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Relativistic lagrangian non-linear field theories supporting non-topological soliton solutions

Diego Rubiera García

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To my beloved parents and brother

"The aim of science is not to open the door to infinite wisdom,
but to set a limit to infinite error"

Bertolt Brecht

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Oviedo, 30 de Octubre de 2008

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Introduction and outline

This thesis describes the work I have done during my PhD, in collaboration with Prof. Joaquin Diaz Alonso, on the study of non-linear field theories supporting non-topological soliton solutions and their possible applications in several branches of modern theoretical physics. Before proceeding to the detailed exposition of the subject, let us sketch in this introductory section a basic overview of the context and nature of the physical problems tackled, as well as our own contributions and the perspectives for future developments. This introduction should be intended as a summary of the contents of the thesis, allowing the reader to reach a global perspective before (or without) reading the whole manuscript.

Scientific context of the work

In the context of field theory the interest in extended configurations describing fields associated to point-like particles (particle-like solutions) dates back to the introduction of non-linear field theories in the early twentieth century aimed to solve the problem of the divergent self-energy of the electron field in Classical Electrodynamics. The models of Mie [1], Born [2] and Born-Infeld (BI) [3] were attempts at this regard. The original aim of Born and Infeld was to give, at a purely classical level, an electromagnetic origin for the masses of the charged particles, through a modification of the Maxwell action. Their idea was to regard elementary particles as “lumps” of the field (“solitons” in the modern sense of the term) in such a way that the notion of mass could be expressed in terms of the energy of the field configuration (an electromagnetic mass). However, the discovery of electrically neutral particles showed that mass is not *necessarily* related to electromagnetic charge. With the advent and success of quantum field theories (QFT) this kind of models were forgotten for a long time. Almost half a century later there was a renewed interest on Born-Infeld theory, as well as other non-linear electrodynamics and their non-abelian generalizations, since it was realized that such non-linear theories arise in several physical contexts as the low-energy limits of string and D-Brane physics or as effective models in QFT. Today this kind of problems, related to non-linear field theories and their particle-like solutions, fall inside the large domain of Soliton Physics, whose methods and applications concern most branches of physical sciences [4].

There is not a universally accepted definition of the concept of soliton. While restrictive definitions require the stability in collision processes between two or more of this class of solutions, a more useful one amounts to require a set of minimal properties which seem to be widely accepted in most contexts: Solitons are *stable, finite-energy solutions of conservative non-linear differential equations.* However, the accepted meaning and content of the term *stable* is not universal. In a *strong* sense, it refers to the existence of soliton entities which can be identified (if present) in field configurations and are preserved by the dynamic evolution of the system. With this definition, the analysis of such configurations in terms of many solitons, interacting via radiative field exchanges, becomes possible. This kind of stability arises in some field theoretical models (most of them in one-space dimension) exhibiting topological conservation laws, related to a non-trivial structure of the vacuum [5]. In these cases the conserved topological charges identify the presence of the *topological solitons* and ensure their preservation. Examples of topological solitons in three-space dimensions are the monopole of 't Hooft and Polyakov [6, 7] or the topological solitons of the Skyrme model [8, 9]. In the *weak* sense stability is identified with linear stability, which demands the preservation of the soliton identity against a certain class of small perturbations for which the soliton configuration is a minimum of the energy functional. This restricted class of perturbations is defined through boundary conditions which amount, in general, to the preservation of (*non-topological*) charges associated with the soliton. The conservation laws of these charges may be implicitly contained within the structure of the field equations (as in electrodynamics) or be consequences of constraints imposed on the external sources, to which the field is coupled. In these thesis we shall be only concerned with this class of non-topological, finite-energy, weakly-stable soliton solutions which arise in classical, local relativistic field theories in three-space dimensions.

What in early times raised lots of interest in Soliton Physics from the theoretical point of view was the fact that solitons, despite being solutions of classical, non-linear wave equations, exhibit some features usually ascribed to particles. In fact, in the early sixties Skyrme proposed a phenomenological model for the description of the hadron structure, identified as a topological soliton (the *Skyrmion*) of the non-linear sigma model with a stabilizer quartic term added [8, 9]. In the eighties there was a revival in this subject when the works of t'Hooft [10, 11] and Witten [12] showed that the Skyrme model can be interpreted within the large number of colors (N_C) limit of Quantum Chromodynamics (QCD). It was shown that the non-linear sigma model is the leading term of an effective lagrangian accounting for the relevant (hadronic) structures in the low-energy limit of QCD, where perturbative expansions cannot be applied. In nuclear physics, the analysis of high-density hadronic matter and its chiral phase structure have been performed in terms of skyrmions in the framework of effective field theories implemented with the large N_C behaviour of QCD [13, 14, 15].

