**Corollary 4.12.** *Let $F \cong K_{1,m} \cup 2nK_2$, where $m$ and $n$ are positive integers such that $m + 2n$ and $2n + 1$ are relatively prime. Then only the valences $2m + 9n + 4$, $3m + 9n + 3$ and $4m + 9n + 2$ are attained by the magic labelings of $F$.*

*Proof.*
To prove this we use the facts and notation of the proof of the previous theorem. First, notice that the vertex labeling $h : V(F) \to \{1, 2, \ldots, p\}$ such that $h(v) = p + q + 1 - f(v)$ extends to a magic labeling of $F$ with valence $4m + 9n + 2$. Next, if we allow magic labelings of $F$, the value of $\alpha$ in the proof can also be $-2$, and thus, only one further valence is attained. □

## 4.2 Crown Products of Some Super Magic Graphs

### 4.2.1 General Results

Unless stated otherwise, the results on this section are due to Figueroa et al. [12]. In this section, we first provide a construction that shows that $G \odot \overline{K}_n$ is super magic whenever $G$ is a graph of odd order at least 3 and admits certain super magic labelings.

**Theorem 4.13.** *Let $G$ be a graph of odd order $p \geq 3$ for which there exists a super magic labeling $f$ with the property that*

$$\max \{f(u) + f(v) \mid uv \in E(G)\} = \frac{3p+1}{2},$$

*then, $G \odot \overline{K}_n$ is super magic for every positive integer $n$.*

*Proof.*
Let $f$ be a super magic labeling of $G$ with valence $k$, and assume that $f$ has the property that $f(v_i) = i$ for every integer $i$ with $1 \leq i \leq p$, where $V(G) = \{v_i \mid 1 \leq i \leq p\}$. Further, let

$$V\left(G \odot \overline{K}_n\right) = V(G) \cup \left\{w_i^j \mid 1 \leq i \leq p \text{ and } 1 \leq j \leq n\right\}$$

and

$$E\left(G \odot \overline{K}_n\right) = E(G) \cup \left\{v_i w_i^j \mid 1 \leq i \leq p \text{ and } 1 \leq j \leq n\right\}.$$

Now, define the vertex labeling

$$g : V\left(G \odot \overline{K}_n\right) \to \{1, 2, \ldots, p(n+1)\}$$

such that $g(v) = f(v)$ for every vertex $v$ of $G$, and

$$g\left(w_i^j\right) = \begin{cases} p + i + \frac{p(2j-1)+1}{2}, & \text{if } 1 \le i \le \frac{p-1}{2} \text{ and } 1 \le j \le n; \\ i + \frac{p(2j-1)+1}{2}, & \text{if } \frac{p+1}{2} \le i \le p \text{ and } 1 \le j \le n. \end{cases}$$

To show that $g$ extends to a super magic labeling of $G \odot \overline{K}_n$, consider the set

$$S_i^j = \left\{ f(v_i) + f\left(w_i^j\right) \mid 1 \le i \le p \text{ and } 1 \le j \le n \right\},$$

and let

$$m_j = \min\left\{S_i^j \mid 1 \le i \le p\right\} = p + 1 + \frac{p(2j-1)+1}{2},$$

$$M_j = \max\left\{S_i^j \mid 1 \le i \le p\right\} = 2p + \frac{p(2j-1)+1}{2}.$$

Finally, observe that $m_1 = (3p+1)/2+1, M_j+1 = m_{j+1}$ $(1 \le j \le n-1)$, and $S_i^j$ is a set of consecutive integers ($1 \le i \le p$ and $1 \le j \le n$), which implies that $g$ is the canonical form of a super magic labeling of $G \odot \overline{K}_n$ with valence $k + 2np$. □

In the previous chapter, we proved that every super magic $(p, q)$ graph is harmonious sequential and hence feliticious when either $G$ is a tree or satisfies $q \ge p$. Hence, we obtain the next corollary.

**Corollary 4.14.** *Let $G$ be a graph of odd order $p \ge 3$ for which there exists a super magic labeling $f$ with the property that*

$$\max\{f(u) + f(v) \mid uv \in E(G)\} = \frac{3p+1}{2}.$$

*Then $G \odot \overline{K}_n$ is harmonious, sequential and felicitous for every positive integer $n$.*

We now provide a similar, though in a sense weaker, result for graphs with even order at least 4.

