Universitat Jaume I

Departament de Matemàtiques

BOUNDED SETS IN TOPOLOGICAL GROUPS

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To my mother

Contents

Abstract

A boundedness structure (bornology) on a topological space is an ideal of subsets (that is, closed under taking subsets and unions of finitely many elements) containing all singletons. In this paper, we introduce certain functions (boundedness maps) as a tool in order to deal with the global properties of families of bounded subsets rather than the specific properties of its members. Our motivation is twofold: on the one hand, we obtain useful information about the structural features of certain remarkable classes of boundedness systems, cofinality, local properties, etc. For example, we estimate the cofinality of these boundedness notions. In the second part of the paper, we apply duality methods in order to estimate the size of a local base for important classes of groups. This translation, which is well-known in the Pontryagin-van Kampen duality theory of locally compact abelian groups, is often very useful and has been extended by many authors to more general classes of topological groups. In this work we follow basically the pattern and terminology given by Vilenkin in 1998.

Memoria

Resumen

Este proyecto investigador se estructura en torno a un doble objetivo final. Se trata por una parte de desarrollar una investigación cuyo propósito principal son los grupos topológicos y sus aplicaciones y, por otro lado, de estudiar la estructura de los grupos y sus cardinales topológicos invariantes mediante la aplicación de técnicas de dualidad. El primer objetivo es continuación del trabajo que se viene realizando durante muchos años sobre teoría de la dualidad, compactación de Bohr, espacios de funciones, etc. La segunda dirección de trabajo se fundamenta en una serie de publicaciones recientes que introducen la aplicación sistemática de técnicas de dualidad en el estudio de ciertos invariantes topológicos. Este planteamiento no puede trasladarse de forma simple a los grupos no abelianos puesto que la dualidad de los grupos no conmutativos requiere un gran bagaje matemático previo. No existe además una descripción de la dualidad de los grupos no conmutativos universalmente aceptada, sino que pueden encontrarse variantes muy diversas a la hora de presentarla, todas ellas técnicamente complicadas. Nuestro objetivo futuro

es extender las técnicas de dualidad para grupos no abelianos, analizando qué versiones de la dualidad encuentran mejor aplicación en la teoría.

1. Objeto y objetivos de la investigación: Uno de los logros fundamentales en el estudio de los grupos topológicos abelianos es la teoría de dualidad de Pontryagin-van Kampen. En breve, el resultado principal de esta teoría afirma que si G es un grupo abeliano localmente compacto y $\widehat{G}$ denota al grupo formado por los caracteres (homomorfismos que van a la circunferencia unidad del plano complejo y dotada del producto que hereda del plano complejo) continuos definidos sobre G y se le dota de la topología compacto abierta, entonces $\widehat{G}$ es también un grupo abeliano localmente compacto llamado grupo dual de G y además su dual, es decir, el grupo $\widehat{\widehat{G}}$ es topológicamente isomorfo a G. Este resultado ha sido extendido con posterioridad a varias clases de grupos topológicos abelianos y ha encontrado aplicación en otras partes de las Matemáticas, como la Teoría de Números y el Análisis Armónico (véase [74]).

La teoría de la dualidad de los grupos topológicos abelianos ha conducido a muchos e importantes resultados. Por ejemplo, proporciona una herramienta básica para la clasificación de los grupos localmente compactos abelianos y es la base sobre la que se asienta el Análisis Armónico Abstracto para los grupos conmutativos.

Es imposible dar un recuento exhaustivo de lo que se ha hecho en este campo a lo largo de los últimos años. Mencionamos simplemente algunas aportaciones recientes que ilustran el importante desarrollo que tiene este área de investigación. Banaszczyk (1994) extiende, en [7], la teoría de la dualidad a los grupos nucleares. Banaszczyk, Chasco y Martín-Peinador (1994), [9], demuestran que la reflexividad de un grupo topológico abeliano queda caracterizada por la reflexividad de sus subgrupos abiertos. Pestov (1986) [83] proporciona condiciones necesarias o suficientes para la reflexividad de un grupo topológico abeliano libre y descubre el importante papel desempeñado por los grupos de cohomotopía de los espacios considerados en el estudio de la dualidad de estos grupos. Posteriormente, Galindo y Hernández [43] obtienen una caracterización completa de la reflexividad de los grupos topológicos abelianos libres. En [57] se estudia la reflexividad de los grupos $C_p(X)$ de funciones continuas definidas sobre un espacio topológico cualquiera. Se demuestra que la reflexividad de estos grupos no puede caracterizarse en la axiomática de Zermelo-Franklyn y el axioma de elección. Uniendo los resultados obtenidos en Chasco [21], Galindo [41], Hernández [53], y Hernández y Trigos [56] se obtiene la caracterización definitiva de los grupos topológicos abelianos reflexivos y se presentan ejemplos que corrijen algunas caraterizaciones erróneas que habían aparecido previamente. Otras contribuciones recientes se dan en [6, 21, 23]. Finalmente, en [62], Hofmann y Morris estudian los grupos que son límites proyectivos de grupos abelianos localmente compactos (*pro-LCA*, para abreviar)

El objetivo más inmediato del proyecto investigador es estudiar cuando el conjunto $\Psi(G)$ es dominante (siendo $\Psi : G \to \omega^\omega$ definida como sigue: a todo x de G se le asocia la aplicación f_x, donde $f_x(n) = min\{m : x \in m{*}U_n\}$), ya que la resolución de este problema proporcionaría nuevas cuotas de la cofinalidad de la familia de acotados de un grupo topológico y mediante la teoría de dualidad de Pontryagin-van Kampen puede aplicarse al estudio de la estructura de los grupos topológicos abelianos (caracter, peso). Se han obtenido ya algunos resultados para grupos polacos, pero no se comentaron aquí por estar todavía en fase preliminar.

Las cuestiones que nos hemos propuesto investigar abordan problemas esenciales de la teoría de la dualidad de los grupos topológicos y su resolución supondría un gran avance en este área de las matemáticas.

2. Planteamiento y metodologia:

Metodología

La metodología que se siguió es la habitual del trabajo en Matemáticas :

1. Estudio riguroso de los conceptos, resultados y métodos relativos a cada problema.

2. Revisión profunda de los libros y artículos más recientes sobre el tema

3. Contactos periódicos con los investigadores del grupo de Análisis Matemático para discutir los problemas que se han ido encontrando.

4. Asistencia, con participación activa, a reuniones y congresos científicos de interés para nuestros temas de investigación.

En cuanto a las técnicas que se han utilizado en la investigación de los problemas que nos hemos propuesto, nos hemos apoyado principalmente en los métodos de los grupos topológicos, análisis armónico y funcional y la topología general.

Plan de trabajo

Para avanzar en los objetivos que nos hemos propuesto, el programa de trabajo a realizar fue el siguiente:

1. Estudio pormenorizado de las publicaciones más recientes relacionadas con la teoría de la dualidad, espacios métricos (acotaciones) y teoría de conjuntos.

2. Asistencia a los seminarios de investigación organizados por los miembros del grupo de Análisis Matemático del departamento de matemáticas.

3. Consultas técnicas, con el director de tesis y otros miembros del grupo de investigación.

4. Investigar nuevas técnicas que permitan obtener resultados originales que mejoren sustancialmente los ya conocidos.

5. Escribir los resultados que se obtengan y enviarlos para su publicación en revistas internacionales.

6. Exposición de los resultados más importantes en los congresos nacionales e internacionales relacionados con el tema de investigación.

Detallando por objetivos concretos el plan de trabajo que se propuso para poder atacar con garantías los problemas incluidos en el proyecto investigador, reseñaremos lo siguiente:

1. Para los problemas relacionados con la dualidad de los grupos abelianos, fueron fundamentales los trabajos de Hofmann y Morris [61, 62]. La tarea aquí consistió en establecer las bases teóricas suficientes para poder diferenciar claramente qué grupos pro-localmente compactos satisfacen la dualidad de Pontryagin-van Kampen.

2. En los problemas relacionados con la dualidad de los grupos no abelianos, fueron básicos los trabajos de Bekka [13], Doplicher y Roberts [27], French, Luukainen y Price [38], de la Harpe y Valette [51], Heyer [59], Hofmann y Morris [60], Galindo y Hernández [42] y Hernández y Wu [54].

Tratándose de un proyecto cuyos resultados previsibles están inmersos en el desarrollo de la ciencia básica, la aplicabilidad y utilidad práctica de los mismos debe entenderse en ese contexto principalmente. Sin embargo, el interés de los problemas tratados se relaciona también con la repercusión que tiene la teoría de grupos topológicos en la formulación y resolución de diversos problemas en distintas áreas de las Matemáticas, la Física, y Teoría de la Información.

La difusión de los resultados que se han ido obteniendo se hizo de la forma habitual en el campo de las matemáticas. En primer lugar, los resultados principales se enviarán para su publicación en revistas internacionales de amplia difusión. Además, fueron expuestos en ponencias presentadas a congresos nacionales e internacionales y otras reuniones científicas relacionadas con nuestra área de investigación. Finalmente, una parte de los resultados serán asimismo expuestos en conferencias especializadas en departamentos de matemáticas con los que mantenemos contacto.

Dada la experiencia investigadora de los miembros del equipo, los grupos nacionales o internacionales con los que mantenemos contacto se sitúan principalmente en el ámbito de los grupos topológicos, la topología general. Nuestra presencia en esos foros fue intensa intercambiando visitas frecuentes con otros investigadores y participando en todo tipo de actividades: organización de congresos, participación en congresos, etc. A modo de ejemplo, cabe mencionar que los resultados obtenidos fueron presentados en los siguientes congresos internacionales : *VII Congreso Iberoamericano de Topología y sus aplicaciones*, Valencia, 2008; *Set Theory, Topology and Banach Spaces: Second International Topology Conference in Kielce*, Polonia, 2008; *Advances in Set Theoric Topology*, Sicilia, 2008.

3. Aportaciones originales: La actividad investigadora que se ha llevado a cabo se ha centrado en el campo de la topología general y del análisis funcional. En particular, se han estudiado distintas bornologias (acotaciones) definidas en grupos topológicos metrizables y se han combinado estos resultados con la teoría de dualidad de Pontryagin-van Kampen para estimar algunos cardinales (peso, caracter, estrechez...) asociados a distintas clases de grupos topológicos y sus respectivos subconjuntos precompactos.

Respecto a los resultados que se han obtenido, se han investigado fundamentalmente problemas relacionadas con diferentes bornologias (acotaciones) que se pueden definir en un grupo topológico metrizable, centrando el estudio en la estructura de estas bornologias. Hemos avanzado en dos direcciones: por una parte, se ha obtenido información valiosa sobre las características topológicas del grupo y, por otra parte, la teoría de la dualidad de los grupos topológicos abelianos permite trasladar, o dualizar, la información obtenida para estudiar (caracterizar) las propiedades topológicas del grupo dual. Esta traslación se ha presentado ampliamente en la Teoría de la dualidad de Pontryagin-van Kampen de los grupos abelianos localmente compactos. N.Ya.Vilenkin define la noción de grupo con una acotación para grupos abelianos arbitrarios con el propósito de extender la dualidad de Pontryagin a grupos no localmente compactos (véase [35]). En esta investigación se ha trabajado con una definición más débil que la de Vilenkin, para grupos no necesariamente abelianos.

Sea G un grupo. Una *acotación* (o *bornología*) en G es una familia de subconjuntos de G, llamados *conjuntos acotados*, que satisfacen las siguientes condiciones:

(i) Todo subconjunto de un conjunto acotado es acotado.

(ii) Si A y B son acotados $A \cup B$ es acotado.

(iii) Todo conjunto finito es acotado.

Sea G un grupo y sea $\mathcal{A}$ una acotación en G. Entonces $(G, \mathcal{A})$ se llama *grupo con acotación*. Algunos ejemplos de estructuras de acotación son: la familia de subconjuntos finitos de cualquier grupo; la familia de subconjuntos precompactos de un grupo topológico; la familia de subconjuntos acotados (en el sentido usual) del grupo aditivo de un espacio vectorial topológico, etc. La acotación se preserva en subgrupos, cocientes y productos y sumas directas, de un modo natural (Véase [44]).

En nuestra investigación trabajamos con un tipo de estructuras de acotación que tienen asociadas ciertas funciones, muy útiles a la hora de caracterizar las propiedades de la acotación.

Burke y Todorcevic han estudiado en [17] las acotaciones de espacios vectoriales localmente convexos y metrizables. Resulta que el conjunto $\mathbb{N}^{\mathbb{N}}$ tiene un papel esencial en sus resultados, especialmente en aquellos relacionados con la cofinalidad. Nosotros hemos generalizado sus resultados estudiando la cofinalidad de las estructuras de acotación para grupos topológicos metrizables. En lo que sigue G será un grupo metrizable e (I, b) una bornología en G.

Sea $(\mathbb{P}, \leq)$ un conjunto parcialmente ordenado. Un subconjunto D se dice *cofinal* en $\mathbb{P}$ si $D \subseteq \mathbb{P}$, y para todo $p \in \mathbb{P}$ existe $d \in D$ tal que $p \leq d$. La *cofinalidad* de $\mathbb{P}$, denotada por $cof(\mathbb{P})$, es la mínima cofinalidad de un subconjunto cofinal en $\mathbb{P}$. Es bien conocido que, si se considera el orden puntual, es decir, para $f, g \in \mathbb{N}^{\mathbb{N}}$, $f \leq g$ significa que: $f(n) \leq g(n)$ para todo n, entonces $\mathfrak{d} = cof(\mathbb{N}^{\mathbb{N}})$.

Para una relación binaria R en $\kappa^{\mathbb{N}}$, fR^*g significa que: $f(n)Rg(n)$ para todos excepto un número finito de n. En particular $\mathfrak{b}$ denota el mínimo número cardinal de un subconjunto no acotado de $(\mathbb{N}^{\mathbb{N}}, <^*)$.

Véase [14] para más información sobre $\mathfrak{d}$ y otros cardinales relacionados. En particular, $\aleph_1 \leq \mathfrak{d} \leq 2^{\aleph_0}$, y para todo κ, λ dado, con $\aleph_1 \leq \kappa \leq \lambda$ y $cof(\lambda) > \aleph_0$, es consistente que $\mathfrak{d} = \kappa$ y $2^{\aleph_0} = \lambda$.

Dados dos conjuntos parcialmente ordenados $(X, \leq)$ e $(Y, \widetilde{\leq})$ se dice que $(X, \leq)$ tiene *cofinalidad mayor o igual que* $(Y, \widetilde{\leq})$ cuando existe una aplicación $\Phi : X \to Y$ que preserva el orden y tal que $\Phi(X)$ sea cofinal en Y. La *equivalencia cofinal* (cofinalmente equivalente) se define del mismo modo.

En lo que sigue presentaré algunos resultados destacados de nuestro trabajo. Aplicando varios resultados de Shelah y Gitik hemos obtenido la siguiente estimación de la cofinalidad de una estructura de acotación.

Theorem 1 *Sea G un grupo metrizable e (I, b) un sistema de acotación (bornología) en G. Entonces se cumplen las siguientes afirmaciones:*

1. *Si G es trivial o acotado, entonces $\mathcal{A}$ es cofinalmente equivalente a $\{0\}$.*

2. *Si G es localmente acotado tenemos que $\mathcal{A}$ es cofinalmente equivalente a $\mathcal{P}_{Fin}(I)$. Como consequencia $cof(\mathcal{A}) = |I|$.*

3. *Si G no es localmente acotado, entonces*

$$|I| \leq cof(\mathcal{A}) \leq max(\mathfrak{d}, cof(\mathcal{P}_{\aleph_0}(I)) \leq |I|^{\aleph_0}$$

   Además, cúando $\aleph_0 \leq |I| < \aleph_\omega$, se tiene que

$$|I| \leq cof(\mathcal{A}) \leq max(\mathfrak{d}, |I|),$$

   siendo I un conjunto de índices asociado a la bornología.

Corollary 1 *Sea G grupo de Baire separable y metrizable, recubierto por menos de $\mathfrak{b}$ subconjuntos precompactos. Entonces G es localmente precompacto.*

Entre los más destacados resultados, está el siguiente teorema sobre el caracter de algunos grupos abelianos libres.

Theorem 2 *Sea* $X = \bigcup_{n<\omega} X_n$ *un* κ_ω*-espacio. Si* $\kappa = sup\{w(X_n) : n < \omega\}$*, entonces se cumplen las siguientes afirmaciones:*

(i) $\kappa \leq \chi(A(X)) \leq max(\mathfrak{d}, cof(\mathcal{P}_{\aleph_0}(\kappa))) \leq \kappa^{\aleph_0}$.

(ii) Si $\aleph_0 \leq \kappa < \aleph_\omega$*, entonces se cumple*

$$\kappa \leq \chi(A(X)) \leq max(\mathfrak{d}, \kappa).$$

Además, para todo subconjunto precompacto (respectivamente Bohr compacto) P *de* $A(X)$ *se tiene que* $w(P) \leq \kappa$.

4. Conclusiones obtenidas y futuras líneas de investigación:

Dualidad no conmutativa: Al contrario de lo que ocurre con los grupos abelianos, la dualidad de los grupos no conmutativos presenta importantes dificultades adicionales. Las cuestiones más importantes que se tratan en este proyecto están relacionadas con el estudio de las representaciones unitarias. Hablando en términos generales, se trata de averiguar hasta qué punto las representaciones de un grupo topológico determinan la estructura algebraica y topológica del mismo.

Un elemento imprescindible en la definición de grupo dual de un grupo abeliano es el grupo formado por los números complejos de módulo unidad con la topología y producto usual (llamado *toro* de una dimensión y denotado por $\mathbb{T}$). Sin embargo, si consideramos cualquier grupo finito simple no abeliano, G, el único homomorfismo que puede definirse entre el grupo G y el toro $\mathbb{T}$ es el trivial. Esto obliga a reemplazar el grupo $\mathbb{T}$ por los grupos $U(n, \mathbb{C})$ de las matrices unitarias $n \times n$. Incluso, para algunos grupos, es necesario utilizar las representaciones en grupos de operadores unitarios de dimensión infinita. Otra dificultad muy importante que aparece en el establecimiento de la dualidad para grupos no conmutativos estriba en que el *objeto dual* asociado a un grupo topológico no abeliano ya no admite una estructura de grupo. Por ello, la teoría se divide en varias direcciones dependiendo del punto de vista adoptado para definir y estudiar el objeto dual a un grupo topológico. Por ejemplo, la dualidad de Tannaka-Kreĭn está ampliamente aceptada para los grupos compactos. Nuestra aproximación al problema se basa en la teoría introducida por Chu [25] que extiende y unifica la de Pontryagin-van Kampen y la de Tannaka-Kreĭn.

En 1963, Chu definió una dualidad para grupos maximalmente casi periódicos que contiene varios de los ingredientes que están presentes en la dualidad de Pontryagin-van Kampen. Esbozamos a continuación los elementos básicos de la teoría iniciada por Chu en [25].

En lo que sigue, $GL(n, \mathbb{C})$ denota al *grupo general lineal de orden* n, $\mathcal{U}(n)$ es el *grupo unitario de orden* n y se define $\mathcal{U} = \sqcup_{n<\omega} \mathcal{U}(n)$ (suma topológica). Si G es un grupo topológico, el símbolo $Rep_n(G)$ denota al conjunto de todas las representaciones (homomorfismos continuos) de G en $\mathcal{U}(n)$ provisto de la topología compacta abierta. Finalmente, el espacio

$Rep(G) = \sqcup_{n<\omega} Rep_n(G)$ (suma topológica) se llama *dual de Chu* de G. Una *casi representación* de G es una aplicación continua $Q : Rep(G) \longrightarrow \mathcal{U}$ que preserva las propiedades algebraicas de $Rep(G)$. El conjunto de todas las casi representaciones de G equipado con la topología compacto abierta tiene estructura de grupo topológico (con la operación producto punto a punto). Se llama *grupo casi dual de Chu* de G y se denota por $Rep(G)^\vee$. Se comprueba sin dificultad que la aplicación evaluación $\epsilon : G \longrightarrow Rep(G)^\vee$ es un homomorfismo de grupos que es inyectivo si y sólo si G es maximalmente casi periódico. Se dice que el grupo localmente compacto G satisface la dualidad de Chu si la aplicación evaluación ϵ es un isomorfismo de grupos topológicos (Chu [25], 1966). El problema de la dualidad de los grupos no conmutativos ha sido tratado por muchos autores a partir del trabajo de Chu [25]. Los primeros resultados fueron recogidos por Heyer en [59]. Posteriormente, la teoría de la dualidad fue investigada, entre otros, por Poguntke [84] tratando de resolver las aparentemente difíciles cuestiones que quedan abiertas. Entre las aportaciones recientes, destacamos las de W. French, J. Luukkainen, J. Price [38], S. Doplicher, J. Roberts [27] y M. Enock, J. Schwartz [31]. Bekka [13] ha obtenido recientemente resultados que relacionan la propiedad de Kazhdan de un grupo localmente compacto con su compactación de Bohr.

Recientemente, Galindo y Hernández, y [42], Hernández y Wu en [54] estudian cuestiones relacionadas con la dualidad y la topología de Bohr de los grupos discretos no abelianos, resolviendo algunas de las muchas cuestiones que permanecen abiertas en esta área de investigación. Como consecuencia, se han encontrado resultados que relacionan la teoría de la dualidad y la topología de Bohr con la de los grupos profinitos (esto es, grupos compactos y cero dimensionales). Esto abre una nueva línea de investigación de especial interés. Nuestra intención es aplicar la experiencia y los resultados obtenidos sobre la teoría de la dualidad en el estudio de algunas cuestiones relacionadas con nuestra investigación. Por ejemplo, muchos de los resultados obtenidos en la segunda parte de esta memoria se pueden generalizar al caso no conmutativo.

Códigos de grupo: La investigación realizada durante el primer año de disfrute de la beca ha dado lugar al estudio de las aplicaciones de la dualidad de Pontryagin a los códigos de grupo. El fruto inmediato más destacable que se ha derivado de ello es la realización del trabajo de investigación correspondiente al programa de doctorado del departamento de Matemáticas, el cuál fue defendido el 30 de mayo de 2007 ante un tribunal del departamento. Esta memoria está dedicada al estudio de los códigos de grupo según Forney y el objetivo principal es la aplicación de las técnicas de la teoría de dualidad de Pontryagin.

El teorema de dualidad de Pontryagin-van Kampen ha sido extendido por Kaplan a productos y sumas directas: el dual del producto directo de una familia de grupos topológicos abelianos es topológicamente isomorfo a la suma directa y viceversa. Aunque estos resultados tuvieron en su origen un interés dentro de la teoría de los grupos topológicos, se puede ver a lo largo de la memoria como tienen una aplicación a la teoría de códigos de grupo.

En el año 1948 Claude Shannon sentó las bases de una teoría basada en una nueva forma de modelar la información, dándole a la misma el carácter de magnitud cuantificable. Esta teoría, universalmente conocida bajo el nombre de Teoría de la Información, permitió abordar el procesamiento de los datos mediante procedimientos íntegramente matemáticos, ateniéndose únicamente a su contenido, sin necesidad de adentrarse en la problemática de su implementación. El estudio de la teoría de códigos está íntimamente ligado a una serie de nociones propias de las matemáticas tales como retículos, empaquetamientos de esferas, teoría de grupos, sumas exponenciales, la teoría de grafos o la geometría aritmética, así como otra serie de temas de procedencia variada tales como la teoría de la información, la criptografía, el álgebra computacional o el análisis armónico.

En este proyecto investigador se tratan principalmente los llamados *códigos de grupo*. Un código de este tipo es un subgrupo de un grupo $\mathcal{W}$, llamado *de Laurent*, que tiene la forma genérica $\mathcal{W} = \mathcal{U} \times \mathcal{V}$, donde $\mathcal{U}$ es un producto directo de grupos (en general localmente compactos) y $\mathcal{V}$ es una suma directa. Obviamente, la ley de composición es aquí el producto componente a componente (véase [36, 73]). Por ejemplo, sea G un grupo cualquiera y sea $G^{\mathbb{Z}}$ el producto directo cuyos elementos son sucesiones bi-infinitas con elementos de G. Entonces, todo subgrupo $\mathcal{C}$ de $G^{\mathbb{Z}}$ es un código de grupo. Un código de grupo puede verse también como un sistema dinámico en el sentido de Willems (véase [99, 102]). Muchas de las propiedades fundamentales de los códigos y sistemas lineales dependen únicamente de su estructura de grupo (véase [36, 72, 73]). En general, las técnicas que se considerarán en este proyecto se aplican a un cierto tipo de códigos, llamados *convolucionales*. En este tipo de códigos, el output (la salida) de un sistema es la convolución del input (entrada) con los estados del codificador convolucional, llamados registros.

La relevancia de la dualidad en estos códigos se basa en que, dado un código de grupo abeliano $\mathcal{C}$, existe siempre un *código dual* asociado $\mathcal{C}^{\perp}$ que se construye utilizando la teoría de dualidad de Pontryagin-van Kampen (cf. [37]). De este modo, los conceptos fundamentales de la teoría pueden *dualizarse*. Es decir, las propiedades de $\mathcal{C}$ se reflejan en propiedades duales de $\mathcal{C}^{\perp}$, siguiendo la tradición de dualización de propiedades en la teoría de dualidad de Pontryagin-van Kampen.

