Directed hereditary species
and
Decomposition spaces of intervals

Wilson Forero

DIRECTED HEREDITARY SPECIES
and
DECOMPOSITION SPACES OF INTERVALS

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Abstract

In the present thesis, we study the theory of decomposition spaces, focusing on the interval construction for decomposition spaces and the decomposition space of subdivided intervals U, which was constructed by Gálvez, Kock, and Tonks [60] as a recipient of Lawvere's interval construction. Our interest in U is due to the Gálvez–Kock–Tonks conjecture, which states that U enjoys a certain universal property: for every complete decomposition space X, the space of culf functors from X to U is contractible.

The first main contribution, developed in collaboration with Alex Cebrian, is to introduce the concept of connected and non-connected directed hereditary species and show that they have associated monoidal decomposition spaces, comodule bialgebras, and operadic categories. The notion subsumes Schmitt's hereditary species, Gálvez–Kock–Tonks directed restrictions species, and a directed version of Carlier's construction of monoidal decomposition spaces and comodule bialgebras. In addition to all the examples of Schmitt, Gálvez–Kock–Tonks and Carlier, the new construction covers also the Fauvet–Foissy–Manchon comodule bialgebra of finite topological spaces, the Calaque–Ebrahimi-Fard–Manchon comodule bialgebra of rooted trees, and the Faà di Bruno comodule bialgebra of linear trees.

The second main contribution is to prove the Gálvez–Kock–Tonks conjecture. First, we proved the conjecture for the discrete case. However, as kindly pointed out by the anonymous referee of [47], the proof works the same for a broader class of decomposition spaces, namely those 1-truncated decomposition spaces with the property that all their intervals are discrete. This locally discrete case is general enough to cover all locally finite posets, Cartier–Foata monoids, Möbius categories, and strict (directed) restriction species. The proof is 2-categorical. First, we construct a local strict model of U, which is then used to show by hand that the Lawvere interval construction, considered as a natural transformation, does not admit other self-modifications than the identity.

It is natural to ask whether the techniques developed in the proof of the discrete case of the conjecture can be applied or refined to prove the conjecture in full generality. Unfortunately, this is not very likely, since the proof relies on explicit strictification. Therefore, we prefer to study the conjecture from another perspective by imposing cardinal bounds through the Möbius condition for decomposition spaces. This is a certain finiteness condition ensuring that the general Möbius inversion principle admits a homotopy cardinality. From this perspective proving the conjecture is equivalent to proving that the decomposition space of subdivided Möbius intervals U_{Mob} is a terminal object in the ∞-category of Möbius decomposition spaces and culf maps. The proof is given by combining $(\infty, 2)$-category theory of 2-colimits, the interval construction, and the straightening-unstraightening equivalence of ∞-categories. The Möbius case, together with the fact that the ∞-category of decomposition spaces and culf maps is locally an ∞-topos imply that the ∞-category of Möbius decomposition spaces and culf maps is an ∞-topos.

Keywords: decomposition spaces, topos, interval, directed hereditary species, incidence coalgebra.

MSC classes: 18N10, 18N50, 06A11, 16T15.

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This thesis brings together dozens of stories, innumerable words of encouragement, and countless moments in front of a blackboard or a computer. It is the culmination of a formative process that led me to grow as a human being and realise that I could never have done it alone. So I want to dedicate a few lines to mention the people and institutions that, with their support, company, and knowledge, made it possible for a child of limited resources in an underdeveloped country to become a doctor.

En esta tesis convergen decenas de historias, sin fin de palabras de aliento e innumerables momentos frente a una pizarra o un ordenador. Es la culminación de un proceso formativo que me llevó a crecer como ser humano y darme cuenta que jamás lo hubiera podido hacer solo. Así que dedicaré unas cuantas líneas para mencionar a personas e instituciones que con su apoyo, compañía y conocimientos hicieron posible que un niño de escasos recursos de un país subdesarrollado lograra convertirse en doctor.

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Contents

Introduction

One of the earliest forms of cognitive evolution of our species was the symbolic representation of our environment. This allowed the transmission of information from generation to generation more efficiently than that provided by memory and oral tradition [3].

Over the millennia, our mind evolved and, along with it, the symbols we used, to the point that we could give a more precise meaning to those first scribbles, and we created writing, opening a new era for humanity. When studying the first traces of writing, it is interesting that some of the oldest texts contain elementary counting techniques, from keeping the inventory of material possessions to state taxation [89]. These counting techniques were perfected, from counting animals to the combinations of syllables of the Greek alphabet, from finding out how many different combinations of flavors we can choose to the formulation of the binomial coefficients.

In recent decades, these techniques were framed in the enumerative combinatorics, which addresses the problem of how to count the number of elements of a finite set given by some combinatorial conditions [4]. One of the branches of enumerative combinatorics is algebraic enumeration which deals with exact results, either explicit formulas for the numbers in question, or more often, generating functions or recurrences from which the numbers can be computed [62].

However, it has been shown that working with objects rather than numbers provides more insight into the problems at hand. For example, with the theory of species [69], Joyal showed that many manipulations with generating functions could be carried out directly on the combinatorial structures themselves. If we now work with objects instead of numbers, we move to objective combinatorics, where we can obtain bijective proofs instead of algebraic proofs, leading to a deeper understanding.

A nice feature of the objective level is that it is possible to use techniques from category theory. (This theory is a mechanism of abstraction, transversal to all mathematics, in which we understand an object by how it relates to its environment [96].) We can use linear algebra with coefficients in the category of sets instead of rational numbers, linear maps are replaced by spans, and equalities are expressed by bijections [77]. But in recent years, homotopy theory has taught us, that one retains more information and that certain constructions become better behaved when sets are replaced by spaces and bijections by homotopy equivalences. This allowed the incorporation of coalgebras and related structures that came from combinatorics to the world of higher categories through the theory of decomposition spaces, formulated by Gálvez, Kock, and Tonks [58–60] to frame mathematical structures that canonically can be associated with an algebraic structure that has a strong connection with combinatorics. This theory will be the central focus of this thesis, so it is necessary to delve deeper into its history.

Background and motivation

Decomposition spaces [58–60] were created as a far-reaching generalisation of posets for the purpose of defining incidence algebras and Möbius inversion, covering the classical theory for posets by Rota [84], for monoids (Cartier–Foata [25]), Möbius categories (Leroux [32, 78]), as well as various constructions with operads ([28], [91], [92], [90]). Decomposition spaces are certain simplicial ∞-groupoids, and the theory becomes homotopical in nature. Independently Dyckerhoff and Kapranov [35] had arrived at the equivalent notion of 2-Segal spaces (see Feller et al. [41] for the last piece in the equivalence), from the viewpoint of representation theory, homological algebra, and K-theory, where Hall algebras and Waldhausen's S-construction are the main motivating examples.

It was suggested by Gálvez, Kock, and Tonks [56] that virtually all combinatorial co- bi- and Hopf algebra should be incidence algebras of decomposition spaces, whereas many are not incidence algebras of posets. This idea is difficult to state as a precise theorem, but several recent papers have contributed with classes of examples vindicating the principle.

A series of combinatorial co- bi- and Hopf algebras which are not directly incidence algebras of posets were given in the seminal paper of Schmitt [85], where he identified large significant families of combinatorial coalgebras coming from extra structures on combinatorial species in the sense of Joyal [69] (see Aguiar–Mahajan [2] for further treatment). The two main such structures are restriction species, as exemplified by the chromatic Hopf algebra of graphs [86] (see also [43]) and hereditary species, as for example the Faà di Bruno Hopf algebra (see also [72]). Restriction species are presheaves on the category of finite sets and injections. Hereditary species are presheaves on the category of finite sets and partial surjections.

The constructions of these two families of examples have been assimilated into decomposition-space theory: Gálvez, Kock, and Tonks [57] showed how Schmitt's coalgebra of a restriction species is a special case of the general incidence-coalgebra construction of decomposition spaces, and generalised it to *directed restriction species* (presheaves on the category of finite posets and convex inclusions); this generalisation includes for example the Butcher–Connes–Kreimer Hopf algebra from numerical analysis [21] and renormalisation theory [73], [31]. Shortly after, Carlier [23] showed that also Schmitt's construction of bialgebras from hereditary species is a special case of the incidence bialgebra of monoidal decomposition spaces. He went on to establish that these are actually *comodule bialgebras*, an intricate interaction between two bialgebra structures which is of importance in certain areas of analysis (see Manchon [81], and also [44] and [70]). Carlier also discovered that hereditary species provide a new class of examples of the operadic categories of Batanin and Markl [11–13]: while there is a clear operadic flavour in Schmitt's hereditary species, they are not always operads. They turn out to be operadic categories.

The unification of the above examples relates to the idea that all combinatorial co- bi- and Hopf algebra should be incidence algebras of decomposition spaces. However, we must remember that the principal idea of decomposition-space theory is to create an environment where we have a Möbius formula from a simplicial point of view.

The first antecedent of this formula at the objective level comes from Lawvere [76], who discovered in the 1980s that there is a universal Möbius function which induces all other Möbius functions. It is an 'arithmetic function' on a certain Hopf algebra of Möbius intervals. A category is an interval if it has an initial and a terminal object [75], and the Möbius condition is a certain finiteness condition. This Hopf algebra has the property that it receives a canonical coalgebra homomorphism from every incidence coalgebra of a Möbius category. This includes all locally finite posets and all the monoids considered in [25]. Lawvere's work remained unpublished for some decades, but it is cited in influential texts from that time, such as Joyal [69] and Joni–Rota [84]. Independently, Ehrenborg [39] constructed a closely related Hopf algebra, but less universal. It only accounts for intervals in posets. In both cases, the universal object can be interpreted as the colimit of all incidence coalgebras of intervals. The possibility of this is closely related to the local nature of coalgebras, expressed for example in the well-known fact that every coalgebra is the colimit of its finite-dimensional subcoalgebras, see Sweedler [87].

Lawvere's discovery did not appear in print until Lawvere–Menni [77] in 2010. In that work the authors took an important step towards explaining the universal property by lifting the construction of the Hopf algebra of Möbius intervals to the objective level. This means that its comultiplication is realised as something called a pro-comonoidal structure on certain extensive categories. The original Hopf algebra is exhibited as being only a numerical shadow of this categorical construction. There are at least two precursors to the idea of a more objective approach to incidence algebras. One is given by Joyal [69]. In his foundational paper on species, there is a final section where he considers certain decomposition structures on categories (that final section has little to do with species). Another is in the work of Dür [34] who constructed incidence coalgebras of certain categorical and simplicial structures.

However, many coalgebras, bialgebras and Hopf algebras in combinatorics are not of incidence type, meaning that they cannot arise directly as the incidence coalgebra of any Möbius category. In fact the Lawvere–Menni Hopf algebra is not of incidence type. This gives the somewhat unsatisfactory situation that the universal object is not of the same type as the objects it is universal for.

A solution to this problem was found by Gálvez, Kock, and Tonks [58–60] with the theory of decomposition spaces, since they showed that the Lawvere–Menni Hopf algebra is the incidence coalgebra of the decomposition space of subdivided intervals U. With this discovery the universal property could be stated, showing its nature as a moduli space:

Gálvez–Kock–Tonks Conjecture [60, §5.4] For each decomposition space X, the space of culf maps $\operatorname{map}(X, U)$ is contractible.

The conjecture will be studied in detail in Chapters 3 and 4. Gálvez, Kock, and Tonks [60] observed that the decomposition space of subdivided intervals U takes values in the very large ∞-category of large ∞-categories. This size issue prevents U from being a terminal object in the ∞-category of decomposition spaces and culf maps.

Lawvere's original work (suitably upgraded to the new context) shows that $\operatorname{map}(X, U)$ is not empty: it contains $I \colon X \to U$, which is essentially Lawvere's interval construction. Gálvez, Kock and Tonks [60] were able to establish one further ingredient of the conjecture, namely that $\operatorname{map}(X, U)$ is connected, meaning that every map is homotopy equivalent to I. The finer property of being contractible is the full homotopy uniqueness statement, that not only is every map equivalent to I: it is so uniquely (in a coherent homotopy sense).

The homotopy content was one of the reasons for Gálvez, Kock and Tonks to develop the whole theory in a homotopy setting: decomposition spaces are defined to be certain simplicial ∞-groupoids, and everything is fully homotopy invariant. It is an important insight of higher category theory (see for example Lurie [79]) that a universal object cannot exist in any truncated situation. The most famous example is the fact that the topos of sets (0-types) contains a classifier for monomorphisms ((-1)-types) but cannot contain a classifier for sets (0-types), and that for these to be classified one needs the 2-topos of groupoids (1-types), and to classify 1-types one needs to 3-topos of 2-types, and so on. Only in the limit is it possible to find a classifier for general homotopy types (∞-groupoids) in the ∞-topos of ∞-groupoids.

The conjecture acquires further interest in connection with ∞-topos theory. In case we work with discrete complete decomposition spaces, Kock and Spivak [71] proved that the slice category over any discrete decomposition space is a presheaf topos. In other words, the category of discrete complete decomposition spaces and culf maps is locally a topos. Hackney and Kock [63] extended Kock–Spivak's main result by showing that for any simplicial space X the ∞-category of culf maps over X is equivalent to the ∞-category of right fibrations over the edgewise subdivision of X. A consequence of this result is that the ∞-category of decomposition spaces and culf maps $\mathbf{cDcmp}_{\mathrm{culf}}$ is locally an ∞-topos.

The condition of being locally an ∞-topos of $\mathbf{cDcmp}_{\mathrm{culf}}$ and the solution of the conjecture give the tools to prove that the ∞-category of Möbius decomposition spaces and culf maps is an ∞-topos, which will be the final result of this thesis.

Contribution of the present thesis

Two overall topics are treated.

- Connected and non-connected directed hereditary species as monoidal decomposition spaces, comodule bialgebras and operadic categories, developed in collaboration with Alex Cebrian (Chapter 2).
- The Gálvez–Kock–Tonks conjecture for decomposition spaces (Chapters 3 and 4).

The new contributions of this work have already been released in the form of three papers at different publication stages.

1. Directed hereditary species and decomposition spaces, coauthored with Alex Cebrian [26].
2. The Gálvez–Kock–Tonks conjecture for locally discrete decomposition spaces [47].
3. The Gálvez–Kock–Tonks conjecture for Möbius decomposition spaces [46].

The first has just submitted for possible publication in International Mathematics Research Notices. The second has just been accepted for publication in Communications in Contemporary Mathematics, pending approval of the changes suggested by the referee. The third is ready to be submitted for publication.

We will list by chapters the new contributions of this thesis:

Chapter 2: Connected and non-connected directed hereditary species

In the quest to develop a directed version of the theory of Carlier [23] that covers new examples, we found two different routes called: *connected directed hereditary species* and *non-connected directed hereditary species*. In Chapter 2, we focus on the connected variant since it covers two important examples: the Fauvet–Foissy–Manchon (comodule) bialgebra of finite topologies [40] and the Calaque–Ebrahimi-Fard–Manchon (comodule) bialgebra of trees [22].

The bialgebra of finite topologies is essentially the base case in our setting, namely corresponding to the connected directed hereditary species of posets. Modulo the difference between posets and preorders, this turns out to be the same construction, in view of an equivalence we establish between admissible maps into a poset T (in the sense of [40]) and contractions out of T (2.4).

The construction in Fauvet–Foissy–Manchon [40] was inspired by Écalle's mould calculus in dynamical systems (see [33], [37]), and more precisely by the more elaborate notion of *ormould* [38]. The construction is subtle, and they write in fact that it would have been difficult to guess the comultiplication formula without the inspiration from Écalle's work. From the present viewpoint of connected directed hereditary species, it comes out very naturally from general principles. (The slight difference between preorders and posets might in fact be in the latter's favour: it seems that posets are closer to the ormoulds of Écalle [38]. Fauvet–Foissy–Manchon [40] had to introduce the notion of *quasi-ormould*.)

The Fauvet–Foissy–Manchon bialgebra constitutes a comodule bialgebra in conjunction with the bialgebra of finite topologies of Foissy–Malvenuto–Patras [45]. Again, the subtle algebraic conditions to be verified are now a direct consequence of the general theory.

The other important example, the Calaque–Ebrahimi-Fard-Manchon comodule bialgebra [22], is the connected directed hereditary species of trees (2.1.4). It originates in numerical analysis: its restriction part is the Butcher–Connes–Kreimer Hopf algebra corresponding to composition of B-series [21]. The hereditary part corresponds to the *substitution*, a second operation on B-series discovered by Chartier–Hairer–Vilmart [29]. Again, the comodule-bialgebra condition is now a formal consequence of the theory. Previously, Kock [70] had given a decomposition-space interpretation (in fact in terms of operads) of the Calaque–Ebrahimi-Fard-Manchon comodule bialgebra fitting it into the general framework of the Baez–Dolan construction [8], but his construction in fact gives an operad version with operadic trees rather than the combinatorial trees actually relevant in numerical analysis. An ad hoc quotient construction was required to precisely recover those. The machinery of connected directed hereditary species delivers the Calaque–Ebrahimi-Fard-Manchon comodule bialgebra directly.

Finally, in the context of the theory of operadic categories, our proof that connected directed hereditary species induce operadic categories is very different from Carlier's proof in the discrete case. Where he simply verified the 9 axioms for operadic categories one by one by hand, in the present work we exploit a recent conceptual simplicial approach to operadic categories by Batanin, Kock, and Weber [10]. They reinterpret the operadic-category axioms in simplicial terms in such a way that all 9 axioms end up as simplicial identities. In the end the category of operadic categories can be described as a strict pullback, involving small categories with chosen local terminal objects (in the style of [54]) and certain simplicial groupoids. With this formalism in hand, we can establish the functor from connected directed hereditary species to operadic categories by exploiting the universal property of the pullback, without having to check any axioms by hand. This result is interesting because it constitutes a new family of examples of operadic categories that had not been observed before.

On the other hand, Schmitt's hereditary species are not connected directed hereditary species, as the fibres along a surjection between discrete posets are not necessarily connected. To cover these examples, we introduce the notion of collapse map of posets (2.8.1.1) and directed hereditary species (2.8.2.1).

Chapter 3: The Gálvez–Kock–Tonks conjecture for rigid decomposition spaces

In Chapter 3, the first case of the conjecture is proved. Working at the level of 1-types, we define the simplicial groupoid $\mathcal{U}$ of discrete intervals (i.e. intervals that are simplicial sets rather than simplicial spaces), and show that:

Theorem 3.5.1.5. $\operatorname{map}(X, \mathcal{U})$ *is a contractible* 1*-groupoid for every* 1*-truncated locally discrete decomposition space* X.

This is the first substantial evidence for the full conjecture. The expected generality for X is that of discrete decomposition spaces, but in fact (as kindly pointed out by the anonymous referee of [47]) the proofs work the same for a broader class of decomposition spaces, namely those 1-truncated decomposition spaces with the property that all their intervals are discrete. Clearly, discrete decomposition spaces have this property, so the level of generality already covers all the classical theory of incidence algebras and Möbius inversion in combinatorics, since locally finite posets, Cartier–Foata monoids, Möbius categories, and Schmitt's examples are all 0-truncated simplicial spaces. In particular it gives finally a firm formalisation of Lawvere's intuition that the interval construction should be universal in some sense. As a particular case it establishes also the universal property of the Ehrenborg Hopf algebra.

The idea of the proof of the contractibility of the 1-grupoid $\operatorname{map}(X, \mathcal{U})$ is based on 2-categorical theory. However, a direct verification of the statement seems intractable, due to coherence problems. The difficulty is that $\mathcal{U} \colon \mathbb{\Delta}^{\text{op}} \to \mathbf{Grpd}$ is only a pseudo-simplicial groupoid. Jardine [64] has identified all the 2-cell structure and the 17 coherence conditions for pseudo-simplicial groupoids. The definition of modification in this context requires compatibility with all that. The strategy to overcome this difficulty is to build a local strict model, a kind of neighbourhood $\mathcal{U}_X \subset \mathcal{U}$ around the intervals of a given locally discrete decomposition space X. The bulk of the chapter is concerned with setting up this local model and showing that it is strict. To construct this, we introduce a stricter algebraic notion of interval, where the initial and terminal objects are not just given as properties of a discrete decomposition space, but are carried around as data, in the notion of chosen initial and terminal objects. This focus is inspired by the work in another context of Batanin and Markl on operadic categories [13]. This is quite technical, but the benefit is to achieve a strict local model $\mathcal{U}_X$ which is shown to be a strict simplicial groupoid and a complete decomposition groupoid, and to receive a strict version of the interval construction. With this strict local model in place, the local version of the contractibility of $\operatorname{map}(X, \mathcal{U}_X)$ can be established with 2-category theory by showing that $I \colon X \to \mathcal{U}_X$, interpreted as a natural transformation, does not admit other self-modifications than the identity modification. In the end this check is not so difficult.

Chapter 4: The Gálvez–Kock–Tonks conjecture for Möbius decomposition spaces

It is natural to ask whether the techniques developed in Chapter 3 can be applied or refined to prove the conjecture in full generality. Unfortunately this is not very likely, since the proof relies on an explicit strictification. Therefore, we prefer to study the conjecture from another perspective.

The size issue of U prevents to formulate the conjecture as: U is a terminal object in the ∞-category of complete decomposition spaces and culf maps. But a more refined analysis of the conjecture is possible by standard techniques by imposing cardinal bounds through the Möbius condition for decomposition spaces [59]. This is a certain finiteness condition ensuring that the general Möbius inversion principle admits a homotopy cardinality. The decomposition space of Möbius subdivided intervals $\mathsf{U}_{\mathrm{Mob}}$ is small, so it is a genuine object in the ∞-category **MobDcmp** of Möbius decomposition spaces and culf maps. So to prove the conjecture is to show that:

Theorem 4.2.0.14: The decomposition space of subdivided Möbius intervals $\mathsf{U}_{\mathrm{Mob}}$ is a terminal object in the ∞-category **MobDcmp** of Möbius decomposition spaces and culf maps .

The proof of Theorem 4.2.0.14 is the combination of several results. In the first place, we show that the canonical codomain projection map $\mathbb{\Delta} \downarrow \mathbf{cDcmp} \to \mathbf{cDcmp}$ is a weighted colimit that takes values in the ∞-category $\mathbf{Cat}_\infty$ of ∞-categories (4.2.0.1). On the other hand, we have that any weighted colimit that takes values in $\mathbf{Cat}_\infty$ induces a slice $(\infty, 2)$-category of the ∞-bicategory $\mathbb{C}\mathbf{at}_\infty$ of ∞-categories (§A). So the canonical codomain projection map $\mathbb{\Delta} \downarrow \mathbf{cDcmp} \to \mathbf{cDcmp}$ induces a functor $\mathbf{cDcmp} \to \mathbb{C}\mathbf{at}_{\infty//\mathbb{\Delta}\downarrow\mathbf{cDcmp}}$ (4.2.0.2). The next step is to show that this functor factors through $\mathbf{MobDcmp}_{/\mathsf{U}_{\mathrm{Mob}}}$ using the interval-factorisation construction (4.2.0.3) and the straightening-unstraightening equivalence of ∞-categories (4.2.0.4). After this step, it is not difficult to prove that $\mathsf{U}_{\mathrm{Mob}}$ is a terminal object in **MobDcmp**.

The ∞-category $\mathbf{cDcmp}_{\mathrm{culf}}$ of decomposition spaces and culf maps is locally an ∞-topos [63]. This locally property combining with the solution of the conjecture imply that the ∞-category of Möbius decomposition spaces and culf maps is an ∞-topos (Theorem 4.3.4.1). The proof is easy: $\mathsf{U}_{\mathrm{Mob}}$ is a terminal object in **MobDcmp** (by the solution of the Gálvez–Kock–Tonks conjeture), and therefore the canonical map $\mathbf{MobDcmp}_{/\mathsf{U}_{\mathrm{Mob}}} \to \mathbf{MobDcmp}$ is an equivalence and $\mathbf{MobDcmp}_{/\mathsf{U}_{\mathrm{Mob}}}$ is an ∞-topos (special case of Hackney–Kock's main result [63]).

Summary

Chapter 1: Preliminaries

In Section 1.1, we recall from [58–60] some basic notions and results of the theory of decomposition spaces. Furthermore, we review the notions of incidence coalgebras of decomposition spaces and the Möbius condition (1.1.1), culf and culfy maps (1.1.2), decalage (1.1.3), monoidal decomposition spaces (1.1.4), and full and faithful maps (1.1.5). In 1.2, we give a brief review of basic notions and some results on homotopy pullbacks of groupoids.

Chapter 2: Connected and non-connected directed hereditary species

The material of this chapter is a joint work with Alex Cebrian [26]. In Section 2.1, we introduce the notion of contraction (2.1.1.2), and the connected directed hereditary species as a **Grpd**-valued presheaf on the category of finite connected posets and partially defined contractions (2.1.2.1). Examples of this notion are the Fauvet–Foissy–Manchon Hopf algebra of finite topologies, the Calaque–Ebrahimi-Fard–Manchon comodule bialgebra of rooted trees (2.1.4) and the Faà di Bruno comodule bialgebra of linear trees (2.1.4.8).

In Section 2.2, we define the monoidal decomposition space of contractions **K**. In addition, we show that **K** is complete (2.2.0.9), locally finite, locally discrete, and of locally finite length (2.2.0.10), and monoidal (2.2.0.11).

In Section 2.3, we define the monoidal decomposition space of admissible maps of preorders **A**. In 2.3.1, we show that the incidence bialgebra of **A** corresponds to the Fauvet–Foissy–Manchon bialgebra of finite topologies.

In Section 2.4, we relate the notions of admissible maps of preorders (due to [40]) and of contractions of posets through a culf map (2.4.0.8). This map explains the connection between admissible maps of preorders and contractions of posets. In 2.4.1, we define the double category **AdCon** of finite preorders, admissible maps as horizontal morphisms and contractions as vertical morphisms and show that the Waldhausen $S_\bullet$-construction in the sense of Bergner et al. [14] of **AdCon** is equivalent to the decomposition space of admissible maps **A**.

In Section 2.5, we show that every connected directed hereditary species has an associated monoidal decomposition space (2.5.0.2) which is locally finite, locally discrete, and of locally finite length (2.5.0.3).

Every hereditary species has an underlying monoidal restriction species; therefore, we have two bialgebra structures associated with a hereditary species. Carlier [23] showed that these bialgebras are compatible in the sense that the incidence bialgebra associated with the restriction species is a left comodule bialgebra over the incidence bialgebra of the hereditary species. In Section 2.6, we apply Carlier's ideas to the connected directed case. In 2.6.3, we show that the comodule bialgebra of the connected directed hereditary species of rooted trees is the Calaque–Ebrahimi-Fard–Manchon comodule bialgebra of rooted trees [22], and the comodule bialgebra of the connected directed hereditary species of linear trees is the Faá di Bruno comodule bialgebra of linear trees [70].

In Section 2.7, we construct a functor from the category of connected directed hereditary species to the category of operadic categories in the sense Batanin and Markl [13], using a new approach by Batanin, Kock, and Weber [10].

In Section 2.8, we introduce the notion of collapse map of posets (2.8.1.1) and define the decomposition space of collapse maps **D** (2.8.3). Furthermore, we prove that **D** is complete (2.8.3.8), locally finite, locally discrete and of locally finite length (2.8.3.9). Similarly to the connected case, we prove that directed hereditary species induce monoidal decomposition spaces (2.8.4), comodule bialgebras (2.8.5), and operadic categories (2.8.6).

Chapter 3: The Gálvez–Kock–Tonks conjecture for rigid decomposition spaces

The material of this chapter is the main part of [47]. In Section 3.1, we introduce some necessary material relating to the notion of slice and coslice of decomposition groupoids. Furthermore, we give the definition of interval (3.1.0.14).

In Section 3.2, we work with strict simplicial groupoids such that all active-inert squares are strict pullbacks and such that d_1 is a discrete isofibration. (It follows that all the strict pullbacks are also homotopy pullbacks.) For short we shall call such decomposition groupoids *rigid* (3.2.0.1). Furthermore, we explain the concept of chosen initial and chosen terminal object (3.2.0.5). Also, the notion of discrete interval (3.2.0.7) and some results for discrete intervals are given, in particular a lifting property (3.2.0.17 and 3.2.0.19). In Section 3.3, we construct the stretched-culf factorisation system in the category of discrete intervals. Furthermore, we introduce important working tools (3.3.0.3 and 3.3.0.5) that will be useful in next sections.

In Section 3.4, we define the decomposition groupoid of all discrete intervals U [60]. In 3.4.1, we construct a strict simplicial groupoid U_X (3.4.1.5) that only contains the information about the discrete intervals of a fixed rigid decomposition groupoid X and prove that U_X is a complete decomposition groupoid (3.4.1.9 and 3.4.1.10). Furthermore, we define a simplicial map $I\colon X \to U_X$ and prove that I is culf (3.4.1.12). In 3.4.3, we explain the interval construction of an interval. Furthermore, we compare U_X with a strictification $\widetilde{U}$ of U suggested by the referee in 3.4.4.

In Section 3.5, we finally address the Gálvez–Kock–Tonks conjecture as formulated in [60], and we prove a partial result (3.5.0.1) about the connectedness of the mapping space $\mathrm{map}_{\mathbf{cDcmp}}(X, U)$ in the case of rigid decomposition groupoids. In 3.5.1, we use modifications (3.5.1.2) to prove a truncated version of the conjecture, the case of rigid decomposition groupoids. We first show that $\mathrm{map}_{\mathbf{cDcmp}}(X, U_X)$ is contractible (3.5.1.4) and from this we deduce that the groupoid $\mathrm{map}_{\mathbf{cDcmp}}(X, U)$ is contractible (3.5.1.5). This version of the Gálvez–Kock–Tonks conjecture is the main result of this chapter.

Chapter 4: The Gálvez–Kock–Tonks conjecture for Möbius decomposition spaces

The material of this chapter is the main part of [46]. In Section 4.1, we start by studying the notion of stretched maps (4.1.1.3), flanked decomposition spaces (4.1.0.2), and algebraic intervals 4.1.1.2 from [60]. In 4.1.2, we define the large complete decomposition space of sub-divided intervals U and the Möbius decomposition space of sub-divided Möbius intervals U_{Mob}. In 4.1.3, we give the interval factorisation construction I that allows to construct a culf map $I_X\colon X \to U$ for any complete decomposition space X.

In Section 4.2, we state the Gálvez–Kock–Tonks conjecture for Möbius decomposition spaces (4.2.0.5) and give a proof (4.2.0.14). The proof involves a mixture of results so it is divided into a series of steps (4.2.0.1, 4.2.0.2, 4.2.0.3 and 4.2.0.4) to get a cleaner picture of the proof. In 4.2.1, we compare the ideas behind the proof given in Chapter 4 for locally discrete decomposition spaces with the proof for Möbius decomposition spaces.

In Section 4.3, we expose the main result of Hackney and Kock in [63]: the ∞-category of decomposition spaces and culf maps $\mathbf{cDcmp}_{\mathrm{culf}}$ is locally an ∞-topos (4.3.3.6). The locally ∞-topos condition of $\mathbf{cDcmp}_{\mathrm{culf}}$ and the solution of the conjecture imply that the ∞-category of Möbius decomposition spaces and culf maps is an ∞-topos (4.3.4.1).

In order to improve the readability we have deferred the rather technical result (4.2.0.7) into the appendix.

1

Preliminaries

This chapter establishes a few background facts and notation for the reader. These results are not new.

1.1 *Decomposition spaces*

Our theoretical results in Chapter 4 are formulated in the setting of homotopy pullbacks, ∞-categories and simplicial spaces. Although these theories are extensive we will mention in this section some tools that we will need.

Given a map of ∞-groupoids $p\colon X \to S$ and an object $s \in S$, the *homotopy fibre* X_s of p over s is the homotopy pullback

$$\begin{tikzcd}
X_s \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
1 \arrow[r, "\ulcorner s \urcorner"'] & S.
\end{tikzcd}$$

Lemma 1.1.0.1. *[24] A square of ∞-groupoids*

$$\begin{tikzcd}
P \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\
X \arrow[r, "f"'] & S
\end{tikzcd}$$

is a homotopy pullback if and only if for each $x \in X$ the induced comparison map $u_x\colon P_x \to Y_{fx}$ is an equivalence.

Lemma 1.1.0.2. *[79, Lemma 4.4.2.1] Given a prism diagram of ∞-groupoids*

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

where the right square is a homotopy pullback. Then the left square is a homotopy pullback if and only if the outer diagram is a homotopy pullback.

The *simplex category* $\mathbb{\Delta}$ is the category whose objects are the nonempty finite ordinals and whose morphisms are the monotone maps. These are generated by *coface* maps $d^i\colon [n-1] \to [n]$, which are the monotone injective functions for which $i \in [n]$ is not in the image, and *codegeneracy* maps $s^i\colon [n+1] \to [n]$, which are monotone surjective functions for which $i \in [n]$ has a double preimage. We write $d^\perp := d^0$ and $d^\top := d^n$ for the outer coface maps. In this thesis, we assume a (large) ∞-category $\mathbf{Cat}_\infty$ of all small ∞-categories, with a full ∞-subcategory $\mathcal{S}$ of ∞-groupoids, which we called spaces. Let **sSpaces** denote the functor ∞-category of $\mathcal{S}$-valued presheaves on $\mathbb{\Delta}$.

Definition 1.1.0.3. [58, §2.9][35, §2.1] A simplicial space $X\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ is called a *Segal space* if it satisfies the Segal condition,

$$X_n \xrightarrow{\simeq} X_1 \times_{X_0} \cdots \times_{X_0} X_1 \quad \text{for all } n \geqslant 0.$$

This is equivalent [58, Lemma 2.10] to requiring that for each $n > 0$ the following diagram is a homotopy pullback

$$\begin{tikzcd}
X_{n+1} \arrow[r, "d_\top"] \arrow[d, "d_\bot"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[d, "d_\bot"] \\
X_n \arrow[r, "d_\top"'] & X_{n-1}.
\end{tikzcd}$$

An arrow of $\mathbb{\Delta}$ is termed *active*, and written $g\colon [m] \rightarrow\!\!\!| \; [n]$, if it preserves end-points, $g(0) = 0$ and $g(m) = n$. An arrow is termed *inert*, and written $f\colon [m] \rightarrowtail [n]$, if it is distance preserving, $f(i+1) = f(i) + 1$ for $0 \leqslant i < m$. The category $\mathbb{\Delta}^{\mathrm{act}}$ is the subcategory of $\mathbb{\Delta}$ whose objects are the nonempty finite ordinals and whose morphisms are the active maps.

Definition 1.1.0.4. [58, Definition 3.1][35, Definition 2.3.1] A *decomposition space* is a simplicial space

$$X\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$$

such that the image of any pushout diagram in $\mathbb{\Delta}$ of an active map g along an inert map f is a homotopy pullback of groupoids,

$$X\left(\begin{tikzcd}
{[p]} \arrow[dr, phantom, "\lrcorner", very near start] & {[m]} \arrow[l, "g'"', ->|] \\
{[q]} \arrow[u, "f'", tail] & {[n]} \arrow[l, "g", ->|] \arrow[u, "f"', tail]
\end{tikzcd}\right) = \begin{tikzcd}
X_p \arrow[r, "g'^*"] \arrow[d, "f'^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_m \arrow[d, "f^*"] \\
X_q \arrow[r, "g^*"'] & X_n.
\end{tikzcd}$$

This is equivalent [58, Proposition 3.5] to requiring that the following diagrams are homotopy pullbacks for all $0 < i < n$:

$$\begin{tikzcd}
X_{n+1} \arrow[r, "d_{i+1}"] \arrow[d, "d_\bot"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[d, "d_\bot"] \\
X_n \arrow[r, "d_i"'] & X_{n-1}
\end{tikzcd} \qquad \begin{tikzcd}
X_{n+1} \arrow[r, "d_i"] \arrow[d, "d_\top"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[d, "d_\top"] \\
X_n \arrow[r, "d_i"'] & X_{n-1}.
\end{tikzcd}$$

The notion of decomposition space is equivalent to the notion of 2-Segal space [35, Proposition 2.3.2] (see [41] for the last piece of this equivalence) introduced by Dyckerhoff and Kapranov [35, Definition 2.3.1, Definition 2.5.2, Remark 5.2.4].

Example 1.1.0.5. The decomposition space of rooted trees **RT** is defined as follows [58]. Recall that a *forest* is a disjoint union of rooted trees. An *admissible cut* of a rooted tree is a splitting of the set of nodes into two subsets such that the second forms a subtree containing the root node or is the empty forest. $\mathbf{RT}_1$ denotes the groupoid of forests, and $\mathbf{RT}_2$ denotes the groupoid of forests with an admissible cut. More generally, $\mathbf{RT}_0$ is defined to be a point, and $\mathbf{RT}_k$ is the groupoid of forests with $k-1$ compatible admissible cuts. These form a simplicial groupoid in which the inner face maps forget a cut, and the outer face maps project away stuff: $d_\bot$ deletes the crown and $d_\top$ deletes the bottom layer. It is readily seen that **RT** is not a Segal groupoid: a tree with a cut cannot be reconstructed from its crown and its bottom tree, which is to say that $\mathbf{RT}_2$ is not equivalent to $\mathbf{RT}_1 \times_{\mathbf{RT}_0} \mathbf{RT}_1$. It is straightforward to check that it is a decomposition groupoid [58].

Proposition 1.1.0.6. *[35, Proposition 2.3.4][58, Proposition 3.7] Any Segal space is a decomposition space.*

Certain pullbacks in $\mathbb{\Delta}^{\mathrm{op}}$ are preserved by general decomposition spaces, which is the content of the following result.

Lemma 1.1.0.7. *[58, Lemma 3.10] Let X be a decomposition space. For all $0 < i < j < n$, the following squares of active face and degeneracy maps are homotopy pullbacks*

$$\begin{tikzcd} X_{n+1} \arrow[r, "d_i"] \arrow[d, "d_{j+1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[d, "d_j"] \\ X_n \arrow[r, "d_i"'] & X_{n-1} \end{tikzcd} \qquad \begin{tikzcd} X_{n-3} \arrow[r, "s_{i-1}"] \arrow[d, "s_{j-2}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_{n-2} \arrow[d, "s_{j-1}"] \\ X_{n-2} \arrow[r, "s_{i-1}"'] & X_{n-1}. \end{tikzcd}$$

A map of ∞-groupoids $f\colon X \to Y$ is a *monomorphism* when its homotopy fibres are either empty or contractible.

Definition 1.1.0.8. [59] A decomposition space X is *complete* when $s_0\colon X_0 \to X_1$ is a monomorphism (i.e. is (-1)-truncated). It follows from the decomposition space axiom that in this case all degeneracy maps are monomorphisms [59, Lemma 2.5].

1.1.1 *The incidence coalgebra and the Möbius condition*

Let X be a decomposition space. The span

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1$$

defines a linear functor, the comultiplication

$$\begin{aligned} \Delta\colon \mathcal{S}_{/X_1} &\to \mathcal{S}_{/X_1 \times X_1} \\ f &\mapsto (d_2, d_0)_! \circ d_1^*(f). \end{aligned}$$

Likewise, the span

$$X_1 \xleftarrow{s_0} X_0 \xrightarrow{t} 1$$

defines a linear functor, the counit

$$\begin{aligned} \delta\colon \mathcal{S}_{/X_1} &\to \mathcal{S} \\ f &\mapsto t_! \circ s_0^*(f). \end{aligned}$$

The decomposition space axioms serve to ensure that Δ is coassociative with counit δ, up to coherent homotopy [58, §5.3]. This coalgebra $(\mathcal{S}_{/X_1}, \Delta, \delta)$ is called the *incidence coalgebra*, at the objective level of slices and linear functors. The connection to ordinary coalgebras in the category of vector spaces is given by taking homotopy cardinality. This is an important aspect of the theory (see [61] for details), but it is not a focus point of this thesis, and will be used only in some examples. Here we only briefly explain it. In order to take homotopy cardinality, some finiteness conditions are required:

Definition 1.1.1.1. [59] A decomposition space X is *locally finite* if X_1 is a locally finite ∞-groupoid (meaning that its homotopy groups are finite and trivial from certain dimension on), and the maps $s_0\colon X_0 \to X_1$ and $d_1 : X_1 \to X_0$ have finite homotopy fibres.

If T is a locally finite ∞-groupoid, the cardinality of the slice $\mathbf{spaces}_{/T}$ is the vector space $\mathbb{Q}_{\pi_0 T}$ spanned by one basis vector $\mathbf{e}_t$ for each $t \in \pi_0 T$. This basis vector is the homotopy cardinality of the object $\ulcorner t \urcorner : 1 \to T$, the 'name' of t. In this way the homotopy cardinality of a linear functor given by a span becomes matrix multiplication at the vector space level. The condition of local finiteness is required for the sum in the formula for comultiplication to be finite. Otherwise it does not admit a cardinality.

Example 1.1.1.2. [58, §5.1] If X is the nerve of a category (for example, a poset) then X_2 is the set of all composable pairs of arrows. Since X_1 is discrete in this case, it is automatically locally finite, and the basis vectors of $\mathbb{Q}_{X_1}$ are just the arrows of the category. For X to be locally finite, the category must satisfy the condition that for each arrow there are only finitely many ways to factor it into the composite of two arrows. The comultiplication at the vector space level then becomes

$$\Delta(f) = \sum_{b \circ a = f} a \otimes b,$$

and the counit sends identity arrows to 1 and other arrows to 0. This is the incidence coalgebra of a locally finite category of Leroux [78].

Example 1.1.1.3. The decomposition space **RT** of Example 1.1.0.5 is locally finite. Indeed, any forest has only a finite automorphism group, so the groupoid $\mathbf{RT}_1$ is locally finite (since it is only a 1-groupoid, there are no higher homotopy groups). The simplicial groupoid **RT** is locally finite because for a given forest there is only a finite number of possible admissible cuts. At the groupoid slice level, the comultiplication is

$$\begin{aligned} \mathcal{S}_{/\mathbf{RT}_1} &\longrightarrow \mathcal{S}_{/\mathbf{RT}_1} \otimes \mathcal{S}_{/\mathbf{RT}_1} \\ (\ulcorner T \urcorner \colon 1 \to \mathbf{RT}_1) &\longmapsto (\ulcorner P_c \urcorner \colon 1 \to \mathbf{RT}_1) \otimes (\ulcorner R_c \urcorner \colon 1 \to \mathbf{RT}_1). \end{aligned}$$

which means that after taking homotopy cardinality, the comultiplication at the vector space level becomes

$$\Delta(T) = \sum_{c \in \text{admi.cuts}(T)} P_c \otimes R_c,$$

where P_c is the forest above the cut and R_c is the forest below the cut. After taking homotopy cardinality the comultiplication at the vector space level thus becomes the famous Butcher–Connes–Kreimer Hopf algebra of rooted trees from perturbative renormalisation [73] and numerical analysis [21].

These two examples satisfy two further finiteness conditions typical for examples coming from combinatorics. Namely, they are locally discrete and of locally finite length. Most examples in this thesis (all the examples in Chapter 2) will be both locally discrete and of locally finite length. The universal decomposition space U_{Mob} of Chapter 4) will be locally finite and of locally finite length (i.e. is Möbius).

Definition 1.1.1.4. [59] A decomposition space X is *locally discrete* if the maps $s_0 \colon X_0 \to X_1$ and $d_1 \colon X_1 \to X_0$ have discrete fibres, and X is of *locally finite length* if for each $a \in X_1$ there is an upper bound on the n for which the map $X_n \twoheadrightarrow X_1$ has non-degenerate elements in the fibre.

Definition 1.1.1.5. [60] A decomposition space X which is both locally finite and of locally finite length is called a *Möbius decomposition space*, because they admit a Möbius inversion principle, not just at the objective slice level but also at the vector space level.

1.1.2 *Culf and culfy maps*

A map $F \colon Y \to X$ of simplicial spaces is *cartesian* on an arrow $[n] \to [k]$ in $\mathbb{\Delta}$, if the naturality square for f with respect to this arrow is a pullback.

Definition 1.1.2.1. [58, §4] A simplicial map $F \colon X \to Y$ is called *culf* if F is cartesian on each active map.

Culf stands for 'conservative' and 'unique lifting of factorisations' where *conservative* means cartesian on all codegeneracy maps, and *unique lifting factorisations* means cartesian on all coface maps. The culf condition can be seen as an abstraction of coalgebra homomorphism: the conservative condition corresponds to counit preservation, and ulf corresponds to comultiplicativity.

Lemma 1.1.2.2. *[58, Lemma 4.3] A simplicial map between decomposition spaces is culf if and only if it is cartesian on* $d^1\colon [1] \to [2]$.

Lemma 1.1.2.3. *[58, Lemma 4.6] If* X *is a decomposition space and* $F\colon Y \to X$ *is a culf map, then also* Y *is a decomposition space.*

Since the culfness condition refers to active maps, just as the finiteness conditions stated for locally discrete decomposition spaces, we have the following result.

Lemma 1.1.2.4. *[57, Lemma 1.12] If* X *is a locally discrete (resp. locally finite length) decomposition space and* $F\colon Y \to X$ *is a culf map, also* Y *is locally discrete (resp. locally finite length) decomposition space. This also the case for locally finite, provided we check separately that* Y_1 *is locally finite.*

In many cases it is easier to work with right fibrations than with presheaves, so it is necessary to introduce the notion of culf for right fibrations, which we will call *culfy* [63].

Definition 1.1.2.5. A simplicial map $F\colon X \to Y$ is called a *right fibration* if F is cartesian on all last-point-preserving maps.

Remark 1.1.2.6. The right fibration condition can be rewritten as follows: F is a right fibration if for all terminal-object-preserving maps $\ell\colon \Delta^m \to \Delta^n$ and commutative squares

$$\begin{tikzcd}
\Delta^m \arrow[r] \arrow[d, "\ell"'] & X \arrow[d, "F"] \\
\Delta^n \arrow[r] \arrow[ur, dotted, "\exists!"] & Y,
\end{tikzcd}$$

the space of fillers is contractible. A simplicial map $F\colon X \to Y$ between decomposition spaces is a right fibration if it is cartesian on all bottom coface maps $d_\perp$.

Definition 1.1.2.7. A simplicial map $F\colon X \to Y$ is called a *left fibration* when it is cartesian on all initial-point-preserving maps, or equivalently, if it is right orthogonal to all initial-object-preserving maps $\hbar\colon \Delta^m \to \Delta^n$. In case X and Y are decomposition spaces, F is a left fibration if it is cartesian on $d_\top$.

Lemma 1.1.2.8. *Let* Y *be a Segal space and let* $F\colon X \to Y$ *be a simplicial map that is a left or a right fibration, then also* X *is a Segal space.*

Let $X\colon \mathbb{\Delta}^{op} \to \mathcal{S}$ be a simplicial space. The *category of elements* of X is by definition $\mathrm{el}(X) := \mathbb{\Delta} \downarrow X$. Its objects and arrows are, respectively, diagrams of the form

$$\begin{tikzcd}
\Delta^m \arrow[d, "\lambda"'] \\
X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\Delta^m \arrow[rr, "\alpha"] \arrow[dr, "\lambda"'] & & \Delta^n \arrow[dl, "\gamma"] \\
& X. &
\end{tikzcd}$$

The category of elements is the domain of the right fibration corresponding to X under the straightening-unstraightening equivalence of ∞-categories $\mathrm{RFib}(\mathbb{\Delta}) \simeq \mathbf{PrSh}(\mathbb{\Delta})$ (due to Lurie [79], see [7, Theorem 3,4,6] for a model-independent statement).

Definition 1.1.2.9. Let $\mathcal{E} \to \mathbb{\Delta}$ and $\mathcal{E}' \to \mathbb{\Delta}$ be right fibrations over $\mathbb{\Delta}$. A map $p\colon \mathcal{E} \to \mathcal{E}'$ between right fibrations is called *culfy* if it is a left fibration after restriction to $\mathbb{\Delta}_{\mathrm{act}} \subset \mathbb{\Delta}$.

Proposition 1.1.2.10. *[63, §6] Let* $F\colon Y \to X$ *be a simplicial map between decomposition spaces. The map* F *is culf if and only if the corresponding right fibration*

$$\mathrm{el}(Y) \xrightarrow{\mathrm{el}(F)} \mathrm{el}(X) \longrightarrow \mathbb{\Delta}$$

is culfy.

Proof. Suppose that $\mathrm{el}(F)$ is a left fibration after restriction to $\mathbb{\Delta}_{\mathrm{act}} \subset \mathbb{\Delta}$. Since Y and X are decomposition spaces, to prove that F is culf, it is enough to show that F is cartesian to the active map $d^1 \colon [1] \to [2]$ as a consequence of Lemma 1.1.2.2. In other words, we have to prove that the square

$$\begin{tikzcd} Y_2 \arrow[r, "d_1"] \arrow[d, "F_2"'] & Y_1 \arrow[d, "F_1"] \\ X_2 \arrow[r, "d_1"'] & X_1 \end{tikzcd} \tag{1}$$

is a pullback. The pullback condition of (1) is equivalent to prove that $\mathrm{el}(F)$ is right orthogonal to $d^1 \colon \Delta^2 \to \Delta^1$. But, this is a consequence of the assumption that $\mathrm{el}(F)$ is a left fibration after restriction to $\mathbb{\Delta}_{\mathrm{act}} \subset \mathbb{\Delta}$ since d^1 is an initial-object-preserving map. The other direction use analogous arguments. ☐

1.1.3 *Decalage construction*

Given a simplicial space X, the *lower dec* $\mathrm{Dec}_\perp X$ is a new simplicial space obtained by deleting X_0 and shifting everything one place down, deleting also all d_0 face maps and all s_0 degeneracy maps. It comes equipped with a simplicial map, called the lower dec map, $d_\perp \colon \mathrm{Dec}_\perp X \to X$ given by the original d_0. Similarly, the *upper dec* $\mathrm{Dec}_\top X$ is obtained by instead deleting, in each degree, the top face map $d_\top$ and the top degeneracy map $s_\top$. The deleted top face maps becomes the upper dec map $d_\top \colon \mathrm{Dec}_\top X \to X$.

Proposition 1.1.3.1. *[58, Proposition 4.9] If X is a decomposition space then the dec maps $d_\top \colon \mathrm{Dec}_\top X \to X$ and $d_\perp \colon \mathrm{Dec}_\perp X \to X$ are culf.*

The decomposition property can be characterised in terms of decalage:

Theorem 1.1.3.2. *[35, 41, 58] A simplicial space $X \colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ is a decomposition space if and only if both $\mathrm{Dec}_\perp X$ and $\mathrm{Dec}_\top X$ are Segal spaces.*

1.1.4 *Monoidal decomposition spaces*

There is a natural notion of monoidal decomposition space [58, §9]; which leads to bialgebras. It is a decomposition space X with two functors $\eta \colon 1 \to X$ and $\otimes \colon X \times X \to X$ required to be a monoidal structure and culf. In examples coming from combinatorics, the monoidal structure will typically be given by disjoint union.

1.1.5 *Full and faithful maps*

A map $F \colon X \to Y$ between simplicial spaces is called *full and faithful* if for each $n \geqslant 1$, the diagram

$$\begin{tikzcd} X_n \arrow[r, "F_n"] \arrow[d] & Y_n \arrow[d] \\ X_0^{\times(n+1)} \arrow[r, "F_0^{\times(n+1)}"'] & Y_0^{\times(n+1)} \end{tikzcd}$$

is a pullback.

Remark 1.1.5.1. It is not difficult to prove that this definition agrees with the usual definition in the case of Segal spaces. A map $F \colon X \to Y$ between Segal spaces is full and faithful if and only if the diagram

$$\begin{tikzcd}
X_1 \arrow[r, "F_1"] \arrow[d, "(d_\top, d_\bot)"'] & Y_1 \arrow[d, "(d_\top, d_\bot)"] \\
X_0^{\times 2} \arrow[r, "F_0^{\times 2}"'] & Y_0^{\times 2}
\end{tikzcd}$$

is a pullback.

Lemma 1.1.5.2. *Let* $F\colon X \to Y$ *be a full and faithful map between simplicial spaces. Then for any simplicial space* Z*, the map* $F_!\colon \mathrm{map}_{\mathbf{PrSh}(\mathbb{\Delta})}(Z, X) \to \mathrm{map}_{\mathbf{PrSh}(\mathbb{\Delta})}(Z, Y)$ *is full and faithful.*

Proof. The diagram

$$\begin{tikzcd}
X_n \arrow[r, "F_n"] \arrow[d] & Y_n \arrow[d] \\
X_0^{\times(n+1)} \arrow[r, "F_0^{\times(n+1)}"'] & Y_0^{\times(n+1)}
\end{tikzcd} \tag{1}$$

is a pullback since F is full and faithful. Applying the mapping space functor $\mathrm{map}_{\mathbf{PrSh}(\mathbb{\Delta})}(Z, -)$ to (1), we obtain the diagram

$$\begin{tikzcd}
\mathrm{map}(Z, \mathrm{map}(\Delta^n, X)) \arrow[r, "F_!"] \arrow[d] & \mathrm{map}(Z, \mathrm{map}(\Delta^n, Y)) \arrow[d] \\
\mathrm{map}(Z, \mathrm{map}(\Delta^0, X)^{\times(n+1)}) \arrow[r, "F_!"'] & \mathrm{map}(Z, \mathrm{map}(\Delta^0, Y)^{\times(n+1)})
\end{tikzcd} \tag{2}$$

which is a pullback since $\mathrm{map}_{\mathbf{PrSh}(\mathbb{\Delta})}(Z, -)$ preserves limits. Using the currying isomorphism $\mathrm{map}(Z, \mathrm{map}(\Delta^n, X)) \cong \mathrm{map}(Z \times \Delta^n, X)$, the pullback (2) can be rewritten as the pullback

$$\begin{tikzcd}
\mathrm{map}(Z \times \Delta^n, X)) \arrow[r, "F_!"] \arrow[d] & \mathrm{map}(Z \times \Delta^n, Y)) \arrow[d] \\
\mathrm{map}(Z, X)_0^{\times(n+1)} \arrow[r, "F_!"'] & \mathrm{map}(Z, Y)_0^{\times(n+1)}
\end{tikzcd}$$

and therefore $F_!$ is full and faithful. ☐

Lemma 1.1.5.2 was proved in the context of ∞-categories by Gepner, Haugseng and Nikolaus [55, Lemma 5.2].

