Universitat Autònoma de Barcelona

The range problem and dimension theory for the Cuntz semigroup

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Memoria presentada a
Universitat Autònoma de Barcelona

per aspirar al grau de

Doctor en Ciències
Matemàtiques

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Universitat Autònoma de Barcelona
Departament de Matemàtiques
2022

Certifico que la present Memòria ha
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Departament de Matemàtiques
Universitat Autònoma de Barcelona

Bellaterra, Juny de 2022

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Introduction

Towards the end of the 19th century and the beginning of the 20th physicists noted that, in a number of microscopic experiments with spectrometer, one could not explain the observed phenomena by using their current theoretical framework, which was based on Newton's mechanics and Maxwell's laws. In fact, some of the results deduced using this framework were contradicted by experimental evidence.

This disparity between the theoretical framework and the observed phenomena led Heisenberg in the 20's to note that, when studying microscopic systems, the observable quantites do not commute in a product. With this realization, which marked the beginning of quantum mechanics, came a paradigm shift: The classical notion of 'real variable' (formally associated to a real valued function) had to be changed. The right substitute, as it turned out, was a self-adjoint operator in a separable, infinite-dimensional Hilbert space. This started the theory of operator algebras, which since its very inception has provided new and interesting approaches to problems both in Mathematics and Physics. This has led to the fact that, for example, every physical system can be described by a unital C^*-algebra; see [25].

The need to formalize these new structures of operators prompted Murray and von Neumann to introduce, in their seminal papers [61, 62, 63], the notion of '*rings of operators*'. These algebras, which are now known as *von Neumann algebras*, are defined as subalgebras of bounded operators over a Hilbert space that are closed under the involution and agree with their double commutant. Von Neumann algebras, which are only briefly discussed in this dissertation, were the first operator algebras to be studied in detail, and they constitute a well developed and rich field.

The theory of operator algebras can be divided into two main branches: The analysis of the von Neumann algebras defined above, and the study and classification of *C^*-algebras*. The latter class, which constitutes the main focus of the thesis, were first introduced by Gelfand and Naimark in 1943 [39] as norm and involution closed subalgebras of bounded operators over a Hilbert space. Von Neumann algebra techniques have become indispensable in the theory of C^*-algebras (e.g. in the breakthrough of parts of the Toms-Winter conjecture). Conversely, C^*-techniques can have deep applications to the structure theory of von Neumann algebras. This is why, however different, these theories are intertwined and conform an indivisible unit.

C^*-algebras can also be defined abstractly, without the use of bounded operators, as normed algebras equipped with a norm-compatible involution in a suitable sense. This definition, which is the most widely used today, is equivalent to the one given above as a consequence of the celebrated Gelfand-Naimark theorem. With this new definition at hand, one can readily show that the algebra of continuous, complex valued functions vanishing at infinity from a locally compact, Hausdorff space is a commutative C^*-algebra. In fact, Gelfand and Naimark proved that, up to isomorphism, these are the only commutative examples. The class of C^*-algebras is closed under many natural constructions, such as considering finite matrices over a C^*-algebra, inductive limits, quotients, and tensor products.

As in many areas of mathematics, classification of certain objects in the field provides great insights on their structure, and often constitutes the backbone of many other theoretical advances. This is no different in the theory of operator algebras, where classification is a complex and unsolved problem.

For von Neumann algebras, Connes' groundbreaking classification of injective factors [24] is arguably the most relevant in the field. In analogy to this result, and inspired by Glimm's classification of *UHF-algebras* [42], Elliott conjectured that simple, unital, separable, nuclear C^*-algebras could be classified by means of a complete invariant consisting of K-theory and tracial data. That is, whenever there is an isomorphism between the invariant of two such C^*-algebras, one must be able to lift the isomorphism to an isomorphism between the algebras.

After a number of reformulations, the proposed invariant to tackle the classification program, nowadays known as the *Elliott invariant*, is a tuple consisting of the ordered K_0-group, the K_0-class of the unit, the K_1-group, the tracial space, and its pairing with the states of the K_0-group. Elliott's conjecture began what became to be known as the *Elliott classification program*, the first instance of which is Elliott's own classification of *AF-algebras* [30]; see [35] and the references therein for a survey on the classification program.

Loosely speaking, Elliott showed in [30] that, in order to determine the isomorphism class of an AF-algebra A, it is enough to study the *projections* in arbitrary matrices over A. More concretely, in any C^*-algebra one can use the notion of *Murray-von Neumann equivalence* of projections to define the *ordered K_0-group* of the algebra. What Elliott proved is that two (unital) AF-algebras are isomorphic if and only if their ordered K_0-groups (and the K_0-classes of their units) agree. Equivalently, two unital AF-algebras are isomorphic if and only if their *Murray-von Neumann semigroups*, together with the classes of their units, agree.

The development of this program, and the reformulations of the Elliott invariant itself, came hand in hand with the construction of counterexamples (primarily due to Rørdam [79] and Toms [99]) to Elliott's original conjecture: There exist pairs of simple, unital, separable, nuclear, nonisomorphic C^*-algebras that nevertheless agree on their Elliott invariant. Toms' counterexample has the additional feature that his two algebras not only agree on their Elliott invariant, but also on a large list of other well known invariants, such as the stable rank or the real rank. In order to prove that the pair of C^*-algebras are not isomorphic, Toms showed that these algebras can be distinguished by an order-theoretical property of their *Cuntz semigroup* (see below), termed *almost unperforation*.

In light of these examples, two natural questions impose themselves: Which class of C^*-algebras can one hope to classify using the Elliott invariant? And, restricted to such class, what is the possible range of the invariant?

The second question, which is known as the *range problem*, was answered for the class of AF-algebras by Effros, Handelman, and Shen in [28]. Their result, which is one of the first (and arguably the most celebrated) solution to the range problem, states that a countable, ordered, abelian group is order isomorphic to the ordered K_0-group of a separable AF-algebra if and only if it is unperforated and satisfies the Riesz decomposition property.

Since the official start of the classification program in 1994, [31], and as a result by many hands and years of work, a classification result was reached for a restricted, well-behaved class of simple C^*-algebras. Mainly, the class of simple, unital, separable, nuclear, $\mathcal{Z}$-stable C^*-algebras satisfying the Universal Coefficient Theorem are classifiable by means of the Elliott invariant; see, amongst many others, [32] and [98]. There has also been progress in the non-unital setting; see e.g. [43]. The fruits of the classification program have a broader impact, since many of its theoretical advances, such as *$\mathcal{Z}$-stability* [105] and *nuclear dimension* [109], have been proved to be useful in other situations; see e.g. [18], [80], [84], [91], [100] and [106].

Both of these notions appear in the *Toms-Winter conjecture* (see [106]), which states that, in the setting of (non-trivial) separable, simple, unital, nuclear C^*-algebras, being $\mathcal{Z}$-stable, having finite nuclear dimension, and having almost unperforation on the Cuntz semigroup should all be equivalent. This conjecture has been studied extensively (see for example [65], [85], [14], [55]) and, as a result, most of the implications between its statements are nowadays known. What remains to be proved is that almost unperforation implies $\mathcal{Z}$-stability.

As explained, the Cuntz semigroup appears as a decisive object on two keys aspects of the theory. This invariant is the main object of study of this thesis.

Introduced by Cuntz in 1978 [27], the Cuntz semigroup of a C^*-algebra is a generalization of the Murray-von Neumann semigroup where, instead of projections, one uses positive elements in its construction. Since these elements, which encompass all projections, are abundant in any C^*-algebra, the Cuntz semigroup contains far more information than its Murray-von Neumann counterpart.

This invariant is a positively ordered monoid which, as shown by Coward, Elliott and Ivanescu in [26], satisfies four additional properties. Such properties were used to define abstract Cuntz semigroups, or Cu-*semigroups* for short, and their associated category Cu. By also studying the behavior of the Cuntz semigroup on morphisms between C^*-algebras, one obtains a functor from the category of C^*-algebras to Cu and, as showcased below, the study of said functor has been proved to be very valuable to understand C^*-algebras. An intensive study of the category Cu has been subsequently carried out in [3], [4], [6], [7] and [8], amongst others.

As mentioned before, the structure of the Cuntz semigroup of a C^*-algebra is much richer than the Murray-von Neumann semigroup. For example, it carries the ideal structure of its underlying C^*-algebra. Moreover, when A is separable, simple, unital, finite and $\mathcal{Z}$-stable, Antoine, Dadarlat, Perera and Santiago showed in [2] that the Cuntz semigroup functor is equivalent, if appropriately interpreted, to the Elliott invariant when thought of as a functor. These results suggest that the Cuntz semigroup is a natural candidate to extend classification to the non-simple case, and certain classification results have already been obtained in this direction, as we discuss next.

The class of (separable) *approximate interval algebras*, or *AI-algebras* for short, is a generalization of the class of AF-algebras, and are the noncommutative analog of the algebras of continuous, complex-valued functions on the inverse limit of possibly increasing disjoint copies of the unit interval. Formally, an AI-algebra is a C^*-algebra isomorphic to an inductive limit of finite direct sums of matrices over $C([0,1])$, the continuous, complex-valued functions on $[0,1]$. Ciuperca and Elliott [21] proved that the Cuntz semigroup classifies all such algebras. This result was later generalized by Ciuperca, Elliott and Santiago in [22] to include all *approximate tree algebras*. Further, Robert proved in [72] that *1-dimensional NCCW complexes* with trivial K_1-group can also be classified by their Cuntz semigroup.

Surprisingly, very few results addressing the range problem for the Cuntz semigroup exist. A notable exception is the translation of the Effros-Handelman-Shen theorem to the setting of Cuntz semigroups obtained in [6], which provides a list of abstract properties that determine when a Cu-semigroup is isomorphic to the Cuntz semigroup of an AF-algebra.

Thus, an important problem is to establish range results for wider classes of C^*-algebras. A natural class to consider is that of AI-algebras, since a range result for these algebras would complete the classification obtained by Ciuperca and Elliott.

Precisely because of the richness of its structure alluded to above, the drawback of the Cuntz semigroup is that it is often hard to fully compute. However, without computing it directly, the study of properties that the Cuntz semigroup of certain classes of C^*-algebras (or all of them) satisfies has been proved to be successful in many different scenarios; see [3], [12], [71], [82] and [106]. Two recent examples of such instances are the results published in [91] and [4]:

It is known that a $\mathcal{Z}$-stable C^*-algebra has an almost unperforated and *almost divisible* Cuntz semigroup. Thus, a positive solution to the Toms-Winter conjecture would imply that, in the setting of separable, simple, unital, nuclear C^*-algebras, almost divisibility follows from almost unperforation. In [91], Thiel proves that, under the additional assumption of stable rank one, this is indeed the case.

Because of its relation with the Toms-Winter conjecture, another interesting problem in the theory of Cuntz semigroups is to define and study a notion of dimension for Cu-semigroups in analogy to the nuclear dimension for C^*-algebras.

The other example consists of the study of the Cuntz semigroup of (not necessarily simple) stable rank one C^*-algebras performed in [4], which led to the solution of three open problems for this class of algebras.

One of these problems is the *Global Glimm Problem*. In their study of purely infinite C^*-algebras [54], Kirchberg and Rørdam introduced the *Global Glimm Property* by saying that a C^*-algebra A has this property if, for every $a \in A_+$ and $\varepsilon > 0$, there exists a *-homomorphism $\varphi\colon M_2(C_0((0,1])) \to \overline{aAa}$ such that $(a-\varepsilon)_+$ is in the ideal generated by the image of the morphism. One can then show that every C^*-algebra having the Global Glimm Property has no nonzero elementary ideal-quotients. The Global Glimm Problem asks whether these two conditions are equivalent. That is, if every C^*-algebra with no nonzero elementary ideal-quotients has the Global Glimm Property.

The C^*-algebras that satisfy the Global Glimm Property have many interesting properties. For instance, every unital C^*-algebra with the Global Glimm Property has a full square-zero element. As showcased in [19], this has a number of implications on the unitary group of the algebra. For example, if A, B are unital, separable, prime, traceless C^*-algebras with a full square-zero element such that $U_0(A) \cong U_0(B)$, it follows then that A is isomorphic, or anti-isomorphic, to B.

Using the results from [4] (together with those in this thesis), it follows that every C^*-algebra of stable rank one with no elementary ideal-quotients has the Global Glimm Property. Thus, following the task already initiated in [4], another interesting topic is to translate the Global Glimm Problem into a question about the Cuntz semigroup by using (weak) divisibility conditions defined by Robert and Rørdam in [75].

In this regard, this thesis aims to provide a better understanding of the Cuntz semigroup, both through the study of this invariant for certain classes of C^*-algebras and the definition of second-level invariants, such as the introduction of a notion of dimension for Cu-semigroups. The body of the text has been divided in seven chapters. We outline their contents below.

In Chapter 1 we provide all the necessary results and preliminaries needed to understand the subsequent material. More concretely, it contains an introduction to the theory of C^*-algebras (as well as a brief introductory section on Elliott's classification program), an overview of the main results on Cuntz semigroups of C^*-algebras and abstract semigroups, and a survey on the Effros-Handelman-Shen theorem which, in some sense, is the focus of Chapters 2 and 3. The goal of these chapters is to study the range problem for the class of AI-algebras, thus completing their classification from [21].

We begin this study in Chapter 2, where we focus on commutative, separable, unital AI-algebras. First, we provide a topological characterization for the underlying space of such algebras, which generalizes well known results in continuum theory. Further, we identify the abstract conditions that a Cu-semigroup must satisfy to be isomorphic to the semigroup of lower-semicontinuous functions over a T_1-space. Combining these two results we obtain an abstract characterization for the Cuntz semigroup of commutative, separable, unital AI-algebras. We also list a number of new properties satisfied by the Cuntz semigroup of every separable AI-algebra.

Chapter 3 continues the study initiated in Chapter 2 by providing a local characterization for the Cuntz semigroup of separable AI-algebras. The result is reminiscent of Shen's theorem for AF-algebras, a key ingredient in the Effros-Handelman-Shen theorem. To prove such characterization we show that, given a Cu-semigroup S satisfying certain (mild) properties, Cauchy sequences of morphisms from the Cuntz semigroup of $C([0,1])$ to S have a unique limit. Paired with ideas from continuum theory, such as that of almost commutative diagrams, this allows us to prove the desired characterization. With this local characterization at hand, we provide a list of necessary and sufficient conditions for a countably based Cu-semigroup to be isomorphic to the Cuntz semigroup of an AI-algebra. The results of Chapters 2 and 3 have appeared in [101] and [102] respectively, but their presentation (particularly those in Chapter 3) has been improved considerably.

Chapter 4 is devoted to the introduction of the notion of *covering dimension* of abstract Cuntz semigroups and to apply it to Cuntz semigroups of C^*-algebras. In analogy to Lebesgue's covering dimension for topological spaces (and Winter and Zacharias' nuclear dimension for C^*-algebras), we assign to each Cu-semigroup a value in $\mathbb{N}\cup\{\infty\}$ that reflects certain properties of the semigroup. To a certain extent, this can be interpreted as a measure of the amount of Riesz decomposition that the semigroup has. For commutative C^*-algebras, the local dimension of the spectrum coincides with the covering dimension of the Cuntz semigroup of the algebra. Moreover, we show that for a C^*-algebra A, the dimension of its Cuntz semigroup is always bounded by the nuclear dimension of A. However, these two notions are not equivalent since, for example, every real rank zero C^*-algebra has a zero-dimensional Cuntz semigroup whilst there are examples of real rank zero C^*-algebras with non-zero nuclear dimension, such as the irrational rotation algebra.

In Chapter 5 we continue the study of the covering dimension while also introducing the notion of approximation for abstract Cuntz semigroups and the Löwenheim-Skolem condition for properties of Cu-semigroups. These two new notions allow one to assume, in many instances, that the Cu-semigroup with which one is working is countably based, a property satisfied by the Cuntz semigroup of all separable C^*-algebras. Using this, we provide new results for the covering dimension of Cuntz semigroups of arbitrary C^*-algebras. For example, if a C^*-algebra is approximated by a family of sub-C^*-algebras, we show that the covering dimension of the Cuntz semigroup of the C^*-algebra is always bounded by the supremum of the dimensions of the Cuntz semigroups of the sub-algebras. Almost all of the results on covering dimension presented in this thesis have been published in [92] and [94].

Finally, in Chapters 6 and 7 we introduce the class of *nowhere scattered C^*-algebras* and study their relation to the Global Glimm Property. *Scattered C^*-algebras* were defined by Jensen in [48] as the noncommutative analogues of *scattered spaces*: A topological space is scattered if every nonempty closed subset has an isolated point, while a C^*-algebra is said to be scattered if every nonzero quotient contains a minimal projection (in a suitable sense). One can show that a commutative C^*-algebra is scattered if and only if its spectrum is, and that the notion of scatteredness for C^*-algebras admits a number of interesting characterizations and permanence properties.

At the other extreme of the scale we have nowhere scatteredness: A C^*-algebra A is said to be nowhere scattered if no quotient of A contains a minimal projection. We provide a number characterizations for this notion, amongst them the absence of nonzero elementary ideal-quotients. In addition, we also introduce a new property, termed (O8), that the Cuntz semigroup of every C^*-algebra satisfies. With the help of this new property, we show that a C^*-algebra is nowhere scattered if and only if its Cuntz semigroup is weakly $(2, \omega)$-divisible in the sense of [75].

In Chapter 7, we recall the Global Glimm Property for C^*-algebras and provide a number of characterizations for it. In particular, we show that a C^*-algebra has the Global Glimm Property if and only if its Cuntz semigroup is $(2, \omega)$-divisible, a strengthening of weak $(2, \omega)$-divisibility also introduced in [75]. With this characterization, one can further reformulate the Global Glimm Problem as follows: Is every weakly $(2, \omega)$-divisible Cuntz semigroup of a C^*-algebra $(2, \omega)$-divisible?

This question is studied abstractly in the chapter, where we introduce the notion of *ideal-filtered* Cu-semigroups and three families of soft elements: *strongly*, *weakly* and *functionally* soft. Informally, an abstract Cuntz semigroup is ideal-filtered if the Cuntz classes generating an ideal are downward-directed, and the three definitions of softness are all tailored notions from the soft elements introduced in [6]. In the presence of ideal-filteredness, $(2, \omega)$-divisibility is equivalent to having an abundance of strongly soft elements, while weakly soft elements are deeply related to the notion of weak $(2, \omega)$-divisibility.

Using these new notions, we show that the Global Glimm Problem has a positive answer in the real rank zero or the stable rank one setting. This recovers results from [34] and [4] respectively. We also prove that the problem has a positive solution for the class of separable, residually stably finite C^*-algebras with topological dimension zero. The results correspond to those in [93] and [95], although the main results of Chapter 7 are weaker than those that will appear in [95].

Agraïments

Ara que acabo d'escriure la tesi, volia donar les gràcies a tots aquells que m'heu ajudat, d'alguna manera o d'una altra, a escriure-la:

Primer de tot, vull agrair a tots els companys de doctorat els incomptables cafès, dinars i converses pels passadissos. Tant quan eren de matemàtiques com quan eren més personals, tenir aquestes xerrades sempre ha estat un plaer, i van ser un suport emocional diari durant les temporades més dures del doctorat. D'aquesta època me n'enduc amics nous, com el Salvador, el Juanlu, el Laurent, la Clara, el Ricard i el Damian. Estic segur que, fins i tot quan deixem de treballar junts, mantindrem el contacte.

Continuant a la UAB, també vull donar les gràcies a tots els professors del departament, en particular a tots aquells que vaig tenir com alumne de grau. La seva professionalitat i dedicació em van permetre gaudir molt de la carrera, i han estat un dels motius principals pels que vaig acabar-me apuntant al programa de doctorat.

En especial, vull agrair doblement a tots els membres del grup de recerca en Teoria d'Anells. A molts d'ells ja els vaig tenir com a professors, i durant aquesta etapa els he pogut conèixer més personalment. A més d'excel·lents professors, també són excel·lents persones. Espero poder continuar treballant amb el Joan, el Ramon i el Pere (que són amb qui comparteixo més interessos matemàtics) durant molts anys.

Naturalment, he volgut reservar un paràgraf pel Francesc. Si comencem a comptar des de l'inici del meu treball de fi de grau, d'aquí poc farà més de sis anys que el Francesc m'està tutoritzant! Sincerament, no hauria pogut escollir un tutor millor. Entre moltes altres coses, vull agrair-li la seva dedicació i paciència sense la qual, sens dubte, no hagués pogut escriure aquesta tesi.

I originally planned to write this next paragraph in German but, since I was scared of messing it up, I ultimately chose the safe approach: Of course, I want to deeply thank Hannes for his support and for the many Skype talks. Though not in name, his role has been no less than that of an advisor. For that, I want to express my utmost gratitude.

I also want to thank everyone who I met during my visits in Münster, Prague and Kiel. Though my stays were short, I will always remember the extremely kind treatment that I recieved there.

Finalment, voldria donar el meu més profund agraïment als meus pares i al meu germà. Al meu germà per aguantar-me cada dia, i per distreure'm sempre que estic encaboriat.

Tots els meus mèrits els dec a ell i als meus pares. Són ells qui, amb el seu exemple, em van ensenyar la importància de l'esforç i de dedicar-se a allò que a cadascú més li agrada. Sense ells, de ben segur que no hagués fet mai ni cap doctorat ni cap carrera.

Tampoc em vull descuidar dels altres membres de la meva família. En especial els meus avis i àvies, qui sempre han estat un gran suport per mi.

Contents

Chapter 1

Preliminaries and prerequisites

In this first chapter we introduce the required notions and results needed to understand the chapters that follow.

For Section 1.1, the proofs and a more thorough exposition of the results can be found in [11] and [60]. A good reference for Section 1.2 is [6].

1.1 C^*-algebras

Definition 1.1.1. Let A be a Banach algebra over $\mathbb{C}$ with an antimultiplicative and antilinear involution $*$. We say that A is a *C^*-algebra* if the following equality, known as the C^*-equality, holds:

$$\|a\|^2 = \|aa^*\| \text{ for all } a \in A.$$

We will say that a subalgebra B of A is a *sub-C^*-algebra* if B is closed under the involution and complete with respect to the norm of A. A bilateral ideal of A will be called an *ideal* if it is closed under the involution and norm.

Definition 1.1.2. An algebra homomorphism φ between two C^*-algebras A, B is said to be a **-homomorphism* if $\varphi(a^*) = \varphi(a)^*$ for every $a \in A$.

A bijective *-homomorphism will be called **-isomorphism*.

Examples 1.1.3.

(i) $\mathbb{C}$ is a C^*-algebra with the conjugation as its involution.

(ii) Given a Hilbert space H, the algebra of bounded operators $\mathcal{B}(H)$ is C^*-algebra with the usual norm and involution.

The algebra $\mathcal{K}(H)$ of compact operators on H is in fact a sub-C^*-algebra of $\mathcal{B}(H)$, where recall that a compact operator is the inductive limit of finite-rank operators. We will denote by $\mathcal{K}$ the algebra of compact operators on a separable, infinite-dimensional Hilbert space.

(iii) Given a compact metric space X, the algebra $C(X)$ of continuous functions from X to $\mathbb{C}$ is a C^*-algebra when equipped with the pointwise involution and the supremum norm.

(iv) For any ideal I of a C^*-algebra A, its quotient A/I with the norm $\|a+I\| = \inf\{\|a+x\| \mid x \in I\}$ is again a C^*-algebra.

(v) Let H be a Hilbert space. A $*$-subalgebra M of $\mathcal{B}(H)$ is said to be a *von Neumann algebra* if it is equal to its double commutant. Every von Neumann algebra is a C^*-algebra.

As noted in Examples 1.1.3 above, every algebra of bounded operators over a Hilbert space H is C^*-algebra. Consequently, every sub-C^*-algebra of $\mathcal{B}(H)$ is itself a C^*-algebra. By Theorem 1.1.4 below, all C^*-algebras may be characterized in this way.

Theorem 1.1.4 (Gelfand-Naimark)**.** *For every C^*-algebra A there exists a Hilbert space H such that A is isometrically isomorphic to a sub-C^*-algebra of $\mathcal{B}(H)$.*

1.1.5 (Matrices and direct sums)**.** Given a C^*-algebra A, considering its matrix algebra or its direct sum with another C^*-algebra are among the most basic procedures for creating new C^*-algebras from existing ones. We define these two constructions below:

Given a positive integer n, consider the matrix algebra $M_n(A)$ with the involution given by transposition and applying the involution of A componentwise. By Theorem 1.1.4, there exists an isometric injective $*$-homomorphism φ from A to $\mathcal{B}(H)$ for some Hilbert space H. Given $a \in M_n(A)$, we define $\|a\| := \|\varphi_n(a)\|$, where φ_n is the componentwise morphism from $M_n(A)$ to $\mathcal{B}(H^n)$ induced by φ. One can check that $M_n(A)$ is a C^*-algebra with this involution and norm.

The direct sum between A and another C^*-algebra B is defined as the algebraic direct sum $A \oplus B$ equipped with componentwise involution and the norm $\|(a,b)\| = \max\{\|a\|, \|b\|\}$.

Using these two constructions, one can show that a C^*-algebra A is *finite dimensional* if and only if A is $*$-isomorphic to a finite direct sum of matrices over $\mathbb{C}$.

A C^*-algebra need not have a unit, but one can always attach one to it by a process known as *unitization*. We briefly recall its definition below.

Definition 1.1.6. Let A be a C^*-algebra. We will denote by $A^\dagger$ the C^*-algebra $A \times \mathbb{C}$ with componentwise sum and involution, product given by

$$(a,\lambda)(b,\mu) = (ab + \mu a + \lambda b, \lambda\mu),$$

and norm

$$\|(a,\lambda)\| = \max\{|\lambda|, \sup_{\|x\|=1} \|ax + \lambda x\|\}.$$

The C^*-algebra $A^\dagger$ is always unital, with unit $(0,1)$, and we denote its elements by $a + \lambda 1_{A^\dagger}$.

Definition 1.1.7. Given a C^*-algebra A, we define its *minimal unitization*, denoted by $\tilde{A}$, as $A^\dagger$ if A does not have a unit, and as A otherwise.

Remark 1.1.8. The above unitization process is called minimal since, whenever A is contained in a unital C^*-algebra B with unit 1_B, the minimal unitization $\tilde{A}$ is isomorphic to the sub-C^*-algebra $A + \mathbb{C}1_B$ of B.

Another widely used construction to produce a unital C^*-algebra containing A as an ideal is the *multiplier algebra* $M(A)$, which can be seen as the maximal unitization of A.

This C^*-algebra is characterized (up to isomorphism) by the following property: Whenever A is contained as an ideal in a unital C^*-algebra B, the identity map on A extends uniquely to a *-homomorphism $B \to M(A)$ with kernel $\{b \in B \mid Ab = \{0\}\}$; see [11, II.7.3]

Spectrum and functional calculus

As in the case of bounded operators over a Hilbert space, there are several distinct classes of elements that one needs to consider in order to study a C^*-algebra. To introduce them, we first generalize the concept of *spectrum* that one has in $M_n(\mathbb{C})$:

Definition 1.1.9. Let A be a C^*-algebra and let a be an element in A. We define the spectrum of x, denoted by $\sigma(a)$, as the subset of elements $\lambda \in \mathbb{C}$ such that $a - \lambda 1$ is not invertible in $\tilde{A}$.

Using the notion of spectrum, one can define a variety of distinguished elements in a C^*-algebra. The equivalences listed in the following definition can be found in [11].

Definition 1.1.10. Let a be an element in a C^*-algebra A. We say that a is

(i) *self-adjoint* if $a = a^*$.

(ii) *normal* if $aa^* = a^*a$.

(iii) *positive* if $a = xx^*$ for some $x \in A$ or, equivalently, if a is normal and $\sigma(a) \subseteq \mathbb{R}^+ \cup \{0\}$. The subset of all positive elements of A is denoted by A_+, and we write $a \leq b$ if $b - a \in A_+$.

(iv) a *projection* if $a = a^2 = a^*$ or, equivalently, if $\sigma(a) \subseteq \{0, 1\}$. We denote by $P(A)$ the set of all projections.

(v) a *unitary* if $aa^* = a^*a = 1$ or, equivalently, if $\sigma(a) \subseteq \mathbb{T}$.

Whenever an element a is normal, Theorem 1.1.11 below allows us to identify the continuous functions over $\sigma(a)$ with elements in $\tilde{A}$. This procedure is known as functional calculus, and is a key concept in the study of C^*-algebras.

Theorem 1.1.11 (Gelfand-Naimark)**.** *Let A be a C^*-algebra and let $a \in \tilde{A}$ be a normal element. Then, there exists an isometric *-isomorphism γ between $C^*(1, a)$, the smallest sub-C^*-algebra of $\tilde{A}$ containing 1 and a, and $C(\sigma(a))$ such that $\gamma(z \mapsto z) = a$.*

Remark 1.1.12. Let a be a positive element in a C^*-algebra A, and let γ be the map from Theorem 1.1.11 above. Given a continuous map $f \in C(\mathbb{R})$ such that $f(0) = 0$, we have $\gamma(f) \in A$.

Indeed, we know by the Stone-Weierstrass theorem that f can be written as a limit of polynomials that map 0 to 0 or, equivalently, that can be written as a linear combination of powers of $z \mapsto z$. Since $a \in A$, this shows that the image through γ

of each of these polynomials belongs in A. Using that γ is continuous, it follows that $\gamma(f) \in A$.

We will denote $\gamma(f)$ by $f(a)$. For example, ones writes $\sqrt{a}$ and $(a-\varepsilon)_+$ instead of $\gamma(z \mapsto \sqrt{z})$ and $\gamma(z \mapsto (z-\varepsilon)_+)$.

The following notions are of importance in the study of C^*-algebras and Cuntz semigroups; see, for example, [11], [78], and [81] for more details.

Definition 1.1.13. A C^*-algebra A is

(i) *separable* if it contains a countable dense subset.

(ii) *stably finite* if, for every $n \in \mathbb{N}$, every left-invertible element in $M_n(\tilde{A})$ is invertible.

(iii) *residually stably finite* if every quotient of A is stably finite.

(iv) *purely infinite* if every nonzero positive element in A is *properly infinite*. That is to say, if for every nonzero $a \in A_+$ there exists a sequence $(r_n)_n$ in $M_2(A)$ such that $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) = \lim_n r_n \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right) r_n^*$ in $M_2(A)$.

Definition 1.1.14. A C^*-algebra A is said to be of

(i) *stable rank one* if the set of invertible elements in $\tilde{A}$ is norm dense in $\tilde{A}$.

(ii) *real rank zero* if the set of invertible self-adjoint elements in $\tilde{A}$ is norm dense in the set of self-adjoint elements of $\tilde{A}$.

One can prove that every stable rank one C^*-algebra is stably finite, and that no purely infinite C^*-algebra has stable rank one; see [11, Section V.3]

Inductive limits and the spatial tensor product

In addition to direct sums and matrix expansions, as defined in Paragraph 1.1.5, two of the most useful constructions to introduce C^*-algebras are inductive limits and (spatial) tensor products, as defined in Definition 1.1.15 and Definition 1.1.18 below.

Definition 1.1.15. Let $((A_\lambda)_{\lambda\in\Lambda}, (\varphi_{\mu,\lambda})_{\lambda\leq\mu \text{ in } \Lambda})$ be a directed system with A_λ C^*-algebras and $\varphi_{\mu,\lambda}$ *-homomorphisms from A_λ to A_μ. For every $a \in A_\lambda$, define the seminorm

$$\|a\| = \inf_{\lambda\leq\mu} \|\varphi_{\mu,\lambda}(a)\|.$$

Let $\operatorname{alg}\lim_\lambda A_\lambda$ be the algebraic limit of the system, which is a *-algebra with seminorm $\|\cdot\|$ induced by the above seminorm. The *limit* of the directed system, denoted by $\lim_\lambda A_\lambda$, is the C^*-algebra defined as the completion over $\|\cdot\|$ of the *-algebra $(\operatorname{alg}\lim_\lambda A_\lambda)/\{a \mid \|a\| = 0\}$.

We say that an inductive limit is *sequential*, and write $\lim_n A_n$, if $\Lambda = \mathbb{N}$.

Examples 1.1.16.

(i) A C^*-algebra A is an *approximately finite* C^*-algebra, *AF-algebra* for short, if A is *-isomorphic to a sequential inductive limit of finite dimensional C^*-algebras, as defined in Paragraph 1.1.5.

(ii) Recall from Examples 1.1.3 that $C([0,1])$ is a C^*-algebra. We will say that a C^*-algebra A is a (separable) *AI-algebra* if it is *-isomorphic to a sequential inductive limit of C^*-algebras of the form $\oplus_{i=1}^n M_{k_n}(C([0,1]))$. These algebras and their Cuntz semigroup will studied extensively in Chapter 2 and Chapter 3.

(iii) A *prime dimension drop* algebra is a C^*-algebra of the form

$$\{f \in C([0,1], M_{n_1n_2}(\mathbb{C})) \mid f(0) \in M_{n_1} \otimes \mathrm{id}_{n_2},\ f(1) \in M_{n_2} \otimes \mathrm{id}_{n_1}\}$$

with n_1, n_2 relatively prime.

Jiang and Su constructed in [50] a simple C^*-algebra $\mathcal{Z}$ as a sequential inductive limit of prime dimension drop algebras with unit preserving connecting maps. Such a C^*-algebra is known as the *Jiang-Su algebra*.

(iv) Another important simple C^*-algebra which can also be described as an inductive limit, with slighlty different blocks, is the *Jacelon-Razak algebra* $\mathcal{W}$; see [47]. In this case, the blocks are of the form

$$\left\{ f \in C([0,1], M_{n'}(\mathbb{C})) \;\middle|\; \begin{array}{l} f(0) = \mathrm{diag}(c, \ldots_a, c, 0_n) \\ f(1) = \mathrm{diag}(c, \ldots_{a+1}, c) \end{array} \quad c \in M_n(\mathbb{C}) \right\},$$

where $n, n' \in \mathbb{N}$ are such that $n | n'$, and we set $a := \frac{n'}{n} - 1$.

1.1.17 (C^*-norms on tensor products)**.** Given a *-algebra A and a norm $\|\cdot\|$ on A, we say that $\|\cdot\|$ is a *C^*-norm* if it satisfies the C^*-equality from Definition 1.1.1.

Given two C^*-algebras A and B, we can equip their algebraic tensor product $A \odot B$ with the multiplication $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2$ and the involution $(a \otimes b)^* = a^* \otimes b^*$, thus making $A \odot B$ into an *-algebra.

One can see that there exists a minimal C^*-norm $\|\cdot\|_{\min}$ on $A \odot B$, that is, if γ is another C^*-norm on $A \odot B$, we have $\|\cdot\|_{\min} \leq \gamma$. This minimal norm is known as the *minimal* or *spatial norm*; see [11, II.9.1.3].

Definition 1.1.18. Let A and B be two C^*-algebras, and let $A \odot B$ denote their algebraic tensor product with the multiplication and the involution defined above. We define the *minimal* or *spatial tensor product*, denoted by $A \otimes B$, as the completion of $A \odot B$ by the minimal C^*-norm on $A \odot B$.

Remark 1.1.19. Since there usually is a variety of C^*-norms that one can define in the algebraic tensor product of two C^*-algebras, in the literature the minimal tensor product of A and B is usually denoted by $A \otimes_{\min} B$.

However, in order to ease the notation, and since this the only tensor product that we will use, we will denote it by $A \otimes B$.

Examples 1.1.20.

1. Given a C^*-algebra A and a positive integer $n \in \mathbb{N}$, we have $A \otimes M_n(\mathbb{C}) \cong M_n(A)$.

   One can easily see that the spatial tensor product distributes over direct sums. Thus, a C^*-algebra A is an AI-algebra, as defined in Examples 1.1.16 (ii), if and only if $A \cong \lim_n C([0,1]) \otimes F_n$ with F_n finite dimensional for every n.

2. For any C^*-algebra A, we define the *stabilization of* A as the tensor product $A \otimes \mathcal{K}$. This algebra might be viewed as the smallest C^*-algebra containing all row- and column-finite infinite matrices over A.

3. Given a C^*-algebra A and a Hilbert space H, we say that a *-homomorphism $\varphi \colon A \to \mathcal{B}(H)$ is an *irreducible representation of dimension* d if H is d-dimensional and has no nontrivial closed invariant subspaces under $\varphi(A)$.

   A C^*-algebra A is said to be *d-homogeneous* if every irreducible representation of A is d-dimensional. Similarly, a C^*-algebra is *d-subhomogeneous* if its irreducible representations are of dimension at most d; see [11, Sections IV.1.4, IV.1.7] for their main structure theorems.

4. Let $\mathcal{Z}$ and $\mathcal{W}$ be the Jiang-Su and Jacelon-Razak algebras respectively; see Examples 1.1.16, (iii)-(iv). We say that a C^*-algebra A is *$\mathcal{Z}$-stable* if $A \otimes \mathcal{Z} \cong A$. Similarly, one says that A is *$\mathcal{W}$-stable* if $A \otimes \mathcal{W} \cong A$. Such C^*-algebras are of importance in the classification theory of C^*-algebras (see the discussion below), and will play a role in Chapter 4.

The classification program

A C^*-algebra A is said to be *nuclear* if, for every C^*-algebra B, there exists a unique norm on the algebraic tensor product $A \odot B$ such that its completion is a C^*-algebra. Suitably interpreted, nuclearity can be seen as the noncommutative analogue of the existence of partitions of unity; see [20] and [52].

Inspired by the classification of UHF-algebras due to Glimm ([42]) and his own classification of AF-algebras ([30]), Elliott conjectured in the early 90s that the class of simple, unital, separable, nuclear C^*-algebras might be classifiable by means of a K-theoretic invariant. That is to say, he conjectured that one could find an invariant such that, given any isomorphism between the invariant of two simple, separable, nuclear C^*-algebras A and B, one could lift such isomorphism to a *-isomorphism between A and B. The candidate invariant, which underwent some changes throughout the years, is now known as the *Elliott invariant*. For any simple, unital C^*-algebra A, one defines

$$\mathrm{Ell}(A) := (K_0(A), K_0(A)_+, [1_A]_0, K_1(A), T(A), r_A \colon T(A) \to S(K_0(A))).$$

Let us briefly recall the main components of $\mathrm{Ell}(A)$:

We say that a pair of projections p, q of a C^*-algebra A are *Murray-von Neumann equivalent*, in symbols $p \sim_{MvN} q$, if there exists $v \in A$ such that $p = vv^*$ and $q = v^*v$. One can show that this is an equivalence relation, and can thus define the quotient $V(A) := P(A \otimes \mathcal{K}) / \sim_{MvN}$ which, equipped with the addition induced by diagonal addition, is known as the *Murray-von Neumann semigroup*.

For a unital C^*-algebra A, the Grothendieck group of $V(A)$ is $K_0(A)$, and its *positive cone* $K_0(A)_+$ corresponds to the elements coming from the Murray-von Neumann semigroup of A; in the non-unital case the definition of $K_0(A)$ differs, see for example [81, Definition 4.1.1]. The pair $(K_0(A), K_0(A)_+)$ is known as the *ordered K_0-group of A*. These are the first two elements in $\mathrm{Ell}(A)$, while the third is simply the class $[1_A]_0 \in K_0(A)$.

Given a C^*-algebra A, consider $SA := \{f \in C(\mathbb{T}, A) \mid f(1) = 0\}$. One defines the *$K_1$-group of A* as $K_1(A) := K_0(SA)$. Alternatively, $K_1(A)$ can also be constructed using unitaries; see [81, Definition 8.1.3].

For a unital C^*-algebra A, a linear functional $\tau\colon A \to \mathbb{C}$ is called a *state* if $\tau(a) \geq 0$ whenever $a \in A_+$ and $\tau(1_A) = 1$. A state τ is a *tracial state* if $\tau(xx^*) = \tau(x^*x)$ for every $x \in A$. The set of all tracial states is the *trace space* of A, denoted by $T(A)$.

Every tracial state on A gives rise to a *state* on $K_0(A)$, that is to say a group morphism $\phi\colon K_0(A) \to \mathbb{R}$ such that $\phi(K_0(A)_+) \subseteq \mathbb{R}_+$ and $\phi([1_A]_0) = 1$; see [81, Section 5.2] and [11, V.2.4.26]. This is precisely the pairing $r_A\colon T(A) \to S(K_0(A))$.

As a result by many hands (see [107] for a short overview), we now know:

Theorem 1.1.21. *The Elliott invariant classifies separable, simple, unital, nuclear, $\mathcal{Z}$-stable C^*-algebras satisfying the UCT.*

Since it is not the focus of this thesis, we do not introduce the UCT (which stands for *Universal Coefficient Theorem*). Vaguely, let us just say that a C^*-algebra satisfies the UCT if and only if it is, in some weak sense, homotopically equivalent to a commutative C^*-algebra; see, for example, [11, Theorem V.1.5.8]. It is an open problem if every separable, nuclear C^*-algebra satisfies the UCT.

Although the result is not yet published, one can drop the unital assumption from Theorem 1.1.21 above. However, and as witnessed by Rørdam ([79]) and Toms ([99]), there exist pairs of unital, simple, separable, nuclear (but not $\mathcal{Z}$-stable), nonisomorphic C^*-algebras that agree on their Elliott invariant. Thus, one cannot remove $\mathcal{Z}$-stability from Theorem 1.1.21.

To prove that the pair of C^*-algebras given in [99] were not isomorphic, Toms showed that they could be distinguished by their Cuntz semigroup, a refinement of the Murray-von Neumann semigroup introduced in Section 1.2 below.

1.2 Cuntz semigroups and the category Cu

We introduce in this section the main object of study of this thesis, the Cuntz semigroup. This powerful invariant was introduced by Cuntz in [27], and has been succesfully used in the classification of some non-simple C^*-algebras; see [21], [22] and [72].

The abstract study of the semigroup, started in [26], has unearthed several new structural properties; see Paragraph 1.2.14. In [4], some of these new properties were used to solve three open problems for the class of stable rank one C^*-algebras.

Let us first define the Cuntz subequivalence.

Definition 1.2.1. Let a, b be positive elements in a C^*-algebra A. We say that a is *Cuntz subequivalent* to b, and write $a \precsim b$, if there exists a sequence $(r_n)_n$ in A such that $a = \lim_n r_n b r_n^*$. We also say that a is *Cuntz equivalent* to b, in symbols $a \sim b$, if $a \precsim b$ and $b \precsim a$.

Even though it might not be clear from its definition, condition (iii) in Lemma 1.2.2 below shows that the Cuntz subequivalence can be seen as a generalization of the Murray-von Neumann subequivalence for projections, where recall that given two projections p, q, we say that p is *Murray-von Neumann subequivalent* to q if there exists an element v such that $p = vv^*$ and $v^*v \leq q$.

Lemma 1.2.2 ([77, Proposition 2.4]). *Given two positive elements a, b in a C^*-algebra A, the following are equivalent:*

(1) a is Cuntz subequivalent to b;

(2) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $(a-\varepsilon)_+ \precsim (b-\delta)_+$;

(3) For every $\varepsilon > 0$ there exist $\delta > 0$ and $x \in A$ such that $(a-\varepsilon)_+ = xx^*$ and $x^*x \in \overline{(b-\delta)_+ A (b-\delta)_+}$;

(4) For every $\varepsilon > 0$ there exist $\delta > 0$ and $s \in A$ such that $(a-\varepsilon)_+ = s(b-\delta)_+ s^*$.

Definition 1.2.3. Given a C^*-algebra A, we define its *Cuntz semigroup* $\mathrm{Cu}(A)$ as the quotient $(A \otimes \mathcal{K})_+/\sim$, and we denote the class of an element a by $[a]$.

Further, for each *-homomorphism $\varphi\colon A \to B$, we define the map $\mathrm{Cu}(\varphi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ by $\mathrm{Cu}(\varphi)([a]) = [\varphi(a)]$, where φ also denotes the amplified map $\varphi\colon A \otimes \mathcal{K} \to B \otimes \mathcal{K}$.

Note that, up to unitary equivalence, there is a unique *-isomorphism $\mathcal{K} \otimes M_2 \to \mathcal{K}$. That is to say, if ϕ_1, ϕ_2 are *-isomorphisms from $\mathcal{K} \otimes M_2$ to $\mathcal{K}$, there exists a unitary element u such that $\phi_1(a) = u\phi_2(a)u^*$ for every $a \in \mathcal{K} \otimes M_2$. In particular, for any C^*-algebra A and any pair of elements $a, b \in A \otimes \mathcal{K}$, the element $a \oplus b := \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ satisfies

$$[\mathrm{id}_A \otimes \phi_1(a \oplus b)] = [\mathrm{id}_A \otimes \phi_2(a \oplus b)].$$

We denote by $[\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)]$ the class $[\mathrm{id}_A \otimes \phi_1(a \oplus b)]$ in $\mathrm{Cu}(A)$. This allows us to define an addition in the Cuntz semigroup; see Theorem 1.2.5 below.

1.2.4. Given two elements x, y in a partially ordered set, we say that x is *way-below* y, in symbols $x \ll y$, if for every increasing sequence $(z_n)_n$ whose supremum exists and is greater than or equal to y, there exists $n \in \mathbb{N}$ such that $x \leq z_n$.

Whenever this condition is satisfied, we will often also say that x is *compactly contained* in y. The reason behind this terminology will be explained later, in Examples 1.2.8 (v); see also [41].

Given a C^*-algebra A, the relevance of this notion sits on the fact that $[(a-\epsilon)_+] \ll [a]$ for every $\epsilon > 0$ and $[a] \in \mathrm{Cu}(A)$.

Recall that a *positively ordered monoid* is a commutative monoid S with a partial order $\leq$ such that $0 \leq s$ for every element s in S and such that $s' + t' \leq s + t$ whenever $s' \leq s$ and $t' \leq t$.

Theorem 1.2.5 ([26, Theorem 1]). *Let A be a C^*-algebra. Equipped with the order induced by $\precsim$ and the addition induced by $[a]+[b] = [\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)]$, the Cuntz semigroup $\mathrm{Cu}(A)$ becomes a positively ordered monoid satisfying the following conditions:*

(O1) Every increasing sequence has a supremum.

(O2) Every element can be written as the supremum of a $\ll$-increasing sequence.

(O3) Given $x' \ll x$ and $y' \ll y$, we have $x' + y' \ll x + y$.

(O4) *For any pair of increasing sequences* $(x_n)_n$ *and* $(y_n)_n$*, we have* $\sup_n(x_n + y_n) = \sup_n x_n + \sup_n y_n$.

Note that, in Theorem 1.2.5, property (O2) follows from Paragraph 1.2.4 and using that, for every $[a] \in \mathrm{Cu}(A)$, one has $[a] = \sup_n[(a - 1/n)_+]$.

Definition 1.2.6. A positively ordered monoid S will be said to be a Cu-*semigroup* if it satisfies conditions (O1)-(O4) above.

A monoid morphism between Cu-semigroups will be called a Cu-*morphism* if it preserves order, suprema of increasing sequences and the way-below relation.

We denote by Cu the category whose objects and morphisms are Cu-semigroups and Cu-morphisms respectively.

Further, a submonoid T of S will be a *sub*-Cu-*semigroup* if it is a Cu-semigroup with the induced order and addition, and if the inclusion map $T \to S$ is a Cu-morphism.

We will say that two Cu-semigroups S, T are Cu-*isomorphic*, in symbols $S \cong T$, if there exists Cu-morphisms $\varphi\colon S \to T$ and $\phi\colon T \to S$ such that $\varphi\phi = \mathrm{id}_T$ and $\phi\varphi = \mathrm{id}_S$.

Definition 1.2.7. A monoid morphism between Cu-semigroups is called a *generalized* Cu-*morphism* if it preserves order and suprema of increasing sequences.

In the C^*-algebraic setting, every *-homomorphism $\varphi\colon A \to B$ induces a Cu-morphism $\mathrm{Cu}(\varphi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ (defined in Definition 1.2.3). Moreover, every completely positive, contractive, order-zero map induces a generalized Cu-morphism; see [108].

Given a Cu-semigroup S, we will denote by $S_\ll$ the subset of S consisting of elements x such that $x \ll y$ for some $y \in S$.

Examples 1.2.8.

(i) The monoid of nonzero integers and ∞, denoted by $\overline{\mathbb{N}}$, is a Cu-semigroup with the natural order. Here, one has $n \ll m$ if and only if $n < m$ or if $n = m \neq \infty$.

(ii) Similarly, the monoids $E_k = \{0, 1, \dots, k, \infty\}$ endowed with the natural order and with $n + m = \infty$ if $n + m > k$ are Cu-semigroups. In this case, we have $n \ll m$ if and only if $n \leq m$.

Together with $\overline{\mathbb{N}}$, these semigroups are called the *elementary* Cu-*semigroups* in [6, Paragraph 1.16], where it is shown that only $\overline{\mathbb{N}}$ and $E_0 = \{0, \infty\}$ are Cu-isomorphic to the Cuntz semigroup of a C^*-algebra. More explicitly, they are Cu-isomorphic to the Cuntz semigroup of $\mathbb{C}$ and of a nonzero, simple, separable, purely infinite C^*-algebra respectively.

In Chapter 6 we redefine the notion of elementary Cu-semigroups; see Paragraph 6.3.1.

(iii) The set of nonnegative real numbers and ∞, denoted by $[0, \infty]$, is also a Cu-semigroup with the usual addition and order. In fact, $\mathrm{Cu}(\mathcal{W}) \cong [0, \infty]$, where recall that $\mathcal{W}$ is the Jacelon-Razak algebra.

In this semigroup, one has $x \ll y$ if and only if $x < y$.

(iv) Let Z denote the monoid $(0,\infty]\sqcup\mathbb{N}$, where addition in each component is defined naturally and given $n\in\mathbb{N}$ and $x\in(0,\infty]$ we define $n+x$ as $n+x\in(0,\infty]$. Given $n\in\mathbb{N}$, one usually denotes by n' its corresponding element in $(0,\infty]$.

Define a positive order in Z as follows: The order in each component is the usual one, and if $x\in(0,\infty)$, $n\in\mathbb{N}$, we write $x\leq n$ whenever $x\leq n'$; and $n\leq x$ whenever $n'<x$. Then, Z is a Cu-semigroup where $n\ll n$ for every $n\in\mathbb{N}$ and $x\ll y$ with $x,y\in(0,\infty]$ if and only if $x<y$.

It was shown in [70, Theorem 3.1] that Z is Cu-isomorphic to the Cuntz semigroup of the Jiang-Su algebra $\mathcal{Z}$.

(v) Given a topological space X, recall that a map $f\colon X\mapsto\overline{\mathbb{N}}$ is said to be *lower-semicontinuous* if $\{x\in X\mid f(x)\geq n\}$ is open for every $n\in\overline{\mathbb{N}}$. Whenever the space is compact, metric and finite-dimensional (in the sense of Definition 2.1.1), it is a consequence of [5, Theorem 5.15] that the positively ordered monoid of lower-semicontinuous functions from X to $\overline{\mathbb{N}}$ with pointwise addition and order, denoted by $\mathrm{Lsc}(X,\overline{\mathbb{N}})$, is a Cu-semigroup.

In particular, given two open sets $U,V\subseteq X$ and their respective indicator functions χ_U,χ_V, it follows from [5, Theorem 5.15] that $\chi_U\ll\chi_V$ if and only if $\overline{U}\subseteq V$. This justifies the naming of *compact containment*.

We will prove in Corollary 2.2.21 that $\mathrm{Lsc}(X,\overline{\mathbb{N}})$ is in fact a Cu-semigroup for every compact, metric space X.

Robert showed in [74, Theorem 1.1] that, whenever $\dim(X)\leq 2$, the semigroup $\mathrm{Lsc}(X,\overline{\mathbb{N}})$ is Cu-isomorphic to the Cuntz semigroup of $C(X)$.

1.2.9 (Direct sums). Given two Cu-semigroups S and T, their direct sum $S\oplus T$ as positively ordered monoids is also a Cu-semigroup. Indeed, since the order is componentwise we have $(s',t')\ll(s,t)$ in $S\oplus T$ if and only if $s'\ll s$ in S and $t'\ll t$ in T. Using this fact, it is clear that the semigroup $S\oplus T$ satisfies (O1)-(O4).

In fact, one can also see that given two C^*-algebras A,B, the Cuntz semigroup $\mathrm{Cu}(A\oplus B)$ is Cu-isomorphic to the Cu-semigroup $\mathrm{Cu}(A)\oplus\mathrm{Cu}(B)$.

1.2.10 (Inductive limits). Let $((S_\lambda)_{\lambda\in\Lambda},(\varphi_{\mu,\lambda})_{\lambda\leq\mu\text{ in }\Lambda})$ be a directed system in Cu, that is, let Λ be a directed set, $(S_\lambda)_{\lambda\in\Lambda}$ a family of Cu-semigroups, and $\varphi_{\mu,\lambda}\colon S_\lambda\to S_\mu$ Cu-morphisms such that $\varphi_{\nu,\mu}\circ\varphi_{\mu,\lambda}=\varphi_{\nu,\lambda}$ whenever $\lambda\leq\mu\leq\nu$ and $\varphi_{\lambda,\lambda}=\mathrm{id}_{S_\lambda}$ for every $\lambda\in\Lambda$.

The category Cu has inductive limits by [6, Corollary 3.1.11] (the sequential case had already been proved in [26, Theorem 2]). More explicitly, we will see in Lemma 4.1.8 that a Cu-semigroup S together with Cu-morphisms $\varphi_\lambda\colon S_\lambda\to S$ for $\lambda\in\Lambda$ is the inductive limit in Cu of the system $((S_\lambda)_{\lambda\in\Lambda},(\varphi_{\mu,\lambda})_{\lambda\leq\mu\text{ in }\Lambda})$ if and only if the following conditions are satisfied:

(L0) we have $\varphi_\mu\circ\varphi_{\mu,\lambda}=\varphi_\lambda$ for all $\lambda\leq\mu$ in Λ;

(L1) if $x_\lambda\in S_\lambda$ and $x_\mu\in S_\mu$ satisfy $\varphi_\lambda(x_\lambda)\ll\varphi_\mu(x_\mu)$, then there exists $\nu\geq\lambda,\mu$ such that $\varphi_{\nu,\lambda}(x_\lambda)\ll\varphi_{\nu,\mu}(x_\mu)$;

(L2) for all $x',x\in S$ satisfying $x'\ll x$ there exists $x_\lambda\in S_\lambda$ such that $x'\ll\varphi_\lambda(x_\lambda)\ll x$.

Given an element $x \in S_\lambda$, we will usualy denote its image $\varphi_\lambda(x)$ in S by $[x]$.

The following theorem was first proved for sequential inductive limits in [26, Theorem 2], and was proved for general limits in [6, Corollary 3.2.9].

Theorem 1.2.11. *Let C^* denote the category of C^*-algebras. Then, mapping each C^*-algebra A to its Cuntz semigroup* $\mathrm{Cu}(A)$ *and each *-homomorphism φ to* $\mathrm{Cu}(\varphi)$ *defines a continuous functor* $\mathrm{Cu}\colon C^* \to \mathrm{Cu}$, *where C^* denotes the category of C^*-algebra and *-homomorphisms.*

1.2.12 (Ideals and quotients)**.** An *ideal I* of a Cu-semigroup S is a downward-hereditary submonoid closed under suprema of increasing sequences; see [6, Section 5] and [23].

Given $x, y \in S$, we write $x \leq_I y$ if there exists $z \in I$ such that $x \leq y + z$. We set $x \sim_I y$ if $x \leq_I y$ and $y \leq_I x$. The quotient $S/\sim_I$ endowed with the induced sum and order $\leq_I$ is denoted by S/I.

As shown in [6, Lemma 5.1.2], S/I is a Cu-semigroup and the quotient map $S \to S/I$ is a Cu-morphism.

That the Cuntz semigroup is a natural carrier of the ideal structure of the C^*-algebra is testified by the following result:

Proposition 1.2.13 ([23])**.** *Let A be a C^*-algebra and let I be an ideal of A. Then,* $\mathrm{Cu}(I)$ *is an ideal of* $\mathrm{Cu}(A)$*. Conversely, for every ideal J of* $\mathrm{Cu}(A)$ *there exists an ideal I in A such that $J \cong \mathrm{Cu}(I)$.*

Moreover, we also have $\mathrm{Cu}(A/I) \cong \mathrm{Cu}(A)/\mathrm{Cu}(I)$.

1.2.14 (Additional properties)**.** When working with Cu-semigroups, it is often useful to assume that these satisfy additional properties that are known to hold for certain families of C^*-algebras.

For example, the following properties were proved to be satisfied for the Cuntz semigroup of any C^*-algebra; see [6, Proposition 4.6] (and [82]), [73] and [3, Proposition 2.2] respectively:

(O5) Given elements x, y, z such that $x + y \leq z$, $x' \ll x$ and $y' \ll y$, there exists c such that $x' + c \leq z \leq x + c$ and $y' \ll c$.

(O6) Given $x' \ll x \leq y + z$ there exist elements v, w with $v \leq x, y$, $w \leq x, z$ and such that $x' \leq v + w$.

(O7) Given $x_1' \ll x_1 \leq w$ and $x_2' \ll x_2 \leq w$ there exists x such that $x_1', x_2' \ll x \leq w, x_1 + x_2$.

A common application of (O5) is when $y = 0$. That is, when we have $x' \ll x \leq z$. In this case, (O5) implies that there exists an element c such that $x' + c \leq z \leq x + c$.

In Definition 6.2.1 we will introduce a new property, termed (O8), that the Cuntz semigroup of every C^*-algebra also satisfies.

We will say that a Cu-semigroup is *weakly cancellative* if $x \ll y$ whenever there exists an element z such that $x + z \ll y + z$. In [82, Theorem 4.3], it was shown that the Cuntz semigroup of any stable rank one C^*-algebra has weak cancellation.

An element p in a Cu-semigroup is said to be *compact* if $p \ll p$. In every C^*-algebra A, the Cuntz class of a projection is a natural example of a compact element in $\mathrm{Cu}(A)$. For stably finite C^*-algebras, these are the only examples; see [17].

Given a compact element p in a weakly cancellative Cu-semigroup, one can check that we have $p+x \leq p+y$ if and only if $x \leq y$ and, in particular, that $p+x = p+y$ if and only if $x = y$. This will be extensively used in Chapter 3.

In addition to being weak cancellative, the Cuntz semigroup of any separable stable rank one C^*-algebra is also known to be *inf-semilattice ordered*, that is, infima of arbitrary pairs exists and the equation $x \wedge y + z = (x+z) \wedge (y+z)$ is always satisfied; see [4, Theorem 3.8]. In Chapter 2 we will study Cu-semigroups that are distributive lattice ordered; see Definition 2.2.1.

1.2.15 (Sup-dense subsets)**.** Given a Cu-semigroup S, we will say that a subset D of S is *sup-dense* if whenever $x', x \in S$ satisfy $x' \ll x$, there exists $y \in D$ with $x' \leq y \ll x$. Equivalently, it follows from (O2) that D is sup-dense if and only if every element in S is the supremum of an increasing sequence of elements in D.

We will say that a Cu-semigroup is *countably based* if it contains a countable sup-dense subset. Cuntz semigroups of separable C*-algebras are countably based; see, for example, [5].

A Cu-semigroup whose set of compact elements is sup-dense is said to be *algebraic*. As shown in [26], the Cuntz semigroup of any real rank zero C^*-algebra is algebraic.

We will see in Section 5.1 that, when cheking whether or not a Cu-semigroup satisfies one of the properties in Paragraph 1.2.14 above, one can restrict to the study of sup-dense subsets.

1.2.16 (Functionals)**.** Given a Cu-semigroup S, a map $\lambda\colon S \to [0,\infty]$ is said to be a *functional* if it is a generalized Cu-morphism.

Denote by $F(S)$ the set of all the functionals in S. Then, for every $x \in S$, we define $\widehat{x}\colon F(S) \to [0,\infty]$ as $\widehat{x}(\lambda) = \lambda(x)$.

For a C^*-algebra A, the set $F(\mathrm{Cu}(A))$ can be identified with the set of *lower-semicontinuous 2-quasitraces* of A; see [33].

1.2.17 (Cu-semirings and tensor products)**.** As defined in [6, Definition 6.3.1], given S, T and H Cu-semigroups, we say that a map $f\colon S\times T \to H$ is a Cu-*bimorphism* if it is a positively ordered monoid morphism in each variable such that $f(x', y') \ll f(x, y)$ whenever $x' \ll x$ in S and $y' \ll y$ in T, and such that $\sup_n f(x_n, y_n) = f(\sup_n x_n, \sup_n y_n)$ for every pair of increasing sequences $(x_n)_n$ in S and $(y_n)_n$ in T.

A Cu-*semiring* R is a Cu-semigroup together with an associative, commutative Cu-bimorphism, $(x, y) \mapsto xy$, and an element $1 \in R$ such that $1x = x$ for every $x \in R$.

The Cu-semigroups $\overline{\mathbb{N}}$, $[0,\infty]$, $\{0,\infty\}$ and Z as defined in Examples 1.2.8 are examples of Cu-semirings; see [6, Chapter 7].

Given a Cu-semiring R, we say that a Cu-semigroup S has *R-multiplication* if there exists a Cu-bimorphism $R \times S \to S$, which we denote by $(r, s) \mapsto rs$, such that $1s = s$ and $r_1(r_2 s) = (r_1 r_2)s$ for every $r_1, r_2 \in R$ and $s \in S$

In [6, Chapter 6], a notion of tensor product for Cu-semigroups is introduced. This notion is strongly related to having a multiplication as defined above. For example, we know that the Cuntz semigroup of a $\mathcal{Z}$-stable C^*-algebra always has Z-multiplication, and that the Cuntz semigroup of a $\mathcal{W}$-stable C^*-algebra has $[0,\infty]$-multiplication; see [6, Section 7.3-7.5]. This will be used in Chapter 4.

1.3 The Effros-Handelman-Shen theorem and its Cu-analogue

In the sequel, by a range problem we will mean the following:

1.3.1. Given a family of C^*-algebras $\mathcal{F}$, a category $\mathcal{C}$, and a functor $I\colon C^* \to \mathcal{C}$, the *range problem* for I of $\mathcal{F}$ consists in determining a natural set of properties that an object in $\mathcal{C}$ satisfies if and only if it is isomorphic to $I(A)$ for some $A \in \mathcal{F}$.

In the theory of C^*-algebras, the most well known result addressing this type of problem is the celebrated Effros-Handelman-Shen theorem, which gives a list of properties that characterize when a countable, ordered abelian group is isomorphic to the K_0-group of an AF-algebra; see Theorem 1.3.2 below, and [81] for an introduction to C^*-algebraic K-theory.

A positively ordered monoid is said to satisfy the *Riesz decomposition property* if whenever $x \leq y + z$ there exist $y' \leq y$ and $z' \leq z$ such that $x = y' + z'$. We say that a partially ordered abelian group has such a property if its positive cone does.

Theorem 1.3.2 ([28, Theorem 2.2])**.** *A countable ordered (abelian) group G is order isomorphic to the ordered K_0-group of an AF-algebra if and only if it is unperforated and it satisfies the Riesz decomposition property.*

This results uses prominently Shen's theorem, which had been proved previously in [87, Theorem 3.1]. Regarding the Effros-Handelman-Shen theorem as an abstract characterization of the K_0-groups under study, the result due to Shen can be seen as a local characterization of such groups.

Theorem 1.3.3 ([87, Theorem 3.1])**.** *A countable unperforated ordered (abelian) group G is order isomorphic to the ordered K_0-group of an AF-algebra if and only if, for every ordered homomorphism $\varphi\colon \mathbb{Z}^r \to G$ and any element $\alpha \in \ker(\varphi)$, there exist $s \geq 0$, and ordered homomorphisms θ, ϕ such that the diagram*

$$\begin{tikzcd} \mathbb{Z}^r \arrow[r, "\varphi"] \arrow[d, "\theta"'] & G \\ \mathbb{Z}^s \arrow[ur, "\phi"'] & \end{tikzcd}$$

commutes and $\alpha \in \ker(\theta)$.

Using that an algebraic Cu-semigroup is always of the form $\mathrm{Cu}(M)$ with M a positively ordered monoid, the following Cu-version of the Effros-Handelman-Shen theorem was obtained in [6]. Recall that a positively ordered monoid is *unperforated* if $x \leq y$ whenever $nx \leq ny$ for some $n \in \mathbb{N}$.

Theorem 1.3.4 ([6, Corollary 5.5.13]). *Let S be a countably based* Cu*-semigroup. Then, S is* Cu*-isomorphic to the Cuntz semigroup of an AF-algebra if and only if S is weakly cancellative, unperforated, algebraic and satisfies (O5) and (O6).*

The goal of Chapters 2 and 3 will be to obtain results on the range problem for the Cuntz semigroup of AI-algebras building upon Theorem 1.3.2 and Theorem 1.3.3.

A Cu-generalization of Shen's theorem

We will now prove, using Theorem 1.3.3, a Cu-version of Shen's theorem which, despite not being used in Chapters 2 and 3, provides some insight to the problems that one encounters when trying to generalize this type of result.

We begin with a lemma that will prove to be useful throughout Chapters 2 and 3.

Lemma 1.3.5. *Let S,T be* Cu*-semigroups, and let N be a subset of S. Assume that a map $\phi\colon N\to T$ satisfies*

(i) for every pair $x,y\in N$, one has $\phi(x)\ll\phi(y)$ whenever $x\ll y$ in S;

(ii) for every element $x\in N$, there exists a $\ll$-increasing sequence $(x_n)_n$ in S with supremum x in S such that $x_n\in N$ for every $n\in\mathbb{N}$ and $\phi(x)=\sup_n\phi(x_n)$.

Then, ϕ is order preserving and $\sup_n\phi(x_n)=\sup_n\phi(y_n)$ for every pair of increasing sequences $(x_n)_n,(y_n)_n$ in N such that $\sup_n x_n=\sup_n y_n$ in S.

If, additionally, N is sup-dense in S and ϕ is additive, ϕ extends to a Cu*-morphism $S\to T$.*

Proof. Let $x,y\in N$ be such that $x\leq y$, and let $(x_n)_n$ be a $\ll$-increasing sequence given by (ii). Then, $x_n\ll y$ for every n.

By (i), we have $\phi(x_n)\ll\phi(y)$. Taking supremum on n, we get

$$\phi(x)=\sup_n\phi(x_n)\leq\phi(y),$$

which shows that ϕ is order preserving.

Now let $(x_n)_n$ be an increasing sequence in N, and let x be the supremum of the sequence in S. Using (ii), each x_n can be written as the supremum of a $\ll$-increasing sequence in S formed by elements in N. An standard diagonal argument shows that x itself can be written as such.

Thus, let $(s_n)_n$ be a $\ll$-increasing sequence in S with $s_n\in N$ for each $n\in\mathbb{N}$ and such that $x=\sup_n s_n$ in S. In particular, we have that for every m there exists n such that $s_m\leq x_n$.

Using that ϕ is order preserving, we get $\phi(s_m)\leq\phi(x_n)$ and, consequently,

$$\sup_m\phi(s_m)\leq\sup_n\phi(x_n).$$

Conversely, let $m\in\mathbb{N}$ and take $x'\in N$ such that $x'\ll x_m$ in S. Using that $x_m\leq x=\sup_n s_n$, this implies that $x'\ll s_n$ for some n and, by (i), that

$$\phi(x')\ll\phi(s_n)\leq\sup_n\phi(s_n).$$

Since this holds for each $x' \ll x_m$, it follows from (ii) that $\phi(x_m) \leq \sup_n \phi(s_n)$. Taking supremum on m, one gets

$$\sup_m \phi(x_m) \leq \sup_n \phi(s_n),$$

which shows $\sup_n \phi(x_n) = \sup_m \phi(s_m)$.

Given now two increasing sequences $(x_n)_n, (y_n)_n$ in N with $\sup_n x_n = x = \sup_n y_n$ in S, let $(s_m)_m$ be a $\ll$-increasing sequence in S with $s_m \in N$ for each m and $\sup_m s_m = x$ in S. By the argument above, we get

$$\sup_n \phi(x_n) = \sup_m \phi(s_m) = \sup_n \phi(y_n),$$

as required.

For the second part of the lemma, assume that N is sup-dense in S. For each $x \in S$, let $(x_n)_n$ be an increasing sequence in N with supremum x in S. Then, the map $x \mapsto \sup_n \phi(x_n)$ is well-defined. Moreover, it follows from (ii) that it extends ϕ.

Now take $x', x \in S$ such that $x' \ll x$, and consider $\ll$-increasing sequences $(x'_m)_m$ and $(x_n)_n$ in S of elements in N with supremum x' and x respectively. Then, there exists $n \in \mathbb{N}$ such that $x'_m \ll x_n$ for every m, which implies $\sup_m \phi(x'_m) \leq \phi(x_n) \ll \sup_n \phi(x_n)$. It follows that our map is $\ll$-preserving.

Thus, conditions (i)-(ii) of the first part of the lemma are satisfied for $N = S$. This implies that the map $s \mapsto \sup_n \phi(x_n)$ is order and suprema preserving.

Finally, it is readily checked using (O4) that the map is additive whenever ϕ is. $\square$

Proposition 1.3.6. *Let S be a countably based and algebraic* Cu*-semigroup. Then, S is* Cu*-isomorphic to the Cuntz semigroup of an AF-algebra if and only if, for every* Cu*-morphism $\varphi\colon \overline{\mathbb{N}}^r \to S$ and triple $x, x', y \in \overline{\mathbb{N}}^r$ such that $x \ll x'$ and $\varphi(x') \ll \varphi(y)$, there exist $s \in \mathbb{N}$ and* Cu*-morphisms $\theta\colon \overline{\mathbb{N}}^r \to \overline{\mathbb{N}}^s$ and $\phi\colon \overline{\mathbb{N}}^s \to S$ such that the diagram*

$$\begin{tikzcd} \overline{\mathbb{N}}^r \arrow[r, "\varphi"] \arrow[d, "\theta"'] & S \\ \overline{\mathbb{N}}^s \arrow[ur, "\phi"'] & \end{tikzcd}$$

commutes and $\theta(x) \ll \theta(y)$.

Proof. If S is Cu-isomorphic to the Cuntz semigroup of an AF-algebra, there exists an inductive system $(\overline{\mathbb{N}}^{r_i}, \phi_{i+1,i})$ with limit T such that $S \cong T$; see [6, Corollary 5.5.13].

Let $x, y \in \overline{\mathbb{N}}^r$ be such that $\varphi(x) \ll \varphi(y)$, and take $y' \ll y$ satisfying $\varphi(x) \ll \varphi(y')$. Since the elements $\varphi(x), \varphi(y')$ and $\varphi(1_j)$ are compact for every $j \leq r$, there exists a large enough i and elements $q_x, q_{y'}, p_j \in \overline{\mathbb{N}}^{r_i}$ such that

$$q_x \ll q_{y'}, \quad \varphi(x) = \phi_i(q_x), \quad \varphi(y') = \phi_i(q_{y'}) \quad \text{and} \quad \varphi(1_j) = \phi_i(p_j)$$

for each j, where $\phi_i\colon \overline{\mathbb{N}}^{r_i} \to T$ is the the canonical limit Cu-morphism.

Define the map $\theta\colon \overline{\mathbb{N}}^r \to \overline{\mathbb{N}}^{r_i}$ additively by setting $\theta(1_j) = p_j$. Note that, since the additive span of the 1_j's is dense in $\overline{\mathbb{N}}^r$ and the $\ll$-increasing sequence $(p)_n$ trivially satisfies $\sup_n \theta(p) = \theta(p)$ for every $p \in \mathbb{N}^r$, θ is a Cu-morphism by Lemma 1.3.5 above.

Set $\phi = \phi_i$. These two morphisms satisfy (i) and (ii), as desired.

Conversely, assume that the condition in the statement is satisfied. A direct application of Theorem 1.3.3 shows that a positively ordered monoid H is isomorphic to a limit of the form $\lim_i(\mathbb{N}^{r_i}, \phi_{i+1,i})$ if and only if, for every morphism $\varphi\colon \mathbb{N}^r \to H$ and pair $x, y \in \mathbb{N}^r$ such that $\varphi(x) = \varphi(y)$, there exist morphisms $\theta\colon \mathbb{N}^r \to \mathbb{N}^s$ and $\phi\colon \mathbb{N}^s \to H$ such that $\theta(x) = \theta(y)$ and $\phi\theta = \varphi$.

Let S_c denote the monoid of compact elements in S, and let $\varphi\colon \mathbb{N}^r \to S_c$ be a monoid morphism and $x, y \in \mathbb{N}^r$ be such that $\varphi(x) = \varphi(y)$. By Lemma 1.3.5, there exists a Cu-morphism $\bar{\varphi}\colon \overline{\mathbb{N}}^r \to S$ extending φ. Using our assumption, we find θ, ϕ satisfying conditions (i)-(ii) for $\bar{\varphi}$ and the triple x, x, y. That is to say, $\theta(x) \ll \theta(y)$.

Since $\phi(\theta(x)) = \varphi(x) = \varphi(y) = \phi(\theta(y))$, we can apply the assumption once again to obtain θ', ϕ' satisfying (i)-(ii) for ϕ and the triple $\theta(y), \theta(y), \theta(x)$. In particular, using $\theta(x) \ll \theta(y)$ at the first step, we have

$$\theta'\theta(x) \ll \theta'\theta(y) \ll \theta'\theta(x),$$

and so $\theta'\theta(x) = \theta'\theta(y)$.

Using that the restrictions of $\theta'\theta$ and ϕ' at $\mathbb{N}^r$ and $\mathbb{N}^s$ respectively satisfy the required conditions, it follows that $S_c \cong \lim_i(\mathbb{N}^{r_i}, \phi_{i+1,i})$.

This implies that $S \cong \lim_i(\overline{\mathbb{N}}^{r_i}, \bar{\phi}_{i+1,i})$, where $\bar{\phi}_{i+1,i}\colon \overline{\mathbb{N}}^{r_i} \to \overline{\mathbb{N}}^{r_{i+1}}$ denotes the extension of $\phi_{i+1,i}$ given by Lemma 1.3.5. The desired result now follows from Theorem 1.3.4. $\square$

Chapter 2

The Cuntz semigroup of unital commutative AI-algebras

In this chapter we give a solution to the range problem for the Cuntz semigroup of unital, commutative AI-algebras in the sense of Paragraph 1.3.1. That is to say, we provide a natural set of properties that a Cu-semigroup satisfies if and only if it is isomorphic to $\mathrm{Cu}(A)$ for some AI-algebra A; see Theorem 2.5.12

Recall that, as defined in Examples 1.1.16 (ii), a C^*-algebra A is said to be a (separable) AI-algebra if A is *-isomorphic to an inductive limit of the form $\lim_n C[0,1]\otimes F_n$ with F_n finite dimensional for every n. Moreover, if A is unital and commutative, the results in [97] imply that A is isomorphic to $C(X)$ with X an inverse limit of (possibly increasing) finite disjoint unions of unit intervals. Conversely, any such inverse limit X gives rises to a unital, commutative AI-algebra $C(X)$.

Our solution consists of three parts:

First, and in analogy to the notion of chainable space from continuum theory, we introduce in Section 2.1 almost chainable topological spaces. We show that, whenever X is compact and metric, X is almost chainable if and only if $C(X)$ is an AI-algebra.

From Sections 2.2 to 2.4 we characterize those Cu-semigroups that are of the form $\mathrm{Lsc}(X,\overline{\mathbb{N}})$ for some T_1-space X. More concretely, we define the notion of Lsc-like Cu-semigroup and see that, given a Lsc-like Cu-semigroup S, we can associate to it a T_1-space X_S. It is shown in Theorem 2.4.5 that, in fact, $S\cong \mathrm{Lsc}(X_S,\overline{\mathbb{N}})$.

Given a Lsc-like Cu-semigroup S, in Section 2.5 we give a list of provides that S satisfies if and only if X_S is almost chainable. Paired with the results from the previous sections and Robert's computation of the Cuntz semigroup of certain commutative C^*-algebras in [74, Theorem 1.1], we obtain our desired result; see Theorem 2.5.12.

Using this characterization, we uncover in Section 2.6 new properies that the Cuntz semigroup of every separable C^*-algebra satisfies.

The results in this chapter have appeared in [101]. We also provide a new proof for Theorem 2.4.8, which is otherwise a corollary in [94].

2.1 Chainable and almost chainable spaces

We prove in this section that a compact metric space X is homeomorphic to an inverse limit of finite disjoint copies of unit intervals (i.e. $C(X)$ is an AI-algebra) if and only if X is *almost chainable*, an abstract property introduced in Definition 2.1.7; see Theorem 2.1.20.

As we will see in Section 2.5, almost chainability can be translated to a property of Cu-semigroups (see Definition 2.5.5).

We begin by recalling some notions and results from general topology and continuum theory (see, for example, [64, Chapter 12] for an introduction). We will then generalize such notions to obtain our desired results.

Definition 2.1.1. Let X be a topological space. The *(covering) dimension* of X is the minimum $n \in \mathbb{N}$ such that every open cover of X has a refinement where each point is contained at most in $n+1$ sets of the refinement. If no such n exists, we say that X is infinite-dimensional.

2.1.2. Recall that a *compactum* is a compact metric space, and that a *continuum* is a connected compactum. Moreover, given a finite nonempty indexed collection $C = \{C_1, \ldots, C_k\}$ of open subsets of a topological space X, we will say that C is a *chain* if

$$C_i \cap C_j \neq \emptyset \text{ if and only if } |i-j| \leq 1.$$

Additionally, if X is metric, the *mesh* of C, in symbols $\mathrm{mesh}(C)$, is defined as

$$\mathrm{mesh}(C) = \max\{\mathrm{diam}(C_i)\}.$$

An ε-*chain* is a chain of mesh less than ε.

The following definition coincides with the notion of chainability from continuum theory ([64, Chapter 12]) whenever X is a continuum; see Lemma 2.1.8 below. Note that, whenever the space is a compactum, the definition implies that X is connected and of dimension at most one.

Definition 2.1.3. Let X be a topological space. We say that X is *chainable* if any finite open cover of X can be refined by a chain.

Proposition 2.1.4 (cf. [64, Chapter 12])**.** *Let X be a compactum. Then, X is chainable if and only if X is homeomorphic to the inverse limit of unit intervals.*

Proof. Assume that X is chainable, which in particular implies that X is connected and, therefore, a continuum.

Recall that X is said to be *degenerate* if it only consists of a point. Thus, either X is degenerate (in which case we are done) or non-degenerate. By [64, Theorem 12.11], a non-degenerate chainable continuum is an inverse limit of unit intervals, as required.

Conversely, if X is the inverse limit of unit intervals, note that X is a continuum since the unit interval is compact and connected.

If X is degenerate, it is trivially chainable, so we may assume otherwise. Then, [64, Theorems 12.11, 12.19] and the comments following Theorem 12.19 of [64] imply that X is chainable, as desired. □

Definition 2.1.5. We will say that a unital AI-algebra A is *block-stable* if it is isomorphic to a finite direct sum of the form $\oplus_{k=1}^{n} C(X_k)$ with X_k a chainable continuum for each k.

As explained in the introduction, the aim of this section is to characterize inverse limits of (possibly increasing) finite disjoint unions of unit intervals. In light of Proposition 2.1.4, our approach to obtain such a characterization is to generalize the above mentioned definitions and results.

We begin by weakening the notions of chain and chainable space.

2.1.6. Let X be a topological space. An *almost chain* in X will be a finite nonempty indexed collection of open subsets $C = \{C_1, \ldots, C_k\}$ such that

$$C_i \cap C_j = \emptyset \text{ whenever } |i-j| \geq 2.$$

If X is a metric space, the *mesh* of an almost chain is $\mathrm{mesh}(C) = \max\{\mathrm{diam}(C_i)\}$, and an almost chain of mesh less than ε will be called an *ε-almost chain.*

Definition 2.1.7. A topological space X will be said to be *almost chainable* if any finite open cover of X can be refined by an almost chain.

Lemma 2.1.8. *Let X be a compactum. Then, X is almost chainable if and only if, for every $\varepsilon > 0$, there exists an ε-almost chain covering X*

Similarly, a continuum X is chainable if and only if for every $\varepsilon > 0$ there exists an ε-chain covering X.

Proof. Assume first that X is almost chainable and take $\varepsilon > 0$. Since X is compact, there exist finitely many points $x_1, \ldots, x_n$ such that their ε-balls cover X.

Using that X is almost chainable, we can refine this finite open cover by open subsets $C_1, \ldots, C_k$ such that $C_i \cap C_j = \emptyset$ whenever $|i-j| \geq 2$. Since each C_i is contained in some ε-ball, it follows that $C_1, \ldots, C_k$ is an ε-almost chain, as required.

Conversely, assume that X satisfies the stated condition and let $U_1, \ldots, U_n$ be a finite open cover of X. Then, since X is a compactum, the cover has a nonzero Lebesgue number δ. That is to say, that every open subset of X of diameter less than δ is contained in U_j for some j.

Set $\varepsilon < \delta$ and consider an ε-almost chain $C = \{C_1, \ldots, C_k\}$ covering X. Since the mesh of C is strictly less than δ, it follows that each C_i is contained in some U_j. Thus, $C_1, \ldots, C_k$ is an open refinement of $U_1, \ldots, U_n$ with the required property. ☐

Let us now define the notion of *(ε, δ)-maps* and *generalized arc-like spaces*. This is done in analogy with [64, Definition 2.12], where recall that a continuous map $f \colon X \to Y$ is said to be an *ε-map* if $\mathrm{diam}(f^{-1}(y)) \leq \varepsilon$ for every $y \in Y$; and a continuum X is said to be *arc-like* if for every $\varepsilon > 0$ there exists an ε-map from X onto $[0,1]$

Definition 2.1.9. Let X, Y be compacta, and let ε, δ be positive real numbers. A continuous map $f \colon X \to Y$ is an *(ε, δ)-map* if $\mathrm{diam}(f^{-1}(Z)) < \varepsilon$ for every $Z \subseteq Y$ with $\mathrm{diam}(Z) < \delta$.

Definition 2.1.10. A compactum X will be said to be a *generalized arc-like space* if, for every $\varepsilon > 0$, there exists $\delta > 0$ and an (ε, δ)-map $f \colon X \to [0,1] \sqcup \ldots_n \sqcup [0,1]$ for some $n \in \mathbb{N}$.

Remark 2.1.11. Given a finite disjoint union of unit intervals $[0,1]\sqcup\ldots_n\sqcup[0,1]$ and $\delta>0$, one can construct a (δ,δ')-map $r\colon [0,1]\sqcup\ldots_n\sqcup[0,1]\to[0,1]$ by simply rescaling $[0,1]\sqcup\ldots_n\sqcup[0,1]$ until it fits in $[0,1]$.

Consequently, a compactum X is a generalized arc-like space if and only if for every $\varepsilon>0$ there exists $\delta>0$ and an (ε,δ)-map $f\colon X\to[0,1]$.

Lemma 2.1.12. *Let X be an inverse limit of finite disjoint unions of unit intervals. Then, X is a generalized arc-like space.*

Proof. As stated, let X be an inverse limit of finite disjoint unions of unit intervals, and let $([0,1]\sqcup\ldots\sqcup[0,1], f_{i,j})$ be its associated inverse system. Recall that the metric on X is defined as

$$\mathrm{d}((x_i)_i,(y_i)_i) := \sum_{i=1}^{\infty}\frac{1}{2^i}\frac{\mathrm{d}_i(x_i,y_i)}{1+\mathrm{d}_i(x_i,y_i)},$$

where d_i is the distance in the i-th component of the inverse system.

Also recall that, for any two points in a finite disjoint union $[0,1]\sqcup\ldots\sqcup[0,1]$, their distance is defined as the usual distance if both points belong to the same connected component or 2 otherwise.

Let $\varepsilon>0$, and choose $n\in\mathbb{N}$ such that $\sum_{i>n}2/2^{-i}<\varepsilon/2$. Then, since the maps $f_{i,n}$ are uniformly continuous for each $i\leq n$, one can find $\delta>0$ with

$$\mathrm{d}_i(f_{i,n}(x),f_{i,n}(y))\leq\frac{\varepsilon}{2n}$$

whenever $\mathrm{d}_n(x,y)\leq\delta$.

Let $\pi_n\colon X\to[0,1]\sqcup\ldots\sqcup[0,1]$ denote the n-th canonical projection map, and let Z be a subset of $\pi_n(X)$ with $\mathrm{diam}_n(Z)\leq\delta$. Using that any two points are at distance at most two, we have that, for each $x,y\in\pi_n^{-1}(Z)$,

$$\mathrm{d}(x,y)=\sum_{i=1}^{n}\frac{1}{2^i}\frac{\mathrm{d}_i(f_{i,n}(x_n),f_{i,n}(y_n))}{1+\mathrm{d}_i(f_{i,n}(x_n),f_{i,n}(y_n))}+\sum_{i>n}\frac{1}{2^i}\frac{\mathrm{d}_i(x_i,y_i)}{1+\mathrm{d}_i(x_i,y_i)}\leq n\frac{\varepsilon}{2n}+\sum_{i>n}\frac{1}{2^i}2\leq\varepsilon.$$

This implies $\mathrm{diam}(\pi_n^{-1}(Z))\leq\varepsilon$ and, consequently, that π_n is an (ε,δ)-map. □

Proposition 2.1.13. *Let X be a compactum. Then, X is a generalized arc-like space if and only if X is almost chainable.*

Proof. We reproduce the proof of [64, Theorem 12.11] while making some minor adjustments.

Assume that X is almost chainable and take $\varepsilon>0$. Let C be an $\varepsilon/2$-almost chain covering X, and decompose it as $C=C_1\sqcup\ldots\sqcup C_r$ with $C_j=\{C_{i,j}\}_i$ an $\varepsilon/2$-chain for each $j\leq r$. Note that, since C covers X, one has

$$X=\sqcup_j(\cup_i C_{i,j}).$$

Let $j\leq r$. Then, if $|C_j|\leq 2$, let $f_j\colon \cup_i C_{i,j}\to[0,1]$ denote the map that sends every element to 0, which clearly satisfies $\mathrm{diam}(f_j^{-1}(Z))<\varepsilon$ whenever $\mathrm{diam}(Z)<1$.

If $|C_j|\geq 3$, one can use the techniques in the proof of [64, Theorem 12.11] to obtain an (ε,δ_j)-map $f_j\colon \cup_i C_{i,j}\to[0,1]$. Define $f:=f_1\sqcup\ldots\sqcup f_r\colon X\to[0,1]\sqcup\ldots_r\sqcup[0,1]$, which is continuous by construction.

Set $\delta < \min\{1, \delta_j\}$. Then, given $Z \subseteq [0,1] \sqcup \ldots_r \sqcup [0,1]$ with diameter less than δ, we can see Z as a subset of the j-th copy of $[0,1]$ for some j. Thus, $f^{-1}(Z) = f_j^{-1}(Z)$ has diameter at most ε, as desired.

Conversely, if X is a generalized arc-like space, for any $\varepsilon > 0$ there exists $\delta > 0$ and an (ε, δ)-map $f\colon X \to [0,1] \sqcup \ldots_r \sqcup [0,1]$. For each copy of $[0,1]$, we can find a δ-chain. The inverse image of these chains through f is an ε-almost chain covering X. □

Corollary 2.1.14. *Let X be an inverse limit of disjoint unions of unit intervals. Then, X is almost chainable.*

Proof. This follows as a combination of Lemma 2.1.12 and Proposition 2.1.13. □

We now prove the converse of Corollary 2.1.14. That is, every almost chainable compactum is an inverse limit of disjoint unions of unit intervals; see Theorem 2.1.20.

By a *closed interval* we will mean a possibly degenerate, closed interval of $[0,1]$ (that is to say, either a point or a non-degenerate closed interval).

Lemma 2.1.15 ([38]). *Let C be a closed, nonempty subset of $[0,1]$. Then, for every $\varepsilon > 0$, there exist a finite disjoint union of closed intervals Y and a map $\alpha\colon C \to Y$ such that $\alpha(y) = y$ for every $y \in Y$ and $|\alpha(x) - x| < \varepsilon$ for every $x \in C$.*

Remark 2.1.16. With the notation of Lemma 2.1.15 above, α is an onto 2ε-map from C to Y. Indeed, for every $y \in Y$ and $x \in C$ with $\alpha(x) = y$, we must have $|x - y| < \varepsilon$. Since $y \in \alpha^{-1}(y)$, we get $\operatorname{diam}(\alpha^{-1}(y)) < 2\varepsilon$, as desired.

2.1.17. Let X be a compactum. For any (ε, δ)-map $f\colon X \to [0,1]$, one can consider its induced onto ε-map f from X to $\operatorname{Im}(f)$. Thus, since $\operatorname{Im}(f)$ is compact, we know from Lemma 2.1.15 that there exists an onto δ-map $\alpha\colon \operatorname{Im}(f) \to Y$ with Y the finite disjoint union of closed intervals. This shows that the composition $\alpha f\colon X \to Y$ is an onto ε-map.

This implies, by Remark 2.1.11, that given a generalized arc-like compactum X (equivalently, an almost chainable compactum by Proposition 2.1.13) and any $\varepsilon > 0$, there exists an onto ε-map $f\colon X \to Y$ with Y a finite disjoint union of closed intervals.

Lemma 2.1.18 below is a natural generalization of [64, Lemma 12.17]. We follow both the structure and notation of its proof.

Lemma 2.1.18. *Let X be a compactum and Y_1 be a finite disjoint union of closed intervals. Take $\eta > 0$ and let $g_1\colon X \to Y_1$ be an onto continuous map. Then, there exists $\varepsilon > 0$ such that, for any onto ε-map $g_2\colon X \to Y_2$ with Y_2 a finite disjoint union of closed intervals, there exists a continuous map $\varphi\colon Y_2 \to Y_1$ such that*

$$|g_1(x) - \varphi g_2(x)| < \eta$$

for every $x \in X$.

Proof. Since Y_1 is the disjoint union of closed intervals, we can write

$$Y_1 = J_1 \sqcup \ldots \sqcup J_{n_1} \sqcup \{q_1\} \sqcup \ldots \sqcup \{q_{m_1}\}$$

with J_k closed non-degenerate intervals for each k.

Fix $m \in \mathbb{N}$ such that $1/m < \eta/2$ and such that the distance between the connected components of Y_1 is greater than $1/m$. Define $s_i := i/m$ for each $0 \leq i \leq m$. Using that g_1 is uniformly continuous, there exists $\gamma > 0$ such that $\operatorname{diam}(g_1(A)) < 1/m$ whenever $\operatorname{diam}(A) < \gamma$.

Set $\varepsilon := \gamma/2$ and fix an onto ε-map $g_2 \colon X \to Y_2$ as in the statement of the lemma. Recall that there exists $\delta > 0$ such that $\operatorname{diam}(g_2^{-1}(Z)) < 2\varepsilon = \gamma$ whenever $\operatorname{diam}(Z) < \delta$.

Let $n \in \mathbb{N}$ be such that $1/n < \delta/2$ and define $t_j := j/n$ for every $0 \leq j \leq n$. As above, we can write Y_2 as

$$Y_2 = I_1 \sqcup \ldots \sqcup I_{n_2} \sqcup \{p_1\} \sqcup \ldots \sqcup \{p_{m_2}\}$$

with I_l closed non-degenerate intervals for every l.

For each l and k, consider the subsets

$$\begin{aligned} A_1^k &= [s_0, s_1)\,,\ A_i^k = (s_{i-1}, s_{i+1})\,,\ A_m^k = (s_{m-1}, s_m] \subseteq J_k, \\ B_1^l &= [t_0, t_1)\,,\ B_j^l = (t_{j-1}, t_{j+1})\,,\ B_n^l = (t_{n-1}, t_n] \subseteq I_l. \end{aligned}$$

By construction, we know that $\operatorname{diam}(B_j^l) < \delta$ for each j, l. Thus, $\operatorname{diam}(g_2^{-1}(B_j^l)) < \gamma$ and, consequently,

$$g_1(g_2^{-1}(B_j^l)) \subseteq A_i^k \text{ or } g_1(g_2^{-1}(B_j^l)) \subseteq \{q_{r'}\}$$

for some i, k, r'.

Analogously, one also gets that, for every $r \leq m_2$, we can find i, k such that $g_1(g_2^{-1}(p_r)) \subseteq A_i^k$ or r' with $g_1(g_2^{-1}(p_r)) \subseteq \{q_{r'}\}$. Further, since g_2 is onto and for each l, j we have $B_j^l \cap B_{j+1}^l \neq \emptyset$, one has

$$\emptyset \neq g_1(g_2^{-1}(B_j^l)) \cap g_1(g_2^{-1}(B_{j+1}^l)).$$

Thus, for each fixed l, the sets $g_1(g_2^{-1}(B_j^l))$ belong to the same connected component of Y_1. This implies that there exists k such that $g_1(g_2^{-1}(I_l)) \subseteq J_k$ or that there exists r' with $g_1(g_2^{-1}(I_l)) = \{q_{r'}\}$.

For each connected component Y of Y_2, we now define the map $\varphi_Y \colon Y \to Y_1$ as follows:

If $g_1(g_2^{-1}(Y)) = \{q_{r'}\}$ for some r', let $\varphi_Y \colon Y \to \{q_{r'}\}$ be the constant map.

Else, there exists some k such that $g_1(g_2^{-1}(Y)) \subseteq J_k$. If Y is degenerate, we can find $A_i^k \subseteq J_k$ such that $g_1(g_2^{-1}(Y)) \subseteq A_i^k$ for some i, k. Define $\varphi_Y \colon Y \to A_i^k \subseteq J_k$ as the constant map $\varphi_Y \equiv s_i$.

Finally, if Y is non-degenerate, it is of the form $Y = I_l$. Then, for each j fix $i(j)$ such that $g_1(g_2^{-1}(B_j^l)) \subseteq A_{i(j)}^k$, and recall that

$$\emptyset \neq g_1(g_2^{-1}(B_j^l)) \cap g_1(g_2^{-1}(B_{j+1}^l))$$

for each j. This implies $|i(j) - i(j+1)| \leq 1$ and, consequently, we can define $\varphi_{I_l} \colon I_l \to J_k$ as $\varphi_{I_l}(t_j) = s_{i(j)}$ and extend it linearly.

Let φ be the map $\varphi_{I_1} \sqcup \ldots \sqcup \varphi_{I_{n_2}} \sqcup \varphi_{p_1} \sqcup \ldots \sqcup \varphi_{p_{m_2}} \colon Y_2 \to Y_1$, which is continuous by construction. We will now show that $|g_1(x) - \varphi g_2(x)| < \eta$ for each $x \in X$.

Thus, let $x \in X$ and let $B \subseteq Y_2$ such that $g_2(x) \in B$ with B being either B^l_j for some l, j or $\{p_r\}$ for some r. Note that $g_1(x) \in g_1(g_2^{-1}(B))$.

Thus, if $g_1(g_2^{-1}(B)) = \{q_{r'}\}$, we get $g_1(x) = q_{r'}$ and, consequently,

$$|g_1(x) - \varphi g_2(x)| = |q_{r'} - q_{r'}| = 0$$

by the definition of φ.

