Chapter 7

Other Related Labelings

7.1 Special Super Magic Labelings of Bipartite Graphs

7.1.1 Introduction

Unless otherwise stated the results on this section are found in [31]. In our study of super magic labelings the concept of special super magic labelings of bipartite graphs has emerged naturally. We define a special super magic labeling of a bipartite graph G with bipartite sets V_1 and V_2 to be a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that

I. f is a super magic labeling of G,

II. $f(V_1) = \{1, 2, \ldots, |V_1|\}$, and

III. $f(V_2) = \{|V_1| + 1, |V_2| + 2, \ldots, |V(G)|\}$.

If a graph G admits a special super magic labeling, then we say that G is special super magic.

Note that it is possible to redefine special super magic labelings of bipartite graphs in such a way that only the vertices of the graph are labeled, and we do this in the next lemma.

Lemma 7.1. *A bipartite graph G with bipartite sets V_1 and V_2 is special super magic if and only if there exits a bijective function $g : V(G) \to \{1, 2, \ldots, |V(G)|\}$ with the properties*

I. $g(V_1) = \{1, 2, \ldots, |V_1|\}$,

II. $g(V_2) = \{|V_1| + 1, |V_1| + 2, \ldots, |V(G)|\}$, *III. the set* $\{g(u)+g(v) \mid uv \in (G)\}$ *is a set of* $|E(G)|$ *consecutive integers.*

Proof.

For the necessity, assume that a bipartite graph G with bipartite sets V_1 and V_2 is special super magic. Let f be a special super magic labeling of G. Then, the function $f\mid_{V(G)}$ defined to be the restriction of the function f to the set $V(G)$, meets the properties of the function g as described in the statement of the lemma. Therefore, take $g = f\mid_{V(G)}$. For the sufficiency, assume that g is a bijective function defined on the vertex set of a bipartite graph G, meeting the requirements of the statements of the lemma. Define the function $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ as follows

$$f(x) = \begin{cases} g(x) & \text{if } x \in V(G) \\ k - g(u) - g(v) & \text{if } x = uv \in E(G) \end{cases}$$

where $k = |V(G)| + |E(G)| + \min\{g(a) + g(b) \mid ab \in E(G)\}$.

Then, the function f is, in fact, a special super magic labeling of G. Therefore G is a special super magic graph. □

Next, we prove the following lemma concerning special super magic bipartite graphs.

Theorem 7.2. *If G is a special super magic bipartite graph with bipartite sets V_1 and V_2, then $|E(G)| \leq |V(G)| - 1$.*

Proof.

Let G be a special super magic bipartite graph with bipartite sets V_1 and V_2 then, by Lemma 7.1, there exists a bijective function $g : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that

I. $g(V_1) = \{1, 2, \ldots, |V_1|\}$,

II. $g(V_2) = \{|V_1| + 1, |V_1| + 2, \ldots, |V(G)|\}$,

III. $\{g(u) + g(v) \mid uv \in (G)\}$ is a set of $|E(G)|$ consecutive integers.

Now,

$$\min\{g(u) + g(u) \mid uv \in E(G)\} \geq 2 + |V_1|$$

and

$$\max\{g(u) + g(v) \mid uv \in E(G)\} \leq 2|V_1| + |V_2|$$

hence, $|\{g(u) + g(v) \mid uv \in E(G)\}| \leq 2|V_1| + |V_2| - |V_1| - 2 + 1 = |V(G)| - 1$; but, $|\{g(u) + g(v) \mid uv \in E(G)\}| = |E(G)|$. Therefore, $|E(G) \leq |V(G)|| - 1$. □

7.1.2 Special Super Magic and Super Magic Labelings, Chessboards, 1-Regular and 2-Regular Graphs

The main goal of this section is to establish relations between super magic graphs and special super magic bipartite graphs. However, in order to do this, we need to define what in the near future will prove to be the "link" between the two concepts. This link is what we call an $n \times n$ chessboard. An $n \times n$ chessboard is defined to be a square that contains inside of it n rows and columns shaping n^2 new little squares inside of the original one. Figure 1 shows a 3×3 and a 5×5 chessboard.