Uno de los objetivos de este trabajo es la profundización en los métodos de dualidad en el estudio de las propiedades y estructura de los códigos de grupo, desde la perspectiva ofrecida por la Teoría de dualidad de Pontryagin. Pretendemos ofrecer una aportación novedosa que, en aras del estudio de los códigos de grupo convolucionales, ayude a obtener una mejor comprensión de las características fundamentales de los mismos, dado el creciente interés que presentan en la actualidad.

Ya que un código de grupo $\mathcal{C}$ es básicamente un subgrupo de un cierto grupo $\mathcal{W}$, descrito anteriormente, se deduce que los elementos de $\mathcal{C}$ pueden verse como sucesiones cuyos índices están en $\mathbb{Z}$. Además, la dualidad de Pontryagin-van Kampen permite caracterizar $\mathcal{C}$ como el subgrupo formado por todas las sucesiones de $\mathcal{W}$ que están en el anulador del código dual $\mathcal{C}^{\perp}$. Es decir, $\mathcal{C}$ puede caracterizarse por medio de un conjunto de condiciones (*checks*).

Una consecuencia inmediata de esta relación de dualidad es que el dual de un código *completo* (es decir, un subgrupo cerrado de $\mathcal{U}$, por ejemplo $G^{\mathbb{Z}}$) es un código *finito* (es decir, un código que está contenido en $\mathcal{V}$, cuyas sucesiones tienen un número finito de elementos no nulos).

Utilizando la teoría de Pontryagin-van Kampen, se deducen otras relaciones fundamentales de dualidad entre las dinámicas de $\mathcal{C}$ y las dinámicas de $\mathcal{C}^{\perp}$. Por ejemplo, los espacios de estado de $\mathcal{C}$ actúan como grupos de caracteres de los espacios de estado de $\mathcal{C}^{\perp}$, y las propiedades de observabilidad de $\mathcal{C}$ coinciden con las propiedades de controlabilidad de $\mathcal{C}^{\perp}$. (Aquí, la observabilidad se define, al igual que en [72], como propiedad de un código y no de una representación de un espacio de estados, véase [100]).

Uno de los objetivos más importantes en la teoría de los códigos de grupo es la construcción de un codificador observacional minimal y de un generador minimal de síndromes/observador de estados para $\mathcal{C}$, basado en su estructura de observabilidad. Forney y Trott han demostrado en [37] que las técnicas de dualidad permiten realizar esta construcción utilizando técnicas de dualidad para grupos localmente compactos abelianos. Por otra parte, Fagnani y Zampieri en [32] han obtenido tales construcciones para códigos de grupo definidos sobre grupos finitos no abelianos en un contexto puramente algebraico. Asimismo, la construcción del codificador minimal presentada en [37], se aplica a códigos definidos sobre grupos no abelianos (finitos). Forney y Trott proponen en [37] el problema de extender los métodos de dualidad también a los grupos no abelianos. Uno de los objetivos más importantes de este proyecto es el de abordar este problema utilizando la experiencia de los miembros del grupo investigador en algunas líneas de investigación relacionadas con la dualidad de los grupos no conmutativos. Un caso particularmente interesante se presenta cuando consideramos códigos de convolución como duales de un *comportamineto dinámico* en el sentido de Willems (véase [98, 101]) tal como se desarrolla en [89] donde un código convolucional lineal de soporte finito se define como un *right-shift-invariant* subespacio de $\mathbb{F}^n[z]$ y su dual se describe como un subespacio *left-shift-invaiant* de $\mathbb{F}^n[[z]]$ (siendo $\mathbb{F}$ el grupo de Galois de orden 2). En esta situación se sabe que propiedades del código pueden describirse desde un punto de vista topológico. Por ejemplo, la completitud del código dual puede caracterizarse utilizando la topología de la convergencia puntual (véase [93]). Nuestra meta aquí es estudiar cómo las popiedades dinámicas de la aplicación *shift* restringida al código dual $\mathcal{C}^{\perp}$ inducen propiedades del código $\mathcal{C}$ o la existencia de algoritmos de decodificación.

Introduction and Motivation

The historical development of what is now called commutative and non-commutative harmonic analysis, i.e. the study of the representations of locally compact groups in Hilbert spaces is closely interwoven with three other themes, which were developed simultaneously: probability theory, number theory and theoretical physics.

The history of probability, number theory and mathematical physics is prior to the development of the idea of group, and there are parts of these theories where characters or Fourier theory intervene, such as Dirichlet series, the Poisson summation formula, solutions of partial differential equations with constant coefficients, generating functions and the Cauchy theory. Later the emergence of the group concept took place. Three developments were without any influence on one another until 1924: the history of theoretical physics in the second half of the XIXth century, up to the "old quantum theory"; the Lebesgue integral, integral equations and Hilbert's spectral theory; and finally the theory of group representations and their characters, for finite groups.

Topological groups were considered for the first time, in a particular case, by Lie. In the interval 1900-1910, Hilbert and others have studied the theory of topological groups in a more general case. The structure of topological groups represent a mixture of algebra and topology. They are groups and topological spaces at the same time, in such a way that group properties and the topological ones are compatible. Topological groups are a cornerstone subject of topological algebra and were the object of study of many mathematical researchers such as A.D.Alexandroff, N.Bourbaki, E.van Kampen, L.S.Pontryagin, W.W.Comfort, E.van Douwen, J.van Mill and others.

One important topic of the theory is that of free topological groups on Tychonoff spaces, with contributions like those of A.A.Markov, S.Kakutani, S.S.Morris, P.Nickolas and V.G.Pestov. Every Tychonoff space X can be considered as a closed subspace of a topological group $F(X)$ (called *the free group* on X) such that every continuous mapping of the space X into a space Y can be extended uniquely to a continuous homomorphism from $F(X)$ to $F(Y)$. The space X algebraically generates $F(X)$. If all groups involved are abelian, the group $F(X)$ is called *the free abelian group* on X and is denoted by $A(X)$. If X is a compact topological space and $C(X,\mathbb{T})$ is endowed with the compact open topology, then the dual group of $A(X)$ (the family of all continuous homomorphisms from $A(X)$ into $\mathbb{T}$ with the same topology) is topologically isomorphic to $C(X,\mathbb{T})$.

For an arbitrary topological group G, the Bohr compactification of G is a pair (bG, b) where bG is a compact Hausdorff group and b is a continuous homomorphism from G onto a dense subgroup of bG with the following universal property: for every continuous homomorphism h from G into a compact group K there is a continuous homomorphism h^b from bG into K extending h in the sense that $h = h^b \circ b$, that is, making the following diagram commutative:

$$\begin{tikzcd} G \arrow[rr, "b"] \arrow[dr, "h"'] & & bG \arrow[dl, "h^b"] \\ & K & \end{tikzcd}$$

The group bG is called the *Bohr compactification* of G and the topology that G receives from bG is called *Bohr topology*.

In the abelian case, the duality theory allows us to represent the Bohr compactification as a group of homomorphisms: if G is a topological abelian group and $\widehat{G}_d$ denotes its dual group equipped with the discrete topology then bG is topologically isomorphic to the dual group of $\widehat{G}_d$. The Bohr topology of an Abelian discrete group G coincides with the largest totally bounded group topology of the group. This topology coincides with the topology induced on G by the group $\mathrm{Hom}(G, \mathbb{T})$ of homomorphisms from G into the usual circle group $\mathbb{T}$.

One of the cornerstones of the study of abelian topological groups is Pontryagin-van Kampen duality theory. Let (G, τ) be an arbitrary topological abelian group. A *character* on (G,τ) is a continuous homomorphism χ from G to the complex numbers of modulus one $\mathbb{T}$. The pointwise product of two characters is again a character, and the set $\widehat{G}$ of all characters is a group with pointwise multiplication as the composition law. If $\widehat{G}$ is equipped with the compact open topology, it becomes a topological group $(\widehat{G},\widehat{\tau})$ which is called the *dual group* of (G,τ). There is a natural evaluation homomorphism ϵ: $G \longrightarrow \widehat{\widehat{G}}$ of G into its bidual group. We say that a topological abelian group (G,τ) satisfies *Pontryagin-van Kampen duality* or, equivalently, is *Pontryagin reflexive* (*reflexive* for short,) if the evaluation map ϵ is a topological isomorphism onto. Polar reflexivity has been characterized by Köthe in [69, §23 9]) and Kye [70], although the poof in the second paper is incomplete.

The Pontryagin-van Kampen theorems establish that defining $A = B = \widehat{\ }$, the dual functor on the category $\mathcal{L}$ of locally compact abelian groups, we obtain that (A,B) is a duality on $\mathcal{L}$. In other words, every LCA group is reflexive. An important consequence is that the subcategories of the compact abelian groups and of the discrete abelian groups are brought into duality by means of the functor $\widehat{\ }$. The Pontryagin-van Kampen duality theory is essential in the study of the topological structure of locally compact abelian groups and provides the basis for developing harmonic analysis on LCA groups.

P-groups are topological groups in which every G_δ-subset is open. Lindeloff P-spaces behave in many respects as compact Hausdorff spaces and are Raǐkov complete (see [4]).

In 1981, I.I.Guran introduced the class of $\aleph_0$-bounded topological groups and established their main properties (see [49]). This class is closed under products, taking subgroups and continuous homomorphic images being, thus, considered as an analogue of the Tychonoff spaces. It contains the class of topological subgroups of Lindeloff topological groups and was described as the class of subgroups of topological groups of countable cellularity (see [2]), that is, topological subgroups of arbitrary topological products of second-countable topological groups.

A. Weil introduced in 1937 precompact topological groups (called also totally bounded by analogy with metric spaces) in [96] as topological groups which can be covered by a finite number of translations of every neighborhood of the identity. It is immediate that compact topological groups are precompact and these, in turn, are $\aleph_0$-bounded. Discrete groups are precompact if and only if they are finite. In the same paper A. Weil established that the Weil completion of every precompact group is again a group and coincides with the Raǐkov completion.

Many interesting examples of topological groups are expressed as topological products of topological groups or subobjects of products, like direct sums or Σ-products. The *direct product* of a family $\{G_i\}_{i\in I}$ of groups, denoted by $\prod_{i\in I} G_i$, is the cartesian product of the groups, endowed with coordinatewise multiplication. The *direct sum* of a family $\{G_i\}_{i\in I}$ of groups, denoted by $\bigoplus_{i\in I} G_i$, is the subgroup of the direct product consisting of those elements which have a finite quantity of coordinates distinct from the neutral element. One can define the *Σ-product of* $\{G_i\}_{i\in I}$ as the subspace of $\prod_{i\in I} G_i$ formed by all points $x \in \prod_{i\in I} G_i$ with countably many nonzero coordinates.

The investigation in this memory was focused on the field of general topology and functional analysis. In particular, different bornologies defined on metrizable topological groups have been studied and the results obtained have been merged with the well-known Pontryagin-van Kampen duality theory in order to estimate several cardinals like weight, character and tightness, associated with certain classes of topological groups and their respective precompact subsets. Such translations are fairly common in the Pontryagin-van Kampen theory.

In view of correspondences like these, we have studied several notions related to different bornologies, focusing our attention on the structure of these objects.

A major goal of our research was to obtain valuable information on the topological features of the groups involved. A second goal, which we touch only briefly here, was to translate the information obtained and apply it to the study (characterization) of topological properties of the dual group, by means of the duality theory of abelian topological groups.

We digress for a moment to comment on the meaning of "bornology" in a Tychonoff space in the name of our subject. In principle, "bornology" refers to a cover of the space which is closed under finite unions and is hereditary.

We remark in particular that the first approach to this notion was that of S.-T.Hu in [64] and [65], although his definition is more general (he didn't require the family to be a cover of the space).

A major application of bornologies has been as useful tools in the study of several important structures. One example is the topologization of the group of characters of a given group endowed with the pointwise convergence topology in which as bounded there are declared only the finite subsets (see [39]).

A different range of applications are the generalizations of the Attouch-Wets convergence (see [71] and [11]), as well as the development of a unified theory of differentiation (see [15] and [16]).

That bornologies have a relevant importance both in abstract approximation theory and in the theory of locally convex spaces follows already from [12] and [63], respectively. For a detailed description we refer to the introduction of [10].

Another aspect of the relevance of the bounded sets also deserves a brief digression. The extension of the Pontryagin duality to non-locally compact groups encounters a series of obstacles. Yet, the introduction by Vilenkin in [97] of groups with boundedness allows one to obtain analogs of the well-known duality theorems for groups which are not Cech-complete.

In the realm of metric spaces, bounded sets are defined in an obvious way. Nevertheless, recalling that every metric space can be re-metrized to become bounded without changing its uniform structure, we see that in metric spaces boundedness is not an invariant notion even if we consider uniformly continuous one-to-one mappings. Therefore, in this context, the usual concept of boundedness is not a relevant one.

Another environment where bounded sets have been defined involves locally convex topological linear spaces, where the equivalence of the definitions given by J.v.Neumann in [77] and A. Kolmogorov in [68] can be easily seen.

Along the years, there have been introduced more useful definitions of boundedness, like those which appear in [5] and [52]. Their definition coincides with the usual boundedness of the Euclidean spaces and locally convex topological linear spaces and remains valid for arbitrary uniform spaces. The boundedness defined like this contains totally bounded, hence compact or finite sets.

Following Vilenkin, we define here a convenient concept of boundedness for topological groups which generalizes all so far-known notions of boundedness. Further, it is shown that the theory of characters of locally quasi convex abelian groups leads to results on several topological cardinals.

Our methods allow us to consider both locally bounded and non-locally bounded groups, in a sense that will be explained further. Obviously, the class of locally bounded groups include locally compact as well as locally precompact groups.

There are certain constraints in the theory. For example, one of the disadvantages of local boundedness is that it is not hereditary, that is this property is not preserved in subspaces. This fact is witnessed by the existence of normed linear spaces containing a group which is not locally bounded in itself (see [67]). However, the property is preserved in factor groups.

More important, there is a locally bounded group (in the usual sense, see [52]) whose character group is not locally bounded, therefore it is not possible to develop the theory of characters of locally bounded groups in the same way as for locally compact groups. For this reason, as an application, we consider the dualizing methods on certain classes of abelian topological groups, with emphasis on the so called locally pk_ω groups.

Summary

The present memory is divided into three parts.

Chapter 1. Preliminary results and terminology.

The first chapter has a preliminary nature. It incorporates the notation, definitions and basic notions which are going to be used as tools along this investigation, and recalls mainly concepts well-known for most readers. In this chapter we are not going to impose any conditions of algebraic nature on the bornology. Our basic references are [4] for topological groups, [10],[52], [64], [97] for boundedness, [7] for locally quasi convex groups (there are lots of results in the field) and [1], [14] for cofinality, where all assertions given in this introduction are proved. It is pointed out when using a different terminology in this memory could lead to misunderstandings due to the fact that other authors might have assigned a different sense to the same term.

Chapter 2. Cofinality of boundedness structures.

In this chapter we introduce bounded sets in a topological group by means of certain functions which allow us to characterize and finally *dualize* their fundamental properties. We shall see, in due course, that this definition generalizes most so far known concepts of boundedness available in the literature. Let G be a topological group, I an abstract set and consider a map $b : I \times \mathcal{P}(G) \to \mathcal{P}(G)$. We say that $B \subseteq G$ is *bounded* when $\forall U \in \mathcal{N}(e_G)$

there is a finite set $F_U \subseteq I$ such that $B \subseteq b(F_U, U)$. The family $Bdd(G)$ of all bounded subsets of G contains all finite and compact subsets, while the precompact subsets need not be bounded when G is not complete.

By a *boundedness system* in G we mean a pair (I, b) with several restrictions on the map b. If the map b satisfies these axioms, then the family $Bdd(G)$ of all bounded subsets of G is a boundedness (or bornology in the sense of [10]), that is a family of subsets of G which is closed under taking subsets and unions of finitely many elements and contains all finite subsets of G.

In Subsection 2.2.1, following Saxon and Sánchez-Ruíz in [91] and, independently, Burke and Todorcevic in [17], we extend the study of the boundedness structures from metrizable locally convex vector spaces to the class of metrizable topological groups. It turns out that the set $\mathbb{N}^{\mathbb{N}}$ plays a crucial role in these results, especially those dependent on the cofinality. We obtain that for a metrizable group G endowed with a boundedness system (I, b) $cof(Bdd(G)) \geq |I|$.

In Subsection 2.2.2 we define κ*-bounded* groups. In case G is metrizable, we assume without loss of generality that G is κ-bounded, with $\kappa = |I|$. With this in mind, we proceed to the study of the cofinality of the collection $Bdd(G)$ of all bounded subsets of G.

In Subsections 2.2.3 and 2.2.4 we make a few remarks about the so-called *locally bounded* and *non locally bounded* groups. In Subsection 2.2.3 we define *locally bounded* groups and prove that if G is a metrizable locally bounded group endowed with a boundedness system (I, b), then $cof(Bdd(G)) = |I|$. For metrizable non locally bounded groups, the cofinality of the collection of bounded subsets may also be estimated in some cases.

Further, in Subsection 2.2.5 we present our main result on the cofinality of a boundedness structure defined on an arbitrary metrizable topological group G:

Let G be a metrizable group endowed with a boundedness system (I, b). Then the following assertions are true.

1. If G is trivial or bounded then $Bdd(G)$ is cofinally equivalent to $\{0\}$.

2. If G is locally bounded then $Bdd(G)$ is confinally equivalent to $\mathcal{P}_{Fin}(I)$. As a consequence we have $cof(Bdd(G)) = |I|$.

3. If G is a non locally bounded group then we have

$$|I| \leq cof(Bdd(G)) \leq max(\mathfrak{d}, cof(\mathcal{P}_{\aleph_0}(I)) \leq |I|^{\mathbb{N}}$$

   Furthermore, in case $\aleph_0 \leq |I| < \aleph_\omega$, we have

$$|I| \leq cof(Bdd(G)) \leq max(\mathfrak{d}, |I|).$$

In Section 2.3 we have considered an interesting property which can arise in the context $H \subset G$ with G a topological group and H a dense subgroup. It is shown how the bounded subsets of every metrizable topological group are completely determined by the bounded subsets of any dense subgroup. Special cases of it were proved in [48] and [17]. It is proved that the metrizability condition is essential, providing a counterexample.

In Section 2.4, as an immediate consequence of the cofinality results obtained so far, we have an interesting characterization of a special class of metrizable topological groups called Baire-like. We assume two additional conditions on the map b and show that if G is a separable, metrizable, Baire-like group that is covered by less than $\mathfrak{b}$ precompact subsets, then G is locally precompact.

Chapter 3. Topological invariants of topological abelian groups.

In this third chapter we take a topological approach in order to establish the relationship between the boundedness structures object of our investigation and three invariant cardinals characterizing the dual group: weight, character and tightness. We shall see how every piece of information about the boundedness structure of a locally quasi-convex topological group G is contained in some piece of information about its dual group and viceversa. For example, if the dual group is κ-bounded, we shall see that the bounded subsets of G have weight less then or equal to κ.

In this chapter, the Pontryagin-van Kampen duality, defined for topological abelian groups, is given in terms of the precompact-open topology, that is, we consider the duality of topological abelian groups when the topology of the dual is the precompact-open topology. We characterize the $p$recompact weakly reflexive groups as those topological groups satisfying the weak group duality defined in terms of the precompact-open topology. We shall consider only locally quasi-convex topological groups.

In Section 3.2 we obtain that for a locally quasi-convex group (G, τ) with a boundedness $\mathcal{A}$, if $(\widehat{G}, \tau_{\mathcal{A}}(G))$ is κ-bounded, then $w(A) \leq \kappa$ for every $A \in \mathcal{A}$. This result is a variant of the approach given in [33, 34] for locally convex spaces and countable cardinality. Applied to the boundedness of precompact subsets, the result above states that the precompact subsets of a locally quasi-convex group G have weight less than or equal to a cardinal κ if and only if the group $(\widehat{G}, \tau_{pc}(G))$ is κ-bounded.

Unfortunately the structure of locally quasi-convex topological groups in general is not so pleasant, so we are going to apply our methods on certain subclasses of these objects.

In Section 3.3 we study weakly reflexive groups, with emphasis on *pk-groups*, which are a particular case of the well-known concept of a k-group, defined in [81] for general topological groups. A group G is a *pk-group* if the topology of G is determined by its precompact subsets, that is, $C \subseteq G$ is closed if and only if $C \cap P$ is closed in P for each precompact subset P of G. We introduce pk_γ-groups and show that a locally quasi-convex group G is weakly reflexive if and only if it is a pk_γ-group. We obtain that for a weakly reflexive locally quasi-convex abelian group G such that $\widehat{G}_{pc}$ is metrizable and locally precompact $\chi(G) = sup\{w(P) : P \in Prec(\widehat{G})\}$.

In Subsection 3.4.2 we study hemi-precompact groups. The main result states that:

If G is a locally hemi-precompact group and G_0 is an open subgroup of G such that $G_0 = \bigcup\limits_{n<\omega} P_n$, where $\{P_n\}_{n<\omega}$ is a co-base for the precompact subsets in G_0, then if $\kappa = sup\{w(P_n) : n < \omega\}$, the following assertions are true:

(i) If G_0 is locally precompact then $\chi(G) = \kappa$.

(ii) If G_0 is not locally precompact then we have:

(a) $\kappa \leq \chi(G) \leq max(\mathfrak{d}, cof(\mathcal{P}_{\aleph_0}(\kappa))) \leq \kappa^{\aleph_0}$,

(b) finally, in case $\aleph_0 \leq \kappa < \aleph_\omega$, we have

$$\kappa \leq \chi(G) \leq max(\mathfrak{d}, \kappa).$$

Afterwards, we have applied the results above in order to estimate the character of some free abelian groups. A completely different approach to this question has been given by Nickolas and Tkachenko in [79, 80].

In Section 3.4.3 we are going to apply our methods on certain topological groups that contain the well-known class of k_ω-groups. By a *k-space* we understand a Hausdorff topological space X admitting a cover Ω by compact subspaces, generating the topology of X in the sense that a subset $O \subset X$ is open in X if and only if $O \cap K$ is relatively open in K for every $K \in \Omega$. In this case we say that X *is determined by* the cover Ω. Further, a space X is called a *k_ω-space* (see [75]) if it is determined by a countable cover of compact subsets.

We say that a locally quasi-convex group G is a *pk_ω-group* when it is hemi-precompact and pk-group. The class of locally pk_ω abelian groups contains among others locally k_ω-groups and locally precompact groups. In this section we extend a result of Saxon and Sanchez-Ruiz [91, Corollary 2] for the strong dual of metrizable spaces to topological abelian groups.

As for Section 3.5, it is well-known that there are difficulties in translating results that hold for locally convex spaces to topological groups, and viceversa, mostly because of the lack of a result like the Hahn-Banach theorem for groups and because few groups are divisible. We however have been able to translate to the field of topological groups some results obtained previously for locally convex spaces.

Chapter 1

Preliminary results and terminology

In this section we introduce notation and some results required along the presentation. Our basic references are [4] for topological groups, [10],[52], [64], [97] for boundedness, [7] for locally quasi convex groups (there are lots of results in the field) and [1], [14] for cofinality, where all assertions given in this introduction are proved. Some basic definitions are recalled and it is pointed out when using a different terminology in this memory could lead to misunderstandings due to the fact that other authors might have assigned a different sense to the same term.

1.1 Topological groups

The one dimensional torus $\mathbb{T}$ will be identified with the set of complex numbers of modulus 1, endowed with the usual product of complex numbers. From here on 0 will usually represent the neutral element of any abelian group, except for the group $\mathbb{T}$ that is usually treated with multiplicative notation (his identity element will by denoted by 1). For the sake of simplicity, we will use both additive and multiplicative notations depending on the nature of the specific structure we deal with at any point. We shall also use the notation e_G to denote the neutral element of an arbitrary group G. By $\mathcal{N}_G(g)$ (or $\mathcal{N}(g)$ when there is no possibility of confusion) we denote the filter of neighborhoods of g in G.

For every subset A of a group G, we shall denote by $< A >$ the subgroup of G generated by A, that is the smallest subgroup of G that contains A.

A subset A of a topological group G is called *dense* in G if and only if $\overline{A} = G$ if and only if every non-empty open subset of G contains points of A. Discrete topological groups have no proper dense subgroups. The *density* $d(G)$ of a group G is the minimal cardinality of a dense subset. A topological group G is called *separable* if it contains a countable dense subset, that is to say, if $d(G) = \aleph_0$.

The set of all cardinal numbers of the form $\mid \mathcal{B} \mid$, where $\mathcal{B}$ is a base for a topological space (X, τ), is well-ordered by $<$ and has a smallest element. This cardinal number is called the *weight* of the topological space (X, τ) and is denoted by $w(X, \tau)$ (or, if there is no possibility of confusion, by $w(X)$).

We refer to [30] for unexplained topological definitions.

1.2 Inductive limits

The general concept of inductive limit of a family of topological groups was introduced by Varapoulos in [95]. Consider an abstract group G. A family of topological groups $\{(H_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ and a family of homomorphisms $h_\alpha : H_\alpha \longrightarrow G, \alpha \in \Lambda$, induce a group topology over G : the finest group topology that makes all h_α continuous. This topology is called *the inductive limit of the topologies* τ_α and shall be denoted by $\lim\limits_{\rightarrow} \tau_\alpha$.

There are several examples or restrictions of this concept. When $H_\alpha = G$ for every $\alpha \in \Lambda$ (and so each h_α is the identity homomorphism), the inductive limit is called *intersection of topologies* and is denoted by $\bigcap_{\alpha \in \Lambda} \tau_\alpha$.