**Theorem 4.15.** *Let $G$ be a graph of even order $p \ge 4$ having a super magic labeling $f$ with the property that*

$$\max\{f(u) + f(v) \mid uv \in E(G)\} = \frac{3p}{2}.$$

*Then, the graph $H$ obtained by attaching $n$ pendant edges to each vertex of $G$ except the vertex $v$ with $f(v) = p$ is super magic for every positive integer $n$.*

*Proof.*

Let $V(G) = \{v_1, v_2, \ldots, v_p\}$, then take a super magic labeling $f$ of $G$ with valence $k$ satisfying the property that $f(v_i) = i$ for $i = 1, 2, \ldots, p$. Next, define the graph $H$ as follows:

$$V(H) = V(G) \cup \left\{w_i^j \mid 1 \leq i \leq p-1 \text{ and } 1 \leq j \leq n\right\}$$

and

$$E(H) = E(G) \cup \left\{v_i w_i^j \mid 1 \leq i \leq p-1 \text{ and } 1 \leq j \leq n\right\}.$$

Consequently, through an analogous argument to the one used in the proof of the previous theorem, the vertex labeling

$$g : V(H) \to \{1, 2, \ldots, p(n+1) - n\}$$

such that $g(v) = f(v)$ for every vertex $v$ of $G$, and when $1 \leq j \leq n$ we have that

$$g\left(w_i^j\right) = \begin{cases} i + \frac{p}{2} + (p-1)j + 1, & \text{if } 1 \leq i \leq \frac{p}{2} - 1; \\ i + \frac{p}{2} + (p-1)(j-1) + 1, & \text{if } \frac{p}{2} \leq i \leq p-1; \end{cases}$$

is the canonical form of a super magic labeling of $H$ with valence $k + 2n(p-1)$. □

Again, we have the following corollary.

**Corollary 4.16.** *Let $G$ be a graph of order $p \geq 4$ having a super magic labeling $f$ with the property that*

$$\max\{f(u) + f(v) \mid uv \in E(G)\} = \frac{3p}{2}.$$

*Then the graph $H$ obtained by attaching $n$ pendant edges to each vertex of $G$ except the vertex $v$ with $f(v) = p$ is harmonious, sequential and felicitous for every positive integer $n$.*

### 4.2.2 Super Magic Labelings of $n$-Crowns of 2-regular graphs

We now proceed to study the super magicness of $n$-crowns of 2-regular graphs. The results of this section can be found in [12].

**Theorem 4.17.** *If $G$ is a (super) magic 2-regular graph, then $G \odot \overline{K}_n$ is (super) magic for every positive integer $n$.*

*Proof.*

Let $f$ be a (super) magic labeling of $G$ with valence $k$. Assume that $H$ is a component of $G \odot \overline{K}_n$. Then $H \cong C_r \odot \overline{K}_n$ for some integer $r \geq 3$. Let

$$V(H) = \{v_i \mid i \in \mathbb{Z}_r\} \cup \{u_{i,j} \mid i \in \mathbb{Z}_r \text{ and } 1 \leq j \leq n\}$$

and

$$E(H) = \{v_i v_{i+1} \mid i \in \mathbb{Z}_r\} \cup \{v_i u_{i,j} \mid i \in \mathbb{Z}_r \text{ and } 1 \leq j \leq n\},$$

where $\mathbb{Z}_r$ denotes the set of integers modulo $r$.

Then $f\mid_H$ extends to a labeling $g$ of $H$ as follows.

$$\begin{aligned} g(v_i) &= (n+1) f(v_i) - n, \\ g(v_{i-1} v_j) &= n f(v_{i-1} v_i), \\ g(u_{i,j}) &= (n+1) f(v_{i-1}) - n + j, \\ g(v_i u_{i,j}) &= n f(v_{i-1} v_i) - j, \end{aligned}$$

where $i \in \mathbb{Z}_r$ and $1 \leq j \leq n$. Therefore, $f$ extends likewise in every component of $G \odot \overline{K}_n$, and a (super) magic labeling of $G \odot \overline{K}_n$ is obtained with valence $n(k-2)+2$. □

Next, recall the following super magic characterization of the $n$-cycle $C_n$ found by Enomoto et al. [7].

**Theorem 4.18.** *The $n$-cycle $C_n$ is super magic if and only if $n \geq 3$ is odd.*

Hence, by the previous theorem, we know that the $n$-crowns with cycle length $m$ are super magic when $m \geq 3$ is odd. In the following result, we show with considerable more effort that the $n$-crowns with cycle length $m$ are also super magic when $m \geq 4$ is even.

