoted  $L(v)$  is defined as the intersection of a horosphere centered at  $v$  with the tetrahedron. The link  $L(v)$  is an Euclidean rectangle well-defined up to similarity. Its similarity class is characterized by the *vertex invariant* of any of its three vertices  $w_1, w_2, w_3$  (written using the right hand rule towards  $v$ ). The vertex invariant is defined by embedding the triangle in  $\mathbb{C}$  as the ratio  $z(w_i) = (w_{i+2} - w_i) / (w_{i+1} - w_i)$ , where the subindices are modulo 3. The correspondence between edge invariant and vertex invariant is that the vertex invariant of  $w_i$  is equal to the edge invariant of the edge containing  $w_i$ .

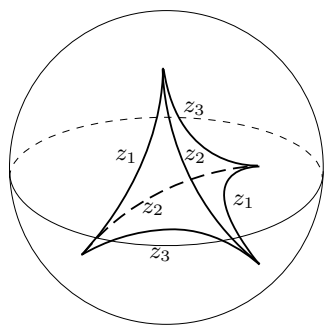


Figure 1.1: Ideal tetrahedron with edge invariants.

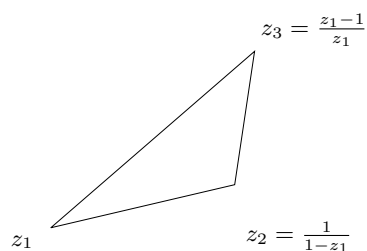


Figure 1.2: Tetrahedron relations.

Let  $M$  be a possibly non-orientable complete hyperbolic 3-manifold of finite volume, which admits a geometric ideal triangulation,  $\Delta = \{A_1, \dots, A_n\}$ . The usual parameterization of the triangulation goes as follows: we fix an

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edge  $e_i$  in each tetrahedron  $A_i$ , and consider its edge invariant,  $z_i$ . We will denote the parameters of the complete triangulation  $\{z_1^0, \dots, z_n^0\}$ . The *deformation space of  $M$  with respect to  $\Delta$* ,  $\text{Def}(M, \Delta)$ , is defined as the set of parameters  $\{z_1, \dots, z_n\}$  in a small enough neighborhood of the complete structure for which the gluing bestows a hyperbolic structure on  $M$ . However, we find that the equations defining the deformation space are easier to work with if we use  $3n$  parameters (one for each edge after taking into account the duplicity in opposite edges) and ask them to satisfy the second tetrahedron relation too.

When  $M$  is orientable, in order for the gluing to be geometric it is necessary and sufficient that around each edge cycle  $[e] = \{e_{i_1, j_1}, \dots, e_{i_m, j_m}\}$  the following two compatibility conditions are satisfied:

$$\prod_{l=1}^m z(e_{i_l, j_l}) = 1, \quad (1.1)$$

$$\sum_{l=1}^m \arg(z(e_{i_l, j_l})) = 2\pi. \quad (1.2)$$

The parameters which correspond to the complete hyperbolic structure are denoted by  $\{z^0(e_{1,1}), \dots, z^0(e_{n,3})\}$ . In a small enough neighborhood of  $\{z^0(e_{1,1}), \dots, z^0(e_{n,3})\}$ , fulfillment of (1.1) implies (1.2).

Motivation for equations (1.1) and (1.2) can be found in figure 1.3, which intuitively expresses that in order for a geometric structure to exist around an edge, the triangulation must close out around it forming a  $2\pi$  angle without any shearing.

For reference, in terms of  $n$  parameters  $z_i$  the compatibility conditions are laid out as ([37]):

$$\prod_{\nu=1}^n z_\nu^{r'_{j\nu}} (1 - z_\nu)^{r''_{j\nu}} = \pm 1 \quad (j = 1, \dots, n), \quad (1.3)$$

where  $\nu$  runs through all of the tetrahedra,  $j$  does so through every edge cycle (whose number equals the one of tetrahedra due to an Euler characteristic argument), and  $(r'_{j\nu}, r''_{j\nu})$  is a sum of  $(1, 0)$ ,  $(-1, 1)$ ,  $(0, -1)$  or  $(0, 0)$ . This sum depends on the edge cycle containing the edge with parameter  $z_\nu$ ,

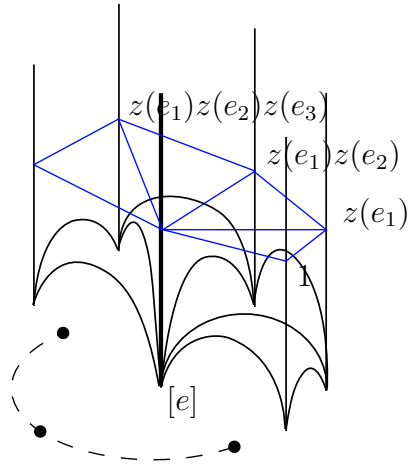


Figure 1.3: Triangulation around the edge cycle  $[e]$ .

the one with parameter  $\frac{z_\nu-1}{z_\nu}$ , the one with  $\frac{1}{1-z_\nu}$  or not having any edge of the tetrahedron  $A_\nu$  at all, respectively.

We end the overview of the orientable case with the theorem we want to extend to the non-orientable case:

**Theorem 1.0.1** (W. Thurston [43], W. Neumann - D. Zagier [37]). *Let  $M$  be connected, oriented, hyperbolic, of finite volume with  $l$  cusps. Then  $\text{Def}(M, \Delta)$  is bi-holomorphic to an open set of  $\mathbb{C}^l$ .*

The proof of the theorem comes from finding an intrinsic parametrization of the deformation space, as the one we have defined depends on the particular ideal triangulation. To any hyperbolic 3-manifold  $M^3$ , there is an associated *holonomy representation*:

$$\text{hol} : \pi_1(M^3) \rightarrow \text{Isom}(\mathbb{H}^3).$$

In particular, we can restrict the holonomy representation to the cusps (or more generally, ends) of  $M^3$ . Thus, assuming  $M^3$  orientable, we have for each end, a representation

$$\text{hol} : \pi_1(T^2) \rightarrow \text{Isom}(\mathbb{H}^3).$$

By conjugating so that the geodesic invariant by the holonomy is the one going from 0 to  $\infty$  in the upper half space model, there is a well-defined derivative of the holonomy:

$$\text{hol}' : \pi_1(T^2) \rightarrow \mathbb{C}^*.$$

Let us fix a longitude-meridian pair in each end,  $l_i$  and  $m_i$ . The map

$$\begin{aligned} \text{Def}(M^3, \Delta) &\longrightarrow (\mathbb{C}^*)^l \\ (z_{i,j}) &\longmapsto (\text{hol}'(l_i)) \end{aligned}$$

is the bi-holomorphism in Theorem 1.0.1. The same holds when we define the map with the meridians. When the manifold admits an ideal triangulation, the derivative of the holonomy can be computed by means of the edge parameters and it is a fractional polynomial in terms of them.

Furthermore, Neumann and Zagier show that we can consider the *generalized Dehn filling coefficients* associated to a deformation,  $(p_i, q_i) \in (\mathbb{R}^2 \cup \{\infty\})^l$ , and the map from a deformation to its Dehn coefficients is a homeomorphism. The generalized Dehn filling coefficients are related to Thurston's hyperbolic Dehn filling Theorem ([43]).

We will explore in Section 1.2 the interplay between the covering transformations group of the orientation covering and the derivative of the holonomy and the generalized Dehn filling coefficients, which leads to our main result of the chapter in Theorem 1.2.16.

## 1.1 The Deformation Space of a non-orientable manifold

When we deal with non-orientable manifolds, again the problem of the gluing being geometric lives within a neighborhood of the edges.

**Lemma 1.1.1.** *Let  $e$  be an edge of an ideal tetrahedron  $A \subset \mathbb{H}^3$  with edge invariant  $z(e)$ , and let  $\iota$  be a non orientable isometry of  $\mathbb{H}^3$ . The edge invariant of the edge  $\iota(e)$  in  $\iota(A)$  is  $1/\overline{z(e)}$ .*

*Proof.* Let us first notice that the edge invariant is preserved under orientation preserving isometries. Thus, by writing  $\iota$  as the composition  $\iota = cg$  with  $g \in \text{Isom}^+(\mathbb{H}^3)$ ,  $c \in \text{Isom}^-(\mathbb{H}^3)$ , we can assume without loss of generality that  $\iota = c$ , where  $c$  denotes the Poincaré extension of the complex conjugation in the upper-space model, i.e., the reflection on a fixed hyperplane of  $\mathbb{H}^3$ . Moreover, by applying another orientation preserving isometry, we may also assume that the tetrahedron  $A$  has vertices  $0, 1, z(e)$  and  $\infty$ .

Under these assumptions, the tetrahedron  $\iota(A) = c(A)$  has vertices  $0, \overline{z(e)}, 1$  and  $\infty$ , and the edge  $e$  is fixed by  $\iota$ ,  $\iota(e) = c(e) = e$ . From here, it is clear that the edge invariant of  $z(\iota(e)) \subset \iota(A)$  is

$$z(\iota(e)) = 1/\overline{z(e)}.$$

□

**Proposition 1.1.2.** *Let  $N$  be a non-orientable manifold triangulated by a finite number of ideal tetrahedra  $A_i$ . The triangulation bestows a hyperbolic structure around the edge cycle  $[e] = \{e_{i_1, j_1}, \dots, e_{i_n, j_n}\}$  if and only if the following compatibility equations are satisfied :*

$$\prod_{l=1}^n \frac{z(e_{i_l, j_l})^{\epsilon_l}}{z(e_{i_l, j_l})^{1-\epsilon_l}} = 1, \quad (1.4)$$

$$\sum_{l=1}^n \arg(z(e_{i_l, j_l})) = 2\pi, \quad (1.5)$$

where  $z(e_{i_l, j_l})$  is the edge invariant of  $e_{i_l, j_l}$ , and  $\epsilon_l = 0, 1$  in such a way that, in the gluing around the edge cycle  $[e]$ , a coherent orientation of the tetrahedra is obtained by gluing a copy of  $A_{i_l}$  with its orientation reversed if  $\epsilon_l = 0$ , (or kept the original one if  $\epsilon_l = 1$ ), and with the initial condition that the orientation of the tetrahedron  $A_{i_1}$  is kept as given.

*Proof.* When we follow a cycle of side identifications around an edge, we can always reorient the tetrahedra (maybe more than once) so that the gluing is done by orientation preserving isometries. The compatibility equations for the orientable case can be then applied and, hence, for the neighborhood of

the edge cycle to inherit a hyperbolic structure, Equations (1.1) and (1.2) must be satisfied, with the corresponding edge invariants.

Now, let us consider an edge  $e_{i,j} \in A_i$  with parameter  $z(e_{i,j})$ . We can apply Lemma 1.1.1 to conclude that the edge invariant of  $\iota(e_{i,j}) \subset \iota(A_i)$  is  $1/\overline{z(e_{i,j})}$ . Notice that  $\arg(\overline{z}^{-1}) = \arg(z)$  and, thus, the proposition follows with ease after changing the orientation of some tetrahedra.  $\square$

**Definition 1.1.3.** *Let  $N$  be a connected, complete, non-orientable, hyperbolic 3-manifold of finite volume. Let  $\Delta$  be an ideal triangulation of  $N$ . The deformation space of  $N$  related to the triangulation  $\Delta$  is*

$$\text{Def}(N, \Delta) = \{ (z_{1,1}, \dots, z_{n,3}) \in U \cap \mathbb{C}^{3n} \text{ satisfying the compatibility} \\ \text{Equations (1.4) and (1.5) and the tetrahedron relations} \},$$

where  $U$  is a small enough neighborhood of the parameters  $(z_{i,j}^0)$  of the complete structure.

**Remark 1.1.4.** As in the orientable case, in a small enough neighbourhood  $U$ , fulfillment of Equation (1.4) implies (1.5).

Let  $\hat{N}$  be the orientation covering of  $N$ . The ideal triangulation on  $N$ ,  $\Delta$ , can be lifted to an ideal triangulation  $\hat{\Delta}$  on  $\hat{N}$ . There is an orientation reversing homeomorphism,  $\iota$ , acting on  $\hat{N}$  such that  $N = \hat{N}/\iota$  and  $\iota^2 = \text{Id}$ , that is,  $\iota$  is a covering transformation. The triangulation on  $\hat{N}$  is constructed in the usual way: for every tetrahedron  $A_i$  we take another tetrahedron with the opposite orientation,  $\overline{A}_i$ , and re-glue all the tetrahedra so that the orientation is coherent. Then, the action of the covering transformation on the triangulation permutes them, that is,  $\iota(A_i) = \overline{A}_i$ . For every edge,  $e_{i,j} \in A_i$ , let  $z(e_{i,j})$  or  $z_{i,j}$  denote its edge invariant. Analogously,  $w(\iota(e_{i,j}))$  or  $w_{i,j}$  will denote the edge invariant of  $\iota(e_{i,j}) \in \iota(A_i)$ .

**Remark 1.1.5.** The compatibility equations (1.4) and (1.5) around  $[e] \in N$  are precisely the (orientable) compatibility equations in any lift of  $[e]$  to the orientation covering. To see this, we only have to realize that the choice of  $\epsilon_l$  in proposition 1.1.2 actually describes how to orient the tetrahedra



coherently around an edge cycle, which is how they are oriented in one of the two lifts of the aforementioned edge cycle to the orientation covering. There is still a nuance about the chosen lift of the edge, but Equations (1.4) and (1.5) are equivalent in both lifts.

The orientation reversing homeomorphism acts on  $\text{Def}(\hat{N}, \hat{\Delta})$  by pulling-back (equivalently, pushing-forward) the associated hyperbolic metric on each tetrahedron. Combinatorially, the action is described in the following lemma:

**Lemma 1.1.6.** *Let  $N = \hat{N}/\iota$ , where  $\iota$  is an orientation reversing homeomorphism. Let  $N$  admit an ideal triangulation  $\Delta$ . Then,  $\iota$  acts on  $\text{Def}(\hat{N}, \hat{\Delta})$  as*

$$\iota_*((z_{i,j}, w_{i,j})) = \left( \frac{1}{w_{i,j}}, \frac{1}{z_{i,j}} \right). \quad (1.6)$$

*Proof.* The proof follows easily from the fact that  $\iota$  permutes the edges and, for  $e_{i,j} \in A_i$  with invariant  $z(e_{i,j})$ , the edge invariant of  $\iota(e_{i,j}) \in \iota(A_i)$  is  $1/\overline{z(e_{i,j})}$ , by Lemma 1.1.1. This describes the push-forward of the metric. The action is well-defined, for  $(z_{i,j}, w_{i,j}) \in \text{Def}(\hat{N}, \hat{\Delta})$ , Equation (1.4) at edge cycles  $[e]$  and  $[\iota(e)]$  is, respectively,

$$\prod_{k \in I_1} z_{i_k, j_k} \prod_{l \in I_2} w_{i_l, j_l} = 1, \quad \prod_{k \in I_1} w_{i_k, j_k} \prod_{l \in I_2} z_{i_l, j_l} = 1.$$

Hence, for  $\iota_*((z_{i,j}, w_{i,j}))$ , equation (1.4) at  $[e]$  is

$$\prod_{k \in I_1} \overline{w_{i_k, j_k}}^{-1} \prod_{l \in I_2} \overline{z_{i_l, j_l}}^{-1} = 1,$$

which is equivalent to the compatibility equation at  $[\iota(e)]$ . Therefore,  $\iota_*((z_{i,j}, w_{i,j}))$  satisfies equations (1.4) and (1.5). There is a small nuance to be noted: the order in the gluing is different around  $[e]$  and  $[\iota(e)]$  due to the change of orientation, however this is feature does not appear in the equations due to the commutativity of the multiplication.

As an action of  $\mathbb{C}^{3n}$ ,  $\iota_*$  is a homeomorphism, and a neighbourhood  $U$  can be chosen so that the image  $\iota_*((z_{i,j}, w_{i,j})) = \left( \frac{1}{w_{i,j}}, \frac{1}{z_{i,j}} \right)$  belongs to  $\text{Def}(\hat{N}, \hat{\Delta})$ .

Take for instance any neighbourhood  $U_0$  of the complete structure and define  $U := U_0 \cap \iota_*(U_0)$ .  $\square$

**Remark 1.1.7.** Metrics on tetrahedra are considered up to isotopy.

We will denote the subset of fixed points of  $\text{Def}(\hat{N}, \hat{\Delta})$  under the action of  $\iota_*$  as  $\text{Def}(\hat{N}, \hat{\Delta})^\iota$ .

**Corollary 1.1.8.** *The map  $(z_{i,j}) \in \text{Def}(N, \Delta) \mapsto (z_{i,j}, 1/\overline{z_{i,j}}) \in \text{Def}(\hat{N}, \hat{\Delta})^\iota$  is a real analytic isomorphism.*

*Proof.* Remark 1.1.5 shows that the map goes to  $\text{Def}(\hat{N}, \hat{\Delta})$  and Lemma 1.1.6, that it is a fixed point under  $\iota_*$ . The inverse of the map exists again due to Remark 1.1.5. The fact that it is a real analytic isomorphism is clear from the definition.  $\square$

**Remark 1.1.9.** Before ending the section there is a last small observation to be made. Let  $A$  be some tetrahedron of the triangulation  $\hat{\Delta}$  and three edges  $e_1, e_2, e_3$  with a common ideal vertex  $v$  indexed following the right hand rule as in the second tetrahedron relation. The corresponding edges at  $\iota(A), \iota(e_1), \iota(e_2), \iota(e_3)$ , no longer follow the right hand rule and we have to take into account that what satisfies the tetrahedron relation is  $w(\iota(e_3)), w(\iota(e_2)), w(\iota(e_1))$ .

## 1.2 The dimension of the Deformation Space $\text{Def}(N, \Delta)$

Our goal is to use Corollary 1.1.8 and Theorem 1.0.1 in order to identify the deformation space of  $N$  with the fixed points under an action on  $\mathbb{C}^k$ . Let us suppose for the time being that  $N$  has only one cusp which is non-orientable. The section of this cusp must be a Klein bottle. In order to define the bi-holomorphism via the holonomy representation we must first fix a longitude-meridian pair in the peripheral torus in the orientation covering  $\hat{N}$ . As we will see, there is a canonical choice. Afterwards, we will

compute the derivative of the holonomy,  $\text{hol}'$ , and translate the action of  $\iota$  over there and finally, to the generalized Dehn filling coefficients.

**Fixing a longitude-meridian pair.** Let  $K^2$  be Klein bottle, its fundamental group admits a presentation

$$\pi_1(K^2) = \langle a, b \mid aba^{-1} = b^{-1} \rangle.$$

The elements  $a^2, b$  in the orientation covering  $T^2$  are generators of  $\pi_1(T^2)$  and are represented by the unique homotopy classes of loops in the orientation covering that are invariant by the deck transformation (as unoriented curves). From now on, we will choose as longitude-meridian pair the elements:

$$\begin{aligned} l &:= a^2, \\ m &:= b. \end{aligned}$$

**Definition 1.2.1.** *The previous generators of  $\pi_1(T^2)$  are called distinguished elements.*

Let  $\iota$  denote the involution on  $T^2$  obtained as a restriction of the involution on  $N$ . The deck group of the covering  $T^2 \rightarrow K^2$  is  $\mathbb{Z}_2$ , thus the action of  $\iota$  on  $T^2$  is determined. It can be identified with the conjugation by the loop  $a$ , which is

$$\iota_*(l) = l, \tag{1.7}$$

$$\iota_*(m) = m^{-1}. \tag{1.8}$$

**Lemma 1.2.2.** *Let  $[\alpha] \in \pi_1(T)$ , let  $\iota$  be the involution in the orientation covering  $\hat{N}$ , that is,  $N = \hat{N}/\iota$ . We also denote by  $\iota$  the restriction of  $\iota$  to the peripheral torus  $T$ . If*

$$\text{hol}'(\alpha) = \prod_{r \in I} z(e_{i_r, j_r})^{\epsilon_r} \prod_{s \in J} w(\iota(e_{i_s, j_s}))^{\epsilon_s},$$

1.2. DIMENSION OF THE DEFORMATION SPACE  $\text{DEF}(N, \Delta)$

where  $\epsilon_r, \epsilon_s \in \{\pm 1\}$ , then

$$\text{hol}'(\iota(\alpha)) = \prod_{r \in I} w(\iota(e_{i_r, j_r}))^{-\epsilon_r} \prod_{s \in J} z(e_{i_s, j_s})^{-\epsilon_s}.$$

*Proof.* We use Thurston's method for computing the holonomy through the developing of triangles in  $\mathbb{C}$  (see [43]). A representant of  $[\alpha]$  can be taken so that it crosses the boundary of tetrahedra at faces. Moreover, we can assume that when it enters inside a tetrahedron it exists through a different face. Otherwise, we can push out the loop from the tetrahedron and take another representant that does not go into it. This way, the loop goes through tetrahedra a finite number of times and each time it does so, it crosses two faces with a common edge. The product of the edge invariants of the previous edges computes the derivative of the holonomy  $\text{hol}'(\alpha)$ .

In order to see the behaviour of the holonomy under the action of  $\iota$ , we only need to observe what happens to a piece of the loop  $\alpha$  when it passes through a tetrahedron. If a piece of loop contributes to the holonomy in a factor of  $z_{i,j} = z(e_{i,j})$ , we can assume as usual that the tetrahedron has vertices  $0, 1, z_{i,j}$  and  $\infty$ , with the isolated edge is the geodesic with ideal endpoints  $0$  and  $\infty$ , and that  $\iota = c$ . Then, as shown in Figure 1.4, the corresponding piece of  $\iota(\alpha)$  contributes in a factor  $w_{i,j}^{-1} = w(\iota(e_{i,j}))^{-1}$ . The same occurs if the piece of  $\alpha$  adds a factor of  $w_{i,j}$ .  $\square$

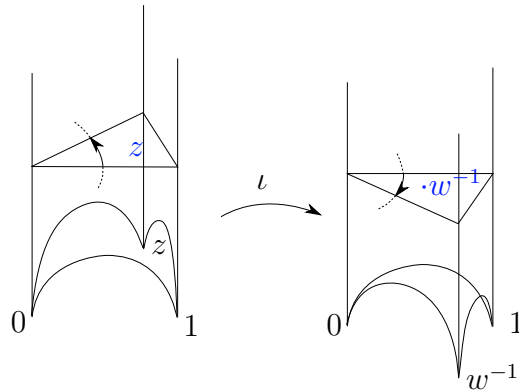


Figure 1.4: Change under the action of  $\iota$ .

**Remark 1.2.3.** Lemma 1.2.2 displays the fact that if we write a loop  $\alpha$  as a concatenation of paths  $\alpha = \alpha_1 * \alpha_2 * \cdots * \alpha_n$  ( $*$  denoting the concatenation), then the derivative of the holonomy of  $\alpha$ , can be computed as a product of *partial holonomies*

$$\text{hol}'(\alpha) = \text{hol}'(\alpha_1) \cdots \text{hol}'(\alpha_n).$$

In order to make sense of this partial holonomy we have to ask each path  $\alpha_i$  to start and end at faces of the triangulation. It is clear that if this happen the partial-holonomy is well-defined up to homotopy. Moreover, Lemma 1.2.2 can be applied to the partial holonomy too.

**Lemma 1.2.4.** *The derivative of the holonomy of the chosen longitude-meridian pair has the following features:*

$$\text{hol}'(m) = \prod_{r \in I_m} z(e_{i_r, j_r})^{\epsilon_r} \prod_{s \in J_m} w(\iota(e_{i_s, j_s}))^{\epsilon_s} = \prod_{r \in I_m} w(\iota(e_{i_r, j_r}))^{\epsilon_r} \prod_{s \in J_m} z(e_{i_s, j_s})^{\epsilon_s}, \quad (1.9)$$

$$\text{hol}'(l) = \prod_{r \in I_l} (z(e_{i_r, j_r}) w(\iota(e_{i_r, j_r}))^{-1})^{\epsilon_r}. \quad (1.10)$$

*Proof.* This is a consequence of previous Lemma 1.2.2. First, let us recall that the action of  $\iota$  on the longitude-meridian pair is (1.7)

$$\iota_*(l) = l, \quad \iota_*(m) = m^{-1}.$$

In general, the holonomy of any loop  $\alpha$  can be written down as

$$\text{hol}'(\alpha) = \prod_{r \in I} z(e_{i_r, j_r})^{\epsilon_r} \prod_{s \in J} w(\iota(e_{i_s, j_s}))^{\epsilon_s}.$$

Thus, from  $\iota_*(m) = m^{-1}$ , we obtain

$$\text{hol}'(\iota(m)) = \text{hol}'(m)^{-1}.$$

The first assertion follows after applying Lemma 1.2.2 to the expression of  $\text{hol}'(m)$ .

Regarding the holonomy of the longitude, we have to notice that  $l = a^2$  is short-hand for  $\hat{a} * \iota(\hat{a})$ , where  $\hat{a}$  is any lift of the loop  $a$  to  $T^2$ . As noted in Remark 1.2.3, we can compute the holonomy of  $l$  as the product of two partial holonomies

$$\text{hol}'(l) = \text{hol}'(a)\text{hol}'(\iota(a)),$$

and applying Lemma 1.2.2 to the second factor we obtain the second assertion. □

**Proposition 1.2.5.** *For the chosen longitude-meridian pair, the action of  $\iota$  on  $\text{Im}(\text{hol}') \subset \mathbb{C}^2$  is*

$$\iota_*(L, M) = (\bar{L}, \bar{M}^{-1}), \quad (1.11)$$

where  $L = \text{hol}'(l)$ ,  $M = \text{hol}'(m)$ .

*Proof.* Let  $(z_{i,j}, w_{i,j}) \in \text{Def}(\hat{N}, \hat{\Delta})$  and let us write the derivate of holonomy of this structure as

$$\text{hol}'_{(z_{i,j}, w_{i,j})}$$

in order to highlight the dependance on the parameter  $(z_{i,j}, w_{i,j})$ . For any loop  $\alpha$  in  $\hat{N}$ , the action of  $\iota_*$  on  $\text{Def}(\hat{N}, \hat{\Delta})$  is translated to  $\text{hol}'(\alpha)$  as

$$\iota_*(\text{hol}'_{(z_{i,j}, w_{i,j})}(\alpha)) = \text{hol}'_{\iota_*(z_{i,j}, w_{i,j})}(\alpha) = \text{hol}'_{(\frac{1}{w_{i,j}}, \frac{1}{z_{i,j}})}(\alpha).$$

In the meridian case,

$$\iota_*(\text{hol}'_{(z_{i,j}, w_{i,j})}(m)) = \text{hol}'_{(1/\overline{w_{i,j}}, 1/\overline{z_{i,j}})}(m) = \prod_{r \in I_m} \overline{w_{i_r, j_r}}^{-\epsilon_r} \prod_{s \in J_m} \overline{z_{i_s, j_s}}^{-\epsilon_s},$$

with  $\epsilon_r, \epsilon_s \in \{\pm 1\}$ , which by Lemma 1.2.4 is equal to  $\overline{\text{hol}'(m)}^{-1}$ . Similarly, by Lemma 1.2.4, we can assume

$$\text{hol}'_{(z_{i,j}, w_{i,j})}(l) = \prod_{r \in I_l} (z_{i_r, j_r} w_{i_r, j_r}^{-1})^{\epsilon_r},$$

and by Lemma 1.2.2 we obtain

$$\iota_*(\text{hol}'_{(z_{i,j}, w_{i,j})}(l)) = \text{hol}'_{(1/\overline{w_{i,j}}, 1/\overline{z_{i,j}})}(l) = \prod_{r \in I_l} (\overline{z_{i_r, j_r} w_{i_r, j_r}}^{-1})^{\epsilon_r} = \overline{\text{hol}'_{(z_{i,j}, w_{i,j})}(l)}.$$

□

**Remark 1.2.6.** Following the notation of Proposition 1.2.5,  $(L, M) \in (\mathbb{C}^2)^\iota$  if and only if  $L \in \mathbb{R}$ ,  $|M| = 1$ .

Let us denote by  $u := \log \text{hol}'(l)$ ,  $v := \log \text{hol}'(m)$ .

**Proposition 1.2.7.** *Let  $N$  be a one cusped, non-orientable, complete, connected hyperbolic 3-manifold with horospherical Klein bottle. Let  $N$  admit a geometric ideal triangulation  $\Delta$ . Then, its deformation space is real bi-analytic to an open subset of  $\mathbb{R}$ .*

*Proof.* Thurston and Neumann-Zagier Theorem 1.0.1 prove that the map

$$\begin{aligned} \text{Def}(\hat{N}, \hat{\Delta}) &\longrightarrow U \subset \mathbb{C} \\ (z_{i,j}, w_{i,j}) &\longmapsto u = \log \text{hol}'(l) \end{aligned}$$

is a biholomorphism onto an open subset  $U$  of  $\mathbb{C}$ . Therefore, by Corollary 1.1.8, the map

$$\text{Def}(N, \Delta) \longrightarrow U^\iota \subset \mathbb{C}$$

is real bi-analytic to  $U^\iota$ . Finally, Remark 1.2.6 shows that  $U^\iota$  is an open subset of  $\mathbb{R} \times \{0\} \subset \mathbb{C}$ .

The same can be proved considering the meridian instead. □

The *generalized Dehn coefficients* are the solutions in  $\mathbb{R}^2 \cup \{\infty\}$  to *Thurston's equation*

$$pu + qv = 2\pi i. \tag{1.12}$$

Indeed, Thurston and Neumann-Zagier Theorem 1.0.1 states that, for  $M^3$  orientable, the map  $(z_{i,j}) \in \text{Def}(M^3, \Delta) \mapsto (p_k, q_k)$  is a homeomorphism onto its image, which is an open neighborhood of  $(\infty, \dots, \infty) \in \overline{\mathbb{C}}^l$ , where  $l$  is the number of cusps of  $M^3$ .

**Remark 1.2.8.** As noted in [37], for an orientable manifold with  $n$  cusps, for deformations  $(u_i, v_i) \in \mathbb{C}^{2n}$  small enough, the solution  $(p_i, q_i) \in (\mathbb{R}^2 \cup \{\infty\})^n$  to Thurston's equation (1.12) is unique.

**Proposition 1.2.9.** *The action of  $\iota$  on  $(p, q) \in U \cap \mathbb{R}^2 \cup \{\infty\}$ , where  $(p, q)$  are the generalized Dehn coefficients, is*

$$\iota_*(p, q) = (-p, q). \quad (1.13)$$

*Proof.* The action of  $\iota$  can be translated through the logarithm to  $(u, v)$  from the action on the holonomy (1.11) as

$$\iota_*(u, v) = (\log \iota_*(L), \log \iota_*(M)) = (\log \bar{L}, \log \bar{M}^{-1}) = (\bar{u}, -\bar{v}),$$

where  $L = \text{hol}'(l)$ ,  $M = \text{hol}'(m)$ . Then, to find the action on generalized Dehn coefficients, we have to solve Thurston's equation (1.12) with  $\bar{u}$  and  $-\bar{v}$ , that is,

$$p'\bar{u} - q'\bar{v} = 2\pi i, \quad (1.14)$$

where  $(p', q') \in \mathbb{R}^2 \cup \{\infty\}$ . The solution  $(p', q')$  will be the result of the action of  $\iota$  on  $(p, q)$ , that is,

$$\iota_*(p, q) = (p', q').$$

From here, it is straightforward to check that  $(p', q') = (-p, q)$  is a solution. By continuity of the action of  $\iota_*$  on deformations,  $\iota_*(u, v)$  is a small deformation, so Remark 1.2.8 applies and, hence, the solution is unique.  $\square$

**Remark 1.2.10.** In [43], Thurston draws a fixed-point free orientation reversing involution on the figure eight knot exterior (see Figure 1.5) whose action on the Dehn coefficients is  $(p, q) \mapsto (-p, q)$ . This action is the deck transformation on the figure eight knot exterior as the orientation covering of the Gieseking manifold.

**Corollary 1.2.11.** *The fixed points under  $\iota$ , which are in correspondence with  $\text{Def}(N, \Delta)$ , are those whose generalized Dehn filling coefficients are of*



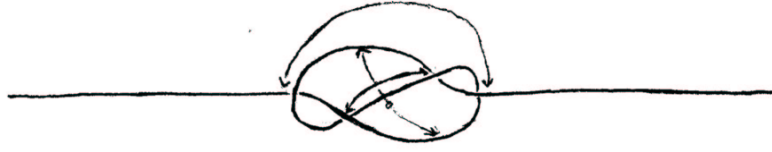


Figure 1.5: Involution on the figure eight knot exterior. Image from Thurston's Notes [43].

type  $(0, q)$ .

Before proving the main theorem of this chapter, we have to show how the involution  $\iota$  behaves on orientable ends. Although it may seem unintuitive, examples of nonorientable hyperbolic 3-manifolds of finite volume with one cusp which is orientable can be easily be constructed. A way to construct an example is the following one: start from a closed nonorientable manifold (for instance, take one from [29]) and choose a orientation reversing loop in it. By Cartan's Theorem on closed geodesics (see [12]), there exists a geodesic free homotopic to the chosen loop. By removing said geodesic, we obtain a manifold with one peripheral torus.

Due to the fact that the cusp  $T^2 \times [0, \infty)$  is orientable in  $N$ , when we consider the orientation covering  $\hat{N}$ , it will correspond with two cusps  $T_1^2 \times [0, \infty)$  and  $T_2^2 \times [0, \infty)$ . The involution  $\iota$  permutes the peripheral tori

$$\iota : T_1^2 \rightarrow T_2^2.$$

**Fixing longitude-meridian pairs in the peripheral tori.** Let us fix any longitude-meridian pair in one of the peripheral tori,  $l_1, m_1 \in \pi_1(T_1)$ . We then choose the following longitude-meridian pair in  $T_2$ :

$$l_2 := \iota_*(l_1), \quad m_2 := \iota_*(m_1) \in \pi_1(T_2).$$

**Proposition 1.2.12.** *For the chosen longitude-meridian pairs for two peripheral tori, the action of  $\iota$  on  $\text{Im}(\text{hol}') \subset \mathbb{C}^4$  is*

$$\iota_*(L_1, M_1, L_2, M_2) = (\overline{L_2}, \overline{M_2}, \overline{L_1}, \overline{M_1}), \quad (1.15)$$

where  $L_i = \text{hol}'(l_i)$ ,  $M_i = \text{hol}'(m_i)$ ,  $i = 1, 2$ .

*Proof.* We only need to apply Lemma 1.2.2 to our choice of longitude-meridian pairs. For instance, if

$$\text{hol}'(l_1) = \prod_{r \in I} z_{i_r, j_r}^{\epsilon_r} \prod_{s \in J} w_{i_s, j_s}^{\epsilon_s},$$

with  $\epsilon_r, \epsilon_s \in \{\pm 1\}$ , then

$$\text{hol}'(l_2) = \text{hol}'(\iota(l_1)) = \prod_{r \in I} w_{i_r, j_r}^{-\epsilon_r} \prod_{s \in J} z_{i_s, j_s}^{-\epsilon_s}.$$

Thus,

$$\iota_*(\text{hol}'(l_1)) = \prod_{r \in I} \overline{w_{i_r, j_r}}^{-\epsilon_r} \prod_{s \in J} \overline{z_{i_s, j_s}}^{-\epsilon_s} = \overline{\text{hol}'(l_2)}.$$

□

**Proposition 1.2.13.** *Let  $N$  be a one cusped, non-orientable, complete, connected hyperbolic 3-manifold with horospherical torus. Let  $N$  admit a geometric ideal triangulation  $\Delta$ . Then, its deformation space is real bi-analytic to an open subset of  $\mathbb{C}$ .*

*Proof.* The proof is analogous to the one of Proposition 1.2.7 by taking into account that now the action of  $\iota$  is given by Proposition 1.2.12. □

**Proposition 1.2.14.** *The action of  $\iota$  on  $(p_1, q_1, p_2, q_2) \in U \cap (\mathbb{R}^2 \cup \{\infty\})^2$ , where  $(p_1, q_1, p_2, q_2)$  are the generalized Dehn coefficients, is*

$$\iota_*(p_1, q_1, p_2, q_2) = -(p_2, q_2, p_1, q_1) \quad (1.16)$$

*Proof.* Let  $u_i := \log \text{hol}'(l_i)$ ,  $v_i := \log \text{hol}'(m_i)$ , by proposition 1.2.14,  $\iota$  acts on  $(u_1, v_1, u_2, v_2)$  as

$$\iota_*(u_1, v_1, u_2, v_2) = (\overline{u_2}, \overline{v_2}, \overline{u_1}, \overline{v_2}).$$

Hence, it is a matter of straightforward verification that the unique solution to the new Thurston's equation is  $-(p_2, q_2, p_1, q_1)$ , so we conclude

$$\iota_*(p_1, q_1, p_2, q_2) = -(p_2, q_2, p_1, q_1). \quad \square$$

**Corollary 1.2.15.** *The fixed points under  $\iota$ , which are in correspondence with  $\text{Def}(N, \Delta)$ , are those whose generalized Dehn filling coefficients are of type  $(p, q, -p, -q)$ , with  $(p, q) \in \mathbb{R}^2 \cup \{\infty\}$ .*

We can now prove the general statement:

**Theorem 1.2.16.** *Let  $N$  be a connected, complete, non-orientable, hyperbolic 3-manifold of finite volume. Let  $N$  have  $k$  non-orientable cusps and  $l$  orientable ones and let it admit an ideal triangulation  $\Delta$ . Then  $\text{Def}(N, \Delta)$  is real bi-analytic to an open set of  $\mathbb{R}^{k+2l}$ .*

*Proof.* We have already proved the theorem for  $(k, l) = (1, 0)$  and  $(k, l) = (0, 1)$  in Propositions 1.2.7 and 1.2.13.