If the family $\{(H_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ is a direct system of topological groups, that is if Λ is a directed set with a partial order $\preceq$ and if for every $\alpha_1 \preceq \alpha_2 \in \Lambda$ we can define a continuous homomorphism $p_{\alpha_1\alpha_2} : (H_{\alpha_1}, \tau_{\alpha_1}) \to (H_{\alpha_2}, \tau_{\alpha_2})$ such that $\alpha_1 \preceq \alpha_2 \preceq \alpha_3$ implies $p_{\alpha_2\alpha_3} \circ p_{\alpha_1\alpha_2} = p_{\alpha_1\alpha_3}$, then we can associate to $\{(H_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ the following set

$$S :=< \{x_\alpha - p_{\alpha\beta}(x_\alpha) : x_\alpha \in H_\alpha \text{ and } \alpha \preceq \beta \in \Lambda\} >\subseteq \bigoplus_{\alpha\in\Lambda} H_\alpha,$$

which is a subgroup of the direct sum of the groups H_α. Then, the quotient $(\bigoplus_{\alpha\in\Lambda} H_\alpha)/\overline{S}$ will be called *the direct limit of the system* $\{(H_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$.

Further, if we define for every $\alpha \in \Lambda$ the map $h_\alpha : H_\alpha \to (\bigoplus_{\alpha\in\Lambda} H_\alpha)/\overline{S}$ as the composition of the injection from H_α to $\bigoplus_{\alpha\in\Lambda} H_\alpha$ with the canonical epimorphism of $\bigoplus_{\alpha\in\Lambda} H_\alpha$ into $(\bigoplus_{\alpha\in\Lambda} H_\alpha)/\overline{S}$, then the inductive limit topology $\varinjlim \tau_\alpha$ (defined previously) determined by the direct system $\{(H_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ and the homomorphisms h_α coincides with the topology of the direct limit of the system. Moreover, if the topology τ_α is discrete for every $\alpha \in \Lambda$, the inductive limit of the family $\{(H_\alpha, \tau_\alpha)\}$ coincides with the usual concept of direct or inductive limit of groups (see [40]).

When instead of Λ we consider the set $\mathbb{N}$, each H_n is a closed subgroup of H_{n+1}, $G = \bigcup_{n=1}^\infty H_n$ and for every $n < \omega$ we take the inclusion $i_n : H_n \to G$, then the inductive limit determined by the applications $i_n : H_n \to G$ is *the strict inductive limit* of the family $\{H_n\}_{n=1}^\infty$. The strict inductive limit topology is the finest topology which makes all injections from H_n to G continuous. An example of strict inductive limit is the direct sum $\bigoplus_{n=1}^\infty G_n$ of a countable collection of topological groups, which coincides with the strict inductive limit of the family of groups $\{\bigoplus_{j=1}^n G_j\}_{n=1}^\infty$.

It was proved by Varapoulos in [95] that if $G = \cup_{n=1}^\infty G_n$ is the strict inductive limit of a family$\{G_n\}_{n=1}^\infty$ of topological groups and $\mathcal{U}_n$ is a neighborhood basis of the identity in each G_n for $n \prec \omega$, then a basis of neighborhoods of the identity in G is given by

$$\mathcal{U} := \{\cup_{p=1}^{\infty}(V_1 + ... + V_p) : V_n \in \mathcal{U}_n,\ for\ every\ n < \omega\}.$$

If $\{(G_n, \tau_n) : n < \omega\}$ is a sequence of topological groups such that $G_n \subseteq G_{n+1}$ and $\tau_{n+1|G_n} = \tau_n$ and if (G, τ) is the strict inductive limit of the family G_n, then the topology $\tau_{|G_n}$ coincides with the topology τ_n of G_n.

1.3 Boundedness

N. Ya. Vilenkin introduced the notion of group with a boundedness for an arbitrary abelian group in order to extend Pontryagin duality to non locally compact groups (see [97]). Next we consider here a notion weaker than the definition given by Vilenkin for non necessarily abelian groups.

Let G be a group. Vilenkin defines a *boundedness* on G as a family $\mathcal{A}$ of subsets of G, called *bounded sets*, satisfying the following conditions:

(i) Subsets of bounded sets are bounded.

(ii) If A and B are bounded so is $A \cup B$.

(iii) Finite sets are bounded.

If a group G is endowed with a boundedness $\mathcal{A}$, then $(G, \mathcal{A})$ is called a *group with a boundedness*. Examples of boundedness structures are: the family of all finite subsets of any group; the family of precompact subsets of a topological group; the family of all bounded sets (in the usual sense) of the additive group of a topological vector space, and so on.

The concept of boundedness is preserved in subgroups, quotients, products and direct sums in a natural way. If $(G, \mathcal{A})$ is a group with a boundedness and H is a subgroup of G, then we denote by $\mathcal{A}_{|H} := \{B \cap H : B \in \mathcal{A}\}$ the boundedness induced by G over H.

If, moreover, H is closed and $p : G \to G/H$ is the canonical epimorphism, then $(G, \mathcal{A})$ induces over G/H a boundedness $\mathcal{A}(G/H) := \{p(B) : B \in \mathcal{A}\}$. If we consider a family $\{(G_i, \mathcal{A}_i) : i \in I\}$ of groups with boundedness, then a boundedness can by defined in the group $\sqcap_{i \in I} G_i$ consisting of those subsets $B \subseteq \sqcap_{i \in I} G_i$ such that $\pi_i(B) \in \mathcal{A}_i$ for every $i \in I$ and denoted by $\sqcap_{i \in I} \mathcal{A}_i$. Similarly, we can define a boundedness $\bigoplus_{i \in I} \mathcal{A}_i$ on the direct sum $\bigoplus_{i \in I} G_i$, considering those subsets $B \subseteq \bigoplus_{i \in I} G_i$ such that $\pi_i(B) \in \mathcal{A}_i$ for every $i \in I$ and $\pi_j(B) = \{0\}$ for all but a finite number of indices $j \in I$.

For every sequence $\{(G_n, \mathcal{A}_n) : n < \omega\}$ of groups with boundedness, we can define on $G = \bigcup_{n=1}^{\infty} G_n$ a boundedness $\underrightarrow{\lim}\, \mathcal{A}_n$ consisting of those subsets $B \subseteq G$ such that there is $n_0 \in \mathbb{N}$ with $B \subseteq G_{n_0}$ and $B \in \mathcal{A}_{n_0}$.

1.4 Duality for groups

Let (G, τ) be an abelian topological group, with underlying group G and topology τ. A character of G is a τ-continuous group homomorphism from G into the unit circle $\mathbb{T}$, the latter equipped with the usual product as composition law and with the topology inherited from the usual complex plane. The *character (dual) group of* G, denoted by $\widehat{G}$, is defined by

$$\widehat{G} := \{\chi : G \to \mathbb{T} \,|\, \chi \text{ is a character}\},$$

with group operation defined pointwise:

$$(\chi_1\chi_2)(x) := \chi_1(x)\chi_2(x)\; \forall x \in G.$$

The topology on $\widehat{G}$ of *uniform convergence on the compact sets* (denoted by τ_c and also known as the compact-open topology) is the topology whose basic open sets are of the form

$$(K, U) := \{\chi \in \widehat{G} : \chi[K] \subset U\},$$

where K is a compact subset of G and U is open in $\mathbb{T}$.

If instead of compact we consider K to be precompact, then we obtain the topology τ_{pc} of *uniform convergence on the precompact sets* (or the precompact-open topology). Both $(\widehat{G}, \tau_c)$ and $(\widehat{G}, \tau_{pc})$ are abelian topological groups.

It is said that G satisfies *Pontryagin-van Kampen duality* if the evaluation map

$$E_G : G \to (\widehat{(\widehat{G}, \tau_c)}, \tau_c),$$

defined by

$$E_G(g)(\chi) := \chi(g) \ \ \forall g \in G,$$

is a surjective topological isomorphism. The famous theorem of *Pontryagin-van Kampen* states that every locally compact abelian group satisfies Pontryagin-van Kampen duality. If G satisfies Pontryagin-van Kampen duality, we say that G is Pontryagin-van Kampen-*reflexive.* The class just defined contains also some non locally compact groups since it is closed under arbitrary products, as proved by KAPLAN [66].

In our investigation, we focus instead on those topological groups G such that the map

$$E_G : G \to (\widehat{(\widehat{G}, \tau_{pc})}, \tau_{pc}),$$

defined by

$$E_G(g)(\chi) := \chi(g) \ \ \forall g \in G,$$

is a surjective topological group isomorphism. These groups are called *precompact reflexive.*

1.5 Locally quasi - convex groups

Locally quasi-convex groups were defined by Vilenkin in [97]. This theory started with locally convex spaces, that is topological vector spaces which have a basis of neighborhoods of zero consisting of convex sets. Applying the Hahn-Banach theorem, if F is a closed convex subset of a locally convex space E, for every x in $E\backslash F$ there is a continuous linear functional f and a real number α such that $f(x) > \alpha$ and $f(y) \leqslant \alpha$, for every $y \in F$. Although topological vector spaces considered with their additive structure are abelian topological groups, convexity is not meaningful in the realm of topological groups.

Nevertheless, there has been defined a similar notion called "quasi-convexity". A subset A of a topological abelian group G is *quasi-convex* if every point of $G \setminus A$ can by separated from A through a continuous character, that is,

$$\forall g \in G \setminus A, \exists \chi \in A^0 \text{ such that } Re(\chi(g)) < 0.$$

We denote by A^0 the set $\{\chi \in \widehat{G} : Re\chi(a) \geq 0,\ \forall a \in A\}$.

The *quasi-convex hull* of A in G is defined as

$$^0(A^0) := \{g \in G : Re(\chi(g)) \geq 0, \forall \chi \in A^0\}.$$

It follows that A is quasi-convex if and only if $A =^0 (A^0)$. We also have the following property :

$$A \subseteq B \Rightarrow B^0 \subseteq A^0 \Rightarrow^0 (A^0) \subseteq^0 (B^0).$$

Further, a topological group is called *locally quasi-convex* if it admits a basis of quasi-convex neighborhoods of zero. It follows from the definition that a Hausdorff locally quasi-convex group has sufficiently many continuous characters.

Banaszczyk proved that a topological vector space is locally convex if and only if, considered with its additive structure, it is a Hausdorff locally quasi-convex group (see [7, Proposition 2.4]).

If a topological abelian group G is topologically isomorphic to the dual of some topological group H, then G is locally quasi-convex. Examples of locally quasi-convex groups are: nuclear groups, reflexive groups, locally compact abelian groups and locally convex spaces. It can be easily proved that the class of locally quasi-convex groups is closed under taking subgroups and arbitrary products but not under taking quotients, as we can see by the following example. Every normed space X contains a discrete, weakly dense subgroup Y and therefore the quotient X/Y is nontrivial and verifies $(X/Y)^\wedge = \{0\}$ ([8]).

1.6 Infinite cardinals

The classical theorem of Cantor in [18] states that the cardinality $\mathfrak{c} = 2^{\aleph_0}$ of the continuum is strictly larger then the cardinality $\aleph_0$ of a countably infinite set.

In what follows, we shall use the notation ZFC for Zermelo-Fraenkel set theory including the axiom of choice, CH for the continuum hypothesis ($\mathfrak{c} = \aleph_1$) and GCH for the generalized continuum hypothesis ($2^{\aleph_l} = \aleph_{l+1}$ for each cardinal $\aleph_l$). If CH is false, then there are cardinals strictly between $\aleph_0$ and $\mathfrak{c}$.

Following [94], consider the set of functions $\mathbb{N}^{\mathbb{N}}$ from $\mathbb{N}$ into $\mathbb{N}$ endowed with the quasi-order $\leq^*$ defined by

$$f \leq^* g \; if \; \{n \in \mathbb{N} : f(n) > g(n)\} \; is \; finite.$$

A subset C of $\mathbb{N}^{\mathbb{N}}$ is said to be *cofinal* if for each $f \in \mathbb{N}^{\mathbb{N}}$ there is some $g \in C$ with $f \leq^* g$. A subset of $\mathbb{N}^{\mathbb{N}}$ is said to be *unbounded* (respectively *dominating*) if it is unbounded (respectively cofinal) in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. The cardinality, or *size*, of a subset A is denoted by $|A|$. One defines

$$\mathfrak{b} = min\{|B| : B \text{ is an unbounded subset of } \mathbb{N}^{\mathbb{N}}\}$$

and

$$\mathfrak{d} = min\{|D| : D \text{ is a dominating subset of } \mathbb{N}^{\mathbb{N}}\},$$

yielding $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$.

If instead of $f \leq^* g$ we consider $f \leq g$, that is $f(n) \leq g(n)$ for all $n \in \mathbb{N}$, the value of $\mathfrak{b}$ would be $\aleph_0$. As for $\mathfrak{d}$, it wouldn't change its value. Indeed, let D be a $\mathfrak{d}$-sized dominating subset of $\mathbb{N}^{\mathbb{N}}$. Thus, given any $f \in \mathbb{N}^{\mathbb{N}}$, there exists $g \in D$ with $f(n) \leq g(n)$ for almost all $n \in \mathbb{N}$. Now the set $\mathbb{D} = \{mg : m \in \mathbb{N} \textit{ and } g \in D\}$ still has size $\aleph_0 \cdot \mathfrak{d} = \mathfrak{d}$.

All that is known about cardinals $\mathfrak{b}$ and $\mathfrak{d}$ in ZFC is resumed in the following result.

$$\aleph_1 \leq cof(\mathfrak{b}) = \mathfrak{b} \leq cof(\mathfrak{d}) \leq \mathfrak{d} \leq \mathfrak{c}.$$

In what follows, we are going to present briefly some basic results on cofinalities that will be used later in this memory (see[24]). The *cofinality* of a set is the minimal cardinality of a cofinal subset. In particular, $\mathfrak{d} = cof(\mathbb{N}^{\mathbb{N}})$. We shall denote by $\mathcal{P}_{Fin}(I)$ the set of finite parts of I while $\mathcal{P}_{Fin}(I)^{\mathbb{N}}$ stands for the family of all functions from $\mathbb{N}$ to $\mathcal{P}_{Fin}(I)$.

See [14] for a survey concerning $\mathfrak{d}$ and related cardinals. In particular, $\aleph_1 \leq \mathfrak{d} \leq 2^{\aleph_0}$, and for all prescribed κ, λ with $\aleph_1 \leq \kappa \leq \lambda$ and $cof(\lambda) > \aleph_0$, it is consistent that $\kappa = \mathfrak{d}$ and $\lambda = 2^{\aleph_0}$.

In order to estimate the cofinality of the set $\mathcal{P}_{Fin}(I)^{\mathbb{N}}$ notice that $cof(\mathcal{P}_{Fin}(I)^{\mathbb{N}})$ depends only on $\kappa = |I|$ and when I is finite, $cof(\mathcal{P}_{Fin}(I)^{\mathbb{N}}) = 1$. Thus, we consider only infinite cardinalities, and by *cardinal* we always mean an infinite cardinal. The crucial observation is that $cof(\mathcal{P}_{Fin}(\kappa)^{\mathbb{N}})$ can be expressed in terms of two well studied cofinalities of partial orders.

Definition 1.6.1 Let κ and λ be two cardinal numbers and define

$$\mathcal{P}_{\lambda}(\kappa) = \{A \subseteq \kappa : |A| \leq \lambda\}.$$

This family is partially ordered by $\subseteq$.

Lemma 1.6.2 *Let λ and κ be cardinal numbers such that $\lambda \leq \kappa$. Then*

$$cof(\mathcal{P}_{Fin}(\lambda)^{\mathbb{N}}) \leq cof(\mathcal{P}_{Fin}(\kappa)^{\mathbb{N}}),$$

and

$$cof(\mathcal{P}_{\aleph_0}(\lambda)) \leq cof(\mathcal{P}_{\aleph_0}(\kappa)).$$

Proof. Let D be cofinal in $\mathcal{P}_{Fin}(\kappa)^{\mathbb{N}}$, that is $|D| = cof(\mathcal{P}_{Fin}(\kappa)^{\mathbb{N}})$. For each $f \in D$, define $g_f \in \mathcal{P}_{Fin}(\lambda)^{\mathbb{N}}$ by $g_f(n) = f(n) \cap \lambda$ for all n. Then $\{g_f : f \in D\}$ is cofinal in $\mathcal{P}_{Fin}(\lambda)^{\mathbb{N}}$, that is

$$cof(\mathcal{P}_{Fin}(\kappa)^{\mathbb{N}}) \geq |\{g_f : f \in D\}| = |D| = cof(\mathcal{P}_{Fin}(\kappa)^{\mathbb{N}}).$$

The proof for $\mathcal{P}_{\aleph_0}(\kappa)$ is similar.

Lemma 1.6.3 *For each cardinal κ we have $cof(\mathcal{P}_{\aleph_0}(\kappa)) \leq cof(\mathcal{P}_{Fin}(\kappa)^{\mathbb{N}})$.*

Proof. Assume that D is cofinal in $\mathcal{P}_{Fin}(\kappa)^{\mathbb{N}}$, that is $|D| = cof(\mathcal{P}_{Fin}(\kappa)^{\mathbb{N}})$. Define

$$E := \left\{ \bigcup_{n\in\mathbb{N}} f(n) : f \in D \right\}.$$

Then E is cofinal in $\mathcal{P}_{\aleph_0}(\kappa)$: Given $A \in \mathcal{P}_{\aleph_0}(\kappa)$, choose a surgection $f : \mathbb{N} \to A$, and define $g \in \mathcal{P}_{Fin}(\kappa)^{\mathbb{N}}$ by $g(n) = \{f(n)\}$ for all $n \in \mathbb{N}$. Since D is cofinal in $\mathcal{P}_{Fin}(\kappa)^{\mathbb{N}}$, there must be $h \in D$ be such that $h \supseteq g$. Recalling that f is a surgection, we obtain $A = \bigcup_n f(n) = \bigcup_n g(n) \subseteq \bigcup_n h(n) \in E$.

It is now proved that E is cofinal in $\mathcal{P}_{\aleph_0}(\kappa)$, that is

$$cof(\mathcal{P}_{\aleph_0}(\kappa)) \leq |E| = |D| = cof(\mathcal{P}_{Fin}(\kappa)^{\mathbb{N}}).$$

For $f \in \mathcal{P}_{Fin}(\mathbb{N})^{\mathbb{N}}$, let $\max f \in \mathbb{N}^{\mathbb{N}}$ be the function with value $\max f(n)$ at each n, where $\max \emptyset$ is defined to be 0.

Lemma 1.6.4 $\mathfrak{d} \leq cof(\mathcal{P}_{Fin}(\mathbb{N})^{\mathbb{N}})$.

Proof.

Let D be a cofinal subset of $\mathcal{P}_{Fin}(\mathbb{N})^{\mathbb{N}}$ such that $|D| < cof(\mathbb{N}^{\mathbb{N}})$. Then $E := \{\max f : f \in D\}$ is not cofinal in $\mathbb{N}^{\mathbb{N}}$. Let $g \in \mathbb{N}^{\mathbb{N}}$ be a witness for that. Define $h \in \mathcal{P}_{Fin}(\mathbb{N})^{\mathbb{N}}$ by $h(n) = \{g(n)\}$. Since D is cofinal in $\mathcal{P}_{Fin}(\mathbb{N})^{\mathbb{N}}$, there must be $d \in D$ such that $d \supseteq h$. It follows that $g \subset max\ d \in E$ which is absurd.

Theorem 1.6.5 *For each cardinal number κ we have that $cof(\mathcal{P}_{Fin}(\kappa)^{\mathbb{N}}) = \max(\mathfrak{d}, cof(\mathcal{P}_{\aleph_0}(\kappa)))$.*

Proof.

In order to prove this theorem, observe firstly that the mentioned cofinalities are monotonically nondecreasing with κ.

Lemmas 1.6.2 and 1.6.4 imply that for each κ, $cof(\mathcal{P}_{Fin}(\kappa)^{\mathbb{N}}) \geq \mathfrak{d}$. It remains to show that $cof(\mathcal{P}_{Fin}(\kappa)^{\mathbb{N}}) \leq \mathfrak{d} \cdot cof(\mathcal{P}_{\aleph_0}(\kappa))$.

For each $s \in \mathcal{P}_{\aleph_0}(\kappa)$, fix a surjection $r_s : \mathbb{N} \to s$. Let $F \subseteq \mathbb{N}^{\mathbb{N}}$ and $S \subseteq \mathcal{P}_{\aleph_0}(\kappa)$ both be cofinal. For each $f \in \mathbb{N}^{\mathbb{N}}$ and each $s \in \mathcal{P}_{\aleph_0}(\kappa)$, define

$$g_{f,s}(n) = \{r_s(0), r_s(1), \ldots, r_s(f(n))\}$$

for all n.

We claim that $D := \{g_{f,s} : f \in F, s \in S\}$ is cofinal in $\mathcal{P}_{Fin}(\kappa)^{\mathbb{N}}$. Indeed, given $h \in \mathcal{P}_{Fin}(\kappa)^{\mathbb{N}}$, choose $s \in S$ such that $\bigcup_n h(n) \subseteq s$. Choose $f \in F$ such that

$$\min\{m : h(n) \subseteq \{r_s(0), \ldots, r_s(m)\}\} \leq f(n)$$

for all n. Fix n. Let m be the minimal such that $h(n) \subseteq \{r_s(0), \ldots, r_s(m)\}$. Then $m \leq f(n)$, and thus

$$h(n) \subseteq \{r_s(0), \ldots, r_s(m)\} \subseteq \{r_s(0), r_s(1), \ldots, r_s(f(n))\} = g_{f,s}(n).$$

This completes the proof of the theorem.

It remains to estimate $cof(\mathcal{P}_{\aleph_0}(\kappa))$.

Lemma 1.6.6 *Let κ be a cardinal number such that $\kappa > \aleph_0$. Then, $\kappa \leq cof(\mathcal{P}_{\aleph_0}(\kappa)) \leq \kappa^{\aleph_0}$.*

Proof. Generally,

$$cof(\mathcal{P}_{\aleph_0}(\kappa)) \leq |\mathcal{P}_{\aleph_0}(\kappa)| = \kappa^{\aleph_0}.$$

For the other inequality, note that if $\mathcal{F} \subseteq \mathcal{P}_{\aleph_0}(\kappa)$ and $|\mathcal{F}| < \kappa$, then $|\bigcup \mathcal{F}| \leq |\mathcal{F}| \cdot \aleph_0 < \kappa$, and thus $\bigcup \mathcal{F} \neq \kappa$. In particular, $\mathcal{F}$ is not cofinal in $\mathcal{P}_{\aleph_0}(\kappa)$.

Recall that if λ is an arbitrary cardinal number and $\kappa = \lambda^{\aleph_0}$, then $\kappa^{\aleph_0} = \kappa$. In particular, $\kappa = 2^\lambda$ implies that $\kappa^{\aleph_0} = \kappa$. Moreover, $\kappa^{\aleph_0} = \kappa$ implies that $(\kappa^+)^{\aleph_0} = \kappa^+$ (where κ^+ denotes the cardinal subsequent to κ).

Corollary 1.6.7 *Let κ be a cardinal number such that $\kappa^{\aleph_0} = \kappa$. Then, we have*

$$cof(\mathcal{P}_{Fin}(\kappa)^{\mathbb{N}}) = cof(\mathcal{P}_{\aleph_0}(\kappa)) = \kappa.$$

Proof. If $\kappa^{\aleph_0} = \kappa$, then $\kappa \geq 2^{\aleph_0} \geq \mathfrak{d}$. Apply Theorem 1.6.5 and Lemma 1.6.6.

Lemma 1.6.8 *Let κ be a cardinal number such that $\kappa > \aleph_0$. Then, we have*

$$cof(\mathcal{P}_{\aleph_0}(\kappa)) = \kappa \cdot \sup\{cof(\mathcal{P}_{\aleph_0}(\lambda)) : \lambda \leq \kappa, cof(\lambda) = \aleph_0\}.$$

Proof. $(\geq)$ Apply monotonicity and Lemma 1.6.6.

$(\leq)$ If $cof(\kappa) = \aleph_0$, this is just Lemma 1.6.6.

If $cof(\kappa) > \aleph_0$, then each countable subset of κ is bounded in κ. Thus,

$$\mathcal{P}_{\aleph_0}(\kappa) = \bigcup_{\alpha<\kappa} \mathcal{P}_{\aleph_0}(\alpha),$$

and therefore $cof(\mathcal{P}_{\aleph_0}(\kappa)) \leq \kappa \cdot \sup\{cof(\mathcal{P}_{\aleph_0}(\lambda)) : \lambda < \kappa\}$. The statement for $\kappa = \aleph_1$ follows, and by induction, for each $\lambda < \kappa$ with $\lambda \geq \aleph_1$,

$$\begin{aligned} cof(\mathcal{P}_{\aleph_0}(\lambda)) &= \lambda \cdot \sup\{cof(\mathcal{P}_{\aleph_0}(\mu)) : \mu \leq \lambda, cof(\mu) = \aleph_0\} \leq \\ &\leq \kappa \cdot \sup\{cof(\mathcal{P}_{\aleph_0}(\mu)) : \mu \leq \kappa, cof(\mu) = \aleph_0\}. \end{aligned}$$

Corollary 1.6.9 *For each cardinal number κ such that $\aleph_0 < \kappa < \aleph_\omega$ we have that:*

1. $cof(\mathcal{P}_{\aleph_0}(\kappa)) = \kappa$.
2. $cof(\mathcal{P}_{Fin}(\kappa)^{\mathbb{N}}) = \max(\mathfrak{d}, \kappa)$.