**Theorem 4.19.** *For every two integers $m \geq 3$ and $n \geq 1$, the $n$-crown $G \cong C_m \odot \overline{K}_n$ is super magic.*

*Proof.*

Let $G \cong C_m \odot \overline{K}_n$ be the $n$-crown with

$$V(G) = \{u_i \mid 1 \leq i \leq m\} \cup \{v_{i,j} \mid 1 \leq i \leq m \text{ and } 1 \leq i \leq n\}$$

and

$$\begin{aligned} E(G) &= \{u_1 u_m\} \cup \{u_i u_{i+1} \mid 1 \leq i \leq m-1\} \\ &\quad \cup \{u_i v_{i,j} \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}. \end{aligned}$$

Now, notice that if $m \geq 3$ is odd, then the result follows from the previous two theorems. Thus, assume that $m \geq 4$ is even for the remainder of the proof, and proceed by cases.
Case 1: For $m = 4$, define the vertex labeling

$$f : V(G) \to \{1, 2, \ldots, 4(n+1)\}$$

such that

$$\begin{aligned} f(u_{2i-1}) &= i; & f(u_{2i}) &= 3i; \\ f(v_{2i-1,1}) &= 2i+3; & f(v_{2i,1}) &= 12-4i \end{aligned}$$

when $i = 1$ or 2; and $f(v_{i,j}) = 4j - i + 5$ when $1 \leq i \leq 4$ and $2 \leq j \leq n$.
Case 2: For $m = 6$, define the vertex labeling

$$f : V(G) \to \{1, 2, \ldots, 6(n+1)\}$$

such that

$$\begin{aligned} f(u_1) &= 9; & f(u_2) &= 1; & f(u_3) &= 4; \\ f(u_4) &= 2; & f(u_5) &= 5; & f(u_6) &= 3; \\ f(v_{1,1}) &= 6; & f(v_{2,1}) &= 8; & f(v_{3,1}) &= 7; \\ f(v_{4,1}) &= 12; & f(v_{5,1}) &= 11; & f(v_{6,1}) &= 10; \end{aligned}$$

and

$$f(v_{i,j}) = \begin{cases} 5i + 6j - 4, & \text{if } 1 \leq i \leq 2 \text{ and } 2 \leq j \leq n; \\ i + 6j - 1, & \text{if } 3 \leq i \leq 6 \text{ and } 2 \leq j \leq n. \end{cases}$$

Case 3: For $m = 8$, define the vertex labeling

$$f : V(G) \to \{1, 2, \ldots, 8(n+1)\}$$

such that

$$\begin{aligned} f(u_1) &= 1; & f(u_2) &= 5; & f(u_3) &= 2; \\ f(u_4) &= 6; & f(u_5) &= 3; & f(u_6) &= 7; \\ f(u_7) &= 4; & f(u_8) &= 12; & & \\ f(v_{1,1}) &= 11; & f(v_{2,1}) &= 13; & f(v_{3,1}) &= 15; \\ f(v_{4,1}) &= 14; & f(v_{5,1}) &= 16; & f(v_{6,1}) &= 8; \\ f(v_{7,1}) &= 10; & f(v_{8,1}) &= 9; & & \end{aligned}$$

and $f(v_{i,j}) = 8j - i + 9$, if $1 \leq i \leq 8$ and $2 \leq j \leq n$.
Case 4: Let $m = 8k + 2$, where $k$ is a positive integer, and define the vertex labeling

$$f : V(G) \to \{1, 2, \ldots, (8k+2)(n+1)\}$$

such that

$$f(u_l)=\begin{cases}12k+3, & \text{if } l=1;\\ 4k+i, & \text{if } l=2i-1 \text{ and } 2\le i\le 4k+1;\\ i, & \text{if } l=2i \text{ and } 1\le i\le 4k+1;\end{cases}$$

$$f(v_{l,1})=\begin{cases}8k+i+1, & \text{if } l=2i-1 \text{ and } 1\le i\le 2k+2;\\ 12k+2, & \text{if } l=2;\\ 12k+i+2, & \text{if } l=2i \text{ and } 2\le i\le 2k;\\ 14k+2i+4, & \text{if } l=4k+4i-2 \text{ and } 1\le i\le k;\\ 14k-i+5, & \text{if } l=4k+i+3 \text{ and } 1\le i\le 2;\\ 10k+2i+2, & \text{if } l=4k+4i+3 \text{ and } 1\le i\le k-1;\\ 14k+2i+3, & \text{if } l=4k+4i+4 \text{ and } 1\le i\le k-1;\\ 10k+2i+3, & \text{if } l=4k+4i+5 \text{ and } 1\le i\le k-1;\\ 16k+3, & \text{if } l=8k+2;\end{cases}$$