Finally, if $g_1(g_2^{-1}(B)) \subseteq A^k_i$, we have $g_1(x) \in A^k_i$. Therefore, one gets

$$|g_1(x) - s_i| < \frac{1}{m}.$$

If $B = \{p_r\}$ for some r, we have defined φ_{p_r} as the constant map s_i. Thus, one gets

$$|g_1(x) - \varphi g_2(x)| = |g_1(x) - s_i| < 1/m < \frac{\eta}{2}.$$

Else, if $B = B^l_j$ for some l, j, let $i(j)$ be the previously fixed integer satisfying $g_1(g_2^{-1}(B)) \subseteq A^k_{i(j)}$. Then, since $g_2(x) \in B$, we either have $t_{j-1} \leq g_2(x) \leq t_j$ or $t_j \leq g_2(x) \leq t_{j+1}$. Thus, $\varphi(g_2(x))$ is either between $s_{i(j-1)}$ and $s_{i(j)}$ or between $s_{i(j)}$ and $s_{i(j+1)}$.

Since $|i(j) - i(j+1)| \leq 1$, the triangle inequality implies

$$|g_1(x) - \varphi g_2(x)| \leq |g_1(x) - s_{i(j)}| + |s_{i(j)} - \varphi g_2(x)| \leq \frac{2}{m} < \eta,$$

as desired. □

Proposition 2.1.19 ([64, Proposition 12.18])**.** *Let (X, d) be a compactum and let $Y = \varprojlim(Y_i, f_i)$ be an inverse limit of compacta (Y_i, d_i) with $f_i \colon Y_{i+1} \to Y_i$.*

Assume that there exist two sequences of strictly positive real numbers $(\delta_i)_i$, $(\varepsilon_i)_i$ with $\lim \varepsilon_i = 0$ and a family of onto ε_i-maps $g_i \colon X \to Y_i$ such that the following conditions hold

(i) For every pair $i < j$, we have $\operatorname{diam}(f_{i,j}(A)) \leq \delta_i/2^{j-i}$ for any $A \subseteq Y_j$ with $\operatorname{diam}(A) \leq \delta_j$;

(ii) $d_i(g_i(x), g_i(y)) > 2\delta_i$ whenever $d(x, y) \geq 2\varepsilon_i$;

(iii) $d_i(g_i, f_i g_{i+1}) \leq \delta_i/2$.

Then, X is homeomorphic to Y.

The following statement summarizes the results in this section.

Theorem 2.1.20. *Let X be a compactum. Then, the following are equivalent:*

(1) X is almost chainable;

(2) X is a generalized arc-like space;

(3) X is homeomorphic to an inverse limit of finite disjoint copies of unit intervals;

(4) $C(X)$ is an AI-algebra.

Proof. Proposition 2.1.13 shows (1) if and only if (2), while (3) is equivalent to (4) by the results in [97]. Moreover, (3) implies (1) by Corollary 2.1.14.

To prove that (1) implies (3), let X be an almost chainable compactum. Following the proof from [64, Theorem 12.19], we inductively construct sequences of maps and real numbers satisfying the conditions in Proposition 2.1.19. We provide the construction for the sake of completeness, although the only difference with the original is that we replace $[0,1]$ with Y:

Take a positive number $\varepsilon_1 \leq 1$, and let g_1 be an onto ε_1-map from X to a finite disjoint union of closed intervals Y_1. Such a map exists because X is almost chainable.

In particular, note that $g_1(x) \neq g_1(y)$ whenever $d(x,y) \geq 2\varepsilon_1$. Using that X is compact, there exists $\delta_1 > 0$ such that $|g_1(x) - g_1(y)| > 2\delta_1$ whenever $d(x,y) \geq 2\varepsilon_1$.

Set $\eta := \delta_1/2$, let $\varepsilon > 0$ be the bound given by Lemma 2.1.18 and define $\varepsilon_2 := \min\{1/2, \varepsilon\}$. Since X is almost chainable, there exists an onto ε_2-map $g_2\colon X \to Y_2$.

By the choice of ε_2 and Lemma 2.1.18, there exists $f_1\colon Y_2 \to Y_1$ with

$$|g_1(x) - f_1 g_2(x)| < \eta = \delta_1/2$$

for every $x \in X$.

Proceeding as above, we find $\delta_2 > 0$ with $|g_2(x) - g_2(y)| > 2\delta_2$ whenever $d(x,y) \geq 2\varepsilon_2$. By the uniform continuity of f_1, we can choose δ_2 so that $\operatorname{diam}(f_1(A)) \leq \eta$ whenever $\operatorname{diam}(A) \leq \delta_2$. This implies that g_1, g_2 satisfy (i)-(iii) in Proposition 2.1.19.

Following [64, Theorem 12.19], one can now inductively define $\varepsilon_i, \delta_i > 0$ and $g_i\colon X \to Y_i$, $f_i\colon Y_{i+1} \to Y_i$ satisfying conditions (i)-(iii) from Proposition 2.1.19.

Thus, we get $X \cong \lim(Y_k, f_k)$ with Y_k finite disjoint unions of closed intervals. This implies that $C(X)$ is an AI-algebra, as desired. □

2.2 Lsc-like Cu-semigroups

We begin the section by defining *distributively lattice ordered* Cu-semigroups, which are a natural generalization of inf-semilattice ordered Cu-semigroups (as defined in Paragraph 1.2.14). Using this new notion, we define the class of Lsc-like Cu-semigroups (Definition 2.2.5) and study some of their properties. As we will see in the latter sections of the chapter, these Cu-semigroups are exactly those that are Cu-isomorphic to the Cu-semigroup of lower-semicontinuous functions $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ for some T_1 topological space; see Theorem 2.4.5.

Using Proposition 2.2.19, we also show in Corollary 2.2.21 that the semigroup $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ is a Cu-semigroup whenever X is compact and metric.

Definition 2.2.1. Let S be a Cu-semigroup. We will say that S is *distributively lattice ordered* if S is a distributive lattice such that $x + y = (x \vee y) + (x \wedge y)$ for any pair $x, y \in S$.

Additionally, we will say that S is *complete* if suprema of arbitrary sets exist.

Remark 2.2.2. We note that, in a complete distributively lattice ordered Cu-semigroup S, one has $(\sup_n x_n) \vee (\sup_n y_n) = \sup_n (x_n \vee y_n)$ whenever $(x_n)_n$ and $(y_n)_n$ are increasing sequences in S.

Indeed, since $\sup_n z_n = \vee_{n=1}^{\infty} z_n$ for any increasing sequence, we get

$$(\sup_n x_n) \vee (\sup_n y_n) = (\vee_{n=1}^{\infty} x_n) \vee (\vee_{n=1}^{\infty} y_n) = \vee_{n=1}^{\infty}(x_n \vee y_n) = \sup_n (x_n \vee y_n),$$

as required.

Throughout this chapter, a sum of finitely many indexed elements $x_1 + \ldots + x_m$ will be said to be *decreasingly* (resp. *increasingly*) *ordered* if the sequence $(x_i)_{i=1}^m$ is decreasing (resp. increasing).

Lemma 2.2.3. *Let S be a distributively lattice ordered* Cu*-semigroup, and take two decreasing sequences $(x_i)_{i=1}^m, (y_i)_{i=1}^m$ in S. Then,*

$$\sum_{i=1}^{m} (x_i + y_i) = \sum_{i=1}^{2m} \vee_{j=(i-m)_+}^{m} (x_j \wedge y_{i-j})$$

where, on the right hand side, we set $x_i \wedge y_k := x_i$ and $x_k \wedge y_i := y_i$ whenever $k \leq 0$.

In particular, every finite sum in a distributively lattice ordered semigroup can be written as a decreasingly ordered sum.

Proof. We prove the result by induction on m, where we note that the case $m = 1$ follows directly from Definition 2.2.1.

Thus, fix $m \in \mathbb{N}$ and assume that the result holds for every $k < m$. In particular, one has

$$\sum_{i=1}^{k} z_i + s = z_1 \vee s + z_k \wedge s + \sum_{i=2}^{k-1} (z_i \vee (z_{i-1} \wedge s))$$

for any $k < m$, $s \in S$ and decreasing sequence $(z_i)_i$ in S.

Now take two decreasing sequences $(x_i)_{i=1}^m, (y_i)_{i=1}^m$, and write

$$\sum_{i=1}^{m} (x_i + y_i) = \sum_{i=1}^{m-1} (x_i + y_i) + x_m \vee y_m + x_m \wedge y_m.$$

Set $z_i := \vee_{j=(i+1-m)_+}^{m-1} (x_j \wedge y_{i-j})$. Using our induction hypothesis, one gets

$$\begin{aligned}
\sum_{i=1}^{m} (x_i + y_i) &= \sum_{i=1}^{2m-2} \vee_{j=(i+1-m)_+}^{m-1} (x_j \wedge y_{i-j}) + x_m \vee y_m + x_m \wedge y_m \\
&= \sum_{i=1}^{m-1} z_i + \left(\sum_{i=m}^{2m-2} z_i + x_m \vee y_m \right) + x_m \wedge y_m,
\end{aligned}$$

where note that $z_i \geq x_m, y_m$ for each $i \leq m-1$. Thus, we have $z_i = \vee_{j=(i-m)_+}^{m} (x_j \wedge y_{i-j})$ for every $i \leq m-1$.

Using the induction hypothesis once again to the second summand above, we see that it is equal to

$$z_m \vee (x_m \vee y_m) + z_{2m-2} \wedge (x_m \vee y_m) + \sum_{i=m+1}^{2m-3} z_i \vee (z_{i-1} \wedge (x_m \vee y_m))$$

with

$$z_i \vee (z_{i-1} \wedge (x_m \vee y_m)) = z_i \vee \left(\vee_{j=(i-m)_+}^{m-1}((x_m \wedge y_{(i-1)-j}) \vee (x_j \wedge y_m))\right)$$
$$= z_i \vee ((x_m \wedge y_{i-m}) \vee (x_{i-m} \wedge y_m)) = \vee_{j=(i-m)_+}^{m}(x_j \wedge y_{i-j}).$$

for every $i = m+1, \ldots, 2m-3$.

Doing a similar argument with $z_m \vee (x_m \vee y_m)$ and $z_{2m-2} \wedge (x_m \vee y_m)$, we see that

$$\sum_{i=m}^{2m-2} z_i + x_m \vee y_m = \sum_{i=m}^{2m-1} \vee_{j=(i-m)_+}^{m}(x_j \wedge y_{i-j}).$$

Since $x_m \wedge y_m = \vee_{j=(2m-m)_+}^{m}(x_j \wedge y_{2m-j})$, this finishes the induction argument.

For the second part of the statement, let $x_1 + \ldots + x_m$ be a finite sum in S. Applying the first part of the lemma, we can write $x_1 + x_2$ as an ordered sum $z_1 + z_2$. Then, applying once again the first part of the lemma to $(z_1 + z_2) + (x_3 + 0)$, we can write it as an ordered sum.

Prooceding in this manner, the desired result follows. □

Definition 2.2.4. Let H be a subset of a Cu-semigroup S. We say that H is *topological* if, given two finite decreasing sequences $(x_i)_{i=1}^m, (y_i)_{i=1}^m$ in H, we have

$$\sum_{i=1}^{m} x_i \leq \sum_{i=1}^{m} y_i$$

if and only if $x_i \leq y_i$ for each i.

Recall that, for any element r in partially ordered set P, we denote by $\downarrow r$ the set $\{s \in P \mid s \leq r\}$; see, for example, [41, Definition O-1.3].

Also, given an element y in a Cu-semigroup S, we write $\infty y := \sup_n ny$. Further, if S has a greatest element, we denote it by ∞.

Definition 2.2.5. A Cu-semigroup S will be said to be Lsc-*like* if it is a complete distributively lattice ordered Cu-semigroup such that the following conditions hold:

(C1) For every pair of idempotent elements y, z in S, $y \geq z$ if and only if

$$\{x < \infty \mid x \text{maximal idempotent}, \; x \geq y\} \subseteq \{x < \infty \mid x \text{maximal idempotent}, \; x \geq z\}.$$

(C2) There exists a topological subset of the form $\downarrow e$ such that the finite sums of elements in $\downarrow e$ are sup-dense in S.

The following example justifies our terminology.

Example 2.2.6. Let X be a T_1 topological space. Then, $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ is Lsc-like whenever it is a Cu-semigroup (for example, whenever X is compact and metric, see Corollary 2.2.21 below).

Indeed, first note that $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ is clearly a complete distributively lattice ordered semigroup, where suprema and infima are taken pointwise. Further, we know that the maximal idempotent elements $s < \infty$ are of the form $s = \infty\chi_{X\setminus\{x\}}$. Given any pair of idempotents $\infty f = \infty\chi_{\mathrm{supp}(f)}$ and $\infty g = \infty\chi_{\mathrm{supp}(g)}$, we know that $\infty\chi_{\mathrm{supp}(f)} \leq \infty\chi_{\mathrm{supp}(g)}$ if and only if $\mathrm{supp}(f) \subseteq \mathrm{supp}(g)$. That is, if and only if for every $x \in X$ such that $\mathrm{supp}(g) \subseteq X \setminus \{x\}$ we have $\mathrm{supp}(f) \subseteq X \setminus \{x\}$. This shows that $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ satisfies (C1).

For (C2), it is readily checked that the order in $\downarrow 1$ is topological. Moreover, the subset $\downarrow 1$ clearly generates a dense semigroup in $\mathrm{Lsc}(X, \overline{\mathbb{N}})$.

An element e in a Cu-semigroup S is an *order unit* if $x \leq \infty e$ for every $x \in S$. That is to say, if $\infty e = \infty$.

Remark 2.2.7 (Order units in Lsc-like Cu-semigroups)**.** Let S be an Lsc-like Cu-semigroup, and let e be the element given by (C2). Then, e is an order unit because the semigroup generated by $\downarrow e$ is dense in S.

Further, given any order unit f, take $e' \ll e$. Since $e \leq \infty f$, we can find $f' \ll f$ and $k \in \mathbb{N}$ such that $e' \leq kf'$. Using that the set of finite sums of elements under e is dense in S, and that e is an order unit, we may assume that f' is one such sum.

By Lemma 2.2.3, we can write $f' = \sum_{i=1}^{m} g_i$ with $(g_i)_i$ decreasing and bounded by e. Thus, we have $e' \leq \sum_{i=1}^{m} kg_i$, where note that the right hand side of the inequality is a decreasingly ordered sum of km elements below e (the first k greatest elements are g_1, the next k are g_2, and so on).

Using that the order on $\downarrow e$ is topological, it follows that $e' \leq g_1 \leq f' \ll f$. Since this can be done for each $e' \ll e$, we get $e \leq f$ and, consequently, e is the least order unit of S.

Remark 2.2.8 (Idempotent elements in an Lsc-like Cu-semigroups)**.** Let S be an Lsc-like Cu-semigroup with no maximal idempotent elements $x < \infty$. Then, $S = \{0\}$.

Indeed, assuming that S has no maximal idempotents, it follows from (C1) that $z \leq 0$ for each idempotent z. Thus, for any element $s \in S$, we get $s \leq \infty s = 0$, and it follows that $s = 0$.

Using a similar argument, one can show that an element $s \in S$ satisfies $\infty s = \infty$ if and only if there are no maximal idempotents $x < \infty$ with $\infty s \leq x$.

Lemma 2.2.9. *Let $n \in \mathbb{N}$ and let S be an* Lsc*-like* Cu*-semigroup with least order unit e. Then, given any element $y \leq ne$, y can be written as an ordered sum of at most n nonzero terms below e.*

Proof. By Lemma 2.2.3 and condition (C2), there exists an increasing sequence $(y_k)_k$ with supremum y such that each y_k can be written as a finite ordered sum of elements below e.

Further, since $y_k \leq ne$ for every k and the order in $\downarrow e$ is topological, it follows that each y_k has at most n nonzero summands. Using once again that the order in $\downarrow e$ is topological, we see that the i-th summands of $(y_k)_k$ form an increasing sequence for every $i \leq n$.

One can check that the sum of their suprema is a finite ordered sum of at most n nonzero elements below e, as desired. □

Corollary 2.2.10. *Let S be an* Lsc*-like* Cu*-semigroup with least order unit e, and let $s \leq e$. Then, for any $y_1, \ldots, y_m \in S$ with $s \leq y_1 + \ldots + y_m$, we have $s \leq y_1 \vee \ldots \vee y_m$.*

Proof. Take $s' \ll s$ and let $y_i' \ll y_i$ be such that $s' \leq y_1' + \ldots + y_m'$. Since e is an order unit, there exists $n \in \mathbb{N}$ with $y_1' + \ldots + y_m' \leq ne$.

Applying Lemma 2.2.9 above, each y_i' can be written as an ordered sum of at most n nonzero elements below e. For each i, let $y_{1,i}'$ denote the largest element in the ordered sum of y_i'. Then, it follows from Lemma 2.2.3 that $y_1' + \ldots + y_m'$ can be written as a finite ordered sum of elements below e, with the largest summand being $y_{1,1}' \vee \ldots \vee y_{1,m}'$.

Using that the order in $\downarrow e$ is topological, we get $s' \leq y_{1,1}' \vee \ldots \vee y_{1,m}'$ and, consequently,

$$s' \leq y_{1,1}' \vee \ldots \vee y_{1,m}' \leq y_1' \vee \ldots \vee y_m' \leq y_1 \vee \ldots \vee y_m.$$

Since this holds for every $s' \ll s$, we have $s \leq y_1 \vee \ldots \vee y_m$ as desired. □

The following lemmas will play an important role in the study of the induced topology of an Lsc-like Cu-semigroup S; see Definition 2.3.1. Lemma 2.2.12 below gives an alternative version of (C1) in Definition 2.2.5; see Remark 2.2.13. In particular, it shows that $y < e$, with e the least order unit of S, if and only if there exists a maximal element $x < e$ with $y \leq x$.

Lemma 2.2.11. *Let S be an* Lsc*-like* Cu*-semigroup with least order unit e, and let $x \in S$. Then,*

$$\infty x = \infty(x \wedge e) \quad \textit{and} \quad (\infty x) \wedge e = x \wedge e.$$

In particular, if $x < \infty$ is a maximal idempotent, we must have $x \wedge e \neq e$.

Proof. It follows from Lemma 2.2.9 and condition (C2) in Definition 2.2.5 that x can be written as the supremum of finite ordered sums of elements in $\downarrow e$. Moreover, using that the order in $\downarrow e$ is topological, the sequence formed by the greatest element of each ordered sum is increasing. Let x' denote the supremum of this sequence.

Then, it is clear that $\infty x = \infty x'$ and $x \wedge e = x'$. Therefore, we have

$$\infty x = \infty x' = \infty(x \wedge e) \quad \text{and} \quad (\infty x) \wedge e = (\infty x') \wedge e = x' = x \wedge e,$$

as desired. □

Lemma 2.2.12. *Let S be an* Lsc*-like* Cu*-semigroup with least order unit e, and let y, z be elements in $\downarrow e$. Then, $z \leq y$ whenever $z \leq x$ for every maximal element $x < e$ such that $y \leq x$.*

Proof. We may assume that there exist maximal idempotents $x < \infty$, since otherwise $S = \{0\}$ by Remark 2.2.8 and there is nothing to prove.

We claim that the maximal idempotent elements $x < \infty$ are precisely the elements ∞s with $s < e$ maximal. To prove this, let $x < \infty$ be a maximal idempotent and take $s = x \wedge e$. By Lemma 2.2.11 we have $\infty s = \infty(x \wedge e) = \infty x = x$. To see that s is maximal, let $t \in S$ be such that $s \leq t \leq e$.

Using that x is maximal, we either have $\infty t = \infty$ or $\infty t = x$. Then, it follows from Lemma 2.2.11 that we either have $t = (\infty t) \wedge e = \infty \wedge e = e$ or $t = (\infty t) \wedge e = x \wedge e = s$, as required.

Conversely, let $s < e$ be maximal and consider the element $\infty s = \sup_n ns$. Let $(s_k)_k$ be a $\ll$-increasing sequence with supremum s.

Given an idempotent element x such that $\infty s \leq x \leq \infty$, we know by Lemma 2.2.9 that there exists a $\ll$-increasing sequence $(x_m)_m$ with supremum x such that each x_m can be written as a finite increasing sum of elements in $\downarrow e$.

Let $(m_k)_k$ be a strictly increasing sequence of integers such that $ks_k \leq x_{m_k}$ for every k. Using that $\downarrow e$ is topological, each s_k is less than or equal to each of the first k summands of x_{m_k}. As in the proof of Lemma 2.2.11, note that the largest summands of each sum of x_{m_k} form an increasing sequence of elements below e. Letting x' be the supremum of this sequence, we get $s \leq x' \leq e$, where note that one also has $\infty x' = x$ by construction. Indeed, the largest summand of each x_{m_k} is $x_{m_k} \wedge e$, and Lemma 2.2.11 above implies that $\infty x_{m_k} = \infty(x_{m_k} \wedge e)$. Since $(x_{m_k})_k$ has supremum $x = \infty x$, we have

$$x = \infty x = \sup_k \infty(x_{m_k} \wedge e) = \infty x'.$$

By maximality of s, we either have $s = x'$ or $x' = e$. Thus, we either have $\infty s = x$ or $x = \infty x' = \infty$. This proves that ∞s is a maximal idempotent, as desired.

A similar argument shows that for any $y, z \leq e$ we have $y \leq z \leq e$ if and only if $\infty y \leq \infty z$. Consequently, if $z \leq s$ for every maximal element $s < e$ such that $y \leq s$, we know that $\infty z \leq \infty y$. Taking the infimum with e, one gets $z = (\infty z) \wedge e \leq (\infty y) \wedge e = y$ as required. $\square$

Remark 2.2.13. We note that Lemma 2.2.12 above gives an alternative version of (C1). That is to say, an equivalent definition of an Lsc-like Cu-semigroup S can be given by changing (C1) in Definition 2.2.5 by the property stated in Lemma 2.2.12. Indeed, given two idempotents $y, z \in S$ satisfying

$$\{x < \infty \mid x \text{maximal idempotent}, \ x \geq y\} \subseteq \{x < \infty \mid x \text{maximal idempotent}, \ x \geq z\},$$

consider the elements $y \wedge e$ and $z \wedge e$.

By the proof of Lemma 2.2.12, maximal idempotents are in bijective correspondence with the maximal elements below e, and the bijection is given by $x \mapsto x \wedge e$.

Using Lemma 2.2.11, it follows that $x \geq y$ (resp. $x \geq z$) if and only if $x \wedge e \geq y \wedge e$ (resp. $x \wedge e \geq z \wedge e$). This implies that $y \wedge e$ and $z \wedge e$ satisfy the condition in Lemma 2.2.12 and, consequently, that $y \wedge e \geq z \wedge e$.

Applying Lemma 2.2.11 once again, we get $y \geq z$, as desired.

Recall that a complete lattice $(P, \leq)$ is said to be a *complete Heyting algebra* if, for every $T \subseteq P$ and $s \in P$, we have

$$s \wedge (\vee_{t \in T} t) = \vee_{t \in T}(s \wedge t).$$

Lemma 2.2.14. *Let S be an* Lsc*-like* Cu*-semigroup with least order unit e. Then, $\downarrow e$ is a complete Heyting algebra.*

Proof. We have to show that $s \wedge (\vee_{t \in T} t) = \vee_{t \in T}(s \wedge t)$ for every subset $T \subseteq \downarrow e$ and $s \leq e$. Thus, let $x < e$ be maximal with $\vee_{t \in T}(s \wedge t) \leq x$, which happens if and only if $s \wedge t \leq x$ for every $t \in T$.

Since $x \neq e$ and $(x \vee s) \wedge (x \vee t) = x \vee (s \wedge t) \leq x \vee x = x$, we either have $x \vee s = x$ or $x \vee t = x$ (since otherwise both of these unions would be equal to e and then $e \leq x$, a contradiction).

If $x = x \vee s$, we have $s \wedge (\vee_{t\in T} t) \leq x$. Else, if $t \leq x$ for each $t \in T$, so we also get $s \wedge (\vee_{t\in T} t) \leq x$. It now follows from Lemma 2.2.12 that $s \wedge (\vee_{t\in T} t) \leq \vee_{t\in T}(s \wedge t)$.

The other inequality holds in any lattice. $\square$

Definition 2.2.15. Let S be a Cu-semigroup. We say that S is *sup-semilattice ordered* if suprema of finite sets exist and $x + (y \vee z) = (x + y) \vee (x + z)$ for every $x, y, z \in S$.

Since an Lsc-like Cu-semigroup S is a lattice, it is natural to ask if S is sup-semilattice ordered. Lemma 2.2.16 below shows that this is indeed the case.

Lemma 2.2.16. *Let S be an* Lsc*-like* Cu*-semigroup with least order unit e, and let $n \in \mathbb{N}$ and $y, z \in S$ be such that $y \leq z, ne$. Then, there exists an element $y \setminus z \in S$ such that, for each $x \in S$, we have $x + y \leq z$ if and only if $x \leq y \setminus z$.*

In particular, $x + (y \vee z) = (x + y) \vee (x + z)$ for every $x, y, z \in S$.

Proof. We construct the 'almost-complement' $y \setminus z$ in three steps:

Step 1. *Let $y, z \in S$ be such that $y \leq z \leq e$.*

Then, consider the subset $T := \{x \in S \mid y + x \leq z\}$. Using that arbitrary suprema exist in S, we define

$$y \setminus z := \vee\{x \in S \mid y + x \leq z\} = \vee_{x\in T} x.$$

Further, since $\downarrow e$ is topological and $y + x = (y \vee x) + (y \wedge x) \leq z$ for each $x \in T$, we have $y \vee x \leq z$ and $y \wedge x = 0$.

Thus, Lemma 2.2.14 implies that $(\vee_{x\in T} x) \vee y = \vee_{x\in T}(x \vee y) \leq z$ and $(\vee_{x\in T} x) \wedge y = \vee_{x\in T}(x \wedge y) = 0$. Consequently, one gets

$$y + (y \setminus z) = y \vee (y \setminus z) + y \wedge (y \setminus z) = y \vee (y \setminus z) \leq z,$$

which shows that $x \leq y \setminus z$ if and only if $y + x \leq z$.

Step 2. *Assume that $y, z \in S$ satisfy $y \leq z \leq ne$*

By Lemma 2.2.9, we can write y and z as finite ordered sums $y = \sum_{i=1}^n y_i$ and $z = \sum_{j=1}^n z_j$ of elements below e. Thus, since $\downarrow e$ is topological, $y_i \leq z_i \leq e$ for each i, and we can use Step 1 to define $y \setminus z := \sum_{i=1}^n y_i \setminus z_i$.

Given $x \in S$ with $y + x \leq z \leq ne$, we have $x \leq ne$. Using Lemma 2.2.9, x can be written as a finite ordered sum $\sum_{i=1}^n x_i$ with $x_i \leq e$ for each i. Additionally, one gets

$$\sum_{i=1}^n y_i \vee x_i \leq y + x \leq \sum_{i=1}^n z_i,$$

which implies $y_i \vee x_i \leq z_i$. Note that this holds if and only if $x_i \leq y_i \setminus z_i$.

Thus, $x + y \leq z \leq ne$ if and only if $x \leq y \setminus z$.

Step 3. *Let $y \leq z, ne$, that is $y \leq z \wedge ne$.*

Since $y \leq z \wedge ne$, we have $y \leq z \wedge me$ for every $m \geq n$. Thus, Step 2 allows us to consider $y \setminus (z \wedge me)$. Further, one can easily check that

$$y \setminus (z \wedge me) \leq y \setminus (z \wedge (m + 1)e)$$

for each m.

We define $y \setminus z := \sup_m y \setminus (z \wedge me)$, which satisfies the desired property.

Finally, let $x, y, z \in S$ and note that

$$x + (y \vee z) \geq (x + y) \vee (x + z)$$

is clearly satisfied.

To see that $x + (y \vee z) \leq (x + y) \vee (x + z)$, let $x' \ll x$ and take $s \in S$ such that $x' + (y \vee z) \leq s$.

Since $x' \ll x \leq \infty e$, there exists $n \in \mathbb{N}$ with $x' \leq ne$. Thus, we know that $x' + (y \vee z) \leq s$ holds if and only if $y \vee z \leq x' \setminus s$, which in turn holds if and only if $x' + y, x' + z \leq s$. Consequently, one has

$$x' + (y \vee z) = (x' + y) \vee (x' + z).$$

Since this holds for each $x' \ll x$, it also holds for x, as required. □

Lemma 2.2.17. *Let S be an* Lsc*-like* Cu*-semigroup with least order unit e, and let $x, y, z \leq e$ with $x + y \leq x + z$. Then, $y \leq z$.*

Proof. Since $y \leq x + y \leq x + z$, it follows from Corollary 2.2.10 that $y \leq x \vee z$. Thus, the sum $(x \vee z) + y$ is decreasingly ordered.

Using Lemma 2.2.16 at the first and third step, the inequality $x + y \leq x + z$ at the second step, and that S is distributively lattice ordered at the last step, one obtains

$$\begin{aligned}(x \vee z) + y &= (y + z) \vee (y + x) \leq (y + z) \vee (z + x) = z + (y \vee x) \\ &= (z \vee y \vee x) + z \wedge (y \vee x) \leq (z \vee x) + z.\end{aligned}$$

Since the order in $\downarrow e$ is topological, it follows that $y \leq z$, as required. □

We now show that having a topological order is also reflected in the way below relation.

Proposition 2.2.18. *Let S be an* Lsc*-like* Cu*-semigroup with least order unit e, and let $(x_i)_{i=1}^m, (y_i)_{i=1}^m$ be decreasing sequence of elements below e. Assume that $(x_i)_{i=1}^m, (y_i)_{i=1}^m$ are such that*

$$\sum_{i=1}^m x_i \ll \sum_{i=1}^m y_i.$$

Then, we have $x_i \ll y_i$ for each i.

Proof. Let $y'_m \ll y_m$ be such that $\sum_{i=1}^m x_i \ll \sum_{i=1}^{m-1} y_i + y'_m$. Since $y'_m \ll y_m \leq y_{m-1}$, we can find $y'_{m-1} \ll y_{m-1}$ such that $y'_m \leq y'_{m-1}$ and

$$\sum_{i=1}^m x_i \ll \sum_{i=1}^{m-2} y_i + y'_{m-1} + y'_m.$$

Proceeding in this manner, we obtain a decreasing sequence $(y'_i)_{i=1}^m$ with $y'_i \ll y_i$ for each i, and $\sum_{i=1}^m x_i \ll \sum_{i=1}^m y'_i$.

Using that the order in $\downarrow e$ is topological, one gets $x_i \leq y'_i \ll y_i$ for every i, as desired. □

Proposition 2.2.19 below is probably well known to experts, but no proof has been found in the literature.

As shown in Corollary 2.2.21, this result implies that $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ is a Cu-semigroup whenever X is a compact, metric space. Recall that, for compact, metric, *finite-dimensionals* spaces, this already followed from a much more general result; see [5, Theorem 5.15].

Given $f \in \mathrm{Lsc}(X, \overline{\mathbb{N}})$ and $n \in \mathbb{N}$, we write $\{f \geq n\}$ to denote the open set $f^{-1}([n, \infty])$. For an open set $U \subseteq X$, we denote by χ_U the indicator function of U.

The supremum of f is denoted by $\sup(f)$, while $\mathrm{supp}(f)$ is the support of f.

Proposition 2.2.19. *Let X be a topological space, and let $f, g \in \mathrm{Lsc}(X, \overline{\mathbb{N}})$. Then, $f \ll g$ if and only if $\sup(f) < \infty$ and*

$$\chi_{\{f \geq n\}} \ll \chi_{\{g \geq n\}}$$

for every $n \in \mathbb{N}$.

Proof. First, let us assume that $f \ll g$, which clearly implies $\sup(f) < \infty$. Fix $n \in \mathbb{N}$ and consider an increasing sequence $(h_k)_k$ such that

$$\chi_{\{g \geq n\}} \leq \sup_k h_k,$$

which happens if and only if $\chi_{\{g \geq n\}} \leq \chi_{\cup_k \mathrm{supp}(h_k)}$.

Define the increasing sequence of functions

$$G_k := (n-1) + \chi_{\mathrm{supp}(h_k)} \left(\sum_{r=0}^{\infty} \chi_{\{g \geq n+r\}} \right),$$

and note that $g \leq \sup_k G_k$.

Since $f \ll g$, we can find $k \in \mathbb{N}$ with $f \leq G_k$ and, consequently,

$$\{f \geq n\} \subseteq \{G_k \geq n\} = \mathrm{supp}(h_k) \cap \{g \geq n\} \subseteq \mathrm{supp}(h_k).$$

This shows $\chi_{\{f \geq n\}} \leq \chi_{\mathrm{supp}(h_k)} \leq h_k$. Thus, $\chi_{\{f \geq n\}} \ll \chi_{\{g \geq n\}}$ as required.

Conversely, assume that $\sup(f) < \infty$ and that $\chi_{\{f \geq n\}} \ll \chi_{\{g \geq n\}}$ for each n. Using our first assumption, there exists $m < \infty$ such that $f = \sum_{i=1}^{m} \chi_{\{f \geq i\}}$.

Now let $(h_k)_k$ be an increasing sequence with $g \leq \sup_k h_k$. This implies

$$\{g \geq n\} \subseteq \bigcup_k \{h_k \geq n\}$$

and, therefore, we get $\chi_{\{g \geq n\}} \leq \sup_k \chi_{\{h_k \geq n\}}$.

Since $\chi_{\{f \geq n\}} \ll \chi_{\{g \geq n\}}$ for each n, it follows that for every i there exists $k_i \in \mathbb{N}$ such that $\chi_{\{f \geq i\}} \leq \chi_{\{h_{k_i} \geq i\}}$. Set $k := \max_{i=1,\ldots,m}\{k_i\}$. Then,

$$f \leq \sum_{i=1}^{m} \chi_{\{h_k \geq i\}} \leq h_k,$$

as desired. $\square$

Remark 2.2.20. Given a topological space X and two open subsets $U, V \subseteq X$, we write $U \Subset V$ if $\overline{U} \subseteq V$.

If X is a compact metric space, then $U \Subset V$ if and only if $\chi_U \ll \chi_V$ in $\mathrm{Lsc}(X, \overline{\mathbb{N}})$. Indeed, if $\chi_U \ll \chi_V$, we can write V as a countable increasing union of open sets V_n such that V_n is compactly contained in V_{n+1} for every n. Thus, one gets $\chi_U \ll \sup_n \chi_{V_n}$, which implies that U is contained in V_n for some n. Conversely, if $U \Subset V$ and $(W_n)_n$ is an increasing sequence of open sets with $V = \cup_n W_n$, it is clear that $U \subseteq W_n$ for some n. This shows $\chi_U \ll \chi_V$, as required.

Corollary 2.2.21. *Let X be a compact metric space. Then, $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ is a Cu-semigroup with pointwise order and addition.*

Proof. Axioms (O1) and (O4) are satisfied for every topological space X, so we are left to prove (O2) and (O3).

By Remark 2.2.20, we know that $\chi_U \ll \chi_V$ if and only if U is compactly contained in V. In particular, every indicator function can be written as the supremum of a $\ll$-increasing sequence. Using that every element in $S = \mathrm{Lsc}(X, \overline{\mathbb{N}})$ is the supremum of finite sums of indicator functions, it follows that S satisfies (O2).

Now let $f' \ll f$ and $g' \ll g$ in S, which by Proposition 2.2.19 implies that $\sup(f'), \sup(g') \leq m < \infty$ and that $\{f' \geq n\}$ and $\{g' \geq n\}$ are compactly contained in $\{f \geq n\}$ and $\{g \geq n\}$ respectively for every $n \in \mathbb{N}$. Thus, we have

$$\bigcup_{k=0}^{m} \overline{(\{f' \geq k\} \cap \{g' \geq n-k\})} \subseteq \bigcup_{k=0}^{m} (\{f \geq k\} \cap \{g \geq n-k\})$$

for every $n \leq \sup(f) + \sup(g)$, where note that the left hand side is equal to $\overline{\{f' + g' \geq n\}}$ and the right hand side is contained in $\{f + g \geq n\}$. Proposition 2.2.19 implies $f' + g' \ll f + g$ and, consequently, S satisfies (O3). $\square$

2.3 The topological space of a Lsc-like Cu-semigroup

We begin this section by associating with each Lsc-like Cu-semigroup S a topological space X_S; see Definition 2.3.1. The properties of this space, studied in Proposition 2.3.3, are used to prove that $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ is always a Cu-semigroup; see Theorem 2.3.10. In fact, we will show in Theorem 2.4.5 that, if S is Lsc-like, then S and $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ are Cu-isomorphic.

We also introduce notions for Cu-semigroups that have a topological equivalent whenever the semigroup is Lsc-like. More explicitly, we characterize when X_S is second countable, normal and metric in terms of algebraic properties of the Lsc-like Cu-semigroup S; see Proposition 2.3.7.

Definition 2.3.1. Given an Lsc-like Cu-semigroup S with least order unit e, we define its *topological space* X_S as

$$X_S := \{x \in S \mid x < e \text{ maximal}\},$$

with closed subsets

$$C_y := \{x \in X_S \mid y \leq x\}, \quad y \leq e.$$

We will also denote by U_y the open subset $X_S \setminus C_y$.

Let us first check that the subsets in Definition 2.3.1 above do indeed define a topology in X_S.

Lemma 2.3.2. *Let S be an* Lsc*-like* Cu*-semigroup with least order unit e. Then, the family $\{X_S \setminus C_y \mid y \leq e\}$ is a T_1-topology for X_S.*

Proof. Note that $C_0 = X_S$, $C_e = \emptyset$ and $C_x = \{x\}$ for each $x \in X_S$. Thus, our topology will be T_1.

We now need to prove that arbitrary intersections and finite unions of subsets of the form C_{y_i} are again of the form C_z for some $z \leq e$.

First, for any family of subsets of the form C_{y_i}, we get

$$\bigcap_i C_{y_i} = C_{\vee_i y_i},$$

which implies the first desired property.

Further, one also has

$$\bigcup_{i=1}^{n} C_{y_i} = C_{\wedge_{i=1}^{n} y_i}.$$

Indeed, the inclusion $\cup_i C_{y_i} \subseteq C_{\wedge_i y_i}$ is clear and, for $x \in X_S$ with $\wedge_{i=1}^{n} y_i \leq x$, we have

$$(x \vee y_1) \wedge \ldots \wedge (x \vee y_n) = x \vee (\wedge_{i=1}^{n} y_i) \leq x \vee x = x.$$

Using the maximality of x, we get $x \vee y_i = x$ or $x \vee y_i = e$ for each i. Note, however, that there must exist at least one i such that $x = x \vee y_i$ since otherwise we would have $e \leq x$, a contradiction.

Thus, $y_i \leq x \vee y_i = x$ for some i, which shows that $x \in \cup_i C_{y_i}$ as desired. □

Recall from Lemma 2.2.16 that, for every pair of elements $y \leq z \leq e$, the element $y \setminus z$ denotes the almost complement of z by y. That is, for any $x \in S$ we have $x + y \leq z$ if and only if $x \leq y \setminus z$.

Proposition 2.3.3. *Let S be an* Lsc*-like* Cu*-semigroup with least order unit e. Then,*

(i) For every $y, z \leq e$, $C_y \subseteq C_z$ if and only if $z \leq y$.

(ii) For every $y \leq e$, $U_y = \{x \in X_S \mid y \vee x = e\}$.

(iii) Given $y, z \leq e$ such that $U_y \subseteq C_z$, we have $y \wedge z = 0$.

(iv) For every $y \leq e$, the closure of U_y, denoted by $\overline{U_y}$, is $C_{y\setminus e}$.

(v) Given $y \leq e$, we have $\mathrm{Int}(C_y) = X_S \setminus \overline{(X_S \setminus C_y)} = U_{y\setminus e}$, where $\mathrm{Int}(C_y)$ denotes the interior of C_y.

(vi) For every $y, z \leq e$, $C_y \subseteq U_z$ if and only if $y \vee z = e$.

Proof. To see (i), let $z, y \leq e$. Using Lemma 2.2.12, we know that $z \leq y$ if and only if $z \leq x$ for every $x \geq y$ with $x < e$ maximal. By definition, this is equivalent to $C_y \subseteq C_z$, as required.

For (ii), let $y \leq e$ and take $x < e$ be maximal. Since $x \leq y \vee x \leq e$ and x is maximal, we either have $y \vee x = x$ (i.e. $x \in C_y$) or $y \vee x = e$. Thus, $U_y = \{x \in X_S \mid y \vee x = e\}$ as desired.

To prove (iii), let y, z be as in the statement and assume, for the sake of contradiction, that $y \wedge z \neq 0$. This implies that $U_{y \wedge z}$ is a nonempty subset of X_S.

By (ii), there exists $x \in U_{y \wedge z}$ with $x \vee (y \wedge z) = e$. Then, since $y \wedge z \leq y, z$, one has $x \vee z = e$ and $x \vee y = e$. Using our second equality and the inclusion $U_y \subseteq C_z$, it follows that $x \vee z = x$, which contradicts $x \vee z = e \neq x$, as required.

Let us now prove (iv) and, consequently, (v). First, take $x \in U_y$ and note that, by (ii), one gets

$$y \setminus e + y \leq e \leq (y \vee x) + (y \wedge x) = x + y.$$

Using Lemma 2.2.17 we have $y \setminus e \leq x$, which shows that $U_y \subseteq C_{y \setminus e}$.

Conversely, let z be such that $U_y \subseteq C_z$. This implies, by (iii), that $y \wedge z = 0$. Thus, one has $y + z \leq e$, and Lemma 2.2.16 implies $z \leq y \setminus e$. That is, we get $C_{y \setminus e} \subseteq C_z$ as desired.

Therefore, we have $\mathrm{Int}(C_y) = X_S \setminus \overline{U_y} = X_S \setminus C_{y \setminus e} = U_{y \setminus e}$, which shows (v).

Finally, for (vi), assume first that $y \vee z = e$ and take $x \in C_y$. We have $e = y \vee z \leq x \vee z$. This implies $x \in U_z$.

Conversely, let $y, z \leq e$ be such that $C_y \subseteq U_z$ and assume, for the sake of contradiction, that $y \vee z \neq e$. Then, there exists $x \in X_S$ with $x \geq y \vee z$. This implies $x \geq y$ and, consequently, $x \vee z = e$ from $C_y \subseteq U_z$. However, we have

$$x = x \vee y \vee z \geq x \vee z = e$$

which is a contradiction. □

Example 2.3.4. Let X be a T_1 topological space such that $S = \mathrm{Lsc}(X, \overline{\mathbb{N}})$ is a Cu-semigroup. Recall from Example 2.2.6 that S is Lsc-like with least order unit 1. Then, the topological space of S is homeomorphic to X.

Indeed, note that the maximal elements below 1 are of the form $\chi_{X \setminus \{x\}}$ with $x \in X$. Thus, we have

$$X_S = \{\chi_{X \setminus \{x\}} \mid x \in X\}$$

and

$$\begin{aligned} U_{\chi_{\mathcal{U}}} &= X_S \setminus C_{\chi_{\mathcal{U}}} = X_S \setminus \{\chi_{X \setminus \{x\}} \mid \chi_{X \setminus \{x\}} \geq \chi_{\mathcal{U}}\} \\ &= X_S \setminus \{\chi_{X \setminus \{x\}} \mid x \in X \setminus \mathcal{U}\} = \{\chi_{X \setminus \{x\}} \mid x \in \mathcal{U}\} \end{aligned}$$

for every open subset $\mathcal{U} \subseteq X$.

Let $\varphi\colon X_S \to X$ be the map defined as $\chi_{X \setminus \{x\}} \mapsto x$. Using the above equalities, it is easy to check that φ is a homeomorphism between X and X_S.

Following our study of X_S, Lemma 2.3.5 below provides an analog of Remark 2.2.20.

Lemma 2.3.5. *Let S be an* Lsc*-like* Cu*-semigroup with least order unit e. Assume that e is compact and that S satisfies (O5). Then, X_S is normal.*

Further, given $x, y \leq e$, we have $U_x \Subset U_y$ if and only if $x \ll y$.

Proof. To see that X_S is normal, let C_x and C_y be two closed disjoint subsets in X_S. By Lemma 2.3.2, we have $C_{x\vee y} = C_x \cap C_y = \emptyset = C_e$. In terms of the elements $x, y \in S$, Proposition 2.3.3 (i) implies $e \ll e \leq x \vee y \leq x + y$. Take $x' \ll x$ and $y' \ll y$ such that $e \ll x' + y'$.

Since S satisfies (O5), there exist elements $c, d \leq e$ satisfying

$$x' + c \leq e \leq x + c \quad \text{and} \quad y' + d \leq e \leq y + d.$$

This implies, in particular, that $x' + c + y' + d \leq e + e$ with $e \ll x' + y'$. By Lemma 2.2.17, we get $c + d \leq e$. Using that the order in $\downarrow e$ is topological, one has $c \wedge d = 0$, $x \vee c = e$ and $y \vee d = e$.

It now follows from Proposition 2.3.3 that the above equalities are equivalent to $C_x \subseteq U_c$, $C_y \subseteq U_d$ and $U_c \cap U_d = U_{c\wedge d} = \emptyset$. This implies that X_S is normal, as desired.

Now let $x, y \leq e$ and assume that $U_x \Subset U_y$. Using (iv) and (vi) in Proposition 2.3.3, this happens if and only if $(x \setminus e) \vee y = e$. Thus, we have $e \leq x \setminus e + y$ and, since $x \setminus e + x \leq e \ll e$, we obtain $x \ll y$ by Lemma 2.2.17.

Conversely, if $x \ll y$, we can use (O5) to obtain an element $c \leq e$ such that $x + c \leq e \leq y + c$. Thus, one gets $c \leq x \setminus e$, which implies

$$e \leq y + c \leq y + x \setminus e.$$

Using (iv) and (vi) in Proposition 2.3.3 once again, it follows that $U_x \Subset U_y$, as desired. □

Recall the definition of inf-semilattice ordered Cu-semigroup from Paragraph 1.2.14.

Definition 2.3.6. An inf-semilattice ordered Cu-semigroup S is *normal* if there exists an order unit $z \in S$ such that, for any pair $x, y \in S$ satisfying $z \leq x + y$, there exist $s, t \in S$ with

$$z \leq x + s, \quad z \leq y + t \quad \text{and} \quad s \wedge t = 0.$$

We show in Proposition 2.3.7 below that a number of topological properties can be translated to algebraic properties of Cu-semigroups. In Theorem 2.4.8, we will continue this study and prove an abstract characterization of covering dimension.

Proposition 2.3.7. *Let S be an* Lsc*-like* Cu*-semigroup and let X_S be its associated topological space. Then,*

(i) X_S is second countable if and only if S is countably based.

(ii) X_S is countably compact if and only if S has a compact order unit.

(iii) X_S is normal if and only if S is normal.

(iv) X_S is a metric space whenever S is countably based and normal.

(v) X_S is a compact metric space whenever S is countably based, has a compact order unit and satisfies (O5).

Proof. We will prove each claim separately.

(i) Let e be the least order unit of S, and assume that S is countably based. Let B denote a countable basis of S and denote by E the monoid generated by the elements in $\downarrow e$. Since both B and E are bases, $B' := B \cap E$ is also a countable basis for S.

For every open set U_y with $y \leq e$, we can write $y = \sup_n y_n$ with $y_n \in B'$. Thus, we have $\cup_n U_{y_n} = U_y$, which implies that X_S is second countable.

Conversely, assume that X_S is second countable with basis $C = \{U_{z_n}\}_n$, which we may assume to be closed under finite unions. Then, every open subset U_y can be written as the countable union of increasing open subsets U_{z_m} in C.

Since this is equivalent to $y = \sup_m z_m$, it follows that the monoid generated by $\{z_n\}_n$ is both countable and dense in S, as desired.

(ii) Note that $e \in S$ is compact if and only if X_S is countably compact. Thus, we are left to prove that e is compact whenever there exists a compact order unit in S.

Let p be a compact order unit, which implies $p \leq ne$ for some $n \in \mathbb{N}$. By Lemma 2.2.9, we can write $p = p_1 + \ldots + p_n$ with $p_{i+1} \leq p_i \leq e$ for each i. It then follows from Proposition 2.2.18 that each p_i is compact. Since p_1 is an order unit below e, we must have $p_1 = e$ compact, as desired.

(iii) Assume first that S is normal and let z be the order unit given by Definition 2.3.6. Take disjoint closed subsets C_x, C_y of X_S. We have $x \vee y = e$.

Using that e is an order unit, one gets $z \leq \infty = \infty x + \infty y$. By the definition of normality, we can find $s, t \in S$ with $z \leq \infty x + s$, $z \leq \infty y + t$ and $s \wedge t = 0$. Then, it follows from Corollary 2.2.10 that $e \leq \infty x \vee s$ and $e \leq \infty y \vee t$.

By Lemma 2.2.11, one has

$$e = (\infty x \wedge e) \vee (s \wedge e) = x \vee (s \wedge e), \quad e = y \vee (t \wedge e) \quad \text{and} \quad (s \wedge e) \wedge (t \wedge e) = 0.$$

Using (vi) in Proposition 2.3.3, we obtain $C_x \subseteq U_{s\wedge e}$, $C_y \subseteq U_{t\wedge e}$ and $U_{s\wedge e} \cap U_{t\wedge e} = \emptyset$. Thus, X_S is normal.

Conversely, if X_S is normal, it is readily checked that S is normal by taking $z = e$ in Definition 2.3.6.

(iv) Using (i) and (iii), X_S is a second countable, normal and Hausdorff space. The result now follows from Urysohn's metrization theorem; see, for example, [59, Theorem 34.1].

(v) Note that $e \ll e$ by the argument in (ii). Thus, we know from Lemma 2.3.5 and (iii) that S is normal. Using (iv), we obtain the desired result. □

We now prove that $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ is a Cu-semigroup for every Lsc-like Cu-semigroup S. As in Corollary 2.2.21, note that (O1) and (O4) are always satisfied, so we only need to show (O2) and (O3).

Lemma 2.3.8. *Let S be an* Lsc-*like* Cu-*semigroup with least order unit e, and take $y, z \leq e$. Then, $y \ll z$ if and only if $\chi_{U_y} \ll \chi_{U_z}$ in* $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})$.

Proof. Let us first assume that $y \ll z$, and let $(f_n)_n$ be an increasing sequence in $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ such that $\chi_{U_z} \leq \sup_n f_n$. In particular, note that this holds if and only if $\chi_{U_z} \leq \chi_{\cup_n \mathrm{supp}(f_n)}$ or, equivalently, if

$$\bigcap_n (X_S \setminus \mathrm{supp}(f_n)) = X_S \setminus \bigcup_n \mathrm{supp}(f_n)` \subseteq X_S \setminus U_z = C_z.$$

Let z_n be the elements in $\downarrow e$ with $C_{z_n} = X_S \backslash \mathrm{supp}(f_n)$. Since $\mathrm{supp}(f_n) \subseteq \mathrm{supp}(f_{n+1})$ for each n, (i) in Proposition 2.3.3 implies that $(z_n)_n$ is increasing.

By (the proof of) Lemma 2.3.2, we get

$$C_{\sup_n(z_n)} = \bigcap_n C_{z_n} \subseteq C_z.$$

Using (i) in Proposition 2.3.3 once again, it follows that $z \leq \sup_n z_n$ and, since $y \ll z$, there exists $n \in \mathbb{N}$ with $y \leq z_n$. Thus, $U_y \subseteq U_{z_n}$ or, equivalently, $\chi_{U_y} \leq \chi_{U_{z_n}} = \chi_{\mathrm{supp}(f_n)} \leq f_n$. This shows $\chi_{U_y} \ll \chi_{U_z}$, as required.

Conversely, take $y, z \leq e$ such that $\chi_{U_y} \ll \chi_{U_z}$, and consider an increasing sequence $(h_n)_n$ in S with $z \leq \sup_n h_n$. Note that, by taking $z \wedge h_n$ instead of h_n, we may assume $h_n \leq e$ for each n.

It follows from (the proof of) Lemma 2.3.2 that

$$\bigcap_n C_{h_n} = C_{\sup_n(h_n)} \subseteq C_z$$

and, consequently, we have $\chi_{U_y} \ll \chi_{U_z} \leq \sup_n \chi_{U_{h_n}}$. This implies that there exists $n \in \mathbb{N}$ with $\chi_{U_y} \leq \chi_{U_{h_n}}$, that is to say $C_{h_n} \subseteq C_y$.

Using (i) in Proposition 2.3.3, one has $y \leq h_n$ as desired. □

Proposition 2.3.9. *Let S be an* Lsc*-like* Cu*-semigroup. Then,* $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ *satisfies (O2).*

Proof. To verify that $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ satisfies (O2), let $f \in \mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ and consider the sequence $(y_i)_i$ in $\downarrow e$ satisfying

$$\{f \geq i\} = U_{y_i},$$

which is decreasing by (i) in Proposition 2.3.3.

Using that S satisfies (O2), we can choose $\ll$-increasing sequences $(y_{i,n})_n$ such that $y_i = \sup_n y_{i,n}$ for each n. Thus, for every fixed k, we get

$$y_{k,n} \ll y_k \leq \ldots \leq y_1$$

for all n.

Consequently, for every i one can inductively choose $n_{i,k}$ with $k \geq i$ such that

$$y_{i,n_{i,k}} \ll y_{i,n_{i,k+1}} \quad \text{and} \quad y_{k,n_{k,k}} \leq \ldots \leq y_{1,n_{1,k}}.$$

Indeed, we begin by setting $n_{1,1} = 1$. Then, assuming that we have defined $n_{i,k}$ for every $i, k \leq m-1$ (and $k \geq i$) for some fixed m, we set $n_{m,m} = 1$ and choose $n_{m-1,m}$ large enough so that $y_{m,n_{m,m}} \leq y_{m-1,n_{m-1,m}}$ and $n_{m-1,m-1} \leq n_{m-1,m}$. Similarly, we choose $n_{m-2,m} \geq n_{m-2,m-1}$ such that $y_{m-1,n_{m-1,m}} \leq y_{m-2,n_{m-2,m}}$ and define $n_{i,m}$ for each $i \leq m-2$ in the same fashion.

Now consider the elements $f_k = \sum_{i=1}^{k} \chi_{U_{y_{i,n_{i,k}}}}$, which are ordered by construction. Thus, one has $U_{y_{i,n_{i,k}}} = \{f_k \geq i\}$ for every i. Using Lemma 2.3.8, Proposition 2.2.19 and the fact that $y_{i,n_{i,k}} \ll y_{i,n_{i,k+1}}$ for each i, we get

$$f_k = \sum_{i=1}^{k} \chi_{U_{y_{i,n_{i,k}}}} \ll \sum_{i=1}^{k} \chi_{U_{y_{i,n_{i,k+1}}}} \leq f_{k+1}.$$

One can now check that $\sup_k f_k = f$, as desired. □

Theorem 2.3.10. *Let S be an* Lsc*-like* Cu*-semigroup. Then, the ordered monoid* $\mathrm{Lsc}(X_S,\overline{\mathbb{N}})$ *is a* Cu*-semigroup.*

Proof. As mentioned above, $\mathrm{Lsc}(X_S,\overline{\mathbb{N}})$ always satisfies (O1) and (O4), and we know from Proposition 2.3.9 that it satisfies (O2). Thus, we are left to prove (O3).

Let $f \ll f'$ and $g \ll g'$ in $\mathrm{Lsc}(X_S,\overline{\mathbb{N}})$. By Proposition 2.2.19, we have

$$\chi_{\{f\geq i\}} \ll \chi_{\{f'\geq i\}} \quad \text{and} \quad \chi_{\{g\geq i\}} \ll \chi_{\{g'\geq i\}}$$

for each i. Also, there exists $m<\infty$ such that $\sup(f),\sup(g)\leq m$.

Take $y_i, y_i', z_i, z_i' \in\downarrow e$ with

$$U_{y_i}=\{f\geq i\},\quad U_{y_i'}=\{f'\geq i\},\quad U_{z_i}=\{g\geq i\} \quad\text{and}\quad U_{z_i'}=\{g'\geq i\}.$$

Using Lemma 2.3.8 we see that $y_i \ll y_i'$ and $z_i \ll z_i'$ for every i. Since S satisfies (O3), we have

$$\sum_{i=1}^{m}(y_i+z_i) \ll \sum_{i=1}^{m}(y_i'+z_i').$$

By Lemma 2.2.3, we can rewrite these sums as

$$\sum_{i=1}^{2m} \vee_{j=0}^{m}(y_j\wedge z_{i-j}) \ll \sum_{i=1}^{2m} \vee_{j=0}^{m}(y_j'\wedge z_{i-j}'),$$

where note that both sides are now ordered. Thus, it follows from Proposition 2.2.18 that

$$\vee_{j=0}^{m}(y_j\wedge z_{i-j}) \ll \vee_{j=0}^{m}(y_j'\wedge z_{i-j}')$$

for each i.

Since

$$\{f+g\geq i\}=\bigcup_{j=0}^{m}(\{f\geq j\}\cap\{g\geq i-j\}),$$

one gets, using Lemma 2.3.2 and its proof at the last two steps, that

$$X_S\backslash\{f+g\geq i\}=\bigcap_{j=0}^{m}((X_S\backslash\{f\geq j\})\cup(X_S\backslash\{g\geq i-j\}))=\bigcap_{j=0}^{m}(C_{y_j\wedge z_{i-j}})=C_{\vee_{j=0}^{m}(y_j\wedge z_{i-j})}$$

and, consequently, $\chi_{\{f+g\geq i\}}=\chi_{U_{\vee_{j=0}^{m}(y_j\wedge z_{i-j})}}$.

A similar argument also shows $\chi_{U_{\vee_{j=0}^{m}(y_j'\wedge z_{i-j}')}}\leq \chi_{\{f'+g'\geq i\}}$.

Using Lemma 2.3.8, we have $\chi_{\{f+g\geq i\}} \ll \chi_{\{f'+g'\geq i\}}$ for each i which, by Proposition 2.2.19, implies that $f+g\ll f'+g'$, as required. □

2.4 An abstract characterization of $\mathrm{Lsc}(X,\overline{\mathbb{N}})$

We prove that every Lsc-like Cu-semigroup S is Cu-isomorphic to its associated semigroup of lower-semicontinuous functions $\mathrm{Lsc}(X_S,\overline{\mathbb{N}})$; see Theorem 2.4.5. To do so, we first define in Definition 2.4.2 a map $\varphi_0\colon \mathrm{Lsc}(X_S,\overline{\mathbb{N}})_{\ll}\to S$, which we then extend to a Cu-isomorphism $\varphi\colon \mathrm{Lsc}(X_S,\overline{\mathbb{N}})\to S$.

Lemma 2.4.1. *Let S and H be a pair of* Cu*-semigroups, and let $\varphi\colon S \to H$ be a* Cu*-morphism between them such that*

(i) φ is an order embedding on a basis B of S;

(ii) $\varphi(S)$ is a basis for H.

Then, φ is a Cu*-isomorphism.*

Proof. First, let x, y in S be such that $\varphi(x) \leq \varphi(y)$, and take $x' \ll x$. Since B is a basis, we can find $s \in B$ such that $x' \ll s \ll x$. Thus, since $\varphi(s) \ll \varphi(y)$, we can also find $t \in B$ with $\varphi(s) \ll \varphi(t)$ and $t \ll y$.

Using that φ is an order embedding in B, we obtain $s \ll t$ and, consequently, $x' \ll y$. Since this holds for each $x' \ll x$, we obtain $x \leq y$. This shows that φ is a global order embedding.

To prove surjectivity, let $h \in H$. Since $\varphi(S)$ is a basis for H, we can write $h = \sup_n \varphi(s_n)$ for some $s_n \in S$. Further, as we know that φ is an order embedding, the sequence $(s_n)_n$ is increasing in S, so

$$h = \sup_n \varphi(s_n) = \varphi(\sup_n s_n) \in \varphi(S),$$

as desired. □

We now define the map φ_0, where note that $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})_{\ll}$ can be identified with the elements in $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ with finite supremum.

Definition 2.4.2. Let S be an Lsc-like Cu-semigroup. We define

$$\varphi_0\colon \mathrm{Lsc}(X_S, \overline{\mathbb{N}})_{\ll} \to S$$

by $\varphi_0(f) = \sum_{i=1}^{\sup(f)} z_i$, where $\{f \geq i\} = U_{z_i}$ for each i.

Lemma 2.4.3. *For any* Lsc*-like* Cu*-semigroup S, the map φ_0 defined above is a positively ordered monoid morphism and an order embedding.*

Further, for any pair $f, g \in \mathrm{Lsc}(X_S, \overline{\mathbb{N}})_{\ll}$, we have $\varphi_0(f) \ll \varphi_0(g)$ if and only if $f \ll g$.

Proof. Let $f, g \in \mathrm{Lsc}(X_S, \overline{\mathbb{N}})_{\ll}$ with $\sup(f) = m \in \mathbb{N}$, and take $z_i, y_j \in S$ such that $\{f \geq i\} = U_{z_i}$ and $\{g \geq j\} = U_{y_j}$ for each $i \leq m$ and $j \leq \sup(g)$. We will first prove by induction on m that $\varphi_0(f + g) = \varphi_0(f) + \varphi_0(g)$.

For $m = 1$, f is of the form χ_{U_z} for some open subset U_z. Since $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ is distributively lattice ordered and $f \geq 0 \geq \ldots^{(\sup(g)-1} \ldots \geq 0$ is a decreasing sequence, it follows from Lemma 2.2.3 that

$$f + g = \chi_{U_z} + \sum_{j=1}^{n} \chi_{U_{y_j}} = \chi_{U_z \cup U_{y_1}} + \chi_{(U_z \cap U_{y_1}) \cup U_{y_2}} + \ldots + \chi_{U_z \cap (\cap_j U_{y_j})}.$$

Applying φ_0 and the equalities in the proof of Lemma 2.3.2 at the first step, and Lemma 2.2.3 at the second step, we get

$$\begin{aligned} \varphi_0(f+g) &= (z \vee y_1) + ((z \wedge y_1) \vee y_2) + \ldots + (z \wedge y_1 \wedge \ldots \wedge y_n) \\ &= z + (y_1 + \ldots + y_n) = \varphi_0(f) + \varphi_0(g), \end{aligned}$$

as desired.

Now fix some finite m and assume that the result has been proven for any $k \leq m-1$. Then, using the induction hypothesis at the second step, and the case $m = 1$ at the third and fourth steps, we have

$$\begin{aligned}\varphi_0(f+g) &= \varphi_0((f-\chi_{U_{z_m}}) + (g+\chi_{U_{z_m}})) = \varphi_0(f-\chi_{U_{z_m}}) + \varphi_0(g+\chi_{U_{z_m}}) \\ &= \varphi_0(f-\chi_{U_{z_m}}) + \varphi_0(g) + \varphi_0(\chi_{U_{z_m}}) = \varphi_0(f-\chi_{U_{z_m}} + \chi_{U_{z_m}}) + \varphi_0(g) \\ &= \varphi_0(f) + \varphi_0(g)\end{aligned}$$

as required.

To see that φ_0 is order preserving and an order embedding, note that $f \leq g$ in $\mathrm{Lsc}(X_S,\overline{\mathbb{N}})_\ll$ if and only if $\{f \geq i\} \subseteq \{g \geq i\}$ for each $i \leq \sup(f) = m$.

Using the same notation as above, (i) in Proposition 2.3.3 shows that

$$U_{z_i} = \{f \geq i\} \subseteq \{g \geq i\} = U_{y_i}$$

if and only if $z_i \leq y_i$.

Further, since the sequences $(z_i)_{i=1}^m, (y_i)_{i=1}^m$ are both decreasing and the order in $\downarrow e$ is topological, we have that $z_i \leq y_i$ for each i if and only if

$$\varphi_0(f) = \sum_{i=1}^m z_i \leq \sum_{i=1}^m y_i \leq \varphi_0(g).$$

Finally, if $f \ll g$, it follows from Proposition 2.2.19 and Lemma 2.3.8 that this holds if and only if $z_i \ll y_i$ for each i. As above, and using Proposition 2.2.18, this is equivalent to

$$\varphi_0(f) = \sum_{i=1}^m z_i \ll \sum_{i=1}^m y_i \leq \varphi_0(g),$$

as desired. □

Theorem 2.4.4. *Let S be an* Lsc*-like* Cu*-semigroup. Then, the morphism φ_0 from $\mathrm{Lsc}(X_S,\overline{\mathbb{N}})_\ll$ to S extends to a* Cu*-isomorphism $\varphi\colon \mathrm{Lsc}(X_S,\overline{\mathbb{N}}) \to S$.*

Proof. We will first check that φ_0 satisfies the properties of Lemma 1.3.5 for $N = \mathrm{Lsc}(X_S,\overline{\mathbb{N}})_\ll$. By Lemma 2.4.3 above, we only need to prove that for every element $f \in \mathrm{Lsc}(X_S,\overline{\mathbb{N}})_\ll$ there exists a $\ll$-increasing sequence $(f_n)_n$ in $\mathrm{Lsc}(X_S,\overline{\mathbb{N}})_\ll$ with $\sup_n \varphi_0(f_n) = \varphi_0(f)$.

Let $f \in \mathrm{Lsc}(X_S,\overline{\mathbb{N}})_\ll$ with $\sup(f) = m$. Since $\mathrm{Lsc}(X_S,\overline{\mathbb{N}})$ is a Cu-semigroup, we can find $\ll$-increasing sequences of indicator functions $(h_{i,n})_n$ with $\sup_n h_{i,n} = \chi_{\{f\geq i\}}$.

It is readily checked that suprema of indicator functions is preserved by φ_0. Thus, consider the elements $f_n := \sum_{i=1}^m h_{i,n}$ for each n, which form a $\ll$-increasing sequence with supremum f. We have

$$\sup_n \varphi_0(f_n) = \sum_{i=1}^m \sup_n \varphi_0(h_{i,n}) = \sum_{i=1}^m \varphi_0(\chi_{\{f\geq i\}}) = \varphi_0(f),$$

which shows that the desired property is satisfied.

Using Lemma 1.3.5, φ_0 can be extended to a Cu-morphism $\varphi\colon \mathrm{Lsc}(X_S,\overline{\mathbb{N}}) \to S$.

Further, note that for every $z \leq e$ there exists f such that $\varphi_0(f) = e$, which implies that the image of φ is dense in S. Thus, it follows from Lemma 2.4.1 and Lemma 2.4.3 that φ is an isomorphism, as required. □

Theorem 2.4.5. *Let S be a* Cu*-semigroup. Then, S is* Lsc*-like if and only if S is* Cu*-isomorphic to* $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ *for a T_1 topological space X.*

Proof. Combine Example 2.2.6 and Theorem 2.4.4 above. □

Recall that a functor $T\colon \mathcal{A} \to \mathcal{B}$ is said to be *essentially surjective* if every object b of $\mathcal{B}$ is isomorphic to Ta for some a of $\mathcal{A}$. We also say that T is *full on isomorphisms* if for every pair of isomorphic objects b_1, b_2 of $\mathcal{B}$ there exist a pair of isomorphic objects a_1, a_2 of $\mathcal{A}$ such that Ta_1 is isomorphic to b_1 and Ta_2 is isomorphic to b_2.

Let Top be the category of topological spaces. We denote by $\mathcal{T}_1^{\mathrm{Cu}}$ the subcategory of Top whose objects X are the T_1 spaces such that $\mathrm{Lsc}(X, \overline{\mathbb{N}}) \in \mathrm{Cu}$. Note that, by Corollary 2.2.21, this includes all compact, metric spaces.

Theorem 2.4.6. *Let* Lsc *be the subcategory of* Cu *consisting of* Lsc*-like* Cu*-semigroups. Then, there exists a faithful and essentially surjective contravariant functor $T\colon \mathcal{T}_1^{\mathrm{Cu}} \to$* Lsc *that is full on isomorphisms.*

Proof. Given a topological space $X \in \mathcal{T}_1^{\mathrm{Cu}}$ we define $T(X) = \mathrm{Lsc}(X, \overline{\mathbb{N}})$, where note that $T(X) \in \mathrm{Lsc}$ by construction.

For any continuous map $f\colon X \to Y$, set $T(f)\colon \mathrm{Lsc}(Y, \overline{\mathbb{N}}) \to \mathrm{Lsc}(X, \overline{\mathbb{N}})$ as the unique Cu-morphism such that $T(f)(\chi_U) = \chi_{f^{-1}(U)}$ for every open subset U of Y.

Given $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{T}_1^{\mathrm{Cu}}$, we have

$$T(g \circ f)(\chi_U) = \chi_{(g\circ f)^{-1}(U)} = \chi_{f^{-1}g^{-1}(U)} = (T(f) \circ T(g))(\chi_U),$$

which shows that T is a contravariant functor. It is readily checked that T is faithful by construction.

Further, given any Lsc-like Cu-semigroup S, we know from Theorem 2.4.4 that $S \cong \mathrm{Lsc}(X_S, \overline{\mathbb{N}})$. Thus, T is essentially surjective.

Now let S and T be a pair of isomorphic Lsc-like Cu-semigroups. By Theorem 2.4.4, there exists a Cu-isomorphism of the form $\phi\colon \mathrm{Lsc}(X_S, \overline{\mathbb{N}}) \to \mathrm{Lsc}(X_T, \overline{\mathbb{N}})$.

Since the constant map 1 is a minimal compact element in both $\mathrm{Lsc}(X_S, \overline{\mathbb{N}})$ and $\mathrm{Lsc}(X_T, \overline{\mathbb{N}})$, it follows that $\phi(1) = 1$. This implies that every element below 1 maps to an element below 1. That is to say, indicator functions map to indicator functions. Moreover, using that ϕ is a Cu-isomorphism, maximal elements below 1 must map to maximal elements below 1. Thus, as in Example 2.2.6, for every $x \in X_S$ there exists $y \in X_T$ such that $\phi(\chi_{X_S\setminus\{x\}}) = \chi_{X_T\setminus\{y\}}$.

Define the map $f\colon X_T \to X_S$ as $y \mapsto x$. Since ϕ is a Cu-isomorphism, we have that f is bijective.

To prove that it is also continuous, let U be an open subset of X_S and take $V \subseteq X_T$ such that $\phi(\chi_U) = \chi_V$. For any $y \in X_T$, we have $y \in V$ if and only if

$$1 \leq \chi_{X_T\setminus\{y\}} + \chi_V = \phi(\chi_{X_S\setminus\{f(y)\}} + \chi_U).$$

Using once again that ϕ is a Cu-isomorphism, this holds if and only if

$$(X_S \setminus \{f(y)\}) \cup U = X_S$$

or, equivalently, if $f(y) \in U$. This implies $f^{-1}(U) = V$ and, therefore, that f is continuous.

Finally, let $V \subseteq X_T$ be an open subset and take $U \subseteq X_S$ such that $\chi_V = \phi(\chi_U)$. By the argument above, we have $V = f^{-1}(U)$ and, since f is bijective, it follows that $f(V) = f(f^{-1}(U)) = U$. This shows that f is open.

Thus, f is a homeomorphism between X_T and X_S, as desired. □

Let $n \in \mathbb{N}$. To finish this section, we continue the study of Proposition 2.3.7 and translate the property of having covering dimension at most n to the language of Cu-semigroups; see Theorem 2.4.8. This result is one of the driving reasons for defining a notion of covering dimension for Cu-semigroups; see Chapter 4.

We recall the following characterization of topological covering dimension.

Proposition 2.4.7 (cf [56, Proposition 1.5])**.** *Let $n \in \mathbb{N}$. A metric space X satisfies $\dim(X) \leq n$ if and only if every open cover $X = U_1 \cup \ldots \cup U_r$ admits an $(n+1)$-colorable, finite, open refinement. In other words, there exist open subsets $V_{j,k}$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ such that*

(i) $V_{j,k} \Subset U_j$ *for each j and k;*

(ii) $X = \bigcup_{j,k} V_{j,k}$*;*

(iii) $V_{j,k} \cap V_{j',k} = \emptyset$ *for every $j \neq j'$ and k.*

Theorem 2.4.8. *Let X be a compact, metric space, and let $n \in \overline{\mathbb{N}}$. Then,* $\dim(X) \leq n$ *if and only if, whenever $f' \ll f \ll g_1 + \ldots + g_r$ in* $\mathrm{Lsc}(X,\overline{\mathbb{N}})$*, there exist $h_{j,k}$ with $j = 1, \ldots, r$ and $k = 0, \ldots, n$ such that*

(i) $h_{j,k} \ll g_j$ *for each j and k;*

(ii) $f' \ll \sum_{j,k} h_{j,k}$*;*

(iii) $\sum_{j=1}^{r} h_{j,k} \ll f$ *for each $k = 0, \ldots, n$.*

Proof. First, let n be such that the stated property for $\mathrm{Lsc}(X,\overline{\mathbb{N}})$ is satisfied, and take an open cover $U_1, \ldots, U_r$ of X. Then, we have

$$\chi_X \ll \chi_X \ll \chi_{U_1} + \ldots + \chi_{U_r}.$$

Thus, we can find elements $h_{j,k} \ll \chi_{U_j}$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ satisfying (i)-(iii). In particular, (i) implies that each $h_{j,k}$ is of the form $\chi_{V_{j,k}}$ for some open subset $V_{j,k} \Subset U_j$.

It follows from (ii) that $\{V_{j,k}\}_{j,k}$ is a cover for X, and (iii) implies that the $V_{j,k}$'s are pairwise disjoint on j. Thus, Proposition 2.4.7 above shows that $\dim(X) \leq n$.

Conversely, let $n \geq \dim(X)$, which we may assume to be finite since otherwise there is nothing to prove. Take $f' \ll f \ll g_1 + \ldots + g_r$ in $\mathrm{Lsc}(X,\overline{\mathbb{N}})$, and let f'' and $g_j' \ll g_j$ for each j be such that $f' \ll f'' \ll f \ll g_1' + \ldots + g_r'$.

We may assume that $f', f'', g_1', \ldots, g_r'$ are all bounded by some $m \in \mathbb{N}$, and that there exists $\varepsilon > 0$ such that

$$N_\varepsilon(\{f'' \geq i\}) \Subset \{f'' \geq i-1\}, \{f \geq i\} \quad \text{and} \quad N_\varepsilon(\{g_j' \geq i\}) \subseteq \{g_j \geq i\}$$

for each i, where $N_\varepsilon()$ denotes the open ε-neighbourhood of a set. This can be assumed because such maps are dense in $\mathrm{Lsc}(X,\overline{\mathbb{N}})$.

For every $i \leq m$, set

$$Y_i := N_\varepsilon(\{f'' \geq i\} \setminus N_\varepsilon(\{f'' \geq i+1\})),$$

which satisfy $x \notin Y_i$ whenever $f''(x) > i$.

Moreover, for every $N \in \mathbb{N}^r$, we denote the sum $\sum_j N_j$ by $|N|$ and set

$$U_N := Y_{|N|} \cap N_\varepsilon(\cap_j \{g'_j \geq N_j\}),$$

where note that there are finitely many nonempty U_N's, and that $U_N \subseteq \{g_j \geq N_j\}$ for each j.

One has

$$Y_i = Y_i \cap N_\varepsilon(\{f'' \geq i\}) \subseteq Y_i \cap N_\varepsilon(\cup_{|N|=i} \cap_j \{g'_j \geq N_j\}) = \cup_{|N|=i} U_N \subseteq Y_i$$

and, consequently,

$$N_\varepsilon(\{f'' \geq 1\}) = \cup_i Y_i = \cup_N U_N.$$

By Proposition 2.4.7, we can find open subsets $V_N^{(0)}, \ldots, V_N^{(n)} \Subset U_N$ such that $\{V_N^{(k)}\}_{k,N}$ is an open cover for $N_\varepsilon(\{f'' \geq 1\})$, and such that the $V_N^{(k)}$'s are pairwise disjoint on N. Since there are finitely many nonempty U_N's, we also have finitely many nonempty $V_N^{(k)}$'s.

For each j and k, we set

$$h_{j,k} := \sum_{N \in \mathbb{N}^r} N_j \chi_{V_N^{(k)}},$$

where note that such sums are finite because there are only finitely many nonempty U_N's.

Since $V_N^{(k)} \Subset U_N \subseteq \{g_j \geq N_j\}$, we have $N_j \chi_{V_N^{(k)}} \ll g_j$ and, since the $V_N^{(k)}$'s are pairwise disjoint on N, we obtain $h_{j,k} \ll g_j$ for each j,k.

Further, we have

$$\sum_j h_{j,k} = \sum_N \left(\sum_j N_j \right) \chi_{V_N^{(k)}} = \sum_{i=0}^m i \chi_{\cup_{|N|=i} V_N^{(k)}}$$

and, using that $\cup_{|N|=i} V_N^{(k)} \subseteq \cup_{|N|=i} U_N = Y_i \Subset \{f \geq i\}$, one gets

$$\sum_j h_{j,k} \ll f$$

for each k.

Finally, we note that for every $x \in N_\varepsilon(\{f'' \geq 1\}) \setminus \cup_{i'<i} Y_{i'}$ there exist k, N such that $|N| \geq i$ and $x \in V_N^{(k)}$. Thus, one has

$$\{f' \geq i\} \Subset \{f'' \geq i\} \subseteq N_\varepsilon(\{f'' \geq 1\}) \setminus \cup_{i'<i} Y_{i'} \subseteq \cup_k \cup_{|N| \geq i} V_N^{(k)}$$

and, consequently,

$$f' \ll \sum_{i=0}^m i \chi_{\cup_k \cup_{|N|=i} V_N^{(k)}} \leq \sum_k \sum_{i=0}^m i \chi_{\cup_{|N|=i} V_N^{(k)}} = \sum_{j,k} h_{j,k},$$

as desired. □

2.5 Chain conditions: The Cuntz semigroup of commutative AI-algebras

In this section, we give an abstract characterization for the Cuntz semigroup of a unital commutative block stable AI-algebra (in the sense of Definition 2.1.5; see Theorem 2.5.4) and, more generally, a unital commutative AI-algebra (Theorem 2.5.12) by using the notions of *piecewise chainable* and *weakly chainable* Cu-*semigroups*; see Definitions 2.5.1 and 2.5.5 respectively.

We also show that the Cuntz semigroup of any AI-algebra is weakly chainable, thus uncovering a new propery that the Cuntz semigroup of any AI-algebra satisfies; see Corollary 2.5.10.

Definition 2.5.1. Let S be an inf-semilattice ordered Cu-semigroup, and let $x \in S$. We say that x is *chainable* if, whenever $x \leq y_1 + \ldots + y_n$, there exist $z_1, \ldots, z_m \in S$ such that

(i) For every $i \leq m$, we have $z_i \leq y_k$ for some $k \leq n$;

(ii) $z_i \wedge z_j \neq 0$ if and only if $|i - j| \leq 1$;

(iii) $x \leq z_1 + \ldots + z_m$.

The Cu-semigroup S is said to be *chainable* if it has a chainable order unit.

We say that S is *piecewise chainable* if there exist chainable elements $s_1, \ldots, s_n$ with $s_i \wedge s_j = 0$ whenever $i \neq j$ such that $s_1 + \ldots + s_n$ is an order unit.

Recall the definition of chainability from Definition 2.1.3.

Lemma 2.5.2. *Let S be an* Lsc-*like* Cu-*semigroup S with least order unit e, and let $y \leq e$. Then, U_y is chainable if and only if y is chainable.*

In particular, S is chainable if and only if X_S is chainable.

Proof. First, assume that y is chainable and let $U_{y_1}, \ldots, U_{y_n}$ be a finite open cover of U_y, which implies that

$$y = y_1 \vee \ldots \vee y_n \leq y_1 + \ldots + y_n.$$

Applying that y is chainable, we obtain $z_1, \ldots, z_m$ satisfying (i)-(iii) in Definition 2.5.1 above. In particular, since each z_i is bounded by some y_k, we have $z_1 \vee \ldots \vee z_m \leq y$. By Corollary 2.2.10 and (iii) in Definition 2.5.1, we also get $y \leq z_1 \vee \ldots \vee z_m$ and, consequently, $z_1 \vee \ldots \vee z_m = y$. Therefore, $U_{z_1}, \ldots, U_{z_m}$ is a cover for U_y.

Using (the proof of) Lemma 2.3.2, one can check that $U_{z_1}, \ldots, U_{z_m}$ form a chain in the sense of Definition 2.1.3, as desired.

Conversely, if U_y is chainable, let $y_1, \ldots, y_n \in S$ be such that $y \leq y_1 + \ldots + y_n$. By Corollary 2.2.10, this implies

$$(y_1 \wedge y) \vee \ldots \vee (y_n \wedge y) = y.$$

Thus, we have $U_{y_1 \wedge y} \cup \ldots \cup U_{y_n \wedge y} = U_y$. By the chainability of U_y, we obtain a chain refining this cover. Using Lemma 2.3.2, it is easy to check that the elements below e corresponding to the open subsets of the chain satisfy conditions (i)-(iii) in Definition 2.5.1.

Note, in particular, that this shows that e is chainable whenever X_S is chainable. By definition, this implies that S is chainable.

If, conversely, S is chainable, let s be a chainable order unit. Given $y_1, \ldots, y_n \in S$ with $e \leq y_1 + \ldots + y_n$, we know from Corollary 2.2.10 that $e \leq y_1 \vee \ldots \vee y_n$. This implies that

$$s \leq \infty s = \infty = \infty e \leq \infty y_1 \vee \ldots \vee \infty y_n \leq \infty y_1 + \ldots + \infty y_n.$$

Since s is chainable, we get $z_1, \ldots, z_m \in S$ satisfying (i)-(iii) in Definition 2.5.1. Using that for every i there exists k with $z_i \leq \infty y_k$, it follows from Lemma 2.2.11 that

$$z_i \wedge e \leq (\infty y_k) \wedge e = y_k \wedge e \leq y_k.$$

Further, since e is the least order unit in S and $e \leq s \leq z_1 + \ldots + z_m$, we know by Corollary 2.2.10 that $e \leq z_1 \vee \ldots \vee z_m$. Thus, Corollary 2.2.10 implies that $e \leq z_1 \wedge e + \ldots + z_m \wedge e$.

Therefore, the elements $z_i \wedge e$ satisfy conditions (i)-(iii) in Definition 2.5.1 for $e \leq y_1 \vee \ldots \vee y_n$ and, consequently, e is a chainable order unit. By the first part of the lemma, $U_e = X_S$ is chainable, as required. □

Lemma 2.5.3. *Let S be a countably based* Lsc*-like* Cu*-semigroup with a compact order unit. Then, S is piecewise chainable if and only if X_S is a finite disjoint union of chainable subsets.*

Proof. If X_S is piecewise chainable, there exist $n \in \mathbb{N}$ and chainable subsets $X_1, \ldots, X_n$ of X_S such that $X_S = X_1 \sqcup \ldots \sqcup X_n$. Since chainability implies connectedness (whenever the space is compact), there is a finite number of connected components, and so these are clopen.

It follows from Lemma 2.5.2 that these disjoint chainable components correspond to chainable elements in S with null pairwise infima. Thus, S is piecewise chainable by definition.

Conversely, if S is piecewise chainable, each element s_i in the definition of chainable corresponds to a chainable open subset of X_S, which is disjoint from the other chainable open subsets by construction. □

Theorem 2.5.4. *Let S be a* Cu*-semigroup. Then, S is* Cu*-isomorphic to the Cuntz semigroup of a unital commutative block-stable AI-algebra if and only if S is countably based,* Lsc*-like, piecewise chainable, has a compact order unit, and satisfies (O5).*

Proof. Let S be Cu-isomorphic to the Cuntz semigroup of a unital commutative block-stable AI-algebra. Then, it follows from [74, Theorem 1.1], Proposition 2.1.4 and Definition 2.1.5 that $S \cong \oplus_{k=1}^{n} \mathrm{Lsc}(X_k, \overline{\mathbb{N}}) \cong \mathrm{Lsc}(\sqcup_k X_k, \overline{\mathbb{N}})$ with X_k a chainable continuum for each k. In particular, S satisfies (O5), has a compact order unit, is countably based and Lsc-like. Using Lemma 2.5.3, it also follows that S is piecewise chainable.

Conversely, assume that S is a Cu-semigroup satisfying all the conditions in the statement. Thus, $S \cong \mathrm{Lsc}(X, \overline{\mathbb{N}})$ with X a compactum by Theorem 2.4.5 and (v) in Proposition 2.3.7.

It follows from Example 2.3.4 and Lemma 2.5.3 that X is piecewise chainable. This implies that its dimension is less than or equal to one and, consequently, $S \cong \mathrm{Lsc}(X, \overline{\mathbb{N}}) \cong \mathrm{Cu}(C(X))$ by [74, Theorem 1.1].

This shows that S is isomorphic to the Cuntz semigroup of a unital commutative block-stable AI-algebra, as required. □

We now introduce weakly chainable Cu-semigroups and show that every AI-algebra has such a Cuntz semigroup; see Corollary 2.5.10. We also prove in Proposition 2.5.6 that, under certain mild conditions, an Lsc-like Cu-semigroup S is weakly chainable if and only if X_S is almost chainable.

Recall that, for every pair of elements x, y in a Cu-semigroup, we write $x \propto y$ if $x \leq ny$ for some $n \in \mathbb{N}$.

Definition 2.5.5. A Cu-semigroup S will be said to be *weakly chainable*, or that it satisfies the *weak chainability condition* if, whenever $x' \ll x \ll y_1 + \ldots + y_n$, there exist $z, z_1, \ldots, z_m \in S$ such that $x' \propto z \leq x$ and

(i) For every $i \leq m$, we have $z_i \leq y_k$ for some $k \leq n$;

(ii) $z_i + z_j \leq z$ whenever $|i - j| \geq 2$;

(iii) $z \leq z_1 + \ldots + z_m$.

Proposition 2.5.6. *Let S be an* Lsc*-like* Cu*-semigroup with a compact order unit. Then, X_S is almost chainable whenever S is weakly chainable.*

Conversely, if S also satisfies (O5), S is weakly chainable whenever X_S is almost chainable.

Proof. Assume first that S is weakly chainable, and let $U_{y_1}, \ldots, U_{y_n}$ be a cover of X_S. Then, the elements $y_1, \ldots, y_n \leq e$ satisfy

$$e \leq y_1 \vee \ldots \vee y_n \leq y_1 + \ldots + y_n.$$

We know from (ii) in Proposition 2.3.7 that e is compact. Thus, set $x' = x = e$ and apply Definition 2.5.5 to obtain elements $z, z_1, \ldots, z_m$ satisfying (i)-(iii) in the definition. In particular, since $z \leq e \propto z$, it follows that z is an order unit bounded by e. Using that e is the least order unit of S, this implies $z = e$.

Let $U_{z_1}, \ldots, U_{z_m}$ be the open subsets of X_S corresponding to $z_1, \ldots, z_m$ respectively. Using (i)-(iii) in Definition 2.5.5, it is readily checked that these sets cover $U_z = U_e = X_S$ and form an almost chain refining our original cover. Thus, X_S is almost chainable.

Conversely, assume that X_S is almost chainable and let $x' \ll x \ll y_1 + \ldots + y_n$ in S. Set $z = x' \wedge e$. Then, Corollary 2.2.10 implies that $z \ll y_1 \vee \ldots \vee y_n$. Thus, one gets $z \ll (y_1 \wedge e) \vee \ldots \vee (y_n \wedge e)$, and note that z also satisfies $x' \propto z \leq x$.

Since e is compact and S satisfies (O5), it follows from $z \ll y_1 \vee \ldots \vee y_n$, Lemma 2.3.2 and Lemma 2.3.5 that

$$\overline{U_z} \subseteq U_{y_1} \cup \ldots \cup U_{y_n}.$$

In particular, the open sets $X \setminus \overline{U_z}, U_{y_1}, \ldots, U_{y_n}$ form a cover of X_S and, since X_S is almost chainable, we can find an almost chain $C_1, \ldots, C_m$ refining the cover. For each

$i \leq m$, let $z_i \leq e$ be such that $U_{z_i} = C_i \cap U_z$. By construction, $U_{z_1}, \ldots, U_{z_m}$ is a cover for U_z and, consequently, we have $z \leq z_1 + \ldots + z_m$.

Note that $\{U_{z_i}\}_i$ is a family of open subsets of U_z, which implies that $z_i + z_j \leq z$ if and only if $z_i \wedge z_j = 0$. Thus, since $\{U_{z_i}\}_i$ is also an almost chain, it follows that $z_i + z_j \leq z$ whenever $|i - j| \geq 2$.

This shows that $z, z_1, \ldots, z_m$ satisfy conditions (i)-(iii) in Definition 2.5.5, as required. □

Corollary 2.5.7. *Let X be a compact metric space. Then,* $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ *is weakly chainable if and only if X is almost chainable.*

If, additionally, X is also connected, $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ *is weakly chainable if and only if X is chainable.*

Proof. By Corollary 2.2.21, we have that $S := \mathrm{Lsc}(X, \overline{\mathbb{N}})$ is a Cu-semigroup and, since X is a T_1 space, it follows from Example 2.2.6 and Example 2.3.4 that S is Lsc-like and $X \cong X_S$. Moreover, $1 \ll 1$ in S by (ii) in Proposition 2.3.7.

If S is weakly chainable, Proposition 2.5.6 above implies that $X \cong X_S$ is almost chainable. Conversely, if $X \cong X_S$ is almost chainable, we know that $\dim(X_S) \leq 1$. Thus, it follows from [74, Theorem 1.1] that $\mathrm{Cu}(C(X)) \cong S$. In particular, this implies that S satisfies (O5).

Using Proposition 2.5.6 once again, we get that S is weakly chainable, as desired.

The second part of the statement follows trivially. □

Lemma 2.5.8. *Let S, T be weakly chainable* Cu*-semigroups. Then, $S \oplus T$ is weakly chainable.*

Proof. Let $x' \ll x \ll y_1 + \ldots + y_n$ in $S \oplus T$, and write

$$x' = (x'_1, x'_2), \quad x = (x_1, x_2) \quad \text{and} \quad y_j = (y_{j,1}, y_{j,2})$$

for each $j \leq n$, where $x'_1, x_1, y_{j,1} \in S$ and $x'_2, x_2, y_{j,2} \in T$.

Using that S and T are both weakly chainable, we find elements $z_1, z_{1,1}, \ldots, z_{m,1} \in S$ and $z_2, z_{1,2}, \ldots, z_{m',2} \in T$ satisfying (i)-(iii) in Definition 2.5.5 for the first and second components of x', x and y_j respectively. Set $z := (z_1, z_2)$ and let z_i be $(z_{i,1}, 0)$ for $i \leq m$ and $(0, z_{i-m+1,2})$ whenever $i > m$. Then, we have

$$\begin{aligned} z = (z_1, 0) + (0, z_2) &\leq ((z_{1,1}, 0) + \ldots + (z_{m,1}, 0)) + ((0, z_{1,2}) + \ldots + (0, z_{m',2})) \\ &= z_1 + \ldots + z_{m+m'-1}, \end{aligned}$$

which is condition (iii).

Also note that, for each z_i, we get $z_i \leq y_k$ for some k by construction.

Finally, let z_i, z_j with $|i - j| \geq 2$. If $i, j \leq m$, we have $z_i + z_j \leq (z_1, 0) \leq z$. Similarly, $z_i + z_j \leq (0, z_2) \leq z$ whenever $i, j > m$. If $i \leq m < j$, one gets

$$z_i + z_j = (z_{i,1}, 0) + (0, z_{j-m+1,2}) \leq (z_1, 0) + (0, z_2) = z.$$

This shows that $z_1, \ldots, z_{m+m'-1}$ satisfy conditions (i)-(iii) and, consequently, that S is weakly chainable, as required. □

In the proof of Proposition 2.5.9 below, we will use the characterization of inductive limits from Paragraph 1.2.10.

Proposition 2.5.9. *Let $S = \lim_{\lambda\in\Lambda} S_\lambda$ be an inductive limit of* Cu*-semigroups. Assume that S_λ is weakly chainable for each $\lambda \in \Lambda$. Then, S is also weakly chainable.*

Proof. Given $x' \ll x \ll y_1 + \ldots + y_n$ in S, it follows from Paragraph 1.2.10 that there exist $\lambda \in \Lambda$ and elements $u', u, v_1, \ldots, v_n \in S_\lambda$ with $\varphi_\lambda(v_j) \ll y_j$, $u' \ll u \ll v_1 + \ldots + v_n$ and

$$x' \ll \varphi_\lambda(u') \ll \varphi_\lambda(u) \ll x \ll \varphi_\lambda(v_1) + \ldots + \varphi_\lambda(v_n),$$

where φ_λ is the canonical map $S_\lambda \to S$.

Using that S_λ is weakly chainable, we find $z, z_1, \ldots, z_m \in S_\lambda$ satisfying (i)-(iii) in Definition 2.5.5. Thus, we have

(i) $u' \propto z \leq u$. This implies $x' \ll \varphi_\lambda(u') \propto \varphi_\lambda(z) \leq \varphi_\lambda(u) \ll x$;

(ii) For any i there exists j such that $z_i \leq v_j$, which shows that $\varphi_\lambda(z_i) \leq \varphi_\lambda(v_j) \leq y_j$;

(iii) $z_i + z_j \leq z$ whenever $|i-j| \geq 2$. Consequently, $\varphi_\lambda(z_i) + \varphi_\lambda(z_j) \leq \varphi_\lambda(z)$ whenever $|i-j| \geq 2$.

Using that $z \leq z_1 + \ldots + z_m$, one has $\varphi_\lambda(z) \leq \varphi_\lambda(z_1) + \ldots + \varphi_\lambda(z_m)$. Thus, S is weakly chainable, as required. □

Corollary 2.5.10. *Let A be an AI-algebra. Then,* Cu(A) *is weakly chainable.*

Example 2.5.11. The Cu-semigroups $\mathrm{Lsc}(\mathbb{T}, \overline{\mathbb{N}})$ and $\mathrm{Lsc}([0,1]^2, \overline{\mathbb{N}})$ do not satisfy the weak chainability condition. Indeed, $\mathbb{T}$ and $[0,1]^2$ are not chainable continua, so it follows from Corollary 2.5.7 that the semigroups cannot be weakly chainable.

Using an analogous proof to that of Theorem 2.5.4, we obtain the following result.

Theorem 2.5.12. *Let S be a* Cu*-semigroup. Then, S is* Cu*-isomorphic to the Cuntz semigroup of a unital commutative AI-algebra if and only if S is countably based,* Lsc*-like, weakly chainable, has a compact order unit, and satisfies (O5).*

2.6 New properties of the Cuntz semigroup of an AI-algebra

Following the results obtained in the previous section, we now introduce Cu-semigroups with *refinable* and *almost ordered sums*; see Definitions 2.6.1 and 2.6.5. As in Corollary 2.5.10 and Example 2.5.11, we show that all AI-algebras have Cuntz semigroups satisfying such properties, but that there are well known Cu-semigroups that do not satisfy them; see Example 2.6.3 and Theorem 2.6.9.