Proof. (1) Induction on κ, using Lemma 1.6.8.

(2) By (1) and Theorem 1.6.5.

We have now that for each cardinal number κ, the following relation holds:

$$cof(\mathcal{P}_{Fin}(\kappa)^{\mathbb{N}}) = \max(\mathfrak{d}, cof(\mathcal{P}_{\aleph_0}(\kappa))).$$

Chapter 2

Cofinality of boundedness structures

In this chapter we define bounded sets in topological groups and consider their fundamental properties. In our research we look at a kind of boundedness structures that are associated with certain maps which are very useful in order to understand the properties of the boundedness.

2.1 Basic facts

Let G be a topological group. Suppose that $\mathcal{N}(e_G)$ denotes the neighborhood filter of the identity e_G in G and $\mathcal{P}(G)$ is the set of parts of G.

Let I be an abstract set and consider a map $b : I \times \mathcal{P}(G) \to \mathcal{P}(G)$. For each $A \subseteq G$ and each $J \subseteq I$, define $b(J, A) := \bigcup_{i \in J} b(i, A)$.

Definition 2.1.1 We say that $B \subseteq G$ is *b-bounded* (or simply B is *bounded*, if there is no possibility of confusion) when $\forall U \in \mathcal{N}(e_G)$ there is a finite set $F_U \subseteq I$ such that $B \subseteq b(F_U, U)$.

The family $Bdd(G)$ of all bounded subsets of G is considered with the partial order $\subseteq$. Obviously, finite and compact subsets are always bounded, while the precompact subsets need not be bounded when G is not complete.

Definition 2.1.2 By a *boundedness system* in G we mean a pair (I, b) consisting in an abstract set I and a map $b : I \times \mathcal{P}(G) \to \mathcal{P}(G)$ with the following properties:

(Bdd 1) If A is open then $b(i, A)$ is open for each $i \in I$.

(Bdd 2) $b(I, U) = G$, for each $U \in \mathcal{N}(e_G)$.

(Bdd 3) If A is bounded then $b(j, A)$ is bounded $\forall j \in I$, that is,

$A \subseteq b(F_U, U)$, with $|F_U| < \aleph_0$ $\forall U \in \mathcal{N}(e_G)$ implies that

$b(j, A) \subseteq b(F_j^U, U),$ *with* $|F_j^U| < \aleph_0$ $\forall U \in \mathcal{N}(e_G)$ $\forall j \in I$.

(Bdd 4) If $A \subseteq B \subseteq G$ then $b(i, A) \subseteq b(i, B)$ $\forall i \in I$.

Remark 1 If the map b satisfies axioms $(Bdd1) - (Bdd4)$, then the family $Bdd(G)$ of all bounded subsets of G is a boundedness (or bornology in the sense of [10]), that is a family of subsets of G which is closed under taking subsets and unions of finitely many elements and contains all finite subsets of G.

Remark 2 When considering a boundedness system in a group G, we assume w.l.o.g. that the set I has minimal cardinality in the sense that we can restrict b to a set $J \subseteq I$ ($| J | < | I |$) of minimal cardinality such that $b(J, U) = G \ \forall U \in \mathcal{N}(e_G)$.

Remark 3 In order to avoid trivialities, we shall assume that G is not bounded, that is the set I is infinite.

We shall now consider some examples of boundedness structures defined by maps.

Example 2.1.3 Let G be a topological group. Let D be a dense subset of G. Consider the map $b : D \times \mathcal{P}(G) \to \mathcal{P}(G)$ defined by translations $b(d, A) = dA = \{da : a \in A\}$. Then the pair (D, b) defines the boundedness $Prec(G)$ of all precompact subsets of G. Note that G can be endowed with a boundedness system (I, b) if the density of G is less than or equal to $|I|$.

Example 2.1.4 Let G be a connected multiplicative topological group. Consider the map $b : \mathbb{N} \times \mathcal{P}(G) \to \mathcal{P}(G)$ defined by $b(n, A) = A^n = \{a_1 a_2 ... a_n : a_1, a_2, ..., a_n \in A\}$. The subset $H = \bigcup\limits_{n\in\mathbb{N}} U^n = \bigcup\limits_{n\in\mathbb{N}} b(n, U) = b(\mathbb{N}, U)$ is an open (therefore, closed) subgroup of G, for all open and symmetric $U \in \mathcal{N}(e_G)$. Since G is connected, we have $H = G$. It can be easily verified that the pair $(\mathbb{N}, b)$ defines a boundedness on G.

Example 2.1.5 Let E be a topological vector space. Take $I = \mathbb{N}$ and $b(n, A) = \{na, a \in A\}$ for each $A \subseteq E$. It is well known that the map $b : \mathbb{N} \times \mathcal{P}(E) \to \mathcal{P}(E)$ defines the usual bounded subsets of E. Another option is to consider E a locally convex topological vector space and define $b(n, A) = nA = \{a_1 + a_2 + ... + a_n : a_1, a_2, ..., a_n \in A\}$ for each $A \subseteq E$. The resulting bounded sets are the same as before.

Example 2.1.6 In [52] Hejcman combines the first and third cases above and associates with every topological group G a canonical boundeness that can be described by the map $b : (D \times \mathbb{N}) \times \mathcal{P}(G) \to \mathcal{P}(G)$ such that $b(d, n, A) = dA^n$, where D is a dense subset of G. The bounded sets are the standard bounded sets when G is a topological vector space, and the precompact subsets when G is a locally compact group.

Definition 2.1.7 Two boundedness systems are said to be *equivalent* when they define the same collection of bounded subsets.

Remark 4 For instance, in examples 2.1.3 and 2.1.6, all dense subsets give equivalent boundedness systems.

2.2 Cofinality of boundedness structures

We begin by recalling some definitions that will be needed along this presentation.

Definition 2.2.1 Let $(\mathbb{P}, \leq)$ be a partially ordered set. A subset $D \subseteq \mathbb{P}$ is *cofinal*(in $\mathbb{P}$) if for each $p \in \mathbb{P}$ there is $d \in D$ such that $p \leq d$. The *cofinality* of $\mathbb{P}$, denoted $cof(\mathbb{P})$, is the minimal cardinality of a cofinal set in $\mathbb{P}$.

It is well known that $\mathfrak{d} = cof(\mathbb{N}^{\mathbb{N}})$ with respect to the pointwise order, that is, for $f, g \in \mathbb{N}^{\mathbb{N}}$, $f \leq g$ means: $f(n) \leq g(n)$ for all n.

For a binary relation R on $\kappa^{\mathbb{N}}$, fR^*g means: $f(n)Rg(n)$ for all but finitely many n. In particular, $\mathfrak{b}$ denotes the minimal cardinal number of an unbounded subset in $(\mathbb{N}^{\mathbb{N}}, <^*)$, as mentioned before.

Definition 2.2.2 Given two partially ordered sets $(X, \leq)$ and $(Y, \widetilde{\leq})$ we say that $(X, \leq)$ has *cofinality greater than or equal to* $(Y, \widetilde{\leq})$ when there is a map $\Phi : X \to Y$ which preserves the order and such that $\Phi(X)$ is cofinal in Y. *Cofinal equivalence* is defined accordingly.

2.2.1 Metrizable groups

Saxon and Sánchez-Ruíz in [91] and, independently, Burke and Todorcevic in [17] have studied the boundedness of metrizable locally convex vector spaces. It turns out that the set $\mathbb{N}^{\mathbb{N}}$ plays a crucial role in their results, especially those dependent on the cofinality. In this section, we extend their approach to look at the cofinality of boundedness structures for metrizable topological groups. In the sequel G will be a metrizable group and (I, b) a boundedness system on G.

Let G be a metrizable topological group endowed with a boundedness system (I, b). Consider a neighborhood base $\mathfrak{B} = \{V_m\}_{m<\omega}$ of the identity e_G and let $Bdd(G)$ be the collection of all bounded subsets of G associated to (I, b). Recall that $\mathcal{P}_{Fin}(I)$ denotes the family of (non empty) finite subsets of I and $\mathcal{P}_{Fin}(I)^{\mathbb{N}}$ is the family of all functions $f : \mathbb{N} \to \mathcal{P}_{Fin}(I)$.

Then we have defined the map

$$\Psi_{\mathfrak{B}} : \mathcal{P}_{Fin}(I)^{\mathbb{N}} \to Bdd(G)$$

as follows

$$\Psi_{\mathfrak{B}}(\alpha) = \bigcap_{m<\omega} b(\alpha(m), V_m).$$

If we consider the inclusion order in $Bdd(G)$ and the pointwise inclusion order " $\subseteq$ " in $\mathcal{P}_{Fin}(I)^{\mathbb{N}}$ (that is, for $\alpha, \beta \in \mathcal{P}_{Fin}(I)^{\mathbb{N}}$, $\alpha \subseteq \beta$ means: $\alpha(n) \subseteq \beta(n)$ for all $n < \omega$), then we have the following properties.

Lemma 2.2.3 *The map $\Psi_{\mathfrak{B}}$ preserves the order and $\Psi_{\mathfrak{B}}(\mathcal{P}_{Fin}(I)^{\mathbb{N}})$ is a cofinal subset of $Bdd(G)$.*

Proof. It is clear that $\Psi_{\mathfrak{B}}$ preserves the order. Now, let $A \in Bdd(G)$. Then, for all $m < \omega$, there is $F_m \in \mathcal{P}_{Fin}(I)$ such that $A \subseteq b(F_m, V_m)$. Take $\alpha : \mathbb{N} \longrightarrow \mathcal{P}_{Fin}(I)$ defined by $\alpha(m) = F_m$. Then we have that $\Psi_{\mathfrak{B}}(\alpha) \supseteq A$.

Lemma 2.2.4 *Let G be a metrizable group endowed with a boundedness system (I, b). Then, $cof(Bdd(G)) \geq |I|$.*

Proof. Let $\kappa = |I|$, let $\lambda < \kappa$ and $\{B_\alpha : \alpha < \lambda\} \subseteq Bdd(G)$. Let $\{U_n\}_{n>\omega}$ be a neighborhood base of e_G. Since $\{B_\alpha : \alpha < \lambda\} \subseteq Bdd(G)$ we have that for each $\alpha < \lambda$ and each $n < \omega$ there is a finite set $F_{n,\alpha} \subseteq \kappa$ such that $B_\alpha \subseteq b(F_{n,\alpha}, U_n)$. Then $\bigcup_{n<\omega} \bigcup_{\alpha<\lambda} F_{n,\alpha} =: J$ has cardinality less than κ. Applying Remark 2, there must be $n < \omega$ such that $b(J, U_n) \neq G$. Due to the fact that $\bigcup_{\alpha<\lambda} B_\alpha \subseteq b(J, U_n)$, it follows that $\{B_\alpha : \alpha < \lambda\}$ can not be cofinal in $Bdd(G)$.

2.2.2 $\aleph_0$-bounded and κ-bounded groups

All topological vector spaces can be endowed with a boundedness system (I, b), where $|I| = \aleph_0$. The same occurs for metrizable and separable groups or, more general, for each group whose density is $\aleph_0$.

Definition 2.2.5 Let G be a topological group with a boundedness system (I, b). We say that G is *κ-b-bounded* (or simply G is *κ-bounded*, if there is no possibility of confusion) when for each neighborhood U of e_G there is $J_U \subset I$ such that $|J_U| \leq \kappa$ and $G = b(J_U, U)$.

Definition 2.2.6 The minimal κ such that G is κ-bounded will be called *the boundedness number* of G and shall be denoted by $\flat(G)$.

Remark 5 In view of Remark 4, for topological groups mentioned in Example 2.1.3 the boundedness number does not depend on the choice of the dense subset, i.e. the previous definition is equivalent to saying that for each neighborhood U of e_G there is $J_U \subset G$ such that $|J_U| \leq \kappa$ and $G = J_U U$. Moreover, a topological group G is endowed with a boundedness system (I, b) if and only if $d(G) = |I|$ and thus $\flat(G) \leq d(G)$. In particular, when G is metrizable $\flat(G) = d(G)$. Therefore, when G is metrizable we may assume that G is $d(G)$-bounded.

In case G is metrizable, we assume without loss of generality that G is κ-bounded, with $\kappa = |I|$. This is done in order to study the cofinality of the collection $Bdd(G)$ of all bounded subsets of G.

2.2.3 Locally bounded groups

Definition 2.2.7 We say that a topological group G is *locally bounded* when there is a bounded neighborhood U of e_G. In this case, there is a neighborhood base of the identity consisting of bounded sets.

Remark 6 If U is a bounded neighborhood of the identity in G, then the set

$$Bdd(G)_U = \{b(F,U) \,:\, F \in \mathcal{P}_{Fin}(I)\}$$

is cofinal en $Bdd(G)$.

Indeed, let us denote our bounded neighborhood by U^*. Let $B \in Bdd(G)$. Then, there is $F_{U^*} \in \mathcal{P}_{Fin}(I)$ such that $B \subset b(F_{U^*}, U^*)$ and the proof is done.

As a consequence, we have the following result.

Proposition 2.2.8 *Let G be a locally bounded and $\aleph_0$-bounded group. Then $Bdd(G)$ is cofinally equivalent to $\mathbb{N}$.*

Proof. Since G is $\aleph_0$-bounded there must be a bounded neighborhood $U \in \mathcal{N}(e_G)$ such that for each finite subset $F \subseteq \mathbb{N}$, $b(F,U) \neq G$.
Define $\phi : G \longrightarrow \mathbb{N}$ by

$$\phi(g) = min\{n : g \in b(n,U)\}.$$

The functions $K \mapsto max\ \phi[K]$ and $n \mapsto \phi^{-1}[\{1,...,n\}]$ establish the required cofinal equivalence (for a function $f : X \to Y$ and $A \subseteq X$, $B \subseteq Y$, we use the notation $f[A] = \{f(a) : a \in A\}$, and $f^{-1}[B] = \{x \in X : f(x) \in B\}$).

Moreover, if G is locally bounded and κ-bounded, then $Bdd(G)$ is cofinally equivalent to $\mathcal{P}_{Fin}(I)$.

Proposition 2.2.9 *If G is a metrizable locally bounded group endowed with a boundedness system (I, b), then $cof(Bdd(G)) = |I|$.*

Proof. Since G is locally bounded, there is a bounded neighborhood U of e_G and the set

$$Bdd(G)_U = \{b(F, U) \,:\, F \in \mathcal{P}_{Fin}(I)\}$$

is cofinal en $Bdd(G)$. The cofinality of $Bdd(G)$ is therefore less then or equal to $|\mathcal{P}_{Fin}(I)| = |I|$.

On the other hand, applying Lemma 2.2.4 we have that $cof(Bdd(G)) \geq |I|$ and we are done.

2.2.4 Non locally bounded groups

For metrizable non locally bounded groups, the cofinality of the collection of bounded subsets may also be estimated in some cases.

In the first place we consider the case $I = \mathbb{N}$. It is easily verified that the map $\Theta(\alpha)(n) = max\{j : j \in \alpha(n)\}$ establishes the cofinality equivalence between the sets $\mathcal{P}_{Fin}(\mathbb{N})^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$, the later equipped with the canonical pointwise order (that is, $\alpha \leq \beta$ iff $\alpha(n) \leq \beta(n)\ \forall n < \omega$). Hence, in this case, we shall consider here the set $\mathbb{N}^{\mathbb{N}}$ (not $\mathcal{P}_{Fin}(\mathbb{N})^{\mathbb{N}}$) for the sake of simplicity.

In addition to the already defined map $\Psi_{\mathfrak{B}}$, it will be also useful to consider the map

$$\Phi_{\mathfrak{B}} : Bdd(G) \to \mathbb{N}^{\mathbb{N}}$$

defined by the rule

$$\Phi_{\mathfrak{B}}(K)(m) := min\left\{ n \,:\, K \subseteq \bigcup_{i \leq n} b(i, V_m) \right\}.$$

Obviously,

$$\Phi_{\mathfrak{B}}(K) := \{\Phi_{\mathfrak{B}}(K)(m)\}_{m \leq \omega}.$$

This map is order preserving and relates the confinality of $Bdd(G)$ and $\mathbb{N}^{\mathbb{N}}$.

Lemma 2.2.10 *If G is metrizable, non locally bounded with a boundedness system $(\mathbb{N}, b)$, then there is a neighborhood base $\mathfrak{B} = \{U_n\}_{n<\omega}$ such that $\Phi_{\mathfrak{B}}(Bdd(G))$ is cofinal in $\mathbb{N}^{\mathbb{N}}$.*

Proof. Let $\mathfrak{B}_0 = \{V_m\}_{m<\omega}$ be any countable neighborhood base of e_G. Since V_1 is not bounded, there is $V_{m_0} \in \mathfrak{B}_0$ such that $V_1 \not\subseteq \bigcup\limits_{i \leq n} b(i, V_{m_0})$ $\forall n < \omega$. Now we define inductively the neighborhood base $\mathfrak{B} = \{U_n\}_{n<\omega}$. $U_1 = V_1 \bigcap V_{m_0}$. Suppose U_n has been defined. Since U_n is not bounded, there is $V_{m_n} \in \mathfrak{B}_0$ such that $U_n \not\subseteq \bigcup\limits_{i \leq l} b(i, V_{m_n})$ $\forall l < \omega$. Then we define $U_{n+1} = V_{n+1} \bigcap V_{m_n} \bigcap U_n$. This defines $\mathfrak{B}$ inductively. Let us see now that $\Phi_{\mathfrak{B}}(Bdd(G))$ is cofinal in $\mathbb{N}^{\mathbb{N}}$. Suppose that $\alpha \in \mathbb{N}^{\mathbb{N}}$. Then we take $x_1 \in G \setminus \bigcup\limits_{n=1}^{\alpha(1)} b(n, U_1)$ and for every $m \geq 2$ we take $x_m \in U_{m-1}$ such that $x_m \notin \bigcup\limits_{n=1}^{\alpha(m)} b(n, U_m)$. Set $K = \{x_m\}_{m<\omega}$, which is a sequence convergent to the identity e_G. Then, since $K\bigcup\{e_G\}$ is bounded and $\Phi_{\mathfrak{B}}(K)(m) = min\left\{n : K \subseteq \bigcup_{i=1}^{n} b(i, U_m)\right\}$ it follows that $\alpha \leq \Phi_{\mathfrak{B}}(K)$. This completes the proof.

In general, when I is an arbitrary index set, we have that

$$\mathfrak{d} = cof(\ \mathbb{N}^{\mathbb{N}}) = cof(\ \mathcal{P}_{Fin}(\mathbb{N})^{\mathbb{N}}) \leq cof(\ \mathcal{P}_{Fin}(I)^{\mathbb{N}}).$$

2.2.5 Cofinality of $Bdd(G)$

We want to estimate the cofinality of an arbitrary boundedness structure. Previously, we need two simple lemmas.

Lemma 2.2.11 *Let G be a metrizable group with a boundedness system (I, b) such that G is not locally bounded. Suppose that $\mathcal{F}$ is a collection of bounded subsets such that $G = \bigcup_{A\in\mathcal{F}} A$. Then $|I| \leq cof(\mathcal{F})$.*

Proof. Let $\{A_j : j \in J\}$ be a cofinal subset of $\mathcal{F}$. If $\mathfrak{B} = \{U_n\}_{n<\omega}$ is a neighborhood base of e_G, for every $j \in J$ there is $F_j^n \in \mathcal{P}_{Fin}(I)^{\mathbb{N}}$ such that $A_j \subseteq b(F_j^n, U_n)$. Furthermore, if $L_n = \bigcup_{j\in J} F_j^n$ then $|L_n| \leq |J|$ and $b(L_n, U_n) \supseteq \bigcup_{j\in J} A_j = G$. Set $L = \bigcup_{n<\omega} L_n$. Then $|L| \leq |J|$ and for every $U_n \in \mathfrak{B}$ we have $b(L, U_n) = G$. Since G is $|I|$-bounded then $|I| \leq |L|$. This means that $|I| \leq |J|$ which completes the proof.

Lemma 2.2.12 *Let G be a metrizable group endowed with a boundedness system (I, b) such that G is not locally bounded. Then we have*

$$|I| \leq cof(Bdd(G)) \leq cof(\ \mathcal{P}_{Fin}(I)^{\mathbb{N}}).$$

Proof. We have proved that if I is countable, the cofinality of $Bdd(G)$ coincides with the cofinality of $\mathbb{N}^{\mathbb{N}}$, which is $\mathfrak{d}$. Assume that I is uncountable. By Lemma 2.2.3 we know that the cofinality of $Bdd(G)$ is less than or equal to the cofinality of $\mathcal{P}_{Fin}(I)^{\mathbb{N}}$. Finally, Lemma 2.2.4 verifies that $|I| \leq cof(Bdd(G))$.

Recall that $\mathfrak{b}$(respectively $\mathfrak{d}$) is the smallest cardinal of an unbounded (respectively dominating) subset of $\mathbb{N}^{\mathbb{N}}$ with the eventual dominance ordering that is denoted by $\leq^*$ (cf. [17])

Theorem 2.2.13 *Let G be a metrizable group with a boundedness system $(\mathbb{N}, b)$. Then $Bdd(G)$ is cofinally equivalent to one of the following ordered sets, $\{0\}$, $\mathbb{N}$, or $\mathbb{N}^{\mathbb{N}}$, depending on whether G is either trivial or bounded, locally bounded, or non locally bounded respectively. Hence, if the group G is not locally bounded, it follows that $cof(Bdd(G)) = \mathfrak{d}$, where $\mathfrak{d}$ is the cofinality of $\mathbb{N}^{\mathbb{N}}$.*

Proof. The proof is trivial if G is bounded or locally bounded. In case G is not locally bounded, lemmas 2.2.3 and 2.2.10 prove that $Bdd(G)$ is cofinally equivalent to $\mathbb{N}^{\mathbb{N}}$.

The following result states that every family of bounded subsets of cardinality less then $\mathfrak{b}$ contains a countable cofinal subfamily or, equivalently, has countable cofinality.

Corollary 2.2.14 *Let G be a metrizable group with a boundedness system $(\mathbb{N}, b)$ and let $\mathcal{B} \subseteq Bdd(G)$ such that $|\mathcal{B}| < \mathfrak{b}$. Then there is a countable family $\mathfrak{F}$ of bounded subsets such that every element of $\mathcal{B}$ is contained in an element of $\mathfrak{F}$.*

Proof. The proof is clear if G is bounded or locally bounded. Assume G is not locally bounded. Then $\mathcal{B}$ has cofinality less than or equal to the cardinality of $\mathbb{N}^{\mathbb{N}}$ by Theorem 2.2.13 and $\mathfrak{b}$ is the smallest cardinal of an unbounded subset of $\mathbb{N}^{\mathbb{N}}$ equipped with the eventual dominance ordering that is denoted by $\leq^*$ (cf. [17]). By Lemma 2.2.10, we know that the map $\Phi_{\mathfrak{B}} : Bdd(G) \to \mathbb{N}^{\mathbb{N}}$, defined above, has cofinal range for an appropriately defined neighborhood base $\mathfrak{B}$ of e_G. Therefore, there is an element $\alpha \in \mathbb{N}^{\mathbb{N}}$ that is an upper bound of $\{\Phi_{\mathfrak{B}}(A) : A \in \mathcal{B}\}$. This means that $\Phi_{\mathfrak{B}}(A) \leq^* \alpha \quad \forall A \in \mathcal{B}$. Thus for every $A \in \mathcal{B} \quad \exists n_A$ such that $\Phi_{\mathfrak{B}}(A)(n) \leq \alpha(n) \quad \forall n \geq n_A$. Define the sequence $\{\beta_n^{(m)}\}_{m,n<\omega}$ as follows, $\beta_n^{(m)}(i) = m$ if $i < n$ and $\beta_n^{(m)}(i) = \alpha(n)$ if $i \geq n$. Then $\{\beta_n^{(m)}\}_{m,n<\omega}$ is cofinal in $\Phi_{\mathfrak{B}}(\mathcal{B})$, since if $A \in \mathcal{B}$, $\Phi_{\mathfrak{B}}(A)(i) \leq \alpha(i) \quad \forall i \geq n_A$, and if $m_A = max\{\Phi_{\mathfrak{B}}(A)(i) : 1 \leq i < n_A\}$ then $\Phi_{\mathfrak{B}}(A) < \beta_{n_A}^{(m_A)}$. Since $\Psi_{\mathfrak{B}}(\Phi_{\mathfrak{B}}(A)) \supseteq A \quad \forall A \in \mathcal{B}$, it suffices to take $\mathfrak{F} = \left\{\Psi_{\mathfrak{B}}(\beta_n^{(m)}) : n < \omega, m < \omega\right\}$.

Corollary 2.2.15 *If G is a metrizable group whose density is less than $\mathfrak{b}$ endowed with a boundedness system $(\mathbb{N}, b)$, then every non separable subset of G contains a bounded non separable subset.*

Proof. Let X be a non separable subset of G (this implies $\aleph_1 < \mathfrak{b}$) and take a discrete subset Y of X such that $|Y| = \aleph_1$. Take $\mathcal{B} = \{\{y\} \,/\, y \in Y\} \subseteq Bdd(G)$ and, since $|\mathcal{B}| = \aleph_1 < \mathfrak{b}$, we can apply Corollary 2.2.14 to obtain a countable family $\mathfrak{F} = \{F_n\}_{n<\omega}$ of bounded subsets such that $\forall y \in Y$ there is $F_n \in \mathfrak{F}$ such that $y \in F_n$. Therefore, there is a $n_0 < \omega$ such that F_{n_0} contains uncountably many elements of Y. Since Y is discrete, this means that F_{n_0} is non separable.