and for $2\le j\le n$, we have that

$$f(v_{2i-1,j})=\begin{cases}2(4k+1)j+i, & \text{if } 1\le i\le 2k+1;\\ 2(4k+1)j+i+1, & \text{if } 2k+2\le i\le 4k+1;\end{cases}$$

$$f(v_{2i,j})=\begin{cases}(4k+1)(2j+1)+i, & \text{if } 2\le i\le 2k;\\ (4k+1)(2j+1)+i-1, & \text{if } 2k+2\le i\le 4k+1;\end{cases}$$

$f(v_{2,j})=2(4k+1)(j+1)$ and $f(v_{4k+2,j})=2(k+1)+2(4k+1)j$.

Case 5: Let $m=8k+4$, where $k$ is a positive integer, and define the vertex labeling

$$f:V(G)\to\{1,2,\dots,(8k+4)(n+1)\}$$

such that

$$f(u_l)=\begin{cases}i, & \text{if } l=2i-1 \text{ and } 1\le i\le 4k+2;\\ 4k+i+2, & \text{if } l=2i \text{ and } 1\le i\le 4k+1;\\ 12k+6, & \text{if } l=8k+4;\end{cases}$$

$$f(v_{l,1})=\begin{cases}12k+5, & \text{if } l=1;\\ 16k-4i+8, & \text{if } l=4i-2 \text{ and } 1\le i\le k;\\ 16k-4i+9, & \text{if } l=4i-1 \text{ and } 1\le i\le k;\\ 16k-4i+10, & \text{if } l=4i \text{ and } 1\le i\le k;\\ 16k-4i+7, & \text{if } l=4i+1 \text{ and } 1\le i\le k;\\ 8k+4, & \text{if } l=4k+2;\\ 16k+8, & \text{if } l=4k+3;\\ 16k+7, & \text{if } l=8k+3;\\ 8k+5, & \text{if } l=8k+4;\\ 12k-i+5, & \text{if } l=4k+i+3 \text{ and } 1\le i\le 4k-1;\end{cases}$$

and $f(v_{i,j}) = 4(2k+1)(j+1) - i + 1$, if $1 \leq i \leq 8k+4$ and $2 \leq j \leq n$.
Case 6: Let $m = 8k+6$, where $k$ is a positive integer, and define the vertex labeling

$$f : V(G) \to \{1, 2, \ldots, (8k+6)(n+1)\}$$

such that

$$f(u_1) = \begin{cases} 12k+9, & \text{if } l = 1; \\ 4k+i+2, & \text{if } l = 2i-1 \text{ and } 2 \leq i \leq 4k+3; \\ i, & \text{if } l = 2i \text{ and } 1 \leq i \leq 4k+3; \end{cases}$$

$$f(v_{l,1}) = \begin{cases} 8k+i+5, & \text{if } l = 2i-1 \text{ and } 1 \leq i \leq 2k+3; \\ 12k+8, & \text{if } l = 2; \\ 12k+i+8, & \text{if } l = 2i \text{ and } 2 \leq i \leq 2k+1; \\ 14k-2i+14, & \text{if } l = 4k+3i+1 \text{ and } 1 \leq i \leq 2; \\ 14k+2i+12, & \text{if } l = 4k+4i+2 \text{ and } 1 \leq i \leq k; \\ 14k+2i+9, & \text{if } l = 4k+4i+4 \text{ and } 1 \leq i \leq k; \\ 10k+2i+7, & \text{if } l = 4k+4i+5 \text{ and } 1 \leq i \leq k; \\ 10k+2i+8, & \text{if } l = 4k+4i+7 \text{ and } 1 \leq i \leq k-1; \\ 16k+11 & \text{if } l = 8k+6; \end{cases}$$

and for $2 \leq j \leq n$, we have that

$$f(v_{l,j}) = \begin{cases} 2(4k+3)j+i, & \text{if } l = 2i-1, 1 \leq i \leq 4k+3; \\ (4k+3)(2j+1)+i, & \text{if } l = 2i, 1 \leq i \leq 4k+3. \end{cases}$$