Other approaches to the phenomenological description of both the hadronic structure and the hadronic interactions have been performed in terms of *non-topological* solitons through the effective model for the low-energy regime of QCD given by Friedberg and Lee [16, 17, 18, 19] and related theories. In terms of topological solitons this analysis has been also carried out within the generalized chiral-invariant model of Deser-Duff-Isham (DDI) [20] whose lagrangian density is a rational power (3/2) of the non-linear sigma model lagrangian, chosen in order to circumvent Derrick's theorem [21]. This model and its extensions support topological solitons [22, 23]. In the same context let us also mention the toroidal solitons of Refs.[24, 25], which might describe glueball collective states (a particle-like field configuration constructed entirely from gauge fields) in the low-energy limit of QCD, as suggested in Refs.[26, 27, 28]. Glueball collective states can be shown to exist as soliton solutions of non-abelian BI gauge field models [29]. The introduction of these generalized gauge models supporting soliton solutions and their extensions to higher dimensions is suggested by string theory, since some of them arise in the low-energy physics of D-Branes [30]-[34]. Moreover, it has been shown that the BI extension of the basic lagrangian of the Skyrme model leads to stable solitons removing the need of any "ad hoc" stabilizer term [35] (see also [36] for other BI-like extensions of the Skyrme model).

In the last two decades there has been an increasingly amount of works on self-gravitating field configurations. The aforementioned presence of BI actions in the low-energy physics of D-branes, whose fundamental excitation is gravity, is one of the motivations for this renewed interest (see e.g [37]). But the search for self-gravitating field configurations, as solutions of the Einstein equations for gravity coupled to different kinds of fields, is an older topic [38]. In many works non-linear field theories (in particular Born-Infeld theory) coupled to gravitation have been considered (see e.g. [39]-[45]). Indeed, in a four dimensional flat space-time, Derrick's theorem [21] and other non-existence theorems [46, 47, 48, 49] limit drastically the class of lagrangian field theories supporting soliton solutions. However, the coupling to gravitation can remove these obstructions and thus allow for particle-like solutions in some cases. One example of this is the pure Yang-Mills theory, which does not support glueball solutions in Minkowski space but exhibits particle-like solutions in curved space [50]. Moreover, theories supporting soliton solutions in flat space, as the abelian and non-abelian BI models, have been extended to curved space, leading to black-hole-like solutions [51, 52, 53]. On the other hand, although the existence of singularities in General Relativity seems to be an inherent feature of most of the physically relevant solutions, it is possible to find models describing regular, electrically charged black hole solutions for some particular non-linear electrodynamics [54, 55, 56]. Unfortunately, the lagrangian densities of these theories suffer *"branching"* phenomena since they correspond to multivalued functions.

In the fast-evolving context of modern Cosmology there are some problems for which non-linear field theories have been invoked. It has been suggested that time-dependent but non-dispersive solitons (Q-Balls [57, 58]) may account for the behaviour of self-interacting dark matter [59]. There is also the suggestion that non-topological solitons might have been formed in a second-order phase transition in the early Universe, and contribute significantly to the present mass density [60]. As another example, the Born-Infeld generalization of $SU(2)$ non-abelian gauge field theory, coupled to tensor-scalar gravitation, has been used for the description of dark energy [61, 62, 63]. Also scalar field models with lagrangian densities which are generalized functions of the kinetic term have been used to drive inflationary evolution in the early Universe (k-inflation) [64]. Solitonic configurations of these k-essence fields have been used to reproduce some properties of dark matter as well [65, 66].

This thesis: aims, approaches and objectives

All these considerations underline the relevance of non-linear field theories and their associated soliton solutions, mainly for the cases of (one and many-components) scalar fields and generalized gauge-invariant field theories. By the term *generalized* we mean models for gauge fields of compact semi-simple Lie groups, with lagrangian densities defined as general functions $\varphi(X,Y)$ of the two *standard* first-order gauge invariants, namely, $X = tr\,(F_{\mu\nu}F^{\mu\nu})$ and $Y = tr\,(F_{\mu\nu}F^{*\mu\nu})$. Aside from the aforementioned Born-Infeld-like gauge models, defined by the very particular BI choice of the lagrangian density, there are not in the literature systematic studies on solitons for general scalar and gauge-invariant field theories. In this thesis we have performed the analysis of a large class of relativistic field theories (defined for scalar, electromagnetic and gauge fields) supporting elementary solutions [1] which are non-topological solitons.

The concept of *admissibility*, adopted for dealing with physically consistent models, will play a central role in the sequel. We shall define a lagrangian field theory as *admissible* if it satisfies the requirements of the positive-definite character of the energy, the vanishing of the vacuum energy and the regularity, uniqueness and definiteness in all space of their elementary solutions.