In general, the action of  $\iota$  on  $\text{Im}(\text{hol}') \subset \mathbb{C}^{k+2l}$  can be understood as a product of  $k + l$  actions,  $\iota_1 \times \cdots \times \iota_l$ , the first  $k$ ,  $\iota_i, i = 1, \dots, k$  acting on  $\mathbb{C}$  as in the case for a Klein bottle cusp, and the subsequent  $l$ ,  $\iota_j, j = k + 1, \dots, k + l$ , acting on  $\mathbb{C}^2$  as in the case for a peripheral torus.  $\square$

### 1.3 Example: The Gieseking manifold

The Gieseking manifold  $M$  is a non-orientable hyperbolic 3-manifold with finite volume and one cusp, with horospherical section a Klein bottle. It has an ideal triangulation with a single tetrahedron. The orientation cover of the Gieseking manifold is the figure eight knot exterior, and the ideal triangulation with one simplex lifts to Thurston's ideal triangulation with two ideal simplices in [43].

The Gieseking manifold  $M$  was constructed by Gieseking in 1912 in his doctoral thesis as a student of Dehn. Here we follow the description of [33], using the notation of [1]. Start with the regular ideal vertex  $\Delta$  in  $\mathbb{H}^3$ , with vertices  $\{0, 1, \infty, \frac{1-i\sqrt{3}}{2}\}$ , Figure 1.6. The side identifications are the non-orientable isometries defined by the Möbius transformations

$$U(z) = \frac{1}{1 + \frac{1+i\sqrt{3}}{2}z} \quad \text{and} \quad V(z) = -\frac{1+i\sqrt{3}}{2}\bar{z} + 1.$$

### 1.3. EXAMPLE: THE GIESEKING MANIFOLD

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The identifications of the faces are defined by their action on vertices:

$$U: \left(\frac{1-i\sqrt{3}}{2}, 0, \infty\right) \mapsto \left(\frac{1-i\sqrt{3}}{2}, 1, 0\right) \quad \text{and} \quad V: (1, 0, \infty) \mapsto \left(\frac{1-i\sqrt{3}}{2}, 1, \infty\right).$$

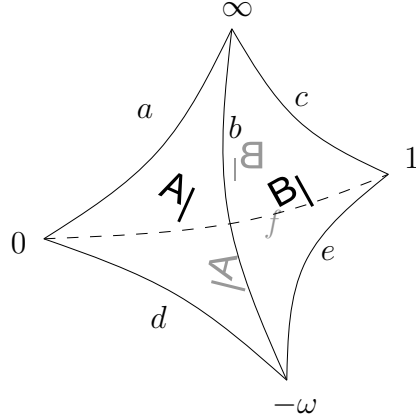


Figure 1.6: Gieseking Manifold with labelled edges.

By applying Poincaré's fundamental theorem

$$\pi_1(M) \cong \langle U, V \mid VU = U^2V^2 \rangle \quad (1.17)$$

The relation  $VU = U^2V^2$  corresponds to a cycle of length 6 around the edge.

We compute the deformation space of the triangulation with a single tetrahedron  $\text{Def}(M, \Delta)$ . We will do it first by checking when the pairing is proper and, afterwards, we will compare the result with our definition of  $\text{Def}(M, \Delta)$ .

For any ideal tetrahedron in  $\mathbb{H}_3$ , we set its ideal vertices in  $0, 1, \infty$  and  $-\omega$ , where  $\omega$  is in  $\mathbb{C}_+$ , the upper half-space of  $\mathbb{C}$ . The role played by  $-\omega$  will be the one of  $\frac{1-i\sqrt{3}}{2}$  in the complete structure. For any such  $\omega$  it is possible to glue the faces of the tetrahedron in the same pattern as in the Gieseking manifold via two orientation-reversing hyperbolic isometries, which we will call likewise  $U$  and  $V$ .

For the gluing to follow the same pattern, it must map  $U: (-\omega, 0, \infty) \mapsto$

$(-\omega, 1, 0)$  and  $V : (1, 0, \infty) \mapsto (-\omega, 1, \infty)$ . The orientation-reversing isometries  $U$  and  $V$  satisfying this are:

$$U(z) = \frac{1}{\frac{1+\omega}{|\omega|^2}\bar{z} + 1} \quad \text{and} \quad V(z) = -(1 + \omega)\bar{z} + 1.$$

Although it is always possible to glue the faces in the same pattern as in the Gieseking manifold, not for all them the gluing will have a hyperbolic structure.

We include the details of the computation here, for a completeness sake:

Let  $U = U'_c$ ,  $V = V'_c$  with  $U', V' \in \text{PSL}(2, \mathbb{C})$ ,  $c$  the Poincaré extension of the complex conjugation. Then,  $U' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and it must satisfy:

$$1. \ U'(\bar{0}) = 1 \Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{ which implies that } b = d.$$

$$2. \ U'(\infty) = 0 \Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ hence } a = 0.$$

$$3. \ U'(\overline{-\omega}) = -\omega \Leftrightarrow \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \begin{pmatrix} -\bar{\omega} \\ 1 \end{pmatrix} = \begin{pmatrix} -\omega \\ 1 \end{pmatrix} \Leftrightarrow \frac{b}{-c\bar{\omega}+d} = -\omega. \text{ Thus,}$$

$$c = b \frac{1 + \omega}{|\omega|^2}.$$

We do not bother about normalization, so we substitute  $b = 1$  in order to get the expression of  $U$  as the fractional linear transformation we wrote before.

Similarly, for  $V'$  we must ask  $V'(0) = 1$ ,  $V'(\infty) = \infty$ ,  $V'(1) = -\omega$ . These imply that

$$V' = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}, \text{ where } b' = d' \text{ and } a' = -b'(1 + \omega).$$

Again, if we substitute  $b' = 1$  we obtain the fractional linear transformation form of  $V$ .

### 1.3. EXAMPLE: THE GIESEKING MANIFOLD

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Let us label the edges as in Figure 1.6. For the topological manifold to be geometric, we only have to check that the pairing is proper (see [41]). In this case, the only condition which we need to satisfy is that the isometry that goes through the only edge cycle is the identity. This is given by:

$$a \xrightarrow{V} c \xrightarrow{V} b \xrightarrow{U} d \xrightarrow{U} e \xrightarrow{V^{-1}} f \xrightarrow{U^{-1}} a,$$

and, therefore, we will have a hyperbolic structure if and only if  $U^{-1}V^{-1}U^2V^2 = \text{Id}$ . Doing this computation, we obtain the equation

$$|\omega(1 + \omega)| = 1. \quad (1.18)$$

Let us show that this equation matches the one obtained from Definition 1.1.3. If we denote by  $z(a)$  the edge invariant of  $a$  and analogously for the rest of the edges, we have that the equation describing the deformation space of the manifold  $\text{Def}(M, \Delta)$  in terms of this triangulation is

$$\frac{z(a)z(b)z(e)}{z(c)z(d)z(f)} = 1. \quad (1.19)$$

We can now write down all of the edge invariants in terms of  $z(a)$  by means of the tetrahedron relations, that is,

$$z(b) = \frac{1}{1 - z(a)}, \quad z(c) = \frac{z(a) - 1}{z(a)},$$

and

$$z(d) = z(c) = \frac{z(a) - 1}{z(a)}, \quad z(e) = z(a), \quad z(f) = z(b) = \frac{1}{1 - z(a)}.$$

Thus, Equation (1.19) is laid out as

$$\frac{z(a)^2 \overline{z(a)}^2}{(1 - z(a))(1 - \overline{z(a)})} = \frac{|z(a)|^4}{|1 - z(a)|^2} = 1. \quad (1.20)$$

If we substitute  $z(a) = -1/\omega$ , we obtain

$$\frac{1}{\omega\bar{\omega}(\omega+1)(\bar{\omega}+1)} = 1,$$

which is equivalent to Equation (1.18).

**Remark 1.3.1.** *The set  $\{w \in \mathbb{C} \mid |w(1+w)| = 1\}$  is homeomorphic to  $S^1$ , and the deformation space  $\{w \in \mathbb{C} \mid |w(1+w)| = 1 \text{ and } \text{Im}(w) > 0\}$  is homeomorphic to an open interval, see Figure 1.7.*

We justify the remark and Figure 1.7. Firstly, to prove that set of algebraic solutions is homeomorphic to a circle, we write the defining equation  $|w(1+w)| = 1$  as

$$\left| \left( w + \frac{1}{2} \right)^2 - \frac{1}{4} \right| = 1.$$

Thus  $\left( w + \frac{1}{2} \right)^2$  lies in the circle of center  $\frac{1}{4}$  and radius 1. As this circle separates 0 from  $\infty$ , the equation defines a connected covering of degree two of the circle. Secondly, the set of algebraic solutions is invariant by the involutions  $w \mapsto \bar{w}$  and  $w \mapsto -1-w$  (hence symmetric with respect to the real line and the line defined by real part equal to  $-\frac{1}{2}$ ). Furthermore it intersects the real line at  $w = \frac{-1 \pm \sqrt{5}}{2}$  and the line with real part  $-\frac{1}{2}$  at  $\frac{-1 \pm i\sqrt{3}}{2}$ .

Let us construct the link of the cusp. We denote the link of each cusp point as in Figure 1.8 and glue them to obtain the link as in Figure 1.9, which is a Klein bottle.

Now we take two tetrahedra and construct the orientation covering of  $M$  (the figure eight knot exterior). This can be done explicitly by considering the tetrahedra  $\Delta$  and  $U(\Delta)$ , which is a lift of the triangulation to  $\hat{M}$ . Thus, the orientation reversing isometry which we have usually denoted by  $\iota$  will be realized by  $U$ . The ideal tetrahedron  $U(\Delta)$  has vertices  $0, 1, -\omega$  and  $\omega' := U(-\omega)$ . The resulting side-pairing of tetrahedron is shown in Figures 1.10- 1.13.

For the first tetrahedron, we will denote by  $z_1 := z(a)$ , and  $z_2, z_3$  so that they follow the cyclic order described in the tetrahedron relations.

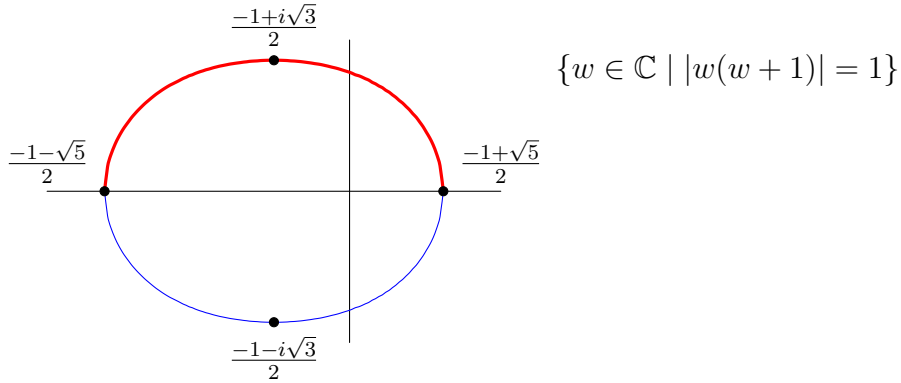


Figure 1.7: The set of solutions of the compatibility equations and  $\text{Def}(M, \Delta)$  (the top half).

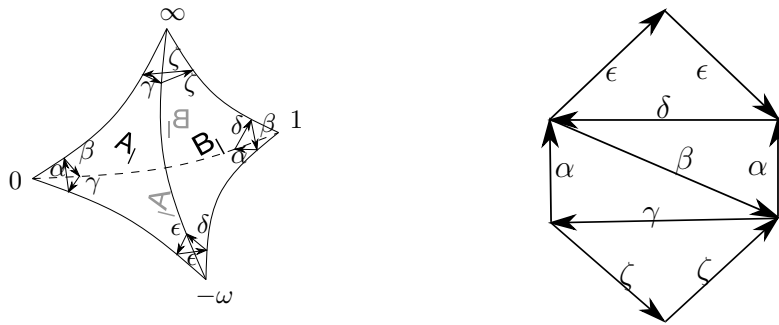


Figure 1.8: Gieseking manifold with link. Figure 1.9: Link of the cusp point.

Afterwards, in the second tetrahedron, we denote by  $w_i$  the edge invariant of the corresponding edge after applying an orientation reversing isometry to the tetrahedron, that is,  $w_i = \frac{1}{z_i}$ .

We consider the link of the orientation covering. The derivative of the holonomy of the two loops in the link of the orientation covering,  $l_1, l_2$ , depicted in Figure 1.14 (which are free homotopic) is  $\frac{w_1}{z_1} = \frac{1}{|z_1|^2}$  and  $\frac{w_3}{z_3} = \frac{1}{|z_3|^2}$ . For the manifold to be complete,  $\text{hol}'(l_i) = 1$  for  $i = 1, 2$ , which happens if and only if  $z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ . This corresponds to the regular ideal tetrahedron, which, as expected, is the manifold originally given by Gieseking. Notice that the upper loop (the one going through the side  $\epsilon$ )



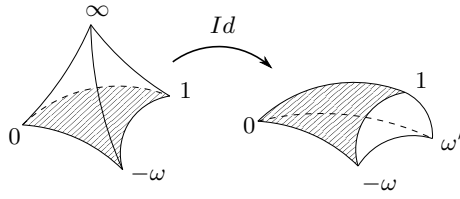


Figure 1.10: Faces glued by the Identity.

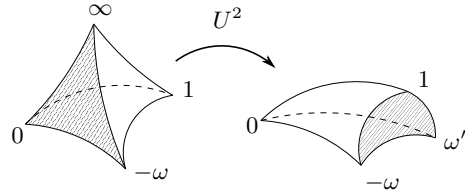


Figure 1.11: Faces glued by  $U^2$ .

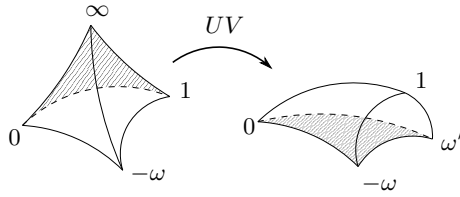


Figure 1.12: Faces glued by  $UV$ .

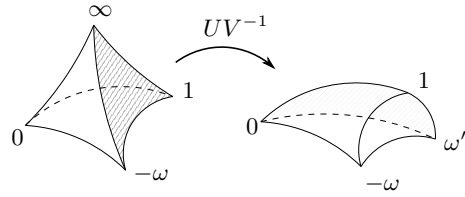


Figure 1.13: Faces glued by  $UV^{-1}$ .

can be taken as a *distinguished* longitude. A suitable meridian is drawn in Figure 1.15.

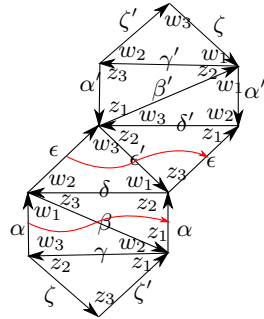


Figure 1.14: Two free homotopic loops.

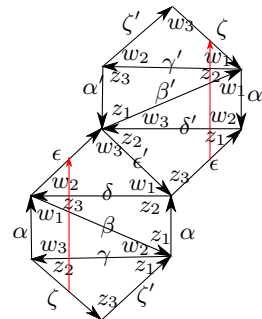


Figure 1.15: Meridian in the link of the cover.

Let us check that both the longitude and the meridian satisfy the conditions we stated for their holonomy in Remark 1.2.6, that is  $\text{hol}'(l) \in \mathbb{R}$ ,  $|\text{hol}'(m)| = 1$ . We have already shown it for the longitude. Regarding the meridian,

$$\text{hol}'(m) = \frac{z_2 z_3 w_2 w_3}{w_2 z_1 z_2 w_1} = \frac{z_1 w_1}{z_3 w_3} = \frac{z_1 \bar{z}_3}{\bar{z}_1 z_3},$$

therefore  $|\text{hol}'(m)| = 1$ . This leads to the result that the generalized Dehn

filling coefficients of a lifted structure have the form  $(0, q)$ , after an appropriate choice of longitude-meridian pair.

**Remark 1.3.2.** The result involving the generalized Dehn filling coefficient can also be obtained from Thurston's triangulation. By rotating the tetrahedra, our triangulation can be related with his as shown in Figure 1.16, and the parameters identified. We can then check that in his choice of longitude and meridian, the holonomy has the same features if the structure is a lift from Gieseking manifold. We will follow the notation in [41], and denote the parameters used as  $z_{\text{Rat}}, w_{\text{Rat}}$ , which are the edge invariants of  $[0, -\omega]$  and  $[1, 0] = U([0, \infty])$ , respectively.

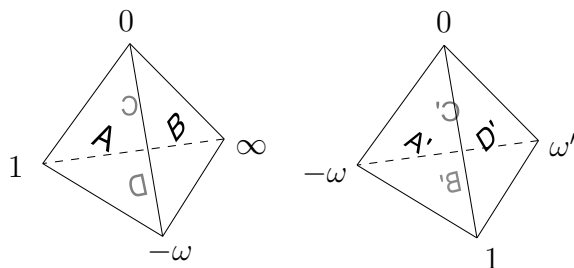


Figure 1.16: Triangulation  $\Delta$  and  $U(\Delta)$  rotated to match Thurston's.

From their choice of longitude-meridian pair, the generalized Dehn surgery invariant is the solution to the equation ([41], (10.5.16-17))

$$p \log(z_{\text{Rat}}(1 - w_{\text{Rat}})) + 2q \log(z_{\text{Rat}}(1 - z_{\text{Rat}})) = 2\pi i, \quad (1.21)$$

where  $z_{\text{Rat}}, w_{\text{Rat}}$  denote the choice of edge invariants made in [41].

We can relate  $z_{\text{Rat}}, w_{\text{Rat}}$  to  $w$ . In our choice of edge parameter,  $z_{\text{Rat}} = z_3 = \frac{z_1 - 1}{z_1}$ ,  $w_{\text{Rat}} = w_1 = 1/\bar{z}_1$ , hence, using the fact that  $z_1 = -1/\omega$ ,

$$z_{\text{Rat}} = \omega + 1, \quad w_{\text{Rat}} = -\bar{\omega}.$$

Substituting in Equation (1.21),

$$\begin{aligned} 2\pi i &= p \log((\omega + 1)(1 + \bar{\omega})) + 2q \log((\omega + 1)(-\omega)) \\ &= p \log |\omega + 1| + 2q \log(-\omega(\omega + 1)). \end{aligned}$$

We see that  $\log |\omega + 1| \in \mathbb{R}$  and, by Equation (1.18), we have that  $\log(-\omega(\omega + 1)) \in i\mathbb{R}$ . Thus,  $p = 0$ ,  $q = \pi i / \log(-\omega(\omega + 1))$ . Moreover,

$$\log(-\omega(\omega + 1)) = \arg(-\omega(\omega + 1)) = \arg(\omega) + \arg(\omega + 1) - \pi.$$

and by Figure 1.7 we see that  $\arg(\omega) + \arg(\omega + 1)$  is a strictly decreasing function on the real part of  $\omega$ , so  $\arg(\omega) + \arg(\omega + 1)$  has values in the interval  $(0, 2\pi)$ . Therefore,  $q$  ranges in the interval  $(1, \infty) \cup \{\infty\} \cup (\infty, -1)$ , where  $\infty$  corresponds to the complete structure.

# 2

## Varieties of representations

---

The group of isometries of hyperbolic space is denoted by  $G$ , and we have the following well known isomorphisms:

$$G = \text{Isom}(\mathbb{H}^3) \cong \text{PO}(3, 1) \cong \text{PSL}(2, \mathbb{C}) \rtimes \mathbb{Z}_2.$$

The group  $G$  has two connected components, according to whether the isometries preserve or reverse the orientation.

For a finitely generated group  $\Gamma$ , the variety of representations of  $\Gamma$  in  $G$  is denoted by

$$\text{hom}(\Gamma, G).$$

As  $G$  is algebraic, it has a natural structure of algebraic set [30], but we consider only its topological structure. We are interested in the set of conjugacy classes of representations:

$$\mathcal{R}(\Gamma, G) = \text{hom}(\Gamma, G)/G.$$

When  $M^3$  is hyperbolic, we write  $\Gamma = \pi_1(M^3)$ . The holonomy of  $M^3$

$$\text{hol}: \Gamma \rightarrow G$$

is well defined up to conjugacy, hence  $[\text{hol}] \in \mathcal{R}(\Gamma, G)$ . To understand deformations, we analyze a neighborhood of the holonomy in  $\mathcal{R}(\Gamma, G)$ . The main result of this chapter is:

**Theorem 2.0.1.** *Let  $M^3$  be a hyperbolic manifold of finite volume. Assume that it has  $k$  non-orientable cusps and  $l$  orientable cusps. Then there exists a neighborhood of  $[\text{hol}]$  in  $\mathcal{R}(\Gamma, G)$  homeomorphic to  $\mathbb{R}^{k+2l}$ .*

When  $M^3$  is orientable, this result is well known [8, 31], hence we assume that  $M^3$  is non-orientable. We will prove a more precise result in Theorem 2.3.2, as for our purposes it is relevant to describe local coordinates in terms of the geometry of holonomy structures at the ends.

Before starting the proof, we need a lemma on varieties of representations. The projection to the quotient  $\pi: \text{hom}(\Gamma, G) \rightarrow \mathcal{R}(\Gamma, G)$  can have quite bad properties, for instance even if  $\text{hom}(\Gamma, G)$  is Hausdorff, in general  $\mathcal{R}(\Gamma, G)$  is not.

Johnson and Millson define in [30] the properties of *stable* representation and *good* representation:

**Definition 2.0.2.** *A representation  $\rho \in \text{hom}(\Gamma, G)$  is stable if the orbit  $\text{Ad}(G)\rho \subset \text{hom}(\Gamma, G)$  is closed in  $\text{hom}(\Gamma, G)$  and the isotropy subgroup of  $\rho$ ,  $Z(\rho)$ , is finite in  $G$ .*

*A stable representation is good if  $Z(\rho) = Z(G)$ , the center of  $G$ .*

By a theorem in [30], the property of being stable is equivalent to the property that the image of  $\rho$  is not contained in any proper parabolic subgroup of  $G$ . Moreover, notice that in our case,  $Z(G) = \{Id\}$ , so a stable representation will be good if it has trivial isotropy group. Let us denote the subsets of stable and good representations by

$$\text{hom}^{st}(\Gamma, G) \quad \text{and} \quad \text{hom}^{good}(\Gamma, G),$$

respectively. They satisfy a couple of important lemmas ([30]):

**Lemma 2.0.3.** *The subset of good representations  $\text{hom}^{good}(\Gamma, G)$  is open in  $\text{hom}(\Gamma, G)$ .*

**Lemma 2.0.4.** *The action of  $G$  by conjugation on  $\text{hom}^{st}(\Gamma, G)$  is proper.*

We will only consider manifolds whose holonomy satisfies the property of being irreducible, which under these definitions is equivalent to being a good representation. We have the following key lemma:

**Lemma 2.0.5.** *There exists a neighborhood  $V \subset \mathcal{R}(\Gamma, G)$  of [hol] such that:*

- (a) *If  $[\rho] = [\rho'] \in V$ , then the matrix  $A \in G$  satisfying  $A\rho(\gamma)A^{-1} = \rho'(\gamma)$ ,  $\forall \gamma \in \Gamma$ , is unique.*
- (b)  *$V$  is Hausdorff and the projection  $\pi: \pi^{-1}(V) \rightarrow V$  is open.*
- (c) *If  $[\rho] \in V$ , then  $\forall \gamma \in \Gamma$ ,  $\rho(\gamma)$  preserves the orientation of  $\mathbb{H}^3$  if and only if  $\gamma$  is represented by a loop that preserves the orientation of  $M^3$*

*Proof.* Assertions (a) and (b) are proved in [30]. The holonomy is a good representation, thus we can apply Lemma 2.0.3 to ensure that we can assume that in a neighbourhood of [hol] every representation is good. Assertion (a) is a direct consequence of the definition of good representation and the fact that  $Z(G) = \{Id\}$ , whereas assertion (b) results from Lemma 2.0.4 and local compactness of  $G$ . Openness in (b) is due to the action of  $G$  being by homeomorphisms.

Finally, the holonomy representation, hol, satisfies the property stated in (c). It is clear by continuity and the decomposition of  $G$  in two connected components according to the orientation, that every representation  $\rho$  in a small enough neighbourhood of hol satisfies (c) too.

□

To describe the neighborhood of the holonomy in  $\mathcal{R}(\Gamma, G)$  we use the orientation covering.

## 2.1 Orientation covering and the involution on representations

As mentioned, we assume  $M^3$  non-orientable. Let

$$M_+^3 \rightarrow M^3$$

denote the orientation covering, with fundamental group  $\Gamma_+ = \pi_1(M_+^3)$ . In particular we have a short exact sequence:

$$1 \rightarrow \Gamma_+ \rightarrow \Gamma \rightarrow \mathbb{Z}_2 \rightarrow 1.$$

**Lemma 2.1.1.** *Let  $\Gamma$  admit a presentation*

$$\Gamma = \langle a_1, \dots, a_n, \zeta \mid R_1 = \dots = R_m = 1 \rangle,$$

where the generators  $a_i$ ,  $i = 1, \dots, n$  are represented by orientation-preserving loops whereas  $\zeta$  is represented by an orientation-reversing one. Then  $\Gamma_+$  is generated by the elements

$$a_1, \dots, a_n, \zeta a_1 \zeta^{-1}, \dots, \zeta a_n \zeta^{-1}, \zeta^2.$$

*Proof.* Let  $\alpha \in \Gamma_+ < \Gamma$ , it can be expressed as

$$\alpha = \zeta^{k_1} w_1(a_1, \dots, a_n, \zeta^2) \dots \zeta^{k_s} w_s(a_1, \dots, a_n, \zeta^2),$$

where  $k_i \in \{0, \pm 1\}$  and  $w_i(\cdot)$  denote a word in terms of the arguments (or the emptyset). Notice that

$$\zeta w(a_1, \dots, a_n, \zeta^2) \zeta^{-1} = w(\zeta a_1 \zeta^{-1}, \dots, \zeta a_n \zeta^{-1}, \zeta^2),$$

$$\zeta w(a_1, \dots, a_n, \zeta^2) \zeta = \zeta w(a_1, \dots, a_n, \zeta^2) \zeta^{-1} (\zeta^2).$$

Therefore, we can assume without loss of generality that either

$$\alpha = \zeta^{\pm 1} w'(a_1, \dots, a_n, \zeta a_1 \zeta^{-1}, \dots, \zeta a_n \zeta^{-1}, \zeta^2),$$

or

$$\alpha = w'(a_1, \dots, a_n, \zeta a_1 \zeta^{-1}, \dots, \zeta a_n \zeta^{-1}, \zeta^2).$$

The first case is not possible because the elements  $\alpha$  and  $a_i$  are represented by orientation-preserving loops in  $M$ , so there cannot be a relation in  $\Gamma$  where  $\zeta$  is a word in the rest of the generators.  $\square$

**Definition 2.1.2.** For  $\zeta \in \Gamma \setminus \Gamma_+$ , define the group automorphism

$$\begin{aligned}\sigma_*: \Gamma_+ &\rightarrow \Gamma_+ \\ \gamma &\mapsto \zeta\gamma\zeta^{-1}\end{aligned}$$

The automorphism  $\sigma_*$  depends on the choice of  $\zeta \in \Gamma \setminus \Gamma_+$ , the automorphisms corresponding to different choices of  $\zeta$  differ by composition (or pre-composition) with an inner automorphism of  $\Gamma_+$ ; furthermore  $\sigma_*^2$  is an inner automorphism because  $\zeta^2 \in \Gamma_+$ . This automorphism  $\sigma_*$  is the map induced by the deck transformation of the orientation covering  $M_+^3 \rightarrow M^3$ .

The map induced by  $\sigma_*$  in the variety of representations is denoted by

$$\begin{aligned}\sigma^*: \mathcal{R}(\Gamma, G) &\rightarrow \mathcal{R}(\Gamma, G) \\ [\rho] &\mapsto [\rho \circ \sigma_*]\end{aligned}$$

and  $\sigma^*$  does not depend on the choice of  $\zeta$ , because  $\sigma_*$  is well defined up to inner automorphism. That is, for two different choices  $\zeta_1, \zeta_2 \in \Gamma \setminus \Gamma_+$ , they are related by an element  $\xi \in \Gamma_+$  as  $\zeta_2 = \xi\zeta_1$ , hence, if we denote the respective automorphisms by  $\sigma_{1,*}$  and  $\sigma_{2,*}$ , and for any  $\gamma \in \Gamma_+$ ,

$$(\rho \circ \sigma_{2,*})(\gamma) = \rho(\zeta_2\gamma\zeta_2^{-1}) = \rho(\xi)\rho(\zeta_1\gamma\zeta_1^{-1})\rho(\xi)^{-1} = \rho(\xi)(\rho \circ \sigma_{1,*})(\gamma)\rho(\xi)^{-1}.$$

Furthermore  $\sigma^*$  is an involution,  $(\sigma^*)^2 = \text{Id}$ .

Consider the restriction map:

$$\text{res}: \mathcal{R}(\Gamma, G) \mapsto \mathcal{R}(\Gamma_+, G)$$

that maps the conjugacy class of a representation of  $\Gamma$  to the conjugacy class of its restriction to  $\Gamma_+$ .

**Lemma 2.1.3.** *There exist  $U \subset \mathcal{R}(\Gamma, G)$  neighborhood of  $[\text{hol}]$  and  $V \subset \mathcal{R}(\Gamma_+, G)$  neighborhood of  $\text{res}([\text{hol}])$  such that*

$$\text{res}: U \xrightarrow{\cong} \{[\rho] \in V \mid \sigma^*([\rho]) = [\rho]\}$$

*is a homeomorphism.*



*Proof.* We show first that  $\text{res}(\mathcal{R}(\Gamma, G)) \subset \{[\rho] \in \mathcal{R}(\Gamma_+, G) \mid \sigma^*([\rho]) = [\rho]\}$ : if  $\rho_+ = \text{res}(\rho)$ , then  $\forall \gamma \in \Gamma_+$ ,

$$\sigma^*(\rho_+)(\gamma) = \rho_+(\sigma_*(\gamma)) = \rho_+(\zeta\gamma\zeta^{-1}) = \rho(\zeta)\rho_+(\gamma)\rho(\zeta)^{-1}.$$

Hence  $\sigma^*([\text{res}(\rho)]) = [\text{res}(\rho)]$ .

Next, given  $[\rho_+] \in \mathcal{R}(\Gamma_+, G)$  satisfying  $\sigma^*([\rho_+]) = [\rho_+]$ , by construction there exists some  $A \in G$ , not necessarily unique, that conjugates  $\rho_+$  and  $\rho_+ \circ \sigma_*$ . If  $\zeta \in \Gamma \setminus \Gamma_+$  is the element such that  $\sigma_*$  is conjugation by  $\zeta$  then, by choosing  $\rho(\zeta) = A$ , the representation  $\rho_+$  extend to  $\rho: \Gamma \rightarrow G$  (we shall prove this at the end). Hence:

$$\text{res}(\mathcal{R}(\Gamma, G)) = \{[\rho] \in \mathcal{R}(\Gamma_+, G) \mid \sigma^*([\rho]) = [\rho]\}.$$

We chose the neighborhood  $V$  so that Lemma 2.0.5 applies (hence  $A \in G$  as above conjugating  $\rho_+$  and  $\rho_+ \circ \sigma_*$  is unique). Let  $U = \text{res}^{-1}(V)$ . With this choice of  $U$  and  $V$ ,

$$\text{res}: U \rightarrow \{[\rho] \in V \mid \sigma^*([\rho]) = [\rho]\}$$

is a continuous bijection.

Now, let us show that the map

$$\pi^{-1}(V) \longrightarrow \pi^{-1}(U)$$

extending a representation to  $\Gamma$  is continuous, hence the map  $\text{res}^{-1}$  in the quotient will be continuous too. This map is defined by the choice  $\rho(\zeta) = A$ , where  $A \in G$  is the unique element satisfying  $\sigma^* \circ \rho_+ = \text{Ad}(A)(\rho_+)$ . Proving that this choice can be made in a continuous manner is enough to prove continuity of the map.

The action of  $G$  on  $\text{hom}^{st}(\Gamma_+, G)$  admits analytic slices at any point (see [30], and [9] for a definition), therefore, a neighbourhood of a representation  $\rho \in \pi^{-1}(V)$  is diffeomorphic to  $G \times S/Z(\rho)$ , where  $S$  is the slice, and  $Z(\rho)$  the isotropy group of  $\rho$ , hence  $Z(\rho) = \text{Id}$ . Thus, a neighbourhood

which we can assume to contain  $\pi^{-1}(V)$  is diffeomorphic to the product  $G \times S$ , where the diffeomorphism is given by the map  $(A, s) \mapsto \text{Ad}(A)(s)$ . Let us consider now a convergent sequence of representations  $\rho_n \mapsto \rho_\infty$ . The sequence yields via the diffeomorphism a convergent sequence of elements  $A_n \mapsto A_\infty$  in  $G$ . As a consequence, the choice  $\rho_n(\zeta)$  is made continuously and the inverse map is continuous. This could had also been proved as a consequence of Lemma 2.0.4.

Finally, we show that  $\rho$  is indeed a representation of  $\Gamma$ . A presentation of  $\Gamma$  can be chosen

$$\Gamma = \langle a_1, \dots, a_n, \zeta \mid R_1 = \dots = R_m = 1 \rangle,$$

so that  $\Gamma_+$  is generated by  $a_1, \dots, a_n, \zeta a_1 \zeta^{-1}, \dots, \zeta a_n \zeta^{-1}, \zeta^2$ , by Lemma 2.1.1. In order to ensure that  $\rho_+$  extends to  $\rho$  by choosing  $\rho(\zeta) = A$ , we have to show that the relations are satisfied and  $\rho$  is well-defined.

The relation  $R_i(a_1, \dots, a_n)$  represents an orientation-preserving loop (as it is homotopy equivalent to the constant loop) for  $i = 1, \dots, m$ , hence it is satisfied in  $\Gamma_+$ . Therefore, the image of the generators by  $\rho_+$  satisfies the relations too.

On the other hand,  $\rho$  is well-defined as long as its coherent with  $\rho_+(\zeta a_i \zeta^{-1})$ ,  $i = 1, \dots, n$  and  $\rho_+(\zeta^2)$ . The first case is immediate due to the choice of  $\rho(\zeta) = A$ . The second case can be verified by conjugating  $\rho_+$  by  $\rho_+(\zeta^2)$ : for any  $\gamma \in \Gamma_+$ ,

$$\rho(\zeta^2)\rho(\gamma)\rho(\zeta^{-2}) = \rho((\sigma_*)^2\gamma) = A^2\rho(\gamma)A^{-2},$$

so by Lemma 2.0.5 (a),  $\rho(\zeta^2) = A^2$ . □

As  $\Gamma_+$  preserves the orientation, next we use the complex structure of the identity component  $G_0 = \text{Isom}^+(\mathbb{H}^3) \cong \text{PSL}(2, \mathbb{C})$ .

## 2.2 Representations in $\mathrm{PSL}(2, \mathbb{C})$

The holonomy of the orientation covering  $M_+^3$  is contained in  $\mathrm{PSL}(2, \mathbb{C})$ , and it is well defined up to the action of  $G = \mathrm{PSL}(2, \mathbb{C}) \rtimes \mathbb{Z}_2$  by conjugation. If we furthermore choose an orientation on  $M_+^3$ , then the holonomy is unique up to the action by conjugation by  $G_0 = \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$ , and complex conjugation corresponds to changing the orientation. We call the conjugacy class in  $\mathrm{PSL}(2, \mathbb{C})$  of the holonomy of  $M_+^3$  the *oriented holonomy*.

We consider

$$\mathcal{R}(\Gamma_+, \mathrm{PSL}(2, \mathbb{C})) = \mathrm{hom}(\Gamma_+, \mathrm{PSL}(2, \mathbb{C}))/\mathrm{PSL}(2, \mathbb{C}).$$

Its local structure is well known:

**Theorem 2.2.1.** *A neighborhood of the oriented holonomy of  $M_+^3$  in the variety of representations  $\mathcal{R}(\Gamma_+, \mathrm{PSL}(2, \mathbb{C}))$  has a natural structure of  $\mathbb{C}$ -analytic variety defined over  $\mathbb{R}$ .*

The fact that it is  $\mathbb{C}$ -analytic follows for instance from [30] or [31]. In Theorem 2.3.1 we precise  $\mathbb{C}$ -analytic coordinates, for the moment this is sufficient for our purposes.