Case 7: Let $m = 16k$, where $k$ is a positive integer, and define the vertex labeling

$$f : V(G) \to \{1, 2, \ldots, 16k(n+1)\}$$

such that

$$f(u_l) = \begin{cases} i, & \text{if } l = 2i-1 \text{ and } 1 \leq i \leq 8k; \\ 8k+i, & \text{if } l = 2i \text{ and } 1 \leq i \leq 8k-1; \\ 24k, & \text{if } l = 16k; \end{cases}$$

$f(v_{1,1}) = 24k - 1; f(v_{2,1}) = 16k + 3; f(v_{3,1}) = 32k - 1;$

$$f(v_{l,1}) = \begin{cases} 32k - 2i + 1, & \text{if } l = 2i - 1 \text{ and } 3 \leq i \leq 4k; \\ 32k - 2i + 2, & \text{if } l = 2i \text{ and } 2 \leq i \leq 4k; \\ 32k - 3i + 3, & \text{if } l = 8k + 2i - 1 \text{ and } 1 \leq i \leq 2; \\ 24k - 8i + 5, & \text{if } l = 8k + 8i - 6 \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 6, & \text{if } l = 8k + 8i - 4 \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 4, & \text{if } l = 8k + 8i - 3 \text{ and } 1 \leq i \leq k; \\ 24k - 8i, & \text{if } l = 8k + 8i - 2 \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 2, & \text{if } l = 8k + 8i - 1 \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 1, & \text{if } l = 8k + 8i \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 3, & \text{if } l = 8k + 8i + 1 \text{ and } 1 \leq i \leq k - 1; \\ 24k - 8i - 1, & \text{if } l = 8k + 8i + 3 \text{ and } 1 \leq i \leq k - 1; \end{cases}$$

and $f(v_{i,j}) = 16k(j + 1) - i + 1$, if $1 \leq i \leq 16k$ and $2 \leq j \leq n$.
Case 8: Let $m = 16k + 8$, where $k$ is a positive integer, and define the vertex labeling

$$f : V(G) \to \{1, 2, \ldots, (16k + 8)(n + 1)\}$$

such that

$$f(u_i) = \begin{cases} i, & \text{if } l = 2i - 1 \text{ and } 1 \leq i \leq 8k + 4; \\ 8k + i + 4, & \text{if } l = 2i \text{ and } 1 \leq i \leq 8k + 3; \\ 24k + 12, & \text{if } l = 16k + 8; \end{cases}$$

$f(v_{1,1}) = 24k + 11; f(v_{2,1}) = 16k + 11; f(v_{3,1}) = 32k + 15;$

$$f(v_{l,1}) = \begin{cases} 32k - 2i + 17, & \text{if } l = 2i - 1 \text{ and } 3 \leq i \leq 4k + 2; \\ 32k - 2i + 18, & \text{if } l = 2i \text{ and } 2 \leq i \leq 4k + 2; \\ 32k - 3i + 19, & \text{if } l = 8k + 2i + 3 \text{ and } 1 \leq i \leq 2; \\ 24k - 8i + 16, & \text{if } l = 8k + 8i - 2 \text{ and } 1 \leq i \leq k + 1; \\ 24k - i + 11, & \text{if } l = 8k + i + 7 \text{ and } 1 \leq i \leq 2; \\ 24k - 8i + 13, & \text{if } l = 8k + 8i + 2 \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 15, & \text{if } l = 8k + 8i + 3 \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 14, & \text{if } l = 8k + 8i + 4 \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 12, & \text{if } l = 8k + 8i + 5 \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 10, & \text{if } l = 8k + 8i + 7 \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 9, & \text{if } l = 8k + 8i + 8 \text{ and } 1 \leq i \leq k; \\ 24k - 8i + 11, & \text{if } l = 8k + 8i + 9 \text{ and } 1 \leq i \leq k - 1; \end{cases}$$

and $f(v_{i,j}) = (16k + 8)(j + 1) - i + 1$, if $1 \leq i \leq 16k + 8$ and $2 \leq j \leq n$.

Therefore, $f$ is the canonical form of a super magic labeling of $G$ with valence $m(4n + 5)/2 + 2$. □

Using the relationships between super magic labelings and other labelings mentioned in the introduction, we finish this chapter with the following corollary, which settles a conjecture by Yegnanarayanan [41].

**Corollary 4.20.** *For every two integers $m \geq 3$ and $n \geq 1$, the $n$-crown $G \cong C_m \odot \overline{K}_n$ is harmonious, sequential and felicitous.*