Applying the results on cofinality mentioned so far, we obtain the following estimation of the cofinality of a boundedness structure.

Theorem 2.2.16 *Let G be a metrizable group endowed with a boundedness system (I, b). Then the following assertions are true.*

1. *If G is trivial or bounded then $Bdd(G)$ is cofinally equivalent to $\{0\}$.*

2. *If G is locally bounded then $Bdd(G)$ is confinally equivalent to $\mathcal{P}_{Fin}(I)$. As a consequence we have $cof(Bdd(G)) = |I|$.*

3. *If G is a non locally bounded group then we have*

$$|I| \leq cof(Bdd(G)) \leq max(\mathfrak{d}, cof(\mathcal{P}_{\aleph_0}(I))) \leq |I|^{\aleph_0}$$

   Furthermore, in case $\aleph_0 \leq |I| < \aleph_\omega$, we have

$$|I| \leq cof(Bdd(G)) \leq max(\mathfrak{d}, |I|).$$

Recall that a topological group is called *almost metrizable* if it contains a compact subset of countable character, i.e. the compact subset has a countable base of neighborhoods (see [82]).

Remark 7 The results of this section are also true for almost metrizable groups using the fact that, when G is almost metrizable, there is a precompact closed subgroup K such that G/K is metrizable and such that the natural mapping $f : G \longrightarrow G/K$ is open and perfect.

2.3 Dense subgroups

In [48] Grothendieck proved that, when E is a metrizable and separable locally convex space, the bounded subsets of E are completely determined by the bounded subsets of any dense subspace. This result has been extended by Burke and Todorcevic for some non separable spaces (see [17]). As we show next, the same assertion holds for metrizable topological groups.

Theorem 2.3.1 *Let G be a metrizable group endowed with a boundedness system $(\mathbb{N}, b)$. Let H be a dense subgroup of G. For each bounded $K \subseteq G$ whose density is less than $\mathfrak{b}$, there is a bounded $P \subseteq H$ such that $\overline{P} \supseteq K$.*

Proof. There exists $D \subseteq K$ such that $|D| < \mathfrak{b}$ and $\overline{D}^K = K$. Furthermore, K is bounded so that, if $\mathfrak{B} = \{V_m\}_{m<\omega}$ is a open neighborhood base of e_G and $\Phi_{\mathfrak{B}}$ is the map defined in Lemma 2.2.10 above, we have $K \subseteq \bigcup_{n=1}^{\Phi_{\mathfrak{B}}(K)(m)} b(n, V_m)$

for all $m < \omega$. On the other hand, since H is dense in G, for all $d \in D \subseteq K$, there is a sequence $S_d \subseteq H$ which converges to d. Therefore, since $S_d \bigcup \{d\}$ is compact, we have $\overline{S_d} = S_d \bigcup \{d\} \subseteq \bigcup_{n=1}^{\Phi_{\mathfrak{B}}(\overline{S_d})(m)} b(n, V_m)$ for all $m < \omega$. So, we have a family $\left\{\Phi_{\mathfrak{B}}(\overline{S_d})\right\}_{d \in D} \subseteq \mathbb{N}^{\mathbb{N}}$ of cardinality less than $\mathfrak{b}$, then it is bounded in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. Therefore, there is $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\Phi_{\mathfrak{B}}(\overline{S_d}) \leq^* \alpha \quad \forall d \in D$. That is, if $d \in D$, then there is $m_d < \omega$ with $\Phi_{\mathfrak{B}}(\overline{S_d})(m) \leq \alpha(m) \quad \forall m \geq m_d$. We also assume that $\Phi_{\mathfrak{B}}(K)(m) \leq \alpha(m) \quad \forall m < \omega$. Pick now a fixed element $d \in D$. If $m < m_d$, we have $K \subseteq \bigcup_{n=1}^{\Phi_{\mathfrak{B}}(K)(m)} b(n, V_m) \subseteq \bigcup_{n=1}^{\alpha(m)} b(n, V_m)$. Therefore, $K \subseteq \bigcap_{m=1}^{m_d - 1} \left(\bigcup_{n=1}^{\alpha(m)} b(n, V_m) \right) = A_d$ that is an open set. Since this open set contains the element $d \in D$ and the sequence S_d converges to d, there is $S'_d = S_d \smallsetminus \{\text{a finite subset}\}$ such that $S'_d \subseteq A_d$. Consider now $P := \bigcup_{d \in D} S'_d \subseteq H$ and let us verify that P is bounded. Take an open neighborhood V of e_G, then there is $V_m \in \mathfrak{B}$ such that $V_m \subseteq V$. For each $d \in D$ we have one of the following two options:

1. $m < m_d$, which implies $S'_d \subseteq A_d \subseteq \bigcup_{n=1}^{\alpha(m)} b(n, V_m)$.

2. $m \geq m_d$, then $S'_d \subseteq S_d \subseteq \bigcup_{n=1}^{\Phi_{\mathfrak{B}}(\overline{S_d})(m)} b(n, V_m) \subseteq \bigcup_{n=1}^{\alpha(m)} b(n, V_m)$.

In both cases, $S'_d \subseteq \bigcup_{n=1}^{\alpha(m)} b(n, V_m) \subseteq \bigcup_{n=1}^{\alpha(m)} b(n, V)$.

Therefore, $P = \bigcup_{d \in D} S'_d \subseteq \bigcup_{n=1}^{\alpha(m)} b(n, V)$, and since V is arbitrary this means that P is bounded.

It is readily seen that $\overline{P} \supseteq K$.

A consequence of this theorem is the following.

Corollary 2.3.2 *If G is a metrizable group endowed with a boundedness system $(\mathbb{N}, b)$ having a dense subset of cardinality less than $\mathfrak{b}$ and D is a dense subset of G, then for each bounded $K \subseteq G$, there is a bounded $P \subseteq D$ such that $\overline{P} \supseteq K$.*

As an application of Theorem 2.3.1 we introduce the following result which can be obtained, in the abelian case, from a result of Chasco and independently Aussenhofer.

Corollary 2.3.3 *Let G be a metrizable group and let H be a dense subgroup of G. If $K \subseteq G$ is precompact, then there is a precompact subset $P \subseteq H$ such that $K \subseteq \overline{P}$.*

Proof. Observe that K is separable because it is metrizable and precompact. Let D be a countable dense subset of K. For every $d \in D$, there is a sequence $S_d \subseteq H$ converging to d. Consider the countable subset $E = D \bigcup \left(\bigcup_{d \in D} S_d \right) = \bigcup_{d \in D} \overline{S_d} = \{y_i\}_{i=1}^{\infty}$ and the set $G_E = \overline{< E >}$ with the topology inherited from G. We have that $K \subseteq G_E$, and G_E is separable and metrizable. We take the boundedness map of the example 2.1.3, $b_E : E \times \mathcal{P}(G_E) \to \mathcal{P}(G_E)$ such that $b_E(y_n, A) = y_n \cdot A$. On the other hand, $H \bigcap G_E$ is countable and dense in G_E and K is clearly b_E-bounded. Applying the theorem 2.3.1, we deduce that there is $P \subseteq H \bigcap G_E$ which is bounded and $K \subseteq \overline{P}^{G_E} \subseteq \overline{P}$. It is readily seen that P is precompact in H.

The metrizability condition in the previous theorem is essential, as one can see from the following example. But first we need a previous lemma.

Lemma 2.3.4 *Let G be a complete P-group having a proper dense subset D. Then, there is a precompact subset P in G such that $P \nsubseteq \overline{K}$ for any precompact subset K in D.*

Proof. Let F be a finite precompact subset of $G \setminus D$. Assume that there is a precompact subset K in D such that $F \subseteq \overline{K}^G$. By completeness of G, $\overline{K}^G$ is compact, thus finite (see [45, 4K.2]). It follows that $F \subseteq \overline{K} = K \subseteq D$ and we have reached a contradiction.

Example 2.3.5 There is a non metrizable group G and a dense subgroup H such that G contains a precompact subset which may not be reached from precompact subsets of H.

Actually, any complete P-group which admits a proper dense subset is a valid example, by the previous lemma. For instance take any of the following two examples.

1) Let $G = \prod\limits_{i \in I} G_i$ (in particular, $G_i = \mathbb{Z}(2)$ for each $i \in I$) endowed with the *countable box topology* $\tau_{\aleph_0}$ defined as follows:

$$\tau_{\aleph_0} = \{N_A \mid A \in \mathcal{P}_{\aleph_0}(I)\}$$

where

$$N_A = \{(x_i) \in \prod_{i \in I} G_i \mid x_i = 0 \text{ if } i \in A\}.$$

Consider the Σ-product $H = \sum_{i\in I} G_i$ with the topology inherited from G. Further, G is a P-group and the Σ-product $H := \sum_{i\in I} G_i$ is dense in G (see [4]).

2) Consider the space W of all countable ordinals, and denote by $W(\alpha)$ the set of all ordinals less than a given ordinal α: $W(\alpha) = \{\sigma : \sigma < \alpha\}$. The set $W(\alpha)$ is well-ordered. Let X be the subspace of $W(\omega_2)$ obtained by deleting all nonisolated points having a countable base. X is a P-space, but it is not realcompact (see [45, 9L.5]). By [45, 8A.5] υX is a P-space and it can be easily seen that $A(\upsilon X)$ is a P-group. Take now the free abelian group $A(X)$ as topological subgroup of $A(\upsilon X)$. Applying [4, 7.9.8], $A(X)$ is dense in $A(\upsilon X)$.

2.4 Baire-like groups

In this section G is a metrizable group with a boundedness $Bdd(G)$ indexed by $\mathbb{N}$. We assume further that the map $b : \mathbb{N} \times \mathcal{P}(G) \to \mathcal{P}(G)$ satisfies two additional conditions.

(Bdd 4') The map $b(i, \ ) : \mathcal{P}(G) \to \mathcal{P}(G)$ is 1-to-1 for all $i \in \mathbb{N}$

(Bdd 5) For all $m_1, m_2 < \omega$, and every neighborhood U of e_G,

we have

$$\bigcup_{i \leq max\{m_1,m_2\}} b(i,U) \subseteq \left(\bigcup_{i\leq m_1} b(i, \bigcup_{j\leq m_2} b(j,U)) \right) \bigcap \left(\bigcup_{j\leq m_2} b(j, \bigcup_{i\leq m_1} b(i,U)) \right).$$

Definition 2.4.1 We say that a boundedness with (4') and (5) is *well behaved.*

Remark 8 It is readily seen that the ordinary boundedness of locally convex spaces and the precompact boundedness of topological groups satisfy this property.

Definition 2.4.2 We say that $A \subseteq G$ is *b-absorbent* (or simply A is *absorbent*, if there is no possibility of confusion) when $\bigcup_{i<\omega} b(i, A) = G$. A topological group G is *Baire-like* when for every closed absorbent subset Q there is an index $i \in \mathbb{N}$ such that $b(i, Q)$ is a neighborhood of e_G.

Theorem 2.4.3 *Suppose G is a Baire-like metrizable group equipped with a well behaved boundedness indexed by $\mathbb{N}$. If G can be covered by less than $\mathfrak{b}$ bounded subsets, then G is locally bounded.*

Proof. Let $\mathfrak{B} = \{V_m\}_{m<\omega}$ be a neighborhood base of e_G and let $b : \mathbb{N} \times \mathcal{P}(G) \to \mathcal{P}(G)$ the boundedness map. For every $m \in \mathbb{N}$ we define the map

$$p_m : G \to \mathbb{N} \quad \text{by} \quad p_m(x) = min\{n \, : \, x \in b(n, V_m)\}.$$

As a consequence, every element $x \in G$ defines a sequence $\{p_m(x)\}_{m<\omega}$ and, therefore, we have defined the map $p : G \to \mathbb{N}^{\mathbb{N}}$ as $p(x) = \{p_m(x)\}_{m<\omega}$ so that $p(x)[m] = p_m(x)$. By hypothesis there is a subset $\mathcal{B} \subseteq Bdd(G)$ such that $|\mathcal{B}| < \mathfrak{b}$ and $G = \bigcup_{P \in \mathcal{B}} P$. Every $P \in \mathcal{B}$ is associated with a map $\Phi_{\mathfrak{B}}(P) \in \mathbb{N}^{\mathbb{N}}$ defined previously; that is $\Phi_{\mathfrak{B}}(P)(m) = min\{n \, : \, P \subseteq \bigcup_{j \leq n} b(j, V_m)\}$.

Take $x \in G$. Then, there is $P \subseteq \mathcal{B}$ such that $x \in P$. Therefore $p(x) \leq \Phi_{\mathfrak{B}}(P)$. Since $|\mathcal{B}| < \mathfrak{b}$ it follows that $\Phi_{\mathfrak{B}}(\mathcal{B}) = \{\Phi_{\mathfrak{B}}(P) : P \in \mathcal{B}\}$ is bounded in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. Thus, there is $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\Phi_{\mathfrak{B}}(P) \leq^* \alpha$ and, since $p(x) \leq \Phi_{\mathfrak{B}}(P)$, we have $p(x) \leq^* \alpha$ for all $x \in G$. So, for every $x \in G$, there is $m_x < \omega$ such that $p_m(x) \leq \alpha(m)$ for all $m \geq m_x$.

Define

$$Q_\alpha = \{x \in G : p_m(x) \leq \alpha(m) \quad \forall m < \omega\} = \bigcap_{m<\omega} \left(\bigcup_{j \leq \alpha(m)} b(j, V_m) \right).$$

Clearly, the set Q_α is bounded. Let us verify that Q_α is also absorbent. Take $x \in G$. Then, since $p_m(x) \leq \alpha(m) \quad \forall m \geq m_x$, we have

$$x \in \bigcap_{m \geq m_x} \left(\bigcup_{j \leq \alpha(m)} b(j, V_m) \right).$$

Thus, $x \in Q_\alpha \bigcup \left(\bigcap_{m<m_x} b(p_m(x), V_m) \right)$. Set $I_x = \{m \in \mathbb{N} : p_m(x) > \alpha(m), 1 \leq m < m_x\}$ and define $F_x = \{j \in \mathbb{N} : j \leq p_m(x), m \in I_x\}$. We claim that $x \in b(F_x, Q_\alpha)$.

Indeed, since the map b is well behaved, it follows that

$$b(F_x, Q_\alpha) = b(F_x, \bigcap_{m<\omega} \bigcup_{j \leq \alpha(m)} b(j, V_m)) = \bigcap_{m<\omega} b(F_x, \bigcup_{j \leq \alpha(m)} b(j, V_m)) =$$

$$= \bigcap_{m<\omega} \left(\bigcup_{i \leq p_m(x)} b(i, \bigcup_{j \leq \alpha(m)} b(j, V_m) \right) \supseteq \bigcap_{m<\omega} \left(\bigcup_{i \leq max\{p_m(x), \alpha(m)\}} b(i, V_m) \right).$$

Now, we have two options for every $m < \omega$

(i) $m \in I_x$, then $max\{p_m(x), \alpha(x)\} = p_m(x)$ and therefore

$$x \in \bigcup_{i \leq p_m(x)} b(i, V_m).$$

(ii) $m \notin I_x$, then $max\{p_m(x), \alpha(m)\} = \alpha(m)$ and therefore

$$x \in \bigcup_{i \leq p_m(x)} b(i, V_m) \subseteq \bigcup_{i \leq \alpha(m))} b(i, V_m).$$

In both cases, $x \in \bigcup_{i \leq max\{p_m(x), \alpha(m)\}} b(i, V_m)$ for all $m < \omega$. Thus, $x \in b(F_x, Q_\alpha)$. This proves that Q_α is absorbent. Therefore $\overline{Q_\alpha}$ is absorbent too and, since G is Baire-like, the subset $b(i, \overline{Q_\alpha})$ is a neighborhood of e_G for some $i \in \mathbb{N}$. Therefore G is locally bounded.

Corollary 2.4.4 *Let G be a separable, metrizable, Baire-like group that is covered by less than $\mathfrak{b}$ precompact subsets. Then G is locally precompact.*

Proof. It suffices to consider the boundedness system (D, b) of Example 2.1.3, where D is an appropriately selected countable dense subset of G and apply Theorem 2.4.3.

Chapter 3

Topological invariants of topological abelian groups

In the rest of the paper we are going to apply the results obtained previously in order to estimate some topological cardinals. Our main tool here is Pontryagin-van Kampen duality and the results of the previous sections on boundedness structures. From here on, we shall use additive notation and all groups G are assumed to be abelian and locally quasi-convex. Moreover, all boundedness structures $\mathcal{A}$ are supposed to satisfy two additional conditions:

(iv) if $A \in \mathcal{A}$ then $-A \in \mathcal{A}$;

(v) if $A, B \in \mathcal{A}$ then $A + B \in \mathcal{A}$.

In this section we extend to topological groups, and arbitrary cardinals, some results given firstly by Cascales and Orihuela [20] for locally convex vector spaces and countable cardinality. Our approach is also based on the papers by Robertson [87] and Ferrando, Kaçol, and López-Pellicer [33, 34].

3.1 Basic facts

Definition 3.1.1 Given a topological abelian group G, we denote by $\widehat{G}$ its *dual* group. If we consider a boundedness $\mathcal{A}$ on G, then we can equip $\widehat{G}$ with the *topology* $\tau_{\mathcal{A}}(G)$ *of uniform convergence on the elements of* $\mathcal{A}$ and we denote by $\widehat{G}_{\mathcal{A}}$ the topological group obtained in this manner. For the sake of simplicity, when $\mathcal{A}$ is the boundedness of precompact subsets, the expression $\tau_{\mathcal{A}}(G)$ will be replaced by $\tau_{pc}(G)$. Furthermore, the compact open topology and the pointwise convergence topology will be denoted by $\tau_c(G)$ and $\omega(G, \widehat{G})$ respectively. As in the case of locally convex spaces, it is readily verified that $\omega(G, \widehat{G})$ is the weakest locally quasi convex topology on G whose dual group is $\widehat{G}$. Notice that $(G, \omega(G, \widehat{G}))$ is always a precompact group, that is, its completion is a compact group.

To simplify matters, $\mathbb{T}$ will be identified with the interval $[-1/2, 1/2)$ equipped with the canonical quotient topology of $\mathbb{R}/\mathbb{Z}$. Thus, using additive notation in the torus $\mathbb{T}$, a *neighborhood base* at the identity in $(\widehat{G}, \tau_{\mathcal{A}}(G))$ consists of sets of the form

$$N(e_{\widehat{G}}, A, \varepsilon) = \{\chi \in \widehat{G} : |\chi(a)| \leq \varepsilon, \forall a \in A\},\ A \in \mathcal{A},\ \varepsilon > 0.$$

If $B \subseteq G$ we define $B^0 = N(e_{\widehat{G}}, B, \frac{1}{4})$. Alternatively, if $X \subseteq \widehat{G}$, we define

$$^0X = \{g \in G : |\chi(g)| \leq \frac{1}{4}, \forall \chi \in X\}.$$

This set operator behaves in many aspects like the polar operator in vector spaces. For instance, it is easily verified that $(^0(A^0))^0 = A^0$ for any $A \subset G$.

Definition 3.1.2 A subset A of G is said to be *quasi-convex* if ${}^0(A^0) = A$. A group is called *locally quasi-convex* if it has a neighborhood base of e_G consisting of quasi-convex sets.

From here on, all topological groups are assumed to be locally quasi-convex.

Definition 3.1.3 A topological group G is said to be κ*-bounded*, without reference to any boundedness, when G is κ-b-bounded for the boundedness of precompact subsets. That is, for every neighborhood U of e_G there is a subset $A \subseteq G$ such that $|A| \leq \kappa$ and $A + U = G$.

The following proposition extends to arbitrary cardinals a previous result given by Robertson when the index set is countable. (cf. [87])

Proposition 3.1.4 *Suppose that G is a topological group that contains a family of subsets $\{A_\alpha : \alpha \in \mathcal{P}_{Fin}(I)^{\mathbb{N}}\}$, for some infinite set I, satisfying the following conditions:*

1. $G = \bigcup\{A_\alpha : \alpha \in \mathcal{P}_{Fin}(I)^{\mathbb{N}}\}$,
2. $A_\alpha \subseteq A_\beta$ *if* $\alpha \subseteq \beta$,
3. *every* A_α *is precompact.*

Then G is $|I|$-bounded.

Proof. Let κ denote the cardinality of I and assume that there is a neighborhood U of e_G such that $\forall A \subseteq G$ with $|A| \leq \kappa$, we have $A + U \neq G$.

We define the set $\mathcal{S} = \{A \subseteq G : \forall\ h_1,\ h_2 \in A\ \textit{with}\ h_1 - h_2 \in U,\ \textit{it holds}\ h_1 = h_2\}$ and we consider the inclusion order on it. It is easily seen that $(\mathcal{S}, \subseteq)$ is a non-empty inductive set. Therefore, there exist maximal members in $\mathcal{S}$. Let B be a maximal member. We shall verify that $|B| > \kappa$. Indeed, if $|B| \leq \kappa$, then $B + U \neq G$ which yields $g \in G \setminus (B + U)$. Clearly $g \notin B$, and $B \cup \{g\}$ is an element of $\mathcal{S}$ that contains B, which is a contradiction. So we assume that $|B| > \kappa$.

Now, let $\gamma = \{\gamma(i)\}_{n<\omega}$ be an arbitrarily chosen member of $\mathcal{P}_{Fin}(I)^{\mathbb{N}}$. The following subsets of G are canonically associated to γ

$$C_{\gamma(1)\ldots\gamma(k)} = \bigcup_{\beta \in\ \mathcal{P}_{Fin}(I)^{\mathbb{N}}} \{A_\beta : \beta(n) = \gamma(n), 1 \leq n \leq k\}$$

We have $C_{\gamma(1)} \supseteq C_{\gamma(1)\gamma(2)} \supseteq \ldots \supseteq C_{\gamma(1)\ldots\gamma(k)} \supseteq \ldots$

We will select now an element $\alpha \in\ \mathcal{P}_{Fin}(I)^{\mathbb{N}}$ and a sequence $\{x_n\}_{n<\omega} \subseteq G$ such that $x_k \in C_{\alpha(1)\ldots\alpha(k)}$ and, if $x_n - x_m \in U$, then $n = m$. Indeed, for $k = 1$, we take $\alpha(1) = D_1 \in \mathcal{P}_{Fin}(I)$ such that $|B \cap C_{\alpha(1)}| > \kappa$ or, equivalently $|B \cap C_{D_1}| > \kappa$. This subset D_1 exists because otherwise we would have $B = \bigcup_{D \in \mathcal{P}_{Fin}(I)} (B \cap C_D)$. Since $|\mathcal{P}_{Fin}(I)| = \kappa$ and $|B \cap C_D| \leq \kappa, \forall D \in \mathcal{P}_{Fin}(I)$ it would follow that $|B| \leq \kappa$, which is a contradiction.

Take $x_1 \in B \cap C_{\alpha(1)}$ arbitrarily selected. Applying an inductive argument, suppose that $C_{\alpha(1)\ldots\alpha(j)}$ and x_j have been defined for $1 \leq j \leq k - 1$. We repeat the procedure in order to take $\alpha(k)$ such that $|B \cap C_{\alpha(1)\ldots\alpha(k)}| > \kappa$

and select $x_k \in B \cap C_{\alpha(1)\ldots\alpha(k)} \setminus \{x_1, ..., x_{k-1}\}$. Thus $x_n - x_m \in U$ implies $n = m$.

Let us verify now that there is $\gamma \in \mathcal{P}_{Fin}(I)^{\mathbb{N}}$ such that $\{x_n\}_{n<\omega} \subseteq A_\gamma$. Indeed, by construction $x_k \in C_{\alpha(1)\ldots\alpha(k)}$, this implies the existence of $\beta_k \in \mathcal{P}_{Fin}(I)^{\mathbb{N}}$ such that $\beta_k(n) = \alpha(n)$, $1 \leq n \leq k$, and $x_k \in A_{\beta_k}$. We define $\gamma(n) = \bigcup_{k=1}^{n-1} \beta_k(n) \bigcup \alpha(n) \in \mathcal{P}_{Fin}(I), \forall n < \omega$. Clearly $\beta_k(n) \subseteq \gamma(n)$, $\forall n < \omega$, $\forall k < \omega$. Therefore, we have $\beta_k \subseteq \gamma$, $\forall k < \omega$. As a consequence, the sequence $\{x_n\}_{n<\omega} \subseteq A_\gamma$ must be a precompact subset of G. But this is impossible since $x_n - x_m \in U$ implies that $n = m$. $\square$

3.2 Weight of precompact subsets

Next result is a variant of the approach given in [33, 34] for locally convex spaces and countable cardinality. In the sequel, the symbol $w(X)$ denotes the weight of the topological space X.

Proposition 3.2.1 *Let (G, τ) be a locally quasi-convex group endowed with a boundedness $\mathcal{A}$. If $(\widehat{G}, \tau_{\mathcal{A}}(G))$ is κ-bounded, then $w(A) \leq \kappa$ for every $A \in \mathcal{A}$.*

Proof. Let A be an arbitrary element of $\mathcal{A}$. Replacing A by $A \cup (-A) \cup \{e_G\}$ if it were necessary, we may assume w.l.o.g. that A is symmetric and contains the neutral element. Set $n[A] = (A+A) \cup 2(A+A) \cup 3(A+A) \cup ... \cup n(A+A)$, which is a bounded subset of G. We have $N(e_{\widehat{G}}, n[A], \frac{1}{4}) = N(e_{\widehat{G}}, A+A, \frac{1}{4n})$.