Definition 2.6.1. A Cu-semigroup S is said to have *refinable sums* if, for every finite $\ll$-increasing sequence $x_1 \ll \ldots \ll x_n$ and elements $x'_1, \ldots, x'_n$ such that $x_i \propto x'_i$ for each i, there exist sequences $(y^i_j)_{j=1}^m$ such that

(i) $y_m^i \leq \ldots \leq y_1^i \leq x'_{i+1}$ for every i;

(ii) $y_j^i \ll y_j^{i+1}$ for each i and j;

(iii) $x_i \ll y_1^i + \ldots + y_l^i \ll x_{i+1}$ for every i.

Example 2.6.2. Let S be an Lsc-like Cu-semigroup. Then, S has refinable sums.

Indeed, take x_i, x'_i as in Definition 2.6.1, and choose y_i such that

$$x_1 \ll y_1 \ll x_2 \ll y_2 \ll x_3 \ll \ldots \ll x_n.$$

Let e denote the least order unit of S. Using Lemma 2.2.9, we see that each y_i can be written as a finite ordered sum of elements below e. Further, by possibly adding zeros, we may assume that all sums have the same number of summands. Thus, we get

$$x_1 \ll y_1 = y_1^1 + \ldots + y_m^1 \ll x_2 \ll y_2 = y_1^2 + \ldots + y_m^2 \ll x_3 \ll \ldots \ll x_n,$$

where each sum is decreasingly ordered.

By Proposition 2.2.18, we have that $y_j^i \ll y_j^{i+1}$ for each i, j. Additionally, using that $y_i \ll x_{i+1} \propto x'_{i+1}$, the topological order in $\downarrow e$ implies that $y_1^i \leq x'_{i+1} \wedge 1 \leq x'_{i+1}$, as desired.

Example 2.6.3. Recall the definition of Z, the Cuntz semigroup of the Jiang-Su algebra, from Examples 1.2.8 (iv). We show that Z does not have refinable sums:

Set $x_1 = x'_1 = x_2 = x'_2 = 1$, $x_3 = 1.1$ and $x'_3 = 0.5$, which satisfy $x_1 \ll x_2 \ll x_3$ and $x_i \propto x'_i$ for each $i = 1, 2, 3$. Then, assuming for the sake of contradiction that Z has refinable sums, we obtain sequences $(y_j^i)_{j=1}^m$ satisfying (i)-(iii) in Definition 2.6.1. In particular, we have

$$1 \ll y_1^1 + \ldots + y_l^1 \ll 1 \ll y_1^2 + \ldots + y_l^2 \ll 1.1,$$

which shows that $y_1^1 = 1$ and $y_i^1 = 0$ whenever $i \geq 2$.

However, we know by (ii) that $1 = y_1^1 \ll y_1^2 \leq 0.5$, which is a contradiction, as required.

Proposition 2.6.4. *Let $S = \lim_{\lambda \in \Lambda} S_\lambda$ be an inductive limit of* Cu*-semigroups S_λ that have refinable sums. Then, S has refinable sums.*

Proof. Let $x_1, x'_1, \ldots, x_n, x'_n \in S$ be as in Definition 2.6.1. By Paragraph 1.2.10, there exist $\lambda \in \Lambda$ and elements $u_j, u'_j \in S_\lambda$ for $j \leq 2n-2$ such that $u_1 \ll \ldots \ll u_{2n-2}$ and $u_j \propto u'_j$ in S_λ, and

$$x_i \ll \varphi_\lambda(u_{2i-1}) \ll \varphi_\lambda(u_{2i}) \ll x_{i+1} \quad \text{and} \quad \varphi_\lambda(u'_{2i}) \leq x'_{i+1} \quad \text{in } S$$

for each $i \leq n-1$.

Using that S_λ has refinable sums, we obtain sequences $(y_j^i)_{j=1}^m$ for $i \leq 2n-2$ satisfying (i)-(iii) in Definition 2.6.1. Thus, we have

$$x_i \ll \varphi_\lambda(u_{2i-1}) \ll \varphi_\lambda(y_1^{2i-1}) + \ldots + \varphi_\lambda(y_1^{2i-1}) \ll \varphi_\lambda(u_{2i}) \ll x_{i+1},$$

and $\varphi_\lambda(y_1^{2i-1}) \leq \varphi_\lambda(u'_{i+1}) \leq x'_{i+1}$.

This shows that S has refinable sums, as desired. ☐

Definition 2.6.5. A Cu-semigroup S has *almost ordered sums* if, for any finite number of elements $x_1, \ldots, x_n$ in S, there exist sequences $(y_{i,j})_i$ with $j = 1, \ldots, n$ such that

(i) $y_{i,n} \leq \ldots \leq y_{i,1}$ for each i;

(ii) $(y_{i,1} + \ldots + y_{i,n})_i$ is increasing and $\sup_i (y_{i,1} + \ldots + y_{i,n}) = x_1 + \ldots + x_n$;

(iii) $(y_{i,n})_i$ is increasing and bounded by $x_1, \ldots, x_n$;

(iv) Given $x', x \in S$ and $J \subseteq \{1, \ldots, n\}$ such that $x' \ll x_j \leq x$ for each $j \in J$, there exists i_0 such that $x' \leq y_{i,|J|}$ and $y_{i,n+1-|J|} \leq x$ whenever $i_0 \leq i$.

Example 2.6.6. Let S be a distributively lattice ordered Cu-semigroup (for example, an Lsc-like Cu-semigroup). Then, S has almost ordered sums.

Indeed, given $x_1, \ldots, x_n \in S$, set

$$\begin{aligned} y_{i,1} &= x_1 \vee \ldots \vee x_n, \\ y_{i,2} &= (x_1 \wedge x_2) \vee \ldots \vee (x_{n-1} \wedge x_n), \\ &\ \vdots \\ y_{i,n} &= x_1 \wedge \ldots \wedge x_n, \end{aligned}$$

for each i.

Using Lemma 2.2.3, we have

$$x_1 + \ldots + x_n = y_{i,1} + \ldots + y_{i,n},$$

and it is now easy to check that S has almost ordered sums.

Example 2.6.7. Let $Z' = Z \sqcup \{1''\}$ with $1''$ a compact element not comparable with 1 such that $1 + x = 1'' + x$ for every $x \in Z \setminus \{0\}$ and $n1'' = n$ for each $n \in \mathbb{N}$; see [6, Chapter 9 (8)]. Then, Z' does not have almost ordered sums.

Indeed, set $x_1 = 1$, $x_2 = 1''$ and assume, for the sake of contradiction, that Z' has refinable sums. Thus, we find sequences $(y_{i,j})_i$ for $j = 1, 2$ with $1 + 1'' = \sup_i (y_{i,1} + y_{i,2})$.

Using that $1 + 1'' = 2$ is compact, we have $1 + 1'' = y_{i,1} + y_{i,2}$ for every sufficiently large i. Consequently, for every large enough i, we have $y_{i,1} = 2$ and $y_{i,2} = 0$, since $1, 1'' \leq y_{i,1}$ and $1, 1''$ are not comparable.

However, since $1, 1'' \leq 1.5$, (iv) in Definition 2.6.5 implies that $2 = y_{1,i} \leq 1.5$, a contradiction.

Proposition 2.6.8. *Let $S = \lim_{\lambda \in \Lambda} S_\lambda$ be an inductive limit of distributively lattice ordered* Cu-*semigroups. Then, S has almost ordered sums.*

Proof. Let $x_1, \ldots, x_n \in S$. As in Paragraph 1.2.10, denote by $\varphi_{\mu,\lambda} \colon S_\lambda \to S_\mu$ and $\varphi_\lambda \colon S_\lambda \to S$ the Cu-morphisms of the limit. Then, there exist an increasing sequence of integers $(\lambda_i)_i$ and elements $x_{i,j} \in S_{\lambda_i}$ for $j = 1, \ldots, n$ such that $\varphi_{\lambda_{i+1},\lambda_i}(x_{i,j}) \ll x_{i+1,j}$ for each i, j and $\sup \varphi_{\lambda_i}(x_{i,j}) = x_j$ for each j.

By Example 2.6.6 applied to $x_{i,1}, \ldots, x_{i,n} \in S_{\lambda_i}$, there exist $y_{i,1}, \ldots, y_{i,n} \in S_{\lambda_i}$ such that $x_{i,1} + \ldots + x_{i,n} = y_{i,1} + \ldots + y_{i,n}$ and such that the properties in Definition 2.6.5 are satisfied.

We claim that $\varphi_{\lambda_i}(y_{i,j})$ satisfy (i)-(iv) in Definition 2.6.5 for $x_1,\dots,x_n$. Note, in particular, that we already have $\sup_i(\varphi_{\lambda_i}(y_{i,1})+\dots+\varphi_{\lambda_i}(y_{i,n}))=x_1+\dots+x_n$.

Moreover, since $y_{i,n}\leq\dots\leq y_{i,1}$ for each i, condition (i) is also satisfied.

Now take $i\in\mathbb{N}$ and note that $y_{i,n}\leq x_{i,j}$ for each j. Since $\varphi_{\lambda_i}(x_{i,j})\leq x_j$ for each i and j, we have $\varphi_{\lambda_i}(y_{i,n})\leq x_1,\dots,x_n$. In fact, one gets $\varphi_{\lambda_{i+1},\lambda_i}(y_{i,n})\ll x_{i+1,j}$ for every $j\leq n$.

Thus, it follows from (iv) in Definition 2.6.5 applied to $S_{\lambda_{i+1}}$ that $\varphi_{\lambda_{i+1},\lambda_i}(y_{i,n})\leq y_{i+1,n}$. This shows that (iii) is satisfied.

Finally, to prove (iv), let $x',x\in S$ and $J\subseteq\{1,\dots,n\}$ be such that $x'\ll x_j\leq x$ for each $j\in J$. Then, there exist $i_0\in\mathbb{N}$ and $u\in S_{\lambda_{i_0}}$ such that $u\ll x_{i_0,j}$ for each $j\in J$ and $x'\ll\varphi_{\lambda_{i_0}}(u)$ in S. By (iv) in $S_{\lambda_{i_0}}$, we get $u\leq y_{i_0,|J|}$ and, consequently, $x'\leq\varphi_{\lambda_{i_0}}(u)\leq\varphi_\mu(y_{i,|J|})$ for each $\mu\geq\lambda_{i_0}$.

Similarly, since $x_j\leq x$ for each $j\in J$, for every large enough i we can find some element $v_i\in S_{\lambda_i}$ such that $x_{i,j}\leq v_i$ for every $j\in J$. Using the same argument as above, it follows that condition (iv) is also satisfied, which shows that S has almost ordered sums, as desired. □

Theorem 2.6.9. *Let A be an AI-algebra. Then, its Cuntz semigroup* $\mathrm{Cu}(A)$ *is weakly chainable, has refinable sums, and almost ordered sums.*

Proof. The Cuntz semigroup $\mathrm{Cu}(A)$ is weakly chainable by Corollary 2.5.10.

In analogy with Lemma 2.5.8, one can check that finite direct sums of Cu-semigroups having refinable or almost ordered sums have refinable or almost ordered sums respectively.

Thus, since Lsc-like Cu-semigroups have refinable sums by Example 2.6.2, it follows from Proposition 2.6.4 that every inductive limit of finite direct sums of such Cu-semigroups will also have refinable sums. In particular, this applies to $\mathrm{Cu}(A)$.

Similarly, we know from Example 2.6.6 that every Lsc-like Cu-semigroup has almost ordered sums. Using Proposition 2.6.8, every inductive limit of finite direct sums of Lsc-like semigroups also has them. Consequently, $\mathrm{Cu}(A)$ has almost ordered sums. □

Chapter 3

A local characterization for the Cuntz semigroup of AI-algebras

This chapter is devoted to the study of the Cuntz semigroup of (separable) AI-algebras. More explicitly, we provide a local characterization for such semigroups resembling Shen's characterization for the ordered K_0-group of AF-algebras, which we recall below:

Theorem (1.3.3)**.** *A countable unperforated ordered abelian group G is order isomorphic to the ordered K_0-group of an AF-algebra if, and only if, for every ordered homomorphism $\varphi\colon \mathbb{Z}^r \to G$ and any element $\alpha \in \ker(\varphi)$, there exist $s \geq 0$, and ordered homomorphisms θ, ϕ such that the diagram*

$$\begin{tikzcd} \mathbb{Z}^r \arrow[r, "\varphi"] \arrow[d, "\theta"'] & G \\ \mathbb{Z}^s \arrow[ur, "\phi"'] & \end{tikzcd}$$

commutes and $\alpha \in \ker(\theta)$.

In order to obtain our result, in Sections 3.2 and 3.3 we generalize to the setting of Cu-semigroups the key ingredients used in the proof of the abovementioned characterization.

For example, and as already witnessed by Proposition 1.3.6, we will see that the right 'substitute' for the kernel in Theorem 1.3.3 is similar to the notion of kernel introduced in [23]. Further, we also note that, since countably based Cu-semigroups are not necessarily countable, one will rarely obtain commutative diagrams.

This lack of commutativity will ultimately be bipased in Section 3.4 by considering a metric version of Shen's theorem; see Theorem 3.4.5 and Proposition 3.4.6. Instead of equalities, we will ask the morphisms to be metrically close.

In Section 3.5, we introduce property I. This notion, together with a discrete version of Theorem 3.4.5 (see Theorem 3.4.8), allows us to provide an abstract characterization for the Cuntz semigroup of AI-algebras; see Theorem 3.5.34.

The results in this chapter can be found in [102], but the presentation we offer here has been changed substantially.

3.1 Preliminaries and notation

In this first section we introduce and prove results about the Cu-semigroup $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ that will be used throughout the chapter. We also prove Theorem 3.1.6, which reduces the range problem to the study of inductive limits of the form $\lim_i \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^{n_i}$.

Definition 3.1.1. Let f be an element in $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$. We say that f is a *basic indicator function* if f is of the form

$$\chi_{(s,1]}, \quad \chi_{(s,t)}, \quad \chi_{[0,t)} \quad \text{or} \quad 1$$

for some s,t.

We will also say that f is a *basic* element if f can be written as a finite sum of basic indicator functions.

Given $n\in\mathbb{N}$, an element in $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})^n$ will be said to be a *basic indicator function* (resp. *basic*) if it is so componentwise.

Remark 3.1.2. It is readily checked that the set of basic elements (together with 0) forms a basis for $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$. Moreover, this monoid can be characterized as follows:

Let F denote the free abelian semigroup generated by the basic indicator functions of $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ as symbols. In F, we write $f\sim_0 g$ if

$$f = h+\chi_U+\chi_V \quad \text{and} \quad g = h+\chi_{U\cup V}+\chi_{U\cap V}$$

for some $h,\chi_U,\chi_V\in F$.

Given $f,g\in F$, we also write $f\sim g$ if $f=g$, $f\sim_0 g$ or $g\sim_0 f$. Let $\simeq$ be the transitive relation induced by $\sim$. Then, $F/{\simeq}$ is isomorphic to the monoid of basic elements in $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$.

Indeed, let $\varphi\colon F\to\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ be the additive extension of the map that sends $\chi_U\in F$ to $\chi_U\in\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ for every basic indicator function χ_U. Since every basic element in $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ is the finite sum of basic indicator functions, it follows that the image of φ is the monoid of basic elements.

Using that $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ is a distributive lattice ordered semigroup (see Definition 2.2.1), we know that $\varphi(f)=\varphi(g)$ whenever $f\sim_0 g$. Consequently, $\varphi(f)=\varphi(g)$ whenever $f\simeq g$. This proves that the map $[f]\mapsto\varphi(f)$ from $F/\simeq$ to the monoid of basic elements of $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ is an isomorphism, as desired.

Note that a basic element is increasing if and only if it can be written as a finite sum of elements of the form $\chi_{(\cdot,1]}$ and 1.

Lemma 3.1.3. *Let $n\in\mathbb{N}$ and let $f,g\in\mathrm{Lsc}([0,1],\overline{\mathbb{N}})^n$ be basic elements such that $f\ll g$. Then, there exist basic increasing elements h,d such that $f+h\ll d\ll g+h$.*

Proof. We first note that it suffices to prove the result for $n=1$. Thus, let $f\ll g$ be basic elements in $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$.

If g is a basic indicator function with associated interval $V\subseteq[0,1]$, the result is clear. Indeed, since $f\ll g$, it follows that $f=\chi_U$ with $U\Subset V$. Then, let W be the (possibly empty) interval $(\sup(U),1]$, where note that $\sup(U)\in V$. Thus, one has

$$U\cap W=\emptyset \quad \text{and} \quad U\cup W\Subset V\cup W.$$

This implies that $f + \chi_W \ll \chi_{V\cup W} \leq \chi_V + \chi_W$. Setting $h = \chi_W$ and $d \ll \chi_{V\cup W}$ such that $f + g \ll d$, the result follows.

Now let $f \ll g$ be basic elements, and write $f = \sum_i \chi_{U_i}$, $g = \sum_i \chi_{V_i}$ such that $U_i \Subset V_i$ with χ_{V_i} a basic indicator. Using the result above, we obtain for each i increasing, basic indicators h_i, d_i such that

$$\chi_{U_i} + h_i \ll d_i \ll \chi_{V_i} + h_i.$$

Let $d = \sum_i d_i$ and $h = \sum_i h_i$, where note that both h and d are basic increasing elements in $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$. It follows by construction that these elements satisfy the required conditions. $\square$

3.1.4 (Retractions)**.** Let $s, t \in [0,1]$ be such that $0 \leq s < t \leq 1$. Given $\varepsilon > 0$, we define the *ε-retraction* of $\emptyset$, $[0,t)$, (s,t), $(s,1]$ and $[0,1]$ as

$$\emptyset, \quad [0, t-\varepsilon), \quad (s+\varepsilon, t-\varepsilon), \quad (s+\varepsilon, 1] \quad \text{and} \quad [0,1]$$

respectively.

If U is a finite disjoint union of intervals, we define its ε-retraction, denoted by $R_\varepsilon(U)$, to be the disjoint union of the ε-retracted intervals. The *ε-retraction* of the associated indicator function χ_U is then defined as $R_\varepsilon(\chi_U) = \chi_{R_\varepsilon(U)}$, where we set $\chi_\emptyset = 0$ by definition.

Similarly, given any basic element $f \in \mathrm{Lsc}([0,1], \overline{\mathbb{N}})$, we know that it can be written as $f = \sum_{i=1}^n \chi_{\{f\geq i\}}$ with $\{f \geq i\}$ a finite disjoint union of intervals. We define the ε-retraction of f as $R_\varepsilon(f) = \sum_{i=1}^n R_\varepsilon(\chi_{\{f\geq i\}})$.

Theorem 3.1.6 below will allow us to reduce the range problem to the study of inductive limits of the form $\lim_i \mathrm{Lsc}([0,1], \overline{\mathbb{N}}^{n_i})$. This result is no doubt well known, but we have not been able to find an explicit proof in the literature. We provide one for the convenience of the reader:

For every finite tuple of positive integers $N = (n_j)_{j\leq k}$, we denote by

$$\xi_N\colon \mathrm{Cu}(\oplus_{j=1}^k C[0,1] \otimes M_{n_j}) \to \oplus_{j=1}^k \mathrm{Cu}(C[0,1] \otimes M_{n_j}).$$

the isomorphism given by Paragraph 1.2.9.

We also let α_N be the Cu-isomorphism from $\oplus_{j=1}^k \mathrm{Cu}(C[0,1] \otimes M_{n_j})$ to $\mathrm{Lsc}([0,1], \overline{\mathbb{N}}^k)$ defined in [5, Corollary 2.7]. That is to say,

$$\alpha_N([(a_1, \ldots, a_k)])(t) = (\mathrm{rk}(a_1(t)), \ldots, \mathrm{rk}(a_k(t)))$$

for every $t \in [0,1]$.

Proposition 3.1.5. *Let $N = (n_j)_{j\leq k}$ and $M = (m_i)_{i\leq s}$ be finite tuples of positive integers. Assume that $\varphi\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}}^k) \to \mathrm{Lsc}([0,1], \overline{\mathbb{N}}^s)$ is a* Cu*-morphism satisfying $\varphi(N) \leq M$. Then, there exists a *-homomorphism*

$$\phi\colon \oplus_{j\leq k} C[0,1] \otimes M_{n_j} \to \oplus_{i\leq s} C[0,1] \otimes M_{m_i}$$

such that $\mathrm{Cu}(\phi) = \xi_M^{-1}\alpha_M^{-1}\varphi\alpha_N\xi_N$.

Proof. We may assume without loss of generality that $\varphi \neq 0$, since it is otherwise trivial to find a lift.

For every finite tuple of positive integers $S = (s_j)$, we will denote by F_S the AF-algebra $\oplus_j M_{s_j}$.

Fix $1 \leq j \leq k$, and let $\iota_j \colon C[0,1] \otimes M_{n_j} \to C[0,1] \otimes F_N$ denote the inclusion to the j-th component. We define the *-homomorphism $\tau_j \colon C[0,1] \to C[0,1] \otimes F_N$ as

$$\tau_j(f) = \iota_j(f \oplus 0_{n_j - 1}),$$

which satisfies $\xi_N \operatorname{Cu}(\tau_j)(x) = (0, \ldots, 0, x, 0 \ldots, 0)$ for every element $x \in \operatorname{Cu}(C[0,1])$. Moreover, one has

$$n_j \varphi(1_j) \leq \varphi(N) \leq M.$$

Set $S = \lfloor M/n_j \rfloor$, and let φ_S be the composition $\xi_S^{-1} \alpha_S^{-1} \varphi \alpha_N \xi_N$. We have

$$\begin{aligned} \varphi_S \operatorname{Cu}(\tau_j)([1]) &= \xi_S^{-1} \alpha_S^{-1} \varphi(1_j) \leq \xi_S^{-1} \alpha_S^{-1}(\lfloor M/n_j \rfloor) \\ &= \xi_S^{-1}([1_{\lfloor m_1/n_j \rfloor}], \ldots, [1_{\lfloor m_s/n_j \rfloor}]) \\ &= [(1_{\lfloor m_1/n_j \rfloor}, \ldots, 1_{\lfloor m_s/n_j \rfloor})]. \end{aligned}$$

Thus, it follows from [21, Theorem 4.1] that there exists an *-homomorphism ψ_j from $C[0,1]$ to $C[0,1] \otimes F_S$ such that $\operatorname{Cu}(\psi_j) = \varphi_S \operatorname{Cu}(\tau_j)$.

Let $\iota \colon C[0,1] \otimes F_{n_j S} \to C[0,1] \otimes F_M$ be the canonical inclusion, and let ϕ_j be the *-homomorphism $\iota \circ (\psi_j \otimes 1_{n_j}) \colon C[0,1] \otimes M_{n_j} \to C[0,1] \otimes F_M$. Then, one has

$$\operatorname{Cu}(\phi_j) = \left(\xi_M^{-1} \alpha_M^{-1} \varphi \alpha_N \xi_N \right) \operatorname{Cu}(\iota_j).$$

Since the previous argument can be done for each j, we obtain *-homomorphisms ϕ_j such that

$$\begin{aligned} \sum_{j=1}^{k} [\phi_j(1_{n_j})] &= \sum_{j=1}^{k} \left(\xi_M^{-1} \alpha_M^{-1} \varphi \alpha_N \xi_N \right) \operatorname{Cu}(\iota_j)[1_{n_j}] = \sum_{j=1}^{k} \left(\xi_M^{-1} \alpha_M^{-1} \varphi \alpha_N \xi_N \right) [1_j] \\ &= \xi_M^{-1} \alpha_M^{-1} \varphi(N) \leq \xi_M^{-1} \alpha_M^{-1}(M) = [(1_{m_1}, \ldots, 1_{m_s})]. \end{aligned}$$

We claim that there exists a family of *-homomorphisms $\phi_j^{\perp} \colon C[0,1] \otimes M_{n_j} \to C[0,1] \otimes F_M$ with pairwise orthogonal ranges such that $\operatorname{Cu}(\phi_j^{\perp}) = \operatorname{Cu}(\phi_j)$. Indeed, we follow the argument from [72, Theorem 3.2.2 (iv)]:

Identifying $C[0,1] \otimes F_M$ with the upper-left corner of $M_k(C[0,1] \otimes F_M)$ and using that both $\phi_1(1_{n_1}) \oplus \ldots \oplus \phi_k(1_{n_k})$ and $(1_{m_1}, \ldots, 1_{m_s})$ are projections, Rørdam's Lemma (see Lemma 1.2.2) implies that we can find $x \in M_k(C[0,1] \otimes F_M)$ such that

$$x^* x = \phi_1(1_{n_1}) \oplus \ldots \oplus \phi_k(1_{n_k}) \quad \text{and} \quad x x^* \in C[0,1] \otimes F_M.$$

Given the polar decomposition $x = v|x|$ of x, we define the *-homomorphisms $\phi_1^{\perp}, \ldots, \phi_k^{\perp}$ as

$$\phi_1^{\perp} = v(\phi_1 \otimes e_{1,1}) v^*, \quad \phi_2^{\perp} = v(\phi_2 \otimes e_{2,2}) v^*, \quad \ldots \quad \text{and} \quad \phi_k^{\perp} = v(\phi_k \otimes e_{k,k}) v^*,$$

where $e_{i,j}$ denotes the matrix whose only nonzero entry is 1 in the position (i,j).

To see that these maps are pairwise orthogonal, take $i \neq j$ and note that

$$\begin{aligned}\phi_i^{\perp}\phi_j^{\perp} &= v(\phi_i \otimes e_{i,i})|x|v^*v|x|(\phi_j \otimes e_{j,j})v^* \\ &= v(\phi_i \otimes e_{i,i})x^*x(\phi_j \otimes e_{j,j})v^* = v(\phi_i \otimes e_{i,i})(\phi_j \otimes e_{j,j})v^* = 0.\end{aligned}$$

Moreover, we have $\mathrm{Cu}(\phi_j^{\perp}) = \mathrm{Cu}(\phi_j)$ for every j. Thus, given the canonical projections $\pi_j\colon C[0,1] \otimes F_N \to C[0,1] \otimes M_{n_j}$ for $j = 1, \ldots, k$, we can consider the map

$$\phi := \phi_1^{\perp}\pi_1 + \ldots + \phi_k^{\perp}\pi_k,$$

which is seen to satisfy $\mathrm{Cu}(\phi) = \xi_M^{-1}\alpha_M^{-1}\varphi\alpha_N\xi_N$. □

Theorem 3.1.6. *The Cuntz semigroup of an AI-algebra is* Cu*-isomorphic to the inductive limit of a system* $(\mathrm{Lsc}([0,1], \overline{\mathbb{N}}^{n_i}), \varphi_i)$*.*

Conversely, for every inductive system $(\mathrm{Lsc}([0,1], \overline{\mathbb{N}}^{n_i}), \varphi_i)$ *there exists an AI-algebra such that its Cuntz semigroup is* Cu*-isomorphic to the limit of the system.*

Proof. The first statement follows from the fact that Cu is a continuous functor [26, Section 2] and that $\mathrm{Cu}(C[0,1]) \cong \mathrm{Lsc}([0,1], \overline{\mathbb{N}})$; see [5, Corollary 2.7].

Now let $(\mathrm{Lsc}([0,1], \overline{\mathbb{N}}^{n_i}), \varphi_i)$ be an inductive system in Cu with limit S, and choose inductively $N_i \in \mathbb{N}^{n_i}$ such that N_i has no zero components and $\varphi_i(N_i) \leq N_{i+1}$.

Using Proposition 3.1.5, for every i let $\phi_{N_i}\colon C[0,1] \otimes F_{N_i} \to C[0,1] \otimes F_{N_{i+1}}$ be a *-homomorphism such that $\mathrm{Cu}(\phi_{N_i}) = \xi_{N_{i+1}}^{-1}\alpha_{N_{i+1}}^{-1}\varphi_i\alpha_{N_i}\xi_{N_i}$.

Thus, we get the following commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r, "\mathrm{Cu}(\phi_{N_{i-1}})"] & \mathrm{Cu}(C[0,1] \otimes F_{N_i}) \arrow[r, "\mathrm{Cu}(\phi_{N_i})"] \arrow[d, "\alpha_{N_i}\xi_{N_i}"'] & \mathrm{Cu}(C[0,1] \otimes F_{N_{i+1}}) \arrow[r, "\mathrm{Cu}(\phi_{N_{i+1}})"] & \ldots \\
\ldots \arrow[r, "\varphi_{i-1}"] & \mathrm{Lsc}([0,1], \overline{\mathbb{N}}^{n_i}) \arrow[r, "\varphi_i"] & \mathrm{Lsc}([0,1], \overline{\mathbb{N}}^{n_{i+1}}) \arrow[r, "\varphi_{i+1}"] \arrow[u, "\xi_{N_{i+1}}^{-1}\alpha_{N_{i+1}}^{-1}"'] & \ldots
\end{tikzcd}$$

where recall that the columns are isomorphisms.

Using once again that Cu is continuous, this implies that

$$S = \lim \mathrm{Lsc}([0,1], \overline{\mathbb{N}}^{n_i}) \cong \lim_i \mathrm{Cu}(C[0,1] \otimes F_{N_i}) \cong \mathrm{Cu}(\lim_i C[0,1] \otimes F_{N_i}),$$

as desired. □

3.2 Extending morphisms

Let S be a Cu-semigroup and let $s_1 \ll \ldots \ll s_n \leq p \ll p$ be a finite increasing sequence in S. We will prove in this section that, under certain assumptions on the Cu-semigroup S, there always exists a Cu-morphism $\phi\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}}) \to S$ such that $\phi(1) = p$ and $\phi\left(\chi_{\left(\frac{n-k}{n},1\right]}\right) = s_k$ for each k; see Theorem 3.2.6.

This is in analogy to the fact that, given any element g in an abelian, ordered group G, there always exists a morphism $\mathbb{Z} \to G$ mapping 1 to g. Despite this being trivial, it is used repeatedly in the proof of Shen's theorem ([87, Theorem 3.1]) and, as we will see later, the Cu-version proven in this section (Theorem 3.2.6) will be one of the key ingredients in the proof of our local characterization; see the proof of Theorem 3.4.5.

3.2.1. As shown in [86, Section 5.2], the sub-Cu-semigroup $\mathcal{G}$ of $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ defined as

$$\mathcal{G} = \{f \in \mathrm{Lsc}([0,1],\overline{\mathbb{N}}) \mid f(0) = 0,\ f \text{ increasing}\}$$

is a generator for the category Cu. That is to say, for every pair of distinct Cu-morphisms $\varphi, \phi\colon S \to T$, there exists a Cu-morphism $\theta\colon \mathcal{G} \to S$ such that $\varphi\theta \neq \phi\theta$.

In particular, Proposition 3.2.2 below follows as a combination of [86, Lemma 5.16] and [7, Proposition 2.10].

Proposition 3.2.2. *Let $s_1 \ll \ldots \ll s_n$ be a finite $\ll$-increasing sequence in a* Cu*-semigroup S. Then, there exists a* Cu*-morphism $\phi\colon \mathcal{G} \to S$ such that $\phi\left(\chi_{(\frac{n-k}{n},1]}\right) = s_k$ for each k.*

Proof. Using [7, Proposition 2.10], we obtain a family $(t_\lambda)_{\lambda\in(0,1]}$ in S with $t_{k/n} = s_k$ for each k such that $t_{\lambda'} \ll t_\lambda$ whenever $\lambda' < \lambda$ and $\sup_{\lambda'<\lambda} t_{\lambda'} = t_\lambda$.

The assignment $\lambda \mapsto t_\lambda$ is a continuous path (in the sense of [86, Definition 5.16]). By [86, Lemma 5.18], there exists a (unique) Cu-morphism $\phi\colon \mathcal{G} \to S$ such that $\phi\left(\chi_{(\lambda,1]}\right) = t_{1-\lambda}$ for every λ. This map satisfies the required condition. □

Thus, the desired result will follow if we show that certain Cu-morphisms $\mathcal{G} \to S$ can be extended to Cu-morphisms of the form $\mathrm{Lsc}([0,1],\overline{\mathbb{N}}) \to S$.

Remarks 3.2.3. The following remarks show that a Cu-morphism $\phi\colon \mathcal{G} \to S$ may not always lift to $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ and that, even if it does, the lift may not be unique.

1. Let $S = \mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ and take $\infty \in S$. Then, by Proposition 3.2.2 (applied to only one element), there exists a Cu-morphism $\phi\colon \mathcal{G} \to S$ such that $\phi(\chi_{(0,1]}) = \infty$.

   Note that ϕ cannot be lifted to a Cu-morphism $\phi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}}) \to S$. Indeed, if such a lift existed we would have

   $$\infty = \phi(\chi_{(0,1]}) \ll \phi(1) = \infty,$$

   a contradiction.

2. Consider the inclusion $\iota\colon \mathcal{G} \to \mathrm{Lsc}([0,1],\overline{\mathbb{N}})$. For every $n \in \mathbb{N}$, define the Cu-morphism $\iota_n\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}}) \to \mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ as

   $$\iota_n(\chi_{(s,1]}) = \chi_{(s,1]}, \quad \iota_n(\chi_{[0,t)}) = (n-1) + \chi_{[0,t)}, \quad \iota_n(\chi_{(s,t)}) = \chi_{(s,t)},$$

   and $\iota_n(1) = n$.

   It is readily checked that ι_n is a Cu-morphism for each n. This shows that ι_n is a lift for ι for every n and, in particular, that ι does not lift uniquely.

Lemma 3.2.4. *Let S be a weakly cancellative* Cu*-semigroup satisfying (O5), and let $\phi\colon \{\chi_{(t,1]}, \chi_{[0,t)}, 1\}_{t\in[0,1]} \to S$ be an order and suprema preserving map. Assume that, for every $t \leq s < t'$, one has*

$$\phi(\chi_{[0,t)}) + \phi(\chi_{(s,1]}) \leq \phi(1) \ll \phi(1) \leq \phi(\chi_{[0,t')}) + \phi(\chi_{(s,1]}).$$

Then, there is a unique Cu*-morphism $\mathrm{Lsc}([0,1],\overline{\mathbb{N}}) \to S$ extending ϕ. Conversely, the restriction of any* Cu*-morphism $\mathrm{Lsc}([0,1],\overline{\mathbb{N}}) \to S$ to $\{\chi_{(t,1]}, \chi_{[0,t)}, 1\}$ satisfies the displayed condition.*

Proof. Necessity is clear, so we are left to prove sufficency. Let B_I and B denote the subsets of basic indicator functions and basic functions in $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ respectively, as defined in Definition 3.1.1.

First, let $s<t$, and note that $\phi(1) \ll \phi(\chi_{[0,t)})+\phi(\chi_{(s,1]})$. Since S satisfies (O5) and $\phi(1)$ is compact, there exists an element $x\in S$ such that

$$\phi(1)+x=\phi(\chi_{[0,t)})+\phi(\chi_{(s,1]}).$$

Moreover, the element x is unique by weak cancellation; see Paragraph 1.2.14. We extend ϕ to a map $\phi\colon B_I\to S$ by sending $\chi_{(s,t)}$ to x. Therefore we get $\phi(1)+\phi(\chi_{(s,t)}) = \phi(\chi_{[0,t)})+\phi(\chi_{(s,1]})$.

For every $n\in\mathbb{N}$ set $\varepsilon_n=1/n$. Note that for a large enough n we have $s+\varepsilon_n<t-\varepsilon_n$. Thus, one gets

$$\begin{aligned}\phi(1)+\phi\left(R_{\varepsilon_{n+1}}(\chi_{(s,t)})\right) &= \phi\left(R_{\varepsilon_{n+1}}(\chi_{[0,t)})\right)+\phi\left(R_{\varepsilon_{n+1}}(\chi_{(s,1]})\right)\\ &\leq \phi\left(R_{\varepsilon_{n}}(\chi_{[0,t)})\right)+\phi\left(R_{\varepsilon_{n}}(\chi_{(s,1]})\right)=\phi(1)+\phi\left(R_{\varepsilon_{n}}(\chi_{(s,t)})\right),\end{aligned}$$

where recall that $R(\cdot)$ denotes the retraction of the function; see Paragraph 3.1.4.

It follows by weak cancellation that the sequence $\left(\phi\left(R_{\varepsilon_n}(\chi_{(s,t)})\right)\right)_n$ is increasing. Further, since

$$\begin{aligned}\phi(1)+\sup_n\phi\left(R_{\varepsilon_n}(\chi_{(s,t)})\right) &= \sup_n\left(\phi\left(R_{\varepsilon_n}(\chi_{[0,t)})\right)+\phi\left(R_{\varepsilon_n}(\chi_{(s,1]})\right)\right)\\ &= \phi(1)+\phi(\chi_{(s,t)}),\end{aligned}$$

we have, by another usage of weak cancellation, that $\sup_n\left(\phi\left(R_{\varepsilon_n}(\chi_{(s,t)})\right)\right)_n=\phi(\chi_{(s,t)})$.

Claim 1. *Let $\chi_U,\chi_V\in B_I$. Then, $\phi(\chi_U)+\phi(\chi_V)=\phi(\chi_{U\cup V})+\phi(\chi_{U\cap V})$.*

We prove the claim for U and V of the form (s,t) and (s',t') respectively with $s\leq s'<t\leq t'$. The other cases are proven similarly.

One has that

$$\begin{aligned}\phi(\chi_U)+\phi(\chi_V)+2\phi(1) &= \phi(\chi_{[0,t)})+\phi(\chi_{(s,1]})+\phi(\chi_{[0,t')})+\phi(\chi_{(s',1]})\\ &= \phi(\chi_{(s,t')})+\phi(\chi_{(s',t)})+2\phi(1),\end{aligned}$$

which implies $\phi(\chi_U)+\phi(\chi_V)=\phi(\chi_{U\cup V})+\phi(\chi_{U\cap V})$ by weak cancellation.

Using Claim 1, it follows from Remark 3.1.2 that the map $\phi\colon B_I\to S$ can be lifted to a monoid morphism $\phi\colon B\to S$. Moreover, since any $\chi_U\in B$ is a finite sum of elements in B_I, we know that the sequence $(R_{\varepsilon_n}(\chi_U))_n$ is increasing and that $\sup_n\phi(R_{\varepsilon_n}(\chi_U))=\phi(\chi_U)$.

Claim 2. *Let $s_0<\ldots<s_n$ be a strictly increasing sequence in $[0,1]$. We have*

$$\begin{aligned}&\phi(\chi_{[0,s_0)})+\phi(\chi_{(s_0,s_1)})+\ldots+\phi(\chi_{(s_{n-1},s_n)})+\phi(\chi_{(s_n,1]})\leq\phi(1)\\ &\quad\leq\phi(\chi_{[0,s_1)})+\phi(\chi_{(s_0,s_2)})+\ldots+\phi(\chi_{(s_{n-2},s_n)})+\phi(\chi_{(s_{n-1},1]}).\end{aligned}$$

It follows from this that, given $\chi_U,\chi_V\in B$, one gets $\phi(\chi_U)\leq\phi(\chi_V)$ whenever $\chi_U\leq\chi_V$ and $\phi(\chi_U)\ll\phi(\chi_V)$ whenever $\chi_U\ll\chi_V$.

For each i, we have $\phi(\chi_{(s_{i-1},s_i)}) + \phi(1) = \phi(\chi_{[0,s_i)}) + \phi(\chi_{(s_{i-1},1]})$. Using this at the first step, and that $\phi(\chi_{[0,s_i)}) + \phi(\chi_{(s_i,1]}) \leq \phi(1)$ at the second step, one gets

$$\begin{aligned}\phi(\chi_{[0,s_0)}) + \phi(\chi_{(s_0,s_1)}) + \ldots + \phi(\chi_{(s_{n-1},s_n)}) + \phi(\chi_{(s_n,1]}) + n\phi(1) \\ = \sum_{i=0}^{n} \phi(\chi_{[0,s_i)}) + \phi(\chi_{(s_i,1]}) \leq (n+1)\phi(1).\end{aligned}$$

Since, for every i, we also have $\phi(\chi_{[0,s_{i+1})}) + \phi(\chi_{(s_{i-1},1]}) \geq \phi(1)$, it follows that

$$\begin{aligned}\phi(\chi_{[0,s_1)}) + \phi(\chi_{(s_0,s_2)}) + \ldots + \phi(\chi_{(s_{n-2},s_n)}) + \phi(\chi_{(s_{n-1},1]}) + (n-1)\phi(1) \\ = \sum_{i=1}^{n} \phi(\chi_{[0,s_{i+1})}) + \phi(\chi_{(s_i-1,1]}) \geq n\phi(1).\end{aligned}$$

Using weak cancellation to cancel $n\phi(1)$ in the first inequality and $(n-1)\phi(1)$ in the second, we get the desired inequalities.

Now assume that $\chi_U, \chi_V \in B$ satisfy $\chi_U \ll \chi_V$. Since χ_U and χ_V are basic elements, we can write them as finite sums of basic indicator functions. That is, U and V can be written as finite disjoint unions of intervals. In particular, the interior of $[0,1] \setminus U$, denoted by $\mathrm{Int}([0,1] \setminus U)$, can also be written as such.

This implies that we can find a sequence $s_0 < \ldots < s_n$ in $[0,1]$ such that

$$\phi(\chi_U) + \phi(\chi_{\mathrm{Int}([0,1]\setminus U)}) = \phi(\chi_{[0,s_0)}) + \phi(\chi_{(s_0,s_1)}) + \ldots + \phi(\chi_{(s_n,1]}).$$

Similarly, since $\chi_U \ll \chi_V$, there also exists a sequence $t_0 < \ldots < t_m$ with

$$\phi(\chi_V) + \phi(\chi_{\mathrm{Int}([0,1]\setminus U)}) = \phi(\chi_{[0,t_1)}) + \phi(\chi_{(t_0,t_2)}) + \ldots + \phi(\chi_{(t_{m-2},t_m)}) + \phi(\chi_{(t_{m-1},1]}).$$

It follows from the first part of the claim that

$$\phi(\chi_U) + \phi(\chi_{\mathrm{Int}([0,1]\setminus U)}) \leq \phi(1) \leq \phi(\chi_V) + \phi(\chi_{\mathrm{Int}([0,1]\setminus U)})$$

and, by weak cancellation, we get $\phi(\chi_U) \ll \phi(\chi_V)$, as desired.

Finally, given any pair $\chi_U \leq \chi_V$ in B, note that $\phi(R_{\varepsilon_n}(\chi_U)) \ll \phi(\chi_V)$ for every n. Taking suprema on n, we get $\phi(\chi_U) \leq \phi(\chi_V)$.

Now, for any $f \in B$, consider the sequence $(\phi(R_{\varepsilon_n}(f)))_n$ where $R_{\varepsilon_n}(f)$ is the ε_n-retraction as defined in Paragraph 3.1.4. Using Claim 2 and the fact that ϕ is additive, it follows that this sequence is $\ll$-increasing with supremum $\phi(f)$.

Since every element in B is a finite sum of indicator functions in B, it follows from Claim 2 that ϕ is order preserving and $\ll$-preserving. Thus, Lemma 1.3.5 implies that the map $\phi\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}}) \to S$ defined as $\phi(g) = \sup_n \phi(g_n)$ for $(g_n)_n$ a $\ll$-increasing sequence in B with supremum g is a Cu-morphism extending ϕ. □

Proposition 3.2.5. *Let S be a weakly cancellative* Cu*-semigroup satisfying (O5), and let $\phi\colon \mathcal{G} \to S$ be a* Cu*-morphism. Assume that $\phi(\chi_{(0,1]}) \leq p$ for some compact element $p \in S$. Then, there exists a unique* Cu*-morphism* $\mathrm{Lsc}([0,1], \overline{\mathbb{N}}) \to S$ *extending ϕ and sending* 1 *to p.*

Proof. Set $\phi(1)=p$ and, for every $n\in\mathbb{N}$, put $\varepsilon_n=1/n$. Then, for every $t\in(0,1]$, one has

$$\phi(\chi_{(t-\varepsilon_{n+1},1]})\ll\phi(\chi_{(t-\varepsilon_n,1]})\ll\phi(1).$$

Since S satisfies (O5), there exists $x_n=x_n(t)$ such that

$$\phi(\chi_{(t-\varepsilon_{n+1},1]})+x_n\leq\phi(1)\leq\phi(\chi_{(t-\varepsilon_n,1]})+x_n.$$

for each n.

In particular, one gets

$$\phi(\chi_{(t-\varepsilon_{n+1},1]})+x_n\leq\phi(1)\ll\phi(1)\leq\phi(\chi_{(t-\varepsilon_{n+1},1]})+x_{n+1}$$

and, by weak cancellation, it follows that $x_n\ll x_{n+1}$ for every n.

Further, if another increasing sequence $(x'_n)_n$ satisfies

$$\phi(\chi_{(t-\varepsilon_{n+1},1]})+x'_n\leq\phi(1)\leq\phi(\chi_{(t-\varepsilon_{n+1},1]})+x'_{n+1}$$

for each n, we can apply weak cancellation to

$$\phi(\chi_{(t-\varepsilon_{n+1},1]})+x'_n\ll\phi(\chi_{(t-\varepsilon_{n+1},1]})+x_{n+1}\quad\text{and}\quad\phi(\chi_{(t-\varepsilon_{n+1},1]})+x_n\ll\phi(\chi_{(t-\varepsilon_{n+1},1]})+x'_{n+1}$$

to deduce that $\sup_n x_n=\sup_n x'_n$. We set $\phi(\chi_{[0,t)})=\sup_n x_n$.

Also note that, since $t>t-\varepsilon_n$ for every n, we have

$$\phi(\chi_{(t,1]})+x_n\leq\phi(\chi_{(t-\varepsilon_{n+1},1]})+x_n\leq\phi(1)$$

for every n.

By taking suprema on n, one gets $\phi(\chi_{[0,t)})+\phi(\chi_{(t,1]})\leq\phi(1)$. Similarly, since for every $\varepsilon>0$ there exists $n\in\mathbb{N}$ with $\varepsilon_n<\varepsilon$, we deduce that $\phi(1)\leq\phi(\chi_{[0,t)})+\phi(\chi_{(t-\varepsilon,1]})$.

Thus, for every $t\leq s<t'$, we have

$$\phi(\chi_{[0,t)})+\phi(\chi_{(s,1]})\leq\phi(1)\ll\phi(1)\leq\phi(\chi_{[0,t')})+\phi(\chi_{(s,1]}).$$

We will now show that the map $\chi_{[0,t)}\mapsto\phi(\chi_{[0,t)})$ is order and suprema preserving, thus proving that $\phi\colon\{\chi_{(t,1]},\chi_{[0,t)},1\}_{t\in[0,1]}\to S$ satisfies the conditions of Lemma 3.2.4. This will give us the desired result.

Let $t<t'$ in $[0,1]$, and let $\varepsilon>0$ be such that $t+\varepsilon=t'-\varepsilon$. Then, we get

$$\phi(\chi_{[0,t)})+\phi(\chi_{(t+\varepsilon,1]})\leq\phi(1)\ll\phi(1)\leq\phi(\chi_{[0,t')})+\phi(\chi_{(t'-\varepsilon,1]})$$

and, using weak cancellation once again, we obtain $\phi(\chi_{[0,t)})\ll\phi(\chi_{[0,t')})$.

Now take $t\in(0,1]$ and consider the $\ll$-increasing sequence $(\chi_{[0,t-\varepsilon_n)})_n$. Then, $(\phi(\chi_{[0,t-\varepsilon_n)}))_n$ is a $\ll$-increasing sequence in S satisfying $\sup_n\phi(\chi_{[0,t-\varepsilon_n)})\leq\phi(\chi_{[0,t)})$.

Moreover, using the same notation as above, we have that

$$\phi(\chi_{(t-\varepsilon_{n+1},1]})+x_n\ll\phi(1)\leq\phi(\chi_{(t-\varepsilon_{n+1},1]})+\phi(\chi_{[0,t-\varepsilon_{n+2})})$$

for every n.

By weak cancellation, one gets $x_n\ll\phi(\chi_{[0,t-\varepsilon_{n+2})})$ and, taking supremum on n, we have $\phi(\chi_{[0,t)})=\sup_n x_n\leq\sup_n\phi(\chi_{[0,t-\varepsilon_n)})$.

This shows that the map $\phi\colon\{\chi_{(t,1]},\chi_{[0,t)},1\}_{t\in[0,1]}\to S$ satisfies conditions (i)-(ii) in Lemma 1.3.5. Consequently, ϕ is order and suprema preserving, as desired. □

Theorem 3.2.6. *Let S be a weakly cancellative* Cu*-semigroup satisfying (O5). Given a finite sequence of the form $s_1 \ll \ldots \ll s_n \leq p \ll p$, there exists a* Cu*-morphism $\phi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}}) \to S$ such that $\phi(1) = p$ and $\phi\left(\chi_{(\frac{n-k}{n},1]}\right) = s_k$ for each k.*

Proof. Take a finite sequence of the form $s_1 \ll \ldots \ll s_n \leq p \ll p$. Then, it follows from Proposition 3.2.2 that there exists a Cu-morphism $\phi\colon \mathcal{G} \to S$ such that

$$\phi\left(\chi_{(\frac{n-k}{n},1]}\right) = s_k$$

for each k.

Note that $\phi(\chi_{(0,1]}) \leq p$. Thus, we know from Proposition 3.2.5 that the Cu-morphism can be lifted to a Cu-morphism $\mathrm{Lsc}([0,1],\overline{\mathbb{N}}) \to S$ sending 1 to p. This lift has the desired properties. □

3.3 Cauchy sequences and their limits

Inspired by [21], [22] and [76], we introduce a notion of distance between Cu-morphisms $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to S$ that agree on their compact elements; see Definition 3.3.1 below. Using this notion, we prove in Theorem 3.3.10 that certain Cauchy sequences of such Cu-morphisms have a unique limit.

Definition 3.3.1. Let S be a Cu-semigroup. Given two Cu-morphisms φ_1, φ_2 from $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ to S with $\varphi_1(1) = \varphi_2(1)$, we define the distance between them as

$$d(\varphi_1,\varphi_2) := \inf\left\{ \varepsilon \in [0,1] \;\middle|\; \begin{array}{l} \varphi_1(\chi_{(t+\varepsilon,1]}) \leq \varphi_2(\chi_{(t,1]}) \\ \varphi_2(\chi_{(t+\varepsilon,1]}) \leq \varphi_1(\chi_{(t,1]}) \end{array} \quad \forall t \in [0,1] \right\}.$$

More generally, if $\varphi_1, \varphi_2\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to S$ are Cu-morphisms with $\varphi_1(1_i) = \varphi_2(1_i)$ for each $i \leq r$, we define

$$d(\varphi_1,\varphi_2) := \sup_{1\leq i\leq r} d(\varphi_1\iota_i, \varphi_2\iota_i),$$

where $\iota_i\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}}) \to \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r$ denotes the canonical inclusion in the i-th component.

Note that the distance between φ_1, φ_2 defined above is precisely the distance between $\varphi_1|_{\mathrm{Lsc}((0,1],\overline{\mathbb{N}})}, \varphi_2|_{\mathrm{Lsc}((0,1],\overline{\mathbb{N}})}$ considered in [21] and [22] (see also [76]).

Remark 3.3.2. Given two Cu-morphisms $\varphi_1, \varphi_2\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}}) \to S$ with $\varphi_1(1) = \varphi_2(1)$ at distance $d(\varphi_1,\varphi_2)$ and two elements $s,t \in [0,1]$ with $s - t > d(\varphi_1,\varphi_2)$, we have

$$\varphi_1(\chi_{(s,1]}) \ll \varphi_2(\chi_{(t,1]}) \quad \text{and} \quad \varphi_2(\chi_{(s,1]}) \ll \varphi_1(\chi_{(t,1]}).$$

Indeed, since $s - t > d(\varphi_1,\varphi_2)$, we can find $\eta > 0$ such that $s - t > \eta > d(\varphi_1,\varphi_2)$. Applying Definition 3.3.1, one gets

$$\varphi_1(\chi_{(s,1]}) \ll \varphi_1(\chi_{(t+\eta,1]}) \leq \varphi_2(\chi_{(t,1]}) \quad \text{and} \quad \varphi_2(\chi_{(s,1]}) \ll \varphi_2(\chi_{(t+\eta,1]}) \leq \varphi_1(\chi_{(t,1]}).$$

As in [21, Theorem 4.1] we now prove that, under the hypothesis of weak cancellation, the distance from Definition 3.3.1 above is a metric.

Lemma 3.3.3. *Let S be a weakly cancellative* Cu*-semigroup satisfying (O5), and let $r \in \mathbb{N}$. Assume that two* Cu*-morphisms $\varphi_1, \varphi_2 \colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^r \to S$ satisfy $d(\varphi_1, \varphi_2) = 0$ and $\varphi_1(1_i) = \varphi_2(1_i)$ for every i. Then, $\varphi_1 = \varphi_2$.*

Proof. We note that it is enough to prove the result for $r = 1$, since the general case then follows from a componentwise application of this case.

Thus, let $\varphi_1, \varphi_2 \colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}}) \to S$ be two Cu-morphisms at distance 0 such that $\varphi_1(1) = \varphi_2(1)$. Since $d(\varphi_1, \varphi_2) = 0$, the morphisms agree on $\chi_{(t,1]}$ for every t. This implies that they agree on $\mathcal{G}$, the sub-Cu-semigroup of $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$ defined in Paragraph 3.2.1.

Let $\phi = \varphi_1|_{\mathcal{G}} = \varphi_2|_{\mathcal{G}}$, and note that both φ_1 and φ_2 are extensions of ϕ sending 1 to $\varphi_1(1) = \varphi_2(1)$.

However, we know by Proposition 3.2.5 that ϕ has a unique extension mapping 1 to $\varphi_1(1)$. This shows $\varphi_1 = \varphi_2$. □

Lemma 3.3.4. *Let S be a* Cu*-semigroup satisfying weak cancellation, and let φ_1, φ_2 be* Cu*-morphisms from $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$ to S with $\varphi_1(1) = \varphi_2(1)$ and $d(\varphi_1, \varphi_2) \leq \varepsilon$. Then, for every basic indicator function $f \in \mathrm{Lsc}([0,1], \overline{\mathbb{N}})$, as defined in Definition 3.1.1, we have*

$$\varphi_1(R_\varepsilon(f)) \leq \varphi_2(f) \quad \text{and} \quad \varphi_2(R_\varepsilon(f)) \leq \varphi_1(f),$$

where $R_\varepsilon(f)$ denotes the ε-retraction of f; see Paragraph 3.1.4.

Proof. We first note that, if $f = 1$ or $f = \chi_{(s,1]}$, the result follows from Definition 3.3.1.

If $f = \chi_{(s,t)}$ for some $s < t$, take $\eta > 0$ and note that

$$R_{\varepsilon+\eta}(f) + \chi_{(t-\varepsilon-\eta,1]} = \chi_{(s+\eta+\varepsilon,t-\eta-\varepsilon)} + \chi_{(t-\eta-\varepsilon,1]} \ll \chi_{(s+\varepsilon,1]} = R_\varepsilon(\chi_{(s,1]}).$$

Using that φ_1, φ_2 are Cu-morphisms at the first and third steps, and $d(\varphi_1, \varphi_2) \leq \varepsilon$ at the second and fourth steps, one gets

$$\begin{aligned} \varphi_1(R_{\varepsilon+\eta}(f)) + \varphi_1(\chi_{(t-\varepsilon-\eta,1]}) &\ll \varphi_1(R_\varepsilon(\chi_{(s,1]}) \leq \varphi_2(\chi_{(s,1]}) \\ &\leq \varphi_2(\chi_{(s,t)}) + \varphi_2(\chi_{(t-\eta,1]}) \leq \varphi_2(f) + \varphi_1(\chi_{(t-\varepsilon-\eta,1]}). \end{aligned}$$

Thus, we get $\varphi_1(R_{\varepsilon+\eta}(f)) \ll \varphi_2(f)$ by weak cancellation. Since this holds for every $\eta > 0$, we obtain $\varphi_1(R_\varepsilon(f)) \leq \varphi_2(f)$ as required.

Finally, if $f = \chi_{[0,t)}$ for some $t \in [0,1]$, let $\eta > 0$. Arguing as above and using $\varphi_1(1) = \varphi_2(1)$ at the third step, we have

$$\begin{aligned} \varphi_1(R_{\varepsilon+\eta}(f)) + \varphi_2(\chi_{(t-\eta,1]}) &\leq \varphi_1(R_{\varepsilon+\eta}(f)) + \varphi_1(\chi_{(t-\varepsilon-\eta,1]}) \ll \varphi_1(1) \\ &= \varphi_2(1) \leq \varphi_2(\chi_{[0,t)}) + \varphi_2(\chi_{(t-\varepsilon,1]}) = \varphi_2(f) + \varphi_2(\chi_{(t-\varepsilon,1]}) \end{aligned}$$

It follows from weak cancellation that $\varphi_1(R_{\varepsilon+\eta}(f)) \ll \varphi_2(f)$ for every $\eta > 0$ and, consequently, we have that $\varphi_1(R_\varepsilon(f)) \leq \varphi_2(f)$.

The inequality $\varphi_2(R_\varepsilon(f)) \leq \varphi_1(f)$ is proved analogously. □

Lemma 3.3.5. *Let S be a weakly cancellative* Cu*-semigroup satisfying (O5) and let $\phi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to S$ be a* Cu*-morphism. Then, for every pair $f' \ll f$ in* $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r$*, there exists $\varepsilon = \varepsilon(f,f') > 0$ and $f'' \in \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r$ such that*

(i) $f' \ll f'' \ll f$;

(ii) For every Cu*-morphism $\varphi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to S$ we have $\phi(f') \ll \varphi(f'') \ll \phi(f)$ whenever $d(\phi,\varphi) < \varepsilon$ and $\phi(1_i) = \varphi(1_i)$ for each i.*

In particular, one gets $\phi(f') \ll \varphi(f)$ and $\varphi(f') \ll \phi(f)$.

Proof. First, note that it is enough to prove the result for $r = 1$, since a componentwise application of this case shows the general result.

Thus, let us assume that f is a basic indicator function in $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$. Then, there exists $\varepsilon = \varepsilon(f,f') > 0$ such that $f' \ll R_{3\varepsilon}(f) \ll f$.

Set $f'' = R_{2\varepsilon}(f)$ and let $\varphi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}}) \to S$ be a Cu-morphism with $d(\phi,\varphi) < \varepsilon$ and $\phi(1) = \varphi(1)$. Applying that ϕ is a Cu-morphism in the first and third step, and Lemma 3.3.4 in the second and last step, we have

$$\phi(f') \ll \phi(R_{3\varepsilon}(f)) \leq \varphi(f'') \ll \varphi(R_{\varepsilon}(f)) \leq \phi(f).$$

Now, given $f' \ll f$ in $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$, let g, g' be basic elements in $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ such that $f' \ll g' \ll g \ll f$; see Definition 3.1.1. Thus, we know that there exist elements h_i', basic indicator functions h_i and $n \in \mathbb{N}$ such that

$$g' = \sum_{i=1}^{n} h_i', \quad g = \sum_{i=1}^{n} h_i \quad \text{and} \quad h_i' \ll h_i \text{ for each } i.$$

It follows from the case above that for each i there exist $\varepsilon_i = \varepsilon_i(h_i, h_i')$ and h_i'' satisfying (i)-(ii). Set $\varepsilon = \varepsilon(f,f') = \min(\varepsilon_i)$ and $f'' = \sum_i h_i''$, where note that $\varepsilon > 0$ because there are finitely many ε_i's.

Given a Cu-morphism $\varphi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}}) \to S$ such that $d(\varphi,\phi) < \varepsilon$ and $\phi(1) = \varphi(1)$, we have $\phi(h_i') \ll \varphi(h_i'') \ll \phi(h_i)$ for each i and, consequently,

$$\phi(f') \ll \phi(g') = \sum_i \phi(h_i') \ll \sum_i \varphi(h_i'') = \varphi(f'') \ll \sum_i \phi(h_i) = \phi(g) \ll \phi(f),$$

as required. □

Corollary 3.3.6. *Let S be a* Cu*-semigroup satisfying (O5) and weak cancellation, and let $f, g, h \in \mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ be such that $f \ll h$. Assume that there exists a* Cu*-morphism $\phi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}}) \to S$ satisfying $\phi(h) \ll \phi(g)$.*

Then, there exists $\varepsilon = \varepsilon(f,g,h,\phi) > 0$ such that, for every $\varphi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}}) \to S$ such that $\phi(1) = \varphi(1)$ and $d(\phi,\varphi) < \varepsilon$, there exists f'' with $f \ll f''$ and $\varphi(f'') \ll \varphi(g)$.

In particular, we have $\varphi(f) \ll \varphi(g)$.

Proof. Let $\varepsilon(h,f) > 0$ and $\varepsilon(g,g') > 0$ be the numbers given by Lemma 3.3.5 applied to $f \ll h$ and $g' \ll g$ respectively.

Set

$$\varepsilon := \varepsilon(f,g,h,\phi) = \min(\varepsilon(h,f)\,,\, \sup\{\varepsilon(g,g') \mid g' \ll g \text{ and } \phi(h) \ll \phi(g')\}),$$

and let f'' be the element given by Lemma 3.3.5 associated with $f \ll h$.

Now take a Cu-morphism $\varphi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}}) \to S$ with $\phi(1) = \varphi(1)$ and $d(\phi,\varphi) < \varepsilon$. Choose $g' \ll g$ with $\phi(h) \ll \phi(g')$ and note that $d(\phi,\varphi) < \varepsilon(h,f), \varepsilon(g,g')$. Let g'' be the element in $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ given by Lemma 3.3.5 applied to $g' \ll g$.

Using Lemma 3.3.5 at the first and third step, we get

$$\varphi(f'') \ll \phi(h) \ll \phi(g') \ll \varphi(g'') \ll \varphi(g)$$

as required. □

The following lemma, which states that our notion of distance can be discretized, will be of importance throughout the chapter.

Lemma 3.3.7. *Let S be a* Cu*-semigroup and let $\varphi_1, \varphi_2\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}}) \to S$ be* Cu*-morphisms satisfying $\varphi_1(1) = \varphi_2(1)$. Assume that there exist $n \in \mathbb{N}$ and $\varepsilon \in (0,1]$ such that*

$$\varphi_1\left(\chi_{(\frac{i}{n}+\varepsilon,1]}\right) \leq \varphi_2\left(\chi_{(\frac{i}{n},1]}\right) \quad \text{and} \quad \varphi_2\left(\chi_{(\frac{i}{n}+\varepsilon,1]}\right) \leq \varphi_1\left(\chi_{(\frac{i}{n},1]}\right)$$

for every $0 \leq i \leq n$.

Then, $d(\varphi_1,\varphi_2) \leq \varepsilon + 1/n$.

Proof. Let $t \in [0,1]$ and take i such that $t \leq \frac{i}{n} \leq t + \frac{1}{n}$.

Using that φ_1, φ_2 are Cu-morphisms at the first and last steps, and our assumption at the second step, we get

$$\varphi_1(\chi_{(t+\varepsilon+\frac{1}{n},1]}) \leq \varphi_1(\chi_{(\frac{i}{n}+\varepsilon,1]}) \leq \varphi_2(\chi_{(\frac{i}{n},1]}) \leq \varphi_2(\chi_{(t,1]}).$$

An analogous argument shows that $\varphi_2(\chi_{(t+\varepsilon+\frac{1}{n},1]}) \leq \varphi_1(\chi_{(t,1]})$. Thus, $d(\varphi_1,\varphi_2) \leq \varepsilon + 1/n$ as desired. □

Proposition 3.3.8. *Let S be a weakly cancellative* Cu*-semigroup satisfying (O5), let $\varepsilon > 0$, and let $\phi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^s$ be a* Cu*-morphism.*

Then, there exists $\varepsilon' > 0$ such that, for any pair of Cu*-morphisms φ_1, φ_2 from $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})^s$ to S with $d(\varphi_1,\varphi_2) < \varepsilon'$ and $\varphi_1(1_i) = \varphi_2(1_i)$ for every i, we have*

$$d(\varphi_1\phi, \varphi_2\phi) < \varepsilon.$$

Proof. Note that it is enough to prove the result for $r = 1$. Thus, let $\varepsilon > 0$ and let $\phi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}}) \to \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^s$.

For each $t \in [0,1]$, consider the pair $\chi_{(t+\varepsilon/2,1]} \ll \chi_{(t,1]}$ in $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$. In particular, we have $\phi(\chi_{(t+\varepsilon/2,1]}) \ll \phi(\chi_{(t,1]})$ in $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})^s$.

By Lemma 3.3.5 applied to $\phi(\chi_{(t+\varepsilon/2,1]}) \ll \phi(\chi_{(t,1]})$, there exists $\varepsilon'_t > 0$ such that, for any pair of Cu-morphisms $\varphi_1, \varphi_2\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^s \to S$ with $d(\varphi_1,\varphi_2) < \varepsilon'_t$ and $\varphi_1(1_i) = \varphi_2(1_i)$ for every i, we have

$$\varphi_1(\phi(\chi_{(t+\varepsilon/2,1]})) \ll \varphi_2(\phi(\chi_{(t,1]})) \quad \text{and} \quad \varphi_2(\phi(\chi_{(t+\varepsilon/2,1]})) \ll \varphi_1(\phi(\chi_{(t,1]})).$$

Now let $n \in \mathbb{N}$ be such that $1/n \leq \varepsilon/2$, and set $t_j = j/n$ for each $j \leq n$. Define $\varepsilon' = \min(\varepsilon'_{t_j}) > 0$.

Then, for any pair of Cu-morphisms $\varphi_1, \varphi_2 \colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^s \to S$ with $d(\varphi_1, \varphi_2) < \varepsilon'$ and $\varphi_1(1_i) = \varphi_2(1_i)$ for every i, we have

$$\varphi_1(\phi(\chi_{(t_j+\varepsilon/2,1]})) \ll \varphi_2(\phi(\chi_{(t_j,1]})) \quad \text{and} \quad \varphi_2(\phi(\chi_{(t_j+\varepsilon/2,1]})) \ll \varphi_1(\phi(\chi_{(t_j,1]})).$$

for each j, since $d(\varphi_1, \varphi_2) < \varepsilon' \leq \varepsilon'_{t_j}$.

Lemma 3.3.7 shows that $d(\varphi_1\phi, \varphi_2\phi) \leq 1/n + \varepsilon/2 \leq \varepsilon$, as desired. □

We are now ready to prove the main result of this section, which states that certain Cauchy sequences of Cu-morphisms $\mathrm{Lsc}([0,1], \overline{\mathbb{N}}) \to S$ have a unique limit. This result will also be generalized to Cu-morphisms $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})^r \to S$; see Theorem 3.3.10.

Proposition 3.3.9. *Let S be a* Cu*-semigroup satisfying (O5) and weak cancellation, and let $\varphi_i \colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}}) \to S$ be* Cu*-morphisms such that $\varphi_i(1) = \varphi_{i+1}(1)$ for every i. Assume that $d(\varphi_i, \varphi_{i+1}) < \varepsilon_i$ with $(\varepsilon_i)_i$ strictly decreasing and $\sum_{i=1}^{\infty} \varepsilon_i < \infty$.*

Then,

(i) The sequence $(\varphi_i)_i$ induces a Cu*-morphism $\varphi \colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}}) \to S$ satisfying $\varphi(1) = \varphi_1(1)$;*

(ii) $d(\varphi, \varphi_i) \to 0$ as i tends to infinity.

Proof. We prove each claim separately:

(i) Let $R = \sum_{i=1}^{\infty} \varepsilon_i < \infty$, and set $R_i = \sum_{k=1}^{i} \varepsilon_k$ for each i. Given $t \in [0,1]$, it follows from Remark 3.3.2 that

$$\varphi_1(\chi_{(t+R,1]}) \ll \varphi_2(\chi_{(t+R-R_1,1]}) \ll \varphi_3(\chi_{(t+R-R_2,1]}) \ll \ldots \ll \varphi_{i+1}(\chi_{(t+R-R_i,1]}) \ll \ldots,$$

since $(t + R - R_i) - (t + R - R_{i+1}) = \varepsilon_i > d(\varphi_i, \varphi_{i+1})$.

Thus, the sequence $(\varphi_{i+1}(\chi_{(t+R-R_i,1]}))_i$ is $\ll$-increasing for each t.

Let $\varphi \colon \{\chi_{(t,1]}\}_t \to S$ be the map defined as $\varphi(\chi_{(t,1]}) := \sup_i \varphi_{i+1}(\chi_{(t+R-R_i,1]})$. We will now see that φ satisfies (i)-(ii) from Lemma 1.3.5.

First, let $\chi_{(s,1]} \ll \chi_{(t,1]}$, which implies that $s - t > 0$. Thus, since $R - R_i \to 0$, there exists some $k \in \mathbb{N}$ such that $s - t > 2(R - R_{k-1})$. For every $i > k$, we get

$$\begin{aligned} d(\varphi_{i+1}, \varphi_k) &\leq d(\varphi_{i+1}, \varphi_i) + \ldots + d(\varphi_{k+1}, \varphi_k) \\ &< \varepsilon_i + \ldots + \varepsilon_k = R_i - R_{k-1} \leq R - R_{k-1}, \end{aligned}$$

which implies that, again by Remark 3.3.2,

$$\varphi_{i+1}(\chi_{(s+R-R_i,1]}) \ll \varphi_k(\chi_{(s+R_{k-1}-R_i,1]}) \leq \varphi_k(\chi_{(t+R-R_{k-1},1]}),$$

where in the first step we have used $(s+R-R_i)-(s+R_{k-1}-R_i) = R-R_{k-1} > d(\varphi_i, \varphi_k)$, and in the second step we have used $s - t \geq 2(R - R_{k-1}) \geq (R - R_{k-1}) + (R_i - R_{k-1})$.

This shows that $\varphi(\chi_{(s,1]}) \leq \varphi_k(\chi_{(t+R-R_{k-1},1]}) \ll \varphi(\chi_{(t,1]})$ and, consequently, that φ preserves the way-below relation.

Now let $(t_n)_n$ be a strictly decreasing sequence converging to t. Since $\chi_{(t_n,1]} \ll \chi_{(t,1]}$ for each n, we have $\varphi(\chi_{(t_n,1]}) \ll \varphi(\chi_{(t,1]})$ and, consequently, $\sup_n \varphi(\chi_{(t_n,1]}) \leq \varphi(\chi_{(t,1]})$.

Conversely, let $k \in \mathbb{N}$ and note that $d(\varphi_{k+2}, \varphi_{k+1}) < R_{k+1} - R_k$. Set $\varepsilon = R_{k+1} - R_k - d(\varphi_{k+2}, \varphi_{k+1}) > 0$.

Since $t_n - t$ is positive and tends to zero, we can find some n such that $t_n - t < \varepsilon$. Thus, we have

$$(t+R-R_k)-(t_n+R-R_{k+1}) = (t-t_n)+R_{k+1}-R_k > -\varepsilon+R_{k+1}-R_k = d(\varphi_{k+2},\varphi_{k+1}).$$

Using Remark 3.3.2 at the first step, one gets

$$\varphi_{k+1}(\chi_{(t+R-R_k,1]}) \ll \varphi_{k+2}(\chi_{(t_n+R-R_{k+1},1]}) \leq \varphi(\chi_{(t_n,1]}) \leq \sup_n \varphi(\chi_{(t_n,1]}).$$

Since this holds for each k, it follows that $\varphi(\chi_{(t,1]}) \leq \sup_n \varphi(\chi_{(t_n,1]})$ and, consequently, $\sup_n \varphi(\chi_{(t_n,1]}) = \varphi(\chi_{(t,1]})$.

Now note that $\mathcal{G}$, the sub-Cu-semigroup of increasing lower-semicontinuous functions in $(0,1]$ from Paragraph 3.2.1, can be seen as the sup-completion of the monoid N generated by $\{\chi_{(t,1]}\}_t$.

Extend φ additively to a morphism $N \to S$. Since every nonzero element in N can be written uniquely as a finite sum of nonzero elements in $\{\chi_{(t,1]}\}_t$, this extension also satisfies (i)-(ii) in Lemma 1.3.5. Thus, it can be extended further to a Cu-morphism $\mathcal{G} \to S$.

Finally, it follows from Proposition 3.2.5 that the map $\mathcal{G} \to S$ has a unique extension $\varphi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}}) \to S$ such that $\varphi(1) = \varphi_i(1)$ for all i.

(ii) Let $\varepsilon \in (0,1]$. We will show that $d(\varphi,\varphi_i) \leq \varepsilon$ for every sufficiently large i.

Since R_i tends to R, there exists some i_0 such that $0 \leq R - R_i \leq \varepsilon/2$ for every $i \geq i_0$. Thus, we have

$$\varphi_i(\chi_{(t+\varepsilon/2,1]}) \leq \varphi_i(\chi_{(t+R-R_i,1]}) \leq \varphi(\chi_{(t,1]})$$

for every $i \geq i_0$ and $t \in [0,1]$.

Now let $n \in \mathbb{N}$ be such that $1/n \leq \varepsilon/2$. One gets

$$\varphi(\chi_{(t_j+\varepsilon/2,1]}) \ll \varphi(\chi_{(t_j,1]})$$

for every j and, by the definition of φ, we can find i_j such that

$$\varphi(\chi_{(t_j+\varepsilon/2,1]}) \leq \varphi_{i_j}(\chi_{(t_j+R-R_{i_j-1},1]}).$$

Let $i = \max(i_0, i_1, \ldots, i_n)$. Then, since $i \geq i_0$, we know that $\varphi_i(\chi_{(t+\varepsilon/2,1]}) \leq \varphi(\chi_{(t,1]})$ for every t. Further, we also have

$$\varphi(\chi_{(t_j+\varepsilon/2,1]}) \leq \varphi_{i_j}(\chi_{(t_j+R-R_{i_j-1},1]}) \leq \varphi_i(\chi_{(t_j+R-R_{i-1},1]}) \leq \varphi_i(\chi_{(t_j,1]}).$$

Thus, it follows from Lemma 3.3.7 that $d(\varphi,\varphi_i) \leq \varepsilon/2 + 1/n \leq \varepsilon$, as desired. $\square$

As a consequence of Proposition 3.3.9, we can now prove Theorem 3.3.10 below.

Theorem 3.3.10. *Let S be a weakly cancellative* Cu*-semigroup satisfying (O5), and let $\varphi_i\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to S$ be* Cu*-morphisms such that $d(\varphi_i,\varphi_{i+1}) < \varepsilon_i$ with $(\varepsilon_i)_i$ strictly decreasing and $\sum_{i=1}^{\infty} \varepsilon_i < \infty$. Also, assume that $\varphi_i(1_j) = \varphi_{i+1}(1_j)$ for each i,j.*

Then, there exists a unique Cu*-morphism $\varphi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to S$ such that $\varphi(1_j) = \varphi_i(1_j)$ for every i,j and such that $(d(\varphi,\varphi_i))_i$ tends to zero 0.*

Proof. Let $\varphi_i\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to S$ be as stated, and for each $j \leq r$ let ι_j denote the canonical inclusion from $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ to the j-th component of $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r$.

For each fixed $j \leq r$, we apply Proposition 3.3.9 above to the sequence $(\varphi_i\iota_j)_i$. This produces a Cu-morphism $\varphi^{(j)}$ satisfying $d(\varphi_i\iota_j, \varphi^{(j)}) \to 0$ and $\varphi_i(1_j) = \varphi^{(j)}(1)$.

Set $\varphi = \varphi^{(1)} \oplus \ldots \oplus \varphi^{(r)}\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to S$, which satisfies $d(\varphi,\varphi_i) \to 0$ by the definition of distance (Definition 3.3.1) and $\varphi_i(1_j) = \varphi(1_j)$ for every i,j.

To prove uniqueness, let $\phi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to S$ be a Cu-morphism with $d(\varphi_i,\phi) \to 0$ and $\phi(1_j) = \varphi_i(1_j)$ for every i,j. Using the triangle inequality, we obtain

$$d(\varphi,\phi) \leq d(\varphi,\varphi_i) + d(\varphi_i,\phi) \to 0.$$

This shows that $d(\varphi,\phi) = 0$ and, consequently, $\varphi = \phi$ by Lemma 3.3.3. □

3.4 A local characterization

Using the results developed thus far, in this section we obtain a local characterization for the Cuntz semigroup of AI-algebras; see Theorem 3.4.5. We also prove a discretized version of the result (Theorem 3.4.8) and use it to show that the Cuntz semigroup of the Jiang-Su algebra, as defined in Examples 1.2.8 (iv), is not the Cuntz semigroup of any AI-algebra; see Example 3.4.9. Finally, we also show in Proposition 3.4.10 that $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ is, under a suitable definition, 'semiprojective'.

Lemma 3.4.1. *Given a* Cu*-semigroup S satisfying (O5) and weak cancellation, let $\varphi_i\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^{n_i} \to S$ and $\sigma_{i+1,i}\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^{n_i} \to \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^{n_{i+1}}$ be a pair of sequences of* Cu*-morphisms. Assume that there exists a strictly decreasing sequence $(\varepsilon_i)_i$ such that*

$$d(\varphi_j\sigma_{j,i}, \varphi_{j+1}\sigma_{j+1,i}) < \varepsilon_i/2^j \quad \text{and} \quad \varphi_{j+1}\sigma_{j+1,i}(1_k) = \varphi_j\sigma_{j,i}(1_k) \text{ for each } k \leq n_i,$$

where $\sigma_{j,i}$ denotes the composition $\sigma_{j,j-1} \circ \ldots \circ \sigma_{i+1,i}$.

Then, we can find a Cu*-morphism $\phi\colon \lim(\mathrm{Lsc}([0,1],\overline{\mathbb{N}})^{n_i},\sigma_{i+1,i}) \to S$ such that its canonical morphisms $\phi_i\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^{n_i} \to S$ are the limits of the sequences $(\varphi_j\sigma_{j,i})_j$.*

Proof. By Theorem 3.3.10, each sequence $(\varphi_j\sigma_{j,i})_j$ has a unique limit, which we denote by $\phi_i\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^{n_i} \to S$. We will see that $\phi_{i+1}\sigma_{i+1,i} = \phi_i$ for each i, which will give us a Cu-morphism $\phi\colon \lim \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^{n_i} \to S$, as required.

Thus, let $i \in \mathbb{N}$ be fixed and take $\varepsilon > 0$. Also, let $\varepsilon' > 0$ be the number given by Proposition 3.3.8 applied to ε and $\sigma_{i+1,i}$. Since $d(\phi_i,\varphi_j\sigma_{j,i})$ tends to 0, there exists some j such that

$$d(\phi_i,\varphi_j\sigma_{j,i}) < \varepsilon \quad \text{and} \quad d(\varphi_j\sigma_{j,i+1},\phi_{i+1}) < \varepsilon'.$$

Using Proposition 3.3.8 at the last step, we have

$$\begin{aligned} d(\phi_i,\phi_{i+1}\sigma_{i+1,i}) &\leq d(\phi_i,\varphi_j\sigma_{j,i}) + d(\varphi_j\sigma_{j,i},\phi_{i+1}\sigma_{i+1,i}) \\ &= d(\phi_i,\varphi_j\sigma_{j,i}) + d(\varphi_j\sigma_{j,i+1}\sigma_{i+1,i},\phi_{i+1}\sigma_{i+1,i}) < 2\varepsilon. \end{aligned}$$

Since this holds for every $\varepsilon > 0$, we have $\phi_i = \phi_{i+1}\sigma_{i+1,i}$ for every i. This induces a morphism ϕ from $\lim(\mathrm{Lsc}([0,1],\overline{\mathbb{N}})^{n_i},\sigma_{i+1,i})$ to S, as desired. □

Lemma 3.4.2. *Let A be the limit of an inductive system of C^*-algebras $(A_i, f_{i+1,i})_{i\in\mathbb{N}}$ where, for every i, we denote the i-th canonical map by $f_i\colon A_i \to A$. Then, for any $\varepsilon > 0$ and *-homomorphism $\varphi\colon C[0,1]\otimes M_m \to A$, there exist $i \in \mathbb{N}$ and a *-homomorphism $h\colon C[0,1]\otimes M_m \to A_i$ such that*

$$d(\mathrm{Cu}(\varphi), \mathrm{Cu}(f_i)\,\mathrm{Cu}(h)) \leq \varepsilon$$

and $\mathrm{Cu}(\varphi)([1\oplus 0_{m-1}]) = \mathrm{Cu}(f_i)\,\mathrm{Cu}(h)([1\oplus 0_{m-1}])$.

Proof. We begin the proof with the following claim.

Claim. *Let A, B be C^*-algebras, and let a, b be elements in B. Assume that $[a] \ll [b]$ in $\mathrm{Cu}(B)$. Then, there exists $\delta = \delta(a,b) > 0$ such that, for every pair of *-homomorphisms $\varphi, \phi\colon B \to A$ with $\|\phi(b) - \varphi(b)\| \leq \delta$, we have*

$$[\varphi(a)] \leq [\phi(b)] \quad \textit{and} \quad [\phi(a)] \leq [\varphi(b)]$$

in $\mathrm{Cu}(A)$.

Using that $[a] \ll [b]$, we obtain $\delta = \delta(a,b) > 0$ such that $[a] \leq [(b-\delta)_+]$ in $\mathrm{Cu}(B)$; see, for example, [6, Remarks 3.2.4]. Further, it follows from [77, Lemma 2.2] that, given any pair of *-homomorphisms $\varphi, \phi\colon B \to A$ with $\|\phi(b) - \varphi(b)\| \leq \delta$, we have

$$[(\phi(b)-\delta)_+] \leq [\varphi(b)] \quad \text{and} \quad [(\varphi(b)-\delta)_+] \leq [\phi(b)]$$

in $\mathrm{Cu}(A)$.

Thus, since *-homomorphisms preserve continuous functional calculus, one gets

$$[\varphi(a)] \leq [\varphi((b-\delta)_+)] = [(\varphi(b)-\delta)_+] \leq [\phi(b)]$$

and, similarly, $[\phi(a)] \leq [\varphi(b)]$, as required.

Now let $\varphi\colon C[0,1]\otimes M_m \to A$ be a *-homomorphism and take $\varepsilon > 0$.

Let $n \in \mathbb{N}$ be such that $2/n \leq \varepsilon$ and, for each $k < n$, let $g'_k \in C[0,1]$ be a function with support $(k/n, 1]$ and norm 1. Set $g_k = g'_k \oplus 0_{m-1} \in C[0,1]\otimes M_m$, whose Cuntz class corresponds to the element $\chi_{(k/n,1]}$ under the identification $\mathrm{Cu}(C[0,1]\otimes M_m) \cong \mathrm{Lsc}([0,1], \overline{\mathbb{N}})$.

Applying the Claim to each pair g_{k+1}, g_k, we obtain a positive number δ_k for each k. Define $\delta = \min(\delta_k, 1)$, and let F be the finite set $\{g_k\}_{k<n} \cup \{1\oplus 0_{m-1}\}$.

Since $C[0,1]\otimes M_m$ is projective (see, for example, [29, Section 3]), there exists $i \in \mathbb{N}$ and a $*$-homomorphism $h\colon C[0,1]\otimes M_m \to A_i$ such that

$$\|\varphi(x) - f_i h(x)\| < \delta$$

for every $x \in F$.

In particular, since $\|\varphi(1\oplus 0_{m-1}) - f_i h(1\oplus 0_{m-1})\| < \delta \leq 1$, we get that $\varphi(1\oplus 0_{m-1})$ is Murray-von Neumann equivalent to $f_i h(1\oplus 0_{m-1})$. This in turn implies that

$$[\varphi(1\oplus 0_{m-1})] = [f_i h(1\oplus 0_{m-1})]$$

in $\mathrm{Cu}(A)$.

Moreover, using that $\|\varphi(g_k) - f_i h(g_k)\| \leq \delta_k$ for each k, the Claim implies that

$$[\varphi(g_{k+1})] \leq [f_i h(g_k)] \quad \text{and} \quad [f_i h(g_{k+1})] \leq [\varphi(g_k)]$$

for each k.

It now follows from Lemma 3.3.7 that $d(\mathrm{Cu}(\varphi), \mathrm{Cu}(f_i)\,\mathrm{Cu}(h)) \leq 2/n \leq \varepsilon$, as desired. □

To state Theorem 3.4.5 we will use a the following notion for Cu-semigroup, which may well be of importance in other scenarios; see, for example, Proposition 4.3.6.

Definition 3.4.3. A Cu-semigroup S is said to be *compactly bounded* if every element compactly contained in another is bounded by a compact. That is to say, if for every element $x \in S_{\ll}$ there exists a compact element $p \in S$ such that $x \leq p$.

Remark 3.4.4. It is readily checked that the Cuntz semigroup of a unital C^*-algebra is always compactly bounded, since every compactly contained element is bounded by a multiple of the class of the unit.

Similarly, one can also check that an inductive limit of Cuntz semigroups of unital C^*-algebras is always compactly bounded.

The proof of Theorem 3.4.5 below adapts some ideas from Shen's theorem ([87, Theorem 3.1]) and continuum theory ([64, Chapter 12, Section 3]).

Theorem 3.4.5. *Let S be a countably based, weakly cancellative, compactly bounded* Cu*-semigroup satisfying (O5). Then, S is* Cu*-isomorphic to the Cuntz semigroup of an AI-algebra if and only if for every* Cu*-morphism $\varphi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to S$, finite subset $F \subseteq \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r$ and $\varepsilon > 0$, there exist a natural number s and* Cu*-morphisms $\theta\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^s$, $\phi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^s \to S$ such that the diagram*

$$\begin{tikzcd} \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \arrow[r, "\varphi"] \arrow[d, "\theta"'] & S \\ \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^s \arrow[ur, "\phi"'] & \end{tikzcd}$$

satisfies

(i) $d(\phi\theta, \varphi) < \varepsilon$;

(ii) For every $x, x', y \in F$, we have $\theta(x) \ll \theta(y)$ whenever $x \ll x'$ and $\varphi(x') \ll \varphi(y)$;

(iii) $\varphi(1_j) = \phi\theta(1_j)$ *for every* $1 \leq j \leq r$.

Proof. Let S be isomorphic the Cuntz semigroup of an AI-algebra A, and let φ, F and ε be as in the statement of the theorem. Since $S \cong \mathrm{Cu}(A)$, we know by [21, Theorem 12.1] that φ lifts to a *-homomorphism $g\colon C[0,1] \otimes M_m \to A$, where $m \in \mathbb{N}$ and A is the limit of an inductive system $(A_i, f_{i+1,i})$ with A_i a direct sum of interval algebras for each i.

By Corollary 3.3.6, for every triple of elements $x, x', y \in F$ with $x \ll x'$ and $\varphi(x') \ll \varphi(y)$, there exists $\varepsilon(x, x', y, \varphi)$ such that for any Cu-morphism $\psi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to S$ with $d(\varphi, \psi) < \varepsilon(x, x', y, \varphi)$ and $\varphi(1_j) = \psi(1_j)$ for every j, we have $\psi(x) \ll \psi(y)$.

Since the cardinality of F is finite, so is the number of way-below relations between its elements. This means that the number

$$\varepsilon_F := \min(\varepsilon, 1, \min\{\varepsilon(x, x', y, \varphi) \mid x, x', y \in F \text{ such that } x \ll x' \text{ and } \varphi(x') \ll \varphi(y)\})$$

is strictly positive.

For each i, let $f_i\colon A_i \to A$ denote the canonical map. Using Lemma 3.4.2 above applied to g (and the definition of distance; Definition 3.3.1) one can find $i \in \mathbb{N}$ and a *-homomorphism $h\colon C[0,1] \otimes M_m \to A_i$ such that $d(\varphi, \mathrm{Cu}(f_i)\,\mathrm{Cu}(h)) < \varepsilon_F$ and such that $\varphi(1_k) = \mathrm{Cu}(f_i)\,\mathrm{Cu}(h)(1_k)$ for every $k \leq r$.

By the choice of ε_F, we also have that $\mathrm{Cu}(f_i)\,\mathrm{Cu}(h)(x) \ll \mathrm{Cu}(f_i)\,\mathrm{Cu}(h)(y)$ for every triple $x, x', y \in F$ with $x \ll x'$ and $\varphi(x') \ll \varphi(y)$.

Finally, note that $\mathrm{Cu}(f_i)\colon \mathrm{Cu}(A_i) \to S$ is the canonical morphism from $\mathrm{Cu}(A_i)$ to the limit $\lim \mathrm{Cu}(A_i) \cong \mathrm{Cu}(A) \cong S$. Thus, we know that $\mathrm{Cu}(f_i)\,\mathrm{Cu}(h)(x) \ll \mathrm{Cu}(f_i)\,\mathrm{Cu}(h)(y)$ if and only if there exists $i(x,y) \geq i$ with $\mathrm{Cu}(f_{i(x,y),i})\,\mathrm{Cu}(h)(x) \ll \mathrm{Cu}(f_{i(x,y),i})\,\mathrm{Cu}(h)(y)$.

Since F is finite, so is the supremum j of all the $i(x,y)$'s with $x, y \in F$. Setting $\theta := \mathrm{Cu}(f_{j,i})\,\mathrm{Cu}(h)$ and $\phi := \mathrm{Cu}(f_j)$, we have

(i) $d(\phi\theta, \varphi) = d(\mathrm{Cu}(f_j)\,\mathrm{Cu}(f_{j,i})\,\mathrm{Cu}(h), \varphi) = d(\mathrm{Cu}(f_i)\,\mathrm{Cu}(h), \varphi) < \varepsilon$.

(ii) For every triple $x, x', y \in F$ such that $x \ll x'$ and $\varphi(x') \ll \varphi(y)$, we get

$$\mathrm{Cu}(f_i)\,\mathrm{Cu}(h)(x) \ll \mathrm{Cu}(f_i)\,\mathrm{Cu}(h)(y).$$

By our choice of j, it follows that

$$\theta(x) = \mathrm{Cu}(f_{j,i})\,\mathrm{Cu}(h)(x) \ll \mathrm{Cu}(f_{j,i})\,\mathrm{Cu}(h)(y) = \theta(y).$$

(iii) $\phi\theta(1_k) = \mathrm{Cu}(f_i)\,\mathrm{Cu}(h)(1_k) = \varphi(1_k)$ for every $k \leq r$.

as required.

We are now left to prove the other implication.

Let $s_1, s_2, \ldots$ be a countable basis for S, where we may assume $s_i \in S_{\ll}$ for each i, and consider a Cu-morphism $\psi_i\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}}) \to S$ such that $\psi_i(\chi_{(0,1]}) = s_i$. Such a morphism can always be found by Theorem 3.2.6 and the fact that all compactly contained elements in S are bounded by a compact. Also, denote by $\rho_i\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^i \to S$ the direct sum $\rho_i = \psi_1 \oplus \ldots \oplus \psi_i$.

For every $j \in \mathbb{N}$, fix a countable and ordered basis for $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})^j$. By '*the first i basis elements in* $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})^j$' we will mean the first i elements of the fixed ordered basis of $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})^j$.

The idea of the proof is as follows:

We will define inductively Cu-morphisms $\sigma_{i+1,i}\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^{n_i} \to \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^{n_{i+1}}$ and $\varphi_i\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^{n_i} \to S$ such that

(i)' There exists a decreasing sequence of positive elements $(\varepsilon_i)_{i\geq 1}$ tending to 0 such that, for every $i \geq 2$, there exists a Cu-morphism

$$\theta_i\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^{n_{i-1}} \oplus \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^i \to \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^{n_i}$$

with $d(\varphi_{i-1} \oplus \rho_i, \varphi_i\theta_i) < \varepsilon_i$;

(ii)' For every fixed $i \geq 1$ and any $j \geq i$, we have

$$d(\varphi_j\sigma_{j,i}, \varphi_{j+1}\sigma_{j+1,i}) < \varepsilon_i/2^j \quad \text{and} \quad \varphi_{j+1}\sigma_{j+1,i}(1_k) = \varphi_j\sigma_{j,i}(1_k)$$

for each $k \leq n_i$. Here, $\sigma_{i,i} = \mathrm{id}$ and $\sigma_{j,i}$ denotes the composition $\sigma_{j,j-1} \circ \ldots \circ \sigma_{i+1,i}$;

(iii)' For every fixed $k \geq 2$ and any $j \geq k$, we have $d(\varphi_j\sigma_{j,k}\theta_k, \varphi_{j+1}\sigma_{j+1,k}\theta_k) < \varepsilon_k/2^j$;

(iv)' For each $i \geq 1$, let F_i be the finite set consisting of the images of the first i basis elements of $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})^{n_r}$ through $\sigma_{i,r}$ for each $r \leq i$. Then, for every $x, x', y \in F_i$ satisfying $\varphi_i(x') \ll \varphi_i(y)$ with $x \ll x'$, we have $\sigma_{i+1,i}(x) \ll \sigma_{i+1,i}(y)$.

Condition (ii)' and Lemma 3.4.1 will provide a limit morphism

$$\phi\colon \lim \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^{n_i} \to S$$

with the canonical morphisms $\phi_i\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^{n_i} \to S$ being the limits of the sequences $(\varphi_j\sigma_{j,i})_j$.

Conditions (i)' and (iii)' will imply that ϕ is surjective. Condition (iv)' will be used to prove that ϕ is also an order embedding, thus showing the desired result.

Set $\varepsilon_1 = 1$ and $\varphi_1 := \rho_1$. Fix some $i \geq 2$ and assume that, for each $k \geq 2$ with $k \leq i-1$, the element ε_k and the morphisms $\sigma_{k,k-1}$, φ_k and θ_k have been defined so that conditions (i)'-(iv)' above are satisfied. Note that, for $i = 2$, nothing is assumed.

For every $k \leq i-1$, let δ_k be the distance given in Proposition 3.3.8 such that for any pair of morphisms $\zeta_1, \zeta_2\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^{n_{i-1}} \to S$ at distance less than δ_k, we have

$$d(\zeta_1\sigma_{i-1,k}, \zeta_2\sigma_{i-1,k}) < \varepsilon_k/2^i \quad \text{and} \quad d(\zeta_1\sigma_{i-1,k}\theta_k, \zeta_2\sigma_{i-1,k}\theta_k) < \varepsilon_k/2^i,$$

where the second bound is only asked if $k \geq 2$. In particular, if $i = 2$, we can take $\delta_1 = \varepsilon_1/4$.

Set $\varepsilon_i := \min_{1\leq k\leq i-1}\{\delta_k, \varepsilon_k\}/2$, which is positive and strictly less than ε_{i-1}. As defined above, let F_{i-1} be the set that contains, for each $r \leq i-1$, the image through $\sigma_{i-1,r}$ of $i-1$ distinct basis elements of $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})^{n_r}$.

Let $\tau_{i-1}\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^{n_{i-1}} \to \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^{n_{i-1}} \oplus \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^i$ be the canonical inclusion, and let $F = \tau_{i-1}(F_{i-1})$.