In principle, generalized gauge-invariant field theories are candidates to describe the dynamics of the gauge fields in gauge theories of fundamental interactions. If one accepts the fundamental character of string theory and the aforementioned results, referenced in [30]-[34], then the description of the gauge-field dynamics, through some generalized lagrangians regarded as effective field models of string theory, could be more "fundamental" that the usual Maxwell-like choice $\varphi(X,Y) \sim X$, a "minimal prescription" which should be understood as a low-energy (or weak-field) approximation limit. Nevertheless, from the field theory point of view, this minimal prescription is generally assumed to describe the fundamental dynamics of the gauge fields when gauge-invariant lagrangians are coupled to other (generally fermionic) sectors. In this case, when the

[1] Through this work we shall call "elementary solutions" the static spherically symmetric (**SSS**) solutions of scalar field equations and the electrostatic spherically symmetric (**ESS**) solutions of generalized gauge field equations.

high-energy degrees of freedom are integrated out in the path integral of the original action, generalized gauge-invariant models emerge as *effective lagrangians* [67], containing new non-linear self-coupling terms accounting, at a classical phenomenological level, for quantum effects and interactions with the removed heavy-mode sector. Historically, the first example of this kind of effective lagrangians was obtained by Heisenberg and Euler [68], in the context of Quantum Electrodynamics (QED). It accounts for the non-linear effects of the Dirac vacuum on low-energy wave propagation, calculated to lowest order in the fine structure constant. When higher order operators are included we are lead to a sequence of effective lagrangians which take the form of polynomials in the field invariants, arranged as an expansion in operators of increasing dimensions [69]. An interesting question arises here, related to the possibility that these effective lagrangians could exhibit soliton solutions, even though the bare lagrangian does not. In chapter 6 we shall give explicit examples for which the soliton elementary solutions of an effective model may be interpreted as finite-energy fields of point-charges screened by the vacuum effects, whereas the elementary field of the bare theory is energy-divergent.

For scalar fields, Derrick's theorem [21] imposes severe restrictions on the class of lagrangian models supporting time-independent soliton solutions in three space dimensions. The choice of the lagrangian densities as general functions of the kinetic term alone [70] (see Eq.(3.48) below) allows us to circumvent the constraints of Derrick's theorem. When these functions are properly chosen the resulting models exhibit static, spherically symmetric (SSS) solutions which are of finite-energy and stable. This choice seems rather arbitrary from the physical point of view. Nevertheless it is the natural restriction for scalar fields of the lagrangian densities of the generalized gauge-invariant models. Moreover, as we shall see in the sequel, many of the necessary results in the analysis of the soliton problem in generalized (abelian and non-abelian) gauge-invariant theories (characterization of the families of soliton-supporting lagrangians, explicit determination of such models, analysis of stability, etc) can be obtained from similar results more easily established for scalar models. Consequently the detailed study of these scalar models will take an important place in this work.

Document organization

Let us summarize the main contents of this thesis.

Chapters 1 and 2 are introductory and summarize some basic results concerning the physics of solitons in field theories (relativistic or not). In chapter 1 we discuss some main results on soliton theory, especially those connected with the non-existence theorems and the different ways to evade them. In chapter 2 we analyze generalized field theories, defined as relativistic field models whose lagrangian densities are general functions of the fields and their first-order derivatives through invariant kinetic terms. These functions can be chosen in order for the new theories to support soliton solutions. We discuss some important examples, their origin and their applications in several contexts

In chapter 3 we consider the scalar field models in detail. We analyze lagrangians depending on the standard kinetic term, i.e. $X = \partial_\mu\phi\partial^\mu\phi$ through a general function $f(X)$. After an initial discussion on the admissibility conditions to be imposed on these functions, we solve the field equations for SSS solutions of all the generic models. This is achieved by obtaining the generic expression of a first-integral, which allows the determination of the field strength once the form of the lagrangian function is specified. We analyze the expression of the integral of energy for these solutions and determine the conditions that must be satisfied (at $r = 0$ and as $r \to \infty$) in order for them to be of finite-energy. These conditions imply supplementary restrictions to be imposed on the lagrangian functions which allow for an exhaustive characterization of the admissible models supporting finite-energy SSS solutions. We also extend these results to the case of N-components scalar fields. We shall show that for a given form of the lagrangian density as a function of the rotationally-invariant kinetic term ($X = \sum_{i=1}^{N}\partial_\mu\phi_i\partial^\mu\phi_i$) the N components of the SSS solutions have the same form, as functions of r, as the SSS solution of the one-component model associated to the same form of the lagrangian density function. This chapter is based on results contained in Refs.[71, 73].