Since $\widehat{G}$ is κ-bounded and $n[A]^0$ is a neighborhood of the identity, there is $H_n \subseteq \widehat{G}$ with $|H_n| \leq \kappa$ such that $\widehat{G} = H_n + n[A]^0$. Let H be the subgroup generated by $\bigcup_{n<\omega} H_n$. Then $|H| \leq \kappa$.

On the other hand, since A^0 is a neighborhood of the identity in $\widehat{G}$, it follows by [7, Pr. 1.5] that ${}^0A^0$ (and, therefore, also A) is $\tau_{pc}(\widehat{G})$-precompact. Since the topology $\tau_{pc}(\widehat{G})$ is finer than τ, it follows that the closure of A in the completion of (G, τ) must be compact. Since a group and its completion have the same algebraic dual group, it follows that the set A and, therefore, also $n[A]$ are equipped with the topology $\omega(G, \widehat{G})$.

It will suffice now to verify that H separates the points of A. Indeed, take $a_1, a_2 \in A$ such that $a_1 - a_2 \neq e_G$. Since G is locally quasi-convex, it follows that there is $\chi \in \widehat{G}$ such that $|\chi(a_1 - a_2)| = a \neq 0$. Take n large enough so that $\frac{1}{n}$ is small compared to a. Then $\chi \in \widehat{G} = H_n + n[A]^0$. That is $\chi = \alpha_n + \beta_n$, $\alpha_n \in H_n$ and $\beta_n \in n[A]^0$. Then $|\beta_n(a_1 - a_2)| \leq \frac{1}{4n} < min(\frac{a}{4}, \frac{1}{4})$ because $k(a_1 - a_2) \in n[A]$, $1 \leq k \leq n$. As a consequence $|\alpha_n(a_1 - a_2)| \geq \frac{a}{4} > 0$. This verifies that H separates the points of A, which implies $w(A) \leq \kappa$.

Lemma 3.2.2 *Let K be a compact space with $w(K) = \kappa$. Then the group $C_u(K, \mathbb{T})$ is κ-bounded.*

Proof. Let $\mathfrak{B}$ be a open base for K of cardinality κ. For every pair of elements U, V in $\mathfrak{B}$ such that $U \subset \overline{U} \subset V$, take a continuous real-valued map $f_{(U,V)}$ such that $0 \leq f_{(U,V)} \leq 1$, $f_{(U,V)}(U) = 0$ and $f_{(U,V)}(K \setminus V) = 1$. If $\mathcal{F}$ denotes the self-adjoint subalgebra of $C(K, \mathbb{C})$ generated by the functions

$f_{(U,V)}$ and the rational constants, then it is clear that $\mathcal{F}$ separates the points of K. Thus, Stone-Weierstrass Theorem yields the density of $\mathcal{F}$ in $C(K,\mathbb{C})$ for the topology of uniform convergence. Now, define

$$\mathcal{D} = \{\frac{f}{|f|} : f \in \mathcal{F}, f(x) \neq 0 \ \forall x \in K\}$$

We shall verify next that $\mathcal{D}$ is dense in $C_u(K,\mathbb{T})$. Indeed, let $f \in C(K,\mathbb{T}) \subseteq C(K,\mathbb{C})$. Then, there is a sequence $\{f_n\}_{n<\omega}$ in $\mathcal{F}$ converging uniformly to f. As a consequence, there is $n_0 \in \omega$ such that $|f_n(x)| > 0$ for all $x \in K$ and for all $n \geq n_0$. Assuming without loss of generality that $|f_n(x)| > 0$ for all $x \in K$ $n < \omega$, it is easily verified that the sequence $\left\{\frac{f_n}{|f_n|}\right\}_{n<\omega}$ converges to f. Since $\mathcal{D}$ has cardinality κ, this means that the group $C_u(K,\mathbb{T})$ is κ-bounded.

The following result also holds in the case of non-abelian topological groups.

Lemma 3.2.3 *Let G be a topological group endowed with a boundedness system (D,t), where D denotes a dense subset of G and t stands for translations. Then for each subgroup H of G, we have $\flat(H) \leq \flat(G)$.*

Proof. Let U be a fixed symmetric neighborhood of the identity in G. Let $J \subseteq G$ such that $|J| \leq \flat(G)$ and $JU = G$. Define $J' := (HU) \cap J$. Then $J'U = ((HU)\cap J)U = HUU \cap JU = HUU^{-1} \cap JU \supseteq H \cap JU = H \cap G = H$. Now fix $h \in H$. Since $G = JU$, there must be $j \in J$ and $u \in U$ so that $h = ju$. Then $j = hu^{-1} \in hU^{-1} \subseteq HU^{-1} = HU$ and therefore $j \in J'$. Finally, $h = ju \in jU \subseteq J'U$.

Applying the results above to the boundedness of precompact subsets, we obtain:

Theorem 3.2.4 *Let (G, τ) be a locally quasi-convex group. Then precompact subsets of G have weight less than or equal to a cardinal κ if and only if the group $(\widehat{G}, \tau_{pc}(G))$ is κ-bounded.*

Proof. *Sufficiency:* This is clear if we take the boundedness of precompact subsets in (G, τ) and apply Proposition 3.2.1.

Necessity: Let $Prec(G)$ be the boundedness of all precompact subsets of G and suppose that $w(P) \leq \kappa$ for all $P \in Prec(G)$. Let K_P denote the closure of $P \in Prec(G)$ in the completion of G, which is compact. By Lemma 3.2.2 above, the group $C_u(K_P, \mathbb{T})$ is κ-bounded. Moreover, since the latter group is metrizable, it follows that κ is also the density of $C_u(K_P, \mathbb{T})$. As a consequence, it is a well-known fact that the product $\prod\limits_{P \in Prec(G)} C(K_P, \mathbb{T})$ must also have density equal to κ. Now, it is readily seen that $(\widehat{G}, \tau_{pc}(G))$ is topologically embedded in $\prod\limits_{P \in Prec(G)} C(K_P, \mathbb{T})$ and, by Lemma 3.2.3, this means that $(\widehat{G}, \tau_{pc}(G))$ is κ-bounded, which completes the proof.

Corollary 3.2.5 *Let (G, τ) be a locally quasi-convex group. Then $\flat(\widehat{G}, \tau_{pc}(G)) = sup\{w(P) : P \in Prec(G)\}$.*

Proof. $(\geq)$ Assume that $\flat(\widehat{G}, \tau_{pc}(G)) = \kappa$. It follows that $(\widehat{G}, \tau_{pc}(G))$ is κ-bounded. Applying Theorem 3.2.4, it follows that $sup\{w(P) : P \in Prec(G)\} \leq \kappa$ and we are done.

($\leq$) Let's assume now that $sup\{w(P) : P \in Prec(G)\} \leq \kappa$. Then, $(\widehat{G}, \tau_{pc}(G))$ is κ-bounded by Theorem 3.2.4. It follows that $\flat(\widehat{G}, \tau_{pc}(G)) \leq \kappa$. Finally, $\flat(\widehat{G}, \tau_{pc}(G)) \leq sup\{w(P) : P \in Prec(G)\}$ and the proof is done.

Corollary 3.2.6 *Let (G, τ) be a locally quasi-convex group. If $\widehat{G}$ is metrizable, then its density equals $sup\{w(P) : P \in Prec(G)\}$.*

Proof. Apply 5 and 3.2.5.

We finish this section with a result that extends to locally quasi-convex groups and arbitrary cardinality, a metrizability condition for the precompact subsets of certain locally convex vector spaces given in [20].

Theorem 3.2.7 *Let G be a group and suppose that $\widehat{G}$ has associated a family $\{A_\alpha : \alpha \in \mathcal{P}_{Fin}(I)^{\mathbb{N}}\}$, for some I, that satisfies the properties:*

1. *$\widehat{G} = \bigcup\{A_\alpha : \alpha \in \mathcal{P}_{Fin}(I)^{\mathbb{N}}\}$,*
2. *$A_\alpha \subseteq A_\beta$ if $\alpha \subseteq \beta$,*
3. *every countable subset in A_α is equicontinuous.*

Then precompact subsets of G have weight less than or equal to $|I|$.

Proof. Observe that we may assume G is complete, since the dual group does not change. Then, taking into account that every equicontinuous in $\widehat{G}$ is $\tau_c(G)$-precompact, it is enough to apply Propositions 3.1.4 and 3.2.1 to the boundedness of precompact subsets on G.

3.3 Weakly reflexive groups: pk-groups

Let G be a topological group and let $(\widehat{G}, \tau_{pc}(G))$ be the dual group equipped with the topology of uniform convergence on the precompact subsets of G. We can repeat the process in order to define the *bidual group* $(\widehat{\widehat{G}}, \tau_{pc}(\widehat{G}))$. There is a canonically defined *evaluation map* $E_G : G \longrightarrow \widehat{\widehat{G}}$ of G into its bidual group.

When the group G is locally quasi-convex, its character group separates the points of G and thus the evaluation map E_G is injective. Applying [7, Proposition 1.5], it can be proved that this map is open onto its image. If E_G is also continuous, then G and $E_G(G)$ are topologically isomorphic, that is, the topology of G coincides with the topology $\tau_{pc}(\widehat{G})$, inherited from the bidual group. In this case we say that G is *weakly reflexive* and we may identify the group G with its continuous injection $E_G(G)$.

We say that a map $f : X \longrightarrow Y$ between topological spaces is *pk-continuous* when for each precompact subset P of X, the restriction $f_{|P} : P \longrightarrow Y$ is continuous, considering P with the induced topology. Further, a topological space is a *pk-space* whenever, for each topological space Y and each application $f : X \longrightarrow Y$ pk-continuous, F is continuous.

Similarly, an abelian topological group G is said to be a *pk-group* if each pk-continuous group homomorphism $h : G \longrightarrow H$ (H an arbitrary abelian topological group) is continuous.

Definition 3.3.1 Let (G, τ) be a locally quasi-convex topological group. We say that $W \subseteq G$ is a *pk-neighborhood* of the identity if for any τ-precompact subset P of G containing e_G, there exists a neighborhood U of e_G such that $U \cap P \subseteq W \cap P$ (or, equivalently, $W \cap P$ is a neighborhood of e_G in P).

Note that G is a *pk-group* if the topology of G is determined by its precompact subsets, that is, $C \subseteq G$ is closed if and only if $C \cap P$ is closed in P for each precompact subset P of G. This notion is a particular case of the well-known concept of a k-group, defined in [81] for general topological groups. It can be easily seen that each abelian topological group which is a pk-space is, in particular, a pk-group.

Following [56], we introduce the following two lemmas.

Lemma 3.3.2 *In a pk-group every quasi-convex pk-neighborhood of the identity is a neighborhood of the identity.*

Proof. The proof is similar to the one in [56], replacing compact by precompact.

Lemma 3.3.3 *Let G be a locally quasi-convex topological group. A quasi-convex subset $U \subseteq G$ is a pk-neighborhood of the identity if and only if U^0 is a precompact subset of $\widehat{G}$.*

Proof. Let U be a quasi-convex pk-neighborhood of the identity in G. The group $\widehat{G}$ endowed with the pointwise convergence topology on elements of G is bounded as a subspace of the Tychonoff product $\mathbb{T}^G$. Its completion $b\widehat{G}$

can be identified with the group of all homomorphisms from G to $\mathbb{T}$. Since $b\widehat{G}$ is compact, the polar of U in $b\widehat{G}$, say U^b, is compact and $U^0 \subset U^b$.

Claim: The topologies τ_{pc} and τ_p coincide on U^0. Indeed, it is obvious that $\tau_p \subseteq \tau_{pc}$. Take $\eta \in U^0$. Let W be a pk-neighborhood of η in U^0 ($\eta \in W \subseteq U^0$). Since $\{P^0 : P \in Prec(G)\}$ is a neighborhood base of the identity in $\tau_{pc}(G)$, there must be $P \in Prec(G)$ such that $(P^0+\eta) \cap U^0 \subset W$.

By hypothesis, U is quasi-convex so $U =^0 (U^0)$. Define ${}^0(U^0)_3 = N(e_G, U^0, \frac{1}{12})$. By [56, Lemma 4], $\forall g \in P$ we have that $(g +^0 (U^0)_3) \cap P \in \mathcal{N}_P(g)$. Since P is precompact, there must be $\{g_1, g_2, ..., g_n\} \subseteq P$ such that $P \subseteq \{g_1, g_2, ..., g_n\} +^0 (U^0)_3$ and therefore $\forall g \in P$ there is $h \in^0 (U^0)_3$ such that $g = g_i + h$ for some $i \in \{1, ..., n\}$. Define $V := N(\eta, \{g_1, ..., g_n\}, \frac{1}{12}) = \{\chi \in \widehat{G} : (\forall i \in \{1, ..., n\}) \mid \chi(g_i) - \eta(g_i) \mid \leq \frac{1}{12}\} \in \mathcal{N}_{\widehat{G}}(\eta)$. That is, $\eta \in V \in \tau_p(G)$. In order to prove that $V \cap U^0 \subset (P^0 + \eta) \cap U^0$, take $\chi_0 \in V \cap U^0$. Since $\eta \in U^0$,

$$\mid \chi_0(g) - \eta(g) \mid \leq \mid \chi_0(g) - \chi_0(g_i) \mid + \mid \chi_0(g_i) - \eta(g_i) \mid + \mid \eta(g_i) - \eta(g) \mid =$$

$$= \mid \chi_0(h) \mid + \mid \chi_0(g_i) - \eta(g_i) \mid + \mid \eta(h) \mid \leq \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{1}{4}.$$

Since this is true $\forall g \in P$, it follows that $\chi_0 - \eta \in P^0$ and therefore $\chi_0 \in P^0 + \eta$.

On the other hand, let U be a quasi-convex subset of G such that $U^0 \in Prec(\widehat{G})$. Let $e_G \in P \in Prec(G)$. Then $(U^0)_{|P}$ is a precompact subset of $C_u(P, \mathbb{T})$. If K is the closure of P in the completion of G, applying Ascoli's theorem $(U^0)_{|K}$ is an equicontinuous subset of $C(K, \mathbb{T})$. Thus $U \cap K \in \mathcal{N}_K(e_G)$ and finally $U \cap P \in \mathcal{N}_P(e_G)$.

Definition 3.3.4 We say that the group G is a *pk_γ-group* if any quasi-convex set that is a pk-neighborhood of the identity is also a neighborhood of the identity in the original topology of G (cf. [56]). Obviously, the class of pk_γ-groups contains the class of pk-groups.

Theorem 3.3.5 *A locally quasi-convex group G is weakly reflexive if and only if it is a pk_γ-group.*

Proof. ($\Rightarrow$) Let G be a locally quasi-convex weakly reflexive group. Take an arbitrary quasi-convex pk-neighborhood U of the identity in G. Applying Lemma 3.3.3, U^0 is precompact in $\widehat{G}$. By the weak reflexivity of G, it follows that $U =^0 (U^0)$ is a neighborhood of the identity in G.

($\Leftarrow$) On the other hand, let's consider precompact subsets P in G and F in $\widehat{G}$, and let K be the closure of P in the completion of G. Applying Ascoli's theorem, $F_{|K}$ is equicontinuous in $C_u(K, \mathbb{T})$. If $0 \in K$, then ${}^0F \cap K \in \mathcal{N}_K(0)$ and hence ${}^0F \cap P \in \mathcal{N}_P(0)$. By hypothesis, ${}^0F \in \mathcal{N}(0)$.

Corollary 3.3.6 *Let G be a locally quasi-convex pk-group. Then, G is weakly reflexive.*

Proof. Every locally quasi-convex pk-group is pk_γ-group and we may apply Theorem 3.3.5.

3.4 Character

In this section we estimate the character of certain topological groups that contain the well-known class of k_ω-groups.

Definition 3.4.1 We recall that the *character* $\chi(G)$ of a topological group G is the smallest cardinal possible of a neighborhood base of the identity. The character is an important invariant cardinal of topological groups. A group is metrizable if and only if its character is $\aleph_0$.

3.4.1 Character of pk_γ-groups

For the proof of the following result, one argues *mutatis mutandis* as in the proof of Proposition 1 in [56], replacing compact by precompact subsets.

Lemma 3.4.2 *Let G be a locally quasi-convex pk_γ-group and let B be a subset of $\widehat{G}$ such that B^0 is a pk-neighborhood of e_G. Then B^0 is a neighborhood of e_G in G.*

Lemma 3.4.3 *Let G be a metrizable group and let $\overline{G}$ be its completion. Then the groups $(\widehat{G}, \tau_{pc}(G))$ and $(\widehat{G}, \tau_c(\overline{G}))$ are topologically isomorphic.*

Proof. The following inclusions hold trivially

$$\tau_{pc}(G) \subseteq \tau_{pc}(\overline{G}) = \tau_c(\overline{G}).$$

On the other hand, Corollary 2.3.3 yields $\tau_{pc}(\overline{G}) \subseteq \tau_{pc}(G)$, which implies $\tau_{pc}(G) = \tau_c(\overline{G})$. $\square$

Lemma 3.4.4 *Assume that G is a weakly reflexive group. Then the family $\{^0P : P \in Prec(\widehat{G})\}$ is a neighborhood base of the identity in G.*

Proof. Since G is weakly reflexive, the set ^0P is a neighborhood of e_G for each $P \in Prec(\widehat{G})$. Let U be an arbitrary neighborhood of e_G. As G is locally quasi-convex, we may assume w.l.o.g. that U is quasi-convex. The set $U^0 \in Prec(G)$ (see [7]) and its inverse polar $^0(U^0) = U$ by the quasi-convexity of U. $\square$

Theorem 3.4.5 *Let G be a weakly reflexive locally quasi-convex abelian group. Then the family of all neighborhoods of the identity $(\mathcal{N}(e_G), \supseteq)$ is cofinally equivalent to $(Prec(\widehat{G}), \subseteq)$, and therefore $\chi(G) = cof(Prec(\widehat{G}))$.*

Proof. The map that assigns to each precompact subset P of $\widehat{G}$ its inverse polar ^0P is order preserving and, by Lemma 3.4.4, it is also cofinal in the family of all neighborhoods of e_G considered with the inclusion order $\supseteq$. On the other hand, the map that assigns to each neighborhood U of e_G , its polar U^0 is order preserving. To see that it is also cofinal, consider an arbitrary precompact subset P of $\widehat{G}$. Its inverse polar ^0P is a neighborhood of e_G, by Lemma 3.4.4 and contains a quasi-convex neighborhood of e_G, let's say U. Finally, $U^0 \supseteq (^0P)^0 \supseteq P$.

Corollary 3.4.6 *Let G be a weakly reflexive locally quasi-convex abelian group, such that $\widehat{G}_{pc}$ is metrizable and locally precompact. Then $\chi(G) = sup\{w(P) : P \in Prec(\widehat{G})\}$.*

Proof. Let $\kappa := sup\{w(P) : P \in Prec(\widehat{G})\}$. Since $\widehat{G}$ is metrizable, the density of $\widehat{G}$ is κ, by Corollary 3.2.6. Therefore $\widehat{G}$ can be endowed with a boundedness system (κ, b). As $\widehat{G}$ is also locally precompact, we may apply Proposition 2.2.9 and obtain that $\kappa = cof(Prec(\widehat{G}))$. Finally, by Theorem 3.4.5, $\chi(G) = cof(Prec(\widehat{G}))$ and we are done.

Corollary 3.4.7 *Let G be a weakly reflexive locally quasi-convex abelian group, such that $\widehat{G}_{pc}$ is metrizable and not locally precompact. Let $\kappa := sup\{w(P) : P \in Prec(\widehat{G})\}$. Then $\kappa \leq \chi(G) \leq \mathfrak{d} \cdot cof(\mathcal{P}_{\aleph_0}(\kappa))$. In particular,*

(1) $\kappa \leq \chi(G) \leq \mathfrak{d} \cdot \kappa^{\aleph_0}$,

(2) If $\kappa = \kappa^{\aleph_0}$, then $\kappa \leq \chi(G) \leq \mathfrak{d} \cdot \kappa$,

(3) If $\kappa = \aleph_n$, $n \in \mathbb{N}$, then $\aleph_n \leq \chi(G) \leq \mathfrak{d} \cdot \aleph_n$.

Proof. Since $\widehat{G}$ is metrizable, the density of $\widehat{G}$ is κ, by Corollary 3.2.6. It follows that $\widehat{G}$ can be endowed with a boundedness system (κ, b). Further, as $\widehat{G}$ is non locally precompact, applying Lemma 2.2.12 we have that $\kappa \leq cof(Prec(\widehat{G})) \leq cof(\mathcal{P}_{Fin}(\kappa)^{\mathbb{N}})$. Then, Theorem 3.5.3 applies to obtain $\kappa \leq cof(Prec(\widehat{G})) \leq \mathfrak{d} \cdot cof(\mathcal{P}_{\aleph_0}(\kappa))$. Finally, Theorem 3.4.5 asserts that $\chi(G) = cof(Prec(\widehat{G}))$ and the proof is done. Statements $(1)-(3)$ follow from section *Infinite cardinals.*

Comment: In [24], due to further investigations of Professor Boaz Tsaban, there have been included extensions of these results to larger cardinals.

3.4.2 Character of hemi-precompact groups

Definition 3.4.8 Given a locally quasi-convex group G, we say that G is *hemi-precompact* when it contains a sequence $\{P_n\}_{n<\omega}$ of precompact subsets (that we will name *co-base*, for short) satisfying the following two properties:

1. $G = \bigcup\limits_{n<\omega} P_n$.
2. For each precompact subset $P \subseteq G$, there is $n < \omega$ such that $P_n \supseteq P$.

In this case $cof(Prec(G)) = \aleph_0$ and $\widehat{G}$ is metrizable.

Definition 3.4.9 We say that a group G is *locally hemi-precompact* when it contains an open subgroup G_0 that is hemi-precompact.

Theorem 3.4.10 *Let (G, τ) be a non locally precompact hemi-precompact group and suppose that the dual group $(\widehat{G}, \tau_{pc}(G))$ is κ-bounded. Then the following assertions are true:*

(i) $\kappa \leq \chi(G) \leq max(\mathfrak{d}, cof(\mathcal{P}_{\aleph_0}(\kappa))) \leq \kappa^{\aleph_0}$.

(ii) Furthermore, in case $\aleph_0 \leq \kappa < \aleph_\omega$, we have

$$\kappa \leq \chi(G) \leq max(\mathfrak{d}, \kappa).$$

Proof. We may assume w.l.o.g. that $G = \bigcup_{n<\omega} P_n$, where $\{P_n\}_{n<\omega}$ is a co-base for the precompact subsets that satisfies the following properties:

1. $e_G \in P_1$;
2. $P_n = -P_n$; and
3. $P_n + P_n \subseteq P_{n+1}$ for all $n < \omega$.

Let $\widehat{G}_{pc}$ denote the dual group of G equipped with the topology $\tau_{pc}(G)$. We have that $\widehat{G}_{pc}$ is metrizable. Applying Theorem 2.2.16, it follows that

$$\kappa \leq cof(Prec(\widehat{G})) \leq max(\mathfrak{d}, cof(\mathcal{P}_{\aleph_0}(\kappa))) \leq \kappa^{\aleph_0}.$$

Now, since G is locally quasi-convex, the evaluation mapping E_G of G into the bidual group $\widehat{\widehat{G}}$ is open onto its image. Therefore, if $\mathcal{F} \subseteq Prec(G)$ denotes the family of all equicontinuous subsets F of $\widehat{G}$, it follows that the collection $\{\,^0F : F \in \mathcal{F}\}$ is a neighborhood base of the identity in G. Hence, we may dualize the result of Lemma 2.2.11 to the collection $\mathcal{F}$ of all equicontinuous subsets of $\widehat{G}$ in order to obtain $\kappa \leq \chi(G)$. On the other hand, since the topology $\tau_{pc}(G)$ is finer than τ and, by duality, the character of $\tau_{pc}(G)$ coincides with the cofinality of $Prec(\widehat{G})$, it follows that

$$\kappa \leq \chi(G) \leq cof(Prec(\widehat{G})) \leq max(\mathfrak{d}, cof(\mathcal{P}_{\aleph_0}(\kappa))) \leq \kappa^{\aleph_0}.$$

This verifies assertion (i).

As for assertion (ii), let $\overline{G}$ denote the completion of G and set $K_n = \overline{P_n}^{\overline{G}}$. Then, taking into account the constraints on the co-base $\{P_n\}_{n<\omega}$, it follows that $\widetilde{G} = \bigcup_{n<\omega} K_n$ is a subgroup of $\overline{G}$. Moreover, let $E_G : G \longrightarrow (\widehat{\widehat{G}}, \tau_c(\widehat{G}))$ be the canonical evaluation map, we claim that the canonical extension of E_G to $\widetilde{G}$ takes values in $\widetilde{\widehat{\widehat{G}}}$. Indeed, suppose that $\{\chi_n\}_{n<\omega}$ is a sequence in $\widehat{G}$ converging to $e_{\widehat{G}}$ in the topology τ_{pc}. This means $\{\chi_n\}_{n<\omega}$ converges uniformly on P_m for all $m < \omega$. Hence $\{\chi_n\}_{n<\omega}$ also converges uniformly on K_m for all $m < \omega$ and we are done because $\widetilde{G} = \bigcup_{n<\omega} K_n$. Thus, we have a well defined map $E_G : \widetilde{G} \longrightarrow (\widehat{\widehat{G}}, \tau_c(\widehat{G}))$.