1.1.6 *Factorisation system*

In Chapters 3 and 4, to define the decomposition space of subdivided intervals U, we use properties of factorisation systems. This section introduces the notions we will need in the future.

A *factorisation system* in an ∞-category $\mathcal{D}$ consists of two classes E and F of maps, that we shall depict as $\twoheadrightarrow$ and $\rightarrowtail$, such that

1. The class E and F is closed under isomorphism.
2. The classes E and F are orthogonal, $E \perp F$. That is, given $e \in E$ and $f \in F$, for every solid square

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, two heads, "e"'] & \cdot \arrow[d, tail, "f"] \\
\cdot \arrow[r] \arrow[ur, dashed] & \cdot
\end{tikzcd}$$

the space of fillers is contractible.

3. Every map h admits a factorisation

$$\begin{tikzcd}
\cdot \arrow[rr, "h"] \arrow[dr, two heads, "e"'] & & \cdot \\
& \cdot \arrow[ur, tail, "f"'] &
\end{tikzcd}$$

with $e \in E$ and $f \in F$.

Remark 1.1.6.1. The classical notion of orthogonal factorisation system requires that E be closed under isomorphism. In Chapter 3 is not require. In case E is not closed under isomorphism we can always saturate it.

Let $\mathrm{Ar}^E(\mathcal{D}) \subset \mathrm{Ar}(\mathcal{D})$ denote the full subcategory spanned by the arrows in the left-hand class E.

Lemma 1.1.6.2. *[60, Lemma 1.3] The domain projection $\mathrm{Ar}^E(\mathcal{D}) \to \mathcal{D}$ is a cartesian fibration. The cartesian arrows in $\mathrm{Ar}^E(\mathcal{D})$ are given by squares of the form*

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, two heads] & \cdot \arrow[d, two heads] \\
\cdot \arrow[r, tail] & \cdot
\end{tikzcd}$$

1.2 *Groupoids and homotopy pullbacks*

Our theoretical results in Chapters 2 and 3 are formulated in the setting of groupoids. A *groupoid* is a category where all the arrows are invertible. Let's take a look at some of the notions we will need.

Homotopy pullbacks are important for the results that will be developed in the following chapters. They are examples of homotopy limits, and as such are defined only up to equivalence. A particular case of homotopy pullbacks is given by the homotopy fibres. Given a map of groupoids $p \colon X \to S$ and an object $s \in S$, the *homotopy fibre* X_s of p over s is the homotopy pullback

$$\begin{tikzcd}
X_s \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
1 \arrow[r, "\ulcorner s \urcorner"'] & S.
\end{tikzcd}$$

We use the following standard lemma many times.

Lemma 1.2.0.1. *[24] A square of groupoids*

$$\begin{tikzcd}
P \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\
X \arrow[r, "f"'] & S
\end{tikzcd}$$

is a homotopy pullback if and only if for each $x \in X$ the induced comparison map $u_x \colon P_x \to Y_{fx}$ is an equivalence.

Since homotopy pullback is defined up to equivalence, for some calculations it is important to work with a specific model. Let's look at one of the models to be used: the homotopy fibre product of a pair of functors $f\colon A \to C$ and $g\colon B \to C$ between groupoids is the groupoid H whose objects are triples (a, θ, b) consisting of objects $a \in A$, $b \in B$ and an isomorphism $\theta\colon f(a) \to g(b)$ in C, and whose arrows $(\alpha, \beta)\colon (a, \theta, b) \to (a', \theta', b')$ consist of arrows $\alpha\colon a \to a' \in A$ and $\beta\colon b \to b' \in B$ such that $g(\beta) \circ \theta = \theta' \circ f(\alpha)$. The groupoid H fits into a homotopy commutative square

$$\begin{tikzcd}
H \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C,
\end{tikzcd}$$

where $\pi_A\colon H \to A$ and $\pi_B\colon H \to B$ are the canonical projections, and the components of the natural isomorphism is given by θ itself. Note that the projections always are isofibrations [24]. Another model is possible when one of the two legs f and g is an isofibration. In that case, the strict pullback is also a homotopy pullback [67, Theorem 1].

The most used result for homotopy pullbacks is the prism lemma.

Lemma 1.2.0.2. *Consider a diagram of groupoids*

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

where the right square is a homotopy pullback. Then the left square is a homotopy pullback if and only if the outer diagram is a homotopy pullback.

We will use the following variation of the prism lemma in Chapter 3.

Lemma 1.2.0.3. *[24] Consider a diagram of groupoids*

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

where the right square is a homotopy fibre product. Then the left square is a strict pullback if and only if the outer diagram is a homotopy fibre product.

A map of groupoids $f\colon X \to Y$ is a *monomorphism* when it is fully faithful. Equivalently, its homotopy fibres are (-1)-groupoids, that is, are either empty or contractible.

1.2.1 *Fat nerve*

The *fat nerve* of a category $\mathbb{X}$ is the simplicial groupoid

$$\begin{aligned} \mathbf{N}\mathbb{X} : & \mathbb{\Delta}^{\mathrm{op}} \to \mathbf{Grpd} \\ & [n] \mapsto \mathrm{map}([n], \mathbb{X}). \end{aligned}$$

where $\mathrm{map}([n], \mathbb{X})$ is the mapping space, defined as the maximal subgroupoid of the functor category $\mathrm{Fun}([n], \mathbb{X})$. The fat nerve of a category is always a Segal space [58, §2.14].

1.2.2 *Symmetric monoidal category functor*

Let $\mathbf{FinSet}^{\mathrm{bij}}$ denote the groupoid of all finite sets and bijections. The symmetric monoidal category functor $S\colon \mathbf{Grpd} \to \mathbf{Grpd}$ [72] is defined as follows: given a groupoid X, the objects of SX are functors from $B \to X$, where $B \in \mathbf{FinSet}^{\mathrm{bij}}$. The morphisms are homotopy-commutative diagrams

$$\begin{tikzcd} B \arrow[rr, "\alpha"] \arrow[dr] & & B' \arrow[dl] \\ & X & \end{tikzcd}$$

where α is a morphism in $\mathbf{FinSet}^{\mathrm{bij}}$. In other words, SX can be interpreted as the comma category $\mathbf{FinSet}^{\mathrm{bij}}_{/X}$. To simplify notation, when we refer to an object SX we will write $\{P_i\}_{i \in I}$ where I is a finite set except in Section 2.7, where we will use finite ordinals in order to have precise constructions in the context of operadic categories.

The algebras over S are symmetric monoidal categories. The unit sends an element x to the list with one element (x), and the multiplication takes disjoint union of index sets. Furthermore, S preserves homotopy pullbacks and fibrations [94].

Remark 1.2.2.1. The functor S is normally defined as follows: given a groupoid X, the objects of SX are finite lists of objects of X, and a morphism $(x_1, x_2, ..., x_n) \to (y_1, y_2, ..., y_m)$ consists of a bijection $\sigma\colon \underline{n} \to \underline{m}$ and morphisms $x_i \to y_{\sigma(i)}$.

Note that a finite list of objects in X is a functor $a\colon \underline{n} \to X$. A morphism in the groupoid $S(X)$ is a bijection $\alpha\colon \underline{n} \simeq \underline{n}'$ together with arrows in X from $a(i)$ to $a(\sigma(i))$. We can package that data into saying that it's a homotopy-commutative triangle

$$\begin{tikzcd} n \arrow[rr, "\alpha"] \arrow[dr] & & n' \arrow[dl] \\ & X. & \end{tikzcd}$$

The components of the natural transformation α are then precisely the arrows from $a(i)$ to $a(\sigma(i))$. Altogether, if $\mathbb{B}$ denotes the skeletal category of finite ordinals and all bijections, then we have that SX is the comma category (weak slice) $\mathbb{B}_{/X}$.

Now that we know this, it is very easy to see the connexion with our presentation of S from skeletal $\mathbb{B}$ to $\mathbf{FinSet}^{\mathrm{bij}}$: It is still just $\mathbf{FinSet}^{\mathrm{bij}}_{/X}$.

2

Connected and non-connected directed hereditary species

In this chapter we introduce the notion of *connected and non-connected directed hereditary species*, which subsumes Schmitt's hereditary species, Gálvez–Kock–Tonks directed restrictions species, and a directed version of Carlier's construction of monoidal decomposition spaces and comodule bialgebras. In addition to all the examples of Schmitt, Gálvez–Kock–Tonks and Carlier, the new construction covers also the Fauvet–Foissy–Manchon Hopf algebra of finite topological spaces, the Calaque–Ebrahimi-Fard–Manchon comodule bialgebra of rooted trees, and the Faà di Bruno comodule bialgebra of linear trees. Finally, we show that directed hereditary species induce a new family of examples of operadic categories. The results of this chapter are a joint work with Alex Cebrian [26].

2.1 *Connected directed hereditary species*

In this section, we will introduce the concept of connected directed hereditary species, but first, we will provide a series of tools necessary to define this notion.

2.1.1 *Contractions*

A map of posets $f\colon P \to Q$ is *convex* if for all $x, y \in P$ and $f(x) \leqslant w \leqslant f(y)$ in Q there is a unique $p \in P$ with $x \leqslant p \leqslant y$ and $f(p) = w$.

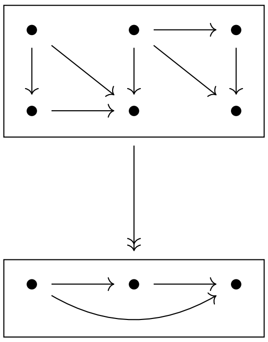
Lemma 2.1.1.1. *In the category of posets, convex maps are stable under pullback.*

In a poset P, we say that p' covers p, written $p \lessdot p'$, if $p < p'$ and there is no element x such that $p < x < p'$.

Definition 2.1.1.2. A map of posets $f\colon P \to Q$ is a *contraction* if:

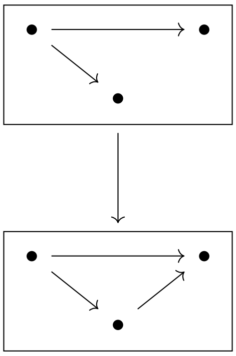
1. f is a monotone surjection;
2. for each $q \in Q$, the fibre P_q is a connected convex subposet of P;
3. for any cover $q \lessdot q'$ in Q, there exists a cover $p \lessdot p'$ in P such that $f(p) = q$ and $f(p') = q'$.

Remark 2.1.1.3. The following picture gives an illustration of a contraction



Note the importance of demand only to lift covers $\lessdot$; if we had demanded to be able to lift $<$, then the above map would not be an example of contraction.

Remark 2.1.1.4. The following picture gives an illustration of a monotone surjection that satisfies condition (2) in Definition 2.1.1.2 but does not lift covers



If we allow this map to be a contraction, Lemma 2.2.0.6 would be false. This lemma is a key ingredient for developing the connection between directed hereditary species and decomposition spaces, so cover lifting is necessary.

Lemma 2.1.1.5. *In the category of posets, contractions are stable under pullback along convex maps.*

Proof. Let P, Q and V be posets. Let $f\colon P \twoheadrightarrow V$ be a contraction. Let $g\colon Q \to V$ be a convex map, and let

$$\begin{tikzcd} P \times_V Q \arrow[r, "\pi_Q"] \arrow[d, "\pi_P"'] \arrow[dr, phantom, "\lrcorner\ (1)"] & Q \arrow[d, "g"] \\ P \arrow[r, two heads, "f"'] & V \end{tikzcd}$$

be a pullback diagram. Since monotone surjections are stable under pullback, we have that π_Q is a monotone surjection. Let us see that for each $q \in Q$, the fibre $(P \times_V Q)_q$ is a connected convex subposet. Since the diagram (1) is a pullback and g is a convex map, for each $q \in Q$, the map $\pi_P : (P \times_V Q)_q \to P_{g(v)}$ is a convex map. Furthermore, $P_{g(v)}$ is a connected convex subposet of P since f is a contraction. Combining this with the convex property of π_P, it follows that $(P \times_V Q)_q$ is a connected convex subposet of $P \times_V Q$. It only remains to prove that for any cover $q \lessdot q'$ in Q, there exists a cover $(p, q) \lessdot (p', q')$ in $P \times_V Q$. Let $q \lessdot q'$ be a cover in Q. Since g is convex, we have that $g(q) \lessdot g(q')$. Since f is a contraction, there exists p and p' in P such that $p \lessdot p'$, and $f(p) = g(q)$ and $f(p') = g(q')$. This means that $(p, q) \lessdot (p', q')$. $\square$

2.1.2 *Partially defined contractions*

The theory of species was introduced by Joyal [69] as a combinatorial theory of formal power series. Through this notion, Joyal showed that manipulations with generating functions can be carried out directly on the combinatorial structures themselves. A *species* is a functor

$$F\colon \mathbb{B} \to \mathbf{Set}$$

from the category of finite sets and bijections to the category of sets. Usual types of species include graphs, trees, endomorphisms, permutations, etc.

Schmitt [85] extended the notion of species in such a way that the combinatorial structures are classified according to their functorial properties. Let $\mathbb{I}$ be the category of finite sets and injections: a *restriction species* [85] is a functor $R\colon \mathbb{I}^{op} \to \mathbf{Set}$. Let $\mathbb{S}_p$ be the category of finite sets and partial surjections: a *hereditary species* [85] is a functor $H\colon \mathbb{S}_p \to \mathbf{Set}$. If partial surjections are portrayed as spans $P \overset{\iota}{\leftarrowtail} P' \overset{f}{\twoheadrightarrow} Q$, we can say that H is contravariant in injections and covariant in surjections.

Gálvez, Kock, and Tonks [57] generalised the notion of restriction species to cover examples with posets, trees and many related structures. A *directed restriction species* [57] is a functor $R\colon \mathbb{C}^{op} \to \mathbf{Grpd}$, where $\mathbb{C}$ is the category of finite posets and convex maps. We introduce the new notion of *connected directed hereditary species* to cover examples related to the Fauvet–Foissy–Manchon comodule bialgebra of finite topologies and admissible maps; and the Calaque–Ebrahimi-Fard–Manchon comodule bialgebra of rooted trees. But it does not cover Schmitt hereditary species. For such species, we need the non-connected case, which will be defined in Section 2.8.

A *partially defined contraction* $P \to Q$ consists of a convex map $\iota\colon P' \to P$ and a contraction $f\colon P' \twoheadrightarrow Q$ depicted in the diagram below:

$$\begin{tikzcd}
& P' \arrow[dl, tail, "\iota"'] \arrow[dr, two heads, "f"] & \\
P & & Q.
\end{tikzcd}$$

Let $\mathbb{K}_p$ denote the category of finite connected non-empty posets, and whose morphisms are partially defined contractions. The composite of $P \overset{\iota}{\leftarrowtail} P' \overset{f}{\twoheadrightarrow} Q$ and $Q \overset{\iota'}{\leftarrowtail} Q' \overset{f'}{\twoheadrightarrow} V$ in $\mathbb{K}_p$ is obtained from the diagram

$$\begin{tikzcd}
& & V' \arrow[dl, tail, "pr_1"'] \arrow[dr, two heads, "pr_2"] \arrow[d, phantom, "\lrcorner"] & & \\
& P' \arrow[dl, tail, "\iota"'] \arrow[dr, two heads, "f"] & {} & Q' \arrow[dl, tail, "\iota'"'] \arrow[dr, two heads, "f'"] & \\
P & & Q & & V
\end{tikzcd}$$

as the span $P \xleftarrow{\iota' pr_1} V' \xrightarrow{f' pr_2} V$. Here V' is the pullback of f and ι' in the category of posets. By the stability property of convex maps (2.1.1.1) and contractions under pullbacks (2.1.1.5), we have that pr_1 is convex and pr_2 is a contraction.

Definition 2.1.2.1. A *connected directed hereditary species* is a functor $H\colon \mathbb{K}_p \to \mathbf{Grpd}$.

Note that H is covariant in contractions and contravariant in convex maps. An element of $H[P]$ is called a H-structure on the finite poset P. An H-structure on a poset P also induces a H structure on any quotient poset and on any convex subposet. Furthermore, these functorialities are compatible in the sense that for any pullback diagram

$$\begin{tikzcd}
P \arrow[d, two heads, "f"'] & P' \arrow[l, tail, "u"'] \arrow[d, two heads, "f'"] \\
Q & Q' \arrow[l, tail, "v"]
\end{tikzcd}$$

we have

$$H[f'] \circ H[u] = H[v] \circ H[f].$$

This ‘Beck–Chevalley’ law is a consequence of the fact that H must respect the composition of spans.

2.1.3 *Coalgebras from directed connected hereditary species*

As in the case of Schmitt hereditary species, connected directed hereditary species give rise to coalgebras. Let P be a poset and X an H-structure on P. The comultiplication of X is given by

$$\Delta(X) = \sum_{f\colon P \twoheadrightarrow Q} \{X|P_q\}_{q\in Q} \otimes X_! Q,$$

where the sum ranges over isomorphism classes of contractions $P \twoheadrightarrow Q$. Here $X|P_q$ is the restriction of X to the fibre P_q for every $q \in Q$, and $X_! Q$ is the H-structure $H(f)(X)$ in $H(Q)$.

2.1.4 *Calaque–Ebrahimi-Fard–Manchon comodule bialgebra of rooted trees: part I*

We will see that the coalgebra of rooted trees studied by Calaque, Ebrahimi-Fard, and Manchon [22] comes from a connected directed hereditary species. For this, we need some preliminaries.

Let P be a poset. For p, p' in P, an interval $[p, p']$ is *linear* if for each $z, w \in [p, p']$, we have that $z \leqslant w$ or $w \leqslant z$.

Definition 2.1.4.1. A *tree* T is a poset with a terminal object where every interval is linear.

A *forest* is a disjoint union of trees. We need the following results to obtain a connected directed hereditary species with connected posets and contractions. Let P be a poset. For each $x \in P$, we define $P_{/x} := \{p \in P | p \leqslant x\}$.

Lemma 2.1.4.2. *Let P be a tree. For each $x \in P$, the poset $P_{/x}$ is a tree with root x.*

Lemma 2.1.4.3. *Let P be a tree. Let S be a convex subposet of P, put $P_{/S} = \sum_{x\in S} P_{/x}$. Then $P_{/S}$ is a forest.*

Proof. Since every interval in P is linear, it is straightforward to see that $P_{/S}$ is a disjoint union of posets. Moreover, for each $x \in S$, the poset $P_{/x}$ is a tree by Lemma 2.1.4.2. Hence, $P_{/S}$ is a forest. ☐

Lemma 2.1.4.4. *Let T be a forest. Let S be a convex subposet of T. Then S is a forest.*

Lemma 2.1.4.5. *Let $f\colon P \twoheadrightarrow Q$ be a contraction between posets. If P is a tree, then Q is a tree.*

Proof. Let $\top_P$ denote the terminal object of P. The object $f(\top_P)$ is terminal in Q. Indeed, let q be an object in Q. Since f is a surjection, there exists $p \in P$ such that $f(p) = q$. Furthermore, $p < \top_P$ since $\top_P$ is terminal, and therefore $f(p) < f(\top_P)$ by the monotonicity condition of f. Now we will prove that every interval in Q is linear. Let $[q_1, q_2]$ be an interval in Q, and let $\iota\colon [q_1, q_2] \to Q$ denote the canonical convex map. Consider the pullback diagram

$$\begin{tikzcd}
f^{-1}([q_1, q_2]) \arrow[r, tail] \arrow[d, two heads, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, two heads, "f"] \\
{[q_1, q_2]} \arrow[r, tail, "\iota"'] & Q.
\end{tikzcd}$$

Here f' is a contraction since contractions are stable under pullback along convex maps (2.1.1.5). Since f is a contraction and ι is a convex map, we have that $f^{-1}([q_1, q_2])$ is a connected convex subposet of P. For each z and w in $[q_1, q_2]$, there exists a and b in $f^{-1}([q_1, q_2])$ such that $f'(a) = z$ and $f'(b) = w$ since f' is surjective. By the connectivity property of $f^{-1}([q_1, q_2])$, we have that a and b are connected by a ziz-zag. The contraction property of f' forces that the ziz-zag is of the form: $a < \cdots < b$ or $b < \cdots < a$. By the monotonicity property of f' follows that $z < w$ or $w < z$. Hence, $[q_1, q_2]$ is linear. ☐

Lemma 2.1.4.6. *Let* $f \colon P \twoheadrightarrow Q$ *be a contraction between posets. Suppose, moreover, that* P *is a forest, then* Q *is a forest.*

The functor $H_{CEM} : \mathbb{K}_p \to \mathbf{Grpd}$ is defined as follows: if P is a tree $H_{CEM}(P)$ has one object, which is P itself, and if P is not a tree $H_{CEM}(P)$ is empty. By Lemma 2.1.4.4, we have that H_{CEM} is contravariant in convex maps. By Lemma 2.1.4.6, the functor H_{CEM} is covariant in contractions. Therefore, H_{CEM} is a connected directed hereditary species.

Remark 2.1.4.7. The comultiplication $\Delta_{H_{CEM}}$ of the incidence coalgebra of H_{CEM} is given by:

$$\Delta_{H_{CEM}}(T) = \sum_{f : T \twoheadrightarrow Q} \{T_q\}_{q \in Q} \otimes Q.$$

This coalgebra is the Calaque–Ebrahimi-Fard–Manchon coalgebra of rooted trees [22] .

Example 2.1.4.8 (Faà di Bruno comodule bialgebra of linear trees, part I). A *linear tree* is one in which every node has precisely one input edge. The functor $H_{FB} : \mathbb{K}_p \to \mathbf{Grpd}$ is defined as follows: if P is a linear tree $H_{FB}(P)$ has one object, which is P itself, and if P is not a linear tree $H_{FB}(P)$ is empty. By Lemmas 2.1.4.4 and 2.1.4.6, we have that H_{FB} is contravariant in convex maps and covariant in contractions. Therefore, H_{FB} is a connected directed hereditary species. The bialgebra associated to H_{CEM} is the Faà di Bruno bialgebra of linear trees since there is a one-to-one correspondence between comultiplying contractions of linear trees and comultiplying monotone surjections $[n] \twoheadrightarrow 1$. The way it appears here is like in algebraic topology where it is the (dual) Landweber–Novikov bialgebra (see [82, §3]), whereas the usual presentation of the Faà di Bruno bialgebra is with (non monotone) surjections (see for example [27] and [72]) or with partitions, as in [42]. Over the rational numbers the two are isomorphic (see for example [36]).

2.2 *The decomposition space of contractions*

In this section we will introduce the decomposition space of contractions $\mathbf{K}$, but first we need some preliminaries.

Let $\mathbf{Cat}_{lt}$ denote the category of categories with chosen local terminals, or equivalently upper-dec coalgebras [54]. Let $\mathbb{K}$ denote the category of finite connected non-empty posets and contractions.

Example 2.2.0.1. In the category $\mathbb{K}$ of connected finite non-empty posets and contractions a chosen terminal object is a poset with one element.

The *t-simplex category* $\mathbb{\Delta}^t$ is the category whose objects are the nonempty finite ordinals and whose morphisms are the monotone maps that preserve the top element.

Definition 2.2.0.2. For $\mathcal{C}$ a category with chosen local terminals, its *fat* lt-*nerve* $\mathbf{N}^{lt}(\mathcal{C})$ is the $\mathbf{Grpd}$-valued $\mathbb{\Delta}^t$-presheaf describe as follows: for $n \geqslant 1$, the groupoid $\mathbf{N}^{lt}(\mathcal{C})_n$ is the same as the groupoid $\mathbf{N}(\mathcal{C})_{n-1}$. The groupoid $\mathbf{N}^{lt}(\mathcal{C})_0$ is the groupoid of chosen local terminal objects in $\mathcal{C}$. The face and degeneracy maps act as the usual fat nerve construction except in $d_\perp \colon \mathbf{N}^{lt}(\mathcal{C})_1 \to \mathbf{N}^{lt}(\mathcal{C})_0$ that sends each object in $\mathcal{C}$ to its corresponding chosen local terminal object. The degeneracy map $s_0 \colon \mathbf{N}^{lt}(\mathcal{C})_0 \to \mathbf{N}^{lt}(\mathcal{C})_1$ is the inclusion.

To simplify the notation we define $\mathbf{K}^\circ := \mathbf{N}^{lt}(\mathbb{K})$. Thus $\mathbf{K}^\circ_2$ is the groupoid of contractions, and $\mathbf{K}^\circ_1$ is the groupoid of finite connected non-empty posets and monotone bijections. $\mathbf{K}^\circ_0$ is the terminal groupoid. To obtain the top face maps, it is necessary to introduce families through the symmetric monoidal category functor. We define $\mathbf{K} := \mathsf{S}\mathbf{K}^\circ$ to be the symmetric monoidal category functor S applied to $\mathbf{K}^\circ$. All the face maps (except the missing top ones) and degeneracy maps are just S applied to the face and degeneracy maps of $\mathbf{N}^{lt}(\mathbb{K})$. The top face map is given by:

- For $n \geqslant 2$, the top face map $d_\top \colon \mathbf{K}_n \to \mathbf{K}_{n-1}$ is defined as follows: given a $(n-1)$-chain of contractions
$$\begin{tikzcd} P_0 \arrow[r, two heads, "f_0"] & P_1 \arrow[r, two heads, "f_1"] & P_2 \arrow[r, two heads] & \cdots \arrow[r, two heads] & P_{n-2} \arrow[r, two heads, "f_{n-2}"] & Q, \end{tikzcd}$$
for each element $q \in Q$, we can form the fibres over q. We end up with a family
$$\begin{tikzcd} \{(P_0)_q \arrow[r, two heads, "(f_0)_q"] & (P_1)_q \arrow[r, two heads, "(f_1)_q"] & (P_2)_q \arrow[r, two heads] & \cdots \arrow[r, two heads] & (P_{n-3})_q \arrow[r, two heads, "(f_{n-2})_q"] & (P_{n-2})_q\}_{q \in Q}. \end{tikzcd}$$
For each $0 \leqslant i \leqslant n-2$, the map $(f_i)_q$ is a contraction by the stability under pullback of the contractions (Lemma 2.1.1.5).
- $d_\top \colon \mathbf{K}_1 \to \mathbf{K}_0$ sends a family of finite posets $\{Q_i\}_{i \in I}$ to the family whose components are the terminal poset 1 indexed by the disjoint union $\sum_{i \in I} Q_i$.

Note that the fibres are convex non-empty subposets since we only consider contractions, not arbitrary maps. The simplicial identities will be verified in Proposition 2.2.0.5.

To simplify the proof that $\mathbf{K}$ is a decomposition space, we need some preliminaries.

Proposition 2.2.0.3. *We have an equality* $\mathrm{Dec}_\top \mathbf{K} = \mathbf{S}\mathbf{N}\mathbb{K}$.

Proof. The proof follows from combining that $\mathbf{K} = \mathbf{S}\mathbf{N}^{\mathrm{lt}}(\mathbb{K})$, the definition of the fat lt-nerve, and that taking upper decalage is deleting, in each degree, the top face map and the top degeneracy map. ☐

Remark 2.2.0.4. Since $\mathbb{K}$ is a category, its fat nerve $\mathbf{N}\mathbb{K}$ is a Segal space 1.2.1, and therefore $\mathbf{S}\mathbf{N}\mathbb{K}$ is a Segal space, as $\mathbf{S}$ preserves pullbacks and hence Segal objects. By Proposition 2.2.0.3, we have that $\mathrm{Dec}_\top \mathbf{K} = \mathbf{S}\mathbf{N}\mathbb{K}$. Combining everything, we have that $\mathrm{Dec}_\top \mathbf{K}$ is a Segal space and therefore for each $n \geqslant 2$ the following diagram is a pullback for $0 < i < n$:
$$\begin{tikzcd} \mathbf{K}_{n+1} \arrow[r, "d_{i+1}"] \arrow[d, "d_\perp"'] & \mathbf{K}_n \arrow[d, "d_\perp"] \\ \mathbf{K}_n \arrow[r, "d_i"'] & \mathbf{K}_{n-1}. \end{tikzcd}$$

Proposition 2.2.0.5. *The groupoids* $\mathbf{K}_n$ *and the degeneracy and face maps given above form a pseudosimplicial groupoid* $\mathbf{K}$.

Proof. The only pseudosimplicial identity is $d_\top d_\top \simeq d_\top d_{\top-1}$. The other simplicial identities are strict and follows from equality $\mathrm{Dec}_\top \mathbf{K} = \mathbf{S}\mathbf{N}\mathbb{K}$ of Proposition 2.2.0.3. Let us prove $d_\top d_\top(\mathbf{K}_n) \simeq d_\top d_{\top-1}(\mathbf{K}_n)$ for $n = 2$. For greater n the proof is completely analogous. Given an object
$$\begin{tikzcd} P \arrow[r, two heads, "f"] & Q \arrow[r, two heads, "g"] & V \end{tikzcd}$$
in $\mathbf{K}_3$, we consider the following commutative diagram for each $v \in V$
$$\begin{tikzcd} (P_v)_q \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P_v \arrow[r, tail] \arrow[d, two heads, "f_v"] \arrow[dr, phantom, "\lrcorner"] & P \arrow[d, two heads, "f"] \\ 1 \arrow[r, "\ulcorner q \urcorner"'] & Q_v \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Q \arrow[d, "g"] \\ & 1 \arrow[r, "\ulcorner v \urcorner"'] & V. \end{tikzcd}$$
It is easy to see that $d_\top d_\top(\mathbf{K}_3) = \{(P_v)_q\}_{q \in Q}$. Furthermore, note that the top horizontal rectangle is isomorphic to P_q for each $q \in Q$. This implies that $d_\top d_\top(\mathbf{K}_3) \simeq d_\top d_{\top-1}(\mathbf{K}_3)$ since $d_\top d_{\top-1}(\mathbf{K}_3) = \{P_q\}_{q \in Q}$.

☐

Lemma 2.2.0.6. *Suppose we have a contraction $f \colon P \twoheadrightarrow Q$ between connected posets, and a family of contractions $\{h_q \colon P_q \twoheadrightarrow W_q\}_{q \in Q}$ where each P_q and W_q are connected posets. Then there exists a unique connected poset W and contractions h and g such that the diagram*

$$\begin{tikzcd}
P \arrow[r, dotted, two heads, "h"'] \arrow[rr, bend left, two heads, "f"] & W \arrow[r, dotted, two heads, "g"'] & Q \\
P_q \arrow[u, tail] \arrow[r, two heads, "h_q"'] & W_q \arrow[u, dotted, tail] &
\end{tikzcd}$$

commutes. Here the vertical arrows are convex inclusions.

Proof. We will do the proof in three steps: in the first place, we will construct the underlying set of the poset W and the functions $h \colon P \to W$ and $g \colon W \to V$. After that, we will construct a partial order $<_W$ on W forced by the requirement that h and g are monotone maps. To conclude, we will prove that f and g are contractions.

1. Put $W := \sum_{q \in Q} W_q$ and $h := \sum_{q \in Q} h_q$. The map $g \colon W \to Q$ is defined as $g(w) = q$ for $w \in W_q$. Furthermore, the diagram

   $$\begin{tikzcd}
   P \arrow[r, dotted, two heads, "h"'] \arrow[rr, bend left, two heads, "f"] & W \arrow[r, dotted, two heads, "g"'] & Q \\
   P_q \arrow[u, tail] \arrow[r, two heads, "h_q"'] & W_q \arrow[u, dotted, tail] \arrow[r] & 1 \arrow[u, "\ulcorner q \urcorner"']
   \end{tikzcd}$$

   commutes at the level of sets by the way h and g were defined.

2. The partial order $<_W$ on W is defined by taking transitive closure in the following relation: for $w, w' \in W$, we declare that $w <_W w'$ if one of the following conditions is satisfied:

   (a) In case $w, w' \in W_q$ and $w <_{W_q} w'$;

   (b) There exist $p <_P p'$ in P such that $h(p) = w$ and $h(p') = w'$.

   The condition (b) is necessary for h and g to be monotone maps.

3. We will prove that h is a contraction. Let $w \in W$, by the way W was defined, we have that $w \in W_q$ for some $q \in Q$. Recall that contractions and convex maps are stable under pullback by Lemmas 2.1.1.1 and 2.1.1.5. These imply that in the commutative diagram

   $$\begin{tikzcd}
   P \arrow[r, two heads, "h"] & W \\
   P_q \arrow[u, tail] \arrow[r, two heads, "h_q"] & W_q \arrow[u, tail] \\
   P_w \arrow[u, tail] \arrow[r] \arrow[ur, phantom, "\urcorner", very near start] & 1, \arrow[u, "\ulcorner w \urcorner"']
   \end{tikzcd}$$

   for each $w \in W$, the poset P_w is connected and convex since h_q is a contraction. Hence, h is a contraction. By analogous arguments, we have that g is a contraction.

The poset W is connected since g is a contraction and Q is connected. □

Lemma 2.2.0.7. *For each $0 < i < n$, the map $d_i \colon \mathbf{K}_n \to \mathbf{K}_{n-1}$ is a fibration.*

Proof. Since S preserves fibrations and $d_i\colon \mathbf{K}_n \to \mathbf{K}_{n-1}$ is equal to $\mathsf{S}(d_i)\colon \mathsf{S}(\mathbf{K}^\circ_n) \to \mathsf{S}(\mathbf{K}^\circ_{n-1})$, it is enough to prove that $d_i\colon \mathbf{K}^\circ_n \to \mathbf{K}^\circ_{n-1}$ is a fibration. We will only check that $d_1 : \mathbf{K}^\circ_2 \to \mathbf{K}^\circ_1$ is a fibration, the other cases use similar arguments. Let $f\colon P \twoheadrightarrow Q$ be an object in $\mathbf{K}^\circ_2$. Let $u : P' \to P$ be a morphism in $\mathbf{K}^\circ_1$. It is straightforward to check that the morphism $\sigma\colon f \circ u \to f$, pictured in the diagram

$$\begin{tikzcd}
P' \arrow[r, "u"] \arrow[d, two heads, "f\circ u"'] \arrow[dr, phantom, "\sigma"] & P \arrow[d, two heads, "f"] \\
Q \arrow[r, "\mathrm{id}_Q"'] & Q,
\end{tikzcd}$$

is a lift of the morphism u in $\mathbf{K}^\circ_2$ such that $d_1(\sigma) = u$. □

Proposition 2.2.0.8. *The pseudosimplicial groupoid* $\mathbf{K}$ *is a decomposition space.*

Proof. We will prove that the following diagrams are pullbacks for $0 < i < n$:

$$\begin{tikzcd}
\mathbf{K}_{n+1} \arrow[r, "d_{i+1}"] \arrow[d, "d_\perp"'] \arrow[dr, phantom, "(1)"] & \mathbf{K}_n \arrow[d, "d_\perp"] \\
\mathbf{K}_n \arrow[r, "d_i"'] & \mathbf{K}_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{K}_{n+1} \arrow[r, "d_i"] \arrow[d, "d_\top"'] \arrow[dr, phantom, "(2)"] & \mathbf{K}_n \arrow[d, "d_\top"] \\
\mathbf{K}_n \arrow[r, "d_i"'] & \mathbf{K}_{n-1}.
\end{tikzcd}$$

Let us prove it for $n = 2$. For greater n the proof is completely analogous. The square (1) is a pullback by Remark (2.2.0.4). To prove that the square (2) is a pullback we will use Lemma 1.2.0.1. This means that (2) is a pullback if and only if for each object $f\colon P \twoheadrightarrow Q$ in $\mathbf{K}_2$, the map $d_\top\colon \mathrm{Fib}_f\, d_1 \to \mathrm{Fib}_{d_\top f}\, d_1$ is an equivalence of groupoids. For an object $\{h_q\colon P_q \twoheadrightarrow W_q\}_{q\in Q}$ in $\mathrm{Fib}_{d_\top f}\, d_1$, Lemma 2.2.0.6 gives contractions h and g such that the diagram commutes

$$\begin{tikzcd}
P \arrow[r, two heads, "h"'] \arrow[rr, bend left, two heads, "f"] & W \arrow[r, two heads, "g"'] & Q \\
P_q \arrow[u, tail] \arrow[r, two heads, "h_q"'] & W_q. \arrow[u, tail] &
\end{tikzcd}$$

The commutativity of the diagram implies that $P \overset{f}{\twoheadrightarrow} W \overset{g}{\twoheadrightarrow} Q$ is an object in $\mathrm{Fib}_f\, d_1$. Therefore, $d_\top$ is surjective on objects. The map $d_\top$ is full. Indeed, for any morphism $u_q r_q\colon f_q \to f'_q$ in $\mathrm{Fib}_{d_\top f}\, d_1$, put $u = \sum_{q\in Q} u_q$ and $r = \sum_{q\in Q} r_q$. The map $u r\, \mathrm{id}_Q$ satisfies that $d_\top(u r\, \mathrm{id}_Q) = u_q r_q$. Furthermore, the diagram

$$\begin{tikzcd}
 & P \arrow[rr, two heads, "h'"] & & W' \arrow[dr, two heads, "g'"] & \\
P_q \arrow[ur, tail] \arrow[rr, two heads, "h'_q" near end] & & W'_q \arrow[ur, tail] & & Q \\
 & P \arrow[uu, "u"' near start] \arrow[rr, two heads, "h" near end] & & W \arrow[uu, "r"'] \arrow[ur, two heads, "g"'] & \\
P_q \arrow[uu, "u_q"] \arrow[ur, tail] \arrow[rr, two heads, "h_q"'] & & W_q \arrow[uu, "r_q" near start] \arrow[ur, tail] & &
\end{tikzcd}$$

commutes by Lemma 2.2.0.6 and the definitions of u and r. This implies that $u r\, \mathrm{id}_Q$ is a morphism in $\mathrm{Fib}_f\, d_1$ and hence $d_\top$ is full. To prove that $d_\top$ is faithful, let $u q\, \mathrm{id}_Q$ and $u' q'\, \mathrm{id}_Q$ be morphisms in $\mathrm{Fib}_f\, d_1$ such that $d_\top u q\, \mathrm{id}_Q = d_\top u' q'\, \mathrm{id}_Q$ in $\mathrm{Fib}_{d_\top f}\, d_1$. This means that for each $q \in Q$, we have $u_q = u'_q$ and $r_q = r'_q$, but ur and $u'r'$ are determined by $u_q r_q$ and $u'{}_q r'{}_q$, hence $u = u'$ and $r = r'$. □

Recall that a decomposition space X is complete when $s_0\colon X_0 \to X_1$ is a monomorphism.

Proposition 2.2.0.9. *The decomposition space* $\mathbf{K}$ *is complete.*

Proof. Note that $s_0 \colon \mathbf{K}_0 \to \mathbf{K}_1$ is actually S of the map $s_0 \colon \mathbf{K}^\circ_0 \to \mathbf{K}^\circ_1$, and S preserves pullbacks and hence monomorphism. This means that $s_0 \colon \mathbf{K}_0 \to \mathbf{K}_1$ is a monomorphism if $s_0 \colon \mathbf{K}^\circ_0 \to \mathbf{K}^\circ_1$ is a monomorphism, but this is clear since $\mathbf{K}^\circ_0$ is the terminal groupoid consisting of only the poset with one element and s_0 sends the terminal groupoid to the poset with one element.

□

Proposition 2.2.0.10. *The decomposition space* $\mathbf{K}$ *is locally finite, locally discrete and of locally finite length.*

Proof. Since S respects finite maps and $\mathbf{K} = S\mathbf{K}^\circ$, we will prove that $\mathbf{K}^\circ$ is locally finite, locally discrete and of locally finite length:

- Note that a connected finite non-empty poset has only a finite number of automorphisms. This means that each object in $\mathbf{K}^\circ_1$ has a finite number of automorphisms, and therefore, $\mathbf{K}^\circ_1$ is locally finite.
- In the proof of Proposition 2.2.0.9, we see that $s_0 \colon \mathbf{K}^\circ_0 \to \mathbf{K}^\circ_1$ is finite and discrete. Since $\mathbf{K}^\circ_1$ is locally finite and s_0 is finite, to prove that $\mathbf{K}^\circ$ is locally discrete and locally finite, we have to check that $d_1 \colon \mathbf{K}^\circ_2 \to \mathbf{K}^\circ_1$ is discrete and finite. In the proof of Lemma 2.2.0.7, we showed that d_1 is a fibration so we will use strict fibres. In the strict pullback diagram

$$\begin{tikzcd}
\mathrm{Fib}_{d_1}(P) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\ulcorner P \urcorner"] \\
\mathbf{K}^\circ_2 \arrow[r, "d_1"'] & \mathbf{K}^\circ_1,
\end{tikzcd}$$

  let $f \colon P \twoheadrightarrow Q$ and $f' \colon P \twoheadrightarrow Q'$ be objects in $\mathrm{Fib}_{d_1}(P)$. A morphism $u \colon f \to f'$ in $\mathrm{Fib}_{d_1}(P)$, it is in fact a monotone bijection $u \colon Q \to Q'$ such that $u \circ f = f'$. This equality with the monotone surjection condition of f and f' force that u is unique. Therefore, $\mathrm{Fib}_{d_1}(P)$ is discrete and $\mathbf{K}^\circ$ is locally discrete. Furthermore, the discrete groupoid $\mathrm{Fib}_{d_1}(P)$ is finite since we have a finite number of contractions whose source is P by the finite condition of P. Therefore, $\mathbf{K}^\circ$ is locally finite.
- $\mathbf{K}^\circ$ is of locally finite length. Indeed, the fibre of P along $\mathbf{K}^\circ_n \mathrel{\rightarrow\!\!\!\mid} \mathbf{K}^\circ_1$ has no degenerate simplices for n greater than the cardinality of P.

□

The decomposition space $\mathbf{K}$ has a monoidal structure given by disjoint union. Recall $\mathbf{K}_n$ is the groupoid of families of $(n-1)$-chains of contractions. The disjoint union of two such families is just the family whose components are the objects of the families index by the disjoint union of the two index sets. This clearly defines a simplicial map $+_{\mathbf{K}} \colon \mathbf{K} \times \mathbf{K} \to \mathbf{K}$. So $\mathbf{K}$ is a monoidal decomposition space if the map $+_{\mathbf{K}}$ is culf [58, §9].

Proposition 2.2.0.11. *The map* $+_{\mathbf{K}} \colon \mathbf{K} \times \mathbf{K} \to \mathbf{K}$ *is culf.*

Proof. By Lemma 1.1.2.2, the map $+_{\mathbf{K}}$ is culf, if the diagram

$$\begin{tikzcd}
\mathbf{K}_2 \times \mathbf{K}_2 \arrow[r, "d_1"] \arrow[d, "+_{\mathbf{K}}"'] & \mathbf{K}_1 \times \mathbf{K}_1 \arrow[d, "+_{\mathbf{K}}"] \\
\mathbf{K}_2 \arrow[r, "d_1"'] & \mathbf{K}_1
\end{tikzcd}$$

is a pullback since $\mathbf{K}$ is a decomposition space. But this is clear: a pair of families of contractions (an object in $\mathbf{K}_2 \times \mathbf{K}_2$) can be uniquely reconstructed if we know what the two source families of posets are (an object in $\mathbf{K}_1 \times \mathbf{K}_1$) and we know how the disjoint union is contract (an object in $\mathbf{K}_2$). This is subject to identifying the disjoint union of those two source families of posets with the source of the disjoint union of the two families of contractions (which is to say that the data agree down in $\mathbf{K}_1$). ☐

Since $\mathbf{K}$ is a monoidal decomposition space, it follows that the resulting incidence coalgebra is also a bialgebra [58, §9].

2.3 *The decomposition space of admissible maps*

In this section we will introduce the decomposition space of admissible maps $\mathbf{A}$.

Definition 2.3.0.1. [40, §2.2] Let $T' \to T$ be an identity-on-objects monotone map between two preorders. We define T/T' to be the preorder with the same objects as T and T', and the preorder is defined by closing the relation $\mathcal{R}$ by transitivity:

$$x\mathcal{R}y \iff (x \leqslant_T y \text{ or } y \leqslant_{T'} x).$$

In other words, we obtain T/T' by inverting those arrows of T that also belong to T', and then closing by transitivity. Later in this section, we will use the following more categorical characterisation of quotients.

The underlying set of a preorder T will be denoted as $\underline{T}$, and $\overline{T}$ denotes the discrete preorder of connected components of T. We have a natural map $\mathrm{comp}\colon T \to \overline{T}$ that sends each object to its corresponding connected component.

Lemma 2.3.0.2. *Let $T' \to T$ be an identity-on-objects monotone map between two preorders. Then T/T' is the pushout in category of preorders*

$$\begin{tikzcd}
T' \arrow[r] \arrow[d] & T \arrow[d] \\
T'/T' \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & T/T'.
\end{tikzcd}$$

Proof. Since the maps are identity-on-objects then $\underline{T/T'} = \underline{T}$. This means that at the level of sets the square is a pushout. Consider the preorder given by closing the relation $\mathcal{R}$ by transitivity:

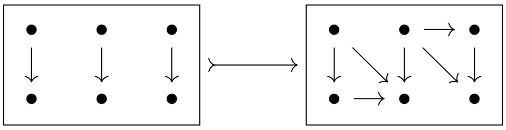
$$x\mathcal{R}y \iff (x \leqslant_T y \text{ or } y \leqslant_{T'} x).$$

It is straightforward to see that the diagram commutes at the level of preorders. In case that we have another preorder $<'$ in T/T' that made the square commutes, the monotonicity of the maps force that $<'$ is equal to the transitive closure of $\mathcal{R}$. ☐

Definition 2.3.0.3. [40, Definition 2.2.] A map of preorders $T' \to T$ is *admissible* if it satisfies the following:

1. it is the identity-on-objects,
2. for every connected sub-preorder Y in T', we have that $T_{|\underline{Y}} = Y$,
3. $x \sim_{T/T'} y$ if and only if $x \sim_{T'/T'} y$.

The following picture gives an illustration of an admissible map of preorders:



Lemma 2.3.0.4. *In the category of preorders, admissible maps are stable under pullback along identity-on-objects monotone maps.*

Proof. Let $\beta\colon V' \rightarrowtail V$ be an admissible map and let $f\colon T \to V$ be an identity-on-objects monotone map. Consider the pullback diagram

$$\begin{tikzcd}
V' \times_V T \arrow[r, "\pi_T"] \arrow[d, "\pi_{V'}"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "f"] \\
V' \arrow[r, tail, "\beta"'] & V.
\end{tikzcd}$$

It is easy to see that π_T is an identity-on-objects monotone map. For a connected sub-preorder Y of $V' \times_V T$, we have that $\pi_{V'}(Y)$ is a connected sub-preorder of V' since $\pi_{V'}$ is a projection map. By the admissible condition of β, it follows that $\beta(\pi_{V'})$ is a connected sub-preorder of V and $V|_Y = Y$. Combining this with the commutative of the square and the identity-on-objects monotone condition of the arrows it follows that $\pi_T(Y)$ is a connected sub-preorder of T and $T|_Y = Y$. To prove that π_T is admissible, all that remains is to check $x \sim_{T/(V' \times_V T)} y$ if and only if $x \sim_{(V' \times_V T)/(V' \times_V T)} y$, but the commutative condition of the square implies that $x \sim_{T/(V' \times_V T)} y$ if $x \sim_{V/V'} y$ and $x \sim_{(V' \times_V T)/(V' \times_V T)} y$ if $x \sim_{V'/V'} y$. In other words $x \sim_{V/V'} y$ if and only if $x \sim_{V'/V'} y$ but this is true as consequence of the admissible condition of β. Hence π_T is admissible. ☐

The posetification of a preorder T will be denoted as $\widetilde{T}$. Let $\overline{P}$ denote the discrete poset of connected components of a poset P, which is the same as $\widetilde{P/P}$.

Lemma 2.3.0.5. *An identity-on-objects monotone map of preorders* $T' \to T$ *is admissible if and only if the pushout square*

$$\begin{tikzcd}
T' \arrow[r] \arrow[d] & T \arrow[d] \\
\overline{V'} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & V,
\end{tikzcd}$$

satisfies the following properties:

1. $\underline{V} = \overline{V'}$
2. *it is also a pullback square.*

Proof. First, note that condition 1 is equivalent to condition 3 of Definition 2.3.0.3. Indeed, if $x \sim_{T/T'} y \Leftrightarrow x \sim_{T'/T'} y$ then necessarily $V = \widetilde{T/T'}$ and $\overline{V'} = \widetilde{T'/T'}$ have the same objects, and vice-versa, because the objects of V are given by the equivalence classes of objects of T/T', and the objects of $\overline{T'}$ are given by the equivalence classes of objects of T'/T'. On the other hand, assuming these two conditions hold, condition 2 is equivalent to condition 2 of Definition 2.3.0.3. Indeed, the connected components of T' are the preimages of $\underline{V}$, and the fact that the square is a pullback is equivalent to the fact that the map $T' \to T$ is full on each connected component of T', which is precisely condition 2 of Definition 2.3.0.3. ☐

2.3.1 *Fauvet–Foissy–Manchon Hopf algebra of finite topologies and admissible maps*

Fauvet, Foissy, and Manchon ([40], §3.1) introduced the notion of quotient of a topology $\mathcal{T}$ on a finite set X by another topology $\mathcal{T}'$ finer than $\mathcal{T}$. The quotient topology $\mathcal{T}/\mathcal{T}'$ thus obtained lives on the same set. Furthermore, they introduced the relation $\circledless$ on the topologies on X defined by

$\mathcal{T}' \circledless \mathcal{T}$ if and only if $\mathcal{T}'$ is finer than $\mathcal{T}$ and fulfills the technical condition of $\mathcal{T}$-admissibility. This enabled them to give the internal coproduct

$$\Delta(\mathcal{T}) = \sum_{\mathcal{T}' \circledless \mathcal{T}} \mathcal{T}' \otimes \mathcal{T}/\mathcal{T}'.$$

Since there is a natural bijection between topologies and quasi-orders on a finite set X, Fauvet, Foissy, and Manchon also expressed the $\mathcal{T}$-admissibility in the context of preorders. This corresponds to the one we use in this section (2.3.0.1). The above coproduct is just rewritten in the context of preorders as:

$$\Delta(T) = \sum_{T' \rightarrowtail T} T' \otimes T/T'.$$

We shall see in 2.3.2.7 that this coproduct corresponds to the coproduct $\Delta_{\mathbf{A}}$ of the incidence coalgebra of the decomposition space of admissible maps $\mathbf{A}$.

2.3.2 *The decomposition space* A

A *groupoid preorder* is a preorder where all its morphisms are invertible. Given a finite preorder T, we denote by T^{inv} the groupoid preorder that contains the same objects that T but only the invertible morphisms of T.

We describe a pseudo simplicial groupoid (2.3.2.3) of admissible maps which we call $\mathbf{A}$. Let $\mathbf{A}_n$ denote the groupoid of $(n-1)$-chains of admissible maps between non-empty finite preorders. Thus, $\mathbf{A}_2$ is the groupoid of admissible maps and $\mathbf{A}_1$ is the groupoid of finite preorders whose underlying sets are ordinals and monotone bijections. $\mathbf{A}_0$ is the groupoid whose objects are the groupoid preorders. Face maps are given by:

- For $n \geqslant 2$, the bottom face map $d_\bot \colon \mathbf{A}_n \to \mathbf{A}_{n-1}$ is defined as follows: given $(n-1)$-chain of admissible maps

  $$T_0 \rightarrowtail T_1 \rightarrowtail T_2 \rightarrowtail \cdots \rightarrowtail T_{n-2} \rightarrowtail T_{n-1},$$

  for each poset in the chain, we can form the quotient T_i/T_0 by Lemma 2.3.0.2. We end up with a $(n-2)$-chain of admissible maps

  $$T_1/T_0 \rightarrowtail T_2/T_0 \rightarrowtail T_3/T_0 \rightarrowtail \cdots \rightarrowtail T_{n-2}/T_0 \rightarrowtail T_{n-1}/T_0;$$

- $d_\bot \colon \mathbf{A}_1 \to \mathbf{A}_0$ sends a preorder T to T^{inv}.
- d_1 forgets the first preorder in the chain;
- $d_i \colon \mathbf{A}_n \to \mathbf{A}_{n-1}$ composes the ith and $(i+1)$th admissible map, for $1 < i < n-1$;
- $d_\top$ forgets the last preorder in the chain.

Degeneracy maps are given by:

- $s_\bot \colon \mathbf{A}_n \to \mathbf{A}_{n+1}$ is given by appending with the map whose source is the underlying groupoid preorder of the first preorder of the chain.
- $s_i \colon \mathbf{A}_n \to \mathbf{A}_{n+1}$ inserts an identity arrow at object number i, for $0 < i \leqslant n$.

The simplicial identities will be verified in 2.3.2.3. Let $\mathbb{A}$ denote the category of finite preorders and admissible maps.