In chapter 4, we consider the abelian and non-abelian generalized gauge-invariant theories. After defining the dynamic problem and the admissibility conditions, we analyze the field equations for ESS fields. We prove that the ESS solutions of these equations, in the abelian and non-abelian cases, can be built from SSS solutions of associated one and many-components scalar field problems, respectively. As a consequence there is a correspondence between scalar models and families of generalized gauge models in such a way that the SSS solutions and the corresponding ESS solutions have the same functional form. Moreover, if the energy of a SSS solution is finite, so is the energy of the ESS solutions of the corresponding generalized gauge family. The results of chapter 3 characterizing the admissible scalar models with finite-energy SSS soliton solutions, characterize also the admissible generalized gauge models with finite-energy ESS solutions, but the latter ones are not always stable. Stability requires now supplementary conditions to be satisfied by the lagrangian densities, which will be determined in chapter 5. The main content of this chapter has been published in Refs.[72, 73] but we also make use of some results of [74].

Chapter 5 is devoted to the analysis of the stability of the elementary solutions for scalar and gauge field models. This analysis leads to necessary and sufficient conditions for the linear static and dynamic stability of the soliton solutions, going beyond the necessary conditions demanded by Derrick's theorem. From the variational study of the energy functional and the spectral analysis of the small perturbations around the elementary solutions we shall prove that the admissibility and the finite-energy conditions imposed upon the admissible (multi-) scalar models supporting SSS solutions are *necessary and sufficient* for the stability of these solutions. In the gauge field case a similar analysis determines supplementary conditions, aside from admissibility, which must be imposed on the lagrangian densities in order for their finite-energy ESS solutions to reach stability. The contents of this chapter are based on results of reference [73].

In chapter 6 we present some examples of field theories belonging to the different classes analyzed in previous chapters, which exhibit soliton solutions and could be physically significant. We shall determine each model for the simpler case of a scalar field theory (this question was extensively studied in reference [73]), before proceeding further to the generalizations to the gauge field case, in such a way that the additional stability constrains be fulfilled by the extended models. Among the models introduced and analyzed in this chapter there are lagrangians of polynomial form in the field invariants, which include those obtained in different effective approaches to QED. Also lagrangians generalizing the abelian and non-abelian Born-Infeld models in two different ways. Finally, lagrangians describing short-ranged interactions without any symmetry breaking mechanism. We also extend here some results outlined in reference [75].

We conclude in chapter 7 by drawing some perspectives and future developments. Among these, let us mention

1. The possibility of elaborating phenomenological models for the internal structure of the hadrons, based on the results of the analysis of dynamic stability in chapter 5. In this picture, quarks and gluons appear as quasi-particles resulting from the quantization of the oscillation modes of a soliton solution of a generalized model, coupled to a fermionic sector and preserving the appropriate symmetries. The confinement should be a consequence of this quasi-particle nature which allows the existence of quarks and gluons only inside the hadron.

2. The use of short-ranged elementary solutions of effective gauge-invariant lagrangian models for the description of weak interactions, without appealing to the Higgs mechanism.

3. The result obtained in chapter 6 on the existence of finite-energy ESS solutions for effective lagrangians of QED (of polynomial form in the field invariants) suggests an interpretation of the renormalization procedure of the self-energy of a point-like charge, in terms of the screening effects of the Dirac vacuum on the field of this charge. Indeed, the non-linear terms in the effective lagrangian account for the vacuum corrections to the bare Maxwell lagrangian and lead to a renormalized charge-field with finite-energy. Unfortunately, the approximation in which these effective lagrangians are obtained fails for the strong values of the field near the center of the charge and the validity of this interpretation requires a similar analysis in terms of effective lagrangians including the vacuum effects in presence of strong fields.

4. The extension of the methods and results of this work in presence of gravitational fields. This problem is outlined in Appendix A, where we show that the generalized field equations in presence of gravitation have first-integrals of the same (or similar) form than in the flat-space case, to which our methods can be applied. This issue is currently in progress.

Our analysis deals with fields in three-space dimensions, but most of our results can be straightforwardly generalized to other spatial dimensions.

Chapter 1

Solitons in relativistic field theories

In this chapter we shall give a short review of history, definitions and properties, as well as some of the main results on solitons in relativistic field theories. Since the early developments in the 60s, this research field has undergone highly significant advances. Many solitons have been studied in great detail both analytically and numerically and several links among them have been discovered. A broad analysis on the subject of solitons in field theory lies beyond the scope of this thesis. Instead we will concentrate on presenting some key ideas that are going to be used throughout subsequent theoretical developments, performed in chapters 3 through 6. More details about the world of solitons can be found in the textbooks of Lee [76], Rajaraman [77] and Manton [78].

1.1 Historical remarks

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put it in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its curse along the channel apparently without change of form or diminution of speed..."