Figure 7.1: Chessboards.

Now, notice that assigning different numbers to the columns and different numbers to the rows of an $n \times n$ chessboard, every square of the chessboard is uniquely determined by an ordered pair (i, j) where i denotes the column to which the square belongs and j denotes the row to which the square belongs. Any function that assigns different numbers to the rows and different numbers to the columns of a chessboard will be called a numbering of the chessboard.

We are ready to present a construction which allows us to transform every super magic labeling of a 2-regular graph of order p into a special super magic labeling of the 1-regular graph of size p.

Let $G \cong \bigcup_{i=1}^{k} C_i$ be a 2-regular super magic graph of order p and k components, such that $l_i = |V(C_i)|$ for $i = 1, 2, \ldots, k$ and define the vertex and edge sets of G in the following manner

$$V(G) = \bigcup_{j=1}^{k} \{v_i^j \mid 1 \leq i \leq l_j\}$$

and

$$E(G) = \left(\bigcup_{i=1}^{k} \{v_1^i v_{l_i}^i\} \right) \cup \left(\bigcup_{j=1}^{k} \{v_i^j v_{i+1}^j \mid 1 \leq i \leq l_k - 1\} \right)$$

and consider the vertex labeling $g : V(G) \to \{1, 2, \ldots, |V(G)|\}$ with the properties of the function g defined in Lemma 7.1. Next, construct a $p \times p$

chessboard with the associated numbering N_1, which assigns to the columns and rows of the chessboard the numbers 1 through p consecutively, from left to right and from top to bottom, respectively. That is to say, the column that is most to the left receives number 1, the column that is next to this one receives number 2, and so on until we get to the column that is most to the right which receives the number p. Also, the row that is located most to the bottom receives number 1. The row immediately on top of this one, receives number 2, and so on, until we reach the top most row, that receives number p. See the Figure 7.2, for geometrical clarification.

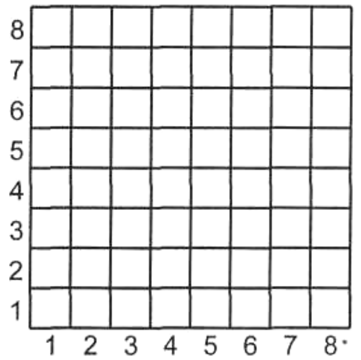

Figure 7.2: Numbered chessboard.

Next, we explain how to represent the function g on the $p \times p$ chessboard. In order to do this we will define the set S_1 in which each ordered pair belonging to S_1 represents a square of the representation of the function g on the $p \times p$ chessboard with numbering N_1. Thus,

$$\begin{aligned} S_1 \;=\; & \left\{(g(v_1^1), g(v_2^1)), (g(v_2^1), g(v_3^1)), \ldots, (g(v_{l_1-1}^2), g(v_{l_1}^1)), (g(v_{l_1}^1), g(v_1^1)),\right. \\ & (g(v_1^2), g(v_2^2)), (g(v_2^2), g(v_2^3)), \ldots, (g(v_{l_2-1}^2), g(v_{l_2}^2)), (g(v_{l_2}^2), g(v_1^2)), \ldots \\ & \left.(g(v_1^k), g(v_2^k)), ((g(v_2^k), g(v_3^k)), \ldots, (g(v_{l_k}^k - 1), g(v_{l_k}^k)), (g(v_{l_k}^k), g(v_1^k))\right\}. \end{aligned}$$

