Chapter 3

The Place of Super Magic Labelings Among Other Labelings of Graphs

3.1 Relationship with Other Labelings

Before starting with the material of this section, we will say that the results 3.1 through 3.7 are found in [17]. The results 3.8, 3.9, 3.10, 3.11 in [14], and those found in the section regarding counting appear in [17], unless stated otherwise.

This chapter places super magic labelings in their proper place among other classes of labelings that have been previously well studied. The order in which we present these relations is the one that we feel is most conducive to a coherent and brief presentation (as opposed to one that lists each kind of labeling by its "relative" importance). With this in mind, we start defining sequential labelings.

The definition of sequential labelings was introduced by Grace [21] and is inspired by the concept of harmonious labelings (which we will discuss shortly). A sequential labeling of a (p,q)-graph G is an injective function $f : V(G) \to \{0, 1, \ldots, q-1\}$ (with the label q allowed if G is a tree) such that the induced edge labeling given by the rule $f(uv) = f(u) + f(v)$ has the property that

$$\{f(uv) \mid uv \in E(G)\} = \{m, m+1, m+2, \ldots, m+q-1\}$$

for some integer m. Moreover, G is said to be sequential if such a labeling exists.

With this definition in hand, we present the following result.

Theorem 3.1. *If G is a super magic graph of order p and size q which is either a tree or with $p \geq q$, then G is sequential.*

Proof.
Let f be a super magic labeling of G with valence k, then

$$\{f(u) + f(v) \mid uv \in E(G)\} = \{k - p - 1, k - p - 2, \ldots, k - p - q)\}$$

by Lemma 2.3.

Now, define the function $g : V(G) \to \{0, 1, \ldots, p - 1\}$ to be the injective function such that $g(v) = f(v) - 1$ for each vertex v of G. Hence,

$$\{g(u) + g(v) \mid uv \in E(G)\} = \{m, m + 1, \ldots, m + q - 1\},$$

where $m = k - (p + q + 2)$, which implies that g is a sequential labeling of G. □

Harmonious labelings have been defined and studied by Graham and Sloane [20] as part of their study of additive bases and are applicable to error-correcting codes. A harmonious labeling of a (p, q)-graph G with $q \geq p$ is an injective function $f : V(G) \to \{0, 1, \ldots, q - 1\}$ satisfying the condition that the induced edge labeling given by the rule $f(uv) \equiv f(u) + f(v) \pmod{q}$ for any edge uv of G is also an injective function. Furthermore, G is said to be harmonious if such a labeling exists. This definition extends to trees (for which $q = p - 1$) if at most one vertex label is allowed to be repeated.

The previous theorem, together with the fact that Grace [21] showed that sequential (p, q)-graphs with $q \geq p$ are harmonious yields the following result.

Theorem 3.2. *If a (p, q)-graph G with $q \geq p$ is super magic, then G is harmonious.*

This theorem extends easily to trees as the next result shows.

Theorem 3.3. *If a tree T of order p and size q is super magic, then G is harmonious.*

Proof.
Recall that $q = p - 1$ and then reduce the edge labels modulo $p - 1$. □

This result implies that the conjecture by Enomoto et al. [7] that all trees are super magic is at least as hard as the conjecture by Graham and Sloane that all trees are harmonious!

The oldest and most famous graph labeling problem that has been studied is that of finding graceful labelings of graphs, which were defined by Rosa [35]. These arose naturally out of the study of graph decompositions and the subsequent Ringel-Kotzig conjecture that all trees are graceful.

Let G be a (p, q)-graph and $f : V(G) \cup E(G) \to \{0, 1, \ldots, q\}$ such that $f(uv) = |f(u) - f(v)|$ for any edge uv of G and $f \mid_{V(G)}$ and $f \mid_{E(G)}$ *are* injective. Then f is a graceful labeling of G and G is called a graceful graph. Also, as a result of Rosa's interest on graph decompositions, he defined what he called an α-valuation of a graph [35]. A graceful labeling f of a (p, q)-graph G is said to be an α-valuation of G if there exist an integer k with $0 \leq k < q$, called the characteristic of f, such that $\min\{f(u), f(v)\} \leq k < \max\{f(u), f(v)\}$ for every edge uv of G.

The next two theorems establish the relationships between super magic labelings and α-valuations.

Theorem 3.4. *Suppose that G is a super magic bipartite $(p, p-1)$-graph with partite sets V_1 and V_2, where $p_1 = |V_1|$ and $p_2 = |V_2|$ and let*

$$f : V(G) \cup E(G) \to \{1, 2, \ldots, 2p - 1\}$$

be a super magic labeling of G such that $f(V_1) = \{1, 2, \ldots, p_1\}$, then G admits an α-valuation.