The above cite was written by the Scottish engineer John Scott Russell and it is part of a report given in the *British Association for the Advancement of Science* (published in 1844) where he describes his observations concerning an odd phenomenon. Around 1834 Russell was travelling on horseback near the bank of an Edinburgh canal when he noticed the existence of a wave running down the canal. Intrigued, he followed it for several miles, observing no changes in shape and velocity of the wave at all. The phenomenon was called at first by Russell as "wave of translation" and later as "solitary wave" (*"a solitary wave travels without changing its shape, size or speed..."*). This historical episode is usually recognized as the first mention of the *"singular and beautiful phenomenon"*, in Russell's own words of what is known today as *soliton*.

Russell suggested that this phenomenon represented a general class of solutions, completely different from the common plane waves in viscous media which tend to dissipate, fading away in time. Unfortunately, he did not succeed in convincing his colleagues and for a long time the subject of the solitary wave was in dispute. Finally, in 1895 Korteweg and de Vries found an equation for the motion of water waves, giving in this way an analytic explanation of Russell's solitary waves as solutions of this hydrodynamical Korteweg-de Vries equation (KdV) [79].

However, the subject lay fallow for almost half a century and it was not until the 60s that some authors started to perform intensive investigations on this and similar phenomena. One of the questions to be answered was if such non-dispersive phenomena could exist in physical contexts other than hydrodynamics. Around 1965 Zabusky and Kruskal, aside from providing a mechanism which explained the way non-linearity balances the dispersive character of the KdV equation, showed that this equation can be analytically solved despite its complexity. Among the solutions of the KdV equation there exist solitary waves maintaining their shapes, heights and speeds during their propagation, giving in this way an explanation for the water "lumps" of Russell. Moreover, they showed how two waves combine in such a way that cannot be merely explained in terms of linear superpositions: here non-linearity is essential.

The impact of these works was deep, giving rise to a vast and powerful variety of mathematical techniques (inverse scattering method, Lax pairs, classical spectral transforms, etc. [80]) for the analysis of solitons in $(1+1)$ dimensions. In the last decades solitons have played an important role in many physical and mathematical areas; their applications cover diverse fields such as engineering, biology, optics, condensed matter physics, cosmology, elementary particle theory, high energy physics including string theory and so on [4]. On the other hand the mathematical methods for the analysis of these soliton-supporting equations is a very active research field, with fruitful connections between different mathematical and physical branches.

1.2 Particle-like solutions

The concept of "particle-like" solutions arose as an attempt to solve some inherent problems associated with the theories regarding elementary particles as mathematical points, which are sources of fields. In this approach, the search for new theories is connected with the hope of finding exact, regular and "localized" (the meaning of the term will be discussed below) finite-energy solutions of classical non-linear field equations, which allow to describe the physics of these elementary particles. Non-linearity is then a mechanism not only for removing divergences associated with classical solutions of linear equations but also for allowing the description of interactions between fields and particles. If this goal were achieved it would not be necessary to postulate the existence of force-laws, which should emerge from the non-linearities of the field equations, thus providing (at least at the classical level) a unified picture of the internal structure of elementary particles as well as of their dynamics and interactions.

It is possible for some non-linear field equations, where both dispersive and non-linear terms are present, that their effects balance each other in such a way that some non-dissipative special solutions hold themselves due to their own auto-interaction. In classical field theory there exist many non-linear models supporting this kind of solutions, including vortices [81], monopoles [6, 7, 82] and instantons [83]. Several names have been coined for these non-dissipative solutions, including localized solutions, solitary waves, solitons, particle-like solutions and "lumps". Depending on the features of the solutions that one is concerned with, some names are more suitable than others and this have led researchers in the field to make slightly different definitions. Then, the next task to be faced is the choice of a precise nomenclature.

1.3 Solitons

1.3.1 Definition

It is fair to say that *there is not a universally accepted definition of what a soliton is.* In the following definitions the term *localized* will refer to those solutions of the field equations with an energy density $\rho(\vec{x}, t)$ localized in space, i.e. non-vanishing in some region of space and falling to zero at infinity, following the definitions of [77]. First of all, it is necessary to clarify the difference between the concepts of *solitary wave* and *soliton*:

A solitary wave is a modernized version of Russell's wave of translation, defined as a localized non-singular classical solution of any non-linear field equation, which energy density propagates without any distortion in time.

On the other hand, a soliton[1] is a solitary wave which has a finite energy and is stable under collisions with similar configurations. The latter condition implies that in a scattering process between two or more solitary waves, initially well-separated (for asymptotically negative times $t \to -\infty$), they recover their initial shapes, at least asymptotically ($t \to \infty$). It is therefore clear that solitons represent a small subset of the wider class of solitary waves.