Let F be a precompact subset of $\widehat{G}_{pc}$. Then, for every $m < \omega$, we can consider F as a precompact subset of the group $C_u(K_m, \mathbb{T})$, which is complete. Applying Ascoli's theorem, it follows that F is an equicontinuous set and, since K_m contains the neutral element, we deduce that ${}^0F \cap K_m$ is a neighborhood of e_G. Taking intersections, we have obtained that ${}^0F \cap P_m$ is a neighborhood of e_G for every $m < \omega$.

As G is a pk_γ-group, it follows that ${}^0F$ is a neighborhood of e_G in G for every compact subset F of $\widehat{G}_{pc}$ (see [56, Ex. 2 and Prop. 1] where this assertion is widely treated for the k-topology, although the same arguments apply to the pk-topology). In the end, we have proved that the evaluation mapping E_G is continuous on G. Thus, the family $\{ {}^0F : F \in Prec(\widehat{G})\}$ is a neighborhood base of the identity in G. This yields the equality $\chi(G) = cof(Prec(\widehat{G}))$ and, applying Theorem 2.2.16, we are done with the proof of (ii).

Finally, observe that, in case G is a pk_γ-group, the evaluation mapping E_G is also continuous on $\widetilde{G}$. This implies that $\widetilde{G}$ is topologically embedded in $\widehat{\widehat{G}}$. That is to say, the group $\widetilde{G}$ is equipped with the topology $\tau_{pc}(\widehat{G})_{|\widetilde{G}}$. Since the group $(\widehat{G}, \tau_{pc}(G))$ is κ-bounded and metrizable, it follows that there is a dense subset D of $\widehat{G}$ such that $|D| = \kappa$. This means that every compact subset of $\widetilde{G}$ is equipped with the topology $\sigma(\widetilde{G}, D)$. In particular, we have $w(K_m) \leq \kappa$ for all $m < \omega$. As a consequence, all precompact subsets of G have weight less than or equal to κ and this completes the proof.

We introduce the following lemma which will be used along the presentation.

Lemma 3.4.11 *Let* $X = \bigcup\limits_{n<\omega} K_n$ *be a* k_ω*-space. Assume that* $w(K_n) = \kappa_n$ *for all* $n < \omega$ *and let* $\kappa = \sup\{\kappa_n : n < \omega\}$. *Then the group* $C_c(X, \mathbb{T})$ *is* κ*-bounded.*

Proof. The group $C_c(X, \mathbb{T})$ is isomorphically embedded into the product $\prod\limits_{n<\omega} C_u(K_n, \mathbb{T})$. Since every group $C_u(K_n, \mathbb{T})$ is κ_n-bounded, it follows that the group $C_c(X, \mathbb{T})$ is κ-bounded.

Indeed, since the groups considered are all metrizable, the proof can be translated to questions about the density of the groups involved and it is well known that density is an hereditary notion for metrizable spaces.

Next follows the main result of this section.

Theorem 3.4.12 *Let G be a locally hemi-precompact group and suppose that G_0 is an open subgroup of G such that $G_0 = \bigcup_{n<\omega} P_n$, where $\{P_n\}_{n<\omega}$ is a co-base for the precompact subsets in G_0. If $\kappa = sup\{w(P_n) : n < \omega\}$ then the following assertions are true:*

(i) If G_0 is locally precompact then $\chi(G) = \kappa$.

(ii) If G_0 is not locally precompact then we have:

(a) $\kappa \leq \chi(G) \leq max(\mathfrak{d}, cof(\mathcal{P}_{\aleph_0}(\kappa))) \leq \kappa^{\aleph_0}$,

(b) finally, in case $\aleph_0 \leq \kappa < \aleph_\omega$, we have

$$\kappa \leq \chi(G) \leq max(\mathfrak{d}, \kappa).$$

Proof. If G is locally precompact, its completion is locally compact. Therefore, assertion (i) follows directly from the theory of locally compact abelian groups (see [26, th. 3.9]).

So, we may assume w.l.o.g. that $G = G_0$ and that G is not locally precompact. We now select a co-base $\{P_n\}_{n<\omega}$ of precompact subsets of G that satisfies the same properties $(1) - (3)$ of Theorem 3.4.10. Let $\overline{G}$ denote the completion of G, $K_n = \overline{P_n}^{\overline{G}}$, and define the set $\widetilde{G} = \bigcup_{n<\omega} K_n$, which is a subgroup of $\overline{G}$. It is readily seen that the dual group $(\widehat{G}, \tau_{pc}(G))$ is topologically embedded in $C_c(\widetilde{G}, \mathbb{T})$, which is κ-bounded by Lemma 3.4.11. Thus, the dual group is κ-bounded and the proof of assertion (ii) follows immediately from Theorem 3.4.10.

Next we shall apply the results above to estimate the character of some free abelian groups. A completely different approach to this question has been given by Nickolas and Tkachenko in [79, 80].

Theorem 3.4.13 *Let* $X = \bigcup_{n<\omega} X_n$ *be a* κ_ω*-space. If* $\kappa = sup\{w(X_n) : n < \omega\}$*, then the following assertions are true:*

(i) $\kappa \leq \chi(A(X)) \leq max(\mathfrak{d}, cof(\mathcal{P}_{\aleph_0}(\kappa))) \leq \kappa^{\aleph_0}$.

(ii) Furthermore, in case $\aleph_0 \leq \kappa < \aleph_\omega$*, we have*

$$\kappa \leq \chi(A(X)) \leq max(\mathfrak{d}, \kappa).$$

In addition, for every precompact (resp. Bohr compact) subset P *of* $A(X)$ *we have* $w(P) \leq \kappa$.

Proof. Observe that $A(X)$ is an hemi-precompact group if we select the co-base of precompact subsets $\{P_{nm}\}_{n<\omega}$ consisting of words of length less than or equal to m generated by the elements of $\bigcup_{i=1}^{n} X_i$. Therefore, assertion (i) is consequence of Theorem 3.4.12.
On the other hand, if the topology of X is determined by its functionally bounded subsets, it follows that $A(X)$ is a pk_γ-group and again the proof of (ii) is a direct consequence of Theorem 3.4.12.

3.4.3 Character of locally pk_ω groups

All the results presented in this section can be applied with small modifications to locally convex vector spaces which contain a countable cofinal family of bounded subsets. In particular, this is the case for countable inductive limits of DF-spaces.

Definition 3.4.14 Given a locally quasi-convex group G, we say that G is a *pk_ω-group* when it is:

1. hemi-precompact and

2. pk-group.

Definition 3.4.15 We say that a group G is *locally pk_ω-group* when it contains an open subgroup G_0 that is a pk_ω-group.

According to the definition, every pk_ω-group G is weakly reflexive and its dual group $\widehat{G}$ is metrizable. Moreover, since G_0 is an open subgroup of G, we know that $\chi(G_0) = \chi(G)$.

The class of locally pk_ω abelian groups contains locally k_ω-groups, locally precompact groups and their countable direct limits, free abelian groups on pseudocompact spaces and any dual group of a countable projective limit of Cech-complete groups, like dual groups of abelian pro-Lie groups defined by a countable system.

Corollary 3.4.16 *Let G be a locally pk_ω group and suppose that G_0 is an open subgroup of G such that $G_0 = \bigcup_{n<\omega} P_n$, where $\{P_n\}_{n<\omega}$ is a co-base for the precompact subsets in G_0. If $\kappa = sup\{w(P_n) : n < \omega\}$ then the following assertions are true:*

(i) If G_0 is locally precompact then $\chi(G) = \kappa$.

(ii) If G_0 is not locally precompact then we have:

(a) $\kappa \leq \chi(G) \leq max(\mathfrak{d}, cof(\mathcal{P}_{\aleph_0}(\kappa))) \leq \mathfrak{d} \cdot \kappa^{\aleph_0}$,

(b) finally, in case $\aleph_0 \leq \kappa < \aleph_\omega$, we have

$$\kappa \leq \chi(G) \leq max(\mathfrak{d}, \kappa).$$

Proof. Apply Corollary 3.4.7. □

Lemma 3.4.17 *A metrizable locally quasi-convex topological group has the same density as its bidual group.*

Proof. Let G be a metrizable group. Then G is weakly reflexive and thus, it is topologically embedded in $\widehat{\widehat{G}}$, which yields $d(G) \leq d(\widehat{\widehat{G}})$.

On the other hand, by Lemma 3.4.3, we may assume w.l.o.g. that G is complete and $\widehat{G}$ is equipped with the compact open topology on G. Hence, $\widehat{G}$ is complete. By Corollary 3.2.6, we have $sup\{w(P) : P \in Prec(\widehat{G})\} = d(\widehat{\widehat{G}})$. Since $\widehat{G}$ is complete $d(\widehat{\widehat{G}}) = sup\{w(P) : P \in Prec(\widehat{G})\} = sup\{w(K) : K \in \mathcal{K}(\widehat{G})\}$. Now, given $K \in \mathcal{K}(\widehat{G})$, since G separates the points of $\widehat{G}$ and K is compact, it follows that $w(K) \leq d(G)$. This implies $d(\widehat{\widehat{G}}) \leq d(G)$.

Next theorem extends a result of Saxon and Sanchez-Ruiz [91, Corollary 2] for the strong dual of metrizable spaces to topological abelian groups.

Theorem 3.4.18 *Let Γ be the dual of a metrizable group of density κ. Then the following assertions are true:*

(i) If Γ is locally precompact then $\chi(\Gamma) = \kappa$.

(ii) If Γ is not locally precompact then we have:

(a) $\kappa \leq \chi(\Gamma) \leq max(\mathfrak{d}, cof(P_{\aleph_0}(\kappa))) \leq \mathfrak{d} \cdot \kappa^{\aleph_0}$,

(b) finally, in case $\aleph_0 \leq \kappa < \aleph_\omega$, we have

$$\kappa \leq \chi(\Gamma) \leq max(\mathfrak{d}, \kappa).$$

Proof. Let G be a group and let Γ be its dual group. The group G is locally quasi-convex and therefore the evaluation map E_G is injective and open onto its image. Moreover, since G is metrizable E_G is also continuous and thus G is weakly reflexive.

Applying Theorem 3.4.5, the cofinality of $Prec(\Gamma)$ is $\aleph_0$ and it follows that Γ is hemi-precompact, that is, $\Gamma = \bigcup\limits_{n<\omega} P_n$, where $\{P_n\}_{n<\omega}$ is a co-base for the precompact subsets of Γ. By a result of Chasco [21] and independently Aussenhofer [6], the dual of a metrizable group is, in particular, a pk-space. Thus, Γ is a pk_ω-group.

The metrizability of G yields, by Lemma 3.4.17, that $d(\widehat{\Gamma}) = d(G) = \kappa$. Now, $\widehat{\Gamma}$ being metrizable, by Corollary 3.2.6, we have $sup\{w(P) : P \in$

$Prec(\Gamma)\} = d(\widehat{\Gamma}) = \kappa$. Finally, $sup\{w(P_n) : n < \omega\} = \kappa$ and we may apply Corollary 3.4.16 to the group Γ. $\square$

3.5 Tightness

Definition 3.5.1 The *tightness* of a group G, denoted by $t(G)$, is the smallest infinite cardinal number l such that for any subset A of G and any element $g \in \overline{A}^G$ there is a subset $B \subseteq A$ such that $|B| \leq l$ and $g \in \overline{B}^G$.

Proposition 3.5.2 *Let $l(X^n) \leq \tau$ for all $n \in \mathbb{N}^+$. Then $t(C_p(X, \mathbb{T})) \leq \tau$.*

Proof. It is similar to the proof of [3, Theorem II 1.1].

Recall that if we consider two Hausdorff topological spaces X and Y and f a map from X into the power set of Y, then the map f is called *upper semi-continuous* (*u.s.c.* for short) if for each neighborhood V of $f(x)$ there is a neighborhood U of x such that $f(y) \subset V$ for $y \in U$. Further, a topological space X is said to be *K-analytic* if it is the image of a Polish space (that is, a separable completely metrizable topological space) under a compact valued upper semi-continuous map.

Following [22], we are going to define two important group topologies associated to a family of group homomorphisms.

If we consider an abelian group G, a family of topological abelian groups $\{G_s\}_{s\in S}$ indexed by a nonempty set S and a collection $\{h_s : G_s \to G\}_{s\in S}$ of group homomorphisms, the finest topology in G making all h_s continuous is not in general a group topology. However, the family of group topologies on G which make all h_s continuous contains its supremum, which is referred to as the *final topology* on G corresponding to the family of homomorphisms $\{h_s : s \in S\}$ and is denoted by τ_f.

Consider the set

$$U_f := \bigcup_{N\in\mathbb{N}} \bigcup_{(s_1,\dots,s_N)\in S^N} \sum_{n=1}^{N} h_{s_n}(U_{s_n,n}), U_{s_n,n} \in \mathfrak{B}_{s_n}$$

where $\mathfrak{B}_s$ is a neighborhood basis of 0 in G_s for all $s \in S$.

A neighborhood basis of 0 for the topology τ_f is defined by the family

$$\{U_f : \{U_{s_n,n} : s_n \in S, n \in \mathbb{N}\} \in \prod_{s\in S} \mathfrak{B}_s^{\mathbb{N}}\}.$$

If we take

$$U_r = \bigcup_{F\in\mathcal{P}_{Fin}(S)} \sum_{s\in F} h_s(U_s)$$

then the family

$$\{U_r : (U_s)_{s\in S} \in \prod_{s\in S} \mathcal{B}_s\}$$

defines a neighborhood basis of 0 for another group topology on G which makes all h_s continuous, i.e. the *box (or rectangular) topology* associated to the family $\{h_s : s \in S\}$ and denoted by τ_r. When $|S| \leqslant \aleph_0$, the final topology τ_f in G coincides with the box topology τ_r.

Theorem 3.5.3 *Let $(G_s, \tau_s)_{s\in S}$ be a family of abelian topological groups with character at most m, let $\{h_s : G_s \to G\}_{s\in S}$ be group homomorphisms and G an abelian topological group endowed with the final topology τ_f corresponding to the family of homomorphisms $\{h_s : s \in S\}$, such that $G = <\bigcup_{s\in S} h_s(G_s)>$. Then we have :*

(i) if $|S| \leqslant m$, then (G, τ_f) and $(G, \omega(G, \widehat{G}))$ have tightness at most m;

(ii) if $|S| \leqslant \aleph_0$, then the precompact subsets of (G, τ_f) have weight at most m.

Proof. For every $s \in S$, let 0_s be the identity element in G_s. Since $\chi(G_s, \tau_s) \leqslant m$, there is a basis $\mathfrak{B}_s$ of neighborhoods of 0_s in G_s such that $|\mathfrak{B}_s| \leqslant m$. Let's assume $|S| \leqslant m$.

(i) In order to prove that $t(G, \tau_f) \leqslant m$, we have to see that for every A in G with $0_G \in \overline{A}^{\tau_f}$, there is a subset B in A of cardinality at most m such that $0_G \in \overline{B}^{\tau_f}$.

Define

$$U_f := \bigcup_{N\in\mathbb{N}} \bigcup_{(s_1,\dots,s_N)\in S^N} \sum_{n=1}^{N} h_{s_n}(U_{s_n,n}), U_{s_n,n} \in \mathfrak{B}_{s_n}.$$

The family

$$\mathfrak{B} := \{U_f : \{U_{s_n,n} : s_n \in S, n \in \mathbb{N}\} \in \prod_{s\in S} \mathfrak{B}_s^{\mathbb{N}}\}$$

is a neighborhood basis of 0_G for the final group topology τ_f on G (see [22, Proposition 5]), and if we define

$$U_0 := U_0(J,N) := \bigcup_{(s_1,\ldots,s_N)\in J^N} \sum_{n=1}^{N} h_{s_n}(U_{s_n,n}),$$

then the family

$$\mathfrak{B}_0 := \{U_0 : (U_{s,n})_{s\in J, 1\leqslant n\leqslant N} \in \prod_{s\in J} \mathfrak{B}_s^N, J \in \mathcal{P}_{Fin}(S), N \in \mathbb{N}\}$$

has at most m elements. Indeed, we have that $|\mathfrak{B}_0| \leqslant |\mathcal{P}_{Fin}(S)| \cdot |\mathfrak{B}_s| \leqslant m \cdot m = m$.

Let A be a subset of G with $0_G \in \overline{A}^{\tau_f}$ and define

$$B := \{x_{U_0} : x_{U_0} \textit{ is a point in } U_0 \bigcap A \textit{ if } U_0 \bigcap A \neq \emptyset, U_0 \in \mathfrak{B}_0\}.$$

Clearly, $B \subseteq A$ and $|B| \leqslant m$ ($|B| \leqslant |\mathfrak{B}_0| \leqslant m$).

In order to see that $0_G \in \overline{B}^{\tau_f}$, take an arbitrary $U_f \in \mathfrak{B}$. As $0_G \in \overline{A}^{\tau_f}$, we have that $U_f \bigcap A \neq \emptyset$. Take $y \in U_f \bigcap A$. There must be $N \in \mathbb{N}$ and $(s_1, \ldots, s_N) \in S^N$ such that $y = \sum_{n=1}^{N} h_{s_n}(x_{s_n,n})$, with $x_{s_n,n} \in U_{s_n,n}$, $n \in \{1, \ldots, N\}$. Define $J := \{s_1, \ldots, s_N\}$ (notice that $|J| \leqslant N$) and define $U := (U_{j,n})_{j\in J, n\leqslant N}$. Then $y \in U_0 = \bigcup_{(j_1,\ldots,j_N)\in J^N} \sum_{n=1}^{N} h_{j_n}(U_{j_n,n})$. Obviously, $U_0 \in \mathfrak{B}_0$ and $U_0 \subset U_f$. Now take the corresponding $x_{U_0} = U_0 \bigcap B \subset U_f \bigcap B$.

Since U_f is an arbitrary element of $\mathfrak{B}$, we obtain that $B \bigcap U_f \neq \emptyset$ for every $U_f \in \mathfrak{B}$, which means that $0_G \in \overline{B}^{\,\tau_f}$.

Thus, we have that $t(G, \tau_f) = |B| \leqslant m$.

To see that $(G, \omega(G, \widehat{G}))$ has tightness at most m, notice firstly that this group is homeomorphically embedded in $C_p(\widehat{G}_\omega, \mathbb{T})$, so

$$t(G, \omega(G, \widehat{G})) \leqslant t(C_p(\widehat{G}_\omega, \mathbb{T})) \qquad (1)$$

Secondly, by Proposition 3.5.2 $t(C_p(X, \mathbb{T})) \leq sup_n \; l(X^n)$, so

$$l((\widehat{G}, \omega(\widehat{G}, G))^n) \leqslant m \; \forall n \in \mathbb{N} \Rightarrow t(C_p(\widehat{G}_\omega, \mathbb{T})) \leqslant m. \qquad (2)$$

We are going to prove that $l((\widehat{G}_\omega)^n) \leqslant m \; \forall n \in \mathbb{N}$ applying [19, Corollary 2.2.].

Consider every $\mathfrak{B}_s$ as a discrete space. Define for every $s \in S$

$$\Psi_s : \mathfrak{B}_s \to \mathcal{P}(\widehat{G}_s, \omega(\widehat{G}_s, G_s))$$

such that $U \mapsto U^0, \; \forall U \in \mathfrak{B}_s$

Claim: The map Ψ_s is compact-valued.

Indeed, by [7, Proposition 1.5], for every $U \in \mathcal{B}_s$, the polar U^0 is compact in $\widehat{G}_{pc}$. Since $t_p(G) \subset t_{pc}(G)$, we have that U^0 is compact in $\widehat{G}_p = (\widehat{G}, \omega(\widehat{G}, G))$. $\square$

Claim: The map Ψ_s is upper semi-continuous.

Indeed, the map Ψ_s is upper semi-continuous if for every $U \in \mathcal{B}_s$ and every open subset O in $(\widehat{G_s}, \omega(\widehat{G_s}, G_s))$ with $O \supset U^0$, there is an open set V

in $\mathcal{B}_s$ such that $U \in V$ and $V^0 \subset O$. Let $U \in \mathcal{B}_s$ and let O be an open subset of $(\widehat{G_s}, \omega(\widehat{G_s}, G_s))$ such that $U^0 \subset O$. Since $\mathcal{B}_s$ is discrete, we can consider U both as a point $U \in \mathcal{B}_s$ and as an open neighborhood $\{U\}$ of itself. So, the open set $\{U\}$ contains the point U and obviously $\Psi_s(\{U\}) = U^0 \subseteq O$. $\square$

Claim: $Y_s := (\widehat{G_s}, \omega(\widehat{G_s}, G_s)) = \bigcup_{U \in \mathcal{B}_s} U^0$.

Indeed, $\supseteq$ is trivial. In order to prove $\subseteq$, let χ be an arbitrary element of $(\widehat{G_s}, \omega(\widehat{G_s}, G_s))$. Then χ is continuous and $\{\chi\}^0 \in \mathcal{N}(0_s)$. Now $\mathcal{B}_s$ is a basis of neighborhoods of 0_s so, there must be $U \in \mathcal{B}_s$ such that $U \subseteq \{\chi\}^0$. Taking polars, $U^0 \supset \{\chi\}^{00} \ni \chi$. Since $\chi \in U^0$, for some $U \in \mathcal{B}_s$, we have that $\chi \in \bigcup_{U \in \mathcal{B}_s} U^0$. It follows that $Y_s \subset \bigcup_{U \in \mathcal{B}_s} U^0$. Finally, we obtain that $Y_s = \bigcup_{U \in \mathcal{B}_s} \{U^0\}$ and this is true for every $s \in S$. $\square$

Now consider $\prod_{s \in S} \mathcal{B}_s$ endowed with the product topology and define the map

$$\Psi : \prod_{s \in S} \mathcal{B}_s \to \mathcal{P}(\prod_{s \in S} (\widehat{G_s}, \omega(\widehat{G_s}, G_s)))$$

such that

$$(U_s)_s \mapsto \prod_{s \in S} \Psi_s(U_s) = \prod_{s \in S} U_s^0, \ \forall (U_s)_s \in \prod_{s \in S} \mathcal{B}_s.$$

This map is compact-valued and upper semi-continuous (see [29, Proposition 3.6]) and satisfies $\prod_{s \in S} \widehat{G_s} = \bigcup \{\Psi((U_s)_s) : (U_s)_s \in \prod_{s \in S} \mathcal{B}_s\}$, as it is proved below.

Claim: Ψ is compact - valued.

Indeed, for every $s \in S$ and every $U_s \in \mathcal{B}_s$, the polar $(U_s)^0$ is compact. Applying Tychonoff's theorem the product $\prod_{s \in S} (U_s)^0$ is also compact. (It is included in the proof of [29, Proposition 3.6].) $\square$

Claim: Ψ is upper semi-continuous.

Indeed, for every $s \in S$, every $U_s \in \mathcal{B}_s$ and every open set O in Y_s such that $O \supset (U_s)^0$, there is $V_s := U_s$ such that $\Psi_s(V_s) \subset O$.

Define for every $s \in S$ the map $\Psi'_s := \Psi_s 0 \pi_s$

$$\begin{tikzcd} \prod_{s\in S}\mathcal{B}_s \arrow[rr, "\Psi'_s"] \arrow[dr, "\pi_s"'] & & \mathcal{P}((\widehat{G_s}, \omega(\widehat{G_s}, G_s))) \\ & \mathcal{B}_s \arrow[ur, "\Psi_s"'] & \end{tikzcd}$$

such that $\Psi'_s((U_s)_{s\in S}) = U_s^0$. This map is still compact-valued and upper semi-continuous, being the composition of the upper semi-continuous map Ψ_s with the continuous projection π_s. Now, applying [29, Proposition 3.6.], we obtain that the product map

$$\Psi : \prod_{s\in S}\mathcal{B}_s \to \prod_{s\in S} Y_s$$

defined by $\Psi(x) = \prod_{s\in S}\Psi_s(x)$ is also compact-valued and upper semi-continuous. $\square$

Claim: $Y := \prod_{s\in S}(\widehat{G_s}, \omega(\widehat{G_s}, G_s)) = \bigcup\{\Psi((U_s)_s) : (U_s)_s \in \prod_{s\in S}\mathcal{B}_s\}$.

Indeed, $\supseteq$ is trivial. As for $\subseteq$, by definition, we have $\Psi((U_s)_s) = \prod_{s\in S}\Psi_s(U_s)$ and it follows that $\prod_{s\in S}(\widehat{G_s}, \omega(\widehat{G_s}, G_s)) = \prod_{s\in S}\bigcup\{\Psi_s(U) : U \in \mathcal{B}_s\} \subseteq \bigcup\{\prod_{s\in S}\Psi_s(U_s) : (U_s)_s \in \prod_{s\in S}\mathcal{B}_s\} = \bigcup\{\Psi((U_s)_s) : (U_s)_s \in \prod_{s\in S}\mathcal{B}_s\}$. $\square$

By [30, Theorem 2.3.13] we know that $w(\prod_{s\in S}\mathcal{B}_s) \leqslant m$. To see that $(\widehat{G},\omega(\widehat{G},G))$ is a closed subgroup of $\prod_{s\in S}((\widehat{G_s},\omega(\widehat{G_s},G_s)))$ consider the following diagram, where $\tau_f^{\oplus}$ denotes the final topology on the direct sum $\bigoplus G_s$ corresponding to the family $\{i_s : s\in S\}$ of group homomorphisms, i.e. the coproduct topology (see [22, pag.7]).

$$\begin{tikzcd} G_s \arrow[rr, "h_s"] \arrow[dr, "i_s"'] & & h_s(G_s) \subseteq (G,\tau_f) := \langle \bigcup h_s(G_s)\rangle \\ & (\bigoplus G_s,\tau_f^{\oplus}) \arrow[ur, "h"'] & \end{tikzcd}$$

The map h which appears in the previous diagram is a quotient if and only if the map $\overline{h}$ defined as follows is a homeomorphism . Consider the diagram

$$\begin{tikzcd} (\bigoplus G_s,\tau_f^{\oplus}) \arrow[rr, "h"] \arrow[dr, "\pi"'] & & (G,\tau_f) \\ & (\bigoplus G_s,\tau_f^{\oplus})/ker\ h \arrow[ur, "\overline{h}"'] & \end{tikzcd}$$

such that $\overline{h}(\overline{x}) := h(x)$.