Proposition 2.3.2.1. *We have an equality* $\operatorname{Dec}_\bot \mathbf{A} = \mathbf{N}\mathbb{A}$.

Proposition 2.3.2.1 implies the compatibility of the face and degeneracy maps in $\mathbf{A}$ except the face maps $d_\perp$. We need the following result to verify the simplicial identities for $d_\perp$.

Lemma 2.3.2.2. *Let* $T_0 \rightarrowtail T_1$ *and* $T_1 \rightarrowtail T_2$ *be admissible maps. The following diagram*

$$\begin{tikzcd}
T_0 \arrow[r, tail] \arrow[d] & T_1 \arrow[r, tail] \arrow[d] & T_2 \arrow[d] \\
T_0/T_0 \arrow[r, tail] & T_1/T_0 \arrow[r, tail] \arrow[d] & T_2/T_0 \arrow[d] \\
 & T_1/T_1 \arrow[r, tail] & T_2/T_1
\end{tikzcd}$$

commutes and the squares are pushouts.

Proof. By the prism Lemma, the outer diagram

$$\begin{tikzcd}
T_1 \arrow[r, tail] \arrow[d] & T_2 \arrow[d] \\
T_1/T_0 \arrow[r, tail] \arrow[d] \arrow[ur, phantom, "\ulcorner", very near end] & T_2/T_0 \arrow[d] \\
\frac{(T_1/T_0)}{(T_1/T_0)} \arrow[r, tail] \arrow[ur, phantom, "\ulcorner", very near end] & \frac{(T_2/T_0)}{(T_1/T_0)}
\end{tikzcd}$$

is a pushout. This combining with the fact that $\frac{(T_1/T_0)}{(T_1/T_0)} = T_1/T_1$ implies that $\frac{(T_2/T_0)}{(T_1/T_0)}$ is the pushout of $T_1 \rightarrowtail T_2$ along $T_1 \to T_1/T_1$ but T_2/T_1 is also a pushout over the same diagram. Therefore, $\frac{(T_2/T_0)}{(T_1/T_0)} \cong T_2/T_1$. ☐

Proposition 2.3.2.3. *The groupoids* $\mathbf{A}_n$ *and the degeneracy and face maps given above form a pseudosimplicial groupoid* $\mathbf{A}$.

Proof. We only need to verify the simplicial identities that involve $d_\perp$. The others follow from the fact that $\mathrm{Dec}_\perp \mathbf{A} = \mathbf{N}\mathbb{A}$ (2.3.2.1). Let us prove that the following diagram

$$\begin{tikzcd}
\mathbf{A}_3 \arrow[r, "d_\perp"] \arrow[d, "d_1"'] & \mathbf{A}_2 \arrow[d, "d_\perp"] \\
\mathbf{A}_2 \arrow[r, "d_\perp"'] & \mathbf{A}_1
\end{tikzcd}$$

commutes, for the other cases the proof follows the same arguments but the notation becomes much heavier. Let $T_0 \rightarrowtail T_1 \rightarrowtail T_2$ be an object in $\mathbf{A}_3$. It easy to check that $d_\perp d_\perp(T_0 \rightarrowtail T_1 \rightarrowtail T_2) = \frac{T_2/T_0}{T_1/T_0}$ and $d_\perp d_1(T_0 \rightarrowtail T_1 \rightarrowtail T_2) = T_2/T_1$. So the square commutes when $\frac{T_2/T_0}{T_1/T_0} \cong T_2/T_1$, but in the proof of Lemma 2.3.2.2 we gave this isomorphism. ☐

Lemma 2.3.2.4. *Suppose we have admissible maps* $T' \rightarrowtail T$ *and* $Q \rightarrowtail T/T'$*, there exists a unique preorder* P *and admissible maps* $T' \rightarrowtail P$ *and* $P \rightarrowtail T$ *such that the diagram*

$$\begin{tikzcd}
T' \arrow[r, tail, dotted] \arrow[rr, tail, bend left] & P \arrow[r, tail, dotted] \arrow[d, dotted] & T \arrow[d] \\
 & Q \arrow[r, tail] & T/T'
\end{tikzcd}$$

commutes.

Proof. Put $P = Q \times_{T/T'} T$. Since admissible maps are stable under pullbacks, the map $P \to T$ is admissible. Furthermore, we have a natural map from $T'/T' \to Q$ which is admissible since $T'/T' \rightarrowtail T/T'$ and $Q \rightarrowtail T/T'$ are admissible. Therefore, the outer diagram

$$\begin{tikzcd}
T' \arrow[r, dotted] \arrow[rr, tail, bend left] \arrow[d] & P \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d] \\
T'/T' \arrow[r, tail] \arrow[rr, tail, bend right] & Q \arrow[r, tail] & T/T'
\end{tikzcd}$$

commutes. The dotted arrow then exists by the pullback property of P. By the prism Lemma 1.2.0.2, the left square is a pullback since the outer diagram is a pullback by Lemma 2.3.0.5. The map $T' \to P$ is admissible since admissible maps are stable under pullbacks and $T'/T' \rightarrowtail Q$ is admissible. ☐

Proposition 2.3.2.5. *The pseudosimplicial groupoid* $\mathbf{A}$ *is a decomposition space.*

Proof. We will prove that the following diagrams are pullbacks for $0 < i < n$:

$$\begin{tikzcd}
\mathbf{A}_{n+1} \arrow[r, "d_{i+1}"] \arrow[d, "d_\perp"'] \arrow[dr, phantom, "(1)"] & \mathbf{A}_n \arrow[d, "d_\perp"] \\
\mathbf{A}_n \arrow[r, "d_i"'] & \mathbf{A}_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{A}_{n+1} \arrow[r, "d_i"] \arrow[d, "d_\top"'] \arrow[dr, phantom, "(2)"] & \mathbf{A}_n \arrow[d, "d_\top"] \\
\mathbf{A}_n \arrow[r, "d_i"'] & \mathbf{A}_{n-1}.
\end{tikzcd}$$

Let us prove it for $n = 2$. For greater n the proof is completely analogous. The square (2) is a pullback as a consequence of Lemma 2.3.2.1. To prove that the square (1) is a pullback we will use Lemma 1.2.0.1. This means that (1) is a pullback if and only if for each object $\alpha \colon T' \rightarrowtail T$ in $\mathbf{A}_2$, the map $d_\perp \colon \operatorname{Fib}_\alpha d_2 \to \operatorname{Fib}_{d_\perp \alpha} d_1$ is an equivalence of groupoids. For an object $V \rightarrowtail T/T'$ in $\operatorname{Fib}_{d_\perp \alpha} d_1$, Lemma 2.3.2.4 gives the following commutative diagram

$$\begin{tikzcd}
T' \arrow[r, tail] \arrow[rr, tail, bend left, "\alpha"] \arrow[d] & V \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d] \\
T'/T' \arrow[r, tail] \arrow[rr, tail, bend right] & V' \arrow[r, tail] & T/T'
\end{tikzcd}$$

where the horizontal arrows are admissible maps. The commutativity of the diagram implies that $T' \rightarrowtail V \rightarrowtail T$ is an object in $\operatorname{Fib}_\alpha d_2$ and hence $d_\perp$ is surjective on objects. For the full condition of $d_\perp$, let $p' \colon V' \rightarrowtail R'$ be an admissible map such that the lower square

$$\begin{tikzcd}
T \arrow[rr, tail] \arrow[rrrr, tail, bend left, "\alpha"] \arrow[dr, "id"'] & & V \arrow[rr, tail] \arrow[dd] \arrow[dr, dotted, "p"] & & T' \arrow[dr, "id"] \arrow[dd] & \\
& T \arrow[rr, tail] & & R \arrow[rr, tail] \arrow[dd] & & T' \arrow[dd] \\
& & V' \arrow[rr, tail] \arrow[dr, "p'"'] & & T/T' \arrow[dr, "id"] & \\
& & & R' \arrow[rr, tail] & & T/T'
\end{tikzcd}$$

commutes. This means that $\{p', \mathrm{id}_{T/T'}\}$ is a morphism in $\operatorname{Fib}_{d_\perp \alpha} d_1$. Lemma 2.3.2.4 applied to V' and R' gives the top admissible maps. The pullback property of R gives the dotted arrow $p \colon V \to R$. The commutativity of the diagram follows from the pullback condition of V and R, and therefore $\{\mathrm{id}_T, p, \mathrm{id}_{T'}\}$ is a morphism in $\operatorname{Fib}_\alpha d_2$. This implies that $d_\perp$ is full. The faithful condition of $d_\perp$ is straightforward to check using Lemma 2.3.2.4. Indeed, let $\{\mathrm{id}_T, p, \mathrm{id}_{T'}\}$ and $\{\mathrm{id}_T, q, \mathrm{id}_{T'}\}$ be morphisms in $\mathrm{Fib}_\alpha\, d_2$ such that $d_\perp\{\mathrm{id}_T, p, \mathrm{id}_{T'}\} = d_\perp\{\mathrm{id}_T, q, \mathrm{id}_{T'}\}$. This means that the following diagram commutes

$$\begin{tikzcd}
T \arrow[rr, tail] \arrow[rrrr, bend left, tail, "\alpha"] \arrow[dr, "\mathrm{id}"'] & & V \arrow[rr, tail] \arrow[dr, shift left, "q"] \arrow[dr, shift right, "p"'] \arrow[dd] & & T' \arrow[dr, "\mathrm{id}"] \arrow[dd] & \\
& T \arrow[rr, tail] & & R \arrow[rr, tail] \arrow[dd] & & T' \arrow[dd] \\
& & V' \arrow[rr, tail] \arrow[dr, "p'"'] & & T/T' \arrow[dr, "\mathrm{id}"'] & \\
& & & R' \arrow[rr, tail] & & T/T'
\end{tikzcd}$$

By the pullback property of R, it follows that $p = q$. ☐

Proposition 2.3.2.6. *The decomposition space* $\mathbf{A}$ *is complete.*

Proof. Let T be an object in $\mathbf{A}_1$. Consider the pullback diagram

$$\begin{tikzcd}
\mathrm{Fib}_{s_0} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\ulcorner T \urcorner"] \\
\mathbf{A}_0 \arrow[r, "s_0"'] & \mathbf{A}_1 .
\end{tikzcd}$$

We have that $\mathrm{Fib}_{s_0} T$ is empty if T is not a groupoid preorder. In case T is a groupoid preorder, we have that $\mathrm{Fib}_{s_0} T \simeq T^{\mathrm{inv}}$. Therefore, $s_0\colon \mathbf{A}_0 \to \mathbf{A}_1$ is a monomorphism. ☐

Recall that the comultiplication of the incidence coalgebra of $\mathbf{A}$ is given by the formula:

$$\Delta_{\mathbf{A}}(\alpha) = \sum_{\left(\substack{\alpha\in \mathbf{A}_2 \\ d_1(\alpha)=T}\right)} d_2(\alpha) \otimes d_0(\alpha)$$

which means that we sum over all admissible maps $\alpha\colon T' \rightarrowtail T$ that have target T and return T' and T/T'. But this is precisely the comultiplication of the Fauvet–Foissy–Manchon coalgebra of finite topological spaces after using the bijection between finite topological spaces and finite preorders (see §2.3.1). Hence, we have the following result.

Lemma 2.3.2.7. *The incidence coalgebra* $\Delta_{\mathbf{A}}$ *is the Fauvet–Foissy–Manchon coalgebra of finite topological spaces.*

Remark 2.3.2.8. There is a natural bijection between finite T_0-topological spaces and finite posets. It would be interesting to explore the constructions given above from a topological point of view. The starting point would be to translate the notion of contraction between posets into a contraction between finite T_0-spaces. We will decide to leave this point of view for future work.

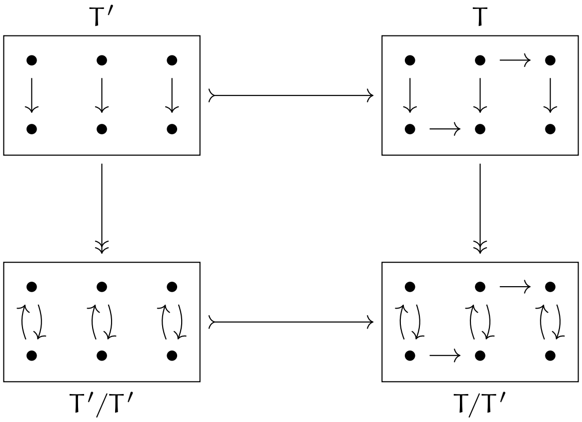
2.4 *Admissible maps and contractions*

In this section, we will relate the notions of admissible maps of preorders (due to [40]) and of contractions for posets through a culf map (see 2.4.0.8). This provides a deeper explanation of both classes of maps and simultaneously allows us to relate the Fauvet–Foissy–Manchon bialgebra of finite topological spaces with the Calaque–Ebrahimi–Fard–Manchon bialgebra of rooted trees (see §2.3.1). To relate admissible maps between preorders and contractions of posets, it is necessary to introduce some results.

Definition 2.4.0.1. A monotone map of preorders $f\colon T \to V$ is a *contraction* if:

1. its is identity on objects;
2. for each $x \in V$, the fibre $f^{-1}(x^{\simeq})$ is a connected sub-preorder of T, where $x^{\simeq}$ denotes the subpreorder of objects equivalent to x in V;
3. for any cover $a \lessdot a'$ in V, there exists a cover $t \lessdot t'$ in T such that $f(t) = a$ and $f(t') = a'$.

Given a preorder T there is a canonical contraction $T \to T/T$. We will show that for any admissible map $T' \rightarrowtail T$, we can construct a contraction $T \twoheadrightarrow T/T'$. An illustration of the canonical contraction from an admissible map is given by the following pushout diagram



Recall that $\widetilde{T}$ is the posetification of a preorder T and let $\text{postf}\colon T \to \widetilde{T}$ denote the posetification monotone map.

Lemma 2.4.0.2. *Let T be a preorder. There is a bijection between the set of admissible maps onto T and the set of admissible maps onto $\widetilde{T}$.*

Proof. The conditions of Definition 2.3.0.3 imply that if $x \sim_T y$ then $x \sim_{T'} y$, so that contract T to $\widetilde{T}$ does not have any effect in the set of admissible maps. ☐

Lemma 2.4.0.3. *Let $f\colon T \to V$ be a contraction between preorders. Then $\widetilde{f}\colon \widetilde{T} \to \widetilde{V}$ is a contraction of posets.*

Proof. The map $\widetilde{f}\colon \widetilde{T} \to \widetilde{V}$ is monotone since for each $[t] \in \widetilde{T}$, we have that $\widetilde{f}([t]) = [f(t)]$. In other words, the diagram

$$\begin{tikzcd}
T \arrow[r, "\text{postf}_T"] \arrow[d, two heads, "f"'] \arrow[dr, phantom, "(1)"] & \widetilde{T} \arrow[d, "\widetilde{f}"] \\
V \arrow[r, "\text{postf}_V"'] & \widetilde{V}
\end{tikzcd}$$

commutes. The commutativity of the square together with the monotone surjection condition of f and postf_T and postf_V implies that $\widetilde{f}$ is a monotone surjection. Let's prove that for each $[v] \in \widetilde{V}$, the fibre $\widetilde{f}_{[v]}$ is a connected convex subposet of $\widetilde{T}$. Let $[a]$ and $[b]$ be objects in $\widetilde{f}_{[v]}$. This implies that

$$f(a) \sim_V v \sim_V f(b),$$

and therefore a and b are objects in $f^{-1}(v^{\simeq})$. Furthermore, a and b are connected since $f^{-1}(v^{\simeq})$ is a connected preorder in T by the contraction condition of f. Therefore, $[a]$ and $[b]$ are connected. Let's prove that $\widetilde{f}$ lifts covers: let $[v] \lessdot [v']$ be a cover in $\widetilde{V}$. Since the posetification map respects covers, we have that $v \lessdot v'$ in V. Recall that f lifts covers since it is a contraction. This means that there exists a cover $t \lessdot t'$ in T such that $f(t) = v$ and $f(t') = v'$. Applying the posetification map to $t \lessdot t'$, we have that $[t] \lessdot [t']$ in $\widetilde{T}$, and by the commutative of the square (1), we get that $\widetilde{f}([t]) = [v]$ and $\widetilde{f}([t']) = [v']$. ☐

Lemma 2.4.0.4. *For an admissible map of preorders* $\alpha\colon T' \rightarrowtail T$, *the map* $T \to T/T'$ *is a contraction.*

Proof. The map $f\colon T \to T/T'$ is given by the pushout in the category of preorders

$$\begin{tikzcd} T' \arrow[r, tail, "\alpha"] \arrow[d] & T \arrow[d, "f"] \\ T'/T' \arrow[r, tail] \arrow[ur, phantom, "\ulcorner", very near end] & T/T'. \end{tikzcd}$$

The map f is clearly identity-on-objects. For each $x \in T/T'$, we have that $f^{-1}(x^{\simeq})$ is connected. Indeed, for each $y \simeq x$ in T/T', we have three possibilities: $y \simeq_T x$ in T, or $x <_{T'} y$ or $y <_{T'} x$. By the admissible property of α, we have that $x <_T y$ or $y <_T x$ if $x < y$ or $y < x$ in T' since α respects the connected subpreorders from T' to T. Furthermore, f lifts covers. Indeed, if $x \lessdot x'$ in T/T' then $x < x'$ in T since f only inverts the relations that are in T' and preserves all other relations in T. Hence, f is a contraction of preorders. ☐

Lemma 2.4.0.5. *Let* T *be a finite preorder. There is a bijection between the set of admissible maps with target* T *and the set of contractions with source* T.

Proof. Given an admissible map $T' \rightarrowtail T$, the map $T \twoheadrightarrow T/T'$ is a contraction by Lemma 2.4.0.4. For a contraction $T \twoheadrightarrow V$, consider the pullback diagram:

$$\begin{tikzcd} T \times_V V^{\mathrm{inv}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, two heads] \\ V^{\mathrm{inv}} \arrow[r] & V. \end{tikzcd}$$

The map $V^{\mathrm{inv}} \to V$ is an identity-on-objects monotone map. Furthermore, this map is canonical admissible since V^{inv} only contains the invertible morphisms in V, which is equivalent to condition (3) of the requirement to be admissible (2.3.0.3). The condition (2) to be admissible is automatic since we only consider invertible morphism in V^{inv}. Therefore, $V^{\mathrm{inv}} \to V$ is admissible. Since admissible maps are stable under pullback along identity-on-objects monotone maps (2.3.0.4), the map $T \times_V V^{\mathrm{inv}} \to T$ is admissible. ☐

Construction 2.4.0.6. *We will construct a simplicial map* $\mathfrak{r}$ *from the decomposition space of admissible maps* **A** *to the decomposition space of contractions* **K**.
The functor $\mathfrak{r}_1 : \mathbf{A}_1 \to \mathbf{K}_1$ *sends a preorder* T *to the family of the connected components* $\{\widetilde{T}_i\}_{i \in \widetilde{T/T}}$ *of the posetification of* T, *where each element of the family is defined by the pullback:*

$$\begin{tikzcd} \widetilde{T}_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{T} \arrow[d, two heads] \\ 1 \arrow[r, "\ulcorner i \urcorner"'] & \widetilde{T/T}. \end{tikzcd}$$

For $n \geqslant 2$, *let*

$$\begin{tikzcd} T_0 \arrow[r, tail] & T_1 \arrow[r, tail] & T_2 \arrow[r, tail] & \cdots \arrow[r, tail] & T_{n-2} \arrow[r, tail] & T_{n-1} \end{tikzcd}$$

be an object in $\mathbf{A}_n$. *Consider the following diagram consisting of pushout squares*

$$\begin{tikzcd}
T_0 \arrow[r, tail] \arrow[d, two heads] & T_1 \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] \arrow[d] & T_{n-2} \arrow[r, tail] \arrow[d, two heads] & T_{n-1} \arrow[d, two heads] \\
T_0/T_0 \arrow[r, tail] & T_1/T_0 \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] \arrow[d] & T_{n-2}/T_0 \arrow[r, tail] \arrow[d, two heads] & T_{n-1}/T_0 \arrow[d, two heads] \\
 & T_1/T_1 \arrow[r, tail] & \cdots \arrow[r, tail] & \vdots \arrow[r, tail] \arrow[d, two heads] & \vdots \arrow[d, two heads] \\
 & & & T_{n-2}/T_{n-2} \arrow[r, tail] & T_{n-1}/T_{n-2} \arrow[d, two heads] \\
 & & & & T_{n-1}/T_{n-1}.
\end{tikzcd} \tag{1}$$

Since the top horizontal arrows are admissible maps and the canonical map $T_0 \rightarrowtail T_0/T_0$ *is a contraction, we have that the horizontal maps are admissible and the verticals are contractions by the stability property of admissible maps and contractions under pushout (2.3.0.4 and 2.4.0.4). Taking the posetification of the last column of the diagram (1), we obtain the chain*

$$\widetilde{T_{n-1}} \twoheadrightarrow \widetilde{T_{n-1}/T_0} \twoheadrightarrow \widetilde{T_{n-1}/T_1} \twoheadrightarrow \cdots \twoheadrightarrow \widetilde{T_{n-1}/T_{n-2}} \twoheadrightarrow \widetilde{T_{n-1}/T_{n-1}} \tag{2}$$

of contractions between posets. Since the posets in the chain are not necessarily connected, for each $i \in \widetilde{T_{n-1}/T_{n-1}}$, *we take the pullback of (2) along* $\ulcorner i \urcorner \colon 1 \to \widetilde{T_{n-1}/T_{n-1}}$ *to obtain the diagram*

$$\begin{tikzcd}
\widetilde{T_{n-1}} \arrow[r, two heads] & \widetilde{T_{n-1}/T_0} \arrow[r, two heads] & \cdots \arrow[r, two heads] & \widetilde{T_{n-1}/T_{n-2}} \arrow[r, two heads] & \widetilde{T_{n-1}/T_{n-1}} \\
(\widetilde{T_{n-1}})_i \arrow[r, two heads] \arrow[u] \arrow[ur, phantom, "\urcorner", very near start] & (\widetilde{T_{n-1}/T_0})_i \arrow[r, two heads] \arrow[u] & \cdots \arrow[r, two heads] \arrow[u] \arrow[ur, phantom, "\urcorner", very near start] & (\widetilde{T_{n-1}/T_{n-2}})_i \arrow[r] \arrow[u] \arrow[ur, phantom, "\urcorner", very near start] & 1 \arrow[u, "\ulcorner i \urcorner"']
\end{tikzcd} \tag{3}$$

where the lower part is a chain of contractions between connected posets since the top part of the diagram is given by a chain of contractions.

For $n \geqslant 2$, *the functor* $\mathfrak{r}_n \colon \mathbf{A}_n \to \mathbf{K}_n$ *sends an* $(n-1)$*-chain of admissible maps to the family of* $(n-1)$*-chains of contractions*

$$\{(\widetilde{T_{n-1}})_i \twoheadrightarrow (\widetilde{T_{n-1}/T_0})_i \twoheadrightarrow (\widetilde{T_{n-1}/T_1})_i \twoheadrightarrow \cdots \twoheadrightarrow (\widetilde{T_{n-1}/T_{n-2}})_i\}_{i \in \widetilde{T_{n-1}/T_{n-1}}} \tag{4}$$

To prove that $\mathfrak{r}$ is a simplicial map, we need the following result:

Lemma 2.4.0.7. *For each admissible map* $T' \rightarrowtail T$, *we have that*

$$\{\widetilde{T}_i\}_{i \in \widetilde{T/T'}} = \{\widetilde{T'}_i\}_{i \in \widetilde{T'/T'}}.$$

Proof. Since $T' \rightarrowtail T$ is admissible, we have that $\underline{T'/T'} = \underline{T/T'}$ and the right square

$$\begin{tikzcd}
\widetilde{T}_i \arrow[rrrd, bend left] \arrow[dr, dotted] \arrow[rdd, bend right] & & & \\
 & \widetilde{T'}_i \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{T'} \arrow[r, tail] \arrow[d, two heads] & \widetilde{T} \arrow[d, two heads] \\
 & 1 \arrow[r, "\ulcorner i \urcorner"'] & \widetilde{T'/T'} \arrow[r, tail] & \widetilde{T/T'}
\end{tikzcd}$$

is a pullback as a consequence of Lemma 2.3.0.5. By the prism Lemma 1.2.0.2 the outer rectangle is a pullback since the left square is a pullback by definition. So by the pullback property of $\widetilde{T'}_i$, the dotted arrow exists and is an equality since the outer diagram is a pullback. □

To see that $\mathfrak{r}$ is a simplicial map, we have to verify that for $0 \leqslant i < n$, the diagram

$$\begin{tikzcd}
\mathbf{A}_n \arrow[r, "d_i"] \arrow[d, "\mathfrak{r}_n"'] & \mathbf{A}_{n-1} \arrow[d, "\mathfrak{r}_{n-1}"] \\
\mathbf{K}_n \arrow[r, "d_i"'] & \mathbf{K}_{n-1}
\end{tikzcd}$$

commutes, but this follows from Construction 2.4.0.6. For the case $i = n$, we have to use Lemma 2.4.0.7. Let us prove it for $n = 2$. For greater n the proof is completely analogous, but the notation becomes much heavier. To prove that the diagram

$$\begin{tikzcd}
\mathbf{A}_2 \arrow[r, "d_\top"] \arrow[d, "\mathfrak{r}_2"'] & \mathbf{A}_1 \arrow[d, "\mathfrak{r}_1"] \\
\mathbf{K}_2 \arrow[r, "d_\top"'] & \mathbf{K}_{n-1}
\end{tikzcd}$$

commutes. It is enough to verify that for each admissible map $T' \rightarrowtail T$, we have that

$$\{\widetilde{T}_i\}_{i \in \widetilde{T/T'}} = \{\widetilde{T'}_i\}_{i \in \widetilde{T'/T'}}$$

since $d_\top(T' \rightarrowtail T) = T'$ and $d_\top(\widetilde{T} \twoheadrightarrow \widetilde{T/T'}) = \{\widetilde{T}_i\}_{i \in \widetilde{T/T'}}$. But this follows from Lemma 2.4.0.7.

Proposition 2.4.0.8. *The map* $\mathfrak{r}\colon \mathbf{A} \to \mathbf{K}$ *is culf.*

Proof. Since $\mathbf{A}$ and $\mathbf{K}$ are decomposition spaces, to prove that $\mathfrak{r}$ is culf, it is enough to check that the diagram

$$\begin{tikzcd}
\mathbf{A}_2 \arrow[r, "d_1"] \arrow[d, "\mathfrak{r}_2"'] & \mathbf{A}_1 \arrow[d, "\mathfrak{r}_1"] \\
\mathbf{K}_2 \arrow[r, "d_1"'] & \mathbf{K}_1
\end{tikzcd}$$

is a pullback by Lemma 1.1.2.2. To prove that the square is a pullback, we will use Lemma 1.2.0.1. This means that for each object $T \in \mathbf{A}_1$, we have to show that the map $\mathfrak{r}_2\colon \mathrm{Fib}_T(d_1) \to \mathrm{Fib}_{\mathfrak{r}_1 T}(d_1)$ is an equivalence. We will divide the proof into three steps: first we will prove that is $\mathfrak{r}_2\colon \mathrm{Fib}_T(d_1) \to \mathrm{Fib}_{\mathfrak{r}_1 T}(d_1)$ is essentially surjective on objects. After we will show that $\mathfrak{r}_2$ is full and finally that it is faithful.

- Let $f\colon \widetilde{T} \twoheadrightarrow P$ be an object in $\mathrm{Fib}_{\mathfrak{r}_1 T}(d_1)$. Consider the following pullback diagram:

$$\begin{tikzcd}
\underline{P} \times_P T \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, two heads, "\mathrm{postf}"] \\
\underline{P} \times_P \widetilde{T} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{T} \arrow[d, two heads, "f"] \\
\underline{P} \arrow[r, tail] & P.
\end{tikzcd}$$

Since $\underline{P} \to P$ is admissible, the map α is admissible by the stability property of admissible maps under pullback (Lemma 2.3.0.4). It is easy to check that α is an object in $\mathrm{Fib}_T(d_1)$. To verify that $\mathfrak{r}_2(\alpha) \cong f$, consider the commutative diagram:

$$\begin{tikzcd}
\underline{P} \times_P T \arrow[r, tail, "\alpha"] \arrow[d, two heads] \arrow[dr, phantom, "(1)"] & T \arrow[d, two heads] \arrow[dr, two heads, "\mathrm{postf}"] & \\
(\underline{P} \times_P T)/\underline{P} \times_P T \arrow[r] \arrow[dr] & T/(\underline{P} \times_P T) \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, phantom, "(2)"] \arrow[dr, dashed, "u"'] & \widetilde{T} \arrow[d, two heads, "f"] \\
 & \underline{P} \arrow[r, tail] & P.
\end{tikzcd}$$

The square (1) is a pushout by definition of $T/(\underline{P} \times_P T)$. Since the outer diagram commutes, the pushout property of $T/(\underline{P} \times_P T)$ gives the dotted arrow u. Applying the posetification functor to the square (2), we obtain the commutative diagram:

$$\begin{tikzcd}
\widetilde{T} \arrow[r, "\mathrm{id}"] \arrow[d, two heads, "\mathfrak{r}_2(\alpha)"'] \arrow[dr, phantom, "(3)"] & \widetilde{T} \arrow[d, two heads, "f"] \\
T/\widetilde{(\underline{P} \times_P T)} \arrow[r, "u'"'] & P.
\end{tikzcd}$$

Furthermore, u' is a bijection. Indeed, u' is a monotone surjection since id and $\mathfrak{r}_2(\alpha)$ and f are monotone surjections, and the diagram (3) commutes. The injectivity condition of u' follows from the fact that the diagrams (1) and (2) commute and u' is obtained from the posetification of u. This implies that $\mathfrak{r}_2(\alpha) \simeq f$, and therefore $\mathfrak{r}_2$ is essentially surjective on objects.

- Let's prove that $\mathfrak{r}_2$ is full. Let u be a morphism in $\mathrm{Fib}_{\mathfrak{r}_1 T}(d_1)$. The morphism u can be illustrated by the following diagram

$$\begin{tikzcd}
 & \widetilde{T} \arrow[dl, two heads, "f"'] \arrow[dr, two heads, "f'"] & \\
P \arrow[rr, "u"'] & & P'.
\end{tikzcd}$$

Since $\mathfrak{r}_2$ is essentially surjective on objects, for f and f' in $\mathrm{Fib}_{\mathfrak{r}_1 T}(d_1)$, we have objects α and α' in $\mathrm{Fib}_T(d_1)$ such that $\mathfrak{r}_2(\alpha) = f$, and $\mathfrak{r}_2(\alpha') = f'$, and the following diagram commutes

$$\begin{tikzcd}
W \arrow[rr, tail, "\alpha"] \arrow[dr, dotted, "\overline{u}"'] \arrow[dd, two heads] & & T \arrow[dr, "\mathrm{id}"] \arrow[dd, two heads] & \\
 & W' \arrow[rr, tail, "\alpha'"' near start] \arrow[dd, two heads] & & T \arrow[dd, two heads] \\
\widetilde{W} \arrow[rr, tail] \arrow[dd] & & \widetilde{T} \arrow[dr, "\mathrm{id}"] \arrow[dd, two heads, "f"' near end] & \\
 & \widetilde{W'} \arrow[rr, tail] \arrow[dd] & & \widetilde{T} \arrow[dd, two heads, "f'"] \\
\underline{P} \arrow[rr] \arrow[dr, "u"'] & & P \arrow[dr, "u"] & \\
 & \underline{P}' \arrow[rr] & & P'.
\end{tikzcd}$$

The pullback property of W' gives the dotted arrow $\overline{u}$, and the top square commutes. This implies that $\overline{u}\colon \alpha \to \alpha'$ is a morphism in $\mathrm{Fib}_T(d_1)$ and $\mathfrak{r}_2(\overline{u}) = u$.

- It only remains to prove that $\mathfrak{r}_2$ is faithful. Let ν and ν' be morphisms in $\mathrm{Fib}_T(d_1)$ such that $\mathfrak{r}_2(\nu) = \mathfrak{r}_2(\nu')$. The objects ν and ν' can be illustrated by the following diagram

$$\begin{tikzcd}
W \arrow[rr, shift left, "\nu'"] \arrow[rr, shift right, "\nu"'] \arrow[dr, tail, "\alpha"'] & & W' \arrow[dl, tail, "\alpha'"] \\
& T. &
\end{tikzcd} \tag{3}$$

Applying $\mathfrak{r}_2$ to (3), we have the following commutative diagram

$$\begin{tikzcd}
W \arrow[rr, tail, "\alpha"] \arrow[dr, shift left, tail, "\nu"] \arrow[dr, shift right, tail, "\nu'"'] \arrow[dd, two heads] & & T \arrow[dr, "\mathrm{id}"] \arrow[dd, two heads] & \\
& W' \arrow[rr, tail, "\alpha'"'] \arrow[dd, two heads] & & T \arrow[dd, two heads] \\
\widetilde{W} \arrow[rr, tail] \arrow[dd] & & \widetilde{T} \arrow[dr, "\mathrm{id}"] \arrow[dd, two heads, "\mathfrak{r}_2(\alpha)"'] & \\
& \widetilde{W}' \arrow[rr, tail] \arrow[dd] & & \widetilde{T} \arrow[dd, two heads, "\mathfrak{r}_2(\alpha')"] \\
\underline{P} \arrow[rr] \arrow[dr, "\mathfrak{r}_2(\nu)"'] & & P \arrow[dr, "\mathfrak{r}_2(\nu)"'] & \\
& \underline{P}' \arrow[rr] & & P'.
\end{tikzcd}$$

By the pullback property of W', there exists a unique map from W to W' such that the above diagram commutes. This forces $\nu = \nu'$. Hence, $\mathfrak{r}_2$ is faithful.

□

Remark 2.4.0.9. The culf map $\mathfrak{r}\colon \mathbf{A} \to \mathbf{K}$ induces a homomorphism from the Fauvet–Foissy–Manchon bialgebra of admissible maps $\Delta_{\mathbf{A}}$ to the incidence bialgebra of contractions $\Delta_{\mathbf{K}}$.

2.4.1 *Admissible maps and the Waldhausen construction*

Bergner, Osorno, Ozornova, Rovelli, and Scheimbauer [14] showed an equivalence between decomposition objects and augmented stable double Segal objects, which is given by an $S_\bullet$-construction. In this section, we will construct a stable augmented double category using admissible maps and contractions between preorders.

A *double category* is a category internal to categories. More informally, it consists of the following data, subject to some axioms: a set of objects; two different classes of morphisms (horizontal and vertical); and squares, which are connected by various source, target, identity, and composition maps.

Example 2.4.1.1. Let **AdCon** denote the double category of finite preorders, admissible maps as horizontal morphisms, and contractions as vertical morphisms. The squares are pushout diagrams of admissible maps along contractions of preorders.

An *augmentation* of a double category $\mathbf{C}$ consists of a set of objects C satisfying the condition that for every object d of $\mathbf{C}$, there are a unique horizontal morphism $c \rightarrowtail d$ and a unique vertical morphism $d \twoheadrightarrow c'$ such that c and c' are in C [14]. We need some results to show an augmentation in **AdCon**.

Lemma 2.4.1.2. *The category of finite preorders and admissible maps has chosen initial objects given by the groupoid finite preorders.*

Proof. Given a finite preorder T, we have a canonical groupoid finite preorder T^{inv} and a canonical admissible map $T^{\mathrm{inv}} \to T$. □

Lemma 2.4.1.3. *The category of finite preorders and contractions of preorders has chosen terminal objects given by the groupoid finite preorders.*

Proof. Given a finite preorder T, we have a groupoid finite preorder T/T and a canonical identity-on-objects monotone map $T \to T/T$, which is a contraction since we invert all the morphism in T. Indeed, the lift of covers is automatic. The connection of the fibres is given by the fact that T/T is the preorder of connected components of T. ☐

Proposition 2.4.1.4. *The double category* **AdCon** *has an augmentation given by the set of groupoid finite preorders*

Proof. For each preorder T, we have a canonical admissible map $T^{\mathrm{inv}} \rightarrowtail T$ (2.4.1.2) and a canonical contraction $T \twoheadrightarrow T/T$ (2.4.1.3). It is easy to see that T^{inv} and T/T are groupoid preorders, and hence the set of groupoid finite preorders is an augmentation of **AdCon**. ☐

A double category is *stable* if every square is uniquely determined by its span of source morphisms and, independently, by its cospan of target morphisms [14].

Proposition 2.4.1.5. *The double category* **AdCon** *is stable.*

Proof. By definition of **AdCon** the squares are pushout diagrams as follows:

$$\begin{tikzcd}
\cdot \arrow[r, tail] \arrow[d, two heads] & \cdot \arrow[d, two heads] \\
\cdot \arrow[r, tail] & \cdot \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

where the horizontal morphisms are admissible maps and the vertical are contractions of preorders. Since the top horizontal arrow is admissible, the square is also a pullback as a consequence of Lemma 2.3.0.5. This means that all the squares in **AdCon** are bipullbacks, and therefore **AdCon** is stable. ☐

In an augmented stable double category, there is a bijection between the set of horizontal morphisms and the set of vertical arrows [14]. This implies that in **AdCon**, we have a bijection between admissible maps and contractions of preorders. This bijection was also proved in Lemma 2.4.0.5.

Bergner, Osorno, Ozornova, Rovelli, and Scheimbauer [14] expressed the Waldhausen $S_\bullet$-construction in terms of a functor $S_\bullet$ from the category of augmented double categories to the category of simplicial groupoids. If we only consider stable augmented double categories, they showed that the functor $S_\bullet$ sends stable augmented double categories to decomposition spaces.

Theorem 2.4.1.6. *[14, Theorem 4.8] The Waldhausen $S_\bullet$-construction restricts to the functor*

$$S_\bullet \colon \mathrm{DCat}^{\mathrm{st}}_{\mathrm{aug}} \to \mathbf{Dcmp},$$

where $\mathrm{DCat}^{\mathrm{st}}_{\mathrm{aug}}$ denotes the category of augmented stable double categories.

The construction of $S_\bullet$ and the proof of Theorem 2.4.1.6 require some preliminaries that are beyond the scope of this thesis but which can be found in detail in [14]. So we will choose to give a detailed description of the example we are interested in, the decomposition space $S_\bullet(\mathbf{AdCon})$. The objects of the groupoid $(S_\bullet\mathbf{AdCon})_0$ are finite preorders and the morphisms monotone bijections. The objects of the groupoid $(S_\bullet\mathbf{AdCon})_n$ are diagrams of the form:

$$\begin{tikzcd}
V_{00} \arrow[r, tail] & T_{01} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & T_{02} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & T_{0(n-1)} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & T_{0n} \arrow[d, two heads] \\
 & V_{11} \arrow[r, tail] & T_{12} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & T_{1(n-1)} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & T_{1n} \arrow[d, two heads] \\
 & & V_{22} \arrow[r, tail] & \cdots \arrow[r, tail] & \vdots \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \vdots \arrow[d, two heads] \\
 & & & & V_{(n-1)(n-1)} \arrow[r, tail] & T_{(n-1)n} \arrow[d, two heads] \\
 & & & & & V_{nn}
\end{tikzcd} \tag{1}$$

where each V_{ii} is a groupoid finite preorder and the squares are bipullbacks. The morphisms in $(S_\bullet\mathbf{AdCon})_n$ are families of monotone bijections between the finite preorders of the diagrams. The face maps $d_i\colon (S_\bullet\mathbf{AdCon})_n \to (S_\bullet\mathbf{AdCon})_{n-1}$ acts by 'erasing' all the finite preorders that containing an index i. The degeneracy maps $s_i\colon (S_\bullet\mathbf{AdCon})_n \to (S_\bullet\mathbf{AdCon})_{n+1}$ 'repeat' the ith row and the ith column.

We have a canonical map from $\pi_{\text{firstrow}}\colon S_\bullet(\mathbf{AdCon}) \to \mathbf{A}$, that forgets the extra information and only conserves the first row of the diagram (1) deleting the groupoid finite preorder V_{00} in the chain. In other words, its sends the diagram (1) to the chain

$$T_{01} \rightarrowtail T_{02} \rightarrowtail \cdots \rightarrowtail T_{0(n-1)} \rightarrowtail T_{0n}.$$

Proposition 2.4.1.7. *The map* $\pi_{\text{firstrow}}\colon S_\bullet(\mathbf{AdCon}) \to \mathbf{A}$ *is an equivalence of simplicial spaces.*

Proof. The map π_{firstrow} is essentially surjective on objects. Let us prove it for an 3-simplex, for a n-simplex the proof is completely analogous. For a 3-simplex

$$T_0 \rightarrowtail T_1 \rightarrowtail T_2$$

in $\mathbf{A}$, we construct the following diagram

$$\begin{tikzcd}
T_0 \arrow[r, tail] \arrow[d, two heads] & T_1 \arrow[r, tail] \arrow[d, two heads] & T_2 \arrow[d, two heads] \\
T_0/T_0 \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & T_1/T_0 \arrow[r, tail] \arrow[d, two heads] \arrow[ur, phantom, "\ulcorner" very near end] & T_2/T_0 \arrow[d, two heads] \\
 & T_1/T_1 \arrow[r, tail] \arrow[ur, phantom, "\ulcorner" very near end] & T_2/T_1
\end{tikzcd}$$

where each square is a pushout, and since the top arrows are admissible it follows that the squares are bipullbacks by Lemma 2.3.0.5. Furthermore, adding to the top row the canonical admissible map $T_0^{\text{inv}} \rightarrowtail T_0$ and the end of the last column the contraction $T_2/T_1 \twoheadrightarrow (T_2/T_1)/(T_2/T_1)$, we obtain a 3-simplex in $S_\bullet(\mathbf{AdCon})$. The map π_{firstrow} is full and faithful since any morphism between n-simpleces in $S_\bullet(\mathbf{AdCon})$ is unique determined by families of monotone bijections between the preorders of the top row of the n-simpleces in $S_\bullet(\mathbf{AdCon})$. Since π_{firstrow} is full and faithful, and essentially surjective on objects, then it is an equivalence of decomposition spaces. ☐

Note that $S_\bullet(\mathbf{AdCon})$ is a strict simplicial space while $\mathbf{A}$ is a pseudo simplicial space. So the equivalence $\pi_{\text{firstrow}}\colon S_\bullet(\mathbf{AdCon}) \to \mathbf{A}$ allows us to see $S_\bullet(\mathbf{AdCon})$ as the strictification of $\mathbf{A}$ in the sense of Gambino [50, §6.4].

2.5 *Connected directed hereditary species as decomposition spaces*

We will show how to obtain a decomposition space $\mathbf{H}$ from a connected directed hereditary species $H\colon \mathbb{K}_p \to \mathbf{Grpd}$.

Given a connected directed hereditary species $H\colon \mathbb{K}_p \to \mathbf{Grpd}$, we consider first its Grothendieck construction. It is a left fibration $\int H \to \mathbb{K}_p$. The objects of $\int H$ are pairs (P, x) where P is a finite poset and $x \in H[P]$. The morphisms are pairs $(p, \alpha)\colon (P, x) \to (Q, y)$ where $p\colon P \to Q$ is a partially defined contraction and $\alpha\colon H(p)(x) \to y$ is a morphism in $H[Q]$. We are interested in the groupoid of connected finite non-empty posets and genuine contractions $\mathbb{K}$, not all partially defined contractions. For that, we consider $\mathbb{H}$ as the pullback

$$\begin{tikzcd}
\mathbb{H} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \int H \arrow[d] \\
\mathbb{K} \arrow[r, hook] & \mathbb{K}_p .
\end{tikzcd}$$

Furthermore, we define $\mathbf{H} := \mathbf{S}\mathbf{N}^{\mathrm{lt}}\mathbb{H}$. Thus, an object of $\mathbf{H}_n$ is a family of $(n-1)$-chains of contractions

$$\begin{tikzcd}
P_0 \arrow[r, two heads, "f_0"] & P_1 \arrow[r, two heads] & \cdots \arrow[r, two heads] & P_{n-2} \arrow[r, two heads, "f_{n-2}"] & P_{n-1}
\end{tikzcd}$$

with a H-structure on P_0 and $\mathbf{H}_0$ is the groupoid of families of H-structure over the poset with one element. We define the bottom face map as follows: we remove the first element of the chain, P_0, and take the H-structure $H(f_0)$ on P_1. To define the top face map, we use (contravariant) functoriality on convex map: for each element $a \in P_{n-1}$, we can form the fibres over a. We end up with a family

$$\begin{tikzcd}
\{(P_0)_a \arrow[r, two heads, "(f_0)_a"] & (P_1)_a \arrow[r, two heads, "(f_1)_a"] & (P_2)_a \arrow[r, two heads] & \cdots \arrow[r, two heads] & (P_{n-3})_a \arrow[r, two heads, "(f_{n-3})_a"] & (P_{n-2})_a\}_{a \in P_{n-1}},
\end{tikzcd}$$

and we restrict the H-structure on P_0 to the corresponding H-structure on the fibre $(P_0)_a$, in the same way as for $\mathbf{K}$.

Proposition 2.5.0.1. *The groupoids* $\mathbf{H}_n$ *form a simplicial groupoid* $\mathbf{H}$.

Proof. It is straightforward to see that all the simplicial identities involving inner face maps and degeneracy maps are satisfied since $\mathbf{H} = \mathbf{S}\mathbf{N}^{\mathrm{lt}}\mathbb{H}^{\simeq}$. Let us see the identity $d_\top d_\bot \simeq d_\bot d_\top$: consider an element

$$\begin{tikzcd}
P_0 \arrow[r, two heads, "f_0"] & P_1 \arrow[r, two heads] & \cdots \arrow[r, two heads] & P_{n-2} \arrow[r, two heads, "f_{n-2}"] & P_{n-1}
\end{tikzcd}$$

of $\mathbf{H}_n$. Clearly the identity is satisfied at the level of posets, since it is satisfied in $\mathbf{K}$. Hence we only need to see that the final H-structures coincide, but this is just a consequence of the Beck–Chevalley rule of 2.1.2 (in turn coming from the functoriality of H) applied to the square

$$\begin{tikzcd}
P_0 \arrow[d, two heads] & (P_0)_p \arrow[l, tail] \arrow[d, two heads] \arrow[dl, phantom, "\llcorner", very near start] \\
P_1 & (P_1)_p . \arrow[l, tail]
\end{tikzcd}$$

for every $p \in P_{n-1}$. Similarly, the identities $d_\top d_\top \simeq d_\top d_{\top-1}$ and $d_\bot d_\bot \simeq d_\bot d_{\bot+1}$ come from functoriality of H in composition of convex maps and composition of contractions, respectively. □

Since the map $\mathbb{H} \to \mathbb{K}$ is a left fibration and S preserves pullbacks, we have a canonical culf map $\mathbf{H} \to \mathbf{K}$.

Proposition 2.5.0.2. *The simplicial groupoid* $\mathbf{H}$ *is a monoidal decomposition space.*

Proof. Since there is a culf map $\mathbf{H} \to \mathbf{K}$ and $\mathbf{K}$ is a decomposition space, so is $\mathbf{H}$ by Lemma 1.1.2.3. The monoidal structure is again given by the disjoint union. ☐

Proposition 2.5.0.3. *The decomposition space* $\mathbf{H}$ *is complete, locally finite, locally discrete and of locally finite length.*

Proof. Note that $\mathbf{H}_1$ is the groupoid of finite families of non-empty H-structures. The automorphisms of an object in $\mathbf{H}_1$ are given by permutations of the family and by the automorphisms of the underlying finite posets. Therefore each object can only have finitely many automorphisms. The rest follows from Propositions 1.1.2.4, 2.2.0.9, and 2.2.0.10. ☐

Since $\mathbf{H}$ is locally finite by Lemma 2.5.0.3, the homotopy sum resulting from $\mathbf{H}$ is just an ordinary sum, as in 2.1.3. Therefore, we have the following result:

Proposition 2.5.0.4. *The homotopy cardinality of the incidence bialgebra of* $\mathbf{H}$ *is isomorphic to the incidence bialgebra of* H.

2.5.1 *Calaque–Ebrahimi-Fard–Manchon comodule bialgebra of rooted trees: part II*

In 2.1.4, we constructed the connected directed hereditary species H_{CEM} of trees and contractions. Let $\mathbf{H}_{\mathrm{CEM}}$ denote the decomposition space associated with H_{CEM} which is defined as follows:

- $(\mathbf{H}_{\mathrm{CEM}})_0$ is the groupoid of families of the tree with one element.
- $(\mathbf{H}_{\mathrm{CEM}})_1$ is the groupoid of families of trees and monotone bijections;
- $(\mathbf{H}_{\mathrm{CEM}})_2$ is the groupoid whose objects are families of contractions between trees and whose morphisms are monotone bijections;
- $(\mathbf{H}_{\mathrm{CEM}})_n$ is the groupoid whose objects are families of $(n-1)$-chains of contractions between trees and whose morphisms are monotone bijections.

The face and degeneracy maps of $\mathbf{H}_{\mathrm{CEM}}$ are defined as in $\mathbf{K}$. The incidence coalgebra of $\mathbf{H}_{\mathrm{CEM}}$ is described in Remark 2.1.4.7.

Example 2.5.1.1 (Faà di Bruno comodule bialgebra of linear trees: part II). Let $\mathbf{H}_{\mathrm{FB}}$ denote the decomposition space induced by the directed hereditary species of linear trees H_{FB} (2.1.4.8). The description of $\mathbf{H}_{\mathrm{FB}}$ is similar to the description of $\mathbf{H}_{\mathrm{CEM}}$ but we consider linear trees instead of arbitrary trees.

2.5.2 *Comodule structure*

The theory of comodules in the context of decomposition spaces (2-Segal spaces) has been developed by Walde [93], and independently by Young [95], both in the context of Hall algebras. Carlier [23] gave a conceptual way to reformulate their definitions using linear functors. Given a map between two simplicial groupoids $F\colon C \to X$, the span

$$C_0 \xleftarrow{d_\top} C_1 \xrightarrow{(F_1, d_\perp)} X_1 \times C_0$$

defines a linear functor

$$\gamma\colon \mathbf{Grpd}_{/C_0} \to \mathbf{Grpd}_{/X_1} \otimes \mathbf{Grpd}_{/C_0}.$$

Proposition 2.5.2.1. *[23, Proposition 2.1.1] Let $F\colon C \to X$ be a map between simplicial groupoids. Suppose moreover that C is Segal, X is a decomposition space, and the map $F\colon C \to X$ is culf, then the span*

$$\begin{tikzcd} C_0 & C_1 \arrow[l, "d_\top"'] \arrow[r, "(F_1{,}d_\bot)"] & X_1 \times C_0 \end{tikzcd}$$

induces on the slice category $\mathbf{Grpd}_{/C_0}$ the structure of a left $\mathbf{Grpd}_{/X_1}$-comodule.

Given a connected directed hereditary species H, the upper decalage of $\mathbf{H}$ induces a left comodule over $\mathbf{Grpd}_{/\mathbf{H}_1}$.

Lemma 2.5.2.2. *The slice category $\mathbf{Grpd}_{/\mathbf{H}_1}$ is a left comodule over $\mathbf{Grpd}_{/\mathbf{H}_1}$.*

Proof. Note that $\mathrm{Dec}_\top \mathbf{H}$ is a Segal space since $\mathbf{H}$ is a decomposition space. Furthermore, the decalage map $d_\top\colon \mathrm{Dec}_\top \mathbf{H} \to \mathbf{H}$ is culf. Since $\mathrm{Dec}_\top \mathbf{H}$ is Segal and $d_\top$ is culf, it follows that

$$\begin{tikzcd} \mathbf{H}_1 & \mathbf{H}_2 \arrow[l, "d_1"'] \arrow[r, "(d_\top{,}d_0)"] & \mathbf{H}_1 \times \mathbf{H}_1 \end{tikzcd}$$

induces a left comodule structure of $\mathbf{Grpd}_{/\mathbf{H}_1}$ over $\mathbf{Grpd}_{/\mathbf{H}_1}$ by Lemma 2.5.2.1. ☐

2.6 *The incidence comodule bialgebra of a connected directed hereditary species*

Every hereditary species is, in particular, a restriction species by precomposition with the inclusion $\mathbb{I}^{\mathrm{op}} \to \mathbb{S}_p$. Therefore we have two bialgebra structures associated to a hereditary species H: the incidence bialgebra B of H and the incidence bialgebra A of the restriction species R associated to H. Carlier [23] showed that A is a left comodule bialgebra over B. The main result of this section is to apply the Carlier ideas to the directed case.

2.6.1 *Directed restrictions species*

Recall from Subsection 2.1.2 that a directed restriction species in the sense of Gálvez–Kock–Tonks [57] is a functor

$$R\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Grpd}.$$

As usual, the idea is that the value of a poset S is the groupoid of all possible R-structures that have S as the underlying poset.