**Lemma 2.2.2.** *Let  $\mathrm{hol}_+$  be the oriented holonomy of  $M_+^3$ . Then*

$$[\mathrm{hol}_+] \neq \overline{[\mathrm{hol}_+]} \in \mathcal{R}(\Gamma_+, \mathrm{PSL}(2, \mathbb{C})).$$

*Namely, the oriented holonomy and its complex conjugate are not conjugate by a matrix in  $\mathrm{PSL}(2, \mathbb{C})$ .*

*Proof.* By contradiction, assume that  $\mathrm{hol}_+$  and  $\overline{\mathrm{hol}_+}$  are conjugate by a matrix in  $\mathrm{PSL}(2, \mathbb{C})$ : there exists an orientation-preserving isometry  $A \in \mathrm{PSL}(2, \mathbb{C})$  such that

$$A \mathrm{hol}_+(\gamma) A^{-1} = \overline{\mathrm{hol}_+(\gamma)}, \quad \forall \gamma \in \Gamma_+.$$

Consider the orientation reversing isometry  $B = c \circ A$ , where  $c$  is the isometry with Möbius transformation the complex conjugation. The previous equation is equivalent to

$$B \mathrm{hol}_+(\gamma) B^{-1} = \mathrm{hol}_+(\gamma), \quad \forall \gamma \in \Gamma_+. \quad (2.1)$$

Brower's fixed point theorem yields that the fixed point set of  $B$  in the ball compactification  $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$  is non-empty:

$$\mathrm{Fix}(B) = \{x \in \mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3 \mid B(x) = x\} \neq \emptyset.$$

By (2.1)  $\mathrm{hol}_+(\Gamma_+)$  preserves  $\mathrm{Fix}(B)$ . Thus, by minimality of the limit set of a Kleinian group, since  $\mathrm{Fix}(B) \neq \emptyset$  is closed and  $\mathrm{hol}_+(\Gamma_+)$ -invariant, it contains the whole ideal boundary:  $\partial_\infty \mathbb{H}^3 \subset \mathrm{Fix}(B)$ . Hence  $B$  is the identity, contradicting that  $B$  reverses the orientation.  $\square$

From Lemma 2.2.2 and Theorem 2.2.1:

**Corollary 2.2.3.** *There exists a neighborhood  $W \subset \mathcal{R}(\Gamma_+, \mathrm{PSL}(2, \mathbb{C}))$  of the conjugacy class of the oriented holonomy of  $M_+$  that is disjoint from its complex conjugate:*

$$\overline{W} \cap W = \emptyset.$$

By choosing the neighborhood  $W \subset \mathcal{R}(\Gamma_+, \mathrm{PSL}(2, \mathbb{C}))$  sufficiently small, we may assume that its projection to  $\mathcal{R}(\Gamma_+, G)$  is contained in  $V$  as in Lemma 2.1.3. The neighborhood  $V$  can also be chosen smaller, to be equal to the projection of  $W$ , as this map is open. Namely the neighborhoods can be chosen so that  $\mathcal{R}(\Gamma_+, \mathrm{PSL}(2, \mathbb{C})) \rightarrow \mathcal{R}(\Gamma_+, G)$  restricts to a homeomorphism between  $W$  (or  $\overline{W}$ ) and  $V$ . In particular we can lift to  $W$  the restriction map from  $U$  to  $V$ :

$$\begin{array}{ccc} & & W \\ & \nearrow \widetilde{\mathrm{res}} & \downarrow \cong \\ U & \xrightarrow{\mathrm{res}} & V \end{array}$$

**Lemma 2.2.4.** *For  $U \subset \mathcal{R}(\Gamma, G)$  and  $W \subset \mathcal{R}(\Gamma_+, \mathrm{PSL}(2, \mathbb{C}))$  as above, the lift of the restriction map yields an homeomorphism:*

$$\widetilde{\mathrm{res}}: U \xrightarrow{\cong} \{[\rho] \in W \mid [\rho \circ \sigma_*] = [\bar{\rho}]\}.$$

This lemma has same proof as Lemma 2.1.3, just taking into account that  $\rho(\zeta) \in G$  reverses the orientation, for  $[\rho] \in U$  and  $\zeta \in \Gamma \setminus \Gamma_+$ .

## 2.3 Local coordinates

Here we give the local coordinates of Theorem 2.2.1 and we prove a stronger version of Theorem 2.0.1.

For  $\gamma \in \Gamma_+$  and  $[\rho] \in \mathcal{R}(\Gamma_+, \mathrm{PSL}(2, \mathbb{C}))$ , as in [11] define

$$I_\gamma([\rho]) = (\mathrm{trace}(\rho(\gamma)))^2 - 4. \quad (2.2)$$

Thus  $I_\gamma$  is a function from  $\mathcal{R}(\Gamma_+, \mathrm{PSL}(2, \mathbb{C}))$  to  $\mathbb{C}$ . This function plays a role in the generalization of Theorem 2.0.1.

**Theorem 2.3.1.** *Let  $M_+^3$  be as above and assume that it has  $N$  cusps. Choose  $\gamma_1, \dots, \gamma_N \in \Gamma_+$  a non-trivial element for each peripheral subgroup. Then, for a neighborhood  $W \subset \mathcal{R}(\Gamma_+, \mathrm{PSL}(2, \mathbb{C}))$  of the oriented holonomy,*

$$(I_{\gamma_1}, \dots, I_{\gamma_N}): W \rightarrow \mathbb{C}^N$$

*defines a bi-analytic map between  $W$  and a neighborhood of the origin.*

This theorem holds for any orientable hyperbolic manifold of finite volume, though we only use it for the orientation covering. Again, see [8, 31] for a proof. As explained in these references, this is the algebraic part of the proof of Thurston's hyperbolic Dehn filling theorem using varieties of representations.

For a Klein bottle  $K^2$ , in Definition 1.2.1 we considered the presentation

of its fundamental group:

$$\pi_1(K^2) = \langle a, b \mid aba^{-1} = b^{-1} \rangle.$$

The elements  $a^2$  and  $b$  are called *distinguished* elements. Recall that, in terms of paths, those are represented by the unique homotopy classes of loops in the orientation covering that are invariant by the deck transformation (as unoriented curves).

Here we prove the following generalization of Theorem 2.0.1:

**Theorem 2.3.2.** *Let  $M^3$  be a non-orientable manifold of finite volume with  $k$  non-orientable cusps and  $l$  orientable cusps. For each horospherical Klein bottle,  $K_i^2$ , choose  $\gamma_i \in \pi_1(K_i^2)$  distinguished,  $i = 1, \dots, k$ . For each horospherical torus,  $T_j^2$ , choose a nontrivial  $\mu_j \in \pi_1(T_j^2)$ ,  $j = 1, \dots, l$ .*

*There exists a neighborhood  $U \subset \mathcal{R}(\Gamma, G)$  of the holonomy of  $M^3$  such that the map*

$$(I_{\gamma_1}, \dots, I_{\gamma_k}, I_{\mu_1}, \dots, I_{\mu_l}) \circ \widetilde{\text{res}} : U \rightarrow \mathbb{R}^k \times \mathbb{C}^l$$

*defines a homeomorphism between  $U$  and a neighborhood of the origin in  $\mathbb{R}^k \times \mathbb{C}^l$ .*

*Proof.* Let  $M_+^3 \rightarrow M^3$  be the orientation covering. By construction, by the choice of distinguished elements in the peripheral Klein bottles,  $\gamma_i \in \Gamma_+$ . Furthermore, as the peripheral tori are orientable,  $\mu_j \in \Gamma_+$ . Hence

$$\{\gamma_1, \dots, \gamma_k, \mu_1, \dots, \mu_l, \sigma_*(\mu_1), \dots, \sigma_*(\mu_l)\}$$

gives a nontrivial element for each peripheral subgroup of  $\Gamma_+$ . We apply Theorem 2.3.1:

$$I = (I_{\gamma_1}, \dots, I_{\gamma_k}, I_{\mu_1}, \dots, I_{\mu_l}, I_{\sigma_*(\mu_1)}, \dots, I_{\sigma_*(\mu_l)}): W \rightarrow \mathbb{C}^{k+2l}$$

is a bi-analytic map with a neighborhood of the origin.

Let  $[\rho] = I^{-1}(x_1, \dots, x_k, y_1, \dots, y_l, z_1, \dots, z_l)$ , then as  $\sigma_*(\gamma_j) = \gamma_j^{\pm 1}$  and  $\text{trace}(\rho) = \text{trace}(\rho^{-1})$ ,

$$I_{\gamma_j} \circ \sigma^*(\rho) = (\text{trace}(\rho(\gamma_j^{\pm 1})))^2 - 4 = x_j,$$

and, furthermore, since  $(\sigma^*)^2 = \text{Id}$ ,

$$\begin{aligned} I_{\mu_j} \circ \sigma^*(\rho) &= (\text{trace}(\rho(\sigma_*(\mu_j))))^2 - 4 = z_j, \\ I_{\sigma_*(\mu_j)} \circ \sigma^*(\rho) &= (\text{trace}(\rho((\sigma_*)^2(\mu_j))))^2 - 4 = y_j. \end{aligned}$$

Thus,

$$I \circ \sigma^* \circ I^{-1}(x_1, \dots, x_k, y_1, \dots, y_l, z_1, \dots, z_l) = (x_1, \dots, x_k, z_1, \dots, z_l, y_1, \dots, y_l).$$

By Lemma 2.2.4, the image  $\widetilde{\text{res}}(U)$  is the subset satisfying  $\sigma^*[\rho] = [\bar{\rho}]$ , therefore, the image  $(I \circ \widetilde{\text{res}})(U)$  is characterized by  $\sigma^*I^{-1}(x_1, \dots, z_l) = [\bar{\rho}]$ . By construction  $I$  commutes with complex conjugation, hence by the previous computation of  $I \circ \sigma^* \circ I^{-1}$ , the image  $(I \circ \widetilde{\text{res}})(U)$  is the subset of a neighbourhood of the origin in  $\mathbb{C}^{k+2l}$  defined by

$$\begin{cases} x_i = \bar{x}_i, & \forall i = 1, \dots, k, \text{ and} \\ z_j = \bar{y}_j, & \forall j = 1, \dots, l. \end{cases}$$

Finally, by combining Theorem 2.3.1 and Lemma 2.2.4, the map  $I \circ \widetilde{\text{res}}$  is a homeomorphism between  $U$  and its image.  $\square$

### 2.3.1 The relation between $\text{Def}(M^3, \Delta)$ and $\mathcal{R}(\Gamma, G)$

The underlying object of our interest in Chapters 1 and the current one is the deformation space of a hyperbolic 3-manifold  $M^3$ . Let us denote the manifold with the original hyperbolic structure by  $M_0^3$ , then, the deformation space can be understood as a neighbourhood of hyperbolic structures near  $M_0^3$ . We will consider deformations up to isotopy to the identity. The relation between the deformation space and the variety of representation is

described next.

**Proposition 2.3.3** ([43], [10]). *The deformation space around  $M_0^3$  is identified with a neighbourhood of  $\mathcal{R}(\Gamma, G)$  around the holonomy representation corresponding to  $M_0^3$ .*

Thus, the results of this chapter compute the deformation space of a non-orientable hyperbolic manifold of finite volume around the complete structure.

The precise relation of the deformation space with the space of deformations of the triangulation  $\text{Def}(M^3, \Delta)$  is a little more subtle. In Thurston's hyperbolic Dehn filling theorem ([43]), Thurston introduced the generalized Dehn filling coefficients, which serve as a parametrization of  $\text{Def}(M^3, \Delta)$  as shown in Chapter 1. If  $M^3$  is orientable and has  $l$  ends, then there is a  $2^l$  to 1 branched covering map between  $\text{Def}(M^3, \Delta)$  and  $\mathcal{R}(\Gamma, G)$  with branching point the complete structure. In terms of the generalized Dehn coefficients for one fixed end,  $(p, q)$  and  $-(p, q)$  induce the same hyperbolic structure. This is due to the fact that the representations of  $\pi_1(T^2)$

$$\begin{pmatrix} e^{u/2} & 1 \\ 0 & e^{-u/2} \end{pmatrix}, \quad \begin{pmatrix} e^{v/2} & \tau \\ 0 & e^{-v/2} \end{pmatrix}$$

and

$$\begin{pmatrix} e^{-u/2} & 1 \\ 0 & e^{u/2} \end{pmatrix}, \quad \begin{pmatrix} e^{-v/2} & \tau \\ 0 & e^{v/2} \end{pmatrix}$$

are conjugate. The choice of sign indicates the spinning direction in the developing map around the axis of the holonomy representation.

Applying the branched covering map to Theorems 2.3.2 and 1.2.16, they can be summarized as follows:

**Theorem 2.3.4.** *Let  $M^3$  be a complete, non-orientable, hyperbolic 3-manifold of finite volume with  $l$  orientable ends and  $k$  non-orientable ones.*

(a) *If  $M^3$  admits an ideal triangulation  $\Delta$ , then,  $\text{Def}(M^3, \Delta) \cong (-1, 1)^k \times B(1)^l$ , where  $B(1) \subset \mathbb{C}$  denotes the unit ball centered at 0, and where*



the parameters  $(\pm t_1, \dots, \pm t_{k+l}) \in (-1, 1)^k \times B(1)^l$  correspond to the same structure.

(b) A neighborhood of the holonomy in  $\mathcal{R}(\pi_1(M^3), \text{Isom}(\mathbb{H}^3))$  is homeomorphic to  $(-1, 1)^k \times B(1)^l$ .

Furthermore, the holonomy map  $\text{Def}(M^3, \Delta) \rightarrow \mathcal{R}(\pi_1(M^3), \text{Isom}(\mathbb{H}^3))$  is a  $2^{k+l}$  branched covering on the image and folds each interval  $(-1, 1)$  at 0. Its image is the product of half-open intervals and open balls  $[0, 1)^k \times B(1)^l$ , where  $(0, \dots, 0)$  corresponds to the complete structure.

If we assume  $M^3$  to have only one non-orientable end, then we have  $\mathcal{R}(\pi_1(M^3), \text{Isom}(\mathbb{H}^3)) \cong (-1, 1)$ . Structures in the subinterval  $[0, 1) \subset (-1, 1)$  in the variety of representations are realized by  $\text{Def}(M^3, \Delta)$  if  $M^3$  admits an ideal triangulation, whereas structures in  $(-1, 0)$  are not. This is related to two different kinds of representations on the peripheral Klein bottles, which we will study in Chapter 3.

## 2.4 The Gieseking manifold revisited

We make use of the Gieseking manifold once again to illustrate the results of the chapter. In this occasion we choose to use the description of the Gieseking manifold as a punctured torus bundle, which we will introduce next. Furthermore, we compare the results with the ones obtained in Section 1.3.

### 2.4.1 The Gieseking manifold as a punctured torus bundle

The Gieseking manifold  $M$  is fibered over the circle with fibre a punctured torus  $T^2 \setminus \{*\}$  (see [1] or [38]). We use this structure to compute the variety of representations. The monodromy of the fibration is an automorphism

$$\phi: T^2 \setminus \{*\} \rightarrow T^2 \setminus \{*\}.$$

It is the restriction of a map of the compact torus  $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$  that lifts to the linear map of  $\mathbb{R}^2$  with matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

This matrix also describes the action on the first homology group  $H_1(T^2 \setminus \{*\}, \mathbb{Z}) \cong \mathbb{Z}^2$ . The map  $\phi$  is orientation reversing (the matrix has determinant  $-1$ ) and  $\phi^2$  is the monodromy of the orientation covering of  $M$ , the figure eight knot exterior.

The fibration induces a presentation of the fundamental group of  $M$ :

$$\pi_1(M) \cong \langle r, s, t \mid trt^{-1} = \phi(r), tst^{-1} = \phi(s) \rangle$$

where  $\langle r, s \mid \rangle = \pi_1(T^2 \setminus \{*\}) \cong F_2$ , and

$$\begin{aligned} \phi_*: F_2 &\rightarrow F_2 \\ r &\mapsto s \\ s &\mapsto rs \end{aligned}$$

is the algebraic monodromy, the map induced by  $\phi$  on the fundamental group.

The relationship with the presentation (1.17) of  $\pi_1(M)$  from the triangulation is given by

$$r = UV, \quad s = VU, \quad t = U^{-1}.$$

Furthermore, a peripheral group is given by  $\langle rsr^{-1}s^{-1}, t \rangle$ , which is the group of the Klein bottle.

We use this fibered structure to compute the variety of conjugacy classes of representations. Set

$$G = \text{Isom}(\mathbb{H}^3) \cong \text{PO}(3, 1) \cong \text{PSL}(2, \mathbb{C}) \rtimes \mathbb{Z}_2,$$

let

$$\text{hom}^{\text{irr}}(\pi_1(M), G)$$

denote the space of *irreducible* representations (i.e. that have no invariant line in  $\mathbb{C}^2$ ). As we are interested in deformations, we restrict to representations  $\rho$  that preserve the orientation type:  $\rho(\gamma)$  is an orientation preserving isometry iff  $\gamma \in \pi_1(M)$  is represented by a loop that preserves the orientation of  $M$ ,  $\forall \gamma \in \pi_1(M)$ . We denote the subspace of representations that preserve the orientation type by

$$\text{hom}_+^{\text{irr}}(\pi_1(M), G).$$

Let

$$\text{hom}_+^{\text{irr}}(\pi_1(M), G)/G$$

be their the space of their conjugacy classes.

**Proposition 2.4.1.** *We have an homeomorphism, via the trace of  $\rho(s)$ :*

$$\begin{aligned} \text{hom}_+^{\text{irr}}(\pi_1(M), G)/G &\rightarrow (\{x \in \mathbb{C} \mid |x - 1| = 1 \text{ and } x \neq 2\})/\sim \\ [\rho] &\mapsto \text{trace}(\rho(s)) \end{aligned}$$

where  $\sim$  is the relation by complex conjugation.

*In particular,  $\text{hom}_+^{\text{irr}}(\pi_1(M), G)/G$  is homeomorphic to a half-open interval.*

*Proof.* Let  $\rho: \pi_1(M) \rightarrow G$  be an irreducible representation. The fibre  $T^2 \setminus \{*\}$  is orientable, so the restriction of  $\rho$  to the free group  $\langle r, s \mid \rangle \cong F_2$  is contained in  $\text{PSL}(2, \mathbb{C})$ . The subgroup  $\langle r, s \mid \rangle$  is precisely the commutator subgroup  $[\pi_1(M), \pi_1(M)]$ : let  $\pi_1(M)^{\text{ab}}$  denote the abelianization of  $\pi_1(M)$ , so that the commutator  $[\pi_1(M), \pi_1(M)]$  satisfies the short exact sequence

$$1 \rightarrow [\pi_1(M), \pi_1(M)] \rightarrow \pi_1(M) \rightarrow \pi_1(M)^{\text{ab}} \rightarrow 1.$$

Let us denote the image of the generators of  $\pi_1(M)$  by  $\bar{r}, \bar{s}, \bar{t} \in \pi_1(M)^{\text{ab}}$ . From the presentation of  $\pi_1(M)$ , and the commutativity of elements in

$\pi_1(M)^{\text{ab}}$ , we obtain a presentation

$$\pi_1(M)^{\text{ab}} \cong \langle \bar{r}, \bar{s}, \bar{t} \mid \bar{r} = \bar{s}, \bar{s} = \bar{r}\bar{s} \rangle.$$

Hence,  $\bar{r} = 1$ ,  $\bar{s} = 1$ , and

$$[\pi_1(M), \pi_1(M)] = \text{Ker}(\pi_1(M) \rightarrow \pi_1(M)^{\text{ab}}) = \langle r, s \mid \rangle.$$

As a consequence, for any element  $u \in \langle r, s \mid \rangle$ , the element  $\rho(u)$  is well-defined in  $\text{SL}(2, \mathbb{C})$ , therefore we may assume that the image  $\rho(\langle r, s \mid \rangle) \subset \text{SL}(2, \mathbb{C})$  [24].

We consider the variety of characters  $X(F_2, \text{SL}(2, \mathbb{C}))$  and the action of the algebraic monodromy  $\phi_*$  on the variety of characters:

$$\begin{aligned} \phi^* : X(F_2, \text{SL}(2, \mathbb{C})) &\rightarrow X(F_2, \text{SL}(2, \mathbb{C})) \\ \chi &\mapsto \chi \circ \phi_* \end{aligned}$$

**Lemma 2.4.2.** *The restriction of  $\text{hom}_+^{\text{irr}}(\pi_1(M), G)/G$  to  $X(F_2, \text{SL}(2, \mathbb{C}))$  is contained in*

$$\{\chi \in X(F_2, \text{SL}(2, \mathbb{C})) \mid \phi^*(\chi) = \bar{\chi}\}$$

*Proof of Lemma 2.4.2.* Let  $\rho \in \text{hom}_+^{\text{irr}}(\pi_1(M), G)$ . If we write  $\rho(t) = A \circ c$  for  $A \in \text{PSL}(2, \mathbb{C})$  and  $c$  complex conjugation, from the relation

$$t\gamma t^{-1} = \phi_*(\gamma) \quad \forall \gamma \in F_2,$$

we get

$$\overline{A\rho(\gamma)A^{-1}} = \rho(\phi_*(\gamma)) \quad \forall \gamma \in F_2.$$

Hence if  $\rho_0$  denotes the restriction of  $\rho$  to  $F_2$ , it satisfies that  $\bar{\rho}_0$  and  $\rho_0 \circ \phi_*$  are conjugate, hence they have the same character and the lemma follows.  $\square$

Lemma 2.4.2 motivates the following computation:

**Lemma 2.4.3.** *We have a homeomorphism:*

$$\{\chi_\rho \in X(F_2, \text{SL}(2, \mathbb{C})) \mid \phi^*(\chi_\rho) = \bar{\chi}_\rho\} \cong \{x \in \mathbb{C} \mid |x - 1| = 1\}$$

by setting  $x = \text{trace}(\rho(s)) = \chi_\rho(s)$ .

*Proof of Lemma 2.4.3.* First at all we describe coordinates for  $X(F_2, \text{SL}(2, \mathbb{C}))$ . Let  $\tau_r, \tau_s$  and  $\tau_{rs}$  denote the trace functions, ie.  $\tau_r(\chi_\rho) = \chi_\rho(r) = \text{trace}(\rho(r))$ , and similarly for  $s$  and  $rs$ . Fricke-Klein's theorem yields an isomorphism

$$(\tau_r, \tau_s, \tau_{rs}): X(F_2, \text{SL}(2, \mathbb{C})) \cong \mathbb{C}^3$$

(see for [19] for a proof). From the relations

$$\phi_*(r) = s, \quad \phi_*(s) = rs, \quad \phi_*(rs) = srs,$$

the equality  $\phi^*(\chi_\rho) = \overline{\chi_\rho}$  is equivalent to:

$$\overline{\tau_r} = \tau_s, \quad \overline{\tau_s} = \tau_{rs}, \quad \overline{\tau_{rs}} = \tau_{srs}.$$

In order to derive the expression for  $\tau_{srs}$  we need the relations

$$\text{trace}(A) = \text{trace}(A^{-1})$$

and

$$\text{tr}(AB) = \text{tr}(A)\text{tr}(B) - \text{tr}(AB^{-1}) \quad \text{for } A, B \in \text{SL}(2, \mathbb{C}),$$

where the latter is obtained from the Cayley-Hamilton theorem  $A^2 - \text{trace}(A)A + \text{Id} = 0$  by multiplying by  $A^{-1}B$  and taking traces. Applying it to  $A = \rho(r)$  and  $B = \rho(rs)$ , we arrive to

$$\tau_{srs} = \tau_s \tau_{rs} - \tau_{ss^{-1}r^{-1}} = \tau_s \tau_{rs} - \tau_r.$$

We take  $x = \tau_r$ , then  $\tau_{rs} = x$  and  $\tau_s = \overline{x}$ . Thus, the defining equation is  $x + \overline{x} = x\overline{x}$ . Namely, the circle  $|x - 1| = 1$ .  $\square$

To prove Proposition 2.4.1, we need to know which conjugacy classes of representations of  $F_2$  are irreducible. By [11], a character  $\chi_\rho$  in  $X(F_2, \text{SL}(2, \mathbb{C}))$

is reducible iff  $\chi_\rho([r, s]) = \text{tr}(\rho([r, s])) = 2$ . The trace of  $\rho([r, s])$  is ([19])

$$\text{tr}(\rho([r, s])) = \tau_r^2 + \tau_s^2 + \tau_{rs}^2 - \tau_r \tau_s \tau_{rs} - 2,$$

and from the identification  $\tau_r = x$  and  $x\bar{x} = x + \bar{x}$ , we can rewrite it as

$$\begin{aligned} 2x^2 + \bar{x}^2 - x^2\bar{x} - 2 &= 2x^2 + \bar{x}^2 - x(x + \bar{x}) - 2 = x^2 + \bar{x}^2 - x\bar{x} - 2 = \\ (x + \bar{x})^2 - 3(x + \bar{x}) - 2 &= (x + \bar{x})((x + \bar{x}) - 3) - 2 = 4\text{Re}(x)^2 - 6\text{Re}(x) - 2. \end{aligned}$$

This is a parabola in terms of  $\text{Re}(x)$ , where  $\text{Re}(x) \in [0, 2]$ . The maximum in this interval is in  $\text{Re}(x) = 2$ , where  $\tau_{[r,s]} = 2$ . This shows that, in the circle  $|x - 1| = 1$ , the only reducible representation is precisely the point  $x = 2$ .

Now, let  $\rho$  be a representation of  $F_2$  in  $\text{SL}(2, \mathbb{C})$  whose character  $\chi_\rho$  satisfies  $\phi^*(\chi_\rho) = \overline{\chi_\rho}$ . Assume  $\rho$  is irreducible, then  $\rho \circ \phi_*$  and  $\bar{\rho}$  are conjugate by a unique matrix  $A \in \text{PSL}(2, \mathbb{C})$ :

$$Ac\rho(\gamma)cA^{-1} = A\overline{\rho(\gamma)}A^{-1} = \rho(\phi_*(\gamma)), \quad \forall \gamma \in F_2,$$

where  $c$  means complex conjugation. Thus, by defining  $\rho(t) = A \circ c$  this gives a unique way to extend  $\rho$  to  $\pi_1(M)$ .

When  $\chi_\rho$  is reducible, then  $x = 2$  and the character  $\chi_\rho$  is trivial. Then either  $\rho$  is trivial or parabolic. We will exhibit that all possible extensions to  $\pi_1(M)$  yield reducible representations. In order to extend a representation  $\langle r, s \mid \rangle$  to a representation of  $\pi_1(M)$ , we only need to choose  $\rho(t)$  such that it verifies the relations

$$\rho(t)\rho(r)\rho(t)^{-1} = \rho(s), \quad \rho(t)\rho(s)\rho(t)^{-1} = \rho(rs).$$

Let  $\rho(r) = \begin{pmatrix} 1 & x_r \\ 0 & 1 \end{pmatrix}$ ,  $\rho(s) = \begin{pmatrix} 1 & x_s \\ 0 & 1 \end{pmatrix} \in \text{PSL}(2, \mathbb{C})$ ,  $\rho(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ c \in \text{PSL}(2, \mathbb{C}) \rtimes \mathbb{Z}_2$ . From  $\rho(t)\rho(r) = \rho(s)\rho(t)$ ,

$$\begin{pmatrix} a & a\bar{x}_r + b \\ c & c\bar{x}_r + d \end{pmatrix} = \begin{pmatrix} a + cx_s & b + dx_s \\ c & d \end{pmatrix},$$

and therefore, either  $x_r = x_s = 0$  ( $\rho$  in the fiber was trivial) or  $c = 0$ . In any of the two cases,  $\rho$  fixes some point in  $\mathbf{CP}^1$ , so the extension to  $\pi_1(M)$  is reducible.  $\square$

## 2.4.2 Comparing both ways of computing deformation spaces

We relate both ways of computing deformation spaces, via the ideal simplex and via the fibration:

**Lemma 2.4.4.** *Given a triangulated structure with parameter  $w$  as in (1.18), the parameter  $x$  of its holonomy as in Proposition 2.4.1 is*

$$x = 1 + w + |w|^2$$

(or  $x = 1 + \bar{w} + |w|^2$ , because  $x$  is only defined up to complex conjugation).

*Proof.* As  $r = UV$ , a straightforward computation yields

$$\rho(r) = \begin{pmatrix} 0 & |w|^2 \\ -\frac{1}{|w|^2} & 1 + w + |w|^2 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}).$$

Then the lemma follows from  $x = \mathrm{trace}(\rho(r))$   $\square$

The fact that not all deformations are obtained from triangulations is reflected in the following remark.

**Remark 2.4.5.** *The image of map*

$$\begin{aligned} \{w \in \mathbb{C} \mid |w(1+w)| = 1\} &\rightarrow \{x \in \mathbb{C} \mid |x-1| = 1\} \\ w &\mapsto x = 1 + w + |w|^2 \end{aligned}$$

is  $\{|x-1| = 1\} \cap \{\mathrm{Re}(x) \geq \frac{3}{2}\}$ , ie. the arc of circle bounded by the image of the holonomy structure (and its conjugate). See Figure 2.1.

*Proof of Remark 2.4.5.* The imaginary part of the image  $1 + w + |w|^2$  is equal to  $\mathrm{Im}w$ . From Figure 1.7, we see that it attains the maximum and

minimum values at points  $w = \frac{-1 \pm i\sqrt{3}}{2}$ . Hence, we see that the image is in the set

$$\{|x - 1| = 1\} \cap \{z \in \mathbb{C} \mid -\sqrt{3}/2 \leq \text{Im } z \leq \sqrt{3}/2\}.$$

This set has two connected components, which are two symmetric arcs with respect to the line  $\{\text{Re } z = 1\}$ . As the domain is connected and  $\text{Im}(1 + w + |w|^2) = [-\sqrt{3}/2, \sqrt{3}/2]$ , the image is precisely one of the two arcs. For  $w = \frac{-1 + i\sqrt{3}}{2}$ ,  $1 + w + |w|^2 = 2$ , hence the image is the arc whose real part is greater or equal than  $\frac{3}{2}$ , as depicted in Figure 2.1.

□

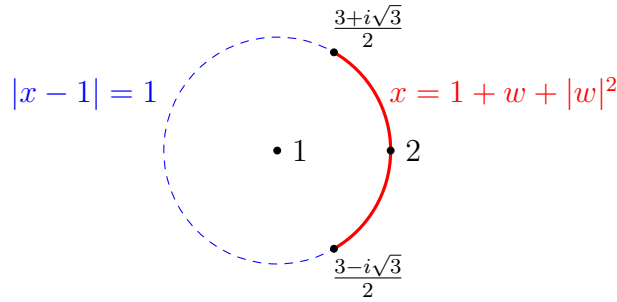


Figure 2.1: The image of  $x = 1 + w + |w|^2$  in the circle  $|x - 1| = 1$ .

To be precise on the type of structures at the peripheral Klein bottle, we compute the trace of the peripheral element  $[r, s]$  for each method. This peripheral element is distinguished (recall Definition 1.2.1) and corresponds to our choice of meridian in the orientation cover (the longitude would be  $t^2$ ).

- We compute it from the variety of representations, i.e. from  $x$ . Using the notation of the proof of Proposition 2.4.1:

$$\begin{aligned} \tau_{[r,s]} &= x_1^2 + x_2^2 + x_3^2 - x_1 x_2 x_3 - 2 = (x + \bar{x})^2 - 3(x + \bar{x}) - 2 = \\ &= (x + \bar{x})((x + \bar{x}) - 3) - 2. \end{aligned}$$



The complete hyperbolic structure corresponds to  $\tau_{[r,s]} = \pm 2$ , which occurs when  $(x + \bar{x}) = 3$ , hence, by deforming  $x$  we may have either  $\tau_{[r,s]} > -2$  or  $\tau_{[r,s]} < -2$ .

- Next we compute it from the ideal triangulation, i.e. from  $w$ . As  $x = 1 + w + |w|^2$ , we get

$$\begin{aligned}\tau_{[r,s]} &= (2 + w + \bar{w} + 2|w|^2)((2 + w + \bar{w} + 2|w|^2) - 3) - 2 \\ &= -4 + w^2 + \bar{w}^2 + 4|w|^4 + w + \bar{w} + 4|w|^2 + 4w|w|^2 + 4\bar{w}|w|^2 \\ &= 2\operatorname{Re}(w + w^2) + 4|w|^2(1 + w + \bar{w} + |w|^2) - 4 \\ &= 2\operatorname{Re}(w + w^2) + 4|w + w^2|^2 - 4.\end{aligned}$$

Because  $|w + w^2| = 1$ , we conclude

$$\tau_{[r,s]} = 2\operatorname{Re}(w + w^2) \geq -2.$$

# 3

## The variety of representations of the Klein bottle

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Deformations of hyperbolic structures in non compact 3-manifolds of finite volume give rise to (possibly non-discrete) representations of the ends into  $\text{Isom}(\mathbb{H}^3)$ . There are two kind of ends when the manifold is non-orientable, the ones related to peripheral tori and the ones related to Klein bottles. Representations of  $\pi_1(T^2)$  are well understood, so we will inspect here the variety of representations for  $\pi_1(K^2)$ . This is an interesting object by itself, but it will also serve us to construct the metric completion of the deformations of the whole  $M^3$  in Chapter 4.

Let  $\pi_1(K^2) = \langle a, b \mid aba^{-1} = b^{-1} \rangle$  be a presentation of the fundamental group of the Klein bottle, and  $G = \text{Isom}(\mathbb{H}^3) \cong \text{PSL}(2, \mathbb{C}) \rtimes \mathbb{Z}_2$ . The variety of representations  $\text{hom}(\pi_1(K^2), G)$  is identified with

$$\text{hom}(\pi_1(K^2), G) = \{A, B \in G \mid ABA^{-1} = B^{-1}\}.$$

Topologically, according to the orientable behaviour of  $A$  and  $B$ , we should be able to distinguish 4 connected components of  $\text{hom}(\pi_1(K^2), G)$ . In fact, we will see that there are actually 8 connected components. We can consider lifts of  $A$  and  $B$  to  $\text{SL}(2, \mathbb{C}) \rtimes \mathbb{Z}_2$ , and obtain a well-defined lift of the relation  $ABA^{-1}B^{-1} = [Id]$  to  $\text{SL}(2, \mathbb{C})$ . Thus, we can further distinguish among the aforementioned 4 connected components according to the Klein bottle relation lifting to  $+Id$  or  $-Id$  in  $\text{SL}(2, \mathbb{C})$ .

Our main interest lies in the following connected component:

**Definition 3.0.1.** A representation  $\rho \in \text{hom}(\pi_1(K^2), G)$  is said to preserve the orientation type if, for every  $\gamma \in \pi_1(K^2)$ ,  $\rho(\gamma)$  is an orientation-preserving isometry if and only if  $\gamma$  is represented by an orientation-preserving loop of  $K^2$ . We denote this subspace of representations by

$$\text{hom}_+(\pi_1(K^2), G).$$

According to our discussion on lifts of the relation, we shall see that  $\text{hom}_+(\pi_1(K^2), G)$  consists of two connected components.

Let  $T^2 \rightarrow K^2$  be the orientation covering. The restriction map on the varieties of representations (without quotienting by conjugation) is:

$$\text{res}: \text{hom}(\pi_1(K^2), G) \rightarrow \text{hom}(\pi_1(T^2), \text{PSL}(2, \mathbb{C})). \quad (3.1)$$

Given the previous presentation of  $\pi_1(K^2)$ , we can choose  $a^2, b$ , the distinguished generators of  $\pi_1(T^2)$ , so that  $\pi_1(T^2) \cong \langle a^2, b \mid a^2b = ba^2 \rangle$ . We denote  $l = a^2$ ,  $m = b$  and call them *distinguished* longitude and meridian, respectively (as in Definition 1.2.1). In terms of the distinguished longitude and meridian the restriction map on the variety of representations can be written as

$$\text{res}(A, B) = (A^2, B). \quad (3.2)$$

The chapter is organized as follows. The first step in order to compute representations will be to study the behaviour of the square map  $A \mapsto A^2$ , which appears as the first coordinate of the restriction map (cf. (3.2)). In particular, we will study the fibers, which on the one hand will serve as very useful tool to compute representations in Section 3.2 and, on the other hand, it will also serve to classify the orientation reversing isometries of  $\mathbb{H}^3$ .

In Section 3.2 we give a list of non-degenerate orientation type preserving representations. Afterwards, in Section 3.3 we show that the restriction map (cf. (3.1)) is a local homeomorphism around a non-degenerate parabolic representation.

Section 3.4 is devoted to compute the homology of the Klein bottle with

twisted coefficients by a parabolic representation and the group cohomology of  $\pi_1(K^2)$  (which is dual to the twisted homology). As a consequence, we show that the representation variety is smooth around said representation and it has real dimension 7.

We end the chapter with a comprehensive list of representations, which serves to prove our claim on the existence of precisely 8 connected components.

### 3.1 The square map

Let  $G$  denote the group of isometries of hyperbolic 3-space,  $\mathbb{H}^3$ . We will denote by  $G_+$  the connected component of the identity, that is, the subgroup of orientation preserving isometries,

$$G_+ = \text{Isom}^+(\mathbb{H}^3) \cong \text{PSL}(2, \mathbb{C}),$$

and by  $G_-$ , the subset of orientation reversing ones,

$$G_- = \text{Isom}^-(\mathbb{H}^3) \subset \text{PSL}(2, \mathbb{C}) \rtimes \mathbb{Z}_2,$$

hence  $G = G_- \sqcup G_+$ .

Given an orientation preserving isometry  $A \in G_+$ ,  $A_c = A \circ c \in G_-$  will be the composition of  $A$  and the complex conjugation  $c$ . Any orientation reversing isometry can be written down as the composition of a orientation preserving isometry and  $c$ .

The universal cover of  $G_+$  is the group  $\text{SL}(2, \mathbb{C})$  and will be denoted by  $\widetilde{G}_+$ . There is a natural map we are interested in:

$$\begin{aligned} Q : G &\rightarrow \widetilde{G}_+ \\ [A] &\mapsto A^2, \end{aligned} \tag{3.3}$$

which is well defined as  $(\pm A)^2 = A^2$ .

The behaviour of  $Q$  changes drastically in each connected component  $G_-$  and  $G_+$ . We will be interested first in its restriction to  $G_-$ . The following

proposition describes the fiber of the square map  $Q$ .