It is easy to see that the set $A_1 = \{i + j \mid (i, j) \in S_1\}$ is a set of $|S_1|$ consecutive integers, since the function g has the property that the set $\{g(u) + g(v) \mid uv \in E(G)\}$ is a set of $|E(G)|$ consecutive integers. Next, define a new numbering N_2 of the $p \times p$ chessboard as follows. For any column C of the chessboard, let $N_2(C) = N_1(C)$ and for any row R of the chessboard, let $N_2(R) = N_1(R) + p$. Then, let $S_2 = \{(x, y + p) \mid (x, y) \in S_1\}$ and observe that the set $A_2 = \{x + y \mid (x, y) \in S_2\}$ is also a set of $|S_2|$ consecutive integers since A_1, is a set $|S_1|$ consecutive integers. The next step is to define the graph $H \cong pK_2$ in the following way, let

$$V(H) = \{v_i \mid 1 \leq i \leq p\} \cup \{u_i \mid 1 \leq i \leq p\}$$

and

$$E(H) = \{u_i v_i \mid 1 \le i \le p\}.$$

Now, let h be any bijective function from the set C of connected components of H to the set S_2, $h : C \to S_2$ and define the bijective function $\bar{f} : V(H) \to \{1, 2, \ldots, |V(H)|\}$ as described below. If $h(u_i v_i) = (x, y)$ then $\bar{f}(u_i) = x$ and $\bar{f}(v_i) = y$. Then, the function $\bar{f}$ is extendable to a special super magic labeling of the graph H, since $\bar{f}$ meets the conditions of the function g described in Lemma 7.1.

The next example illustrates the construction described above.

Example 1: Consider the cycle C_5 with the bijective vertex labeling g shown in Figure 7.3.

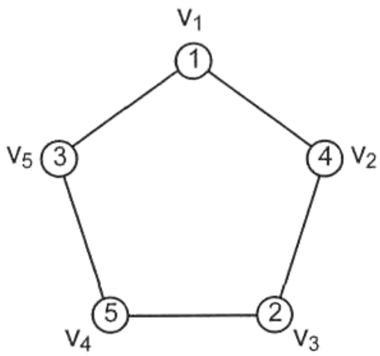

Figure 7.3: Labeled C_5

It is easy to see that $g : V(C_5) \to \{1, 2, 3, 4, 5\}$ is a bijective function with the property that $\{g(u) + g(v) \mid uv \in (G)\} = \{4, 5, 6, 7, 8\}$. Now, we construct a 5×5 chessboard, with numbering N_1 and we represent the function g on the chessboard after obtaining the set

$$S_1 = \{(1, 4), (4, 2), (2, 5), (5, 3), (3, 1)\}.$$

See Figure 7.4.

Next, we define the numbering N_2 on the 5×5 chessboard keeping the representation of g as shown in Figure 7.5.

Now, $S_2 = \{(1, 9), (2, 10), (3, 6), (4, 7), (5, 8)\}$

Consider the graph $5K_2$ represented in Figure 7.6.

And let h be the bijective function from the components of $5K_2$ to the set S_2 defined by the rule,

$$h((u_1 v_1)) = (1, 9)$$

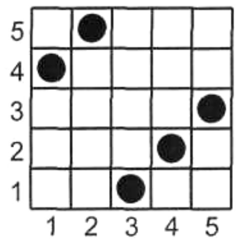

Figure 7.4: Chessboard representing g, numbered with N_1.

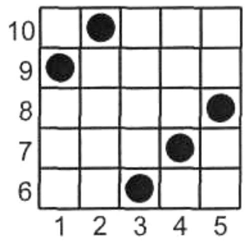

Figure 7.5: Chessboard representing g, numbered with N_2.

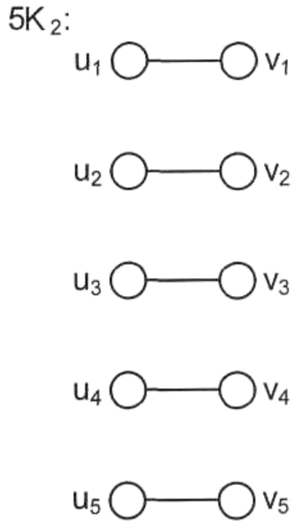

Figure 7.6

$$\begin{aligned} h((u_2v_2)) &= (2,10) \\ h((u_3v_3)) &= (3,6) \\ h((u_4v_4)) &= (4,7) \\ h((u_5v_5)) &= (5,8) \end{aligned}$$

hence, we get the labeling $\bar{f}$ of the vertices of $5K_2$ defined next, $\bar{f}(u_1) = 1$, $\bar{f}(u_2) = 2$, $\bar{f}(u_3) = 3$, $\bar{f}(u_4) = 4$, $\bar{f}(u_5) = 5$, $\bar{f}(v_1) = 9$, $\bar{f}(v_2) = 10$, $\bar{f}(v_3) = 6$, $\bar{f}(v_4) = 7$, $\bar{f}(v_5) = 8$.