Obviously, the above stability requirement is too stringent at least, concerning its

[1] Apparently the name was coined by Zabusky and Kruskal in the sixties, to describe a solitary wave with particle-like features.

usefulness in field theory. Indeed, most of the known solitons appear not to fulfil this criterion[2]. Only for very special models, namely, those where the presence of infinitely many conservation laws guarantees that solitons re-emerge with their original shapes restored after a scattering process (integrable models in (1+1) dimensions), this strong stability criterion has been shown to hold. For the purposes of this thesis, we shall "relax" the above definition of solitons instead of maintaining this restrictive requirement, and adopt the following definition:

A **soliton** is a stable [3], finite-energy classical solution of conservative non-linear differential field equations in a d dimensional Minkowski space. This is the soliton definition in the spirit of Coleman's "lump" of energy [84].

Although in this work we shall only consider *static* solitons, let us mention the existence of localized time-dependent solutions of non-topological nature, the so-called "Q-Balls" [4]. The term *Q-Ball* was introduced by Coleman [57] (although the concept was developed earlier, see Friedberg et al. [85]) and includes a large class of non-topological solitons in (3+1) dimensions which can be associated with several symmetries [86, 87, 88].

1.3.2 Classification

Following Lee's scheme [76], in relativistic field theories all soliton solutions can be classified into two general categories, according to the mechanism by which their stability is ensured.

- *Topological solitons:* Solitons whose stability is supported by the existence of non-trivial boundary conditions giving rise to a non-trivial homotopy group preserved by the field equations. Soliton solutions of the field equations are then classified according to homotopy classes and characterized by a topological invariant ("charge" of the soliton). This topological charge prevents the destruction of the isolated soliton (as well as transitions between solitons belonging to different homotopy sectors).

  Some known examples of this class of solitons include vortices [81], the 't Hooft-Polyakov monopole [6, 7, 82], instantons [83] and skyrmions (see [89] and references therein). The term *"topological defects"* is also frequently used for referring to a class of topological solitons arising as a result of symmetry-breaking phase transitions, which find applications, in particular, in cosmological contexts (see e.g. [90, 91]).

[2] In general it is hard to tell, because of the inherent mathematical difficulties associated with the non-linear field equations, whether a particular solution satisfies the above requirement or not.

[3] In the weak sense, i.e. linearly stable against small charge-preserving perturbations. See chapter 5 for a more clarifying discussion.

[4] See [58] for a review.

- *Non-topological solitons:* These solitons are stable because of the existence of conserved charges of non-topological origin, carried by the soliton. For the class of solitons analyzed in this thesis the mathematical structure of the field equations, or the constraints imposed on the external sources (to which the field is coupled) allows the definition of a charge carried by the soliton. This issue will be extensively discussed in chapter 5.

1.3.3 Solitons in (1+1) dimensions

To illustrate some important results on soliton theory we start from the simpler and better studied case of solitons in one spatial dimension. Here one takes advantage of the large variety of available powerful mathematical tools. These methods are strongly connected with the integrability property of the models, which implies the existence of infinitely many conserved charges. As a result, the models which are exactly integrable both at classical and quantum levels can be analytically treated and soliton dynamics explicitly studied. A key result in the analysis of (1+1) dimensional solitons is that in the so-called theories of Bogomolny type the second order partial differential field equations can be reduced to first order equations, thus greatly simplifying their resolution [92].

Let then be a $(1+1)$ dimensional one-component scalar field theory of the form

$$L(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi), \tag{1.1}$$

where ϕ is a real scalar field and $U(\phi)$ a potential term. The static solutions are determined by $\frac{d^2\phi}{dx^2} = \frac{\partial U(\phi)}{\partial\phi}$ and the finite-energy and stability features of the associated solutions are related to the asymptotic form of the field configurations. To support soliton solutions this asymptotic behaviour must be different in positive and negative spatial directions and identified to different zeros of the potential.

To solve this system one makes a formal mathematical analogy identifying the lagrangian (1.1) with the Newton equation of motion for a particle of unit mass: $L(x(t)) = \frac{1}{2}\dot{x}^2 - U(x)$. For this analogy to work properly the lagrangian (1.1) must be formally interpreted as the trajectory of a point particle moving in an inverted potential $-U(\phi)$. Thus the field equation can be solved through a quadrature

$$x - x_0 = \int_{\phi(x_0)}^{\phi(x)} d\widetilde{\phi}\frac{1}{\sqrt{2U(\widetilde{\phi})}}. \tag{1.2}$$

It is possible to obtain a lower bound for the energy of any field configuration following an argument developed by Bogomolny [92], which leads to the inequality

$$\begin{aligned}\epsilon &= \int T_{00}dx = \int_{-\infty}^{+\infty}\left(\frac{\phi'^2}{2}+U(\phi)\right)dx \geq \left|\int_{\phi(x=-\infty)}^{\phi(x=+\infty)}\sqrt{2U(\widetilde{\phi})}d\widetilde{\phi}\right| = \\ &= |W(\phi=+\infty)-W(\phi=-\infty)|. \end{aligned} \tag{1.3}$$

To attain the equality in the Bogomolny bound (1.3) the field must be static and satisfy the first order equations $\phi'(x) = \pm\frac{\partial W}{\partial \phi}$ with $W(\phi) = \int \sqrt{2U(\widetilde{\phi})}d\widetilde{\phi}$, and thus also solves the second order equations by construction.