The notion of directed restriction species needs to be modified to fit our theory of directed hereditary species since we only work with finite connected posets. Let $\mathbb{C}^\circ$ denote the category of connected finite posets and convex maps and $\mathbb{C}' := \mathsf{S}\mathbb{C}^\circ$. A *directed restriction species* is a functor

$$R\colon (\mathbb{C}')^{\mathrm{op}} \to \mathbf{Grpd}.$$

Every hereditary species H is, in particular, a directed restriction species. Indeed, let R° denote the precomposition of H with the inclusion $(\mathbb{C}^\circ)^{\mathrm{op}} \to \mathbb{K}_p$ which is identity on objects and sends a convex map $\iota\colon P' \to P$ to the span $P \xleftarrow{\iota} P' \xrightarrow{\mathrm{id}} P'$. The directed restriction species R is the monoidal extension of R°. In other words, R is the unique functor that makes the following diagram commutes:

$$\begin{tikzcd} (\mathbb{C}^\circ)^{\mathrm{op}} \arrow[r, "R^\circ"] \arrow[d] & \mathbb{K}_p \\ (\mathbb{C}')^{\mathrm{op}}. \arrow[ur, dotted, "R"'] & \end{tikzcd}$$

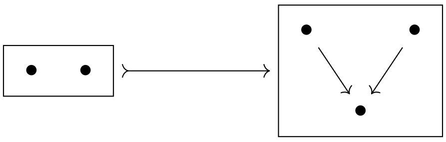
Every directed restriction species R induces a decomposition space $\mathbf{R}$ where an n-simplex is a family of n-layered posets with an R-structure on the underlying posets. In other words, the objects of $\mathbf{R}_2$ are families of maps of posets $P \to 2$ with a R-structure on P, and $\mathbf{R}_n$ is the groupoid of families of R-structures $P \to n$. The construction of $\mathbf{R}$ follows from the theory developed by Gálvez, Kock, and Tonks in [57, §7] considering only finite connected posets.

The comultiplication $\Delta_R : \mathbf{Grpd}_{/\mathbf{R}_1} \to \mathbf{Grpd}_{/\mathbf{R}_1} \otimes \mathbf{Grpd}_{/\mathbf{R}_1}$ is given by the span

$$\mathbf{R}_1 \xleftarrow{d_1} \mathbf{R}_2 \xrightarrow{(d_2,d_0)} \mathbf{R}_1 \times \mathbf{R}_1$$

where d_1 joins the two layeres of the 2-simplex and $d_\top$ and $d_\perp$ return the first and second layers respectively.

Remark 2.6.1.1. Our notion of directed restriction species is less general than the Gálvez–Kock–Tonks notion [57]. For example, the map picture in the following diagram



is a morphism in $\mathbb{C}$ but not in $\mathbb{C}'$.

Example 2.6.1.2. The Butcher–Connes–Kreimer Hopf algebra [31, 34] comes from the directed restriction species of trees R_{BCK}: a forest has an underlying poset, whose convex subposets inherit a tree structure (see Lemma 2.1.4.4). The comultiplication $\Delta_{R_{BCK}}$ is defined by summing over certain admissible cuts c:

$$\Delta_{R_{BCK}}(T) = \sum_{c \in \text{admi.cuts}(T)} P_c \otimes R_c.$$

Note that R_{BCK} is the ordinary directed restriction species associated with the connected directed hereditary species H_{CEM}.

Remark 2.6.1.3. Recall that for a connected directed hereditary species H, we have that $\mathbf{R}_1 = \mathbf{H}_1$, so that by Lemma 2.5.2.2, the slice category $\mathbf{Grpd}_{/\mathbf{R}_1}$ is a left $\mathbf{Grpd}_{/\mathbf{H}_1}$-comodule. The left $\mathbf{Grpd}_{/\mathbf{H}_1}$-coaction is the linear functor $\gamma_{\mathbf{R}_1} : \mathbf{Grpd}_{/\mathbf{R}_1} \to \mathbf{Grpd}_{/\mathbf{H}_1} \otimes \mathbf{Grpd}_{/\mathbf{R}_1}$ given by the span

$$\mathbf{R}_1 \xleftarrow{d_1} \mathbf{H}_2 \xrightarrow{(d_\top,d_0)} \mathbf{H}_1 \times \mathbf{R}_1.$$

The decomposition space $\mathbf{R}$ has a monoidal structure given by disjoint union. Recall $\mathbf{R}_n$ is the groupoid of families of finite connected posets with $n-1$ compatible cuts. The disjoint union of two such structures is given by taking the disjoint union of the underlying posets, with the cuts concatenated. This defines a simplicial map $+_{\mathbf{R}} \colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$. So $\mathbf{R}$ is a monoidal decomposition space if the map $+_{\mathbf{R}}$ is culf [58, §9].

Proposition 2.6.1.4. *The map* $+_{\mathbf{R}} \colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ *is culf.*

Proof. By Lemma 1.1.2.2, the map $+_{\mathbf{R}}$ is culf, if the diagram

$$\begin{tikzcd}
\mathbf{R}_2 \times \mathbf{R}_2 \arrow[r, "d_1"] \arrow[d, "+_{\mathbf{R}}"'] & \mathbf{R}_1 \times \mathbf{R}_1 \arrow[d, "+_{\mathbf{R}}"] \\
\mathbf{R}_2 \arrow[r, "d_1"'] & \mathbf{R}_1
\end{tikzcd}$$

is a pullback since $\mathbf{R}$ is a decomposition space. But this is clear: a pair of families of finite connected posets with a cut (an object in $\mathbf{R}_2 \times \mathbf{R}_2$) can be uniquely reconstructed if we know what are the families of the underlying finite connected posets are (an object in $\mathbf{R}_1 \times \mathbf{R}_1$) and we know how the disjoint union is cut (an object in $\mathbf{R}_2$). This provided of course that we can identify the disjoint union of those two families of posets with the family of the underlying posets of the disjoint union (which is to say that the data agree down in $\mathbf{R}_1$). ☐

Since $\mathbf{R}$ is a monoidal decomposition space, it follows that the resulting incidence coalgebra is also a bialgebra [58, §9].

2.6.2 *Comodule bialgebra*

For a background on comodule bialgebras, see for example [1, §3.2], [23, §5], and [81]. We follow the terminology of Carlier [23, §5]. Let B be a bialgebra. We can associate B with a canonical braided monoidal category of left B-comodules. The categorical structure comes from the coalgebra structure of B, and the (braided) monoidal structure arises from the algebra structure of B. A *comodule bialgebra* over B is a bialgebra object in the (braided) monoidal category of left B-comodules, where a *bialgebra object* in the braided monoidal category of left B-comodules is a B-comodule M together with structure maps

$$\begin{array}{ll} \Delta_M \colon M \to M \otimes M & \qquad\qquad \epsilon_M \colon M \to \mathbb{Q} \\ \mu_M \colon M \otimes M \to M & \qquad\qquad \eta_M \colon \mathbb{Q} \to M \end{array}$$

which are all required to be B-comodules maps and to satisfy the usual bialgebra axioms. We shall be concerned in particular with the requirement that Δ_M and ϵ_M be B-comodule maps which is to say that they are compatible with the coaction $\gamma \colon M \to B \otimes M$:

$$\begin{tikzcd}
M \arrow[r, "\Delta_M"] \arrow[dd, "\gamma"'] & M \otimes M \arrow[d, "\gamma\otimes\gamma"] \\
& B \otimes M \otimes B \otimes M \arrow[d, "\omega"] \\
B \otimes M \arrow[r, "\mathrm{id}_B \otimes \Delta_M"'] & B \otimes M \otimes M
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "\epsilon_M"] \arrow[d, "\gamma"'] & \mathbb{Q} \arrow[d, "\eta_B"] \\
B \otimes M \arrow[r, "\mathrm{id}_B \otimes \epsilon"'] & B
\end{tikzcd}$$

where the map ω is given by first swapping the two middle tensor factors and then using the multiplication of B in the two now adjacent B-factors. In Proposition 2.6.2.3, the two other axioms will be automatically satisfied, because as comodule, M coincides with B itself and the algebra structure of M coincides with that of B.

Recall that every connected directed hereditary species is also a directed restriction species. Let H be a connected directed hereditary species. We denote by R the induced directed restriction species and, as usual, $\mathbf{H}$ and $\mathbf{R}$ correspond to decomposition spaces. Also, the incidence coalgebra of $\mathbf{R}$ is denoted A, and the incidence bialgebra of $\mathbf{H}$ is denoted B.

Lemma 2.6.2.1. *The comultiplication structure of A is a B-comodule map.*

Proof. We need to check that the following diagram commutes, and the squares (1) and (2) are pullbacks:

$$\begin{tikzcd}
\mathbf{R}_1 & \mathbf{R}_2 \arrow[l, "d_1"'] \arrow[r, "(d_2{,}d_0)"] & \mathbf{R}_1 \times \mathbf{R}_1 \\
\mathbf{H}_2 \arrow[u, "d_1"] \arrow[dd, "(d_\top{,}d_0)"'] \arrow[rd, phantom, "(2)"] & Z \arrow[u, dashed, "\overline{d}_1"] \arrow[l, dashed, "\overline{d}_2"] \arrow[r, dashed, "\overline{d}_3"] \arrow[dd, dashed, "(g{,}\overline{d_0})"'] \arrow[ru, phantom, "(1)"] & \mathbf{H}_2 \times \mathbf{H}_2 \arrow[u, "d_1 \otimes d_1"'] \arrow[d, "(d_\top{,}d_0)\otimes(d_\top{,}d_0)"'] \\
 & & \mathbf{H}_1 \times \mathbf{R}_1 \times \mathbf{H}_1 \times \mathbf{R}_1 \arrow[d, "\omega"'] \\
\mathbf{H}_1 \times \mathbf{R}_1 & \mathbf{H}_1 \times \mathbf{R}_2 \arrow[l, "\mathrm{id} \otimes d_1"] \arrow[r, "\mathrm{id} \otimes (d_2{,}d_0)"'] & \mathbf{H}_1 \times \mathbf{R}_1 \times \mathbf{R}_1.
\end{tikzcd}$$

The objects of the groupoid Z are families of pairs of 2-chains of maps $P \twoheadrightarrow Q \to 2$, such that the first one is a contraction and the other is a cut, and with a H-structure on P. The map $\overline{d}_0$ forgets the contraction, the map $\overline{d}_1$ composes the contraction and the cut, the map $\overline{d}_2$ forgets the cut, the map $\overline{d}_3$ gives the pair of contractions $(P_1 \twoheadrightarrow Q_1, P_2 \twoheadrightarrow Q_2)$, and the map g sends the 2-chain to the family $\{P_q\}_{q \in Q}$ of the fibres of the contraction $P \twoheadrightarrow Q$.

By Lemma 1.2.0.1, the square (1) is a pullback if for any pair of contractions $(f_1 \colon P_1 \twoheadrightarrow Q_1, f_2 \colon P_2 \twoheadrightarrow Q_2) \in \mathbf{H}_2 \times \mathbf{H}_2$ and cut $P \to 2 \in \mathbf{R}_2$ such that

$$(d_2, d_0)(P \to 2) = (P_1, P_2) = (d_1 \otimes d_1)(P_1 \twoheadrightarrow Q_1, P_2 \twoheadrightarrow Q_2),$$

there exists a finite connected poset Q, a contraction $f \colon P \twoheadrightarrow Q$, and a cut $Q \to 2$ such that the diagram

$$\begin{tikzcd}
P \arrow[r, two heads, "f"] & Q \arrow[r] & 2 \\
P_i \arrow[u, tail] \arrow[r, two heads, "f_i"'] & Q_i \arrow[u, tail] &
\end{tikzcd}$$

commutes for $i = 1, 2$. But this is easy to show if we put $Q := \sum_{i \in 2} Q_i$, and $f := \sum_{i \in 2} f_i$, and consider the partial order $<_Q$ on Q given by taking transitive closure in the following relation: for $q, q' \in Q$, we declare that $q <_Q q'$ if $q, q' \in Q_i$ and $q <_{Q_i} q'$ or there exists $p <_P p'$ in P such that $f(p) = q$ and $f(p') = q'$.

We will prove that (2) is a pullback. By the prism Lemma 1.2.0.2, (2) is a pullback if the outer diagram

$$\begin{tikzcd}
Z \arrow[r, "(g{,}d_0)"'] \arrow[rr, bend left, "\overline{d}_0"] \arrow[d, "\overline{d}_2"'] & \mathbf{H}_1 \times \mathbf{R}_2 \arrow[r] \arrow[d, "\mathrm{id} \otimes d_1"] \arrow[rd, phantom, "(3)"] & \mathbf{R}_2 \arrow[d, "d_1"] \\
\mathbf{H}_2 \arrow[r, "(d_\top{,}d_0)"] \arrow[rr, bend right, "d_0"'] & \mathbf{H}_1 \times \mathbf{R}_1 \arrow[r] & \mathbf{R}_1
\end{tikzcd}$$

and the square (3) are pullbacks. The square (3) is obtained after projecting away $\mathbf{H}_1$ so it is straightforward to see that it is a pullback. By Lemma 1.2.0.1, the outer diagram is a pullback since for any contraction $P \twoheadrightarrow Q \in \mathbf{H}_2$ and cut $Q \to 2 \in \mathbf{R}_2$, we can form the 2-chain $P \twoheadrightarrow Q \to 2$, which is an object in Z, such that it makes the outer diagram commutes. □

Lemma 2.6.2.2. *The counit structure of A is a B-comodule map.*

Proof. We must show that the following diagram commutes and the squares (4) and (5) are pullbacks:

$$\begin{tikzcd}
\mathbf{R}_1 & \mathbf{R}_0 \arrow[l, "s_0"'] \arrow[r] \arrow[dr, phantom, "(4)"] & 1 \\
\mathbf{H}_2 \arrow[u, "d_1"] \arrow[d, "{(d_\top,d_0)}"'] \arrow[dr, phantom, "(5)"] & \mathbf{R}_0 \arrow[l, dotted] \arrow[r, dotted] \arrow[u, dotted] \arrow[d, dotted] & 1 \arrow[u] \arrow[d] \\
\mathbf{H}_1 \times \mathbf{R}_1 & \mathbf{H}_1 \times \mathbf{R}_0 \arrow[l, "\mathrm{id} \otimes s_0"] \arrow[r] & \mathbf{H}_1 .
\end{tikzcd}$$

The pullback of $\mathrm{id} \otimes s_0$ along $(d_\top, d_0)$ is the groupoid of families of contractions $P \twoheadrightarrow Q$ with a H-structure on P, such that the induced H-structure on Q is an empty H-structure. This forces that both P and Q are empty, and therefore it is any H-structure on the empty poset, which is $\mathbf{R}_0$. This means that (5) is a pullback. ☐

Proposition 2.6.2.3. *A is naturally a left B-comodule bialgebra.*

Proof. Combining Lemmas 2.6.2.1 and 2.6.2.2, it follows that A is naturally a left B-comodule coalgebra. The bialgebraic part follows from the monoidal property of $\mathbf{H}$ and $\mathbf{R}$, and that the multiplication of both is the same for how $\mathbf{R}$ was constructed. ☐

Remark 2.6.2.4. The arguments used in the proof that A is naturally a left B-comodule coalgebra in Proposition 2.6.2.3 are the same as those given in the proof of Proposition 5.3 in [23] considering contractions instead of monotone surjections and families of finite connected posets instead of sets. We prefer to add the proof to make the document as self-contained as possible, but in any case, the ideas come from Carlier [23].

2.6.3 *Calaque–Ebrahimi-Fard–Manchon comodule bialgebra of rooted trees: part III*

The connected directed hereditary species H_{CEM} of trees, described in §2.1.4, induces a comodule bialgebra. The comultiplication $\Delta_{H_{\mathrm{CEM}}}$ is explained in Remark 2.1.4.7. The second comultiplication is given by the directed restriction species of trees R_{BCK} (§2.6.1). By Proposition 2.6.2.3, this is a comodule bialgebra usually known as the Calaque–Ebrahimi-Fard–Manchon comodule bialgebra of rooted trees [22].

On the other hand, Kock [70, §5.4] showed another presentation of the Calaque–Ebrahimi-Fard–Manchon comodule bialgebra of rooted trees related to the reduced Baez–Dolan construction of the terminal operad. In the operadic setting, operadic trees have an input edge (the leaf edge) and an output edge (the root edge), and there is a tree without nodes (where leaf=root). To relate operadic trees and combinatorial trees, we have to forget all decorations and shave off leaf edges and root edge. This construction is known as the *core* of an operadic tree.

Since the trees involved in the Calaque–Ebrahimi-Fard–Manchon comodule bialgebra are combinatorial trees, it is not possible to realise this bialgebra as the incidence bialgebra of an operad, but this can be solved using the core construction. Kock [70, Proposition 5.4.7] proved that taking core of the incidence comodule bialgebra of the reduced Baez–Dolan construction of the terminal operad, we obtain the Calaque–Ebrahimi-Fard–Manchon comodule bialgebra of rooted trees. Note that our approach to this comodule bialgebra is more straightforward since it follows from the connected directed hereditary species H_{CEM} of trees.

Example 2.6.3.1 (Faà di Bruno comodule bialgebra of linear trees: part III). The connected directed hereditary species H_{FB} of linear trees, described in Example 2.1.4.8, induces a comodule bialgebra: the comultiplication $\Delta_{H_{\mathrm{FB}}}$ is explained in Example 2.1.4.8. The second comultiplication is given by the directed restriction species of linear trees R_{FB}, which is similar to R_{BCK} but we consider linear trees instead of arbitrary trees. By Proposition 2.6.2.3, this is a comodule bialgebra.

In fact, it is the Faà di Bruno comodule bialgebra of linear trees. The form in which it arises here is very similar to that shown in [70, Section 5.2] from the reduced Baez–Dolan construction on the identity monad. The difference between these two presentations of the Faà di Bruno comodule bialgebra is analogous to the difference between the Calaque–Ebrahimi-Fard–Manchon bialgebra via the reduced Baez–Dolan construction [70] and the construction from the directed hereditary species H_{CEM}.

2.7 *Connected directed hereditary species and operadic categories*

The goal of this section is to construct a functor from the category of connected directed hereditary species to the category of operadic categories.

Operadic categories were introduced by Batanin and Markl [13] and used to prove the duoidal Deligne conjecture. An operadic category is a kind of combinatorial structure whose 'algebras' are operads of various kinds depending on the operadic category. For example, Δ is an operadic category and its operads are nonsymmetric operads.

An operadic category $\mathcal{C}$ has chosen local terminal objects (that is, in each connected component there is a chosen terminal object), a cardinality functor $|-|: \mathcal{C} \to \mathbf{FinSet}$ to the standard skeleton of the category of finite sets, and a notion of fibre: this is an assignment that for each morphism $F\colon Y \to X$ in the operadic category, and each $i \in |X|$ gives a new object denoted $f^{-1}(i)$, but this is abstract and does not have to be a fibre in the usual sense of the word. For example, it is not necessarily a subobject of Y. These data are subject to many axioms, which can be formulated in various ways [13], [74], [54]. The Carlier proof [23] that hereditary species induce operadic categories consisted in checking the whole list of axioms.

Garner, Kock, and Weber [54] observed that the chosen-local-terminals structure amounts precisely to be a coalgebra for the upper decalage comonad, and went on to give a characterisation of operadic categories in terms of a certain modified decalage comononad.

Recently Batanin, Kock, and Weber [10] have found a more conceptual characterisation, where all the axioms end up formulated as simplicial identities. Their discovery is that just as the chosen-local-terminals structure amounts to an extra top degeneracy map, the fibre structures amount to an extra top face map, except that this extra top face map lives in the Kleisli category for the free-symmetric-monoidal-category monad. To understand this point of view, we will introduce a few concepts.

2.7.1 *The lt-nerve*

Recall that $\mathbf{Cat}_{\mathrm{lt}}$ is the category of categories with chosen local terminals. Given a opfibration $p\colon \mathcal{E} \to \mathcal{B}$ and an object $x \in \mathcal{E}$, we will denote by $f_!(x)$ the opcartesian lift for a map $f\colon p(x) \to y$ in $\mathcal{B}$.

Lemma 2.7.1.1. *Let $\mathcal{B}$ be a category with chosen local terminal objects and let $p\colon \mathcal{E} \to \mathcal{B}$ be a discrete opfibration. Consider the pullback diagram*

$$\begin{tikzcd}
\mathrm{Fib}_{\mathcal{B}_{\mathrm{lt}}}(p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{B}_{\mathrm{lt}} \arrow[r, hook] & \mathcal{B}.
\end{tikzcd}$$

The objects in $\mathrm{Fib}_{\mathcal{B}_{\mathrm{lt}}}(p)$ *equip $\mathcal{E}$ with chosen local terminal objects.*

Proof. Let x be an object in $\mathcal{E}$. Since $\mathcal{B}$ has chosen local terminal objects, we have a unique map $t_{p(x)}: p(x) \to c_{p(x)}$ in $\mathcal{E}$, where $c_{p(x)}$ is a chosen local terminal object. Furthermore, we have a unique lift $(t_{p(x)})_!(x) : x \to c_x$ for the map $t_{p(x)}$ in $\mathcal{E}$ since p is a discrete opfibration. Note that $c_x \in \mathrm{Fib}_{\mathcal{B}_{\mathrm{lt}}}(p)$. To prove that c_x is a chosen local terminal object in $\mathcal{E}$ it is enough to prove that any map $f\colon x \to y$ in $\mathcal{E}$ forces that $c_x = c_y$. Indeed, the objects $c_{p(x)}$ and $c_{p(y)}$ are connected by the zig-zag illustrated in the following diagram

$$\begin{tikzcd}
p(x) \arrow[r, "p(f)"] \arrow[d, "t_{p(x)}"'] & p(y) \arrow[d, "t_{p(y)}"] \\
c_{p(x)} & c_{p(y)}.
\end{tikzcd}$$

Since $c_{p(x)}$ and $c_{p(y)}$ are in the same connected component in $\mathcal{B}$, the chosen local terminal object property forces that $c_{p(x)} = c_{p(y)}$. This implies that the maps $(t_{p(y)})_!(y) \circ f\colon x \to c_y$ and $(t_{p(x)})_!(x)\colon x \to c_x$ are two lifts for the map $t_{p(x)}$. But the discrete opfibration property of p forces that $(t_{p(y)})_!(y) \circ f = (t_{p(x)})_!(x)$, and hence $c_x = c_y$. ☐

We now work with posets whose underlying sets are ordinals and strict pullbacks. These assumptions are necessary to follow the work of Batanin and Markl [13, §1] on operadic categories.

Moreover, we work only with **Set**-valued species, as in the Schmitt theory of hereditary species. This is necessary to ensure that given a connected directed hereditary species $H\colon \mathbb{K}_p \to \mathbf{Set}$, its Grothendieck construction $\int H \to \mathbb{K}_p$ is a discrete opfibration. For a connected directed hereditary species, the category $\mathbb{H}$ is the pullback of the Grothendieck construction $\int H \to \mathbb{K}_p$ along the inclusion $\mathbb{K} \to \mathbb{K}_p$.

Lemma 2.7.1.2. *Let* $H\colon \mathbb{K}_p \to \mathbf{Set}$ *be a connected directed hereditary species. Then* $\mathbb{H}$ *is a category with chosen local terminal objects.*

Proof. Since H is a presheaf with values in **Set**, we have that $\int H \to \mathbb{K}_p$ is a discrete opfibration. This implies by the construction of $\mathbb{H}$ and the stability of discrete opfibrations under pullback that $\mathbb{H} \to \mathbb{K}$ is a discrete opfibration. Combining this with the fact that $\mathbb{K}$ has a terminal object (the poset with one element), it follows that the set $H[1]$ of H-structures of the poset with one element is a set of chosen local terminal objects in $\mathbb{H}$ as a consequence of Lemma 2.7.1.1. ☐

Recall that $\mathbb{\Delta}^t$ is the category whose objects are the non-empty finite ordinals and whose morphisms are the monotone maps that preserve the top element. Let **tsGrpd** denote the category of **Grpd**-valued $\mathbb{\Delta}^t$-presheaves.

Definition 2.7.1.3. For $\mathcal{C}$ a category with chosen local terminals, its lt-*nerve* $N^{lt}(\mathcal{C})$ is the $\mathbb{\Delta}^t$-presheaf

$$N^{lt}(\mathcal{C})\colon (\mathbb{\Delta}^t)^{op} \hookrightarrow \mathbf{Cat}^{op} \xrightarrow{\mathbf{Cat}(-,\mathcal{C})} \mathbf{Set}$$

Let us describe the lt-nerve $N^{lt}(\mathcal{C})$: for $n \geqslant 0$, the set $N^{lt}(\mathcal{C})_n$ is the same as the set $N(\mathcal{C})_n$. The set $N^{lt}(\mathcal{C})_{[-1]}$ is the set of chosen local terminal objects in $\mathcal{C}$. The face and degeneracy maps act as the usual nerve construction except in $d_\perp\colon N^{lt}(\mathcal{C})_0 \to N^{lt}(\mathcal{C})_{[-1]}$ that sends each object in $\mathcal{C}$ to its corresponding chosen local terminal object. The degeneracy map $s_0\colon N^{lt}(\mathcal{C})_{[-1]} \to N^{lt}(\mathcal{C})_0$ is the inclusion.

Example 2.7.1.4. Given $H\colon \mathbb{K}_p \to \mathbf{Set}$ a connected directed hereditary species, we have that $\mathbb{H}$ is a category with chosen local terminal objects by Lemma 2.7.1.2. The lt-nerve of $\mathbb{H}$ is described as follows: $(N^{lt}\,\mathbb{H})_{[-1]}$ is the set $H[1]$ of H-structures over the poset with one element. $(N^{lt}\,\mathbb{H})_0$ is the set of finite posets with a H-structure. $(N^{lt}\,\mathbb{H})_1$ is the set of contractions with a H-structure on the first poset. For $n \geqslant 2$, the elements of the set $(N^{lt}\,\mathbb{H})_n$ are $(n-1)$-chains of contractions with a H-structure on the first poset in the chain.

2.7.2 *Half decalage*

Let $\mathbf{sGrpd}^{\mathrm{tps}}$ denote the subcategory of pseudosimplicial groupoids that the unique pseudosimplicial identities involve the top face maps. Given a pseudo simplicial space X in $\mathbf{sGrpd}^{\mathrm{tps}}$, the *half upper dec* $\mathrm{HDec}_\top X$ is a $\mathbb{\Delta}^{\mathrm{t}}$-presheaf obtained by deleting the top face maps and shifting everything one position down (for $n \geqslant -1$, we have that $(\mathrm{HDec}_\top X)_n = X_{n+1}$). Note that $\mathrm{HDec}_\top$ throws away the top face maps but keeps the top degeneracy maps to get a $\mathbb{\Delta}^{\mathrm{t}}$-presheaf (with values in groupoids). This gives a functor $\mathrm{HDec}_\top \colon \mathbf{sGrpd}^{\mathrm{tps}} \to \mathbf{tsGrpd}$.

A $\mathbb{\Delta}^{\mathrm{t}}$-presheaf A is a $\mathbb{\Delta}^{\mathrm{t}}$*-Segal space* if the simplicial groupoid obtained after eliminating $A_{[-1]}$ is Segal. Since we are working in this section with **Set**-valued species, we have to modify the definition of $\mathbf{H}$ to $\mathbf{H} := \mathrm{S}\,\mathrm{N}^{\mathrm{lt}}\,\mathbb{H}$.

Lemma 2.7.2.1. *Let* $H \colon \mathbb{K}_p \to \mathbf{Set}$ *be a connected directed hereditary species. Then* $\mathrm{HDec}_\top \mathbf{H}$ *is a* $\mathbb{\Delta}^{\mathrm{t}}$*-Segal space.*

Proof. Since $\mathbb{H}$ is a category, its nerve $\mathrm{N}\,\mathbb{H}$ is a Segal space, and therefore $\mathrm{S}\,\mathrm{N}\,\mathbb{H}$ is a Segal space, as S preserves pullbacks and hence Segal objects. This means that $\mathrm{S}\,\mathrm{N}^{\mathrm{lt}}\,\mathbb{H}$ is a $\mathbb{\Delta}^{\mathrm{t}}$-Segal space. □

2.7.3 *The category of connected directed hereditary species and* **OpCat**

For Batanin, Kock, and Weber [10] an *operadic category* is a pseudosimplicial groupoid X whose half upper dec $\mathrm{HDec}_\top X$ is equal to the symmetrical monoidal functor of the lt-nerve of some category with chosen local terminal objects. In short: it is a pair $(\mathcal{C}, X)$ such that $\mathrm{S}\,\mathrm{N}^{\mathrm{lt}}\,\mathcal{C} = \mathrm{HDec}_\top X$, where $X \in \mathbf{sGrpd}^{\mathrm{tps}}$ and $\mathcal{C} \in \mathbf{Cat}_{\mathrm{lt}}$. To be more precise, they prove that the diagram

$$\begin{tikzcd}
\mathbf{OpCat} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{sGrpd}^{\mathrm{tps}} \arrow[d, "\mathrm{HDec}_\top"] \\
\mathbf{Cat}_{\mathrm{lt}} \arrow[r, "\mathrm{S}\,\mathrm{N}^{\mathrm{lt}}"'] & \mathbf{tsGrpd}
\end{tikzcd}$$

is a strict pullback of categories.

Let **ConDirHerSp** denote the category of connected directed hereditary species. For each connected directed hereditary species H, we have a pseudosimplicial groupoid $\mathbf{H}$. For a map $f \colon H' \to H$ between connected directed hereditary species, the Grothendieck construction of f gives a map $\int f \colon \mathbb{H}' \to \mathbb{H}$. This map induces a map from $\mathbf{H}'$ to $\mathbf{H}$ that we denote as $\int^{\mathbf{K}} f \colon \mathbf{H}' \to \mathbf{H}$. Since f is simplicial map for each $P \in \mathbb{K}$, we have a map $f_P \colon H'[P] \to H[P]$. So the map $\int^{\mathbf{K}} f$ sends an n-simplex in $\mathbf{H}'$ which is a n-chain of contractions

$$P_0 \twoheadrightarrow P_1 \twoheadrightarrow \dots \longrightarrow P_{n-2} \twoheadrightarrow P_{n-1}$$

with an H'-structure X in P_0 to the same n-chain but with the H-structure $f_{P_0}(X)$ in P_0 which is a n-simplex in $\mathbf{H}$.

We define the functor $\int^{\mathbf{K}} \colon \mathbf{ConDirHerSp} \to \mathbf{sGrpd}^{\mathrm{tps}}$ as follows: $\int^{\mathbf{K}}(H) = \mathbf{H}$, for each object $H \in \mathbf{ConDirHerSp}$ and for a morphism $f \colon H' \to H$, the functor sends it to $\int^{\mathbf{K}} f$. Furthermore, $\mathrm{HDec}_\top \circ \int^{\mathbf{K}}(H)$ is a $\mathbb{\Delta}^{\mathrm{t}}$-Segal space by Lemma 2.7.2.1.

On the other hand, Lemma 2.7.1.2 established that $\mathbb{H}$ is a category with chosen local terminal objects given by the set $H[1]$ of H-structures of the poset with one element. This gives a functor $\int^{*} \colon \mathbf{ConDirHerSp} \to \mathbf{Cat}_{\mathrm{lt}}$ that sends H to the category $\mathbb{H}$.

Lemma 2.7.3.1. *The diagram*

$$\begin{tikzcd}
\mathbf{ConDirHerSp} \arrow[r, "\int^{\mathbf{K}}"] \arrow[d, "\int^{*}"'] & \mathbf{sGrpd}^{\mathrm{tps}} \arrow[d, "\mathrm{HDec}_{\top}"] \\
\mathbf{Cat}_{\mathrm{lt}} \arrow[r, "\mathrm{S}\,\mathrm{N}^{\mathrm{lt}}"'] & \mathbf{tsGrpd}
\end{tikzcd}$$

strictly commutes.

Proof. Let H be an object in **ConDirHerSp**. The commutativity of the diagram follows from the fact that $\int^{*}(\mathrm{H}) = \mathbb{H}$ and $\int^{\mathbf{K}}(\mathrm{H}) = \mathbf{H} = \mathrm{S}\,\mathrm{N}^{\mathrm{lt}}(\mathbb{H})$. ☐

Proposition 2.7.3.2. *There exists a canonical functor from the category of connected directed hereditary species* **ConDirHerSp** *to the category of operadic categories* **OpCat***.*

Proof. Consider the diagram

$$\begin{tikzcd}
\mathbf{ConDirHerSp} \arrow[rrd, bend left, "\int^{\mathbf{K}}"] \arrow[rd, dotted] \arrow[rdd, bend right, "\int^{*}"'] & & \\
& \mathbf{OpCat} \arrow[r] \arrow[d] \arrow[rd, phantom, "\lrcorner", very near start] & \mathbf{sGrpd}^{\mathrm{tps}} \arrow[d, "\mathrm{HDec}_{\top}"] \\
& \mathbf{Cat}_{\mathrm{lt}} \arrow[r, "\mathrm{S}\,\mathrm{N}^{\mathrm{lt}}"'] & \mathbf{tsGrpd}.
\end{tikzcd}$$

Batanin, Kock, and Weber [10] proved that the square is a pullback. The outer diagram commutes by Lemma 2.7.3.1. The dotted arrow then exists by the pullback property of **OpCat**. The functor **ConDirHerSp** → **OpCat** sends a connected directed hereditary species H to the pair $(\mathbb{H}, \mathbf{H})$. ☐

Connected directed hereditary species constitute a new family of examples of operadic categories. In fact, the connected directed hereditary species associated to the Fauvet–Foissy–Manchon comodule bialgebra of finite topologies and admissible maps; and the connected directed hereditary species $\mathrm{H}_{\mathrm{CEM}}$ associated to the Calaque–Ebrahimi-Fard–Manchon comodule bialgebra of rooted trees are now covered by the theory of operadic categories.

2.8 *Directed hereditary species as monoidal decomposition spaces, comodule bialgebras and operadic categories*

Schmitt hereditary species are not connected directed hereditary species, as the fibres along a surjection between discrete posets are not necessarily connected. To cover these examples, in this section, we introduce the notion of collapse, which allows for non-connected fibres. This leads to the notion of (not-necessarily-connected) directed hereditary species. Furthermore, each directed hereditary species induces a decomposition space (2.8.4), a comodule bialgebra (2.8.5), and a operadic category (2.8.6).

2.8.1 *Partially reflecting maps*

Definition 2.8.1.1. A map of posets $f \colon P \to Q$ is *partially reflecting* if $f(x) < f(y)$ in Q implies that $x < y$ in P. Partially reflecting monotone surjections are called *collapse maps.*

Lemma 2.8.1.2. *In the category of posets, collapse maps are stable under pullback.*

Proof. Let P, Q and V be posets. Let $f\colon P \twoheadrightarrow V$ be a collapse. Let $g\colon Q \to V$ be a monotone map and let

$$\begin{tikzcd} P \times_V Q \arrow[r, "\pi_Q"] \arrow[d, "\pi_P"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q \arrow[d, "g"] \\ P \arrow[r, two heads, "f"'] & V \end{tikzcd}$$

be a pullback diagram. Since monotone surjections are stable under pullback, we have that π_Q is a monotone surjection. It remains to prove that π_Q is a partially reflecting map. Let (p, q) and (p', q') be objects in $P \times_V Q$. This means

$$f(p) = g(q) \text{ and } f(p') = g(q'). \tag{1}$$

Assuming that $\pi_Q(p, q) <_Q \pi_Q(p', q')$, we get that

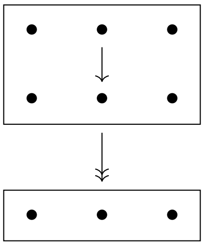
$$q <_Q q'. \tag{2}$$

Since g is a monotone map, $g(q) <_V g(q')$. This together with Eq. (1) implies that

$$f(p) <_V f(p'). \tag{3}$$

Since f is partially reflecting, we have that $p <_P p'$. This combining with (2) implies $(p, q) < (p', q')$. Hence, π_Q is partially reflecting. □

An illustration of a collapse is given by the following picture



2.8.2 *Directed Hereditary Species*

We can also define the notion of directed hereditary species for the non-connected case by substituting the category $\mathbb{K}_p$ of partially defined contractions by the category $\mathbb{D}_p$ of partially defined collapse maps in Definition 2.1.2.1.

A *partially defined collapse map* $P \to Q$ consists of a convex subposet P' of P and a collapse $P' \twoheadrightarrow Q$

$$\begin{tikzcd} & P' \arrow[dl, tail, "i"'] \arrow[dr, two heads, "f"] & \\ P, & & Q \end{tikzcd}$$

where i is a convex map and f is a collapse. Partially defined collapse maps are composed by pullback composition of spans in the category $\mathbb{D}$.

Definition 2.8.2.1. A *directed hereditary species* is a functor $H\colon \mathbb{D}_p \to \mathbf{Grpd}$.

Example 2.8.2.2. Any Schmitt hereditary species is a directed hereditary species since any set can be regarded as a discrete poset, and any partial surjection of sets is then a partially defined collapse map of discrete posets.

2.8.3 *Pseudosimplicial groupoid of collapse maps*

In this subsection, the monoidal decomposition space $\mathbf{D}$ of finite non-empty posets and collapse maps is defined in analogy with $\mathbf{K}$.

Let $\mathbb{D}$ denote the category of finite posets and collapse maps. We define $\mathbf{D} := \mathsf{S}\mathbf{N}^{\mathrm{lt}}(\mathbb{D})$ to be the symmetric monoidal category functor S applied to the fat lt-nerve of $\mathbb{D}$. All the face maps (except the missing top ones) and degeneracy maps are just S applied to the face and degeneracy maps of $\mathbf{N}^{\mathrm{lt}}(\mathbb{D})$. The top face map is given by:

- For $n \geqslant 2$, the top face map $d_\top \colon \mathbf{D}_n \to \mathbf{D}_{n-1}$ is defined as follows: given a $(n-1)$-chain of collapse maps

$$\begin{tikzcd} P_0 \arrow[r, two heads, "f_0"] & P_1 \arrow[r, two heads, "f_1"] & P_2 \arrow[r, two heads] & \cdots \arrow[r, two heads] & P_{n-2} \arrow[r, two heads, "f_{n-2}"] & Q, \end{tikzcd}$$

  for each element q of the last target poset, we can form the fibres over q of the source posets. We end up with a family

$$\begin{tikzcd} \{(P_0)_q \arrow[r, two heads, "(f_0)_q"] & (P_1)_q \arrow[r, two heads, "(f_1)_q"] & (P_2)_q \arrow[r, two heads] & \cdots \arrow[r, two heads] & (P_{n-3})_q \arrow[r, two heads, "(f_{n-2})_q"] & (P_{n-2})_q\}_{q\in Q}, \end{tikzcd}$$

  for each $0 \leqslant i \leqslant n-2$, the map $(f_i)_q$ is a collapse by Lemma 2.8.1.2
- $d_\top \colon \mathbf{D}_1 \to \mathbf{D}_0$ sends a family of finite posets $\{Q_i\}_{i\in I}$ to the underlying set of the disjoint union $\sum_{i\in I} Q_i$.

Note that the fibres are non-empty convex subposets since we only consider collapse maps, not arbitrary maps.

Proposition 2.8.3.1. *The groupoids $\mathbf{D}_n$ and the degeneracy and face maps given above form a pseudosimplicial groupoid $\mathbf{D}$.*

Proof. The proof is analogous to that of Proposition 2.2.0.5. ☐

To simplify the proof that $\mathbf{D}$ is a decomposition space, we need some preliminaries.

Proposition 2.8.3.2. *We have an equality* $\mathrm{Dec}_\top \mathbf{D} = \mathsf{S}\mathbf{N}\mathbb{D}$.

Proof. Note that the objects of $(\mathsf{S}\mathbf{N}\mathbb{D})_n$ are finite families of $(n-1)$-chains of collapse maps between non-empty finite posets, which is the same as the description given above of $(\mathrm{Dec}_\top \mathbf{D})_n$. ☐

Remark 2.8.3.3. Since S preserves pullbacks and $\mathbf{N}\mathbb{D}$ is a Segal space, Proposition 2.8.3.2 implies that $\mathrm{Dec}_\top \mathbf{D}$ is a Segal space. This is equivalent to saying that for each $n \geqslant 2$ the following diagram is a pullback for $0 < i < n$:

$$\begin{tikzcd} \mathbf{D}_{n+1} \arrow[r, "d_{i+1}"] \arrow[d, "d_\perp"'] & \mathbf{D}_n \arrow[d, "d_\perp"] \\ \mathbf{D}_n \arrow[r, "d_i"'] & \mathbf{D}_{n-1}. \end{tikzcd}$$

Lemma 2.8.3.4. *Suppose we have a collapse $f \colon P \twoheadrightarrow Q$ and a family of collapse maps $\{h_q \colon P_q \twoheadrightarrow W_q\}_{q\in Q}$. Then there exists a unique poset W and collapse maps h and g such that the diagram*

$$\begin{tikzcd} P \arrow[r, dotted, two heads, "h"'] \arrow[rr, bend left, two heads, "f"] & W \arrow[r, dotted, two heads, "g"'] & Q \\ P_q \arrow[u, tail] \arrow[r, two heads, "h_q"'] & W_q \arrow[u, dotted, tail] & \end{tikzcd}$$

commutes. Here the vertical arrows are convex inclusions.

Proof. We will do the proof in two steps: in the first place, we will construct the underlying set of the poset W and the functions $h\colon P \to W$ and $g\colon W \to Q$. After that, we will construct a partial order $<_W$ on W forced by the requirement that h and g are collapse maps.

- Put $W := \sum_{q\in Q} W_q$ and $h := \sum_{q\in Q} h_q$. The map $g\colon W \to Q$ is defined as $g(w) = q$ for $w \in W_q$. Furthermore, the diagram

$$\begin{tikzcd}
P \arrow[r, dotted, two heads, "h"'] \arrow[rr, bend left, "f"] & W \arrow[r, dotted, two heads, "g"'] & Q \\
P_q \arrow[u, hook] \arrow[r, two heads, "h_q"'] & W_q \arrow[u, dotted, tail] \arrow[r] & 1 \arrow[u, "\ulcorner q \urcorner"']
\end{tikzcd}$$

  commutes at the level of sets by the way h and g were defined.

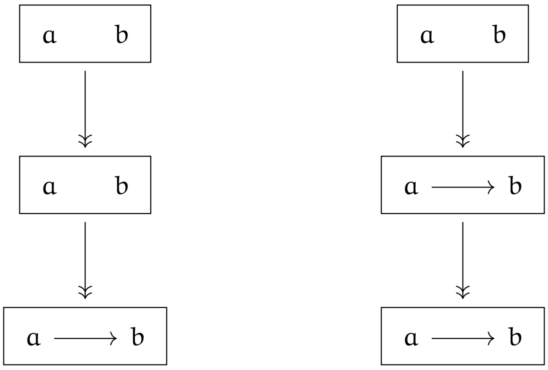
- The partial order $<_W$ on W is defined as follows: for $w, w' \in W$, we declare that $w <_W w'$ if one of the following conditions is satisfied:

  1. In case $w, w' \in W_q$ and $w <_{W_q} w'$;
  2. In case $g(w) \neq g(w')$ and $g(w) <_Q g(w')$.

  The condition (b) is required to make $g\colon W \to Q$ to be a collapse. Let p and p' be objects in P_i such that $h(p) <_W h(p')$. Applying g, we have that $g(h(p)) <_Q g(h(p'))$. Since $f = g \circ h$, it follows that $f(p) <_Q f(p')$. By hypothesis, f is partially reflecting. This implies that $p <_P p'$. Therefore, h is a partially reflecting map. Furthermore, h is a monotone surjection. Indeed, let p and p' be objects in P such that $p <_P p'$. Applying $g \circ h$, we have that $g(h(p)) <_Q g(h(p'))$. Since g is partially reflecting, it follows that $h(p) <_W h(p')$.

☐

Remark 2.8.3.5. Lemma 2.8.3.4 would not hold for general monotone surjections instead of collapse maps. Suppose we have monotone surjections illustrated in the following picture:



If the inclusion map from $\{a, b\}$ to $\{a \to b\}$ plays the role of f in Lemma 2.8.3.4, the collapse maps illustrated above are two solutions to the problem described in Lemma 2.8.3.4 and therefore the lemma would be false.

Lemma 2.8.3.6. *For each $0 < i < n$, the map $d_i\colon \mathbf{D}_n \to \mathbf{D}_{n-1}$ is a fibration.*

Proof. The proof is analogous to that of Proposition 2.2.0.7, but using the category $\mathbb{D}$ instead of $\mathbb{K}$. ☐

Proposition 2.8.3.7. *The pseudosimplicial groupoid $\mathbf{D}$ is a decomposition space.*

Proof. The proof is analogous to that of Proposition 2.2.0.8, but applying Remark 2.8.3.2 instead of Remark 2.2.0.4 and Lemma 2.8.3.4 instead of Lemma 2.2.0.6. ☐

Proposition 2.8.3.8. *The decomposition space* **D** *is complete.*

Proof. The proof is analogous to that of Proposition 2.2.0.9, but using $\mathbb{D}$ instead of $\mathbb{K}$. ☐

Proposition 2.8.3.9. *The decomposition space* **D** *is locally finite, locally discrete, and of locally finite length.*

Proof. The proof is analogous to that of Proposition 2.2.0.10. ☐

Since the simplicial groupoid **D** is equal to $\mathbf{SN}^{\mathrm{lt}}\mathbb{D}$, the decomposition space **D** is a monoidal decomposition space. The monoidal structure is obtained by categorical sum as in **K**.

2.8.4 *Directed hereditary species as decomposition spaces*

We can also construct a decomposition space **H** from a directed hereditary species $\mathsf{H}\colon \mathbb{D}_p \to \mathbf{Grpd}$ similarly to the connected case. We define $\mathbf{H}_1$ as the groupoid of families of non-empty H-structures. An object of $\mathbf{H}_n$ is a family of chains of collapses

$$\begin{tikzcd} P_0 \arrow[r, two heads, "f_0"] & P_1 \arrow[r, two heads] & \cdots \arrow[r, two heads] & P_{n-2} \arrow[r, two heads, "f_{n-2}"] & P_{n-1} \end{tikzcd}$$

with an H-structure on each P_0. The inner face maps and degeneracy maps are induced by the maps in **D**. For the bottom face map we use functoriality of H along collapse maps. Similarly, we use (contravariant) functoriality in convex map to define the top face map. The groupoid $\mathbf{H}_0$ is defined as the groupoid of families of H-structures over the poset with one element.

Proposition 2.8.4.1. *The groupoids* $\mathbf{H}_n$ *form a monoidal decomposition space* **H**.

Proof. The proof is analogous to that of Proposition 2.5.0.2. ☐

Proposition 2.8.4.2. *The decomposition space* **H** *is complete, locally finite, locally discrete, and of locally finite length.*

Proof. The proof is analogous to that of Proposition 2.5.0.3 ☐

2.8.5 *The incidence comodule bialgebra of non-connected directed hereditary species*

Recall that every directed hereditary species is also a directed restriction species, in the sense of Gálvez–Kock–Tonks [57], through precomposition with the inclusion $\mathbb{C}^{\mathrm{op}} \to \mathbb{D}_p$. Let H be a directed hereditary species. We denote by R the induced directed restriction species and, as usual, **H** and **R** the corresponding to decomposition spaces. Also, the incidence coalgebra of **R** is denoted A, and the incidence bialgebra of **H** is denoted B. The following result is a consequence of the theory developed in Section 2.6 but using collapse maps instead of contractions.

Proposition 2.8.5.1. A *is a left comodule bialgebra over* B.

2.8.6 *Directed hereditary species as operadic categories*

We now work with **Set**-valued species as in the classical theory and posets whose underlying sets are ordinals. These are necessary to ensure that given a directed hereditary species $\mathsf{H}\colon \mathbb{D}_p \to \mathbf{Set}$, its Grothendieck construction $\int \mathsf{H} \to \mathbb{D}_p$ is a discrete opfibration and have precise constructions in the context of operadic categories.

Recall that for a directed hereditary species H, the category $\mathbb{H}$ is the pullback of the Grothendieck construction $\int \mathsf{H} \to \mathbb{D}_p$ along the inclusion $\mathbb{D} \to \mathbb{D}_p$.

Lemma 2.8.6.1. *Let* $\mathrm{H}\colon \mathbb{D}_{\mathrm{p}} \to \mathbf{Set}$ *be a directed hereditary species. Then* $\mathbb{H}$ *is a category with local terminal objects.*

Proof. Since H is a presheaf with values in $\mathbf{Set}$, we have that $\int \mathrm{H} \to \mathbb{D}_{\mathrm{p}}$ is a discrete opfibration. This implies by the construction of $\mathbb{H}$ and the stability of discrete opfibration under pullback that $\mathbb{H} \to \mathbb{D}$ is a discrete opfibration. Combining this with the fact that $\mathbb{D}$ has a terminal object (the poset with one element), it follows that the set $\mathrm{H}[1]$ of H-structures of the poset with one element is the set of local terminal objects in $\mathbb{H}$ by Lemma 2.7.1.1. ☐

Example 2.8.6.2. Given $\mathrm{H}\colon \mathbb{D}_{\mathrm{p}} \to \mathbf{Set}$ a directed hereditary species, we have that $\mathbb{H}$ is a category with local terminal objects by Lemma 2.8.6.1. The lt-nerve of $\mathbb{H}$ is described similarly to the connected case (Example 2.7.1.4).

Lemma 2.8.6.3. *Let* $\mathrm{H}\colon \mathbb{D}_{\mathrm{p}} \to \mathbf{Set}$ *be a directed hereditary species. Then* $\mathrm{HDec}_{\top} \mathrm{S}\,\mathrm{N}^{\mathrm{lt}}\,\mathbb{H}$ *is a* $\mathbb{\Delta}^{\mathrm{t}}$*-Segal space.*

Proof. The proof is the same as Lemma 2.7.2.1. ☐

Let **DirHerSp** denote the category of directed hereditary species. For each directed hereditary species H, we have a pseudosimplicial groupoid $\mathbf{H} = \mathrm{S}\,\mathrm{N}^{\mathrm{lt}}\,\mathbb{H}$. This construction gives a functor $\int^{\mathbf{D}}\colon \mathbf{DirHerSp} \to \mathbf{sGrpd}^{\mathrm{tps}}$ defined by $\int^{\mathbf{D}}(\mathrm{H}) = \mathbf{H}$. Furthermore, $\mathrm{HDec}_{\top} \circ \int^{\mathbf{D}}(\mathrm{H})$ is a $\mathbb{\Delta}^{\mathrm{t}}$-Segal space by Lemma 2.8.6.3.

On the other hand, Lemma 2.8.6.1 established that $\mathbb{H}$ is a category with local terminal objects given by the set of H-structures of the poset with one element $\mathrm{H}[1]$. This gives a functor $\int^{*}\colon \mathbf{DirHerSp} \to \mathbf{Cat}_{\mathrm{lt}}$ that sends H to the category $\mathbb{H}$.

Lemma 2.8.6.4. *The diagram*

$$\begin{tikzcd}
\mathbf{DirHerSp} \arrow[r, "\int^{\mathbf{D}}"] \arrow[d, "\int^{*}"'] & \mathbf{sGrpd}^{\mathrm{tps}} \arrow[d, "\mathrm{HDec}_{\top}"] \\
\mathbf{Cat}_{\mathrm{lt}} \arrow[r, "\mathrm{S}\,\mathrm{N}^{\mathrm{lt}}"'] & \mathbf{tsGrpd}
\end{tikzcd}$$

strictly commutes.

Proof. The proof is analogous to that Proposition 2.7.3.1. ☐

Proposition 2.8.6.5. *There exists a canonical functor from the category of directed hereditary species* **DirHerSp** *to the category of operadic categories* **OpCat***.*

Proof. The proof is analogous to that Proposition 2.7.3.2, but using Lemma 2.8.6.4 instead of Lemma 2.7.3.1. ☐

3

The Gálvez–Kock–Tonks conjecture for rigid decomposition spaces

In this chapter, we work at the level of homotopy 1-types to prove the first case of the conjecture, namely for locally discrete decomposition spaces. This provides also the first substantial evidence for the general conjecture. This case is general enough to cover all locally finite posets, Cartier–Foata monoids, Möbius categories and strict (directed) restriction species. The proof is 2-categorical. First, we construct a local strict model of U (3.4.1), which is then used to show by hand that the Lawvere interval construction, considered as a natural transformation, does not admit other self-modifications than the identity (3.5.1). The material of this chapter is the main part of [47].

3.1 *Slices and intervals*

In this section, we introduce some constructions with slice and coslice of decomposition groupoids required to introduce the concept of interval.

Throughout we write $\delta A\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathbf{Grpd}$ for the constant simplicial space on a groupoid A. We have a natural transformation $\pi_{\mathrm{last}}\colon \mathrm{Dec}_\top X \to \delta(X_0)$ defined as follows: the map $\pi_{\mathrm{last}}\colon \mathrm{Dec}_\top X \to \delta(X_0)$ sends an n-simplex λ in $\mathrm{Dec}_\top X$ to $d_\perp^{n+1}(\lambda)$ in X_0 and an arrow $\alpha\colon \lambda \to \eta$ in $(\mathrm{Dec}_\top X)_n$ to $d_\perp^{n+1}(\alpha)$ in X_0. Since $[0]$ is terminal in $(\mathbb{\Delta}^t)^{\mathrm{op}}$, the map π_{last} is a simplicial map.

Lemma 3.1.0.1. *The natural transformation π_{last} is cartesian on right fibrations. That is, given a right fibration $p\colon X \to Y$, the square*

$$\begin{tikzcd}
\mathrm{Dec}_\top X \arrow[r, "\pi_{\mathrm{last}}"] \arrow[d, "\mathrm{Dec}_\top p"'] \arrow[dr, phantom, "(1)"] & \delta(X_0) \arrow[d, "\delta(p_0)"] \\
\mathrm{Dec}_\top Y \arrow[r, "\pi_{\mathrm{last}}"'] & \delta(Y_0)
\end{tikzcd}$$

is a homotopy pullback.