Since the set $\{\bar{f}(u) + \bar{f}(v) \mid uv \in E(5K_2)\}$ is the set $\{9, 10, 11, 12, 13\}$ we can extend the function to a super magic labeling of $5K_2$, as shown in Figure 7.7.

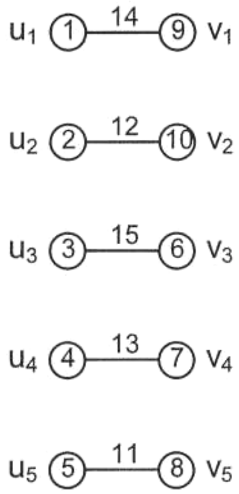

Figure 7.7

7.1.3 A Relation Between Super Magic Labelings of Graphs and Pseudo Graphs and Special Super Magic Labelings of Bipartite Graphs

It has become customary to define labelings of graphs only for simple graphs with no loops nor multiple edges. The goal in this section is to show that it may be useful to study super magic labelings of some types of pseudographs, that is to say graphs with loops. However, before doing this, it is necessary to extend the definition of super magic labelings of graphs to pseudographs. We do this in the obvious way.

Let P be a pseudograph, then a bijective function $f : V(P) \cup E(P) \rightarrow$

$\{1, 2, \ldots, |E(P)|\}$ is called a super magic labeling of P if the following two conditions hold,

I. $f(u) + f(uv) + f(v) = k$ for every uv in the set $E(P)$, and

II. $f(V(P)) = \{1, 2, \ldots, |V(P)|\}$.

We observe that if a pseudograph has attached more than a single loop to any of its vertices, then the pseudograph is not super magic. Thus, the pseudographs that we will consider are basically graphs with at most one loop attached to any given vertex. Some examples of super magic pseudographs (Pseudographs that admit super magic labelings) are showed in Figure 7.8, with their corresponding super magic labelings.

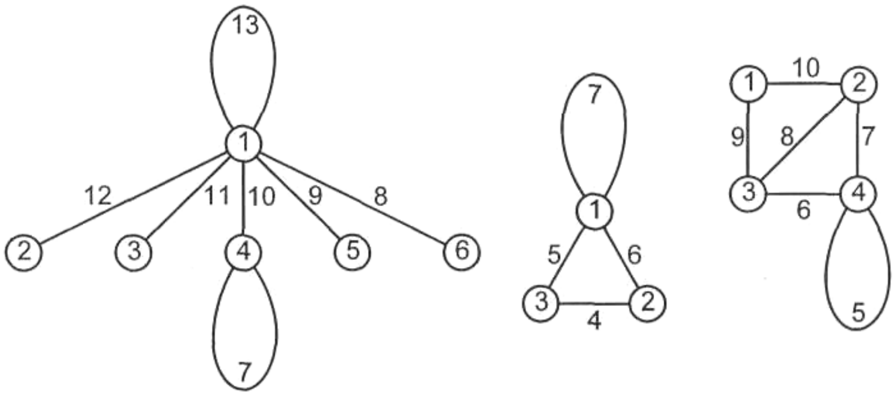

Figure 7.8

After looking at the previous examples, it is obvious that Lemma 7.1 generalizes immediately to pseudographs. Also, Theorem 7.2, generalizes to pseudographs as shown in Corollary 1, the proof of which will be omitted since it is similar to the proof of Lemma 2.2.

Corollary 7.3. *If a pseudograph P is super magic, then*

$$|E(P)| \leq 2\,|V(P)| - 1.$$

At this point, we are ready to establish the connection between super magic labelings of pseudographs and special super magic labelings of bipartite graphs.