For these systems a topological current can be defined as $j_\mu = \frac{1}{2}\varepsilon_{\mu\nu}\partial^\nu\phi$ so a topological charge $q = \int_{-\infty}^{+\infty} j_0 = \frac{1}{2}\left(\phi(x=+\infty)-\phi(x=-\infty)\right)$ prevents non-trivial field configurations with different asymptotic behaviours from decaying to the vacuum state. A familiar system having the form (1.1) is the so-called *Klein-Gordon kink*, which corresponds to the Mexican hat potential $U(\phi) = \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2$, $v^2 = \frac{m^2}{\lambda}$, of the ϕ^4 model [93]. The solution of topological number unit represents a soliton interpolating between the vacua $\phi_- = -v = -\frac{m}{\sqrt{\lambda}}(x \to -\infty)$ and $\phi_+ = +v = +\frac{m}{\sqrt{\lambda}}(x \to +\infty)$ (there also exists another solution interpolating between ϕ_+ and ϕ_- called antikink) with a shape (see figure 1.1)

$$\phi = \pm v \tanh\left(\frac{m}{\sqrt{2}}\left(x - x_0\right)\right), \tag{1.4}$$

and energy $\epsilon = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda} = \frac{2\sqrt{2}}{3}\frac{m}{g}$, where $g = \frac{\lambda}{m^2}$, so the mass of the kink increases when the coupling constant is reduced. However, these solutions do not retain their shapes after collision processes [77].

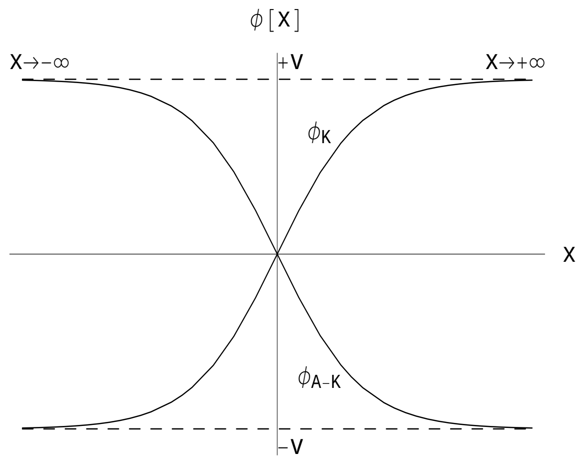

Figure 1.1: One dimensional kink and antikink solutions centered at $x_0 = 0$.

A specially interesting example is the *Sine-Gordon* soliton, defined by a potential $U(\phi) = \frac{\alpha}{\beta^2}(1-\cos(\beta\phi))$. The zeros are given by $\phi = \frac{2\pi n}{\beta}$ so the system has an infinite number of discrete vacua and, consequently, infinitely many non-trivial soliton solutions. This model gives two remarkable results. First, their static solutions survive to collision processes with similar configurations, thus providing examples of true solitons. Second, the works of Coleman [94] and Madelstam [95] showed that the quantized Sine-Gordon model is related to the massive Thirring model (see also [96])

$$L = \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - \frac{1}{2}g\left(\overline{\psi}\gamma^\mu\psi\right)^2, \tag{1.5}$$

when the coupling constants g and β fulfill the relation $\frac{\beta^2}{4\pi} = \frac{1}{1+g/\pi}$. This implies that solitons arising in the *bosonic* sine-Gordon theory correspond to *fermion* states of the Thirring model. This phenomenon also exists in solitons arising in higher dimensional models as, for example, the three-dimensional Skyrme model [8, 9, 97].

1.4 Solitons in higher dimensions

As already mentioned, there exist a large number of mathematical results in the literature concerning solitons in (1+1) dimensions. However, in higher dimensions these methods are not so useful and one is left, in general, with non-integrable systems whose analysis is a complicated task even at the classical level. Moreover, much of the non-linear differential equations admit soliton solutions which do not have a closed, analytic form and, in most cases, they do not admit soliton solutions at all. Indeed, the existence of soliton solutions in higher dimensions is severely restricted by the existence of several no-go theorems which rule out many solutions both in scalar and gauge field theories.