By our assumptions, we can find morphisms φ_i, θ_i such that the diagram

$$\begin{tikzcd}
\mathrm{Lsc}([0,1], \overline{\mathbb{N}})^{n_{i-1}} \oplus \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^i \arrow[r, "\varphi_{i-1}\oplus\rho_i"] \arrow[d, "\theta_i"'] & S \\
\mathrm{Lsc}([0,1], \overline{\mathbb{N}})^{n_i} \arrow[ur, "\varphi_i"'] &
\end{tikzcd}$$

satisfies conditions (i)-(iii) in the statement of the theorem with distance ε_i and finite set F. Note that condition (i)' is immediately satisfied.

Define $\sigma_{i,i-1} := \theta_i \circ \tau_{i-1}$. Condition (iv)' is satisfied by construction.

$$\begin{tikzcd}
\mathrm{Lsc}([0,1], \overline{\mathbb{N}})^{n_{i-1}} \arrow[r, "\tau_{i-1}"] \arrow[rr, bend left, dashed, "\varphi_{i-1}"] \arrow[dr, dashed, "\sigma_{i,i-1}"'] & \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^{n_{i-1}} \oplus \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^i \arrow[r, "\varphi_{i-1}\oplus\rho_i"] \arrow[d, "\theta_i"] & S \\
& \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^{n_i} \arrow[ur, "\varphi_i"'] &
\end{tikzcd}$$

Furthermore, since τ_{i-1} is an inclusion, we have by Definition 3.3.1 and $\rho_i\tau_{i-1} = 0$ that

$$d(\varphi_i\sigma_{i,i-1}, \varphi_{i-1}) = d(\varphi_i\theta_i\tau_{i-1}, (\varphi_{i-1} \oplus \rho_i)\tau_{i-1}) \leq d(\varphi_i\theta_i, \varphi_{i-1} \oplus \rho_i) < \varepsilon_i \leq \delta_k$$

for each k.

By the choice of δ_k, one gets

$$d(\varphi_i\sigma_{i,k}, \varphi_{i-1}\sigma_{i-1,k}) = d(\varphi_i\sigma_{i,i-1}\sigma_{i-1,k}, \varphi_{i-1}\sigma_{i-1,k}) < \varepsilon_k/2^i.$$

Moreover, as $\varphi_i\sigma_{i,i-1}(1_j) = \varphi_{i-1}(1_j)$ for every $j \leq n_{i-1}$, condition (ii)' also holds. An analogous argument shows that (iii)' holds.

This finishes the inductive argument.

By Lemma 3.4.1, condition (ii)' induces a Cu-morphism $\phi\colon \lim \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^{n_i} \to S$ with the canonical morphisms $\phi_i\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^{n_i} \to S$ being the limits of the sequences $(\varphi_j\sigma_{j,i})_j$.

To see that ϕ is surjective, first note that, for every $i \in \mathbb{N}$ and for every $\varepsilon > 0$, there exists $j \in \mathbb{N}$ such that $d(\phi_i\theta_i, \varphi_j\sigma_{j,i}\theta_i) < \varepsilon$. This is due to Proposition 3.3.8 and because $d(\phi_i, \varphi_j\sigma_{j,i})$ tends to 0 as j tends to infinity.

Thus, we get

$$\begin{aligned} d(\varphi_i\theta_i, \phi_i\theta_i) &\leq d(\varphi_i\theta_i, \varphi_{i+1}\sigma_{i+1,i}\theta_i) + \ldots + d(\varphi_{j-1}\sigma_{j-1,i}\theta_i, \varphi_j\sigma_{j,i}\theta_i) + d(\varphi_j\sigma_{j,i}\theta_i, \phi_i\theta_i) \\ &\leq \frac{\varepsilon_i}{2^i} + \ldots + \frac{\varepsilon_i}{2^{j-1}} + \varepsilon \leq 2\varepsilon_i + \varepsilon, \end{aligned}$$

where we have used property (iii)' to bound all but the last element.

Since this holds for every ε, one gets $d(\varphi_i\theta_i, \phi_i\theta_i) \leq 2\varepsilon_i$. In particular, the distance tends to 0 on i.

Now let s_i be a basis element of S and let $x \in S$ be such that $x \ll s_i$. We have $x \ll \psi_i(\chi_{(0,1]}) = s_i$ and, consequently, there exists $s, t \in (0,1]$ such that

$$x \ll \psi_i(\chi_{(t+2s,1]}) \ll \psi_i(\chi_{(t,1]}) \ll \psi_i(\chi_{(0,1]}).$$

Note that, for every $k > i$, we have

$$d(\varphi_{k-1} \oplus \rho_k, \phi_k\theta_k) \leq d(\varphi_{k-1} \oplus \rho_k, \varphi_k\theta_k) + d(\varphi_k\theta_k, \phi_k\theta_k) < \varepsilon_k + 2\varepsilon_k = 3\varepsilon_k,$$

where in the previous bound we have used condition (i)' and the inequality obtained above.

Thus, there exists a large enough k so that $d(\varphi_{k-1} \oplus \rho_k, \phi_k\theta_k) < s$.

Since $k > i$, we know that $(\varphi_{k-1} \oplus \rho_k)\iota_i = \psi_i$, where ι_i denotes the canonical inclusion from $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$ to the $(n_{k-1} + i)$-th component of $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})^{n_{k-1}} \oplus \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^k$.

Using Remark 3.3.2, we have

$$\begin{aligned} x \ll \psi_i(\chi_{(t+2s,1]}) &= (\varphi_{k-1} \oplus \rho_k)\iota_i(\chi_{(t+2s,1]}) \ll \phi_k\theta_k\eta_i(\chi_{(t+s,1]}) \\ &\ll (\varphi_{k-1} \oplus \rho_k)\iota_i(\chi_{(t,1]}) = \psi_i(\chi_{(t,1]}) \ll \psi_i(\chi_{(0,1]}) = s_i. \end{aligned}$$

This shows that for every i and every $x \ll s_i$, there exist some k such that

$$x \ll \phi_k(l) \ll s_i$$

with $l \in \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^{n_k}$. Consequently, ϕ is surjective.

We will now prove that ϕ is an order-embedding. To do this, we will denote by $[v]$ the elements in $\lim \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^{n_i}$ coming from some block $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})^{n_i}$ of the direct limit.

Take $x, y \in \lim \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^{n_i}$ such that $\phi(x) \leq \phi(y)$. Let $v \in \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^{n_s}$ be a basis element with $[v] \ll x$. Also, take $[z], [z']$ such that $[v] \ll [z'] \ll [z] \ll x$ with $z, z' \in \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^{n_{s'}}$ basis elements as well. Finally, take $w \in \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^{n_k}$ a basis element such that $[w] \ll y$ and $\phi([z]) \ll \phi([w]) \ll \phi(y)$.

We can assume that, for a large enough i, we have

$$\sigma_{i,s}(v), \sigma_{i,k}(w), \sigma_{i,s'}(z), \sigma_{i,s'}(z') \in \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^{n_i}$$

with

$$\sigma_{i,s}(v) \ll \sigma_{i,s'}(z') \ll \sigma_{i,s'}(z) \quad \text{and} \quad \phi_i(\sigma_{i,s'}(z)) \ll \phi_i(\sigma_{i,k}(w)).$$

Thus, since we have $d(\phi_i, \varphi_j \sigma_{j,i}) \to 0$, it follows from Corollary 3.3.6 applied to $\sigma_{i,s'}(z')$, $\sigma_{i,k}(w)$, $\sigma_{i,s'}(z)$ and ϕ_i that

$$\varphi_j \sigma_{j,i}(\sigma_{i,s'}(z')) \ll \varphi_j \sigma_{j,i}(\sigma_{i,k}(w))$$

for every sufficiently large j.

Also, since z, z', v, w are basis elements in their respective blocks, we can take j large enough so that we also have $\sigma_{j,s'}(z), \sigma_{j,s'}(z'), \sigma_{j,s}(v), \sigma_{j,k}(w) \in F_j$.

Therefore, since $\sigma_{j,s}(v) \ll \sigma_{j,s'}(z')$ and $\varphi_j(\sigma_{j,s'}(z')) \ll \varphi_j(\sigma_{j,k}(w))$, it follows from condition (iv)' that $\sigma_{j+1,j}(\sigma_{j,s}(v)) \ll \sigma_{j+1,j}(\sigma_{j,k}(w))$. That is to say, we have

$$\sigma_{j+1,s}(v) \ll \sigma_{j+1,k}(w)$$

and thus $[v] \ll [w] \ll y$.

Since x can be written as the supremum of a $\ll$-increasing sequence $([x_n])$ with x_n basis elements, it follows from the previous argument that $[x_n] \ll y$ for every n. Taking the supremum, one gets $x \leq y$ as required.

We now have $S \cong \lim_i \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^{n_i}$, and the desired result follows from Theorem 3.1.6. $\square$

We now show that in Theorem 3.4.5 above one only needs to check conditions (i)-(iii) for triples x, x', y such that $x \ll x'$ and $\varphi(x') \ll \varphi(y)$.

Proposition 3.4.6. *Let S be a countably based, compactly bounded and weakly cancellative* Cu*-semigroup satisfying (O5). Then, S is* Cu*-isomorphic to the Cuntz semigroup of an AI-algebra if and only if for every* Cu*-morphism $\varphi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to S$, $\varepsilon > 0$ and any triple x, x', y in $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r$ with $x \ll x'$ and $\varphi(x') \ll \varphi(y)$, there exist* Cu*-morphisms θ, ϕ and a natural number s such that the diagram*

$$\begin{tikzcd} \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \arrow[r, "\varphi"] \arrow[d, "\theta"'] & S \\ \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^s \arrow[ur, "\phi"'] & \end{tikzcd}$$

satisfies

(i) $d(\phi\theta, \varphi) < \varepsilon$;

(ii) $\theta(x) \ll \theta(y)$;

(iii) $\varphi(1_j) = \phi\theta(1_j)$ *for every* $1 \leq j \leq r$.

Proof. The forward implication follows trivially from Theorem 3.4.5. Thus, we only need to show that (i)-(iii) in the statement of the proposition imply (i)-(iii) in Theorem 3.4.5.

For any Cu-morphism $\varphi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to S$, let $\mathcal{R}\varphi$ denote the set

$$\mathcal{R}\varphi = \{(x, x', y) \in (\mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r)^3 \mid x \ll x' \text{ with } \varphi(x') \ll \varphi(y)\}.$$

Given any $\varepsilon > 0$ and finite subset $R \subseteq \mathcal{R}\varphi$, we will show by induction on $|R|$ that there exist Cu-morphisms θ,ϕ such that the following conditions are satisfied:

(i)' $d(\phi\theta, \varphi) < \varepsilon$;

(ii)' $\theta(x) \ll \theta(y)$ for every $(x, x', y) \in R$;

(iii)' $\varphi(1_j) = \phi\theta(1_j)$ for every $1 \leq j \leq r$.

In particular, since $F^3 \cap \mathcal{R}\varphi$ is finite for every finite subset $F \subseteq \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r$, this will imply the desired result.

Note that, for $n = 1$, the result holds by assumption.

Thus, assume that the result has been proven for every $k \leq n-1$, Cu-morphism $\varphi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to S$, $\varepsilon > 0$ and finite subset $R \subseteq \mathcal{R}\varphi$ with $|R| = k$.

Let $\varphi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to S$ be a Cu-morphism, and take $\varepsilon > 0$ and a finite subset $R \subseteq \mathcal{R}\varphi$ with cardinality n.

Write R as $R = \{(x_1, x'_1, y_1), \ldots, (x_n, x'_n, y_n)\}$, where note that some elements x_i, x'_i, y_i might be repeated. Using Corollary 3.3.6, we find x''_i and $\varepsilon'_i > 0$ such that $x_i \ll x''_i \ll x'_i$ and, for any Cu-morphism $\psi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to S$ with $d(\psi, \varphi) < \varepsilon'_i$ and $\psi(1_j) = \phi(1_j)$ for every j, we have $\psi(x''_i) \ll \psi(y_i)$.

Set $\varepsilon' := \min\{\varepsilon/2, \varepsilon'_i\} > 0$, and use the induction hypothesis on φ, ε' and the set $\{(x_1, x''_1, y_1)\}$ to get Cu-morphisms

$$\theta_1\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^t \quad \text{and} \quad \phi_1\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^t \to S$$

such that

(i) $d(\phi_1\theta_1, \varphi) < \varepsilon'$;

(ii) $\theta_1(x_1) \ll \theta_1(y_1)$;

(iii) $\varphi(1_j) = \phi_1\theta_1(1_j)$ for every $1 \leq j \leq r$.

In particular, note that (i) and (iii) imply that $(\theta_1(x_i), \theta_1(x''_i), \theta_1(y_i)) \in \mathcal{R}\phi_1$ for every $i > 1$. Indeed, since $d(\phi_1\theta_1, \varphi) < \varepsilon' < \varepsilon'_i$ and $x_i \ll x''_i$, we get

$$\theta_1(x_i) \ll \theta_1(x''_i) \quad \text{and} \quad \phi_1\theta_1(x''_i) \ll \phi_1\theta(y_i),$$

which shows that $(\theta_1(x_i), \theta_1(x_i''), \theta_1(y_i)) \in \mathcal{R}\phi_1$ for each $i > 1$.

Set $R' = \{(\theta_1(x_i), \theta_1(x_i''), \theta_1(y_i))\}_{i>1}$, which has cardinality $n-1$. Let ν be the bound given in Proposition 3.3.8 applied to the morphism θ_1 and the constant $\varepsilon/2$.

Using the induction hypothesis to ϕ_1, $\nu > 0$, and R' we get Cu-morphisms θ_2, ϕ_2 such that

(i) $d(\phi_2\theta_2, \phi_1) < \nu$;

(ii) $\theta_2\theta_1(x_i) \ll \theta_2\theta_1(y_i)$ for every $1 \leq i \leq n$ (for $i = 1$, this is because $\theta_1(x_1) \ll \theta_1(y_1)$);

(iii) $\phi_1(1_j) = \phi_2\theta_2(1_j)$ for every $1 \leq j \leq t$.

Set $\phi := \phi_2$ and $\theta := \theta_2\theta_1$. It follows from Proposition 3.3.8 and (i) that $d(\phi\theta, \phi_1\theta_1) < \varepsilon/2$. Thus, we have

$$d(\varphi, \phi\theta) \leq d(\varphi, \phi_1\theta_1) + d(\phi\theta, \phi_1\theta_1) < \varepsilon,$$

which shows that condition (i)' is satisfied. Moreover, note that (ii) above is equivalent to (ii)'.

Finally, given $1_j \in \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^r$, let $k_1, \ldots, k_t \in \mathbb{N}$ be such that

$$\theta_1(1_j) = k_1 1_1 + \ldots + k_t 1_t$$

in $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})^t$.

Using (iii) at the fourth step, we get

$$\begin{aligned} \varphi(1_j) = \phi_1\theta_1(1_j) &= \phi_1(k_1 1_1 + \ldots + k_t 1_t) = k_1\phi_1(1_1) + \ldots + k_t\phi_1(1_t) \\ &= k_1\phi_2\theta_2(1_1) + \ldots + k_t\phi_2\theta_2(1_t) = \phi_2\theta_2\theta_1(1_j). \end{aligned}$$

Thus, (iii)' is also satisfied, as desired. □

Proposition 3.4.6 can be weakened further by only considering basic increasing elements. That is to say, by only considering basic elements (as defined in Definition 3.1.1) that are increasing as functions.

Proposition 3.4.7. *Let S be a countably based, compactly bounded and weakly cancellative* Cu*-semigroup satisfying (O5). Then, S is* Cu*-isomorphic to the Cuntz semigroup of an AI-algebra if and only if, for every* Cu*-morphism $\varphi\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^r \to S$, $\varepsilon > 0$, and any triple of basic increasing elements x, x', y such that $x \ll x'$ with $\varphi(x') \ll \varphi(y)$, conditions (i)-(iii) in Proposition 3.4.6 are satisfied.*

Proof. The forward implication is clear (using, for example, Proposition 3.4.6), so we are left to prove the backward implication.

Given (possibly non-increasing) basic elements x, x', y in $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})^r$ with $x \ll x'$ and $\varphi(x') \ll \varphi(y)$, we will show that (i)-(iii) in Proposition 3.4.6 are satisfied. Since such elements are dense in $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})^r$, the result will follow.

Thus, let $\varphi\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^r \to S$ be a Cu-morphism and let $\varepsilon > 0$. Take x, x', y be basic elements such that $x \ll x'$ and $\varphi(x') \ll \varphi(y)$.

Since $\varphi(x') \ll \varphi(y)$, we can find $y' \ll y$ satisfying $\varphi(x') \ll \varphi(y')$. By Lemma 3.1.3, we obtain basic increasing elements $d, f, h, g \in \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^r$ such that

$$x + h \ll d \ll x' + h \quad \text{and} \quad y' + g \ll f \ll y + g.$$

Let $\delta, \nu, \eta > 0$ satisfy

$$x + h \ll R_\eta(d) \ll d \leq x' + R_{2\delta}(h) \quad \text{and} \quad y' + g \ll f \leq y + R_{2\nu}(g)$$

and such that $R_\eta(d), R_{2\delta}(h), R_{2\nu}(g)$ are still basic increasing elements.

Therefore, using that $\varphi(x') \ll \varphi(y')$ at the second step, and that $y' + g \ll f$ at the third step, one has

$$\varphi(d+g) \leq \varphi(x' + R_{2\delta}(h) + g) \leq \varphi(y' + R_{2\delta}(h) + g) \ll \varphi(f + R_\delta(h)),$$

where note that, by construction, $R_\eta(d)+R_\nu(g)$, $d+g$ and $f+R_\delta(h)$ are basic increasing elements satisfying

$$R_\eta(d) + R_\nu(g) \ll d + g \quad \text{and} \quad \varphi(d+g) \ll \varphi(f + R_\delta(h)).$$

Applying our hypothesis, we obtain Cu-morphisms θ, ϕ satisfying conditions (i)-(iii) in Proposition 3.4.6 for $R_\eta(d) + R_\nu(g)$, $d + g$, and $f + R_\delta(h)$. Using this at the third step, we get

$$\begin{aligned} \theta(x) + \theta(h + R_{2\nu}(g)) = \theta(x + h + R_{2\nu}(g)) \ll \theta(R_\eta(d) + R_\nu(g)) \ll \theta(f + R_\delta(h)) \\ \leq \theta(y + R_{2\nu}(g) + h) = \theta(y) + \theta(R_{2\nu}(g) + h). \end{aligned}$$

It follows from weak cancellation that $\theta(x) \ll \theta(y)$, as required. □

Let $n \in \mathbb{N}$. We denote by B_n the additive span of the set $\{1\} \cup \{\chi_{(i/n,1]}\}_i$ in $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$, and for every $r \in \mathbb{N}$ we let B_n^r be the direct sum $B_n \oplus \overset{(r}{\cdots} \oplus B_n$ in $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})^r$.

Theorem 3.4.8. *Let S be a countably based, compactly bounded and weakly cancellative* Cu*-semigroup satisfying (O5). Then, S is isomorphic to the Cuntz semigroup of an AI-algebra if and only if for every* Cu*-morphism $\varphi\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^r \to S$, $n \in \mathbb{N}$ and triple $x, x', y \in B_n^r$ such that $x \ll x'$ with $\varphi(x') \ll \varphi(y)$, there exist* Cu*-morphisms θ, ϕ and a natural number s such that the diagram*

$$\begin{tikzcd} \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^r \arrow[r, "\varphi"] \arrow[d, "\theta"] & S \\ \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^s \arrow[ur, "\phi"'] & \end{tikzcd}$$

satisfies

(i) for every $i \leq n$ and $j \leq r$, we have

$$\varphi\iota_j\left(\chi_{\left(\frac{i}{n}+\frac{1}{n},1\right]}\right) \ll \phi\theta\iota_j\left(\chi_{\left(\frac{i}{n},1\right]}\right) \quad \text{and} \quad \phi\theta\iota_j\left(\chi_{\left(\frac{i}{n}+\frac{1}{n},1\right]}\right) \ll \varphi\iota_j\left(\chi_{\left(\frac{i}{n},1\right]}\right),$$

where ι_j denotes the canonical inclusion from $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$ *to the j-th component of* $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})^r$*;*

(ii) $\theta(x) \ll \theta(y)$;

(iii) $\varphi(1_j) = \phi\theta(1_j)$ *for every* $1 \leq j \leq r$.

Proof. As in the proof of Proposition 3.4.7, the forward implication follows clearly from Proposition 3.4.6 combined with Lemma 3.3.7 applied to $\varepsilon < 1/n$.

Conversely, it follows from Proposition 3.4.7 that it is enough to prove the existence of Cu-morphisms θ, ϕ satisfying (i)-(iii) in Proposition 3.4.6 for every Cu-morphism $\varphi\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^r \to S$, $\varepsilon > 0$ and triple of basic increasing elements x, x', y such that $x \ll x'$ with $\varphi(x') \ll \varphi(y)$.

Thus, let $n \in \mathbb{N}$ be such that $2/n < \varepsilon$ and such that there exist $v, v', w \in B_n^r$ such that

$$x \ll v \ll v' \ll x', \quad w \ll y \quad \text{and} \quad \varphi(x') \ll \varphi(w).$$

By assumption, we obtain Cu-morphisms ϕ, θ satisfying conditions (i)-(iii) for the elements v, v', w. Thus, we have

$$\theta(x) \ll \theta(v) \ll \theta(w) \ll \theta(y),$$

which shows that ϕ, θ satisfy (ii) in the statement (which is the same as (ii) in Proposition 3.4.6) for x, x', y.

Moreover, condition (i) of the statement implies, by Lemma 3.3.7, that

$$d(\varphi, \phi\theta) < \frac{1}{n} + \frac{1}{n} = \frac{2}{n} < \varepsilon$$

This shows that condition (i) of Proposition 3.4.6 is satisfied.

Finally, note that (iii) in Proposition 3.4.6 coincides with (iii) in our statement. Using Proposition 3.4.7, it follows that S is Cu-isomorphic to the Cuntz semigroup of an AI-algebra, as required. □

Example 3.4.9. Let $\mathcal{Z}$ be the Jiang-Su algebra as defined in Examples 1.2.8. Recall that its Cuntz semigroup, denoted by Z, is Cu-isomorphic to $\mathbb{N} \sqcup (0, \infty]$.

The results developed by Robert in [72] show that Z is not isomorphic to the Cuntz semigroup of an AI-algebra. We now use Theorem 3.4.8 to recover this result:

Take $n \geq 3$, and consider the $\ll$-decreasing sequence $(1' - k/n)_k$ in Z. Since this sequence is bounded by 1, it follows from Theorem 3.2.6 that there exists a Cu-morphism $\varphi\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}}) \to Z$ such that

$$\varphi(\chi_{(k/n,1]}) = 1' - k/n \quad \text{and} \quad \varphi(1) = 1,$$

where note that $\varphi(1) = 1 \ll 3/2 = \varphi(3 \cdot \chi_{(1/2,1]})$.

Assume for the sake of contradiction that Z is the Cuntz semigroup of an AI-algebra. Applying Theorem 3.4.8 to φ, $1/2$, and the triple $1, 1, 3 \cdot \chi_{(1/2,1]} \in Z$, we find Cu-morphisms $\theta\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}}) \to \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^s$ and $\phi\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^s \to Z$ satisfying

$$d(\phi\theta, \varphi) < 1/2, \quad \theta(1) \ll \theta(3 \cdot \chi_{(1/2,1]}) \quad \text{and} \quad \phi\theta(1) = \varphi(1).$$

Let $\bar{1}$ denote the element $(1, \ldots, 1)$ in $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})^s$. Then, since $\theta(1) \ll 3\theta(\chi_{(1/2,1]})$, we get $\theta(1) \wedge \bar{1} \ll 3\theta(\chi_{(1/2,1]}) \wedge \bar{1}$ and, consequently,

$$\mathrm{supp}(\theta(1) \wedge \bar{1}) \subseteq \mathrm{supp}(\theta(\chi_{(1/2,1]}) \wedge \bar{1}),$$

which shows that $\theta(1) \wedge \bar{1} = \theta(\chi_{(1/2,1]}) \wedge \bar{1}$.

Using at the second step that $d(\varphi, \phi\theta) < 1/2$, we now have

$$\begin{aligned}\phi(\theta(\chi_{(1/2,1]}) \wedge 1) \leq \phi\theta(\chi_{(1/2,1]}) \ll \varphi(\chi_{(0,1]}) \\ \ll \varphi(1) = \phi(\theta(1) \wedge 1) = \phi(\theta(\chi_{(1/2,1]}) \wedge 1).\end{aligned}$$

This implies that $\varphi(\chi_{(0,1]}) = 1'$ is compact, a contradiction.

To finish this section, we prove that $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$ is '*semiprojective*' in the sense stated in Proposition 3.4.10 below. Note, in particular, that this result could be used to show the forward implication of Theorem 3.4.5.

Recall that the set $B_n^r \subset \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^r$ denotes the subset of finite sums of elements of the form 1 and $\chi_{(i/n,1]}$.

Proposition 3.4.10. *Let $S = \lim S_m$ be an inductive limit of weakly cancellative* Cu-*semigroups satisfying (O5), and let $\varphi\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^r \to S$ be a* Cu-*morphism.*

Then, for any $\varepsilon > 0$, $n \in \mathbb{N}$ and $x, x', y \in B_n^r$ with $x \ll x'$ and $\varphi(x') \ll \varphi(y)$, there exist $m \in \mathbb{N}$ and a Cu-*morphism $\theta\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}})^r \to S_m$ such that*

(i) $d(\tau_m\theta, \varphi) < \varepsilon$, where $\tau_m\colon S_m \to S$ denotes the canonical inclusion;

(ii) $\theta(x) \ll \theta(y)$;

(iii) $\tau_m\theta(1_j) = \varphi(1_j)$ for each $j \leq r$.

Proof. Note that it is enough to prove the result for $r = 1$, since the general result follows from a componentwise application of this case.

Thus, fix $\varepsilon > 0$ and let $N \in \mathbb{N}$ be large enough so that $2/N < \varepsilon$, $B_n \subseteq B_N$ and such that there exist $y', y'' \in B_N$ with $y' \ll y'' \ll y$ and $\varphi(x') \ll \varphi(y')$. Since φ is a Cu-morphism, one has

$$\varphi(0) \ll \varphi(\chi_{((N-1)/N,1]}) \ll \ldots \ll \varphi(\chi_{(1/N,1]}) \ll \varphi(\chi_{(0,1]}) \ll \varphi(1).$$

Now let $m \in \mathbb{N}$ be large enough so that there exist elements $s_0, \ldots, s_N, p \in S_m$ such that $\tau_m(p) = \varphi(1)$ and

$$\varphi(\chi_{(i/N,1]}) \ll \tau_m(s_i) \ll \varphi(\chi_{((i-1)/N,1]}) \quad \text{and} \quad s_i \ll s_{i-1} \ll p \ll p$$

for every i.

Since $(s_i)_i$ is a $\ll$-decreasing sequence in S_m bounded by p, we know by Theorem 3.2.6 that there exists a Cu-morphism $\theta\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}}) \to S_m$ mapping 1 to p and $\chi_{(i/N,1]}$ to s_i for each i. In particular, since $\tau_m\theta(1) = \varphi(1)$, condition (iii) holds.

Moreover, it follows that

$$\tau_m\theta(\chi_{(i/N,1]}) = \tau_m(s_i) \ll \varphi(\chi_{((i-1)/N,1]})$$

and

$$\varphi(\chi_{(i/N,1]}) \ll \tau_m(s_{i-1}) = \tau_m\theta(\chi_{((i-1)/N,1]}).$$

Using Lemma 3.3.7, this implies $d(\tau_m\theta, \varphi) < 1/N + 1/N < \varepsilon$, which shows that condition (i) is also satisfied.

By conditions (i) and (iii) we know that, for every pair $v, v' \in \{1, \chi_{(i/N,1]}\}_N$ such that $v \ll v'$, one gets

$$\tau_m\theta(v) \ll \varphi(v') \quad \text{and} \quad \varphi(v) \ll \tau_m\theta(v').$$

Since $x, x' \in B_N$, we can write $x = \sum_{k=1}^{\sup(x')} v_k$ and $x' = \sum_{k=1}^{\sup(x')} v'_k$ with $v_k \ll v'_k$ in B_N. This implies

$$\tau_m\theta(x) = \sum_{k=1}^{\sup(x')} \tau_m\theta(v_k) \ll \sum_{k=1}^{\sup(x')} \tau_m\varphi(v'_k) = \varphi(x')$$

and, similarly, one sees that $\varphi(y') \ll \tau_m\theta(y'')$.

Thus, we get

$$\tau_m\theta(x) \ll \varphi(x') \ll \varphi(y') \ll \tau_m\theta(y'') \ll \tau_m\theta(y).$$

Taking m large enough, one has $\theta(x) \ll \theta(y)$, as desired. $\square$

3.5 An abstract characterization

In this section we introduce Property I (see Definition 3.5.29), which we use to provide an abstract characterization for the Cuntz semigroups of AI-algebras as an application of Theorem 3.4.8; see Theorem 3.5.34.

Definition 3.5.1. For every $n \in \mathbb{N}$, set $\Omega_n = \{-\infty, 0, \ldots, n, \infty\}$. We define X_n as the free abelian monoid on

$$\{(\alpha, \beta) \mid n \neq \alpha \lneq \beta \neq 0, \quad \alpha, \beta \in \Omega_n\}.$$

We denote the unit of X_n by 0.

3.5.2 (Relations on X_n)**.** Let $n \in \mathbb{N}$. Given $\alpha, \beta \in \Omega_n$, we define $\alpha \prec \beta$ if $\alpha \lneq \beta$, or $\alpha = \beta = \infty$, or $\alpha = \beta = -\infty$.

We define relations $\prec$ and $\simeq$ in X_n as follows:

(i) For every pair $w, (\alpha, \beta) \in X_n$, we write $w \prec (\alpha, \beta)$ if and only if $w = 0$ or else if there exist $(\alpha_i, \beta_i) \in X_n$ such that

$$\alpha \prec \alpha_1 \prec \beta_1 \prec \alpha_2 \prec \ldots \prec \alpha_m \prec \beta_m \prec \beta$$

and $w = \sum(\alpha_i, \beta_i)$. In particular, $(\alpha', \beta') \prec (\alpha, \beta)$ if and only if $\alpha \prec \alpha' \prec \beta' \prec \beta$.

Set $0 \prec 0$. Given $w, v \in X_n$ we define $w \prec v$ if there exist (possibly zero and repeated) elements $w_i, (\alpha_i, \beta_i) \in X_n$ such that $w = \sum w_i$, $v = \sum(\alpha_i, \beta_i)$ and $w_i \prec (\alpha_i, \beta_i)$ as above for each i.

Note that $\prec$ is compatible with the addition in X_n.

(ii) Given $v, w \in X_n$, we write $w \approxeq v$ if $w = v$ or else if there exist $z \in X_n$ and $\alpha \prec \gamma \prec \beta \prec \delta$ in Ω_n such that

$$w = z + (\alpha, \beta) + (\gamma, \delta) \quad \text{and} \quad v = z + (\alpha, \delta) + (\gamma, \beta)$$

or

$$v = z + (\alpha, \beta) + (\gamma, \delta) \quad \text{and} \quad w = z + (\alpha, \delta) + (\gamma, \beta).$$

Given two elements $w, v \in X_n$, we define $w \simeq v$ if and only if there exist $w_1, \dots, w_m \in X_n$ such that $w \approxeq w_1 \approxeq \dots \approxeq w_m \approxeq v$.

The definition of $\simeq$ tries to mimic the relation introduced in Remark 3.1.2.

Lemma 3.5.3. *Let $n \in \mathbb{N}$, and take $w, v \in X_n$. Then, we have*

(i) If $w = \sum(\gamma_j, \delta_j) \prec (-\infty, \infty)$, there exists at most one j with $\gamma_j = -\infty$ (resp. at most one j with $\delta_j = \infty$).

(ii) $w + (-\infty, \infty) \prec (-\infty, \infty)$ if and only if $w = 0$.

(iii) $w + (-\infty, \infty) \prec v + (-\infty, \infty)$ if and only if $w \prec v$.

Proof. To prove (i), we may assume that $w = \sum_{j \le m}(\gamma_j, \delta_j) \prec (-\infty, \infty)$ with

$$-\infty \prec \gamma_1 \prec \dots \prec \delta_m \prec \infty$$

as in Paragraph 3.5.2 above.

If j is such that $\gamma_j = -\infty$, we have

$$-\infty = \gamma_1 = \delta_1 = \dots = \gamma_j = -\infty$$

and, since $\gamma_1 \neq \delta_1$, it follows that $j = 1$. In particular, j is unique.

Similarly, we see that there is at most one j such that $\delta_j = \infty$ for some j.

The argument in (i) shows that, whenever $w \prec (\alpha, \beta)$, w has a summand of the form $(-\infty, \infty)$ if and only if $w = (-\infty, \infty) = (\alpha, \beta)$.

Thus, we get that $w + (-\infty, \infty) \prec (-\infty, \infty)$ if and only if $w = 0$, and (ii) follows.

Finally, to see (iii), let $v' = v + (-\infty, \infty)$ and write $v' = \sum_{i \le k}(\alpha_i, \beta_i)$ and $w = \sum_{i \le k} w_i$ in such a way that $w_1 + (-\infty, \infty) \prec (\alpha_1, \beta_1)$ and $w_i \prec (\alpha_i, \beta_i)$ for $i \ge 2$.

From the first inequality and (ii), we have $w_1 = 0$ and $(\alpha_1, \beta_1) = (-\infty, \infty)$. Therefore, one gets $v = \sum_{i \ge 2}(\alpha_i, \beta_i)$.

Since $\prec$ is compatible with addition, we have

$$w = \sum_{i \ge 1} w_i = \sum_{i \ge 2} w_i \prec \sum_{i \ge 2} v_i = v,$$

as desired. □

Proposition 3.5.4. *Let $n \in \mathbb{N}$. Then, $\prec$ is a transitive and antisymmetric relation on X_n.*

Proof. To show that $\prec$ is transitive, let $w, v, u \in X_n$ be such that $w \prec v \prec u$. By definition (Paragraph 3.5.2), this means that $w = \sum w_i$, $v = \sum(\alpha_i, \beta_i) = \sum_j v_j$ and $u = \sum_j(\gamma_j, \delta_j)$ such that

$$w_i \prec (\alpha_i, \beta_i) \quad \text{and} \quad v_j \prec (\gamma_j, \delta_j)$$

for every i, j.

For each j, let I_j be the (possibly empty) set such that $\sum_{i\in I_j}(\alpha_i, \beta_i) = v_j$. Using that $\prec$ is compatible with addition, we get

$$\sum_{i\in I_j} w_i \prec \sum_{i\in I_j}(\alpha_i, \beta_i) = v_j \prec (\gamma_j, \delta_j).$$

Thus, since $w = \sum_j \sum_{i\in I_j} w_i$, we have $w \prec u$ using once again that $\prec$ is compatible with addition. This shows that $\prec$ is transitive.

Now let $v, w \in X_n$, and let $m \in \mathbb{N}$ be the number of nonzero summands of w. We will see by induction on m that $w \prec v \prec w$ if and only if $w = v = m(-\infty, \infty)$, where it is understood that $0(-\infty, \infty) = 0$. In particular, this will show that $\prec$ is antisymmetric.

If $m = 0$ or $m = 1$, it follows from Lemma 3.5.3 (i)-(ii) that $w \prec v \prec w$ if and only if $w = v = 0$ or $w = v = (-\infty, \infty)$. Thus, let $m > 1$ and assume that for a fixed m we have proven the result for any $v \in X_n$ and any w having $m - 1$ nonzero summands.

Take $w = \sum_{i\leq m}(\alpha_i, \beta_i)$ and $v = \sum_{j\leq k}(\gamma_j, \delta_j)$ such that $w \prec v \prec w$, where we assume that all summands are nonzero. Set $i_1 = 1$ and find j_1 such that $(\alpha_{i_1}, \beta_{i_1}) \prec (\gamma_{j_1}, \delta_{j_1})$. This can be done because $w \prec v$.

Now let i_2 be such that

$$(\alpha_{i_1}, \beta_{i_1}) \prec (\gamma_{j_1}, \delta_{j_1}) \prec (\alpha_{i_2}, \beta_{i_2}),$$

which exists because $v \prec w$.

If $i_2 = i_1$, we have

$$(\alpha_{i_2}, \beta_{i_2}) \prec (\gamma_{j_1}, \delta_{j_1}) \prec (\alpha_{i_2}, \beta_{i_2}),$$

which implies $(\alpha_{i_2}, \beta_{i_2}) = (\gamma_{j_1}, \delta_{j_1}) = (-\infty, \infty)$. Consequently, we can cancel these summands from $v \prec w \prec v$ by Lemma 3.5.3 (iii) and apply the induction hypothesis to deduce $w = v = m(-\infty, \infty)$.

Thus, we may assume $i_1 \neq i_2$. Proceeding as above, we obtain an ordering $i_1, \ldots, i_n$ of $\{1, \ldots, n\}$ and pairwise different integers $j_1, \ldots, j_n$ such that

$$(\alpha_{i_1}, \beta_{i_1}) \prec (\gamma_{j_1}, \delta_{j_1}) \prec \ldots \prec (\alpha_{i_n}, \beta_{i_n}) \prec (\gamma_{j_n}, \delta_{j_n}).$$

Since $v \prec w$, there exists $s \leq n$ such that $(\gamma_{j_n}, \delta_{j_n}) \prec (\alpha_s, \beta_s)$ and, since $i_1, \ldots, i_n$ is an ordering of $\{1, \ldots, n\}$, we must have $s = j_l$ for some $l \leq n$.

This implies

$$(\alpha_{i_l}, \beta_{i_l}) \prec (\gamma_{j_l}, \delta_{j_l}) \prec (\alpha_{i_l}, \beta_{i_l})$$

and, by the same argument as above, we get $w = v = m(-\infty, \infty)$, as required. ◻

Proposition 3.5.5. *Let $n \in \mathbb{N}$. Then, $\simeq$ is an equivalence relation on X_n compatible with addition.*

Proof. First note that, by definition (Paragraph 3.5.2), $\simeq$ is transitive, symmetric and reflexive. Thus, we only need to check that it is compatible with addition.

First, let $w \approxeq v$ with $w \neq v$, where we may assume that there exist $z \in X_n$ and $\alpha \prec \gamma \prec \beta \prec \delta$ in Ω_n such that

$$w = z + (\alpha, \beta) + (\gamma, \delta) \quad \text{and} \quad v = z + (\alpha, \delta) + (\gamma, \beta).$$

For any $w' \in X_n$, we have

$$w + w' = (z + w') + (\alpha, \beta) + (\gamma, \delta) \quad \text{and} \quad v + w' = (z + w') + (\alpha, \delta) + (\gamma, \beta),$$

which shows $w + w' \approxeq v + w'$. Trivially, if $w = v$, we also have $w + w' \approxeq v + w'$ for any w'.

This implies that, given $w \approxeq v$ and $w' \approxeq v'$, we get

$$w + w' \approxeq v + w' \approxeq v + v'$$

and, consequently, $w + w' \simeq v + v'$.

Now let $w \simeq v$ and $w' \simeq v'$. By definition, there exist elements $w_1, \ldots, w_m$ and $w'_1, \ldots, w'_{m'}$ in X_n such that

$$w \approxeq w_1 \approxeq \ldots \approxeq w_m \approxeq v \quad \text{and} \quad w' \approxeq w'_1 \approxeq \ldots \approxeq w'_{m'} \approxeq v',$$

where we may assume that $m = m'$ because $w \approxeq w$ for any $w \in X_n$.

Thus, one gets

$$w + w' \approxeq w_1 + w'_1 \approxeq \ldots \approxeq w_m + w'_m \approxeq v + v',$$

which implies $w + w' \simeq v + v'$ as desired. $\square$

Chainable subsets

We now use the set X_n and the relations $\prec, \simeq$ defined in Paragraph 3.5.2 to introduce the notion of chainable subsemigroups. This is related to topological chainability and the chainable Cu-semigroups from Chapter 2.

To introduce such notion, we first need a way to think of X_n as a subset of a Cu-semigroup. This is achieved using I-morphisms, defined below.

Definition 3.5.6. Let S be a Cu-semigroup and let $F\colon X_n \to S$ be an additive map. We will say that F is an *I-morphism* if $F(v) \ll F(w)$ whenever $v \prec w$ and $F(v) = F(w)$ whenever $v \simeq w$.

Definition 3.5.7. Let e be a compact element in a Cu-semigroup S, and let $n \in \mathbb{N}$. A subsemigroup H of S containing e is said to be an (n, e)*-chainable subset* if there exists an I-morphism $F\colon X_n \to S$ with $F(X_n) = H$ satisfying the following properties:

(i) $F((-\infty, \infty)) = e$;

(ii) $F((\alpha, \beta)) + F((\beta, \infty)) \leq F((\alpha, \infty))$;

(iii) For every integer $m \geq 1$ there exists an I-morphism $F'\colon X_{mn} \to S$ such that $F'((m\alpha, m\beta)) = F((\alpha,\beta))$ for every $(\alpha,\beta) \in X_n$.

Remark 3.5.8. Let S be a Cu-semigroup. Given a Cu-morphism $\varphi\colon S \to T$ and an (n,e)-chainable subsemigroup H of S, it is readily checked that the subsemigroup $\varphi(H)$ is an $(n,\varphi(e))$-chainable subset of T.

Also note that, if S is a weakly cancellative Cu-semigroup and $F\colon X_n \to S$ is an I-morphism, the inequality

$$F(0) + F((-\infty,\infty)) = F((-\infty,\infty)) \ll F((-\infty,\infty))$$

implies, by weak cancellation, that $F(0) = 0$.

Example 3.5.9. Given any compact element e in a Cu-semigroup S, the additive span H of e (i.e. the set of finite multiples of e) is (n,e)-chainable for every n. Indeed, let $n \in \mathbb{N}$ and define the additive map $F_n\colon X_n \to H$ as $F_n((\alpha,\infty)) = e$ and $F_n((\alpha,\beta)) = 0$ whenever $\beta \neq \infty$.

It follows from Lemma 3.5.3 (i) that, if $w, v \in X_n$ satisfy $w \prec v$, then the number of summands of the form (α,∞) of w must be less than or equal to that of v. In particular, this implies that $F_n(w) \ll F_n(v)$ whenever $w \prec v$.

Now let $\alpha \prec \gamma \prec \beta \prec \delta$ in Ω_n. If $\delta \neq \infty$ we have

$$F_n((\alpha,\beta) + (\gamma,\delta)) = 0 = F_n((\alpha,\delta) + (\gamma,\beta))$$

and, if $\delta = \infty$, one gets

$$F_n((\alpha,\beta) + (\gamma,\delta)) = F_n((\alpha,\delta) + (\gamma,\beta)).$$

Using that F_n is additive, this implies $F_n(w) = F_n(v)$ whenever $w \simeq v$. Consequently, we have that F_n is an I-morphism.

Properties (i) and (ii) of Definition 3.5.7 follow by construction. To see (iii), let $m \in \mathbb{N}$ and consider the I-morphism $F_{nm}\colon X_{nm} \to H$.

Example 3.5.10. For every $n \in \mathbb{N}$, let L_n be the additive span of

$$\{0, 1, \chi_{(i/n,j/n)}, \chi_{(i/n,1]}, \chi_{[0,j/n)}\}_{i,j}$$

in $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$. Then, L_n is an $(n,1)$-chainable subset of $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$.

Let $(\alpha,\beta) \in X_n$. If $\alpha \neq -\infty$ and $\beta \neq \infty$, let $(\alpha/n, \beta/n)$ denote the corresponding interval in $[0,1]$. If $\alpha = -\infty$ but $\beta \neq \infty$, $(-\infty/n, \beta/n)$ denotes the interval $[0,\beta/n)$. Similarly, if $\alpha \neq -\infty$ and $\beta = \infty$, $(\alpha/n, \infty/n)$ corresponds to $(\alpha/n, 1]$, while $(-\infty/n, \infty/n)$ is the unit interval $[0,1]$.

Define the additive map $G_n\colon X_n \to L_n$ as $G_n((\alpha,\beta)) = \chi_{(\alpha/n,\beta/n)}$, which we now check is an I-morphism.

First, let $\alpha \prec \gamma \prec \beta \prec \delta$ in Ω_n, which implies that

$$(\alpha/n,\beta/n) \cup (\gamma/n,\delta/n) = (\alpha/n,\delta/n) \cup (\gamma/n,\beta/n),$$
$$(\alpha/n,\beta/n) \cap (\gamma/n,\delta/n) = (\alpha/n,\delta/n) \cap (\gamma/n,\beta/n).$$

Thus, one gets

$$\begin{aligned} G_n((\alpha,\beta)+(\gamma,\delta)) &= \chi_{(\alpha/n,\beta/n)} + \chi_{(\gamma/n,\delta/n)} = \chi_{(\alpha/n,\beta/n)\cup(\gamma/n,\delta/n)} + \chi_{(\alpha/n,\beta/n)\cap(\gamma/n,\delta/n)} \\ &= \chi_{(\alpha/n,\delta/n)\cup(\gamma/n,\beta/n)} + \chi_{(\alpha/n,\delta/n)\cap(\gamma/n,\beta/n)} = \chi_{(\alpha/n,\delta/n)} + \chi_{(\gamma/n,\beta/n)} \\ &= G_n((\alpha,\delta)+(\gamma,\beta)). \end{aligned}$$

Using the additivity of G_n once again, we have $G_n(v) = G_n(w)$ whenever $v \simeq w$ in X_n.

Now, given $w, (\alpha,\beta) \in X_n$ such that $w \prec (\alpha,\beta)$, we know by definition (see Paragraph 3.5.2) that $w = \sum_{i=1}^m (\alpha_i,\beta_i)$ with

$$\alpha \prec \alpha_1 \prec \beta_1 \prec \alpha_2 \prec \ldots \prec \alpha_m \prec \beta_m \prec \beta.$$

This in turn implies

$$\cup(\alpha_i/n,\beta_i/n) \Subset (\alpha/n,\beta/n) \quad \text{and} \quad (\alpha_i/n,\beta_i/n) \cap (\alpha_j/n,\beta_j/n) = \emptyset$$

for every pair $i \neq j$.

Consequently, we have

$$G_n(w) = \sum_{i=1}^m \chi_{(\alpha_i/n,\beta_i/n)} \ll \chi_{(\alpha/n,\beta/n)} = G_n((\alpha,\beta)),$$

which implies, using the additivity of G_n and the definition of $\prec$, that $G_n(v) \ll G_n(w)$ whenever $v \prec w$.

By definition, one gets $G_n((-\infty,\infty)) = \chi_{[0,1]} = 1$ and

$$G_n((\alpha,\beta)) + G_n((\beta,\infty)) = \chi_{(\alpha/n,\beta/n)} + \chi_{(\beta/n,\infty/n)} \leq \chi_{(\alpha/n,\infty/n)} = G_n((\alpha,\infty)),$$

which shows that (i) and (ii) in Definition 3.5.7 are satisfies

Further, for every $m \geq 1$ we can consider the map $G_{nm}\colon X_{nm} \to L_{nm}$ to check condition (iii). This implies that L_n is $(n,1)$-chainable, as desired.

Moreover, note that G_n is clearly surjective for every n, since every indicator function in L_n is in the image of G_n.

As we have seen in Example 3.5.10 above, the subsemigroup L_n of $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ is $(n,1)$-chainable. As this chainable subsemigroup will of importance for the remainder of the chapter, we study its associated I-morphism in detail; see Proposition 3.5.15.

Lemma 3.5.11. *Let $n \in \mathbb{N}$ and $f \in L_n$. Denote by G_n the additive map $X_n \to L_n$ defined as $G_n((\alpha,\beta)) = \chi_{(\alpha/n,\beta/n)}$. Then, there exists $q_f \in X_n$ such that $w \simeq q_f$ whenever $G_n(w) = f$.*

Proof. Given (α,β) and $w = \sum_{i=1}^m (\alpha_i,\beta_i)$ in X_n, we define

$$\Lambda_w((\alpha,\beta)) = |\{i \leq m \mid \alpha < \alpha_i < \beta < \beta_i\}|$$

and set $\Lambda(w) = \sum_i \Lambda_w((\alpha_i,\beta_i))$.

Claim. *For every $w \in X_n$, there exists $w' \simeq w$ such that $\Lambda(w') = 0$.*

To prove the claim, note that it is enough to find $w' \simeq w$ such that $\Lambda(w') < \Lambda(w)$, since a repeated application of this fact will give us the desired element.

Thus, write $w = \sum_{i=1}^{m}(\alpha_i, \beta_i)$ with $\Lambda_w((\alpha_1, \beta_1))$ maximal. Note that we may assume $\Lambda_w((\alpha_1, \beta_1))$ to be nonzero, since otherwise we set $w' = w$.

Let j be such that $\alpha_1 < \alpha_j < \beta_1 < \beta_j$. Then, one has

$$\Lambda_w((\alpha_1, \beta_j)) + \Lambda_w((\alpha_j, \beta_1)) \leq \Lambda_w((\alpha_1, \beta_1)) + \Lambda_w((\alpha_j, \beta_j)) - 1.$$

Indeed, let (α_i, β_i) be such that $\alpha_1 < \alpha_i < \beta_j < \beta_i$. If $\alpha_j < \alpha_i < \beta_1 < \beta_i$, we have

$$\alpha_1 < \alpha_i < \beta_1 < \beta_i \quad \text{and} \quad \alpha_j < \alpha_i < \beta_j < \beta_i.$$

Else, one gets $\alpha_1 < \alpha_i < \beta_1 < \beta_i$ or $\alpha_j < \alpha_i < \beta_j < \beta_i$.

Similarly, if (α_i, β_i) satisfies $\alpha_j < \alpha_i < \beta_1 < \beta_i$, we have $\alpha_1 < \alpha_i < \beta_1 < \beta_i$ or $\alpha_j < \alpha_i < \beta_j < \beta_i$. Moreover, note that j is in $\{i \leq m \mid \alpha_1 < \alpha_i < \beta_1 < \beta_i\}$, while it is not in the set corresponding to (α_1, β_j) nor (α_j, β_1). This shows the desired inequality.

To finish the proof of the Claim, note that $w' = (\alpha_1, \beta_j) + (\alpha_j, \beta_1) + \sum_{i \neq 1,j}(\alpha_i, \beta_i)$ satisfies

$$w' \simeq w \quad \text{and} \quad \Lambda(w') < \Lambda(w),$$

as required.

Now let $f \in X_n$ and take $v \in X_n$ such that $G_n(v) = f$. We let q_f be the element given by the Claim applied to v. Since $v \simeq q_f$ and G_n is a I-morphism by Example 3.5.10, we have $f = G_n(q_f)$.

Write $q_f = \sum_{i=1}^{m}(\alpha_i, \beta_i)$. Following the notation from Example 3.5.10, note that for every pair $i, j \leq m$ we either have that $(\alpha_i/n, \beta_i/n)$ and $(\alpha_j/n, \beta_j/n)$ are disjoint, or that one interval is contained inside the other. This shows that there exist pairwise disjoint subsets $I_k \subseteq \{1, \ldots, m\}$ such that

$$\{G_n(w) \geq k\} = \sqcup_{i \in I_k}(\alpha_i/n, \beta_i/n)$$

for every k.

Now take $w \in X_n$ such that $G_n(w) = f$, and let w' be the element given by the Claim applied to w. Using that $G_n(w') = f = G_n(q_f)$ and writing $w' = \sum_j(\gamma_j, \delta_j)$, we know that there exist pairwise disjoint sets J_k such that

$$\sqcup_{j \in J_k}(\gamma_j/n, \delta_j/n) = \{f \geq k\} = \sqcup_{i \in I_k}(\alpha_i/n, \beta_i/n),$$

for every k, which implies that $w' = q_f$.

Therefore, we get $w \simeq w' = q_f$, as desired. □

3.5.12 (The subset L_n^0 of L_n)**.** Following the notation of Lemma 3.5.11 above, note that we do not always have $v \simeq v' \prec w' \simeq w$ in X_n whenever $G_n(v) \ll G_n(w)$.

For example, let $v = (1,2) + (2,3)$ and $w = (0,4)$ in X_4. We have $G_4(v) \ll G_4(w)$, but there are no elements $v', w' \in X_4$ such that $v \simeq v' \prec w' \simeq w$. Indeed, such elements would satisfy $w' = w$ and $v' = v$, and it is clear that v is not $\prec$-below w.

To ammend this, let L_n^0 be the subset of L_n consisting of the functions $f \in L_n$ such that, for every $k \leq \sup(f)$, the connected components of $\{f \geq k\}$ are at pairwise distance at least $2/n$.

Proposition 3.5.15 (iii) below shows that one can find v', w' with $v \simeq v' \prec w' \simeq w$ whenever $G_n(v) \ll G_n(w)$ with $G_n(v), G_n(w)$ in L_n^0.

Recall that, for every $\varepsilon > 0$ and basic element $f \in \mathrm{Lsc}([0,1], \overline{\mathbb{N}})$, $R_\varepsilon(f)$ denotes its ε-retraction; see Paragraph 3.1.4.

Lemma 3.5.13. *Let $n \in \mathbb{N}$ and take $f \in L_n$. For every rational $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that $R_\varepsilon(f) \in L_m^0$.*

Proof. Write $\varepsilon = m_1/m_2$ for some integers m_1 and $m_2 > 1$, and let m be nm_2. It follows that $R_\varepsilon(f) \in L_m$.

Further, since $\{f \geq k\}$ is a finite disjoint union of open intervals for each $k \leq \sup(f)$, the ε-retractions of these intervals are at pairwise distance at least $2\varepsilon = 2m_1 n/m \geq 2/m$, as desired. □

Lemma 3.5.14. *Let $n \in \mathbb{N}$ and let $f, g \in L_n^0$ be such that $f \ll g$. Let G_n be the I-morphism $X_n \to L_n$ defined as $G_n((\alpha,\beta)) = \chi_{(\alpha/n,\beta/n)}$. Then, there exist elements $v, w \in X_n$ such that $v \prec w$, $G_n(v) = f$ and $G_n(w) = g$.*

Proof. Given $f, g \in L_n^0$, there exist I_k, J_k finite sets and $(\alpha_{i,k}, \beta_{i,k})$, $(\gamma_{j,k}, \delta_{j,k})$ elements of X_n such that

$$\{f \geq k\} = \sqcup_{i\in I_k}(\alpha_{i,k}/n, \beta_{i,k}/n) \quad \text{and} \quad \{g \geq k\} = \sqcup_{j\leq J_k}(\gamma_{j,k}/n, \delta_{j,k}/n)$$

for every $k \leq \max\{\sup(f), \sup(g)\}$.

Set $v = \sum_k \sum_{i\in I_k}(\alpha_{i,k}, \beta_{i,k})$ and $w = \sum_k \sum_{j\in J_k}(\gamma_{j,k}, \delta_{j,k})$ in X_n, which satisfy $G_n(v) = f$ and $G_n(w) = g$ by construction.

Since $f \ll g$, we know from Proposition 2.2.19 that $\{f \geq k\} \Subset \{g \geq k\}$ for every k. Therefore, for every fixed k there exists a partition $\sqcup_{j\in J_k} B_{j,k} = I_k$ satisfying

$$\sqcup_{i\in B_{j,k}}(\alpha_{i,k}/n, \beta_{i,k}/n) \Subset (\gamma_{j,k}/n, \delta_{j,k}/n)$$

for each $j \in J_k$.

Thus, there is an ordering $i_1, \ldots, i_{|B_{j,k}|}$ of $B_{j,k}$ such that

$$\gamma_{j,k} \prec \alpha_{i_1,k} \prec \beta_{i_1,k} \leq \alpha_{i_2,k} \prec \ldots \prec \beta_{i_{|B_{j,k}|-1},k} \leq \alpha_{i_{|B_{j,k}|},k} \prec \beta_{i_{|B_{j,k}|},k} \prec \delta_{j,k}.$$

Using that f is an element in L_n^0, one gets

$$\gamma_{j,k} \prec \alpha_{i_1,k} \prec \beta_{i_1,k} \prec \alpha_{i_2,k} \prec \ldots \prec \beta_{i_{|B_{j,k}|-1},k} \prec \alpha_{i_{|B_{j,k}|},k} \prec \beta_{i_{|B_{j,k}|},k} \prec \delta_{j,k},$$

since we cannot have $\beta_{i_s,k} = \alpha_{i_{s+1},k}$ for any $s < |B_{j,k}|$.

By the definition of $\prec$ in Paragraph 3.5.2, we have

$$\sum_{i\in B_{j,k}} (\alpha_{i,k}, \beta_{i,k}) \prec (\gamma_{j,k}, \delta_{j,k})$$

and, using that $\prec$ is compatible with addition by definition, we get

$$v = \sum_k \sum_{i\in I_k}(\alpha_{i,k}, \beta_{i,k}) = \sum_k \sum_{j\in J_k} \sum_{i\in B_{j,k}} (\alpha_{i,k}, \beta_{i,k}) \prec \sum_k \sum_{j\in J_k}(\gamma_{j,k}, \delta_{j,k}) = w,$$

as desired. □

Proposition 3.5.15. *Let $n \in \mathbb{N}$, and denote by G_n the I-morphism $X_n \to L_n$ defined as $G_n((\alpha,\beta)) = \chi_{(\alpha/n,\beta/n)}$. Let $f, g \in L_n$. Then, there exist elements $q_f, q_g \in X_n$ satisfying*

(i) $G_n(q_f) = f$ *and* $G_n(q_g) = g$;

(ii) $v' \simeq v \simeq q_f$ *if and only if* $G_n(v') = G_n(v) = f$;

(iii) assuming $f, g \in L_n^0$, $v \simeq q_f \prec q_g \simeq w$ *if and only if* $G_n(v) = f \ll g = G_n(w)$.

Proof. Let q_f, q_g be the elements given by Lemma 3.5.11 applied to f and g respectively. It follows that (i) is satisfied. Also note that necessity in (ii) and (iii) follows from the fact that G_n is an I-morphism; see Example 3.5.10.

Moreover, given $v, v' \in X_n$ such that $G_n(v') = G_n(v) = f$, we know that $v' \simeq q_f$ and $v \simeq q_f$. This shows condition (ii).

Finally, to see (iii), apply Lemma 3.5.14 to $f \ll g$ to obtain v', w' such that $v' \prec w'$, $G_n(v') = f$ and $G_n(w') = g$.

Using the notation from the proof of Lemma 3.5.11 note that, by construction, we have $\Lambda(v') = \Lambda(w') = 0$. Since we also have $\Lambda(q_f) = \Lambda(q_g) = 0$, the argument in the proof of Lemma 3.5.11 shows $v' = q_f$ and $w' = q_g$.

Condition (iii) now follows from (ii). □

Corollary 3.5.16. *Given $n \in \mathbb{N}$, let G_n be the I-morphism $X_n \to L_n$ defined as $G_n((\alpha,\beta)) = \chi_{(\alpha/n,\beta/n)}$, and let $v \simeq \sum_i (\alpha_i, \beta_i)$ in X_n. Then, for every $0 < \varepsilon < 1/2n$, we have $R_\varepsilon(G_n(v)) = \sum_i R_\varepsilon(G_n((\alpha_i,\beta_i)))$.*

Proof. Let U, V be intervals of the form $(\alpha/n, \beta/n)$ for some $\alpha, \beta \in \Omega_n$, and take $0 < \varepsilon < 1/2n$.

A straightforward computation shows that

$$R_\varepsilon(U \cup V) = R_\varepsilon(U) \cup R_\varepsilon(V) \quad \text{and} \quad R_\varepsilon(U \cap V) = R_\varepsilon(U) \cap R_\varepsilon(V).$$

Thus, given $v, (\alpha_i, \beta_i) \in X_n$ as in the statement of the corollary, we get

$$\cup_{|I|=k} \cap_{i\in I} R_\varepsilon((\alpha_i/n, \beta_i/n)) = R_\varepsilon(\cup_{|I|=k} \cap_{i\in I} (\alpha_i/n, \beta_i/n)) = R_\varepsilon(\{G_n(v) \geq k\}),$$

and, therefore,

$$\begin{aligned}\sum_i R_\varepsilon(G_n((\alpha_i,\beta_i))) &= \sum_i \chi_{R_\varepsilon((\alpha_i/n,\beta_i/n))} = \sum_k \chi_{\cup_{|I|=k}\cap_{i\in I} R_\varepsilon((\alpha_i/n,\beta_i/n))} \\ &= \sum_k \chi_{R_\varepsilon(\{G_n(v)\geq k\})} = G_n(v),\end{aligned}$$

as desired. □

With Proposition 3.5.15 above at hand, we will now prove that (n,e)-chainable subsemigroups are deeply related to the Cu-semigroup $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$; see Proposition 3.5.18.

Throughout the remainder of this subsection, H will denote an (n,e)-chainable subsemigroup of a weakly cancellative Cu-semigroup S satisfying (O5). We let $F\colon X_n \to S$ be the I-morphism associated to H.

Note that, for any $m \geq 1$ and $F' \colon X_{nm} \to S$ satisfying (iii) in Definition 3.5.7 for F, the sequence $(F'((\alpha,\infty)))_{\alpha\in\Omega_{nm}}$ is bounded by e and is $\ll$-decreasing. Indeed, F' is an I-morphism and $(\alpha,\infty) \prec (\alpha-1,\infty)$ for every α.

Thus, using the notation from Example 3.5.10, we know by Theorem 3.2.6 that there exists a Cu-morphism $\mathrm{Lsc}([0,1],\overline{\mathbb{N}}) \to S$ mapping $\chi_{(\alpha/nm,1]}$ to $F'((\alpha,\infty))$ for every $\alpha \in \Omega_{nm}$. We fix one such Cu-morphism, which we will denote by ρ_m.

In particular, we note that (iii) in Definition 3.5.7 implies

$$\rho_m(\chi_{(\alpha/n,1]}) = \rho_m(\chi_{(m\alpha/mn,1]}) = F'((m\alpha,\infty)) = F((\alpha,\infty))$$

for every $m \geq 1$ and $\alpha \in \Omega_n$.

Let $\varepsilon > 0$. Recall from Paragraph 3.1.4 that, given any basic indicator function χ_U, we denote by $R_\varepsilon(\chi_U)$ the element $\chi_{R_\varepsilon(U)}$, where $R_\varepsilon(U)$ is the ε-retraction of U. Similarly, we let $N_\varepsilon(\chi_U)$ denote $\chi_{N_\varepsilon(U)}$, where $N_\varepsilon(U)$ is the open ε-neighbourhood of U.

Lemma 3.5.17. *Let G_n denote the I-morphism $X_n \to L_n$ defined as $G_n((\alpha,\beta)) = \chi_{(\alpha/n,\beta/n)}$. Then, for every $\varepsilon > 0$, $m \geq \varepsilon^{-1}$ and $(\alpha,\beta) \in X_n$, one has*

$$\rho_m(R_\varepsilon(G_n((\alpha,\beta)))) \ll F((\alpha,\beta)) \ll \rho_m(N_\varepsilon(G_n((\alpha,\beta)))).$$

Proof. Note that, for $\beta = \infty$, the result follows from the fact that ρ_m is a Cu-morphism and that $\rho_m(\chi_{(\alpha/n,1]}) = F((\alpha,\infty))$. Thus, we assume otherwise.

Using (ii) in Definition 3.5.7 and the observation prior to this lemma, we know that

$$F((\alpha,\beta)) + \rho_m(\chi_{(\beta/n,1]}) = F((\alpha,\beta)) + F((\beta,\infty)) \leq F((\alpha,\infty)) = \rho_m(\chi_{(\alpha/n,1]})$$

and, consequently,

$$\begin{aligned} F((\alpha,\beta)) + \rho_m(\chi_{(\beta/n,1]}) &\leq \rho_m(\chi_{(\alpha/n,1]}) \ll \rho_m(N_\varepsilon(\chi_{(\alpha/n,1]})) \\ &\leq \rho_m(N_\varepsilon(G_n((\alpha,\beta)))) + \rho_m(\chi_{(\beta/n,1]}) \end{aligned}$$

for any $\varepsilon > 0$.

It follows from weak cancellation that $F((\alpha,\beta)) \ll \rho_m(N_\varepsilon(G_n((\alpha,\beta))))$. Moreover, since

$$(m\alpha, m\beta) + (m\beta - n, \infty) \simeq (m\alpha,\infty) + (m\beta - n, m\beta),$$

we get

$$\begin{aligned} F((\alpha,\beta)) + F'((m\beta-n,\infty)) &= F'((m\alpha,m\beta)) + F'((m\beta-n,\infty)) \\ &= F'((m\alpha,\infty)) + F'(m\beta-n,m\beta) \\ &\geq F'((m\alpha,\infty)) = F((\alpha,\infty)). \end{aligned}$$

Using that ρ_m is a Cu-morphism at the first step and the above inequality at the third step, we have

$$\begin{aligned} \rho_m(R_{1/m}(G_n((\alpha,\beta)))) + \rho_m(\chi_{(\beta/n-1/m,1]}) &\ll \rho_m(\chi_{(\alpha/n,1]}) = F((\alpha,\infty)) \\ &\leq F((\alpha,\beta)) + \rho_m(\chi_{(\beta/n-1/m,1]}). \end{aligned}$$

Thus, since $\varepsilon > 1/m$, it follows from weak cancellation that

$$\rho_m(R_\varepsilon(G_n((\alpha,\beta)))) \leq \rho_m(R_{1/m}(G_n((\alpha,\beta)))) \ll F((\alpha,\beta)),$$

as required. □

Proposition 3.5.18. *Let $\{v_i \prec w_i\}_{i=1,\dots,N}$ be a finite family of $\prec$-relations in X_n, and let $G_n\colon X_n \to L_n$ be the I-morphism defined as $G_n((\alpha,\beta)) = \chi_{(\alpha/n,\beta/n)}$. Take $\varepsilon < 1/2n$ positive, and set $f_i = R_\varepsilon(G_n(v_i))$ and $g_i = R_\varepsilon(G_n(w_i))$ for each i. Then, the pairs f_i, g_i satisfy*

(i) $f_i \ll g_i$ for each i;

(ii) For every sufficiently large m, $\rho_m(f_i) \ll F(v_i) \ll \rho_m(g_i) \ll F(w_i)$ for each i;

(iii) $f_i = f_j$ (resp. $f_i = g_j$, $g_i = g_j$) whenever $v_i = v_j$ (resp. $v_i = w_j$, $w_i = w_j$) for every pair i, j.

Proof. Since $v_i \prec w_i$ for each i, we know from Paragraph 3.5.2 that we can write $v_i = \sum_{j\leq M, k\leq K}(\alpha^i_{j,k}, \beta^i_{j,k})$ and $w_i = \sum_{j\leq M'}(\alpha^i_j, \beta^i_j)$ in such a way that

$$\alpha^i_j \prec \alpha^i_{j,1} \prec \beta^i_{j,1} \prec \ldots \prec \beta^i_{j,K} \prec \beta^i_j$$

for every j, that is to say $\sum_k(\alpha^i_{j,k}, \beta^i_{j,k}) \prec (\alpha^i_j, \beta^i_j)$ for each j. Here, it is understood that K depends on both i and j, and that M, M' depend on i.

Thus, since G_n is an I-morphism, we have

$$\sum_k G_n((\alpha^i_{j,k}, \beta^i_{j,k})) \ll G_n((\alpha^i_j, \beta^i_j))$$

for every i, j.

Now let $\varepsilon < 1/2n$ positive, and note that, for every $\alpha \prec \alpha' \prec \beta' \prec \beta$ in Ω_n, we have $(\alpha'/n - \varepsilon, \beta'/n + \varepsilon) \Subset (\alpha/n + \varepsilon, \beta/n - \varepsilon)$ in $[0,1]$. Thus, we have

$$\sum_k G_n((\alpha^i_{j,k}, \beta^i_{j,k})) \ll \sum_k N_\varepsilon(G_n((\alpha^i_{j,k}, \beta^i_{j,k}))) \ll R_\varepsilon(G_n((\alpha^i_j, \beta^i_j)))$$

for each pair i, j.

Now let $m > 2n$. Then, using Lemma 3.5.17 at the first, second, and last step, and that ρ_m is a Cu-morphism at the third step, we get

$$\begin{aligned}\sum_k \rho_m(R_\varepsilon(G_n((\alpha^i_{j,k}, \beta^i_{j,k})))) &\ll \sum_k F((\alpha^i_{j,k}, \beta^i_{j,k})) \ll \sum_k \rho_m(N_\varepsilon(G_n((\alpha^i_{j,k}, \beta^i_{j,k}))))\\ &\ll \rho_m(R_\varepsilon(G_n((\alpha^i_j, \beta^i_j)))) \ll F((\alpha^i_j, \beta^i_j)).\end{aligned}$$

Adding on j, and using Corollary 3.5.16 at the first and fourth steps, we have

$$\begin{aligned}\rho_m(f_i) = \sum_{j,k} \rho_m(R_\varepsilon(G_n((\alpha^i_{j,k}, \beta^i_{j,k})))) &\ll F(v_i)\\ \ll \rho_m(g_i) = \sum_j \rho_m(R_\varepsilon(G_n((\alpha^i_j, \beta^i_j)))) &\ll F(w_i),\end{aligned}$$

which is condition (ii).

By the comments above, we also have

$$\sum_k R_\varepsilon(G_n((\alpha^i_{j,k}, \beta^i_{j,k}))) \ll \sum_k G_n((\alpha^i_{j,k}, \beta^i_{j,k})) \ll R_\varepsilon(G_n((\alpha^i_j, \beta^i_j))),$$

which implies condition (i) in the same fashion.

Condition (iii) follows by construction, since we have applied the same retraction to all the elements. $\square$

Properties I_0 and I

In this last subsection we introduce property I (Definition 3.5.29), which may be seen as a generalization of the properties of weak chainability and being Lsc-like introduced in Chapter 2. In this sense, and understanding the property of being compactly bounded (Definition 3.4.3) as a weakening of having a compact order unit, Theorem 3.5.34 below is a direct generalization of the abstract characterization for the Cuntz semigroup of unital commutative AI-algebras obtained in Theorem 2.5.12.

3.5.19 (Sequences on Ω_m)**.** Let $m \in \mathbb{N}$. Given a sequence $(z_\alpha)_{\alpha\in\Omega_m}$ in a Cu-semigroup S, we will say that the sequence is $\ll$*-decreasing* if $z_\infty = 0$ and $z_\alpha \ll z_\beta$ whenever $\beta \prec \alpha$. In particular, note that $z_{-\infty} \ll z_{-\infty}$.

Similarly, a sequence $(v_\alpha)_{\alpha\in\Omega_m}$ in X_n will be said to be $\prec$*-decreasing* if $v_\infty = 0$ and $v_\alpha \prec v_\beta$ whenever $\beta \prec \alpha$.

Property I_0

We begin by introducing property I_0 and its reduced version. Even though these two notions will not be used in Theorem 3.5.34, they ilustrate the main ideas of its proof.

By a *multiset* we mean a collection of elements that allows multiple instances of each element. For example, $\{0, 0, 1\}$ is a multiset in $\mathbb{N}$.

Definition 3.5.20. We say that a Cu-semigroup S has the *reduced* I_0 *property* if, for any $m \in \mathbb{N}$, any $\ll$-decreasing sequence $(z_\alpha)_{\alpha\in\Omega_m}$, and finite multisets $\mathcal{A}, \mathcal{B}$ of Ω_m such that $\sum_{\alpha\in\mathcal{A}} z_\alpha \ll \sum_{\beta\in\mathcal{B}} z_\beta$, there exists a chainable subsemigroup H of S with associated I-morphism $F\colon X_n \to S$ such that

(i) There exists a $\prec$-decreasing sequence $(v_\alpha)_{\alpha\in\Omega_m}$ in X_n satisfying

$$F(v_\alpha) \ll z_\alpha \ll F(v_\beta)$$

for every $\beta \prec \alpha$;

(ii) There exist $v, w \in X_n$ such that $\sum_{\alpha\in\mathcal{A}} v_\alpha \simeq v \prec w \simeq \sum_{\beta\in\mathcal{B}} v_\beta$.

Remark 3.5.21. Note that condition (i) in Definition 3.5.20 above is shorthand notation for

$$0 = z_\infty \ll F(v_\infty) \ll z_m \ll F(v_m) \ll z_{m-1} \ll \ldots \ll F(v_0) \ll z_0 \ll F(v_{-\infty}) = z_{-\infty}$$

with $F(v_{-\infty}) = z_{-\infty}$ compact.

Recall that we denote by B_m the additive span of the set $\{1\} \cup \{\chi_{(i/m,1]}\}_i$ in $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$. With the notation of Example 3.5.10, we can see B_m as the additive span of $\{\chi_{(\alpha/m,1]}\}_{\alpha\in\Omega_m}$.

Proposition 3.5.22. *Let S be a weakly cancellative* Cu*-semigroup satisfying (O5) and the reduced* I_0 *property, and let $\varphi\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}}) \to S$ be a* Cu*-morphism. Then, for every $m \in \mathbb{N}$ and $x, x', y \in B_m$ with $x \ll x'$ and $\varphi(x') \ll \varphi(y)$, there exist* Cu*-morphisms $\theta\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}}) \to \mathrm{Lsc}([0,1], \overline{\mathbb{N}})$ and $\phi\colon \mathrm{Lsc}([0,1], \overline{\mathbb{N}}) \to S$ such that*

(i) *for every* $i \leq m$,

$$\varphi\left(\chi_{\left(\frac{i}{m}+\frac{1}{m},1\right]}\right) \ll \phi\theta\left(\chi_{\left(\frac{i}{m},1\right]}\right) \quad \textit{and} \quad \phi\theta\left(\chi_{\left(\frac{i}{m}+\frac{1}{m},1\right]}\right) \ll \varphi\left(\chi_{\left(\frac{i}{m},1\right]}\right);$$

(ii) $\theta(x) \ll \theta(y)$;

(iii) $\varphi(1) = \phi\theta(1)$.

Proof. Let x, x', y be as in the statement. Since $x', y \in B_m$, there exist multisets $\mathcal{A}, \mathcal{B}$ of Ω_m such that $x' = \sum_{\alpha\in\mathcal{A}} \chi_{(\alpha/m,1]}$ and $y = \sum_{\beta\in\mathcal{B}} \chi_{(\beta/m,1]}$.

Set $z_\alpha = \varphi(\chi_{(\alpha/2m,1]})$ for every $\alpha \in \Omega_{2m}$, where note that $(z_\alpha)_\alpha$ is $\ll$-decreasing. Moreover, we have

$$\sum_{\alpha\in\mathcal{A}} z_{2\alpha} = \varphi(x') \ll \varphi(y) = \sum_{\beta\in\mathcal{B}} z_{2\beta}$$

By the reduced I_0 property, there is a chainable subsemigroup H with associated I-morphism $F\colon X_n \to S$ such that there exists a sequence $(v_\alpha)_{\alpha\in\Omega_{2m}}$ in X_n and $v, w \in X_n$ satisfying (i)-(iii) in Definition 3.5.20.

In particular, we have

$$\varphi(\chi_{(i/m,1]}) = z_{2i} \ll F(v_{2i-1}) \ll F(v_{2(i-1)}) \ll z_{2(i-1)} = \varphi(\chi_{((i-1)/m,1]})$$

for every $1 \leq i \leq m$.

Consider the finite family of $\prec$-relations

$$\{v_\alpha \prec v_{\alpha-1}\}_{\alpha\geq 1} \cup \{v_0 \prec v_{-\infty}\} \cup \{v_{-\infty} \prec v_{-\infty}\} \cup \{v \prec w\}.$$

Let $G_n\colon X_n \to L_n$ be the I-morphism defined as $G_n((\alpha,\beta)) = \chi_{(\alpha/n,\beta/n)}$, and take $\varepsilon < 1/2n$. Then, we know by Proposition 3.5.18 that there exists a large enough $N \in \mathbb{N}$ such that the functions $f_\alpha = R_\varepsilon(G_n(v_\alpha))$, $f = R_\varepsilon(G_n(v))$ and $g = R_\varepsilon(G_n(w))$ satisfy

$$\begin{aligned} f_\alpha \ll f_{\alpha-1} \ll f_{-\infty}, \quad f \ll g,\\ \rho_N(f_\alpha) \ll F(v_\alpha) \ll \rho_N(f_{\alpha-1}) \ll F(v_{\alpha-1}),\\ \rho_N(f_{-\infty}) = F(v_{-\infty}),\\ \rho_N(f) \ll F(v) \ll \rho_N(g) \ll F(w) \end{aligned}$$

for every $\alpha \geq 1$, where recall that ρ_N is the Cu-morphism defined prior to Lemma 3.5.17. Note, in particular, that $f_{-\infty}$ is compact because $v_{-\infty} \prec v_{-\infty}$.

Furthermore, since $\sum_{\alpha\in\mathcal{A}} v_{2\alpha} \simeq v$, it follows from Corollary 3.5.16 that

$$f = R_\varepsilon(G_n(v)) = \sum_{\alpha\in\mathcal{A}} R_\varepsilon(G_n(v_{2\alpha})) = \sum_{\alpha\in\mathcal{A}} f_{2\alpha},$$

and, similarly, that $g = \sum_{\beta\in\mathcal{B}} f_{2\beta}$.

Now note that the sequence $(f_{2\alpha})_{\alpha\in\Omega_m}$ is $\ll$-decreasing and bounded by the compact $f_{-\infty}$. Thus, we know from Theorem 3.2.6 that there exists a Cu-morphism $\theta\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}}) \to \mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ satisfying $\theta(1) = f_{-\infty}$ and $\theta(\chi_{(i/m,1]}) = f_{2i}$ for every $0 \leq i \leq m$.

By construction, we get

$$\theta(x) \ll \theta(x') = \sum_{\alpha \in \mathcal{A}} f_{2\alpha} = f \ll g = \sum_{\beta \in \mathcal{B}} f_{2\beta} = \theta(y),$$

which is condition (ii).

Set $\phi = \rho_N$. Then, we have $\phi\theta(1) = \rho_N(f_{-\infty}) = F(v_{-\infty}) = z_{-\infty} = \varphi(1)$, which is condition (iii).

Further, one also has

$$\begin{aligned}\varphi(\chi_{(i/m,1]}) \ll F(v_{2i-1}) \ll \rho_m(f_{2(i-1)}) &= \phi(\theta(\chi_{((i-1)/m,1]})) \\ &\ll F(v_{2(i-1)}) \ll \varphi(\chi_{((i-1)/m,1]}),\end{aligned}$$

for every $1 \leq i \leq m$, which implies condition (i) in the proposition. □

We now strengthen the previous property.

Definition 3.5.23. A Cu-semigroup S is said to satisfy *property* I_0 if, for any $m, M \in \mathbb{N}$, any $\ll$-decreasing sequences $(z_{\alpha,j})_{\alpha \in \Omega_m}$ for $j \leq M$, and finite multisets $\mathcal{A}, \mathcal{B}$ of Ω_m such that $\sum_{\alpha \in \mathcal{A}, j \leq M} z_{\alpha,j} \ll \sum_{\beta \in \mathcal{B}, j \leq M} z_{\beta,j}$, there is a chainable subsemigroup H of S with associated I-morphism $F\colon X_n \to S$ such that

(i) For each $j \leq M$, there exists a $\prec$-decreasing sequence $(v_{\alpha,j})_{\alpha \in \Omega_m}$ in X_n satisfying

$$F(v_{\alpha,j}) \ll z_{\alpha,j} \ll F(v_{\beta,j})$$

for every j and $\beta \prec \alpha$;

(ii) There exist $v, w \in X_n$ such that $\sum_{\alpha \in \mathcal{A}, j \leq M} v_{\alpha,j} \simeq v \prec w \simeq \sum_{\beta \in \mathcal{B}, j \leq M} v_{\beta,j}$.

Lemma 3.5.24. *Property* I_0 *passes to inductive limits.*

Proof. Let $S = \lim_k S_k$ be an inductive limit of Cu-semigroups S_m satisfying property I_0, and let $(z_{\alpha,j})_{\alpha \in \Omega_m}$, $\mathcal{A}$ and $\mathcal{B}$ be as in Definition 3.5.23. Given $f \in S_k$, we will denote by $[f]$ the image of f through the limit morphism from S_k to S.

There exists a large enough k and elements $x_{\alpha,j}, y_{\alpha,j} \in S_k$ such that

$$x_{\alpha,j} \ll y_{\alpha,j} \ll x_{\beta,j}, \qquad \sum_{\alpha \in \mathcal{A}, j} y_{\alpha,j} \ll \sum_{\beta \in \mathcal{B}, j} y_{\beta,j}$$

in S_k, and

$$z_{\alpha,j} \ll [x_{\beta,j}] \ll [y_{\beta,j}] \ll z_{\beta,j}, \qquad \sum_{\alpha \in \mathcal{A}, j} z_{\alpha,j} \ll \sum_{\beta \in \mathcal{B}, j} [y_{\beta,j}] \ll \sum_{\beta \in \mathcal{B}, j} z_{\beta,j}$$

for every $\beta \prec \alpha$ in Ω_m and $j \leq M$.

Since S_k satisfies property I_0, we can find a chainable subset H with associated I-morphism $F\colon X_n \to S_k$ such that:

(i) There exist $\prec$-decreasing sequences

$$v_{\infty,j} \prec w_{\infty,j} \prec v_{m,j} \prec \ldots \prec v_{-\infty,j} = w_{-\infty,j} \prec w_{-\infty,j}$$

in X_n such that

$$x_{\alpha,j} \ll F(v_{\alpha,j}) \ll y_{\alpha,j} \ll F(w_{\alpha,j}) \ll x_{\beta,j}$$

in S_m for every $\beta \prec \alpha$ and j;

(ii) There exist $v, w \in X_n$ such that $\sum_{\alpha\in\mathcal{A},j\leq M} v_{\alpha,j} \simeq v \prec w \simeq \sum_{\beta\in\mathcal{B},j\leq M} v_{\beta,j}$.

Therefore, one gets $z_{\alpha,j} \ll [F(v_{\beta,j})] \ll z_{\beta,j}$ for every j and $\beta \prec \alpha$.

By Remark 3.5.8, $[H]$ is a chainable subsemigroup of S, and it satisfies the required properties by the previous considerations. □

Example 3.5.25. Let S be an inductive limit of the form $S = \lim_m S_m$, with $S_m = \overline{\mathbb{N}}$ for every m. Then S satisfies property I_0.

To prove this, we know from Lemma 3.5.24 that it is enough to see that $\overline{\mathbb{N}}$ satisfies property I_0.

Thus, let $(z_{\alpha,j})_{\alpha\in\Omega_m}$, $\mathcal{A}$ and $\mathcal{B}$ be as in Definition 3.5.23. Since $z_{\alpha,j}$ belongs to $\overline{\mathbb{N}}_{\ll}$ for every α, j, it follows that $z_{\alpha,j} \in \mathbb{N}$.

Set $H = \mathbb{N}$, which is the additive span of 1. Then, it follows from Example 3.5.9 that H is $(n,1)$-chainable for every n. Take $n = 1$ and let $F\colon X_1 \to H$ be as in Example 3.5.9.

Set $v_{\alpha,j} = z_{\alpha,j}(-\infty,\infty)$, whose image through F is $z_{\alpha,j}$. This implies condition (i) in Definition 3.5.23.

Also, since $\sum_{\alpha\in\mathcal{A},j\leq M} z_{\alpha,j} \ll \sum_{\beta\in\mathcal{B},j\leq M} z_{\beta,j}$, we clearly have

$$\sum_{\alpha\in\mathcal{A},j\leq M} z_{\alpha,j}(-\infty,\infty) \prec \sum_{\beta\in\mathcal{B},j\leq M} z_{\beta,j}(-\infty,\infty).$$

Letting $v = \sum_{\alpha\in\mathcal{A},j\leq M} v_{\alpha,j}$ and $w = \sum_{\beta\in\mathcal{B},j\leq M} v_{\beta,j}$, condition (ii) also follows.

Example 3.5.26. Every Cu-semigroup S of the form $S = \lim_m S_m$, with $S_m = \mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ for every m, satisfies property I_0.

Indeed, we first note that, as in Example 3.5.25 above, it is enough to prove that $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ satisfies property I_0. Thus, take $(z_{\alpha,j})_{\alpha\in\Omega_m}$, $\mathcal{A}$ and $\mathcal{B}$ as in Definition 3.5.23, and let L_n be the subsets of $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ defined in Example 3.5.10.

Since $\cup_n L_n$ is dense in $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$, there exist $n' \in \mathbb{N}$ and $\ll$-decreasing sequences $(f_{\alpha,j})_{\alpha\in\Omega_m}$ in $L_{n'}$ such that $z_{\alpha,j} \ll f_{\beta,j} \ll z_{\beta,j}$ for every $\beta \prec \alpha$ and

$$\sum_{\alpha\in\mathcal{A},j\leq M} f_{\alpha,j} \ll \sum_{\beta\in\mathcal{B},j\leq M} f_{\beta,j}.$$

Take a small enough positive rational $\varepsilon < 1/2n'$ such that

$$z_{\alpha,j} \ll R_\varepsilon(f_{\beta,j}) \ll z_{\beta,j}$$

for each $\beta \prec \alpha$ and

$$\sum_{\alpha\in\mathcal{A},j} R_\varepsilon(f_{\alpha,j}) = R_\varepsilon\left(\sum_{\alpha\in\mathcal{A},j} f_{\alpha,j}\right) \ll R_\varepsilon\left(\sum_{\beta\in\mathcal{B},j} f_{\beta,j}\right) = \sum_{\beta\in\mathcal{B},j} R_\varepsilon(f_{\beta,j}),$$

where the equalities follow from Corollary 3.5.16 since each $f_{\alpha,j}$ is in the image of the I-morphism associated to $L_{n'}$.

Define $g_{\alpha,j} = R_\varepsilon(f_{\alpha,j})$ and note that, by Lemma 3.5.13, there exists some $n \in \mathbb{N}$ such that $g_{\alpha,j} \in L_n^0$ for each α, j.

Now recall from Example 3.5.10 that L_n is $(n,1)$-chainable with an associated I-morphism $G_n\colon X_n \to L_n$ induced by $G_n((\alpha,\beta)) = \chi_{(\alpha/n,\beta/n)}$.

Set $g_{\mathcal{A}} = \sum_{\alpha\in\mathcal{A},j} g_{\alpha,j}$ and $g_{\mathcal{B}} = \sum_{\beta\in\mathcal{B},j} g_{\beta,j}$, so that $g_{\mathcal{A}} \ll g_{\mathcal{B}}$. Let $q_{g_{\alpha,j}}$, $q_{g_{\mathcal{A}}}$ and $q_{g_{\mathcal{B}}}$ be the elements in X_n given by Proposition 3.5.15.

Note that, by Proposition 3.5.15, $(q_{g_{\alpha,j}})_\alpha$ is $\prec$-decreasing for every j and that, additionally, we have

$$z_{\alpha,j} \ll G_n(q_{g_{\beta,j}}) \ll z_{\beta,j}$$

for every $\beta \prec \alpha$ in Ω_m and, since $G_n(\sum_{\alpha\in\mathcal{A},j} q_{g_{\alpha,j}}) = g_{\mathcal{A}}$ and $G_n(\sum_{\beta\in\mathcal{B},j} q_{g_{\beta,j}}) = g_{\mathcal{B}}$, we also get

$$\sum_{\alpha\in\mathcal{A},j} q_{g_{\alpha,j}} \simeq q_{g_{\mathcal{A}}} \prec q_{g_{\mathcal{B}}} \simeq \sum_{\beta\in\mathcal{B},j} q_{g_{\beta,j}}$$

which are conditions (i) and (ii) in Definition 3.5.23, as desired.

Example 3.5.27. The Cu-semigroup $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})\oplus\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ does not satisfy property I_0. Indeed, consider the sequences $(0,0) \ll (1,0) \ll (1,0)$ and $(0,0) \ll (0,1) \ll (0,1)$ indexed in $\Omega_0 = \{-\infty, 0, \infty\}$. Set $\mathcal{A} = \mathcal{B} = \emptyset$.

If $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})\oplus\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ were to satisfy property I_0, we could find in particular elements $v_{-\infty,1}$ and $v_{-\infty,2}$ in X_0 such that

$$v_{-\infty,1} \prec v_{-\infty,1}, \quad v_{-\infty,2} \prec v_{-\infty,2}, \quad (1,0) = F(v_{-\infty,1}) \quad \text{and} \quad (0,1) = F(v_{-\infty,2})$$

for some I-morphism F.

Thus, setting $e = F(-\infty,\infty)$, and using that $v_{-\infty,1} = k_1(-\infty,\infty)$ and $v_{-\infty,2} = k_2(-\infty,\infty)$ for some $k_1, k_2 \in \mathbb{N}$, we see that e is a compact element satisfying $(1,0) = k_1 e$ and $(0,1) = k_2 e$. Clearly, this is not possible.

Note that the same argument works for $\overline{\mathbb{N}} \oplus \overline{\mathbb{N}}$.

Theorem 3.5.28. *Let S be a* Cu*-semigroup. Then, S is* Cu*-isomorphic to an inductive limit of the form* $\lim \mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ *if and only if S is countably based, weakly cancellative, compactly bounded (see Definition 3.4.3) and it satisfies (O5), (O6), and property* I_0.

Proof. The proof of one implication is analogous to the proof of Proposition 3.5.22, so we omit it.

The other implication is Example 3.5.26 above, together with the well known fact that (O5), (O6), weak cancellation and being countably based pass to inductive limits (see, for example, [6, Chapter 4]). Since it is clear that being compactly bounded also passes to inductive limits, the result follows. □

Property I

Definition 3.5.29. We say that a Cu-semigroup S satisfies *property* I if, for any $m, M \in \mathbb{N}$, $\ll$-decreasing sequences $(z_{\alpha,j})_{\alpha\in\Omega_m}$ for $j \leq M$, and finite multisets $\mathcal{A}, \mathcal{B}$ of Ω_m such that $\sum_{\alpha\in\mathcal{A},j\leq M} z_{\alpha,j} \ll \sum_{\beta\in\mathcal{B},j\leq M} z_{\beta,j}$, there exist finitely many chainable subsemigroups H_k of S with associated I-morphisms $F_k\colon X_{n_k} \to S$ such that

(i) For each $j \leq M$, there exists a $\prec$-decreasing sequence $(v_{\alpha,j}^k)_{\alpha\in\Omega_m}$ in X_{n_k} satisfying

$$\sum_k F_k(v_{\alpha,j}^k) \ll z_{\alpha,j} \ll \sum_k F_k(v_{\beta,j}^k)$$

for every j and $\beta \prec \alpha$;

(ii) For each k, there exist $v_k, w_k \in X_{n_k}$ such that

$$\sum_{\alpha\in\mathcal{A},j} v_{\alpha,j}^k \simeq v_k \prec w_k \simeq \sum_{\beta\in\mathcal{B},j} v_{\beta,j}^k.$$

Remark 3.5.30. Note that a Cu-semigroup satisfying property I_0 will also satisfy property I.

Lemma 3.5.31. *Property* I *passes to inductive limits.*

Proof. Following the proof of Lemma 3.5.24, let $S = \lim S_t$ with S_t Cu-semigroups satisfying property I. Take $(z_{\alpha,j})_{\alpha\in\Omega_m}$ for $j \leq M$, and multisets $\mathcal{A}, \mathcal{B}$ of Ω_m as in Definition 3.5.29.

Then, for a large enough t, there exist elements $x_{\alpha,j}, y_{\alpha,j} \in S_t$ such that

$$x_{\alpha,j} \ll y_{\alpha,j} \ll x_{\beta,j}, \quad \sum_{\alpha\in\mathcal{A},j} y_{\alpha,j} \ll \sum_{\beta\in\mathcal{B},j} y_{\beta,j}$$

in S_t, and

$$z_{\alpha,j} \ll [x_{\beta,j}] \ll [y_{\beta,j}] \ll z_{\beta,j}, \quad \sum_{\alpha\in\mathcal{A},j} z_{\alpha,j} \ll \sum_{\beta\in\mathcal{B},j} [y_{\beta,j}] \ll \sum_{\beta\in\mathcal{B},j} z_{\beta,j}$$

for each pair $\beta \prec \alpha$ in Ω_m and $j \leq M$.

Since S_t satisfies property I, we can find a chainable subsets H_k with I-morphism $F_k\colon X_{n_k} \to S_t$ such that

(i) There exist $\prec$-decreasing sequences

$$v_{\infty,j}^k \prec w_{\infty,j}^k \prec v_{m,j}^k \prec \ldots \prec v_{-\infty,j}^k = w_{-\infty,j}^k \prec w_{-\infty,j}^k$$

in X_{n_k} such that

$$x_{\alpha,j} \ll \sum_k F_k(v_{\alpha,j}^k) \ll y_{\alpha,j} \ll \sum_k F_k(w_{\alpha,j}^k) \ll x_{\beta,j}$$

in S_t for every $\beta \prec \alpha$ and j;

(ii) There exist $v_k, w_k \in X_{n_k}$ such that $\sum_{\alpha\in\mathcal{A},j} v_{\alpha,j}^k \simeq v \prec w \simeq \sum_{\beta\in\mathcal{B},j} v_{\beta,j}^k$.

One can now check that the subsemigroups $[H_k]$ and the elements $v_{\alpha,j}^k$ satisfy the desired properties for our original pair of multisets and sequences.

Recall that $[H_k]$ is a chainable subsemigroup of S by Remark 3.5.8. □

Recall from Example 3.5.27 that property I_0 is not preserved under direct sums. Lemma 3.5.32 below exemplifies one of the main differences between property I_0 and property I, since the latter is preserved under such sums.

Lemma 3.5.32. *Let S, T be* Cu*-semigroups satisfying property* I. *Then, $S \oplus T$ satisfies property* I.

Proof. As before, let $(z_{\alpha,j})_{\alpha\in\Omega_m}$ be $\ll$-decreasing in $S \oplus T$ for $j \leq M$, and let $\mathcal{A}, \mathcal{B}$ be multisets of Ω_m as in Definition 3.5.29.

Let $(x_{\alpha,j})_{\alpha\in\Omega_m}$ in S and $(y_{\alpha,j})_{\alpha\in\Omega_m}$ in T be such that $z_{\alpha,j} = (x_{\alpha,j}, y_{\alpha,j})$ for every α, j. Note that, since $(z_{\alpha,j})_{\alpha\in\Omega_m}$ is $\ll$-decreasing, $(x_{\alpha,j})_{\alpha\in\Omega_m}$ and $(y_{\alpha,j})_{\alpha\in\Omega_m}$ are as well.

Using that both S and T satisfy property I, we obtain chainable subsemigroups H_k of S and T_l of T, and $\prec$-decreasing sequences $(v^k_{\alpha,j})_{\alpha\in\Omega_m}$ in X_{n_k} and $(w^k_{\alpha,j})_{\alpha\in\Omega_m}$ in X_{n_l} satisfying the required properties for $(x_{\alpha,j})_{\alpha\in\Omega_m}$ and $(y_{\alpha,j})_{\alpha\in\Omega_m}$ respectively.