Proof.

Consider a $(p, p-1)$-graph G and a super magic labeling f of G such that both satisfy the hypothesis of the theorem. Furthermore, select the vertices of G so that $V(G) = \{v_i \in V(G) \mid f(v_i) = i\}$. Then

$$f(V_1) = \{1, 2, \ldots, p_1\} \text{ and } f(V_2) = \{p_1 + 1, p_1 + 2, \ldots, p_1 + p_2\}.$$

Now, let $g : V(G) \cup E(G) \to \{0, 1, \ldots, p - 1\}$ be the labeling with the property that

$$g(v) = \begin{cases} f(v) - 1, & \text{if } v \in V_1; \\ 2p_1 + p_2 - f(v), & \text{if } v \in V_2. \end{cases}$$

We next prove that g is an α-valuation of G with characteristics $p_1 - 1$. First, observe that

$$g(V_1) = \{0, 1, \ldots, p_1 - 1\} \text{ and } g(V_2) = \{p_1, p_1 + 1, \ldots, p_1 + p_2 - 1\}.$$

Also, if $u \in V_2$ and $v \in V_1$, then

$$\begin{aligned} |g(u) - g(v)| &= g(u) - g(v) \\ &= 2p_1 + p_2 + 1 - (f(u) + f(v)). \end{aligned}$$

Hence, $1 \leq |g(u) - g(v)| \leq p - 1$, since $p_1 + 2 \leq f(u) + f(v) \leq 2p_1 + p_2$.

Finally, since $u \in V_2$ and $v \in V_1$ are arbitrary vertices of G, it suffices to observe that $\{f(u) + f(v) \mid uv \in E(G)\}$ is a set of $p - 1$ consecutive integers by Lemma 2.3, which implies that $g(E(G)) = \{1, 2, \ldots, p - 1\}$. □

We comment here that Rosa [35] has shown that all graphs that admit α-valuations are bipartite. Therefore, we have the converse of the previous theorem, which we state without proof.

Theorem 3.5. *Let G be a bipartite $(p, p-1)$-graph with an α-valuation f such that there exists partite sets V_1 and V_2, where $p_1 = |V_1|$, $p_2 = |V_2|$ and $f(V_1) = \{0, 1, \ldots, p_1 - 1\}$, then G is super magic.*

This theorem is important due to the following corollary.

Corollary 3.6. *If T is a tree having an α-valuation, then T is super magic.*

A number of techniques to construct trees from smaller ones with α-valuations have been shown to yield α-valuations in the resulting trees. The reader is referred to the survey paper by Gallian [18] for references to these methods.

Cahit [5] defined cordial labelings of graphs as a way of stating a weaker condition that would reflect the spirit of both graceful and harmonious labelings. A cordial labeling of G is a function $f : V(G) \to \mathbb{Z}_2$ with an induced edge labeling $f(uv) \equiv f(u) - f(v) \pmod 2$ such that if $v_f(i)$ and $e_f(i)$ are the number of vertices v and edges e satisfying that $f(v) = i$ and $f(e) = i$ for all $i \in \mathbb{Z}_2$, respectively, then $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph that admits a cordial labeling is said to be cordial.

With this definition in mind, we are able to show the next relationship between labelings.

Theorem 3.7. *If a graph G is super magic, then G is cordial.*

Proof.

Let G be super magic with a super magic labeling f. Then consider the function $g : V(G) \cup E(G) \to \mathbb{Z}_2$ such that $g(v) \equiv f(v) \pmod 2$ for every vertex v of G and $g(uv) \equiv g(u) - g(v) \pmod 2$ for any edge uv of G. Notice that

$$g(uv) \equiv g(u) - g(v) \equiv g(u) + g(v) \equiv f(u) + f(v) \pmod 2.$$

Also, since $f(V(G))$ and $\{f(u) + f(v) \mid uv \in E(G)\}$ are sets of consecutive integers by definition of super magic graph and Lemma 2.3, respectively, it follows that $|v_g(0) - v_g(1)| \leq 1$ and $|e_g(0) - e_g(1)| \leq 1$. □

Ringel [33] has also provided the definition for edge-antimagic graphs.

For a (p, q)-graph G, a bijective function $f : V(G) \to \{1, 2, \ldots, p\}$ is an *edge-antimagic labeling* of G if

$$|\{f(u) + f(v) \mid uv \in E(G)\}| = q.$$

If such a labeling exists, then G is called an *edge antimagic graph.*

In this section, we present some relationships between super magic and edge-antimagic graphs.

The following is an immediate consequence of Lemma 2.3.