Proof. The pullback property can be checked level-wise. Note that $(\mathrm{Dec}_\top X)_n = X_{n+1}$. In level $n \geqslant 0$, the square (1) is

$$\begin{tikzcd}
X_{n+1} \arrow[r, "d_\perp^{n+1}"] \arrow[d, "p_{n+1}"'] & X_0 \arrow[d, "p_0"] \\
Y_{n+1} \arrow[r, "d_\perp^{n+1}"'] & Y_0,
\end{tikzcd}$$

which is a homotopy pullback since p is a right fibration. □

We also have a natural transformation $\pi_{\mathrm{first}}\colon \mathrm{Dec}_\perp X \to \delta(X_0)$ defined as follows: the simplicial map $\pi_{\mathrm{first}}\colon \mathrm{Dec}_\perp X \to \delta(X_0)$ sends an n-simplex λ in $\mathrm{Dec}_\perp X$ to $d_\top^{n+1}(\lambda)$ in X_0 and an arrow $\alpha\colon \lambda \to \eta$ in $(\mathrm{Dec}_\perp X)_n$ to $d_\top^{n}(\alpha)$ in X_0. The proof of the following result is analogous to that of Lemma 3.1.0.1.

Lemma 3.1.0.2. *The natural transformation π_{first} is cartesian on left fibrations.*

Lemma 3.1.0.3. *[41, Proposition 2.1] Let X be a decomposition groupoid. For all $0 \leqslant i \leqslant n$ the following squares are homotopy pullbacks:*

$$\begin{tikzcd} X_{n+1} \arrow[r, "s_{i+1}"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_{n+2} \arrow[d, "d_0"] & X_{n+1} \arrow[r, "s_i"] \arrow[d, "d_{n+1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_{n+2} \arrow[d, "d_{n+2}"] \\ X_n \arrow[r, "s_i"'] & X_{n+1}. & X_n \arrow[r, "s_i"'] & X_{n+1}. \end{tikzcd}$$

The pullbacks of Lemma 3.1.0.3 are called the upper and lower unital condition.

Definition 3.1.0.4. Let X be a decomposition groupoid. For an object y in X_0, the *slice* $X_{/y}$ is defined as the homotopy pullback

$$\begin{tikzcd} X_{/y} \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\ulcorner y \urcorner"] \\ \mathrm{Dec}_\top X \arrow[r, "\pi_{\mathrm{last}}"'] & \delta(X_0). \end{tikzcd}$$

Remark 3.1.0.5. Taking the upper decalage construction of X gives a simplicial object starting in X_1, but equipped with an augmentation $d_0 \colon X_1 \to X_0$. Pulling back this simplicial object along $\ulcorner y \urcorner \colon 1 \to X_0$, yields a new simplicial object which is $X_{/y}$. The map u is cartesian since $1 \to \delta(X_0)$ is cartesian and cartesian maps are stable under pullback. Therefore, u is a right fibration, and as a consequence, $X_{/y}$ is Segal by 1.1.2.8.

Definition 3.1.0.6. Let X be a Segal groupoid. An object $b \in X_0$ is called *terminal* if the projection map $X_{/b} \to X$ is a levelwise equivalence.

Proposition 3.1.0.7. *Let X be a decomposition groupoid. Then for an object y in X_0, the object $s_0(y)$ is terminal in $X_{/y}$.*

Proof. In the diagram

$$\begin{tikzcd} (X_{/y})_{/s_0(y)} \arrow[r] \arrow[d, "u'"'] \arrow[dr, phantom, "(1)"] & 1 \arrow[d, "\ulcorner s_0 y \urcorner"] \\ \mathrm{Dec}_\top X_{/y} \arrow[r, "\pi_{\mathrm{last}}"'] \arrow[d, "\mathrm{Dec}_\top u"'] \arrow[dr, phantom, "(2)"] & \delta((X_{/y})_0) \arrow[d, "\delta(u_0)"] \\ \mathrm{Dec}_\top \mathrm{Dec}_\top X \arrow[r, "\pi_{\mathrm{last}}"'] & \delta((\mathrm{Dec}_\top X)_0), \end{tikzcd}$$

the square (1) is a homotopy pullback by definition of $(X_{/y})_{/s_0(y)}$. Since $u \colon X_{/y} \to \mathrm{Dec}_\top X$ is a right fibration, we have that (2) is a homotopy pullback by Lemma 3.1.0.1. Therefore, the outer diagram is a homotopy pullback. Furthermore, note that $\delta(u_0)(s_0(y)) = s_0(y)$. This means that $(X_{/y})_{/s_0(y)}$ is the homotopy pullback of π_{last} along $\ulcorner s_0 y \urcorner \colon 1 \to \delta((\mathrm{Dec}_\top X)_0)$. Note that in the diagram

$$\begin{tikzcd} X_{/y} \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "(3)"] & 1 \arrow[d, "\ulcorner y \urcorner"] \\ \mathrm{Dec}_\top X \arrow[r, "\pi_{\mathrm{last}}"'] \arrow[d, "H"'] \arrow[dr, phantom, "(4)"] & \delta(X_0) \arrow[d, "\delta(s_0)"] \\ \mathrm{Dec}_\top \mathrm{Dec}_\top X \arrow[r, "\pi_{\mathrm{last}}"'] & \delta((\mathrm{Dec}_\top X)_0) \end{tikzcd}$$

the square (3) is a homotopy pullback by definition of $X_{/y}$. The map $H \colon \mathrm{Dec}_\top X \to \mathrm{Dec}_\top \mathrm{Dec}_\top X$ is defined by $H((\mathrm{Dec}_\top X)_n) = s_{n+1}(X_{n+1})$. The square (4) is a pullback as a consequence of Lemma 3.1.0.3 and the definition of H. Combining (3) and (4), the outer diagram is a homotopy pullback. Furthermore, note that $\delta(s_0)(y) = s_0(y)$. This means that $X_{/y}$ is the homotopy pullback of π_{last} along $\ulcorner s_0 y \urcorner \colon 1 \to \delta((\mathrm{Dec}_\top X)_0)$. Since $X_{/y}$ and $(X_{/y})_{/s_0(y)}$ are homotopy pullbacks over the same diagram, we get a canonical identification $(X_{/y})_{/s_0(y)} \cong X_{/y}$. Furthermore, this identification is given by the canonical projection map $\mathrm{pr}\colon (X_{/y})_{/s_0(y)} \to X_{/y}$ since $H \circ u \circ \mathrm{pr} = u' \circ \mathrm{Dec}_\top u$. ☐

Definition 3.1.0.8. Let X be a decomposition groupoid. For an object x in X_0, the *coslice* $X_{x/}$ is defined as the homotopy pullback

$$\begin{tikzcd}
X_{x/} \arrow[r] \arrow[d, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\ulcorner x \urcorner"] \\
\mathrm{Dec}_\perp X \arrow[r, "\pi_{\text{first}}"'] & \delta(X_0).
\end{tikzcd}$$

We write $v\colon X_{x/} \to \mathrm{Dec}_\perp X$ for the canonical map of Definition 3.1.0.8. Note that for each x in X_0, the coslice $X_{x/}$ is Segal. Indeed, $\mathrm{Dec}_\perp X$ is Segal and $X_{x/}$ is a left fibration over $\mathrm{Dec}_\perp X$, and is therefore Segal too by Lemma 1.1.2.8.

Definition 3.1.0.9. Let X be a Segal groupoid. An object $a \in X_0$ is called *initial* if the projection map $X_{a/} \to X$ is a levelwise equivalence.

Proposition 3.1.0.10. *Let X be a decomposition groupoid. For an object x in X_0, the object $s_0(x)$ is an initial object in $X_{x/}$.*

Proof. The proof is analogous to that of Proposition 3.1.0.7. ☐

Lemma 3.1.0.11. *Let $\mathcal{C}$ be a Segal groupoid with an initial object $\perp_{\mathcal{C}}$. Then for each object y in $\mathcal{C}$, the slice $\mathcal{C}_{/y}$ has an initial object.*

Proof. Since $\perp_{\mathcal{C}}$ is an initial object, we have a map $f_{\perp_{\mathcal{C}}}\colon \perp_{\mathcal{C}} \to y$. This map can be regarded as an object in $\mathcal{C}_{/y}$ or in $\mathcal{C}_{\perp/}$, and after two pullbacks of $\mathrm{Dec}_\top \mathrm{Dec}_\perp = \mathrm{Dec}_\perp \mathrm{Dec}_\top$ we get the natural identification $(\mathcal{C}_{/y})_{f_{\perp_{\mathcal{C}}/}} \cong (\mathcal{C}_{\perp_{\mathcal{C}}/})_{/f_{\perp_{\mathcal{C}}}}$. Furthermore, in the diagram

$$\begin{tikzcd}
(\mathcal{C}_{\perp_{\mathcal{C}}/})_{/f_{\perp_{\mathcal{C}}}} \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "(1)"] & 1 \arrow[d, "\ulcorner f_{\perp_{\mathcal{C}}} \urcorner"] \\
\mathrm{Dec}_\top \mathcal{C}_{\perp_{\mathcal{C}}/} \arrow[r, "\pi_{\text{last}}"'] \arrow[d, "\mathrm{Dec}_\top d_\perp"'] \arrow[dr, phantom, "(2)"] & \delta((\mathcal{C}_{\perp_{\mathcal{C}}/})_0) \arrow[d, "\delta(d_\perp)"] \\
\mathrm{Dec}_\top \mathcal{C} \arrow[r, "\pi_{\text{last}}"'] & \delta(\mathcal{C}_0)
\end{tikzcd}$$

the square (1) is a homotopy pullback by definition of $(\mathcal{C}_{\perp_{\mathcal{C}}/})_{/f_{\perp_{\mathcal{C}}}}$. Since $\perp_{\mathcal{C}}$ is an initial object, we have that $d_\perp \colon \mathcal{C}_{\perp/} \to \mathcal{C}$ is a levelwise equivalence. This implies that (2) is a homotopy pullback. Combining (1) and (2), we have that the outer diagram is a homotopy pullback. Furthermore, note that $d_\perp(f_{\perp_{\mathcal{C}}}) = y$. This means that $(\mathcal{C}_{\perp_{\mathcal{C}}/})_{/f_{\perp_{\mathcal{C}}}}$ is the homotopy pullback of π_{last} along $\ulcorner y \urcorner \colon 1 \to \delta(\mathcal{C}_0)$. But this is precisely the definition of $\mathcal{C}_{/y}$. This implies that $(\mathcal{C}_{\perp_{\mathcal{C}}/})_{/f_{\perp_{\mathcal{C}}}} \cong \mathcal{C}_{/y}$ and therefore $(\mathcal{C}_{/y})_{f_{\perp_{\mathcal{C}}/}} \cong \mathcal{C}_{/y}$. Furthermore, this isomorphism is given by the canonical projection map $\mathrm{pr}\colon (\mathcal{C}_{/y})_{f_{\perp_{\mathcal{C}}/}} \to \mathcal{C}_{/y}$ since $u'' \circ \mathrm{pr} = \mathrm{Dec}_\top d_\perp \circ u$, where $u''\colon \mathcal{C}_{/y} \to \mathrm{Dec}_\top \mathcal{C}$ denotes the canonical map of Definition 3.1.0.4. ☐

Lemma 3.1.0.12. *Let $\mathcal{C}$ be a Segal groupoid with a terminal object. Then for each object x in $\mathcal{C}$, the coslice $\mathcal{C}_{x/}$ has a terminal object.*

Proof. The proof is analogous to that of Lemma 3.1.0.11. □

Let X be a decomposition groupoid. For $\lambda\colon \Delta^n \to X$, we denote by $\operatorname{long}(\lambda)$ the 1-simplex $\Delta^1 \twoheadrightarrow \Delta^n \xrightarrow{\lambda} X$. Applying lower and upper decalage to X, we obtain a new decomposition groupoid $\operatorname{Dec}_\top \operatorname{Dec}_\bot X$ and a map $\epsilon\colon \operatorname{Dec}_\top \operatorname{Dec}_\bot X \to X$ which is culf by Proposition 1.1.3.1. Furthermore, we have a natural transformation $\pi_{\mathrm{long}}\colon \operatorname{Dec}_\top \operatorname{Dec}_\bot X \to \delta(X_1)$ defined as follows: the map $\pi_{\mathrm{long}}\colon \operatorname{Dec}_\top \operatorname{Dec}_\bot X \to \delta(X_1)$ sends an n-simplex λ in $\operatorname{Dec}_\top \operatorname{Dec}_\bot X$ to $\operatorname{long}(\lambda)$ in X_1 and an arrow $\alpha\colon \lambda \to \eta$ in $(\operatorname{Dec}_\top \operatorname{Dec}_\bot X)_n$ to $\operatorname{long}(\alpha)$ in X_1. Recall that the category $\mathbb{\Delta}^{\mathrm{act}}$ is the subcategory of $\mathbb{\Delta}$ whose objects are the nonempty finite ordinals and whose morphisms are the active maps. Since $[1]$ is terminal in $(\mathbb{\Delta}^{\mathrm{act}})^{\mathrm{op}}$, the map π_{long} is a simplicial map.

Lemma 3.1.0.13. *The natural transformation* π_{long} *is cartesian on culf maps. That is, given a culf map* $F\colon X \to Y$ *between decomposition groupoids, the square*

$$\begin{tikzcd}
\operatorname{Dec}_\top \operatorname{Dec}_\bot X \arrow[r, "\pi_{\mathrm{long}}"] \arrow[d, "\operatorname{Dec}_\top \operatorname{Dec}_\bot F"'] & \delta(X_1) \arrow[d, "\delta(F_1)"] \\
\operatorname{Dec}_\top \operatorname{Dec}_\bot Y \arrow[r, "\pi_{\mathrm{long}}"'] & \delta(Y_1)
\end{tikzcd}$$

is a homotopy pullback.

Proof. The pullback property can be checked levelwise. Note that $(\operatorname{Dec}_\top \operatorname{Dec}_\bot X)_n = X_{n+2}$. For $n \geqslant 0$, the square

$$\begin{tikzcd}
X_{n+2} \arrow[r, "d_1^{n+1}"] \arrow[d, "F_{n+2}"'] & X_1 \arrow[d, "F_1"] \\
Y_{n+2} \arrow[r, "d_1^{n+1}"'] & Y_1
\end{tikzcd}$$

is a homotopy pullback since F is culf. □

Definition 3.1.0.14. Let X be a decomposition groupoid and let f be an object in X_1. The Segal groupoid I_f is defined as the homotopy pullback, called the *interval* of f,

$$\begin{tikzcd}
I_f \arrow[r] \arrow[d, "w"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\ulcorner f \urcorner"] \\
\operatorname{Dec}_\top \operatorname{Dec}_\bot X \arrow[r, "\pi_{\mathrm{long}}"'] & \delta(X_1).
\end{tikzcd}$$

We write $w\colon I_f \to \operatorname{Dec}_\top \operatorname{Dec}_\bot X$ for the simplicial map obtained in this way. From its construction as a pullback of a map between constant simplicial groupoids, it is clear that w is culf. The double decalage construction induces a culf map $M_f\colon I_f \to X$, defined by the composition of w and the canonical map $\epsilon\colon \operatorname{Dec}_\top \operatorname{Dec}_\bot X \to X$.

Remark 3.1.0.15. When X is the ordinary nerve of a category, the description of I_f is due to Lawvere [77]: the objects of I_f are two-step factorisations of f. The 1-cells are arrows between such factorisations, or equivalently 3-step factorisations, and so on. More generally, let X be a decomposition set and $f \in X_1$. The Segal set I_f is described as follows:

1. An object of I_f is any $\sigma \in X_2$ such that $d_1(\sigma) = f$.
2. Given two objects σ and σ' in I_f, a morphism $\gamma\colon \sigma \to \sigma'$ in I_f is any object $\gamma \in X_3$, such that $d_2(\gamma) = \sigma$ and $d_1(\gamma) = \sigma'$.

3. Given two morphisms $\gamma\colon \sigma \to \sigma'$ and $\gamma'\colon \sigma' \to \sigma''$ of I_f, the composition is defined by $\gamma' \circ \gamma \colon = d_2(\eta)$, where $\eta \in X_4$ satisfies that $d_1(\eta) = \gamma'$ and $d_3(\eta) = \gamma$. The unique existence of η is a consequence of the decomposition-groupoid axioms in the form of Lemma 1.1.0.7. Associativity also follows by Lemma 1.1.0.7.

Applying lower and upper decalage to X generate two sections on $\mathrm{Dec}_\top \mathrm{Dec}_\perp X$. The first one is induced by $s_\perp\colon X \to \mathrm{Dec}_\perp X$ and the other is induced by $s_\top\colon X \to \mathrm{Dec}_\top X$. We shall see later that $s_\perp(f)$ is an initial object and $s_\top(f)$ is a terminal object in I_f. Recall that we write $u\colon X_{/y} \to \mathrm{Dec}_\top X$ for the canonical map of Definition 3.1.0.4 and $v\colon X_{x/} \to \mathrm{Dec}_\perp X$ for the canonical map of Definition 3.1.0.8. When further (co)slicing is used we decorate the u or v with a prime.

Lemma 3.1.0.16. *Let X be a decomposition groupoid. For $f \in X_1$, put $x = d_1(f)$ and $d_0(f) = y$. There are canonical equivalences $(X_{x/})_{/f} \to I_f$ and $(X_{/y})_{f/} \to I_f$ such that the following diagram commutes up to isomorphism*

$$\begin{tikzcd}
(X_{x/})_{/f} \arrow[r, "\simeq"] \arrow[d, "d_\top \circ u"'] \arrow[dr, phantom, "(1)"] & I_f \arrow[d, "M_f"] \arrow[dr, phantom, "(2)"] & (X_{/y})_{f/} \arrow[l, "\simeq"'] \arrow[d, "d_\perp \circ v'"] \\
X_{x/} \arrow[r, "d_\perp \circ v"'] & X & X_{/y}. \arrow[l, "d_\top \circ u'"]
\end{tikzcd}$$

Proof. In the diagram

$$\begin{tikzcd}
(X_{x/})_{/f} \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "(3)"] & 1 \arrow[d, "\ulcorner f \urcorner"] \\
\mathrm{Dec}_\top X_{x/} \arrow[r, "\pi_{\mathrm{last}}"'] \arrow[d, "\mathrm{Dec}_\top v"'] \arrow[dr, phantom, "(4)"] & \delta((X_{x/})_0) \arrow[d, "\delta(v_0)"] \\
\mathrm{Dec}_\top \mathrm{Dec}_\perp X \arrow[r, "\pi_{\mathrm{long}}"'] & \delta(X_1)
\end{tikzcd}$$

the square (3) is a homotopy pullback by construction of $(X_{x/})_{/f}$. Since v is a right fibration the square (4) is a homotopy pullback by Lemma 3.1.0.2. Therefore, the outer diagram is a homotopy pullback. Note that $\delta(v_0)(f) = y$. This implies that $(X_{x/})_{/f}$ is the homotopy pullback of π_{long} along $\ulcorner f \urcorner\colon 1 \to \delta(X_1)$. But this is precisely the definition of I_f. This gives us an equivalence $G\colon (X_{x/})_{/f} \to I_f$ such that $w \circ G \simeq \mathrm{Dec}_\top v \circ u$, which is the upper square in the diagram

$$\begin{tikzcd}
(X_{x/})_{/f} \arrow[rr, "G"] \arrow[d, "u"'] & & I_f \arrow[dl, "w"'] \arrow[dd, "M_f"] \\
\mathrm{Dec}_\top X_{x/} \arrow[r, "\mathrm{Dec}_\top v"] \arrow[d, "d_\top"'] & \mathrm{Dec}_\top \mathrm{Dec}_\perp X \arrow[d, "d_\top"'] \arrow[dr, "\epsilon"] & \\
X_{x/} \arrow[r, "v"'] & \mathrm{Dec}_\perp X \arrow[r, "d_\perp"'] & X
\end{tikzcd}$$

Since the other regions in the diagram commute strictly (by functoriality of upper decalage and by definition of ϵ and M_f), we get a natural isomorphism for the outer square, which is precisely (1). By analogous arguments, (2) commutes up to isomorphism. □

When X is the ordinary nerve of a category, Lemma 3.1.0.16 is the same as Lemma 3.2 in [77].

Lemma 3.1.0.17. *Let X be a Segal groupoid with an initial object $\perp$ and a terminal object $\top$. Let $h\colon \perp \to \top$ be a map from $\perp$ to $\top$, then $X \simeq I_h$.*

Proof. Applying Lemma 3.1.0.16 to h, we have that $(X_{\perp/})_{/h} \simeq I_h$. Applying Lemma 3.1.0.11 to $\top$, it follows that $X_{/\top} \simeq (X_{\perp/})_{/h}$. Furthermore, $X_{/\top} \simeq X$ since $\top$ is a terminal object. Combining these equivalences, we get that $X \simeq I_h$. □

When X is the ordinary nerve of a category, Lemma 3.1.0.17 is the same as Lemma 3.3 in [77].

Proposition 3.1.0.18. *Let X be a complete decomposition groupoid. Then for each $f \in X_1$, the Segal groupoid I_f is complete in the sense of decomposition groupoids, meaning that $s_0 \colon (I_f)_0 \to (I_f)_1$ is a monomorphism.*

Proof. By construction of I_f, we have the following diagram

$$\begin{tikzcd}
(I_f)_0 \arrow[r, "s_0"] \arrow[d, "w_0"'] & (I_f)_1 \arrow[r, "d_0"] \arrow[d, "w_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & (I_f)_0 \arrow[d, "w_0"] \\
X_1 \arrow[r, "s_1"'] & X_2 \arrow[r, "d_1"'] & X_1 .
\end{tikzcd}$$

Since X is complete, the map $s_1 : X_1 \to X_2$ is a monomorphism, and therefore also its pullback $s_0 \colon (I_f)_0 \to (I_f)_1$ is a monomorphism, which is to say that I_f is complete. □

3.2 *Discrete intervals and rigid decomposition groupoids*

To study the first case of the Gálvez–Kock–Tonks conjecture, the obvious level of generality would be discrete decomposition groupoids, but the proofs to be presented in this section work for locally discrete decomposition groupoids of the kind featured in the following definition:

Definition 3.2.0.1. A *rigid decomposition groupoid* is a strict simplicial groupoid X such that $d_1 : X_2 \to X_1$ is a discrete fibration, $s_0 \colon X_0 \to X_1$ is a monomorphism and the active-inert squares are strict pullbacks.

The point, as we shall see, is for a rigid decomposition groupoid X, we have that for all $f \in X_1$, the Segal groupoid I_f (3.1.0.14) is discrete. Note that every discrete decomposition groupoid is a rigid decomposition groupoid. This means that the rigid decomposition groupoids already cover locally finite posets, Cartier–Foata monoids and Möbius categories. The importance of locally discrete is to cover also strict (directed) restriction species as shown in the following example:

Example 3.2.0.2. A directed restriction species $R \colon \mathbb{C}^{op} \to \mathbf{Grpd}$ induces a decomposition groupoid $\mathbf{R}$ [57], where $\mathbf{R}_n$ is the groupoid of R structures with an n-layering of the underlying poset P, that is a monotone map $P \to \underline{n}$, the linear order with n elements. The map $d_1 : \mathbf{R}_2 \to \mathbf{R}_1$ forgets the layering and is clearly a discrete fibration. Altogether, the decomposition groupoid $\mathbf{R}$ is rigid.

Example 3.2.0.3. Recall that for the decomposition groupoid of rooted trees $\mathbf{RT}$ of Example 1.1.0.5, $\mathbf{RT}_2$ is the groupoid of forests with an admissible cut, $\mathbf{RT}_1$ is the groupoid of forests and the map $d_1 : \mathbf{RT}_2 \to \mathbf{RT}_1$ forgets the admissible cut. It is straightforward to see that d_1 is a discrete fibration. Therefore, $\mathbf{RT}$ is a rigid decomposition groupoid.

Dür [34] gave an incidence-coalgebra construction of the Butcher–Connes–Kreimer coalgebra by starting with the category of forests and root-preserving inclusions, generating a coalgebra and imposing the equivalence relation that identifies two root-preserving forest inclusions if their complement crowns are isomorphic forests.

Consider the rigid decomposition groupoid of rooted trees $\mathbf{RT}$. We can consider a tree T as an object in $\mathbf{RT}_1$. The interval I_T can be described as follows: $(I_T)_0$ is the set of all isoclasses of admissible cuts of T, and $(I_T)_k$ is the set of isoclasses of all $k+1$ compatible admissible cuts of T.

We can relate the construction of Dür with the interval construction of a rooted tree as follows: note that admissible cuts are essentially the same thing as root-preserving forest inclusions: then the cut is interpreted as the division between the included forest and the forest induced on the nodes in its complement. In this way we see that $(I_T)_k$ is the discrete groupoid of $k+1$ consecutive root-preserving inclusions ending in T.

Remark 3.2.0.4. In a decomposition groupoid X, every active face map is a pullback of $d_1 \colon X_2 \to X_1$ [59, Lemma 1.10]. Therefore, in the case where X is rigid we have that π_{long} is a levelwise discrete fibration since in each level it is the long-edge map, which is a composition of active face maps, and these are all discrete fibrations. Therefore, the strict pullback of π_{long} along $\ulcorner f \urcorner \colon 1 \to \delta(X_1)$ is also a homotopy pullback. Furthermore, for every $f \in X_1$, the Segal groupoid I_f is discrete.

Definition 3.2.0.5. Let X be a discrete Segal groupoid. A *chosen terminal* object is the choice of a terminal object b. A *chosen initial* is the choice of an initial object a.

Example 3.2.0.6. Let X be a discrete decomposition groupoid. Let y be an object in X_0. We already know from Proposition 3.1.0.7 that $s_0(y)$ is a terminal object in $X_{/y}$, which we take as the chosen one. In this way $X_{/y}$ acquires a canonical chosen terminal. Similarly, $s_0(y)$ is an initial object in $X_{y/}$ (Proposition 3.1.0.10), which we take as the chosen one. In this way $X_{x/}$ acquires a canonical chosen initial.

Definition 3.2.0.7. A *discrete interval* is a discrete Segal groupoid $\mathcal{C}$ with a chosen initial object $\perp$ and a chosen terminal object $\top$. We denote the map from the chosen initial to the chosen terminal by $\varpi \colon \perp \to \top$.

Remark 3.2.0.8. In the case of the nondiscrete intervals, further structure is required in the notion of chosen terminal, namely the choice of a section $s \colon X \to X_{/b}$ for the canonical map $d_\top u \colon X_{/b} \to X$. This will not be needed in the present paper, but the interested reader can find the theory of these worked out in Version 1 of this paper on arXiv.

Remark 3.2.0.9. Batanin and Markl [13] used the notion of a category with chosen local terminal objects, meaning a category which in each connected component is provided with a chosen terminal object. This notion plays an important role in their theory of operadic categories. Garner, Kock and Weber [54] observed that the structure of chosen local terminal objects is precisely to be a coalgebra for the upper-Dec comonad. This in turn amounts to having an extra top degeneracy map for the nerve of the category. When we insist on having a chosen terminal object, it is inspired by this decalage viewpoint on chosen terminals. Similarly of course, the notion of chosen local initial object amounts to coalgebra structure for the lower-Dec comonad, via extra bottom degeneracy maps, as the chosen initial object in our definition. Finally, the main point here is the combination of the two ideas. A discrete interval structure is in particular a coalgebra for the two-sided-Dec comonad. This is very much in line with the notion of flanking of Gálvez–Kock–Tonks [60, §1].

Definition 3.1.0.14 can be rewritten in terms of rigid decomposition groupoid as follows:

Definition 3.2.0.10. Let X be a rigid decomposition groupoid and let f be an object in X_1. The Segal groupoid I_f is defined as the strict pullback

$$\begin{tikzcd}
I_f \arrow[r] \arrow[d, "w"'] \arrow[dr, phantom, very near start, "\lrcorner"] & 1 \arrow[d, "\ulcorner f \urcorner"] \\
\mathrm{Dec}_\top \mathrm{Dec}_\perp X \arrow[r, "\pi_{\mathrm{long}}"'] & \delta(X_1).
\end{tikzcd}$$

In fact this strict pullback is also a homotopy pullback since π_{long} is a discrete fibration as a consequence of the fact that d_1 is a discrete fibration.

Lemma 3.2.0.11. *Let* X *be a rigid decomposition groupoid and* $f \in X_1$*. The Segal groupoid* I_f *has a canonical structure of an interval, where the chosen initial object is* $s_0(f)$ *and the chosen terminal object is* $s_1(f)$*.*

Proof. The object $s_1(f)$ is a terminal object in I_f as a consequence of Lemma 3.1.0.7, which we take as the chosen one. On the other hand, the object $s_0(f)$ is a initial object in I_f as a consequence of Lemma 3.1.0.10, which we take as the chosen initial. □

A simplicial map $F\colon X \to Y$ between rigid decomposition groupoids is called *strict culf* if the naturality square for F with respect any active map $[n] \rightarrowtail [k]$ in $\mathbb{\Delta}$ is a strict pullback. Since the active maps are fibrations in rigid decomposition groupoids, it follows that the strict pullbacks of a strict culf map are also homotopy pullbacks, so that strict culf map is in fact culf in the usual homotopy invariant sense. Lemma 3.1.0.13 can be rewritten in terms of the strict condition as follows:

Lemma 3.2.0.12. *The natural transformation* π_{long} *is cartesian on strict culf maps. That is, given a strict culf map* $F\colon X \to Y$ *between rigid decomposition groupoids, the square*

$$\begin{tikzcd}
\mathrm{Dec}_\top \mathrm{Dec}_\bot X \arrow[r, "\pi_{\text{long}}"] \arrow[d, "\mathrm{Dec}_\top \mathrm{Dec}_\bot F"'] & \delta(X_1) \arrow[d, "\delta(F_1)"] \\
\mathrm{Dec}_\top \mathrm{Dec}_\bot Y \arrow[r, "\pi_{\text{long}}"'] & \delta(Y_1)
\end{tikzcd}$$

is a strict pullback.

Proof. The proof is analogous to that of Lemma 3.1.0.13. Furthermore, the strict pullback is also a homotopy pullback since π_{long} is a discrete fibration. □

The culf maps preserve the algebraic structure of a decomposition groupoid, but do not necessarily preserve the chosen initial and chosen terminal objects for a discrete interval. The maps that preserve this structure is the content of the following definition:

Definition 3.2.0.13. A simplicial map between discrete intervals is termed *stretched* , and written $\mathcal{C} \to\!\!\!\shortmid \mathcal{D}$, if it preserves the chosen initial object $\bot_{\mathcal{C}}$ and the chosen terminal object $\top_{\mathcal{C}}$.

Lemma 3.2.0.14. *Let* X *be a rigid decomposition groupoid and let* f *be a* 1*-simplex in* X*. The unique stretched map* $\varpi\colon \Delta^1 \to\!\!\!\shortmid I_f$ *is compatible with* M_f*, meaning that we have a commutative triangle*

$$\begin{tikzcd}
\Delta^1 \arrow[r, "\varpi", -{>|}] \arrow[dr, "f"'] & I_f \arrow[d, "M_f"] \\
& X.
\end{tikzcd}$$

Proof. Put $x := d_\top(f)$ and $y := d_\bot(f)$ (the domain and codomain of f). Recall that the objects of I_f are 2-simplices with long edge f. The arrows in I_f are 3-simplices with long edge f. We know that the (chosen) initial object is $s_\bot(f)$ (which can be thought of as the triangle with short sides id_x and f) and the (chosen) terminal object is $s_\top(f)$ (which can be thought of as the triangle with short sides f and id_y). The unique arrow ϖ from the initial to the terminal is the tetrahedron $s_\bot s_\top(f)$ (which we can think of as the tetrahedron with short sides id_x, f, and id_y). By definition $M_f = d_\top d_\bot w$. Since I_f is a discrete interval the map $w\colon I_f \to \mathrm{Dec}_\top \mathrm{Dec}_\bot X$ is level-wise injective on objects. So what M_f does is that it applies $d_\top d_\bot$. In conclusion we have $M_f(\varpi) = d_\top d_\bot s_\top s_\bot(f) = f$, which is what we wanted to prove. □

Example 3.2.0.15. In the situation of Lemma 3.2.0.14, if X is already an interval and f is its long edge, we see from the argument in the proof that M_f is stretched in this case we will see in 3.3.0.2 that M_f is actually invertible in this case.

Remark 3.2.0.16. Δ^n is an interval for each n. The stretched maps $\Delta^m \mathrel{\rightarrow\!\!\!\mid} \Delta^n$ are precisely the active maps. Every interval A receives a unique stretched map from Δ^1. A simplicial map between intervals $A \to B$ is stretched if and only if it commutes with the stretched maps from Δ^1.

Let $\mathcal{C}$ be a discrete interval. Let $\mathcal{C}_n$ denote the set of n-simplices in $\mathcal{C}$ and let $\mathcal{C}_n^{\mathrm{str}}$ be the subset of stretched $(n+2)$-simplices in $\mathcal{C}$.

Lemma 3.2.0.17. *The map* $d_\top d_\perp \colon \mathcal{C}_{n+2}^{\mathrm{str}} \to \mathcal{C}_n$ *is a bijection.*

Proof. We will construct an inverse $t\colon \mathcal{C}_n \to \mathcal{C}_{n+2}^{\mathrm{str}}$ of $d_\top d_\perp$ as follows. Let λ be an object in $\mathcal{C}_n$, put $a = d_\top(\mathrm{long}(\lambda))$ and $b = d_\perp(\mathrm{long}(\lambda))$. Since $\mathcal{C}$ is a discrete interval, we have a chosen edge $f_\perp \colon \Delta^1 \to \mathcal{C}$ such that $d_\top(f_\perp) = \perp_{\mathcal{C}}$ and $d_\perp(f_\perp) = a$. By the same argument, we have a chosen edge $f_\top \colon \Delta^1 \to \mathcal{C}$ such that $d_\perp(f_\top) = \top_{\mathcal{C}}$ and $d_\top(f_\top) = b$. In the diagram

$$\begin{tikzcd}
1 \arrow[drr, bend left, "\lambda"] \arrow[dr, dashed, "\mu"] \arrow[dddr, bend right, "f_\perp"'] & {} \arrow[dr, phantom, "(1)"] & \\
& \mathcal{C}_{n+1} \arrow[r, "d_\perp"] \arrow[d, dotted, "d_2^{n-1}"'] \arrow[dr, phantom, "(2)"] & \mathcal{C}_n \arrow[d, "d_1^{n-1}"] \\
{} \arrow[r, phantom, "(4)"] & \mathcal{C}_2 \arrow[r, "d_\perp"] \arrow[d, "d_\top"'] \arrow[dr, phantom, "(3)"] & \mathcal{C}_1 \arrow[d, "d_\top"] \\
& \mathcal{C}_1 \arrow[r, "d_\perp"'] & \mathcal{C}_0
\end{tikzcd}$$

the squares (2) and (3) are strict pullbacks since $\mathcal{C}$ is a discrete Segal groupoid. Therefore, the outer rectangle is a strict pullback. By the pullback property of $\mathcal{C}_{n+1}$, there exists a unique map $\mu\colon 1 \to \mathcal{C}_{n+1}$ such that the diagram commutes. Since the square (2) commutes and $d_\perp \mu = \lambda$, we have that $d_\perp \mathrm{long}(\mu) = b$. Furthermore, in the diagram

$$\begin{tikzcd}
1 \arrow[drr, bend left, "\mu"] \arrow[dr, dashed, "\eta"] \arrow[dddr, bend right, "f_\top"'] & {} \arrow[dr, phantom, "(5)"] & \\
& \mathcal{C}_{n+2} \arrow[r, "d_\top"] \arrow[d, "d_1^{n}"'] \arrow[dr, phantom, "(6)"] & \mathcal{C}_{n+1} \arrow[d, "d_1^{n}"] \\
{} \arrow[r, phantom, "(8)"] & \mathcal{C}_2 \arrow[r, "d_\top"] \arrow[d, "d_\perp"'] \arrow[dr, phantom, "(7)"] & \mathcal{C}_1 \arrow[d, "d_\perp"] \\
& \mathcal{C}_1 \arrow[r, "d_\top"'] & \mathcal{C}_0
\end{tikzcd}$$

the squares (6) and (7) are strict pullbacks since $\mathcal{C}$ is a discrete Segal groupoid. Therefore, the outer rectangle is a strict pullback. By the pullback property of $\mathcal{C}_{n+2}$, there exists a unique map $\ulcorner\eta\urcorner\colon 1 \to \mathcal{C}_{n+2}$ such that the diagram commutes. Note that $d_\top(\mathrm{long}(\eta)) = \perp_{\mathcal{C}}$ since (4) commutes and $d_\top(f_\perp) = \perp_{\mathcal{C}}$. Since (8) commutes and $d_\perp(f_\top) = \top_{\mathcal{C}}$, we have that $d_\perp(\mathrm{long}(\eta)) = \top_{\mathcal{C}}$. This together with $d_\top(\mathrm{long}(\eta)) = \perp_{\mathcal{C}}$ implies that $\mathrm{long}(\eta) = \varpi_{\mathcal{C}}$ since $\mathcal{C}$ is a discrete interval. We define

$$t(\lambda) := \eta.$$

Combining (1) and (5), we have that $d_\perp d_\top \eta = \lambda$. This means that $d_\top d_\perp(t(\lambda)) = \lambda$. Now we will check that $t \circ d_\top d_\perp = \mathrm{id}_{\mathcal{C}_{n+2}^{\mathrm{str}}}$. Let ψ be an object in $\mathcal{C}_{n+2}^{\mathrm{str}}$. Since $\mathcal{C}$ is a discrete interval, we have a chosen edge $f'_\perp \colon \Delta^1 \to \mathcal{C}$ such that $d_\top(f'_\perp) = \perp_{\mathcal{C}}$ and $d_\perp(f'_\perp) = d_\perp(\mathrm{long}(d_\top d_\perp \psi))$.

By the same argument, we have a chosen edge $f'_\top \colon \Delta^1 \to \mathcal{C}$ such that $d_\perp(f'_\top) = \top_\mathcal{C}$ and $d_\top(f'_\top) = d_\perp(\mathrm{long}(d_\top d_\perp \psi))$. The commutative diagrams

$$\begin{tikzcd}
1 \arrow[rrd, bend left, "d_\top d_\perp \psi"] \arrow[rd, dashed, "\mu'"'] \arrow[rdd, bend right, "f'_\perp"'] & & \\
& \mathcal{C}_{n+1} \arrow[r, "d_\perp"] \arrow[d, dashed, "d_\top d_2^{n-1}"'] \arrow[rd, phantom, very near start, "\lrcorner"] & \mathcal{C}_n \arrow[d, "d_\top d_1^{n-1}"] \\
& \mathcal{C}_1 \arrow[r, "d_\perp"'] & \mathcal{C}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[rrd, bend left, "\mu'"] \arrow[rd, dashed, "\eta'"'] \arrow[rdd, bend right, "f'_\top"'] & & \\
& \mathcal{C}_{n+2} \arrow[r, "d_\top"] \arrow[d, dashed, "d_\top d_1^n"'] \arrow[rd, phantom, very near start, "\lrcorner"] & \mathcal{C}_{n+1} \arrow[d, "d_\perp d_1^n"] \\
& \mathcal{C}_1 \arrow[r, "d_\top"'] & \mathcal{C}_0
\end{tikzcd}$$

are given by the construction of t. Furthermore, $t(d_\top d_\perp(\psi)) = \eta'$. If we substitute $d_\top \psi$ by μ', we have that the left diagram commutes. By the uniqueness of μ', it follows that $\mu' = d_\top \psi$. This together with the stretched condition of ψ implies that the right diagram commutes if we substitute ψ by η'. Therefore, by the uniqueness of η', we have that $\eta' = \psi$. This means that $t(d_\top d_\perp(\psi)) = \psi$. □

Suppose X is a rigid decomposition groupoid. For an object $f \colon x \to y$, we have a canonical projection $\pi_m \colon (X_{x/})_{/f} \to X$ defined as the composite

$$(X_{x/})_{/f} \xrightarrow{u} \mathrm{Dec}_\top X_{x/} \xrightarrow{\mathrm{Dec}_\top v} \mathrm{Dec}_\top \mathrm{Dec}_\perp X \xrightarrow{\epsilon} X.$$

Lemma 3.2.0.18. *Let $\mathcal{C}$ be a discrete interval. The canonical projection $\pi_m \colon (\mathcal{C}_{\perp_\mathcal{C}/})_{/\varpi_\mathcal{C}} \to \mathcal{C}$ has an inverse $L \colon \mathcal{C} \to (\mathcal{C}_{\perp_\mathcal{C}/})_{/\varpi_\mathcal{C}}$.*

Proof. Since $\mathcal{C}$ has a chosen initial object, the projection $d_\perp \colon \mathcal{C}_{\perp_\mathcal{C}/} \to \mathcal{C}$ is an equivalence, and therefore an isomorphism since $\mathcal{C}$ and $\mathcal{C}_{\perp_\mathcal{C}/}$ are discrete. The map $p_{\perp_\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}_{\perp_\mathcal{C}/}$ denotes the inverse of $d_\perp$. The object $\varpi_\mathcal{C}$ is terminal in $\mathcal{C}_{\perp_\mathcal{C}/}$ since it is chosen terminal in $\mathcal{C}$. This implies that the projection $d_\top \colon (\mathcal{C}_{\perp_\mathcal{C}/})_{/\varpi_\mathcal{C}} \to \mathcal{C}_{\perp_\mathcal{C}/}$ has an inverse $p_{\varpi_\mathcal{C}} \colon \mathcal{C}_{\perp_\mathcal{C}/} \to (\mathcal{C}_{\perp_\mathcal{C}/})_{/\varpi_\mathcal{C}}$ since $d_\top$ is an equivalence between discrete intervals. So we define L as the composite

$$\mathcal{C} \xrightarrow{p_{\perp_\mathcal{C}}} \mathcal{C}_{\perp_\mathcal{C}/} \xrightarrow{p_{\varpi_\mathcal{C}}} (\mathcal{C}_{\perp_\mathcal{C}/})_{/\varpi_\mathcal{C}}.$$

Since $\mathcal{C}$ is a discrete interval, u and v are level-wise injective on objects. This implies that π_m is equal to $d_\top \circ d_\perp$. Since $p_{\varpi_\mathcal{C}}$ and $p_{\perp_\mathcal{C}}$ are inverse of $d_\top$ and $d_\perp$, it follows that $L \circ \pi_m = \mathrm{id}_{(\mathcal{C}_{\perp_\mathcal{C}/})_{/\varpi_\mathcal{C}}}$ and $\pi_m \circ L = \mathrm{id}_\mathcal{C}$. □

Recall that the class of culf maps can also be characterised as the class right orthogonal to the active maps (between representables). That is, $X \to Y$ is culf if and only if for every active map $p \colon [m] \mathrel{\rightarrow\mkern-9mu\shortmid} [n]$ and every commutative square

$$\begin{tikzcd}
\Delta^m \arrow[r] \arrow[d, "p"'] & X \arrow[d] \\
\Delta^n \arrow[r] \arrow[ru, dotted, "\exists!"] & Y
\end{tikzcd}$$

there is a unique filler. Usually this is about homotopy commutative squares and a contractible space of lifts, but for strict culf, it is actually about strictly commutative squares and truly unique lifts.

Proposition 3.2.0.19. *For any n-simplex $\lambda \colon \Delta^n \to X$ with long edge f, there is a unique lift ϕ_λ for the square*

$$\begin{tikzcd}
\Delta^1 \arrow[r, "" , ->|] \arrow[d] & I_f \arrow[d, "M_f"] \\
\Delta^n \arrow[r, "\lambda"'] \arrow[ru, dotted, "\phi_\lambda"] & X.
\end{tikzcd}$$

Proof. The square commutes by Lemma 3.2.0.14. Indeed the composite $\Delta^1 \to \Delta^n \to X$ is equal to f since f was defined as the long edge, and $\Delta^1 \to I_f \to X$ is equal to f by Lemma 3.2.0.14. Furthermore, since $\Delta^1 \to \Delta^n$ is active and M_f is strict culf, we have a unique filler which we denote as ϕ_λ. ☐

Lemma 3.2.0.20. *Let $G\colon X \to Y$ be a simplicial map between rigid decomposition groupoids. For $f \in X_1$, there is a unique stretched map G_f fitting into the commutative diagram*

$$\begin{tikzcd}
I_f \arrow[r, "G_f", "\shortmid"{marking, pos=0.95}] \arrow[d, "M_f"'] \arrow[dr, phantom, "(1)"] & I_{Gf} \arrow[d, "M_{Gf}"] \\
X \arrow[r, "G"'] & Y.
\end{tikzcd}$$

If G is strict culf then G_f is an isomorphism.

Proof. In the diagram

$$\begin{tikzcd}
I_f \arrow[r] \arrow[d, "w"'] \arrow[dr, phantom, "(2)"] & 1 \arrow[d, "\ulcorner f \urcorner"] \\
\mathrm{Dec}_\top \mathrm{Dec}_\perp X \arrow[r, "\pi_{\mathrm{long}}"'] \arrow[d, "\mathrm{Dec}_\top \mathrm{Dec}_\perp G"'] \arrow[dr, phantom, "(3)"] & \delta(X_1) \arrow[d, "\delta(G_1)"] \\
\mathrm{Dec}_\top \mathrm{Dec}_\perp Y \arrow[r, "\pi_{\mathrm{long}}"'] & \delta(Y_1),
\end{tikzcd}$$

the square (2) is a strict pullback by construction of I_f, and (3) commutes since π_{long} is a natural transformation. Combining (2) and (3), we have that the outer diagram commutes. By the pullback property of I_{Gf}, we have a unique map $G_f\colon I_f \to I_{Gf}$ fitting into a commutative diagram

$$\begin{tikzcd}
I_f \arrow[d, "w"'] \arrow[drr, "G_f"] \arrow[drrr, bend left] & & & \\
\mathrm{Dec}_\top \mathrm{Dec}_\perp X \arrow[d, "\epsilon"'] \arrow[drr, "\mathrm{Dec}_\top \mathrm{Dec}_\perp G" description] & (4) & I_{Gf} \arrow[r] \arrow[d, "w'"'] & 1 \arrow[d, "\ulcorner Gf \urcorner"] \\
X \arrow[drr, "G"'] & (5) & \mathrm{Dec}_\top \mathrm{Dec}_\perp Y \arrow[r, "\pi_{\mathrm{long}}"'] \arrow[d, "\epsilon'"] & \delta(Y_1). \\
 & & Y &
\end{tikzcd}$$

Combining (4) and (5), we get that (1) commutes. This implies that $M_{Gf}\, G_f(\varpi_f) = Gf$. Since X and Y are rigid, the maps w and w' are level-wise injective on objects. The functor G_f is described as follows: for an n-simplex λ in I_f, we have that $(G_f)_n(\lambda) = G_{n+2}(\lambda)$. This description is possible since we work with strict pullbacks, w is level-wise injective on objects and (1) commutes. This implies that $w(\lambda)$ is the same λ but interpreted as an $(n+2)$-simplex in X. Using this description of I_f it is immediate to see that $G_f(s_0(f)) = s_0(Gf)$ and $G_f(s_1(f)) = s_1(Gf)$, this means that G_f sends the chosen initial and terminal objects of I_f to the chosen initial and terminal objects of I_{Gf}. In other words, G_f is stretched.

Recall that the chosen edge $\varpi_{Gf}\colon \perp_{I_{Gf}} \to \top_{I_{Gf}}$ of I_f, satisfying that $M_{Gf}\, \varpi_{Gf} = Gf$. Applying Proposition 3.2.0.19 to the map Gf, we have a unique stretched map ϕ_{Gf} satisfies $M_{Gf}\, \phi_{Gf} = Gf$. But as shown above, $G_f(\varpi_f)$ and ϖ_{Gf} also satisfy this condition. This implies that $G_f(\varpi_f) = \varpi_{Gf}$. Furthermore, if G is strict culf, (3) is a strict pullback by Lemma 3.2.0.12. Therefore, combining (2) and (3), we have that I_f is the strict pullback of $\pi_{\mathrm{long}}\colon \mathrm{Dec}_\top \mathrm{Dec}_\perp Y \to \delta(Y_1)$ along $\ulcorner Gf \urcorner\colon 1 \to \delta(Y_1)$. But this is precisely the definition of I_{Gf}. Since I_f and I_{Gf} are pullbacks over the same diagram, it follows that $I_f \cong I_{Gf}$. Furthermore, this isomorphism is given by G_f since the squares (3) and (4) commute. ☐

Remark 3.2.0.21. The uniqueness of Lemma 3.2.0.20 immediately implies the following 'transitivity' property of the construction $G \mapsto G_f$: Given

$$\begin{tikzcd}
I_f \arrow[r, "G_f"] \arrow[d, "M_f"] & I_{Gf} \arrow[r, "H_{Gf}"] \arrow[d, "M_{Gf}"] & I_{HGf} \arrow[d, "M_{HGf}"] \\
X \arrow[r, "G"'] & Y \arrow[r, "H"'] & Z,
\end{tikzcd}$$

we have

$$(H \circ G)_f = H_{Gf} \circ G_f.$$

3.3 *Stretched-culf factorisation system*

Let **DisInt** be the category whose objects are discrete intervals and whose morphisms are functors. We need some preliminary results to prove that the stretched functors as left-hand class and the culf functors as right-hand class form a factorisation system in **DisInt**.

Lemma 3.3.0.1. *Consider the following commutative diagram of simplicial maps*

$$\begin{tikzcd}
& \mathcal{B} \arrow[dr, "G"] & \\
\mathcal{A} \arrow[ur, "S", "\shortmid"{marking, pos=0.95}] \arrow[rr, "F"'] & & \mathcal{C}.
\end{tikzcd}$$

where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are discrete intervals, and S is stretched. Then F is stretched if and only if G is stretched.

Proof. Let $\varpi_{\mathcal{A}}\colon \perp_{\mathcal{A}} \to \top_{\mathcal{A}}$ be the unique map from $\perp_{\mathcal{A}}$ to $\top_{\mathcal{A}}$. Let $\varpi_{\mathcal{B}}\colon \perp_{\mathcal{B}} \to \top_{\mathcal{B}}$ and $\varpi_{\mathcal{C}}\colon \perp_{\mathcal{C}} \to \top_{\mathcal{C}}$ be the unique maps in $\mathcal{B}$ and $\mathcal{C}$. It is obvious that F is stretched when G is stretched. For the other direction, suppose F stretched. We have that

$$\begin{aligned}
G(\varpi_{\mathcal{B}}) &= G(S(\varpi_{\mathcal{A}})) && \text{(since } S \text{ is stretched)} \\
&= F(\varpi_{\mathcal{A}}) && \text{(since } F = GS\text{)} \\
&= \varpi_{\mathcal{C}}. && \text{(since } F \text{ is stretched)}
\end{aligned}$$

This means that G is stretched. □

Lemma 3.3.0.2. *Let $\mathcal{C}$ be a discrete interval with long edge ϖ. The simplicial map $M_{\varpi_{\mathcal{C}}}\colon I_{\varpi} \to \mathcal{C}$ has an inverse $W\colon \mathcal{C} \to I_{\varpi}$.*

Proof. Since $\mathcal{C}$ is a discrete interval, we have a map $L\colon \mathcal{C} \to (\mathcal{C}_{\perp_{\mathcal{C}}/})_{/\varpi}$ by Lemma 3.2.0.18. Recall that for an n-simplex λ in $\mathcal{C}_n$, the n-simplex $L(\lambda)$ satisfies that $\text{long}(L(\lambda)) = s_\top s_\perp \varpi_{\mathcal{C}}$ and $d_\top d_\perp L(\lambda) = \lambda$. Consider the canonical projections $u\colon (\mathcal{C}_{\perp_{\mathcal{C}}/})_{/\varpi} \to \text{Dec}_\top \mathcal{C}_{\perp_{\mathcal{C}}/}$ and $v\colon \mathcal{C}_{\perp_{\mathcal{C}}/} \to \text{Dec}_\perp \mathcal{C}$. Since $\mathcal{C}$ is a discrete interval, the canonical projections are level-wise injective on objects. So it is straightforward to check that $\pi_{\text{long}}(\text{Dec}_\top v \circ u \circ L(\lambda)) = \varpi_{\mathcal{C}}$. Therefore, the outer diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, dashed, "W"] \arrow[drr] \arrow[ddr, bend right, "\text{Dec}_\top v \circ u \circ L"'] & & \\
& I_{\varpi} \arrow[r] \arrow[d, "w"] & 1 \arrow[d, "\ulcorner \varpi \urcorner"] \\
& \text{Dec}_\top \text{Dec}_\perp \mathcal{C} \arrow[r, "\pi_{\text{long}}"'] & \delta(\mathcal{C}_1)
\end{tikzcd}$$

commutes. By the pullback property of I_{ϖ}, we have a unique map $W\colon \mathcal{C} \to I_{\varpi}$ such that the diagram commutes. Informally, for an n-simplex λ in $\mathcal{C}_n$, the map W only adds to λ the chosen initial edge $\bot_{\mathcal{C}} \to d_{\top}(\mathrm{long}(\lambda))$ by precomposing and the chosen terminal edge $d_{\bot}(\mathrm{long}(\lambda)) \to \top_{\mathcal{C}}$ by postcomposing. Since $w \circ W = \mathrm{Dec}_{\top} v \circ u \circ L$ and $d_{\top} d_{\bot} L(\lambda) = \lambda$, we have that $M_{\varpi} \circ W(\lambda) = \lambda$. By analogous arguments we have that $W \circ M_{\varpi} = \mathrm{id}_{I_{\varpi}}$. ☐

Lemma 3.3.0.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be discrete intervals, and let X be a rigid decomposition groupoid. Suppose we have a fork diagram*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, shift left, "V", ->|] \arrow[r, shift right, "W"', ->|] & \mathcal{B} \arrow[r, "F"] & X
\end{tikzcd}$$

(meaning $FV = FW$) where V and W are stretched and F is strict culf. Then already $V = W$.