**Proposition 3.1.1.** *Let  $B$  be a matrix in Jordan form in  $\widetilde{G}_+$  and let us consider the restriction  $Q : G_- \rightarrow \widetilde{G}_+$ . Then:*

- If  $B = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ , with  $\lambda \in \mathbb{R}_+ \setminus \{1\}$ , the fiber of  $B$  is

$$Q^{-1}(B) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}_c \mid a \in \mathbb{C}^*, |a|^2 = \lambda \right\}.$$

- If  $B = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ ,  $\theta \in \mathbb{R} \setminus \{n\pi \mid n \in \mathbb{Z}\}$ , the fiber of  $B$  is

$$Q^{-1}(B) = \left\{ \begin{pmatrix} 0 & \rho e^{i(\theta+\pi)/2} \\ -\rho^{-1} e^{-i(\theta+\pi)/2} & 0 \end{pmatrix}_c \mid \rho \in \mathbb{R}^* \right\}.$$

- If  $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ , the fiber of  $B$  is

$$Q^{-1}(B) = \left\{ \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}_c \mid \tau \in \mathbb{C}, \operatorname{Re}(\tau) = 1/2 \right\}.$$

- If  $B = Id$ , the fiber of  $B$  is

$$Q^{-1}(Id) = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix}_c \mid a \in \mathbb{C}, b, c \in i\mathbb{R}, |a|^2 - bc = 1 \right\} = \operatorname{Ad}(G_+) \left( \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}_c \right).$$

- If  $B = -Id$ , the fiber of  $B$  is

$$\begin{aligned} Q^{-1}(-Id) &= \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix}_c \mid a \in \mathbb{C}, b, c \in \mathbb{R}, |a|^2 + bc = -1 \right\} = \\ &= \operatorname{Ad}(G_+) \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_c \right). \end{aligned}$$

- Otherwise, the fiber of  $B$  is empty.

Furthermore, in every case the fiber is connected.

*Proof.* The conjugacy class of any element in  $\widetilde{G}_+$  is either diagonal  $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$  or parabolic  $\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ , so we only have to compute the fibers for these two kinds of matrices. This is straightforward but tedious.

Let  $A \in Q^{-1}(B)$ ,  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}_c$ . Then,

$$A^2 = \begin{pmatrix} |a|^2 + b\bar{c} & a\bar{b} + b\bar{d} \\ c\bar{a} + \bar{c}d & |d|^2 + \bar{b}c \end{pmatrix}. \quad (3.4)$$

We will solve for  $A$  in the equation  $A^2 = B$ .

Case 1: If  $B$  is diagonal, we have from the off-diagonals entries,  $a\bar{b} + b\bar{d} = 0$  and  $\bar{c}a + c\bar{d} = 0$ . By multiplying the first expression by  $c$ , the second one by  $b$  and subtracting them we get  $0 = a(2i\text{Im}(c\bar{b}))$ . Similarly, if we multiply the first expression by  $\bar{c}$  and the second one by  $\bar{b}$ , we obtain  $0 = d(2i\text{Im}(\bar{c}b))$ . Hence, either  $\text{Im}(\bar{c}b) = 0$  or  $\text{Im}(\bar{c}b) \neq 0$  (then,  $a = d = 0$ ).

If  $\text{Im}(\bar{c}b) \neq 0$ :  $a = d = 0$ , then the only possibility is  $A = \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}_c$ ,  $B = \begin{pmatrix} -b/\bar{b} & 0 \\ 0 & -\bar{b}/b \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ , for some  $\theta \in \mathbb{R}$ .

If  $\text{Im}(\bar{c}b) = 0$ : Then  $\lambda \in \mathbb{R}$ . We can assume either  $a$  or  $d$  different from zero as this was already covered. Indeed, it is easy to see that if one of them equal to zero, the other one is zero too. We can write  $d = \frac{1+bc}{a}$  and substitute in one of the off-diagonal entries of (3.4) to get  $-b = \bar{b}(|a|^2 + \bar{c}b) = \bar{b}(\lambda)$ . We get a similar equation for  $c$ . We conclude that either  $\lambda = \pm 1$  ( $B = Id$ ) or  $b = c = 0$ .

- If  $b = c = 0$ , we have  $d = a^{-1}$  and  $|a|^2 = \lambda$ .
- If we consider  $\lambda = +1$ ,  $B = Id$ ,  $b, c \in i\mathbb{R}$ , from the off-diagonal equations we get  $d = \bar{a}$  and, from the diagonal entries, it must also be satisfied  $|a|^2 + b\bar{c} = 1$  (which is equivalent to  $\det(A) = 1$ ).
- If we consider  $\lambda = -1$ ,  $B = -Id$ ,  $b, c \in \mathbb{R}$ , and, analogously,  $d = -\bar{a}$  and  $|a|^2 + bc = -1$  (equivalent again to  $\det(A) = 1$ ).

Last two cases correspond to the fiber of  $\pm Id$ . By conjugating first by some parabolic transformation  $\begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}$  and then by a matrix  $\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$  we can see that  $Q^{-1}(Id) = \text{Ad}(G_+) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}_c$  and  $Q^{-1}(-Id) = \text{Ad}(G_+) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_c$ .

Case 2: If  $B$  is parabolic, from the diagonal equations we get  $b\bar{c} \in \mathbb{R}$  and therefore, multiplying the off-diagonal equation  $a\bar{b} + b\bar{d} = \pm 1$  by  $c$  we obtain  $c = 0$ . Hence,  $d = a^{-1}$  and  $|a| = 1$ . Finally, by writing  $a$  and  $b$  in

polar form and focusing on the off-diagonal equation, we obtain  $a = \pm 1$  and  $\operatorname{Re} b = \pm 1/2$ . □

**Remark 3.1.2.** *If  $B$  is not in Jordan form and conjugating it by  $g \in \widetilde{G}_+$  takes it to the Jordan form, then the fiber  $Q^{-1}(B)$  is computed from the cases of Proposition 3.1.1 by conjugating the corresponding fiber by  $g^{-1}$ .*

**Remark 3.1.3.** Let  $A \in G_-$  such that  $Q(A) \neq Id$ . We will say  $A$  is *hyperbolic*, *elliptic* or *parabolic* according to the usual clasification of isometries with respect to the fixed points in  $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$ . Moreover,  $A$  is hyperbolic, elliptic or parabolic iff  $Q(A)$  is hyperbolic, elliptic or parabolic, respectively.

**Corollary 3.1.4.** *The image of  $Q$  is*

$$\mathcal{J} := \{A \in \widetilde{G}_+ \mid \operatorname{tr}(A) \in (-2, \infty)\} \cup \{-Id\},$$

where  $\operatorname{tr}(A)$  denotes the trace of  $A$ .

**Remark 3.1.5.** Proposition 3.1.1 can be extended to the quotient map  $\overline{Q} : G_- \mapsto G_+$ . Then,  $\overline{Q}^{-1}([Id])$  and  $\overline{Q}^{-1}([\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}]))$  have two connected components whereas  $\overline{Q}^{-1}([\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}]))$  and  $\overline{Q}^{-1}([\begin{smallmatrix} 1 & \tau \\ 0 & 1 \end{smallmatrix}]))$  just one.

**Corollary 3.1.6** (Classification of non-orientable isometries of  $\mathbb{H}^3$ ). *Let us consider an orientation reversing isometry of hyperbolic 3-space. Then, up to conjugation, it is one of the following:*

- *Composition of a reflection on a hyperplane with hyperbolic translation in an axis contained in said hyperplane.*
- *Composition of a reflection on a hyperplane with an elliptic transformation with axis perpendicular the aforementioned hyperplane.*
- *Composition of a reflection on a hyperplane with a parabolic transformation with fixed point an ideal point of the hyperplane.*
- *Reflection on a hyperplane.*

- *Inversion through a point.*

*Proof.* Interpret each case of Proposition 3.1.1. □

**Remark 3.1.7.** Let  $B \in \widetilde{G}_+$  and consider its fiber by  $Q$  as in Proposition 3.1.1. According to Corollary 3.1.6, elements in the same fiber  $Q^{-1}(B)$  differ by the choice of hyperplane in which to reflect (or point with respect to do the inversion in the last case). Thus, the elements in a fiber are conjugated. In particular, let  $G_d, G_r < G_+$  be the subgroups of  $G_+$  of dilations and rotations, respectively, leaving invariant the geodesic from 0 to  $\infty$ , that is

$$G_d = \left\{ \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \mid \mu \in \mathbb{R} \right\}, \quad G_r = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \mid \theta \in \mathbb{R} \right\}.$$

Then, it is a straightforward computation that the fiber of a dilation (in Jordan form) satisfies

$$Q^{-1}(B) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}_c \mid |a|^2 = \lambda \right\} = \text{Ad}(G_r) \left( \begin{pmatrix} |a| & 0 \\ 0 & |a|^{-1} \end{pmatrix}_c \right).$$

Similarly, for a rotation, the fiber  $Q^{-1}(B)$  is equal to

$$\left\{ \begin{pmatrix} 0 & \rho e^{i(\theta+\pi)/2} \\ -\rho^{-1} e^{-i(\theta+\pi)/2} & 0 \end{pmatrix}_c \mid \rho \in \mathbb{R}^* \right\} = \text{Ad}(G_d) \left( \begin{pmatrix} 0 & e^{i(\theta+\pi)/2} \\ -e^{-i(\theta+\pi)/2} & 0 \end{pmatrix}_c \right).$$

Finally, in the case of a parabolic transformation (in Jordan form), let  $G_{par} < G_+$  denote the subgroup of parabolic transformations leaving the  $\infty$  point fixed. Then, the fiber of a parabolic transformation is

$$Q^{-1}(B) = \left\{ \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}_c \mid \text{Re}(\tau) = 1/2 \right\} = \text{Ad}(G_{par}) \left( \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}_c \right).$$

If we consider the restriction  $Q : G_+ \rightarrow \widetilde{G}_+$ , which we denote  $Q_+$ , we have the following proposition, of which a version for  $\text{PSL}(2, \mathbb{R})$  can be found in [39].



**Proposition 3.1.8.** 1. For any  $B \in \widetilde{G}_+ \setminus \text{tr}^{-1}(-2)$ , there is a unique  $A \in G_+$  such that  $Q_+(A) = B$ , given by

$$A = \left[ \frac{B + Id}{\sqrt{\text{tr}B + 2}} \right].$$

2. The fiber  $Q_+^{-1}(-Id)$  is the set  $\text{Ad}(G_+) \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right)$ .

3. For  $B = \begin{pmatrix} -1 & \tau \\ 0 & -1 \end{pmatrix}$ ,  $\tau \neq 0$ , the fiber is empty in  $G_+$ .

*Proof.* The proof is analogous to the one of the  $\text{PSL}(2, \mathbb{R})$ -case. First, given any matrix  $A \in M_{2 \times 2}(\mathbb{R})$ ,  $\text{tr}A^2 = (\text{tr}A)^2 - 2\det(A)$ . First assertion is due to the Cayley-Hamilton theorem, that is, for  $A \in \text{SL}(2, \mathbb{C})$ ,  $A^2 - A \text{tr}A + Id = 0$ . Thus, if  $\text{tr}A^2 \neq -2$ ,

$$A = \frac{A^2 + Id}{\text{tr}A} = \pm \frac{A^2 + Id}{\sqrt{\text{tr}A^2 + 2}}.$$

On the other hand, the second assertion comes from the fact that  $\text{tr}(A^2) = -2$  if and only if  $\text{tr}(A) = 0$ . Then, it is a straightforward computation that for any  $A \in G_+$  such that  $\text{tr}(A) = 0$ ,  $A^2 = -Id$ . This also proves the last assertion.  $\square$

## 3.2 Orientation type preserving representations

Our main interest lies in a neighbourhood of non-degenerate representations of the Klein bottle in the quotient  $\text{hom}_+(\pi_1(K^2), G)/G$ . Recall that  $\pi_1(K^2)$  admits a presentation  $\langle a, b \mid aba^{-1} = b^{-1} \rangle$ ; the following theorem describes what kind of ‘non-degenerate’ representations we can expect in terms of the generators  $a$  and  $b$ :

**Theorem 3.2.1.** *Let  $\rho \in \text{hom}_+(\pi_1(K^2), G)$  preserve the orientation type and let  $\rho(b) \neq \text{Id}$ . By writing  $A = \rho(a)$ ,  $B = \rho(b)$  as Möbius transformations, up to conjugation one of the following holds:*

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- a)  $A(z) = \bar{z} + 1, B(z) = z + \tau i, \text{ with } \tau \in \mathbb{R}_{>0}.$
- a')  $A(z) = \bar{z}, B(z) = z + \tau i, \text{ with } \tau \in \mathbb{R}_{>0}.$
- b)  $A(z) = e^l \bar{z}, B(z) = e^{i\theta} z, \text{ with } l \in \mathbb{R}_{\geq 0}, \theta \in (0, \pi].$
- c)  $A(z) = e^{i\theta} / \bar{z}, B(z) = e^l z, \text{ with } l \in \mathbb{R}_{>0}, \theta \in [0, \pi].$
- d)  $A(z) = e^{i\theta} / \bar{z}, B(z) = -e^l z, \text{ with } l \in \mathbb{R}_{>0}, \theta \in [0, \pi].$

*Proof.* The idea of the proof is applying the restriction map (3.1) to a representation on the Klein bottle and look for its image in the variety of representations  $\text{hom}(\pi_1(T^2), G_+)$ .

The variety of representations  $\text{hom}(\pi_1(T^2), G_+)/G_+$  is well known. A representation  $[\rho_0]$  in this variety is the class of either a parabolic representation,  $\rho_0(l)(z) = z + 1, \rho_0(m)(z) = z + \tau, \tau \in \mathbb{C}$ , a parabolic degenerated one,  $\rho_0(l)(z) = z, \rho_0(m)(z) = z + \tau, \tau \in \mathbb{C}$ , or a hyperbolic or elliptic one  $\rho_0(l)(z) = \lambda z, \rho_0(m)(z) = \mu z, \lambda, \mu \in \mathbb{C}$ , where  $\pi_1(T^2) = \langle l, m | lm = ml \rangle$ .

For  $\rho_0 = \text{res}(\rho)$ , let  $A = \rho(a), B = \rho(b)$  where  $a, b$  are generators of  $\pi_1(K^2)$ , and  $L = \rho_0(l), M = \rho_0(m)$ . The following is satisfied:

$$\begin{aligned} (A^2, B) &= (L, M), & (\text{Restriction of a representation to the torus}) \\ ABA^{-1} &= B^{-1}. & (\text{Klein bottle relation}) \end{aligned}$$

In fact, in order for  $\rho_0$  to be a restriction, there must be  $A$  and  $B$  satisfying the previous conditions. We prove the theorem using these equations.

If  $[\rho_0]$  is in the parabolic case, by hypothesis  $\tau \neq 0$ . Then, we can write  $B = M = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$ . The restriction  $A^2 = L$  yields from Proposition 3.1.1,

$$A = \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}_c, \quad \text{where } \text{Re } \nu = 1/2.$$

The elements  $A, B$  must satisfy  $AB = B^{-1}A$ , that is

$$\begin{pmatrix} 1 & \nu + \bar{\tau} \\ 0 & 1 \end{pmatrix} = \pm \begin{pmatrix} 1 & \nu - \tau \\ 0 & 1 \end{pmatrix},$$

where the  $\pm$  sign is due to the matrices actually being in  $\mathrm{PSL}(2, \mathbb{C})$ . The relation implies that  $\tau \in i\mathbb{R}$ . Furthermore, let  $\omega = -\frac{i}{2}\mathrm{Im} \nu$  and  $g = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \in G_+$ , then  $\mathrm{Ad}(g)A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}_c$ ,  $\mathrm{Ad}(g)B = B$ . Thus, in the quotient the solution is unique and,  $A(z) = \bar{z} + 1/2$ ,  $B(z) = z + \tau$ ,  $\tau \in i\mathbb{R} \setminus \{0\}$ , hence  $L(z) = z + 1$ ,  $M(z) = z + \tau$ .

Similarly, for the degenerated parabolic case, let  $A = \begin{pmatrix} x & y \\ w & z \end{pmatrix}_c$ ,  $B = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$ . The Klein bottle relation implies  $w = 0$ , and, from  $A^2 = Id$ , we have  $z = \bar{x}$ ,  $|x| = |z| = 1$  and  $\mathrm{Re} y = 0$ . That is,  $A = \begin{pmatrix} e^{i\theta} & y \\ 0 & e^{-i\theta} \end{pmatrix}_c$ , for some  $\theta \in \mathbb{R}$ , so let now  $g = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$  so that  $\mathrm{Ad}(g)(A) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}_c$ ,  $\mathrm{Ad}(g)(B) = \begin{pmatrix} 1 & \tau' \\ 0 & 1 \end{pmatrix}$ . Therefore, conjugating again by a translation as before and taking into account the Klein bottle relation once more, we see that the representation is conjugated to  $A(z) = \bar{z}$ ,  $B(z) = z + \tau''i$ , for some  $\tau'' \in \mathbb{R}$ .

Otherwise, for  $[\rho]$  hyperbolic or elliptic, according to Proposition 3.1.1, either  $L$  corresponds to a real dilation or to a rotation. If  $L$  is a dilation, let us write (see Remark 3.1.7)  $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}_c$ ,  $B = M = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$ , for  $\lambda \in \mathbb{R}$ ,  $\mu \in \mathbb{C}$ . The Klein bottle relation can be then written as

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \bar{\mu} & 0 \\ 0 & \bar{\mu}^{-1} \end{pmatrix} = \pm \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

which yields  $|\mu| = +1$ , that is  $\mu = e^{i\theta/2}$ , for  $\theta \in (0, 2\pi)$ , hence  $B$  is a rotation. In terms of Möbius transformations, let  $\lambda = e^{l/2}$ ,  $l \in \mathbb{R} \setminus \{0\}$ , then  $A(z) = e^l \bar{z}$ ,  $B(z) = e^{i\theta} z$ .

If  $L$  is a rotation, the situation is similar to the previous one, we can write  $A = \begin{pmatrix} 0 & e^{i\theta/2} \\ -e^{-i\theta/2} & 0 \end{pmatrix}_c$ ,  $B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$ , for  $\theta \in [0, 2\pi]$ ,  $\mu \in \mathbb{C}$  and from the Klein bottle relation deduce that  $\mu = \pm \bar{\mu}$ . Then, either  $\mu \in \mathbb{R}$  or  $\mu \in i\mathbb{R}$ . Thus, on the one hand we can denote  $\mu = e^{l/2}$ ,  $l \in \mathbb{R} \setminus \{0\}$  and we have representations where  $A(z) = e^{i\theta}/\bar{z}$ ,  $B(z) = e^l z$ . On the other hand, if we denote  $\mu = ie^{l/2}$ ,  $l \in \mathbb{R} \setminus \{0\}$ , we obtain representations  $A(z) = e^{i\theta}/\bar{z}$ ,  $B(z) = -e^l z$ .

Thus, we obtain a classification of representations in  $\mathrm{hom}(\pi_1(K^2), G)/G_+$ . To get the classification quotienting by the whole group,  $\mathrm{hom}(\pi_1(K^2), G)/G$ , we only have to see how the complex conjugation  $c$  acts by conjugation on

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each representation: In  $a)$ ,  $a')$ ,  $c$  maps  $z + \tau i \mapsto z - \tau i$ ; in  $b)$ ,  $e^{i\theta} z \mapsto e^{-i\theta} z$ ; and in  $c)$ ,  $d)$ ,  $e^{i\theta} / \bar{z} \mapsto e^{-i\theta} / \bar{z}$ . The choice  $\alpha > 0, l > 0$  in  $b)$ ,  $c)$ ,  $d)$  is obtained by taking into account that  $[\rho] = [\rho^{-1}]$ .

□

**Definition 3.2.2.** *According to the different cases of Theorem 3.2.1, a representation  $\rho \in \text{hom}_+(\pi_1(K^2), G)$  is called:*

- parabolic non-degenerate in case  $a)$  and parabolic degenerate in case  $a')$ ,
- type I in case  $b)$ ,
- type II in case  $c)$ , and
- type III in case  $d)$ .

*Furthermore, type I and type II or III are called non-degenerate if  $l \neq 0$  or  $\theta \neq 0$  respectively, and degenerate otherwise.*

**Remark 3.2.3.** Type III representations belong to a different connected component than the other representations from Theorem 3.2.1 (see Section 3.5 and Chapter 5 for more details).

**Remark 3.2.4.** Let  $M^3$  be a complete, non-orientable hyperbolic 3-manifold of finite volume. The holonomy of a non-orientable cusp restricts to a representation of the Klein bottle that preserves the orientation type and is parabolic non-degenerate.

Furthermore, deformations of this representation still preserve the orientation type and are non-degenerate (possibly of type I or II by Remark 3.2.3), by continuity.

For  $\gamma \in \pi_1(T^2) \triangleleft \pi_1(K^2)$ , recall from (2.2) that

$$\begin{aligned} I_\gamma: \text{hom}(\pi_1(K^2), G) &\rightarrow \mathbb{C} \\ \rho &\mapsto (\text{trace}_{\text{PSL}(2, \mathbb{C})}(\rho(\gamma)))^2 - 4, \end{aligned}$$

where  $\text{trace}_{\text{PSL}(2, \mathbb{C})}$  means trace as matrix in  $\text{PSL}(2, \mathbb{C})$ .

**Lemma 3.2.5.** *Let  $\rho \in \text{hom}(\pi_1(K^2), G)$  preserve the orientation type and  $\rho(b) \neq \text{Id}$ . Then:*

- *If  $\rho$  is parabolic, then  $I_\gamma(\rho) = 0, \forall \gamma \in \pi_1(T^2)$ .*
- *If  $\rho$  is of type I, then  $I_{a^2}(\rho) \geq 0$  and  $I_b(\rho) < 0$ .*
- *If  $\rho$  is of type II, then  $I_{a^2}(\rho) \leq 0$  and  $I_b(\rho) > 0$ .*
- *If  $\rho$  is of type III, then  $I_{a^2}(\rho) \leq 0$  and  $I_b(\rho) \in i\mathbb{R}$ .*

*Proof.* It is a straightforward computation from Theorem 3.2.1. □

**Remark 3.2.6.** In the case of the Gieseking manifold, at the end of Subsection 2.4.2, we computed  $\text{trace}_{\text{SL}(2, \mathbb{C})}(b)$  (which in that case was well defined) for the representations obtained in the cusp. The result was that for deformation of the triangulation  $\text{trace}_{\text{SL}(2, \mathbb{C})}(b) > -2$ , whereas from the variety of representations, either  $\text{trace}_{\text{SL}(2, \mathbb{C})}(b) > -2$  or  $\text{trace}_{\text{SL}(2, \mathbb{C})}(b) < -2$ . Thus,  $I_b(\rho) < 0$  for representations associated to deformations of the triangulation, that is, by Lemma 3.2.5, representations of type I and, in general, either representations of type I or type II could happen.

In fact, we notice that path of deformations of the Gieseking manifold lifts to a path of deformations of the figure-eight knot exterior that is the same considered by Hilden, Lozano and Montesinos in [25] from deformation of a polyhedron. What the authors of [25] call spontaneous surgery corresponds to the transition from type I to type II of the Gieseking manifold.

### 3.3 The restriction map

This section is dedicated to seeing some properties of the restriction map. The goal is proving that, in the quotient, it is a local homeomorphism. Let us start by showing some immediate properties:

**Lemma 3.3.1.** *The maps  $\text{res} : \text{hom}_+(\pi_1(K^2), G) \rightarrow \text{hom}(\pi_1(T^2), G_+)$  and  $\text{res} : \text{hom}_+(\pi_1(K^2), G)/G_+ \rightarrow \text{hom}(\pi_1(T^2), G_+)/G_+$  are continuous.*

*Proof.* The first statement is trivial from (3.2). The second one is a consequence of the universal property of the quotient.  $\square$

Let  $\text{hom}_+^0(\pi_1(K^2), G)$ ,  $\text{hom}^0(\pi_1(T^2), G)$  denote the subsets of non-degenerate representations in their respective varieties. Notice that, for us, a representation of the Klein bottle is degenerate iff its restriction to the torus (its orientation covering) is degenerate, that is,  $\rho(a^2)$  or  $\rho(b)$  equal the identity.

**Proposition 3.3.2.** *The restriction map in the quotient*

$$\text{res} : \text{hom}_+^0(\pi_1(K^2), G)/G_+ \rightarrow \text{hom}^0(\pi_1(T^2), G_+)/G_+$$

*is not injective.*

*Proof.* Let us consider two representations of type II:  $A(z) = e^{i\theta}/\bar{z}$ ,  $B(z) = e^l z$  and  $A'(z) = -e^{i\theta}/\bar{z}$ ,  $B'(z) = B(z)$ . Clearly, their image under the restriction map is the same. On the other hand, they are not conjugate; for instance, under the square map  $Q$  (cf. (3.3)),  $A$  and  $A'$  go to elements with different trace:

$$Q(A) = \begin{pmatrix} -e^{i\theta} & 0 \\ 0 & -e^{-i\theta} \end{pmatrix}, \quad Q(A') = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

The same happens when considering type III representations.  $\square$

We can define an action of  $\mathbb{Z}_2$  on the variety of representations induced by the covering transformation of  $T^2 \rightarrow K^2$ . Let  $a, b$  be our usual generators of  $\pi_1(K^2)$ , that is, they satisfy  $aba^{-1} = b^{-1}$ , and notice that the covering transformation  $\iota : T^2 \rightarrow T^2$  (cf. (1.7)) corresponds to conjugation by  $a$ . Let  $\rho \in \text{hom}_+(\pi_1(K^2), G)$ , then the action by conjugation of  $\rho(a) = c \circ A$ , for  $A \in G_+$ , on  $\rho(a^2)$  and  $\rho(b)$  is

$$\rho(a)\rho(a^2)\rho(a)^{-1} = \rho(a^2), \quad \rho(a)\rho(b)\rho(a)^{-1} = \rho(b)^{-1}.$$

This shows how the action of  $\rho(a)$  by conjugation is on  $\text{Im}(\text{res}) \subset \text{hom}^0(\pi_1(T^2), G_+)$ . We are interested in extending the action to the quo-

tient  $\text{hom}(\pi_1(T^2), G_+)/G_+$ , where given any class of representation  $[\rho]$ , conjugating by  $c \circ A$  equals to the representation  $[\bar{\rho}]$ . Thus, for  $\gamma \in \pi_1(T^2)$ ,  $\rho \in \text{hom}(\pi_1(T^2), G_+)$ , we define the action of  $\iota$  on  $\rho$ ,  $\iota \cdot \rho$ , as follows:

$$(\iota \cdot \rho)(\gamma) := \overline{\rho(\iota_* \gamma)}. \quad (3.5)$$

The action passes down to the quotient, where it will also be denoted by  $\iota$ .

**Lemma 3.3.3.** *Let  $\rho$  be a non-degenerate parabolic representation of the Klein bottle,  $\rho \in \text{hom}_+(\pi_1(K^2), G)$  and let  $\rho_0 = \text{res}(\rho)$  be its restriction to the torus. There exist neighbourhoods  $U \subset \text{hom}_+(\pi_1(K^2), G)/G_+$ ,  $V \subset (\text{hom}^0(\pi_1(T^2), G_+)/G_+)$  of  $[\rho]$  and  $[\rho_0]$ , respectively, such that the image  $U$  under the restriction map is identified with  $V^\iota$ .*

*Proof.* From Theorem 3.2.1, it is clear that

$$\text{res}(\text{hom}_+(\pi_1(K^2), G)/G_+) \subset (\text{hom}^0(\pi_1(T^2), G_+)/G_+)^\iota.$$

On the other direction, let  $[L, M] = [\bar{L}, \bar{M}^{-1}]$ , where  $L, M$  is the image by  $\rho$  of a longitude-meridian pair. If the representation is parabolic, we can assume  $L(z) = z + 1$ ,  $M(z) = z + \tau$ , and we have

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \pm \text{Ad}(g) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} = \pm \text{Ad}(g) \begin{pmatrix} 1 & -\bar{\tau} \\ 0 & 1 \end{pmatrix}, \quad (3.6)$$

where  $g \in G_+$ . The adjoint representation appears because the equation is in the quotient, whereas the signs are due to the equation being in  $\text{PSL}(2, \mathbb{C})$ . It is clear that  $g$  must fix  $\infty$ ,  $g = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ , and that  $x = z = \pm 1$ , so  $g$  acts trivially. Thus, the only possible outcome is  $\tau = -\bar{\tau}$ , that is,  $\tau \in i\mathbb{R}$ . The obtained representation is the image of a parabolic representation of the Klein bottle.

In the non-parabolic case, we can assume  $L(z) = \lambda z$ ,  $M(z) = \mu z$ ,  $\lambda \neq 1, \mu \neq 1$ . Thus, the expression  $[L, M] = [\bar{L}, \bar{M}^{-1}]$  can be written down

as

$$\begin{aligned} \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix} &= \pm \text{Ad}(g) \begin{pmatrix} \bar{\lambda}^{1/2} & 0 \\ 0 & \bar{\lambda}^{-1/2} \end{pmatrix}, \\ \begin{pmatrix} \mu^{1/2} & 0 \\ 0 & \mu^{-1/2} \end{pmatrix} &= \pm \text{Ad}(g) \begin{pmatrix} \bar{\mu}^{-1/2} & 0 \\ 0 & \bar{\mu}^{1/2} \end{pmatrix}, \end{aligned} \quad (3.7)$$

for  $g \in G_+$ . Now  $g$  has to either fix 0 and  $\infty$  or permute them, that is, either  $g$  is hyperbolic or elliptic fixing both points (then the adjoint action is trivial) or  $\begin{pmatrix} 0 & -\omega \\ \omega^{-1} & 0 \end{pmatrix}$  (we can assume  $\omega = 1$ ). Therefore, either  $\lambda = \pm\bar{\lambda}$ ,  $|\mu| = \pm 1$  or  $|\lambda| = \pm 1$ ,  $\mu = \pm\bar{\mu}$ . In the case  $\lambda = \bar{\lambda}$ ,  $|\mu| = 1$ , we have a restriction of a representation of type I. The case  $\lambda = -\bar{\lambda}$ ,  $|\mu| = 1$ , implies  $\lambda \in i\mathbb{R}$ , therefore,  $\text{tr}A \in i\mathbb{R}$ , which can be avoided by choosing  $V$  small enough. On the other hand,  $|\lambda| = 1$ ,  $\mu = \bar{\mu}$  if and only if we have the restriction of a type II representation. The case  $|\lambda| = 1$ ,  $\mu = -\bar{\mu}$  corresponds to the restriction of a type III representation. Finally, the cases  $|\lambda| = -1$  or  $|\mu| = -1$  cannot happen.

As we have seen, as long as  $V$  avoids any (non-zero) pure imaginary number as trace of  $L$ , the lemma is true for  $U = \text{res}^{-1}(V^\iota)$  by Lemma 3.3.1.  $\square$

Lemma 3.3.3 can be further refined to a local homeomorphism if the neighbourhoods are chosen with care.

**Lemma 3.3.4.** *Let  $[\rho] \in \text{hom}_+^0(\pi_1(K^2), G)/G_+$  be a non-degenerate parabolic representation, then there exists a neighbourhood  $U \subset \text{hom}_+^0(\pi_1(K^2), G)/G_+$  of it such that the restriction map  $\text{res}|_U$  is injective.*

*Proof.* As shown in Proposition 3.3.2, the problem lies within the type II and III representations. In addition, the proof of Proposition 3.3.2 shows a way to choose a neighbourhood. Let  $V$  be a ‘small neighbourhood’ (for instance, let us only consider elliptic transformations of ‘angle of rotation’  $< \pi/2$ ) in  $\text{hom}_+^0(\pi_1(T^2), G)/G_+$ , and let  $U = \text{res}^{-1}(V)$ , which is a neighbourhood of  $[\rho]$  by Lemma 3.3.1. If  $V$  is small enough, then we can assume that  $U$  does not contain type III representations (see Remark 3.2.3



and Lemma 3.2.5). Notice that it is not evident that a neighbourhood of a parabolic representation contains representations of type II. We will show that, by our choice of neighbourhood,  $\text{res}^{-1}(V)$  has 2 connected components  $U_1$  and  $U_2$ , the first one containing parabolic representations as well as representations of type I and II, whereas the second one only contains representations of type II.

An elliptic element  $A_\theta = \begin{bmatrix} 0 & e^{i\theta} \\ -e^{-i\theta} & 0 \end{bmatrix}_c \in G$  whose square  $[Q(A)]$  is close to the identity satisfies either  $\theta$  close to  $\pi/2$  or to 0. We will use  $\sim$  to indicate closeness. Thus, for  $[A_\theta, B]$ , where  $B = \begin{pmatrix} e^{v/2} & 0 \\ 0 & e^{-v/2} \end{pmatrix}$ ,  $v \in \mathbb{R}$ , to be close to a parabolic representation,  $\theta$  must be  $\theta \sim \pi/2$  (equivalently,  $3\pi/2$ ) or  $\theta \sim 0$  (equivalently,  $\pi$ ).

Let  $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$ . Hence,

$$gAg^{-1} = \begin{pmatrix} -e^{i\theta}\alpha\bar{\gamma} - e^{-i\theta}\beta\bar{\delta} & e^{i\theta}|\alpha|^2 + e^{-i\theta}|\beta|^2 \\ -e^{i\theta}|\gamma|^2 - e^{-i\theta}|\delta|^2 & e^{i\theta}\bar{\alpha}\gamma + e^{-i\theta}\bar{\delta}\beta \end{pmatrix}_c.$$

We want to check when this can be close to the parabolic element  $\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}_c$ . Let us check some conditions to be met:

- First,  $e^{i\theta}|\alpha|^2 + e^{-i\theta}|\beta|^2 \sim 1/2$ . Looking into real and imaginary parts, we observe that this occurs if and only if  $|\alpha|^2 \sim |\beta|^2 \sim \frac{1}{4\cos\theta}$ .
- Second,  $e^{i\theta}|\gamma|^2 + e^{-i\theta}|\delta|^2 \sim 0$ . Then, if  $\theta$  is close enough to  $\pi/2$ , this adds the condition  $|\gamma| \sim |\delta|$  but they can be anything in a bounded region. The closer  $\pi/2$ , the bigger the region can be. On the other hand, if  $\theta \sim 0$ ,  $|\gamma| \sim |\delta| \sim 0$ .

Putting together the previous conditions, and taking into account that

$$1 = \alpha\delta - \beta\gamma = |\alpha\delta - \beta\gamma| \leq |\alpha||\delta| + |\beta||\gamma|,$$

we see that for  $\theta$  close to 0,  $gAg^{-1}$  cannot be near a parabolic element.

In order to show that for  $\theta \sim \pi/2$  there is a representation in the

conjugacy class which is near some parabolic representation, we take

$$g = \begin{pmatrix} \frac{1}{2\sqrt{|\cos \theta|}} \pm \frac{i}{2\sqrt{|\cos \theta|}} \\ 0 & 2\sqrt{|\cos \theta|} \end{pmatrix},$$

where the sign of the off-diagonal element is opposite to the sign of  $\cos \theta$ . Then, after possibly changing the representant matrix in  $PSL_2(\mathbb{C})$ ,

$$gAg^{-1} = \begin{pmatrix} -e^{-i\theta}i & \frac{1}{2} \\ -e^{-i\theta}4\cos\theta & -e^{-i\theta}i \end{pmatrix}_c, \quad gBg^{-1} = \begin{pmatrix} e^{v/2} & \frac{i}{4\cos\theta}(e^{v/2} - e^{-v/2}) \\ 0 & e^{-v/2} \end{pmatrix},$$

which can be as close as wanted to some fixed parabolic representation.

Therefore, there is a connected component  $U_2$  of  $\text{res}^{-1}(V)$  made up of representations of type II of the form  $[A_\theta, B]$  with  $\theta \sim 0$ ; and another connected component which is  $U_1$ . As  $V$  is small enough,  $U_1 \cap U_2 = \emptyset$ . Now, taking into account Remark 3.1.5, it is clear that for each representation  $\rho \in V$ , there are at most two possible preimages in  $U$ . If  $\text{res}^{-1}(\rho)$  has two preimages, then  $\rho(a)$  is an elliptic transformation and it has one preimage in  $U_1$  and the other one in  $U_2$ . Therefore, the restriction map restricted to  $U_1$  is injective. □

**Theorem 3.3.5.** *Let  $U \subset \text{hom}_+^0(\pi_1(K^2), G)/G_+$  be a neighbourhood of a non-degenerate parabolic representation as in Lemma 3.3.4, then the restriction map restricted to  $U$  is a local homeomorphism onto its image.*

*Proof.* By Lemmas 3.3.1 and 3.3.4 the restriction map is continuous and injective. Therefore, we can consider the map  $\text{res}^{-1}$  restricted to the image. Let  $V = \text{res}(U)$  and let  $p_1, p_2$  be the projections of  $\text{hom}_+(\pi_1(K^2), G)$  and  $\text{hom}(\pi_1(T^2), G_+)$  to the respective quotients. The respective preimages of  $U$  and  $V$  by the projection will be denoted  $\tilde{U} = p_1^{-1}(U)$  and  $\tilde{V} = p_2^{-1}(V)$ . We aim to construct a commutative diagram as follows:

$$\begin{array}{ccc}
 \tilde{U} & \xleftarrow{\quad f \quad} & \tilde{V} \\
 \downarrow p_1 & \swarrow g & \downarrow p_2 \\
 U & \xleftarrow{\quad \text{res}^{-1} \quad} & V
 \end{array}$$

The map  $g$  will be defined so that the diagram commute, hence  $f = g \circ p_2$ . Therefore, if  $g$  is continuous, then  $\text{res}^{-1}$  is also continuous. Due to the quotient universal property,  $g$  being continuous is equivalent to  $f$  being continuous.