Since $\overline{h}\circ\pi = h$ is continuous, we have that $\overline{h}$ must be continuous. If we consider the group G endowed with the final topology corresponding to the map $\overline{h}$, denoted by $\tau_f(\overline{h})$, then $\overline{h}$ is also open. It is now obvious that $\overline{h}$ is a homeomorphism, so (G,τ_f) is topologically isomorphic to a quotient of the direct sum $(\bigoplus G_s,\tau_f^{\oplus})$, that is $(G,\tau_f)\cong(\bigoplus_{s\in S} G_s,\tau_f^{\oplus})/ker\ h$.

Applying a result of Nickolas (see [78, Theorem 4.3]), the groups $(\bigoplus G_s,\tau_f^{\oplus})^{\wedge}$ and $(\prod\widehat{G_s},\tau_{Tych})$ are topologically isomorphic, where τ_{Tych} denotes the product topology on $\prod\widehat{G_s}$.

On the other hand, the anihilator $(ker\ h)^{\perp}$ is a subgroup of $(\prod \widehat{G_s}, \tau_{Tych})$ and it is weakly-closed. It follows that $(G, \tau_f)^{\wedge}$ is topologically isomorphic to a weakly closed subgroup of $(\prod \widehat{G_s}, \tau_{Tych})$.

If we consider the weak topology $\omega(\prod_{s\in S} \widehat{G_s}, \bigoplus_{s\in S} G_s) =: \tau_\omega$ on the product $\prod_{s\in S} \widehat{G_s}$ we have that $\widehat{G}$ endowed with the induced topology $\tau_\omega|_{\widehat{G}}$ is topologically isomorphic to a closed subgroup of $(\prod_{s\in S} \widehat{G_s}, \tau_\omega)$.

Applying [19, Corollary 2.2] for $(\widehat{G}, \tau_\omega|\widehat{G})$ and $(\prod_{s\in S} \widehat{G_s}, \tau_\omega)$ we obtain that $l((\widehat{G}, \tau_\omega|\widehat{G})) \leqslant w(\prod_{s\in S} \mathcal{B}_s) \leqslant m$.

Indeed, we have already seen that the map

$$\Psi : \prod_{s\in S} \mathcal{B}_s \to \mathcal{P}((\prod_{s\in S} \widehat{G_s}, \tau_{Tych}))$$

is compact-valued and upper semi-continuous and satisfies

$$\prod_{s\in S} \widehat{G_s} = \bigcup \{\Psi((U_s)_s) : (U_s)_s \in \prod_{s\in S} \mathcal{B}_s\}.$$

Since the topology τ_ω is coarser then the product topology τ_{Tych}, when considering the product $\prod_{s\in S} \widehat{G_s}$ endowed with the topology τ_ω, the map Ψ remains compact-valued and upper semi-continuous.

Indeed , every τ_{Tych} -compact subset of $\prod_{s\in S} \widehat{G_s}$ is τ_ω -compact. As for the upper semi-continuity we have that for each element $(U_s)_s \in \prod_{s\in S} \mathcal{B}_s$ and each τ_{Tych} -open set G of $\prod_{s\in S} \widehat{G_s}$ containing $\Psi((U_s)_s)$, there is an open neighborhood U of $(U_s)_s$ in $\prod_{s\in S} \mathcal{B}_s$ such that $\Psi(U) \subset G$. Now, each τ_ω - open subset of $\prod_{s\in S} \widehat{G_s}$ being τ_{Tych} -open, the upper semi-continuity of Ψ when $\prod_{s\in S} \widehat{G_s}$ is equipped with the topology τ_ω is also fulfilled. As for the third requirement it does not depend of the topology in $\prod_{s\in S} \widehat{G_s}$.

Obviously, $\omega(\widehat{G}, G) \subseteq \tau_\omega$ implies that $l((\widehat{G}, \omega(\widehat{G}, G))) \leqslant m$. And finally, by (1) and (2) we conclude that $t(G, \omega(G, \widehat{G})) \leqslant m$.

(ii) By [22, Propositon 7], when $|S| \leqslant \aleph_0$ the final topology τ_f in G coincides with the box topology τ_r and we are going to use the latter, for simplicity. Recall that a neighborhood base for the box topology in G is

$$\mathcal{B}_r = \{U_r | (U_s)_{s\in S} \in \prod_{s\in S} \mathcal{B}_s\}$$

with

$$U_r = \bigcup_{\Delta \in \mathcal{P}_F(S)} \sum_{s\in\Delta} h_s(U_s)$$

and a basis for the uniformity $\mathbb{U}$ associated to this topology is

$$\mathcal{B}_{\mathbb{U}} = \{N_{U_r} : U_r \in \mathcal{B}_r\}$$

with

$$N_{U_r} = \{(x, y) \in G \times G \mid x - y \in U_r\}.$$

Consider each $\mathcal{B}_s$ with its natural order. Then

$$U_{r_1} \subseteq U_{r_2} \Rightarrow N_{U_{r_1}} \subseteq N_{U_{r_2}} \text{ for all } U_{r_1} \in \mathcal{B}_{r_1}, U_{r_2} \in \mathcal{B}_{r_2}.$$

We have that $|\mathcal{B}_s| \leqslant m$ $\forall s \in S$ and therefore $sup_{s\in S}|\mathcal{B}_s| \leqslant m$ and we can apply [19, Theorem 3.1.] to obtain that for every precompact subset P of (G, τ), the weight $w(P) \leqslant m$.

Corollary 3.5.4 *Let* $(G, \tau_f) = \varinjlim (G_n, \tau_n)$ *be an inductive limit of metrizable groups. Then,*

(i) (G, τ_f) *and* $(G, \omega(G, \widehat{G}))$ *have countable tightness ;*

(ii) the precompact subsets of (G, τ_f) *are metrizable.*

Proof. It is the former theorem for $m = \aleph_0$.

3.5.1 Class $\mathfrak{G}$ for groups

Definition 3.5.5 *A group* (G, τ) *is said to be in class* $\mathfrak{G}$ *if there is a family* $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\} \subseteq \widehat{G}$ *such that:*

(i) $\widehat{G} = \bigcup \{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$,

(ii) $A_\alpha \subseteq A_\beta$ *if* $\alpha \leq \beta$ *in* $\mathbb{N}^{\mathbb{N}}$,

(iii) in each A_α *sequences are* τ*-equicontinuous.*

Definition 3.5.6 *A family* $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ *in* $\widehat{G}$ *verifying the conditions (i)-(iii) shall be called a* $\mathfrak{G}$*-representation of* G *in* $\widehat{G}$.

As for the stability properties of the class $\mathfrak{G}$ we have the following.

Proposition 3.5.7 *Class* $\mathfrak{G}$ *is preserved in direct sums, topological products, quotients and subgroups, completions and inductive limits.*

Proof. See [20].

Definition 3.5.8 Let G be a topological group and let k be any cardinal number. An homomorphism $h : G \to \mathbb{T}$ is called k *-continuous* if it is continuous on each subset A of G of cardinality $\mid A \mid\leqslant k$. In particular, when $k = \aleph_0$ the homomorphism h will be called *countably continuous.*

In [55, Theorem 3.2] it is proved that for any duality $< G, G' >$, the group $(G, \omega(G, G'))$ is realcompact if and only if each homomorphism $h : (G', \omega(G', G)) \to \mathbb{T}$ that is countably continuous is continuous.

Theorem 3.5.9 *Let (G, τ) be a group in class $\mathfrak{G}$. The following are equivalent:*

(i) $(G, \omega(G, \widehat{G}))$ *has countable tightness;*

(ii) For every topological space (Y, τ_Y), any function from G into Y that is $\omega(G, \widehat{G})$-continuous restricted to $\omega(G, \widehat{G})$-closed and separable subsets of G is $\omega(G, \widehat{G})$-continuous on G;

(iii) Every character on G that is $\omega(G, \widehat{G})$-continuous restricted to $\omega(G, \widehat{G})$-closed and separable subgroups of G is $\omega(G, \widehat{G})$-continuous on G;

(iv) $(\widehat{G}, \omega(\widehat{G}, G))$ *is realcompact;*

(v) $(\widehat{G}, \omega(\widehat{G}, G))$ *is K-analytic;*

(vi) $(\widehat{G}, \omega(\widehat{G}, G))^n$ *is Lindelöf, for every $n = 1, 2, \ldots$;*

(vii) $(\widehat{G}, \omega(\widehat{G}, G))$ *is Lindelöf;*

(viii) Every group homomorphism on G *that is countably* $\omega(G, \widehat{G})$*-continuous is* $\omega(G, \widehat{G})$*-continuous.*

Proof. Let (G, τ) be a group in class $\mathcal{G}$.

(i)⇒(ii) Assume $(G, \omega(G, \widehat{G}))$ has countable tightness, that is $\forall A \subset G, \forall x \in \overline{A}^{\,\omega(G,\widehat{G})}$ there is $B \in A$ with $\mid B \mid \leqslant \aleph_0$ such that $x \in \overline{B}^{\,\omega(G,\widehat{G})}$.

Let (Y, τ_Y) be an arbitrary topological space and let $f : G \to Y$ be a function which is $\omega(G, \widehat{G})$-continuous on every $\omega(G, \widehat{G})$-closed and separable subset of G. In order to see that f is $\omega(G, \widehat{G})$-continuous, let A be an arbitrary subset of G and let $x \in \overline{A}^{\,\omega(G,\widehat{G})}$. Since $t((G, \omega(G, \widehat{G})) \leqslant \aleph_0$, there must be $B \subset A$ with $\mid B \mid \leqslant \aleph_0$, such that $x \in \overline{B}^{\,\omega(G,\widehat{G})}$.

As B is countable, we have that $\overline{B}^{\,\omega(G,\widehat{G})}$ is closed and separable and by the $\omega(G, \widehat{G})$-continuity of f on $\omega(G, \widehat{G})$-closed and separable subsets of G, we obtain that $f(\overline{B}^{\,\omega(G,\widehat{G})}) \subset \overline{f(B)}^{\,\tau_Y}$. Now follows that $f(x) \in \overline{f(B)}^{\,\tau_Y} \subset \overline{f(A)}^{\,\tau_Y}$.

(ii)⇒(iii) In the particular case when $Y = \mathbb{T}$ we have the following. Let $f : G \to \mathbb{T}$ be a character which is $\omega(G, \widehat{G})$- continuous restricted to $\omega(G, \widehat{G})$-closed and separable subgroups of G.

Let S be an arbitrary $\omega(G, \widehat{G})$-closed and separable subset of G and let D be a dense countable subset of S. Then $H := \overline{< D >}$ is an $\omega(G, \widehat{G})$-closed and separable subgroup of G and $S \subseteq H$. By hypothesis, f is $\omega(G, \widehat{G})$-

continuous on H. Since $S \subseteq H$, it follows that f is $\omega(G, \widehat{G})$-continuous on S. Now, S is arbitrary so we have that f is $\omega(G, \widehat{G})$-continuous on every $\omega(G, \widehat{G})$-closed and separable subset of G. By (ii) it follows that f is $\omega(G, \widehat{G})$-continuous on G.

(iii)⇒(iv) Assume that every character on G which is $\omega(G, \widehat{G})$-continuous restricted to $\omega(G, \widehat{G})$-closed and separable subgroups of G is $\omega(G, \widehat{G})$-continuous on G. Applying [55, Theorem 3.2.], in order to see that $(\widehat{G}, \omega(\widehat{G}, G))$ is realcompact we have to see that every countably continuous group homomorphism of $(G, \omega(G, \widehat{G}))$ into $\mathbb{T}$ is continuous. Let $\chi : G \to \mathbb{T}$ be an arbitrary countably continuous group homomorphism of $(G, \omega(G, \widehat{G}))$, that is χ is $\omega(G, \widehat{G})$-continuous on each countable subset of G.

Let S be an arbitrary $\omega(G, \widehat{G})$-closed separable subgroup of G. It follows that there is a countable dense subset D of S and by definition χ is $\omega(G, \widehat{G})$-continuous on D. Let $x \in S$ be an arbitrary element. Then, there must be a sequence $\{x_n\} \subset D$ such that $x_n \to x$. Now $\{x_n\} \bigcup \{x\}$ is countable and so $\chi(x_n) \to \chi(x)$. It follows that χ is not only $\omega(G, \widehat{G})$-continuous on S, but on every $\omega(G, \widehat{G})$-closed and separable subgroup of G. By (iii), this means that χ is $\omega(G, \widehat{G})$-continuous on G.

(iv)⇒(v) Assume $(\widehat{G}, \omega(\widehat{G}, G))$ is realcompact. In order to prove that $(\widehat{G}, \omega(\widehat{G}, G))$ is K-analytic one has to see that there is an upper semi-continuous, compact-valued map $\Psi : \mathbb{N}^{\mathbb{N}} \to \mathcal{P}((\widehat{G}, \omega(\widehat{G}, G)))$ such that $(\widehat{G}, \omega(\widehat{G}, G)) = \{\Psi(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$.

Since (G, τ) is in class $\mathcal{G}$, there is a family $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\} \subset \widehat{G}$ such that:

a) $\widehat{G} = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$,

b) in each A_α sequences are τ-equicontinuous,

c)$A_\alpha \subset A_\beta$ whenever $\alpha \leqslant \beta$ in $\mathbb{N}^\mathbb{N}$.

Let $\alpha \in \mathbb{N}^\mathbb{N}$ and let $\{x_n\}$ be an arbitrary sequence in A_α. By b), we have that $\{x_n\}$ is τ-equicontinuous, so $\{x_n\} \subset U^0$ for some neighborhood U of 0_G. By [7, Proposition 1.5], the polar U^0 is compact in $\widehat{G}_{pc}$. It follows that U^0 is $\omega(\widehat{G}, G)$-compact and $\{x_n\}$ is $\omega(\widehat{G}, G)$-relatively compact.

Hence A_α is $\omega(\widehat{G}, G)$-relatively countably compact for each $\alpha \in \mathbb{N}^\mathbb{N}$. And since $(\widehat{G}, \omega(\widehat{G}, G))$ is realcompact, this implies that for each α in $\mathbb{N}^\mathbb{N}$, A_α is $\omega(\widehat{G}, G)$-relatively compact.

Define $\varphi : \mathbb{N}^\mathbb{N} \to \mathcal{P}(\widehat{G})$ by $\varphi(\alpha) := A_\alpha,\ \alpha \in \mathbb{N}^\mathbb{N}$. By c) it follows that for each convergent sequence $(\alpha_n)_n$ in $\mathbb{N}^\mathbb{N}$, $\alpha_n \to \alpha$ implies that $\bigcup_{n\in\mathbb{N}} \varphi(\alpha_n) = \bigcup_{n\in\mathbb{N}} A_{\alpha_n} \subset A_\beta$, for some $\beta \in \mathbb{N}^\mathbb{N}$. We use the order $\leq^*$ and that the cardinality of an unbounded subset of $(\mathbb{N}, \leq^*)$ is uncountable.

Since A_β is $\omega(\widehat{G}, G)$-relatively compact, so is $\bigcup_{n\in\mathbb{N}} \varphi(\alpha_n)$ and one can apply [19, Theorem 2.3] to get an upper semi-continuous $\omega(\widehat{G}, G)$-compact-valued map $\Psi : \mathbb{N}^\mathbb{N} \to \mathcal{P}(\widehat{G})$ such that $\varphi(\alpha) \subseteq \Psi(\alpha)$ for every $\alpha \in \mathbb{N}^\mathbb{N}$. By a), we have

$$\widehat{G} = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\} = \bigcup\{\varphi(\alpha) : \alpha \in \mathbb{N}^\mathbb{N}\} = \bigcup\{\Psi(\alpha) : \alpha \in \mathbb{N}^\mathbb{N}\}.$$

It follows now that $(\widehat{G}, \omega(\widehat{G}, G))$ is K - analytic.

(v)⇒(vi) Assume $(\widehat{G}, \omega(\widehat{G}, G))$ is K-analytic, that is there exists an upper semi-continuous compact-valued map $\Psi : \mathbb{N}^{\mathbb{N}} \to \mathcal{P}(\widehat{G})$ such that $\widehat{G} = \{\Psi(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Since $\mathbb{N}^{\mathbb{N}}$ is metric and separable, we have $w(\mathbb{N}^{\mathbb{N}}) \leqslant \aleph_0$. Applying [19, Proposition 2.1], it follows that $(\widehat{G}, \omega(\widehat{G}, G))^n$ is Lindelőf for every $n \in \mathbb{N}^*$.

(vi)⇒(i) By Proposition 3.5.2, it follows that $C_p((\widehat{G}, \omega(\widehat{G}, G)), \mathbb{T})$ has countable tightness. Further, countable tightness is preserved in subspaces and thus $(G, \omega(G, \widehat{G})) \hookrightarrow C_p((\widehat{G}, \omega(\widehat{G}, G)), \mathbb{T})$ has countable tightness.

(vi)⇒(vii) Take $n = 1$.

(vii)⇒(iv) Applying [30, Theorem 3.11.12], Lindelőf groups are real-compact.

(viii)⇔(iv) It is [55, Theorem 3.2].

Definition 3.5.10 *Let G be a topological group. Let τ be a topology on G. Assume that every character on G which is τ-continuous on τ-closed and separable subgroups of G is τ-continuous on G. Then we say that G has property $\tau - \mathrm{css}$.*

Lemma 3.5.11 *Let G be a topological group with the property $\tau - css$. Then G has property $\omega(G, \widehat{G}) - css$.*

Proof. Assume that G has property $\tau - css$. Let $\chi : G \to \pi$ be $\omega(G, \widehat{G})$-continuous restricted to $\omega(G, \widehat{G})$-closed and separable subgroups of G and suppose χ isn't $\omega(G, \widehat{G})$-continuous on G, which is equivalent to saying that

χ is not τ-continuous on G. It follows, by hypothesis, that χ is not continuous restricted to τ-closed and separable subgroups of G. Then, there must be a τ-closed and separable subgroup S of G such that χ is not continuous on S. Now since S is τ-separable, there is a countable subset D of S such that $\overline{D}^{\,\tau} = S$. We have that $S = \overline{< D >}^{\,\tau} \subset \overline{< D >}^{\,\omega(G,\widehat{G})}$, being $\overline{< D >}^{\,\omega(G,\widehat{G})}$ an $\omega(G,\widehat{G})$-closed and separable subgroup of G. Now, since χ is not continuous on S, it means that it is not continuous on $\overline{< D >}^{\,\omega(G,\widehat{G})} \supset S$ either, so χ is not continuous restricted to $\omega(G,\widehat{G})$-closed and separable subgroups of G and we have reached a contradiction.

Proposition 3.5.12 *Let (G,τ) be a group in the class $\mathcal{G}$. If (G,τ) has countable tightness then $(G,\omega(G,\widehat{G}))$ has countable tightness.*

Proof. Reasoning just like we did in the proof of $(i) \Rightarrow (ii) \Rightarrow (iii)$ in Theorem 3.5.9, replacing $(G,\omega(G,\widehat{G}))$ by (G,τ), we obtain that G has property $\tau - css$. Applying the previous lemma, G has also property $\omega(G,\widehat{G}) - css$, which is condition (iii) in Theorem 3.5.9 and $(G,\omega(G,\widehat{G}))$ has countable tightness by the very same theorem.

Future Investigation

1). The main goal of this project is twofold. Firstly, we plan to develop a research project aimed to Topological Groups and their applications. On the other hand, we seek the application of duality techniques to a better understanding of the structure of several group characteristics. The first aim is, in continuation of previous work, developed along several years, around Duality Theory, Bohr Compactification, Function Spaces, etc. The second line of work is based on a series of recent publications in which duality methods are applied systematically to the study of certain invariant cardinals of topological spaces. This setting may not be translated in a simple way to non-abelian groups, mainly because the duality theory of non-abelian groups requires a greater mathematical background. Moreover, there are several possible descriptions of non-Abelian group duality, none of them universally accepted but all of them requiring a considerable technical sophistication.

Our objective here is to extend our methods to non-abelian groups, searching the versions of this notion that best suit for their application to our settings. This question appears as a logical continuation of our Chapter 3 where only abelian groups are considered. Due to our previous work and experience in the duality theory of abelian and non-abelian groups, we believe to be well placed to approach this project.

The notion of Pontryagin duality is an essential tool in the results exposed in this memory. The main question now is finding out to what extent the results discussed for abelian groups can be conveyed to the non commutative context. The idea is to consider the topologies defined by finitely dimensional unitary representations on a group G and apply them to describe the structure of the group.

We believe the more appropriate environment for continuing our investigation in the non-abelian case is Chu duality, which is defined as follows. Let $\mathcal{U}(n)$ denote the *unitary group of order* n, namely, the group of all complex-valued $n \times n$ matrices A for which $A^{-1} = \overline{A}^*$. (here, $\overline{A}^*$ denotes the conjugate transpose of A). Then $\mathcal{U}(n)$ is a compact Lie group, and can be realized as the group of isometries of $\mathbb{C}^n$. Now define the topological sum $\mathcal{U} = \sqcup_{n<\omega}\mathcal{U}(n)$.

A *unitary representation* T of the (topological) group G is a (continuous) homomorphism into the group of all linear isometries of a complex Hilbert space $\mathcal{H}$.

When $\dim \mathcal{H} < \infty$, we say that T is a *finite dimensional representation*; in this case, T is a homomorphism into one of the groups $\mathcal{U}(n)$. The symbol $Rep_n(G)$ denotes the set of all representations of G into $\mathcal{U}(n)$:

$$Rep_n(G) = \{f : G \longrightarrow \mathcal{U}(n)|\ f \text{ is a continuous homomorphism}\}$$

The set $Rep_n(G)$ equipped with the compact open topology is a locally compact topological space. The topological sum $Rep(G) = \sqcup_{n<\omega} Rep_n(G)$ is called the *Chu dual* of G.

As unitary representations are linear operators, the following operations can be defined on $Rep(G)$

1. $(D \oplus D')(x) = D(x) \oplus D'(x)$, $D, D' \in Rep(G)$ and $x \in G$;
2. $(D \otimes D')(x) = D(x) \otimes D'(x)$, $D, D' \in Rep(G)$ and $x \in G$;
3. $(U^{-1}DU)(x) = U^{-1}D(x)U$, $D \in Rep_n(G)$, $U \in \mathcal{U}(n)$ and $x \in G$.

A *quasi-representation* of G is a mapping $Q : Rep(G) \longrightarrow \mathcal{U}$ satisfying:

1. $Q[Rep_n(G)] \subset \mathcal{U}(n)$;
2. $Q(D \oplus D') = Q(D) \oplus Q(D')$, $D, D' \in Rep(G)$;
3. $Q(D \otimes D') = Q(D) \otimes Q(D')$, $D, D' \in Rep(G)$;
4. $Q(U^{-1}DU) = U^{-1}Q(D)U$, $D \in Rep_n(G)$, $U \in \mathcal{U}(n)$.

The set of all *continuous* quasi-representations of G equipped with the compact-open topology is a topological group with pointwise multiplication as composition law, called the *Chu quasi-dual group* of G and denoted by $Rep(G)^\vee$. The evaluation mapping $\epsilon : G \longrightarrow Rep(G)^\vee$ is a group homomorphism which is a monomorphism if and only if G is MAP (Recall that a topological group G is said to be *maximally almost periodic*, or *MAP*, if the continuous finite-dimensional unitary representations of G separate points in G. See [?] or [58]). The topology that G receives from $Rep(G)^\vee$ is called

Chu topology. We say that the locally compact group G *satisfies Chu duality* when the evaluation mapping ϵ is an isomorphism of topological groups (Chu, 1966). In his paper, Chu showed that each LCA group and each compact group satisfies Chu duality.

2). Although the scope of this memory is mainly for metrizable groups, we shall dedicate time for applications to the non-metrizable case. Various problems are already encountered, even in the very simple cases. But, obviously, it will be a very interesting research point for the future. In this context, it would be nice to consider our new proposed boundedness structures within the context of balleans as defined by Protasov and analyze the implications of such an investigation.

3). We shall try to have a contribution to the study of the structure of distinguished subsets of $C(X, Y)$ in view of applications to the study of group actions of $G \times X \to X$ and $Hom(G, H)$ between two topological groups.

Among the most general lines of investigation we mention equicontinuous and fragmentably equicontinuous sets, continuing the work of mathematicians such as I.Namioka, J.P.Troallic, A.Bouziad, O.Cascales and Megrelishvili. An especially important role is played by Namioka's theorem and the results of Bourgain, Fremlin and Talagrand. Several results have already been achieved in this direction. However, they are not mentioned here for having, at the moment, a preliminary nature. Again, we leave this action for the next coming research.

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