Theorem 7.4. *Let P be either a pseudograph or a graph of order p and size q, and assume that $L : V(P) \to S \subset \mathbb{N}$ is a bijective vertex labeling of P. Then, there exists a bipartite graph H of order $2p$ and size q, and a bijective* *vertex labeling* $L^* : V(H) \to S \cup \{x + p \mid x \in S\}$ *such that there exists a natural number* γ, *with the property that* $\{L(x) + L(y) + \gamma \mid xy \in E(P)\} = \{L^*(x) + L^*(y) \mid xy \in E(H)\}$.

Proof.

Let P be either a graph or a pseudograph of order p and size q, and let $L : V(P) \to S \subseteq \mathbb{N}$ be a bijective function. Assume that

$$P \cong \left(\bigoplus_{i=1}^{k} P^i_{\alpha_i} \right) \oplus \left(\bigoplus_{i=1}^{r} C^i_{\beta_i} \right)$$

is any edge disjoint decomposition of P into k paths and r cycles (each loop will be considered as a cycle), and orient each path in such a way that we can "travel" from one "end" of the path to its other "end" following the direction of the arrows. Also arbitrarily orient each cycle either clockwise or counterclockwise.

Build $k + r$, $p \times p$, chessboards and assign to each chessboard N_1 the numbering defined next. The numbers $1, 2, \ldots, p$ will be assigned to the columns of the chessboard in such a way that the column placed left most, receives number 1. The column next to this one, receives number 2, and so on until we reach the column right most which will receive number p. Also N_1, assigns the numbers 1 through p to the rows 1 through p of the chessboard with number 1 being assigned to the row that is placed at the bottom. Number 2, being assigned to the row immediately on top of this one, and so on, until we reach the top row that will receive number p.

For each path $P^j_{\alpha_j} (1 \leq j \leq k)$ and for each cycle $C^j_{\beta_j} (1 \leq j \leq r)$ we will represent the functions $L \mid_{V\left(C^j_{\beta_j}\right)}$ and $L \mid_{V\left(P^j_{\alpha_j}\right)}$ respectively on different chessboards as described next. The square $\left(L \mid_{V(X)} (u), L \mid_{V(X)} (v)\right)$ will be chosen if and only if uv is an arc of X (where X is any path or cycle in the decomposition of P. Thus, the set S_1 of all ordered pairs that represent the squares chosen on the chessboards with numbering N_1, is the set

$$\begin{aligned} S_1 \;=\; & \big\{(L \mid_{V(X)} (u), L \mid_{V(X)} (v)) \mid uv \text{ is an arc of } X, \\ & \text{where } X \in \left\{P^i_{\alpha_i} \mid 1 \leq i \leq k\right\} \cup \left\{C^i_{\beta_i} \mid 1 \leq i \leq r\right\}\big\}. \end{aligned}$$

Define the set A_1 to be $A_1 = \{a + b \mid (a, b) \in S_1\}$. It is clear that $A_1 = \{L(u) + L(v) \mid uv \in E(P)\}$. Next, define a new numbering N_2 on the chessboards depending on the numbering N_1 as follows. For any column C of the chessboard, let $N_2(C) = N_1(C)$ and for any row R of the chessboard, let $N_2(R) = N_1(R) + p$. Thus, if we call S_2 to the set of the ordered pairs that represent the squares on the chessboard, but now, with respect to N_2, we have that $S_2 = \{(a, b+p) \mid (a, b) \in S_1\}$. Therefore, we have that the set A_2 defined as $A_2 = \{x + y \mid (x, y) \in S_2\}$, is basically a shift of the set A_1 by p units. That is to say, $A_2 = \{x + p : x \in A_1\}$. With all this information in mind, we are now ready to define the bipartite graph H as follows $V(H) = \{x_i\}_{i=1}^{2p}$, $E(H) = \{x_i x_j \mid (i, j) \in S_2\}$ and let $L^* : V(H) \to \{1, 2, \ldots, 2p\}$ be the bijective function defined by the rule $L^*(x_i) = i$ for every $x_i \in V(H)$. Then, L^* has the properties of the function described in the statement of the theorem. □

In order to clarify the previous proof, we will provide an example.