Proof. Let $\varpi_{\mathcal{A}}$ denote the long edge of $\mathcal{A}$ and $\varpi_{\mathcal{B}}$ the long edge of $\mathcal{B}$, as usual. Since V and W are stretched, we have $V(\varpi_{\mathcal{A}}) = \varpi_{\mathcal{B}} = W(\varpi_{\mathcal{A}})$, so the following diagram is well formed from applying the construction of Lemma 3.2.0.20 (for composable maps as in Remark 3.2.0.21):

$$\begin{tikzcd}
I_{\varpi_{\mathcal{A}}} \arrow[r, shift left, "V_{\varpi_{\mathcal{A}}}", ->|] \arrow[r, shift right, "W_{\varpi_{\mathcal{A}}}"', ->|] \arrow[d, "M_{\varpi_{\mathcal{A}}}"'] & I_{\varpi_{\mathcal{B}}} \arrow[r, "F_{\varpi_{\mathcal{B}}}"] \arrow[d, "M_{\varpi_{\mathcal{B}}}"] & I_{F\varpi_{\mathcal{B}}} \arrow[d, "M_{F\varpi_{\mathcal{B}}}"] \\
\mathcal{A} \arrow[r, shift left, "V", ->|] \arrow[r, shift right, "W"', ->|] & \mathcal{B} \arrow[r, "F"'] & X.
\end{tikzcd}$$

Since $FV = FW$, we also have $F_{\varpi_{\mathcal{B}}} V_{\varpi_{\mathcal{A}}} = F_{\varpi_{\mathcal{B}}} W_{\varpi_{\mathcal{A}}}$. This is a consequence of the uniqueness statement in Lemma 3.2.0.20 as in Remark 3.2.0.21. But since F is strict culf, the map $F_{\varpi_{\mathcal{B}}}$ is an isomorphism by Lemma 3.2.0.20. It follows that $V_{\varpi_{\mathcal{A}}} = W_{\varpi_{\mathcal{A}}}$. Finally, since $\mathcal{A}$ and $\mathcal{B}$ are discrete intervals and $\varpi_{\mathcal{A}}$ and $\varpi_{\mathcal{B}}$ are their long edges, it follows from Lemma 3.3.0.2 that the two vertical maps $M_{\varpi_{\mathcal{A}}}$ and $M_{\varpi_{\mathcal{B}}}$ are isomorphisms, and this implies that $V = W$. ☐

Lemma 3.3.0.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be discrete intervals. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a simplicial map. Then F admits an stretched-culf factorisation.*

Proof. Let $\varpi_{\mathcal{C}}$ be the long 1-simplex of the interval $\mathcal{C}$. By Lemma 3.2.0.20, we have an stretched map $F_{\varpi_{\mathcal{C}}}\colon I_{\varpi_{\mathcal{C}}} \to I_{F\varpi_{\mathcal{C}}}$ fitting into the commutative diagram

$$\begin{tikzcd}
I_{\varpi_{\mathcal{C}}} \arrow[r, "F_{\varpi_{\mathcal{C}}}", ->|] \arrow[d, "M_{\varpi_{\mathcal{C}}}"'] & I_{F\varpi_{\mathcal{C}}} \arrow[d, "M_{F\varpi_{\mathcal{C}}}"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{D}.
\end{tikzcd}$$

Recall that the vertical arrows are strict culf. The map $M_{\varpi_{\mathcal{C}}}$ is stretched by Example 3.2.0.15. Since $\mathcal{C}$ is a discrete interval, we have that $M_{\varpi_{\mathcal{C}}}$ is invertible by Lemma 3.3.0.2. From the diagram

$$\begin{tikzcd}
I_{\varpi_{\mathcal{C}}} \arrow[rr, "\mathrm{id}_{I_{\varpi_{\mathcal{C}}}}", ->|] \arrow[dr, "M_{\varpi_{\mathcal{C}}}"', ->|] & & I_{\varpi_{\mathcal{C}}} \\
& \mathcal{C}, \arrow[ur, "M^{-1}_{\varpi_{\mathcal{C}}}"'] &
\end{tikzcd}$$

it follows that $M^{-1}_{\varpi_{\mathcal{C}}}$ is also stretched, by Lemma 3.3.0.1. So altogether, the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "F_{\varpi_{\mathcal{C}}} \circ M^{-1}_{\varpi_{\mathcal{C}}}"', ->|] & & \mathcal{D} \\
& I_{F\varpi_{\mathcal{C}}} \arrow[ur, "M_{F\varpi_{\mathcal{C}}}"'] &
\end{tikzcd}$$

commutes, where the map $F_{\varpi_{\mathcal{C}}} \circ M^{-1}_{\varpi_{\mathcal{C}}}$ is stretched and $M_{F\varpi_{\mathcal{C}}}$ is culf. ☐

Lemma 3.3.0.5. *Let $\mathcal{E}, \mathcal{E}'$ and $\mathcal{C}$ be discrete intervals. Let X be a rigid decomposition groupoid. For the commutative square of simplicial maps*

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "G"] \arrow[d, "S"'] & \mathcal{C} \arrow[d, "F"] \\
\mathcal{E}' \arrow[r, "H"'] \arrow[ru, dashed] & X
\end{tikzcd}$$

where $S\colon \mathcal{E} \mathrel{\rightarrow\mkern-6mu|} \mathcal{E}'$ is stretched and $F\colon \mathcal{C} \to \mathcal{D}$ is strict culf, there is a unique filler.

Proof. We will first construct a filler $L\colon \mathcal{E}' \to \mathcal{C}$ and then prove it is unique. For each n-simplex $\lambda\colon \Delta^n \to \mathcal{E}'$, Lemma 3.2.0.17 gives an $(n+2)$-simplex $\eta_\lambda\colon \Delta^{n+2} \mathrel{\rightarrow\mkern-6mu|} \mathcal{E}'$ such that

$$d_\perp d_\top(\eta_\lambda) = \lambda, \tag{3.3.1}$$

and

$$\mathrm{long}(\eta_\lambda) = \varpi_{\mathcal{E}'}. \tag{3.3.2}$$

We assumed that S is stretched, so $S(\varpi_\mathcal{E}) = \varpi_{\mathcal{E}'}$. This together with the equation $HS = FG$ and the stretched condition of S are used in the following calculation:

$$\mathrm{long}(H(\eta_\lambda)) = H(\mathrm{long}(\eta_\lambda)) = H(\varpi_{\mathcal{E}'}) = H(S(\varpi_\mathcal{E})) = F(G(\varpi_\mathcal{E})).$$

In other words, the outer diagram

$$\begin{tikzcd}
1 \arrow[rrd, bend left, "\mathrm{id}"] \arrow[rd, dotted, "\overline{H\eta_\lambda}"] \arrow[rdd, bend right, "H(\eta_\lambda)"'] & & \\
& (3)\ (I_{FG\varpi_\mathcal{E}})_n \arrow[r] \arrow[d, "w'_n"] \arrow[rd, phantom, very near start, "\lrcorner"] & 1 \arrow[d, "FG\varpi_\mathcal{E}"] \\
& X_{n+2} \arrow[r, "\mathrm{long}"'] & X_1
\end{tikzcd}$$

commutes. The pullback property of $I_{FG\varpi_\mathcal{E}}$ gives the dotted map $\overline{H\eta_\lambda}\colon \Delta^n \mathrel{\rightarrow\mkern-6mu|} I_{FG\varpi_\mathcal{E}}$ such that the diagram commutes. We define the map $V\colon \mathcal{E}' \to I_{FG\varpi_\mathcal{E}}$ by $V(\lambda) = \overline{H(\eta_\lambda)}$, for each n-simplex $\lambda\colon \Delta^n \to \mathcal{E}'$. It is straightforward to check that V is a simplicial map. Furthermore,

$$\begin{aligned}
M_{FG\varpi_\mathcal{E}} V(\lambda) &= d_\perp d_\top w'_n \overline{H(\eta_\lambda)} && \text{(by def. of } M_{FG\varpi_\mathcal{E}} \text{ and } V) \\
&= d_\perp d_\top H(\eta_\lambda) && \text{(by triangle (3))} \\
&= H d_\perp d_\top(\eta_\lambda) && \text{(since } H \text{ is a sim. map)} \\
&= H(\lambda). && \text{(by Eq. (3.3.1))}
\end{aligned}$$

This means that the following diagram commutes

$$\begin{tikzcd}
& I_{FG\varpi_\mathcal{E}} \arrow[rd, "M_{FG\varpi_\mathcal{E}}"] \arrow[d, phantom, "(4)"] & \\
\mathcal{E}' \arrow[ru, "V"] \arrow[rr, "H"'] & {} & X.
\end{tikzcd}$$

Since F is strict culf, Lemma 3.2.0.20 gives an isomorphism $K\colon I_{G\varpi_\mathcal{E}} \to I_{FG\varpi_\mathcal{E}}$ fitting into the commutative diagram

$$\begin{tikzcd}
I_{G\varpi_\mathcal{E}} \arrow[r, "M_{G\varpi_\mathcal{E}}"] \arrow[rd, phantom, "(5)"] & \mathcal{C} \arrow[d, "F"] \\
I_{FG\varpi_\mathcal{E}} \arrow[u, "K^{-1}"] \arrow[r, "M_{FG\varpi_\mathcal{E}}"'] & X.
\end{tikzcd}$$

Combining the commutativity of (4) and (5) gives $H = F \circ M_{G\varpi_{\mathcal{E}}} \circ K^{-1} \circ V$, and the hypothesis that $H \circ S = F \circ G$, we have that

$$F \circ M_{G\varpi_{\mathcal{E}}} \circ K^{-1} \circ V \circ S = H \circ S = F \circ G. \tag{3.3.3}$$

Lemma 3.3.0.3 says that it is possible to cancel F in Eq. (3.3.3) if F is strict culf, which is given by hypothesis. Therefore, the diagram

$$\begin{tikzcd}
 & \mathcal{E}' \arrow[r, "V"] & I_{FG\varpi_{\mathcal{E}}} \arrow[r, "K^{-1}"] \arrow[d, phantom, "(7)"] & I_{G\varpi_{\mathcal{E}}} \arrow[d, "M_{G\varpi_{\mathcal{E}}}"] \\
\mathcal{E} \arrow[ur, "S", ->|] \arrow[rrr, "G"'] & & & \mathcal{C}
\end{tikzcd}$$

commutes. The pieces now fit together to form the commutative diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "G"] \arrow[ddd, "S"', ->|] & & \mathcal{C} \arrow[ddd, "F"] \\
(7) & I_{G\varpi_{\mathcal{E}}} \arrow[ur, "M_{G\varpi_{\mathcal{E}}}"] & \\
 & I_{FG\varpi_{\mathcal{E}}} \arrow[u, "K^{-1}"] \arrow[dr, "M_{FG\varpi_{\mathcal{E}}}"] \arrow[ur, phantom, "(5)"] \arrow[d, phantom, "(4)"] & \\
\mathcal{E}' \arrow[ur, "V"] \arrow[rr, "H"'] & \, & X.
\end{tikzcd}$$

We define $L \colon \mathcal{E}' \to \mathcal{C}$ as $L := M_{G\varpi_{\mathcal{E}}} \circ K^{-1} \circ V$. Finally we establish uniqueness, exploiting that we already have existence given by the functor L. Suppose we have two fillers

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "G"] \arrow[d, "S"', ->|] & \mathcal{C} \arrow[d, "F"] \\
\mathcal{E}' \arrow[ur, shift left, "L_1"] \arrow[ur, shift right, "L_2"'] \arrow[r, "H"'] & X.
\end{tikzcd}$$

Now factor G as a stretched map G' followed by a strict culf map C,

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "G'", ->|] \arrow[dr, "G"'] & \mathcal{C}' \arrow[d, "C"] \\
 & \mathcal{C}
\end{tikzcd}$$

which is possible by Lemma 3.3.0.4. Now we can invoke existence of lifts to the situation

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "G'", ->|] \arrow[d, "S"', ->|] & \mathcal{C}' \arrow[d, "C"] \\
\mathcal{E}' \arrow[ur, dotted, shift left, "L_1'"] \arrow[ur, dotted, shift right, "L_2'"'] \arrow[r, shift left, "L_1"] \arrow[r, shift right, "L_2"'] & \mathcal{C}
\end{tikzcd}$$

since S is stretched and C is culf. This gives the existence of L_1' and L_2' as indicated, and they are stretched by Lemma 3.3.0.1 since both S and G' are stretched. But now we are in position to apply Lemma 3.3.0.3: Since we have $FL_1 = FL_2$ (as both are equal to H), we also have $FCL_1' = FCL_2'$.

Furthermore, since FC is culf and L_1' and L_2' are stretched, we conclude by Lemma 3.3.0.3 that already $L_1' = L_2'$, and therefore also $L_1 = L_2$.

□

Remark 3.3.0.6. In Lemma 3.3.0.5, we have that the diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r, "G"] \arrow[d, "S"'] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{E}' \arrow[r, "H"'] \arrow[ur, dashed, "L"] & X \end{tikzcd}$$

commutes. By hypothesis F is culf. Therefore, L is a culf if and only if H is culf. On the other hand, by hypothesis S is stretched and applying Lemma 3.3.0.1, we have that L is stretched if and only if G is stretched.

Remark 3.3.0.7. If we had required X to be a discrete interval, then Lemma 3.3.0.5 would say that the stretched and strict culf maps are orthogonal classes of maps in the category of discrete intervals and simplicial maps, as exploited in the following proposition. It will be important later in 3.4.1 that we allow X to be more general than just a discrete interval.

Proposition 3.3.0.8. *The stretched maps as left-hand class and the strict culf functors as right-hand class form a factorisation system in* **DisInt**.

Proof. The strict culf maps are closed under isomorphism. We have that every simplicial map F in **DisInt** admits an stretched-culf factorisation by Lemma 3.3.0.4. Therefore, we only have to prove that the classes are orthogonal, which follows from Lemma 3.3.0.5. □

3.4 *The decomposition groupoid $\mathcal{U}$*

In Section 3.3, the stretched-culf factorisation system was defined in **DisInt**, which we can use to define a fibration that encodes the pseudo-simplicial groupoid of discrete intervals.

Let $\mathrm{Ar}^s(\mathbf{DisInt}) \subset \mathrm{Ar}(\mathbf{DisInt})$ denote the full subcategory spanned by the stretched functors. $\mathrm{Ar}^s(\mathbf{DisInt})$ is a cartesian fibration over **DisInt** via the domain projection by Lemma 1.1.6.2. We now restrict this cartesian fibration to $\mathbb{\Delta} \subset$ **DisInt**

$$\begin{tikzcd} \mathrm{Ar}^s(\mathbf{DisInt})_{|\mathbb{\Delta}} \arrow[r, "\text{f.f.}"] \arrow[d, "\mathrm{dom}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ar}^s(\mathbf{DisInt}) \arrow[d, "\mathrm{dom}"] \\ \mathbb{\Delta} \arrow[r, "\text{f.f.}"'] & \mathbf{Int}. \end{tikzcd}$$

We put

$$\mathcal{U} := \mathrm{Ar}^s(\mathbf{DisInt})_{|\mathbb{\Delta}}.$$

$\mathcal{U} \to \mathbb{\Delta}$ is the cartesian fibration of subdivided discrete intervals. By Lemma 1.1.6.2, the cartesian maps in $\mathcal{U}$ are squares

$$\begin{tikzcd} \Delta^k \arrow[r] \arrow[d] & \Delta^n \arrow[d] \\ \mathcal{C} \arrow[r, "\text{culf}"'] & \mathcal{D}. \end{tikzcd}$$

The cartesian fibration $\mathcal{U} \to \mathbb{\Delta}$ determines a right fibration $\mathcal{U}^{\mathrm{cart}} \to \mathbb{\Delta}$, and hence by straightening [20, Theorem 8.3.1] a simplicial groupoid

$$U \colon \mathbb{\Delta}^{\mathrm{op}} \to \widehat{\mathbf{Grpd}}$$

where $\widehat{\mathbf{Grpd}}$ is the 2-category of large groupoids, functors and natural transformations.

The following result is due to Gálvez–Kock–Tonks [58, Theorem 4.8], who prove it in the more general setting of ∞-groupoids.

Theorem 3.4.0.1. *The simplicial groupoid* $\mathsf{U}\colon \mathbb{\Delta}^{\mathrm{op}} \to \widehat{\mathbf{Grpd}}$ *is a complete decomposition groupoid.*

3.4.1 *The complete decomposition groupoid* U_X

The decomposition groupoid $\mathsf{U}\colon \mathbb{\Delta}^{\mathrm{op}} \to \widehat{\mathbf{Grpd}}$ is not a strict simplicial object but only a pseudo-simplicial object. In a famous paper [64], Jardine figured out all the 2-cell data and 17 coherence laws for pseudo-simplicial objects in terms of face and degeneracy maps. We overcome the difficulty of working with these coherence laws by building a local strict model, a kind of neighbourhood $\mathsf{U}_X \subset \mathsf{U}$ around the discrete intervals of a given rigid decomposition groupoid X.

From the viewpoint of cartesian fibrations, the problem with $\mathcal{U} \to \mathbb{\Delta}$ is that it is not split. It is not possible to define a coherent global choice of cartesian lifts of arrows in $\mathbb{\Delta}$. To fix this, we restrict to a full subcategory $\mathcal{U}_X$ consisting only of the (subdivided) intervals of X (and not even including isomorphic intervals).

Definition 3.4.1.1. Let $\mathcal{U}_X \subset \mathcal{U}$ denote the full subcategory consisting only of the subdivided intervals $\Delta^n \mathrel{\rightarrow\!\!\!\shortmid} I_f$, where $f \in X_1$.

The benefit is that when everything is inside X, we can make canonical choices of cartesian lifts. They are given by the following lemma.

Lemma 3.4.1.2. *Let* X *be a rigid decomposition groupoid, and let* $p\colon \Delta^{n'} \to \Delta^n$ *be a map in* $\mathbb{\Delta}$*. For any* n*-simplex* $\lambda\colon \Delta^n \to X$*, the commutative triangle*

$$\begin{tikzcd}
\Delta^{n'} \arrow[rr, "p"] \arrow[dr, "\lambda'"'] & & \Delta^n \arrow[dl, "\lambda"] \\
& X &
\end{tikzcd}$$

gives the standard factorisations (Proposition 3.2.0.19) as in the solid square

$$\begin{tikzcd}
\Delta^{n'} \arrow[rr, "p"] \arrow[d, "\phi_{\lambda'}"'] & & \Delta^n \arrow[d, "\phi_\lambda"] \\
I_{f'} \arrow[rr, dotted, "c^p_\lambda"] \arrow[dr, "M_{f'}"'] & & I_f \arrow[dl, "M_f"] \\
& X. &
\end{tikzcd}$$

The statement is that there is a unique filler c^p_λ *as indicated with the dotted arrow, and this map is strict culf.*

Proof. Since $\phi_{\lambda'}$ is stretched and M_f is strict culf, the required map c^p_λ is given by Lemma 3.3.0.5, and it is strict culf by Remark 3.3.0.6. ☐

Notice how the ambient X is crucially exploited to characterise the lift uniquely. We also spell out how this choice of lifts act on isomorphisms:

Lemma 3.4.1.3. *Let X be a rigid decomposition groupoid, and consider an isomorphism $F\colon (I_f, \phi_\lambda) \xrightarrow{\sim} (I_g, \phi_\mu)$ in $(\mathcal{U}_X)_n$, as on the right in the following diagram. For any map $p\colon \Delta^{n'} \to \Delta^n$ in $\mathbb{\Delta}$, there is induced an isomorphism $F'\colon (I_{f'}, \phi_{\lambda'}) \xrightarrow{\sim} (I_{g'}, \phi_{\mu'})$ in $(\mathcal{U}_X)_{n'}$, as indicated with the dotted arrow:*

$$\begin{tikzcd}
& \Delta^{n'} \arrow[rr, "p"] \arrow[dl, "\phi_{\lambda'}"'] \arrow[ddr, "\phi_{\mu'}" near start] & & \Delta^n \arrow[dl, "\phi_\lambda"'] \arrow[ddr, "\phi_\mu"] & \\
I_{f'} \arrow[rr, "c^p_\lambda" near end] \arrow[dr, dotted, "\exists!\, F'"'] & & I_f \arrow[dr, "F"] & & \\
& I_{g'} \arrow[rr, "c^p_\mu"'] & & I_g &
\end{tikzcd}$$

This F' is characterised as the unique isomorphism in $(\mathcal{U}_X)_{n'}$ compatible with the canonical interval inclusions c^p_λ and c^p_μ (that is, unique making the whole diagram commute).

Let us explain the notation. The domain and codomain of F are objects in $(\mathcal{U}_X)_n$: as usual, the notation refers to an n-simplex $\lambda\colon \Delta^n \to X$ with long edge $f := \mathrm{long}(\lambda)$ and another n-simplex $\mu\colon \Delta^n \to X$ with long edge $g := \mathrm{long}(\mu)$, and $F\colon I_f \xrightarrow{\sim} I_g$ is an isomorphism of intervals compatible with the subdivisions $\phi_\lambda\colon \Delta^n \to\!\!| \, I_f$ and $\phi_\mu\colon \Delta^n \to\!\!| \, I_g$ provided by Proposition 3.2.0.19.

The map $p\colon \Delta^{n'} \to \Delta^n$ gives rise to n'-simplices λ' and μ' in X:

$$\begin{tikzcd}
\Delta^{n'} \arrow[rr, "p"] \arrow[dr, "\lambda'"'] & & \Delta^n \arrow[dl, "\lambda"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta^{n'} \arrow[rr, "p"] \arrow[dr, "\mu'"'] & & \Delta^n \arrow[dl, "\mu"] \\
& X &
\end{tikzcd}$$

and induced interval inclusions (strict culf maps)

$$I_{f'} \xrightarrow{c^p_\lambda} I_f \qquad\qquad I_{g'} \xrightarrow{c^p_\mu} I_g$$

as in Lemma 3.4.1.2.

Proof Lemma 3.4.1.3. Rearranging the bottom and left part of the diagram as

$$\begin{tikzcd}
\Delta^{n'} \arrow[rr, "\phi_{\mu'}"] \arrow[d, "\phi_{\lambda'}"'] & & I_{g'} \arrow[d, "c^p_\mu"] \\
I_{f'} \arrow[r, "c^p_\lambda"'] \arrow[urr, dotted, "F'"] & I_f \arrow[r, "F"'] & I_g
\end{tikzcd}$$

we see that F' is the unique lift existing by Lemma 3.3.0.5 since $\phi_{\lambda'}$ is stretched and c^p_μ is strict culf. □

Remark 3.4.1.4. In Lemma 3.4.1.3, the diagram

$$\begin{tikzcd}
\Delta^{n'} \arrow[r, "\phi_{\mu'}"] \arrow[d, "\phi_{\lambda'}"'] & I_{g'} \arrow[d, "c^p_\mu"] \\
I_{f'} \arrow[r, "Fc^p_\lambda"'] \arrow[ur, "F'"] & I_g
\end{tikzcd}$$

commutes. When p is active, we have that $f' = f$ and $g' = g$. Furthermore, if we substitute F' by F, the diagram also commutes. By Lemma 3.3.0.5, we have that $F' = F$. Therefore, when we work with an active map, we will use F instead of F'.

Whit these preparations, we can establish that $\mathcal{U}_X$ is split:

Proposition 3.4.1.5. *The cartesian fibration $\mathcal{U}_X \to \mathbb{\Delta}$ is split. The splitting is given by the cartesian arrows chosen in Lemma 3.4.1.2.*

Proof. That this choice of lifts constitutes a splitting means that it is functorial: composites of chosen lifts are lifts of composites, and lift of identity arrows are identity arrows. For composition: given the solid diagram

$$\begin{tikzcd}
\Delta^{n''} \arrow[r, "q"] \arrow[d, "\phi_{\lambda''}"', "\shortmid" marking] & \Delta^{n'} \arrow[r, "p"] \arrow[d, "\phi_{\lambda'}"', "\shortmid" marking] & \Delta^{n} \arrow[d, "\phi_{\lambda}", "\shortmid" marking] \\
I_{f''} \arrow[r, dotted, "c^q_{\lambda'}"] \arrow[dr, "M_{f''}"'] & I_{f'} \arrow[r, dotted, "c^p_{\lambda}"] \arrow[d, "M_{f'}"'] & I_f \arrow[dl, "M_f"] \\
 & X &
\end{tikzcd}$$

there are induced $c^q_{\lambda'}$ and c^p_λ making the whole diagram commute. Now by the uniqueness characterisation of c-maps, the composite $c^p_\lambda \circ c^q_{\lambda'}$ must be equal to c^{pq}_λ, as required. ☐

Knowing that the c-maps provide a splitting for $\mathcal{U}_X \to \mathbb{\Delta}$, there is now induced a strict functor

$$U_X \colon \mathbb{\Delta}^{\mathrm{op}} \to \mathbf{Grpd}$$

(groupoid-valued functor corresponding to the associated right fibration). We can now simply spell out explicitly what this simplicial groupoid is. On objects, we simply have to describe the fibres: $(U_X)_n$ is thus the groupoid whose objects are subdivided intervals of X, say $\phi_\lambda : \Delta^n \rightarrow\!\!\!\!\shortmid\; I_f$ (for some $\lambda \in X_n$ with long edge f), and whose arrows are the vertical arrows in $\mathcal{U}_X$, namely strictly commutative triangles

$$\begin{tikzcd}
 & \Delta^n \arrow[dl, "\phi_\lambda"', "\shortmid" marking] \arrow[dr, "\phi_\mu", "\shortmid" marking] & \\
I_f \arrow[rr, "\sim"] & & I_g .
\end{tikzcd}$$

Note that since $\mathcal{U}_X$ was defined as full inside $\mathcal{U}$, there are no compatibility requirement with the 'inclusions' $M_f \colon I_f \to X$ and $M_g \colon I_g \to X$.

The simplicial operators act via cartesian lifts: the formula for $p : \Delta^{n'} \to \Delta^n$ is

$$p^*(\Delta^n \overset{\phi_\lambda}{\rightarrow\!\!\!\!\shortmid\;} I_f) = (\Delta^{n'} \overset{\phi_{\lambda'}}{\rightarrow\!\!\!\!\shortmid\;} I_{f'})$$

with reference to the chosen cartesian arrow

$$\begin{tikzcd}
\Delta^{n'} \arrow[r, "p"] \arrow[d, "\phi_{\lambda'}"', "\shortmid" marking] & \Delta^n \arrow[d, "\phi_\lambda", "\shortmid" marking] \\
I_{f'} \arrow[r, "c^p_\lambda"'] & I_f .
\end{tikzcd} \tag{3.4.1}$$

The action of the simplicial operator on an isomorphism in $(U_X)_n$, say $F\colon (I_f, \phi_\lambda) \xrightarrow{\sim} (I_g, \phi_\mu)$, is given by Lemma 3.4.1.3. Indeed, this lemma is nothing but the standard description of how a vertical isomorphism is transported along a cartesian lift.

(Note that the construction of the isomorphism F', which in Lemma 3.4.1.3 was given using the stretched-culf factorisation system, can also be regarded as the argument why general arrows in a cartesian fibration factor uniquely through cartesian arrows. Indeed the stretched-culf factorisation system is the abstract reason why we have a cartesian fibration.)

Lemma 3.4.1.6. *Let $p\colon [n] \rightarrow\!\!\!\!\shortmid\; [m]$ be an active map in $\mathbb{\Delta}$. Then $p^*\colon (U_X)_m \to (U_X)_n$ is a discrete fibration.*

Proof. Let (I_g, ϕ_μ) be an object in $(\mathcal{U}_X)_m$ and let $F\colon (I_f, \phi_\lambda) \to p^*(I_g, \phi_\mu)$ be a morphism in $(\mathcal{U}_X)_n$. To provide a lift is to use the same underlying F (by 3.4.1.4, since p is active), but the compatibility which characterises morphisms in $(\mathcal{U}_X)_m$ is now with the ϕ maps from Δ^m instead of from Δ^n. In other words, we need to find the dashed arrow in the diagram

$$\begin{tikzcd}
\Delta^n \arrow[rr, "\phi_\lambda"] \arrow[dr, "p"'] & & I_f \arrow[d, "F"] \\
& \Delta^m \arrow[ur, dashed, "\phi_{\overline{\eta}}"] \arrow[r, "\phi_\mu"'] & I_g,
\end{tikzcd}$$

which is possible since F is invertible, in fact $\overline{\eta} = M_f F^{-1}(\phi_\lambda)$. Therefore p^* is a discrete fibration. □

Example 3.4.1.7. In general, the image of an inert map of $\mathbb{\Delta}^{op}$ under $\mathcal{U}_X$ is not a discrete fibration. Let $\mathcal{C}$ be the category pictured by the following commutative diagram

$$\begin{tikzcd}
& y \arrow[dr, "b"] & & \\
x \arrow[ur, "a"] \arrow[rr, "f" description] \arrow[dr, "a'"'] & & z \arrow[r, "g"'] & w. \\
& y' \arrow[ur, "b'"'] & &
\end{tikzcd}$$

Since x is an initial object and w is a terminal object, we have that $N(\mathcal{C}) \simeq I_{gf}$ by Lemma 3.3.0.2. Let ϕ_{gf} be the 2-simplex induced by the morphisms f and g in $(N(\mathcal{C}))_2$. Let $(N(\mathcal{C}), \phi_{gf}) \in (\mathcal{U}_{N(\mathcal{C})})_2$ be the interval construction of ϕ_{gf}. Applying d_0 to ϕ_{gf}, we have that $d_0(N(\mathcal{C}), \phi_{gf}) = (I_g, \phi_g)$. Let $\mathrm{id}_{I_g}\colon (I_g, \phi_g) \to (I_g, \phi_g)$ be the identity morphism in $(\mathcal{U}_{N(\mathcal{C})})_1$.

We can construct two lifts of id_{I_g} in $(\mathcal{U}_{N(\mathcal{C})})_2$. Let $F\colon (N(\mathcal{C}), \phi_{gf}) \to (N(\mathcal{C}), \phi_{gf})$ be the functor that fixes all the objects in $\mathcal{C}$ except y and y'. It is easy to check that $d_0 F = \mathrm{id}_{I_g}$. On the other hand, it is straightforward to see that the identity morphism $\mathrm{id}_{I_{gf}}$ satisfies $d_0\, \mathrm{id}_{I_{gf}} = \mathrm{id}_{I_g}$. Therefore, F and $\mathrm{id}_{I_{gf}}$ are two different lifts of id_{I_g}.

When S is a simplicial groupoid, we have a simplicial set induced by the *object functor* $\mathrm{Obj}\colon \mathbf{Grpd} \to \mathbf{Set}$, which is defined as forgetting the morphisms. We denote $\mathrm{Obj} \circ S$ as S^0.

Proposition 3.4.1.8. *Let $X\colon \mathbb{\Delta}^{op} \to \mathbf{Grpd}$ be a rigid decomposition groupoid. Then $\mathcal{U}_X^0 \cong X^0$.*

Proof. The proof is easily deduced from the fact that every object (I_f, ϕ_λ) in $(\mathcal{U}_X^0)_n$ corresponds to some λ in X_n^0 by definition of $\mathcal{U}_X$. □

Lemma 3.4.1.9. *Let X be a rigid decomposition groupoid. The simplicial groupoid $\mathcal{U}_X\colon \mathbb{\Delta}^{op} \to \mathbf{Grpd}$ is a decomposition groupoid.*

Proof. We need to show that for an active-inert pullback square in $\mathbb{\Delta}^{op}$, the image under $\mathcal{U}_X$ is a homotopy pullback

$$\begin{tikzcd}
(\mathcal{U}_X)_m \arrow[r, "g", "|"{marking, pos=0.9}] \arrow[d, "h"', tail] & (\mathcal{U}_X)_n \arrow[d, "\overline{h}", tail] \\
(\mathcal{U}_X)_k \arrow[r, "\overline{g}"', "|"{marking, pos=0.9}] & (\mathcal{U}_X)_s.
\end{tikzcd}$$

Here g and $\overline{g}$ are active maps, h and $\overline{h}$ are inert maps. Since g and $\overline{g}$ are active maps, they are discrete fibrations by Lemma 3.4.1.6. Therefore, we can work with strict fibres. By Lemma 1.2.0.1, the previous square is a homotopy pullback if and only if for each object (I_f, ϕ_λ) in $(\mathcal{U}_X)_n$, corresponding to some $\lambda \in X_n$, the morphism $h'\colon \mathrm{Fib}_{(I_f, \phi_\lambda)}(g) \to \mathrm{Fib}_{\overline{h}(I_f, \phi_\lambda)}(\overline{g})$, induced by the morphism h, is an equivalence. Here $\mathrm{Fib}_{(I_f,\phi_\lambda)}(g)$ is the strict fibre of g over (I_f,λ) and $\mathrm{Fib}_{\overline{h}(I_f,\phi_\lambda)}(\overline{g})$ is the strict fibre of $\overline{g}$ over $\overline{h}(I_f,\phi_\lambda)$.

The fibres $\mathrm{Fib}_{(I_f,\phi_\lambda)}(g)$ and $\mathrm{Fib}_{\overline{h}(I_f,\phi_\lambda)}(\overline{g})$ are discrete groupoids since g and g' are discrete fibrations. Furthermore, as a consequence of Proposition 3.4.1.8, we have a bijection between the objects of X and U_X. This implies that the diagram

$$\begin{tikzcd}
\mathrm{Fib}_{(\lambda)}(g) \arrow[r, "h''"] \arrow[d] & \mathrm{Fib}_{\overline{h}\lambda}(\overline{g}) \arrow[d] \\
\mathrm{Fib}_{(I_f,\phi_\lambda)}(g) \arrow[r, "h'"] & \mathrm{Fib}_{\overline{h}(I_f,\phi_\lambda)}(\overline{g})
\end{tikzcd}$$

commutes. Here $h'' \colon \mathrm{Fib}_\lambda(g) \to \mathrm{Fib}_{\overline{h}\lambda}(\overline{g})$ is the morphism induced by $h \colon X_m \to X_k$. Since X is rigid, the morphism h'' is an equivalence. Since $\mathrm{Fib}_{(I_f,\phi_\lambda)}(g)$ and $\mathrm{Fib}_{\overline{h}(I_f,\phi_\lambda)}(\overline{g})$ are discrete groupoids, the vertical maps are equivalences by Proposition 3.4.1.8. Hence, the map $h' \colon \mathrm{Fib}_{(I_f,\phi_\lambda)}(g) \to \mathrm{Fib}_{\overline{h}(I_f,\phi_\lambda)}$ is an equivalence. ☐

Lemma 3.4.1.10. *Let X be a rigid decomposition groupoid. Then the decomposition groupoid U_X is complete.*

Proof. To establish that U_X is complete, we need to check that the map $s_0 \colon (U_X)_0 \to (U_X)_1$ is a monomorphism. This means that we need to show that the fibre is either empty or singleton. Remember that the objects in $(U_X)_0$ are given by 0-simplices of X. Combining this with the fact that the long edge of a 0-simplex is $s_0(x)$, we have that the objects in $(U_X)_0$ are of the form $(I_{s_0(x)}, \phi_x)$. Since s_0 is active, we have that s_0 is a discrete fibration by Lemma 3.4.1.6. Therefore, we will consider strict fibres. For $f \in X_1$ denote by $\phi_f \colon \Delta^1 \rightarrow\mathrel{\mkern-6mu|} I_f$ the unique stretched map. The strict fibre over $(I_f, \phi_f) \in (U_X)_1$ is given by the strict pullback

$$\begin{tikzcd}
\mathrm{Fib}_{(I_f,\phi_f)}(s_0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (U_X)_0 \arrow[d, "s_0"] \\
1 \arrow[r, "\ulcorner (I_f,\phi_f) \urcorner"'] & (U_X)_1 .
\end{tikzcd}$$

Unless f is degenerate, the strict fibre is empty. In the degenerate case, consider $(I_{s_0(x)}, \phi_{s_0(x)})$ and $(I_{s_0(y)}, \phi_{s_0(y)})$ two objects in $\mathrm{Fib}_{(I_f,\phi_f)}(s_0)$ such that $(I_{s_0(x)}, \phi_{s_0(x)}) = (I_{s_0(y)}, \phi_{s_0(y)})$. This means that $\phi_{s_0(x)} = \phi_{s_0(y)}$. This together with the rigid condition of X (the map $s_0 \colon X_0 \to X_1$ is a monomorphism) implies that $x = y$. ☐

To construct a map from X to U_X, the following result is necessary:

Lemma 3.4.1.11. *Given an isomorphism $\alpha \colon \lambda \to \mu$ in X_n, there is induced an isomorphism*

$$\begin{tikzcd}
& \Delta^n \arrow[dl, "\phi_\lambda"'] \arrow[dr, "\phi_\mu"] & \\
I_f \arrow[rr, "F_\alpha"'] & & I_g .
\end{tikzcd}$$

As usual, $f = \mathrm{long}(\lambda)$ *and* $g = \mathrm{long}(\mu)$.

Proof. The main point is to prove it just for 1-simplices: given $\mathrm{long}(\alpha) \colon f \to g$ in X_1, the interval I_f is the fibre over $f \in \delta(X_1)$ of the whole simplicial groupoid $\mathrm{Dec}_\top \mathrm{Dec}_\bot X \to \delta(X_1)$, and I_g is the fibre over g. This is a level-wise fibration over $\delta(X_1)$, since it is formed entirely of active maps. But in a fibration, any isomorphism $f \cong g$ between two objects in the base induces an isomorphism $F_\alpha \colon I_f \to I_g$ between the fibres. Recall the objects of I_f are the 2-simplices with long edge f, and the objects of I_g are the 2-simplices with long edge g. So the isomorphism F_α sends a 2-simplex with long edge f to a 2-simplex with long edge g. This forces $F_\alpha s_0(f) = s_0(g)$ and $F_\alpha s_1(f) = s_1(g)$, which is equivalent to saying that F_α is stretched and therefore $F_\alpha \varpi_{I_f} = \varpi_{I_g}$.

Coming back to the general case, $\lambda \cong \mu$: we have the solid outer square:

$$\begin{tikzcd}
\Delta^1 \arrow[r, "\varpi_{I_f}", ->|] \arrow[rr, bend left, "\varpi_{I_g}", ->|] \arrow[d, ->|] & I_f \arrow[r, "F_\alpha", ->|] & I_g \arrow[d, "M_g"] \\
\Delta^n \arrow[ru, dotted, "\phi_\lambda", ->|] \arrow[rru, dotted, "\phi_\mu"', ->|] \arrow[rr, "\mu"'] & & X.
\end{tikzcd}$$

The curved triangle commutes by the 1-simplex case already treated. The dotted arrows then exist individually by Proposition 3.2.0.19. The triangle that these two dotted arrows form with $I_f \simeq I_g$ is now forced to commute, since $\Delta^1 \to \Delta^n$ is active and M_g is strict culf. □

We define a simplicial map $I\colon X \to U_X$, using the interval construction:

- the map I sends an object $\lambda \in X_n$ to the pair (I_f, ϕ_λ) where $f = \mathrm{long}(\lambda)$ and ϕ_λ is the n-simplex induced by λ of Proposition 3.2.0.19.
- the map I sends an arrow $\alpha\colon \lambda \to \mu$ in X_n to the isomorphism $F_\alpha\colon (I_f, \phi_\lambda) \to (I_g, \phi_\mu)$ induced by α of Lemma 3.4.1.11. (As usual, $f = \mathrm{long}(\lambda)$ and $g = \mathrm{long}(\mu)$.)

Proposition 3.4.1.12. *Let X be a rigid decomposition groupoid. The simplicial map $I\colon X \to U_X$ is strict culf.*

Proof. Since d_1 is active, we have that d_1 is a discrete fibration by Lemma 3.4.1.6. Therefore, as a consequence of Lemma 1.1.2.2, to prove that I is culf it is enough to prove that the following diagram is a strict pullback

$$\begin{tikzcd}
X_2 \arrow[r, "d_1"] \arrow[d, "I_2"'] & X_1 \arrow[d, "I_1"] \\
(U_X)_2 \arrow[r, "d_1"'] & (U_X)_1,
\end{tikzcd}$$

which is equivalent to proving that the functor $G\colon X_2 \to (U_X)_2 \times_{(U_X)_1} X_1$ induced by the pullback property is an isomorphism. For each $\sigma \in X_2$, the object $G(\sigma)$ is equal to $((I_{d_1(\sigma)}, \phi_\sigma), d_1(\sigma))$ where ϕ_σ is given by Proposition 3.2.0.19. For a morphism $\alpha\colon \sigma \to \sigma'$ put $f = d_1(\sigma)$ and $g = d_1(\sigma')$, the morphism $G(\sigma)$ is equal to $(H_\alpha, d_1(\alpha))$. Here $H_\alpha\colon (I_f, \phi_\sigma) \to (I_g, \phi_{\sigma'})$ is the isomorphism given by Lemma 3.4.1.11.

Recall that d_1 is a discrete fibration, this together with Proposition 3.2.0.19 allows to construct a functor $R\colon (U_X)_2 \times_{(U_X)_1} X_1 \to X_2$. For an object $(\phi_\sigma\colon \Delta^2 \twoheadrightarrow I_f, f)$, the object $R(\phi_\sigma, f)$ is defined as $M_f\, \phi_\sigma$ in X_2. For a morphism $(H, \overline{\alpha})$, where $H\colon (I_f, \phi_\sigma) \to (I_g, \phi_{\sigma'})$ and $\overline{\alpha}\colon f \to g$, the morphism $R(H, \overline{\alpha})$ is defined as the morphism $\alpha\colon M_f\, \phi_\sigma \to M_g\, \phi_{\sigma'}$ which is the lifting of the arrow $\overline{\alpha}\colon f \to g$ with respect to $M_f\, \phi_\sigma$ and $M_g\, \phi_{\sigma'}$. The lift is unique since d_1 is a discrete fibration. It is straightforward to verify that R is the inverse of the functor G. Note that the diagram is also a homotopy pullback since it is a strict pullback and d_1 is a discrete fibration. □

3.4.2 *Compatibility of* M*-maps and subdivided intervals*

Given a simplicial map $G\colon X \to Y$, there will be natural relationships between intervals in X and intervals in Y, but to compare them we need to step out to the global U, leaving the realms of U_X and U_Y.

Lemma 3.2.0.20 can be proved in an alternative way as follows:

Lemma 3.4.2.1. *For any simplicial map between decomposition groupoids* $G\colon X \to Y$*, there is a unique stretched map* $G_f\colon I_f \twoheadrightarrow I_{Gf}$*, compatible with* M*-maps. This means that the diagram*

$$\begin{tikzcd}
I_f \arrow[r, "G_f", ->|] \arrow[d, "M_f"'] & I_{Gf} \arrow[d, "M_{Gf}"] \\
X \arrow[r, "G"'] & Y
\end{tikzcd}$$

commutes. If G *is culf then* G_f *is invertible.*

Proof. The unique stretched map is given by Lemma 3.3.0.5:

$$\begin{tikzcd}
\Delta^1 \arrow[rr, ->|] \arrow[d, ->|] & & I_{Gf} \arrow[d, "M_{Gf}"] \\
I_f \arrow[r, "M_f"'] \arrow[urr, dotted, ->|] & X \arrow[r, "G"'] & Y
\end{tikzcd}$$

If G is culf, then the dotted arrow is culf too by Remark 3.3.0.6, and since it is both culf and stretched, it is invertible as a consequence of Proposition 3.3.0.8. □

Lemma 3.4.2.2. *For any simplicial map between decomposition groupoids* $G\colon X \to Y$*, there is a unique stretched map* $G_f\colon I_f \twoheadrightarrow I_{Gf}$*, compatible with subdivision: if we start with* $\lambda : \Delta^n \to X$ *(with long edge* f*), then the triangle*

$$\begin{tikzcd}
 & \Delta^n \arrow[dl, "\phi_\lambda"'] \arrow[dr, "\phi_{G\lambda}"] & \\
I_f \arrow[rr, "G_f"', ->|] \arrow[d, "M_f"'] & & I_{Gf} \arrow[d, "M_{Gf}"] \\
X \arrow[rr, "G"'] & & Y
\end{tikzcd} \tag{3.4.1}$$

commutes.

Proof. Orthogonality (Lemma 3.3.0.5) for the square

$$\begin{tikzcd}
\Delta^1 \arrow[dr] \arrow[rrr, bend left, ->|] \arrow[ddr, bend right] & & & \\
 & \Delta^n \arrow[rr, ->|] \arrow[d, ->|] & & I_{Gf} \arrow[d, "M_{Gf}"] \\
 & I_f \arrow[r, "M_f"'] \arrow[urr, dotted, ->|] & X \arrow[r, "G"'] & Y
\end{tikzcd}$$

gives a unique filler, which has to be G_f, since it is also a filler for the square starting at Δ^1. □

Finally we need to establish also the corresponding result for isomorphisms in X_n: given $\lambda \cong \mu$ in X_n, Lemma 3.4.1.11 gives isomorphisms of (subdivided) intervals

$$\begin{tikzcd}
 & \Delta^n \arrow[dl, "\phi_\lambda"'] \arrow[dr, "\phi_\mu"] & & & \Delta^n \arrow[dl, "\phi_{G\lambda}"'] \arrow[dr, "\phi_{G\mu}"] & \\
I_f \arrow[rr, "\sim"] & & I_g & I_{Gf} \arrow[rr, "\sim"] & & I_{Gg}.
\end{tikzcd} \tag{3.4.2}$$

Lemma 3.4.2.3. *Let* $G\colon X \to Y$ *be a simplicial map between decomposition groupoids. For any* $f \cong g$ *in* X_1, *the diagram*

$$\begin{tikzcd}
I_f \arrow[r, "G_f", ->|] \arrow[d, "\sim"'] & I_{Gf} \arrow[d, "\sim"] \\
I_g \arrow[r, "G_g"', ->|] & I_{Gg}
\end{tikzcd} \tag{3.4.3}$$

commutes. Here the horizontal arrows are given by Lemma 3.4.2.1 and the vertical arrows by Lemma 3.4.1.11.

Proof. The diagram

$$\begin{tikzcd}
\Delta^n \arrow[rr, "\phi_{G\lambda}"] \arrow[rrr, bend right=15, "\phi_{G\mu}"'] \arrow[d, "\phi_\lambda"', ->|] \arrow[dr, "\phi_\mu"] & & I_{Gf} \arrow[r, "\sim"] & I_{Gg} \arrow[d, "M_{Gg}"] \\
I_f \arrow[r, "\sim"] & I_g \arrow[r, "M_g"'] & X \arrow[r, "G"'] & Y
\end{tikzcd}$$

commutes: the middle pentagon region is (3.4.1), and the triangles are (3.4.2). Inside the outer square we have the following two dotted maps:

$$\begin{tikzcd}
\Delta^n \arrow[rr, "\phi_{G\lambda}"] \arrow[d, "\phi_\lambda"', ->|] & & I_{Gf} \arrow[r, "\sim"] & I_{Gg} \arrow[d, "M_{Gg}"] \\
I_f \arrow[r, "\sim"] \arrow[urr, dotted] & I_g \arrow[r, "M_g"'] \arrow[urr, dotted] & X \arrow[r, "G"'] & Y.
\end{tikzcd}$$

The two triangle-shaped regions with dotted arrows also commute: the leftmost triangle is the triangle part of (3.4.1) for λ, and the rightmost 'triangle' is the square part of (3.4.1) for μ. The dotted parallelogram is now forced to commute, since both composites in it are fillers for the outer square, and by orthogonality (Lemma 3.3.0.5) only one filler can exist as ϕ_λ is stretched and M_{Gg} is strict culf. ☐

So now we completely control the G-maps in each simplicial degree individually. We shall also establish the naturality in simplicial operators: We have seen (in Lemma 3.4.1.2) that for any $p\colon \Delta^{n'} \to \Delta^n$, there is induced a canonical culf map $c^p_\lambda\colon I_{f'} \to I_f$ compatible like this:

$$\begin{tikzcd}
\Delta^{n'} \arrow[r, "p"] \arrow[d, "\phi_{\lambda'}"', ->|] & \Delta^n \arrow[d, "\phi_\lambda", ->|] \\
I_{f'} \arrow[r, "c^p_\lambda"'] & I_f.
\end{tikzcd} \tag{3.4.4}$$

The following lemma shows that these functorialities are compatible.

Lemma 3.4.2.4. *Let* $G\colon X \to Y$ *be a simplicial map between decomposition groupoids. From the situation*

$$\begin{tikzcd}
\Delta^{n'} \arrow[rr, "p"] \arrow[dr, "\lambda'"'] & & \Delta^n \arrow[dl, "\lambda"] \\
& X \arrow[d, "G"] & \\
& Y, &
\end{tikzcd}$$

we get a commutative square

$$\begin{tikzcd}
I_{f'} \arrow[r, "c^p_\lambda"] \arrow[d, "G_{f'}"'] & I_f \arrow[d, "G_f"] \\
I_{Gf'} \arrow[r, "c^p_{G\lambda}"'] & I_{Gf}
\end{tikzcd}$$

involving the maps from the previous functorialities.

Proof. The diagram

$$\begin{tikzcd}
\Delta^{n'} \arrow[r, "p"] \arrow[dd, "\phi_{\lambda'}"', ->|] & \Delta^n \arrow[r, "\phi_\lambda", ->|] & I_f \arrow[r, "G_f"] \arrow[d] & I_{Gf} \arrow[dd, "M_{Gf}"] \\
 & & X \arrow[dr] & \\
I_{f'} \arrow[r, "G_{f'}"'] \arrow[urr] & I_{Gf'} \arrow[rr, "M_{Gf'}"'] & & Y
\end{tikzcd}$$

commutes: the pentagon by construction of the ϕ-maps (3.2.0.19), and the two squares by Equation (3.4.1).

The outer square has the following two dotted c-maps:

$$\begin{tikzcd}
\Delta^{n'} \arrow[r, "p"] \arrow[d, "\phi_{\lambda'}"', ->|] & \Delta^n \arrow[r, "\phi_\lambda", ->|] & I_f \arrow[r, "G_f"] & I_{Gf} \arrow[d, "M_{Gf}"] \\
I_{f'} \arrow[r, "G_{f'}"'] \arrow[urr, dotted, ->|] & I_{Gf'} \arrow[rr, "M_{Gf'}"'] \arrow[urr, dotted, ->|] & & Y.
\end{tikzcd}$$

The two triangle-shaped regions with dotted arrows commute by construction of the c-maps (Lemma 3.4.1.2). The dotted parallelogram is now forced to commute, since both composites in it are fillers for the outer square, and only one filler can exist, as $\phi_{\lambda'}$ is stretched and M_{Gf} is strict culf. ☐

3.4.3 *Interval construction of an interval*

Let A be a discrete interval (simplicial set), and consider a subdivision of it, $a : \Delta^n \mathrel{\rightarrow\!\!\!\shortmid} A$. This whole data describes an n-simplex in U, which we denote $\underline{a} : \Delta^n \to U$. Note that the long edge of $\underline{a}$ is A itself.

We can now apply Proposition 3.2.0.19 to $\underline{a}$ (as an n-simplex in U) to get

$$\begin{tikzcd}
\Delta^n \arrow[r, "\phi_{\underline{a}}", ->|] \arrow[dr, "\underline{a}"'] & I^U_A \arrow[d, "M_A"] \\
 & U
\end{tikzcd}$$

Lemma 3.4.3.1. *There is a canonical isomorphism $A \simeq I^U_A$ compatible with the subdivision:*

$$\begin{tikzcd}
 & \Delta^n \arrow[dl, "a"', ->|] \arrow[dr, "\phi_{\underline{a}}", ->|] & \\
A \arrow[rr, "\sim"] & & I^U_A.
\end{tikzcd} \tag{3.4.1}$$

Proof. There is a canonical simplicial map $A \to \mathrm{Dec}_\perp \mathrm{Dec}_\top U$, given by sending an n-simplex $\lambda \colon \Delta^n \to A$ to the corresponding stretched $(n+2)$-simplex $\overline{\lambda} \colon \Delta^{n+2} \mathrel{\rightarrow\!\!\!\shortmid} A$, interpreted as an $(n+2)$-simplex in U. This simplicial map clearly factors through $I^U_A \to \mathrm{Dec}_\perp \mathrm{Dec}_\top U$. We claim that the induced simplicial map $A \to I^{U}_{A}$ is an isomorphism. Indeed, $(I^{U}_{A})_n$ is by definition the strict pullback

$$\begin{tikzcd}
(I^{U}_{A})_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\ulcorner A \urcorner"] \\
U_{n+2} \arrow[r, -{>|}] & U_1
\end{tikzcd}$$

which is to say that it is the groupoid of stretched maps $\Delta^{n+2} \rightarrow\!\!\mid A$, in turn isomorphic to the groupoid of general maps $\Delta^{n} \to A$, which is the groupoid A_n. ☐

Lemma 3.4.3.2. *For any stretched isomorphism of intervals* $A \simeq B$*, we get from Lemma 3.4.1.11 an isomorphism* $I_A \simeq I_B$*. The statement is that these isos are compatible, meaning that the diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & I_A \arrow[d] \\
B \arrow[r] & I_B
\end{tikzcd}$$

commutes.