Since $H_k \oplus 0$ and $0 \oplus T_l$ are chainable subsemigroups of $S \oplus T$, it is readily checked that such subsemigroups together with the sequences $(v^k_{\alpha,j})_{\alpha\in\Omega_m}$ and $(w^k_{\alpha,j})_{\alpha\in\Omega_m}$ satisfy the required properties for $(z_{\alpha,j})_{\alpha\in\Omega_m}$, as desired. □

Example 3.5.33. By a combination of Lemma 3.5.31, Lemma 3.5.32, Remark 3.5.30 and Examples 3.5.25 and 3.5.26, we obtain the following examples:

1. Let S be the Cuntz semigroup of an AF-algebra. Then S satisfies property I.

2. The Cuntz semigroup of an AI-algebra satisfies property I.

Theorem 3.5.34. *Let S be a* Cu*-semigroup. Then, S is* Cu*-isomorphic to the Cuntz semigroup of an AI-algebra if and only if S is countably based, weakly cancellative, compactly bounded, and satisfies (O5), (O6), and property* I.

Proof. If S is Cu-isomorphic to the Cuntz semigroup of an AI-algebra, then it is well known that S is countably based, weakly cancellative, compactly bounded and satisfies (O5) and (O6). That S satisfies property I follows from Example 3.5.33 above.

Conversely, assume that S is a countably based, weakly cancellative, compactly bounded Cu-semigroup that satisfies (O5), (O6), and property I. By Theorem 3.4.8, we know that it is enough to prove that, for every Cu-morphism $\varphi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to S$, $n \in \mathbb{N}$ and triple $x, x', y \in B^r_m$ such that $x \ll x'$ with $\varphi(x') \ll \varphi(y)$, there exist a natural number s and Cu-morphisms

$$\theta\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^s \quad \text{and} \quad \phi\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^s \to S$$

such that the following properties are satisfied

(i) for every $i \leq m$ and $j \leq r$, we have

$$\varphi\iota_j\left(\chi_{\left(\frac{i}{m}+\frac{1}{m},1\right]}\right) \ll \phi\theta\iota_j\left(\chi_{\left(\frac{i}{m},1\right]}\right) \quad \text{and} \quad \phi\theta\iota_j\left(\chi_{\left(\frac{i}{m}+\frac{1}{m},1\right]}\right) \ll \varphi\iota_j\left(\chi_{\left(\frac{i}{m},1\right]}\right),$$

where ι_j denotes the canonical inclusion from $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ to the j-th component of $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r$;

(ii) $\theta(x) \ll \theta(y)$;

(iii) $\varphi(1_j) = \phi\theta(1_j)$ for every $1 \leq j \leq r$.

To see this, we generalize the proof of Proposition 3.5.22. Thus, using the notation from Example 3.5.10, we define $z_{\alpha,j} = \varphi\iota_j(\chi_{(\alpha/2m,1]})$ for every $\alpha \in \Omega_{2m}$ and $j \leq r$.

We also let $\mathcal{A}, \mathcal{B}$ be the multisets in Ω_m such that

$$x' = \sum_{\alpha\in\mathcal{A}, j\leq r} \iota_j(\chi_{(\alpha/m,1]}) \quad \text{and} \quad y = \sum_{\beta\in\mathcal{B}, j\leq r} \iota_j(\chi_{(\beta/m,1]}),$$

which implies that $\sum_{\alpha\in\mathcal{A},j} z_{2\alpha,j} \ll \sum_{\beta\in\mathcal{B},j} z_{2\beta,j}$.

Since $(z_{\alpha,j})_{\alpha\in\Omega_{2m}}$ is $\ll$-decreasing for every j, we can apply property I to such sequences and the multisets $\mathcal{A}, \mathcal{B}$. Thus, we get chainable subsemigroups H_k with associated I-morphisms $F_k\colon X_{n_k} \to S$ and $\prec$-decreasing sequences $(v^k_{\alpha,j})_{\alpha\in\Omega_m}$ in X_{n_k} satisfying (i)-(iii) in Definition 3.5.29.

In particular, we note that

$$\varphi\iota_j(\chi_{(i/m,1]}) = z_{2i,j} \ll \sum_k F_k(v^k_{2i-1,j}) \ll \sum_k F_k(v^k_{2(i-1),j}) \ll z_{2(i-1),j} = \varphi\iota_j(\chi_{((i-1)/m,1]})$$

for every $1 \leq i \leq m$.

Consider, for every k, the family of $\prec$-relations

$$\{v^k_{\alpha,j} \prec v^k_{\alpha-1,j}\}_{\alpha\geq 1,j} \cup \{v^k_{0,j} \prec v^k_{-\infty,j}\}_j \cup \{v^k_{-\infty,j} \prec v^k_{-\infty,j}\}_j \cup \{v_k \prec w_k\}.$$

As in Proposition 3.5.22, we obtain an integer N_k and elements $f^k_{\alpha,j}$, f_k and g_k in $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ such that

$$\begin{aligned} f^k_{\alpha,j} \ll f^k_{\alpha-1,j} \ll f^k_{-\infty,j}, \quad f_k \ll g_k, \\ \rho_{N_k}(f^k_{\alpha,j}) \ll F_k(v^k_{\alpha,j}) \ll \rho_{N_k}(f^k_{\alpha-1,j}) \ll F_k(v^k_{\alpha-1,j}), \\ \rho_{N_k}(f^k_{-\infty,j}) = F_k(v^k_{-\infty,j}), \\ \rho_{N_k}(f_k) \ll F_k(v_k) \ll \rho_{N_k}(g_k) \ll F_k(w_k) \end{aligned}$$

for every $\alpha \geq 1$. Recall that ρ_{N_k} is the Cu-morphism defined prior to Lemma 3.5.17. We also note that $f^k_{-\infty,j}$ is compact.

Moreover, following the proof Proposition 3.5.22, we also have

$$\sum_{\alpha\in\mathcal{A},j} f^k_{2\alpha,j} = f_k \ll g_k = \sum_{\beta\in\mathcal{B},j} f^k_{2\beta,j}.$$

We define $\phi = \oplus_k \rho_{N_k}\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^s \to S$, and set $f_{\alpha,j} = \sum_k \iota_k(f^k_{\alpha,j})$.

Note that $f_{\alpha,s} \ll f_{\alpha-1,s} \ll f_{-\infty,j}$ for each pair α, j. For each $j \leq r$, we let $\theta_j\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}}) \to \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^s$ be a Cu-morphism sending $\chi_{(\alpha/m,1]}$ to $f_{2\alpha,j}$ for each α. Recall that such morphism exists by Theorem 3.2.6.

We define $\theta = \oplus_j \theta_j\colon \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r \to \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^s$. For every $j \leq r$, we get

$$\phi\theta(1_j) = \phi(f_{-\infty,j}) = \sum_k \rho_{N_k}(f^k_{-\infty,j}) = \sum_k F_k(v^k_{-\infty,j}) = z_{-\infty,j} = \varphi(1),$$

which is property (iii).

Further, we also have

$$\begin{aligned}\varphi\iota_j(\chi_{(i/m,1]}) &\ll \sum_k F_k(v^k_{2i-1,j}) \ll \sum_k \rho_{N_k}(f^k_{2(i-1),j}) \\ &= \phi\theta(\chi_{((i-1)/m,1]}) \ll \sum_k F_k(v^k_{2(i-1),j}) \ll \varphi\iota_j(\chi_{((i-1)/m,1]})\end{aligned}$$

for every $1 \leq i \leq m$. This proves property (i).

Finally, we have

$$\theta(x) \ll \theta(x') = \sum_{\alpha\in\mathcal{A},j} f_{2\alpha,j} = \sum_k f_k \ll \sum_k g_k = \sum_{\beta\in\mathcal{B},j} f_{2\beta,j} = \theta(y),$$

as desired. □

During our investigations on the covering dimension of Cuntz semigroups (as defined in Chapter 4 below), in Chapter 5 we introduce a notion of approximation for Cu-semigroups. Using such a notion and Theorem 3.5.34 above, one can prove Theorem 3.5.35 below; see Theorem 5.1.16 for the proof.

Theorem 3.5.35. *Let S be a countably based* Cu*-semigroup. Then, S is* Cu*-isomorphic to the Cuntz semigroup of an AI-algebra if and only if S is approximated by the family* $\{\mathrm{Lsc}([0,1],\overline{\mathbb{N}})^n\}_n$.

Chapter 4

Covering dimension of Cuntz semigroups

We introduce a notion of covering dimension for Cuntz semigroups and, more generally, for Cu-semigroups. The main permanence properties are studied in Section 4.1, where amongst other results we prove that purely infinite C^*-algebras have zero-dimensional Cuntz semigroups, and that the dimension of the Cuntz semigroup of any $\mathcal{Z}$-stable C^*-algebra is at most one; see Proposition 4.1.24.

A connection between the dimension of the Cuntz semigroup of a C^*-algebra and the nuclear dimension is presented in Section 4.2. We also show in Section 4.3 that the Cuntz semigroup of a real rank zero C^*-algebra is zero-dimensional and that, in the stable rank one case, these two notions coincide. Moreover, we prove that the Cuntz semigroup of a simple, $\mathcal{Z}$-stable C^*-algebra is zero-dimensional if and only if the C^*-algebra is of real rank zero or is stably projectionless.

Finally, we study in Sections 4.4 and 4.5 when a simple, soft Cu-semigroup is zero-dimensional.

The results of this chapter have been published in [94]. We also provide additional remarks and comments that were not provided in [94].

Some of the results in this chapter will be generalized in Chapter 5.

4.1 Dimension of Cuntz semigroups

In this first section we define a notion of covering dimension for Cu-semigroups inspired by Theorem 2.4.8; see Definition 4.1.1. We prove that this notion satisfies the expected permanence properties (Proposition 4.1.10), and provide a number of examples where the dimension can be computed; see, for example, Example 4.1.3 and Example 4.1.22.

Further, we introduce the notion of *retracts* (Definition 4.1.15) for Cu-semigroups, and use it to unearth a relation between the dimension of a simple Cu-semigroup and its soft part; see Proposition 4.1.20. We finish the section by studying how the dimension behaves in the presence of a semimodule structure. We then apply the result to the Cuntz semigroups of purely infinite, $\mathcal{W}$-stable and $\mathcal{Z}$-stable C^*-algebras; see Proposition 4.1.24.

Definition 4.1.1. Let S be a Cu-semigroup. Given $n \in \mathbb{N}$, we write $\dim(S) \leq n$ if, whenever $x' \ll x \ll y_1 + \ldots + y_r$ in S, then there exist $z_{j,k} \in S$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ such that

(i) $z_{j,k} \ll y_j$ for each j and k;

(ii) $x' \ll \sum_{j,k} z_{j,k}$;

(iii) $\sum_{j=1}^{r} z_{j,k} \ll x$ for each $k = 0, \ldots, n$.

We set $\dim(S) = \infty$ if there exists no $n \in \mathbb{N}$ with $\dim(S) \leq n$. Otherwise, we let $\dim(S)$ be the smallest $n \in \mathbb{N}$ such that $\dim(S) \leq n$. We call $\dim(S)$ the *(covering) dimension* of S.

We note that, in Definition 4.1.1 above, some of the $\ll$-relations may be changed to $\leq$.

Lemma 4.1.2. *Let S be a* Cu*-semigroup and let $n \in \mathbb{N}$. Then,* $\dim(S) \leq n$ *if and only if, whenever $x' \ll x \ll y_1 + \ldots + y_r$ in S, there exist $z_{j,k} \in S$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ such that:*

(i) $z_{j,k} \leq y_j$ *for each j and k;*

(ii) $x' \leq \sum_{j,k} z_{j,k}$*;*

(iii) $\sum_{j=1}^{r} z_{j,k} \leq x$ *for each $k = 0, \ldots, n$.*

Proof. The forward implication is straightforward, so we are left to prove the converse.

Thus, let $x' \ll x \ll y_1 + \ldots + y_r$ in S. Choose elements $s', s, y_1', \ldots, y_r' \in S$ such that $y_j' \ll y_j$ for each j and

$$x' \ll s' \ll s \ll x \ll y_1' + \ldots + y_r'.$$

Applying the assumption, we obtain elements $z_{j,k}$ satisfying properties (i)-(iii) in the statement for $s' \ll s \ll y_1' + \ldots + y_r'$.

One can verify that such elements satisfy (i)-(iii) in Definition 4.1.1 for $x' \ll x \ll y_1 + \ldots + y_r$, thus showing that $\dim(S) \leq n$, as desired. $\square$

Example 4.1.3. Let X be a compact, metric space. It follows from Theorem 2.4.8 that

$$\dim(\mathrm{Lsc}(X, \overline{\mathbb{N}})) = \dim(X),$$

which justifies our terminology of *covering dimension* for a Cu-semigroup.

As noted in Corollary 4.2.8, this equality will also follow from some more general results developed in Section 4.2.

Recall the definition of ideal and quotient of a Cu-semigroup from Paragraph 1.2.12.

Proposition 4.1.4. *Let S be a* Cu*-semigroup, and let $I \subseteq S$ be an ideal. Then,*

$$\dim(I), \dim(S/I) \leq \dim(S).$$

Proof. Set $n := \dim(S)$, which we may assume to be finite, since otherwise there is nothing to prove. Using that every ideal of a Cu-semigroup is, by definition, downward-hereditary, one can easily check that $\dim(I) \leq n$.

To verify $\dim(S/I) \leq n$, let $\pi\colon S \to S/I$ denote the quotient map, and take elements $x', x, y_1, \ldots, y_r \in S$ such that

$$\pi(x') \ll \pi(x) \ll \pi(y_1) + \ldots + \pi(y_r).$$

Then, there exists $y_{r+1} \in I$ such that $x \leq y_1 + \ldots + y_r + y_{r+1}$ in S. Using that the quotient map $S \to S/I$ preserves suprema of increasing sequences, there exist elements $s', s \in S$ such that

$$s' \ll s \ll x \quad \text{and} \quad \pi(x') \leq \pi(s').$$

Applying the definition of $\dim(S) \leq n$ to $s' \ll s \ll y_1 + \ldots + y_r + y_{r+1}$, we obtain elements $z_{j,k} \in S$ for $j = 1, \ldots, r+1$ and $k = 0, \ldots, n$ such that $z_{j,k} \ll y_j$ for every j, k, such that $s' \ll \sum_{j,k} z_{j,k}$, and such that $\sum_j z_{j,k} \ll s$ for every k.

Since $y_{r+1} \in I$, we have $z_{r+1,k} \in I$ and thus $\pi(z_{r+1,k}) = 0$ in S/I for each k. Using that π is $\ll$-preserving, we see that the elements $\pi(z_{j,k})$ have the desired properties. □

Problem 4.1.5. Given a Cu-semigroup S and an ideal $I \subseteq S$, does there exist a bound of $\dim(S)$ in terms of $\dim(I)$ and $\dim(S/I)$? In particular, do we always have $\dim(S) \leq \dim(I) + \dim(S/I) + 1$? (as in the case of nuclear dimension; see Paragraph 4.2.1 below).

In fact, it is not known whether $\dim(\mathrm{Cu}(\tilde{A})) \leq \dim(\mathrm{Cu}(A)) + 1$ for every C^*-algebra A. We do have, however, examples where $\dim(\mathrm{Cu}(\tilde{A})) \neq \dim(\mathrm{Cu}(A))$; see, for example, Example 5.4.5 and the discussion in Question 5.4.6.

Given two Cu-semigroups S and T, we can consider their direct sum $S \oplus T$ as defined in Paragraph 1.2.9. We provide a proof for Proposition 4.1.6 below, which was ommitted in [94, Propositon 3.7].

Proposition 4.1.6. *Let S and T be* Cu*-semigroups. Then*

$$\dim(S \oplus T) = \max\{\dim(S), \dim(T)\}.$$

Proof. Note that S and T can be seen as ideals of $S \oplus T$, so it follows that

$$\max\{\dim(S), \dim(T)\} \leq \dim(S \oplus T)$$

Conversely, let $n = \dim(S)$ and $m = \dim(T)$, which we may assume to be finite since otherwise there is nothing to prove. Also, we may assume without loss of generality that $n \geq m$.

Take $x' \ll x \ll y_1 + \ldots + y_r$ in $S \oplus T$, and let π_1, π_2 denote the projections from $S \oplus T$ to S and T respectively.

Then, since $\pi_i(x') \ll \pi_i(x) \ll \pi_i(y_1) + \ldots + \pi_i(y_r)$ for $i = 1, 2$, there exist elements $v_{j,k} \in S$ and $w_{j,t} \in T$ for $j = 1, \ldots, r$, $k = 0, \ldots, n$ and $t = 0, \ldots, m$ satisfying conditions (i)-(iii) in Definition 4.1.1.

For each $k > m$, set $w_{j,k} = 0$ and define $z_{j,k} = (v_{j,k}, w_{j,k})$ for every j and $k \leq n$. One can now check that the elements $z_{j,k} \in S \oplus T$ satisfy the desired conditions. □

Lemma 4.1.8 below provides a useful characterization for inductive limits in Cu. In order to prove it, we first recall the definition of W, a category closely related to Cu.

4.1.7 (W-semigroups and W-morphisms)**.** It follows from [8, Theorem 2.9] that Cu is a full, reflective subcategory of W, an abstract category defined in [8, Definition 2.5]. Thus, inductive limits in Cu can be obtained by applying the reflection functor $\mathrm{W} \to \mathrm{Cu}$ to the inductive limits in W. We briefly recall the definition of this category:

Let S be a commutative monoid together with a transitive, binary relation $\prec$, and denote by $z^{\prec}$ the set $\{z' \mid z' \prec z\}$ for every $z \in S$. We say that S is a W-*semigroup* if: $0 \prec x$ for every $x \in S$; for every $x \in S$ there is a $\prec$-increasing, $\prec$-cofinal sequence in $x^{\prec}$; and, for every $x, y \in S$, we have that $x^{\prec} + y^{\prec}$ is contained and $\prec$-cofinal in $(x+y)^{\prec}$.

A map $\varphi\colon S \to T$ between W-semigroups is said to be a W-*morphism* if φ is a $\prec$-preserving monoid morphism such that for every $x \in S$ the set $\varphi(x^{\prec}) \subseteq \varphi(x)^{\prec}$ is $\prec$-cofinal.

The category W is defined as the category of W-semigroups and W-morphisms. Since every Cu-semigroup paired with $\ll$ is a W-semigroup, we obtain an inclusion $\mathrm{Cu} \to \mathrm{W}$.

Given an inductive system $((S_\lambda)_{\lambda\in\Lambda}, (\varphi_{\mu,\lambda})_{\lambda\leq\mu \text{ in } \Lambda})$ in Cu, its inductive limit in W is constructed as follows:

Consider the equivalence relation $\sim$ on the disjoint union $\bigsqcup_\lambda S_\lambda$ given by $x_\lambda \sim x_\mu$ if there exists $\nu \geq \lambda, \mu$ such that $\varphi_{\nu,\lambda}(x_\lambda) = \varphi_{\nu,\mu}(x_\mu)$. We denote by S_{alg} the set of equivalence classes, which is the set-theoretic inductive limit of the system. We write $[x_\lambda]$ for the equivalence class of $x_\lambda \in S_\lambda$.

Further, given $x_\lambda \in S_\lambda$ and $x_\mu \in S_\mu$, we set

$$[x_\lambda] + [x_\mu] := [\varphi_{\nu,\lambda}(x_\lambda) + \varphi_{\nu,\mu}(x_\mu)]$$

in S_{alg} for any $\nu \geq \lambda, \mu$.

We also write $[x_\lambda] \prec [x_\mu]$ if there exists $\nu \geq \lambda, \mu$ such that $\varphi_{\nu,\lambda}(x_\lambda) \ll \varphi_{\nu,\mu}(x_\mu)$ in S_ν. This gives S_{alg} the structure of a W-semigroup, which together with the natural maps $S_\lambda \to S_{\mathrm{alg}}, x_\lambda \mapsto [x_\lambda]$, is the inductive limit in W.

By [6, Theorem 3.1.8], the reflection of S_{alg} in Cu is a Cu-semigroup S together with a (universal) W-morphism $\alpha\colon S_{\mathrm{alg}} \to S$ characterized by the following conditions:

(R1) For every $x_\lambda \in S_\lambda$ and $x_\mu \in S_\mu$, $[x_\lambda]^{\prec} \subseteq [x_\mu]^{\prec}$ if and only if $\alpha([x_\lambda]) \leq \alpha([x_\mu])$;

(R2) For all $x', x \in S$ satisfying $x' \ll x$ there exists $x_\lambda \in S_\lambda$ such that $x' \ll \alpha([x_\lambda]) \ll x$.

Lemma 4.1.8. *Let* $((S_\lambda)_{\lambda\in\Lambda}, (\varphi_{\mu,\lambda})_{\lambda\leq\mu \text{ in } \Lambda})$ *be an inductive system in* Cu *as defined in Paragraph 1.2.10. Then, a* Cu*-semigroup* S *together with* Cu*-morphisms* $\varphi_\lambda\colon S_\lambda \to S$ *for* $\lambda \in \Lambda$ *is the inductive limit of the system* $((S_\lambda)_{\lambda\in\Lambda}, (\varphi_{\mu,\lambda})_{\lambda\leq\mu \text{ in } \Lambda})$ *if and only if the following conditions are satisfied:*

(L0) $\varphi_\mu \circ \varphi_{\mu,\lambda} = \varphi_\lambda$ *for all* $\lambda \leq \mu$ *in* Λ*;*

(L1) if $x'_\lambda, x_\lambda \in S_\lambda$ *and* $x_\mu \in S_\mu$ *satisfy* $x'_\lambda \ll x_\lambda$ *and* $\varphi_\lambda(x_\lambda) \leq \varphi_\mu(x_\mu)$*, then there exists* $\nu \geq \lambda, \mu$ *such that* $\varphi_{\nu,\lambda}(x'_\lambda) \ll \varphi_{\nu,\mu}(x_\mu)$*;*

(L2) for all $x', x \in S$ *satisfying* $x' \ll x$ *there exists* $x_\lambda \in S_\lambda$ *such that* $x' \ll \varphi_\lambda(x_\lambda) \ll x$*.*

Proof. As mentioned in Paragraph 4.1.7 above, the inductive limit of the inductive system $((S_\lambda)_{\lambda\in\Lambda}, (\varphi_{\mu,\lambda})_{\lambda\leq\mu \text{ in } \Lambda})$ in Cu is S, the reflection of S_{alg} in Cu. Note that (L0) is satisfied by construction, where we define $\varphi_\lambda(x_\lambda) := \alpha([x_\lambda])$ for each λ.

Moreover, given $x'_\lambda, x_\lambda \in S_\lambda$ and $x_\mu \in S_\mu$ as in (L1), we have $[x_\lambda]^\prec \subseteq [x_\mu]^\prec$ by (R1). Thus, it follows that $[x'_\lambda] \prec [x_\mu]$ which, by definition, implies that $\varphi_{\nu,\lambda}(x'_\lambda) \ll \varphi_{\nu,\mu}(x_\mu)$ for some $\nu \geq \lambda, \mu$.

To see (L2), take $x' \ll x$ in S. By (R2), there exists some $x_\lambda \in S_\lambda$ such that $x' \ll \alpha([x_\lambda]) \ll x$, where recall that $\alpha([x_\lambda]) = \varphi_\lambda(x_\lambda)$. □

Proposition 4.1.9. *Let $S = \varinjlim_{\lambda\in\Lambda} S_\lambda$ be an inductive limit of* Cu*-semigroups. Then,*

$$\dim(S) \leq \liminf_\lambda \dim(S_\lambda).$$

Proof. Let $n = \liminf_\lambda \dim(S_\lambda)$, which we may assume to be finite.

To see that $\dim(S) \leq n$, take $x' \ll x \ll y_1 + \ldots + y_r$ in S and choose $y'_1, \ldots, y'_r \in S$ such that $y'_j \ll y_j$ for each j and

$$x \ll y'_1 + \ldots + y'_r.$$

For every λ, denote by $\varphi_\lambda \colon S_\lambda \to S$ the Cu-morphism into the inductive limit. By Lemma 4.1.8 above, these morphisms satisfy properties (L0)-(L2).

Combining (L0) and (L2), there exist λ and elements $v, w_1, \ldots, w_r \in S_\lambda$ such that $x' \ll \varphi_\lambda(v) \ll x$ and $y'_j \ll \varphi_\lambda(w_j) \ll y_j$ for each j.

Let $v'', v' \in S_\lambda$ satisfy $v'' \ll v' \ll v$ and

$$x' \ll \varphi_\lambda(v'') \ll \varphi_\lambda(v) \ll x,$$

which exist since φ_λ is a Cu-morphism.

Hence,

$$\varphi_\lambda(v) \ll x \ll y'_1 + \ldots + y'_r \ll \varphi_\lambda(w_1 + \ldots + w_r).$$

Applying (L1), we obtain $\nu \geq \lambda$ such that $\varphi_{\nu,\lambda}(v') \ll \varphi_{\nu,\lambda}(w_1 + \ldots + w_r)$.

Since $\liminf_\lambda \dim(S_\lambda) \leq n$, we may also assume that $\dim(S_\nu) \leq n$. Applying $\dim(S_\nu) \leq n$ to

$$\varphi_{\nu,\lambda}(v'') \ll \varphi_{\nu,\lambda}(v') \ll \varphi_{\nu,\lambda}(w_1 + \ldots + w_r),$$

we obtain elements $z_{j,k} \in S_\nu$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ satisfying properties (i)-(iii) from Definition 4.1.1. Thus, we have $\varphi_\nu(z_{j,k}) \ll \varphi_\lambda(w_j) \ll y_j$ for every j, k. Moreover, one also gets

$$x' \ll \varphi_\lambda(v'') \ll \sum_{j,k} \varphi_\nu(z_{j,k})$$

and

$$\sum_j \varphi_\nu(z_{j,k}) \ll \varphi_\lambda(v') \ll x \quad \text{for each } k,$$

as desired. □

Proposition 4.1.10. *Let A be a C^*-algebra A, and let I be a closed, two-sided ideal I of A. Then,*

$$\dim(\mathrm{Cu}(I)), \dim(\mathrm{Cu}(A/I)) \leq \dim(\mathrm{Cu}(A)).$$

Given C^*-algebras A and B, we have

$$\dim(\mathrm{Cu}(A \oplus B)) = \max\{\dim(\mathrm{Cu}(A)), \dim(\mathrm{Cu}(B))\}.$$

Given an inductive limit of C^*-algebras $A = \varinjlim_\lambda A_\lambda$, we have

$$\dim(\mathrm{Cu}(A)) \leq \liminf_\lambda \dim(\mathrm{Cu}(A_\lambda)).$$

Proof. By [6, Section 5.1], $\mathrm{Cu}(I)$ is naturally isomorphic to an ideal of $\mathrm{Cu}(A)$, and $\mathrm{Cu}(A/I)$ is naturally isomorphic to $\mathrm{Cu}(A)/\mathrm{Cu}(I)$. Thus, the first statement follows from Proposition 4.1.4.

The second statement follows from Proposition 4.1.6, since $\mathrm{Cu}(A\oplus B)$ is isomorphic to $\mathrm{Cu}(A)\oplus\mathrm{Cu}(B)$.

Finally, the third statement follows from Proposition 4.1.9 and the fact that the Cuntz semigroup of an inductive limit of C^*-algebras is naturally isomorphic to the inductive limit of the C^*-algebras; see [6, Corollary 3.2.9]. □

Example 4.1.11. Recall that $\mathrm{Cu}(\mathbb{C})$ is naturally isomorphic to $\overline{\mathbb{N}} := \{0,1,2,\dots,\infty\}$; see Examples 1.2.8 (i)-(ii). Using that $\mathbb{N}$ has Riesz decomposition (as defined in Section 1.3), one can easily check that $\dim(\overline{\mathbb{N}}) = 0$. Thus, it follows from Proposition 4.1.6 that $\dim(\overline{\mathbb{N}}^k) = 0$ for every $k \geq 1$.

Applying Proposition 4.1.9, every inductive limit of such Cu-semigroups has dimension zero. Proposition 4.1.10 also shows that $\dim(\mathrm{Cu}(A)) = 0$ for every AF-algebra A. In Theorem 4.3.8, we will generalize this to C^*-algebras of real rank zero.

Thus, we also know by the Cu-semigroup version of the Effros-Handelman-Shen theorem (Theorem 1.3.4) that every countably based, weakly cancellative, unperforated, algebraic Cu-semigroup satisfying (O5) and (O6) is zero-dimensional. This is because such semigroups are isomorphic to inductive limits of $\overline{\mathbb{N}}^k$'s. In Proposition 4.3.6, we will show that all weakly cancellative, algebraic Cu-semigroups satisfying (O5) and (O6) have dimension zero.

Example 4.1.12. For each $k \in \mathbb{N}$, let E_k be the Cu-semigroups defined in Examples 1.2.8 (ii). As above, one can see that they also have dimension zero.

Note, however, that not all algebraic Cu-semigroups are zero dimensional. For example, consider $S := \overline{\mathbb{N}} \cup \{1'\}$, with $1'$ a compact element not comparable with 1 and such that $1'+1' = 2$ and $1+k = 1'+k$ for every $k \in \overline{\mathbb{N}} \setminus \{0\}$. Then, $\dim(S) = \infty$.

Indeed, assume for the sake of contradiction that $\dim(S) \leq n$ for some $n \in \mathbb{N}$. Then, since $1' \ll 1' \ll 2 = 1+1$, there exist elements $z_{1,k}, z_{2,k} \in S$ for $k = 0,\dots,n$ satisfying (i)-(iii) from Definition 4.1.1.

By (i), we have $z_{j,k} \ll 1$ and therefore $z_{j,k} = 0$ or $z_{j,k} = 1$ for every j,k. By condition (ii), we have $1' \ll \sum_{j,k} z_{j,k}$, and so there exist $j' \in \{1,2\}$ and $k' \in \{0,\dots,n\}$ such that $z_{j',k'} = 1$. However, condition (iii) implies $1 = z_{j',k'} \ll 1'$, which is a contradiction because the elements 1 and $1'$ are not comparable.

Example 4.1.13. Let $k,l \in \mathbb{N}$, and let E_k and E_l be the Cu-semigroups defined in Examples 1.2.8 (ii). Then, their abstract bivariant Cu-semigroup $[\![E_k, E_l]\!]$, as defined in [7], has dimension one whenever $l > k$ and dimension zero otherwise.

Indeed, by [7, Proposition 5.18], we know that $[\![E_k, E_l]\!] = \{0, r, \dots, l, \infty\}$ with $r = \lceil (l+1)/(k+1) \rceil$. Thus, if $l \leq k$, then $[\![E_k, E_l]\!] = E_l$, which is zero-dimensional by Example 4.1.12.

If, conversely, we have $l > k$, that is $r > 1$, consider $r+1 \ll r+1 \ll r+r$. One cannot find $z_1, z_2 \ll r$ such that $r+1 = z_1+z_2$, which shows that $\dim([\![E_k, E_l]\!]) \neq 0$.

To verify $\dim([\![E_k, E_l]\!]) \leq 1$, let $x \ll x \ll y_1 + \ldots + y_r$ in $[\![E_k, E_l]\!]$. We may assume that y_j is nonzero for every j. If there exists $i \in \{1,\dots,r\}$ with $x \leq y_i$, then $z_{i,0} := x$ and $z_{j,k} := 0$ for $j \neq i$ or $k = 1$ have the desired properties.

So we may assume that $y_j < x$ for each j. Let k be the least integer such that $x \leq y_1 + \ldots + y_k$. Define $z_{j,0} := y_j$ for every $j < k$ and $z_{j,0} := 0$ for $j \geq k$. Further, define $z_{k,1} := y_k$ and $z_{j,1} := 0$ for $j \neq k$. By choice of k, we have $\sum_j z_{j,0} \ll x$. We also have $\sum_j z_{j,1} = y_k \ll x$. Finally, $x \ll \sum_j z_{j,0} + \sum_j z_{j,1}$, as desired.

Example 4.1.14. Given a compact, metric space X containing at least two points, let $S = \mathrm{Lsc}(X, \overline{\mathbb{N}})_{++} \cup \{0\}$ be the sub-Cu-semigroup of $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ consisting of strictly positive functions and 0. We claim that $\dim(S) = \infty$.

To see this, assume for the sake of contradiction that $\dim(S) \leq n$ for some $n \in \mathbb{N}$, and take $r > n$. Since X contains at least two points, we can choose open subsets $U', U \subseteq X$ such that $\emptyset \neq U'$, $\overline{U'} \subseteq U$ and $U \neq X$.

Let $\chi_{U'}$ and χ_U denote their corresponding characteristic functions, and consider the elements $x' := 1 + (n+1)\chi_{U'}$ and $x := 1 + (n+1)\chi_U$ in S. We have

$$x' \ll x \ll r + 1 = 1 + \overset{(r+1}{\cdots} + 1.$$

Using $\dim(S) \leq n$, one obtains elements $z_{j,k} \in S$ for $j = 1, \ldots, r+1$ and $k = 0, \ldots, n$ satisfying (i)-(iii) from Definition 4.1.1.

Condition (i) implies $z_{j,k} \ll 1$ and, therefore, $z_{j,k} = 0$ or $z_{j,k} = 1$ for each j, k.

Further, for every $k \leq n$, condition (iii) implies $\sum_j z_{j,k} \ll x$. Consequently, all but possibly one of the elements $z_{1,k}, \ldots, z_{r+1,k}$ are zero. Thus, $\sum_j z_{j,k} \leq 1$. Using this at the last step, and using condition (ii) at the first step, one has

$$x' \ll \sum_{j,k} z_{j,k} = \sum_{k=0}^{n} \left(\sum_{j=1}^{r+1} z_{j,k} \right) \leq n + 1,$$

a contradiction.

Note that S does not arise as the Cuntz semigroup of a C^*-algebra since, for example, it does not satisfy (O5). (Take $1 \ll 1 \ll 1 + \chi_U$ with $U \neq X$.)

The previous example shows that, given a Cu-semigroup S and a sub-Cu-semigroup T of S, one may not have $\dim(T) \leq \dim(S)$. However, we will see in Chapter 5 that there are always plenty of sub-Cu-semigroups for which this bound is satisfied; see Proposition 5.3.7.

Retracts and soft elements

Definition 4.1.15 below mimics the definition of a topological retract. As shown in [10], this notion might also be of importance in the study of the radius of comparison; see [12] for its definition.

Recall from Definition 1.2.7 that a monoid morphism between two Cu-semigroups is a generalized Cu-morphism if it preserves order and suprema of increasing sequences.

Definition 4.1.15. Let S and T be Cu-semigroups. We say that S is a *retract* of T if there exist a Cu-morphism $\iota\colon S \to T$ and a generalized Cu-morphism $\sigma\colon T \to S$ such that $\sigma \circ \iota = \mathrm{id}_S$.

Proposition 4.1.16. *Let S and T be* Cu*-semigroups and assume that S is a retract of T. Then $\dim(S) \leq \dim(T)$.*

Proof. Set $n := \dim(T)$, which we may assume to be finite.

To see that $\dim(S) \leq n$, let $x' \ll x \ll y_1+\ldots+y_r$ in S and let ι, σ be a Cu-morphism and a generalized Cu-morphism such that $\sigma \circ \iota = \mathrm{id}_S$. Then, since ι is a Cu-morphism, one gets

$$\iota(x') \ll \iota(x) \ll \iota(y_1) + \ldots + \iota(y_r)$$

in T.

Using that $\dim(T) \leq n$, we obtain $z_{j,k}$ in T satisfying conditions (i)-(iii) of Definition 4.1.1. Since σ is a generalized Cu-morphism, we see that the elements $\sigma(z_{j,k})$ satisfy conditions (i)-(iii) in Lemma 4.1.2, from which the result follows. $\square$

4.1.17 (Soft elements)**.** As defined in [6, Paragraph 5.2.2], an element x in a Cu-semigroup S is *soft* if for every $x' \ll x$ there exists $n \geq 1$ such that $(n+1)x' \ll nx$. Further, the subset of soft elements, denoted by S_{soft}, is seen to be a sub-Cu-semigroup whenever S is simple, weakly cancellative and satisfies (O5) and (O6); see [6, Proposition 5.3.18].

Given a simple Cu-semigroup S, let us now show that S_{soft} is a retract of S. As we will see in Proposition 4.1.20 below, this sub-Cu-semigroup will play an important role in the study of the dimension of S.

Proposition 4.1.18. *Let S be a simple, countably based, weakly cancellative* Cu*-semigroup satisfying (O5) and (O6). Then S_{soft} is a retract of S.*

Proof. As shown in [6, Proposition 5.3.18], the set S_{soft} is a Cu-semigroup. If S is isomorphic to $\overline{\mathbb{N}}$, the result follows trivially. Thus, we may assume otherwise. Using [91, Proposition 2.9], we see that for each $x \in S$ there exists a unique maximal soft element below x and that the map $\sigma\colon S \to S_{\mathrm{soft}}$ given by

$$\sigma(x) := \max\left\{x' \in S_{\mathrm{soft}} : x' \leq x\right\}$$

is a generalized Cu-morphism.

Note that the canonical inclusion $\iota\colon S_{\mathrm{soft}} \to S$ is a Cu-morphism satisfying $\sigma \circ \iota = \mathrm{id}_{S_{\mathrm{soft}}}$, as desired. $\square$

Lemma 4.1.19 below follows as a combination of [6, Lemma 5.1.18] and [73, Proposition 5.2.1]. However, since this result will be used repeatedly, we include its proof for the convenience of the reader

Lemma 4.1.19. *Let S be a nonzero, simple* Cu*-semigroup satisfying (O5) and (O6), and let $u_0, u_1 \in S$ be nonzero. Assume that S is not isomorphic to $\overline{\mathbb{N}}$ or E_k for any k. Then, there exists a nonzero $w \in S$ such that $2w \ll u_0, u_1$.*

Proof. Let $u_0'', u_0' \in S$ be nonzero elements such that $u_0'' \ll u_0' \ll u_0$. Since S is simple and $u_1 \neq 0$, we get $u_0' \ll u_0 \leq \infty = \infty u_1$ and, consequently, $u_0' \leq n u_1$ for some $n \in \mathbb{N}$.

Applying (O6) to $u_0'' \ll u_0' \leq u_1 + \overset{n}{\ldots} + u_1$, there exist elements $z_1, \ldots, z_n \in S$ such that

$$u_0'' \ll z_1 + \ldots + z_n \quad \text{and} \quad z_1, \ldots, z_n \ll u_0', u_1.$$

Moreover, using that u_0'' is nonzero, there is some $j \leq n$ such that z_j is nonzero. Set $v := z_j$, and note that $v \ll u_0, u_1$.

Since S is not isomorphic to $\overline{\mathbb{N}}$ or E_k for any k, it follows from [6, Proposition 5.1.19] that v is not a minimal nonzero element. Thus, there exist $v'', v' \in S$ with $0 \neq v'' \ll v' \lneq v$. Using (O5), we obtain an element $c \in S$ such that

$$v'' + c \leq v \leq v' + c.$$

Note that, since $v' \neq v$, c is nonzero. Applying the first part of the argument to $\tilde{u}_0 = v''$ and $\tilde{u}_1 = c$, we obtain $w \in S$ such that $0 \neq w \ll v'', c$. Then, w has the desired properties. □

Proposition 4.1.20. *Let S be a simple, countably based, weakly cancellative* Cu*-semigroup satisfying (O5) and (O6). Then,*

$$\dim(S_{\mathrm{soft}}) \leq \dim(S) \leq \dim(S_{\mathrm{soft}}) + 1.$$

Proof. The first inequality follows as a combination of Propositions 4.1.16 and 4.1.18.

For the second inequality, let $n := \dim(S_{\mathrm{soft}})$, which we may assume to be finite. We may also assume that $S \ncong E_k, \overline{\mathbb{N}}$ for every k, since otherwise Examples 4.1.11 and 4.1.12 imply that the dimension of S is zero and the result is trivial.

To verify $\dim(S) \leq n + 1$, let $x' \ll x \ll y_1 + \ldots + y_r$ in S, where we may assume x and y_1 nonzero. By [6, Proposition 5.3.16], every nonzero element of S is either soft or compact. Thus, x is either soft or compact. We study each case separately:

Assume first that x is soft. Then, since S_{soft} is a sub-Cu-semigroup, we find $s \in S_{\mathrm{soft}}$ satisfying $x' \ll s \ll x$. Let $\sigma\colon S \to S_{\mathrm{soft}}$ be as in the proof of Proposition 4.1.18. We have

$$s \ll x = \sigma(x) \leq \sigma(y_1) + \ldots + \sigma(y_r)$$

in S_{soft}.

Using that $\dim(S_{\mathrm{soft}}) \leq n$ and that $\sigma(y_j) \leq y_j$, we obtain elements $z_{j,k}$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ such that $z_{j,k} \ll y_j$,

$$x' \ll s \ll \sum_{j,k} z_{j,k} \quad \text{and} \quad \sum_{j=1}^{r} z_{j,k} \ll \sigma(x) = x,$$

as desired.

Now assume that x is compact. Lemma 4.1.19 implies that there exists a nonzero element $w \in S$ satisfying $w \ll x, y_1$. Thus, we know from [6, Proposition 5.4.4] that $x \ll \sigma(x) + w$, which allows us to choose (using once again that S_{soft} is a sub-Cu-semigroup) elements $s' \ll s$ in S_{soft} such that $s \ll \sigma(x)$ and $x \ll s' + w$. We have

$$s' \ll s \ll \sigma(x) \leq \sigma(y_1) + \ldots + \sigma(y_r)$$

in S_{soft}.

Arguing as above, and using that $\dim(S_{\mathrm{soft}}) \leq n$, we find soft elements $z_{j,k} \in S$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ satisfying

(i) $z_{j,k} \ll \sigma(y_j) \leq y_j$ for each j and k;

(ii) $s' \ll \sum_{j,k} z_{j,k}$;

(iii) $\sum_{j=1}^{r} z_{j,k} \ll s \ll \sigma(x) \leq x$ for each k.

Set $z_{1,n+1} := w$ and $z_{j,n+1} := 0$ for $j = 2, \ldots, r$. Then, $z_{j,n+1} \ll y_j$ for each j and

$$x' \ll x \ll s' + w \leq \left(\sum_{k=0}^{n} \sum_{j=1}^{r} z_{j,k} \right) + \sum_{j=1}^{r} z_{j,n+1} = \sum_{k=0}^{n+1} \sum_{j=1}^{r} z_{j,k}.$$

We also have $\sum_{j=1}^{r} z_{j,n+1} = w \ll x$, which shows that the elements $z_{j,k}$ have the desired properties. □

Remark 4.1.21. Engbers showed in [36] that, for every compact element c in the Cuntz semigroup of a simple, separable, stably finite C^*-algebra A, the set $\{x \in S \mid x < c\}$ has a maximum, called the *predecessor* of c.

The proof of Proposition 4.1.20 above can be generalized to this situation to obtain

$$\dim(\mathrm{Cu}(A)_{\mathrm{soft}}) \leq \dim(\mathrm{Cu}(A)) \leq \dim(\mathrm{Cu}(A)_{\mathrm{soft}}) + 1.$$

Example 4.1.22. Let Z be the Cuntz semigroup of the Jiang-Su algebra $\mathcal{Z}$ as defined in Examples 1.2.8 (iv). Then, $\dim(Z) = 1$.

Indeed, one can easily check, using that $[0, \infty)$ has Riesz decomposition, that the Cu-semigroup $[0, \infty]$ has dimension zero. Since $Z_{\mathrm{soft}} \cong [0, \infty]$, it follows from Proposition 4.1.20 above that $\dim(Z) \leq \dim([0, \infty]) + 1 = 1$.

To see $\dim(Z) \neq 0$, consider the compact element $1 \in Z$ and the soft element $\frac{3}{4} \in Z$, which satisfy $1 \ll 1 \ll \frac{3}{4} + \frac{3}{4}$. Note that there are no elements $z_0, z_1 \in Z$ satisfying $1 = z_0 + z_1$ and $z_0, z_1 \ll \frac{3}{4}$, which implies $\dim(Z) \neq 0$. Alternatively, it will also follow from Corollary 4.3.7 that Z is not zero-dimensional.

Let Z' be the Cu-semigroup $Z' := Z \cup \{1''\}$ with $1''$ a compact element not comparable with 1 and such that $1'' + 1'' = 2$ and $1 + x = 1'' + x$ for every $x \in Z \setminus \{0\}$, which was considered in [6, Question 9(8)]. An analogous argument shows that $\dim(Z') = 1$.

The study of soft elements, and the techniques presented in this section, can be generalized to a broader setting. This will be pursued in [10].

R-multiplications

Recall from Paragraph 1.2.17 that, given a Cu-semiring R, an R-multiplication on a Cu-semigroup S is a scalar multiplication on S with natural compatibility conditions.

As shown in [6, Theorem 7.2.2], a Cu-semigroup has $\{0, \infty\}$-multiplication if and only if every element in the semigroup is idempotent. By [6, Theorem 7.3.8], a Cu-semigroup has Z-multiplication if and only if it is almost unperforated and almost divisible. Further, we know from [6, Theorem 7.5.4] that a Cu-semigroup has $[0, \infty]$-multiplication if and only if it has Z-multiplication and every element in S is soft.

Proposition 4.1.23. *Let S be a* Cu*-semigroup satisfying (O5) and (O6).*

(i) If S has $\{0, \infty\}$-multiplication, then $\dim(S) = 0$.

(ii) If S has $[0, \infty]$-multiplication, then $\dim(S) = 0$.

(iii) If S has Z-multiplication, then $\dim(S) \leq 1$.

Proof. We prove each claim separately:

(i) Given elements $x' \ll x \ll y_1 + \ldots + y_r$ in a Cu-semigroup with $\{0,\infty\}$-multiplication, we can apply (O6) to obtain elements $z_j \in S$ such that

$$x' \leq z_1 + \ldots + z_r$$

and $z_j \leq x, y_j$ for each j.

Since every element in S is idempotent, we get

$$z_1 + \ldots + z_r \leq x + \ldots_r + x = rx = x.$$

This implies, by Lemma 4.1.2, that $\dim(S) = 0$.

(ii) Given a Cu-semigroup S, recall from Paragraph 1.2.16 that $F(S)$ denotes the set of functionals of S. When equipped with a suitable topology, $F(S)$ becomes a compact Hausdorff space; see [73]. One defines the *realification of* S, denoted by S_R, as the smallest subsemigroup of $\mathrm{Lsc}(F(S),[0,\infty])$ closed under suprema of increasing sequences and containing $\frac{1}{n}\widehat{x}$ for every $n \geq 0$ and $x \in S$.

We know from Theorem 7.5.4 and Proposition 7.5.9 in [6] that, if S has $[0,\infty]$-multiplication, S is isomorphic to its realification S_R. Thus, by [73, Theorem 4.1.1], it follows that whenever $x' \ll x \ll y_1 + \ldots + y_r$ there exist elements $z_j \ll y_j$ such that

$$x' \ll z_1 + \ldots + z_r \ll x.$$

This shows that the dimension of S is zero.

(iii) Recall the definition of Z from Examples 1.2.8. By [6, Proposition 7.3.13], an element x in a Cu-semigroup S with Z-multiplication is soft if and only if $x = 1'x$.

Further, it follows from [6, Corollary 7.5.10] that the Cu-semigroup S_{soft} is isomorphic to the realification of S. Consequently, we have $\dim(S_{\mathrm{soft}}) = 0$ by (ii).

Now let $x' \ll x \ll y_1 + \ldots + y_r$ in S. Since S has Z-multiplication, we get

$$\tfrac{5}{8}x' \ll \tfrac{6}{8}x \ll \tfrac{7}{8}y_1 + \ldots + \tfrac{7}{8}y_r,$$

where note that all the elements in the previous expression belong to S_{soft}.

Since $\dim(S_{\mathrm{soft}}) = 0$, we obtain elements $z_1, \ldots, z_r \in S_{\mathrm{soft}}$ such that $z_j \ll \frac{7}{8}y_j$ for each j, and such that

$$\tfrac{5}{8}x' \ll z_1 + \ldots + z_r \ll \tfrac{6}{8}x.$$

Set $z_{j,k} := z_j$ for $j = 1, \ldots, r$ and $k = 0, 1$, and note that we have $z_{j,k} \ll y_j$ for each j and k. Further,

$$x' \leq \tfrac{10}{8}x' \ll 2(z_1 + \ldots + z_r) = \sum_{j,k} z_{j,k},$$

and

$$\sum_j z_{j,k} \ll \tfrac{6}{8}x \leq x.$$

for each k, as desired. □

Proposition 4.1.24. *Let A be a C^*-algebra.*

(i) If A is purely infinite, then $\dim(\mathrm{Cu}(A)) = 0$.

(ii) If A is $\mathcal{W}$-stable, then $\dim(\mathrm{Cu}(A)) = 0$.

(iii) If A is $\mathcal{Z}$-stable, then $\dim(\mathrm{Cu}(A)) \leq 1$.

Proof. Given a C^*-algebra A, we know from [6, Proposition 7.2.8] that A is purely infinite (in the sense of Definition 1.1.13) if and only if $\mathrm{Cu}(A)$ has $\{0,\infty\}$-multiplication. This, together with Proposition 4.1.23, shows (i).

By [6, Proposition 7.6.3], the Cuntz semigroup of A has $[0,\infty]$-multiplication or Z-multiplication whenever A is $\mathcal{W}$-stable or $\mathcal{Z}$-stable respectively. Combined with Proposition 4.1.23 above, this shows (ii) and (iii). □

4.2 Commutative and subhomogeneous C^*-algebras

We now move our attention to the relation between the covering dimension of a Cuntz semigroup and the nuclear dimension of the underlying C^*-algebra. More concretely, we show in Theorem 4.2.2 that the dimension of the Cuntz semigroup of a C^*-algebra A is always bounded by the nuclear dimension of A.

In the case of subhomogeneous C^*-algebras (which include commutative C^*-algebras), we show that both dimensions agree; see Proposition 4.2.7 and Theorem 4.2.12.

4.2.1 (Nuclear dimension)**.** As defined in [109, Definition 1.1], a linear map φ between two C^*-algebras A, B is said to be a cpc order zero map if it is completely positive, contractive, and if, for every pair of orthogonal positive elements $a, b \in A$, their images $\varphi(a), \varphi(b)$ are orthogonal in B.

A cpc order-zero map $\varphi\colon A \to B$ always induces a generalized Cu-morphism, denoted by $\mathrm{Cu}(\varphi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$; see [108, Paragraph 3.5].

In [109], Winter and Zacharias define the nuclear dimension of a C^*-algebra A, in symbols $\dim_{\mathrm{nuc}}(A)$, as the least nonnegative integer n such that there exists a net $(F_\lambda, \psi_\lambda, \varphi_\lambda)$ with F_λ finite dimensional C^*-algebras, $\psi_\lambda\colon A \to F_\lambda$ cpc. maps, and $\varphi_\lambda\colon F_\lambda \to A$ completely positive maps such that

(i) $\psi_\lambda\varphi_\lambda(a)$ tends to a uniformly on finite subsets of A;

(ii) for every λ we can write $F_\lambda = F_\lambda^{(0)} \oplus \ldots \oplus F_\lambda^{(n)}$ in such a way that $\varphi_\lambda|_{F_\lambda^{(k)}}$ is a cpc order-zero map for each k.

If no such n exists, one sets $\dim_{\mathrm{nuc}}(A) = \infty$.

Given an ultrafilter $\mathcal{U}$ on an index set Λ, and a family of C^*-algebras A_λ indexed by Λ, recall that the *ultraproduct with respect to* $\mathcal{U}$, denoted by $A_\mathcal{U}$, is defined to be the quotient of $\prod_{\lambda\in\Lambda} A_\lambda$ by the ideal

$$\left\{ (a_\lambda) \in \prod_{\lambda\in\Lambda} A_\lambda \mid \lim_{\mathcal{U}} \|a_\lambda\| = 0 \right\}.$$

Let A be a C^*-algebra such that $\dim_{\mathrm{nuc}}(A)$ is finite, and let $n \in \mathbb{N}$ be such that $\dim_{\mathrm{nuc}}(A) = n$. Robert shows in [71, Proposition 2.2] that there exist an ultrafilter $\mathcal{U}$

on an index set Λ, finite-dimensional C^*-algebras $F_{\lambda,k}$ for $\lambda \in \Lambda$ and $k = 0, \ldots, n$, and cpc. order-zero maps $\psi_k \colon A \to \prod_{\mathcal{U}} F_{\lambda,k}$ and $\varphi_k \colon \prod_{\mathcal{U}} F_{\lambda,k} \to A_{\mathcal{U}}$ such that

$$\iota = \sum_{k=0}^{n} \varphi_k \circ \psi_k,$$

where $\iota \colon A \to A_{\mathcal{U}}$ denotes the natural inclusion map.

Theorem 4.2.2. *Let A be a C^*-algebra. Then,* $\dim(\mathrm{Cu}(A)) \leq \dim_{\mathrm{nuc}}(A)$.

Proof. Set $n := \dim_{\mathrm{nuc}}(A)$, which we may assume to be finite. As in the comments above, there exist an ultrafilter $\mathcal{U}$ on an index set Λ, finite-dimensional C^*-algebras $F_{\lambda,k}$, and cpc. order-zero maps $\psi_k \colon A \to \prod_{\mathcal{U}} F_{\lambda,k}$ and $\varphi_k \colon \prod_{\mathcal{U}} F_{\lambda,k} \to A_{\mathcal{U}}$ with $k = 0, \ldots, n$ such that:

$$\iota = \sum_{k=0}^{n} \varphi_k \circ \psi_k.$$

We let $\mathrm{Cu}(\psi_k)$ and $\mathrm{Cu}(\varphi_k)$ denote the induced generalized Cu-morphisms by ψ_k and φ_k respectively. Then, the equality $\iota = \sum_{k=0}^{n} \varphi_k \circ \psi_k$ implies

$$\mathrm{Cu}(\varphi_l)(\mathrm{Cu}(\psi_l)(x)) \leq \mathrm{Cu}(\iota)(x) \leq \sum_{k=0}^{n} \mathrm{Cu}(\varphi_k)(\mathrm{Cu}(\psi_k)(x))$$

for every $x \in \mathrm{Cu}(A)$ and each $l \in \{0, \ldots, n\}$.

To show that $\dim(\mathrm{Cu}(A)) \leq n$, let $x' \ll x \ll y_1 + \ldots + y_r$ in $\mathrm{Cu}(A)$ and set, for each $k \leq n$, $x_k := \mathrm{Cu}(\psi_k)(x) \in \mathrm{Cu}(\prod_{\mathcal{U}} F_{\lambda,k})$.

We have

$$\mathrm{Cu}(\iota)(x') \ll \mathrm{Cu}(\iota)(x) \leq \sum_{k=0}^{n} \mathrm{Cu}(\varphi_k)(\mathrm{Cu}(\psi_k)(x)) = \sum_{k=0}^{n} \mathrm{Cu}(\varphi_k)(x_k).$$

In particular, since $\mathrm{Cu}(\varphi_k)$ preserves suprema of increasing sequences, there exist elements $x'_k \ll x_k$ such that

$$\mathrm{Cu}(\iota)(x') \ll \sum_{k=0}^{n} \mathrm{Cu}(\varphi_k)(x'_k).$$

Moreover, for every fixed $k \in \{0, \ldots, n\}$, we have

$$x'_k \ll x_k = \mathrm{Cu}(\psi_k)(x) \leq \mathrm{Cu}(\psi_k)(\sum_{j=1}^{r} y_j) = \sum_{j=1}^{r} \mathrm{Cu}(\psi_k)(y_j).$$

Using that $\prod_{\mathcal{U}} F_{\lambda,k}$ has real rank zero and stable rank one, we know from [6, Corollary 5.5.10] that its Cuntz semigroup has Riesz decomposition. Thus, we obtain elements $z_{1,k}, \ldots, z_{r,k} \in \mathrm{Cu}(\prod_{\mathcal{U}} F_{\lambda,k})$ such that $z_{j,k} \leq \mathrm{Cu}(\psi_k)(y_j)$ for each j and

$$x'_k \leq \sum_{j=1}^{r} z_{j,k} = x_k.$$

Applying $\mathrm{Cu}(\varphi_k)$, one gets

$$\mathrm{Cu}(\varphi_k)(z_{j,k}) \leq \mathrm{Cu}(\varphi_k)(\mathrm{Cu}(\psi_k)(y_j)) \leq \mathrm{Cu}(\iota)(y_j)$$

for each j and k, and

$$\mathrm{Cu}(\iota)(x') \ll \sum_{k=0}^{n} \mathrm{Cu}(\varphi_k)(x'_k) \leq \sum_{k=0}^{n} \mathrm{Cu}(\varphi_k)(\sum_{j=1}^{r} z_{j,k}) = \sum_{k=0}^{n}\sum_{j=1}^{r} \mathrm{Cu}(\varphi_k)(z_{j,k})$$

in $\mathrm{Cu}(A_{\mathcal{U}})$.

Further, we also have

$$\sum_{j=1}^{r} \mathrm{Cu}(\varphi_k)(z_{j,k}) = \mathrm{Cu}(\varphi_k)(\sum_{j=1}^{r} z_{j,k}) \leq \mathrm{Cu}(\varphi_k)(x_k) = \mathrm{Cu}(\varphi_k)(\mathrm{Cu}(\psi_k)(x)) \leq \mathrm{Cu}(\iota)(x).$$

Using that the classes of elements in $\bigcup_{N\in\mathbb{N}}(A_{\mathcal{U}} \otimes M_N)_+$ are sup-dense in $\mathrm{Cu}(A_{\mathcal{U}})$, we find $N \in \mathbb{N}$ and positive elements $c_{j,k} \in A_{\mathcal{U}} \otimes M_N$ such that $[c_{j,k}] \ll \mathrm{Cu}(\varphi_k)(z_{j,k})$ and $\mathrm{Cu}(\iota)(x') \ll \sum_{j,k}[c_{j,k}]$.

Now note that $A_{\mathcal{U}} = \prod_\lambda A/c_{\mathcal{U}}$ with

$$c_{\mathcal{U}} = \{(a_\lambda)_\lambda \in \prod_\lambda A \mid \lim_{\lambda\to\mathcal{U}} \|a_\lambda\| = 0\},$$

and let $\pi\colon \prod_\lambda A \to A_{\mathcal{U}}$ denote the quotient map and also its amplification to matrix algebras, where we have $A_{\mathcal{U}} \otimes M_N \cong (A \otimes M_N)_{\mathcal{U}}$.

Choose positive elements $c_{j,k,\lambda} \in A \otimes M_N$ such that $\pi((c_{j,k,\lambda})_\lambda) = c_{j,k}$. Then, for a sufficiently large λ, the elements $[c_{j,k,\lambda}] \in \mathrm{Cu}(A)$ satisfy the conditions of Lemma 4.1.2 for $x' \ll x \ll y_1 + \ldots + y_r$, as desired. □

As we have just seen in Theorem 4.2.2 above, the nuclear dimension of a C^*-algebra is always a bound for the covering dimension of its Cuntz semigroup. The following results show that, for subhomogeneous C^*-algebras, both dimensions agree.

We note that this is not always the case; see Example 4.3.10.

4.2.3. As defined in Examples 1.1.20 (3), a C^*-algebra is d-homogeneous if all of its irreducible representations are of dimension d. We also say that a C^*-algebra is d-subhomogeneous if its irreducible representations are of dimension at most d. Generally, a C^*-algebra is said to be homogeneous (resp. subhomogeneous) if it is d-homogeneous (resp. d-subhomogeneous) for some d. Commutative C^*-algebras, and algebras of the form $C_0(X, M_n)$ with X locally compact, are examples of homogeneous C^*-algebras. A C^*-algebra is subhomogeneous if and only if it is isomorphic to a sub-C^*-algebra of a unital, homogeneous C^*-algebra; see [11, Proposition IV.1.4.3]

We briefly recall below the definition of the canonical homogeneous ideal-quotients of a subhomogeneous C^*-algebra; see [11, Sections IV.1.4, IV.1.7] for details.

Let A be a d-subhomogeneous C^*-algebra. For each $k \leq d+1$, define $I_{\geq k}$ as the closed, two-sided ideal formed by the elements $a \in A$ such that $\pi(a) = 0$ for some irreducible representation π of dimension at most $k-1$. For $k = 1$, we set $I_{\geq 1} = A$.

By construction, $I_{\geq k+1}$ is contained in $I_{\geq k}$ for each k. The *canonical k-homogeneous ideal-quotient* of A is $A_k := I_{\geq k}/I_{\geq k+1}$. As its name indicates, A_k is k-homogeneous.

4.2.4 (Local and topological dimension)**.** As defined in [67, Definition 5.1.1, p.188], the *local dimension* of a topological space X, in symbols $\operatorname{locdim}(X)$, is the smallest nonnegative integer n such that every point in X has a closed neighbourhood of covering dimension at most n.

If X is a locally compact, Hausdorff space, [67, Proposition 3.5.6] implies that

$$\operatorname{locdim}(X) = \sup\big\{ \dim(K) \mid K \subseteq X \text{ compact } \big\}.$$

In particular, $\operatorname{locdim}(X)$ and $\dim(X)$ agree whenever X is σ-compact, locally compact and Hausdorff. Moreover, $\operatorname{locdim}(X) = \dim(\alpha X)$ whenever X is locally compact, Hausdorff but not compact, where αX is the one-point compactification of X; see [67, Proposition 3.5.6].

Recall that the primitive ideal space $\operatorname{Prim}(A)$ of a C^*-algebra A is the set of ideals that can be expressed as the kernel of an irreducible representation of A equipped with the hull-kernel topology. That is to say, the closure of a subset $\{J_i\}_i$ of $\operatorname{Prim}(A)$ is defined to be

$$\overline{\{J_i\}_i} := \{J \in \operatorname{Prim}(A) \mid \cap_i J_i \subseteq J\}.$$

This topology makes $\operatorname{Prim}(A)$ a locally compact, T_0-space; see [11, Section IV.1.4]. If A is homogeneous, $\operatorname{Prim}(A)$ is Hausdorff.

The notion of *topological dimension* for certain C^*-algebras was defined in [16]. For a homogeneous C^*-algebra A, one lets its topological dimension, in symbols $\operatorname{topdim}(A)$, be the local dimension of its primitive ideal space.

In the subhomogeneous case, the topological dimension of A is defined as

$$\operatorname{topdim}(A) := \max_{k=1,\dots,d} \operatorname{topdim}(A_k) = \max_{k=1,\dots,d} \operatorname{locdim}(\operatorname{Prim}(A_k)),$$

where recall that A_k denotes the canonical k-homogeneous ideal-quotient defined in Paragraph 4.2.3 above.

Let $\operatorname{Sub}_{\mathrm{sep}}(A)$ denote the collection of separable sub-C^*-algebras of a C^*-algebra A, and let $\mathcal{S}$ be a family of $\operatorname{Sub}_{\mathrm{sep}}(A)$. We say that $\mathcal{S}$ is *cofinal* if for every separable sub-C^*-algebra B_0 there exists $B \in \mathcal{S}$ with $B_0 \subseteq B$. We will also say that $\mathcal{S}$ is *σ-complete* if $\overline{\bigcup\{B \mid B \in \mathcal{T}\}} \in \mathcal{S}$ for every countable, directed subfamily $\mathcal{T}$ of $\mathcal{S}$.

Results like Proposition 4.2.5 below are one of the driving reasons behind Chapter 5, where we investigate if the covering dimension of Cuntz semigroups is a noncommutative dimension theory; see Section 5.4.

Proposition 4.2.5. *Let A be a subhomogeneous C^*-algebra such that* $\operatorname{topdim}(A) \leq n$ *for some $n \in \mathbb{N}$. Then, the set*

$$\big\{B \in \operatorname{Sub}_{\mathrm{sep}}(A) \mid \operatorname{topdim}(B) \leq n\big\}$$

is σ-complete and cofinal.

Proof. We will prove the result for d-subhomogeneous C^*-algebras by induction over d. For $d = 1$, note that a 1-subhomogeneous C^*-algebra is 1-homogeneous. Thus, the result follows from the following fact.

Fact 1 ([90, Proposition 3.5]). *For a homogeneous C^*-algebra B of local dimension at most n, the collection*

$$\big\{C \in \mathrm{Sub}_{\mathrm{sep}}(B) \mid \mathrm{topdim}(C) \leq n\big\}$$

is σ-complete and cofinal.

Now let $d \geq 1$ and assume that the result holds for every d-subhomogeneous C^*-algebra. Given a $(d+1)$-subhomogeneous C^*-algebra A, we will prove that $\mathcal{S} := \{B \in \mathrm{Sub}_{\mathrm{sep}}(A) \mid \mathrm{topdim}(B) \leq n\}$ is σ-complete and cofinal.

Given a countable, directed family $\mathcal{T} \subseteq \mathcal{S}$, we note that the sub-$C^*$-algebra $C := \overline{\bigcup\{B \mid B \in \mathcal{T}\}}$ of A is separable and can be approximated by the sub-C^*-algebras $B \in \mathcal{T}$, which satisfy $\mathrm{topdim}(B) \leq n$ by construction. Thus, it follows from [89, Proposition 8] that $\mathrm{topdim}(C) \leq n$. This shows $C \in \mathcal{S}$, as desired.

Fact 2 ([16, Proposition 2.2]). *Let I be an ideal of a subhomogeneous C^*-algebra B. Then,*

$$\mathrm{topdim}(B) = \max\{\mathrm{topdim}(I), \mathrm{topdim}(B/I)\}.$$

Consider $I := I_{\geq d+1} \subseteq A$, the closed, two-sided ideal defined in Paragraph 4.2.3. By definition, I is $(d+1)$-homogeneous and A/I is d-subhomogeneous. Using Fact 2 above, we see that $\mathrm{topdim}(I) \leq n$ and $\mathrm{topdim}(A/I) \leq n$. By Fact 1 and by the induction hypothesis, both of the collections below are σ-complete and cofinal:

$$\mathcal{T}_1 := \big\{C \in \mathrm{Sub}_{\mathrm{sep}}(I) \mid \mathrm{topdim}(C) \leq n\big\},$$
$$\mathcal{T}_2 := \big\{D \in \mathrm{Sub}_{\mathrm{sep}}(A/I) \mid \mathrm{topdim}(D) \leq n\big\}.$$

Thus, it follows from [90, Lemma 3.2] that

$$\mathcal{S}_1 := \big\{B \in \mathrm{Sub}_{\mathrm{sep}}(A) \mid \mathrm{topdim}(B \cap I) \leq n\big\},$$
$$\mathcal{S}_2 := \big\{B \in \mathrm{Sub}_{\mathrm{sep}}(A) \mid \mathrm{topdim}(B/(B \cap I)) \leq n\big\},$$

are also σ-complete and cofinal.

An standard argument (see, for example, Paragraph 5.3.3) shows that $\mathcal{S}_1 \cap \mathcal{S}_2$ is σ-complete and cofinal as well. Moreover, given any $B \in \mathcal{S}_1 \cap \mathcal{S}_2$, it follows from [11, Proposition IV.1.4.3] that B is subhomogeneous. Thus, Fact 2 implies that

$$\mathrm{topdim}(B) = \max\{\mathrm{topdim}(B \cap I), \mathrm{topdim}(B/(B \cap I))\} \leq n.$$

Thus, $\mathcal{S}_1 \cap \mathcal{S}_2 \subseteq \mathcal{S}$. Since $\mathcal{S}_1 \cap \mathcal{S}_2$ is cofinal, so is $\mathcal{S}$. □

The following result is probably well known to the experts but, since it does not appear in the literature, we provide a proof. For separable subhomogeneous C^*-algebras, the equality between $\mathrm{dr}(A)$ and $\mathrm{topdim}(A)$ has already been shown in [104].

Recall that $\mathrm{dr}(A)$ denotes the *decomposition rank* of A, as defined in [56]. Its definition is the same as the nuclear dimension of A, with the additional condition that the maps φ_λ in Paragraph 4.2.1 are contractive.

Theorem 4.2.6. *Let A be a subhomogeneous C^*-algebra. Then*

$$\dim_{\mathrm{nuc}}(A) = \mathrm{dr}(A) = \mathrm{topdim}(A).$$

Proof. We will prove that, for every subhomogeneous C^*-algebra A, one has

$$\mathrm{dr}(A) \leq \mathrm{topdim}(A) \leq \dim_{\mathrm{nuc}}(A).$$

The result will then follow from the fact that $\dim_{\mathrm{nuc}}(B) \leq \mathrm{dr}(B)$ for every C^*-algebra B; see [109, Remarks 2.2(ii)].

Thus, set $n := \mathrm{topdim}(A)$. To see that $\mathrm{dr}(A) \leq n$, we may assume n to be finite. Using Proposition 4.2.5, we know that the family

$$\mathcal{S} := \big\{ B \in \mathrm{Sub}_{\mathrm{sep}}(A) \mid \mathrm{topdim}(B) \leq n \big\}$$

is cofinal and σ-complete.

Since every sub-C^*-algebra $B \in \mathcal{S}$ is separable and subhomogeneous (by [11, Proposition IV.1.4.3]), we know from [104, Theorem 1.6] that $\mathrm{dr}(B) = \mathrm{topdim}(B) \leq n$. It follows that A is approximated by the collection $\mathcal{S}$, which consists of C^*-algebras with decomposition rank at most n. Using this, one can verify that $\mathrm{dr}(A) \leq n$; see, for example, [89, Remark 2].

Now let $m := \dim_{\mathrm{nuc}}(A)$, and we will prove that $\mathrm{topdim}(A) \leq m$. As before, we may assume m to be finite. By [109, Proposition 2.6], the family

$$\mathcal{T} := \big\{ B \in \mathrm{Sub}_{\mathrm{sep}}(A) \mid \dim_{\mathrm{nuc}}(B) \leq m \big\}$$

is cofinal, where note that every $B \in \mathcal{T}$ is a separable and subhomogeneous.

For each $B \in \mathcal{T}$ and $k \geq 1$, let B_k denote the canonical k-homogeneous ideal-quotient of B as defined in Paragraph 4.2.3. Using [109, Corollary 2.10] at the first step, and that the nuclear dimension does not increase when passing to ideals ([109, Proposition 2.5]) or quotients ([109, Proposition 2.3(iv)]) at the second step, one gets

$$\mathrm{topdim}(B_k) = \dim_{\mathrm{nuc}}(B_k) \leq \dim_{\mathrm{nuc}}(B) \leq m.$$

Thus, we have

$$\mathrm{topdim}(B) = \max_k \mathrm{topdim}(B_k) \leq m$$

Since A is approximated by the collection $\mathcal{T}$, which consists of C^*-algebras with topological dimension at most m, it follows from [89, Proposition 8] that $\mathrm{topdim}(A) \leq m$, as desired. □

Proposition 4.2.7. *Let X be a compact, Hausdorff space. Then,*

$$\dim(\mathrm{Cu}(C(X))) = \dim(X).$$

Proof. If X is a second countable, compact, Hausdorff space, we have $\dim(X) = \dim_{\mathrm{nuc}}(C(X))$ by [109, Proposition 2.4]. Thus, since $C(X)$ is a homogeneous C^*-algebra and $\dim(X) = \mathrm{topdim}(C(X))$, Theorem 4.2.6 implies that this equality also holds for any compact, Hausdorff space. Using Theorem 4.2.2, one obtains the inequality $\dim(\mathrm{Cu}(C(X))) \leq \dim(X)$.

Now set $n := \dim(\mathrm{Cu}(C(X)))$. To see that $\dim(X) \leq n$, we may assume n to be finite.

Let $U_1, \ldots, U_r$ be a finite open cover of X. By [56, Proposition 1.5] (see also Proposition 2.4.7), it is enough to find an $(n+1)$-colourable, finite, open refinement of the cover.

Since X is normal, we can find an open cover $V_1, \ldots, V_r$ of X such that $\overline{V_j} \subseteq U_j$ for each j; see, for example, [67, Proposition 1.3.9, p.20]. For every $j \leq r$, it follows from Urysohn's lemma that there exists a continuous function $f_j \colon X \to [0,1]$ taking the value 1 on $\overline{V_j}$ and 0 on $X \setminus U_j$.

In particular, we have $1 \leq f_1 + \ldots + f_r$ and, consequently,

$$[1] \ll [1] \leq [f_1 + \ldots + f_r] \leq [f_1] + \ldots + [f_r]$$

in $\mathrm{Cu}(C(X))$.

Applying $\dim(\mathrm{Cu}(C(X))) \leq n$, we obtain elements $z_{j,k} \in \mathrm{Cu}(C(X))$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ satisfying (i)-(iii) in Definition 4.1.1.

For each pair j, k, let $g_{j,k} \in (C(X) \otimes \mathcal{K})_+$ be such that $z_{j,k} = [g_{j,k}]$. Viewing $g_{j,k}$ as a positive, continuous function from X to $\mathcal{K}$, set

$$W_{j,k} := \{x \in X \mid g_{j,k}(x) \neq 0\},$$

which is an open set.

It follows from condition (i) that $g_{j,k} = \lim_n h_n f_j h_n^*$ for some sequence $(h_n)_n$ in $C(X) \otimes \mathcal{K}$. Thus, $g_{j,k}(x) = 0$ whenever $f_j(x) = 0$, which implies $W_{j,k} \subseteq U_j$.

By condition (ii) in Definition 4.1.1, X is covered by the sets $W_{j,k}$. Thus, the sets $W_{j,k}$ form a finite, open refinement of our original cover.

Finally, let $k \in \{0, \ldots, n\}$ and $x \in X$. It follows from condition (iii) in Definition 4.1.1 that the rank of $g_{1,k}(x) \oplus \ldots \oplus g_{r,k}(x)$ is at most one. This implies that at most one of $g_{1,k}(x), \ldots, g_{r,k}(x)$ is nonzero. Thus, the sets $W_{1,k}, \ldots, W_{r,k}$ are pairwise disjoint, as desired. □

As mentioned in Example 4.1.3, we can now provide a simpler proof to the already known fact that $\dim(\mathrm{Lsc}(X, \overline{\mathbb{N}})) = \dim(X)$.

Corollary 4.2.8. *Let X be a compact, metrizable space. Then*

$$\dim(\mathrm{Lsc}(X, \overline{\mathbb{N}})) = \dim(X).$$

Proof. The inequality '$\geq$' follows easily by considering, for every open cover $U_1, \ldots, U_r$, the inequalities $1 \ll 1 \ll \chi_{U_1} + \ldots + \chi_{U_r}$ in $\mathrm{Lsc}(X, \overline{\mathbb{N}})$; see the first part of Theorem 2.4.8.

For '$\leq$', it is enough to see that $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ is a retract of $\mathrm{Cu}(C(X))$, since then the inequality will follow from Proposition 4.1.16 and Proposition 4.2.7.

Set $S = \mathrm{Lsc}(X, \overline{\mathbb{N}})$ and $T = \mathrm{Cu}(C(X))$, and define $\iota \colon S \to T$ as the unique Cu-morphism mapping each characteristic function χ_U to the class of a positive function in $C(X)$ with support U.

We also define $\sigma \colon T \to S$ as the generalized Cu-morphism mapping the class of an element $a \in C(X) \otimes \mathcal{K}$ to its rank function $\sigma(a) \colon X \to \overline{\mathbb{N}}$, $\sigma(a)(x) = \mathrm{rank}(a(x))$.

It is readily checked that $\sigma \circ \iota = \mathrm{id}_S$, as desired. □

Theorem 4.2.9. *Let X be a locally compact, Hausdorff space. Then,*

$$\dim(\mathrm{Cu}(C_0(X))) = \mathrm{locdim}(X).$$

Proof. For every compact subset $K \subseteq X$, $C(K)$ is a quotient of $C_0(X)$. Thus, using Propositions 4.2.7 and 4.1.10 at the first and second step respectively, one gets

$$\dim(K) = \dim(\mathrm{Cu}(C(K))) \leq \dim(\mathrm{Cu}(C_0(X))),$$

which implies $\mathrm{locdim}(X) \leq \dim(\mathrm{Cu}(C_0(X)))$.

Conversely, recall that $C_0(X)$ is an ideal in $C(\alpha X)$, where αX denotes the one-point compactification of X. Using Proposition 4.1.10 at the first step, and applying Proposition 4.2.7 and the fact that $\dim(\alpha X) = \mathrm{locdim}(X)$ at the second, we have

$$\dim(\mathrm{Cu}(C_0(X))) \leq \dim(\mathrm{Cu}(C(\alpha X))) = \mathrm{locdim}(X),$$

as required. $\square$

4.2.10 (Continuous fields of C^*-algebras)**.** Let X be a locally compact, Hausdorff space, let $\{A(x)\}_{x\in X}$ be a family of C^*-algebras indexed by X, and let $\Gamma := \{a\colon X \mapsto \sqcup_X A(x)\}$ be a set of sections. As defined in [11, Definition IV.1.6.1], the tuple $(X, \{A(x)\}, \Gamma)$ is a *continuous field of C^*-algebras* if the following conditions are satisfied:

(i) For every $x \in X$ and $a \in \Gamma$, one has $a(x) \in A(x)$.

(ii) For every $a \in \Gamma$, the assignment $x \mapsto \|a(x)\|$ is an element in $C_0(X)$.

(iii) Γ is closed under pointwise addition, multiplication and adjoint, and also under scalar multiplication.

(iv) For every $x \in X$ and $b \in A(x)$, there exists $a \in \Gamma$ with $a(x) = b$.

(v) Γ is closed under local uniform limits.

The set Γ becomes a C^*-algebra when equipped with the norm $\|a\| := \sup_x \|a(x)\|$, known as the *C^*-algebra of the continuous field.*

The field Γ is *trivial* if it is isomorphic to $C_0(X, A)$ with $A(x) = A$ for each x. Further, one says that the field is *locally trivial* if, for every $x \in X$, there exists an open neighborhood U of x such that $\Gamma|_U$ is trivial.

If a C^*-algebra A is d-homogeneous, it is isomorphic to a C^*-algebra of a $M_d(\mathbb{C})$-locally trivial field over $X = \mathrm{Prim}(A)$. That is to say, A is isomorphic to a continuous field C^*-algebra Γ as above such that for every point $x \in X$ there exists a neighborhood U with $\Gamma|_U \cong C_0(U) \otimes M_d$; see, for example, [11, Section IV.1.7]

Lemma 4.2.11. *Let A be a homogeneous C^*-algebra. Then,*

$$\dim_{\mathrm{nuc}}(A) \leq \dim(\mathrm{Cu}(A)).$$

Proof. Let A be a d-homogeneous C^*-algebra, and let $X := \mathrm{Prim}(A)$, which we know is locally compact and Hausdorff. Since $\mathrm{topdim}(A) = \mathrm{locdim}(X)$, we have to prove $\mathrm{locdim}(X) \leq \dim(\mathrm{Cu}(A))$.

Thus, let $x \in X$ and, as in the comments above, let us think of A as the algebra of sections vanishing at infinity of a locally trivial $M_d(\mathbb{C})$-bundle over X. This implies that there exists a compact neighbourhood Y of x over which the bundle is trivial.

Let I denote the ideal of A consisting of all sections in A that vanish on $X \setminus Y$. Then, A/I is the algebra of sections of the trivial $M_d(\mathbb{C})$-bundle over Y, which implies $A/I \cong C(Y) \otimes M_d$.

Using Proposition 4.2.7 at the first step, that $C(Y)$ and $C(Y) \otimes M_d$ have isomorphic Cuntz semigroups at the second step, and Proposition 4.1.10 at the last step, we have

$$\dim(Y) = \dim(\mathrm{Cu}(C(Y))) = \dim(\mathrm{Cu}(C(Y) \otimes M_d)) \leq \dim(\mathrm{Cu}(A)).$$

It follows that every point in X has a closed neighbourhood of dimension at most $\dim(\mathrm{Cu}(A))$, whence $\mathrm{locdim}(X) \leq \dim(\mathrm{Cu}(A))$. □

Theorem 4.2.12. *Let A be a subhomogeneous C^*-algebra. Then,*

$$\dim(\mathrm{Cu}(A)) = \dim_{\mathrm{nuc}}(A) = \mathrm{dr}(A) = \mathrm{topdim}(A).$$

Proof. Note that it is enough to verify the inequality $\mathrm{topdim}(A) \leq \dim(\mathrm{Cu}(A))$, since the second and third equalities are shown in Theorem 4.2.6 and we know that the inequality $\dim(\mathrm{Cu}(A)) \leq \dim_{\mathrm{nuc}}(A)$ holds in general by Theorem 4.2.2.

As in Paragraph 4.2.3, let A_k denote the canonical k-homogeneous ideal-quotient of A for each k. Using Lemma 4.2.11 and Theorem 4.2.6 at the first step, and Proposition 4.1.10 at the second, one has

$$\mathrm{topdim}(A_k) \leq \dim(\mathrm{Cu}(A_k)) \leq \dim(\mathrm{Cu}(A))$$

and, consequently,

$$\mathrm{topdim}(A) = \max_{k \geq 1} \mathrm{topdim}(A_k) \leq \dim(\mathrm{Cu}(A)),$$

as required. □

4.3 Algebraic, zero-dimensional Cuntz semigroups

In the remaining sections of this chapter we focus on zero-dimensional Cu-semigroups. We first prove a useful characterization of zero-dimensionality (Lemma 4.3.2), which we use to provide a sufficient criterion for zero-dimensional Cu-semigroups; see Proposition 4.3.4. For weakly cancellative, simple, zero-dimensional Cu-semigroups satisfying (O5), we also prove in Corollary 4.3.7 a dichotomy: either they are algebraic or have no nonzero compact elements.

This section is dedicated to the study of the first case (without assuming simplicity). In particular, we prove that the Cuntz semigroup of every real rank zero C^*-algebra is zero-dimensional and that, conversely, every unital stable rank one C^*-algebra with zero-dimensional Cuntz semigroup has real rank zero; see Theorem 4.3.8.

The second case, that is to say when there are no nonzero compact elements, is covered in Sections 4.4 and 4.5.

4.3.1 (Relation between (O6) and zero-dimensionality)**.** Lemma 4.3.2 below shows that every zero-dimensional Cu-semigroup satisfies (O6). However, the converse does not hold since, for example, the Cuntz semigroup of the Jiang-Su algebra satisfies (O6) but is not zero-dimensional; see Example 4.1.22.

Thus, zero-dimensionality is strictly stronger than (O6).

Lemma 4.3.2. *A* Cu*-semigroup S is zero-dimensional if and only if, whenever $x' \ll x \ll y_1 + y_2$ in S, there exist elements $z_1, z_2 \in S$ such that*

$$z_1 \ll y_1, \quad z_2 \ll y_2 \quad \textit{and} \quad x' \ll z_1 + z_2 \ll x.$$

Proof. The forward implication follows from Definition 4.1.1 by taking $r = 2$.

To prove the converse, let $r \geq 1$ and take $x' \ll x \ll y_1 + \ldots + y_r$ in S. We will prove by induction on r that there exist elements $z_1, \ldots, z_r \in S$ such that

$$z_j \ll y_j \text{ for } j = 1, \ldots, r \quad \text{and} \quad x' \ll z_1 + \ldots + z_r \ll x.$$

Note that the case $r = 1$ is trivial, and that the case $r = 2$ holds by assumption.

Thus, fix $r > 2$, assume that the result holds for $r-1$, and let $x' \ll x \ll y_1 + \ldots + y_r$ in S. Applying the case $r = 2$ to

$$x' \ll x \ll (y_1 + \ldots + y_{r-1}) + y_r,$$

we obtain obtain elements $u_1, u_2 \in S$ such that

$$u_1 \ll y_1 + \ldots + y_{r-1}, \quad u_2 \ll y_r \quad \text{and} \quad x' \ll u_1 + u_2 \ll x.$$

Now choose $u_1' \ll u_1$ such that $x' \ll u_1' + u_2$. Applying the induction hypothesis to

$$u_1' \ll u_1 \ll y_1 + \ldots + y_{r-1},$$

we obtain $z_1, \ldots, z_{r-1} \in S$ such that $z_j \ll y_j$ for each j and

$$u_1' \ll z_1 + \ldots + z_{r-1} \ll u_1.$$

Then $z_1, \ldots, z_{r-1}, u_2$ have the desired properties. □

4.3.3 (Riesz properties)**.** Recall from Section 1.3 that a positively ordered monoid M is said to have the *Riesz decomposition property* if, whenever $x \leq y + z$ in M, there exist $y' \leq y$ and $z' \leq z$ such that $x = y' + z'$.

Further, one says that M has

(i) the *Riesz interpolation property* if, given x_1, x_2, y_1, y_2 such that $x_i \leq y_j$ for every i, j, there exists $z \in M$ such that $x_i \leq z \leq y_j$ for every i, j.

(ii) the *Riesz refinement property* if, whenever $x_1 + x_2 = y_1 + y_2$, there exist $z_{i,j}$ for $i, j = 1, 2$ such that $x_i = z_{i,1} + z_{i,2}$ and $y_j = z_{1,j} + z_{2,j}$ for each i, j.

All three Riesz properties agree whenever the order is cancellative and algebraic; see, for example, [45, Proposition 2.1].

Further, note that M has Riesz interpolation whenever every pair of elements in M has an infimum. Indeed, if $x_i \leq y_j$ for every i, j, we can consider $z = y_1 \wedge y_2$.

Proposition 4.3.4. *Let S be a* Cu*-semigroup, and let $D \subseteq S$ be a dense subsemigroup of S. Assume that D satisfies the Riesz decomposition property for the pre-order induced by $\ll$. Then, S is zero-dimensional.*

Proof. To see that $\dim(S)=0$, let $x' \ll x \ll y_1+y_2$ in S. Since D is sup-dense, there exist elements $\tilde{x}, \tilde{y}_1, \tilde{y}_2 \in D$ with

$$x' \ll \tilde{x} \ll x \leq \tilde{y}_1+\tilde{y}_2, \quad \tilde{y}_1 \ll y_1 \quad \text{and} \quad \tilde{y}_2 \ll y_2.$$

In particular,we have $\tilde{x} \ll \tilde{y}_1+\tilde{y}_2$ and, using that D satisfies the Riesz decomposition property, we get $x_1, x_2 \in D$ such that

$$\tilde{x} = x_1+x_2, \quad x_1 \ll \tilde{y}_1 \quad \text{and} \quad x_2 \ll \tilde{y}_2.$$

One can now check that x_1 and x_2 satisfy the properties in Lemma 4.3.2, as desired. □

Given an element c in a Cu-semigroup S, we let the *ideal generated by* c be the subset $\{x \in S \mid x \leq \infty c\}$, which is an ideal in S.

Lemma 4.3.5. *Let S be a weakly cancellative, zero-dimensional* Cu*-semigroup satisfying (O5), and let $c \in S$ be compact. Then, the ideal generated by c is algebraic.*

Proof. First, note that an element $x \in S$ belongs to the ideal generated by c if and only if $x \leq \infty c$. To verify that the ideal is algebraic, for every pair x', x satisfying $x' \ll x \leq \infty c$ we will find a compact element z such that $x' \ll z \ll x$.

Thus, let x', x be such a pair and choose $x'' \ll x$ with $x' \ll x''$. Then $x'' \ll \infty c$, which implies that there exists $n \in \mathbb{N}$ such that $x'' \leq nc$. Applying (O5) to $x' \ll x'' \leq nc$, we obtain $y \in S$ satisfying

$$x'+y \leq nc \leq x''+y$$

and, applying that S is zero-dimensional to $nc \ll nc \ll x''+y$, we obtain $z_1, z_2 \in S$ such that

$$nc = z_1+z_2, \quad z_1 \ll x'' \quad \text{and} \quad z_2 \ll y.$$

Note that, by weak cancellation, z_1 and z_2 are compact. Further, one has

$$x'+y \ll nc = z_1+z_2 \leq z_1+y.$$

Using weak cancellation once again, we get $x' \ll z_1$, which shows that $z=z_1$ has the desired properties. □

Recall from Definition 3.4.3 that a Cu-semigroup S is said to be compactly bounded if every element in $S_{\ll}$ is bounded by a compact element.

Proposition 4.3.6. *Let S be a* Cu*-semigroup satisfying (O5) and weak cancellation. Then, the following are equivalent:*

(1) S is compactly bounded and zero-dimensional;

(2) S is algebraic and satisfies (O6).

Proof. Assuming (1), let $x' \ll x$ in S and take x'' such that $x' \ll x'' \ll x$. Since S is compactly bounded, it follows that x'' is in the ideal generated by some compact. It follows from Lemma 4.3.5 that S is algebraic.

Further, it is clear that zero-dimensionality implies that S satisfies (O6); see Paragraph 4.3.1.

Conversely, assuming (2), let S_c denote the subsemigroup of compact elements, which by assumption is a dense subsemigroup. By [6, Corollary 5.5.10], S_c satisfies the Riesz decomposition property. Hence, $\dim(S) = 0$ by Proposition 4.3.4.

The fact that S is compactly bounded follows clearly from algebraicity. □

Corollary 4.3.7. *Let S be a weakly cancellative, simple, zero-dimensional* Cu*-semigroup satisfying (O5). Then, S is either algebraic or has no nonzero compact elements.*

Proof. Assume that S has at least one nonzero compact element, since otherwise there is nothing to prove.

Then, since S is simple, S is compactly bounded. It follows from Proposition 4.3.6 that S is algebraic, as desired. □

Theorem 4.3.8. *Let A be a C^*-algebra. Consider the following conditions:*

(1) A has real rank zero;

(2) $\dim(\mathrm{Cu}(A)) = 0$.

Then, (1) ⇒ (2) and, if A is unital and of stable rank one, (2) ⇒ (1).

Proof. Let A be a real rank zero C^*-algebra. By [15, Corollary 3.3], $A\otimes\mathcal{K}$ has real rank zero. Applying [111, Theorem 1.1], the Murray-von Neumann semigroup $V(A)$ satisfies the Riesz decomposition property.

For every projection $p \in A \otimes \mathcal{K}$, denote by $[p]_0$ its class in $V(A)$ and consider the map $[p]_0 \mapsto [p]$ from $V(A)$ to $\mathrm{Cu}(A)_c$. This map is always well-defined, additive and order-preserving. Further, since A has real rank zero, the map is also surjective. Indeed, given $[a] \in \mathrm{Cu}(A)_c$, let $B := \overline{a(A\otimes\mathcal{K})a}$ be the associated hereditary sub-C^*-algebra. Since A has real rank zero, B has an approximate unit consisting of projections (see [15, Theorem 2.6]) and, using that $a \precsim (a-\varepsilon)_+$ for some $\varepsilon > 0$, we can find a projection $p \in \overline{a(A\otimes\mathcal{K})a}$ such that $a \precsim p \precsim (a-\varepsilon)_+$. This shows $[a] = [p]$, as desired.

Now, suppose that $[p] \leq [q] + [r]$ in $\mathrm{Cu}(A)_c$, we have $[p]_0 \leq [q]_0 + [r]_0$ in $V(A)$. Using that $V(A)$ has Riesz decomposition, one can find $p_1, p_2 \in A\otimes\mathcal{K}$ such that $[p]_0 = [p_1]_0 + [p_2]_0$ with $[p_1]_0 \leq [q]_0$ and $[p_2]_0 \leq [r]_0$. This implies $[p] = [p_1] + [p_2]$, $[p_1] \leq [q]$ and $[p_2] \leq [r]$ in $\mathrm{Cu}(A)$. Thus, $\mathrm{Cu}(A)_c$ has the Riesz decomposition property.

Using once again that A has real rank zero, it follows that $\mathrm{Cu}(A)_c$ is sup-dense in $\mathrm{Cu}(A)$. This implies $\dim(\mathrm{Cu}(A)) = 0$ by Proposition 4.3.4.

Conversely, let A be a unital, stable rank one C^*-algebra, and assume that $\mathrm{Cu}(A)$ is zero-dimensional. Since A is unital, it follows from Remark 3.4.4 that $\mathrm{Cu}(A)$ is compactly bounded.

Using Proposition 4.3.6 we have that $\mathrm{Cu}(A)$ is algebraic, and [26, Corollary 5] now implies that A has real rank zero. □

Recall that a C^*-algebra A is said to be *stably projectionless* if $A\otimes\mathcal{K}$ has no nonzero projections.

Corollary 4.3.9. *Let A be a separable, simple, $\mathcal{Z}$-stable C^*-algebra. Then, one has* $\dim(\mathrm{Cu}(A)) \leq 1$*. Moreover,* $\mathrm{Cu}(A)$ *is zero-dimensional if and only if A has real rank zero or if A is stably projectionless.*

Proof. The first part of the corollary follows from Proposition 4.1.24, where note that we only use $\mathcal{Z}$-stability.

To show the forward implication of the second statement, assume that $\mathrm{Cu}(A)$ has dimension zero. By [78, Theorem 4.1.10], a separable, simple, $\mathcal{Z}$-stable C^*-algebra is either purely infinite or stably finite.

If A is purely infinite, it follows from [110] that A has real rank zero, as required. Thus, assume that A is stably finite, and note that we may also assume that A is not stably projectionless, since otherwise there is nothing to prove.

Now let $p \in A \otimes \mathcal{K}$ be a nonzero projection, so that A is stably isomorphic to $p(A \otimes \mathcal{K})p$ and, consequently, $\mathrm{Cu}(A) \cong \mathrm{Cu}(p(A \otimes \mathcal{K})p)$. Thus, $\dim(\mathrm{Cu}(p(A \otimes \mathcal{K})p)) = 0$ and, since $p(A \otimes \mathcal{K})p$ is unital, we deduce from Proposition 4.3.6 that $\mathrm{Cu}(p(A \otimes \mathcal{K})p)$ is algebraic.

By [80, Theorem 6.7], the separable, unital, simple, stably finite, $\mathcal{Z}$-stable C^*-algebra $p(A \otimes \mathcal{K})p$ has stable rank one. Consequently, it follows from Theorem 4.3.8 that $p(A \otimes \mathcal{K})p$ has real rank zero. By [15, Corollary 2.8 and 3.3], a C^*-algebra has real rank zero if and only if its stabilization does. Thus, A has real rank zero.

To show the backward implication of the second statement, note that $\dim(\mathrm{Cu}(A)) = 0$ whenever A has real rank zero by Theorem 4.3.8.

If A is stably projectionless, [17, Theorem 5.8] implies that $\mathrm{Cu}(A)$ has no nonzero compact elements. Thus, it follows from [6, Theorem 7.5.4] that $\mathrm{Cu}(A)$ has $[0, \infty]$-multiplication. Hence, $\dim(\mathrm{Cu}(A)) = 0$ by Proposition 4.1.23. □

Example 4.3.10. We know from Theorem 4.2.12 that the nuclear dimension of subhomogeneous C^*-algebras agrees with the covering dimension of their Cuntz semigroup. However, there are many examples where these two notions are not the same. For example, let A be a separable C^*-algebra. In Theorem 4.3.8 above we have shown that the Cuntz semigroup of A is zero-dimensional whenever A has real rank zero, but [109, Remarks 2.2(iii)] implies that $\dim_{\mathrm{nuc}}(A) = 0$ if and only if A is an AF-algebra.