We will construct  $g$  by parts on the different types of representations in  $U$ , that is, parabolic, type I and type II. Let us consider first a continuous section of  $p_2$  (see [34]); given

$$[L, M] = \left( \begin{bmatrix} e^{u/2} & 0 \\ 0 & e^{-u/2} \end{bmatrix}, \begin{bmatrix} e^{v/2} & 0 \\ 0 & e^{-v/2} \end{bmatrix} \right) \in \text{hom}(\pi_1(T^2), G_+)/G_+,$$

its image by the section is

$$\left( \begin{pmatrix} e^{u/2} & 1 \\ 0 & e^{-u/2} \end{pmatrix}, \begin{pmatrix} e^{v/2} & \tau(u, v) \\ 0 & e^{-v/2} \end{pmatrix} \right),$$

where the relation  $\sinh \frac{v}{2} = \tau(u, v) \sinh \frac{u}{2}$  holds so that the matrices commute. The function  $\tau$  is continuous on both arguments (undefined if  $u = 0$  or  $v = 0$ ). This ensures that if  $(u(t), v(t)) \rightarrow (0, 0)$ , the image of the section is a non-degenerate parabolic representation. The section of a parabolic representation

$$[L, M] = \left[ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \right],$$

is

$$\left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \right).$$

We will construct  $g$  in a way that is reminiscent to the previous section. For  $[L, M] \in V$  parabolic,  $L(z) = z + 1$ ,  $B(z) = z + i\nu$ ,  $\nu \in \mathbb{R}$ ,  $g([L, M]) =$

$(A, B)$  is defined as

$$A = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}_c, \quad B = \begin{pmatrix} 1 & i\nu \\ 0 & 1 \end{pmatrix}.$$

For non-parabolic representations, let us consider first the type I case. Let  $L(z) = e^u z$ ,  $M(z) = e^{i\theta} z$ , where  $u, \theta \in \mathbb{R}$ . Then,  $g([L, M]) = (A, B)$  with

$$A = \begin{pmatrix} e^{u/4} & 1/2 \\ 0 & e^{-u/4} \end{pmatrix}_c, \quad B = \begin{pmatrix} e^{i\theta/2} & \tau(u, i\theta) \cosh \frac{u}{4} \\ 0 & e^{-i\theta/2} \end{pmatrix},$$

where  $\tau$  is defined as in the section of  $p_2$ , that is  $\sinh \frac{v}{2} = \tau(u, v) \sinh \frac{u}{2}$ . In order to check that this satisfies the Klein bottle relation, notice first that  $e^{u/2} = \frac{|\operatorname{tr}(L)| \pm \sqrt{(\operatorname{tr}(L))^2 - 4}}{2}$  and analogously for  $e^{v/2}$ . Thus, for a type I representation,  $\sinh \frac{u}{2} = \sqrt{(\operatorname{tr}(L))^2 - 4} \in \mathbb{R}$ , whereas  $\sinh \frac{v}{2} = \sqrt{(\operatorname{tr}(M))^2 - 4} \in i\mathbb{R}$ , hence,  $\tau(u, i\theta) \in i\mathbb{R}$ . Therefore, the Klein bottle relation is equivalent to

$$\begin{pmatrix} e^{u/4-i\theta/2} & -\tau \cosh \frac{u}{4} + 1/2 e^{i\theta/2} \\ 0 & e^{-u/4+i\theta/2} \end{pmatrix} = \begin{pmatrix} e^{u/4-i\theta/2} & -\tau \cosh \frac{u}{4} e^{-u/4} + 1/2 e^{-i\theta/2} \\ 0 & e^{-u/4+i\theta/2} \end{pmatrix}.$$

Then,  $-\tau \cosh \frac{u}{4} + 1/2 e^{i\theta/2} = -\tau \cosh \frac{u}{4} e^{-u/4} + 1/2 e^{-i\theta/2}$  if and only if

$$\tau 2 \cosh \frac{u}{4} \sinh \frac{u}{4} = \sinh \frac{i\theta}{2},$$

which is true by the relation  $\sinh \frac{u}{2} = 2 \cosh \frac{u}{4} \sinh \frac{u}{4}$  and the definition of  $\tau(u, i\theta)$ .

Regarding the type II case, let  $[L, M] \in V$ ,  $L(z) = e^{i\theta} z$ ,  $M(z) = e^v z$ , where  $\theta, v \in \mathbb{R}$ . Let  $(A, B) = g([L, M])$  be defined as

$$A = \begin{pmatrix} -e^{-i\theta/4} & \frac{1}{2} \\ -e^{-i\theta/4} 4i \sin \theta/4 & -e^{-i\theta/4} \end{pmatrix}_c, \quad B = \begin{pmatrix} e^{v/2} & \frac{-i}{4 \sin \theta/4} (e^{v/2} - e^{-v/2}) \\ 0 & e^{-v/2} \end{pmatrix},$$

where these formulae come from the end of the proof of Lemma 3.3.4, for  $\theta/4 + \pi/2$  and where the relation  $\cos(\phi + \pi/2) = -\sin \phi$  has been taken

into account. Moreover, the off-diagonal entry of  $B$  can be expressed as

$$\begin{aligned} \frac{-i}{4 \sin \theta/4} (e^{v/2} - e^{-v/2}) &= \frac{\sinh v/2}{2i \sin \theta/4} = \frac{\sinh v/2}{-2 \sinh -u/4} = \frac{\sinh v/2 \cosh u/4}{\sinh u/2} \\ &= \tau \cosh \frac{u}{4}, \end{aligned}$$

which matches the type I case.

This definition of  $g$  satisfies  $p_1 \circ g = \text{res}^{-1}$ , so the diagram is commutative. In addition,  $g$  is defined essentially in terms of the traces of  $L$  and  $M$ , so  $f$  is continuous restricted to non-parabolic representations. On the other hand, it is straightforward that  $f$  is sequentially continuous when the limit is a parabolic representation (that is,  $f$  is continuous at every point). Hence,  $g$  is continuous, so is  $\text{res}^{-1}$ .  $\square$

**Corollary 3.3.6.** *There exists a (continuous) local section of the projection*

$$p_1 : \text{hom}_+^0(\pi_1(K^2), G) \rightarrow \text{hom}_+^0(\pi_1(K^2), G)/G_+$$

*around a non-degenerate parabolic representation.*

*Proof.* Consider the map  $g \circ \text{res}$  defined in the proof of Theorem 3.3.5.  $\square$

**Remark 3.3.7.** The results in this section can be extended to the quotient by the whole group  $G$  by adding the identification  $[\bar{A}, \bar{B}] = [A, B]$ . Thus, for instance, the action  $\iota$  (cf. (3.5)) in the quotient  $\text{hom}(\pi_1(T^2), G_+)/G$  is  $\iota \cdot [L, M] = [L, M^{-1}]$ . The analogous of Theorem 3.3.5 can also be obtain from Theorem 3.3.5 by considering a section of the projection

$$\text{hom}(\pi_1(T^2), G_+)/G_+ \rightarrow \text{hom}(\pi_1(T^2), G_+)/G,$$

and constructing from it a commutative diagram

$$\begin{array}{ccc} U & \xleftarrow{\text{res}^{-1}} & V \\ \downarrow p'_1 & \searrow & \downarrow p'_2 \\ U/G & \xleftarrow{\text{res}^{-1}} & V/G \end{array}$$

### 3.4 Homology and cohomology of a parabolic representation

Let us consider the Lie algebra of  $G$ ,  $\mathfrak{sl}(2, \mathbb{C})$ . Let  $\rho \in \text{hom}_+(\pi_1(K^2), G)$  denote the parabolic representation defined as

$$\rho(a) = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}_c, \quad \rho(b) = \begin{pmatrix} 1 & i\nu \\ 0 & 1 \end{pmatrix},$$

where  $a, b$  generate  $\pi_1(K^2)$  and satisfy  $aba^{-1} = b^{-1}$ . The fundamental group  $\pi_1(K^2)$  acts on the universal cover  $\widetilde{K^2} \cong \mathbb{R}^2$  by covering transformations. On the other hand,  $\pi_1(K^2)$  acts on  $\mathfrak{sl}(2, \mathbb{C})$  by the adjoint action through the representation  $\rho$ , that is,

$$\gamma \cdot \eta = \text{Ad}(\rho(\gamma))(\eta), \quad \text{where } \gamma \in \pi_1(K^2), \eta \in \mathfrak{sl}(2, \mathbb{C}).$$

Finally, let us denote  $\Gamma := \pi_1(K^2)$ .

We will compute both the homology and cohomology of  $K^2$  with coefficients in  $\mathfrak{sl}(2, \mathbb{C})$  twisted by  $\rho$ . As  $K^2$  is a  $K(\Gamma, 1)$ -space, we can identify the cohomology of the group  $\Gamma$  with the cohomology dual to the homology of  $K^2$  with twisted coefficients. Moreover, we have the Kronecker pairing

$$\begin{aligned} H_\rho^i(K^2) \times H_i^\rho(K^2) &\longrightarrow \mathbb{R} \\ ([\phi], [g \otimes \alpha]) &\longmapsto B(g, \phi(\alpha)), \end{aligned} \tag{3.8}$$

where  $B$  is the Killing form, which is non-degenerate as  $G$  is semisimple. Therefore, the Kronecker pairing is non-degenerate too and the homology and cohomology are dual.

#### 3.4.1 Homology with twisted coefficients

We will consider here the homology of  $K^2$  with the coefficients twisted by the representation of  $\Gamma$  into  $\mathfrak{sl}(2, \mathbb{C})$  (see [44], [35] for more details). The construction of the chain complex with twisted coefficients goes as follows:

the universal cover  $\widetilde{K^2}$  has a structure of CW-complex. We can consider the action of  $\Gamma$  in this structure by covering transformations, which extends to an action of  $\Gamma$  on the cellular chain groups  $C_*(\widetilde{K^2})$ . Thus, we can consider  $C_k(\widetilde{K^2})$  as a  $\mathbb{R}\Gamma$ -module, whose boundary homomorphism  $\partial$  is linear over  $\mathbb{R}\Gamma$ . The chain complex we have just defined is free over  $\mathbb{R}\Gamma$ .

Now we can twist the coefficients of the previous chain complex. Let us denote by

$$C_i^\rho(K^2) = \mathfrak{sl}(2, \mathbb{C}) \otimes_\rho C_i(\widetilde{K^2})$$

the  $\mathbb{R}\Gamma$ -module generated by pairs

$$\{g \otimes c^i \mid g \in \mathfrak{sl}(2, \mathbb{C}), c^i \in C_i(\widetilde{K^2})\},$$

and subject to the relations

$$g \otimes (\gamma \cdot c^i) = (\rho(\gamma^{-1}) \cdot g) \otimes c^i, \quad \forall \gamma \in \Gamma,$$

where the action on chains is by covering transformations and on  $\mathfrak{sl}(2, \mathbb{C})$ , the adjoint action.

The boundary operator  $\partial$  on  $C_*^\rho(\widetilde{K^2})$  is then defined by linearity and the formula

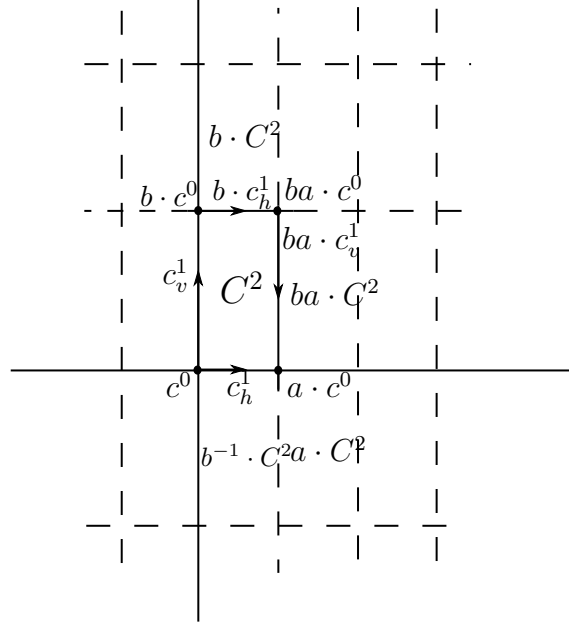
$$\partial(g \otimes c^i) := g \otimes (\partial c^i),$$

where  $\partial(c^i)$  is the boundary operator of  $C(\widetilde{K^2})$ .

We are interested in computing the homology of the chain complex we have just defined. We will utilize the usual notation of cycles and boundaries: a  $i$ -chain  $c^i$  is a cycle if  $c^i \in Z_i^\rho(K^2) := \text{Ker}(\partial_i)$  and it is a boundary if  $c^i \in B_i^\rho(K^2) := \text{Im}(\partial_{i+1})$ . Thus, the  $i$ -th homology group with twisted coefficients is defined as  $H_i^\rho(K^2) := Z_i^\rho(K^2)/B_i^\rho(K^2)$ .

The chain complex of  $\widetilde{K^2}$  is generated by one 2-cell,  $c^2$ , two 1-cells,  $c_v^1$  and  $c_h^1$ , and one 0-cell,  $c^0$ . The boundary operator is given by  $\partial_2(c^2) = (1-b)c_h^1 - (ba+1)c_v^1$ ,  $\partial_1(c_h^1) = (a-1)c^0$ ,  $\partial_1(c_v^1) = (b-1)c^0$  (see Figure 3.4.1).

In order to compute the boundary operator with twisted coefficients, let first write down some auxiliary expressions which are recurrent. Let  $A = \begin{pmatrix} x & y \\ w & -x \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C})$ :


 Figure 3.1: Cell decomposition of  $\widetilde{K}^2$ .

- First of all,  $\rho(ba) = \begin{pmatrix} 1 & 1/2 + i\nu \\ 0 & 1 \end{pmatrix}_c$ ,

- 

$$b^{-1} \cdot A = \begin{pmatrix} 1 & -i\nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ w & -x \end{pmatrix} \begin{pmatrix} 1 & i\nu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x - i\nu w & 2i\nu x + y + \nu^2 w \\ w & i\nu w - x \end{pmatrix},$$

- 

$$\begin{aligned} (ba)^{-1} \cdot A &= \begin{pmatrix} 1 & -1/2 - i\nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ w & -z \end{pmatrix} \begin{pmatrix} 1 & 1/2 + i\nu \\ 0 & 1 \end{pmatrix}_c \\ &= \begin{pmatrix} \bar{x} - 1/2\bar{w} + i\nu\bar{w} & (1 - 2i\nu)\bar{x} + \bar{y} - (1/2 - i\nu)^2\bar{w} \\ \bar{w} & 1/2\bar{w} - i\nu\bar{w} - \bar{x} \end{pmatrix} \end{aligned}$$

- 

$$a^{-1} \cdot A = \begin{pmatrix} 1 & -1/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ w & -x \end{pmatrix} \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}_c = \begin{pmatrix} \bar{x} - 1/2\bar{w} & \bar{x} + \bar{y} - 1/4\bar{w} \\ \bar{w} & 1/2\bar{w} - \bar{x} \end{pmatrix}$$



Let us now compute the boundary operator on 2-chains:

$$\begin{aligned} \partial_2 : C_2^\rho(K^2) &\longrightarrow C_1^\rho(K^2) \\ A \otimes c^2 &\longmapsto A \otimes ((1-b)c_h^1) - (1+ba)c_v^1 \\ &= (A - b^{-1} \cdot A) \otimes c_h^1 - (A + (ba)^{-1} \cdot A) \otimes c_v^1. \end{aligned}$$

Then,

$$\begin{aligned} A - b^{-1} \cdot A &= \begin{pmatrix} i\nu w & -2i\nu x - \nu^2 w \\ 0 & -i\nu w \end{pmatrix}, \\ A + (ba)^{-1} \cdot A &= \begin{pmatrix} 2\operatorname{Re}(x) - 1/2\bar{w} + i\nu\bar{w} & (1 - 2i\nu)\bar{x} + 2\operatorname{Re}(y) - (1/2 - i\nu)^2\bar{w} \\ 2\operatorname{Re}(w) & 1/2\bar{w} - i\nu\bar{w} - 2\operatorname{Re}(x) \end{pmatrix}. \end{aligned}$$

The second homology group is  $H_2^\rho(K^2) = \operatorname{Ker}(\partial_2)$ . Notice that in order for  $A \otimes c^2$  to belong in the kernel, the entries of  $A$  must satisfy  $x = w = 0$ . In addition, this implies  $\operatorname{Re}(y) = 0$ . Thus,  $\operatorname{ker}(\partial_2)$  is generated by  $\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \otimes c^2$  and  $H_2^\rho(K^2) \cong \mathbb{R}$ .

The boundary operator on 1-chains is defined as

$$\begin{aligned} \partial_1 : C_1^\rho(K^2) &\longrightarrow C_0^\rho(K^2) \\ A_h \otimes c_h^1 &\longmapsto A_h \otimes (a-1)c^0 = (a^{-1} \cdot A_h - A_h) \otimes c^0, \\ A_v \otimes c_v^1 &\longmapsto A_v \otimes (b-1)c^0 = (b^{-1} \cdot A_v - A_v) \otimes c^0. \end{aligned}$$

Therefore, in general  $\partial_1(A_h \otimes c_h^1 + A_v \otimes c_v^1) = (a^{-1} \cdot A_h - A_h + b^{-1} \cdot A_v - A_v) \otimes c^0$ . Let  $A_h = \begin{pmatrix} x_h & y_h \\ w_h & -x_h \end{pmatrix}$ ,  $A_v = \begin{pmatrix} x_v & y_v \\ w_v & -x_v \end{pmatrix}$ ; from the previous auxiliary computations, we have that

$$a^{-1} \cdot A_h - A_h = \begin{pmatrix} -2i\operatorname{Im}(x_h) - 1/2\bar{w}_h & \bar{x}_h - 2i\operatorname{Im}(y_h) - 1/4\bar{w}_h \\ -2i\operatorname{Im}(w_h) & 2i\operatorname{Im}(x_h) + 1/2\bar{w}_h \end{pmatrix},$$

and

$$b^{-1} \cdot A_v - A_v = \begin{pmatrix} -i\nu w_v & 2i\nu x_v + \nu^2 w_v \\ 0 & i\nu w_v \end{pmatrix}.$$

It is straightforward that the image of  $\partial_1$  can have at most dimension 5, since  $C_0^\rho(K^2) \cong \mathbb{R}^6$  and the element  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes c^0 \notin \operatorname{Im}(\partial_1)$ . Let us write down 5 linearly independent vectors in the image:

- The boundary operator  $\partial_1$  maps the 1-chain with coefficients  $A_v = \begin{pmatrix} \frac{1}{2i\nu} & 0 \\ 0 & -\frac{1}{2i\nu} \end{pmatrix}$ ,  $A_h = 0$  to  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes c^0$ .
- Similarly,  $A_v = \begin{pmatrix} \frac{1}{2\nu} & 0 \\ 0 & -\frac{1}{2\nu} \end{pmatrix}$ ,  $A_h = 0$  is mapped to  $\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \otimes c^0$ .
- On the other hand, the 1-chain with  $A_v = \begin{pmatrix} -\frac{1}{2} & 0 \\ \frac{1}{-i\nu} & \frac{1}{2} \end{pmatrix}$ ,  $A_h = 0$  is mapped to  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes c^0$ .
- Analogously,  $\partial_1$  maps  $A_v = \begin{pmatrix} -\frac{i}{2} & 0 \\ -\frac{1}{\nu} & \frac{i}{2} \end{pmatrix}$ ,  $A_h = 0$  to  $\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \otimes c^0$ .
- Finally, we remark that as long as  $\text{Im}(w_h) \neq 0$ , the image of the corresponding 1-chain will have as coefficient with a below-the-diagonal entry a pure imaginary number. Hence, taking into account the 4 linearly independent elements we have already written down,  $\begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} \otimes c^0 \in \text{Im}(\partial_1)$ .

Thus,  $H_0^\rho(K^2) \cong \mathbb{R}$ . Moreover, due to the fact that  $\dim(\text{Ker}(\partial_i)) + \dim(\text{Im}(\partial_i)) = \dim(C_i)$ , we obtain that  $Z_1^\rho(K^2) \cong \mathbb{R}^7$  and, by the same formula,  $B_1^\rho(K^2) = \mathbb{R}^5$ . Therefore,  $H_1(E_\rho) \cong \mathbb{R}^2$ .

To conclude, let us find representants of each homology class. Regarding  $H_2^\rho(K^2)$  and  $H_0^\rho(K^2)$  it is immediate; we have already seen that

$$\left\langle \left[ \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \otimes c^2 \right] \right\rangle = H_2(E_\rho), \quad \left\langle \left[ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes c^0 \right] \right\rangle = H_0(E_\rho).$$

Regarding  $H_1^\rho(K^2)$ , we can easily check that the following two cycles do not belong to the same equivalent class, hence they generate  $H_1^\rho(K^2)$ :

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes c_h^1,$$

$$\begin{pmatrix} 0 & 0 \\ 8\nu & 0 \end{pmatrix} \otimes c_h^1 + \begin{pmatrix} -(2\nu + i) & 0 \\ 4i & 2\nu + i \end{pmatrix} \otimes c_v^1.$$

They are not related by a boundary as for any element in  $\text{Im}(\partial_2)$ , the under-the-diagonal entry of the coefficient of  $c_h^1$  is always equal to 0.

### 3.4.2 Cohomology of $\Gamma$

Due to the adjoint action of  $\Gamma$  on the Lie Algebra via the representation,  $\mathfrak{sl}(2, \mathbb{C})$  can be considered a  $\mathbb{R}\Gamma$ -module. We define the cochain complex  $(C^*(\Gamma, \mathfrak{sl}(2, \mathbb{C})), \delta)$  as follows (see [31], §4.5):

$$C^i(\Gamma, \mathfrak{sl}(2, \mathbb{C})) := \mathbb{R}\Gamma - \text{module generated by maps } \Gamma^i \rightarrow \mathfrak{sl}(2, \mathbb{C}),$$

where  $\Gamma^i$  is the  $i$ -fold product of  $\Gamma$ , and  $\Gamma^0 := \{*\}$ . Thus,  $C^0(\Gamma, \mathfrak{sl}(2, \mathbb{C}))$  can be identified with  $\mathfrak{sl}(2, \mathbb{C})$ . The coboundary operator,  $\delta$ , is the dual to the boundary operator of the corresponding chain complex  $C_*(\Gamma)$ . We will write down the coboundary explicitly for the 0-cochains and the 1-cochains:

$$\begin{aligned} \delta_0 : C^0(\Gamma, \mathfrak{sl}(2, \mathbb{C})) \cong \mathfrak{sl}(2, \mathbb{C}) &\longrightarrow C^1(\Gamma, \mathfrak{sl}(2, \mathbb{C})) \\ \xi &\longmapsto \delta_0(\xi)(\gamma) = \xi - \gamma \cdot \xi, \quad \gamma \in \Gamma, \end{aligned}$$

$$\begin{aligned} \delta_1 : C^1(\Gamma, \mathfrak{sl}(2, \mathbb{C})) &\longrightarrow C^2(\Gamma, \mathfrak{sl}(2, \mathbb{C})) \\ c &\longmapsto \delta_1(c)(\alpha, \gamma) = c(\alpha) + \alpha c(\gamma) - c(\alpha\gamma), \quad \alpha, \gamma \in \Gamma. \end{aligned}$$

Therefore,  $H^0(\Gamma, \mathfrak{sl}(2, \mathbb{C})) = \ker(\delta_0) = \mathfrak{sl}(2, \mathbb{C})^\Gamma$ , that is, the elements of  $\mathfrak{sl}(2, \mathbb{C})$  fixed by the action of  $\Gamma$ . Let  $A = \begin{pmatrix} x & y \\ w & -x \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C})^\Gamma$ . The action of  $b$ ,  $b \cdot A = A$  implies  $w = 0$  and  $x = 0$ . Then,  $a \cdot A = A$  if and only if  $y \in \mathbb{R}$ . Hence,  $\mathbb{R} \cong \mathfrak{sl}(2, \mathbb{C})^\Gamma = H^0(\Gamma, \mathfrak{sl}(2, \mathbb{C}))$ .

Let  $c \in \ker(\delta_1)$ , i.e.,  $c$  satisfies  $c(\alpha \cdot \gamma) = c(\alpha) + \alpha \cdot c(\gamma)$ ,  $\forall \alpha, \gamma \in \Gamma$ . Notice that  $c(e) = c(e \cdot e) = 2c(e)$ , then  $c(e) = 0$ , where  $e$  is the identity element, and  $0 = c(\gamma^{-1}\gamma) = c(\gamma^{-1}) + \gamma^{-1} \cdot c(\gamma)$ , so  $c(\gamma^{-1}) = -\gamma^{-1} \cdot c(\gamma)$ ,  $\forall \gamma \in \Gamma$ . This implies that the 1-cocycle  $c$  is determined by its value in  $a$  and  $b$ :  $c(a)$  and  $c(b)$ . Thus, we will be able to identify the kernel  $\ker(\delta_1)$  with a subspace of  $\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$ . The relation  $aba^{-1} = b^{-1}$  can be rewritten as  $b = a^{-1}b^{-1}a$ , then

$$\begin{aligned} c(b) &= c(a^{-1}b^{-1}a) = c(a^{-1}) + a^{-1} \cdot c(b^{-1}a) = \\ &= c(a^{-1}) + a^{-1} \cdot c(b^{-1}) + a^{-1}b^{-1} \cdot c(a) = \\ &= -a^{-1}b^{-1} \cdot c(b) + a^{-1}(b^{-1} - 1)c(a). \end{aligned}$$

Hence,

$$0 = a^{-1}(b^{-1} - 1) \cdot c(a) - (a^{-1}b^{-1} + 1) \cdot c(b).$$

Let us consider the linear maps

$$\begin{aligned} S_1 : \mathfrak{sl}(2, \mathbb{C}) &\longrightarrow \mathfrak{sl}(2, \mathbb{C}) \\ g &\longmapsto S_1(g) := a^{-1}(b^{-1} - 1) \cdot g, \end{aligned}$$

and

$$\begin{aligned} S_2 : \mathfrak{sl}(2, \mathbb{C}) &\longrightarrow \mathfrak{sl}(2, \mathbb{C}) \\ g &\longmapsto S_2(g) := (a^{-1}b^{-1} + 1) \cdot g, \end{aligned}$$

and then define

$$\begin{aligned} S : \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) &\longrightarrow \mathfrak{sl}(2, \mathbb{C}) \\ (g, h) &\longmapsto S(g, h) := S_1(g) - S_2(h) = \\ &= a^{-1}(b^{-1} - 1) \cdot g - (a^{-1}b^{-1} + 1) \cdot h. \end{aligned}$$

Therefore, we have that  $(c(a), c(b)) \in \text{Ker}(S)$ . In order to compute this kernel, let us compute first the image of the maps  $S_1$  and  $S_2$  for some elements  $g, h \in \mathfrak{sl}(2, \mathbb{C})$ . Let  $g = \begin{pmatrix} x_g & y_g \\ w_g & -x_g \end{pmatrix}$ ,  $h = \begin{pmatrix} x_h & y_h \\ w_h & -x_h \end{pmatrix}$ , by using the auxiliary computations of the previous subsection we obtain,

$$S_1(g) = \begin{pmatrix} i\nu\overline{w}_g & -2i\nu\overline{x}_g + (i\nu + \nu^2)\overline{w}_g \\ 0 & -i\nu\overline{w}_g \end{pmatrix},$$

$$S_2(h) = \begin{pmatrix} 2\text{Re}(x_h) - 1/2\overline{w}_h + i\nu\overline{w}_h & (1 - 2i\nu)\overline{x}_h + 2\text{Re}(y_h) - (1/2 - i\nu)^2\overline{w}_h \\ 2\text{Re}(w_h) & 1/2\overline{w}_h - i\nu\overline{w}_h - 2\text{Re}(x_h) \end{pmatrix}.$$

The image of  $S_1$  consists of  $\text{Im}(S_1) = \left\{ \begin{pmatrix} x & y \\ 0 & -x \end{pmatrix} \mid x, y \in \mathbb{C} \right\}$ , so it has real dimension 4. Consequently, from a straightforward inspection of the image of  $S_2$  we can conclude that  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \text{Im}(S)$ , but  $\begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} \notin \text{Im}(S)$ , which implies  $\dim_{\mathbb{R}}(\text{Im}(S)) = 5$  and  $\dim_{\mathbb{R}}(\text{ker}(S)) = \dim_{\mathbb{R}}(Z^1(\Gamma, \mathfrak{sl}(2, \mathbb{C}))) = 7$ .

In order to compute  $\dim(\text{Im}(\delta_0))$ , recall that  $\delta_0(\xi)(\gamma) = \xi - \gamma \cdot \xi$ , where  $\xi \in \mathfrak{sl}(2, \mathbb{C})$  and  $\delta_0(\xi)$  is completely determined by its evaluation in  $a$  and  $b$  (equivalently, in  $a^{-1}$  and  $b^{-1}$ ). Thus,  $\text{Ker}(\delta_1)$  can be identified with a subset of  $\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$ , which corresponds to the image of  $a$  and the image of

b. In consequence, the image of  $\delta_0$  can be identified with the image of the map

$$\begin{aligned} \mathfrak{sl}(2, \mathbb{C}) &\longrightarrow \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) \\ \xi &\longmapsto (\xi - a^{-1} \cdot \xi, \xi - b^{-1} \cdot \xi). \end{aligned}$$

Let  $\xi = \begin{pmatrix} x & y \\ w & -x \end{pmatrix}$  belong to the kernel of the previous map, then, from  $\xi - b^{-1} \cdot \xi = 0$  we conclude  $x = w = 0$  and hence, from  $\xi - a^{-1} \cdot \xi = 0$ ,  $\text{Im}(y) = 0$ . Thus, the kernel is generated by  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  and has dimension 1. Therefore, the image has dimension 5 and  $H^1(\Gamma, \mathfrak{sl}(2, \mathbb{C})) \cong \mathbb{R}^2$ . Finally, by an Euler-Poincaré characteristic argument,  $H^2(\Gamma, \mathfrak{sl}(2, \mathbb{C})) \cong \mathbb{R}$ .

**Remark 3.4.1.** These cohomology groups could have been obtained from the computations in homology and taking into account the Kronecker pairing (cf. (3.8)) as stated at the beginning of the section. Another possible approach could have been to take into account the cohomology of  $\pi_1(T^2)$  via the representation  $\rho_0 = \text{res}(\rho)$ . It is well-known that

$$\begin{aligned} H^0(\pi_1(T^2), \mathfrak{sl}(2, \mathbb{C})) &\cong H^2(\pi_1(T^2), \mathfrak{sl}(2, \mathbb{C})) \cong \mathbb{C}, \\ H^1(\pi_1(T^2), \mathfrak{sl}(2, \mathbb{C})) &\cong \mathbb{C}^2. \end{aligned}$$

For instance,  $H^0(\pi_1(T^2), \mathfrak{sl}(2, \mathbb{C})) \cong \mathfrak{sl}(2, \mathbb{C})^{\pi_1(T^2)} \cong \mathbb{C}$  can be easily seen and the rest follows by Poincaré Duality and Euler-Poincaré characteristic.

As  $T^2 \rightarrow K^2$  is a regular finite-sheeted covering space, the pullback in cohomology is injective. This can be shown by an averaging construction. If we consider the covering transformation  $\iota$  acting on  $H^i(\pi_1(T^2), \mathfrak{sl}(2, \mathbb{C}))$  we can identify  $H^i(\pi_1(T^2), \mathfrak{sl}(2, \mathbb{C}))^\iota \cong H^i(K^2, \mathfrak{sl}(2, \mathbb{C}))$ . It can be checked that the action of  $\iota$  is related to the complex conjugation and halves the (real) dimension of  $H^i(\pi_1(T^2), \mathfrak{sl}(2, \mathbb{C}))$ . Thus, we could conclude

$$\begin{aligned} H^0(\pi_1(K^2), \mathfrak{sl}(2, \mathbb{C})) &\cong H^2(\pi_1(K^2), \mathfrak{sl}(2, \mathbb{C})) \cong \mathbb{R}, \\ H^1(\pi_1(K^2), \mathfrak{sl}(2, \mathbb{C})) &\cong \mathbb{R}^2. \end{aligned}$$

Finally, by taking into account these cohomology groups and the dimension of the space of cochains, we can deduce the dimensions of the coboundaries and cocycles.

### 3.4.3 Smoothness of the representation variety

The variety of representations is tightly related to the cohomology group we computed in the last subsection. Let  $\rho' \in \text{hom}(\Gamma', G')$ , where  $\Gamma'$  is a group and  $G'$  a Lie group with  $\mathfrak{g}'$  as Lie algebra, and let  $T_{\rho'}\text{hom}(\Gamma', G')$  be the Zariski tangent space. Then, the tangent space  $T_{\rho'}\text{hom}(\Gamma', G')$  is isomorphic to the space of 1-cocycles  $Z^1(\Gamma', \mathfrak{g}')$  (see [31], §4.5).

Similarly, we can identify the space of coboundaries  $B^1(\Gamma', \mathfrak{g}')$  with the tangent space of deformations of  $\rho'$  by conjugation. If the action of  $G'$  on  $\text{hom}(\Gamma', G')$  by conjugation is free near  $\rho'$ , then the (Zariski) tangent space of  $\text{hom}(\Gamma', G')/G'$  at  $[\rho']$  is isomorphic to  $H^1(\Gamma', G')$ . Thus, by a result of A. Weil ([45]), if  $H^1(\Gamma', \mathfrak{g}') = 0$  (i.e., the representation is infinitesimally rigid), then locally every representation  $\rho''$  close enough to  $\rho'$  is conjugated to  $\rho'$  and we say that  $\rho'$  is locally rigid.

Our computations of the previous subsection show that the Zariski tangent space  $T_{\rho}\text{hom}(\Gamma, G)$  has dimension 7. Another consideration we want to make about the representation variety is whether it is smooth or not. There are a series of infinite obstructions to this fact given by Goldman and Millson ([21], [22]). These obstructions consist of higher-order Massey products and they live in the second cohomology group  $H^2(\Gamma, \mathfrak{g})$ . Therefore, if  $H^2(\Gamma, \mathfrak{g}) = 0$ , then we can say that the variety of representations is smooth. Nonetheless, in our case of interest this does not happen and we will have to see that the obstructions are null.

**Proposition 3.4.2.** *The representation variety  $\text{hom}(\pi(K^2), G)$  around the parabolic representation  $\rho \in \text{hom}_+^0(\pi_1(K^2), G)$  is smooth and has (real) dimension 7.*

*Proof.* Let us consider the orientation covering  $\pi : T^2 \rightarrow K^2$  and the induced map

$$\pi_* : \pi_1(T^2) \rightarrow \pi_1(K^2).$$

The pullback in cohomology

$$\pi^* : H^i(\pi_1(K^2), \mathfrak{g}) \rightarrow H^i(\pi_1(T^2), \mathfrak{g})$$

is injective as it comes from a regular finite-sheeted covering.

Given a cocycle  $c \in H^1(\pi_1(K^2), \mathfrak{g})$ , the obstructions to the existence of an analytic path of representations passing through  $\rho$  with tangent vector  $c$  are denoted  $o_i(c) \in H^2(\pi_1(K^2), \mathfrak{g})$ . These obstructions are defined inductively and are higher-order Massey products, therefore they satisfy a naturality condition, that is

$$\pi^*(o_i(c)) = o_i(\pi^*(c)).$$

The element  $o_i(\pi^*(c))$  is the  $i$ -th obstruction to the existence of an analytic path passing through  $\text{res}(\rho)$  with tangent vector  $\pi^*(c)$ . The representation variety  $\text{hom}(\pi_1(T^2), \text{PSL}(2, \mathbb{C}))$  is smooth ([31]) and therefore  $o_i(\pi^*(c)) = 0$ . By injectivity of  $\pi^*$ , this implies  $o_i(c) = 0$ . Thus, it is possible to construct the sought-after analytic path in  $\text{hom}(\pi_1(K^2), \text{Isom}(\mathbb{H}^3))$ .

Finally, smoothness implies that the Zariski tangent space is the usual tangent space, and we already computed  $\dim_{\mathbb{R}}(Z^1(\pi_1(K^2), \mathfrak{sl}(2, \mathbb{C}))) = 7$  in Subsection 3.4.2.  $\square$

### 3.5 Degenerate representations and other connected components

We end the section with an exhaustive list of representations that were not considered in the previous subsections, namely, orientation type preserving degenerate representations and representations which are not orientation type preserving. This list is included for completeness sake and it is obtained by straightforward computations. We will make use of Proposition 3.1.1 and Remark 3.1.7 in order to simplify a little bit the computations.

The list can be used to identify the different connected components. As we will see throughout this section, after choosing the orientation behaviour of  $\rho(a) = A$  and  $\rho(b) = B$ , where  $\rho \in \text{hom}(\pi_1(K^2), \text{Isom}(\mathbb{H}^3))$ , and  $a, b \in \pi_1(K^2)$  are our usual choice of generators, we can consider the lifted Klein bottle relation

$$\widetilde{ABA^{-1}B} \in \{\pm Id\} \subset \text{SL}(2, \mathbb{C}).$$

The lift happens to be well-defined as it can be computed by considering individual lifts of  $A$  and  $B$  to  $\mathrm{SL}(2, \mathbb{C}) \rtimes \mathbb{Z}_2$ . Two distinct lifts of  $A$  or  $B$  differ by multiplication by  $-Id$ , hence the lifted Klein bottle relation yields the same result independently of the chosen lift. For each possibility,  $+Id$  or  $-Id$ , we have one connected component. Thus, we obtain:

**Proposition 3.5.1.** *The representation variety  $\mathrm{hom}(\pi_1(K^2), G)$  has 8 connected components.*

This idea of lifting the relator has been extensively used to distinguish connected components and we will consider it once more in Chapter 5 in order to generalize Proposition 3.5.1 to higher genus non-orientable surfaces.

### 3.5.1 Degenerate type preserving representations

There are two main cases we have to account for, either  $A^2 = [Id]$  or  $B = [Id]$ . However these can be subdivided again by means of the square map  $Q$ .