Example 2: Consider the graph G with the vertex labeling L shown in Figure 7.9 and let $G = C_{4,1} \bigoplus C_{4,2} \bigoplus P_{3,1} \bigoplus P_{1,2}$, where $C_{4,1}$, $C_{4,2}$, $P_{3,1}$, $P_{1,2}$ are as shown in Figure 7.10.

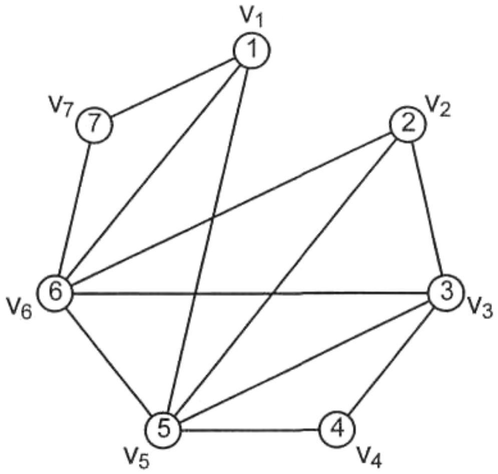

Figure 7.9

Now, give arbitrary orientations to the edges of G in such a way that the cycles are orientated either clockwise or counterclockwise, and the paths are oriented in such a way that we can travel from one end to the other following the direction of the arrows. We do this in Figure 7.11.

Then, construct 4, 7×7 chessboard with numbering N_1 and represent the functions $L\mid_{V(C_{4,1})}$, $L\mid_{V(C_{4,2})}$, $L\mid_{V(P_{3,1})}$ and $L\mid_{V(P_{1,2})}$ as shown in Figure 7.12.

Next, build 4 more 7×7 chessboards and label the rows and the columns with numbering N_2, keeping the same squares chosen. See Figure 7.13.

Finally, we construct in Figure 7.14, the graph H with function g.