Every non-nuclear, real rank zero C^*-algebra A, such as $\mathcal{B}(\ell^2(\mathbb{N}))$, is also an example where both dimensions are not the same. In fact, $\dim(\mathrm{Cu}(A)) = 0$ while $\dim_{\mathrm{nuc}}(A) = \infty$. As a final example, the irrational rotation algebra has a zero-dimensional Cuntz semigroup, but its nuclear dimension is 1.

4.4 Thin boundary and complementable elements

We introduce in this section two classes of elements in a simple Cu-semigroup: elements with thin boundary (Definition 4.4.3) and complementable elements (Definition 4.4.6). Both of these classes behave similarly to compact elements, and they agree whenever S is simple, stably finite, soft, weakly cancellative, and satisfies (O5) and (O6); see Theorem 4.4.9. In particular, this includes the Cuntz semigroup of any simple, stably projectionless C^*-algebra of stable rank one by Proposition 4.4.2.

Following the study of zero-dimensional, simple Cu-semigroups from Section 4.3, we will show in Section 4.5 that zero-dimensionality is characterized by the denseness of complementable elements; see Theorem 4.5.6.

4.4.1 (Residually stably finite Cu-semigroups)**.** A Cu-semigroup S will be said to be *stably finite* if for every pair of elements $x, y \in S$ we have $y = 0$ whenever $x + y \ll x$. Although this definition is more restrictive than the one given in [6], both notions agree whenever S is simple.

Indeed, in [6] one says that a Cu-semigroup S is stably finite if, for every $x \in S_{\ll}$, one has that $x + y = x$ implies $y = 0$. Clearly, if S satifies this condition, S is stably finite in our sense (since $x + y = x$ whenever $x + y \ll x$).

Conversely, if S is simple and nonzero, we know from [6, Proposition 5.2.10] that S is stably finite in the sense of [6] if and only if $\infty \in S$ is not compact. Assume that S is stably finite in our sense. Then, if ∞ was compact, we would have $\infty + \infty \ll \infty$ and, therefore, $\infty = 0$, a contradiction. This shows that both notions agree for simple Cu-semigroups.

In analogy to residually stably finite C^*-algebras, we will say that a Cu-semigroup is *residually stably finite* if all of its quotients are stably finite. Note that every weakly cancellative Cu-semigroup is residually stably finite.

A simple, stably finite (resp. residually stably finite) C^*-algebra has a simple, stably finite (resp. residually stably finite) Cuntz semigroup.

Recall from Paragraph 4.1.17 that an element x in a Cu-semigroup S is soft if for every pair $x' \ll x$ in S there exists $k \in \mathbb{N}$ such that $(k+1)x' \ll kx$, and that we denote by S_{soft} the set of soft elements in S. If S is simple, stably finite and satisfies (O5), it follows from [6, Proposition 5.3.16] that we can decompose S as $S = S_{\mathrm{soft}} \sqcup S_c$, where S_c denotes the set of compact elements in S. Thus, in this case, $x \in S$ is soft if and only if $x = 0$ or if for every $x' \ll x$ there exists a nonzero $t \in S$ such that $x' + t \ll x$.

Indeed, the two previous conditions imply that x is either zero or noncompact and, therefore, x is soft. Conversely, if x is a nonzero soft element, x is not compact. Thus, if $x' \ll x$, we can take $x''', x'' \in S$ with $x' \ll x''' \ll x'' \ll x$ and $x'' \neq x$. Using (O5) we find $c \in S$ such that $x''' + c \leq x \leq x'' + c$. Since $x'' \neq x$, it follows that $c \neq 0$. Taking $t \ll c$ nonzero, the result follows.

We will say that a Cu-semigroup is *soft* if all of its elements are soft.

Proposition 4.4.2. *Let A be a simple, stably projectionless C^*-algebra. Then, the Cuntz semigroup* $\mathrm{Cu}(A)$ *is simple, stably finite, soft, and satisfies (O5) and (O6).*

Proof. Since A is a simple C^*-algebra, it follows from [6, Corollary 5.1.12] that the Cuntz semigroup $\mathrm{Cu}(A)$ is simple, It also satisfies (O5) and (O6); see Paragraph 1.2.14. Moreover, since A is simple and stably projectionless, [17, Theorem 5.8] implies that $\mathrm{Cu}(A)$ has no nonzero compact elements.

By [6, Proposition 5.2.10], a simple Cu-semigroup is stably finite if and only if ∞ is not compact or if S is zero. Thus, the Cuntz semigroup of a stably projectionless C^*-algebra is always stably finite.

Consequently, we have $\mathrm{Cu}(A) = \mathrm{Cu}(A)_{\mathrm{soft}} \sqcup \mathrm{Cu}(A)_c = \mathrm{Cu}(A)_{\mathrm{soft}} \sqcup \{0\}$ by [6, Proposition 5.3.16], as desired. □

Definition 4.4.3. An element x of a simple Cu-semigroup S will be said to have *thin boundary* if $x \ll x+t$ for every nonzero $t \in S$.

We let $S_{\rm tb}$ denote the subset of elements with thin boundary.

Remark 4.4.4. Every compact element has thin boundary. However, the converse does not hold. For instance, every nonzero element in $[0,\infty]$ has thin boundary but is not compact.

The definition of such elements is inspired by the notion of 'small boundary' in dynamical systems:

Let X be a compact, metric space, and take a minimal homeomorphism $T\colon X \to X$. An open subset U of X is said to be of *small boundary* if $\mu(\partial U)=0$ for every T-invariant probability measure μ on X; see [58, Section 3].

Using the dynamical notion of comparison introduced by Kerr in [51, Section 3], one can associate a dynamical version of the Cuntz semigroup to (X,T). One can then show that U has small boundary whenever $[U] \ll [U]+[V]$ for every nonempty open set $V \subseteq X$ in the dynamical Cuntz semigroup.

Thus, in this sense, 'thin boundary' implies 'small boundary'; see Examples 6.4, 6.5 and Remark 6.6 in [94] for more details.

The following result lists some of the main properties of elements with thin boundary. Note that not all the assumptions in the statement are needed for each claim.

Proposition 4.4.5. *Let S be a simple, weakly cancellative* Cu-*semigroup satisfying (O5) and (O6). Then,*

(i) $S_{\rm tb}$ is a submonoid of S.

(ii) $x \leq y$ whenever $x+z \leq y+z$ with $x,y \in S_{\rm soft}$ and $z \in S_{\rm tb}$.

(iii) For every pair $x,y \in S$, x,y have thin boundary whenever $x+y$ has thin boundary.

(iv) $x+s \ll x+t$ whenever $x \in S_{\rm tb}$, t is nonzero and soft, and $s \ll t$.

Proof. We prove each claim separately.

(i): We may assume that S is not isomorphic to $\overline{\mathbb{N}}$ or E_k for any k, since then the result follows trivially.

Thus, let $x,y \in S_{\rm tb}$ and take $t \in S$ nonzero. By Lemma 4.1.19, there is a nonzero element s such that $2s \leq t$. Using that x and y have thin boundary, we have

$$x+y \ll x+s+y+s \leq x+y+t.$$

Since this can be done for any nonzero t, we get that $x+y \in S_{\rm tb}$, as desired.

(ii): Note that the result is clear for $x=0$, so we may assume otherwise. Then, let $x',x'' \in S$ be nonzero and such that $x' \ll x'' \ll x$. Using that x is nonzero and soft, there exists a nonzero $t \in S$ with $x''+t \leq x$. Thus, since $z \ll z+t$ as z has thin boundary, we have

$$x'+z \ll x''+(z+t) \leq x+z \leq y+z.$$

Using weak cancellation, one obtains $x' \ll y$ and, since this holds for every $x' \ll x$, we get $x \leq y$.

(iii): Now assume that $x, y \in S$ are such that $x+y \in S_{\mathrm{tb}}$, and let $t \in S$ be nonzero. Then,

$$x + y \ll (x + y) + t = (x + t) + y.$$

This implies, by weak cancellation, that $x \ll x + t$. As this can be done for any nonzero $t \in S$, it follows that $x \in S_{\mathrm{tb}}$. An analogous argument shows that y has thin boundary.

(iv): Finally, to see (iv), choose $t' \in S$ such that $s \ll t' \ll t$. Since t is nonzero and soft, we can find a nonzero element $c \in S$ such that $t' + c \leq t$. Thus, since $x \in S_{\mathrm{tb}}$ we have $x \ll x + c$ and therefore

$$x + s \ll (x + c) + t' \leq x + t,$$

as required. □

Definition 4.4.6. An element x in a Cu-semigroup S is *complementable* if for every $y \in S$ satisfying $x \ll y$ there exists $z \in S$ such that $x + z = y$.

Proposition 4.4.7 below implies that elements with thin boundary are complementable; see Corollary 4.4.8.

Proposition 4.4.7. *Let S be a simple, soft, stably finite* Cu*-semigroup satisfying (O5) and (O6), and let $x, y \in S$ be such that $x \ll y$. Assume that x has thin boundary. Then, there exists $z \in S$ such that $x + z = y$.*

Proof. We may assume that x is nonzero, since the case $x = 0$ is readily checked by taking $z = y$.

Step 1. *We construct an increasing sequence $(y_n)_n$ and a sequence $(s_n)_n$ of nonzero elements such that* $\sup_n y_n = y$, *$x = y_0$, and*

$$y_n + s_n \ll y_{n+1}$$

for every $n \in \mathbb{N}$.

Set $y_0 := x$, and let $(\bar{y}_n)_n$ be any $\ll$-increasing sequence in S with supremum y. Let $y' \in S$ be such that $y_0 \ll y' \ll y$. Then, since y' is nonzero, we can find a nonzero element s_0 satisfying $y_0 + s_0 \leq y'$.

Using that $y_0 + s_0$ and $\bar{y}_1$ are both way-below y, we can choose y_1 such that

$$y_0 + s_0 \ll y_1, \quad \bar{y}_1 \ll y_1 \quad \text{and} \quad y_1 \ll y.$$

Since $y_1 \ll y$ and y is soft, there is as above $s_1 \neq 0$ such that $y_1 + s_1 \ll y$. Thus, there exists $y_2 \in S$ such that

$$y_1 + s_1 \ll y_2, \quad \bar{y}_2 \ll y_2 \quad \text{and} \quad y_2 \ll y.$$

Continuing in this way, we obtain the desired sequences $(y_n)_n$ and $(s_n)_n$.

Step 2. *We construct a sequence $(r_n)_n$ of nonzero elements such that*

$$2r_{n+1} \ll r_n, s_{n+1} \quad \textit{and} \quad y_n + r_n + r_{n+1} \ll y_{n+1} \tag{4.4.1}$$

for every $n \in \mathbb{N}$.

Using Lemma 4.1.19 to s_0, we get a nonzero element $r_0 \in S$ such that $2r_0 \ll s_0$. Then, applying Lemma 4.1.19 once again to r_0 and s_1, we obtain some nonzero $r_1 \in S$ such that $2r_1 \ll r_0, s_1$.

Continuing in this way, we construct a sequence $(r_n)_n$ such that $2r_{n+1} \ll r_n, s_{n+1}$ for every $n \in \mathbb{N}$.

In particular, we have

$$y_n + r_n + r_{n+1} \leq y_n + 2r_n \leq y_n + s_n \ll y_{n+1}$$

for each $n \in \mathbb{N}$, which shows that $(r_n)_n$ has the desired properties.

Step 3. *We construct a $\ll$-increasing sequence $(w_n)_n$ and a sequence $(v_n)_n$ such that*

$$x + r_{n+1} + v_n \leq y_n \leq x + r_n + v_n, \quad w_n \ll r_n + v_n, v_{n+1} \quad \textit{and} \quad y_{n-1} \leq x + w_n$$

for every $n \geq 1$.

Using Proposition 4.4.5 (iv) at the first step and that S is soft, one gets

$$x + r_2 \ll x + r_1 \leq y_0 + s_0 \leq y_1$$

and, applying (O5), we obtain $v_1 \in S$ such that

$$x + r_2 + v_1 \leq y_1 \leq x + r_1 + v_1.$$

Further, using that $x = y_0 \ll y_1 \ll x + r_1 + v_1$, yields an element w_1 such that

$$x = y_0 \leq x + w_1 \quad \text{and} \quad w_1 \ll r_1 + v_1.$$

Now let $n \geq 1$, and assume that we have already chosen v_n and w_n. Using for the first inequality that $x + r_{n+1} + v_n \leq y_n$ and Step 2, we have

$$x + r_{n+1} + r_n + v_n \leq y_{n+1}, \quad \text{and} \quad x + r_{n+2} \ll x + r_{n+1}.$$

Consequently, since we also have $w_n \ll r_n + v_n$ by hypothesis, we can apply (O5) (in its general setting, as defined in Paragraph 1.2.14) to $x + r_{n+1} + r_n + v_n \leq y_{n+1}$ in order to obtain an element $v_{n+1} \in S$ with

$$x + r_{n+2} + v_{n+1} \leq y_{n+1} \leq x + r_{n+1} + v_{n+1} \quad \text{and} \quad w_n \ll v_{n+1}.$$

Moreover, since $y_n \ll y_{n+1} \leq x + r_{n+1} + v_{n+1}$ and $w_n \ll v_{n+1} \leq r_{n+1} + v_{n+1}$, we get $w_{n+1} \in S$ such that

$$y_n \leq x + w_{n+1} \quad \text{and} \quad w_n \ll w_{n+1} \ll r_{n+1} + v_{n+1}.$$

This ends the proof of Step 3.

Now, since the sequence $(w_n)_n$ is increasing, we can set $z := \sup_n w_n$. For each $n \geq 1$, we have

$$x + w_n \leq x + v_{n+1} \leq y_{n+1} \leq y$$

and therefore $x + z \leq y$. Further, we also have

$$y_n \leq x + w_{n+1} \leq x + z$$

for every n and therefore $y \leq x + z$.

This implies $x + z = y$, as desired. $\square$

Corollary 4.4.8. *Every element with thin boundary in a simple, soft, stably finite* Cu*-semigroup satisfying (O5) and (O6) is complementable.*

Assuming that S is also weakly cancellative, we can now prove that the converse of Corollary 4.4.8 also holds:

Theorem 4.4.9. *Let S be a simple, soft, weakly cancellative* Cu*-semigroup satisfying (O5) and (O6), and let $x \in S$ satisfy $x \ll \infty$. Then, x has thin boundary if and only if x is complementable.*

Proof. The forwards implication follows from Corollary 4.4.8.

To show the backwards implication, assume that x is complementable and let $t \in S$ be nonzero. Since t is nonzero, we can choose a nonzero $t' \in S$ such that $t' \ll t$. Using that S is simple, one has $x \ll \infty = \infty t'$ and, consequently, we get $x \leq nt'$ for some $n \geq 1$.

Choose $t_1, \ldots, t_n \in S$ such that

$$t' \ll t_1 \ll t_2 \ll \ldots \ll t_n \ll t,$$

and set $y := t_1 + \ldots + t_n$.

Since x is complementable and $x \leq nt' \ll y$, we obtain $z \in S$ such that $x + z = y$. Further, note that

$$y = t_1 + t_2 + \ldots + t_{n-1} + t_n \ll t_2 + t_3 + \ldots + t_n + t \leq y + t,$$

which implies

$$x + z = y \ll y + t = x + z + t.$$

Using weak cancellation, we obtain $x \ll x + t$, as desired. □

Theorem 4.4.10. *Let S be a simple, soft, weakly cancellative* Cu*-semigroup satisfying (O5) and (O6). Then, S_{tb} is a cancellative monoid and, for every pair of nonzero elements $x, y \in S_{\mathrm{tb}}$, we have $x \ll y$ if and only if $x + z = y$ for some $z \in S_{\mathrm{tb}}^{\times}$.*

Proof. It follows from Proposition 4.4.5 that S_{tb} is a cancellative monoid.

Next, let $x, y \in S_{\mathrm{tb}}$. If $x \ll y$, Theorem 4.4.9 implies that there exists $z \in S$ such that $x + z = y$. Since y is not compact, we have $z \neq 0$. Further, we have $z \in S_{\mathrm{tb}}$ by Proposition 4.4.5 (iii).

Conversely, if $x + z = y$ for some nonzero $z \in S_{\mathrm{tb}}$, we have $x \ll x + z = y$ by the definition of thin boundary. □

Corollary 4.4.11. *Let A be a simple, stably projectionless C^*-algebra of stable rank one. Then,* $\mathrm{Cu}(A)_{\mathrm{tb}}$ *is a cancellative monoid and, for every pair of nonzero elements $x, y \in \mathrm{Cu}(A)_{\mathrm{tb}}$, we have $x \ll y$ if and only if $x + z = y$ for some $z \in \mathrm{Cu}(A)_{\mathrm{tb}}^{\times}$.*

Proof. Using Proposition 4.4.2, we know that $\mathrm{Cu}(A)$ is simple, soft, and satisfies (O5) and (O6). Moreover, recall from Paragraph 1.2.14 that a stable rank one C^*-algebra has a weakly cancellative Cuntz semigroup.

The result now follows from Theorem 4.4.10 above. □

4.5 Simple, zero-dimensional Cuntz semigroups

As mentioned in Section 4.4, we prove in Theorem 4.5.6 that countably based, simple, weakly cancellative, soft Cu-semigroups that satisfy (O5) and (O6) are zero-dimensional if and only if the complementable elements are dense in the semigroup.

Using this result and Corollary 4.3.7, we deduce that a countably based, simple, weakly cancellative Cu-semigroup S satisfying (O5) and (O6) is zero-dimensional if and only if S is the retract of a simple, algebraic Cu-semigroup; see Theorem 4.5.8.

Let us first show that S_{tb} has the Riesz decomposition property.

Lemma 4.5.1. *Let S be a simple, soft, weakly cancellative* Cu*-semigroup satisfying (O5) and (O6). Assume that S_{tb} is sup-dense. Let $x, y, z \in S$ be such that $x \in S_{\mathrm{tb}}$ and $x \ll y + z$. Then, there exist $v, w \in S_{\mathrm{tb}}$ such that*

$$x = v + w, \quad v \ll y \quad \textit{and} \quad w \ll z.$$

Proof. We first note that, if z is zero, we can set $v = x$ and $w = 0$, which trivially satisfy the required conditions. Thus, we may assume otherwise, i.e. $z, y \neq 0$.

Let $z' \in S$ nonzero be such that

$$x \ll y + z' \quad \text{and} \quad z' \ll z.$$

Since z is nonzero and soft, we obtain a nonzero $t \in S$ such that $z' + t \ll z$. Further, since x has thin boundary, we get $x \ll x + t$, which allows us to choose $x' \ll x$ such that $x \ll x' + t$. By the denseness of S_{tb}, we may assume that x' has thin boundary.

Applying (O6) to $x' \ll x \ll y + z'$, there exist elements $e, f \in S$ such that

$$x' \ll e + f, \quad e \ll x, y \quad \text{and} \quad f \ll x, z'.$$

Using once again that S_{tb} is sup-dense, we can take e havein thin boundary. Thus, e is complementable by Corollary 4.4.8, and we can find $c \in S$ such that $e + c = x$. In particular, it follows from Proposition 4.4.5 that e and c have thin boundary.

This implies

$$e + c = x \ll x' + t \leq e + f + t$$

and, by weak cancellation, we get $c \ll f + t$.

Thus, we have

$$c \ll f + t \leq z' + t \ll z.$$

Hence, $v := e$ and $w := c$ have the desired properties. □

Recall from Paragraph 4.3.3 the definition of Riesz decomposition and Riesz refinement.

Given two elements x, y in a positively ordered monoid M, we write $x \leq_{\mathrm{alg}} y$ if $x + z = y$ for some $z \in M$.

Proposition 4.5.2. *Let S be a simple, soft, weakly cancellative* Cu*-semigroup satisfying (O5) and (O6). Assume that S_{tb} is sup-dense. Then, S_{tb} is a simple, cancellative refinement monoid and* $\dim(S) = 0$.

Proof. The subset of elements with thin boundary S_{tb} is a cancellative monoid by Theorem 4.4.10. Moreover, for any $x, y \neq 0$, $x \ll y$ if and only if there exists $z \in S_{\mathrm{tb}}^{\times}$ with $x + z = y$. Thus, $x \leq_{\mathrm{alg}} y$ in S_{tb} if and only if $x = y$ or $x \ll y$.

Now let $x \leq_{\mathrm{alg}} y + z$ in S_{tb}. Then, either $x = y + z$ or $x \ll y + z$. Setting $v = y$ and $w = z$ in the first case or using Lemma 4.5.1 in the second, note that there always exist elements $v, w \in S_{\mathrm{tb}}$ such that $x = v + w$ with $v \leq_{\mathrm{alg}} y$ and $w \leq_{\mathrm{alg}} z$. That is to say, S_{tb} satisfies the Riesz decomposition property.

Since S_{tb} is cancellative, it follows that S_{tb} is a refinement monoid. Further, since S_{tb} satisfies the Riesz decomposition property, it follows from Proposition 4.3.4 that $\dim(S) = 0$. □

Lemma 4.5.3. *Let S be a weakly cancellative, zero-dimensional* Cu*-semigroup satisfying (O5). Assume that $x', x'', x, e, t \in S$ satisfy*

$$x' \ll x'' \quad \text{and} \quad x'' + t \leq x \leq e \ll e + t.$$

Then, there exists $y \in S$ such that $x' \ll y \ll x$ and $y \ll y + t$.

Proof. Using (O5) at $x' \ll x'' \leq e$, we obtain $c \in S$ satisfying

$$x' + c \leq e \leq x'' + c,$$

which implies that $e \ll e + t \leq x'' + c + t = x'' + (c + t)$.

Moreover, since $\dim(S) = 0$, there exist $u, v \in S$ such that

$$u \ll x'', \quad v \ll c + t \quad \text{and} \quad e \ll u + v \ll e + t.$$

Thus, one gets

$$x' + c \leq e \ll u + v \leq u + c + t$$

and, using weak cancellation, we have $x' \ll u + t$. Since $u + v \ll e + t \leq u + v + t$, we obtain $u \ll u + t$, again by weak cancellation.

Now choose $t' \in S$ such that

$$t' \ll t, \quad x' \ll u + t' \quad \text{and} \quad u \ll u + t'$$

and set $y := u + t'$.

Then,

$$x' \ll u + t' = y \quad \text{and} \quad y = u + t' \ll x'' + t \leq x.$$

Using that $u \ll u + t'$ and $t' \ll t$, we get

$$y = u + t' \ll u + t' + t = y + t.$$

This shows that y has the desired properties. □

Lemma 4.5.4. *Let S be a countably based, simple, soft, weakly cancellative* Cu*-semigroup satisfying (O5) and (O6). Assume that for every $x', x \in S$ with $x' \ll x$ and $t \neq 0$ there exists $y \in S$ such that*

$$x' \ll y \ll x, \quad y \ll y + t.$$

Then, for every $x' \ll x$ there exists $y \in S$ with thin boundary such that $x' \ll y \ll x$.

Proof. Since S is countably based, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ of nonzero elements such that, for every nonzero $t \in S$, $t_n \leq t$ for some $n \in \mathbb{N}$.

Now let $x', x \in S$ be such that $x' \ll x$. Then, it follows from our assumption that we can choose an element y_0 such that $x' \ll y_0 \ll x$ and $y_0 \ll y_0 + t_0$.

Thus, we can take y_0' with

$$x' \ll y_0' \ll y_0 \ll x, \quad y_0 \ll y_0' + t_0.$$

Applying the assumption once again, but now to y_0', y_0, t_1, we obtain y_1 such that $y_0' \ll y_1 \ll y_0$ and $y_1 \ll y_1 + t_1$, and we can choose y_1' satisfying

$$y_0' \ll y_1' \ll y_1 \ll y_0, \quad y_1 \ll y_1' + t_1.$$

Proceeding in this way, we find elements y_n' and y_n such that

$$x' \ll y_0' \ll \ldots \ll y_n' \ll y_n \ll \ldots \ll y_0 \ll x, \quad y_n \ll y_n' + t_n.$$

Set $y := \sup_n y_n'$, which satisfies $x' \ll y_0' \leq y \leq y_0 \ll x$. Further, given some nonzero $t \in S$, there exists n such that $t_n \leq t$. This implies

$$y \leq y_n \ll y_n' + t_n \leq y + t_n \leq y + t$$

and it follows that y has thin boundary. □

Proposition 4.5.5. *Let S be a countably based, simple, soft, weakly cancellative, zero-dimensional* Cu*-semigroup satisfying (O5) and (O6). Then, S_{tb} is sup-dense.*

Proof. It suffices to verify the assumptions of Lemma 4.5.4. Thus, take $x' \ll x$ and $t \neq 0$ in S.

If $x' = 0$, the element $y = 0$ satisfies

$$x' \ll y \ll x \quad \text{and} \quad y \ll y + t,$$

as desired.

If $x' \neq 0$, choose $x'', u \in S$ such that $x' \ll x'' \ll u \ll x$. Since u is soft and nonzero, there exists a nonzero element s with $x'' + s \ll u$. Using Lemma 4.1.19, and noting that $\overline{\mathbb{N}}$ and E_k are not soft, we find $r \in S$ with $0 \neq r \leq s, t$.

Now choose $r' \in S$ such that $0 \neq r' \ll r$. Since r' is nonzero and S is simple, one has $u \ll \infty = \infty r'$ and, consequently, we get $u \leq nr'$ for some $n \in \mathbb{N}$.

Let $r_1, \ldots, r_n \in S$ be such that

$$r' \ll r_1 \ll r_2 \ll \ldots \ll r_n \ll r,$$

and set $e := r_1 + \ldots + r_n$.

As in the proof of Theorem 4.4.9, we get $e \ll e + r \leq e + t$. Thus, one has

$$x' \ll x'' \quad \text{and} \quad x'' + r \leq x'' + s \ll u \leq nr' \leq e \ll e + r.$$

By Lemma 4.5.3, there exists $y \in S$ with $x' \ll y \ll u$ and $y \ll y + r \leq y + t$, as desired. □

Theorem 4.5.6. *Let S be a countably based, simple, soft, weakly cancellative* Cu*-semigroup satisfying (O5) and (O6). Then, the following conditions are equivalent:*

(1) S is zero-dimensional;

(2) the elements with thin boundary are sup-dense;

(3) the complementable elements are sup-dense;

(4) there exists a countably based, simple, algebraic, weakly cancellative Cu*-semigroup T satisfying (O5) and (O6) such that $S \cong T_{\mathrm{soft}}$.*

Proof. By Proposition 4.5.5, (1) implies (2). Conversely, (2) implies (1) by Proposition 4.5.2. That (2) and (3) are equivalent follows from Theorem 4.4.9.

To show that (4) implies (1), let T be as in (4). Using Proposition 4.1.20 and Proposition 4.3.6 at the second and last steps respectively, we have

$$\dim(S) = \dim(T_{\mathrm{soft}}) \leq \dim(T) = 0,$$

as desired.

Now assume (2) and let us prove (4). Since S is countably based and S_{tb} is sup-dense, there exists a countable subset $M_0 \subseteq S_{\mathrm{tb}}$ that is also sup-dense and whose elements are $\ll$-below ∞. Moreover, we know by Proposition 4.5.2 that S_{tb} is a simple, cancellative refinement monoid.

Let M_0' denote the (countable) subset containing all finite sums of elements in M_0, and consider the countable sets

$$C_0 := \{(x,y) \in (M_0')^2 \mid x \leq_{\mathrm{alg}} y \text{ in } S_{\mathrm{tb}}\},$$
$$I_0 := \{(x,y,z,t) \in (M_0')^4 \mid x+y = z+t\}.$$

For each $c = (x,y) \in C_0$, let $z_c \in S_{\mathrm{tb}}$ be such that $x + z_c = y$. Since S_{tb} is cancellative, this element is unique.

Further, for every $i = (x,y,z,t) \in I_0$, let $r_{1,i}, r_{2,i}, r_{3,i}, r_{4,i} \in S_{\mathrm{tb}}$ be a choice of elements satisfying the refinement conditions for $x + y = z + t$.

We set

$$M_1 := M_0' \cup \{z_c\}_{c\in C_0} \cup \{r_{1,i}, r_{2,i}, r_{3,i}, r_{4,i}\}_{i\in I_0},$$

where we note that every element in M_1 is still $\ll$-below ∞.

Proceeding in this manner, we obtain an increasing sequence of subsets

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$$

such that $M := \cup_n M_n$ is a countable, cancellative, refinement submonoid of S_{tb}, such that the inclusion $(M, \leq_{\mathrm{alg}}) \to (S_{\mathrm{tb}}, \leq_{\mathrm{alg}})$ is an order-embedding, and such that every element in M is $\ll$-below ∞. Thus, given $x, y \in M$, we have $y \ll \infty = \infty x$ in S, which implies that there exists some $k \in \mathbb{N}$ with $y \ll kx$. By Theorem 4.4.10, we get $y \leq_{\mathrm{alg}} kx$ and, consequently, that M is simple

We let T be the sequential round ideal completion $\mathrm{Cu}(M, \leq_{\mathrm{alg}})$ of M with respect to the algebraic partial order; see [6, Section 5.5]. Then, T is a countably based, algebraic Cu-semigroup. Since M is cancellative, algebraically ordered and satisfies the

Riesz decomposition property, we know from [6, Proposition 5.5.8] that T is weakly cancellative and satisfies (O5) and (O6). Further, since M is simple, so is T.

To see that S is isomorphic to T_{soft}, we identify T with the set of intervals in M ordered by inclusion. Recall that an interval $I \subseteq M$ is a downward hereditary and upward directed subset.

Moreover, the intervals of the form $\{y \in M \mid y \leq_{\mathrm{alg}} x\}$ for some $x \in M$ are precisely identified with the compact elements in T. Thus, it follows that the nonzero soft elements in T are those intervals that do not contain a largest element. Since every upward directed set in a countably based Cu-semigroup has a supremum, we can define the Cu-morphism $\alpha\colon T_{\mathrm{soft}} \to S$ by $\alpha(I) := \sup I$ for every (soft) interval $I \subseteq M$.

To see that α is surjective, note that every element $s \in S$ can be written as the supremum of a $\ll$-increasing sequence in $M_0 \subseteq M$. Thus, one has

$$\alpha(\{y \in M \mid y \leq_{\mathrm{alg}} s\}) = s.$$

That α is an order-embedding follows from the fact that the natural inclusion $(M, \leq_{\mathrm{alg}}) \to (S_{\mathrm{tb}}, \leq_{\mathrm{alg}})$ is an order-embedding. □

Remark 4.5.7. Note that, with the notation of Theorem 4.5.6 (4) above, one cannot always set $T = \mathrm{Cu}(S, \leq_{\mathrm{alg}})$. Indeed, this semigroup (which satisfies all the other properties) may not be countably based, since every basis of T contains at least all of its compact elements and, consequently, all the elements of S_{tb}.

In fact, there is no canonical choice for T. For example, set $S = [0, \infty]$, and note that, for every supernatural number q satisfying $q = q^2 \neq 1$, its associated UHF-algebra M_q has a countably based, simple, algebraic, weakly cancellative Cu-semigroup satisfying (O5) and (O6) such that $(\mathrm{Cu}(M_q))_{\mathrm{soft}} \cong [0, \infty]$; see [6, Section 7.4].

Using Corollary 4.3.7 and Theorem 4.5.6, we can now prove the following result. One should recall the definition of retract from Definition 4.1.15.

Theorem 4.5.8. *Let S be a countably based, simple, weakly cancellative* Cu*-semigroup satisfying (O5) and (O6). Then, S is zero-dimensional if and only if S is a retract of a countably based, simple, algebraic, weakly cancellative* Cu*-semigroup satisfying (O5) and (O6).*

Proof. We know from Corollary 4.3.7 that, if S is zero-dimensional, S is either algebraic or soft. If it is algebraic, we choose S as its own retract. Else, S is soft. By Theorem 4.5.6, S isomorphic to the soft part of a countaly based, simple, algebraic, weakly cancellative Cu-semigroup T. Using Proposition 4.1.18, we see that T satisfies the required properties.

Conversely, suppose that S is a retract of such an algebraic Cu-semigroup T. Then, it follows from Proposition 4.3.6 that $\dim(T) = 0$. Using Proposition 4.1.20, one obtains $\dim(S) \leq \dim(T) = 0$, as desired. □

Recall the definition of Riesz interpolation from Paragraph 4.3.3. As defined in [6, Definition 7.3.4], a Cu-semigroup S is *almost divisible* if, for every pair $x' \ll x$ in S and $n \in \mathbb{N}$, there exists $y \in S$ such that $ny \leq x$ and $x' \leq (n+1)y$.

Lemma 4.5.9. *Let S be a retract of a* Cu*-semigroup T. Then,* *(i) If T is almost divisible, so is S.*

(ii) If T has the Riesz interpolation property, so does S.

Proof. Since S is a retract of T, there is a Cu-morphism $\iota\colon S \to T$ and a generalized Cu-morphism $\sigma\colon T \to S$ such that $\sigma \circ \iota = \mathrm{id}_S$.

(i): Assume that T is almost divisible, and let $n \in \mathbb{N}$ and $x', x \in S$ satisfying $x' \ll x$. Since ι is a Cu-morphism, we have $\iota(x') \ll \iota(x)$ in T. Using that T is almost divisible, there exists $y \in T$ such that $ny \leq \iota(x)$ and $\iota(x') \leq (n+1)y$. Then, $n\sigma(y) \leq x$ and $x' \leq (n+1)\sigma(y)$, as desired.

(ii): Now assume that T has the Riesz interpolation property, and let $x_1, x_2, y_1, y_2 \in S$ be such that $x_i \leq y_j$ for all $i, j \in \{1, 2\}$. Thus, $\iota(x_i) \leq \iota(y_j)$ in T for each i, j and, consequently, there exists $z \in T$ such that $\iota(x_i) \leq z \leq \iota(y_j)$. This implies $x_i \leq \sigma(z) \leq y_j$ for all $i, j \in \{0, 1\}$, which shows that $\sigma(z)$ has the desired property. □

Proposition 4.5.10. *Let S be a countably based, simple, weakly cancellative, zero-dimensional* Cu*-semigroup satisfying (O5). Then,*

(i) S satisfies the Riesz interpolation property.

(ii) if S is not isomorphic to $\overline{\mathbb{N}}$ or E_k for any k, S is almost divisible.

Proof. Using Theorem 4.5.8, we know that S is the retract of a countably based, simple, algebraic, weakly cancellative Cu-semigroup T satisfying (O5) and (O6).

(i): Since T_c is a simple and cancellative refinement monoid, it follows that T_c has the Riesz interpolation property. Consequently, we know from [6, Proposition 5.5.8 (3)] that T also has Riesz interpolation. By Lemma 4.5.9, S has Riesz interpolation.

(ii): Now assume that S is not isomorphic to $\overline{\mathbb{N}}$ or E_k for any k. We have that T_c is weakly divisible, that is, for every $x \in T_c$ there exist elements $y, z \in T_c$ such that $x = 2y + 3z$; see [9, Theorem 6.7]. This implies that T is almost divisible. By Lemma 4.5.9, S is almost divisible. □

In [96], we will investigate when non-simple, soft Cu-semigroups are have dimension zero.

Question 4.5.11. To what extent does the converse of Proposition 4.5.10 holds? That is to say, when does a countably based, simple, soft, weakly cancellative, almost divisible Cu-semigroup S that satisfies Riesz interpolation, (O5), and (O6) have dimension zero?

We note that the question has a negative answer if softness is not assumed, as pointed out by the referee of [94]. For example, the Cuntz semigroup Z of the Jiang-Su algebra satisfies all of the above properties except for softness, but we know from Example 4.1.22 that it is not zero-dimensional.

Question 4.5.12. Let S be a zero-dimensional, weakly cancellative Cu-semigroup satisfying (O5). Does S have the Riesz interpolation property? If, additionally, no quotient of S is isomorphic to $\overline{\mathbb{N}}$ or E_k for any k, is S almost divisible?

Chapter 5

Approximations and sub-Cu-semigroups

In this chapter we introduce the notion of approximation for Cu-semigroups and the Löwenheim-Skolem condition for properties in Cu. The first one is defined in Section 5.1, where we show that it is compatible with inductive limits in Cu and with the notion of approximations for C^*-algebras. As a result, we prove that, for any C^*-algebra A, the dimension of $\mathrm{Cu}(A)$ can be bounded by the dimension of the Cuntz semigroups of any approximating family of sub-C^*-algebras of A. Moreover, we provide a new characterization for the Cuntz semigroup of AI-algebras; see Theorem 5.1.16.

In Sections 5.2 and 5.3 we investigate the general structure of sub-Cu-semigroups and show that most of the usual properties in Cu (such as the ones in Paragraph 1.2.14) satisfy the Löwenheim-Skolem condition. In particular, the results in this section imply that the covering dimension of a Cu-semigroup is determined by the dimension of its countably based sub-Cu-semigroups. This allows us to drop separability assumptions from some of the results in Chapter 4.

We finish the chapter by discussing if associating to each C^*-algebra the dimension of its Cuntz semigroup is a noncommutative dimension theory in the sense of [89]; see Section 5.4

The results, except Theorem 5.1.16, have appeared in [92].

5.1 Approximation of Cu-semigroups

Inspired by the notion of approximation for C^*-algebras, in this section we introduce in Definition 5.1.1 a notion of approximation for a Cu-semigroup S by a family of Cu-morphisms $S_\lambda \to S$. We prove that properties such as (O5)-(O7) and having dimension at most n pass to the approximated Cu-semigroup; see Proposition 5.1.7 and Proposition 5.1.8.

We also show in Proposition 5.1.12 that, for any family of sub-C^*-algebras A_λ approximating a C^*-algebra A, their Cuntz semigroups $\mathrm{Cu}(A_\lambda)$ approximate $\mathrm{Cu}(A)$. Moreover, if a Cu-semigroup S is the inductive limit of a system of Cu-semigroups S_λ, the family $(S_\lambda)_\lambda$ approximates S (Proposition 5.1.14). Finally, using the results from Chapter 3, it is shown that a countably based Cu-semigroup is the Cuntz semigroup of an AI-algebra if and only if it can be approximated by the Cuntz semigroup of interval algebras; see Theorem 5.1.16.

Definition 5.1.1. Let S be a Cu-semigroup and let $(S_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$ be a family of Cu-semigroups S_λ and Cu-morphisms $\varphi_\lambda\colon S_\lambda \to S$.

We say that the family $(S_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$ *approximates* S if the following holds: Given finite sets J and K, given elements $x_j', x_j \in S$ for $j \in J$, and given functions $m_k, n_k\colon J \to \mathbb{N}$ for $k \in K$, such that $x_j' \ll x_j$ for all $j \in J$ and such that

$$\sum_{j\in J} m_k(j)x_j \ll \sum_{j\in J} n_k(j)x_j'$$

for all $k \in K$, there exist $\lambda \in \Lambda$ and $y_j \in S_\lambda$ for $j \in J$ such that $x_j' \ll \varphi_\lambda(y_j) \ll x_j$ for each $j \in J$, and such that

$$\sum_{j\in J} m_k(j)y_j \ll \sum_{j\in J} n_k(j)y_j$$

for all $k \in K$.

We will also say that a family $\mathcal{F}$ of Cu-semigroups *approximates* S if there exists an approximation $(S_\lambda, \varphi_\lambda)_\lambda$ of S such that $S_\lambda \in \mathcal{F}$ for each λ.

Remark 5.1.2. Note that, in Definition 5.1.1 above, we do not ask the Cu-morphisms $\varphi_\lambda\colon S_\lambda \to S$ to be order-embeddings. This is because our two main sources of approximations (Proposition 5.1.12 and Proposition 5.1.14) would otherwise be excluded, since neither the natural maps $S_\lambda \to S$ to an inductive limit, or the induced Cu-morphisms $\mathrm{Cu}(A_\lambda) \to \mathrm{Cu}(A)$ from sub-C^*-algebras need be order-embeddings. For the second case consider for example $\mathbb{C} \subseteq \mathcal{O}_2$.

Further, one might think of J as the index set for a collection of variables, and for each $k \in K$ we think of the pair (m_k, n_k) as the encoding of a 'formula'. A 'property' for Cu-semigroups could then be seen as a collection of formulas. The notion introduced in Definition 5.1.1 ensures that all such properties pass to the Cu-semigroup that is being approximated.

However, we will not formalize here the notions of 'formula' or 'property'. This goes into the direction of developing a model theory for Cu-semigroups, an elaborate task that will be taken up elsewhere.

In order to prove that properties such as (O5)-(O7) and weak cancellation pass to the approximated Cu-semigroup, we first need to express them with the notation of Definition 5.1.1.

Lemma 5.1.3. *Let S be a* Cu*-semigroup. Then, S satisfies (O5) if and only if there exists a basis $B \subseteq S$ such that, for all $x', x, y', y, z', z \in B$ with $x' \ll x$, $y' \ll y$, $z' \ll z$ and such that $x + y \ll z'$, there exists $c \in B$ such that*

$$x' + c \ll z, \quad z' \ll x + c \quad \textit{and} \quad y' \ll c.$$

Proof. Let B be a basis of S, and assume that S satisfies (O5). Take $x' \ll x$, $y' \ll y$ and $z' \ll z$ in B such that $x + y \ll z'$, and choose $z'' \in B$ with $z' \ll z'' \ll z$. Then, since S satisfies (O5), one finds $a \in S$ with

$$x' + a \leq z'' \leq x + a \quad \text{and} \quad y' \ll a.$$

Using that $y' \ll a$ and $z' \ll z'' \leq x + a$, there exists $a' \ll a$ such that

$$z' \ll x + a' \quad \text{and} \quad y' \ll a'.$$

Thus, since B is a basis, we can find $c \in B$ with $a' \ll c \ll a$. It is readily checked that c has the required properties.

Conversely, assume that there exists a basis B of S with the stated property. We follow a similar argument to that of [6, Theorem 4.4 (1)]:

Let $x', x, y', y, z \in S$ be such that

$$x + y \leq z, \quad x' \ll x \quad \text{and} \quad y' \ll y.$$

Using that B is a basis, we can find elements $c_0', c_0 \in B$ with $y' \ll c_0' \ll c_0 \ll y$ and a $\ll$-decreasing sequence $(x_n)_n$ in B satisfying

$$x' \ll \ldots \ll x_{n+1} \ll x_n \ll \ldots \ll x_1 \ll x_0 \ll x.$$

Since $x_0 + c_0 \ll z$, there exists a $\ll$-increasing sequence $(z_n)_n$ in B with supremum z such that $x_0 + c_0 \ll z_0$.

Thus, we get

$$x_0 + c_0 \ll z_0, \quad x_1 \ll x_0, \quad c_0' \ll c_0 \quad \text{and} \quad z_0 \ll z_1.$$

Using that B satisfies the stated property, we find $c_1 \in B$ satisfying

$$x_1 + c_1 \ll z_1, \quad z_0 \ll x_0 + c_1 \quad \text{and} \quad c_0' \ll c_1.$$

Consequently, we can choose $c_1' \in B$ such that $z_0 \ll x_0 + c_1'$ and $c_0' \ll c_1' \ll c_1$. Then,

$$x_1 + c_1 \ll z_1, \quad x_2 \ll x_1, \quad c_1' \ll c_1 \quad \text{and} \quad z_1 \ll z_2,$$

and we can use the assumption once again to obtain $c_2 \in B$ such that

$$x_2 + c_2 \ll z_2, \quad z_1 \ll x_1 + c_2 \quad \text{and} \quad c_1' \ll c_2.$$

This allows us to choose $c_2' \in B$ such that $z_1 \ll x_1 + c_2'$ and $c_1' \ll c_2' \ll c_2$.

Proceeding in this manner inductively, we find a $\ll$-increasing sequence $(c_n')_n$ satisfying

$$x' + c_n' \leq x_n + c_n \ll z_n \leq z \quad \text{and} \quad z_n \ll x_n + c_{n+1}' \leq x + c_{n+1}'$$

for each n.

Therefore, the supremum $c := \sup_n c_n'$ satisfies $x' + c \leq z \leq x + c$ and $y' \ll c_0 \leq c$, as required. $\square$

Lemma 5.1.4. *Let S be a* Cu*-semigroup. Then, S satisfies (O6) if and only if there exists a basis $B \subseteq S$ such that, for all $x', x, y', y, z', z \in B$ with $x' \ll x$, $y' \ll y$, $z' \ll z$ and such that $x \ll y' + z'$, there exist $v, w \in B$ satisfying*

$$x' \ll v + w, \quad v \ll x, y \quad \textit{and} \quad w \ll x, z.$$

Proof. Given a basis $B \subseteq S$, an analogous argument to that of Lemma 5.1.3 shows that B has the stated property whenever S satisfies (O6).

Conversely, assume that there exists a basis $B \subseteq S$ with the stated property, and let $x', x, y, z \in S$ satisfy
$$x' \ll x \leq y + z.$$

Since B is a basis, we can find $a', a \in B$ such that $x' \ll a' \ll a \ll x$. Thus, since $a \ll y + z$, there exist elements $b', b, c', c \in B$ satisfying

$$a \ll b' + c', \quad b' \ll b \ll y \quad \text{and} \quad c' \ll c \ll z.$$

Using that B has the stated property, we get $v, w \in B$ such that

$$x' \ll a' \ll v + w, \quad v \ll a, b \quad \text{and} \quad w \ll a, c.$$

Note that $a \ll x$, $b \ll y$ and $c \ll z$. Thus, the elements v and w have the required properties. That is, S satisfies (O6). □

We omit the proofs of Lemmas 5.1.5 and 5.1.6 below, since they can be proven with the same methods as Lemma 5.1.4.

Lemma 5.1.5. *Let S be a* Cu*-semigroup. Then, S satisfies (O7) if and only if there exists a basis $B \subseteq S$ such that, for all $x_1', x_1, x_2', x_2, w', w \in B$ with $x_1' \ll x_1$, $x_2' \ll x_2$, $w' \ll w$ and satisfying $x_1 \ll w'$ and $x_2 \ll w'$, there exists $x \in B$ satisfying*

$$x_1', x_2' \ll x \ll w, x_1 + x_2.$$

Lemma 5.1.6. *Let S be a* Cu*-semigroup. Then, S is weakly cancellative if and only if there exists a basis $B \subseteq S$ such that, for all $x', x, y', y, z', z \in B$ with $x' \ll x$, $y' \ll y$, $z' \ll z$, and such that $x + z \ll y' + z'$, we have $x' \ll y$.*

Proposition 5.1.7. *Let S be a* Cu*-semigroup that is approximated by $(S_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$. If each S_λ is weakly cancellative, then so is S. Similarly, if each S_λ satisfies (O5) (respectively, (O6) or (O7)), then so does S.*

Proof. Assume first that S is approximated by $(S_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$ with S_λ weakly cancellative for each $\lambda \in \Lambda$. We will prove that S satisfies the condition in Lemma 5.1.6 for $B = S$, which will imply that S is weakly cancellative.

Thus, let $x', x, y', y, z', z \in S$ be such that $x' \ll x$, $y' \ll y$, $z' \ll z$ and $x+z \ll y'+z'$. Using that S is approximated by $(S_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$, there exist $\lambda \in \Lambda$ and elements $u, v, w \in S_\lambda$ satisfying

$$x' \ll \varphi_\lambda(u) \ll x, \quad y' \ll \varphi_\lambda(v) \ll y, \quad z' \ll \varphi_\lambda(w) \ll z \quad \text{and} \quad u + w \ll v + w.$$

Applying weak cancellation at $u + w \ll v + w$, we get $u \ll v$. Therefore, one has

$$x' \ll \varphi_\lambda(u) \ll \varphi_\lambda(v) \ll y,$$

as desired.

Now assume that S_λ satisfies (O5) for each λ. As before, we will show that S satisfies the property of Lemma 5.1.3 for $B = S$.

Let $x' \ll x$, $y' \ll y$ and $z' \ll z$ in S satisfy $x + y \ll z'$. Since $(S_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$ approximates S, we can find $\lambda \in \Lambda$ and $u, v, w \in S_\lambda$ such that

$$x' \ll \varphi_\lambda(u) \ll x, \quad y' \ll \varphi_\lambda(v) \ll y, \quad z' \ll \varphi_\lambda(w) \ll z \quad \text{and} \quad u + v \ll w.$$

Moreover, using that φ_λ is a Cu-morphism and $u + v \ll w$, there exist $u', v' \in S_\lambda$ satisfying

$$x' \ll \varphi_\lambda(u'), \quad u' \ll u, \quad y' \ll \varphi_\lambda(v') \quad \text{and} \quad v' \ll v.$$

Since S_λ satisfies (O5), we obtain $a \in S_\lambda$ such that

$$u' + a \leq w \leq u + a \quad \text{and} \quad v' \ll a.$$

Thus, the element $c := \varphi_\lambda(a)$ has the required properties, and it follows from Lemma 5.1.3 that S satisfies (O5).

The statements for (O6) and (O7) can be proven analogously, using Lemmas 5.1.4 and 5.1.5 insted of Lemma 5.1.3 respectively. □

Proposition 5.1.8. *Let S be a* Cu*-semigroup approximated by a family $(S_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$. Then,* $\dim(S) \leq \sup_{\lambda\in\Lambda} \dim(S_\lambda)$.

Proof. We may assume that $n := \sup_{\lambda\in\Lambda} \dim(S_\lambda)$ is finite, since otherwise there is nothing to prove.

Thus, let $x' \ll x \ll y_1 + \ldots + y_r$ in S, and choose elements $y_j' \ll y_j$ for each $j \leq r$ such that

$$x' \ll x \ll y_1' + \ldots + y_r'.$$

Since the family $(S_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$ approximates S, there exists $\lambda \in \Lambda$ and elements $v, w_1, \ldots, w_r \in S_\lambda$ such that

$$x' \ll \varphi_\lambda(v) \ll x, \quad y_j' \ll \varphi_\lambda(w_j) \ll y_j$$

for each $j \leq r$ and

$$v \ll w_1 + \ldots + w_r.$$

Using that φ_λ is a Cu-morphism and that $x' \ll \varphi_\lambda(v)$, we can find $v' \in S_\lambda$ with

$$x' \ll \varphi_\lambda(v') \quad \text{and} \quad v' \ll v.$$

Therefore, one has $v' \ll v \ll w_1 + \ldots + w_r$ in S_λ. Since $\dim(S_\lambda) \leq n$, we obtain elements $z_{j,k} \in S_\lambda$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ satisfying conditions (i)-(iii) in Definition 4.1.1.

One can now check that the elements $\varphi_\lambda(z_{j,k})$ in S satisfy conditions (i)-(iii) in Definition 4.1.1 for $x' \ll x \ll y_1 + \ldots + y_r$, as required. □

Recall that we say that a collection of sub-C^*-algebras A_λ of a C^*-algebra A *approximates* A if, for every $\varepsilon > 0$ and finitely many elements $a_1, \ldots, a_n \in A$, there exist $\lambda \in \Lambda$ and $b_1, \ldots, b_n \in A_\lambda$ such that $\|b_j - a_j\| < \varepsilon$ for every j.

In order to prove Proposition 5.1.12, we first need some preliminary lemmas:

Lemma 5.1.9 ([22, Lemma 1])**.** *Let A be a C^*-algebra. For every $\varepsilon > 0$ and $a \in A_+$, there exists $\delta > 0$ such that for every $b \in A$ with $\|b - a\| < \delta$ one has $\|(b^*b)^{1/2} - a\| < \varepsilon$.*

Lemma 5.1.10. *Let A be a C^*-algebra. For every $\varepsilon, \delta > 0$ and $a \in A_+$, there exists $\sigma > 0$ such that*

$$\|(a-\varepsilon)_+ - (b-\varepsilon)_+\| \leq \delta$$

whenever $b \in A_+$ satisfies $\|a-b\| \leq \sigma$.

Proof. Let $\varepsilon, \delta > 0$ and $a \in A_+$, and let $p(t) := \sum_{i=1}^n \lambda_i t^i$ be a polynomial on $[0, 2\|a\|]$ such that

$$\|(t-\varepsilon)_+ - p(t)\| \leq \frac{\delta}{3}.$$

Take $\sigma > 0$ such that

$$\sigma \leq \|a\|, \quad \text{and} \quad \sigma \leq \frac{\delta}{3n|\lambda_i|}\left(4^{i-1} + \frac{4^i - 1}{3}\right)^{-1} \|a\|^{1-i}$$

for each $i \leq n$.

Now let $b \in A_+$ satisfying $\|a-b\| \leq \sigma$. For each $i \leq n$, we have

$$\begin{aligned}
\|a^i - b^i\| &= \|(a+b)(a^{i-1} - b^{i-1}) + b^{i-1}a - a^{i-1}b\| \\
&\leq \|a+b\|\|a^{i-1} - b^{i-1}\| + \|b^{i-1}a - a^{i-1}b\| \\
&\leq (2\|a\| + \sigma)\|a^{i-1} - b^{i-1}\| + \|b^{i-1}a - a^i\| + \|a^i - a^{i-1}b\| \\
&\leq 3\|a\|\|a^{i-1} - b^{i-1}\| + \|a\|\|a^{i-1} - b^{i-1}\| + \|a\|^{i-1}\sigma \\
&= 4\|a\|\|a^{i-1} - b^{i-1}\| + \|a\|^{i-1}\sigma \\
&\leq 4\|a\|(4\|a\|\|a^{i-2} - b^{i-2}\| + \|a\|^{i-2}\sigma) + \|a\|^{i-1}\sigma \\
&= 4^2\|a\|^2\|a^{i-2} - b^{i-2}\| + (4+1)\|a\|^{i-1}\sigma \\
&\leq \ldots \leq 4^{i-1}\|a\|^{i-1}\|a-b\| + \frac{4^{i-1}-1}{3}\|a\|^{i-1}\sigma \\
&\leq \left(4^{i-1} + \frac{4^{i-1}-1}{3}\right)\|a\|^{i-1}\sigma \leq \frac{\delta}{3n|\lambda_i|}.
\end{aligned}$$

Thus, given any $b \in A_+$ satisfying $\|a-b\| \leq \sigma$, one obtains

$$\begin{aligned}
\|(a-\varepsilon)_+ - (b-\varepsilon)_+\| &\leq \frac{2\delta}{3} + \|p(a) - p(b)\| \\
&\leq \frac{2\delta}{3} + \sum_{i=1}^n |\lambda_i|\|a^i - b^i\| \\
&\leq \frac{2\delta}{3} + \sum_{i=1}^n |\lambda_i|\frac{\delta}{3n|\lambda_i|} = \delta,
\end{aligned}$$

as desired. □

Lemma 5.1.11. *Let A be a C^*-algebra, and take $\varepsilon > 0$ and $a, b, r \in A$ such that $\|a - rbr^*\| < \varepsilon$. Then, there exists $\delta > 0$ such that, whenever $c, d, s \in A$ satisfy*

$$\|c-a\| < \delta, \quad \|d-b\| < \delta \quad \textit{and} \quad \|s-r\| < \delta,$$

one has $\|c - sds^*\| < 2\varepsilon$.

Proof. Let $\varepsilon > 0$ and $a, b, r \in A$ satisfy $\|a - rbr^*\| < \varepsilon$. Take $\delta > 0$ such that

$$\delta < \|r\|, \quad \text{and} \quad \delta < \frac{\varepsilon}{1 + 3\|b\|\|r\| + 4\|r\|^2}.$$

Then, given $c, d, s \in A$ such that

$$\|c - a\| < \delta, \quad \|d - b\| < \delta \quad \text{and} \quad \|s - r\| < \delta,$$

we have

$$\begin{aligned}
\|c - sds^*\| &\leq \|c - a\| + \|a - sds^*\| \\
&\leq \delta + \|a - sbs^*\| + \|sbs^* - sds^*\| \\
&\leq \delta + \|a - rbr^*\| + \|rbr^* - sbs^*\| + \|s\|^2\delta \\
&\leq \delta + \varepsilon + \|rbr^* - sbs^*\| + (\delta + \|r\|)^2\delta \\
&\leq \delta + \varepsilon + \|rbr^* - rbs^*\| + \|rbs^* - sbs^*\| + 4\|r\|^2\delta \\
&\leq \delta + \varepsilon + \|b\|\|r\|\delta + \|b\|\|s\|\delta + 4\|r\|^2\delta \\
&\leq \delta + \varepsilon + \|b\|\|r\|\delta + \|b\|(\delta + \|r\|)\delta + 4\|r\|^2\delta \\
&\leq \delta + \varepsilon + \|b\|\|r\|\delta + 2\|b\|\|r\|\delta + 4\|r\|^2\delta \\
&= \varepsilon + (1 + 3\|b\|\|r\| + 4\|r\|^2)\delta \\
&< 2\varepsilon,
\end{aligned}$$

as desired. □

Proposition 5.1.12. *Let A be a C^*-algebra approximated by a family of sub-C^*-algebras $A_\lambda \subseteq A$, and denote by $\iota_\lambda\colon A_\lambda \to A$ the inclusion maps for each $\lambda \in \Lambda$. Then, the system $(\mathrm{Cu}(A_\lambda), \mathrm{Cu}(\iota_\lambda))_{\lambda\in\Lambda}$ approximates $\mathrm{Cu}(A)$.*

Proof. First note that we may assume that A and A_λ are stable for every $\lambda \in \Lambda$.

To see that the system $(\mathrm{Cu}(A_\lambda), \mathrm{Cu}(\iota_\lambda))_{\lambda\in\Lambda}$ approximates $\mathrm{Cu}(A)$, take finite sets J and K, elements $x_j', x_j \in \mathrm{Cu}(A)$ for each $j \in J$, and functions $m_k, n_k\colon J \to \mathbb{N}$ for every $k \in K$, such that $x_j' \ll x_j$ for each $j \in J$ and such that

$$\sum_{j\in J} m_k(j)x_j \ll \sum_{j\in J} n_k(j)x_j'$$

for all $k \in K$, where we may assume that $(\sum_{j\in J} m_k(j))(\sum_{j\in J} n_k(j)) \neq 0$ for each k.

Now let $a_j \in A_+$ be such that $x_j = [a_j]$ for every $j \in J$. Using that J is finite, we can find $\varepsilon > 0$ such that

$$x_j' \ll [(a_j - 2\varepsilon)_+] \ll [a_j] = x_j$$

for all $j \in J$.

Moreover, we have

$$\left[\bigoplus_{j\in J} a_j^{\oplus m_k(j)}\right] = \sum_{j\in J} m_k(j)x_j \ll \sum_{j\in J} n_k(j)x_j' \leq \left[\bigoplus_{j\in J} (a_j - 2\varepsilon)_+^{\oplus n_k(j)}\right]$$

for every $k \in K$, which allows us to take $r_k \in A$ such that

$$\left\| \bigoplus_{j\in J} a_j^{\oplus m_k(j)} - r_k \left(\bigoplus_{j\in J} (a_j - 2\varepsilon)_+^{\oplus n_k(j)} \right) r_k^* \right\| < \frac{\varepsilon}{2}.$$

Applying Lemma 5.1.11 to the previous inequality, we find for each k a bound $\delta_k > 0$. Take $\delta > 0$ be such that

$$\delta < \min_{k\in K} \left\{ \frac{\delta_k}{(\sum_{j\in J} m_k(j))(\sum_{j\in J} n_k(j))} \right\} \quad \text{and} \quad \delta < \varepsilon.$$

Thus, we know from Lemma 5.1.10 that there exists $\sigma > 0$ such that $\sigma \leq \delta$ and such that, for every $j \in J$ and $b \in A_+$ satisfying $\|a_j - b\| \leq \sigma$, we have $\|(a_j - 2\varepsilon)_+ - (b - 2\varepsilon)_+\| \leq \delta$.

Using Lemma 5.1.9 and the fact that the sub-C^*-algebras A_λ approximate A, we find $\lambda \in \Lambda$ and elements $s_k \in A_\lambda$ and $b_j \in (A_\lambda)_+$ such that

$$\|s_k - r_k\| \leq \sigma \quad \text{and} \quad \|b_j - a_j\| \leq \sigma$$

for every $k \in K$ and $j \in J$.

Thus, it follows from the choice of σ that $\|(b_j - 2\varepsilon)_+ - (a_j - 2\varepsilon)_+\| \leq \delta$ for every $j \in J$. Using $\|b_j - a_j\| \leq \delta < \varepsilon$ and $\|(b_j - \varepsilon)_+ - a_j\| < 2\varepsilon$ in the first and second step respectively, we have

$$[(a_j - 2\varepsilon)_+] \ll [(b_j - \varepsilon)_+] \ll [a_j]$$

for each $j \in J$.

Further, for every $k \in K$ one gets

$$\left\| \bigoplus_{j\in J} a_j^{\oplus m_k(j)} - \bigoplus_{j\in J} b_j^{\oplus m_k(j)} \right\| \leq \sum_{j\in J} m_k(j)\|a_j - b_j\| \leq \sum_{j\in J} m_k(j)\delta < \delta_k$$

and, similarly,

$$\left\| \bigoplus_{j\in J} (a_j - 2\varepsilon)_+^{\oplus n_k(j)} - \bigoplus_{j\in J} (b_j - 2\varepsilon)_+^{\oplus n_k(j)} \right\| \leq \sum_{j\in J} n_k(j)\delta < \delta_k.$$

Using Lemma 5.1.11 we get that, for every $k \in K$,

$$\left\| \bigoplus_{j\in J} b_j^{\oplus m_k(j)} - s_k \left(\bigoplus_{j\in J} (b_j - 2\varepsilon)_+^{\oplus n_k(j)} \right) s_k^* \right\| < 2\frac{\varepsilon}{2} = \varepsilon$$

and, consequently,

$$\sum_{j\in J} m_k(j)[(b_j - \varepsilon)_+] \leq \sum_{j\in J} n_k(j)[(b_j - 2\varepsilon)_+]$$

in $\mathrm{Cu}(A_\lambda)$.

Finally, recall that $\iota_\lambda\colon A_\lambda \to A$ denotes the inclusion map. Using that $[(a_j-2\varepsilon)_+] \ll [(b_j-\varepsilon)_+] \ll [a_j]$ in $\mathrm{Cu}(A)$ and that $[(b_j-2\varepsilon)_+] \ll [(b_j-\varepsilon)_+]$ in $\mathrm{Cu}(A_\lambda)$, the elements $[(b_j-\varepsilon)_+] \in \mathrm{Cu}(A_\lambda)$ satisfy

$$x_j' \ll [(a_j-2\varepsilon)_+] \ll \mathrm{Cu}(\iota_\lambda)([(b_j-\varepsilon)_+]) \ll [a_j] = x_j$$

for every $j \in J$, and

$$\sum_{j\in J} m_k(j)[(b_j-\varepsilon)_+] \leq \sum_{j\in J} n_k(j)[(b_j-2\varepsilon)_+] \ll \sum_{j\in J} n_k(j)[(b_j-\varepsilon)_+]$$

for each $k \in K$, as required. □

Theorem 5.1.13. *Let A be a C^*-algebra approximated by a family of sub-C^*-algebras $A_\lambda \subseteq A$, for $\lambda \in \Lambda$. Then,* $\dim(\mathrm{Cu}(A)) \leq \sup_{\lambda\in\Lambda} \dim(\mathrm{Cu}(A_\lambda))$.

Proof. The system $(\mathrm{Cu}(A_\lambda), \mathrm{Cu}(\iota_\lambda))_{\lambda\in\Lambda}$ approximates $\mathrm{Cu}(A)$ by Proposition 5.1.12.
The result now follows from Proposition 5.1.8. □

Proposition 5.1.14. *Let $S = \varinjlim_{\lambda\in\Lambda} S_\lambda$ be an inductive limit of* Cu*-semigroups, and let $\varphi_\lambda\colon S_\lambda \to S$ be the* Cu*-morphisms into the limit. Then, S is approximated by the family $(S_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$.*

Proof. Let $\varphi_{\mu,\lambda}\colon S_\lambda \to S_\mu$ denote the connecting Cu-morphisms of the inductive system for each $\lambda \leq \mu$ in Λ, and recall from Lemma 4.1.8 that the inductive limit S is characterized by the following three properties:

(L0) we have $\varphi_\mu \circ \varphi_{\mu,\lambda} = \varphi_\lambda$ for all $\lambda \leq \mu$ in Λ;

(L1) if $x_\lambda', x_\lambda \in S_\lambda$ and $x_\mu \in S_\mu$ satisfy $x_\lambda' \ll x_\lambda$ and $\varphi_\lambda(x_\lambda) \leq \varphi_\mu(x_\mu)$, then there exists $\nu \geq \lambda, \mu$ such that $\varphi_{\nu,\lambda}(x_\lambda') \ll \varphi_{\nu,\mu}(x_\mu)$;

(L2) for all $x', x \in S$ satisfying $x' \ll x$ there exists $x_\lambda \in S_\lambda$ such that $x' \ll \varphi_\lambda(x_\lambda) \ll x$.

To see that S is approximated by the stated system, let J and K be finite sets, x_j', x_j be elements in S satisfying $x_j' \ll x_j$ for each $j \in J$, and let $m_k, n_k\colon J \to \mathbb{N}$ be maps such that

$$\sum_{j\in J} m_k(j)x_j \ll \sum_{j\in J} n_k(j)x_j'$$

for all $k \in K$.

Using (L2) for each $j \in J$, we get $\lambda_j \in \Lambda$ and an element $w_j \in S_{\lambda_j}$ such that

$$x_j' \ll \varphi_{\lambda_j}(w_j) \ll x_j.$$

Take $\lambda \in \Lambda$ with $\lambda_j \leq \lambda$ for all $j \in J$, and define $z_j := \varphi_{\lambda,\lambda_j}(w_j) \in S_\lambda$ for every j. Thus, we have

$$\varphi_\lambda\left(\sum_{j\in J} m_k(j)z_j\right) \ll \sum_{j\in J} m_k(j)x_j \ll \sum_{j\in J} n_k(j)x_j' \ll \varphi_\lambda\left(\sum_{j\in J} n_k(j)z_j\right)$$

for each $k \in K$.

Moreover, since φ_λ is a Cu-morphism, we can choose elements $z_j' \ll z_j$ such that

$$\varphi_\lambda\left(\sum_{j\in J} m_k(j)z_j\right) \ll \varphi_\lambda\left(\sum_{j\in J} n_k(j)z_j'\right) \quad \text{and} \quad x_j' \ll \varphi_\lambda(z_j') \ll x_j.$$

Using that S satisfies (L1), there exist $\nu_k \in \Lambda$ with $\lambda \leq \nu_k$ such that

$$\varphi_{\nu_k,\lambda}\left(\sum_{j\in J} m_k(j)z_j'\right) \ll \varphi_{\nu_k,\lambda}\left(\sum_{j\in J} n_k(j)z_j'\right)$$

for every k.

Now let $\nu \in \Lambda$ be such that $\nu_k \leq \nu$ for each k, and set $y_j := \varphi_{\nu,\lambda}(z_j') \in S_\nu$ for every $j \in J$. We obtain

$$\begin{aligned}\sum_{j\in J} m_k(j)y_j &= \varphi_{\nu,\nu_k}\left(\varphi_{\nu_k,\lambda}\left(\sum_{j\in J} m_k(j)z_j'\right)\right)\\ &\ll \varphi_{\nu,\nu_k}\left(\varphi_{\nu_k,\lambda}\left(\sum_{j\in J} n_k(j)z_j'\right)\right) = \sum_{j\in J} n_k(j)y_j,\end{aligned}$$

for every $k \in K$ and, for each $j \in J$, one has $x_j' \ll \varphi_\nu(y_j) \ll x_j$, as desired. $\square$

Using the previous result, we can recover [6, Theorem 4.5] and Proposition 4.1.9. Note that the following statement is new for (O7).

Corollary 5.1.15. *Let $S = \varinjlim_{\lambda\in\Lambda} S_\lambda$ be an inductive limit of* Cu*-semigroups. If each S_λ satisfies weak cancellation (respectively, (O5), (O6) or (O7)), then so does S. Further, if* $\dim(S_\lambda) \leq n$ *for each λ for some $n \in \mathbb{N}$, then* $\dim(S) \leq n$.

Proof. This follows by combining Propositions 5.1.7, 5.1.8 and 5.1.14. $\square$

As announced at the end of Section 3.5, the notion of approximation for Cu-semigroups allows us to give another characterization for the Cuntz semigroup of AI-algebras.

Theorem 5.1.16. *Let S be a countably based* Cu*-semigroup. Then, S is* Cu*-isomorphic to the Cuntz semigroup of an AI-algebra if and only if S is approximated by the family* $\{\mathrm{Lsc}([0,1],\overline{\mathbb{N}})^n\}_n$.

That is to say, a countably based Cu*-semigroup S is* Cu*-isomorphic to the Cuntz semigroup of an AI-algebra if and only if S can be approximated by the Cuntz semigroups of interval algebras.*

Proof. If S is Cu-isomorphic to the Cuntz semigroup of an AI-algebra, it follows from Theorem 3.1.6 and Proposition 5.1.14 that S is approximated by the family $\{\mathrm{Lsc}([0,1],\overline{\mathbb{N}})^n\}_n$.

Conversely, assume that S is approximated by such a family, and let $(S_\lambda, \varphi_\lambda)_\lambda$ be an approximation for S such that $S_\lambda = \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^{r_\lambda}$ for each λ.

It follows from Proposition 5.1.7 that S satisfies (O5), (O6) and weak cancellation. Moreover, it is also easy to check that, since each $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})^n$ is compactly bounded, so is S.

By Theorem 3.5.34, it suffices to show that S satisfies property I. Thus, let $m, M \in \mathbb{N}$, take $\ll$-decreasing sequences $(z_{\alpha,j})_{\alpha\in\Omega_m}$ for $j \leq M$, and finite multisets $\mathcal{A}, \mathcal{B}$ of Ω_m satisfying the conditions in Definition 3.5.29.

For each β and j, let $z'_{\beta,j} \ll z_{\beta,j}$ be such that $z_{\alpha,j} \ll z'_{\beta,j}$ whenever $\beta \prec \alpha$ and

$$\sum_{\alpha\in\mathcal{A},j\leq M} z_{\alpha,j} \ll \sum_{\beta\in\mathcal{B},j\leq M} z'_{\beta,j}.$$

Since $(S_\lambda, \varphi_\lambda)_\lambda$ approximates S, there exist λ and elements $y_{\beta,j} \in S_\lambda$ such that $z'_{\beta,j} \ll \varphi_\lambda(y_{\beta,j}) \ll z_{\beta,j}$ for every β, j,

$$y_{\alpha,j} \ll y_{\beta,j} \quad \text{and} \quad \sum_{\alpha\in\mathcal{A},j\leq M} y_{\alpha,j} \ll \sum_{\beta\in\mathcal{B},j\leq M} y_{\beta,j}$$

for each j and $\beta \prec \alpha$.

Now let $r \in \mathbb{N}$ be such that $S_\lambda = \mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r$, and recall that for each n we defined L_n as the additive span of $\{0, 1, \chi_{(i/n,j/n)}, \chi_{(i/n,1]}, \chi_{[0,j/n)}\}_{i,j}$; see Example 3.5.10. Also recall that L_n is a chainable subsemigroup of $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$, as defined in Definition 3.5.7.

Since $\cup_n L_n^r$ is dense in S_λ, we may assume that there exists $n \in \mathbb{N}$ such that each $y_{\beta,j}$ is in L_n^r. For each $k \leq r$, let $F_k\colon X_n \to S_\lambda$ denote the composition of the I-morphism $F\colon X_n \to L_n$ given in Example 3.5.10 with the inclusion to the k-th component in $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})^r$. Also, let $\pi_k\colon S_\lambda \to \mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ denote the projection to the k-th component.

Letting $v^k_{\beta,j}$ be such that $F_k(v^k_{\beta,j}) = \pi_k(y_{\beta,j})$, it is now easy to check that these elements satisfy the required conditions in Definition 3.5.29. $\square$

5.2 The lattice of sub-Cu-semigroups

The aim of this and the next section is to reduce the computation of the covering dimension of a Cu-semigroup to that of its countably based sub-Cu-semigroups.

In order to do this, in this section we study when a submonoid of a Cu-semigroup S is a sub-Cu-semigroup and, given any submonoid $T \subseteq S$, we construct an associated sup-closed submonoid $\overline{T}^{\sup} \subseteq S$ (Definition 5.2.3) and its 'derived' submonoid $T' \subseteq S$ (Definition 5.2.5). With this notation, we prove that T is a sub-Cu-semigroup if and only if $T = T'$; see Proposition 5.2.8.

We also show in Theorem 5.2.11 that the collection of sub-Cu-semigroups of a Cu-semigroup is a complete lattice.

Recall the definition of sub-Cu-semigroup from Definition 1.2.6. We begin by giving two characterizations of when a submonoid is a sub-Cu-semigroup.

Lemma 5.2.1. *Let S be a Cu-semigroup and T be a submonoid of S. Then, T is a sub-Cu-semigroup of S if and only if T is closed under passing to suprema of increasing sequences and, for every $x' \in S$ and $x \in T$ with $x' \ll x$, there exists $y \in T$ such that $x' \ll y \ll x$.*

Proof. Let $\iota\colon T \to S$ denote the inclusion (as a monoid morphism), and assume that T satisfies both of the stated properties. Given an increasing sequence $(x_n)_n$ in T, denote by x_T and x_S its supremum in T and S respectively. Clearly, one has $x_S \leq x_T$.

Now let $(x'_n)_n$ be a $\ll$-increasing sequence in S such that $x'_n \in T$ for each n and $x_T = \sup_n x'_n$. Such a sequence exists by our second assumption. Since $x'_n \ll x_T$ in S, one gets $x'_n \ll x_m$ for some m and, consequently, $x'_n \leq x_S$ for each n. This implies $x_T \leq x_S$ and, therefore, that ι preserves suprema of increasing sequences.

Further, one has $x \ll y$ in T if and only if $x \ll y$ in S. Using this and the fact that ι preserves suprema, it is clear that T satisfies (O1)-(O4).

The inclusion ι is a Cu-morphism by the comments above. □

Lemma 5.2.2. *Let S, T be* Cu*-semigroups, and let $\varphi\colon T \to S$ be a* Cu*-morphism. Then the following are equivalent:*

(1) φ is an order-embedding, that is, $x, y \in T$ satisfy $x \leq y$ if (and only if) $\varphi(x) \leq \varphi(y)$;

(2) $x, y \in T$ satisfy $x \ll y$ if (and only if) $\varphi(x) \ll \varphi(y)$;

*(3) $\varphi(T) \subseteq S$ is a sub-*Cu*-semigroup and $\varphi\colon T \to \varphi(T)$ is an isomorphism.*

Proof. That (1) holds if and only if (2) holds is well known. Further, it is easy to check that (3) implies (2) by using that $\varphi\colon T \to \varphi(T)$ is an isomorphism.

Finally, to see that (1) implies (3), note that $\varphi\colon T \to \varphi(T)$ is a surjective Cu-morphism by construction. Moreover, it is injective by (1).

Let φ^{-1} denote the inverse Cu-morphism of φ. Then, the inclusion map $\iota\colon \varphi(T) \to S$ can be decomposed as $\iota = \varphi\varphi^{-1}$. Since both of the previous maps are Cu-morphisms, it follows that ι is as well. □

Definition 5.2.3. Let T be a subset of a Cu-semigroup S. We define

$$\overline{T}^{\mathrm{seq}} := \left\{ \sup_n x_n \in S \mid (x_n)_n \text{ is an increasing sequence in } T \right\}.$$

By using transfinite induction, for every ordinal α we let $\overline{T}^{(\alpha)}$ be $\overline{T}^{(0)} := T$, $\overline{T}^{(1)} := \overline{T}^{\mathrm{seq}}$, and

$$\begin{aligned} \overline{T}^{(\alpha+1)} &:= \overline{\overline{T}^{(\alpha)}}^{\mathrm{seq}}, \\ \overline{T}^{(\lambda)} &:= \bigcup_{\alpha<\lambda} \overline{T}^{(\alpha)} \text{ if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

The *sup-closure* of T is defined as $\overline{T}^{\mathrm{sup}} := \bigcup_{\alpha \geq 1} \overline{T}^{(\alpha)}$. Further, T will be said to be *sup-closed* if $T = \overline{T}^{\mathrm{sup}}$.

Remark 5.2.4. Using the notation of Definition 5.2.3 above, the sup-closure of T is sup-closed. Indeed, the sequence $(\overline{T}^{(\alpha)})_\alpha$ is an increasing family of subsets of S. Thus, it eventually stabilizes. That is, there exists α_0 such that $\overline{T}^{(\alpha)} = \overline{T}^{(\alpha_0)}$. We get $\overline{T}^{\mathrm{sup}} = \overline{T}^{(\alpha_0)}$ and, consequently, $\overline{\overline{T}^{\mathrm{sup}}}^{\mathrm{seq}} = \overline{T}^{\mathrm{sup}}$, as desired.

Moreover, note that T is sup-closed if and only if $T = \overline{T}^{\mathrm{seq}}$.

Given a subset of a Cu-semigroup, we now introduce what might be seen as the Cu-version of its derived subset and its Cantor-Bendixson derivative; see Remark 5.2.6 below for a discussion.

Definition 5.2.5. Let T be a subset of a Cu-semigroup S. We set

$$T' := \left\{ \sup_n x_n \in S \mid (x_n)_n \text{ is a } \ll\text{-increasing sequence in } T \right\}.$$

We define $T^{(\alpha)}$ for every ordinal α by setting $T^{(0)} := T$, $T^{(1)} := T'$, and

$$\begin{aligned} T^{(\alpha+1)} &:= \left(T^{(\alpha)}\right)', \\ T^{(\lambda)} &:= \bigcap_{\alpha<\lambda} T^{(\alpha)}, \quad \text{if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

Further, we set

$$\delta(T) := \bigcap_{\alpha \geq 1} T^{(\alpha)}.$$

Remark 5.2.6. Given a subset Y of a topological space X, the *derived set* of Y, denoted by Y', is defined as the set of limit points of Y.

In a subset T of a Cu-semigroup S, we may view suprema of $\ll$-increasing sequences in T as the limit points of T. Thus, one can think of T' as the *derived set* of T, and the subsets $T^{(\alpha)}$ as the α-th Cantor-Bendixson derivatives of T.

Moreover, the following statements, which are analogs of well known properties satisfied by the derived subsets of a topological space, are also satisfied by the derived subsets of a Cu-semigroup:

(i) If $x \in T'$ and if x is not compact (that is, $x \not\ll x$), then x also belongs to $(T - \{x\})'$.

(ii) We have $(T \cup H)' = T' \cup H'$.

(iii) If $T \subseteq H$, then $T' \subseteq H'$.

Further, recall that a subset of a topological space is said to be *perfect* if it is equal to its derived set. Continuing with the previous analogy, Proposition 5.2.8 below shows that we may view sub-Cu-semigroups as the perfect submonoids of a Cu-semigroup.

Lemma 5.2.7. *Let T be a submonoid of a* Cu*-semigroup S. Then, T' is a sup-closed submonoid of S.*

Proof. That T' is a submonoid follows from the fact that $0 \ll 0$ and that the way-below relation is additive. Thus, it remains to verify that T' is sup-closed.

Let $(x_n)_n$ be an increasing sequence in T', and let x denote its supremum in S. Since S satisfies (O2), there exists a $\ll$-increasing sequence $(x'_m)_m$ in S with supremum x. Then, there exists $n_1 \in \mathbb{N}$ such that $x'_1 \ll x_{n_1}$. Since $x_{n_1} \in T'$, we can find $y_1 \in T$ such that

$$x'_1 \ll y_1 \ll x_{n_1}.$$

Further, since there is $n_2 \geq n_1$ with $y_1, x'_2 \ll x_{n_2}$, there exists $y_2 \ll x_{n_2}$ in T such that $y_1, x'_2 \ll y_2$.

Proceeding in this manner, we construct a $\ll$-increasing sequence $(y_m)_m$ in T such that

$$x'_m \ll y_m \leq x.$$

This implies that $x = \sup_m x'_m = \sup_m y_m \in T'$, as desired. □

Proposition 5.2.8. *Let T be a submonoid of a* Cu*-semigroup S. Then, T is a sub-*Cu*-semigroup of S if and only if $T = T'$.*

Proof. First, assume that T is a sub-Cu-semigroup. Then, the supremum of any $\ll$-increasing sequence in T is also in T. This shows that $T = T'$.

Conversely, assume that T is a submonoid such that $T = T'$. Using Lemma 5.2.7, we know that T is sup-closed. Thus, it follows from Lemma 5.2.1 that T is a sub-Cu-semigroup. □

We can now prove the following result, which recovers [6, Lemma 5.3.17].

Corollary 5.2.9. *Let T be a submonoid of a* Cu*-semigroup S. Assume that every element in T is the supremum of a $\ll$-increasing sequence in T. Then, $T' = \overline{T}^{\mathrm{seq}} = \overline{T}^{\mathrm{sup}}$, which is a sub-*Cu*-semigroup of S.*

Proof. The inclusions $T' \subseteq \overline{T}^{\mathrm{seq}} \subseteq \overline{T}^{\mathrm{sup}}$ hold generally, and we know by our assumption that $T \subseteq T'$. Using Lemma 5.2.7 at the second step, one has

$$\overline{T}^{\mathrm{sup}} \subseteq \overline{T'}^{\mathrm{sup}} = T',$$

as desired.

To see that T' is a sub-Cu-semigroup, note that $T'' \subseteq T'$ because T' is sup-closed and that, since $T \subseteq T'$, one gets $T' \subseteq T''$. This shows $T' = T''$ and, consequently, $T' \subseteq S$ is a sub-Cu-semigroup by Proposition 5.2.8. □

Theorem 5.2.10. *Let T be a submonoid of a* Cu*-semigroup S. Then, $\delta(T) \subseteq S$ is a sub-*Cu*-semigroup.*

If T is sup-closed, then $\delta(T) \subseteq T$.

Proof. We prove that $T^{(\alpha)}$ is always a sup-closed submonoid using transfinite induction: For $\alpha = 1$ and the successor case, this is Lemma 5.2.7. Moreover, the limit case follows from its definition. This implies that $\delta(T)$ is a submonoid.

Further, we note that $(T^{(\alpha)})_{\alpha \geq 1}$ is a decreasing family of submonoids, which must therefore stabilize. Thus, $\delta(T) = T^{(\alpha)}$ for some $\alpha \geq 1$. In particular, one has

$$\delta(T) = T^{(\alpha)} = T^{(\alpha+1)} = \delta(T)',$$

which implies that $\delta(T)$ is a sub-Cu-semigroup by Proposition 5.2.8.

One clearly also gets $T' \subseteq \overline{T}^{\mathrm{seq}}$, which shows that $\delta(T) \subseteq \overline{T}^{\mathrm{sup}}$. □

Theorem 5.2.11. *Let S be a* Cu*-semigroup. Then, the collection of sub-*Cu*-semigroups of S is a complete lattice when ordered by inclusion.*

*For any collection $(T_j)_{j \in J}$ of sub-*Cu*-semigroups, their supremum is the sup-closure of the submonoid generated by $\bigcup_j T_j$, while their infimum is $\delta(\bigcap_j T_j)$.*

Proof. Let $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{S}$ denote the collection of all subsets, sup-closed submonoids and sub-Cu-semigroups of S respectively. Equip each of these with the partial order given by inclusion, and note that $\mathcal{P}$ is a complete lattice.

Let $\alpha\colon \mathcal{P} \to \mathcal{C}$ denote the order-preserving map that sends an element of $\mathcal{P}$ (that is, a subset of S) to the sup-closure of the submonoid it generates. By considering α as a map $\mathcal{P} \to \mathcal{P}$, it follows from Remark 5.2.4 that α is idempotent and satisfies $X \subseteq \alpha(X)$ for every $X \in \mathcal{P}$. Thus, $\alpha\colon \mathcal{P} \to \mathcal{P}$ is a closure operator in the sense of [41, Definition 0-3.8 (ii)]. This implies, since $\mathcal{P}$ is a complete lattice, that $\mathcal{C}$ is also a complete lattice. Moreover, we also have that α preserves arbitrary suprema, and that the inclusion map $\iota\colon \mathcal{C} \to \mathcal{P}$ preserves arbitrary infima.

Now let $\delta\colon \mathcal{C} \to \mathcal{S}$ be the map that sends $T \in \mathcal{C}$ to $\delta(T)$ as defined in Definition 5.2.5, which is well-defined by Theorem 5.2.10. Also note that the map is order-preserving by construction.

It follows from Proposition 5.2.8 that δ as a map $\mathcal{C} \to \mathcal{C}$ is idempotent and satisfies $\delta(T) \subseteq T$ for every $T \in \mathcal{C}$. Therefore, $\delta\colon \mathcal{C} \to \mathcal{C}$ is a kernel operator in the sense of [41, Definition 0-3.8 (iii)]. This shows that $\mathcal{S}$ is a complete lattice, and that δ preserves arbitrary infima, and the inclusion map $\iota\colon \mathcal{S} \to \mathcal{C}$ preserves arbitrary suprema.

Thus, one gets the following diagram

$$\begin{tikzcd} \mathcal{S} \arrow[r, hook, "\iota"'] & \mathcal{C} \arrow[l, bend right, "\delta"'] \arrow[r, hook, "\iota"'] & \mathcal{P}. \arrow[l, bend right, "\alpha"'] \end{tikzcd}$$

Now let $T = (T_j)_{j\in J}$ be a collection of sub-Cu-semigroups. By the comments above, its infimum in $\mathcal{P}$ is $\inf_{\mathcal{P}} T = \delta(\inf_{\mathcal{C}} T) = \delta(\cap T)$. Similarly, their supremum is $\sup_{\mathcal{P}} T = \sup_{\mathcal{C}} T = \alpha(\cup T)$, as required. $\square$

5.3 Reduction to countably based Cu-semigroups

Given a Cu-semigroup S, we prove in this section that the dimension of its countably based sub-Cu-semigroups determine the dimension of S; see Theorem 5.3.8. This allows us to generalize results from Chapter 4 by dropping the countably based assumption; see Propositions 5.3.9 and 5.3.10.

5.3.1 (Countably based sub-Cu-semigroups)**.** Let S be a Cu-semigroup. We denote by $\mathrm{Sub}_{\mathrm{ctbl}}(S)$ the collection of countably based sub-Cu-semigroups of S.

Given a countable, directed family $\mathcal{T} \subseteq \mathrm{Sub}_{\mathrm{ctbl}}(S)$, its supremum in the complete lattice of sub-Cu-semigroups (see Theorem 5.2.11) is again a countably based sub-Cu-semigroup. Indeed, let $\bigcup \mathcal{T}$ denote the union of all the members of $\mathcal{T}$. Then, every element in the submonoid $\bigcup \mathcal{T}$ is the supremum of a $\ll$-increasing sequence in $\bigcup \mathcal{T}$, and it follows from Corollary 5.2.9 that the sup-closure $\overline{\bigcup \mathcal{T}}^{\sup}$ is a sub-Cu-semigroup. This semigroup is countably based since $\mathcal{T}$ is countable and every member in the family has a countable basis.

Using the results developed in Section 5.2 above, we obtain the following lemma.

Lemma 5.3.2. *Let T_0 be a countable subset of a* Cu*-semigroup S. Then, there exists a countably based sub-*Cu*-semigroup $T \in \mathrm{Sub}_{\mathrm{ctbl}}(S)$ that contains T_0.*

Proof. First, note that the submonoid generated by T_0 is also countable. Thus, we may assume directly that T_0 is a submonoid.

Then, for each $x \in T_0$, choose a $\ll$-increasing sequence in S with supremum x, and let T_1 be the submonoid of S generated by T_0 and the elements in each of the chosen sequences. Note that, once again, T_1 is a countable submonoid.

Proceeding in this manner, we obtain an increasing, countable sequence $(T_k)_k$ of countable submonoids. Therefore, their union $T_\infty := \bigcup_k T_k$ is also a countable submonoid of S. Using that for every element x in T_∞ there exists a $\ll$-increasing sequence in T_∞ with supremum x, it follows from Corollary 5.2.9 that $T := \overline{T_\infty}^{\mathrm{seq}}$ is a sub-Cu-semigroup of S.

By construction, T_∞ is a countable basis for T, as desired. $\square$

5.3.3 (Löwenheim-Skolem condition in $\mathrm{Sub}_{\mathrm{ctbl}}(S)$)**.** Given a Cu-semigroup S and a collection $\mathcal{R} \subseteq \mathrm{Sub}_{\mathrm{ctbl}}(S)$, we say that $\mathcal{R}$ is *cofinal* if for every $T \in \mathrm{Sub}_{\mathrm{ctbl}}(S)$ there exists $R \in \mathcal{R}$ with $T \subseteq R$. We also say that $\mathcal{R}$ is *σ-complete* if $\overline{\bigcup \mathcal{T}}^{\mathrm{sup}}$ belongs to $\mathcal{R}$ for every countable, directed subset $\mathcal{T} \subseteq \mathcal{R}$.

A property $\mathcal{P}$ of Cu-semigroups will be said to satisfy the *Löwenheim-Skolem condition* if, for every Cu-semigroup S satisfying $\mathcal{P}$, there exists a σ-complete, cofinal subcollection $\mathcal{R} \subseteq \mathrm{Sub}_{\mathrm{ctbl}}(S)$ such that $\mathcal{P}$ is satisfied in every $R \in \mathcal{R}$. We will show in Propositions 5.3.4, 5.3.5 and 5.3.7 below that (O5)-(O7), simplicity, weak cancellation and '$\dim(\_\,) \leq n$' (for a fixed $n \in \mathbb{N}$) each satisfy the Löwenheim-Skolem condition.

Note that, given any pair of σ-complete and cofinal collections $\mathcal{R}_1, \mathcal{R}_2 \subseteq \mathrm{Sub}_{\mathrm{ctbl}}(S)$, their intersection $\mathcal{R}_1 \cap \mathcal{R}_2$ is again σ-complete and cofinal. Indeed, to see that $\mathcal{R}_1 \cap \mathcal{R}_2$ is σ-complete, take a countable, directed subset $\mathcal{T} \subseteq \mathcal{R}_1 \cap \mathcal{R}_2$. Then, since both $\mathcal{R}_1$ and $\mathcal{R}_2$ are σ-complete, we see that $\overline{\bigcup \mathcal{T}}^{\mathrm{sup}} \in \mathcal{R}_1 \cap \mathcal{R}_2$, as desired.

Further, to show that $\mathcal{R}_1 \cap \mathcal{R}_2$ is also cofinal, let $T \in \mathrm{Sub}_{\mathrm{ctbl}}(S)$ and define inductively a $\subseteq$-increasing sequence $(R_n)_{n\geq 1}$ in $\mathrm{Sub}_{\mathrm{ctbl}}(S)$ such that $T \subseteq R_1$, $R_{2k} \in \mathcal{R}_1$ and $R_{2k+1} \in \mathcal{R}_2$ for each $k \in \mathbb{N}$. Set $R := \overline{\cup_n R_n}^{\mathrm{sup}}$, and note that $R = \overline{\cup_k R_{2k}}^{\mathrm{sup}} = \overline{\cup_k R_{2k+1}}^{\mathrm{sup}}$. Since both $\mathcal{R}_1$ and $\mathcal{R}_2$ are σ-complete, it follows that $R \in \mathcal{R}_1 \cap \mathcal{R}_2$ with $T \subseteq R$, as required.

In particular, this implies that if a finite number of properties $\mathcal{P}_1, \ldots, \mathcal{P}_n$ of Cu-semigroups all satisfy the Löwenheim-Skolem condition, their intersection (that is, the property of satisfying all properties at once) also satisfies the Löwenheim-Skolem condition.

Proposition 5.3.4. *Let S be a* Cu*-semigroup satisfying (O5) (respectively (O6), (O7)). Then, the countably based sub-*Cu*-semigroups satisfying (O5) (resp. (O6), (O7)) form a σ-complete and cofinal subcollection of* $\mathrm{Sub}_{\mathrm{ctbl}}(S)$.

In particular, (O5)-(O7) each satisfy the Löwenheim-Skolem condition.

Proof. Given a Cu-semigroup S satisfying (O5), let $\mathcal{R}$ denote the subcollection of $\mathrm{Sub}_{\mathrm{ctbl}}(S)$ formed by those countably based sub-Cu-semigroups satisfying (O5).

It follows from [6, Theorem 4.5] (see also Corollary 5.1.15) that (O5) passes to inductive limits. Thus, for any countable, directed subset $\mathcal{T} \subseteq \mathcal{R}$, we have $\overline{\bigcup \mathcal{T}}^{\mathrm{sup}} \in \mathcal{R}$ because $\overline{\bigcup \mathcal{T}}^{\mathrm{sup}}$ can be written as the inductive limit of the elements in $\mathcal{T}$ indexed over $\mathcal{T}$. This shows that $\mathcal{R}$ is σ-complete.

To see that $\mathcal{R}$ is also cofinal, let $T \in \mathrm{Sub}_{\mathrm{ctbl}}(S)$ and choose a countable basis $B_0 \subseteq T$.

Claim. *There exist an increasing sequence $(R_n)_n$ in $\mathrm{Sub}_{\mathrm{ctbl}}(S)$ with $R_0 = T$ and countable basis $B_n \subseteq R_n$ for each n such that, whenever $x + y \ll z' \ll z$, $x' \ll x$ and $y' \ll y$ in B_n, there is $c \in B_{n+1}$ such that $x' + c \ll z$, $z' \ll x + c$ and $y' \ll c$.*

We prove the claim by induction, where note that we already have R_0 and B_0. Now let $n \in \mathbb{N}$ and assume that we have chosen R_k and B_k for all $k \leq n$. Then, consider the following set

$$I_n := \big\{(x', x, y', y, z', z) \in B_n^6 \mid x + y \ll z' \ll z, x' \ll x, y' \ll y\big\},$$

which is countable because B_n^6 is.

Further, since S satisfies (O5), we know that for each $i = (x', x, y', y, z', z) \in I_n$ there exists $c_i \in S$ such that $x' + c_i \ll z$, $z' \ll x + c_i$ and $y' \ll c_i$. By Lemma 5.3.2, we can find a countable sub-Cu-semigroup R_{n+1} containing B_n and $\{c_i : i \in I_n\}$.

Let B_{n+1} be a countable basis for R_{n+1} and note that, since B_n is a basis for R_n, we have $R_n \subseteq R_{n+1}$. This finishes the proof of the Claim.

Now let $B := \bigcup_n B_n$ and $R := \overline{\bigcup_n R_n}^{\sup}$, which is a countably based sub-Cu-semigroup by Paragraph 5.3.1, with countable basis B. Further, note that for every $x', x, y', y, z', z \in B$ with $x' \ll x$, $y' \ll y$, $z' \ll z$ and $x + y \ll z'$, we can find $n \in \mathbb{N}$ with $x', x, y', y, z', z \in B_n$. Thus, since the tuple formed by these elements is in I_n, there exists $c \in B_{n+1}$ such that

$$x' + c \ll z, \quad z' \ll x + c \quad \text{and} \quad y' \ll c.$$

This shows that B satisfies the condition from Lemma 5.1.3 and, consequently, that R satisfies (O5). Thus, we have $R \in \mathcal{R}$, as required.