Proof. This follows since the isomorphisms involved

$$(I^{U}_{A})_n \simeq \mathrm{map}^{\mathrm{str}}(\Delta^{n+2}, A) \simeq \mathrm{map}(\Delta^{n}, A) = A_n$$

are all natural in stretched isomorphisms $A \simeq B$. ☐

A simplicial map $G\colon X \to Y$ between decomposition groupoids is *Full and faithful* if for all objects $x, y \in X$ it induces an equivalence on the mapping groupoids

$$G_{x,y}\colon \ \mathrm{map}_X(x,y) \to \mathrm{map}_Y(Gx, Gy).$$

Recall that we have a canonical simplicial map $\jmath\colon U_X \to U$, defined by $\jmath(I_f, \varphi_\lambda) = (I_f, \varphi_\lambda)$ for $(I_f, \varphi_\lambda) \in (U_X)_n$ and $\jmath F = F$ for $F \in (U_X)_n$. It is straightforward to prove the following result.

Lemma 3.4.3.3. *Let* X *be a rigid decomposition groupoid. Then the simplicial map* $\jmath\colon U_X \to U$ *is full and faithful.*

Remark 3.4.3.4. Gálvez, Kock and Tonks [60] defined the culf *classifying map* $I'\colon \mathbb{\Delta}_{/X} \to \mathfrak{U}$. It takes an n-simplex $\lambda\colon \Delta^{n} \to X$ to an n-subdivided interval $\varphi_\lambda\colon \Delta^{n} \rightarrow\!\!\mid I_f$ in $\mathfrak{U}$ (or to the pair (I_f, φ_λ) in U_n). Here $f = \mathrm{long}(\lambda)$ and $\mathbb{\Delta}_{/X}$ denotes the Grothendieck construction of X. In the present chapter I' is the map $(\jmath \circ I)\colon X \to U$, since for each $\lambda \in X_n$, we have that $(\jmath \circ I)(\lambda) = (I_f, \varphi_\lambda)$ which is the same as $I'(\lambda)$. We will abuse of notation and denote I' as I in Section 3.5.

3.4.4 *Comparison with a strictification of* U

In this section, we briefly compare our local strict model U_X with a global strictification $\widetilde{U}$ of U, proposed by the referee.

There is a well-known construction that replaces a pseudo-functor into **Grpd** with a strict functor (see for example [50, §6.4]). In the present case there is a very explicit combinatorial description of such a strictification. An inert map from Δ^{k} to Δ^{n} is completely determined by the values of 0 and k. So we will denote an inert map $\Delta^{k} \rightarrowtail \Delta^{n}$ as a pair $(i,j)\colon \Delta^{k} \rightarrowtail \Delta^{n}$ such that $0 \mapsto i$ and $k \mapsto j$. We denote by P_n the poset of inert faces of Δ^{n}. We define $\widetilde{U}_n$ to be the groupoid of liftings

$$\begin{tikzcd}
 & \mathfrak{U} \arrow[d, "\mathrm{dom}"] \\
P_n \arrow[r] \arrow[ur, dashed] & \mathbb{\Delta}.
\end{tikzcd}$$

This gives a whole family of stretched maps $\Delta^k \mathrel{\rightarrow\!\!\!\shortmid} \mathcal{C}$, one for each $(i,j) \in P_n$, and squares

$$\begin{tikzcd}
\Delta^k \arrow[r, tail] \arrow[d, "\shortmid" marking] & \Delta^{k'} \arrow[d, "\shortmid" marking] \\
\mathcal{C}_{ij} \arrow[r, swap, "\text{culf}"] & \mathcal{C}_{i'j'}
\end{tikzcd}$$

for each map $(i,j) \rightarrowtail (i',j')$. Here $\mathcal{C}_{ij}$ and $\mathcal{C}_{i'j'}$ are discrete intervals. For example an object in $\widetilde{\mathcal{U}}_2$ is pictured as follows

$$\begin{tikzcd}
 & & & & \Delta^0 \arrow[dd, tail] \arrow[dr, "\shortmid" marking] & \\
 & & & & & \mathcal{C}_{22} \arrow[dd] \\
 & & \Delta^0 \arrow[rr, tail] \arrow[dd, tail] \arrow[dr, "\shortmid" marking] & & \Delta^1 \arrow[dd, tail] \arrow[dr, "\shortmid" marking] & \\
 & & & \mathcal{C}_{11} \arrow[rr] \arrow[dd] & & \mathcal{C}_{12} \arrow[dd] \\
\Delta^0 \arrow[rr, tail] \arrow[dr, "\shortmid" marking] & & \Delta^1 \arrow[rr, tail] \arrow[dr, "\shortmid" marking] & & \Delta^2 \arrow[dr, "\shortmid" marking] & \\
 & \mathcal{C}_{00} \arrow[rr] & & \mathcal{C}_{01} \arrow[rr] & & \mathcal{C}_{02}.
\end{tikzcd}$$

It is possible to define face and degeneracy maps between the groupoids $\widetilde{\mathcal{U}}_n$ to assemble them into a strict simplicial groupoid $\widetilde{\mathcal{U}}$. Informally, the face map d_i acts by 'erasing' all stretched maps containing an index i. The degeneracy maps s_i repeat the ith row and the ith column. We have a canonical equivalence $\pi_{\text{endvertex}} \colon \widetilde{\mathcal{U}} \to \mathcal{U}$ that on objects erases all the stretched maps except the last one. In case we consider only intervals that come from a fixed strict decomposition groupoid X, we have a strict simplicial groupoid $\widetilde{\mathcal{U}}_X$ and a canonical equivalence $\pi'_{\text{endvertex}} \colon \widetilde{\mathcal{U}}_X \to \mathcal{U}_X$.

The interval construction $I \colon X \to \mathcal{U}_X$ from [60] can easily be factored through $\widetilde{\mathcal{U}}_X$ to give a refined interval construction $\widetilde{I} \colon X \to \widetilde{\mathcal{U}}_X$ that sends an n-simplex $\lambda \colon \Delta^n \to X$ to $(I_{\lambda c}, \phi_{\lambda c})$ for each $c \colon \Delta^k \rightarrowtail \Delta^n \in P_n$. Here $\phi_{\lambda c}$ is given by Proposition 3.2.0.19. For example, for a 1-simplex $f \colon \Delta^1 \to X$, the object $\widetilde{I}\,(f)$ in $(\widetilde{\mathcal{U}}_X)_1$ is given by the following diagram

$$\begin{tikzcd}
 & \Delta^0 \arrow[d, tail] \arrow[dr, "\phi_{fd^1}"] & \\
\Delta^0 \arrow[r, tail] \arrow[dr, swap, "\phi_{fd^0}"] & \Delta^1 \arrow[dr, "\phi_f"] & I_{fd^1} \arrow[d] \\
 & I_{fd^0} \arrow[r] & I_f.
\end{tikzcd}$$

Note that since $\mathcal{U}_X$ is already strict, all this refined data is redundant.

The four versions of $\mathcal{U}$ and the four interval constructions are compatible, as indicated in the commutative diagram

$$\begin{tikzcd}
 & \widetilde{\mathcal{U}}_X \arrow[r, hook, "j'"] \arrow[dd, "\pi_{\text{endvertex}}"] & \widetilde{\mathcal{U}} \arrow[dd, "\pi_{\text{endvertex}}"] \\
X \arrow[ur, "\widetilde{I}"] \arrow[dr, swap, "I"] & & \\
 & \mathcal{U}_X \arrow[r, hook, swap, "j"] & \mathcal{U}.
\end{tikzcd}$$

The original $\mathcal{U}$ is hard to work with, as it is pseudo-simplicial instead of strict simplicial. Both $\widetilde{\mathcal{U}}_X$ and $\mathcal{U}_X$ are practical because they are strict. ($\widetilde{\mathcal{U}}_X$ is strict but is too redundant.) In this chapter we prefer to work with U_X since in any case most of the arguments are carried out locally at X, and in this situation it is the most direct approach.

3.5 *Gálvez–Kock–Tonks Conjecture*

Let $\mathbf{cDcmp}_{\text{culf}}$ denote the ∞-category of complete decomposition spaces and culf maps. The construction of the complete decomposition groupoid U was motivated by the following statement:

Gálvez–Kock–Tonks Conjecture [60, §5.4] For each decomposition space X, the mapping space of culf maps $\operatorname{map}(X, U)$ is contractible.

A partial result states that $\operatorname{map}(X, U)$ is connected. An ∞-version of this result is Theorem 5.5 in [60]. We include a proof here for two reasons. Firstly, we need to be more precise regarding strictness conditions, and secondly, the proof in [60] does not actually give any argument for naturality in inert maps. As we shall see, this is a subtle issue, and the lack of argument in [60] may be considered a gap in that proof.

In our setting of rigid decomposition spaces, the relevant maps are the strict culf maps. We are now concerned with culf maps to U. Recall that $U\colon \mathbb{\Delta}^{\text{op}} \to \mathbf{Grpd}$ is only pseudo-simplicial, but that it is actually strict on active maps. Furthermore, for active $[n'] \mathrel{\rightarrow\!\!\!\shortmid} [n]$, the corresponding $U_n \to U_{n'}$ is a fibration. The notion of strict culf map $J\colon X \to U$ is therefore still meaningful: we do allow pseudo-simplicial maps, but they are still required to be strict on the active part, and the naturality squares on active maps are required to be strict pullbacks. This implies in particular that for the unique active map $\text{long}\colon [1] \mathrel{\rightarrow\!\!\!\shortmid} [n]$, and for every n-simplex $\lambda \in X_n$ with long edge $f = \text{long}(\lambda)$, we have a strict equality

$$\text{long}(J_n(\lambda)) = J_1(f). \tag{3.5.1}$$

For general $p\colon [n'] \to [n]$ in $\mathbb{\Delta}$ (not necessarily active) it follows that J takes a strict triangle

$$\begin{tikzcd}
\Delta^{n'} \arrow[rr, "p"] \arrow[dr, "\lambda'"'] & & \Delta^{n} \arrow[dl, "\lambda"] \\
& X &
\end{tikzcd}$$

to a commutative square of the form

$$\begin{tikzcd}
\Delta^{n'} \arrow[r, "p"] \arrow[d, "J_{n'}(\lambda')"'] & \Delta^{n} \arrow[d, "J_n(\lambda)"] \\
J_1(f') \arrow[r, "e"'] & J_1(f)
\end{tikzcd} \tag{3.5.2}$$

with e culf. (This is to say, it is a cartesian morphism for the right fibration $\mathcal{U}^{\text{cart}} \to \mathbb{\Delta}$.)

Theorem 3.5.0.1. *For any rigid decomposition groupoid X, the groupoid $\operatorname{map}_{\mathbf{cDcmp}_{\text{culf}}}(X, U)$ of strict culf maps from X to U is connected. More precisely, for any strict culf map $J\colon X \to U$, there is a natural transformation (actually a modification) $\Gamma\colon J \xrightarrow{\sim} I$.*

Proof. There are three steps in the proof: Step 1 is to establish a canonical isomorphism $J_1(f) \cong I_1(f)$ for each $f \in X_1$, and show that this is natural in arrows in X_1. Step 2 is to exploit culfness to extend this isomorphism canonically to $J_n(\lambda) \cong I_n(\lambda)$ for each $\lambda \in X_n$ (again naturally in λ). The construction in Step 2 actually shows that these isomorphisms are natural in active maps in $\mathbb{\Delta}$.

But in any case, Step 3 consists in showing that the isomorphisms are natural in *all* maps in $\mathbb{\Delta}$, meaning that for any $p\colon [n'] \to [n]$ in $\mathbb{\Delta}$, the naturality square

$$\begin{tikzcd}
\Delta^{n'} \arrow[r, "p"] \arrow[d, "a'"', -{>|}] & \Delta^{n} \arrow[d, "a", -{>|}] \\
J_1(f') \arrow[r] \arrow[d, "\sim"] & J_1(f) \arrow[d, "\sim"] \\
I_1(f') \arrow[r] & I_1(f)
\end{tikzcd}$$

commutes (as will be detailed).

Step 1. Given $f \in X_1$, we construct isomorphisms

$$I^X_f \simeq I^U_{J_1(f)} \simeq J_1(f).$$

Here the first isomorphism is an instance of Lemma 3.4.1.2, where $J\colon X \to U$ plays the role of $G\colon X \to Y$. The second isomorphism is an instance of Lemma 3.4.3.1, where $J_1(f)$ plays the role of A.

We should now argue why these isomorphisms are natural in arrows in X_1: given $f \simeq g$, we need to check that this naturality square commutes:

$$\begin{tikzcd}
J_1(f) \arrow[r] \arrow[d] & I_1(f) \arrow[d] \\
J_1(g) \arrow[r] & I_1(g).
\end{tikzcd}$$

Since the vertical isomorphisms are composites of isos from Lemma 3.4.2.1 and from Lemma 3.4.3.1, the naturality in maps inside X_1 follows from the naturality expressed by Lemmas 3.4.2.3 and 3.4.3.2.

Step 2. We now show that these isomorphisms $J_1(f) \cong I_1(f)$ extend to isomorphisms $J_n(\lambda) \cong I_n(\lambda)$ for each n, using that both I and J are strict culf. We have

$$\begin{tikzcd}
X_1 \arrow[d] & X_n \arrow[l, -{>|}] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
(U_X)_1 & (U_X)_n \arrow[l, -{>|}]
\end{tikzcd}$$

Since these horizontal maps are fibrations, we can describe the objects in $(U_X)_n$ as follows. To give an object $J_n(\lambda)$ in $(U_X)_n$ is to give the underlying interval $J_1(f)$ and an object in the fibre over $J_1(f)$. Since the square is a strict pullback, to give an object in the fibre of the bottom horizontal map is the same as giving an object in the fibre over f of the top horizontal maps, i.e. a subdivision, i.e. an object $\lambda \in X_n$. This same description holds for I. So to give, for a fixed $\lambda \in X_n$, an isomorphism $J_n(\lambda) \overset{\sim}{\to} I_n(\lambda)$ is to give an isomorphism $I_1(f) \overset{\sim}{\to} I_1(f)$, and keep the λ in the fibres fixed.

As in Step 1, we should now argue why these isomorphisms are natural in arrows in X_n: given $\lambda \simeq \mu$ in X_n, we need to check that this naturality square commutes:

$$\begin{tikzcd}
J_n(\lambda) \arrow[r] \arrow[d] & I_n(\lambda) \arrow[d] \\
J_n(\mu) \arrow[r] & I_n(\mu).
\end{tikzcd}$$

The argument is the same as that given in degree 1, but invoking now Lemma 3.4.2.2 instead of Lemma 3.4.2.1.

Note that the isomorphisms are natural in all active maps $[n] \rightarrow\!\!\!\!\shortmid [1]$ by construction, and therefore, by the standard prism-lemma argument, are also natural in all active maps.

Step 3. The final step is to show that the isomorphisms are also natural in inert maps, and in fact we prove uniformly that they are natural in all maps $p : [n'] \to [n]$ in $\mathbb{\Delta}$. Given $\lambda : \Delta^n \to X$ (with long edge f) and a map $p : \Delta^{n'} \to \Delta^n$, put $\lambda' := \lambda \circ p$ (with long edge f'), so that we have

$$\begin{tikzcd}
\Delta^{n'} \arrow[rr, "p"] \arrow[dr, "\lambda'"'] & & \Delta^n \arrow[dl, "\lambda"] \\
& X &
\end{tikzcd}$$

which is sent by J to

$$\begin{tikzcd}
\Delta^{n'} \arrow[r, "p"] \arrow[d, "a'"'] & \Delta^n \arrow[d, "a"] \\
J_1(f') \arrow[r, "e \text{ culf}"'] & J_1(f).
\end{tikzcd} \tag{3.5.3}$$

By Step 1 we have isomorphisms in each simplicial degree, which are strictly compatible with the subdivisions by Step 2, to give commutative triangles

$$\begin{tikzcd}
& \Delta^n \arrow[dl] \arrow[d] \arrow[dr] & & & & \Delta^n \arrow[dl] \arrow[d] \arrow[dr] & \\
J_1(f') \arrow[r, "\sim"] & I^U_{J_1(f')} \arrow[r, "\sim"] & I_1(f') & & J_1(f) \arrow[r, "\sim"] & I^U_{J_1(f)} \arrow[r, "\sim"] & I_1(f).
\end{tikzcd} \tag{3.5.4}$$

These diagrams together with Lemma 3.4.1.2 ensure that the following outer rectangle commutes:

$$\begin{tikzcd}
\Delta^{n'} \arrow[r, "p"] \arrow[d, "a'"'] & \Delta^n \arrow[r, "a"] & J_1(f) \arrow[r, "\sim"] & I^U_{J_1(f)} \arrow[r, "\sim"] & I_1(f) \arrow[d, "M_f"] \\
J_1(f') \arrow[r, "\sim"] \arrow[urr, dotted, "e"] & I^U_{J_1(f')} \arrow[r, "\sim"] & I_1(f') \arrow[rr, "M_{f'}"'] \arrow[urr, dotted, "c^p_\lambda"'] & & X
\end{tikzcd}$$

We furthermore have the two diagonal dotted arrows indicated. The leftmost triangle-shaped region is (3.5.3), and the right-most triangle is given in Lemma 3.4.1.2. The composed dotted parallelogram is now forced to commute, since the composites in it are fillers for the outer square, and only one filler can exist since a' is stretched and M_f is strict culf.

The dotted arrows are the cartesian lifts of p to $J_1(f)$ and $I_1(f)$, and the commutativity of

$$\begin{tikzcd}
\Delta^{n'} \arrow[r, "p"] \arrow[d, "a'"'] & \Delta^n \arrow[d, "a"] \\
J_1(f') \arrow[r, "e"] \arrow[d, "\sim"] & J_1(f) \arrow[d, "\sim"] \\
I_1(f') \arrow[r, "c^p_\lambda"'] & I_1(f)
\end{tikzcd}$$

now shows that the isomorphisms $J_n \xrightarrow{\sim} I_n$ are natural in p (and thereby with the whole simplicial structure). ☐

3.5.1 *Modifications*

Theorem 3.5.0.1 implies that every natural transformation from X to U is isomorphic to I. Therefore, to prove the conjecture, we only need to prove that I does not admit other self-modifications than the identity. Thus, we will introduce the notion of modification in the context in which we need it.

A modification between two natural transformations is a family of 2-cells in the 2-category of (small) categories that satisfies some coherence conditions as indicated in the following definition:

Definition 3.5.1.1. [19, Definition 7.3.1] Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors and $\alpha, \beta\colon F \Rightarrow G$ be two natural transformations from F to G. A modification $\Gamma\colon \alpha \Rrightarrow \beta$ assigns to each object x in $\mathcal{C}$ a 2-cell $\Gamma_x\colon \alpha_x \to \beta_x$ of $\mathcal{D}$ compatibly with the 2-cell components of F and G in the sense of the equation

$$\begin{tikzcd}
F(x) \arrow[r, bend left, ""{name=A, below}] \arrow[r, bend right, ""{name=B}] \arrow[d, "F(f)"'] & G(x) \arrow[d, "G(f)"] \arrow[dl, phantom, "\Downarrow \beta_f"] & = & F(x) \arrow[r] \arrow[d, "F(f)"'] & G(x) \arrow[d, "G(f)"] \arrow[dl, phantom, "\Downarrow \alpha_f"] \\
F(y) \arrow[r] & G(y) & & F(y) \arrow[r, bend left, ""{name=C, below}] \arrow[r, bend right, ""{name=D}] & G(y).
\arrow[Rightarrow, from=A, to=B, "\Gamma_x"]
\arrow[Rightarrow, from=C, to=D, "\Gamma_y"]
\end{tikzcd}$$

We are interested in the case where $\mathcal{C} = \mathbb{\Delta}$ and $\mathcal{D} = \mathbf{Grpd}$, and where $F = X$ and $G = U_X$, and where α and β are both equal to I. In this case, Definition 3.5.1.1 can be written as follows.

Definition 3.5.1.2. A modification $\Gamma\colon I \to I$ assigns to each $[n]$ in $\mathbb{\Delta}$ a natural transformation $\Gamma_n\colon I_n \to I_n$ in **Grpd** such that for each $n \geqslant 1$ the following equations hold for each $0 \leqslant i \leqslant n$ and $0 \leqslant j < n$

$$\begin{tikzcd}
X_n \arrow[r, bend left, ""{name=A, below}] \arrow[r, bend right, ""{name=B}] \arrow[d, "d_i"'] & (U_X)_n \arrow[d, "d_i"] & = & X_n \arrow[r] \arrow[d, "d_i"'] & (U_X)_n \arrow[d, "d_i"] \\
X_{n-1} \arrow[r] & (U_X)_{n-1} & & X_{n-1} \arrow[r, bend left, ""{name=C, below}] \arrow[r, bend right, ""{name=D}] & (U_X)_{n-1}
\arrow[Rightarrow, from=A, to=B, "\Gamma_n"]
\arrow[Rightarrow, from=C, to=D, "\Gamma_{n-1}"]
\end{tikzcd} \qquad (1)$$

$$\begin{tikzcd}
X_{n-1} \arrow[r, bend left, ""{name=A, below}] \arrow[r, bend right, ""{name=B}] \arrow[d, "s_j"'] & (U_X)_{n-1} \arrow[d, "s_j"] & = & X_{n-1} \arrow[r] \arrow[d, "s_j"'] & (U_X)_{n-1} \arrow[d, "s_j"] \\
X_n \arrow[r] & (U_X)_n & & X_n \arrow[r, bend left, ""{name=C, below}] \arrow[r, bend right, ""{name=D}] & (U_X)_n.
\arrow[Rightarrow, from=A, to=B, "\Gamma_{n-1}"]
\arrow[Rightarrow, from=C, to=D, "\Gamma_n"]
\end{tikzcd} \qquad (2)$$

Remark 3.5.1.3. We can define a modification $\Gamma\colon I \to I$ level by level, so let $\Gamma_n\colon I_n \to I_n$ be a component of the modification Γ. Given $\lambda \in X_n$, let ϕ_λ be the n-simplex induced by λ constructed in Proposition 3.2.0.19 and $f = \mathrm{long}(\lambda)$. The modification Γ assigns to λ an invertible morphism $\Gamma_n^\lambda\colon (I_f, \phi_\lambda) \to (I_f, \phi_\lambda)$ in $(U_X)_n$. The morphism Γ_n^λ has associated an underlying map $\overline{\Gamma_n^\lambda}\colon I_f \to I_f$.

Let $p\colon [m] \twoheadrightarrow [n]$ be an active map. By Remark 3.4.1.4, we have that $p^*\overline{\Gamma_n^\lambda} = \overline{\Gamma_n^\lambda}$. This implies that

$$\overline{\Gamma_m^{\lambda p}} = \overline{\Gamma_n^\lambda}$$

where $\overline{\Gamma_m^{\lambda p}}\colon I_f \to I_f$ is the underlying map of $\Gamma_m^{\lambda p}$. The difference between Γ_n^λ and $\Gamma_m^{\lambda p}$ is that the first one respects the n-subdivision ϕ_λ and the other respects the m-subdivision $\phi_{\lambda p}$.

Lemma 3.5.1.4. *Let X be a rigid decomposition groupoid. The mapping groupoid $\mathrm{map}_{\mathbf{cDcmp}_{\mathrm{culf}}}(X, U_X)$ is contractible.*

Proof. Theorem 3.5.0.1 shows that we only have to prove that I does not admit other self-modifications Γ than the identity. Let λ be an n-simplex in X and put $f = \mathrm{long}(\lambda)$. Let Γ a modification, with components $\Gamma_n\colon I_n \to I_n$ and let $\overline{\Gamma_n^\lambda}\colon I_f \to I_f$ be the underlying map of $\Gamma_n^\lambda\colon (I_f, \phi_\lambda) \to (I_f, \phi_\lambda)$ of Remark 3.5.1.3.

Since long: $[1] \nrightarrow [n]$ is an active map in $\mathbb{\Delta}$, by Remark 3.5.1.3, we have that

$$\overline{\Gamma^\lambda_n} = \overline{\Gamma^f_1} \tag{3.5.5}$$

where $\overline{\Gamma^f_1} \colon I_f \to I_f$ is the underlying map of $\Gamma^f_1 \colon (I_f, \phi_f) \to (I_f, \phi_f)$. On the other hand, given a morphism $\alpha \colon \sigma \to \overline{\sigma}$ in I_f, Lemma 3.2.0.17 gives an stretched 3-simplex $\eta_\alpha \colon \Delta^3 \nrightarrow I_f$ such that

$$d_\perp d_\top \eta_\alpha = \alpha. \tag{3.5.6}$$

The modification Γ assigns to $M_f \eta_\alpha$ an invertible map $\Gamma^{\eta_\alpha}_3 \colon (I_f, \eta_\alpha) \to (I_f, \eta_\alpha)$ such that $\overline{\Gamma^{\eta_\alpha}_3}\eta_\alpha = \eta_\alpha$. Furthermore,

$$\begin{aligned}
\overline{\Gamma^{\eta_\alpha}_3}(\alpha) &= \overline{\Gamma^{\eta_\alpha}_3}(d_\perp d_\top \eta_\alpha) && \text{(by Eq. (3.5.6))}\\
&= d_\perp d_\top \overline{\Gamma^{\eta_\alpha}_3}(\eta_\alpha) && \text{(since } \overline{\Gamma^{\eta_\alpha}_3} \text{ is a sim. map)}\\
&= d_\perp d_\top (\eta_\alpha)\\
&= \alpha.
\end{aligned}$$

By Definition 3.5.1.2, we have the equality

$$\begin{tikzcd}
X_3 \arrow[r, bend left, "I_3"] \arrow[r, bend right, "I_3"'] \arrow[r, phantom, "\Gamma_3 \Downarrow"] & (U_X)_3 \arrow[r, "d_1 d_1"] & (U_X)_1 \arrow[r, phantom, "="] & X_3 \arrow[r, "d_1 d_1"] & X_1 \arrow[r, bend left, "I_1"] \arrow[r, bend right, "I_1"'] \arrow[r, phantom, "\Gamma_1 \Downarrow"] & (U_X)_1.
\end{tikzcd}$$

This equation implies that $d_1 d_1(\Gamma^{\eta_\alpha}_3) = \Gamma^f_1$. Since $d^1 d^1$ is active, we have that $\overline{\Gamma^{\eta_\alpha}_3} = \overline{\Gamma^f_1}$ by Remark 3.5.1.3. Hence altogether, for each $\alpha \in I_f$

$$\begin{aligned}
\overline{\Gamma^\lambda_n}(\alpha) &= \overline{\Gamma^f_1}(\alpha) && \text{(by Eq. (3.5.5))}\\
&= \overline{\Gamma^{\eta_\alpha}_3}(\alpha)\\
&= \alpha.
\end{aligned}$$

Since Γ^λ_n is the identity arrow for each $\lambda \in X_n$, we have that Γ is the identity modification. ☐

Theorem 3.5.1.5. *Let X be a rigid decomposition groupoid. The mapping groupoid* $\mathrm{map}_{\mathbf{cDcmp}_{\mathrm{culf}}}(X, U)$ *is contractible.*

Proof. Since $\jmath \colon U_X \to U$ is full and faithful (3.4.3.3), we have that $\jmath_! \colon \mathrm{map}_{\mathbf{cDcmp}_{\mathrm{culf}}}(X, U_X) \to \mathrm{map}_{\mathbf{cDcmp}_{\mathrm{culf}}}(X, U)$ is also full and faithful. Furthermore, each natural transformation from X to U is isomorphic to I by Theorem 3.5.0.1, this implies that $\jmath_!$ is essentially surjective on objects, and therefore $\mathrm{map}_{\mathbf{cDcmp}_{\mathrm{culf}}}(X, U_X) \cong \mathrm{map}_{\mathbf{cDcmp}_{\mathrm{culf}}}(X, U)$. This equivalence combining with the contractibility of $\mathrm{map}_{\mathbf{cDcmp}_{\mathrm{culf}}}(X, U_X)$ (3.5.1.4) implies that $\mathrm{map}_{\mathbf{cDcmp}_{\mathrm{culf}}}(X, U)$ is contractible. ☐

4

The Gálvez–Kock–Tonks conjecture for Möbius decomposition spaces

In Chapter 3, it was studied the conjecture at the level of locally discrete decomposition spaces. With this strict restriction, the contractibility of $\mathrm{map}(X, U_X)$ can be established with 2-category theory by showing that $I_X \colon X \to U_X$, interpreted as a natural transformation, does not admit other self-modifications than the identity modification. It is natural to ask whether the techniques developed in Chapter 3 can be applied or refined to prove the conjecture in full generality. Unfortunately this is not very likely, since the proof relies on explicit strictification. Therefore, we prefer to study the conjecture from another perspective by imposing cardinal bounds through the Möbius condition (1.1.1). The decomposition space of Möbius subdivided intervals U_{Mob} is small, so to prove the conjecture is to show that U_{Mob} is a terminal object in the ∞-category **MobDcmp** of Möbius decomposition spaces and culf maps . This proof is the main result (4.2.0.14) of this chapter. Furthermore, the proof of the conjecture allows together with the fact that the ∞-category of decomposition spaces and culf maps is locally an ∞-topos [63] to prove that **MobDcmp** is an ∞-topos (4.3.4.1).

4.1 *Flanked decomposition spaces*

In this section we recall from [60] some constructions and results required to set up the decomposition space of sub-divided intervals U.

We denote by Ξ the category of finite strict intervals, that is, a skeleton of the category whose objects are nonempty finite linear orders with a bottom and a top elements, required to be distinct, and whose arrows are the maps that preserve both the order and the bottom and top elements. There is a forgetful functor $u \colon \Xi \to \mathbb{\Delta}$ which forgets that there is anything special about the bottom and top elements. This functor has a left adjoint $i \colon \mathbb{\Delta} \to \Xi$ which to a linear order adjoins a bottom and top elements. The two functors can be described in object as $u([k]) = [k+2]$ and $i([k]) = [k]$, and the adjunction is given by the following isomorphism:

$$\Xi([n],[k]) = \mathbb{\Delta}([n],[k+2]) \quad n \geqslant 0, k \geqslant -1. \tag{4.1.1}$$

The objects in the category Ξ are $[-1], [0], [1]$, etc. Furthermore, compared to $\mathbb{\Delta}$ via the inclusion i, the category Ξ has one extra coface map $[-1] \to [0]$. It also has in each degree, two extra *outer degeneracy maps*: $s^{\perp} \colon [n] \to [n-1]$ and $s^{\top} \colon [n] \to [n-1]$.

The adjunction $i \dashv u$ induces an adjunction $i^* \dashv u^*$

$$\begin{tikzcd} \mathbf{Fun}(\Xi^{\mathrm{op}}, \mathcal{S}) \arrow[r, shift left, "i^*"] & \mathbf{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S}). \arrow[l, shift left, "u^*"] \end{tikzcd} \tag{4.1.2}$$

The functor i^* takes the underlying simplicial space A and deletes $A_{[-1]}$ and removes the extra outer degeneracy maps. On the other hand, the functor u^* applied to a simplicial space X, deletes X_0 and removes all outer face maps and then reindexes.

Definition 4.1.0.1. [60, §2.9] A Ξ^{op}-space A is called *flanked* if the extra outer degeneracy maps form cartesian squares with the opposite outer face maps. Precisely, for $n \geqslant 0$

$$\begin{tikzcd}
A_{n-1} \arrow[d, "s_{\bot-1}"'] & A_n \arrow[l, "d_\top"'] \arrow[d, "s_{\bot-1}"] \arrow[dl, phantom, "\llcorner", very near start] \\
A_n & A_{n+1} \arrow[l, "d_\top"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A_{n-1} \arrow[d, "s_{\top+1}"'] & A_n \arrow[l, "d_\bot"'] \arrow[d, "s_{\top+1}"] \arrow[dl, phantom, "\llcorner", very near start] \\
A_n & A_{n+1}. \arrow[l, "d_\bot"]
\end{tikzcd}$$

Definition 4.1.0.2. [60, §2.12] A Ξ^{op}-space A is called *complete flanked decomposition space* if A is flanked and i^*A is a complete decomposition space.

Let **cDcmp** denote the full subcategory of **sSpaces** spanned by the complete decomposition spaces and let **cFD** denote the full subcategory of $\mathbf{Fun}(\Xi^{\mathrm{op}}, \mathcal{S})$ spanned by the complete flanked decomposition spaces. It follows that the adjunction 4.1.2 restricts to the adjunction

$$\begin{tikzcd}
\mathbf{cFD} \arrow[r, shift left, "i^*"] & \mathbf{cDcmp}. \arrow[l, shift left, "u^*"]
\end{tikzcd} \tag{4.1.3}$$

4.1.1 *Algebraic intervals*

Definition 4.1.1.1. [60, §3.2] A Ξ^{op}-space A is called *reduced* when $A_{[-1]} \simeq 1$.

Definition 4.1.1.2. [60, §3.4] A Ξ^{op}-space A is called *algebraic interval* if A is a reduced complete flanked decomposition space.

We denote by **aInt** the full subcategory of $\mathbf{Fun}(\Xi^{\mathrm{op}}, \mathcal{S})$ spanned by algebraic intervals.

Definition 4.1.1.3. [60] An arrow $G \colon A \to B$ in $\mathbf{Fun}(\Xi^{\mathrm{op}}, \mathcal{S})$ is *stretched* if its $[-1]$-component is an equivalence.

Note that every morphism in **aInt** is stretched and all the representables $\Xi[k]$ are algebraic intervals.

Proposition 4.1.1.4. *[60, Proposition 4.2] The stretched maps as left-hand class and the cartesian maps as right-hand class form a factorisation system on* **aInt***.*

Let **Int** denote the image of **aInt** $\subset$ **cFD** under the left adjoint i^* in the adjunction (4.1.3). Say a map in **Int** is *stretched* if it is the i^* image of a map in **aInt**. Furthermore, a *Möbius interval* is an interval which is a Möbius decomposition space.

Proposition 4.1.1.5. *[60, Proposition 4.2] The stretched maps as left-hand class and the culf maps as right-hand class form a factorisation system on* **Int***.*

Given an arrow f in a decomposition space X, we can construct an algebraic interval associate to f. In the case where X is a 1-category the construction is due to Lawvere [77]: the objects are two-steps factorisation of f, with initial object id-followed by-f and the terminal object f-followed by-id. The 1-cells are arrows between such factorisations.

For the general case, by Yoneda, to give an arrow f in X_1 is to give $\Delta^1 \to X$ in **cDcmp**. By adjunction, this equivalent to giving $\Xi_{[-1]} \to u^*X$ in **cFD**. Now factor this map as an stretched map followed by a cartesian map:

$$\begin{tikzcd}
\Xi_{[-1]} \arrow[rr] \arrow[dr, "\text{stretched}"'] & & u^*X \\
& A \arrow[ur, "\text{cartesian}"'] &
\end{tikzcd}$$

The object A is an algebraic interval since it is stretched under $\Xi_{[-1]}$. By definition, the *factorisation interval* of f is $I_f := i^*A$.

Lemma 4.1.1.6. *Consider the following commutative diagram of simplicial maps*

$$\begin{tikzcd} & \mathcal{B} \arrow[dr, "G"] & \\ \mathcal{A} \arrow[ur, two heads, "S"] \arrow[rr, "F"'] & & \mathcal{C}, \end{tikzcd}$$

where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are intervals, and S is stretched. Then F is stretched if and only if G is stretched.

Proof. One direction is trivial. We assumed that F is stretched, so $F(\mathcal{A}_{[-1]}) \simeq \mathcal{C}_{[-1]}$. This together with the commutativity of the triangle and the stretched condition of S are used in the following calculation:

$$G(\mathcal{B}_{[-1]}) \simeq G(S(\mathcal{A}_{[-1]})) \simeq F(\mathcal{A}_{[-1]}) \simeq \mathcal{C}_{[-1]}.$$

Hence, G is stretched. □

4.1.2 *The decomposition spaces of intervals*

Let $\mathrm{Ar}^s(\mathbf{Int}) \subset \mathrm{Ar}(\mathbf{Int})$ denote the full subcategory spanned by the stretched functors. We have an stretched-culf factorisation system in **Int** that we can use to define a fibration that encodes the simplicial space of sub-divided intervals. Recall that $\mathrm{Ar}^s(\mathbf{Int})$ is a cartesian fibration over **Int** via the domain projection by Lemma 1.1.6.2. We now restrict this cartesian fibration to $\mathbb{\Delta} \subset \mathbf{Int}$

$$\begin{tikzcd} \mathrm{Ar}^s(\mathbf{Int})_{|\mathbb{\Delta}} \arrow[r, "\text{f.f.}"] \arrow[d, "\mathrm{dom}"'] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Ar}^s(\mathbf{Int}) \arrow[d, "\mathrm{dom}"] \\ \mathbb{\Delta} \arrow[r, "\text{f.f.}"'] & \mathbf{Int}. \end{tikzcd}$$

We put

$$\mathcal{U} := \mathrm{Ar}^s(\mathbf{Int})_{|\mathbb{\Delta}}.$$

The *cartesian fibration of subdivided intervals* $\mathcal{U} \to \mathbb{\Delta}$ determines a right fibration $\mathcal{U}^{\mathrm{cart}} \to \mathbb{\Delta}$, and hence by straightening [79, §3.2] a simplicial space

$$U\colon \mathbb{\Delta}^{\mathrm{op}} \to \widehat{\mathcal{S}}$$

where $\widehat{\mathcal{S}}$ is the ∞-category of large ∞-groupoids.

Theorem 4.1.2.1. *[60, Theorem 4.8] The simplicial space $U\colon \mathbb{\Delta}^{\mathrm{op}} \to \widehat{\mathcal{S}}$ is a complete decomposition space.*

The objects of the ∞-groupoid U_n are n-subdivided intervals. That is, an interval $\underline{A}$ equipped with an stretched map $\Delta^n \twoheadrightarrow A$. Note that U_1 is equivalent to the ∞-groupoid $\mathbf{Int}^{\mathrm{eq}}$.

The fibres of the right fibration $\mathcal{U} \to \mathbb{\Delta}$ are large ∞-groupoids. Therefore, U takes values in large ∞-groupoids. This means that U can not be literally an object in the ∞-category of complete decomposition spaces and culf maps.

Remark 4.1.2.2. For κ a regular and strong limit cardinal [65, §5], say that a simplicial space X is κ-bounded, when for each $n \in \mathbb{\Delta}$, the space X_n is κ-small. Hence the ∞-category of κ-bounded decomposition spaces and κ-bounded intervals is essentially κ-small. Carrying the κ-bounded and the Möbius condition through in all the constructions, Gálvez, Kock, and Tonks [60, §6] proved that there is an essentially small ∞-category $(U_{\mathrm{Mob}})_1$ of Möbius intervals, and a legitimate presheaf $U_{\mathrm{Mob}}\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ of Möbius intervals.

Theorem 4.1.2.3. *[60, Theorem 6.14] The decomposition space of subdivided Möbius intervals U_{Mob} is Möbius.*

4.1.3 *Interval construction as a coreflection*

Let **Int** ↓ **cDcmp** denote the comma ∞-category whose objects are simplicial maps $F\colon A \to X$ from an interval A to a decomposition space X.

Theorem 4.1.3.1. *[60, Theorem 5.1] The inclusion functor* $\mathrm{Ar}^{s}(\mathbf{Int}) \hookrightarrow \mathbf{Int} \downarrow \mathbf{cDcmp}$ *has a right adjoint*

$$\mathfrak{I}\colon \mathbf{Int} \downarrow \mathbf{cDcmp} \to \mathrm{Ar}^{s}(\mathbf{Int})$$

which takes cartesian arrows to cartesian arrows.

Remark 4.1.3.2. The functor $\mathfrak{I}\colon \mathbf{Int} \downarrow \mathbf{cDcmp} \to \mathrm{Ar}^{s}(\mathbf{Int})$ sends a simplicial map $F\colon A \to X$ to the stretched map in the following diagram

$$\begin{tikzcd}
A \arrow[rr, "F"] \arrow[dr, "\text{stretched}"'] & & X \\
& A' \arrow[ur, "\text{culf}"'] &
\end{tikzcd}$$

which is obtained from the stretched-culf factorisation of the map F.

Let $\mathbb{\Delta}$ ↓ **cDcmp** denote the full ∞-subcategory of **Int** ↓ **cDcmp** whose objects are maps $\Delta^{n} \to X$ with X a complete decomposition space, and the morphisms are squares

$$\begin{tikzcd}
\Delta^{n} \arrow[r] \arrow[d] & \Delta^{m} \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

where $X \to Y$ is a simplicial map. The adjunction of Theorem 4.1.3.1 restricts as follows:

$$\begin{tikzcd}
\mathbb{\Delta} \downarrow \mathbf{cDcmp} \arrow[r, shift left, "\mathfrak{I}"] \arrow[d, hook] & \mathrm{Ar}^{s}(\mathbf{Int})_{|\mathbb{\Delta}} \arrow[l, shift left] \arrow[d, hook] \\
\mathbf{Int} \downarrow \mathbf{cDcmp} \arrow[r, shift left, "\mathfrak{I}"] & \mathrm{Ar}^{s}(\mathbf{Int}). \arrow[l, shift left]
\end{tikzcd}$$

To simplify the notation, we put $\mathcal{W} := \mathbb{\Delta} \downarrow \mathbf{cDcmp}$. We call this restriction the factorisation interval construction $\mathfrak{I}$. Recall that $\mathcal{U} = \mathrm{Ar}^{s}(\mathbf{Int})_{|\mathbb{\Delta}}$ and note that $\mathfrak{I}$ is a morphism of cartesian fibrations over $\mathbb{\Delta}$:

$$\begin{tikzcd}
\mathcal{W} \arrow[rr, shift left, "\mathfrak{I}"] \arrow[dr, "\text{dom}"'] & & \mathcal{U} \arrow[ll, shift left] \arrow[dl, "\text{dom}"] \\
& \mathbb{\Delta}. &
\end{tikzcd}$$

Let $I\colon W \to U$ denote the simplicial map classified by the map $\mathfrak{I}\colon \mathcal{W} \to \mathcal{U}$. Inside $\mathcal{W}$, we have the fibre over X, for the codomain fibration. This fibre is just $\mathrm{el}(X)$, the category of elements of X. This fibre clearly includes into the cartesian part of $\mathcal{W}$.

Lemma 4.1.3.3. *[60, Lemma 5.3] The associated morphism of right fibrations*

$$\mathrm{el}(X) \to \mathcal{W}^{\mathrm{cart}}$$

is culfy, and by composition we get a culfy map

$$\mathfrak{I}_{X}\colon \mathrm{el}(X) \to \mathcal{W}^{\mathrm{cart}} \to \mathcal{U}^{\mathrm{cart}}.$$

Notation 4.1.3.4. *The simplicial map classified by* $\mathfrak{I}_{X}$ *is denoted as* $I_{X}\colon X \to U$*. Furthermore, the map* I_{X} *is culf since* $\mathfrak{I}_{X}$ *is culfy and there exists a bijection between culf and culfy maps (1.1.2).*

4.2 *The Gálvez–Kock–Tonks conjecture for Möbius decomposition spaces*

We need the following results to prove the main theorem of the paper.

Theorem 4.2.0.1. *[66, Proposition 4.2][30, Theorem 4.3.11] Let $\mathcal{C}$ be an ∞-category. We consider an object $\top$ in $\mathcal{C}$ and write $\mathrm{dom}\colon \mathcal{C}_{/\top} \to \mathcal{C}$ for the canonical projection. Then the following conditions are equivalent:*

1. *The object $\top$ is terminal.*
2. *For any object c in $\mathcal{C}$, the ∞-groupoid $\mathrm{map}_{\mathcal{C}}(c, \top)$ is contractible.*
3. *The map $\mathrm{dom}\colon \mathcal{C}_{/\top} \to \mathcal{C}$ has a section which sends $\top$ to a terminal object in $\mathcal{C}_{/\top}$.*
4. *The map $\mathrm{dom}\colon \mathcal{C}_{/\top} \to \mathcal{C}$ is an equivalence.*

Corollary 4.2.0.2. *[30, Corollary 4.3.13] The terminal objects of an ∞-category $\mathcal{C}$ form an ∞-groupoid which is either empty or equivalent to the point.*

Corollary 4.2.0.3. *[30, Corollary 4.3.8] For any object c in an ∞-category $\mathcal{C}$, the object id_c is terminal in the slice $\mathcal{C}_{/c}$.*

Remark 4.2.0.4. The condition (3) of Theorem 4.2.0.1 is formulated in [30, Theorem 4.3.11] as follows: the map $\mathrm{dom}\colon \mathcal{C}_{/\top} \to \mathcal{C}$ has a section which sends $\top$ to $\mathrm{id}_\top$. The subtle modification we use is valid thanks to Corollary 4.2.0.2.

Let $\mathbf{cDcmp}_{\mathrm{culf}}$ denote the ∞-category of complete decomposition spaces and culf maps. The construction of the complete decomposition space of sub-divided intervals U was motivated by the following statement:

Gálvez–Kock–Tonks Conjecture [60, §5.4] For each complete decomposition space X, the space of culf maps $\mathrm{map}(X, U)$ is contractible.

Since U is not a legitimate object in $\mathbf{cDcmp}_{\mathrm{culf}}$, the conjecture does not assert that U is a terminal object. If we impose a cardinality bound using the Möbius condition, U_{Mob} is a legitimate object in the ∞-category **MobDcmp** of Möbius decomposition spaces and culf maps by Remark 4.1.2.2 and Theorem 4.1.2.3. So, in the Möbius case, the conjecture says that U_{Mob} is a terminal object in **MobDcmp**.

Lemma 4.2.0.5. *For each Möbius decomposition space X, the mapping space*

$$\mathrm{map}_{\mathbf{MobDcmp}}(X, U_{\mathrm{Mob}})$$

is contractible if and only if there exists a section map

$$s\colon \mathbf{MobDcmp} \to \mathbf{MobDcmp}_{/U_{\mathrm{Mob}}}$$

of $\mathrm{dom}\colon \mathbf{MobDcmp}_{/U_{\mathrm{Mob}}} \to \mathbf{MobDcmp}$ *such that* $s(U_{\mathrm{Mob}}) \simeq \mathrm{id}_{U_{\mathrm{Mob}}}$.

Proof. The proof follows from conditions (2) and (3) of Theorem 4.2.0.1. □

The goal of this section is to prove the conjecture in the Möbius case. By Lemma 4.2.0.1, this is equivalent to the following theorem, which we prove.

Theorem 4.2.0.13: There exists a section map

$$s\colon \mathbf{MobDcmp} \to \mathbf{MobDcmp}_{/U_{\mathrm{Mob}}}$$

of $\mathrm{dom}\colon \mathbf{MobDcmp}_{/U_{\mathrm{Mob}}} \to \mathbf{MobDcmp}$ such that $s(U_{\mathrm{Mob}}) \simeq \mathrm{id}_{U_{\mathrm{Mob}}}$.

The proof is a bit technical so we will break it down into a series of steps as follows:

4.2.0.1 *Step 1: $\mathcal{W}$ as an oplax colimit*

Let $\Theta\colon \mathbf{cDcmp} \to \mathbf{Cat}_\infty$ denote the functor that sends a complete decomposition space X to its ∞-category of elements $\mathrm{el}(X)$ (see §1.1.2). Recall that $\mathcal{W} = \mathbb{\Delta} \downarrow \mathbf{cDcmp}$ (see §4.1.3). For each $X \in \mathbf{cDcmp}$, the following diagram is a pullback:

$$\begin{tikzcd}
\mathrm{el}(X) \arrow[r] \arrow[d] & \mathcal{W} \arrow[d, "\mathrm{codom}"] \\
1 \arrow[r, "\ulcorner X \urcorner"'] & \mathbf{cDcmp}.
\end{tikzcd}$$

By similar arguments, it is easy to check that a culf map F is classified by the map $\mathrm{el}(F)$. This implies that $\mathrm{codom}\colon \mathcal{W} \to \mathbf{cDcmp}$ is the cocartesian fibration that classifies the functor Θ. By [55, Theorem 7.4], the cocartesian fibration classified by a functor is given by the oplax colimit of the functor. This implies the following result:

Proposition 4.2.0.6. *The oplax colimit of the functor Θ is $\mathcal{W}$.*

4.2.0.2 *Step 2: Slice $(\infty, 2)$-category over $\mathcal{W}$*

Let $\mathbb{C}\mathbf{at}_\infty$ denote the ∞-bicategory of ∞-categories. The goal of this step is to use Proposition 4.2.0.7 which states that cocartesian fibrations that classify $\mathbf{Cat}_\infty$-values functors induce slice $(\infty, 2)$-categories of $\mathbb{C}\mathbf{at}_\infty$. The arguments used in the proof are technical so we prefer to add an appendix (A) with a deeper explanation, but the idea comes from combining the fact that any 2-colimit induces a slice $(\infty, 2)$-category of $\mathbb{C}\mathbf{at}_\infty$ and cocartesian fibrations that classify $\mathbf{Cat}_\infty$-values functors are weighted colimits [55, Theorem 7.4], which are a special case of 2-colimits as a consequence of results of Gagna, Harpaz, and Lanari [49].

Proposition 4.2.0.7. *Let $\mathcal{C}$ be an ∞-category. Let $H\colon \mathcal{C} \to \mathbf{Cat}_\infty$ be a functor and let $\mathcal{H} \to \mathcal{C}$ be a cocartesian fibration that classifies the map H. There exists a functor*

$$\widehat{H}\colon \mathcal{C} \to \mathbb{C}\mathbf{at}_{\infty//\mathcal{H}},$$

that sends an object c in $\mathcal{C}$ to the map $\iota_c\colon H(c) \to \mathcal{H}$ given by the pullback

$$\begin{tikzcd}
H(c) \arrow[r, "\iota_c"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{H} \arrow[d] \\
1 \arrow[r, "\ulcorner c \urcorner"'] & \mathcal{C},
\end{tikzcd}$$

and a morphism $f\colon c \to b$ in $\mathcal{C}$ to the triangle

$$\begin{tikzcd}
H(c) \arrow[dd, "H(f)"'] \arrow[dr, "\iota_c"] & \\
{} & \mathcal{H} \arrow[l, Rightarrow, shorten=6mm, "\alpha_f"] \\
H(b) \arrow[ur, "\iota_b"'] &
\end{tikzcd}$$

where the 2-cell α_f is defined for each morphism $f\colon c \to b$ by the commutative square

$$\begin{tikzcd}
\iota_c(x) \arrow[r, "\alpha_f(x)"] \arrow[d, "\iota_c(g)"'] & \iota_b \circ H(f)(x) \arrow[d, "\iota_b \circ H(f)(g)"] \\
\iota_c(y) \arrow[r, "\alpha_f(y)"'] & \iota_b \circ H(f)(y).
\end{tikzcd}$$

Here $g\colon x \to y$ *is a morphism in* $H(c)$*. Note that the triangle is just the cocone diagram given by the classifying property of* $\mathcal{H}$ *and the 2-cell is part of the data given by the the classifying property of* $\mathcal{H}$*.*

Combining Propositions 4.2.0.6 and 4.2.0.7, we have a map

$$\widehat{\Theta}\colon \mathbf{cDcmp} \to \mathbf{Cat}_{\infty//\mathcal{W}},$$

that sends a complete decomposition space X to the canonical map $\mathrm{el}(X) \to \mathcal{W}$ and a map $F\colon Y \to X$ to the triangle:

$$\begin{tikzcd}
\mathrm{el}(Y) \arrow[dd, "\mathrm{el}(F)"'] \arrow[dr] & \\
& \mathcal{W} \arrow[l, phantom, "\swarrow\!\!\!\swarrow\ \alpha_F"] \\
\mathrm{el}(X) \arrow[ur] &
\end{tikzcd}$$

where the 2-cell α_F is defined for each $([n], \lambda) \in \mathrm{el}(Y)$ by the commutative square

$$\begin{tikzcd}
\Delta^n \arrow[r, "\mathrm{id}"] \arrow[d, "\lambda"'] & \Delta^n \arrow[d, "F(\lambda)"] \\
Y \arrow[r, "F"'] & X.
\end{tikzcd}$$

4.2.0.3 ***Step 3****: Restriction to Möbius decomposition spaces and culf maps*

The idea of Steps 3 (4.2.0.3) and 4 (4.2.0.4) is to show that the functor $\widehat{\Theta}\colon \mathbf{cDcmp} \to \mathbf{Cat}_{\infty//\mathcal{W}}$ factors through $\mathbf{MobDcmp}_{/\mathcal{U}_{\mathrm{Mob}}}$ using the interval-factorisation construction $I\colon \mathcal{W} \to \mathcal{U}_{\mathrm{Mob}}$ and the straightening-unstraightening equivalence of ∞-categories ($\mathrm{RFib}(\mathbb{\Delta}) \simeq \mathbf{sSpaces}$). After these steps, it is easy to prove that U_{Mob} is a terminal object in $\mathbf{MobDcmp}$.

For any n-simplex $\lambda\colon \Delta^n \to X$ of a decomposition space X, there exists a unique stretched n-simplex $\phi_\lambda\colon \Delta^n \mathrel{\to\!\!\!|} I_\lambda$ such that the digram

$$\begin{tikzcd}
\Delta^n \arrow[rr, "\lambda"] \arrow[dr, "\phi_\lambda"', ->|] & & X \\
& I_\lambda \arrow[ur, "M_\lambda"'] &
\end{tikzcd}$$

commutes. Here ϕ_λ and M_λ are the stretched-culf factorisation of λ.