Theorem 3.8. *Every super magic graph is edge antimagic.*

We then note that Lemma 2.2 follows from Theorem 3.8 and a comment by Ringel [33] to the effect that the inequality $q \leq 2p-3$ holds for edge-antimagic (p,q) graphs.

Ringel [33] also mentioned that if a graph G of order p is edge antimagic with an edge-antimagic labeling f, then

$$\{f(u)+f(v) \mid uv \in E(G)\} \subseteq \{3,4,\ldots,2p-1\}.$$

This remark implies the following partial converse of Theorem 3.8.

Theorem 3.9. *If G is an edge-antimagic (p,q)-graph with $q = 2p-3$, then G is super magic.*

Proof.

Let G be an edge antimagic (p,q)-graph such that $q = 2p-3$ with an edge antimagic labeling f. Then

$$\{f(u)+f(v) \mid uv \in E(G)\} = \{3,4,\ldots,2p-1\}$$

so the result follows from Lemma 2.3. □

Ringel [33] presented the following theorem as well.

Theorem 3.10. *If G is a maximal outerplanar graph of order p with exactly two vertices a, b of degree 2 and whose distance $d_H(a,b)$ on the Hamilton cycle H in G is either*

$$\left\lfloor \frac{p}{2} \right\rfloor \text{ or } \left\lfloor \frac{p}{2} \right\rfloor - 1,$$

then G is edge-antimagic.

Since all maximal outerplanar (p,q)-graphs satisfy $q = 2p-3$, we have the following result from Theorems 3.9 and 3.10.

Corollary 3.11. *If G is a maximal outerplanar graph of order p with exactly two vertices a, b of degree 2 and whose distance $d_H(a,b)$ on the Hamilton cycle H in G is*

$$\left\lfloor \frac{p}{2} \right\rfloor \text{ or } \left\lfloor \frac{p}{2} \right\rfloor - 1,$$

then G is super magic.

The previous corollary implies that the upper bound in Lemma 2.2 is also sharp for maximal outerplanar graphs.

3.2 Counting

A well known result by Gilbert [19] states that almost all graphs are connected, which implies that almost all (p, q)-graphs satisfy that $q \geq p$. This combined with Graham and Sloane's result [20] that almost all graphs are not harmonious and Theorem 3.2 leads to the following theorem.

Theorem 3.12. *Almost all graphs are not super magic.*

Next, we will provide the following closed formula for the number of super magic graphs.

Theorem 3.13. *The number of distinct super magic labeled (p, q)-graphs is*

$$\sum_{i=3}^{2p-q} \prod_{j=1}^{i+q-1} a(j)$$

where

$$a(j) = \begin{cases} \lfloor \frac{j-1}{2} \rfloor, & \text{if } 3 \leq j \leq p+1; \\ \lfloor \frac{2p-j+1}{2} \rfloor, & \text{if } p+2 \leq j \leq 2p-1. \end{cases}$$

Proof.
Consider the complete graph K_p with

$$V(K_p) = \{v_i \mid 1 \leq i \leq p\}$$

and the labeling

$$f : V(G) \cup E(G) \to \left\{1, 2, \ldots, p + \frac{p(p-1)}{2}\right\}$$

such that $f(v_i) = i$ for every integer i with $1 \leq i \leq p$ and $f(uv) = f(u)+f(v)$ for any edge uv of G.

Let $A_j = \{uv \in E(G) \mid f(uv) = j\}$ and $a(j) = |A_j|$ for every integer j with $3 \leq j \leq 2p - 1$. Then, by Lemma 2.3, a vertex labeling f of a (p, q)-graph G with $f(V(G))$ extends to a super magic labeling if

$$\{f(u) + f(v) \mid uv \in E(G)\}$$

is a set of q consecutive integers. Thus, a super magic (p, q)-graph G can be constructed from the labeling f of K_p by taking

$$V(G) = V(K_p) \text{ and } E(G) = \{e_j \in A_j \mid i \leq j \leq i + q - 1\}$$

for some fixed integer i with $3 \leq i \leq 2p - q$. Then a super magic labeling of G is obtained by restricting f to $V(G)$. Notice that $E(G)$ can be selected in $\prod_{j=1}^{i+q-1} a(j)$ ways.

Finally, if we take all possible integer values of i such that $3 \leq i \leq 2p-q$, then the result follows immediately. □

The following is a table of the number of super magic labelings of (p, q)-graphs, where $2 \leq p \leq 7$ and $1 \leq q \leq 11$.

Table 2

	1	2	3	4	5	6	7	8	9	10	11
2	1										
3	3	2									
4	6	6	6	4	2						
5	10	14	20	24	24	16	8				
6	15	26	48	80	120	144	144	96	48		
7	21	44	99	212	420	720	1080	1296	1296	864	432