- If  $B = \pm Id$ , then  $A$  can be any element in  $G$ .
- Else if  $Q(A) = Id$ , then  $A = \mathrm{Ad}(g) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}_c$ . Let  $B = \mathrm{Ad}(g) \begin{pmatrix} x & y \\ w & z \end{pmatrix}$ , so that  $AB = B^{-1}A$  is equivalent in  $G$  to

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} \bar{x} & \bar{y} \\ \bar{w} & \bar{z} \end{pmatrix} = \pm \begin{pmatrix} z & -y \\ -w & x \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

This implies either  $x, z \in \mathbb{R}$ ,  $w = -\bar{y}$  or  $x, z \in i\mathbb{R}$ ,  $w = \bar{y}$ . Thus, we have two possibilities:

$$B \in \left\{ \mathrm{Ad}(g) \begin{pmatrix} x & y \\ -\bar{y} & z \end{pmatrix} \mid x, z \in \mathbb{R}, xz + |y|^2 = 1 \right\}, \quad (3.9)$$

or

$$B \in \left\{ \mathrm{Ad}(g) \begin{pmatrix} x & y \\ \bar{y} & z \end{pmatrix} \mid x, z \in i\mathbb{R}, xz - |y|^2 = 1 \right\}. \quad (3.10)$$

We remark that in the first case  $\mathrm{tr}(B)$  can be any real, whereas in the second one,  $\mathrm{tr}(B)$  can be any pure imaginary number.



- Else if  $Q(A) = -Id$ , then  $A = \text{Ad}(g) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_c$ . Let  $B = \text{Ad}(g) \begin{pmatrix} x & y \\ w & z \end{pmatrix}$ , so that  $AB = B^{-1}A$  is equivalent to

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \bar{x} & \bar{y} \\ \bar{w} & \bar{z} \end{pmatrix} = \pm \begin{pmatrix} z & -y \\ -w & x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This implies either  $x, z \in \mathbb{R}$ ,  $w = \bar{y}$  or  $x, z \in i\mathbb{R}$ ,  $w = -\bar{y}$ . Thus, either

$$B \in \{\text{Ad}(g) \begin{pmatrix} x & y \\ \bar{y} & z \end{pmatrix} \mid x, z \in \mathbb{R}, xz - |y|^2 = 1\}, \quad (3.11)$$

or

$$B \in \{\text{Ad}(g) \begin{pmatrix} x & y \\ -\bar{y} & z \end{pmatrix} \mid x, z \in i\mathbb{R}, xz + |y|^2 = 1\}. \quad (3.12)$$

We remark that again either  $\text{tr}(B)$  can be any real or any pure imaginary.

Taking into account the non-degenerate representations, we see that parabolic, type I, type II representations form a connected component together with the cases where  $B = \pm Id$ , as well as (3.9) and (3.11). On the other hand, as noticed in Remark (3.2.3), type III representations belong to another connected component, together with cases (3.10) and (3.12).

### 3.5.2 Representations with both $A, B$ orientation preserving

We will consider two non-exclusive cases.

- Let us assume first that  $A$  and  $B$  commute. This is the case, for instance, if  $Q(A) \neq -Id$  (this is straightforward if we write  $A^2$  and  $B$  in their Jordan form). Hence from the Klein bottle relation,  $Q(B) = \pm Id$ . If  $Q(B) = Id$ ,  $B = Id$  and  $A$  can be any element in  $G_+$ , that is, we have representations of the type

$$(A, Id) \in \text{hom}(\pi_1(K^2), G). \quad (3.13)$$

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Otherwise,  $Q(B) = -Id$ , and  $B = \text{Ad}(g) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ,  $A = \text{Ad}(g) \begin{pmatrix} x & y \\ w & z \end{pmatrix}$ , for some  $g \in G_+$ . Then, the Klein bottle relation can be read as

$$\begin{pmatrix} x & y \\ w & z \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ w & z \end{pmatrix},$$

which yields either  $z = -x, w = y$  or  $z = x, w = -y$ . Therefore, we obtain representations

$$\text{Ad}(G_+) \left( \begin{pmatrix} x & y \\ y & -x \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right), \quad (3.14)$$

and

$$\text{Ad}(G_+) \left( \begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right). \quad (3.15)$$

- If  $Q(A) = -Id$ , we can write  $A = \text{Ad}(g) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ,  $B = \text{Ad}(g) \begin{pmatrix} x & y \\ w & z \end{pmatrix}$ , for some  $g \in G_+$ . Then, the Klein bottle relation states

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ w & z \end{pmatrix} = \pm \begin{pmatrix} z & -y \\ -w & x \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which implies either  $w = y$ , or  $x = z = 0$  and  $w = -y = \pm 1$  (the latter due to the determinant being equal to 1). Thus, we obtain the following kind of representations:

$$\text{Ad}(G_+) \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} x & y \\ y & z \end{pmatrix} \right), \quad (3.16)$$

and

$$\text{Ad}(G_+) \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right). \quad (3.17)$$

Notice that representations (3.13), (3.14) and (3.16) belong to the same connected component. There is a particular case in (3.16) where  $B = Id$ , which indicates that the intersection between cases (3.13) and (3.16) is

non-empty. Moreover, the representations

$$\left( \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right), \quad \text{and} \quad \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right),$$

which can be immediately seen to be respectively in cases (3.14) and (3.16), are conjugated by  $\frac{\sqrt{2}}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ , which shows that the intersection between these two cases is non-empty too.

On the other hand, representations in (3.17) can be seen to be a particular case of (3.15) and they belong to another connected component.

### 3.5.3 Representations with $A$ be orientation preserving, $B$ orientation reversing

Let us consider the fundamental group of the torus  $\pi_1(T^2) = \langle l, m \mid lm = ml \rangle$ . The elements  $l, m^2$  generate a normal subgroup  $\langle l, m^2 \mid lm^2 = ml^2 \rangle$  of index two. Therefore, we have a double cover  $T^2 \rightarrow T^2$ . The restriction of the representation of the Klein bottle to the torus and then this double cover is

$$(A, B) \mapsto (A^2, B) \mapsto (A^2, B^2).$$

We will consider first the cases where (as we will see, a priori) neither  $A^2$ , nor  $B^2$  are degenerate.

- If  $A^2, B^2$  are parabolic, up to conjugation,  $A^2 = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$ ,  $B^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ , where  $\tau \in \mathbb{C}$ . Then, up to conjugation,  $B = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}_c$ , and from  $A^2B = BA^2$  we obtain  $\tau \in \mathbb{R}$ . The Klein bottle relation yields

$$\begin{pmatrix} 1 & \tau/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} = \pm \begin{pmatrix} 1 & -1/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \tau/2 \\ 0 & 1 \end{pmatrix},$$

which leads to a contradiction.

- If  $A^2, B^2$  have a common invariant axis and  $B^2$  is elliptic, up to conjugation,  $A^2 = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{pmatrix}$ ,  $B = \begin{pmatrix} 0 & e^{i\varphi} \\ -e^{-i\varphi} & 0 \end{pmatrix}_c$ . Then,  $A^2$  and  $B$

commute if and only if  $|\lambda| = 1$ , hence,  $\lambda = e^{i\theta}$ . From the Klein bottle relation,

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & e^{i\varphi} \\ -e^{-i\varphi} & 0 \end{pmatrix} = \pm \begin{pmatrix} 0 & -e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix},$$

we obtain  $e^{i\varphi} = \mp e^{-i\varphi}$ , that is, either  $\varphi = \pm\pi/2$  or  $\varphi = 0, \pi$ . In  $G$ , both  $\varphi = \pi/2$  and  $\varphi = -\pi/2$  give rise to the same representation. Analogously,  $\varphi = 0$  and  $\varphi = \pi$  have the same associated representation in  $G$ . Therefore, we get representations

$$\text{Ad}(G_+) \left( \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}_c \right), \quad (3.18)$$

and

$$\text{Ad}(G_+) \left( \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_c \right). \quad (3.19)$$

- If  $A^2, B^2$  have a common invariant axis and  $B^2$  is a hyperbolic translation, up to conjugation,  $A^2 = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{pmatrix}$ ,  $B = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}_c$ , where  $\mu \in \mathbb{R}$ . Then,  $A^2$  and  $B$  commute if and only if either  $\lambda^2 \in \mathbb{R}$  or  $\lambda^2 \in i\mathbb{R}$ . In the first case, if  $\lambda \in \mathbb{R}$ , the Klein bottle relation

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} = \pm \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

implies  $\mu^2 = \pm 1$  and therefore, as  $\mu \in \mathbb{R}$ ,  $\mu = 1$ . If we consider now the case,  $\lambda \in i\mathbb{R}$ , the Klein bottle relation yields  $\mu^2 = \mp 1$ , so again  $\mu = 1$ . Thus, we obtain representations

$$\text{Ad}(G_+) \left( \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_c \right), \quad \text{where } \lambda \in \mathbb{R}, \quad (3.20)$$

and

$$\text{Ad}(G_+)\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_c\right), \quad \text{where } \lambda \in i\mathbb{R}. \quad (3.21)$$

On the other hand, if  $\lambda^2 \in i\mathbb{R}$ , then  $\lambda = re^{\pm i\pi/4}$ , where  $r \in \mathbb{R}$ . We can easily check that this yields no possible solution.

Now we will take into account the cases where either  $A^2$  or  $B^2$  is equal to the identity.

- Let us suppose  $Q(B) = Id$ . Then, we can write  $B = \text{Ad}(g)\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)_c$ ,  $A = \text{Ad}(g)\left(\begin{smallmatrix} x & y \\ w & z \end{smallmatrix}\right)$  for some  $g \in G_+$ . From the Klein bottle relation,

$$\begin{pmatrix} x & y \\ w & z \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} \bar{x} & \bar{y} \\ \bar{w} & \bar{z} \end{pmatrix},$$

we obtain either  $z = \bar{x}, w = \bar{y}$  or  $z = -\bar{x}, w = -\bar{y}$ . Thus, there are two kind of representations in this case:

$$\text{Ad}(G_+)\left(\begin{pmatrix} x & y \\ \bar{y} & \bar{x} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}_c\right), \quad (3.22)$$

and

$$\text{Ad}(G_+)\left(\begin{pmatrix} x & y \\ -\bar{y} & -\bar{x} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}_c\right). \quad (3.23)$$

- If  $Q(B) = -Id$ , then we can write  $B = \text{Ad}(g)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)_c$ ,  $A = \text{Ad}(g)\left(\begin{smallmatrix} x & y \\ w & z \end{smallmatrix}\right)$  for some  $g \in G_+$ . The Klein bottle relation is equivalent to

$$\begin{pmatrix} x & y \\ w & z \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \bar{x} & \bar{y} \\ \bar{w} & \bar{z} \end{pmatrix},$$

which implies either  $w = \bar{y}, z = -\bar{x}$  or  $w = -\bar{y}, z = \bar{x}$ . The case  $w = \bar{y}, z = -\bar{x}$  doesn't belong to  $G$  as  $\det(A) = -(|x|^2 + |y|^2) < 0$ .

That is, we obtain the family of representations:

$$\text{Ad}(G_+)\left(\begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_c\right). \quad (3.24)$$

- The case  $Q(A) = Id$ , implies  $A = Id$  and, from the Klein bottle relation,  $Q(B) = \pm Id$ . Thus, it has already been covered.
- If  $Q(A) = -Id$ , then  $A = \text{Ad}(g)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ,  $B = \text{Ad}(g)\begin{pmatrix} x & y \\ w & z \end{pmatrix}_c$  for some  $g \in G_+$ . From the Klein bottle relation,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ w & z \end{pmatrix} = \pm \begin{pmatrix} \bar{z} & -\bar{y} \\ -\bar{w} & \bar{x} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we obtain either  $w = \bar{y}$ ,  $x, z \in \mathbb{R}$ ; or  $w = -\bar{y}$ ,  $x, z \in i\mathbb{R}$ . Hence, there are two kind of representations in this case, namely,

$$\text{Ad}(G_+)\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} x & y \\ \bar{y} & z \end{pmatrix}_c\right) \quad \text{where } x, z \in \mathbb{R}, y \in \mathbb{C}, \quad (3.25)$$

and

$$\text{Ad}(G_+)\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} x & y \\ -\bar{y} & z \end{pmatrix}_c\right) \quad \text{where } x, z \in i\mathbb{R}, y \in \mathbb{C}. \quad (3.26)$$

Representations in cases (3.22) and (3.25) belong to the same connected component. For instance, the representations

$$\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}_c\right), \quad \text{and} \quad \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_c\right)$$

are conjugated by the element  $\frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$ . Therefore there is overlapping between both cases. Similarly, the representation  $(Id, c)$  belong to both (3.20) and (3.22). Finally, (3.18) is a particular case of (3.22).

On the other hand, representations in (3.23) and (3.24) overlap with (3.26), and, moreover, there is also overlapping between (3.19) and (3.24). More-

over, representations in (3.21) and (3.23) intersect in

$$\left( \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)_c, \quad \text{and} \quad \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right)_c,$$

as they are conjugated by  $\frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$ . Therefore, we obtain another connected component.

### 3.5.4 Representations with both $A, B$ orientation reversing

We will consider again the restriction of the representation to the covers  $T^2 \rightarrow T^2 \rightarrow K^2$ , which in terms of elements in  $G \times G$  is

$$(A, B) \mapsto (A^2, B) \mapsto (A^2, B^2).$$

We will start by considering the cases where *a priori* neither  $A^2$  nor  $B^2$  is the identity:

- If  $A^2, B^2$  are parabolic, after conjugating we can assume  $A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ,  $B^2 = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$ , where  $\tau = re^{i\theta}$ , then  $B = \begin{pmatrix} e^{i\theta} & r\nu \\ 0 & e^{-i\theta} \end{pmatrix}_c$ , with  $\text{Re } \nu = 1/2$ . Moreover,  $A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}_c$ , with  $\text{Re } \alpha = 1/2$ , and the Klein bottle relation  $AB = B^{-1}A$  reads

$$\begin{pmatrix} 1 & \bar{\alpha} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\theta} & r\nu \\ 0 & e^{-i\theta} \end{pmatrix} = \pm \begin{pmatrix} e^{-i\theta} & -r\nu \\ 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}.$$

The expression implies that  $e^{i\theta} = \pm e^{-i\theta}$ ,  $r\nu \pm r\nu = e^{-i\theta}(\alpha \mp \bar{\alpha})$ . When the plus sign is considered, then  $e^{i\theta} = 0, \pi$  and  $r\nu = \pm i \text{Im } \alpha$ . As  $r \in \mathbb{R}$ ,  $\text{Re } \nu = 1/2$ , the only possibility is  $r = 0$ ,  $\text{Im}(\alpha) = 0$ . Thus, we have representations

$$\text{Ad}(G_+) \left( \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}_c, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_c \right), \quad \text{where } \alpha \in \mathbb{R}. \quad (3.27)$$

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Otherwise, if we take the minus sign,  $e^{i\theta} = \pm i$ , and  $\operatorname{Re} \alpha = 0$ . This yields representations

$$\operatorname{Ad}(G_+)\left(\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}_c, \begin{pmatrix} \pm i & r\nu \\ 0 & \mp i \end{pmatrix}_c\right), \quad \text{where } \alpha \in i\mathbb{R}, r \in \mathbb{R}, \operatorname{Re} \nu = \frac{1}{2} \quad (3.28)$$

- If  $B^2$  is an elliptic transformation, then  $A^2$  must be too so that  $A^2$  and  $B$  commute. Thus, up to conjugation,  $A = \begin{pmatrix} 0 & re^{i\theta} \\ -r^{-1}e^{-i\theta} & 0 \end{pmatrix}_c$ ,  $B = \begin{pmatrix} 0 & e^{i\varphi} \\ -e^{-i\varphi} & 0 \end{pmatrix}_c$ , where  $r \in \mathbb{R}$ , and, from the Klein bottle relation,

$$\begin{pmatrix} 0 & re^{-i\theta} \\ -r^{-1}e^{i\theta} & 0 \end{pmatrix} \begin{pmatrix} 0 & e^{i\varphi} \\ -e^{-i\varphi} & 0 \end{pmatrix} = \pm \begin{pmatrix} 0 & -e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix} \begin{pmatrix} 0 & re^{i\theta} \\ -r^{-1}e^{-i\theta} & 0 \end{pmatrix},$$

we obtain  $r^2 = 1$  and  $e^{2i\varphi} = \mp 1$ . Hence,  $r = \pm 1$ , and in  $G$ , the associated representations are the same. Regarding  $\varphi$ , if  $e^{2i\varphi} = -1$ ,  $\varphi = \pm\pi/2$ , (both yield the same representation in  $G$ ). Otherwise, if  $e^{2i\varphi} = 1$ , we get  $\varphi = 0, \pi$ , and again both yield the same representation. Therefore, we obtain representations

$$\operatorname{Ad}(G_+)\left(\begin{pmatrix} 0 & e^{i\theta} \\ -e^{-i\theta} & 0 \end{pmatrix}_c, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}_c\right), \quad (3.29)$$

and

$$\operatorname{Ad}(G_+)\left(\begin{pmatrix} 0 & e^{i\theta} \\ -e^{-i\theta} & 0 \end{pmatrix}_c, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_c\right). \quad (3.30)$$

- If  $B^2$  is a hyperbolic transformation,  $A^2$  must also be one so that  $A^2$  and  $B$  commute. Up to conjugation, we can write  $A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}_c$ ,  $B = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}_c$ , where  $a \in \mathbb{R}$ ,  $b \in \mathbb{C}$ . The Klein bottle relation

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} = \pm \begin{pmatrix} b^{-1} & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

implies  $b^2 = \pm 1$ . If  $b^2 = 1$ ,  $b = \pm 1$ , which in  $G$  gives rise to the same representation. Otherwise, if  $b^2 = -1$ ,  $b = \pm i$ . Thus, we obtain



representations

$$\text{Ad}(G_+)\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}_c, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_c\right), \quad \text{where } a \in \mathbb{R}, \quad (3.31)$$

and

$$\text{Ad}(G_+)\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}_c, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}_c\right), \quad \text{where } a \in \mathbb{R}. \quad (3.32)$$

Let us compute now the cases where either  $A^2$  or  $B^2$  is equal to the identity.

- If  $Q(A) = Id$ , then, up to conjugation we can write  $A = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}_c$ ,  $B = \begin{pmatrix} x & y \\ w & z \end{pmatrix}_c$ . The Klein bottle relation

$$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} x & y \\ w & z \end{pmatrix} = \pm \begin{pmatrix} z & -y \\ -w & x \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

implies that either  $x = z = 0$  and  $w = y = \pm i$  (due to  $\det(B) = 1$ ); or  $w = -y$ . We obtain representations

$$\text{Ad}(G_+)\left(\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}_c, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}_c\right), \quad (3.33)$$

and

$$\text{Ad}(G_+)\left(\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}_c, \begin{pmatrix} x & y \\ -y & z \end{pmatrix}_c\right). \quad (3.34)$$

- If  $Q(A) = -Id$ , then, up to conjugation,  $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_c$ ,  $B = \begin{pmatrix} x & y \\ w & z \end{pmatrix}_c$ . From the Klein bottle relation,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ w & z \end{pmatrix} = \pm \begin{pmatrix} z & -y \\ -w & x \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we obtain either  $w = y$  or  $w = -y$  and  $x = z = 0$  (hence,  $y = \pm 1$ ).

Thus, the representations in this case are

$$\text{Ad}(G_+)\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_c, \begin{pmatrix} x & y \\ y & z \end{pmatrix}_c\right), \quad (3.35)$$

and

$$\text{Ad}(G_+)\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_c, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_c\right). \quad (3.36)$$

- If  $Q(B) = Id$ , let us write, up to conjugation,  $A = \begin{pmatrix} x & y \\ w & z \end{pmatrix}_c$ ,  $B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}_c$ . From the Klein bottle relation,

$$\begin{pmatrix} \bar{x} & \bar{y} \\ \bar{w} & \bar{z} \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \pm \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} x & y \\ w & z \end{pmatrix},$$

we obtain either  $z = -\bar{x}$ ,  $w = -\bar{y}$  or  $z = \bar{x}$ ,  $w = \bar{y}$ . Hence, the respective representations are

$$\text{Ad}(G_+)\left(\begin{pmatrix} x & y \\ -\bar{y} & -\bar{x} \end{pmatrix}_c, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}_c\right), \quad (3.37)$$

and

$$\text{Ad}(G_+)\left(\begin{pmatrix} x & y \\ \bar{y} & \bar{x} \end{pmatrix}_c, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}_c\right). \quad (3.38)$$

- If  $Q(B) = -Id$ , up to conjugation,  $A = \begin{pmatrix} x & y \\ w & z \end{pmatrix}_c$ ,  $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_c$ . The Klein bottle relation

$$\begin{pmatrix} \bar{x} & \bar{y} \\ \bar{w} & \bar{z} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ w & z \end{pmatrix}$$

implies either  $z = -\bar{x}$ ,  $w = \bar{y}$  or  $z = \bar{x}$ ,  $w = -\bar{y}$ . The first case does not lie in  $G$ , as  $\det(A) = -(|x|^2 + |y|^2) < 0$ . Thus, the obtained

representations are

$$\text{Ad}(G_+)\left(\begin{pmatrix} x & y \\ \bar{y} & \bar{x} \end{pmatrix}_c, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_c\right). \quad (3.39)$$

Finally, it is not difficult to see that cases (3.27), (3.29), (3.31), (3.33), (3.35) and (3.37) form one connected component whereas the rest of the cases form another one.

# 4

## Metric Completion

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As we deform non-compact 3-manifolds as in Chapters 1 and 2, the deformations into non-complete manifolds are not unique (eg. one can consider an open subset of a non-complete manifold). The goal of this chapter is not to discuss the different issues related to this non-uniqueness, just the existence of a deformation into a metric that can be completed as a conifold. This will be done by considering a maximal structure which corresponds to the canonical structure when the deformation comes from an ideal triangulation.

The main result of this chapter is Theorem 4.2.14. In the orientable case, the metric completion after deforming an orientable cusp is a singular space with a singularity called of *Dehn type* (that include non-singular manifolds), see [28] and [8, Appendix B]. In the non-orientable case, the singularity is more specific, a so called conifold.

### 4.1 Conifolds and cylindrical coordinates

A *conifold* is a metric length space locally isometric to the metric cone of constant curvature on a spherical conifold of dimension one less, see for instance [7]. When, as topological space, a conifold is homeomorphic to a manifold, it is called a *cone manifold*, but in general it is only a pseudo-manifold. In dimension 2 conifolds are also cone manifolds, but in dimension three there may be points with a neighborhood homeomorphic to the cone

on a projective plane  $P^2$ .

We are interested in three local models of singular spaces, that as conifolds are:

- The hyperbolic cone over a round sphere  $S^2$ . This corresponds to a point with a non-singular hyperbolic metric.
- The hyperbolic cone over  $S^2(\alpha, \alpha)$ , the sphere with two cone points of angle  $\alpha$ , that is the spherical suspension of a circle of perimeter  $\alpha$ . It corresponds to a singular axis of angle  $\alpha$ .
- The hyperbolic cone over  $P^2(\alpha)$ , the projective plane with a cone point of angle  $\alpha$ . This is the quotient of the previous one by a metric involution, which is the antipodal map on each concentric sphere.

Next we describe metrically those local models, by using cylindrical coordinates in the hyperbolic space. These coordinates are defined from a geodesic line  $g$  in  $\mathbb{H}^3$ , and we fix a point in the unit normal bundle to  $g$ , i.e. a vector  $\vec{u}$  of norm 1 and perpendicular to  $g$ . Cylindrical coordinates give a diffeomorphism:

$$\begin{aligned} \mathbb{H}^3 \setminus g &\xrightarrow{\cong} (0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R} \\ p &\longmapsto (r, \theta, h) \end{aligned}$$

where  $r$  is the distance between  $g$  and  $p$ ,  $\theta$  is the angle parameter (the angle between the parallel transport of  $\vec{u}$  and the tangent vector to the orthogonal geodesic from  $g$  to  $p$ ) and  $h$  is the arc parameter of  $g$ , the signed distance between the base point of  $\vec{u}$  and the orthogonal projection from  $p$  to  $g$ , Figure 4.1.

In the upper-half space model of  $\mathbb{H}^3$ , if  $g$  is the geodesic from 0 and  $\infty$ , then there exists a choice of coordinates (a choice of  $\vec{u}$ ) so that the projection from  $g$  to the ideal boundary  $\partial_\infty\mathbb{H}^3$  maps a point with cylindrical coordinates  $(r, \theta, h)$  to  $e^{h+i\theta} \in \mathbb{C}$ , Figure 4.2. A different choice of  $\vec{u}$  would yield instead  $\lambda e^{h+i\theta} \in \mathbb{C}$ , for some  $\lambda \in \mathbb{C} \setminus \{0\}$ .

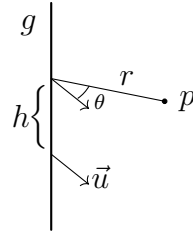


Figure 4.1: Cylindrical coordinates.

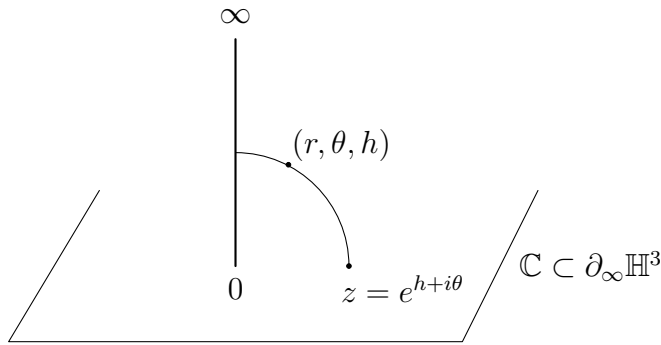


Figure 4.2: Orthogonal projection to  $\partial_\infty \mathbb{H}^3$  with  $g$  the geodesic with ideal end-points  $0$  and  $\infty$ .

The hyperbolic metric on  $\mathbb{H}^3$  with these coordinates is

$$dr^2 + \sinh^2(r)d\theta^2 + \cosh^2(r)dh^2.$$

More precisely,  $\mathbb{H}^3$  is the metric completion of  $(0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$  with this metric.

**Definition 4.1.1.** For  $\alpha \in (0, 2\pi)$ ,  $\mathbb{H}^3(\alpha)$  is the metric completion of  $(0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$  for the metric

$$ds^2 = dr^2 + \left(\frac{\alpha}{2\pi}\right)^2 \sinh^2(r)d\theta^2 + \cosh^2(r)dh^2$$

The metric space  $\mathbb{H}^3(\alpha)$  may be visualized by taking a sector in  $\mathbb{H}^3$  of angle  $\alpha$  and identifying its sides by a rotation. Alternatively, with the

change of coordinates  $\tilde{\theta} = \frac{\alpha}{2\pi}\theta$ ,  $\mathbb{H}^3(\alpha)$  is the metric completion of  $(0, +\infty) \times \mathbb{R}/\alpha\mathbb{Z} \times \mathbb{R}$  for the metric  $dr^2 + \sinh^2(r)d\tilde{\theta}^2 + \cosh^2(r)dh^2$ .

**Remark 4.1.2.** *The metric models are:*

- *For the non-singular case (the cone on the round sphere) it is  $\mathbb{H}^3$ .*
- *For the singular axis (the cone on  $S^2(\alpha, \alpha)$ ) it is  $\mathbb{H}^3(\alpha)$ .*
- *For the cone on  $P^2(\alpha)$ , it is the quotient*

$$\mathbb{H}^3(\alpha)/(r, \theta, h) \sim (r, -\theta, -h).$$

## 4.2 Conifolds bounded by a Klein bottle

We keep the notation of Section 4.1, with cylindrical coordinates. Before discussing conifolds bounded by a Klein bottle, we describe a cone manifold bounded by a torus.

**Definition 4.2.1.** *A solid torus with singular soul is  $\mathbb{H}^3(\alpha)/\sim$ , where  $\sim$  is the relation induced by the isometric action of  $\mathbb{Z}$  generated by*

$$(r, \theta, h) \mapsto (r, \theta + \tau, h + L)$$

for  $\tau \in \mathbb{R}/2\pi\mathbb{Z}$  and  $L > 0$ .

The space  $\mathbb{H}^3(\alpha)/\sim$  is a solid torus of infinite radius with singular soul of cone angle  $\alpha$ , length of the singularity  $L > 0$  and torsion parameter  $\tau \in \mathbb{R}/2\pi\mathbb{Z}$  (the rotation angle induced by parallel transport along the singular geodesic is  $\frac{\alpha}{2\pi}\tau \in \mathbb{R}/\alpha\mathbb{Z}$ ).

By considering the metric neighborhood of radius  $r_0 > 0$  on the singular soul, we get a compact solid torus, bounded by a 2-torus.

We describe two conifolds bounded by a Klein bottle, that are a quotient of this solid torus by an involution.

**Definition 4.2.2.** A solid Klein bottle with singular soul is  $\mathbb{H}^3(\alpha)/\sim$ , where  $\sim$  is the relation induced by the isometric action of  $\mathbb{Z}$  generated by

$$(r, \theta, h) \mapsto (r, -\theta, h + L)$$

for  $L > 0$ .

The space  $\mathbb{H}^3(\alpha)/\sim$  is a solid Klein bottle of infinite radius with singular soul of cone angle  $\alpha$ , and length of the singularity  $L > 0$ . We may consider a metric tubular neighborhood of radius  $r_0$ , bounded by a Klein bottle. Its orientation covering is a solid torus with singular soul, cone angle  $\alpha$ , length of the singularity  $2L$  and torsion parameter  $\tau = 0$ .

**Definition 4.2.3.** The disc orbi-bundle with singular soul is  $\mathbb{H}^3(\alpha)/\sim$ , where  $\sim$  is the relation induced by two isometric involutions:

$$\begin{aligned} (r, \theta, h) &\mapsto (r, \theta + \pi, -h) \\ (r, \theta, h) &\mapsto (r, \theta + \pi, 2L - h) \end{aligned}$$

for  $L > 0$ .

To describe this space, it is useful first to look at the action on the geodesic (corresponding to  $r = 0$ ). These involutions map  $h \in \mathbb{R}$  to  $-h$  and to  $2L - h$  respectively. Thus it is the action of the infinite dihedral group  $\mathbb{Z}_2 * \mathbb{Z}_2$  on a line generated by two reflections. Its orientation preserving subgroup is  $\mathbb{Z}$  acting by translations on  $\mathbb{R}$ . Thus  $\mathbb{R}/\mathbb{Z}$  is a circle, and  $\mathbb{R}/(\mathbb{Z}_2 * \mathbb{Z}_2)$  is an orbifold. The solid torus is a disc bundle over the circle, and our space is an orbifold-bundle over  $\mathbb{R}/(\mathbb{Z}_2 * \mathbb{Z}_2)$  with fibre a disc.

This space is the quotient of an involution on the solid torus. View the solid torus as two 3-balls joined by two 1-handles, Figure 4.3. On each 3-ball apply the antipodal involution (on each concentric sphere of given radius), and extend this involution by permuting the 1-handles. The quotient of each ball is the (topological) cone on  $P^2$ , hence our space is the result of joining two cones on  $P^2$  by a 1-handle. Its boundary is the connected sum  $P^2 \# P^2 \cong K^2$ .



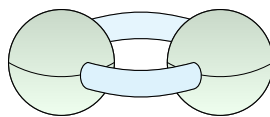


Figure 4.3: A solid torus as two 3-balls joined by two 1-handles.

The singular locus of the disc orbi-bundle  $\mathbb{H}^3(\alpha)/\sim$  is an interval (the underlying space of the orbi-bundle) of length  $L$ . The interior points of the singular locus have cone angle  $\alpha$ , and the boundary points of the interval are precisely the points where it is not a topological manifold.

Again  $\mathbb{H}^3(\alpha)/\sim$  has  $\infty$  radius, and the metric tubular neighborhood of radius  $r$  of the singularity is bounded by a Klein bottle. It is the quotient of a solid torus of length  $2L$  and torsion parameter  $\tau = 0$  by an isometric involution with two fixed points (thus, as an orbifold, its orientation orbi-covering is a solid torus).

**Remark 4.2.4.** The boundary of both, the solid Klein bottle and the disc orbi-bundle, is a Klein bottle. In both cases the holonomy preserves the orientation type, but the type of the presentation is different:

- a) The holonomy of the boundary of a solid Klein bottle with singular soul is a representation of type I.
- b) The holonomy of the boundary of a disc orbi-bundle over a singular interval is of type II.

For a non-orientable end, the holonomy of the peripheral torus is either parabolic non-degenerate, of type I or of type II, also nondegenerate (Remark 3.2.4). The aim of next section is to prove that the deformations can be defined so that the metric completion is either solid Klein bottle with singular soul or a disc orbi-bundle with singular soul, according to the type. This is the content of Theorem 4.2.14, that we prove at the end of the section.

### 4.2.1 The radial structure

Let  $M^3$  be a non-compact hyperbolic 3-manifold of finite volume. We deform its holonomy representation and accordingly we deform its hyperbolic metric. Nonetheless, incomplete metrics are not unique, so here we give a statement about the existence of a maximal structure, which corresponds to the one completed in Theorem 4.2.14.

Let  $[\rho] \in \mathcal{R}(\pi_1(M^3), G)$  be a deformation of its complete structure. There is some nuance in associating to  $[\rho]$  a hyperbolic structure which is made explicit in [10]. Here, the authors conclude that every deformation with a given holonomy representation are related by an isotopy of the inclusion of  $M^3$  in some fixed thickening  $(M^3)^*$ , where a thickening is just another hyperbolic 3-manifold containing ours.

We will start by making clear what we mean by a maximal structure.

**Definition 4.2.5.** *Let  $M$  be a manifold with an analytic  $(G, X)$ -structure. We say that  $M^*$  is an isotopic thickening of  $M$  if it is a thickening and there is a isotopy,  $i'$ , of the inclusion,  $i : M \hookrightarrow M^*$ , such that  $i'(M) = M^*$ .*

Given two isotopic thickenings of  $M$  we say that  $M_1^* \leq M_2^*$  if there is a  $(G, X)$  isomorphism from  $M_1^*$  to some subset of  $M_2^*$  extending the identity on  $M$ . Hence, we say that an isotopic thickening is maximal if it is with respect the partial order relation we have just defined.

Although Canary, Epstein and Green state in [10] that thickenings are unique, we have to remark that this unicity is in a local sense. Without the isotopy condition in the definition, there are plenty of examples of two different thickenings. For instance, take a closed ball in a closed hyperbolic surface, then two possible thickenings are the whole surface and the whole hyperbolic plane, and both of them are in a sense maximal. In general, it is not clear whether maximal isotopic thickenings exist, nor under which circumstances they do exist. However, we will construct in our situation an explicit maximal thickening.

**Lemma 4.2.6.** *Let  $\text{inj}_{M^3}(x)$  denote the injectivity radius at a point  $x \in M^3$ . Then, a necessary condition for a non-trivial thickening of  $M^3$  to exist is that there must exist a Cauchy sequence  $\{x_n\} \subset M^3$  with  $\text{inj}_{M^3}(x_n) \rightarrow 0$ .*

*Proof.* Let us suppose a non-trivial thickening  $(M^3)^*$  exists. Then, take a point  $x \in \partial((M^3)^* \setminus M^3)$ . Any sequence  $\{x_n\} \subset M^3$  such that  $x_n \rightarrow x$  satisfies  $\text{inj}_{M^3}(x_n) \rightarrow 0$  because  $x \notin M^3$ .  $\square$

The purpose of Lemma 4.2.6 is two-fold: first, it gives a condition for a thickening to be maximal (in the sense of the partial order relation we just defined), and second, it shows where a manifold could possibly be thickened. Taking into account a thick-thin decomposition of the manifold, the thickening can only be done in the deformed cusps.

Each cusp of  $M^3$  is diffeomorphic to either  $T^2 \times [0, \infty)$  or  $K^2 \times [0, \infty)$ . Let us consider a proper product compact subset  $K^2 \times [0, \lambda]$  or  $T^2 \times [0, \lambda]$  of an end, for some  $\lambda > 0$ , and let us denote by  $D_\rho$  the developing map of a structure with holonomy  $\rho$  in the equivalence class  $[\rho] \in \mathcal{R}(\Gamma, G)$ .

Before getting into the description of the deformed cusps, we need a technical lemma on *Busemann functions* ([3], [2]). Let  $x_0 \in \mathbb{H}^3$  and let  $\sigma$  be a geodesic at unit speed from  $x_0$  to  $x_\infty \in \partial\mathbb{H}^3$ . The Busemann function at  $x_\infty$  based at  $x_0$  is defined as the limit

$$B_{x_\infty, x_0}(x) := \lim_{t \rightarrow \infty} \text{dist}(x, \sigma(t)) - \text{dist}(x_0, \sigma(t)).$$

The Busemann function can be defined too as a limit of normalized distances when considering some sequence  $\{x_n\} \subset \mathbb{H}^3 \rightarrow x_\infty$  not necessarily in a geodesic. Let us define a normalized distance

$$b_{x_n, x_0}(x) := \text{dist}(x, x_n) - \text{dist}(x_0, x_n).$$

The function  $b_{x_n, x_0}$  converges with respect to the compact-open topology to the Busemann function  $B_{x_\infty, x_0}$  as  $n \rightarrow \infty$  (see [2], Proposition 2.5).