As an immediate consequence to Theorem 7.4, we get that if G is an

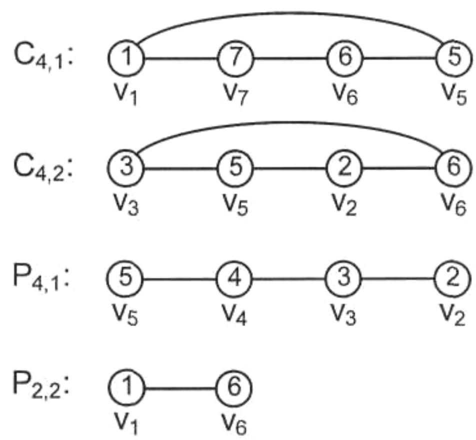

Figure 7.10

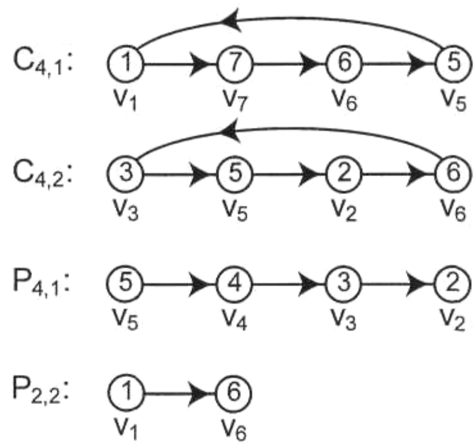

Figure 7.11

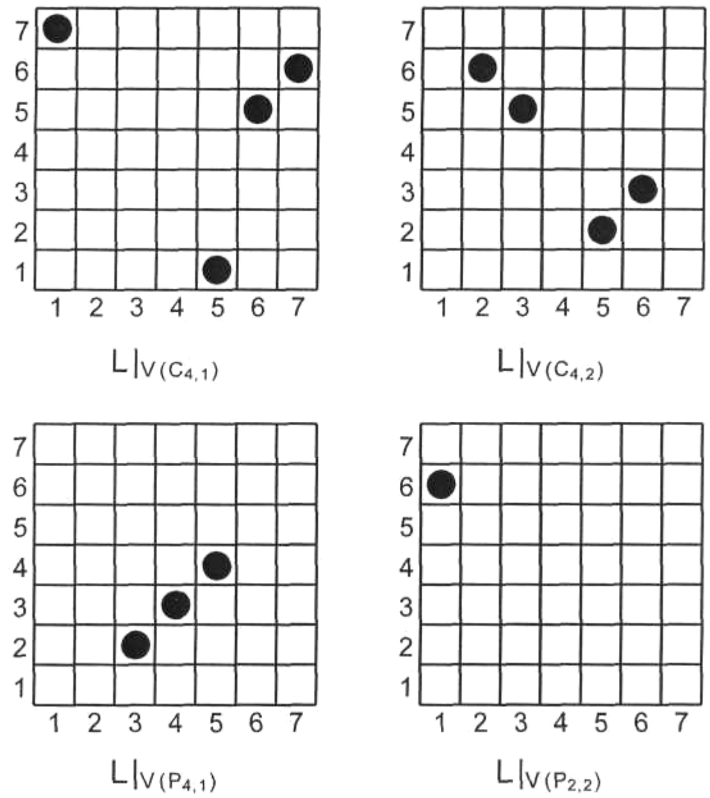

Figure 7.12

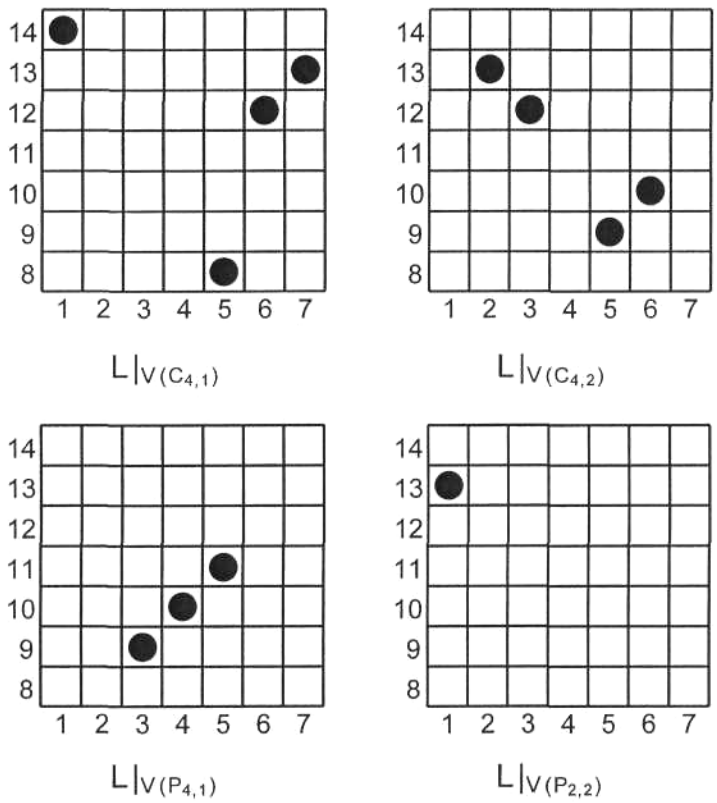

Figure 7.13

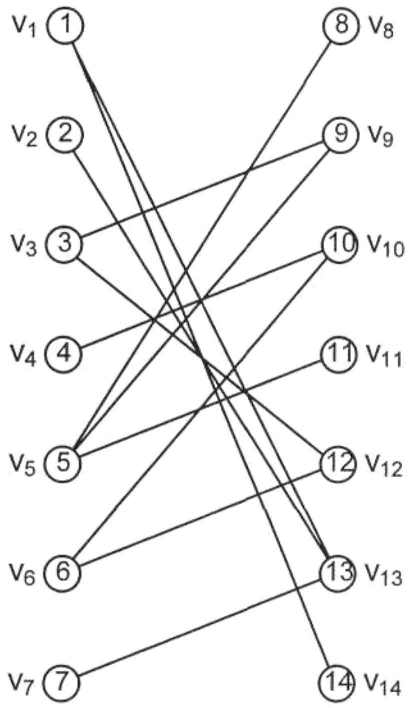

Figure 7.14

orientation of a super magic pseudograph of order p and size q. Then there exists a bipartite graph H of order $2p$ and size q which is special super magic. From now on, we will refer to this graph H obtained from G, as the bipartite graph H induced by G, and we will write it as H_G.

A natural question to ask is whether we can say anything about the bipartite graph H_G, from the properties of G. Although the structure of H_G has not been really studied yet, the following results are immediate from the discussion above.

Theorem 7.5. *If a digraph G contains a eulerian cycle, then H_G contains a perfect matching.*

Theorem 7.6. *If the digraph G can be edge decomposed into two subgraphs that contain eulerian cycles, then H_G contains a 2-regular spanning subgraph.*

Theorem 7.7. *Let G be a digraph with $V(G) = \{v_i \mid 1 \leq i \leq p\}$. Then if V_1 and V_2 are the bipartite sets of the graph H_G, the degree sequence of the vertices of V_1 is $\text{out}(v_1), \text{out}(v_2), \ldots, \text{out}(v_p)$ and the degree sequence of the vertices of V_2 is $\text{in}(v_i), \text{in}(v_2), \ldots, \text{in}(v_p)$.*

The next theorem provides an interesting corollary to the previous theorem.

Theorem 7.8. *If G is a (p,q)-graph with degree sequence $2k_1, 2k_2, \ldots, 2k_p$, then there exists $\vec{G}$ an orientation of G such that the graph $H_{\vec{G}}$ is a bipartite graph with bipartite sets V_1 and V_2 then the degree sequence of the vertices of V_1 is equal to the degree sequence of the vertices of V_2 and equal to $k_1, k_2, \ldots, k_p$.*

Proof.

Since all the degrees of the vertices of G are even, then G is decomposable into cycles (see [40], for example). Orient each cycle either clockwise or counterclockwise and apply the above theorem. □