Using Corollary 5.1.15 (for σ-completeness) and Propositions 5.1.4 and 5.1.5 (for cofinality) an analogous argument shows that (O6) and (O7) also satisfy the Löwenheim-Skolem condition. □

Proposition 5.3.5. *Every sub-*Cu*-semigroup of a simple (resp. weakly cancellative)* Cu*-semigroup S is simple (resp. weakly cancellative).*

In particular, simplicity and weak cancellation each satisfy the Löwenheim-Skolem condition.

Proof. Let T be a sub-Cu-semigroup of a simple Cu-semigroup S, and let $x, y \in T$ be a pair of nonzero elements. Using that S is simple, one gets $x \leq \infty y$ in S. Since the inclusion $T \to S$ is an order-embedding, one also gets $x \leq \infty y$ in T, which implies that T is simple.

Assume now that S is weakly cancellative and that $T \subseteq S$ is a sub-Cu-semigroup. Given $x, y, z \in T$ satisfying $x + z \ll y + z$, it follows from weak cancellation that $x \ll y$ in S. Thus, $x \ll y$ in T, as desired. □

Lemma 5.3.6. *Let R be a countably based sub-*Cu*-semigroup of a* Cu*-semigroup S. Then, there exists $T \in \mathrm{Sub}_{\mathrm{ctbl}}(S)$ such that $R \subseteq T$ and $\dim(T) \leq \dim(S)$.*

Proof. Note that, if $\dim(S) = \infty$, $T := R$ has the desired properties. Thus, let $n := \dim(S)$, which we may assume to be finite.

Claim. *For every $P \in \mathrm{Sub}_{\mathrm{ctbl}}(S)$ there exists $Q \in \mathrm{Sub}_{\mathrm{ctbl}}(S)$ such that $P \subseteq Q$ and such that, whenever $x' \ll x \ll y_1 + \ldots + y_r$ in P, there exist $z_{j,k} \in Q$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ satisfying (i)-(iii) from Definition 4.1.1.*

Let $P \in \mathrm{Sub}_{\mathrm{ctbl}}(S)$ and denote by $B \subseteq P$ a countable basis of P. Consider the set

$$I := \left\{(x', x, y_1, \ldots, y_r) \in \sqcup_{r\geq 1} B^{r+2} \mid x' \ll x \ll y_1 + \ldots + y_r\right\},$$

which is countable since each B^{r+2} is.

Then, since $\dim(S) \leq n$, we know that for each $i = (x', x, y_1, \ldots, y_r) \in I$ there exist elements $z_{i,j,k} \in S$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ satisfying (i)-(iii) from Definition 4.1.1 for $x' \ll x \ll y_1 + \ldots + y_r$. Thus, it follows from Lemma 5.3.2 that we can find $Q \in \mathrm{Sub}_{\mathrm{ctbl}}(S)$ containing B and each $z_{i,j,k}$ for $i \in I$, $j = 1, \ldots, r$, and $k = 0, \ldots, n$. Using that B is a basis for P, we get $P \subseteq Q$.

To see that Q verifies the claimed property, let $x' \ll x \ll y_1 + \ldots + y_r$ in P. Then, since B is a basis, there exist $c', c, d_1, \ldots, d_r \in B$ such that $d_j \ll y_j$ for each j and

$$x' \ll c' \ll c \ll x \ll d_1 + \ldots + d_r.$$

This implies that the tuple $i := (c', c, d_1, \ldots, d_r)$ is in I. Thus, Q contains the elements $z_{i,j,k}$, which satisfy (i)-(iii) from Definition 4.1.1 for $c' \ll c \ll d_1 + \ldots + d_r$. One can now check that these elements also satisfy (i)-(iii) from Definition 4.1.1 for $x' \ll x \ll y_1 + \ldots + y_r$, which finishes the proof of the claim.

Now apply the Claim successively in order to obtain an increasing sequence $(T_k)_k$ in $\mathrm{Sub}_{\mathrm{ctbl}}(S)$ with $T_0 := R$ and such that, for every $k \in \mathbb{N}$ and $x' \ll x \ll y_1 + \ldots + y_r$ in T_k, there exist $z_{j,k} \in T_{k+1}$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ satisfying (i)-(iii) from Definition 4.1.1.

Set $T := \overline{T_\infty}^{\mathrm{seq}}$, which by Corollary 5.2.9 and Paragraph 5.3.1 is a countably based sub-Cu-semigroup of S satisfying $R \subseteq T$. By construction, we have $\dim(T) \leq n$ as desired. □

Proposition 5.3.7. *Let S be a* Cu*-semigroup satisfying* $\dim(S) \leq n$ *for some* $n \in \mathbb{N}$*. Then, the countably based sub-*Cu*-semigroups* $T \subseteq S$ *satisfying* $\dim(T) \leq n$ *form a* σ*-complete and cofinal subcollection of* $\mathrm{Sub}_{\mathrm{ctbl}}(S)$*.*

In particular, the property of having covering dimension at most n satisfies the Löwenheim-Skolem condition.

Proof. Let $\mathcal{R}$ be the subcollection of $\mathrm{Sub}_{\mathrm{ctbl}}(S)$ consisting of the sub-Cu-semigroups T satisfying $\dim(T) \leq n$.

It follows from Proposition 4.1.9 that the property of having dimension at most n passes to inductive limits. In particular, this implies that $\mathcal{R}$ is σ-complete. Lemma 5.3.6 above shows that $\mathcal{R}$ is also cofinal, as required. □

Theorem 5.3.8. *Let S be a* Cu*-semigroup, and let* $n \in \mathbb{N}$*. Then, the following are equivalent:*

(1) $\dim(S) \leq n$*;*

*(2) every countable subset of S is contained in a countably based sub-*Cu*-semigroup* $T \subseteq S$ *satisfying* $\dim(T) \leq n$*;*

*(3) every finite subset of S is contained in a sub-*Cu*-semigroup* $T \subseteq S$ *satisfying* $\dim(T) \leq n$*.*

Proof. Lemmas 5.3.2 and 5.3.6 show that (1) implies (2). Moreover, (2) implies (3) trivially.

Finally, to prove that (3) implies (1), let $\mathcal{T}$ denote the collection of sub-Cu-semigroups of S with dimension at most n. For each $T \in \mathcal{T}$, let $\iota_T \colon T \to S$ be the inclusion map. By our assumption, the family $(T, \iota_T)_{T\in\mathcal{T}}$ approximates S in the sense of Definition 5.1.1. Thus, it follows from Proposition 5.1.8 that $\dim(S) \leq n$. □

With these results at hand, we can now generalize Proposition 4.5.10 and Proposition 4.1.20 by removing the assumption of being countably based. Recall the definition of soft element from Paragraph 4.1.17.

Proposition 5.3.9. *Let S be a simple, weakly cancellative, zero-dimensional* Cu-*semigroup satisfying (O5). Then, S satisfies the Riesz interpolation property. Further, if S is not isomorphic to $\overline{\mathbb{N}}$ or E_k for any k, then S is almost divisible.*

Proof. First note that S has Riesz interpolation whenever it is isomorphic to $\overline{\mathbb{N}}$ or E_k for some k. Thus, we may assume that S is not isomorphic to any of these Cu-semigroups for the remainder of the proof. Thus, it follows from [6, Proposition 5.1.19] that there exists a sequence $(s_n)_n$ in S with $s_0 > s_1 > \ldots$.

Let $\mathcal{R}_{\mathrm{O5}}$, $\mathcal{R}_{\mathrm{simple}}$, $\mathcal{R}_{\mathrm{canc}}$, and $\mathcal{R}_{\mathrm{dim0}}$ denote the subcollections of sub-Cu-semigroups in $\mathrm{Sub}_{\mathrm{ctbl}}(S)$ that satisfy (O5), or that are simple, weakly cancellative, or zero-dimensional respectively. Then, it follows from Paragraph 5.3.3 that the intersection $\mathcal{R} := \mathcal{R}_{\mathrm{O5}} \cap \mathcal{R}_{\mathrm{simple}} \cap \mathcal{R}_{\mathrm{canc}} \cap \mathcal{R}_{\mathrm{dim0}}$ is σ-complete and cofinal, since we know from Propositions 5.3.4, 5.3.5 and 5.3.7 that each of the subcollections is σ-complete and cofinal.

Now let $x_0, x_1, y_0, y_1 \in S$ satisfy $x_j \leq y_k$ for each $j, k \in \{0, 1\}$. Since $\mathcal{R}$ is cofinal, it follows from Lemma 5.3.2 that there exists $R \in \mathcal{R}$ containing x_0, x_1, y_0, y_1 and the sequence $s_0, s_1, \ldots$, which forces R to be nonisomorphic to either $\overline{\mathbb{N}}$ or E_k for any k. By Proposition 4.5.10, we can find $z \in R$ such that $x_j \leq z \leq y_k$ for every $j, k \in \{0, 1\}$. Thus, since $z \in R \subseteq S$, we get that S satisfies the Riesz interpolation property.

Similarly, let $n \in \mathbb{N}$ and take $x', x \in S$ such that $x' \ll x$. Using the same argument as above, we find $R \in \mathcal{R}$ containing x', x and satisfying all the conditions in Proposition 4.5.10. Thus, there is $y \in R$ with $ny \ll x$ and $x' \ll (n+1)y$. This shows that S is almost divisible, as desired. □

Proposition 5.3.10. *Let S be a simple, weakly cancellative* Cu-*semigroup satisfying (O5) and (O6). Then,*

$$\dim(S_{\mathrm{soft}}) \leq \dim(S) \leq \dim(S_{\mathrm{soft}}) + 1.$$

Proof. To prove the first inequality we will verify condition (3) of Theorem 5.3.8. Thus, let H be a finite subset of S_{soft} and set $n := \dim(S)$, which we may assume to be finite.

Proceeding as in the proof of Proposition 5.3.9, it follows from Lemma 5.3.2 and Propositions 5.3.4, 5.3.5 and 5.3.7, that we can find a simple, weakly cancellative sub-Cu-semigroup $T \in \mathrm{Sub}_{\mathrm{ctbl}}(S)$ satisfying (O5) and (O6) and such that $H \subseteq T$ and $\dim(T) \leq n$.

Thus, Proposition 4.1.20 implies $\dim(T_{\mathrm{soft}}) \leq n$, where note that T_{soft} is a sub-Cu-semigroup of S_{soft} containing H. This shows that every finite subset of S_{soft} is contained in a sub-Cu-semigroup of dimension at most n, as desired.

Now set $m := \dim(S_{\mathrm{soft}})$ and let H be a finite subset of S. To prove the second inequality, we will verify condition (3) of Theorem 5.3.8. As above, we may assume m to be finite.

Using Lemma 5.3.2 and Proposition 5.3.4, we find $T^{(1)} \in \mathrm{Sub}_{\mathrm{ctbl}}(S)$ satisfying (O5) and (O6) with $H \subseteq T^{(1)}$. By Proposition 5.3.7, we can choose $R^{(1)} \in \mathrm{Sub}_{\mathrm{ctbl}}(S_{\mathrm{soft}})$ with $T^{(1)}_{\mathrm{soft}} \subseteq R^{(1)}$ and $\dim(R^{(1)}) \leq m$. Since $R^{(1)}$ and $T^{(1)}$ are both countably based, it follows from Lemma 5.3.2 that there exists a countably based sub-Cu-semigroup containing both $R^{(1)}$ and $T^{(1)}$. By Proposition 5.3.4, there exists $T^{(2)} \in \mathrm{Sub}_{\mathrm{ctbl}}(S)$ satisfying (O5) and (O6) and such that $R^{(1)}, T^{(1)} \subseteq T^{(2)}$.

Proceeding in this manner, we obtain an increasing sequence $(T^{(k)})_k$ in $\mathrm{Sub}_{\mathrm{ctbl}}(S)$ of sub-Cu-semigroups satisfying (O5) and (O6), and another increasing sequence $(R^{(k)})_k$ in $\mathrm{Sub}_{\mathrm{ctbl}}(S_{\mathrm{soft}})$ of sub-Cu-semigroups with dimension at most m such that

$$T^{(k)}_{\mathrm{soft}} \subseteq R^{(k)} \subseteq T^{(k+1)}$$

for each k.

Set $T := \overline{\bigcup_k T^{(k)}}^{\mathrm{sup}}$ and $R := \overline{\bigcup_k R^{(k)}}^{\mathrm{sup}}$. It follows from Paragraph 5.3.1, Proposition 5.3.4 and Proposition 5.3.5 that T is a countably based, simple, weakly cancellative sub-Cu-semigroup of S satisfying (O5) and (O6). Moreover, since $\dim(R^{(k)}) \leq m$ for every k, we get that $\dim(R) \leq m$ by Theorem 5.3.8.

Further, note that for every soft element x in T there exists an increasing sequence $(k_i)_i$ of natural numbers and an increasing sequence of elements $x_i \in T^{(k_i)}_{\mathrm{soft}}$ such that $\sup_i x_i = x$. Thus, since $T^{(k_i)}_{\mathrm{soft}} \subseteq R^{(k_i)}$ for each i, we see that $x \in R$. This shows that $T_{\mathrm{soft}} \subseteq R$, and we know by construction that $R \subseteq T_{\mathrm{soft}}$. Thus, $T_{\mathrm{soft}} = R$.

Using Proposition 4.1.20, one gets

$$\dim(T) \leq \dim(T_{\mathrm{soft}}) + 1 = \dim(R) + 1 \leq m + 1.$$

Therefore, every finite subset of S is contained in a sub-Cu-semigroup with dimension at most $m+1$. By Theorem 5.3.8, we have $\dim(S) \leq m+1$, as required. □

5.4 Dimension of the Cuntz semigroup as a noncommutative dimension theory

In this section we study which properties of a noncommutative dimension theory are satisfied by the assignment $A \mapsto \dim \mathrm{Cu}(A)$, for each C^*-algebra A; see Paragraph 5.4.1 below. In particular, we prove in Theorem 5.4.3 that this assignment satisfies the Löwenheim-Skolem condition, which implies that $A \mapsto \dim(\mathrm{Cu}(A))$ satisfies all but one of the properties of a noncommutative dimension theory.

5.4.1. Recall from [89, Definition 1] that an assignment that to each C^*-algebra A associates a number (the dimension) $d(A) \in \{0, 1, 2, \ldots, \infty\}$ is a *noncommutative dimension theory* if the following conditions are satisfied:

(D1) $d(I) \leq d(A)$ for every ideal $I \subseteq A$;

(D2) $d(A/I) \leq d(A)$ for every ideal $I \subseteq A$;

(D3) $d(A \oplus B) = \max\{d(A), d(B)\}$ for any pair of C^*-algebras A and B;

(D4) $d(\widetilde{A}) = d(A)$ for every C^*-algebra A;

(D5) Let $n \in \mathbb{N}$. If a C^*-algebra A is approximated by sub-C^*-algebras $A_\lambda \subseteq A$ with $d(A_\lambda) \leq n$, then $d(A) \leq n$;

(D6) Given a C^*-algebra A and a separable sub-C^*-algebra $B_0 \subseteq A$, there exists a separable sub-C^*-algebra $B \subseteq A$ such that $B_0 \subseteq B$ and $d(B) \leq d(A)$.

Note that, by Proposition 4.1.10 and Theorem 5.1.13, we already know that assigning to a C^*-algebra the dimension of its Cuntz semigroup satisfies conditions (D1)-(D3) and (D5). Moreover, using [89, Proposition 3], one can see that this assignment is in fact *Morita-invariant*, that is, $\dim(\mathrm{Cu}(A)) = \dim(\mathrm{Cu}(B))$ whenever A and B are Morita equivalent.

We will see in Theorem 5.4.3 below that (D6) is also satisfied. However, Example 5.4.5 shows that (D4) does not hold.

It is an open question if assigning to each C^*-algebra the dimension of the Cuntz semigroup of its minimal unitization, $A \mapsto \dim(\mathrm{Cu}(\tilde{A}))$, is a dimension theory; see Question 5.4.6.

Recall that, given a sub-C^*-algebra B of a C^*-algebra A, the Cuntz semigroup of B is not necessarily a sub-Cu-semigroup of $\mathrm{Cu}(A)$. However, the next result shows that there are sufficiently many separable sub-C^*-algebras whose Cuntz semigroups are sub-Cu-semigroups.

Given a C^*-algebra A, we denote by $\mathrm{Sub}_{\mathrm{sep}}(A)$ the collection of separable sub-C^*-algebras of A; see [90, Paragraph 3.1]. A family $\mathcal{S}$ of $\mathrm{Sub}_{\mathrm{sep}}(A)$ is said to be *σ-complete* if for every countable, directed subset $\mathcal{T} \subseteq \mathcal{S}$ we have $\overline{\bigcup \mathcal{T}} \in \mathcal{S}$. We also say that $\mathcal{S}$ is *cofinal* if for every $B \in \mathrm{Sub}_{\mathrm{sep}}(A)$ there is $C \in \mathcal{S}$ with $B \subseteq C$.

Proposition 5.4.2. *Let A be a C^*-algebra. Then,*

$$\mathcal{S} := \big\{ B \in \mathrm{Sub}_{\mathrm{sep}}(A) \mid \mathrm{Cu}(B) \to \mathrm{Cu}(A) \text{ is an order-embedding} \big\}$$

is σ-complete and cofinal.

*The map $\alpha\colon \mathcal{S} \to \mathrm{Sub}_{\mathrm{ctbl}}(\mathrm{Cu}(A))$ that sends $B \in \mathcal{S}$ to the (up to isomorphism) sub-*Cu*-semigroup $\mathrm{Cu}(B) \subseteq \mathrm{Cu}(A)$ preserves the order and the suprema of countable directed subsets, and the image of α is a cofinal subset of $\mathrm{Sub}_{\mathrm{ctbl}}(\mathrm{Cu}(A))$.*

Proof. Let $\mathcal{T}$ be a countable, directed subfamily of $\mathcal{S}$, and set $D := \overline{\bigcup \mathcal{T}}$. To prove that $\mathcal{S}$ is σ-complete, we need to show $D \in \mathcal{S}$. Thus, let $\varphi_{A,D}\colon \mathrm{Cu}(D) \to \mathrm{Cu}(A)$ denote the Cu-morphism induced by the inclusion map $D \to A$, and take $x, y \in \mathrm{Cu}(D)$ such that $\varphi_{A,D}(x) \leq \varphi_{A,D}(y)$.

Choose $x', x'' \in \mathrm{Cu}(D)$ with $x' \ll x'' \ll x$. Since $\varphi_{A,D}$ is a Cu-morphism, there exists $y' \ll y$ satisfying $\varphi_{A,D}(x'') \ll \varphi_{A,D}(y')$.

Using that $D \cong \varinjlim_{B\in\mathcal{T}} B$, it follows from [6, Corollary 3.2.9] that $\mathrm{Cu}(D) \cong \varinjlim_{B\in\mathcal{T}} \mathrm{Cu}(B)$. By (L2) from Lemma 4.1.8, we get $B \in \mathcal{T}$ and $c, d \in \mathrm{Cu}(B)$ such that

$$x' \ll \varphi_{D,B}(c) \ll x'' \quad \text{and} \quad y' \ll \varphi_{D,B}(d) \ll y,$$

where $\varphi_{D,B}\colon \mathrm{Cu}(B) \to \mathrm{Cu}(D)$ denotes the induced Cu-morphism from the inclusion map.

Similarly, denote by $\varphi_{A,B}$ the Cu-morphism induced by the inclusion $B \to A$. Then,

$$\varphi_{A,B}(c) = \varphi_{A,D}(\varphi_{D,B}(c)) \ll \varphi_{A,D}(x'') \ll \varphi_{A,D}(y') \ll \varphi_{A,D}(\varphi_{D,B}(d)) = \varphi_{A,B}(d).$$

Since $\varphi_{A,B}$ is an order-embedding, we obtain $c \ll d$ in $\mathrm{Cu}(B)$. Consequently,

$$x' \ll \varphi_{D,B}(c) \ll \varphi_{D,B}(d) \ll y,$$

which implies that $x' \ll y$ for every x' way-below x or, equivalently, that $x \leq y$. This shows that $\varphi_{A,D}$ is an order-embedding and that $D \in \mathcal{S}$.

Now take $B_0 \in \mathrm{Sub}_{\mathrm{sep}}(A)$. By [37, Theorem 2.6.2], we can find $B \in \mathrm{Sub}_{\mathrm{sep}}(A)$ such that $B_0 \subseteq B$ and such that B is an elementary submodel of A. Using [37, Lemma 8.1.3], one gets that $\mathrm{Cu}(B) \to \mathrm{Cu}(A)$ is an order-embedding. Thus, B belongs to $\mathcal{S}$ and, therefore, $\mathcal{S}$ is cofinal.

Let $\alpha\colon \mathcal{S} \to \mathrm{Sub}_{\mathrm{ctbl}}(\mathrm{Cu}(A))$ be the map that sends each element $B \in \mathcal{S}$ to $\mathrm{Cu}(B)$, which by Lemma 5.2.2 can be identified with a countably based sub-Cu-semigroup of $\mathrm{Cu}(A)$. It is straightforward to check that α is order-preserving.

To see that α preserves suprema of countable directed subsets, let $\mathcal{T} \subseteq \mathcal{S}$ be a countable, directed subset and set $D := \overline{\bigcup \mathcal{T}}$. Note that $D \in \mathcal{S}$ by the first part of the proof. Since α is order preserving, $(\mathrm{Cu}(B))_{B\in\mathcal{T}}$ is a countable, directed family in $\mathrm{Sub}_{\mathrm{ctbl}}(\mathrm{Cu}(A))$. Using Theorem 5.2.11, we know that its supremum is $\overline{\bigcup_{B\in\mathcal{T}} \mathrm{Cu}(B)}^{\mathrm{sup}}$.

Since $\mathrm{Cu}(B)$ is contained in $\mathrm{Cu}(D)$ for each $B \in \mathcal{T}$, one gets

$$\overline{\bigcup_{B\in\mathcal{T}} \mathrm{Cu}(B)}^{\mathrm{sup}} \subseteq \mathrm{Cu}(D).$$

Using that D is the inductive limit of the elements in $\mathcal{T}$ indexed over themselves, and that the Cuntz semigroup preserves inductive limits, we see that $\mathrm{Cu}(D)$ is the inductive limit of $(\mathrm{Cu}(B))_{B\in\mathcal{T}}$, from which the other inclusion follows.

Finally, let us show that the image of α is cofinal in $\mathrm{Sub}_{\mathrm{ctbl}}(\mathrm{Cu}(A))$. Take $T \in \mathrm{Sub}_{\mathrm{ctbl}}(\mathrm{Cu}(A))$, choose a countable basis $D \subseteq T$ and, for each $x \in D$, let $a_x \in (A\otimes\mathcal{K})_+$ be such that $x = [a_x]$.

Choose a separable sub-C^*-algebra $B_0 \subseteq A$ such that each a_x is contained in $B_0\otimes\mathcal{K}$. Since $\mathcal{S}$ is cofinal, there exists $B \in \mathcal{S}$ with $B_0 \subseteq B$. Then, $x \in \mathrm{Cu}(B)$ for each $x \in D$, which implies $T \subseteq \mathrm{Cu}(B)$ as desired. □

Theorem 5.4.3. *Let $n \in \mathbb{N}$, and let A be a C^*-algebra satisfying* $\dim(\mathrm{Cu}(A)) \leq n$. *Then,*

$$\mathcal{S} := \big\{B \in \mathrm{Sub}_{\mathrm{sep}}(A) \mid \mathrm{Cu}(B) \to \mathrm{Cu}(A) \text{ order-embedding}, \dim(\mathrm{Cu}(B)) \leq n\big\}$$

is σ-complete and cofinal.

In particular, for every $B_0 \in \mathrm{Sub}_{\mathrm{sep}}(A)$ there exists $B \in \mathrm{Sub}_{\mathrm{sep}}(A)$ such that $B_0 \subseteq B$ and $\dim(\mathrm{Cu}(B)) \leq n$.

Proof. Consider the subcollections

$$\mathcal{T} := \big\{ T \in \mathrm{Sub}_{\mathrm{ctbl}}(\mathrm{Cu}(A)) \mid \dim(T) \leq n \big\},$$
$$\mathcal{S}_0 := \big\{ B \in \mathrm{Sub}_{\mathrm{sep}}(A) \mid \mathrm{Cu}(B) \to \mathrm{Cu}(A) \text{ is an order-embedding} \big\},$$

which are σ-complete and cofinal in $\mathrm{Sub}_{\mathrm{ctbl}}(\mathrm{Cu}(B))$ and $\mathrm{Sub}_{\mathrm{sep}}(A)$ by Proposition 5.3.7 and Proposition 5.4.2 respectively.

As in Proposition 5.4.2, let $\alpha\colon \mathcal{S}_0 \to \mathrm{Sub}_{\mathrm{ctbl}}(\mathrm{Cu}(A))$ send each $B \in \mathcal{S}_0$ to the sub-Cu-semigroup of $\mathrm{Cu}(A)$ identified with $\mathrm{Cu}(B)$. Then, $\mathcal{S} = \big\{ B \in \mathcal{S}_0 \mid \alpha(B) \in \mathcal{T} \big\}$.

Since both $\mathcal{S}_0$ and $\mathcal{T}$ are σ-complete, and α preserves suprema of countable, directed sets, it follows that $\mathcal{S}$ is σ-complete.

To see that it is also cofinal, take $B_0 \in \mathrm{Sub}_{\mathrm{sep}}(A)$. Using that $\mathcal{T}$ is cofinal, we obtain $T_0 \in \mathcal{T}$ such that $\mathrm{Cu}(B_0) \subseteq T_0$. By Proposition 5.4.2, the image of α is cofinal. Thus, we can find $B_1 \in \mathcal{S}_0$ with $T_0 \subseteq \alpha(B_1)$.

Proceeding in this way, and similar to the proof of Proposition 5.3.10, we get increasing sequences $(T_k)_{k\in\mathbb{N}}$ in $\mathcal{T}$ and $(B_k)_{k\geq 1}$ in $\mathcal{S}_0$ such that

$$\alpha(B_0) \subseteq T_0 \subseteq \alpha(B_1) \subseteq T_1 \subseteq \alpha(B_2) \subseteq T_2 \subseteq \ldots.$$

Setting $B := \overline{\bigcup_k B_k}$ and $T := \overline{\bigcup_k T_k}^{\mathrm{sup}}$, and using that α preserves suprema of countable, directed sets, one gets $\alpha(B) = T$. Then, since $B_0 \subseteq B$, $B \in \mathcal{S}_0$ and $T \in \mathcal{T}$, we obtain $B \in \mathcal{S}$, as required. □

Corollary 5.4.4. *Let A be a C^*-algebra, and let $n \in \mathbb{N}$. Then,* $\dim(\mathrm{Cu}(A)) \leq n$ *if and only if every finite subset of A is contained in a separable sub-C^*-algebra $B \subseteq A$ satisfying* $\dim(\mathrm{Cu}(B)) \leq n$.

Proof. If $\dim(\mathrm{Cu}(A)) \leq n$, it follows from Theorem 5.4.3 above that every finite subset of A is contained in a separable sub-C^*-algebra $B \subseteq A$ satisfying $\dim(\mathrm{Cu}(B)) \leq n$.

Conversely, if this condition is satisfied, the family of separable sub-C^*-algebras whose Cuntz semigroup has dimension at most n approximates A. Thus, the result follows from Theorem 5.1.13. □

Example 5.4.5. The assignment $A \mapsto \dim(\mathrm{Cu}(A))$ does not satisfy property (D4) from Paragraph 5.4.1. Indeed, take for example the Jacelon-Razak algebra $\mathcal{W}$, which satisfies $\dim(\mathrm{Cu}(\mathcal{W})) = 0$ by Proposition 4.1.24 (2).

Since $\mathcal{W}$ has stable rank one and nonzero real rank, so does its minimal unitization $\widetilde{\mathcal{W}}$. Thus, using that $\widetilde{\mathcal{W}}$ is unital, Theorem 4.3.8 implies that $\mathrm{Cu}(\widetilde{\mathcal{W}})$ is not zero-dimensional.

In fact, since we know that the nuclear dimension of $\widetilde{\mathcal{W}}$ is 1, we get $\dim(\mathrm{Cu}(\widetilde{\mathcal{W}})) = 1$ by Theorem 4.2.2.

Question 5.4.6. Do we have $\dim(\mathrm{Cu}(\widetilde{I})) \leq \dim(\mathrm{Cu}(A))$ whenever I is an ideal of a unital C^*-algebra A?

If this question can be answered affirmatively, the assignment $A \mapsto \dim(\mathrm{Cu}(\tilde{A}))$ is a noncommutative dimension theory. Indeed, using the results in this section one can verify that (D2)-(D6) are satisfied, and Question 5.4.6 is asking if (D1) holds.

Chapter 6

Nowhere scattered C^*-algebras

Scattered topological spaces, that is to say, spaces such that every subset contains an isolated point, and their C^*-analog (defined by Jensen in [48]) admit various and interesting characterizations. One such characterization consists of demanding that the spectrum of every self-adjoint element is countable. Another consists of asking every sub-C^*-algebra to have real rank zero; see [46] and [57] respectively.

In this chapter, we study C^*-algebras that, informally speaking, are really far from being scattered. These algebras, which we term nowhere scattered C^*-algebras, have appeared implicitly in the literature but they have never been given a name; see, for example, [34] and [75].

The permanence properties of nowhere scattered C^*-algebras and their associated topological spaces are studied in Section 6.1, while a characterization of nowhere scatteredness in terms of the Cuntz semigroup is found in Section 6.3. More concretely, we see that a C^*-algebra A is nowhere scattered if and only if $\mathrm{Cu}(A)$ is weakly $(2,\omega)$-divisible. To obtain such a characterization, in Section 6.2 we unveil a new property that the Cuntz semigroup of every C^*-algebra satisfies.

Finally, we show in Section 6.4 that, under the additional assumption of real rank zero or stable rank one, the Cuntz semigroup of a nowhere scattered C^*-algebra is $(2,\omega)$-divisible, a strengthening of weak $(2,\omega)$-divisibility. Whether this stronger property is satisfied in the Cuntz semigroup of every nowhere scattered C^*-algebra is unknown, and this question is known as the Global Glimm Problem, studied in Chapter 7.

The results announced in this chapter have appeared in [93].

6.1 Scattered and nowhere scattered C^*-algebras

In this section we define and begin our study on nowhere scattered C^*-algebras; see Definition 6.1.1 and Theorem 6.1.2. Examples of such C^*-algebras include all simple, unital, infinite-dimensional C^*-algebras (Example 6.1.3) and traceless C^*-algebras (in particular, weakly purely infinite C^*-algebras, see Example 6.3.11).

A topological space X is said to be *scattered* if every closed subset of X contains an isolated point. Equivalently, X is scattered if each of its subsets contains an isolated point. Thus, given any $x, y \in X$, the set $\{x, y\}$ has an isolated point. This shows that every scattered space is T_0.

In analogy to this definition, one says that a C^*-algebra is *scattered* if each of its quotients contains a minimal projection, where recall that a nonzero projection p in a C^*-algebra A is said to be *minimal* if $pAp = \mathbb{C}p$. Note that, even though every minimal projection is minimal with respect to the order $\leq$, the converse does not hold: Take, for example, the unit in the Jiang-Su algebra.

Although this is not the original definition of scatteredness for C^*-algebras given in [48, Definition 2.1], it can be seen to be equivalent; see [40, Theorem 1.4]. It is also known that a C^*-algebra is scattered if and only if each of its sub-C^*-algebras has real rank zero ([57, Theorem 2.3]), and if and only if it contains no sub-C^*-algebra of the form $C_0((0,1])$. In particular, every sub-C^*-algebra of a scattered C^*-algebra is once again scattered.

Definition 6.1.1. A C^*-algebra is said to be *nowhere scattered* if none of its quotients contains a minimal projection.

A C^*-algebra is said to be *elementary* if it is isomorphic to the algebra of compact operators on some Hilbert space. An *ideal-quotient* of a C^*-algebra is an ideal of a quotient.

A sub-C^*-algebra $B \subseteq A$ is *hereditary* if, whenever $0 \leq b - a$ with $b \in B$, we have $a \in B$.

We say that a positive element $a \in A_+$ is *abelian* if $\overline{aAa}$ is commutative, and a C^*-algebra is said to be *antiliminal* if it contains no nonzero abelian positive elements; see [11, Definition IV.1.1.6]. Also, a C^*-algebra A is said to be of *type I* if every quotient of A contains a nonzero abelian element.

Recall from Examples 1.1.20 (3) that, given a C^*-algebra A and a Hilbert space H, a *-homomorphism $\pi\colon A \to \mathcal{B}(H)$ is an irreducible representation if H has no nontrivial closed invariant subspaces under $\pi(A)$. We will say that π is *GCR* if $\pi(A) \cap \mathcal{K}(H) \neq \{0\}$.

Theorem 6.1.2. *Let A be a C^*-algebra. Then the following are equivalent:*

(1) A is nowhere scattered;

(2) every quotient of A is antiliminal;

(3) A has no nonzero ideal-quotients of type I*;*

(4) A has no nonzero scattered ideal-quotients;

(5) A has no nonzero elementary ideal-quotients;

(6) A has no nonzero GCR irreducible representation;

(7) no hereditary sub-C^*-algebra of A admits a finite-dimensional irreducible representation;

(8) no hereditary sub-C^*-algebra of A admits a one-dimensional irreducible representation.

Proof. Assume first that there exists an ideal I of A such that A/I contains a nonzero, abelian element b. Then, $\overline{b(A/I)b}$ is commutative and we can find an ideal $J \subseteq \overline{b(A/I)b}$

satisfying $\overline{b(A/I)b}/J \cong \mathbb{C}$. Since $\overline{b(A/I)b}$ is hereditary, we have $K \cap \overline{b(A/I)b} = J$, where K is the ideal of A/I generated by J. Thus,

$$\overline{(b+K)((A/I)/K)(b+K)} \cong \overline{b(A/I)b}/K \cong \overline{b(A/I)b}/J \cong \mathbb{C},$$

which implies that $(A/I)/K$ has a minimal projection. Indeed, the projection p in $\overline{(b+K)((A/I)/K)(b+K)}$ corresponding to $1 \in \mathbb{C}$ through the isomorphism satisfies

$$p((A/I)/K)p = p\overline{(b+K)((A/I)/K)(b+K)}p = \mathbb{C}p.$$

This shows that (1) implies (2).

To see that (2) implies (3), assume that (2) is satisfied and also assume, for the sake of contradiction, that there exist ideals $I \subseteq J$ of A such that J/I is a nonzero, type I ideal-quotient. Since J/I is type I, it contains a nonzero, abelian element. Using that J/I is an ideal of A/I, it follows that A/I is not antiliminal, a contradiction.

The implications '(3)⇒(4)⇒(5)' follow using that every elementary C^*-algebra is scattered, and that every scattered C^*-algebra is type I.

To prove that (5) implies (6), assume that $\pi\colon A \to \mathcal{B}(H)$ is a nonzero irreducible GCR representation, and let I be the kernel of π. By [11, Corollary IV.1.2.5], we have $\mathcal{K}(H) \subseteq \pi(A)$ and, consequently, $\pi^{-1}(\mathcal{K}(H))/I \cong \mathcal{K}(H)$, which is an elementary ideal-quotient of A.

Let us now see '(6)⇒(7)'. Assume, for the sake of contradiction, that there exists a hereditary sub-C^*-algebra $B \subseteq A$ with a nonzero, finite-dimensional irreducible representation $\pi_0\colon B \to \mathcal{B}(H_0)$. By [11, Proposition II.6.4.11], we can find an irreducible representation $\pi\colon A \to \mathcal{B}(H)$ on some Hilbert space H containing H_0, such that H_0 is invariant under $\pi(B)$ and such that $\pi(b)\xi = \pi_0(b)\xi$ for all $b \in B$ and $\xi \in H_0$. Using that B is hereditary and [11, Proposition II.6.1.9], we see that $\pi|_B$ is irreducible on its essential subspace $\{\pi(b)\xi \mid b \in B, \xi \in H\}$. Note that, since π_0 is irreducible, the intersection of H_0 with this subspace is nonzero. Thus, we have $\pi(b)\xi = 0$ for all $\xi \in H_0^\perp$ and $b \in B$. Consequently, $\pi(B) \subseteq \mathcal{K}(H)$. This shows that π is GCR and nonzero.

That (7) implies (8) is clear.

Finally, to show that (8) implies (1), take an ideal I of A such that the quotient A/I contains a minimal projection p. Denote by $\pi\colon A \to A/I$ the quotient map, and consider the hereditary sub-C^*-algebra $\pi^{-1}(\mathbb{C}p)$. This sub-C^*-algebra admits a nonzero, one-dimensional representation. □

Example 6.1.3. By Theorem 6.1.2 (5) above, a C^*-algebra is nowhere scattered if and only if none of its nonzero ideal-quotients is elementary. Thus, a simple C^*-algebra is nowhere scattered if and only if it is not elementary. Consequently, a unital simple C^*-algebra is nowhere scattered if and only it is infinite-dimensional.

Recall the definition of a von Neumann algebra from Examples 1.1.3 (v), and its main structure theorems from, for example, [11, III].

Proposition 6.1.4. *Let M be a von Neumann algebra. Then, M is nowhere scattered if and only if the type* I *summand of M is zero.*

Proof. Assume first that M is nowhere scattered. By Theorem 6.1.2, no hereditary sub-C^*-algebra of M admits a one-dimensional irreducible representation. Thus, M cannot contain a nonzero abelian projection p, since otherwise pMp would be commutative and thus admit a one-dimensional irreducible representation. This shows that the type I summand of M is zero; see, for example, [11, IV.1.1].

Conversely, if the type I summand of M is zero, assume for the sake of contradiction that there exists a hereditary sub-C^*-algebra $B \subseteq M$ admitting a finite-dimensional, irreducible representation π. Then, let $p \in B$ be a projection such that $\pi(p) \neq 0$. By [88, Proposition V.1.35], we know that, for every $n \geq 1$, the projection p can be expressed as the sum of 2^n equivalent and pairwise orthogonal projections. However, $\pi(p)$ is nonzero and of finite rank, a contradiction. Theorem 6.1.2 implies that M is nowhere scattered. □

Let us now prove some permanence properties of nowhere scattered C^*-algebras. More properties, such as passing to limits and satisfying the Löwenheim-Skolem, are proven in Section 6.3.

Proposition 6.1.5. *Nowhere scatteredness passes to hereditary sub-C^*-algebras and quotients.*

Proof. Condition (7) of Theorem 6.1.2 passes to hereditary sub-C^*-algebras, while condition (2) passes to quotients. □

Proposition 6.1.6. *Let A be a C^*-algebra, and let I be an ideal of A. Then, A is nowhere scattered if and only if I and A/I are nowhere scattered.*

Proof. If A is nowhere scattered, it follows from Proposition 6.1.5 above that I and A/I are nowhere scattered.

Conversely, assume that I and A/I are nowhere scattered, and let $J \subseteq K$ be ideals of A such that K/J is scattered. Since $(I \cap K)/(I \cap J)$ is an ideal-quotient of I isomorphic to an ideal of K/J, it follows that $(I \cap K)/(I \cap J)$ is scattered and, therefore, zero by Theorem 6.1.2 applied to I. This shows $I \cap K = I \cap J$.

Moreover, the ideal-quotient $(K/I \cap K)/(J/I \cap J)$ of A/I corresponds to a quotient of K/J and is thus scattered. Since A/I is nowhere scattered, we have $K/I \cap K = J/I \cap J$, which shows $K = J$ or, equivalently, that K/J is zero. Theorem 6.1.2 (4) now implies that A is nowhere scattered. □

Given a sub-C^*-algebra B of a C^*-algebra A, we say that B *separate the ideals* of A if two ideals $I, J \subseteq A$ satisfy $I = J$ whenever $I \cap B = J \cap B$.

Proposition 6.1.7. *Let A be C^*-algebra, and let $B \subseteq A$ be a nowhere scattered sub-C^*-algebra that separates the ideals of A. Then, A is nowhere scattered.*

Proof. Let $I \subseteq J$ be ideals of A such that J/I is scattered. Since the ideal-quotient $(J \cap B)/(I \cap B)$ of B is isomorphic to a sub-C^*-algebra of J/I, it follows that it is scattered. By Theorem 6.1.2, we have $J \cap B = I \cap B$.

Using that B separates ideals, we get $I = J$ and, consequently, that J/I is zero. The result now follows from Theorem 6.1.2. □

Corollary 6.1.8. *Let A and B be Morita equivalent C^*-algebras. Then, A is nowhere scattered if and only if B is.*

Proof. Using [11, Theorem II.7.6.9], there exists a C^*-algebra C and a projection p in the multiplier algebra of C (see Remark 1.1.8) such that both pCp and $(1-p)C(1-p)$ are full, hereditary sub-C^*-algebras of C, and $pCp \cong A$, $(1-p)C(1-p) \cong B$.

Assume that A is nowhere scattered. Then, since pCp is full in C, it follows that pCp separates the ideals of C. By Proposition 6.1.7, C is nowhere scattered.

Consequently, the hereditary sub-C^*-algebra $(1-p)C(1-p)$ is also nowhere scattered by Proposition 6.1.5, which implies that B is nowhere scattered.

The converse follows by symmetry. □

As defined in the beginning of this section, a topological space X is said to be scattered if every closed subset of X contains an isolated point. We now define nowhere scattered topological spaces as those that are very far from being scattered.

As we will see in Theorem 6.1.11 below, a separable C^*-algebra is nowhere scattered if and only if its spectrum is.

Definition 6.1.9. We say that a topological space X is *nowhere scattered* if no closed subset of X contains an isolated point.

Recall that a subset of a topological space is *locally closed* if it can be written as the intersection of an open and a closed subset.

Proposition 6.1.10. *Let X be a topological space. Then the following are equivalent:*

(1) X is nowhere scattered;

(2) X has no nonempty, locally closed, scattered subsets;

(3) X has no nonempty, locally closed T_1 subsets;

(4) X has no locally closed one-element subsets.

Proof. Note first that nowhere scatteredness passes to open and closed subsets. Thus, if X is nowhere scattered, every locally closed subset of X is also nowhere scattered. Since no nonempty scattered space is nowhere scattered, it follows that X has no nonempty, locally closed, scattered subsets. Thus, (1) implies (2).

That (2) implies (3), and that (3) implies (4), is clear. By definition, (4) implies (1). □

Recall that, given a C^*-algebra A, we define its spectrum $\widehat{A}$ as the set of unitary equivalence classes of irreducible representations of A. We equip $\widehat{A}$ with the hull-kernel topology; see [11, Paragraph II.6.5.13] for details.

We know from [49, Corollary 3] that a C^*-algebra A is scattered if and only if A is of type I and $\widehat{A}$ is scattered. Thus, a separable C^*-algebra is scattered if and only if its spectrum is. Indeed, this follows from the fact that every scattered space is T_0, and that, for separable C^*-algebras, being type I is equivalent to having a T_0-spectrum; see [11, Theorem IV.1.5.7].

We note that there are examples of nonseparable, not scattered C^*-algebras with scattered spectrum; see [1].

Theorem 6.1.11. *Let A be a separable C^*-algebra. Then, A is nowhere scattered if and only if $\widehat{A}$ is.*

Proof. It is well known that there is a natural correspondence between ideal-quotients of A and locally closed subsets of $\widehat{A}$; see [11, Paragraph II.6.5.13]. Since A is separable, it follows from [49, Corollary 3] that an ideal-quotient of A is scattered if and only if its corresponding locally closed subset is scattered.

We know from Theorem 6.1.2 that A is nowhere scattered if and only if it has no nonzero scattered ideal-quotients. By the remarks above, this is equivalent to $\widehat{A}$ having no nonempty, scattered, locally closed subsets. By Proposition 6.1.10, this happens if and only if $\widehat{A}$ is nowhere scattered, as desired. $\square$

6.2 A new property for Cuntz semigroups

In order to characterize nowhere scattered C^*-algebras in terms of their Cuntz semigroup, we introduce a new property that the Cuntz semigroup of every C^*-algebra satisfies; see Definition 6.2.1. With it, we prove in Proposition 6.3.8 that a Cu-semigroup satisfying (O5), (O6) and this new property is weakly $(2,\omega)$-divisible (in the sense of Paragraph 6.3.6) if and only if it has no elementary ideal-quotients. This implies that a C^*-algebra is nowhere scattered if and only if its Cuntz semigroup is weakly $(2,\omega)$-divisible; see Theorem 6.3.9 and Remark 6.3.10.

Definition 6.2.1. A Cu-semigroup S is said to satisfy (O8) if, whenever

$$2w = w, \quad x + y \ll z + w, \quad x' \ll x \quad \text{and} \quad y' \ll y \quad \text{in } S,$$

there exist $z_1, z_2 \in S$ such that

$$z_1 + z_2 \ll z, \quad x' \ll z_1 + w, \quad y' \ll z_2 + w, \quad z_1 \ll x + w \quad \text{and} \quad z_2 \ll y + w.$$

Remark 6.2.2. Note that a Cu-semigroup S satisfies (O8) if and only if, whenever

$$2w = w, \quad x + y \ll z + w, \quad x' \ll x \quad \text{and} \quad y' \ll y \quad \text{in } S,$$

there exist $z_1, z_2 \in S$ such that

$$z_1 + z_2 \leq z, \quad x' \leq z_1 + w, \quad y' \leq z_2 + w, \quad z_1 \leq x + w \quad \text{and} \quad z_2 \leq y + w.$$

Indeed, if x', x, y', y, z, w are as above and S satisfies the stated property, let x'', y'' be such that $x' \ll x'' \ll x$ and $y' \ll y'' \ll y$. Applying the property to x', x'', y', y'', z, w, we obtain $v_1, v_2 \in S$ such that

$$v_1 + v_2 \leq z, \quad x' \ll x'' \leq v_1 + w, \quad y' \ll y'' \leq v_2 + w, \quad v_1 \leq x + w \quad \text{and} \quad v_2 \leq y + w.$$

Choose $z_1 \ll v_1$ and $z_2 \ll v_2$ such that $x' \ll z_1 + w$ and $y' \ll z_2 + w$. It follows that z_1, z_2 have the desired properties.

The forward implication is trivial.

Theorem 6.2.3. *The Cuntz semigroup of every C^*-algebra satisfies (O8).*

Proof. Let A be a C^*-algebra, which we may assume to be stable, and take elements $x', x, y', y, z, w \in \mathrm{Cu}(A)$ as in Definition 6.2.1. That is to say, such that

$$2w = w, \quad x + y \ll z + w, \quad x' \ll x \quad \text{and} \quad y' \ll y.$$

Since A is stable, we can choose $a, b, c \in A_+$ and $\varepsilon > 0$ with a orthogonal to b,

$$x = [a], \quad y = [b], \quad z = [c], \quad x' \leq [(a - \varepsilon)_+] \quad \text{and} \quad y' \leq [(b - \varepsilon)_+].$$

Using that the set $\{s \in \mathrm{Cu}(A) : s \leq w\}$ is an ideal of $\mathrm{Cu}(A)$, we can find by Proposition 1.2.13 an ideal I of A with $\mathrm{Cu}(I)$ equal to it. Let $\pi\colon A \to A/I$ denote the quotient map, and note that

$$\pi(a) + \pi(b) = \pi(a + b) \precsim \pi(c).$$

Thus, it follows from Lemma 1.2.2 that we can find $r \in A/I$ satisfying

$$(\pi(a) - \varepsilon)_+ + (\pi(b) - \varepsilon)_+ = ((\pi(a) + \pi(b)) - \varepsilon)_+ = r^* r$$

and $rr^* \in \overline{\pi(c)(A/I)\pi(c)}$.

Now set $e := r(\pi(a) - \varepsilon)_+ r^*$ and $f := r(\pi(b) - \varepsilon)_+ r^*$, which are orthogonal elements contained in $\overline{\pi(c)(A/I)\pi(c)}$. Using that the C^*-algebra $C_0((0, \|e\|]) \oplus C_0((0, \|f\|])$ is projective (see [29, Section 4]), and that π maps $\overline{cAc}$ onto $\overline{\pi(c)(A/I)\pi(c)}$, we find orthogonal positive elements $\tilde{e}, \tilde{f} \in \overline{cAc}$ such that $\pi(\tilde{e}) = e$ and $\pi(\tilde{f}) = f$.

Let z_1 and z_2 be the Cuntz classes of $\tilde{e}$ and $\tilde{f}$ respectively. Since $\tilde{e}$ and $\tilde{f}$ are orthogonal, we get $\tilde{e} + \tilde{f} \in \overline{cAc}$ and, consequently, $z_1 + z_2 \leq [c] = z$.

Further, we know that

$$\pi(\tilde{e}) = e \precsim (\pi(a) - \epsilon)_+ \precsim \pi(a),$$

which implies $z_1 \leq x + w$. Similarly, $z_2 \leq y + w$.

One also has

$$r^* e r = r^* r(\pi(a) - \varepsilon)_+ r^* r = (\pi(a) - \varepsilon)_+^3 \sim (\pi(a) - \varepsilon)_+.$$

This shows $x' \leq z_1 + w$, and an analogous argument proves $y' \leq z_2 + w$. Using Remark 6.2.2, we see that $\mathrm{Cu}(A)$ satisfies (O8), as desired. □

Recall that (O5) can be seen as a weakening of having algebraic order, while (O6) is a weakened form of Riesz decomposition. In this sense, (O8) can be thought of as a weak version of Riesz refinement, and Proposition 6.2.4 below is the Cu-version of the fact that a cancellative, algebraically ordered semigroup with Riesz decomposition has Riesz refinement.

Proposition 6.2.4. *A weakly cancellative* Cu*-semigroup with (O5) and (O6) satisfies (O8).*

Proof. Let S be a Cu-semigroup satisfying (O5), (O6) and weak cancellation. As in Definition 6.2.1, let $x', x, y', y, z, w \in S$ satisfy

$$2w = w, \quad x + y \ll z + w, \quad x' \ll x \quad \text{and} \quad y' \ll y.$$

Since $x' \ll x \ll z + w$, we can use (O6) to obtain an element $\tilde{z}_1$ satisfying

$$x' \ll \tilde{z}_1 + w \quad \text{and} \quad \tilde{z}_1 \ll x, z.$$

Now let $z_1 \in S$ be such that $z_1 \ll \tilde{z}_1$ and $x' \ll z_1 + w$. Applying (O5) to $z_1 \ll \tilde{z}_1 \leq z$, there exists $c \in S$ with

$$z_1 + c \leq z \leq \tilde{z}_1 + c.$$

Thus, one has

$$x + y \ll z + w \leq \tilde{z}_1 + c + w \leq x + c + w$$

and, using that S is weakly cancellative, we get $y \ll c + w$.

Applying (O6) once again to $y' \ll y \ll c + w$, we find $z_2 \in S$ such that

$$y' \ll z_2 + w \quad \text{and} \quad z_2 \ll y, c.$$

In particular, this implies $z_1 + z_2 \leq z_1 + c \leq z$. Thus, the elements z_1 and z_2 satisfy the properties in Remark 6.2.2, which shows that S satisfies (O8). □

With a view towards Proposition 6.2.7 and Theorem 6.3.9, we first need some preliminary results:

Lemma 6.2.5. *Let S be a* Cu*-semigroup satisfying (O8), and let $w, z, x_j', x_j \in S$ for $j = 1, \ldots, n$ be such that*

$$2w = w, \quad x_1 + \ldots + x_n \ll z + w, \quad x_1' \ll x_1, \quad \ldots, \quad \textit{and} \quad x_n' \ll x_n.$$

Then, there exist $z_1, \ldots, z_n \in S$ such that

$$z_1 + \ldots + z_n \ll z, \quad x_j' \ll z_j + w \quad \textit{and} \quad z_j \ll x_j + w$$

for $j = 1, \ldots, n$.

Proof. Let us prove the result by induction over n, where note that the result holds clearly for $n = 1$ and by (O8) for $n = 2$.

Thus, let $n > 2$ be fixed and assume that the result holds for n. Take $w, z, x_j', x_j \in S$ for $j = 1, \ldots, n+1$ such that

$$2w = w, \quad x_1 + \ldots + x_{n+1} \ll z + w, \quad x_1' \ll x_1, \quad \ldots, \quad \text{and} \quad x_{n+1}' \ll x_{n+1}.$$

Let $x_n'', x_{n+1}'' \in S$ satisfy $x_n' \ll x_n'' \ll x_n$ and $x_{n+1}' \ll x_{n+1}'' \ll x_{n+1}$. Since our result holds for n, we can use it on w, z, x_j', x_j for $j = 1, \ldots, n-1$ and $x_n'' + x_{n+1}'', x_n + x_{n+1}$ to obtain elements $z_1, \ldots, z_{n-1}, v \in S$ satisfying

$$z_1 + \ldots + z_{n-1} + v \ll z, \quad x_j' \ll z_j + w \quad \text{and} \quad z_j \ll x_j + w$$

for $j = 1, \ldots, n-1$ and

$$x_n'' + x_{n+1}'' \ll v + w \quad \text{and} \quad v \ll x_n + x_{n+1} + w.$$

Using (O8) on $x_n', x_n'', x_{n+1}', x_{n+1}'', v, w$, we obtain z_n, z_{n+1} with $z_n + z_{n+1} \ll v$ and such that

$$x_n' \ll z_n + w, \quad x_{n+1}' \ll z_{n+1} + w, \quad z_n \ll x_n'' + w \quad \text{and} \quad z_{n+1} \ll x_{n+1}'' + w.$$

One can now check that $z_1, \ldots, z_{n+1}$ have the required properties. □

Proposition 6.2.6. *Let S be a* Cu*-semigroup satisfying (O6) and (O8), and let w, x_j', x_j in S for $j = 1, \dots, n$ be such that*

$$2w = w, \quad x_1 + \ldots + x_n \ll z + w \quad \textit{and} \quad x_1' \ll x_1 \ll x_2' \ll x_2 \ll \ldots \ll x_n' \ll x_n.$$

Then, there exist $z_1, \dots, z_n \in S$ such that

$$z_1 + z_2 + \ldots + z_n \ll z, \quad z_1 \ll \ldots \ll z_n, \quad x_j' \ll z_j + w \quad \textit{and} \quad z_j \ll x_j + w$$

for $j = 1, \dots, n$.

Proof. Using Lemma 6.2.5, we get elements $y_1, \dots, y_n \in S$ satisfying

$$y_1 + \ldots + y_n \ll z, \quad x_j' \ll y_j + w \quad \text{and} \quad y_j \ll x_j + w$$

for $j = 1, \dots, n$.

Set $z_n := y_n$, and let $y_{n-1}' \ll y_{n-1}$ be such that $x_{n-1}' \ll y_{n-1}' + w$. Then, we have

$$y_{n-1}' \ll y_{n-1} \leq x_{n-1} + w \leq x_n' + w \leq y_n + w = z_n + w.$$

Thus, we can apply (O6) to obtain $z_{n-1} \in S$ such that

$$y_{n-1}' \ll z_{n-1} + w \quad \text{and} \quad z_{n-1} \ll y_{n-1}, z_n.$$

Now let $y_{n-2}' \ll y_{n-2}$ satisfy $x_{n-2}' \ll y_{n-2}' + w$, and note that

$$y_{n-2}' \ll y_{n-2} \leq x_{n-2} + w \leq x_{n-1}' + w \leq y_{n-1}' + w \leq z_{n-1} + w,$$

which allows us to apply (O6) once again. Proceeding in this manner, we get elements $z_1, \dots, z_n \in S$ such that

$$y_j' \ll z_j + w \quad \text{and} \quad z_j \ll y_j, z_{j+1}$$

for each $j \leq n - 1$.

These elements satisfy the desired properties. □

Proposition 6.2.7. *Let S be a* Cu*-semigroup satisfying (O6) and (O8). Let x', x, y, w be elements in S be such that*

$$2w = w, \quad nx \ll y + w \quad \textit{and} \quad x' \ll x$$

for some $n \geq 1$.

Then, there exists $z \in S$ such that

$$nz \ll y, \quad x' \ll z + w \quad \textit{and} \quad z \ll x + w.$$

Proof. Since $x' \ll x$, we can choose elements $x_1', x_1, x_2', x_2, \dots, x_n \in S$ satisfying

$$x' \ll x_1' \ll x_1 \ll x_2' \ll x_2 \ll \ldots \ll x_n' \ll x_n \ll x.$$

By Proposition 6.2.6, there exist $z_1, \dots, z_n$ such that

$$z_1 + \ldots + z_n \ll y, \quad z_1 \ll \ldots \ll z_n, \quad x_j' \ll z_j + w \quad \text{and} \quad z_j \ll x_j + w$$

for $j = 1, \dots, n$.

Thus, setting $z := z_1$, we obtain

$$nz \leq z_1 + \ldots + z_n \ll y, \quad x' \leq x_1' \ll z_1 + w = z + w,$$

and $z = z_1 \leq z_n \ll x_n + w \leq x + w$, as desired. □

6.3 Divisibility conditions on the Cuntz semigroup

We show in this section that nowhere scatteredness can be characterized in terms of divisibility conditions on the Cuntz semigroup; see Proposition 6.3.8 and Theorem 6.3.9.

First, let us give a tailored definition of elementary Cu-semigroups.

6.3.1. Recall that a C^*-algebra A is *elementary* if A is isomorphic to the algebra of compact operators on some Hilbert space.

As explained in Examples 1.2.8 (2), elementary Cu-semigroups were defined in [6, Paragraph 5.1.16] to be those simple Cu-semigroups containing a minimal nonzero element. In particular, the Cu-semigroup $\{0,\infty\}$ is elementary in this sense, although it is the Cuntz semigroup of simple, purely infinite C^*-algebras, which are not elementary.

We propose the following revised definition: A Cu-semigroup is *elementary* if it is simple and contains a minimal, nonzero, *finite* element x (that is, $x \neq 2x$).

Lemma 6.3.2 below, which is Theorem 4.4.4 in [36], shows that this ammended definition agrees with the established terminology in C^*-algebras. We include its proof for the convenience of the reader.

Lemma 6.3.2. *Let A be a (nonzero) C^*-algebra. Then, A is elementary if and only if* $\mathrm{Cu}(A)$ *is elementary (if and only if* $\mathrm{Cu}(A) \cong \overline{\mathbb{N}}$*).*

Proof. Assume first that A is elementary. Then, $A \otimes \mathcal{K}(H) \cong \mathcal{K}(H)$ for some infinite-dimensional Hilbert space H and, consequently, $V(A) \cong \mathbb{N}$. Since A has real rank zero, $\mathrm{Cu}(A)$ is the ideal completion of $V(A)$, which implies $\mathrm{Cu}(A) \cong \overline{\mathbb{N}}$ and that $\mathrm{Cu}(A)$ is elementary; see, for example, [6, Remark 5.5.6].

Now assume that $\mathrm{Cu}(A)$ is elementary. Since, with our new definition, $\{0,\infty\}$ is not an elementary Cu-semigroup, we get $\mathrm{Cu}(A) \cong \overline{\mathbb{N}}$ by [6, Paragraph 5.1.16]. In particular, it follows from [6, Corollary 5.1.12] that A is a simple C^*-algebra.

By [17, Theorem 5.8], there exists a projection $p \in A \otimes \mathcal{K}$ such that $[p]$ corresponds to 1 under the identification $\mathrm{Cu}(A) \cong \overline{\mathbb{N}}$. Given any nonzero element $a \in A_+$, we have $[p] \leq [a]$ and, consequently, $p \precsim a$. Thus, there exists a projection $q \in A$ with $[q] = [p]$ and $q \precsim a$ for every $a \in A_+$. This shows that q is minimal, and the existence of such a projection is well-known to imply that A is elementary. □

Lemma 6.3.3. *Let S be a* Cu*-semigroup satisfying (O8) and let I be an ideal of S. Then, I and S/I satisfy (O8).*

Proof. Given an ideal I of S and elements $x', x, y', y, z, w \in I$ as in Definition 6.2.1, note that the elements $z_1, z_2 \in S$ obtained using (O8) satisfy $z_1 + z_2 \ll z$. Since I is downward-hereditary, it follows that both z_1 and z_2 are in I. This shows that I satisfies (O8).

Now let $\pi\colon S \to S/I$ denote the quotient map. If $x', x, y', y, z, w \in S$ satisfy

$$2\pi(w) = \pi(w), \quad \pi(x) + \pi(y) \ll \pi(z) + \pi(w), \quad \pi(x') \ll \pi(x) \quad \text{and} \quad \pi(y') \ll \pi(y)$$

in S/I, there exist elements $r', r, t', t \in S$ such that

$$r' \ll r \ll x, \quad t' \ll t \ll y, \quad \pi(x') \ll \pi(r') \quad \text{and} \quad \pi(y') \ll \pi(t').$$

Thus, we can find an idempotent $v \in I$ satisfying

$$2(w+v) = (w+v) \quad \text{and} \quad r+t \ll z+(w+v).$$

Using that S satisfies (O8), one obtains elements $z_1, z_2 \in S$ satisfying the conditions in Definition 6.2.1 for $r', r, t', t, z, w+v$. The elements $\pi(z_1)$ and $\pi(z_2)$ satisfy the desired conditions in S/I. □

Lemma 6.3.4. *Let S be a* Cu*-semigroup satisfying (O6) and (O8), and let I be an ideal of S. Denote by π the canonical map $\pi\colon S \to S/I$. Then, given $y \in S$ and $e, e' \in S/I$ such that*

$$e' \ll e \quad \textit{and} \quad 2e \leq \pi(y),$$

there exists $z \in S$ satisfying $e' \ll \pi(z) \ll e$ and $2z \ll y$.

Proof. Let $x', x'', x \in S$ be such that $x' \ll x'' \ll x$, $\pi(x) = e$ and $e' \ll \pi(x')$. Then, since $2\pi(x) \leq \pi(y)$, there exists an idempotent element $w \in I$ such that $2x'' \ll 2x \leq y + w$.

Using Proposition 6.2.7, we get $z \in S$ satisfying

$$2z \ll y, \quad x' \ll z + w \quad \text{and} \quad z \ll x'' + w \leq x + w.$$

Consequently, we have

$$e' \ll \pi(x') \ll \pi(z) \ll \pi(x) = e,$$

as desired. □

Lemma 6.3.5. *Let S be a* Cu*-semigroup satisfying (O5), (O6) and (O8) with no nonzero elementary ideal-quotients, and let $x \in S$ be nonzero. Then, there exists $z \in S$ with $0 \neq 2z \leq x$.*

Proof. Let $x'', x' \in S$ be nonzero elements such that $x'' \ll x' \ll x$. Since S satisfies (O5), there exists $c \in S$ satisfying

$$x'' + c \leq x \leq x' + c.$$

Note that, if $x \leq \infty c$, one gets $n \in \mathbb{N}$ such that $x' \leq nc$. As in the proof of [73, Proposition 5.2.1], (O6) allows us to obtain elements $c_1, \ldots, c_n \leq x'', c$ with

$$x'' \leq c_1 + \ldots + c_n.$$

Thus, since x'' is nonzero, there exists j such that $c_j \neq 0$. Such an element satisfies $2c_j \leq x'' + c \leq x$, as required.

Now assume that $x \not\leq \infty c$ or, equivalently, that x does not belong to the ideal I generated by c. Let K be the ideal generated by x. If K/I is not simple, let $J \subsetneq K$ be a maximal ideal containing I. Else, set $J = I$.

Thus, if $\pi\colon K \to K/J$ denotes the quotient map, $\pi(x)$ is a nonzero, compact element in the (nonzero) simple, nonelementary ideal-quotient K/J.

Using that (O5) and (O6) pass to ideals and quotients (see [6, Proposition 5.1.3]), [73, Proposition 5.2.1] implies that there exists a nonzero element $e \in K/J$ such that $2e \leq \pi(x)$. By Lemmas 6.3.3 and 6.3.4, there exists $z \in S$ such that $0 \neq 2z \ll x$, as desired. □

6.3.6. Let $k \in \mathbb{N}$. As defined in [75, Definition 5.1], an element x in a Cu-semigroup S is said to be (k, ω)-*divisible* if, whenever $x' \ll x$, there exists $y \in S$ such that $ky \leq x$ and $x' \leq ny$ for some $n \in \mathbb{N}$. Similarly, x is said to be *weakly* (k, ω)-*divisible* if, whenever $x' \ll x$, there exist $n \in \mathbb{N}$ and $y_1, \ldots, y_n \in S$ such that $ky_1, \ldots, ky_n \leq x$ and $x' \leq y_1 + \ldots + y_n$; see also [4, Paragraph 5.1].

A Cu-semigroup S will be said to be *(weakly)* (k, ω)-*divisible* if each element of S is (weakly) (k, ω)-divisible.

Lemma 6.3.7. *Let S be a* Cu-*semigroup. Then, S is weakly $(2, \omega)$-divisible if and only if S is weakly (k, ω)-divisible for every $k \geq 2$.*

Similarly, S is $(2, \omega)$-divisible if and only if S is (k, ω)-divisible for every $k \geq 2$.

Proof. We only prove the result for weak (k, ω)-divisibility, since the proof for (k, ω)-divisibility is analogous.

Thus, assume that S is weakly $(2, \omega)$-divisible and note that, to see that S is weakly (k, ω)-divisible for each $k \geq 2$, it is enough to prove that S is weakly $(2^k, \omega)$-divisible for every $k \geq 1$. We proceed by induction:

For $k = 1$, the result holds by assumption. Now fix $k \in \mathbb{N}$ and assume that S is weakly $(2^k, \omega)$-divisible. Then, given $x' \ll x$ in S, choose $x'' \ll x$ such that $x' \ll x''$, and use weak $(2, \omega)$-divisibility to obtain $m \in \mathbb{N}$ and $y_1, \ldots, y_m \in S$ such that

$$2y_1, \ldots, 2y_m \leq x \quad \text{and} \quad x'' \leq y_1 + \ldots + y_m.$$

Since $x' \ll x'' \leq y_1 + \ldots + y_m$, we can find elements $y_j' \ll y_j$ for each $j \leq m$ such that $x' \ll y_1' + \ldots + y_m'$. Thus, applying weak $(2^k, \omega)$-divisibility to each pair $y_j' \ll y_j$, we get $n(j) \in \mathbb{N}$ and elements $z_{j,1}, \ldots, z_{j,n(j)} \in S$ satisfying

$$2^k z_{j,1}, \ldots, 2^k z_{j,n(j)} \leq y_j \quad \text{and} \quad y_j' \leq z_{j,1} + \ldots + z_{j,n(j)}$$

for each j.

Thus, one gets

$$2^{k+1} z_{j,n} \leq 2y_j' \leq x$$

for each $j \leq m$ and $n \leq n(j)$, and

$$x' \ll y_1' + \ldots + y_m' \leq (z_{1,1} + \ldots + z_{1,n(1)}) + \ldots + (z_{m,1}, \ldots, z_{m,n(m)}),$$

as desired.

The converse is trivial. $\square$

Proposition 6.3.8. *Let S be a* Cu-*semigroup satisfying (O5), (O6) and (O8). Then, the following are equivalent:*

(1) S has no nonzero elementary ideal-quotients;

(2) S is weakly $(2, \omega)$-divisible;

(3) S is weakly (k, ω)-divisible for every $k \geq 2$.

Proof. Let us first assume that S is weakly $(2,\omega)$-divisible. Since this property passes to ideals and quotients, every ideal-quotient of S is also weakly $(2,\omega)$-divisible. This implies that no nonzero ideal-quotient of S is elementary, since a minimal, finite, nonzero element of such Cu-semigroups is not weakly $(2,\omega)$-divisible. Thus, (2) implies (1).

To see that (1) implies (2), we follow a similar proof to that of [9, Theorem 6.7]. Let $x \in S$ and let I be the ideal generated by the elements $z \in S$ such that $2z \leq x$.

We claim that $x \in I$. Indeed, let $\pi\colon S \to S/I$ denote the quotient map, and note that, by [6, Proposition 5.1.3] and Lemma 6.3.3, S/I satisfies (O5), (O6) and (O8). If $x \notin I$, the element $\pi(x)$ is nonzero. Using Lemma 6.3.5, we find nonzero elements $e' \ll e$ in S/I such that $2e \leq \pi(x)$.

Thus, we know from Lemma 6.3.4 that there exists $z \in S$ with $2z \leq x$ and

$$0 \neq e' \ll \pi(z).$$

Since $2z \leq x$, we get $\pi(z) = 0$, a contradiction. This shows that $x \in I$.

Now let $x' \ll x$ and take $x'' \in S$ such that $x' \ll x'' \ll x$. Since $x \in I$, there exist elements $z_1, \ldots, z_n \in S$ with $2z_1, \ldots, 2z_n \leq x$ and $x' \ll x'' \leq z_1 + \ldots + z_n$. This shows that x is weakly $(2,\omega)$-divisible, as desired.

That (2) is equivalent to (3) follows from Lemma 6.3.7. $\square$

Theorem 6.3.9. *Let A be a C^*-algebra. Then, the following are equivalent:*

(1) A is nowhere scattered;

(2) $\mathrm{Cu}(A)$ is weakly $(2,\omega)$-divisible;

(3) $\mathrm{Cu}(A)$ is weakly (k,ω)-divisible for every $k \geq 2$;

Proof. By Theorem 6.1.2, A is nowhere scattered if and only if it has no nonzero elementary ideal-quotients. Using Proposition 1.2.13 and Lemma 6.3.2, this is equivalent to $\mathrm{Cu}(A)$ having no nonzero elementary ideal-quotients.

Thus, Proposition 6.3.8 shows that A is nowhere scattered if and only if $\mathrm{Cu}(A)$ is weakly $(2,\omega)$-divisible.

The equivalence between (2) and (3) is Lemma 6.3.7. $\square$

Remark 6.3.10. Our proof of Theorem 6.3.9 above is a direct consequence of Proposition 6.3.8. We note that, if one restricts to the study of Cuntz semigroups of C^*-algebras, this can also be proven using results due to Robert and Rørdam from [75].

Indeed, given a C^*-algebra A, Corollary 6.1.8 above implies that A is nowhere scattered if and only if its stabilization is. Thus, we can assume A to be stable.

If A is nowhere scattered, we know from Theorem 6.1.2 that, for any $a \in A_+$, the hereditary sub-C^*-algebra $\overline{aAa}$ does not admit a one-dimensional irreducible representation. By [75, Theorem 5.3(iii)], this is equivalent to $[a]$ being weakly $(2,\omega)$-divisible in $\mathrm{Cu}(\overline{aAa})$. Consequently, $[a]$ is weakly $(2,\omega)$-divisible in $\mathrm{Cu}(A)$. Since this holds for each $a \in A_+$, it follows that $\mathrm{Cu}(A)$ is weakly $(2,\omega)$-divisible.

Conversely, if $\mathrm{Cu}(A)$ is weakly $(2,\omega)$-divisible, assume for the sake of contradiction that A is not nowhere scattered. Thus, Theorem 6.1.2 implies that there exists a hereditary sub-C^*-algebra B with a one-dimensional irreducible representation π. Given $b \in B_+$ such that $\pi(b) \neq 0$, it follows that $\overline{bAb}$ also admits a one-dimensional irreducible representation. Using [75, Theorem 5.3(iii)] once again, we obtain that $[b]$ cannot be weakly $(2,\omega)$-divisible in $\mathrm{Cu}(\overline{bAb})$ and, consequently, not in $\mathrm{Cu}(A)$, a contradiction.

Example 6.3.11. Recall from Definition 1.1.13 (iv) that, as defined in [53, Definition 4.1], a C^*-algebra is *purely infinite* if each element x of its Cuntz semigroup satisfies $x = 2x$. Further, we say that a C^*-algebra is *weakly purely infinite* if there exists $n \in \mathbb{N}$ such that each element x of its Cuntz semigroup satisfies $2nx = nx$; see [54, Definition 4.3].

Every weakly purely infinite C^*-algebra A is *traceless*. That is, for every element $x \in \mathrm{Cu}(A)$, the map $\widehat{x}\colon S \to [0,\infty]$ (defined in Paragraph 1.2.16) satisfies $2\widehat{x} = \widehat{x}$; see [13, Remark 2.27].

Traceless C^*-algebras are nowhere scattered. In particular, every weakly purely infinite C^*-algebra is. To see this, let A be a traceless C^*-algebra and assume, for the sake of contradiction, that its Cuntz semigroup has a nonzero elementary ideal-quotient $I/J \cong \overline{\mathbb{N}}$. Let $\pi\colon I \to I/J$ be the quotient map, and denote by $\sigma\colon I/J \to [0,\infty]$ the nontrivial functional corresponding to the map $\overline{\mathbb{N}} \to [0,\infty]$ given by $1 \mapsto 1$.

Let $\tau\colon \mathrm{Cu}(A) \to [0,\infty]$ be the map defined as $\tau(x) = \sigma\pi(x)$ if $x \in I$ and $\tau(x) = \infty$ otherwise. It is readily checked that τ is a nonzero functional. Take $x \in I$ such that $\pi(x)$ corresponds to $1 \in \overline{\mathbb{N}}$. Then, $\widehat{x}(\tau) = \tau(x) = 1 \neq 2 = 2\widehat{x}(\tau)$, a contradiction.

Proposition 6.3.8 and Theorem 6.3.9 now show that A is nowhere scattered.

Using the characterization given by Theorem 6.3.9 above, we can show more permanence properties that nowhere scatteredness enjoys. The proofs given here differ from those in [93, Section 4].

Recall the definition of approximation from Section 5.1.

Proposition 6.3.12. *Let A be C^*-algebra, and let $(A_\lambda)_{\lambda\in\Lambda}$ be a family of sub-C^*-algebras that approximates A. Assume that each A_λ is nowhere scattered. Then, A is nowhere scattered.*

Proof. Let $\iota_\lambda\colon A_\lambda \to A$ be the inclusion map for each λ. By Theorem 6.3.9 and Proposition 5.1.12, each $\mathrm{Cu}(A_\lambda)$ is weakly $(2,\omega)$-divisible and $(\mathrm{Cu}(A_\lambda), \mathrm{Cu}(\iota_\lambda))_\lambda$ approximates $\mathrm{Cu}(A)$.

Given $x' \ll x$ in $\mathrm{Cu}(A)$, let $\lambda \in \Lambda$ and $z' \ll z$ in $\mathrm{Cu}(A_\lambda)$ be such that

$$x' \ll \mathrm{Cu}(\iota_\lambda)(z') \ll \mathrm{Cu}(\iota_\lambda)(z) \ll x.$$

Using that $\mathrm{Cu}(A_\lambda)$ is weakly $(2,\omega)$-divisible, there exist $n \in \mathbb{N}$ and $y_1, \ldots, y_n$ such that $2y_1, \ldots, 2y_n \leq z$ and $z' \leq y_1 + \ldots + y_n$.

Considering the elements $\mathrm{Cu}(\iota_\lambda)(y_j)$ for $j = 1, \ldots, n$, one sees that $\mathrm{Cu}(A)$ is weakly $(2,\omega)$-divisible. By Theorem 6.3.9, this implies that A is nowhere scattered. $\square$

Proposition 6.3.13. *An inductive limit of nowhere scattered C^*-algebras is nowhere scattered.*

Proof. Let A be an inductive limit of nowhere scattered C^*-algebras A_λ with limit morphisms $\varphi_\lambda\colon A_\lambda \to A$. Since each $\varphi_\lambda(A_\lambda)$ is a quotient of A_λ, it follows from Proposition 6.1.6 that $\varphi_\lambda(A_\lambda)$ is nowhere scattered for every λ.

Using that the family of sub-C^*-algebras $\varphi_\lambda(A_\lambda) \subseteq A$ approximates A, Proposition 6.3.12 above implies that A is nowhere scattered. $\square$

In analogy to Paragraph 5.3.3, we say that a property $\mathcal{P}$ for C^*-algebras satisfies the *Löwenheim-Skolem condition* if for every C^*-algebra A with property $\mathcal{P}$ there exists a σ-complete and cofinal family $\mathcal{S}$ of separable sub-C^*-algebras of A that each have property $\mathcal{P}$

Proposition 6.3.14. *Nowhere scatteredness satisfies the Löwenheim-Skolem condition.*

Proof. Using the techniques from the proof of Proposition 5.3.4, one can show that weak $(2,\omega)$-divisibility satisfies the Löwenheim-Skolem condition for Cu-semigroups.

Now let A be a nowhere scattered C^*-algebra. By Proposition 5.4.2, the family

$$\mathcal{S} = \big\{ B \in \mathrm{Sub}_{\mathrm{sep}}(A) \mid \mathrm{Cu}(B) \to \mathrm{Cu}(A) \text{ is an order-embedding} \big\}$$

is σ-complete and cofinal.

Given a separable sub-C^*-algebra $B \subseteq A$, and proceeding as in Theorem 5.4.3, we obtain an increasing sequence

$$\mathrm{Cu}(B_0) \subseteq T_0 \subseteq \mathrm{Cu}(B_1) \subseteq T_1 \subseteq \mathrm{Cu}(B_2) \subseteq T_2 \subseteq \ldots$$

in $\mathrm{Sub}_{\mathrm{ctbl}}(\mathrm{Cu}(A))$ such that each T_i is weakly $(2,\omega)$-divisible, $(B_i)_i$ is increasing in $\mathcal{S}$ and $B \subseteq B_0$.

Using Theorem 5.2.11 and the second part of Proposition 5.4.2, one gets

$$\mathrm{Cu}(\overline{\bigcup_k B_k}) = \overline{\bigcup_k T_k}^{\mathrm{sup}},$$

which is a $(2,\omega)$-divisible Cuntz semigroup. By Theorem 6.3.9, $\overline{\bigcup_k B_k}$ is a separable, nowhere scattered C^*-algebra that contains B. This shows that the family of separable, nowhere scattered sub-C^*-algebras is cofinal.

It follows from Proposition 6.3.13 that the family of separable, nowhere scattered sub-C^*-algebras is σ-complete, since for any countable, directed family $\{B_i\}_i$ of such sub-C^*-algebras we know that $\overline{\cup_i B_i}$ can be written as an inductive limit of the B_i's indexed over themselves. $\square$

Proposition 6.3.15. *Let $(A_j)_{j\in J}$ be a family of nowhere scattered C^*-algebras. Then, the direct sum $\bigoplus_{j\in J} A_j$ is nowhere scattered.*

Proof. Note that the direct sum of two weakly $(2,\omega)$-divisible Cu-semigroups is again weakly $(2,\omega)$-divisible. By Theorem 6.3.9, this shows that $\bigoplus_{j\in J} A_j$ is nowhere scattered whenever J is finite.

Since, for any J, the sum $\bigoplus_{j\in J} A_j$ is the inductive limit of $\bigoplus_{j\in F} A_j$ indexed over all finite subsets $F \subseteq J$ ordered by inclusion, Proposition 6.3.13 implies that $\bigoplus_{j\in J} A_j$ is nowhere scattered. $\square$

Example 6.3.16. Proposition 6.3.15 above shows that nowhere scatteredness is preserved under direct sums. However, this is not the case for products.

Indeed, [75, Corollary 8.6] shows that there exist unital, simple, infinite-dimensional (thus nowhere scattered) C^*-algebras $(A_k)_{k\in\mathbb{N}}$ such that $\prod_k A_k$ has a one-dimensional irreducible representation. It follows from Theorem 6.1.2 that $\prod_k A_k$ is not nowhere scattered.

Moreover, since each A_k is unital, it follows from [11, II.8.1.3] that the multiplier algebra $M(\bigoplus_k A_k)$ (see Remark 1.1.8) is isomorphic to $\prod_k A_k$. This shows that nowhere scatteredness is not preserved when passing to the multiplier algebra.

Example 6.3.17. Given a type II_1 factor M and a pure state $\varphi\colon M \to \mathbb{C}$, let A be the associated hereditary kernel

$$A = \{a \in M : \varphi(aa^*) = \varphi(a^*a) = 0\}.$$

It follows from [83, Theorem 1] and the remarks after Proposition 2.6 in [68] that A is a simple, nowhere scattered C^*-algebra such that $M(A)/A \cong \mathbb{C}$. Thus, its multiplier algebra is not nowhere scattered.

Question 6.3.18. As witnessed by Examples 6.3.16 and 6.3.17 above, nowhere scatteredness may not be preserved when passing to the multiplier algebra of a separable or simple C^*-algebra.

Does there exist a nonunital, separable, simple, nonelementary C^*-algebra A such that $M(A)$ has a one-dimensional irreducible representation?

6.4 Real rank zero and stable rank one

In this section we give further characterizations of nowhere scatteredness under the additional assumptions of real rank zero or stable rank one; see Theorem 6.4.1 and Theorem 6.4.11 respectively.

Recall from Definition 1.1.14 (ii) that a C^*-algebra A is said to have real rank zero if the self-adjoint, invertible elements of $\tilde{A}$ are dense in the set of self-adjoint elements of $\tilde{A}$.

Theorem 6.4.1. *Let A be a real rank zero C^*-algebra. Then, the following are equivalent:*

(1) A is nowhere scattered;

(2) $V(A)$ is weakly divisible;

(3) $\mathrm{Cu}(A)$ is weakly divisible.

Proof. Assume first that A is nowhere scattered. By Corollary 6.1.8, we may also assume A to be stable.

Thus, Theorem 6.1.2 (7) implies that, for every $[p] \in V(A)$, the hereditary sub-C^*-algebra pAp does not have one-dimensional irreducible representations. It follows from the proof of [9, Corollary 6.8] that there exist $y, z \in V(A)$ such that $[p] = 2y + 3z$. This shows that (1) implies (2).

Now assume that $V(A)$ is weakly divisible, and let $\leq_{\mathrm{alg}}$ denote the algebraic preorder on $V(A)$. Since A has real rank zero, we know from (the proof of) Theorem 4.3.8 that there exists an order preserving, monoid morphism $\alpha\colon V(A) \to \mathrm{Cu}(A)$ with sup-dense image. Further, observe that, whenever $x' = 2y' + 3z' \leq_{\mathrm{alg}} x$ in $V(A)$, there exist $y, z \in V(A)$ satisfying

$$y' \leq_{\mathrm{alg}} y, \quad z' \leq_{\mathrm{alg}} z \quad \text{and} \quad x = 2y + 3z.$$

Indeed, let w be such that $x' + w = x$. Then, we have $w = 2u + 3v$ in $V(A)$, and setting $y = y' + u$ and $z = z' + v$ proves the claim.

Using that $\alpha(V(A))$ is sup-dense, we deduce that every element in $\mathrm{Cu}(A)$ can be written as $\sup_n(2y_n + 3z_n)$ with $(2y_n + 3z_n)_n$ increasing. By the previous observation, we may choose the elements y_n, z_n such that $(y_n)_n$ and $(z_n)_n$ are increasing. Thus, every element in $\mathrm{Cu}(A)$ is of the form

$$\sup_n(2y_n + 3z_n) = 2\sup_n(y_n) + 3\sup_n(z_n),$$

which implies that $\mathrm{Cu}(A)$ is weakly divisible.

Finally, to show that (3) implies (1), assume that $\mathrm{Cu}(A)$ is weakly divisible and let $x' \ll x$ in $\mathrm{Cu}(A)$. By weak divisibility, we find elements y, z such that $x = 2y + 3z$. Thus, we have $2(y+z) \leq x$ and $x' \leq x \leq 3(y+z)$, which implies that x is $(2, \omega)$-divisible.

Since this can be done for every pair $x' \ll x$, it follows from Theorem 6.3.9 that A is nowhere scattered, as desired. □

Remark 6.4.2. Let $\mathcal{Z}$ be the Jiang-Su algebra, which is a simple, non-elementary C^*-algebra. By Example 6.1.3, $\mathcal{Z}$ is nowhere scattered. However, its Murray-von Neumann semigroup is isomorphic to $\mathbb{N}$, which is not weakly divisible.

This shows that, without the real rank zero assumption, weak divisibility of the Murray-von Neumann semigroup is not equivalent to nowhere scatteredness.

Definition 6.4.3. A Cu-semigroup S is said to satisfy the *interval axiom* if, whenever

$$x' \ll x, \quad x \ll y + u \quad \text{and} \quad x \ll y + v \quad \text{in } S,$$

there exists $w \in S$ such that

$$x' \ll y + w \quad \text{and} \quad w \ll u, v.$$

Remark 6.4.4. In [103, Paragraph 1.3], Wehrung defines the 'algebraic interval axiom' for positively ordered monoids. The definition above is the Cu-version of this notion.

Recall the definition of inf-semilattice ordered semigroups and the Riesz interpolation property from Paragraph 1.2.14 and Paragraph 4.3.3 respectively.

Proposition 6.4.5. *Let S be a countably based* Cu*-semigroup. Then, S is an inf-semilattice ordered* Cu*-semigroup if and only if S satisfies the interval axiom and has the Riesz Interpolation Property.*

Proof. Recall first that, if S is inf-semilattice ordered, S satisfies the Riesz Interpolation Property; see Paragraph 4.3.3. Further, given $x', x, y, u, v \in S$ as in Definition 6.4.3, set $\tilde{w} = u \wedge v$. We have

$$x' \ll x \leq (y + u) \wedge (y + v) = y + \tilde{w} \quad \text{and} \quad \tilde{w} \leq u, v,$$

which allows us to choose $w \ll \tilde{w}$ such that $x' \ll y + w$. This shows that S also satisfies the interval axiom.

Conversely, assume that S satisfies both the interval axiom and the Riesz Interpolation Property. Since S is countably based, suprema of upward directed sets exists.

Using that S has interpolation, it follows that the set $\{z \in S : z \leq x, y\}$ is upward directed and, consequently, has a supremum, which is $x \wedge y$. This implies that S is an inf-semilattice.

Now, given $x, y, z \in S$, note that $(x+z) \wedge (y+z) \geq (x \wedge y) + z$ is always satisfied. For the other inequality, set $s = (x+z) \wedge (y+z)$, and take $s' \in S$ such that $s' \ll s$. Since S satisfies the interval axiom, there exists $w \in S$ satisfying

$$s' \leq z + w \quad \text{and} \quad w \leq x \wedge y.$$

This implies $s' \leq z + x \wedge y$. Since this holds for every $s' \ll s$, we obtain

$$(x+z) \wedge (y+z) = s \leq (x \wedge y) + z,$$

as desired. □

The following result can be proven with the methods used in Proposition 5.3.4. We omit its proof.

Proposition 6.4.6. *The interval axiom and the Riesz Interpolation Property both satisfy the Löwenheim-Skolem condition.*

As mentioned in Paragraph 1.2.14, the Cuntz semigroup of any separable, stable rank one C^*-algebra is inf-semilattice ordered. Thus, Proposition 6.4.5 implies that it satisfies the interval axiom. Using the techniques from Chapter 5, we generalize this to the nonseparable case.

Corollary 6.4.7. *Let A be a stable rank one C^*-algebra. Then,* $\mathrm{Cu}(A)$ *satisfies weak cancellation, the Riesz Interpolation Property, and the interval axiom.*

Proof. The Cuntz semigroup of any stable rank one C^*-algebra A is weakly cancellative and has the Riesz interpolation property by [82, Theorem 4.3] and [4, Theorem 3.5] respectively.

To see that $\mathrm{Cu}(A)$ also satisfies the interval axiom, let $x', x, y, u, v \in \mathrm{Cu}(A)$ be as in Definition 6.4.3. Using Proposition 5.4.2 (in conjunction with Lemma 5.3.2) and the fact that having stable rank one satisfies the Löwenheim-Skolem condition, we find a separable, stable rank one sub-C^*-algebra $B \subseteq A$ such that the Cu-morphism $\mathrm{Cu}(B) \to \mathrm{Cu}(A)$ induced by the inclusion is an order-embedding, and such that x', x, y, u, v belong to the image of this map.

Since B is separable and of stable rank one, its Cuntz semigroup $\mathrm{Cu}(B)$ is inf-semilattice ordered. By Proposition 6.4.5 above, $\mathrm{Cu}(B)$ also satisfies the interval axiom.

Thus, an element w with the desired properties can be found in $\mathrm{Cu}(B)$. This Cuntz semigroup can be identified with a sub-Cu-semigroup of $\mathrm{Cu}(A)$, so it follows that w also has the desired properties in $\mathrm{Cu}(A)$, as required. □

Remark 6.4.8. Goodearl constructs in [44] a separable, real rank zero C^*-algebra A such that $K_0(A)$ does not satisfy Riesz decomposition and, consequently, Riesz interpolation. It follows from [69, Lemma 4.2] that $V(A)$ does not satisfy Riesz interpolation either.

One can check that, in an algebraically ordered monoid, the algebraic interval axiom implies Riesz interpolation. Thus, since the order in $V(A)$ is algebraic and $V(A)$ does not have Riesz interpolation, $V(A)$ does not satisfy the algebraic interval axiom either. This implies that $\mathrm{Cu}(A)$ does not satisfy the interval axiom, since $V(A)$ is dense in $\mathrm{Cu}(A)$; see, for example, [6, Remark 5.5.6].

Thus, even in the real rank zero case, the Cuntz semigroup of a separable C^*-algebra might not satisfy the interval axiom.

Question 6.4.9. Which C^*-algebras have a Cuntz semigroup satisfying the interval axiom?

Lemma 6.4.10. *Let S be a weakly cancellative* Cu*-semigroup satisfying (O5), the interval axiom, and the Riesz Interpolation Property. Then, S is weakly $(2,\omega)$-divisible if and only if S is $(2,\omega)$-divisible.*

Proof. Assume first that S is weakly $(2,\omega)$-divisible, and let $x \in S$. Since x is weakly $(2,\omega)$-divisible, we find a sequence $(y_n)_n$ with $2y_n \leq x$ for each n, and such that $x \leq \sum_{n=1}^{\infty} y_n$. Indeed, take a $\ll$-increasing sequence $(x_m)_m$ with supremum x. Then, applying weak $(2,\omega)$-divisibility for each pair $x_m \ll x$, we obtain elements $y_{1,m}, \ldots, y_{r_m,m} \in S$ such that $2y_{j,m} \leq x$ for each j, m and such that $x_m \leq y_{1,m} + \ldots + y_{r_m,m}$. In particular, we have $x \leq \sum_{j,m} y_{j,m}$, as desired.

Now, since weak cancellation, (O5), the interval axiom and Riesz interpolation all satisfy the Löwenheim-Skolem condition, we find a countably based, weakly cancellative sub-Cu-semigroup $T \subseteq S$ satisfying (O5), the interval axiom, and Riesz interpolation which contains the sequence $(y_n)_n$ and the element x.

Thus, since $y_n \in T$ for each n, it follows that x is also weakly $(2,\omega)$-divisible in T. Using that T is inf-semilattice ordered by Proposition 6.4.5, it follows from [4, Theorem 5.5] that x is $(2,\omega)$-divisible in T. Consequently, x is $(2,\omega)$-divisible in S, as desired.

The remaining implication is trivial. □

Theorem 6.4.11. *Let A be a C^*-algebra with stable rank one. Then, the following are equivalent:*

(1) A is nowhere scattered;

(2) $\mathrm{Cu}(A)$ *is $(2,\omega)$-divisible;*

(3) $\mathrm{Cu}(A)$ *is (k,ω)-divisible for every $k \geq 2$.*

Proof. That (2) is equivalent to (3) is Lemma 6.3.7. Further, since A has stable rank one, its Cuntz semigroup $\mathrm{Cu}(A)$ satisfies, by Corollary 6.4.7, the hypothesis of Lemma 6.4.10. This shows that $\mathrm{Cu}(A)$ is $(2,\omega)$-divisible if and only if $\mathrm{Cu}(A)$ is weakly $(2,\omega)$-divisible.

The result now follows from Theorem 6.3.9. □

Note that, as a consequence of Theorem 6.3.9, every C^*-algebra with a $(2,\omega)$-divisible Cuntz semigroup is nowhere scattered. It is unknown if the converse holds, that is to say, if every nowhere scattered C^*-algebra has a $(2,\omega)$-divisible Cuntz semigroup.

This problem, studied in the next chapter, is known as the Global Glimm Problem.

Chapter 7

The Global Glimm Problem

A C^*-algebra A is said to satisfy the Global Glimm Property if, for each $\varepsilon > 0$, each $a \in A_+$, and every natural number $n \geq 2$, there exists a *-homomorphism $\varphi\colon C_0((0,1], M_n) \to \overline{aAa}$ such that the ideal of A generated by the image of φ contains $(a-\varepsilon)_+$. This property, whose definition is inspired by a classical result by Glimm, implies that those C^*-algebras that satisfy it are nowhere scattered; see, for example, Theorem 7.4.6. In this chapter, we focus on the converse. That is to say, we study the *Global Glimm Problem*, which asks if every nowhere scattered C^*-algebra satisfies the Global Glimm Property; see Problem 7.4.3.

Our approach consists on the introduction of a new notion for abstract Cuntz semigroups, which we term ideal-filteredness; see Definition 7.1.1. We show that, although not all Cuntz semigroups have this property (Example 7.1.5), there are a number of interesting families of C^*-algebras that do; see Theorems 7.1.12, 7.1.14 and 7.1.17.

In Sections 7.2 and 7.3 we investigate the connections between $(2,\omega)$-divisibility and weak $(2,\omega)$-divisibility in the presence of ideal-filteredness since, as witnessed by Theorem 7.4.6, the study of these relations is deeply connected to the Global Glimm Problem. With the introduction of strongly and weakly soft elements in Definition 7.2.1, we see that a residually stably finite Cuntz semigroup is $(2,\omega)$-divisible if and only if it is ideal-filtered and weakly $(2,\omega)$-divisible; see Theorem 7.3.10.

Finally, in Section 7.4 we prove that a C^*-algebra A has the Global Glimm Property if and only if its Cuntz semigroup is $(2,\omega)$-divisible. Using the results from the previous sections, we reformulate the Global Glimm Problem in a number of ways and give a positive answer to the problem for some classes of C^*-algebras; see Theorem 7.4.11.

The results in this chapter are from [95]. At the time of writing this thesis [95] is in preparation, and it is possible that the results presented here will be improved further in the paper.

7.1 Ideal-filtered Cuntz semigroups

We introduce in Definition 7.1.1 below the notion of ideal-filteredness for Cu-semigroups. As shown in Theorems 7.1.12 and 7.1.14, the Cuntz semigroup of a stable rank one or real rank zero C^*-algebra is ideal-filtered. Further, we also prove that separable C^*-algebras with zero-dimensional primitive ideal space also have ideal-filtered Cuntz semigroups; see Theorem 7.1.17. We also note that not all Cuntz semigroups satisfy this property; see Example 7.1.5.

Definition 7.1.1. A Cu-semigroup S is said to be *ideal-filtered* if, whenever

$$v' \ll v \ll \infty x, \infty y \quad \text{in } S,$$

there exists $z \in S$ such that

$$v' \ll \infty z \quad \text{and} \quad z \ll x, y.$$

The following notation will be used throughout the chapter:

Notation 7.1.2. Given elements v, x in a Cu-semigroup S, we write $v \lll\!\!\lhd\, x$ if there exists $x' \in S$ satisfying $x' \ll x$ and $v \leq \infty x'$.