Let us consider now a sequence of geodesics  $\gamma_t$  converging to  $x_\infty$  as  $t \rightarrow \infty$  (equivalently, both endpoints of  $\gamma_t$  converge to  $x_\infty$ ) and consider

the function

$$B_{\gamma_t, x_0}(x) := \text{dist}(x, \gamma_t) - \text{dist}(x_0, \gamma_t).$$

**Lemma 4.2.7.** *The function  $B_{\gamma_t, x_0}$  converges to the Busemann function  $B_{x_\infty, x_0}$  with respect to the compact-open topology.*

*Proof.* Let  $K \subset \mathbb{H}^3$  be compact and connected. The projection of  $K$  into  $\gamma_t$  is a geodesic segment  $[a_t, b_t]$ . Let us consider some arbitrary point  $x_t \in [a_t, b_t]$ . The projection into  $\gamma_t$  is a contraction assuming  $K \cap \gamma_t = \emptyset$ ; in particular, the length of the geodesic segment  $[a_t, b_t]$  decreases with the distance to  $K$  (this can be seen, from instance, from the  $\cosh^2(r)dh^2$  part of the metric in cylindrical coordinates). Thus, the difference

$$B_{\gamma_t, x_0}(x) - b_{x_t, x_0}(x) = \text{dist}(\pi_t(x), x_t) \xrightarrow[t \rightarrow \infty]{} 0,$$

where  $\pi_t(x)$  is the projection of the point  $x$  to  $\gamma_t$ . This shows that restricted to the compact  $K$ , the function  $B_{\gamma_t, x_0}$  converges uniformly to the limit of  $b_{x_t, x_0}$ , which is the Busemann function  $B_{x_\infty, x_0}$  ([2], Proposition 2.5). This can be done for any compact  $K$ , hence, the limit of  $B_{\gamma_t, x_0}$  is the Busemann function  $B_{x_\infty, x_0}$  with respect to the compact-open topology.  $\square$

**Lemma 4.2.8.** *The image of the proper product subset by the developing map,  $D_\rho(\widetilde{K}^2 \times [0, \lambda])$ ,  $D_\rho(\widetilde{T}^2 \times [0, \lambda])$  lies within two tubular neighborhoods of a geodesic  $\gamma \in M^3$ , that is, in  $N_{\epsilon_2}(\gamma) \setminus N_{\epsilon_1}(\gamma)$ , where  $N_\epsilon(\gamma) = \{x \in \mathbb{H}^3 \mid d(x, \gamma) < \epsilon\}$ . Moreover, for every geodesic ray exiting orthogonally from  $\gamma$ , the intersection of the ray with  $D_\rho(\widetilde{K}^2 \times [0, \lambda])$  is non-empty and transverse to any section  $D_\rho(\widetilde{K}^2 \times \{\mu\})$ ,  $\mu \in [0, \lambda]$  and, analogously for an orientable end.*

*Proof.* We will use a modified argument of Thurston (see [43]) to prove the lemma for a non-orientable end (the same idea goes for an orientable one). The original argument of Thurston shows that in an ideal triangulated manifold, the image of the universal cover of the end under the developing map is the whole tubular neighborhood but the geodesic. Let  $[\rho_0]$  be the parabolic representation corresponding to the complete structure, then  $D_{\rho_0}(\widetilde{K}^2 \times [0, \lambda])$  is the region between two horospheres centered

at  $p_0 \in \partial_\infty(\mathbb{H}^3)$ . Let  $K \subset \widetilde{K}^2 \times [0, \lambda]$  denote a fundamental domain of  $K^2 \times [0, \lambda]$ . The domain  $K$  can be taken so that  $D_{\rho_0}(\overline{K})$  is a rectangular prism between two horospheres.

In order to understand how  $D_{\rho_0}(\overline{K})$  deforms as we deform the representation, we have to introduce a Busemann function. Let  $\rho_t$  be a path of representations converging to  $\rho_0$ , and let  $\gamma_t$  be invariant geodesic by the holonomy of  $\rho_t$ . Let us consider the following function on  $\mathbb{H}^3$  for a fixed  $x_0 \in \mathbb{H}^3$ :

$$B_{\gamma_t, x_0}(x) := \text{dist}(x, \gamma_t) - \text{dist}(x_0, \gamma_t).$$

We consider the limit

$$B_{p_0, x_0} := \lim_{t \rightarrow 0} B_{\gamma_t, x_0},$$

with respect to the compact-open topology, which by Lemma 4.2.7, is the Busemann function at  $p_0 \in \partial\mathbb{H}^3$  based at  $x_0$ .

The level sets of the Busemann function  $B_{p_0, x_0}$  are horospheres centered at  $p_0$ . On the other hand, the level sets of  $B_{\gamma_t, x_0}$  are the boundaries of tubular neighbourhoods of  $\gamma_t$ . Therefore, tubular neighbourhoods deform into horoballs as  $\rho_t \rightarrow \rho_0$ .

If we consider now  $D_\rho(\overline{K})$ , it remains close to the previous rectangular prism. Let  $\rho_t$  be a path from  $\rho$  to  $\rho_0$ . As the limit is in the compact-open topology, there exist tubular neighbourhoods of  $\gamma$  such that  $D_\rho(\overline{K})$  is between them. By equivariance, the whole  $D_\rho(\widetilde{K}^2 \times [0, \lambda])$  is between the aforementioned tubular neighbourhoods. By equivariance again, any geodesic ray orthogonal to  $\gamma$  can be translated to a ray intersecting  $D_\rho(\overline{K})$ . Finally, the last part of the proposition follows due to transversality being a stable property.  $\square$

**Definition 4.2.9.** *The geodesic of Lemma 4.2.8 is called the soul of the end.*

**Remark 4.2.10.** The face of the section of proper product subset the cusp  $K^2 \times [0, \lambda]$  or  $T^2 \times [0, \lambda]$  that is glued to the thick part of the manifold is the section of the cusp which is further away from the geodesic. Hence, we will only consider thickenings “towards” the soul.

Let  $x$  be a point in a cusp of the manifold and consider its image under the developing  $y = D_\rho(\tilde{x})$  of any lift  $\tilde{x}$ . There is only one geodesic segment in  $\mathbb{H}^3$  such that  $\gamma(0) = y$  and goes towards the soul orthogonally. In cylindrical coordinates around the soul, if  $y = (r, \theta, h)$ , the image of the geodesic consists of a connected subset of  $\{(t, \theta, h) \mid t \in [0, r]\}$ . Let us denote by  $\gamma_x$  the corresponding geodesic in  $M^3$ , which exists due to the developing map being a local isometry.

**Theorem 4.2.11.** *There exists a maximal thickening  $M^*$  of a half-open product  $K^2 \times [0, \lambda)$  or  $T^2 \times [0, \mu)$ . It is characterized by the following property: for every point  $x \in M$ , the geodesic  $\gamma_x$  can be extended in  $M^*$  so that  $D_\rho(\tilde{\gamma}_x)$  is the geodesic orthogonal to the soul of the end and whose cylindrical coordinates with respect to the soul are  $\{(t, \theta, h) \mid t \in (0, r]\}$ .*

*Proof.* Let us consider the following:

- A cusp section  $S := K^2$  or  $T^2$ .
- A product subset of the end  $K := S \times [0, \lambda]$ .
- A fixed fundamental domain  $K_0 \subset \tilde{K}$  of  $K$ .
- A small neighborhood of  $K_0$ ,  $N(K_0)$ .

The set  $T := \{t \in \text{Deck}(\tilde{K}/K) \mid tN(K_0) \cap N(K_0)\}$  is finite, where  $\text{Deck}(\tilde{K}/K)$  denotes the group of covering transformations of the universal cover. Hence, we can suppose that  $D_{\rho|_{(T\tilde{K}_0)}}$  is an embedding.

Let  $\mathcal{U}$  be an open cover of  $K$  by simply connected charts. For each  $U$ , take a lift  $U_0 \in \tilde{\mathcal{U}}$  such that  $U_0 \cap K_0 \neq \emptyset$  and consider  $D_\rho(U_0)$ . Given such a lift  $U_0$ , the other possible lifts that could have non-empty intersection with  $K_0$  are  $tU_0$ , for  $t \in T$ . Furthermore, we can always assume that the chart  $U$  is isometric with the image of  $U_0$  under the developing map,  $D_\rho(U_0)$ . Thus, we can identify

$$K \cong \left( \bigcup_{U \in \mathcal{U}} D_\rho(U_0) \right) / \sim,$$

where the equivalence relation is by the action of  $\text{hol}(t)$ , for  $t \in T$ .

Each  $U \in \mathcal{U}$  can be thickened by first identifying  $U$  with  $D_\rho(U_0)$  and then considering, in cylindrical coordinates, the set of rays

$$R(U) = \{(t, \theta, h) \in \mathbb{H}^3 \setminus \{\text{soul}\} \mid \exists(t_0, \theta, h) \in U, 0 < t < t_0\}.$$

Given two lifts of two thickened charts  $\widetilde{R(U_1)}$  and  $\widetilde{R(U_2)}$  with non-empty intersection with  $K_0$ , we glue them together in the points corresponding to  $t(\widetilde{R(U_1)}) \cap \widetilde{R(U_2)}$ , where  $t \in T$ . We then have the following thickening of the cusp:

$$K^* := \bigcup_{U \in \mathcal{U}} R(U) / \sim,$$

where the equivalence relation is given by the action of  $\text{hol}(t)$ , for  $t \in T$ . By considering the previous identification of  $K$  with the glued charts, we see that  $K \hookrightarrow K^*$  is an embedding.

We have yet to show that it is isotopic to the original (half-open) product subset. Let us consider the section  $S \times \{0\}$  of the cusp, the radial geodesics  $\gamma_x$  for  $x \in S \times \{0\}$  define a foliation of  $K^*$  of finite length. Moreover, due to Lemma 4.2.8, the foliation is transversal to  $S \times \{0\}$ . By re-parameterizing the foliation and considering its flow, we obtain a trivialization of the cusp,  $K^* \cong S \times [0, \mu)$ . Similarly,  $K^* \setminus K$  is also a product. This let us construct an isotopy from  $K^*$  to  $K$ .

This thickening clearly satisfies the property that  $\gamma_x \subset K^*$  can be extended so that  $D_\rho(\tilde{\gamma}_x) = \{(t, \theta, h) \mid t \in (0, r]\}$ . By taking geodesics  $\gamma_x$  to geodesics through the developing map, it is clear our thickening can be mapped into every other thickening satisfying this property. Furthermore, if we consider the thickenings to be isotopic, we obtain an embedding.

Regarding the maximality, we will differentiate between an orientable end and a non-orientable one. The general idea will be the same one, for another isotopic thickening  $(K)^{**}$  to include ours, the developing map should map some open set  $V$  into a ball  $W$  around a point  $y_0$  in the soul (by Lemma 4.2.6), what will led to a contradiction.

If  $K$  is non-orientable, let us denote  $a, b$  the generators of  $\pi_1(K^2)$  satisfying the relation  $aba^{-1} = b^{-1}$ . If  $[\rho]$  is type I,  $y_0$  is fixed by  $\rho(b)$ . Let

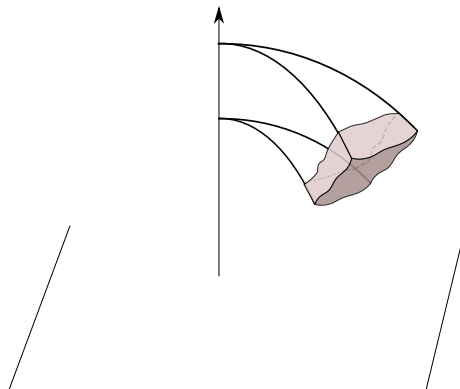


Figure 4.4: The radial thickening.

$y \in W \setminus \{\text{soul}\}$  and  $x \in V$  be its preimage.  $W$  is invariant by  $\rho(b)$  and, in addition, both  $x$  and  $b \cdot x$  belong to  $V$ . Now take the geodesic  $\gamma : I \rightarrow \widehat{(K)}^{**}$  from  $x$  to  $x_0$  which corresponds to the geodesic from  $y$  to  $y_0$ . By equivariance and continuity,  $x_0 = \lim \gamma(t) = \lim b\gamma(t) = bx_0$ . This contradicts  $b$  being a covering transformation. If  $[\rho]$  is type II, the previous argument with  $a^2$  holds.

If  $K$  is orientable, we will follow the same arguments leading to the completion of the cusp (for more details see, for instance, [6]). The deformation  $[\rho]$  is characterized in terms of its generalized Dehn filling coefficients  $\pm(p, q)$ . The case  $p = 0$  or  $q = 0$  are solved as in the non-orientable cusp, so we have the 2 usual cases,  $p/q \in \mathbb{Q}$  or  $p/q \in \mathbb{I}$ . For  $p/q \in \mathbb{Q}$ , there exists  $k > 0$  such that  $k(p, q) \in \mathbb{Q}^2$  and  $(kp)a + (kq)b$  is a trivial loop in the new thickening. If  $p/q \in \mathbb{I}$ , then  $y_0$  is dense in  $\{\text{soul}\} \cap W$ , which is a contradiction.  $\square$

**Definition 4.2.12.** *We call the previous thickening the radial thickening of the cusp.*

**Remark 4.2.13.** If the manifold  $M$  admits an ideal triangulation, the canonical structure coming from the triangulation is precisely the radial thickening of the cusp.



**Theorem 4.2.14.** *For a deformation of the holonomy  $M^3$ , the corresponding deformation of the metric can be chosen so that on a non-orientable end one of the following holds:*

- *It is a cusp, a metrically complete end, if the peripheral holonomy is parabolic.*
- *The metric completion is a solid Klein bottle with singular soul if the peripheral holonomy is of type I.*
- *The metric completion is a disc orbi-bundle with singular soul if it is of type II.*

*Furthermore, the cone angle  $\alpha$  and the length  $L$  of the singular locus is described by the peripheral boundary, so that those parameters start from  $\alpha = L = 0$  for the complete structure and grow continuously when deforming in either direction.*

*Proof of Theorem 4.2.14.* The proof considers the radial thickening and uses the orientation covering and equivariance. More precisely, the deformation is constructed in the complete case for the orientation covering and it can be made equivariant. The holonomy of a torus restricted from a Klein bottle is either parabolic or the holonomy of a solid torus without the singular soul and torsion parameter  $\tau = 0$ . In particular the holonomy of a Klein bottle is parabolic iff its restriction to the orientable covering is parabolic. This corresponds to the complete case.

For a non trivial deformation, we will use the description in cylindrical coordinates (Figure 4.2) around the invariant geodesic by the holonomy of the end. The holonomy is the same as the holonomy of the solid torus of Definition 4.2.1 *without* the singular soul. In terms of generalized Dehn filling coefficients, the deformation has parameters  $(p, 0)$  or  $(0, q)$ , therefore, the conic angle of the torus is  $\alpha = 2\pi/p$  or  $\alpha = 2\pi/q$ . The torsion parameter  $\tau$  is zero due to the fact that either  $\text{hol}(a^2)$  or  $\text{hol}(b)$  is a hyperbolic transformation with real trace. The solid torus with parameters  $\alpha$  and  $\tau = 0$  without the singular geodesic is the radial thickening of the orientation covering of the end.

The radial thickening of the non-orientable end is obtained by quotienting the solid torus without the soul by the action of  $\pi_1(K^2)/\pi_1(T^2) \cong \mathbb{Z}_2$ . This action is given by  $\text{hol}(a)$ . We will obtain the models bounded by a Klein bottle described in Section 4.2, and the two cases are distinguished according to the holonomy of the Klein bottle (see Definition 3.2.2):

- Holonomy of type I: The model space  $\mathbb{H}^3(\alpha)$  is  $\mathbb{H}^3/\text{hol}(b)$  and the corresponding solid torus is

$$\mathbb{H}^3(\alpha)/\text{hol}(a^2).$$

Thus, the action of  $\text{hol}(a)$  in the quotient and in cylindrical coordinates is

$$(r, \theta, h) \mapsto (r, -\theta, h + L),$$

which corresponds to a solid Klein bottle described in Definition 4.2.2. Therefore, the radial thickening of the end is the solid Klein bottle without the singular soul, and the completion consists of adding the soul.

- Holonomy of type II: Now  $\mathbb{H}^3(\alpha) = \mathbb{H}^3/\text{hol}(a^2)$  and the solid torus is

$$\mathbb{H}^3(\alpha)/\text{hol}(b).$$

In cylindrical coordinates, the action of  $\text{hol}(a)$  is

$$(r, \theta, h) \mapsto (r, -\theta, -h),$$

and the other relation of the orbi-bundle in Definition 4.2.3 comes from  $\text{hol}(ab)$ . As before, the radial thickening corresponds to the orbi-bundle without the singular soul and the completion is the whole orbi-bundle.

The description of  $\mathbb{H}^3(\alpha) = \mathbb{H}^3/\text{hol}(a^2)$ , or  $\mathbb{H}^3/\text{hol}(b)$  is only valid as long as the conic angle  $\alpha \leq 2\pi$ . For larger angles, let  $\mathbb{H}^3 \setminus g$  be  $\mathbb{H}^3$  without

the geodesic  $g$  with conic angle  $\alpha$  and consider its universal cover  $\exp : \widetilde{\mathbb{H}^3 \setminus g} \rightarrow \mathbb{H}^3 \setminus g$  and the quotient of  $\widetilde{\mathbb{H}^3 \setminus g}$  under the lifted holonomy. The quotient is non-complete and its completion is  $\mathbb{H}^3(\alpha)$ .  $\square$

# 5

## Connected Components of Representations of Surfaces

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Let  $S$  be a closed surface. We consider the variety of representations  $\text{hom}(\pi_1(S), G)$ , for  $G = \text{Isom}(\mathbb{H}^3)$ . As we have seen in previous chapters, this variety can arise naturally when considering embedded or immersed surfaces in a hyperbolic 3-manifold. Representations in this variety can also be related to complex projective structures under the identification

$$\text{Isom}(\mathbb{H}^3) \cong \text{PSL}(2, \mathbb{C}) \rtimes \mathbb{Z}_2.$$

Namely, if  $S$  is orientable, it is known ([17]) that complex projective structures give rise to non-elementary representations in  $\text{hom}(\pi_1(S), \text{PSL}(2, \mathbb{C}))$  that can be lifted to  $\text{SL}(2, \mathbb{C})$  (as we will see later, this can be associated with a Stiefel-Whitney class). In the non-orientable case, these representations must be orientation type preserving and, evidently, in general this must not be satisfied for every representation in  $\text{hom}(\pi_1(S), \text{Isom}(\mathbb{H}^3))$ .

In this chapter we will compute the number of connected components of the variety of representations  $\text{hom}(\pi_1(S), G)$ , for  $S$  closed and non-orientable. By the classification of surfaces,  $S$  is the connected sum of  $k$  projective planes,  $S = \#^k P^2$ , where  $k$  is the (non-orientable) genus of  $S$ . To any representation  $\phi \in \text{hom}(\pi_1(M), G)$ , there is an associated flat  $G$ -bundle over  $M$  (see [36]). The Stiefel-Whitney classes are a classical invariant of the bundle and can be thought of as invariants of the representation  $\phi$ . In this sense, these cohomological classes are constant on connected components. In fact, it is enough to use them to distinguish different components

and, thus, they are indexed by the first and second Stiefel-Whitney classes. More precisely, we obtain the following result, which is a generalization of Proposition 3.5.1:

**Theorem 5.0.1.** *Let  $N_k$  denote the closed non-orientable surface of genus  $k$ . The representation variety  $\text{hom}(\pi_1(N_k), G)$  has  $2^{k+1}$  connected components. In particular, the connected components are classified by the first and second Stiefel-Whitney class of the associated bundle.*

A key piece for the proof of the theorem is the square map (cf. (3.3)), of which we show that the square map itself and other related maps satisfy a *path-lifting property*, that is, paths can be lifted through them. Our main focus throughout the chapter happens in the orientation type preserving representations, as it is a little bit more involved to work with. The end of the chapter is devoted to the components of the image of the restriction map to the variety of representations of the orientation covering.

## 5.1 The cases of genus 1 and 2

Let  $N_k$  be the closed non-orientable surface of (non-orientable) genus  $k$  and  $\pi_1(N_k)$  its fundamental group. Then,  $\pi_1(N_k)$  admits a presentation

$$\pi_1(N_k) = \langle a_1, \dots, a_k \mid a_1^2 \cdots a_k^2 = 1 \rangle.$$

Let  $\text{hom}(\pi_1(N_k), G)$  be the representation variety; it can be identified with the algebraic set

$$\{A_1, \dots, A_k \in G \mid A_1^2 \cdots A_k^2 = [Id]\}.$$

We will use the notation of Section 3.1. The group of isometries  $G$  is composed of the subgroup of orientation preserving isometries,  $G_+ \cong \text{PSL}(2, \mathbb{C})$  and the subset of orientation reversing ones,  $G_-$ , that is,  $G = G_- \sqcup G_+$ . We can express any element of  $G_-$  as  $A_c$ , where  $A \in G_+$  and  $c$  is the Möbius extension of the complex conjugation. The universal cover of

$G_+$  is

$$\widetilde{G}_+ \cong \mathrm{SL}(2, \mathbb{C}).$$

Therefore, the tangent space at any point of  $G$  can be identified with the Lie algebra  $\mathfrak{sl}(2, \mathbb{C})$ .

Recall that we say that  $\phi \in \mathrm{hom}(\pi_1(N_k), G)$  *preserves the orientation type* if it satisfies  $\phi(\gamma) \in G_-$  if and only if  $\gamma$  is represented by a loop reversing the orientation.

There is a well-defined square map

$$\begin{aligned} Q : G &\rightarrow \widetilde{G}_+ \\ [A] &\mapsto A^2, \end{aligned}$$

whose fibers were studied in Section 3.1. Notice that for  $A_c \in G_-$ ,  $Q(A_c) = A\bar{A}$ , where  $\bar{A}$  is the element with entries the complex conjugate of each entry of  $A$ . We can state two immediate corollaries of Proposition 3.1.1 regarding representation varieties of genus 1 and 2 and using the previous presentation of  $\pi_1(N_k)$ :

**Corollary 5.1.1.** *Let  $N_1$  be a projective plane. The variety of orientation type preserving representations has two connected components.*

**Corollary 5.1.2.** *Let  $N_2$  be a Klein bottle. The variety of orientation type preserving representations has two connected components.*

*Proof.* Let  $A, B \in G_-$  satisfy  $A^2 = B^2$ . Then, in  $\widetilde{G}_+$ ,  $Q(A) = \pm Q(B)$ . If the sign is plus, then  $A$  and  $B$  are in the same fiber of  $Q$ , which is connected by Proposition 3.1.1. Thus, there is a path connecting  $A$  and  $B$  inside the fiber. Moreover, as  $G_-$  is connected, any two representations  $(A_1, B_1), (A_2, B_2)$  with  $A_i = B_i$  and in different fibers can be joined by a path.

Otherwise, if  $Q(A) = -Q(B)$ , then, as noticed in Remark 3.1.5,  $Q(A)$  is either elliptic or  $\pm Id$ . By connectedness of both the fibers of  $Q$  and the subset of elements of  $G_-$  which are neither hyperbolic nor parabolic, we can prove in a similar fashion that the subset of representations such that  $Q(A) = -Q(B)$  is connected too.  $\square$

## 5.2 Path-lifting of the square map

Dealing with connected components of representation varieties is easier if we switch the approach from connectedness to path-connectedness, as shown in the proof of Corollary 5.1.2. As noted in [20] in the frame of representation varieties they are equivalent. A very useful tool in this regard is the *path-lifting property*:

**Definition 5.2.1.** *A map  $f : X \rightarrow Y$  satisfies the path-lifting property if for every point  $x \in X$  and every path  $\gamma : [0, 1] \rightarrow \text{Im } f \subset Y$  with  $\gamma(0) = f(x)$ , there exists, up to reparametrization of  $\gamma$  (precomposition with a non-decreasing surjective map  $\tau : [0, 1] \rightarrow [0, 1]$ ), a lift of  $\gamma$  to a path  $\sigma : [0, 1] \mapsto [0, 1]$  with  $\sigma(0) = x$ .*

Notice that, in general, the path lifting property does not imply uniqueness of the lift.

From the submersion normal form, we can prove:

**Lemma 5.2.2.** *Let  $f : X \rightarrow Y$  be a smooth map between smooth manifolds. If  $f$  is a submersion, then it satisfies the path lifting property.*

An immediate consequence of satisfying the path-lifting property is the following one: let  $f$  satisfy the path-lifting property and let  $\text{Im } f$  be connected. If there exists a point  $y \in \text{Im } f$  whose fiber is path-connected, then the domain of  $f$  is connected.

Recall  $\mathcal{J} = \text{Im}(Q) = \{A \in \widetilde{G}_+ \mid \text{tr}(A) \in (-2, \infty)\} \cup \{-Id\}$ . We will denote

$$\mathcal{J}_0 := \mathcal{J} \setminus \{\pm Id\}.$$

The map  $Q$  restricted to  $Q^{-1}(\mathcal{J}_0)$  has good properties:

**Lemma 5.2.3.** *The set  $\mathcal{J}_0$  is a codimension-1 submanifold of  $\widetilde{G}_+$ . Moreover, the map  $Q$  restricted to  $Q^{-1}(\mathcal{J}_0)$  is a submersion in the image  $\mathcal{J}_0$ . In particular, the square map satisfies the path lifting property.*

*Proof.* Let  $A \in \mathcal{J}_0$ , and let  $U$  be an open neighbourhood around  $A$  in  $\widetilde{G}_+$ . Then, an open neighbourhood around  $A$  in  $\mathcal{J}_0$  can be obtained as

$U \cap (\text{Im} \circ \text{tr})^{-1}(0)$ . It is easy to check that 0 is a regular value of the map  $\text{Im} \circ \text{tr}$  restricted to  $\mathcal{J}_0$ , hence  $\mathcal{J}_0$  is a codimension-1 submanifold.

The differential of  $Q$  at  $A_c \in G_-$ , applied to a tangent vector  $\xi$  is

$$dQ(\xi) = \xi A\bar{A} + A\bar{\xi}\bar{A} = (A\bar{\xi}A^{-1} + \xi)A\bar{A},$$

where we are taking multiplication at right. Thus, from the Lie algebra point of view, it is

$$\begin{aligned} \mathfrak{sl}(2, \mathbb{C}) &\longrightarrow \mathfrak{sl}(2, \mathbb{C}) \\ \xi &\longmapsto A\bar{\xi}A^{-1} + \xi. \end{aligned}$$

As we are only interested on the rank of the map, we can swap  $\xi$  by its conjugate  $\bar{\xi}$ , which leaves the image as  $\text{Ad}(A)\xi + \bar{\xi}$ . Similarly, we can take  $A$  to be any element in its conjugacy class.

The proof will come from computing the adjoint representation for each conjugacy class of elements in  $G_- \setminus Q^{-1}(\pm Id)$ . Proposition 3.1.1 states that, up to conjugation, the element  $A_c$  can be assumed to be either hyperbolic, elliptic or parabolic (see Remark 3.1.3), that is, either

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}_c, \quad \begin{pmatrix} 0 & e^{i(\theta+\pi)/2} \\ -e^{-i(\theta+\pi)/2} & 0 \end{pmatrix}_c, \quad \text{or} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}_c,$$

where  $\lambda \in \mathbb{R}$ ,  $\theta \in (0, \pi)$ . We denote the matrix  $A \in G_+$  of each respective case by  $A_{hyp}$ ,  $A_{ell}$  or  $A_{par}$ . If  $\xi = \begin{pmatrix} x_3 & x_1 \\ x_2 & -x_3 \end{pmatrix}$  belongs to the Lie algebra  $\mathfrak{sl}(2, \mathbb{C})$ , then, the adjoint for each case is:

$$\begin{aligned} \text{Ad}(A_{hyp})\xi &= \begin{pmatrix} x_3 & \lambda^2 x_1 \\ \lambda^{-2} x_2 & -x_3 \end{pmatrix}, & \text{Ad}(A_{ell})\xi &= \begin{pmatrix} -x_3 & x_2 e^{i\theta} \\ x_1 e^{-i\theta} & x_3 \end{pmatrix}, \\ \text{Ad}(A_{par})\xi &= \begin{pmatrix} x_2 + x_3 & x_1 - x_2 - 2x_3 \\ x_2 & -x_2 - x_3 \end{pmatrix}. \end{aligned}$$

Thus, as an action of  $\text{SO}(2, 1)$ , the adjoint representation is, respectively

$$\text{Ad}(A_{hyp}) = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{Ad}(A_{ell}) = \begin{pmatrix} 0 & e^{i\theta} & 0 \\ e^{-i\theta} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$



$$\text{Ad}(A_{par}) = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

We are interested in expressing the Lie algebra as  $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{sl}(2, \mathbb{R}) \oplus i\mathfrak{sl}(2, \mathbb{R})$ , hence, the matrix of the linear map  $\xi \mapsto \text{Ad}(A)\xi + \bar{\xi}$  is:

$$\begin{aligned} \text{Ad}(A_{hyp})\xi + \bar{\xi} &= \begin{pmatrix} \lambda^2 + 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^{-2} + 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^2 - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^{-2} - 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ \text{Ad}(A_{ell})\xi + \bar{\xi} &= \begin{pmatrix} 1 & \cos \theta & 0 & 0 & -\sin \theta & 0 \\ \cos \theta & 1 & 0 & \sin \theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sin \theta & 0 & -1 & \cos \theta & 0 \\ -\sin \theta & 0 & 0 & \cos \theta & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}, \\ \text{Ad}(A_{par})\xi + \bar{\xi} &= \begin{pmatrix} 2 & -1 & -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

We notice that the rank is always five (in the elliptic case, due to  $\theta \neq 0, \pi$ ), which equals the dimension of the image.

Finally, it satisfies the path-lifting property due to Lemma 5.2.2.  $\square$

If we try to extend Lemma 5.2.3 to the whole image  $\mathcal{J}$  we are bound to fail. A geometric interpretation of why these points are troublesome comes from noticing that a rotation in  $S^2$  is given by its unique axis and its angle of rotation. Moreover, the square of an elliptic transformation can be

thought of as a rotation in the boundary at infinity (a sphere). Hence, the ‘square root’ will have the same axis and half the angle. We can consider a sequence of rotations such that the angle goes towards 0 or  $\pi$  (where there is no longer a unique axis) but the sequence of axes does not converge. Thus, the sequence has a limit but the square root will not. This is illustrated in the following example ([39]):

**Example 5.2.4.** Let

$$g_t = \begin{pmatrix} \sqrt{2} + \sin(1/t) & \cos(1/t) \\ \cos(1/t) & \sqrt{2} - \sin(1/t) \end{pmatrix}, \quad R_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

In general,  $\text{Ad}(g)R_{\theta_t}$  tends towards to  $\pm Id$  if we make  $\theta_t$  tend to 0 or  $\pi$ , respectively. Thus, in the particular case  $g_t R_{\theta_t} g_t^{-1}$  with  $\theta_t = \pi - t$ , we obtain

$$g_t R_{\theta_t} g_t^{-1} \xrightarrow{t \rightarrow 0} -Id.$$

On the other hand, the path  $g_t R_{\theta_t} g_t^{-1}$  cannot be lifted along  $Q$ , due to the appearance of the terms  $\sin(1/t)$  and  $\cos(1/t)$  in any possible lift of the path outside of 0.

Same example works with  $\theta_t = t$ , where the limit now is the identity.

Let  $X(F_2, \widetilde{G}_+)$  denote the variety of characters of  $F_2$ , the free group on two elements, and let

$$\begin{aligned} \chi : \widetilde{G}_+ \times \widetilde{G}_+ &\longrightarrow X(F_2, \widetilde{G}_+) \cong \mathbb{C}^3 \\ (A, B) &\longmapsto \chi(A, B) := (\text{tr } A, \text{tr } B, \text{tr } AB) \end{aligned} \tag{5.1}$$

be the character map, where  $\text{tr } A$  denotes the trace of  $A$ . The character map was motivated by Fricke-Klein ([15]). The polynomial map  $\kappa : \mathbb{C}^3 \rightarrow \mathbb{C}$ ,  $\kappa(x, y, z) := x^2 + y^2 + z^2 - xyz - 2$  satisfies  $\kappa(\chi(a, b)) = \text{tr } [a, b]$ , where  $[\cdot, \cdot]$  is the commutator.

Let

$$\begin{aligned} Q_n : G^n &\longrightarrow \widetilde{G}_+^n \\ (A_1, \dots, A_n) &\longmapsto (Q(A_1), \dots, Q(A_n)). \end{aligned}$$

Thus,  $Q_1$  coincides with  $Q$ . The map  $Q_n$  can be restricted to the varieties of representations. For instance, let  $a, b, c$  be generators of  $\pi_1(N_3)$ , then the map  $Q_3$  in the variety of representations  $\text{hom}(\pi_1(N_3), G)$  is

$$\begin{aligned} Q_3 : \text{hom}(\pi_1(N_3), G) &\longrightarrow \{(X, Y, Z) \in (\widetilde{G}_+)^3 \mid XYZ = \pm Id\}, \\ \phi &\longmapsto (Q(\phi(a)), Q(\phi(b)), Q(\phi(c))) \end{aligned}$$

We will often use the following notation: let  $\phi(a) = A_c$  where  $A \in G_+$  and  $c$  is the complex conjugation and, analogously,  $\phi(b) = B_c$ ,  $\phi(c) = C_c$ . Then,  $Q(\phi(a)) = A\bar{A}$ , where  $\bar{A}$  denotes the complex conjugated matrix. Therefore, the map  $Q_3$  can be written down as

$$(A_c, B_c, C_c) \mapsto (A\bar{A}, B\bar{B}, C\bar{C}).$$

Moreover, the image of  $Q_3$  in  $\{(X, Y, Z) \in (\widetilde{G}_+)^3 \mid XYZ = \pm Id\}$  is identified with

$$\text{Im } Q_3 \cong \{(X, Y) \in \mathcal{J} \times \mathcal{J} \mid XY \in \pm \mathcal{J}\}.$$

The following lemma can be found in [20]:

**Lemma 5.2.5.** *Let  $(A, B) \in \widetilde{G}_+ \times \widetilde{G}_+$ . Then, the differential of  $\chi$  (cf. (5.1)) at  $(A, B)$ ,  $d_{(A,B)}\chi$ , is surjective if and only if  $A$  and  $B$  do not commute.*

This can be adapted to the following:

**Lemma 5.2.6.** *Let  $(A, B) \in \mathcal{J}_0 \times \mathcal{J}_0$ . Then, the differential of  $\chi$  restricted to  $\mathcal{J}_0 \times \mathcal{J}_0$  (cf. (5.1)) at  $(A, B)$ ,  $d_{(A,B)}\chi$  is surjective in  $\mathbb{R}^2 \times \mathbb{C}$  if and only if  $A$  and  $B$  do not commute.*

*Proof.* First, we prove that if  $A$  and  $B$  do not commute, then the differential is surjective. Let  $\xi, \eta$  be tangent vectors at  $A$  and  $B$ , respectively. Then,

$$d\chi(\xi, \eta) = (\text{tr } \xi A, \text{tr } \eta B, \text{tr } \xi AB + \text{tr } A\eta B).$$

Let us focus on the traces depending on  $\xi$ . By conjugating by some element in  $\widetilde{G}_+$ , we can assume  $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$  or  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ,  $\xi = \begin{pmatrix} x_3 & x_1 \\ x_2 & -x_3 \end{pmatrix}$  and

$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Moreover,  $\text{Im}(\text{tr } A\xi) = 0$  as  $\xi$  is tangent to  $A$  in  $\mathcal{J}_0$ . Therefore, for  $A = \text{diag}(\lambda, \lambda^{-1})$ ,

$$\text{tr } A\xi = (\lambda - \lambda^{-1})x_3, \quad \text{tr } \xi AB = \lambda^{-1}cx_1 + \lambda bx_2 + (\lambda a - \lambda^{-1}d)x_3.$$

The elements  $A$  and  $B$  do not commute if and only if  $b$  or  $c$  are different from 0, hence, by the previous computations it is evident that the image of  $(\xi, 0)$  is  $\mathbb{R} \times \{0\} \times \mathbb{C}$ . If, instead,  $A$  is parabolic, then

$$\text{tr } A\xi = x_2, \quad \text{tr } \xi AB = cx_1 + cx_2 + (a - d)x_3.$$

As before,  $A$  and  $B$  do not commute if and only if either  $B$  is not parabolic or parabolic with a fixed point at infinity different from the one of  $A$ . It is easy to see that this is equivalent to  $a \neq d$  or  $c \neq 0$  and, thus, if  $A$  and  $B$  do not commute the image of  $(\xi, 0)$  is  $\mathbb{R} \times \{0\} \times \mathbb{C}$  as well. By a similar argument on  $\eta$  for either case, we obtain the whole  $\mathbb{R}^2 \times \mathbb{C}$ .

On the other hand, the same computations also prove the converse statement. □

**Corollary 5.2.7.** *Let  $(A, B) \in G_- \times G_-$ . The differential of  $\chi \circ Q_2$  at  $(A, B)$ ,  $d_{(A,B)}\chi \circ Q_2$ , is surjective over  $\mathbb{R}^2 \times \mathbb{C}$  if and only if  $Q(A)$  and  $Q(B)$  do not commute. In particular, the map  $\chi \circ Q_2$  satisfies the path-lifting property.*

*Proof.* From Lemma 5.2.6 we see that a necessary condition for the differential to be exhaustive is that  $Q(A)$  and  $Q(B)$  do not commute.

In the other direction, if  $Q(A)$  and  $Q(B)$  do not commute, in particular they are different from  $\pm Id$ , then, from Lemma 5.2.3,  $(A, B) \mapsto (Q(A), Q(B))$  is a submersion at  $(A, B)$ . By Lemma 5.2.6,  $\chi \circ Q_2$  is a submersion too. The last assertion is a consequence of Lemma 5.2.2. □

**Lemma 5.2.8.** *The set of regular points of the map*

$$\begin{aligned} Q^{-1}(\mathcal{J}_0) \times Q^{-1}(\mathcal{J}_0) &\longrightarrow \widetilde{G}_+ \\ (A, B) &\longmapsto Q(A)Q(B) \end{aligned}$$

is

$$\{A, B | [Q(A), Q(B)] \neq Id\} \cup \{A, B | (\text{tr}(Q(A)) - 2)(\text{tr}(Q(B)) - 2) < 0\}.$$

More generally,  $A_1, \dots, A_n$  is a regular point of the map  $\text{prod} \circ Q_n$ , where  $\text{prod} : \widetilde{G}_+^n \rightarrow \widetilde{G}_+$  denotes the product, and  $Q_n$  is restricted to  $Q^{-1}(\mathcal{J}_0)^n$ , if and only if there exists  $i, j \in \{1, \dots, n\}$  such that either  $Q(A_i), Q(A_j)$  do not commute or  $(\text{tr}(Q(A_i)) - 2)(\text{tr}(Q(A_j)) - 2) < 0$ .