7.2 Magical and Antimagical Product Labelings

7.2.1 Introduction

An $n \times n$ magic square is an $n \times n$ array consisting of all integers $1, 2, \ldots, n^2$ such that the sum of any row or column in the array is constant. It is known that there is an $n \times n$ magic square for every integer $n \geq 3$, see [2]. Steward [38] was motivated by the notion of magic squares to define vertex magic labelings. A graph G of size q is said to be vertex magic if there is a labeling from $E(G)$ onto $\{1, 2, \ldots, q\}$ such that, at each vertex v, the sum of the labels on the edges incident with v is constant. Such a labeling is called a vertex magic labeling. It is interesting to notice that if an $n \times n$ magic square is given, then it is possible to construct a vertex magic labeling of a complete bipartite graph $K_{n,n}$ for every integer $n \geq 3$, and vice versa.

Hartsfield and Ringel [24] introduced antimagic labelings as follows. A graph G of size q is said to be antimagic if there is a bijective labeling $f : E(G) \to \{1, 2, \ldots, q\}$ such that the sum of all the labels incident with each vertex are distinct given that the vertices are distinct. Such a labeling is called an antimagic labeling. Among the graphs known to be antimagic we find paths, cycles, complete graphs, and wheels. It is also easy to see, that K_2 is not antimagic. In fact, Hartsfield and Ringel [24] conjectured that all graphs other than K_2 are antimagic.

The last definition that will be presented in this introduction is the one given by Ringel and Lladó in [34]. A (p,q)-graph G is defined to be edge antimagic if there exists a bijective labeling $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ such that if v_1u_1 and v_2u_2 are any two different edges of G, then $f(v_1) + f(v_1u_1) + f(u_1) \neq f(v_2) + f(v_2u_2) + f(v_2)$. In their paper they included the result that every connected graph other than K_2 is