With this notation, S is ideal-filtered if and only if, whenever $v' \ll v \lll\!\!\lhd\, x, y$, there exists $z \in S$ with $v' \lll\!\!\lhd\, z \ll x, y$.

Remark 7.1.3. Given a Cu-semigroup S and elements $v, x, y \in S$, note that

(i) $v \lll\!\!\lhd\, x$ and $x \lll\!\!\lhd\, y$ implies $v \lll\!\!\lhd\, y$.

(ii) if $v \ll x$, then $v \lll\!\!\lhd\, x$.

(iii) if $v \lll\!\!\lhd\, x$, there exists $x' \in S$ with $v \lll\!\!\lhd\, x' \ll x$.

Remark 7.1.4. Let v', v be elements in an ideal-filtered Cu-semigroup S such that $v' \ll v \leq \infty v'$. Then, the set $F := \{x \in S : v \leq \infty x\}$ is filtered. That is to say, for every pair of elements $z, t \in F$, one can find $w \in F$ satisfying $w \leq z, t$.

In particular, if S has a compact full element (for example, if S is an ideal-filtered Cuntz semigroup of a unital C^*-algebra), the full elements in S form a filtered set.

Example 7.1.5. Let A be the commutative C^*-algebra $C(S^2)$, where S^2 denotes the two-sphere. Then, $\mathrm{Cu}(A)$ is not ideal-filtered.

Indeed, we know by [74, Theorem 1.2] (or [91, Example 6.7]) that the submonoid of compact elements $\mathrm{Cu}(A)_c$ is isomorphic to

$$V(A) \cong \big\{(0,0)\big\} \cup \big\{(n,m) : m > 0\big\} \subseteq \mathbb{Z} \times \mathbb{Z}.$$

Thus, through this identification, the elements $x = (0,1)$ and $y = (1,1)$ are both elements in $\mathrm{Cu}(A)_c$, full in $\mathrm{Cu}(A)$, that have no full element below them. It follows from Remark 7.1.4 above that $\mathrm{Cu}(A)$ is not ideal-filtered.

Lemma 7.1.6. *Let S be an ideal-filtered* Cu*-semigroup, and let $x', x, y \in S$ and $n \geq 1$ be such that $x' \ll x \ll 2^n y$. Then, there exists $z \in S$ such that*

$$x' \lll\!\!\lhd\, z \ll x, y.$$

Proof. If $n = 1$, take $w \in S$ such that $x' \ll w \ll x$. Thus, we have

$$x' \ll w \lll\!\!\lhd\, x, y$$

and, since S is ideal-filtered, we find $z \in S$ satisfying

$$x' \lll z \ll x, y,$$

as required.

We now prove the result by induction on n. Thus, fix $n \geq 1$ and assume that the result holds for n. Then, given $x', x, y \in S$ satisfying $x' \ll x \ll 2^{n+1}y$, one gets

$$x' \ll x \ll 2^{n+1}y = 2(2^n y).$$

Using the case $n = 1$, we find $z_0 \in S$ with

$$x' \lll z_0 \ll x, 2^n y.$$

Choose $z_0' \ll z_0$ such that $x' \lll z_0'$; see Remark 7.1.3. Then, applying the induction assumption to $z_0' \ll z_0 \ll 2^n y$, there exists $z \in S$ such that

$$z_0' \lll z \ll z_0, y$$

The element z satisfies the required properties by construction. □

Proposition 7.1.7. *A* Cu*-semigroup S is ideal-filtered if and only if it satisfies the following two conditions:*

(i) Whenever $v' \ll v \lll x$ in S, there exists $z \in S$ such that

$$v' \lll z \lll v \quad \textit{and} \quad z \ll x;$$

(ii) Whenever $x' \ll x \ll 2y$, there exists $z \in S$ such that

$$x' \lll z \ll x, y.$$

Proof. Assume first that S is ideal-filtered. If $v', v, x \in S$ are such that $v' \ll v \lll x$, we can find $w \in S$ with $v' \ll w \ll v$. In particular, one gets

$$v' \ll w \lll x, v.$$

Since S is ideal-filtered, there exists an element z in S such that

$$v' \lll z \ll x, v,$$

which shows that S satisfies (i).

Condition (ii) follows from Lemma 7.1.6.

Conversely, assume now that S satisfies (i) and (ii), and let $v', v, x, y \in S$ be such that $v' \ll v \lll x, y$. Using (i), one gets an element $w \in S$ such that

$$v' \lll w \lll v \quad \text{and} \quad w \ll x.$$

Thus, by Remark 7.1.3 there exist $w'', w' \in S$ satisfying $v' \lll w'' \ll w' \ll w$.

Since $w' \ll w \lll v \lll y$, we have $w' \ll \infty y$ and, consequently, there exists $n \geq 1$ such that $w' \ll 2^n y$.

Now note that, in the proof of Lemma 7.1.6, we have only used that S satisfies condition (ii). Thus, one gets $z \in S$ with $w'' \lll z \ll w', y$, which shows that

$$v' \lll w'' \lll z, \quad z \ll w' \ll w \ll x \quad \text{and} \quad z \ll y,$$

as desired. □

We now prove that, if S is the Cuntz semigroup of a C^*-algebra, condition (i) from Proposition 7.1.7 above is always satisfied; see Corollary 7.1.10.

Lemma 7.1.8. *Let v', v, x be elements in a* Cu*-semigroup S that satisfies (O6) and (O7). Assume that $v' \ll v \leq \infty x$. Then, there exists $z \in S$ such that*

$$v' \lll z \lll v \quad \textit{and} \quad z \ll x.$$

Proof. Let $v'' \in S$ be such that $v' \ll v'' \ll v$, and take $n \geq 1$ such that $v'' \ll nx$. Since S satisfies (O6), there exist $z_1, \ldots, z_n \in S$ satisfying

$$v' \ll z_1 + \ldots + z_n, \quad z_1 \ll v'', x, \quad \ldots, \quad z_n \ll v'', x.$$

For each $j \leq n$, let $z_j' \in S$ be such that $z_j' \ll z_j$ and $v' \ll z_1' + \ldots + z_n'$. Using that $z_j' \ll z_j \ll x$ for every j, and that S satisfies (O7), one gets $z \in S$ with

$$z_1', \ldots, z_n' \ll z \ll x, z_1 + \ldots + z_n.$$

In particular, we have

$$v' \ll z_1' + \ldots + z_n' \leq \infty z, \quad z \ll z_1 + \ldots + z_n \leq \infty v'' \leq \infty v \quad \text{and} \quad z \ll x.$$

Using that $v' \ll \infty z$, we can find $z' \in S$ such that $z' \ll z$ and $v' \leq \infty z'$. This implies, by definition, that $v' \lll z$, as desired. □

Proposition 7.1.9. *Let S be a* Cu*-semigroup satisfying (O6) and (O7). Then, S is ideal-filtered if and only if, whenever $x' \ll x \ll 2y$ in S, there exists $z \in S$ such that*

$$x' \lll z \ll x, y.$$

Proof. Combine Proposition 7.1.7 and Lemma 7.1.8. □

Corollary 7.1.10. *Let A be a C^*-algebra. Then,* $\mathrm{Cu}(A)$ *is ideal-filtered if and only if, whenever $x' \ll x \ll 2y$ in* $\mathrm{Cu}(A)$*, there exists $z \in \mathrm{Cu}(A)$ such that*

$$x' \lll z \ll x, y.$$

Lemma 7.1.11. *Let S be a* Cu*-semigroup satisfying (O6) and Riesz interpolation. Then, S is ideal-filtered.*

Proof. Note that, since S has Riesz interpolation, it satisfies (O7). Thus, in order to prove the result, we only need to verify the condition in Proposition 7.1.9.

That is to say, given $x', x, y \in S$ with $x' \ll x \ll 2y$, we have to find $z \in S$ such that $x' \lll z \ll x, y$.

Applying (O6) to $x' \ll x \ll 2y$, we get $z_1, z_2 \in S$ satisfying

$$x' \ll z_1 + z_2, \quad z_1 \ll x, y \quad \text{and} \quad z_2 \ll x, y.$$

Now take $\tilde{x} \ll x$ and $\tilde{y} \ll y$ with $z_1 \ll \tilde{x}, \tilde{y}$ and $z_2 \ll \tilde{x}, \tilde{y}$. Since S has Riesz interpolation, we find $z \in S$ such that

$$z_1, z_2 \leq z \leq \tilde{x}, \tilde{y}$$

and, consequently, one gets

$$x' \ll z_1 + z_2 \leq \infty z \quad \text{and} \quad z \ll x, y,$$

as required. □

Theorem 7.1.12. *Let A be a C^*-algebra with stable rank one. Then,* $\mathrm{Cu}(A)$ *is ideal-filtered.*

Proof. Recall from Paragraph 1.2.14 that the Cuntz semigroup of every C^*-algebra satisfies (O6). Moreover, since A is of stable rank one, $\mathrm{Cu}(A)$ has Riesz interpolation; see [4, Theorem 3.5] or Corollary 6.4.7.

The result now follows from Lemma 7.1.11. □

Lemma 7.1.13. *Let S be a zero-dimensional* Cu*-semigroup satisfying (O7). Then, S is ideal-filtered.*

Proof. Since S is zero-dimensional, we have that S satisfies (O6); see Paragraph 4.3.1. Thus, as in Lemma 7.1.11 above, we only need to verify the condition from Proposition 7.1.9.

Now let x', x, y in S be such that $x' \ll x \ll 2y$. Since $\dim(S) = 0$, we find $z_1, z_2 \ll y$ satisfying

$$x' \ll z_1 + z_2 \ll x.$$

Take $z_1' \ll z_1$ and $z_2' \ll z_2$ in such a way that $x' \ll z_1' + z_2'$. Applying (O7) to $z_j' \ll z_j$ for $j = 1, 2$, one obtains $z \in S$ with

$$z_1', z_2' \ll z \ll y, z_1 + z_2$$

and, therefore,

$$x' \ll z_1' + z_2' \leq \infty z, \quad z \ll z_1 + z_2 \ll x \quad \text{and} \quad z \ll y,$$

as required. □

Theorem 7.1.14. *Let A be a real rank zero C^*-algebra. Then,* $\mathrm{Cu}(A)$ *is ideal-filtered.*

Proof. We know that the Cuntz semigroup of every C^*-algebra satisfies (O7); see Paragraph 1.2.14. Additionally, it follows from Theorem 4.3.8 that $\mathrm{Cu}(A)$ is zero-dimensional.

Applying Lemma 7.1.13, we see that $\mathrm{Cu}(A)$ is ideal-filtered. □

Let S be a Cu-semigroup. Given $x \in S$, we denote by $\langle x \rangle$ the ideal generated by x. That is to say,

$$\langle x \rangle := \{ y \in S : y \leq \infty x \}.$$

We let $\mathrm{Lat}_f(S)$ be the set of singly generated ideals in S, which is a Cu-semigroup with the expected order and addition. It follows from [6, Proposition 5.1.7] that one has $\langle x \rangle \ll \langle y \rangle$ in $\mathrm{Lat}_f(S)$ if and only if there is some element $y' \in S$ with $x \leq \infty y'$ and $y' \ll y$.

Lemma 7.1.15. *Let S be a* Cu*-semigroup satisfying (O6) and (O7). Then, the following are equivalent:*

(1) whenever $x' \ll x$ in S, there exist elements $y', y \in S$ with $x' \ll y' \ll y \ll x$ and $y \ll \infty y'$;

(2) $\mathrm{Lat}_f(S)$ *is algebraic.*

Proof. Assume first that (1) is satisfied. Then, given $I' \ll I$ in $\mathrm{Lat}_f(S)$, we can find $x' \ll x$ in S such that

$$I' \leq \langle x' \rangle \ll \langle x \rangle = I.$$

Applying our assumption at $x' \ll x$, we find elements y', y in S such that $x' \ll y' \ll y \ll x$ and $y \ll \infty y'$. Then, since $y \ll \infty y'$, we obtain $\langle y \rangle \ll \langle y \rangle$. Thus, we have

$$I' \ll \langle y \rangle \ll \langle y \rangle \ll I,$$

as required.

Conversely, assume that $\mathrm{Lat}_f(S)$ is algebraic, and let $x \in S$. Take $x' \in S$ such that $x' \ll x$ and let $x'' \in S$ be such that $x' \ll x'' \ll x$.

Since $\langle x'' \rangle \ll \langle x \rangle$ in $\mathrm{Lat}_f(S)$, we can find $z \in S$ such that $\langle x'' \rangle \ll \langle z \rangle \ll \langle z \rangle \ll \langle x \rangle$. That is to say, there exists $z' \in S$ satisfying

$$x'' \leq \infty z' \quad \text{and} \quad z' \ll z \leq \infty x, \infty z'.$$

Let $z'' \in S$ be such that $z' \ll z'' \ll z$. Then, one gets $n \in \mathbb{N}$ with $z' \ll z'' \leq nx$. Using (O6), we find elements $e_1, \ldots, e_n$ satisfying

$$z' \ll e_1 + \ldots + e_n \quad \text{and} \quad e_j \ll z'', x \text{ for } j = 1, \ldots, n.$$

For each $j \leq n$, let $e_j' \ll e_j$ be such that $z' \ll e_1' + \ldots + e_n'$. Applying (O7) to $x' \ll x'' \ll x$ and $e_j' \ll e_j \ll x$ for $j = 1, \ldots, n$, one finds $y \in S$ with

$$x', e_1', \ldots, e_n' \ll y \ll x, e_1 + \ldots + e_n + x''.$$

Let $y' \in S$ be such that $y' \ll y$ and $x', e_1', \ldots, e_n' \ll y'$. Then, one has

$$x' \ll y' \ll y \ll x$$

and, using that $z'' \ll z \leq \infty z'$ at the third step,

$$y \ll e_1 + \ldots + e_n + x'' \leq \infty z'' \leq \infty z' \leq \infty(e_1' + \ldots + e_n') \leq \infty y',$$

as desired. □

Lemma 7.1.16. *Let S be a* Cu*-semigroup satisfying (O5)-(O7) such that* $\mathrm{Lat}_f(S)$ *is algebraic. Then, S is ideal-filtered.*

Proof. We will show that the condition in Proposition 7.1.9 is satisfied. Thus, let $x', x, y \in S$ be such that $x' \ll x \ll 2y$.

By (O6), there exist elements $e, f \in S$ satisfying

$$x' \ll e + f, \quad e \leq x, y \quad \text{and} \quad f \leq x, y.$$

Thus, we can find $f' \ll f$ and $e' \ll e$ such that $x' \ll e' + f'$. Since $\mathrm{Lat}_f(S)$ is algebraic, Lemma 7.1.15 allows us to choose $e \ll \infty e'$.

Now let $e'' \in S$ be such that $e' \ll e'' \ll e$. Applying (O5) to $e'' \ll e \leq x$ to get an element $c \in S$ with

$$e'' + c \leq x \leq e + c.$$

In particular, one gets $f' \ll f \leq x \leq e+c$ and, since S satisfies (O6), we find $d \in S$ with

$$f' \ll e+d, \quad \text{and} \quad d \leq f, c.$$

Let $d' \ll d$ be such that $f' \ll e+d'$. Then, we have

$$e' \ll e'' \leq e \leq y \quad \text{and} \quad d' \ll d \leq f \leq y.$$

Applying (O7), we find an element $z \in S$ satisfying

$$e', d' \ll z \quad \text{and} \quad z \ll y, e''+d.$$

Since $d \leq c$, we have $z \leq e''+c \leq x$. Further, using that $e \leq \infty e'$, we also have

$$x' \ll e'+f' \leq e'+e+d' \leq \infty(e'+d') \leq \infty z,$$

as required. □

As defined in [16, Remark 2.5(vi)], we say that a C^*-algebra A has *topological dimension zero* if its primitive ideal space $\mathrm{Prim}(A)$ has a basis consisting of compact-open subsets.

Theorem 7.1.17. *Let A be a separable C^*-algebra with topological dimension zero. Then, $\mathrm{Cu}(A)$ is ideal-filtered.*

Proof. Since A is separable and has topological dimension zero, it follows from [66, Corollary 4.3] that $A \otimes \mathcal{O}_2$ has real rank zero. By [26], $\mathrm{Cu}(A \otimes \mathcal{O}_2)$ is algebraic. Thus, using [6, Corollary 7.2.15] at the first step and [6, Proposition 5.1.7] at the second step, we get that

$$\mathrm{Cu}(A \otimes \mathcal{O}_2) \cong \mathrm{Cu}(A) \otimes \{0, \infty\} \cong \mathrm{Lat}_f(\mathrm{Cu}(A))$$

is algebraic.

The result now follows from Lemma 7.1.16. □

Generally, if A is a stable C^*-algebra, ideal-filteredness can be characterized as follows:

Proposition 7.1.18. *Let A be a stable C^*-algebra. Then, $\mathrm{Cu}(A)$ is ideal-filtered if and only if, for every $a \in A_+$, every $b \in (\overline{aAa} \otimes M_2)_+$, and every $\varepsilon > 0$, there exists $c \in \overline{aAa}$ such that*

$$c \precsim b \quad \text{and} \quad (b-\varepsilon)_+ \in \overline{AcA}.$$

Proof. Assume first that $\mathrm{Cu}(A)$ is ideal-filtered, and let a, b, ε as in the statement. Then, using that $b \in \overline{aAa} \otimes M_2$ in the second step, one has

$$[(b-\varepsilon)_+] \ll [(b-\varepsilon/2)_+] \ll 2[a].$$

Since $\mathrm{Cu}(A)$ is ideal-filtered, there exists $d \in A_+$ satisfying

$$[(b-\varepsilon)_+] \lll [d] \ll [(b-\varepsilon/2)_+], [a].$$

Let $\delta > 0$ such that $[(b-\varepsilon)_+] \leq \infty[(d-\delta)_+]$. Since $c \precsim a$, we obtain r such that $(d-\delta)_+ = rr^*$ and $r^*r \in \overline{aAa}$; see Lemma 1.2.2. It follows that $c := r^*r$ has the required properties.

To prove the backward implication, recall from Proposition 7.1.9 that $\mathrm{Cu}(A)$ is ideal-filtered if and only if, whenever $x' \ll x \ll 2y$, there exists an element $z \in \mathrm{Cu}(A)$ with $x' \lll z \leq x, y$.

Thus, take $x', x, y \in \mathrm{Cu}(A)$ such that $x' \ll x \ll 2y$, and let $a, b \in A_+$ with $x = [b]$ and $y = [a]$. Since $x' \ll [b]$, we can find $\varepsilon > 0$ satisfying $x' \ll [(b-2\varepsilon)_+]$.

Now identify a with $a \oplus 0$ in $A \otimes M_2$, and consider the strictly positive element $a \otimes 1$ in $\overline{aAa} \otimes M_2$. Using that $x \ll 2y$, one has $b \precsim a \otimes 1$. It follows from Lemma 1.2.2 that there exists $r \in A \otimes M_2$ satisfying

$$(b-\varepsilon)_+ = rr^* \quad \text{and} \quad r^*r \in \overline{aAa} \otimes M_2.$$

Using our assumption, we find $c \in (\overline{aAa})_+$ with $c \precsim r^*r$ and such that $(r^*r-\varepsilon)_+$ belongs to $\overline{AcA}$. Note that one has

$$[c] \leq [a] = y, \quad [c] \leq [r^*r] = [rr^*] \leq [b] = x$$

and

$$x' \ll [(b-2\varepsilon)_+] = [(rr^*-\varepsilon)_+] = [(r^*r-\varepsilon)_+] \leq \infty[c],$$

as desired. □

Problem 7.1.19. When is the Cuntz semigroup of a C^*-algebra ideal-filtered?

7.2 Three shades of softness

We refine the notion of soft element from [6, Definition 5.3.1] (see Paragraph 4.1.17) to obtain three diferent kinds of softness, termed strong softness, weak softness and functional softness; see Definition 7.2.1. Functional softness agrees with the definition from [6, Definition 5.3.1], and stronlgy soft elements were implicitly considered in [6, Section 5.3].

As shown in Proposition 7.2.3, every strongly soft element is weakly soft, and every weakly soft element is functionally soft. In the residually stably finite case, a functionally soft element is strongly soft, and so all three notions agree.

Definition 7.2.1. An element x in a Cu-semigroup S will be said to be *strongly soft* if, whenever $x' \ll x$ in S, there exists $t \in S$ such that

$$x' + t \ll x \quad \text{and} \quad x' \ll \infty t.$$

We will say that x is *weakly soft* if, whenever $x' \ll x$ in S, there exists $n \geq 1$ and elements $t_1, \ldots, t_n \in S$ such that

$$x' + t_j \ll x \quad \text{for each } j \quad \text{and} \quad x' \ll t_1 + \ldots + t_n.$$

The element will be called *functionally soft* if, whenever $x' \ll x$ in S, there exists $n \geq 1$ such that

$$(n+1)x' \ll nx.$$

The Cu-semigroup S will be said to be *strongly* (resp. *functionally*, *weakly*) *soft* if every element in S is strongly (resp. functionally, weakly) soft.

Lemma 7.2.2. *Let x be an element in a* Cu*-semigroup S. Then, the following are equivalent:*

(1) x is strongly soft;

(2) whenever $x' \ll x$ in S, there exists $t \in S$ such that

$$x' + t \leq x \leq \infty t;$$

(3) whenever $x' \ll x$ in S, there exists $t \in S$ such that

$$x' + t \leq x \quad \text{and} \quad x' \leq \infty t.$$

Proof. Let us first assume that x is strongly soft, and choose $x' \in S$ such that $x' \ll x$. Let $(x_n)_n$ be a $\ll$-increasing sequence with supremum x and $x_0 = x'$. Then, one can inductively find elements y_n and t_n satisfying

$$y_n + t_n \ll x, \quad y_n + t_n \ll y_{n+1}, \quad y_n \ll \infty t_n \quad \text{and} \quad x_{n+1} \ll y_{n+1}$$

for every $n \geq 0$.

Indeed, set $y_0 := x'$. Since x is strongly soft, there exists an element $t_0 \in S$ such that $y_0 + t_0 \ll x$ and $y_0 \ll \infty t_0$.

Now fix $n \in \mathbb{N}$. If the elements y_k and t_k have been chosen for every $k \leq n$, we can use that both $y_n + t_n$ and x_{n+1} are compactly contained in x to find $y_{n+1} \ll x$ satisfying

$$y_n + t_n \ll y_{n+1} \quad \text{and} \quad x_{n+1} \ll y_{n+1}.$$

Using once again that x is strongly soft, we get $t_{n+1} \in S$ such that $y_{n+1} + t_{n+1} \ll x$ and $y_{n+1} \ll \infty t_{n+1}$, as desired.

Set $t := \sum_{k=0}^{\infty} t_k$. For each $n \geq 0$, we have $x_n \leq y_n \leq \infty t$, which implies $x \leq \infty t$. Moreover, for every $n \in \mathbb{N}$, we also get

$$x' + \sum_{k=0}^{n} t_k = y_0 + t_0 + t_1 + \ldots + t_n \leq y_1 + t_1 + \ldots + t_n \leq \ldots \leq y_{n+1} \leq x$$

and, consequently, $x' + t \leq x$. This proves that (1) implies (2).

That (2) implies (3) is clear, so it remains to show that (3) implies (1). Thus, let $x' \in S$ be such that $x' \ll x$, and take x'' satisfying $x' \ll x'' \ll x$. Applying (3) to $x'' \ll x$, one gets $t \in S$ with $x'' + t \leq x$ and $x'' \leq \infty t$.

Since $x' \ll x''$, there exists $t' \in S$ satisfying

$$x' \ll \infty t' \quad \text{and} \quad t' \ll t,$$

which implies $x' + t' \ll x$, as required. □

Recall the definition of a residually stably finite Cu-semigroup from Paragraph 4.4.1.

Proposition 7.2.3. *Let x be an element in a* Cu*-semigroup S, and consider the following properties:*

(1) x is strongly soft;

(2) x is weakly soft;

(3) x is functionally soft.

Then, (1) implies (2), and (2) implies (3). If S is residually stably finite and satisfies (O5), all properties are equivalent.

Proof. It is clear that (1) implies (2). If x is weakly soft and x' is such that $x' \ll x$, there exist $n \geq 1$ and elements $t_1, \ldots, t_n$ in S satisfying

$$x' + t_j \ll x \quad \text{for each } j \leq n, \quad \text{and} \quad x' \ll t_1 + \ldots + t_n.$$

Thus, we get

$$(n+1)x' \leq nx' + t_1 + \ldots + t_n \ll nx,$$

which implies that x is functionally soft. That is to say, (2) implies (3).

Next, assume that S is residually stably finite and satisfies (O5). It follows from [6, Lemma 5.3.8] that (3) implies (1) and, consequently, that all properties are equivalent. □

Corollary 7.2.4. *Let S be a weakly cancellative* Cu*-semigroup satisfying (O5). Then, an element $x \in S$ is strongly soft if and only if x is functionally soft.*

Proof. It is readily checked that every quotient of a weakly cancellative Cu-semigroup is again weakly cancellative. Since stably finiteness is weaker than weak cancellation, the result follows from Proposition 7.2.3. □

Corollary 7.2.5. *Let S be a* Cu*-semigroup, and consider the following properties:*

(1) S is strongly soft;

(2) S is weakly soft;

(3) S is functionally soft.

Then, (1) implies (2), and (2) implies (3). If S is residually stably finite and satisfies (O5), all properties are equivalent.

Example 7.2.6. A Cu-semigroup S is said to be *idempotent* if every element $x \in S$ satisfies $2x = x$. By [6, Section 7.2], a C^*-algebra A is purely infinite if and only if $\mathrm{Cu}(A)$ is idempotent. It follows that the Cuntz semigroup of every purely infinite C^*-algebra is strongly soft.

Similarly, the Cuntz semigroup of every weakly purely infinite C^*-algebra (see Example 6.3.11) is functionally soft. Indeed, let S be the Cuntz semigroup of a weakly purely infinite C^*-algebra and let $n \in \mathbb{N}$ be as in Example 6.3.11. Then, given any pair $x', x \in S$ with $x' \ll x$, we have $nx' \ll nx$ and, since $2nx' = nx'$, the result follows.

It is not known if every weakly purely infinite C^*-algebra is purely infinite; see [54, Question 9.5].

Proposition 7.2.7. *Let x be compact element in a* Cu*-semigroup S. Then, $x = 2x$ if and only if x is strongly soft, if and only if x is weakly soft.*

Proof. If $x = 2x$, then it follows directly from Definition 7.2.1 that x is strongly soft, and this implies that x is weakly soft.

Conversely, if x is weakly soft, we can apply Definition 7.2.1 to $x \ll x$ to get $n \geq 1$ and elements $t_1, \ldots, t_n \in S$ satisfying

$$x + t_1 \ll x, \quad \ldots, \quad x + t_n \ll x \quad \text{and} \quad x \ll t_1 + \ldots + t_n.$$

Thus, we have $x + t_j = x$ for each j. Therefore, one gets

$$x \leq 2x \leq x + t_1 + t_2 + \ldots + t_n = x + t_2 + \ldots + t_n = \ldots = x,$$

as desired. □

Example 7.2.8. Given $k \geq 1$, recall the definition of the Cu-semigroup E_k from Examples 1.2.8 (ii). The element $x := 1$ is compact in E_k and satisfies $(k + 2)x = (k + 1)x$, but not $2x = x$. Thus, x is an example of a functionally soft element that is not weakly soft. In particular, x is not strongly soft either.

Given a Cu-semigroup S, we will denote by S_{soft} the subset of strongly soft elements in S. Note that, in the previous chapters, this notation has already been used to denote the subset of (functionally) soft elements of a weakly cancellative Cu-semigroups. By Corollary 7.2.4, the new and old notation agree.

Theorem 7.2.9. *Let S be a* Cu*-semigroup. Then,*

(i) S_{soft} is a submonoid of S closed under suprema of increasing sequences.

(ii) S_{soft} is absorbing. That is to say, whenever $x \in S$ and $y \in S_{\mathrm{soft}}$ satisfy $x \leq \infty y$, we have $x + y \in S_{\mathrm{soft}}$.

Proof. Let us first show that S_{soft} is closed under addition. Thus, take $x, y \in S_{\mathrm{soft}}$ and let $w \in S$ be such that $w \ll x + y$.

Choose $x' \ll x$ and $y' \ll y$ satisfying $w \ll x' + y'$. Using that x and y are strongly soft, we get elements r, s in S with

$$x' + r \ll x, \quad x' \ll \infty r, \quad y' + s \ll y \quad \text{and} \quad y' \ll \infty s,$$

which implies

$$w + (r + s) \leq x' + y' + r + s \ll x + y \quad \text{and} \quad w \leq x' + y' \ll \infty(r + s).$$

Consequently, $x + y \in S_{\mathrm{soft}}$ and, since 0 is strongly soft, it follows that S_{soft} is a submonoid.

Now let $(x_n)_n$ be an increasing sequence in S_{soft}, and set $x := \sup_n x_n$ in S. To see that $x \in S_{\mathrm{soft}}$, take $x' \in S$ with $x' \ll x$.

Since x is the supremum of $(x_n)_n$, there exists $n \in \mathbb{N}$ satisfying $x' \ll x_n$. Using that x_n is strongly soft, we obtain $r \in S$ with

$$x' + r \ll x_n \quad \text{and} \quad x' \ll \infty r.$$

Thus, we get $x' + r \ll x$ and $x' \ll \infty r$, as desired.

To prove (ii), take $x \in S$ and $y \in S_{\rm soft}$ such that $x \leq \infty y$. Let $w \in S$ satisfy $w \ll x + y$, and choose $y' \ll y$ with $w \ll x + y'$. Since y is strongly soft, we can apply Lemma 7.2.2 to find an element $r \in S$ such that

$$y' + r \leq y \leq \infty r.$$

Using that $x \leq \infty y$ at the third step, one has

$$w + r \leq x + y' + r \leq x + y \leq \infty y \leq \infty r,$$

which shows that condition (2) of Lemma 7.2.2 is verified for $x + y$, as required. □

7.3 Divisibility vs weak divisibility

Recall from Paragraph 6.3.6 that a Cu-semigroup S is said to be weakly $(2,\omega)$-divisible if, whenever $x' \ll x$, there exist $y_1, \ldots, y_n \in S$ such that

$$2y_1, \ldots, 2y_n \leq x \quad \text{and} \quad x' \leq y_1 + \ldots + y_n.$$

Further, one says that S is $(2,\omega)$-divisible if, whenever $x' \ll x$, there exists $y \in S$ with $2y \leq x$ and $x' \leq \infty y$. That is to say, if one can always set $n = 1$ in the previous definition.

In this section, we study when a weakly $(2,\omega)$-divisible Cu-semigroup is $(2,\omega)$-divisible . First, we characterize $(2,\omega)$-divisibility in the presence of ideal-filteredness by the existence of strongly soft elements; see Theorem 7.3.5. We also show in Proposition 7.3.8 that, in an ideal-filtered and weakly $(2,\omega)$-divisible Cu-semigroup S, the number n of elements $y_1, \ldots, y_n$ in the definition of weak $(2,\omega)$-divisibility can always be bounded by 2. If, additionally, one assumes S to be residually stably finite, one can set $n = 1$; see Proposition 7.3.9. In particular, it follows that a residually stably finite C^*-algebra A has a $(2,\omega)$-divisible Cuntz semigroup if and only if $\mathrm{Cu}(A)$ is ideal-filtered and weakly $(2,\omega)$-divisible.

Lemma 7.3.1. *Let S be a $(2,\omega)$-divisible* Cu*-semigroup satisfying (O6) and (O7). Then, S is ideal-filtered.*

Proof. Since S satisfies (O6) and (O7), it is enough to verify the condition in Proposition 7.1.9. Thus, let x', x, y be elements in S such that

$$x' \ll x \ll 2y.$$

Applying that S is $(2,\omega)$-divisible, there exists $d \in S$ satisfying $2d \ll x$ and $x' \ll \infty d$. Thus, we have $x' \lll d$ and, using Remark 7.1.3, we can take $d' \in S$ such that $x' \lll d' \ll d$.

Then, applying (O6) to $d' \ll d \ll 2y$, we find elements $z_1, z_2 \in S$ such that

$$d' \ll z_1 + z_2 \quad \text{and} \quad z_1, z_2 \ll d, y.$$

Choose $z_1' \ll z_1$ and $z_2' \ll z_2$ with $d' \ll z_1' + z_2'$. Since S satisfies (O7) and we have that $z_1' \ll z_1 \ll y$ and $z_2' \ll z_2 \ll y$, there exists $z \in S$ with

$$z_1', z_2' \ll z \ll y, z_1 + z_2,$$

which implies

$$x' \lll d' \ll z_1' + z_2' \lll z, \quad z \ll z_1 + z_2 \leq d + d \ll x \quad \text{and} \quad z \ll y.$$

This shows that z is such that $x' \lll z \ll x, y$, as required. □

Proposition 7.3.2. *Let S be a $(2, \omega)$-divisible* Cu*-semigroup satisfying (O5). Then, for every element $x \in S$ and $k \geq 2$ there exists a strongly soft element $y \in S$ satisfying $ky \leq x \leq \infty y$.*

Proof. First, note that it is enough to verify the result for $k = 2$. We will make use of the following claim:

Claim. *Let $v', v \in S$ be such that $v' \ll v$. Then, there exist $y, w \in S$ satisfying*

$$2y + w \leq v \leq \infty w, \quad v' \leq \infty y \quad \textit{and} \quad y \ll v.$$

To prove the Claim, recall from Lemma 6.3.7 that S is $(3, \omega)$-divisible. Thus, we can find $z \in S$ with $3z \leq v$ and $v' \ll \infty z$.

Since $v' \ll \infty z$, there exist $z', z'' \in S$ such that $z' \ll z'' \ll z$ and $v' \leq \infty z'$. In particular, we get

$$(2z'') + z \leq v, \quad 2z' \ll 2z'' \quad \text{and} \quad z'' \ll z.$$

Using (O5), we find an element $w \in S$ such that $z'' \ll w$ and

$$2z' + w \leq v \leq 2z'' + w.$$

Set $y := z'$, and note that $v' \leq \infty z' = \infty y$. Moreover, we also have

$$2y + w = 2z' + w \leq v \leq 2z'' + w \leq 3w \leq \infty w \quad \text{and} \quad y = z' \ll z \leq v,$$

and this establishes the claim.

Now let $x \in S$, and let $(x_n)_n$ be a $\ll$-increasing sequence with supremum x. Then, one can inductively find elements $w_n \in S$ for $n \in \mathbb{N}$ and $y_n \in S$ for $n \geq 1$ satisfying

$$2y_{n+1} + w_{n+1} \leq w_n, \quad x \leq \infty w_{n+1}, \quad x_{n+1}, y_n \leq \infty y_{n+1} \quad \text{and} \quad y_{n+1} \ll x$$

for all $n \geq 0$.

Indeed, set $w_0 := x$ and $w_0' := x_1$. Applying the claim to $w_0' \ll w_0$, we get $y_1, w_1 \in S$ such that

$$2y_1 + w_1 \leq w_0 \leq \infty w_1, \quad w_0' \leq \infty y_1 \quad \text{and} \quad y_1 \ll w_0,$$

which implies

$$x = w_0 \leq \infty w_1, \quad x_1 = w_0' \leq \infty y_1 \quad \text{and} \quad y_1 \ll w_0 = x.$$

Thus, assume that the elements y_k and w_k have been found for all $k \leq n$ for some fixed $n \in \mathbb{N}$. Since both x_{n+1} and y_n are compactly contained in ∞w_n, there exists $w_n' \ll w_n$ such that

$$x_{n+1}, y_n \leq \infty w_n'.$$

Applying the claim once again to $w_n' \ll w_n$, we get elements $y_{n+1}, w_{n+1} \in S$ satisfying

$$2y_{n+1} + w_{n+1} \leq w_n \leq \infty w_{n+1}, \quad w_n' \leq \infty y_{n+1} \quad \text{and} \quad y_{n+1} \ll w_n,$$

which, as before, implies

$$\begin{aligned} x \leq \infty w_n \leq \infty w_{n+1}, \quad x_{n+1}, y_n \leq \infty w_n' \leq \infty y_{n+1} \quad \text{and} \\ y_{n+1} \ll w_n \leq w_{n-1} \leq \ldots \leq w_0 = x, \end{aligned}$$

as required.

Set $y := \sum_{k=1}^{\infty} y_k$. For every $n \geq 1$, one has

$$2(y_1 + \ldots + y_n) \leq 2(y_1 + \ldots + y_{n-1}) + w_{n-1} \leq \ldots \leq 2y_1 + w_1 \leq w_0 = x,$$

and, consequently, $2y \leq x$.

Further, since $x_n \leq \infty y_n \leq \infty y$, one also gets $x \leq \infty y$.

Finally, to verify that y is strongly soft, let us verify condition (3) of Lemma 7.2.2. Take $y' \in S$ such that $y' \ll y$. Then, there exists $n \in \mathbb{N}$ such that $y' \leq \sum_{k=1}^{n} y_k$. Since $y_k \leq \infty y_{k+1}$ for every k, we get

$$\sum_{k=1}^{n} y_k \leq \infty y_2 + \sum_{k=2}^{n} y_k \leq \infty y_3 + \sum_{k=3}^{n} y_k \leq \ldots \leq \infty y_n \leq \infty y_{n+1}.$$

This implies

$$y' + y_{n+1} \leq \sum_{k=1}^{n} y_k + y_{n+1} \leq y \quad \text{and} \quad y' \leq \sum_{k=1}^{n} y_k \leq \infty y_{n+1},$$

as required. $\square$

Lemma 7.3.3. *Let S be an ideal-filtered* Cu*-semigroup, and let x be an element in S. If x is strongly soft, then x is $(2, \omega)$-divisible.*

Proof. Assume that x is strongly soft, and take $x' \ll x$. Let $v', v \in S$ be such that $x' \ll v' \ll v \ll x$. Then, since x is strongly soft, one can find $t \in S$ satisfying

$$v + t \ll x \quad \text{and} \quad v \ll \infty t.$$

In particular, we have $x' \ll v' \lll v, t$. Using that S is ideal-filtered, there exists an element $z \in S$ with $x' \lll z \ll v, t$.

This implies that

$$2z \ll v + t \ll x \quad \text{and} \quad x' \leq \infty z,$$

as required. $\square$

Lemma 7.3.4. *Let x be a weakly soft element in a* Cu*-semigroup S satisfying (O6). Then, x is weakly $(2,\omega)$-divisible.*

Proof. Let $x' \in S$ be such that $x' \ll x$, and let $x'' \in S$ satisfy $x' \ll x'' \ll x$. Using that x is weakly soft, we find $n \geq 1$ and elements $t_1, \ldots, t_n$ in S with

$$x'' + t_1 \ll x, \quad \ldots, \quad x'' + t_n \ll x \quad \text{and} \quad x'' \ll t_1 + \ldots + t_n.$$

Applying (O6) to $x' \ll x'' \ll t_1 + \ldots + t_n$, we get elements $z_1, \ldots, z_n \in S$ satisfying

$$x' \ll z_1 + \ldots + z_n, \quad z_1 \ll x'', t_1, \quad \ldots \quad \text{and} \quad z_n \ll x'', t_n.$$

Thus, one has $x' \ll z_1 + \ldots + z_n$ and

$$2z_j \ll x'' + t_j \ll x$$

for every $j = 1, \ldots, n$, as desired. □

Theorem 7.3.5. *Let S be a* Cu*-semigroup satisfying (O5), (O6) and (O7). Then, the following are equivalent:*

(1) S is $(2,\omega)$-divisible;

(2) S is ideal-filtered and, for every element x in S, there exists a strongly soft element $y \in S$ satisfying $2y \leq x \leq \infty y$;

(3) S is ideal-filtered and, whenever $x' \ll x$, there exists a strongly soft element $y \in S$ with $x' \lhd\!\!\lhd y \ll x$.

Proof. That (1) implies (2) follows as a combination of Lemma 7.3.1 and Proposition 7.3.2.

Assuming (2), let $x', x \in S$ be such that $x' \ll x$. Take $x'' \in S$ satisfying $x' \ll x'' \ll x$. Then, there exists a strongly soft element $y \in S$ with $2y \leq x'' \leq \infty y$. In particular, it follows that $y \leq x'' \ll x$ and $x' \lhd\!\!\lhd y$. This shows that (2) implies (3).

Finally, to see that (3) implies (1), take $x \in S$ and let $x' \in S$ be such that $x' \ll x$.

Then, we know by our assumption that there exists a strongly soft element $y \in S$ satisfying $x' \lhd\!\!\lhd y \ll x$. Further, we can find $y' \ll y$ with $x' \lhd\!\!\lhd y'$.

Since y is $(2,\omega)$-divisible by Lemma 7.3.3, we obtain $z \in S$ such that $2z \leq y$ and $y' \leq \infty z$. Consequently, we have $2z \ll x$ and $x' \leq \infty z$, as required. □

Corollary 7.3.6. *A strongly soft* Cu*-semigroup satisfying (O5), (O6) and (O7) is $(2,\omega)$-divisible if and only if it is ideal-filtered.*

Proof. Let S be a strongly soft Cu-semigroup that satisfies (O5), (O6) and (O7). If S is $(2,\omega)$-divisible, it follows from Lemma 7.3.1 that S is ideal-filtered.

Conversely, if S is ideal-filtered, we note that S satisfies (3) in Theorem 7.3.5 above. Thus, S is $(2,\omega)$-divisible. □

Lemma 7.3.7. *Let S be an ideal-filtered, weakly $(2,\omega)$-divisible* Cu*-semigroup satisfying (O5). Then, for every pair $x' \ll x$ in S, there exist elements $c, d_1, \ldots, d_n \in S$ such that*

$$c + d_1, \ldots, c + d_n \leq x, \quad x' \ll \infty c \quad \textit{and} \quad x' \ll d_1 + \ldots + d_n.$$

Proof. Let $x', x \in S$ be such that $x' \ll x$. Using that S is weakly $(2,\omega)$-divisible, there exist elements $y_1, \ldots, y_n \in S$ satisfying

$$2y_1, \ldots, 2y_n \ll x \quad \text{and} \quad x' \ll y_1 + \ldots + y_n.$$

For every $j \leq n$, let $y_j'', y_j' \in S$ be such that $y_j'' \ll y_j' \ll y_j$ and $x' \ll y_1'' + \ldots + y_n''$. Applying (O5) to

$$y_j' + y_j \leq x, \quad y_j'' \ll y_j' \quad \text{and} \quad y_j' \ll y_j,$$

we find $c_j \in S$ such that

$$y_j'' + c_j \leq x \leq y_j' + c_j \quad \text{and} \quad y_j' \ll c_j.$$

In particular, one gets $x' \ll x \leq 2c_j \leq \infty c_j$ for each j. Thus, given $x'' \in S$ with $x' \ll x'' \ll x$, we have

$$x' \ll x'' \ll \infty c_1, \ldots, \infty c_n$$

and, applying that S is ideal-filtered, there exists $c \in S$ satisfying

$$x' \ll \infty c \quad \text{and} \quad c \ll c_1, \ldots, c_n.$$

Thus, the elements c and $d_j := y_j''$ have the desired properties. □

Proposition 7.3.8. *Let S be an ideal-filtered, weakly $(2,\omega)$-divisible* Cu*-semigroup satisfying (O5)-(O8). Then, for every pair $x' \ll x$ in S, there exist elements $c, d_1, d_2 \in S$ such that*

$$c + d_1, c + d_2 \leq x, \quad x' \ll \infty c \quad \textit{and} \quad x' \ll \infty(d_1 + d_2).$$

Further, one can find $e_1, e_2 \in S$ satisfying

$$2e_1, 2e_2 \leq x \quad \textit{and} \quad x' \ll \infty(e_1 + e_2).$$

Proof. Let $x', x \in S$ be such that $x' \ll x$. Also, let $x'' \in S$ satisfy $x' \ll x'' \ll x$. Then, using Lemma 7.3.7 above, there exist elements $y, z_1, \ldots, z_n \in S$ such that

$$y + z_1, \ldots, y + z_n \leq x, \quad x'' \ll \infty y \quad \text{and} \quad x'' \ll z_1 + \ldots + z_n.$$

Let $y' \in S$ satisfy $y' \ll y$ and $x'' \ll \infty y'$. Then, applying (O5) to $y' \ll y \leq x$, we find an element $f \in S$ satisfying

$$y' + f \leq x \leq y + f.$$

In particular, we have $z_j + y \leq x \leq y + f$ for each j. Thus, we get

$$\begin{aligned}2x'' &\ll 2(z_1 + \ldots + z_n) \leq 2(z_1 + \ldots + z_n) + y \leq 2(z_1 + \ldots + z_{n-1}) + z_n + y + f \\ &\leq 2(z_1 + \ldots + z_{n-1}) + y + 2f \leq \ldots \leq y + 2nf \leq y + \infty f.\end{aligned}$$

Since S satisfies (O8), we can apply Proposition 6.2.7 to $2x'' \ll y + \infty f$ and $x' \ll x''$. Thus, we find $g \in S$ satisfying

$$2g \ll y \quad \text{and} \quad x' \ll g + \infty f.$$

Let $g', g'' \in S$ be such that $g' \ll g'' \ll g$ and $x' \ll g' + \infty f$. Then, applying (O5) to $g'' + g \leq y$ with $g' \ll g''$ and $g'' \ll g$, there exists $h \in S$ with

$$g' + h \leq y \leq g'' + h \quad \text{and} \quad g'' \leq h.$$

It follows that $y \leq 2h$ and, consequently, that

$$x' \ll x'' \ll \infty y \leq \infty h \quad \text{and} \quad x' \ll x'' \ll \infty y'.$$

Since S is ideal-filtered, there exists an element $c \in S$ with

$$x' \ll \infty c \quad \text{and} \quad c \ll h, y'.$$

Now set $d_1 := f$ and $d_2 := g'$. Then, one has

$$c + d_1 = c + f \leq y' + f \leq x, \quad c + d_2 \leq h + g' \leq y \leq x,$$

and

$$x' \ll g' + \infty f \leq \infty(d_1 + d_2),$$

as required.

To prove the second part of the statement, and abusing notation, we let $x', x \in S$ be such that $x' \ll x$. Take $x'' \in S$ satisfying $x' \ll x'' \ll x$.

Applying the first part of the proof, we find elements $c, d_1, d_2 \in S$ such that

$$c + d_1, c + d_2 \leq x, \quad x'' \ll \infty c \quad \text{and} \quad x'' \ll \infty(d_1 + d_2).$$

Thus, applying (O6) to $x' \ll x'' \ll \infty d_1 + \infty d_2$, one obtains elements w_1, w_2 in such a way that

$$x' \ll w_1 + w_2, \quad w_1 \ll x'', \infty d_1 \quad \text{and} \quad w_2 \ll x'', \infty d_2.$$

Further, one can choose $w_1' \ll w_1$ and $w_2' \ll w_2$ with $x' \ll w_1' + w_2'$. Then, note that for each $j \leq 2$ we have $w_j' \ll w_j \leq \infty c, \infty d_j$.

Thus, it follows from ideal-filteredness that there exists elements $e_1, e_2 \in S$ with

$$w_1' \ll \infty e_1, \quad w_2' \ll \infty e_2, \quad e_1 \ll c, d_1 \quad \text{and} \quad e_2 \ll c, d_2.$$

It follows that

$$2e_1 \leq c + d_1 \leq x, \quad 2e_2 \leq c + d_2 \leq x,$$

and

$$x' \ll w_1' + w_2' \leq \infty(e_1 + e_2),$$

as required. □

Proposition 7.3.9. *Let S be an ideal-filtered, residually stably finite* Cu*-semigroup satisfying (O5). Then, S is $(2, \omega)$-divisible if and only if S is weakly $(2, \omega)$-divisible.*

Proof. We note that the forward implication is trivial.

Thus, assume that S is weakly $(2,\omega)$-divisible, let $x \in S$ and take $x' \ll x$. Also, let $x'' \in S$ be such that $x' \ll x'' \ll x$.

Since S is weakly $(2,\omega)$-divisible, we can apply Lemma 7.3.7 to find elements $c, d_1, \ldots, d_n \in S$ satisfying

$$c + d_1, \ldots, c + d_n \leq x, \quad x'' \ll \infty c \quad \text{and} \quad x'' \ll d_1 + \ldots + d_n.$$

Using that $x'' \ll \infty c$, one can choose elements c'', c' in S such that $c'' \ll c' \ll c$ and $x'' \ll \infty c''$.

Since S satisfies (O5) and $c'' \ll c' \leq x$, there exists $e \in S$ such that $c''+e \leq x \leq c'+e$. Thus, for each $j \in \{1, \ldots, n\}$, we get

$$c + d_j \leq x \leq c' + e$$

with $c' \ll c$.

Let $\pi \colon S \to S/\langle e \rangle$ be the quotient map. Then, one has

$$\pi(c) + \pi(d_j) \leq \pi(c') \ll \pi(c)$$

and, since $S/\langle e \rangle$ is stably finite, it follows that $\pi(d_j)$ is zero in $S/\langle e \rangle$ for every j. That is to say, for each $j \leq n$ we have $d_j \leq \infty e$ in S.

Using that $x'' \ll d_1 + \ldots + d_n$, we obtain $x'' \ll \infty e$ and, consequently, one gets

$$x' \ll \infty c'', \infty e.$$

Applying that S is ideal-filtered, we find $y \in S$ with $x' \ll \infty y$ and $y \ll c'', e$. Thus, we have

$$2y \leq c'' + e \leq x \quad \text{and} \quad x' \ll \infty y,$$

as required. □

Theorem 7.3.10. *Let S be a* Cu*-semigroup satisfying (O5)-(O7), and consider the following statements:*

(1) S is $(2,\omega)$-divisible;

(2) S is ideal-filtered and, for every element x in S, there exists a strongly soft element $y \in S$ satisfying $2y \leq x \leq \infty y$;

(3) S is ideal-filtered and, whenever $x' \ll x$, there exists a strongly soft element $y \in S$ with $x' \lll y \ll x$;

(4) S is ideal-filtered and, whenever $x' \ll x$, there exists a weakly soft element $y \in S$ with $x' \lll y \ll x$;

(5) S is ideal-filtered and weakly $(2,\omega)$-divisible.

Then, (1)-(3) are equivalent and imply (4), which in turn implies (5). If, additionally, S is residually stably finite, (5) implies (1) and all the statements are equivalent.

Proof. The first three statements are equivalent by Theorem 7.3.5, and that (3) implies (4) follows from Proposition 7.2.3.

Assuming (4), let $x \in S$ and take $x' \ll x$. Then, we can find a weakly soft element $y \in S$ with $x' \lll y \ll x$.

Let $y' \ll y$ be such that $x' <\!\lhd\, y'$. Since y is weakly soft and S satisfies (O6), Lemma 7.3.4 implies that y is weakly $(2, \omega)$-divisible. This implies that there exists some $z \in S$ with $y' \leq \infty z$ and $2z \ll y$. In particular, one has $x' \leq \infty z$ and $2z \ll x$, which shows that x is $(2, \omega)$-divisible.

Since this can be done for every $x \in S$, it follows that (4) implies (5).

Finally, if S is also residually stably finite, Proposition 7.3.9 shows that (5) implies (1), as desired. □

7.4 The Global Glimm Property

In this section we study the Global Glimm Property (Definition 7.4.1) and its relation to nowhere scatteredness: In order for a C^*-algebra A to satisfy the Global Glimm Property, it is necessary that A is nowhere scattered. Asking if this condition is also sufficient is known as the *Global Glimm Problem*.

Following the ideas in Section 6.3, we begin the section by characterizing the Global Glimm Property in terms of divisibility conditions in the Cuntz semigroup. With this characterization at hand, we are able to reformulate the Global Glimm Problem in a number of ways; see Theorem 7.4.6, Problem 7.4.7 and Theorem 7.4.8.

Using these different formulations, a positive answer to the Global Glimm Problem is given when the C^*-algebra under study is of stable rank one, real rank zero or residually stably finite and of topological dimension zero; see Theorem 7.4.11 (the first two cases are recovered from [4] and [34] respectively).

Finally, we study some permanence properties of the Global Glimm Property and generalize the results in Theorem 7.4.11.

Let us first indicate what we mean by the Global Glimm Property, which should not be confused with the 'Glimm Halving Property'; see Remark 7.4.2 below.

Definition 7.4.1 ([54, Definition 4.12])**.** A C^*-algebra A is said to have the *Global Glimm Property* if, for every $\varepsilon > 0$, every $a \in A_+$, and every natural number $n \geq 2$, there exists a *-homomorphism $\varphi\colon C_0((0,1], M_n) \to \overline{aAa}$ such that the ideal of A generated by the image of φ contains $(a-\varepsilon)_+$.

Remark 7.4.2. As defined in [75], a unital C^*-algebra A is said to have the 'Glimm Halving Property' if there exists a *-homomorphism $M_2(C_0((0,1])) \to A$ with full image. As testified by $A = M_2(\mathbb{C})$, this is not equivalent to the property defined in Definition 7.4.1 above.

We will see in Theorem 7.4.6 that a C^*-algebra A has the Global Glimm Property if and only if $\mathrm{Cu}(A)$ is $(2, \omega)$-divisible. Using Theorem 6.3.9, it follows that every C^*-algebra having the Global Glimm Property is nowhere scattered. The *Global Glimm Problem* asks if the converse holds:

Problem 7.4.3 (The Global Glimm Problem)**.** Let A be a nowhere scattered C^*-algebra. Does A satisfy the Global Glimm Property?

In order to prove Theorem 7.4.6, let us recall the notion of a *scale* in a Cu-semigroup:

As defined in [8, Definition 4.1], a subset Σ of a Cu-semigroup S is a scale if Σ is downward-hereditary, is closed under suprema of increasing sequences, and generates S as an ideal. For a C^*-algebra A, the subset

$$\Sigma_A := \big\{x \in \mathrm{Cu}(A) : \text{for each } x' \ll x \text{ there exists } a \in A_+ \text{ with } x' \leq [a]\big\}$$

is a scale of $\mathrm{Cu}(A)$; see [8, 4.2].

Lemma 7.4.4. *Let A be a C^*-algebra, and let $x \in \Sigma_A$ and $x' \in \mathrm{Cu}(A)$ be such that $x' \ll x$. Then, there exists $a \in A_+$ such that $x' \ll [a] \ll x$.*

Proof. Let $x'' \in \mathrm{Cu}(A)$ be such that $x' \ll x'' \ll x$, and take $c \in (A \otimes \mathcal{K})_+$ with $x'' = [c]$. Then, since $x \in \Sigma_A$, we can find $b \in A_+$ satisfying $x'' \leq [b]$.

Using that $x' \ll x'' = [c]$, we can find $\varepsilon > 0$ such that $x' \ll [(c-\varepsilon)_+]$. Moreover, we can apply Lemma 1.2.2 to $c \precsim b$ and ε in order to obtain an element $r \in A \otimes \mathcal{K}$ satisfying

$$(c-\varepsilon)_+ = r^*r \quad \text{and} \quad rr^* \in \overline{bAb}.$$

Setting $a := rr^* \in A_+$, we note that

$$x' \ll [(c-\varepsilon)_+] = [r^*r] = [rr^*] = [a] \quad \text{and} \quad [a] = [(c-\varepsilon)_+] \ll [c] = x'' \ll x,$$

as desired. □

Lemma 7.4.5. *Let Σ be a scale of a* Cu*-semigroup S satisfying (O5), (O6) and (O7). Assume that every element in Σ is $(2,\omega)$-divisible. Then, S is $(2,\omega)$-divisible.*

Proof. First note that, using an analoguous argument to that of Lemma 6.3.7, every element in Σ is (k,ω)-divisible for every $k \geq 2$.

Set

$$\Sigma^{(2)} := \big\{x \in S : \text{ for each } x' \ll x \text{ there are } y_1, y_2 \in \Sigma \text{ with } x' \leq y_1 + y_2\big\},$$

which is a scale of S containing Σ.

We claim that, to prove the result, we only need to show that every element in $\Sigma^{(2)}$ is $(2,\omega)$-divisible. Indeed, if $\Sigma^{(2)}$ is $(2,\omega)$-divisible, we can apply the same argument to $\Sigma^{(2)}$ in order to get that $(\Sigma^{(2)})^{(2)}$ is also $(2,\omega)$-divisible. Proceeding in this manner, and using that Σ generates S as an ideal, it follows that S is $(2,\omega)$-divisible.

Thus, let $x \in \Sigma^{(2)}$ and take $x', x'' \in S$ with $x' \ll x'' \ll x$. Also, let $y_1, y_2 \in \Sigma$ be such that $x' \ll x'' \leq y_1 + y_2$.

Since S satisfies (O6), there exist elements $z_1, z_2 \in S$ satisfying

$$x' \ll z_1 + z_2, \quad z_1 \leq x'', y_1 \quad \text{and} \quad z_2 \leq x'', y_2.$$

In particular, since Σ is downward-hereditary, we see that z_1, z_2 are elements in Σ. Thus, given $z_1' \ll z_1$ and $z_2' \ll z_2$ such that $x' \ll z_1' + z_2'$, we can use that every element in Σ is $(3,\omega)$-divisible to find elements $c, d \in S$ with

$$3c \leq z_1, \quad z_1' \ll \infty c, \quad 3d \leq z_2 \quad \text{and} \quad z_2' \ll \infty d.$$

This allows us to find $c', c'', d' \in S$ with $c' \ll c'' \ll c$, $d' \ll d$,

$$z_1' \ll \infty c' \quad \text{and} \quad z_2' \ll \infty d'.$$

Note that, since $z_1 \leq x$ and $3c \leq z_1$, we have $2c'' + c \leq 3c \leq x$. Applying (O5) to $(2c'') + c \leq x$ with $2c' \ll 2c''$ and $c'' \ll c$, there exists an element $e \in S$ such that

$$2c' + e \leq x \leq 2c'' + e \quad \text{and} \quad c'' \leq e,$$

which shows $x \leq 3e$.

Applying (O6) to $d' \ll d \leq z_2 \leq x \leq 3e$, we find elements $e_1, e_2, e_3 \in S$ satisfying

$$d' \ll e_1 + e_2 + e_3 \quad \text{and} \quad e_1, e_2, e_3 \leq d, e.$$

For each $j \leq 3$, let $e_j' \ll e_j$ be such that $d' \ll e_1' + e_2' + e_3'$. Then, applying (O7) to $e_j' \ll e_j \leq e$, we find $f \in S$ with

$$e_1', e_2', e_3' \leq f \leq e, e_1 + e_2 + e_3.$$

One gets

$$z_2' \ll \infty d' \leq \infty(e_1' + e_2' + e_3') \leq \infty f \quad \text{and} \quad f \leq e_1 + e_2 + e_3 \leq 3d \leq z_2,$$

which implies, in particular, that $f \in \Sigma$. Consequently, f is $(2, \omega)$-divisible.

Thus, let $f' \ll f$ be such that $z_2' \ll \infty f'$. Then, there exists an element $g \in S$ with $2g \leq f$ and $f' \ll \infty g$.

Set $h := c' + g$. Then, one has

$$2h = 2c' + 2g \leq 2c' + e \leq x$$

and, since $z_2' \ll \infty f' \leq \infty g$, we also get

$$x' \ll z_1' + z_2' \leq \infty c' + \infty g = \infty h,$$

as required. □

Theorem 7.4.6. *Let A be a C^*-algebra. Then, the following are equivalent:*

(1) A has the Global Glimm Property;

(2) for each $a \in A_+$ and each $\varepsilon > 0$, there exists a *-homomorphism

$$\varphi\colon M_2(C_0((0,1])) \to \overline{aAa}$$

such that $(a-\varepsilon)_+$ belongs to the ideal generated by the image of φ;

(3) for each $a \in A_+$ and each $\varepsilon > 0$, there is an element $r \in \overline{aAa}$ with $r^2 = 0$ such that $(a-\varepsilon)_+ \in \overline{ArA}$;

(4) $\mathrm{Cu}(A)$ is $(2, \omega)$-divisible;

(5) $\mathrm{Cu}(A)$ is (n, ω)-divisible for every $n \geq 2$;

Proof. It is clear that (1) implies (2) and that (4) implies (5). Further, (5) implies (1) by [75, Theorem 5.3 (i)].

To see that (2) implies (3), take $a \in A_+$ and $\varepsilon > 0$. By (2), we can find a *-homomorphism $\varphi\colon M_2(C_0((0,1])) \to \overline{aAa}$ such that $(a-\varepsilon)_+$ belongs to the ideal generated by the image of φ.

Let $r = \varphi((\begin{smallmatrix} 0 & t \\ 0 & 0 \end{smallmatrix}))$, where t denotes the identity map on $C_0((0,1])$. Then, $r^2 = 0$ and $r \in \overline{aAa}$ by construction. Further, both $\varphi((\begin{smallmatrix} t & 0 \\ 0 & 0 \end{smallmatrix}))$ and $\varphi((\begin{smallmatrix} 0 & 0 \\ 0 & t \end{smallmatrix}))$ belong to the ideal generated by r, so it follows that $(a-\varepsilon)_+ \in \overline{ArA}$.

To prove (4) we will apply Lemma 7.4.5 to Σ_A, which is possible since $\mathrm{Cu}(A)$ satisfies (O5)-(O7); see Paragraph 1.2.14. Let $x \in \Sigma_A$ and take $x' \in \mathrm{Cu}(A)$ such that $x' \ll x$. Then, it follows from Lemma 7.4.4 that there is $a \in A_+$ with $x' \ll [a] \ll x$.

Since $x' \ll [a]$, we can find $\varepsilon > 0$ such that $x' \ll [(a-\varepsilon)_+]$. Thus, ussing (3), there exists $r \in \overline{aAa}$ such that $r^2 = 0$ and $(a-\varepsilon)_+ \in \overline{ArA}$.

In particular, using that $r \in \overline{aAa}$ and that $r^2 = 0$, we see that both rr^* and r^*r are orthogonal elements in $\overline{aAa}$. Let $c = [rr^*]$. Then,

$$2c = 2[rr^*] = [rr^* + r^*r] \leq [a] \ll x.$$

Further, since $(a-\varepsilon)_+ \in \overline{ArA}$, it follows that $x' \ll [(a-\epsilon)_+] \leq \infty c$, as desired. □

By Theorem 7.4.6 above, a C^*-algebra A satisfies the Global Glimm Property if and only if $\mathrm{Cu}(A)$ is $(2,\omega)$-divisible. In light of Theorem 6.3.9, this result leads to the following reformulation of the Global Glimm Problem:

Problem 7.4.7 (The Global Glimm Problem)**.** Is a weakly $(2,\omega)$-divisible Cuntz semigroup always $(2,\omega)$-divisible?

Using the results developed thus far, we can reformulate Problem 7.4.7 once again in the residually stably finite case.

Theorem 7.4.8. *Let A be a residually stably finite C^*-algebra. Then, the following statements are equivalent:*

(1) A has the Global Glimm Property;

(2) $\mathrm{Cu}(A)$ is weakly $(2,\omega)$-divisible and ideal-filtered;

(3) A is nowhere scattered and $\mathrm{Cu}(A)$ is ideal-filtered.

Proof. First, assume that A satisfies the Global Glimm Property. Then, it follows from Theorem 7.4.6 that $\mathrm{Cu}(A)$ is $(2,\omega)$-divisible and, consequently, weakly $(2,\omega)$-divisible. Further, Theorem 7.3.10 implies that $\mathrm{Cu}(A)$ is ideal-filtered. This shows that (1) implies (2).

That (2) and (3) are equivalent is Theorem 6.3.9.

Finally, if (3) is satisfied, it follows from Theorem 6.3.9 and Theorem 7.3.10 that $\mathrm{Cu}(A)$ is $(2,\omega)$-divisible. Using Theorem 7.4.6 once again, we deduce that A has the Global Glimm Property. □

As witnessed by Theorem 7.4.8 above, and using Theorem 7.4.6, one can restate the Global Glimm Problem (in the residually stably finite case) as: ‘Does every residually stably finite, nowhere scattered C^*-algebra have an ideal-filtered Cuntz semigroup?’.

Indeed, if every residually stably finite, nowhere scattered C^*-algebra has the Global Glimm Property, it follows from Theorem 7.3.5 that all such C^*-algebras have an ideal-filtered Cuntz semigroup. Conversely, if every residually stably finite, nowhere scattered C^*-algebra has an ideal-filtered Cuntz semigroup, Theorem 7.4.8 above shows that all such C^*-algebras have the Global Glimm Property.

Problem 7.4.9. Is the Cuntz semigroup of every nowhere scattered C^*-algebra ideal-filtered?

Using the results in [75], one can show that a C^*-algebra is nowhere scattered if and only if, for every $a \in A_+$ and each $\varepsilon > 0$, there exist finitely many *-homomorphisms

$$\varphi_1, \ldots, \varphi_n \colon C_0((0,1], M_2) \to \overline{aAa}$$

such that $(a-\varepsilon)_+$ belongs to the ideal of A generated by the union of the images of $\varphi_1, \ldots, \varphi_n$.

As a consequence of Theorem 7.4.6, the Global Glimm Problem is equivalent to asking if one can always set $n = 1$ in the above characterization of nowhere scatteredness.

Proposition 7.4.10. *Let A be a nowhere scattered C^*-algebra with an ideal-filtered Cuntz semigroup. Then, for every $a \in A_+$ and each $\varepsilon > 0$, there exist *-homomorphisms $\varphi_1 \colon C_0((0,1], M_2) \to \overline{aAa}$ and $\varphi_2 \colon C_0((0,1], M_2) \to \overline{aAa}$ such that $(a-\varepsilon)_+$ belongs to the ideal of A generated by the combined images of φ_1 and φ_2.*

Proof. It follows from Theorem 6.3.9 that $\mathrm{Cu}(A)$ is a weakly $(2,\omega)$-divisible Cu-semigroup. Further, $\mathrm{Cu}(A)$ satisfies (O5)-(O8) by Paragraph 1.2.14 and Theorem 6.2.3. By assumption, $\mathrm{Cu}(A)$ is also ideal-filtered.

Let $a \in A_+$ and $\varepsilon > 0$. Then, we have $[(a-\varepsilon/2)_+] \ll [a]$ in $\mathrm{Cu}(A)$. Set $x' = [(a-\varepsilon/2)_+]$ and $x = [a]$. Using the second part of Proposition 7.3.8, there exist $e_1, e_2 \in \mathrm{Cu}(A)$ such that

$$2e_1, 2e_2 \leq x \quad \text{and} \quad x' \ll \infty(e_1 + e_2).$$

Let $e_1' \ll e_1$ be such that $x' \ll \infty(e_1' + e_2)$. Then, it follows from [75, Lemma 2.4] applied to $2e_1 \leq x$ that there exist two orthogonal, equivalent, positive elements $b_1, b_2 \in \overline{a(A \otimes \mathcal{K})a} = \overline{aAa}$ such that $e_1' \ll [b_1] = [b_2] \ll e_1$. Thus, there exists a *-homomorphism $\varphi_1 \colon C_0((0,1], M_2) \to \overline{aAa}$ with $\varphi_1(e_{1,1} \otimes \iota) = b_1$ and $\varphi_2(e_{2,2} \otimes \iota) = b_2$.

Now take $e_2' \ll e_2$ satisfying $x' \ll \infty(e_1' + e_2')$. Arguing as above, we find a *-homomorphism $\varphi_2 \colon C_0((0,1], M_2) \to \overline{aAa}$ such that $\varphi_2(e_{1,1} \otimes \iota) = c_1$ and $\varphi_2(e_{2,2} \otimes \iota) = c_2$ with c_1, c_2 orthogonal satisfying $e_2' \ll [c_1] = [c_2] \ll e_2$.

Thus, one gets $k \in \mathbb{N}$ such that

$$[(a-\varepsilon/2)_+] = x' \leq k[b_1] + k[c_1].$$

Since $(a-\varepsilon/2)_+, b_1, c_1 \in A$, it follows from [75, Lemma 2.3] that $(a-\varepsilon)_+$ belongs in the ideal generated by the images of φ_1 and φ_2, as desired. $\square$

Using Theorem 7.4.8, we recover in Theorem 7.4.11 (i)-(ii) below the positive solutions to the Global Glimm Problem from [4] and [34].

Theorem 7.4.11. *Let A be a nowhere scattered C^*-algebra. Assume that A is either:*

(i) of stable rank one;

(ii) of real rank zero;

(iii) separable, residually stably finite, and with topological dimension zero.

Then, A satisfies the Global Glimm Property.

Proof. First note that, if A is of stable rank one, it is residually stably finite. Thus, assuming (i) or (iii), it follows from Theorem 7.1.12 or 7.1.17 together with Theorem 7.4.8 that A has the Global Glimm Property.

If A is of real rank zero, we know from Theorem 6.4.1 that $\mathrm{Cu}(A)$ is $(2,\omega)$-divisible. Thus, Theorem 7.4.6 shows that A satisfies the Global Glimm Property. $\square$

To finish this section, and as in Section 6.3, let us now prove permanence properties for $(2,\omega)$-divisibility of Cu-semigroups. Using the results in Theorem 7.4.6, these lead to permanence properties for the Global Glimm Property. In particular, we show that a nowhere scattered C^*-algebra with generalized stable rank one or generalized real rank zero has the Global Glimm Property; see Corollary 7.4.18.

Proposition 7.4.12. *The Global Glimm Property is invariant under Morita equivalence.*

Proof. Morita equivalent C^*-algebras have isomorphic Cuntz semigroups. Thus, the result follows from Theorem 7.4.6. $\square$

Arguing as in Proposition 6.3.12 and Proposition 6.3.13, one also has the following results:

Proposition 7.4.13. *Let A be C^*-algebra, and let $(A_\lambda)_{\lambda\in\Lambda}$ be a family of sub-C^*-algebras that approximates A. Assume that each A_λ has the Global Glimm Property. Then, A has the Global Glimm Property.*

Corollary 7.4.14. *The Global Glimm Property passes to inductive limits.*

Proposition 7.4.15. *Let I be an ideal of a* Cu*-semigroup S satisfying (O5)-(O8). Then, S is $(2,\omega)$-divisible if and only if I and S/I are $(2,\omega)$-divisible.*

Proof. Assume first that S is $(2,\omega)$-divisible, and let $\pi\colon S\to S/I$ denote the quotient map. Let $y\in S$ and take $x\in S$ such that $\pi(x)\ll\pi(y)$. Since π is a Cu-morphism, we find $y'\ll y$ with $\pi(y')\ll\pi(y)$.

Using that y is $(2,\omega)$-divisible in S, one gets an element $z\in S$ such that $2z\leq y$ and $y'\ll\infty z$. This implies that $2\pi(z)\leq\pi(y)$ and $\pi(x)\ll\infty\pi(z)$. Consequently, S/I is $(2,\omega)$-divisible.

Clearly, we also have that I is $(2,\omega)$-divisible.

Assume now that I and S/I are both $(2,\omega)$-divisible, and take $x',x\in S$ such that $x'\ll x$. Let $x''\in S$ satisfy $x'\ll x''\ll x$. Then, $\pi(x'')\ll\pi(x)$.

Since S/I is $(3,\omega)$-divisible (see Lemma 6.3.7), there exists $y \in S$ such that

$$3\pi(y) \leq \pi(x) \quad \text{and} \quad \pi(x'') \ll \infty\pi(y).$$

Using once again that π is a Cu-morphism, there exist $y', y'' \in S$ with $y' \ll y'' \ll y$ and $\pi(x'') \leq \infty\pi(y')$. Moreover, since $3\pi(y) \leq \pi(x)$ and $\pi(x'') \leq \infty\pi(y')$, there exist $w_1, w_2 \in I$ such that

$$3y \leq x + w_1 \quad \text{and} \quad x'' \leq \infty y' + w_2.$$

Setting $w := \infty(w_1 + w_2)$, we get

$$2w = w, \quad 3y \leq x + w \quad \text{and} \quad x'' \leq \infty y' + w.$$

Thus, applying Proposition 6.2.7 to $3y'' \ll x + w$ and $y' \ll y''$, we find $z \in S$ such that

$$3z \ll x \quad \text{and} \quad y' \ll z + w.$$

Take $z', z'' \in S$ with $z' \ll z'' \ll z$ and $y' \ll z' + w$. Then, one gets

$$2z'' + z \leq x, \quad 2z' \ll 2z'' \quad \text{and} \quad z'' \ll z.$$

Thus, using (O5), one can find $c \in S$ with

$$2z' + c \leq x \leq 2z'' + c \quad \text{and} \quad z'' \ll c$$

and, consequently, $x \leq 2z'' + c \leq 3c$. Further, since $y' \leq z' + w$, we also have

$$x'' \leq \infty y' + w \leq \infty z' + w$$

Assume first that S is countably based. Then, we know from [3, Theorems 2.4, 2.5] that for every $s \in S$ and $u = 2u \in S$, the infimum $s \wedge u$ exists. Moreover, the map $s \mapsto s \wedge u$ preserves addition, order and suprema of increasing sequences.

Thus, using at the first step that $x'' \leq \infty z' + w$ and $x' \leq \infty c$, and at the last step that $z' \leq c$, one gets

$$x'' \leq (\infty z' + w) \wedge \infty c = (\infty z' \wedge \infty c) + (w \wedge \infty c) = \infty z' + \infty(w \wedge c).$$

Since $x' \ll x''$, we can find $d' \ll w \wedge c$ satisfying $x' \leq \infty z' + \infty d'$. Thus, since $w \wedge c$ is an element in I and I is $(2,\omega)$-divisible, there exists $e \in I$ with

$$2e \leq w \wedge c \quad \text{and} \quad d' \leq \infty e.$$

Note that the element $z' + e$ satisfies

$$2(z' + e) \leq 2z' + (w \wedge c) \leq 2z' + c \leq x$$

and

$$x' \leq \infty z' + \infty d' \leq \infty z' + \infty e = \infty(z' + e),$$

as desired.

In the general case, assume that I is an ideal of S such that I and S/I are both $(2,\omega)$-divisible, and let $x' \ll x$ in S.

Recall from Proposition 5.3.4 that (O5)-(O7) each satisfy the Löwenheim-Skolem condition. Similarly, one can see that (O8) and $(2,\omega)$-divisibility also satisfy this condition. Thus, there exists a countably based sub-Cu-semigroup H of S satisfying (O5)-(O8) such that $x, x' \in H$ and such that $H \cap I$ and $H/(H \cap I)$ are $(2,\omega)$-divisible.

Applying our result to H, we obtain $r \in H$ such that $2r \leq x$ and $x' \ll \infty r$. Since H is a sub-Cu-semigroup of S, the element r satisfies the required properties in S. □

Theorem 7.4.16. *Let A be a C^*-algebra and let I be an ideal of A. Then, A has the Global Glimm Property if and only if I and A/I do.*

Further, the Global Glimm Property passes to hereditary sub-C^*-algebras.

Proof. Let I be an ideal of a C^*-algebra A. Then, as explained in Paragraph 1.2.12, $\mathrm{Cu}(I)$ is naturally identified with an ideal in $\mathrm{Cu}(A)$, and $\mathrm{Cu}(A/I)$ is isomorphic to $\mathrm{Cu}(A)/\mathrm{Cu}(I)$.

Thus, the first part of the statement follows as a combination of Theorem 7.4.6 and Proposition 7.4.15.

Now let B be a hereditary sub-C^*-algebra of a C^*-algebra A satisfying the Global Glimm Property. Then, the Cuntz semigroup of B is isomorphic to the Cuntz semigroup of the ideal generated by B. This implies, by Theorem 7.4.6 and the first part of the theorem, that B satisfies the Global Glimm Property. □

Theorem 7.4.17. *Let A be a C^*-algebra. Then, there exists a largest ideal $I_{\mathrm{Glimm}}(A)$ of A satisfying the Global Glimm Property. That is to say, whenever I is an ideal of A with the Global Glimm Property, one has $I \subseteq I_{\mathrm{Glimm}}(A)$.*

Proof. Denote by $\mathcal{J}$ the family of ideals of A satisfying the Global Glimm Property. Note that, given two ideals $I, J \in \mathcal{J}$, it follows from Theorem 7.4.16 that $I + J$ is also in $\mathcal{J}$. Indeed, I is an ideal of $I + J$, and $(I + J)/I$ is isomorphic to a quotient of J.

Thus, we see that $\mathcal{J}$ is upward directed with respect to the order induced by inclusion. We set $I_{\mathrm{Glimm}}(A) := \overline{\bigcup \mathcal{J}}$, which is an ideal of A that satisfies the Global Glimm Property by Corollary 7.4.14. □

As defined in [16, 2.1(iv)], a C^*-algebra is of *generalized stable rank one* (resp. *generalized real rank zero*) if it admits a composition series $(I_\lambda)_{\lambda \leq \kappa}$ of ideals with $A = I_\kappa$ such that each quotient $I_{\lambda+1}/I_\lambda$ has stable rank one (resp. real rank zero).

Corollary 7.4.18. *Let A be a nowhere scattered C^*-algebra. Assume that A has generalized stable rank one or generalized real rank zero. Then, A has the Global Glimm Property.*

Proof. If A has either generalized stable rank one or generalized real rank zero, the result follows as a combination of Theorem 7.4.11, Corollary 7.4.14 and Theorem 7.4.16. □

Bibliography

[1] C. Akemann and N. Weaver. Consistency of a counterexample to Naimark's problem. *Proc. Natl. Acad. Sci. USA*, 101:7522–7525, 2004.

[2] R. Antoine, M. Dadarlat, F. Perera, and L. Santiago. Recovering the Elliott invariant from the Cuntz semigroup. *Trans. Amer. Math. Soc.*, 366(6):2907–2922, 2014.

[3] R. Antoine, F. Perera, L. Robert, and H. Thiel. Edwards' condition for quasitraces on C^*-algebras. *Proc. Roy. Soc. Edinburgh Sect. A*, 151(2):525–547, 2021.

[4] R. Antoine, F. Perera, L. Robert, and H. Thiel. C^*-algebras of stable rank one and their Cuntz semigroups. *Duke Math. J.*, 171(1):33–99, 2022.

[5] R. Antoine, F. Perera, and L. Santiago. Pullbacks, $C(X)$-algebras, and their Cuntz semigroup. *J. Funct. Anal.*, 260(10):2844–2880, 2011.

[6] R. Antoine, F. Perera, and H. Thiel. Tensor products and regularity properties of Cuntz semigroups. *Mem. Amer. Math. Soc.*, 251(1199):viii+191, 2018.

[7] R. Antoine, F. Perera, and H. Thiel. Abstract bivariant Cuntz semigroups. *Int. Math. Res. Not. IMRN*, (17):5342–5386, 2020.

[8] R. Antoine, F. Perera, and H. Thiel. Cuntz semigroups of ultraproduct C^*-algebras. *J. Lond. Math. Soc.*, 2(102):994–1029, 2020.

[9] G. Aranda Pino, K. R. Goodearl, F. Perera, and M. Siles Molina. Non-simple purely infinite rings. *Amer. J. Math.*, 132(3):563–610, 2010.

[10] M. Asadi-Vasfi, H. Thiel, and E. Vilalta. Dimension and radius of comparison of soft Cuntz semigroups. in preparation, 2022.

[11] B. Blackadar. *Operator algebras*, volume 122 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2006. Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.

[12] B. Blackadar, L. Robert, A. P. Tikuisis, A. S. Toms, and W. Winter. An algebraic approach to the radius of comparison. *Trans. Amer. Math. Soc.*, 364(7):3657–3674, 2012.

[13] E. Blanchard and E. Kirchberg. Non-simple purely infinite C^*-algebras: the Hausdorff case. *J. Funct. Anal.*, 207(2):461–513, 2004.

[14] J. Bosa, N. P. Brown, Y. Sato, A. Tikuisis, S. White, and W. Winter. Covering dimension of C^*-algebras and 2-coloured classification. *Mem. Amer. Math. Soc.*, 257(1233):vii+97, 2019.

[15] L. G. Brown and G. K. Pedersen. C^*-algebras of real rank zero. *J. Funct. Anal.*, 99(1):131–149, 1991.

[16] L. G. Brown and G. K. Pedersen. Limits and C^*-algebras of low rank or dimension. *J. Operator Theory*, 61(2):381–417, 2009.

[17] N. P. Brown and A. Ciuperca. Isomorphism of Hilbert modules over stably finite C^*-algebras. *J. Funct. Anal.*, 257(1):332–339, 2009.

[18] J. Castillejos, S. Evington, A. Tikuisis, S. White, and W. Winter. Nuclear dimension of simple C^*-algebras. *Invent. Math.*, 224:245–290, 2021.

[19] A. Chand and L. Robert. Simplicity, bounded normal generation, and automatic continuity of groups of unitaries. preprint (arXiv:2108.09796 [math.OA]), 2021.

[20] M. D. Choi and G. A. Elliott. Nuclear C^*-algebras and the approximation property. *Amer. J. Math.*, 100:61–79, 1978.

[21] A. Ciuperca and G. A. Elliott. A remark on invariants for C^*-algebras of stable rank one. *Int. Math. Res. Not. IMRN*, 2008, 2008.

[22] A. Ciuperca, G. A. Elliott, and L. Santiago. On inductive limits of type-I C^*-algebras with one-dimensional spectrum. *Int. Math. Res. Not. IMRN*, (11):2577–2615, 2011.

[23] A. Ciuperca, L. Robert, and L. Santiago. The Cuntz semigroup of ideals and quotients and a generalized Kasparov stabilization theorem. *J. Operator Theory*, 64(1):155–169, 2010.

[24] A. Connes. Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$. *Ann. of Math. (2)*, 104(1):73–115, 1976.

[25] A. Connes. *Noncommutative Geometry.* Studies in Advanced Mathematics. Academic Press Inc, 1994.

[26] K. T. Coward, G. A. Elliott, and C. Ivanescu. The Cuntz semigroup as an invariant for C^*-algebras. *J. Reine Angew. Math.*, 623:161–193, 2008.

[27] J. Cuntz. Dimension functions on simple C^*-algebras. *Math. Ann.*, 233(2):145–153, 1978.

[28] E. G. Effros, D. E. Handelman, and C. L. Shen. Dimension groups and their affine representations. *Amer. J. Math.*, 102(2):385–407, 1980.

[29] E. G. Effros and J. Kaminker. Homotopy continuity and shape theory for C^*-algebras. In *Geometric methods in operator algebras (Kyoto, 1983)*, volume 123 of *Pitman Res. Notes Math. Ser.*, pages 152–180. Longman Sci. Tech., Harlow, 1986.

[30] G. A. Elliott. On the classification of inductive limits of sequences of semisimple finite-dimensional algebras. *J. Algebra*, 38(1):29–44, 1976.

[31] G. A. Elliott. The classification problem for amenable C^*-algebras. In *Proceedings of the International Congress of Mathematicians (Zürich, 1994)*, volume 1, pages 922–932. Birkhäuser, Basel, 1995.

[32] G. A. Elliott, G. Gong, H. Lin, and Z. Niu. On the classification of simple amenable C^*-algebras with finite decomposition rank, II. preprint (arXiv:1507.03437 [math.OA]), 2015.

[33] G. A. Elliott, L. Robert, and L. Santiago. The cone of lower semicontinuous traces on a C^*-algebra. *Amer. J. Math.*, 133(4):969–1005, 2011.

[34] G. A. Elliott and M. Rørdam. Perturbation of Hausdorff moment sequences, and an application to the theory of C^*-algebras of real rank zero. In *Operator Algebras: The Abel Symposium 2004*, volume 1 of *Abel Symp.*, pages 97–115. Springer, Berlin, 2006.

[35] G. A. Elliott and A. S. Toms. Regularity properties in the classification program for separable amenable C^*-algebras. *Bull. Amer. Math. Soc. (N.S.)*, 45(2):229–245, 2008.

[36] M. Engbers. Decomposition of simple Cuntz semigroups. Ph.D Thesis, WWU Münster, 2014.

[37] I. Farah, B. Hart, M. Lupini, L. Robert, A. Tikuisis, A. Vignati, and W. Winter. Model theory of C^*-algebras. *Mem. Amer. Math. Soc.*, 271(1324):viii+127, 2021.

[38] H. Freudenthal. Entwicklungen von räumen und ihren gruppen. *Compositio Math.*, 4:145–234, 1937.

[39] I. Gelfand and M. Naimark. On the imbedding of normed rings into the ring of operators in Hilbert space. *Rec. Math. [Mat. Sbornik] N.S.*, 12(54):197–213, 1943.

[40] S. Ghasemi and P. Koszmider. Noncommutative Cantor-Bendixson derivatives and scattered C^*-algebras. *Topology Appl.*, 240:183–209, 2018.

[41] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *Continuous lattices and domains*, volume 93 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2003.

[42] J. G. Glimm. On a certain class of operator algebras. *Trans. Amer. Math. Soc.*, 95(2):318–340, 1960.

[43] G. Gong and H. Lin. On classification of non-unital amenable simple C^*-algebras, III, stably projectionless C^*-algebras. preprint (arXiv:2112.14003 [math.OA]), 2021.

[44] K. Goodearl. C^*-algebras of real rank zero whose K_0's are not riesz groups. *Canad. Math. Bull.*, 39:429–437, 1996.

[45] K. R. Goodearl. *Partially ordered abelian groups with interpolation*, volume 20 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1986.

[46] T. Huruya. A spectral characterization of a class of C^*-algebras. *Sci. Rep. Niigata Univ. Ser. A*, (15):21–24, 1978.

[47] B. Jacelon. A simple, monotracial, stably projectionless C^*-algebra. *J. Lond. Math. Soc. (2)*, 87(2):365–383, 2013.

[48] H. E. Jensen. Scattered C^*-algebras. *Math. Scand.*, 41(2):308–314, 1977.

[49] H. E. Jensen. Scattered C^*-algebras. II. *Math. Scand.*, 43(2):308–310 (1979), 1978.

[50] X. Jiang and H. Su. On a simple unital projectionless C^*-algebra. *Amer. J. Math.*, 121(2):359–413, 1999.

[51] D. Kerr. Dimension, comparison, and almost finiteness. *J. Eur. Math. Soc. (JEMS)*, 22:3697–3745, 2020.

[52] E. Kirchberg. C*-nuclearity implies cpap. *Math. Nachr.*, 76:203–212, 1977.

[53] E. Kirchberg and M. Rørdam. Non-simple purely infinite C^*-algebras. *Amer. J. Math.*, 122(3):637–666, 2000.

[54] E. Kirchberg and M. Rørdam. Infinite non-simple C^*-algebras: absorbing the Cuntz algebra $\mathcal{O}_\infty$. *Adv. Math.*, 167(2):195–264, 2002.

[55] E. Kirchberg and M. Rørdam. Central sequence C^*-algebras and tensorial absorption of the Jiang-Su algebra. *J. Reine Angew. Math.*, 695:175–214, 2014.

[56] E. Kirchberg and W. Winter. Covering dimension and quasidiagonality. *Internat. J. Math.*, 15(1):63–85, 2004.

[57] M. Kusuda. C^*-algebras in which every C^*-subalgebra is AF. *Q. J. Math.*, 63(3):675–680, 2012.

[58] E. Lindenstrauss. Mean dimension, small entropy factors and an embedding theorem. *Inst. Hautes Études Sci. Publ. Math.*, 89:227–262, 1999.

[59] J. Munkres. *Topology.* Pearson College Div; Second Edition, 2000.

[60] G. J. Murphy. *C^*-algebras and operator theory.* Academic Press, Inc., Boston, MA, 1990.

[61] F. J. Murray and J. Von Neumann. On rings of operators. *Ann. of Math. (2)*, 37(1):116–229, 1936.

[62] F. J. Murray and J. von Neumann. On rings of operators. II. *Trans. Amer. Math. Soc.*, 41(2):208–248, 1937.

[63] F. J. Murray and J. von Neumann. On rings of operators. IV. *Ann. of Math. (2)*, 44:716–808, 1943.

[64] S. Nadler. *Continuum theory: An introduction*, volume 158 of *Monographs and textbooks in pure and applied mathematics.* M. Dekker, 1992.

[65] N. Ozawa. Dixmier approximation and symmetric amenability for C^*-algebras. *J. Math. Sci. Univ. Tokyo*, 20(3):349–374, 2013.

[66] C. Pasnicu and M. Rørdam. Purely infinite C^*-algebras of real rank zero. *J. Reine Angew. Math.*, 613:51–73, 2007.

[67] A. R. Pears. *Dimension theory of general spaces.* Cambridge University Press, Cambridge, England-New York-Melbourne, 1975.

[68] G. K. Pedersen. Applications of weak* semicontinuity in C^*-algebra theory. *Duke Math. J.*, 39:431–450, 1972.

[69] F. Perera. The structure of positive elements for C^*-algebras with real rank zero. *Internat. J. Math.*, 8(3):383–405, 1997.

[70] F. Perera and A. S. Toms. Recasting the Elliott conjecture. *Math. Ann.*, 338(3):669–702, 2007.

[71] L. Robert. Nuclear dimension and n-comparison. *Münster J. Math.*, 4:65–71, 2011.

[72] L. Robert. Classification of inductive limits of 1-dimensional NCCW complexes. *Adv. Math.*, 231(5):2802–2836, 2012.

[73] L. Robert. The cone of functionals on the Cuntz semigroup. *Math. Scand.*, 113(2):161–186, 2013.

[74] L. Robert. The Cuntz semigroup of some spaces of dimension at most two. *C. R. Math. Acad. Sci. Soc. R. Can.*, 35(1):22–32, 2013.

[75] L. Robert and M. Rørdam. Divisibility properties for C^*-algebras. *Proc. Lond. Math. Soc. (3)*, 106(6):1330–1370, 2013.

[76] L. Robert and L. Santiago. Classification of C^*-homomorphisms from $C_0(0,1]$ to a C^*-algebra. *J. Funct. Anal.*, 258(3):869–892, 2010.

[77] M. Rørdam. On the structure of simple C^*-algebras tensored with a UHF-algebra. II. *J. Funct. Anal.*, 107(2):255–269, 1992.

[78] M. Rørdam. Classification of nuclear, simple C^*-algebras. In *Classification of nuclear C^*-algebras. Entropy in operator algebras*, volume 126 of *Encyclopaedia Math. Sci.*, pages 1–145. Springer, Berlin, 2002.

[79] M. Rørdam. A simple C^*-algebra with a finite and an infinite projection. *Acta Math.*, 191(1):109–142, 2003.

[80] M. Rørdam. The stable and the real rank of $\mathcal{Z}$-absorbing C^*-algebras. *Internat. J. Math.*, 15(10):1065–1084, 2004.

[81] M. Rørdam, F. Larsen, and N. Laustsen. *An introduction to K-theory for C^*-algebras*, volume 49 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2000.

[82] M. Rørdam and W. Winter. The Jiang-Su algebra revisited. *J. Reine Angew. Math.*, 642:129–155, 2010.

[83] S. Sakai. Derivations of simple C^*-algebras. III. *Tohoku Math. J. (2)*, 23:559–564, 1971.

[84] Y. Sato. Trace spaces of simple nuclear C^*-algebras with finite-dimensional extreme boundary. preprint (arXiv:1209.3000 [math.OA]), 2012.

[85] Y. Sato, S. White, and W. Winter. Nuclear dimension and $\mathcal{Z}$-stability. *Invent. Math.*, 202(2):893–921, 2015.

[86] C. Schons. Categorical aspects of Cuntz semigroups. Master thesis, 2018.

[87] C.-L. Shen. On the classification of the ordered groups associated with the approximately finite dimensional C^*-algebras. *Duke Math. J.*, 46(3):613–633, 1979.

[88] M. Takesaki. *Theory of operator algebras. I*, volume 124 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2002. Reprint of the first (1979) edition, Operator Algebras and Non-commutative Geometry, 5.

[89] H. Thiel. The topological dimension of type I C^*-algebras. In *Operator algebra and dynamics*, volume 58 of *Springer Proc. Math. Stat.*, pages 305–328. Springer, Heidelberg, 2013.

[90] H. Thiel. The generator rank of subhomogeneous C^*-algebras. preprint (arXiv:2006.03624 [math.OA]), 2020.

[91] H. Thiel. Ranks of operators in simple C^*-algebras with stable rank one. *Comm. Math. Phys.*, 377(1):37–76, 2020.

[92] H. Thiel and E. Vilalta. Covering dimension of Cuntz semigroups II. *Internat. J. Math.*, 32(14):27 p., Paper No. 2150100, 2021.

[93] H. Thiel and E. Vilalta. Nowhere scattered C^*-algebras. preprint (arXiv:2112.09877 [math.OA]), 2021.

[94] H. Thiel and E. Vilalta. Covering dimension of Cuntz semigroups. *Adv. Math.*, 394:44 p., Paper No. 108016, 2022.

[95] H. Thiel and E. Vilalta. The global Glimm property. preprint (arXiv:2204.13059 [math.OA]), 2022.

[96] H. Thiel and E. Vilalta. Zero dimensional Cuntz semigroups. in preparation, 2022.

[97] K. Thomsen. Homomorphisms between finite direct sums of circle algebras. *Linear Multilinear Algebra*, 32:33–50, 1992.

[98] A. Tikuisis, S. White, and W. Winter. Quasidiagonality of nuclear C^*-algebras. *Ann. of Math. (2)*, 185(1):229–284, 2017.

[99] A. S. Toms. On the classification problem for nuclear C^*-algebras. *Ann. of Math. (2)*, 167(3):1029–1044, 2008.

[100] A. S. Toms, S. White, and W. Winter. $\mathcal{Z}$-stability and finite-dimensional tracial boundaries. *Int. Math. Res. Not. IMRN*, (10):2702–2727, 2015.

[101] E. Vilalta. The Cuntz semigroup of unital commutative AI-algebras. preprint (arXiv:2104.08165 [math.OA]), 2021.

[102] E. Vilalta. A local characterization for the Cuntz semigroups of AI-algebras. *J. Math. Anal. Appl.*, 506(1):125612, 2022.

[103] F. Wehrung. Monoids of intervals of ordered abelian groups. *J. Algebra*, 182(1):287–328, 1996.

[104] W. Winter. Decomposition rank of subhomogeneous C^*-algebras. *Proc. London Math. Soc. (3)*, 89(2):427–456, 2004.

[105] W. Winter. Decomposition rank and $\mathcal{Z}$-stability. *Invent. Math.*, 179(2):229–301, 2010.

[106] W. Winter. Nuclear dimension and $\mathcal{Z}$-stability of pure C^*-algebras. *Invent. Math.*, 187(2):259–342, 2012.

[107] W. Winter. Structure of nuclear C^*-algebras: From quasidiagonality to classification, and back again. *Proc. ICM, Rio de Janeiro*, 2:1797–1820, 2018.

[108] W. Winter and J. Zacharias. Completely positive maps of order zero. *Münster J. Math.*, 2:311–324, 2009.

[109] W. Winter and J. Zacharias. The nuclear dimension of C^*-algebras. *Adv. Math.*, 224(2):461–498, 2010.

[110] S. Zhang. A property of purely infinite simple C^*-algebras. *Proc. Amer. Math. Soc.*, 109(3):717–720, 1990.

[111] S. Zhang. A Riesz decomposition property and ideal structure of multiplier algebras. *J. Operator Theory*, 24(2):209–225, 1990.