Lemma 4.2.0.8. *Let* $F\colon Y \to X$ *be a simplicial map between decomposition spaces. For any* n*-simplex* $\lambda\colon \Delta^n \to X$*, we have an stretched functor* I_F^λ *from* I_λ *to* $I_{F\lambda}$ *such that the diagram*

$$\begin{tikzcd}
\Delta^n \arrow[r, "\phi_{F\lambda}"] \arrow[d, "\phi_\lambda"'] & I_{F\lambda} \arrow[d, "M_{F\lambda}"] \\
I_\lambda \arrow[r, "M_\lambda \circ F"'] \arrow[ur, "I_F^\lambda"] & Y.
\end{tikzcd}$$

commutes. If F *is culf then* I_F^λ *is an equivalence.*

Proof. Consider the following commutative diagram induced by the stretched-culf factorisation of λ and $F\lambda$:

$$\begin{tikzcd}
\Delta^n \arrow[r, "\phi_{F\lambda}", ->|] \arrow[d, "\phi_\lambda"', ->|] & I_{F\lambda} \arrow[d, "M_{F\lambda}"] \\
I_\lambda \arrow[r, "M_\lambda \circ F"'] \arrow[ur, dotted, "I_F^\lambda", ->|] & Y.
\end{tikzcd}$$

The dotted arrow I_F^λ exists since we have the stretched-culf factorisation system on **Int** (4.1.1.5). Furthermore, I_F^λ is stretched by Lemma 4.1.1.6 since $\phi_{F\lambda}$ is stretched . If F is culf, the map $M_\lambda \circ F$ is culf. This implies, together with the culf condition of $M_{F\lambda}$, that I_F^λ is culf. Hence, I_F^λ is an equivalence since it is stretched. ☐

Recall that **MobDcmp** is the ∞-subcategory of **cDcmp** consisting of Möbius decomposition spaces and culf maps. Let $\widehat{\Theta}_{|_{\mathbf{MobDcmp}}} \colon \mathbf{MobDcmp} \to \mathfrak{Cat}_{\infty//\mathcal{W}}$ denote the restriction of the map $\widehat{\Theta}\colon \mathbf{cDcmp} \to \mathfrak{Cat}_{\infty//\mathcal{W}}$ to the subcategory **MobDcmp**. In 4.1.3 was defined the interval-factorisation construction $\mathfrak{I}\colon \mathcal{W} \to \mathfrak{U}_{\mathrm{Mob}}$. This construction induces a functor

$$\mathfrak{I}_! \colon \mathfrak{Cat}_{\infty//\mathcal{W}} \to \mathfrak{Cat}_{\infty//\mathfrak{U}_{\mathrm{Mob}}}$$

given by postcomposing with I. Therefore, we have a functor

$$\mathfrak{I}_! \circ \widehat{\Theta}_{|_{\mathbf{MobDcmp}}} \colon \mathbf{MobDcmp} \to \mathfrak{Cat}_{\infty//\mathfrak{U}_{\mathrm{Mob}}}$$

that sends a Möbius decomposition space to the map $\mathfrak{I}_X \colon \mathrm{el}(X) \to \mathfrak{U}^{\mathrm{cart}}_{\mathrm{Mob}}$ and a map $F\colon Y \to X$ to the triangle:

$$\begin{tikzcd}
\mathrm{el}(Y) \arrow[dd, "\mathrm{el}(F)"'] \arrow[dr, "\mathfrak{I}_Y"] \arrow[dd, phantom, shift left=5ex, "I(\alpha_F)\ \Downarrow"] & \\
 & \mathfrak{U}^{\mathrm{cart}}_{\mathrm{Mob}} \\
\mathrm{el}(X). \arrow[ur, "\mathfrak{I}_X"'] &
\end{tikzcd}$$

Lemma 4.2.0.9. *For each culf map* $F\colon Y \to X$, *the* 2*-cell* $I(\alpha_F)$ *is an equivalence.*

Proof. The components of the 2-cell $I(\alpha_F)$ are given by squares:

$$\begin{tikzcd}
\Delta^n \arrow[r, "\mathrm{id}"] \arrow[d, "\phi_\lambda"', ->|] & \Delta^n \arrow[d, "\phi(F\lambda)", ->|] \\
I_\lambda \arrow[r, "I_F^\lambda"', ->|] & I_{(F\lambda)}.
\end{tikzcd}$$

To prove that $I(\alpha_F)$ is an equivalence it is enough to show that for each $\lambda \in X_n$, the stretched map $I_F^\lambda \colon I_\lambda \to I_{F(\lambda)}$ is an equivalence. Indeed, since F is culf we have that I_F^λ is an equivalence by Lemma 4.2.0.8. ☐

To simplify the notation, we use $\mathfrak{g} = \mathfrak{I}_! \circ \widehat{\Theta}_{|_{\mathbf{MobDcmp}}}$. Lemma 4.2.0.9 implies that the image of $\mathfrak{g}$ lands in $\mathbf{Cat}_{\infty/\mathfrak{U}^{\mathrm{cart}}_{\mathrm{Mob}}}$.

Proposition 4.2.0.10. *We can factor* $\mathfrak{g}$ *as*

$$\begin{tikzcd}
 & \mathbf{Cat}_{\infty/\mathfrak{U}^{\mathrm{cart}}_{\mathrm{Mob}}} \arrow[d, hook] \\
\mathbf{MobDcmp} \arrow[ur, "\mathfrak{g}'"] \arrow[r, "\mathfrak{g}"'] & \mathfrak{Cat}_{\infty//\mathfrak{U}_{\mathrm{Mob}}}.
\end{tikzcd}$$

Proof. By Lemma 4.2.0.9, we have that the 2-cells $I(\alpha_F)$ are equivalences so the triangles

$$\begin{tikzcd}
\mathrm{el}(Y) \arrow[dd, "\mathrm{el}(F)"'] \arrow[dr, "\mathfrak{I}_Y"] \arrow[dd, phantom, shift left=5ex, "I(\alpha_F)\ \Downarrow"] & \\
 & \mathfrak{U}^{\mathrm{cart}}_{\mathrm{Mob}} \\
\mathrm{el}(X) \arrow[ur, "\mathfrak{I}_X"'] &
\end{tikzcd}$$

are in fact morphisms in $\mathbf{Cat}_{\infty/\mathfrak{U}^{\mathrm{cart}}_{\mathrm{Mob}}}$. Therefore, $\mathfrak{g}$ factors through $\mathbf{Cat}_{\infty/\mathfrak{U}^{\mathrm{cart}}_{\mathrm{Mob}}}$. The map $\mathfrak{g}'$ denotes the first part of this factorisation. ☐

4.2.0.4 *Step 4: Restriction to right fibrations*

Recall that the map $\mathfrak{I}_X \colon \mathrm{el}(X) \to \mathfrak{U}^{\mathrm{cart}}_{\mathrm{Mob}}$ is culfy by Lemma 4.1.3.3.

Proposition 4.2.0.11. *We can factor* $\mathfrak{g}'$ *as*

$$\begin{tikzcd}
& \mathrm{RFib}\left(\mathbb{\Delta}\right)_{/\mathfrak{U}^{\mathrm{cart}}_{\mathrm{Mob}}} \arrow[d, hook] \\
\mathbf{MobDcmp} \arrow[ur, "\mathfrak{s}'"] \arrow[r, "\mathfrak{g}'"'] & \mathbf{Cat}_{\infty/\mathfrak{U}^{\mathrm{cart}}_{\mathrm{Mob}}}.
\end{tikzcd}$$

Proof. For each culf map $F \colon Y \to X$ in **MobDcmp**, the map $\mathfrak{g}'$ sends F to the top triangle in the following diagram:

$$\begin{tikzcd}
\mathrm{el}(Y) \arrow[rr, "\mathrm{el}(F)"] \arrow[dr, "\mathfrak{I}_Y"] \arrow[ddr] & & \mathrm{el}(X) \arrow[dl, "\mathfrak{I}_X"'] \arrow[ddl] \\
& \mathfrak{U}^{\mathrm{cart}}_{\mathrm{Mob}} \arrow[d] & \\
& \mathbb{\Delta}. &
\end{tikzcd}$$

The vertical arrows are the domain projections and therefore right fibrations. This implies that the top triangle is in fact a morphism in $\mathrm{RFib}\left(\mathbb{\Delta}\right)_{/\mathfrak{U}^{\mathrm{cart}}_{\mathrm{Mob}}}$, and therefore $\mathfrak{g}'$ factors through $\mathrm{RFib}\left(\mathbb{\Delta}\right)_{/\mathfrak{U}^{\mathrm{cart}}_{\mathrm{Mob}}}$. The map $\mathfrak{s}'$ denotes the first part of this factorisation. ☐

Using the straightening-unstraightening equivalence of ∞-categories ($\mathrm{RFib}(\mathbb{\Delta}) \simeq \mathbf{sSpaces}$), we obtain the map

$$s' \colon \mathbf{MobDcmp} \to \mathbf{sSpaces}_{/U_{\mathrm{Mob}}}$$

that sends a Möbius decomposition spaces X to the culf map $I_X \colon X \to U_{\mathrm{Mob}}$. Since in Step 3, we restrict our constructions to culf maps we have the following result:

Proposition 4.2.0.12. *We can factor* s' *as*

$$\begin{tikzcd}
& \mathbf{MobDcmp}_{/U_{\mathrm{Mob}}} \arrow[d, hook] \\
\mathbf{MobDcmp} \arrow[ur, "s"] \arrow[r, "s'"'] & \mathbf{sSpaces}_{/U_{\mathrm{Mob}}}.
\end{tikzcd}$$

Proof. Since s' is obtained from the straightening construction of the map $\mathfrak{s}'$, we have that s' sends a culf map $F \colon Y \to X$ to the triangle

$$\begin{tikzcd}
Y \arrow[rr, "F"] \arrow[dr, "I_Y"'] & & X \arrow[dl, "I_X"] \\
& U_{\mathrm{Mob}} &
\end{tikzcd}$$

which is a morphism in $\mathbf{MobDcmp}_{/U_{\mathrm{Mob}}}$ since I_X and I_Y are culf. Therefore, s' factors through $\mathbf{MobDcmp}_{/U_{\mathrm{Mob}}}$. Let s denotes the first part of this factorisation. ☐

Theorem 4.2.0.13. *The map* $s \colon \mathbf{MobDcmp} \to \mathbf{MobDcmp}_{/U_{\mathrm{Mob}}}$ *is a section of the map* $\mathrm{dom} \colon \mathbf{MobDcmp}_{/U_{\mathrm{Mob}}} \to \mathbf{MobDcmp}$ *such that* $s(U_{\mathrm{Mob}}) \simeq \mathrm{id}_{U_{\mathrm{Mob}}}$.

Proof. We will first show that s is a section map and then prove that $s(U_{\mathrm{Mob}}) \simeq \mathrm{id}_{U_{\mathrm{Mob}}}$.

- Let $F\colon Y \to X$ be a culf map. By definition of s (4.2.0.12), $s(F)$ is equal to triangle

  $$\begin{tikzcd} Y \arrow[rr, "F"] \arrow[dr, "I_Y"'] & & X \arrow[dl, "I_X"] \\ & U_{\mathrm{Mob}}. & \end{tikzcd}$$

  The domain map applied to the above triangle consists of deleting U_{Mob} and retaining the map F. In other words, $\mathrm{dom}(s(F)) = F$ and hence s is a section map of dom.

- Note that $s(U_{\mathrm{Mob}}) = I_{U_{\mathrm{Mob}}}\colon U_{\mathrm{Mob}} \to U_{\mathrm{Mob}}$ sends an n-simplex $\lambda\colon \Delta^n \rightarrow\!\!\!\shortmid A$ to the map $\phi_\lambda\colon \Delta^n \rightarrow\!\!\!\shortmid I_\lambda$ given by the stretched-culf factorisation pictured in the following diagram:

  $$\begin{tikzcd} \Delta^n \arrow[rr, "\lambda"] \arrow[dr, "\phi_\lambda"'] & & A \\ & I_\lambda. \arrow[ur, "M_\lambda"'] & \end{tikzcd}$$

  Since λ and ϕ_λ are stretched, we have that M_λ is stretched by Lemma 4.1.1.6. This implies that M_λ is an equivalence since M_λ is culf. Therefore, $I_{U_{\mathrm{Mob}}}(\lambda) \simeq \lambda$ and $s(U_{\mathrm{Mob}}) \simeq \mathrm{id}_{U_{\mathrm{Mob}}}$.

☐

Combining Theorem 4.2.0.13 and Lemma 4.2.0.5, we have a proof of the Gálvez–Kock–Tonks conjecture for Möbius decomposition spaces.

Theorem 4.2.0.14. *The decomposition space of subdivided Möbius intervals* U_{Mob} *is a terminal object in the* ∞*-category* **MobDcmp** *of Möbius decomposition spaces and culf maps .*

Combining Theorems 4.2.0.1 and 4.2.0.14, we have the following result which we will use in Section 4.3.

Corollary 4.2.0.15. *The canonical projection* $\mathrm{dom}\colon \mathbf{MobDcmp}_{/U_{\mathrm{Mob}}} \to \mathbf{MobDcmp}$ *is an equivalence.*

4.2.1 *Comparison with the proof of the locally discrete case*

As we mention before, the fibres of the right fibration $\mathcal{U} \to \mathbb{\Delta}$ are large ∞-groupoids. This means that U can not be literally an object in the ∞-category of complete decomposition spaces and culf maps. We have several alternatives to deal with this problem. One of them is the construction of a kind of neighbourhood $U_X \subset U$ around the intervals of a complete decomposition space X.

Let $\mathcal{U}_X$ denote the full simplicial space of $\mathcal{U}$, whose objects are stretched maps $\phi_\lambda\colon \Delta^n \rightarrow\!\!\!\shortmid I_\lambda$ for some $\lambda \in X_n$. Let $U_X\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$ denote the simplicial space classified by the right fibration $\mathrm{dom}\colon \mathcal{U}_{\mathcal{X}} \to \mathbb{\Delta}$. Since right fibrations are stable under pullback, the map $U_X \to U$ is cartesian and therefore culf. This implies that U_X is a complete decomposition space since U is a complete decomposition space by Lemma 1.1.2.3. Moreover, the canonical inclusion from $\mathcal{U}_X$ to $\mathcal{U}^{\mathrm{cart}}$ induces a full and faithful functor of mapping spaces for any decomposition space as a consequence of Lemma 1.1.5.2.

Proposition 4.2.1.1. *For any decomposition space* X*, the inclusion map* $U_X \hookrightarrow U$ *induces a full and faithful functor* $\mathrm{map}(X, U_X) \to \mathrm{map}(X, U)$.

Gálvez, Kock and Tonks [60] proved a partial result of the conjecture given by the following result:

Theorem 4.2.1.2. *[60, Theorem 5.5] For any complete decomposition space* X*, the mapping space* $\mathrm{map}_{\mathbf{cDcmp}_{\mathrm{culf}}}(X, U)$ *of culf maps from* X *to* U *is connected. More precisely, for any culf map* $J\colon X \to U$*, we have that* $J \simeq I$*.*

Corollary 4.2.1.3. *For any decomposition space* X*. The canonical inclusion* $\jmath\colon U_X \hookrightarrow U$ *induces a essentially surjetive on objects map* $\mathrm{map}_{\mathbf{cDcmp}_{\mathrm{culf}}}(X, U_X) \to \mathrm{map}_{\mathbf{cDcmp}_{\mathrm{culf}}}(X, U)$*.*

Proof. For any culf map J in $\mathrm{map}_{\mathbf{cDcmp}_{\mathrm{culf}}}(X, U)$, we have that $J \simeq I$ by Theorem 4.2.1.2. On the other hand, we have a canonical factorisation

$$\begin{tikzcd}
X \arrow[rr, "I"] \arrow[dr, "I_X"'] & & U \\
& U_X. \arrow[ur, hook, "\jmath"'] &
\end{tikzcd}$$

So, it is easy to see that $\jmath(I_X) \simeq J$. □

Combining Proposition 4.2.1.1 and Corollary 4.2.1.3, we have the following result:

Theorem 4.2.1.4. *For any complete decomposition space* X*, we have the equivalence*

$$\mathrm{map}_{\mathbf{cDcmp}_{\mathrm{culf}}}(X, U_X) \simeq \mathrm{map}_{\mathbf{cDcmp}_{\mathrm{culf}}}(X, U).$$

The most important consequence of Theorem 4.2.1.4 is that $\mathrm{map}_{\mathbf{cDcmp}_{\mathrm{culf}}}(X, U)$ is contractible if and only if $\mathrm{map}_{\mathbf{cDcmp}_{\mathrm{culf}}}(X, U_X)$ is contractible. In Chapter 3, it was proved the conjecture at the level of locally discrete decomposition spaces. With this strict restriction, the contractibility of $\mathrm{map}(X, U_X)$ can be established with 2-category theory by showing that $I_X\colon X \to U_X$, interpreted as a natural transformation, does not admit other self-modifications than the identity modification. The following is the main lemma of Chapter 3.

Lemma 3.5.1.4: For each locally discrete decomposition space X, the mapping space $\mathrm{map}_{\mathbf{cDcmp}_{\mathrm{culf}}}(X, U_X)$ is contractible.

Combining Theorem 4.2.1.4 and Lemma 3.5.1.4, we have a proof of the Gálvez–Kock–Tonks conjecture for locally discrete decomposition spaces:

Theorem 3.5.1.5: For each locally discrete decomposition space X, the mapping space $\mathrm{map}_{\mathbf{cDcmp}_{\mathrm{culf}}}(X, U)$ is contractible.

As we mention in the introduction of this chapter it is natural to ask whether the techniques developed in Chapter 3 can be applied or refined to prove the conjecture in full generality. Unfortunately this is not very likely, since the proof relies on explicit strictification.

4.3 *The ∞-topos of Möbius decomposition spaces and culf maps*

Hackney and Kock [63] proved that for any simplicial space X the ∞-category of culf maps over X is equivalent to the ∞-category of right fibrations over the edgewise subdivision of X. A consequence of this result is that the ∞-category of decomposition spaces and culf maps is locally an ∞-topos. In this section, we will explain the locally ∞-topos condition of $\mathbf{cDcmp}_{\mathrm{culf}}$, and combining with the proof of the Gálvez–Kock–Tonks conjecture in the Möbius case that state that **MobDcmp** has a terminal object, we will show that the ∞-category of Möbius decomposition spaces and culf maps is an ∞-topos (Theorem 4.3.4.1). We recall from [63] some results required to prove Theorem 4.3.4.1.

4.3.1 *Edgewise subdivision*

Consider the functor $Q\colon \mathbb{\Delta} \to \mathbb{\Delta}$ that sends an ordinal $[n]$ to the ordinal $[n]^{\mathrm{op}} \star [n] = [2n+1]$, with the following special notation for the elements of the ordinal $[2n+1]$,

$$\begin{tikzcd}
0 \arrow[r] & 1 \arrow[r] & \cdots \arrow[r] & n \\
0' \arrow[u] & 1' \arrow[l] & \cdots \arrow[l] & n'. \arrow[l]
\end{tikzcd}$$

The functor Q is described on arrows by sending a coface map $d^i\colon [n-1] \to [n]$ to the monotone map that omits the elements i and i', and by sending a codegeneracy map $s^i\colon [n] \to [n+1]$ to the monotone map that repeats both i and i'.

Definition 4.3.1.1. For a simplicial space $X\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}$, the *edgewise subdivision* $\mathrm{Sd}(X)$ is given by precomposing with Q:

$$\mathrm{Sd}(X) := Q^*X = X \circ Q.$$

Remark 4.3.1.2. At the level of right fibrations over $\mathbb{\Delta}$, the edgewise subdivision is the pullback

$$\begin{tikzcd}
Q^*(\mathrm{el}(X)) \arrow[r] \arrow[d] & \mathrm{el}(X) \arrow[d] \\
\mathbb{\Delta} \arrow[r, "Q"'] & \mathbb{\Delta}.
\end{tikzcd}$$

This means that we have the identification

$$Q^*(\mathrm{el}\, X) = \mathrm{el}(\mathrm{Sd}\, X).$$

Decomposition spaces can be characterised in terms of edgewise subdivision, by a result of Begner, Osorno, Ozornova, Rovelli, and Scheimbauer [15] :

Lemma 4.3.1.3. *[15] A simplicial space X is a decomposition space if and only if $\mathrm{Sd}(X)$ is a Segal space.*

Furthermore, culf maps can be characterised in terms of edgewise subdivision, by a result of Hackney and Kock [63]:

Lemma 4.3.1.4. *[63, Lemma 5.3] A simplicial map f is culf if and only if $\mathrm{Sd}(f)\colon \mathrm{Sd}(Y) \to \mathrm{Sd}(X)$ is a right fibration.*

A simplicial map is called *final* if it is left orthogonal to every right fibration. Note that every terminal-object-preserving map between representables $\ell\colon \Delta^m \to \Delta^n$ is final. Hackney and Kock [63] gave a series of adjunctions for a map using the final-right fibration factorisation system in $\mathbf{Cat}_\infty$ that in the case of the endofunctor $Q\colon \mathbb{\Delta} \to \mathbb{\Delta}$ gives the adjunctions:

$$\begin{tikzcd}
\mathbf{sSpaces} \arrow[r, "Q^*" description] & \mathbf{sSpaces}. \arrow[l, bend right=30, "Q_!"'] \arrow[l, bend left=30, "Q_*"] \arrow[l, phantom, shift right=2.5, "\perp"] \arrow[l, phantom, shift left=2.5, "\perp"]
\end{tikzcd}$$

Recall that $Q^*(\mathrm{el}(X)) = \mathrm{el}(\mathrm{Sd}(X))$.

Proposition 4.3.1.5. *[63, Proposition 8.4] The right Kan extension functor $Q_*\colon$ **sSpaces** $\to$ **sSpaces** takes right fibrations to culf maps.*

Proof. Let $p\colon Y \to X$ be a right fibration. The map $Q_*(p)$ is culf when it is right orthogonal to every active map $\Delta^m \rightarrow\!\!\!\shortmid \Delta^n$, so we have to prove that for the square:

$$\begin{tikzcd}
\Delta^m \arrow[r] \arrow[d, "(1)" phantom, shift left=40] \arrow[d] & Q_*(Y) \arrow[d, "Q_*(p)"] \\
\Delta^n \arrow[r] & Q_*(X)
\end{tikzcd}$$

the space of fillers is contractible. By the adjunction $Q^* \dashv Q_*$, the right orthogonality condition of (1) is equivalent to say that the space of fillers of the square

$$\begin{tikzcd}
Q^*\Delta^m \arrow[r] \arrow[d] \arrow[dr, phantom, "(2)"] & Y \arrow[d, "p"] \\
Q^*\Delta^n \arrow[r] & X
\end{tikzcd}$$

is contractible. Since the edgewise subdivision functor Q^* takes active maps $\Delta^m \rightarrow\!\!\!\shortmid \Delta^n$ to terminal-object-preserving maps, which are then final, we have that the map $Q^*\Delta^m \to Q^*\Delta^n$ is final. Hence, it is orthogonal to p by the right fibration condition of p. This implies that the space of fillers is contractible for the square (2). Therefore, $Q_*(p)$ is culf. □

4.3.2 *Slicing adjunctions*

Given an adjunction

$$\begin{tikzcd}
\mathcal{D} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "G"]
\end{tikzcd}$$

where $\mathcal{D}$ has pullbacks, we can obtain an adjunction

$$\begin{tikzcd}
\mathcal{D}_{/d} \arrow[r, bend left, "F_d"] \arrow[r, phantom, "\perp"] & \mathcal{C}_{/Fd} \arrow[l, bend left]
\end{tikzcd}$$

whose right adjoint is given by applying $G_{Fd}\colon \mathcal{C}_{/Fd} \to \mathcal{D}_{/GFd}$ and then pulling back along the unit $\eta_d\colon d \to GFd$ (See [79], Proposition 5.2.5.1).

Lemma 4.3.2.1. *[63] For a decomposition space X, we have the adjunction*

$$\begin{tikzcd}
\mathbf{Dcmp}_{\mathrm{culf}/X} \arrow[r, bend left, "\mathrm{Sd}_X"] \arrow[r, phantom, "\perp"] & \mathrm{RFib}(\mathrm{Sd}\,X). \arrow[l, bend left, "(\eta'_X)^*\circ Q_*"]
\end{tikzcd}$$

Proof. The adjunction $Q^* \dashv Q_*$ induces the sliced adjunction

$$\begin{tikzcd}
\mathbf{sSpaces}_{/X} \arrow[r, bend left, "\mathrm{Sd}_X"] \arrow[r, phantom, "\perp"] & \mathbf{sSpaces}_{/\mathrm{Sd}\,X}. \arrow[l, bend left, "(\eta'_X)^*\circ Q_*"]
\end{tikzcd}$$

By Lemma 4.3.1.4, the map Sd_X sends culf maps to right fibrations and by Lemma 4.3.1.5 takes right fibrations to culf maps. So this adjunction restricts to an adjunction

$$\begin{tikzcd}
\mathbf{Dcmp}_{\mathrm{culf}/X} \arrow[r, bend left, "\mathrm{Sd}_X"] \arrow[r, phantom, "\perp"] & \mathrm{RFib}(\mathrm{Sd}\,X). \arrow[l, bend left, "(\eta'_X)^*\circ Q_*"]
\end{tikzcd}$$

□

Remark 4.3.2.2. The right adjoint of Lemma 4.3.2.1 acts on a given right fibration $W \to Q^*X$ by first applying Q_* to get a culf map $Q_*W \to Q_*Q^*X$ (by Proposition 4.3.1.5), and then pulling back along the unit η'_X of the $Q^* \dashv Q_*$ adjunction to get a culf map $Y \to X$ as in

$$\begin{tikzcd} Y \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & Q_*W \arrow[d, two heads] \\ X \arrow[r, "\eta'_X"'] & Q_*Q^*X. \end{tikzcd}$$

Hackney and Kock [63] proved that the adjunction of Lemma 4.3.2.1 is in fact an adjoint equivalence.

Theorem 4.3.2.3. *[63, Theorem 8.12] For any decomposition space* X*, there is a natural equivalence*

$$\mathbf{Dcmp}_{\mathrm{culf}/X} \simeq \mathrm{RFib}(\mathrm{Sd}\, X).$$

4.3.3 *Rezk completion*

Let X be a decomposition space. Let $X_1^{\mathrm{eq}} \subset X_1$ denote the full sub ∞-groupoid spanned by those $f \colon x \to y$ for which there exists $\sigma, \tau \in X_2$ with $d_0\sigma \simeq f$ and $d_1\sigma \simeq s_0 y$ and $d_2\tau \simeq f$ and $d_1\tau \simeq s_0 x$.

Definition 4.3.3.1. [59] A decomposition space X is called *Rezk complete* when the canonical map $s_0 \colon X_0 \to X^{\mathrm{eq}}$ is a homotopy equivalence.

Remark 4.3.3.2. If X is a Segal space, then Definition 4.3.3.1 is equivalent to the original definition of Rezk [83].

To prove that every Möbius decomposition space is Rezk complete, we need the following result:

Lemma 4.3.3.3. *For any Möbius decomposition space* X*, the following diagram*

$$\begin{tikzcd} X_0 \arrow[r, "\mathrm{id}"] \arrow[d, "s_0 s_0"'] & X_0 \arrow[d, "s_0"] \\ X_2 \arrow[r, "d_1"'] & X_1 \end{tikzcd}$$

is a pullback.

Proof. Given an object x in X_0, the ∞-groupoid $\mathrm{Fib}_x(s_0)$ is contractible. So to prove that the square is a pullback, it is enough to prove that $\mathrm{Fib}_{s_0 x}(s_0 s_0)$ is contractible. The ∞-groupoid $\mathrm{Fib}_{s_0 x}(s_0 s_0)$ is non-empty since $s_0 s_0(x)$ is an object in $\mathrm{Fib}_{s_0 x}(s_0 s_0)$. Now suppose we have another object $\sigma \in \mathrm{Fib}_{s_0 x}(s_0 s_0)$ such that $\sigma \neq s_0 s_0(x)$, this means that $d_0(\sigma)$ or $d_2(\sigma)$ is non-degenerate. Combining that $d_0(\sigma)$ or $d_2(\sigma)$ is non-degenerate with the decomposition axiom satisfying X, we can construct infinite non-degenerate n-simplex with long edge $d_0(\sigma)$ or $d_2(\sigma)$. This implies that the long-edge map

$$\sum \vec{X}_r \to X_1$$

is not finite, but this contradicts the Möbius condition of X. In other words, the Möbius condition of X forces σ to be totally degenerate and equivalent to $s_0 s_0(x)$. Hence, $\mathrm{Fib}_{s_0 x}(s_0 s_0)$ is contractible. ☐

Proposition 4.3.3.4. *[59, §8] Every Möbius decomposition space is Rezk complete.*

Proof. Let X be a Möbius decomposition space. Recall that every Möbius decomposition space is complete, then $s_0 \colon X_0 \to X_1$ is a monomorphism. Since $X^{eq} \to X_1$ is a monomorphism by construction, to show that X_1^{eq} and X_0 are equivalents, it is enough to show that every object $f \colon x \to y$ in X_1^{eq} is degenerate. But if $\sigma \in X_2$ exists with $d_1(\sigma) = s_0(y)$ and $d_0(\sigma) = f$, as in definition of X_1^{eq}, then Lemma 4.3.3.3 implies that f is degenerate. ☐

There is a full and faithful nerve functor

$$\begin{tikzcd} N_{nat} \colon \mathbf{Cat}_\infty \arrow[r] & \mathbf{sSpaces} \\ \mathcal{C} \arrow[r, mapsto] & \mathrm{map}(-, \mathcal{C}) \end{tikzcd}$$

whose essential image is the subcategory of Rezk-complete Segal spaces **CSegal** [68]. Furthermore, for each Segal space X, we can construct a Rezk-complete Segal space $\widehat{X}$. This construction is in fact functorial and gives a map $L_{Rezk} \colon \mathbf{Segal} \to \mathbf{CSegal}$ that is left adjoint to the inclusion map $\mathbf{CSegal} \hookrightarrow \mathbf{Segal}$ (due to [83], see [7] for a model-independent statement). Let $\widehat{X}$ denote the Rezk completion of a Segal space X. The unit $\eta_X^{L_{Rezk}} \colon X \to \widehat{X}$, we will call it *the completion map*.

Proposition 4.3.3.5. *[63, Proposition 9.4] Suppose X is a Segal space. Then pulling back along the completion map $X \to \widehat{X}$ induces an equivalence* $\mathrm{RFib}(\widehat{X}) \to \mathrm{RFib}(X)$.

Theorem 4.3.3.6. *[63, Theorem 9.3] The ∞-category of decomposition spaces and culf maps is locally an ∞-topos. More precisely, for X a decomposition space, we have an equivalence*

$$\mathbf{Dcmp}_{culf/X} \simeq \mathrm{RFib}(\mathrm{Sd}\,X) \simeq \mathrm{RFib}(\widehat{\mathrm{Sd}\,X}) \simeq \mathbf{PrSh}(\widehat{\mathrm{Sd}\,X}).$$

Proof. The first equivalence is Theorem 4.3.2.3. The second equivalence is a consequence of Proposition 4.3.3.5. The last equivalence is the straightening-unstraightening equivalence of ∞-categories. ☐

Remark 4.3.3.7. In the last equivalence of Theorem 4.3.3.6, we use the straightening-unstraightening equivalence of ∞-categories. If we work with $\mathrm{Sd}(X)$ this is not possible since Segal spaces are not a model for ∞-categories whereas Rezk complete Segal spaces are. Therefore, it becomes necessary to take Rezk completion.

Proposition 4.3.3.8. *[63, Proposition 9.10] If X is a Rezk complete decomposition space, and $Y \to X$ is culf, then also Y is a Rezk-complete decomposition space.*

Remark 4.3.3.9. Note that Proposition 4.3.3.8 forces the decomposition space of sub-divided intervals U not to be Rezk complete. This happens since we have for any complete decomposition space Y a culf map $I_Y \colon Y \to U$. So if U were Rezk complete, Proposition 4.3.3.8 would imply that Y is Rezk complete, i.e., that any complete decomposition space is Rezk complete, which is not true by [59, §2.2].

The following result is needed in the proof of the main theorem of this section (4.3.4.1).

Proposition 4.3.3.10. *[63, Proposition 9.12] If X is a Rezk complete decomposition space, then* $\mathrm{Sd}(X)$ *is a Rezk complete Segal space.*

4.3.4 *Toposes*

An ∞-topos is an ∞-category $\mathcal{X}$ that arises as a left exact-localisation of an ∞-category of presheaves [79, Definition 6.1.0.4]. (There are several ways to think about what is an ∞-topos, for example [88], [5].) One of the fundamental persistence properties is that if $\mathcal{X}$ is an ∞-topos, for every object x in $\mathcal{X}$, the slice ∞-category $\mathcal{X}_{/x}$ is an ∞-topos [79, Proposition 6.3.5.1], i.e. $\mathcal{X}$ is locally an ∞-topos. The converse is not always true. This means not every ∞-category that is locally an ∞-topos is an ∞-topos. The converse is true when the ∞-category has a terminal object. Recall that **MobDcmp** is the ∞-category of Möbius decomposition spaces and culf maps.

Theorem 4.3.4.1. *The ∞-category of Möbius decomposition spaces and culf maps is an ∞-topos.*

Proof. By Proposition 4.3.3.4, $\mathcal{U}_{\mathrm{Mob}}$ is Rezk complete since it is Möbius, and therefore $\mathrm{Sd}(\mathcal{U}_{\mathrm{Mob}})$ is Rezk complete by Proposition 4.3.3.10. Moreover, we have the equivalence

$$\mathbf{MobDcmp} \simeq \mathbf{MobDcmp}_{/\mathcal{U}_{\mathrm{Mob}}} \simeq \mathbf{PrSh}(\mathrm{Sd}(\mathcal{U}_{\mathrm{Mob}})).$$

The first equivalence is Corollary 4.2.0.15, the other equivalence is a consequence of Theorem 4.3.3.6 and the Rezk complete property of $\mathrm{Sd}(\mathcal{U}_{\mathrm{Mob}})$. □

A

Oplax colimits as weighted out-colimits

In the ∞-category context, we usually encode the information of diagrams of ∞-categories, indexed by an ∞-category with a fibration. In the context of diagrams of ∞-bicategories, indexed by an ∞-bicategory, we have four main notions of fibrations since we have to encode the change of direction of 2-cells and not just 1-cells. (Op)lax colimits in the ∞-context were studied first by Gepner, Haugseng, and Nikolaus [55], with further contributions provided by Berman [18], García [51] and García–Stern [52, 53]. Here we follow the ideas of Gagna, Harpaz, and Lanari [49]. They identified these four types of fibrations as inner (co)cartesian and outer (co)cartesian based on Lurie's work [80]. This identification allowed them to define 2-(co)limits for diagrams taking values in an ∞-bicategory. (They call them inner and outer (co)limits.) Furthermore, they proved under some technical conditions that weighted 2-(co)limits, and consequently all 2-(co)limits, can be computed in terms of weighted homotopy (co)limits. A directed consequence of Gagna–Harpaz–Lanari's work is that the oplax colimits in the sense of Gepner–Haugseng–Nikolaus [55] are a special case of outer-colimits [49, Proposition 5.2.3, Remark 5.2.5]. The goal of this appendix is to prove the following proposition using outer-colimits:

Proposition 4.2.0.7: Let $\mathcal{C}$ be an ∞-category. Let $H\colon \mathcal{C} \to \mathbf{Cat}_\infty$ be a functor and let $\mathcal{H} \to \mathcal{C}$ be a cocartesian fibration that classifies the map H. There exists a functor

$$\widehat{H}\colon \mathcal{C} \to \mathbf{Cat}_{\infty//\mathcal{H}},$$

that sends an object c in $\mathcal{C}$ to the map $\iota_c\colon H(c) \to \mathcal{H}$ given by the pullback

$$\begin{tikzcd}
H(c) \arrow[r, "\iota_c"] \arrow[d] & \mathcal{H} \arrow[d] \\
1 \arrow[r, "\ulcorner c \urcorner"'] & \mathcal{C},
\end{tikzcd}$$

and a morphism $f\colon c \to b$ in $\mathcal{C}$ to the triangle:

$$\begin{tikzcd}
H(c) \arrow[dd, "H(f)"'] \arrow[dr, "\iota_c"] & \\
{} & \mathcal{H} \arrow[l, Rightarrow, "\alpha_f"] \\
H(b) \arrow[ur, "\iota_b"'] &
\end{tikzcd}$$

where the 2-cell α_f is defined for each morphism $f\colon c \to b$ by the commutative square

$$\begin{tikzcd}
\iota_c(x) \arrow[r, "\alpha_f(x)"] \arrow[d, "\iota_c(g)"'] & \iota_b \circ H(f)(x) \arrow[d, "\iota_b \circ H(f)(g)"] \\
\iota_c(y) \arrow[r, "\alpha_f(y)"'] & \iota_b \circ H(f)(y).
\end{tikzcd}$$

Here $g\colon x \to y$ is a morphism in $H(c)$. Note that the triangle is just the cocone diagram given by the classifying property of $\mathcal{H}$ and the 2-cell is part of the data given by the the classifying property of $\mathcal{H}$.

In the first place, we will show that cocartesian fibrations that classify $\mathbf{Cat}_\infty$-values functors are oplax colimits which are a special case of weighted colimits taking values in $\mathbf{Cat}_\infty$ [55, Theorem 7.4]. After that, we will see that weighted colimits, which take values in $\mathbf{Cat}_\infty$, can be interpreted as 2-colimits landing in $\mathbb{C}\mathbf{at}_\infty$ (A.0.0.6). Finally we will use the fact that 2-colimits induce slice $(\infty, 2)$-categories of $\mathbb{C}\mathbf{at}_\infty$ to describe the slice $(\infty, 2)$-category induced by a cocartesian fibration that classifies a $\mathbf{Cat}_\infty$-value functor.

Remark A.0.0.1. Dealing with out-colimits and oplax colimits help in the proof of the conjecture since we construct the section map s (§4.2) using the property that cocartesian fibrations that classify functors, which take values in $\mathbf{Cat}_\infty$, induce slice $(\infty, 2)$-categories of $\mathbb{C}\mathbf{at}_\infty$.

We will use some results that are too technical to explain in detail in a few lines so we reference them precisely. Assume that $\mathcal{C}$ is an ∞-category. Let $H\colon \mathcal{C} \to \mathbf{Cat}_\infty$ be a functor and let $\mathcal{H} \to \mathcal{C}$ be a cocartesian fibration that classifies the map H. In this section, we follow the terminology of Gagna, Harpaz, and Lanari [49]. Let $\mathcal{C}_{-/}\colon \mathcal{C} \to \mathbf{Cat}_\infty$ denote the functor that sends an object x in $\mathcal{C}$ to the ∞-category $\mathcal{C}_{x/}$.

Definition A.0.0.2. [55, Definition 2.9] Let $\mathcal{C}$ be an ∞-category and let $F\colon \mathcal{C} \to \mathbf{Cat}_\infty$ be a functor. The *oplax colimit* of F is the colimit of F weighted by $\mathcal{C}_{-/}$ i.e.

$$\operatorname{colim}_{\mathrm{Tw}(\mathcal{C})} F(-) \times \mathcal{C}_{-/}.$$

Here $\mathrm{Tw}(\mathcal{C})$ denotes the twisted arrow ∞-category of $\mathcal{C}$.

Lemma A.0.0.3. *[55, Theorem 7.4] The oplax colimit of the functor $H\colon \mathcal{C} \to \mathbf{Cat}_\infty$ is given by the cocartesian fibration $\mathcal{H}$ classified by H.*

There are several models for $(\infty, 2)$-categories, and all of them have been proven to be equivalent in the works of Ara ([6]), Barwick–Schommer-Pries ([9]), Bergner–Rezk ([16, 17]), Gagna–Harpaz–Lanari ([48]), Lurie ([80]) and others. We will focus in Lurie's bicategorical model structure on scaled simplicial sets. A scaled simplicial set is a pair formed by a simplicial set with a subset of 2-simplices. An ∞-bicategory is a scaled simplicial set which admits extensions along generating scaled anodyne maps [49, Definition 1.2.7].

Before introducing the concept of out-colimit, we need the notion of out-coslice simplicial set. Let K and C be scaled simplicial sets, and let $p\colon K \to C$ be an arbitrary simplicial map. The *out-coslice* simplicial set $C^{\mathrm{out}}_{p/}$ is characterised by the mapping property of the form

$$\mathrm{Hom}_{\mathbf{Set}^{+,sc}}(X, C^{\mathrm{out}}_{p/}) \simeq \mathrm{Hom}_p(K \diamond_{\mathrm{out}} X, C).$$

Here $\diamond_{\mathrm{out}}$ is the fat join equipped with some addition decorations which encodes the "laxness" [49, §4.2] and $\mathbf{Set}^{+,sc}$ denotes the category of marked-scaled simplicial sets.

Definition A.0.0.4. [49, Definition 5.1.2] Let $\mathcal{K}$ and $\mathcal{C}$ be ∞-bicategories and let $p\colon \mathcal{K} \to \mathcal{C}$ be a functor. An *out-colimit* for p is an initial object of $\mathcal{C}^{\mathrm{out}}_{p/}$.

Remark A.0.0.5. Gagna, Harpaz, and Lanari [49] defined out-colimits in terms of marked-scaled simplicial sets. A *marked-scaled simplicial set* is a pair formed by a scaled simplicial set with a subset of 1-simplices. In this section we only work with marked-scaled simplicial sets whose marked edges are only the degenerate edges since they correspond to (op)lax cones. We chose to omit the word marked since we work with the canonical marked structure given by the degenerate edges.

Since any ∞-category is an ∞-bicategory, we have that $\mathbf{Cat}_\infty$ is an subcategory of $\mathbb{C}\mathbf{at}_\infty$. Let $\jmath\colon \mathbf{Cat}_\infty \to \mathbb{C}\mathbf{at}_\infty$ denote the canonical inclusion functor. Adapting the notion of weighted out-colimit proposed by Gagna, Harpaz, and Lanari [49, Definition 5.2.1], we have the following notion:

Definition A.0.0.6. The out-colimit of $H\colon \mathcal{C} \to \mathbf{Cat}_\infty$ weighted by $\mathcal{C}_{-/}$ is defined as the out-colimit of

$$\overline{H}\colon \operatorname{Ar}(\mathcal{C}) \to \mathbb{C}\mathbf{at}_\infty,$$

where $\overline{H} = \jmath \circ H \circ \operatorname{dom}$ and $\operatorname{dom}\colon \operatorname{Ar}(\mathcal{C}) \to \mathcal{C}$ is the canonical cartesian fibration.

Remark A.0.0.7. Gagna, Harpaz, and Lanari [49, Proposition 5.2.3, Remark 5.2.5] characterized weighted out-colimits as follows: Let $\mathcal{C}$ be an ∞-bicategory and let $f\colon \mathcal{J} \to \mathcal{C}$ and $w\colon \mathcal{J}^{\mathrm{op}} \to \mathbb{C}\mathbf{at}_\infty$ be functors. Then the out-colimit $c \in \mathcal{C}$ of f weighted by w is characterised by the natural equivalence of ∞-categories

$$\operatorname{Hom}_{\mathcal{C}}(c, x) \simeq \operatorname{Nat}_{\mathcal{J}^{\mathrm{op}}}(w(-), \operatorname{Hom}_{\mathcal{C}}(f(-), x)).$$

Gagna, Harpaz, and Lanari [49, Proposition 5.2.3, Remark 5.2.5] proved that the notion of oplax colimit and out-colimit (in the sense of [49, Definition 5.2.1]) are the same when we work with ∞-categories. Therefore, we have the following result:

Lemma A.0.0.8. *The out-colimit of* $\overline{H}$ *is* $\mathcal{H}$.

Proof. By definition of $\overline{H}$, for each ∞-category $\mathcal{X}$ we have that

$$\operatorname{Fun}_{\mathbb{C}\mathbf{at}_\infty}(\overline{H}(-), \jmath(\mathcal{X}))) \simeq \operatorname{Fun}_{\mathbb{C}\mathbf{at}_\infty}(\jmath \circ H(-), \jmath(\mathcal{X})).$$

Since H lands in $\mathbf{Cat}_\infty$ and $\mathcal{X}$ is an ∞-category, we have that

$$\operatorname{Fun}_{\mathbb{C}\mathbf{at}_\infty}(\jmath \circ H(-), \jmath(\mathcal{X})) \simeq \operatorname{Fun}_{\mathbf{Cat}_\infty}(H(-), \mathcal{X})).$$

Furthermore, considering $w = \mathcal{C}_{-/}$ and $f = \overline{H}$ in Remark A.0.0.7, we have the following result

$$\begin{aligned}
\operatorname{Nat}_{\mathcal{C}^{\mathrm{op}}}(\jmath \circ \mathcal{C}_{-/}, \operatorname{Fun}_{\mathbb{C}\mathbf{at}_\infty}(\overline{H}(-), \jmath(\mathcal{X}))) &\simeq \operatorname{Nat}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{C}_{-/}, \operatorname{Fun}_{\mathbf{Cat}_\infty}(H(-), \mathcal{X})) \\
&\simeq \lim_{\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}} \operatorname{Fun}_{\mathbf{Cat}_\infty}(\mathcal{C}_{-/}, \operatorname{Fun}_{\mathbf{Cat}_\infty}(H(-), \mathcal{X})) \quad \text{([55, §6])} \\
&\simeq \operatorname{Fun}_{\mathbf{Cat}_\infty}(\operatorname{colim}_{\mathrm{Tw}(\mathcal{C})} H(-) \times \mathcal{C}_{-/}, \mathcal{X}) \qquad \text{([55, §7])} \\
&\simeq \operatorname{Fun}_{\mathbf{Cat}_\infty}(\mathcal{H}, \mathcal{X}). \qquad \text{(by Lemma A.0.0.3)}
\end{aligned}$$

This equivalence implies that $\mathcal{H}$ is the out-colimit of $\overline{H}$ by [49, Proposition 5.2.3]. ☐

An out-colimit of a functor $p\colon \mathcal{K} \to \mathcal{C}$ is an object of $\mathcal{C}^{\mathrm{out}}_{p/}$ by Definition A.0.0.4. So we may identify this object with a map $\mathcal{K} \diamond_{\mathrm{out}} \Delta^0 \to \mathcal{C}$ extending p. Therefore, we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Ar}(\mathcal{C}) \arrow[d, hook] \arrow[dr, "\overline{H}"] & \\
\operatorname{Ar}(\mathcal{C}) \diamond_{\mathrm{out}} \Delta^0 \arrow[r, "\overline{H}'"'] & \mathbb{C}\mathbf{at}_\infty
\end{tikzcd} \tag{A.0.1}$$

such that $\overline{H}'|_{\operatorname{Ar}(\mathcal{C})} = \overline{H}$ and $\overline{H}'|_{\Delta^0} = \mathcal{H}$ since $\mathcal{H}$ is the out-colimit of $\overline{H}$. Furthermore, Gagna, Harpaz, and Lanari [49, Proposition 4.2.9] proved that $\operatorname{Ar}(\mathcal{C}) \diamond_{\mathrm{out}} \Delta^0 \simeq \operatorname{Ar}(\mathcal{C}) \star \Delta^0$. Therefore, the diagram (A.0.1) can be rewritten as:

$$\begin{tikzcd}
\operatorname{Ar}(\mathcal{C}) \arrow[d, hook] \arrow[dr, "\overline{H}"] & \\
\operatorname{Ar}(\mathcal{C}) \star \Delta^0 \arrow[r, "\overline{H}'"'] & \mathbb{C}\mathbf{at}_\infty.
\end{tikzcd} \tag{A.0.2}$$

The slice $(\infty,2)$-category $\mathbf{Cat}_{\infty//\mathcal{H}}$ can be described by the universal property: for any ∞-bicategory $\mathcal{C}'$, we have an equivalence

$$\operatorname{map}(\mathcal{C}',\mathbf{Cat}_{\infty//\mathcal{H}}) \simeq \operatorname{map}_{\mathcal{H}}(\mathcal{C}' \star \Delta^0,\mathbf{Cat}_\infty)$$

where the subscript on the right hand side indicates that we consider only those functors $\mathcal{C}' \star \Delta^0 \to \mathbf{Cat}_\infty$ whose restriction to Δ^0 coincides with $\mathcal{H}$ [49, §4.2]. This implies that the map $\overline{H}'$ induces a map:

$$\widetilde{H}\colon \operatorname{Ar}(\mathcal{C}) \to \mathbf{Cat}_{\infty//\mathcal{H}}.$$

Remark A.0.0.9. The functor $\widetilde{H}$ is fully determined by the oplax colimit property of $\mathcal{H}$. For an object $r\colon c \to d$ in $\operatorname{Ar}(\mathcal{C})$, the functor $\widetilde{H}$ sends r to the classifying map $\widetilde{H}_r\colon H(c) \to \mathcal{H}$, which is obtained from the pullback diagram

$$\begin{tikzcd}
H(c) \arrow[r, "\widetilde{H}_r"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{H} \arrow[d] \\
1 \arrow[r, "\ulcorner c \urcorner"'] & \mathcal{C}
\end{tikzcd}$$

given by the fact that $\mathcal{H}$ is a cocartesian fibration that classifies H. For a morphism

$$\begin{tikzcd}
c \arrow[r, "f"] \arrow[d, "r"'] \arrow[dr, phantom, "\sigma"] & c' \arrow[d, "r'"] \\
d \arrow[r] & d'
\end{tikzcd}$$

in $\operatorname{Ar}(\mathcal{C})$, the functor $\widetilde{H}$ sends σ to the cocone diagram

$$\begin{tikzcd}
H(c) \arrow[dd, "H(f)"'] \arrow[dr, "\widetilde{H}_r"] & \\
\arrow[r, phantom, "\overset{\swarrow}{\alpha_\sigma}"] & \mathcal{H} \\
H(c') \arrow[ur, "\widetilde{H}_{r'}"'] &
\end{tikzcd}$$

in $\mathbf{Cat}_\infty$ given by the oplax colimit property of $\mathcal{H}$, which is a morphism in $\mathbf{Cat}_{\infty//\mathcal{H}}$. The natural transformation α_σ is defined as follows: α_σ sends an object $x \in H(c)$ to a morphism $\alpha_\sigma(x)\colon \widetilde{H}_r(x) \to \widetilde{H}_{r'} \circ H(f)(x)$ in $\mathcal{H}$ such that for any morphism $g\colon x \to y$ in $H(c)$, the diagram

$$\begin{tikzcd}
\widetilde{H}_r(x) \arrow[r, "\alpha_\sigma(x)"] \arrow[d, "\widetilde{H}_r(g)"'] & \widetilde{H}_{r'} \circ H(f)(x) \arrow[d, "\widetilde{H}_{r'} \circ H(f)(g)"] \\
\widetilde{H}_r(y) \arrow[r, "\alpha_\sigma(y)"'] & \widetilde{H}_{r'} \circ H(f)(y)
\end{tikzcd}$$

commutes in $\mathcal{H}$. Note that α_σ is part of the data given by the oplax colimit property of $\mathcal{H}$.

On the other hand, the map $\operatorname{dom}\colon \operatorname{Ar}(\mathcal{C}) \to \mathcal{C}$ has a canonical section $\overline{s}\colon \mathcal{C} \to \operatorname{Ar}(\mathcal{C})$ that sends $c \in \mathcal{C}$ to id_c. We define $\widehat{H}$ as the composite

$$\mathcal{C} \xrightarrow{\overline{s}} \operatorname{Ar}(\mathcal{C}) \xrightarrow{\widetilde{H}} \mathbf{Cat}_{\infty//\mathcal{H}}. \tag{A.0.3}$$

We have an explicit description of $\widehat{H}$ by combining Remark A.0.0.9 and that the functor $\overline{s}$ sends a morphism $f\colon c \to b$ in $\mathcal{C}$ to the square:

$$\begin{tikzcd}
c \arrow[r, "f"] \arrow[d, "\mathrm{id}_c"'] & b \arrow[d, "\mathrm{id}_b"] \\
c \arrow[r, "f"'] & b.
\end{tikzcd}$$

Put $\widehat{H}(c)$ as the canonical map $\iota_c\colon H(c) \to \mathcal{H}$. For a morphism $f\colon c \to b$ in $\mathcal{C}$, the map $\widehat{H}(f)$ is the triangle

$$\begin{tikzcd}
H(c) \arrow[dd, "H(f)"'] \arrow[dr, "\iota_c", ""'{name=A}] & \\
 & \mathcal{H} \\
H(b), \arrow[ur, "\iota_b"', ""{name=B}] &
\arrow[Rightarrow, from=A, to=B, "\alpha_f"', shorten=2mm]
\end{tikzcd}$$

where the 2-cell α_f is defined for each morphism $g\colon x \to y$ in $H(c)$ by the commutative square

$$\begin{tikzcd}
\iota_c(x) \arrow[r, "\alpha_f(x)"] \arrow[d, "\iota_c(g)"'] & \iota_b \circ H(f)(x) \arrow[d, "\iota_b \circ H(f)(g)"] \\
\iota_c(y) \arrow[r, "\alpha_f(x')"'] & \iota_b \circ H(f)(y).
\end{tikzcd}$$

Note that the triangle is just the cocone diagram given by the oplax colimit property of $\mathcal{H}$ (Lemma A.0.0.3).

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List of Symbols

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