*Proof.* Let us assume right multiplication in the Lie group. A straightforward computation shows that the differential applied to a tangent vector  $(\xi, \eta)$  is

$$\xi A \bar{A} B \bar{B} + A \bar{\xi} \bar{A} B \bar{B} + A \bar{A} \eta B \bar{B} + A \bar{A} B \bar{\eta} \bar{B}.$$

This corresponds to the vector of the Lie algebra  $\mathfrak{sl}(2, \mathbb{C})$

$$\xi + \text{Ad}(A)\bar{\xi} + \text{Ad}(A\bar{A})\eta + \text{Ad}(A\bar{A}B)\bar{\eta}.$$

We can multiply the expression by  $\text{Ad}(A\bar{A})^{-1}$  and swap  $\xi$  by  $\text{Ad}(A)^{-1}\xi$ , and  $\eta$ , by  $\bar{\eta}$ . We obtain

$$\text{Ad}(\bar{A})^{-1}(\xi) + \bar{\xi} + \text{Ad}(B)(\eta) + \bar{\eta}.$$

We can assume  $A$  to be in its Jordan normal form so that its adjoint representation is easy to compute. On the other hand, with the previous assumption, we will have no control on the adjoint representation of  $B$ , so we need to compute the adjoint representation of any matrix  $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  applied to an element  $\xi = \begin{pmatrix} x_3 & x_1 \\ x_2 & -x_3 \end{pmatrix}$ :

$$\text{Ad}(X)(\xi) = \begin{pmatrix} (ad + bc)x_3 + bdx_2 - acx_1 & -2acx_3 - b^2x_2 + a^2x_1 \\ 2cdx_3 + d^2x_2 - c^2x_1 & -((ad + bc)x_3 + bdx_2 - acx_1) \end{pmatrix}.$$

Thus,

$$\begin{aligned} \text{Ad}(X)(\xi) + \bar{\xi} = \\ \left( \begin{array}{cc} (ad + bc)x_3 + \bar{x}_3 + bdx_2 - acx_1 & -2acx_3 - b^2x_2 + a^2x_1 + \bar{x}_1 \\ 2cdx_3 + d^2x_2 + \bar{x}_2 - c^2x_1 & -((ad + bc)x_3 + \bar{x}_3 + bdx_2 - acx_1) \end{array} \right). \end{aligned}$$

Taking  $A$  in its Jordan form and resting upon the computations in the proof of Lemma 5.2.3, we prove the first assertion.

The general statement follows in a similar way. The differential applied to a tangent vector  $(\xi_1, \dots, \xi_n)$  is

$$\sum_{i=1}^n \left( \text{Ad} \left( \prod_{j=1}^{i-1} A_j \bar{A}_j \right) (\xi_i + \text{Ad}(A_i) \bar{\xi}_i) \right).$$

The image of each summand has rank five as seen in the proof of Lemma 5.2.3, therefore,  $(A_1, \dots, A_n)$  is a regular point iff there exists  $i < j$  such that the respective summands generate the whole space. Let us apply induction on the distance between both matrices  $k := j - i$ .

For  $k = 1$ , the case  $n = 2$  can be applied to conclude that either  $Q(A_i)$  and  $Q(A_{i+1})$  do not commute or  $(\text{tr}(Q(A_i)) - 2)(\text{tr}(Q(A_{i+1})) - 2) < 0$ . If  $k > 1$ , either the image of the tangent vectors  $(0, \dots, 0, \xi_i, \xi_{i+1}, 0, \dots, 0)$  generate the whole tangent space or not. If they generate it, then we can take  $A_i$  and  $A_{i+1}$  instead. Otherwise, the tangent vectors associated to  $A_{i+1}$  and  $A_j$  generate the whole image, and we can apply the induction hypothesis.  $\square$

## 5.3 Representation varieties

In this section we apply the results of the previous section to compute the connected components of the variety of representations  $\text{hom}(\pi_1(N_k), G)$ . They will be indexed by two Stiefel-Whitney classes. The first of them is due to the two connected components of  $G$ , whereas the second one is the second Stiefel-Whitney class of the associated flat principal bundle. We

first focus on computing the second Stiefel-Whitney class in the case of orientation type preserving representations, whose study is slightly more complex. In order to compute the number of connected components, we will initially compute them for non-orientable genera  $k = 2, 3$  and then, the general case follows by induction by (in some sense) cutting the surface in a subsurface of genus  $k - 2$  and another one of genus 2.

### 5.3.1 The orientation type preserving components

We will compute here the connected components of the space of *orientation type preserving* representations of the fundamental group of the closed non-orientable surface  $N_k$  into  $G$ . We will denote the set of orientation type preserving representations by

$$\text{hom}^{tp}(\pi_1(N_k), G).$$

These connected components can be identified as fibers of the Stiefel-Whitney map  $w_2 : \text{hom}^{tp}(\pi_1(N_k), G) \rightarrow \mathbb{Z}_2$ . The algebraic variety can be identified with the set

$$\text{hom}^{tp}(\pi_1(N_k), G) = \{A_1, \dots, A_k \in G_- \mid [Id] = R(A_1, \dots, A_k) = \pi\left(\prod_{i=1}^k Q(A_i)\right)\},$$

where  $\pi : \widetilde{G}_+ \rightarrow G_+$  is the covering projection. The relator map  $R$  can be lifted to  $\tilde{R} : G_-^k \rightarrow \widetilde{G}_+$  as  $\tilde{R} = \prod Q(A_i)$  and its image lies on the set  $\{\pm Id\}$ . This lifted relator map  $\tilde{R}$  is constant on connected components and its image can be identified with the second Stiefel-Whitney class of the associated flat  $G$ -bundle (see [36]).

As in [20], let  $C \in \widetilde{G}_+$ ,  $n \geq 2$  and let us define

$$X_n(C) := \{(A_1, \dots, A_n) \in (G_-)^n \mid \prod_{i=1}^n Q(A_i) = C\}.$$

The set  $X_n(Id)$  is precisely the variety of representations  $\text{hom}^{tp}(\pi_1(N_n), \widetilde{G}_+)$ . The whole representation variety  $\text{hom}^{tp}(\pi_1(N_n), G)$  can be identified with

the set  $X_n(Id) \sqcup X_n(-Id)$ . Each set  $X_n((-Id)^u)$  corresponds to the preimage of the second Stiefel-Whitney class,  $w_2^{-1}(u)$ ,  $u \in \mathbb{Z}_2$ , of the principal bundle. If we prove that  $X_n((-Id)^u)$  is non-empty and connected for  $u = \pm 1$ , then  $\text{hom}^{tp}(\pi_1(N_n), G)$  has two connected components.

Lemma 5.2.8 shows that instead of working with the whole space  $X_n(C)$  it is actually more practical to work with

$$X'_n(C) := \{(A_1, \dots, A_n) \in X_n(C) \mid \exists i, j \text{ such that } [Q(A_i), Q(A_j)] \neq Id\}.$$

In fact, for us it will be more useful to restrict to the following subset, for  $n \geq 4$ ,

$$X''_n(C) := \{(A_1, \dots, A_n) \in X_n(C) \mid \exists i, j \leq n-2 \text{ such that } [Q(A_i), Q(A_j)] \neq Id\}.$$

Notice that  $X'_2(\pm Id) = \emptyset$ ,  $X'_2(C) = X_2(C)$  if  $C \neq \pm Id$ . For  $n = 1$ , let us define  $X_1(C) := Q^{-1}(C)$ . Last, for  $n = 3$ , we define

$$X''_3(C) := \{(A_1, A_2, A_3) \in X_3(C) \mid Q(A_1) \neq \pm Id, [Q(A_2), Q(A_3)] \neq Id\}.$$

As the following lemma shows, from a connectivity point of view it is indifferent considering either  $X_n(C)$ ,  $X'_n(C)$  or  $X''_n(C)$ .

**Remark 5.3.1.** For  $n \geq 3$ ,  $X''_n(C) \subset X'_n(C) \subset X_n(C)$ . Thus, if  $X''_n(C)$  is dense in  $X_n(C)$  and connected, then  $X'_n(C)$  is dense and connected too.

**Lemma 5.3.2.** *Both sets  $X'_n(C)$  and  $X''_n(C)$  are dense in  $X_n(C)$ , when defined, for any  $C \in \widetilde{G}_+$ ,  $n \geq 2$ .*

*Proof.* Let  $(A_1, \dots, A_n) \in X_n(C)$ , we will show that we can find elements  $(B_1, \dots, B_n) \in \mathcal{J}^n$  as close as wanted to  $(Q(A_1), \dots, Q(A_n))$  and such that there exist  $i, j$  such that  $[B_i, B_j] \neq Id$ . If  $(Q(A_1), \dots, Q(A_n)) \in \mathcal{J}_0$ , by Lemma 5.2.3, the square map is open, so there will be suitable preimages of  $(A'_1, \dots, A'_n)$  in  $X'_n(C) \subset (G_-)^n$  as close as desired to  $(A_1, \dots, A_n) \in X_n(C)$ ; otherwise, we will have to perturb the elements  $(A_1, \dots, A_n)$  ad hoc. The main idea consists of swapping two consecutive elements  $A_i, A_{i+1}$  for two



close enough elements  $A'_i, A'_{i+1}$  with non-commuting squares so that the product remains the same,  $Q(A_i)Q(A_{i+1}) = Q(A'_i)Q(A'_{i+1})$ . By choosing the indices  $i, j$  with care and some small modification, the proof will apply to  $X''_n(C)$  too.

Case 1: All of the  $Q(A_i)$  are different from  $\pm Id$ .

Let all of the  $Q(A_i)$  commute, then either all of them can be conjugated to a diagonal matrix or a parabolic one. Let us suppose that all of them are diagonal and let,  $Q(A_1) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ ,  $Q(A_2) = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$ , then if one of the matrix is not the inverse of the other ( $\mu \neq \pm\lambda^{-1}$ ), we can take  $\epsilon, \delta$  as small as wanted so that  $B_1 = \begin{pmatrix} \lambda & \epsilon \\ 0 & \lambda^{-1} \end{pmatrix}$ ,  $B_2 = \begin{pmatrix} \mu & \delta \\ 0 & \mu^{-1} \end{pmatrix}$  do not commute and  $B_1B_2 = Q(A_1)Q(A_2)$ . If there are no two consecutive elements  $A_i, A_{i+1}$  such that  $Q(A_i)Q(A_{i+1}) \neq Id$ , then if we take  $B_1, B_2$  as before, the elements  $B_2$  and  $Q(A_3)$  will not commute.

In the parabolic case, given two matrices  $Q(A_i) = \begin{pmatrix} 1 & x_i \\ 0 & 1 \end{pmatrix}$ ,  $i = 1, 2$  (we can assume  $\text{tr } Q(A_i) = 2$ ) matrices  $B_i = \begin{pmatrix} \lambda & y_i \\ 0 & \lambda^{-1} \end{pmatrix}$  can be chosen as close as wanted to  $Q(A_i)$  such that they do not commute and  $B_1B_2 = Q(A_1)Q(A_2)$ , as long as  $Q(A_1)Q(A_2) \neq Id$ . If they are inverse matrices, after deforming them as before,  $B_2$  and  $Q(A_3)$  will no longer commute.

Case 2: Some  $Q(A_i) = \pm Id$ ,  $Q(A_{i+1})$  or  $Q(A_{i-1})$  different from  $\pm Id$ .

In general, this case may not be problematic if we have enough elements different from  $\pm Id$  as we can ignore the  $\pm Id$  and apply Case 1. On the other hand, in  $X_3(Id)$  it could happen that two of the matrices are inverse of each other and the third one, the identity. We will assume that the problem lies within  $Q(A_1)$  and  $Q(A_2)$ .

We will need to consider Corollary 3.1.6 and the relative position of the fixed points, invariant axes and/or reflection planes:

Case 2.1: Let  $Q(A_1)$  be elliptic,  $Q(A_2) = -Id$  (hence,  $Q(A_1)Q(A_2) = -Q(A_1)$ ).

The isometry  $A_2$  corresponds to an inversion through a point according to Corollary 3.1.6. The fixed point of  $A_2$  together with the invariant axis of  $Q(A_1)$  define a hyperplane  $H \subset \mathbb{H}^3$ . By conjugating by an appropriate

element, we can assume  $H$  to be the hyperplane  $\mathbb{R} \times \{0\} \times \mathbb{R}_{>0}$  in the upper-half space. Moreover, the axis of  $Q(A_1)$  is orthogonal to the plane, so  $Q(A_1)$  acts as a transformation of  $\mathrm{PSL}(2, \mathbb{R})$ . We can further assume that it is the transformation

$$Q(A_1) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

On the other hand, the element  $A_2$  is the isometry (see Proposition 3.1.1)

$$A_2 = \begin{pmatrix} 0 & -\rho \\ \rho^{-1} & 0 \end{pmatrix}_c, \quad \text{where } \rho \in \mathbb{R}^*.$$

Then, we can deform  $A_2$  to

$$A'_2 = \begin{pmatrix} 0 & \rho e^{i\varphi} \\ -\rho^{-1} e^{-i\varphi} & 0 \end{pmatrix}_c, \quad \text{where } \rho \in \mathbb{R}^*, \varphi \in (\pi - \epsilon, \pi + \epsilon).$$

Then,  $Q(A_1)$  is deformed to

$$B_1 := -Q(A_1)Q(A'_2)^{-1} = \begin{pmatrix} -\cos \theta e^{-2i\varphi} & -\sin \theta e^{-2i\varphi} \\ \sin \theta e^{2i\varphi} & -\cos \theta e^{2i\varphi} \end{pmatrix},$$

whose trace is  $2 \cos \theta \cos \varphi \cong 2 \cos \theta$ , therefore  $B_1 \in \mathcal{J}_0$  and clearly is close to  $Q(A_1)$ . By openness of the square map in  $\mathcal{J}_0$ , there is a preimage  $A'_1$  close to  $A_1$ .

Another possible way to deform  $A_1$  and  $A_2$  is to take the orthogonal plane to the axis of  $A_1$  containing the fixed point of  $A_2$ . Then, we can deform  $A_2$  towards an elliptic transformation with axis orthogonal to the previous hyperplane, so that both the axes of  $A_1$  and  $A'_2$  are orthogonal to said hyperplane. Therefore,  $Q(A_1)$  and  $Q(A'_2)$  both act as elements of  $\mathrm{PSL}(2, \mathbb{R})$  on the hyperplane, so  $Q(A'_1)$  will also act as an element of  $\mathrm{PSL}(2, \mathbb{R})$ , hence,  $Q(A'_1) \in \mathcal{J}$ .

Case 2.2: Let  $Q(A_1)$  be hyperbolic,  $Q(A_2) = -Id$ .

Let us consider now the hyperplane containing both the axis of  $Q(A_1)$

and the fixed point of  $A_2$ . Over this plane,  $Q(A_1)$  acts as a transformation of  $\mathrm{PSL}(2, \mathbb{R})$ . As in the alternate solution of the previous case, we can deform  $A_2$  towards an elliptic transformation  $A'_2$  orthogonal to the hyperplane, so it also acts as an element of  $\mathrm{PSL}(2, \mathbb{R})$ . As before,  $Q(A_1)$  can be deformed to  $B_1 := -Q(A_1)Q(A'_2)$  and by openness of the square map, the deformation can be carried over to the preimage.

Case 2.3: Let  $Q(A_1)$  be parabolic,  $Q(A_2) = -Id$ .

Consider the geodesic from the fixed point of  $A_2$  towards the fixed point of  $A_1$  at  $\partial_\infty(\mathbb{H}^3)$ . Up to conjugation, we can assume that said geodesic has endpoints 0 and  $\infty$  in the upper half-space model and  $A_1$  acts as a parabolic transformation with fixed point  $\infty$ . Therefore, we can deform  $A_2$  towards an elliptic transformation with axis the geodesic with endpoints 0 and  $\infty$ . By the same arguments as in the previous cases,  $A_1$  is then deformed too so that  $Q(A'_1)Q(A'_2) = Q(A_1)Q(A_2)$ .

Case 2.4: Let  $Q(A_1)$  be hyperbolic,  $Q(A_2) = Id$ .

Recall that when  $Q(A_2) = Id$ , by Corollary 3.1.6,  $A_2$  is the reflection on a hyperplane of  $\mathbb{H}^3$ . Consider any hyperplane  $H_1$  containing the axis of  $A_1$  and intersecting with the fixed hyperplane  $H_0$  of  $A_2$ . The isometry  $Q(A_1)$  acts as an element of  $\mathrm{PSL}(2, \mathbb{R})$  under the identification  $H_1 \cong \mathbb{H}^2$ . We can deform  $A_2$  towards a hyperbolic transformation  $A'_2$  with axis  $H_0 \cap H_1$ , then  $Q(A'_2)$  acts also as an element of  $\mathrm{PSL}(2, \mathbb{R})$ . Therefore, we can also deform  $Q(A_1)$  in  $\mathcal{J}$ .

Case 2.5: Let  $Q(A_1)$  be elliptic,  $Q(A_2) = Id$ .

Consider any hyperplane perpendicular to the axis of  $A_1$  and intersecting the fixed hyperplane  $H_0$  of  $A_2$ . Then, as in case 2.4, we can deform  $A_2$  towards a hyperbolic transformation so that both  $Q(A_1)$  and  $Q(A'_2)$  behave as elements of  $\mathrm{PSL}(2, \mathbb{R})$  acting on  $H_0 \cong \mathbb{H}^2$ .

Case 2.6: Let  $Q(A_1)$  be parabolic,  $Q(A_2) = Id$ .

Up to changing slightly  $A_1$ , we can assume the fixed point of  $Q(A_1)$  not to be an ideal point of the hyperplane  $H$  fixed by  $A_2$ . Then, we can

consider the unique geodesic orthogonal to  $H$  with endpoint the fixed point of  $A_1$ . We can deform  $A_2$  towards an elliptic transformation with axis said geodesic and, afterwards, deform  $A_1$ .

Case 3: All of the  $A_i$  are  $\pm Id$ .

We only need to deform two of the matrices and then apply Case 2. As before, we will have to consider the relative position of fixed points and hyperplanes.

Case 3.1: Let  $Q(A_1) = Q(A_2) = -Id$ .

We consider the geodesic joining the fixed point of  $A_1$  and the fixed point of  $A_2$ . Then, it is easy to deform both towards elliptic transformations having said geodesic as axis and such that the squares cancel.

Case 3.2: Let  $Q(A_1) = Q(A_2) = Id$ .

Now we have to consider the relative position of the planes  $H_1$  and  $H_2$  fixed by  $A_1$  and  $A_2$ , respectively.

If  $H_1$  and  $H_2$  do not intersect in  $\mathbb{H}^3 \cup \partial_\infty(\mathbb{H}^3)$ , then, there is a geodesic orthogonal to both. We can deform then both  $A_1$  and  $A_2$  toward elliptic transformations with axis said geodesic.

If  $H_1$  and  $H_2$  intersect in  $\mathbb{H}^3$ , then we deform  $A_1$  and  $A_2$  towards hyperbolic transformation with axis  $H_1 \cap H_2$ .

Finally, if  $H_1$  and  $H_2$  only intersect in  $\partial_\infty(\mathbb{H}^3)$ , we can deform both of them towards parabolic transformations with fixed point the previous ideal point.

Case 3.3: Let  $Q(A_1) = -Id$ ,  $Q(A_2) = Id$ :

Let  $H_0$  be the fixed hyperplane by  $A_2$  and consider the geodesic perpendicular to  $H_0$  and passing through the fixed point of  $A_1$ . Then, we can deform both  $A_1$  and  $A_2$  towards elliptic transformations with axis the aforementioned geodesic and such that  $Q(A'_1)Q(A'_2) = -Id$ .

Neither in Case 2 or 3 we have taken into account the possibility that the fixed point, axis or hyperplane (depending on the case) are contained in one another. However, these are very easy to work through.

In order to prove the proposition in the  $X_n''(C)$  case, we notice first that as long as  $n$  is big enough, we can apply the previous ideas to the first  $n - 2$  elements. In fact, for  $n \geq 5$ , this suffices.

In the case  $X_4''(C)$  we have to deal with the possibility where  $Q(A_1), Q(A_2)$  are inverses of one another and  $Q(A_3)$  does not commute with them (so we cannot apply case 1 to solve this). Consider a small element  $S$  of the stabilizer of  $Q(A_2)Q(A_3)$ , then, we deform  $A_2, A_3$  to  $\text{Ad}(S)(A_2), \text{Ad}(S)(A_3)$ . Therefore,  $Q(A_1)$  and  $Q(SA_2S^{-1})$  no longer commute.

Regarding the case  $X_3''(C)$ , we first deform  $A_1, A_2$  so that  $Q(A'_1) \neq \pm Id$  if needed. Afterwards, we can apply the same idea as in the  $X_4''(C)$  case.  $\square$

**Proposition 5.3.3.** *The sets  $X'_3(Id)$  and  $X'_3(-Id)$  are non-empty and connected.*

*Proof.* Let us consider first the case  $X'_3(Id)$ . Let  $(A, B, C) \in X'_3(Id)$ , we can assume without loss of generality that  $A$  and  $B$  satisfy  $[Q(A), Q(B)] \neq Id$ . Let us fix  $(x_1, y_1, z_1) \in (-2, \infty)^3$  such that  $\kappa(x_1, y_1, z_1) \neq 2$ , then  $\kappa^{-1}(x_1, y_1, z_1)$  is composed of a single  $\widetilde{G}_+$ -orbit ([20]). Let  $(A_1, B_1) \in Q_2^{-1}(S)$ , where

$$S = \{(X, Y) \in \mathcal{J}_0 \times \mathcal{J}_0 \mid [X, Y] \neq Id, XY \in \mathcal{J}_0\}$$

and such that  $(Q(A_1), Q(B_1))$  is in the previous  $\widetilde{G}_+$ -orbit.

We will construct a path from  $(A, B)$  to  $(A_1, B_1)$  inside of  $X'_3(Id)$ . Let  $(x_0, y_0, z_0) := \chi(Q(A), Q(B))$ , then a path can be constructed in  $(-2, \infty)^3$  joining  $(x_0, y_0, z_0)$  and  $(x_1, y_1, z_1)$ . By Corollary 5.2.7 the path can be lifted to  $S$  starting at  $(A_0, B_0)$  and ending at the fiber of  $(x_1, y_1, z_1)$  and, as the fiber is connected, it can be continued to  $(A_1, B_1)$ .

The case  $X'_3(-Id)$  can be proved in the same way choosing  $(x_1, y_1, z_1) \in (-2, \infty)^2 \times (-\infty, 2)$  instead.  $\square$

**Corollary 5.3.4.** *The sets  $X_3(Id)$  and  $X_3(-Id)$  are non-empty and connected.*

*Proof.* Apply Lemma 5.3.2 to Proposition 5.3.3. □

**Proposition 5.3.5.** *The set  $X_2(C)$  is non-empty and connected for any  $C \in \widetilde{G}_+$ .*

*Proof.* The case  $C = \pm Id$  is Corollary 5.1.2.

Let  $z = \text{tr } C$  and let us consider  $(x_1, y_1, z) \in (-2, \infty)^2 \times \mathbb{C}$  such that  $\kappa(x_1, y_1, z) \neq 2$ . The fiber  $\kappa^{-1}(x_1, y_1, z)$  is one  $\widetilde{G}_+$ -conjugacy class (see [20]). Let us fix some element  $(A_1, B_1) \in G_- \times G_-$  in the aforementioned conjugacy class such that  $Q(A_1)Q(B_1) = C$ . Now, for any  $(A_0, B_0) \in X'_2(C)$ , with  $\chi(Q(A_0), Q(B_0)) = (x_0, y_0, z)$  a path  $(x_t, y_t, z)$  can be constructed in  $(-2, \infty)^2 \times \mathbb{C}$ . The path can be reparametrized and lifted to  $\{(X, Y) \in \mathcal{J} \times \mathcal{J} \mid [X, Y] \neq Id\}$  starting at  $(Q(A_0), Q(B_0))$  and, as  $Q$  is a submersion in  $\{(X, Y) \in \mathcal{J} \times \mathcal{J} \mid [X, Y] \neq Id\}$  (see Lemma 5.2.3), it can be lifted to  $G_- \times G_-$ ; notice, however, the resulting path  $(X_t, Y_t, C_t)$  does not necessarily satisfy  $C_t = C$ .

We can obtain continuously a path  $g_t \in \widetilde{G}_+$  such that  $C_t = g_t C g_t^{-1}$ , so conjugating by  $g_t^{-1}$  we obtain a path in  $X'_2(C)$ . This can be done by writing the matrix  $C$  in its Jordan canonical form and understanding  $g_t$  as a change of basis matrix to its Jordan form. Due to the fact that the Jordan form of  $C_t$  remains constant during the whole path (otherwise, it wouldn't be true), this basis can be chosen to depend continuously on  $C_t$ : for instance, we can ask for the basis to have constant norm and first coordinate real.

The path we have thus constructed in  $\mathcal{J} \times \mathcal{J}$  ends in the fiber  $\kappa^{-1}(x_1, y_1, z)$ , which is connected (it is a  $\widetilde{G}_+$ -conjugacy class). The elements of the fiber having  $C$  as third coordinate is a  $\text{Stab}(C)$ -conjugacy class, where  $\text{Stab}(C)$  denotes the stabilizer of  $C$ . The stabilizer is connected unless  $C$  is parabolic, then it has two connected components  $\text{Stab}^0(C)$  (the connected component of the identity) and  $-\text{Stab}^0(C)$ . Therefore, in any case, the  $\text{Stab}(C)$ -conjugacy class is connected. Thus, the lifted path to  $G_- \times G_-$  can be joined with  $(A_1, B_1)$ . □

In the same vein as in [20], let us denote by  $f_{n-2}$  the following map:

$$\begin{aligned} f_{n-2} : X_n(C) &\longrightarrow \widetilde{G}_+ \\ (A_1, \dots, A_n) &\longmapsto Q(A_1) \cdots Q(A_{n-2}). \end{aligned}$$

**Proposition 5.3.6.** *The set  $X_n''(C)$  is non-empty and connected for any  $C \in \widetilde{G}_+$  and  $n \geq 3$ . Moreover, the path  $f_{n-2}$  satisfies the path lifting property.*

*Proof.* Let us work by induction on  $n$ . We need then two initial cases, one for  $n$  even and another for  $n$  odd.

The fiber of  $f_1 : X_3''(C) \rightarrow \mathcal{J}_0$  at  $\nu \in \mathcal{J}_0$  is  $Q^{-1}(\nu) \times X_2'(\nu^{-1}C)$ . By Propositions 3.1.1 and 5.3.5 both factors are connected, therefore  $f_1^{-1}(\nu)$  is connected too.

The fiber of  $f_2 : X_4''(C) \rightarrow \widetilde{G}_+$  at  $\nu \in \widetilde{G}_+$  is  $X_2'(\nu) \times X_2(\nu^{-1}C)$ , which, by the same arguments as before, is connected.

By Lemma 5.2.8  $f_{n-2|X_n''(C)}$  is a submersion, thus it satisfies the path lifting property. Moreover, both images  $\mathcal{J}_0$  and  $\widetilde{G}_+$  are connected, thus  $X_n''(C)$  is connected for  $n = 3, 4$ .

For  $n > 4$ , we apply the same argument to  $f_{n-2} : X_n''(C) \rightarrow \widetilde{G}_+$ . The fiber at  $\nu \in \widetilde{G}_+$  is  $X_{n-2}'(\nu) \times X_2(\nu^{-1}C)$ , which by Remark 5.3.1 and the induction hypothesis is connected.  $\square$

**Remark 5.3.7.** The technique to compute connected components of  $X_3'(\pm Id)$  was used in [20] to compute the connected components of  $W(\Sigma_{0,3}) \subset \text{hom}(\pi_1(\Sigma_{0,3}), \text{PSL}(2, \mathbb{R}))$ , where  $\Sigma_{0,3}$  is the three-holed sphere and  $W(\Sigma_{0,3})$  is the subset of non-commuting hyperbolic representations. The three components of  $W(\Sigma_{0,3})$  are distinguished as the fibers of a relative Euler class,  $e^{-1}(n)$ ,  $n = -1, 0, 1$ . Given  $\phi \in W(\Sigma_{0,3})$ , if  $\chi_\phi \in (2, \infty)^3$ , then  $\phi \in e^{-1}(0)$ . Otherwise, if  $\chi_\phi \in (2, \infty)^2 \times (-\infty, -2)$ , then  $\phi \in e^{-1}(\pm 1)$ , where the components  $e^{-1}(-1)$  and  $e^{-1}(1)$  are interchanged if  $\phi$  is conjugated by an element of  $\text{PGL}(2, \mathbb{R})$  not in  $\text{PSL}(2, \mathbb{R})$ .

In the non-orientable case  $N_3$ , by Proposition 5.3.3 we can assume that any representation is hyperbolic and we can cut out the Möbius strips in

order to obtain a representation  $Q_3(\phi) \in \text{hom}(\pi_1(\Sigma_{0,3}), \text{PSL}(2, \mathbb{R}))$ . As the traces are real, by conjugation, we can also assume the representation is actually in  $\text{PSL}(2, \mathbb{R})$  and compute in which component of  $W(\Sigma_{0,3})$  it is (this component is not well defined). When  $\chi_{Q_3(\phi)} \in (2, \infty)^2 \times (-\infty, -2)$  it can be either in  $e^{-1}(1)$  or  $e^{-1}(-1)$  and we can pass from one component to the other by conjugating by the element  $[\text{diag}(i, -i)] \in \text{PSL}(2, \mathbb{C})$ .

This hints that a possible approach could have been closer to the  $\text{PSL}(2, \mathbb{R})$  case worked out in [39], where paths of representations joining  $e^{-1}(n)$  and  $e^{-1}(n+2)$  are constructed.

### 5.3.2 The rest of the connected components

When studying connected components which are not orientation type preserving we have to take into account that  $\pi_0(G) = \mathbb{Z}_2$ . We define the *first Stiefel-Whitney class* of a representation  $\phi \in \text{hom}(\pi_1(N_k), G)$  as the element  $w_1(\phi) \in \text{hom}(\pi_1(N_k), \pi_0(G))$  obtained by postcomposing the representation with the map  $G \rightarrow \pi_0(G)$ . Thus,  $w_1(\phi)$  can be seen as an element of  $H^1(N_k, \mathbb{Z}_2) \cong \mathbb{Z}_2^k$ . For instance, the first Stiefel-Whitney class of an orientation type preserving representation is  $w_1(\phi) = (1, \dots, 1)$ .

In Section 3.1 we defined the square map  $Q : G \rightarrow \widetilde{G}_+$  and up to this point we have been interested in its restriction to  $G_-$ . Here, we need to consider the restriction to  $G_+$  too, which we denote by  $Q_+ : G_+ \rightarrow \widetilde{G}_+$ .

Recall the statement of Proposition 3.1.8 (1.), which proves that the map  $Q_+^{-1} : \widetilde{G}_+ \setminus \text{tr}^{-1}(-2) \rightarrow G_+$  is well-defined and it is smooth. Thus, paths can always (and uniquely) be lifted as long as they avoid  $-Id$ . This shows that the arguments made in Subsection 5.3.1 for the orientation type preserving components can be done in this context for the rest of components with small modifications. Hence, we can prove that each fiber  $w_1^{-1}(\epsilon)$ ,  $\epsilon \in \mathbb{Z}_2^k$  has two connected components. We obtain the following result:

**Theorem 5.3.8.** *The representation variety  $\text{hom}(\pi_1(N_k), G)$  has  $2^{k+1}$  connected components.*



## 5.4 The restriction map

Let  $k > 2$ , the orientation covering of  $N_k$  is the closed surface of genus  $g = k - 1$ ,  $\Sigma_g$ . There is a restriction map to the variety of representations  $\text{hom}(\pi_1(\Sigma_g), G)$ ,

$$\text{res} : \text{hom}(\pi_1(N_k), G) \longrightarrow \text{hom}(\pi_1(\Sigma_g), G).$$

We consider the Stiefel-Whitney classes of the restricted representations,  $w_i(\text{res}(\phi)) \in \text{hom}(\pi_1(\Sigma_g), \mathbb{Z}_2)$ . Let us denote  $G_\phi$  and  $G_{\text{res}(\phi)}$  the associated flat principal  $G$ -bundles over  $N_k$  and  $\Sigma_g$ , respectively. The bundle  $G_{\text{res}(\phi)}$  is the pullback of the bundle  $G_\phi$  by the covering projection  $\pi$ . By the naturality of the Stiefel-Whitney class, they coincide with the pullback through the projection  $\pi : \Sigma_g \rightarrow N_k$ , that is,

$$w_i(\text{res}(\phi)) = \pi^*(w_i(\phi)).$$

### 5.4.1 The second Stiefel-Whitney class

A quick computation of the pullback  $\pi^*$  on the second cohomology group gives us the second Stiefel-Whitney class of a restricted representation:

**Proposition 5.4.1.** *The second Stiefel-Whitney class of a representation  $\phi \in \text{hom}(\pi_1(N_k), G)$  restricted to the orientation covering is  $w_2(\text{res}(\phi)) = 0$ .*

*Proof.* The pullback of  $\pi : \Sigma_g \rightarrow N_k$  is the map

$$\pi^* : H^2(N_k, \mathbb{Z}_2) \rightarrow H^2(\Sigma_g, \mathbb{Z}_2),$$

such that  $(\pi^*(c))(\alpha) = c(\pi_*(\alpha))$ . Now,  $\pi_* : H_2(\Sigma_g, \mathbb{Z}_2) \rightarrow H_2(N_k, \mathbb{Z}_2)$  is 0, hence  $\pi^* = 0$ .  $\square$

### 5.4.2 The first Stiefel-Whitney class

In order to see how the pullback behaves in the first cohomology group, we need to relate generators of both  $\pi_1(N_k)$  and  $\pi_1(\Sigma_g) \triangleleft \pi_1(N_k)$ .

**Lemma 5.4.2.** *Let us consider the presentation*

$$\pi_1(N_k) = \langle a_1, \dots, a_k \mid a_1^2 \cdots a_k^2 = 1 \rangle,$$

then  $\pi_1(\Sigma_g)$  is generated by  $a_1^2, \dots, a_{k-1}^2, a_1a_2, \dots, a_1a_k$ .

*Proof.* Let  $H$  be the subgroup generated by  $a_1^2, \dots, a_{k-1}^2, a_1a_2, \dots, a_1a_k$ , that is,

$$H = \langle a_1^2, \dots, a_{k-1}^2, a_1a_2, \dots, a_1a_k \rangle.$$

It is clear that  $H \subset \pi_1(\Sigma_g)$ . In the other direction, any element  $\gamma \in \pi_1(\Sigma_g)$  can be expressed as an even product of generators  $a_i^{\pm 1}$ , thus, we only need to show that  $a_i^{\pm 1}a_j^{\pm 1} \in H$  for any  $i, j$ .

- The element  $a_k^2$  is obtained by consider the relation  $a_{k-1}^{-2} \cdots a_1^{-2} = a_k^2$ .
- For  $i \neq j$ ,  $a_i a_j = a_i^2 (a_1 a_i)^{-1} (a_1 a_j) \in H$ .
- Finally, by multiplying the previous case by  $a_i^{-2}$ ,  $a_j^{-2}$  if needed, we obtain  $a_i^{\pm 1} a_j^{\pm 1} \in H$ .

□

Let us consider the presentation of  $\pi_1(\Sigma_g)$  generated by  $a_1^2, \dots, a_{k-1}^2, a_1a_2, \dots, a_1a_k$ .

**Proposition 5.4.3.** *Let  $\text{res}(\phi) \in \text{hom}(\pi_1(N_k), G)$  be a representation restricted to the orientation covering. Its first Stiefel-Whitney class admits an expression  $w_1(\text{res}(\phi)) = (0, \dots, 0, \epsilon_1, \dots, \epsilon_{k-1})$ , where  $\epsilon_i \in \mathbb{Z}_2$ . In particular, the representations  $\phi$  such that the restricted representation  $\text{res}(\phi)$  has first Stiefel-Whitney class  $w_1(\text{res}(\phi)) = (0, \dots, 0)$  are the orientation type preserving representations and the representations in  $\text{hom}(\pi_1(N_k), G_+)$ .*

*Proof.* By choosing generators of  $\pi_1(N_k)$  and  $\pi_1(\Sigma_g)$  as in Lemma 5.4.2, the pullback is laid out as

$$\begin{aligned} \pi^* : H^1(\pi_1(N_k), \mathbb{Z}_2) \cong \mathbb{Z}_2^k &\longrightarrow H^1(\pi_1(\Sigma_g), \mathbb{Z}_2) \cong \mathbb{Z}_2^g \\ ([a_1], \dots, [a_k]) &\longmapsto ([a_1^2], \dots, [a_{k-1}^2], [a_1a_2], \dots, [a_1a_k]) \\ &= (0, \dots, 0, [a_1a_2], \dots, [a_1a_k]). \end{aligned}$$

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This shows that the kernel consists of representations where either every element is mapped to an orientation preserving isometry or orientation type preserving representations.  $\square$

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