1.4.1 Derrick's theorem

A simple but powerful result concerning the existence of static solitons in a large class of field theories is the so-called Derrick's theorem [21]. It gives essentially a negative result. In many field theories the variation of the energy functional against certain spatial rescalings is never zero for any static (non-vacuum) field configuration. However, a static field configuration which is a minimum of the energy should be stationary against all admissible variations of the field, including these spatial rescalings. Then, in these theories there do not exist static, finite-energy solutions of the field equations, excepting the vacuum states. In particular, the theorem establishes the non-existence of non-trivial scalar solitons in theories with canonical lagrangians (1.1) in dimensions greater than one. For simplicity let us analyze the one-component scalar field theory with energy functional given (for static configurations) by

$$\epsilon[\phi] = \int d^D x \left(\frac{1}{2} \vec{\nabla}_i \phi \cdot \vec{\nabla}_i \phi + U(\phi(x)) \right) \equiv \epsilon_S[\phi] + \epsilon_U[\phi], \tag{1.6}$$

(D stands for the number of spatial dimensions). Here not only $\epsilon[\phi]$ but also $\epsilon_S[\phi]$ and $\epsilon_U[\phi]$ are non-negative. Following Derrick's argument, consider a static solution $\phi(x)$ and let

$$\phi_\lambda(x) \equiv \phi(\lambda x), \lambda > 0, \tag{1.7}$$

be a uniparametric family of field configurations. The transformation $x \to \lambda x$ through this family will be called hereafter a spatial rescaling (sometimes called *dilation)*. Derrick's argument amounts to a study of what happens to the energy functional when this rescaling is performed. If for an arbitrary (but static) solution with energy $\epsilon[\phi(x)]$ the rescaled energy functional $\epsilon[\phi_\lambda(x)]$ has not stationary points, then the theory does not support static, finite-energy solutions of the field equations, excepting the vacuum. The energy functional (1.6) transforms against these spatial rescalings (1.7) as

$$\epsilon[\phi_\lambda] = \epsilon_S[\phi_\lambda] + \epsilon_U[\phi_\lambda] = \lambda^{2-D} \epsilon_S[\phi] + \lambda^{-D} \epsilon_U[\phi]. \tag{1.8}$$

Then $\epsilon[\phi_\lambda]$ is a simple function of λ, with the coefficients ϵ_S and ϵ_U relying on the initial choice of the field configuration $\phi(x)$ while the character of $\epsilon[\phi_\lambda]$ depends critically on the dimensionality of space. Now one must set the variation of $\epsilon[\phi_\lambda]$ with respect to a rescaling (1.7) to zero, i.e. $\frac{d\epsilon[\phi_\lambda]}{d\lambda} |_{\lambda=1} = 0$. From (1.8) it is trivial to see that this implies

$$(2 - D)\epsilon_S[\phi] = D\epsilon_U[\phi], \tag{1.9}$$

and since $\epsilon_S[\phi]$ and $\epsilon_U[\phi]$ are non-negative, the relation (1.9) cannot be fulfilled for $D > 1$ unless $\epsilon_S[\phi] = \epsilon_U[\phi] = 0$, the trivial vacuum solution. Then, the theorem holds for scalar field theories of the form (1.1) in $D > 1$ dimensions: the only time-independent solutions of finite-energy in these cases are the vacuum states. On the other hand, soliton solutions in $D = 1$ are not ruled out by this theorem and this explains the existence of static solitons for lagrangians of the form (1.1) in this case. The above argument can be also straightforwardly extended to multicomponent scalar fields.

1.4.2 Circumventing Derrick's theorem

To circumvent Derrick's theorem many proposals have been considered in the literature. Let us briefly discuss some of them:

- To include other fields in order to "balance" Derrick's scaling. For example, one can include gauge fields $A^\mu(x)$. In this case, the energy is the sum of terms of the form

$$\epsilon_S[\phi] = \int d^D x T_n\left(D_i\phi\right) \;\; ; \;\; \epsilon_U[\phi] = \int d^D x U(\phi) \;\; ; \;\; \epsilon_G[A^\mu] = \int d^D x F_{ij}F_{ij}, \quad (1.10)$$

where T_n stands for some power of the derivatives of the scalar kinetic term ($n = 2$ for the standard canonical kinetic term). The appropriate transformation against spatial rescalings of a 1-form is $A^\mu_\lambda \equiv \lambda A^\mu(\lambda x)$ whereas the covariant derivative transforms as $D^\mu_\lambda \phi(x) = (d^\mu \phi_\lambda(x) + A^\mu \phi_\lambda(x)) \equiv \lambda D^\mu \phi(\lambda x)$ and the field strength as $F^{\mu\nu}_\lambda(x) \equiv \lambda^2 F^{\mu\nu}(\lambda x)$. Then, the energy scaling will be

$$\epsilon[\phi_\lambda] = \epsilon_S[\phi_\lambda] + \epsilon_U[\phi_\lambda] + \epsilon_G[A^\mu_\lambda] = \lambda^{n-D}\epsilon_S[\phi] + \lambda^{-D}\epsilon_U[\phi] + \lambda^{4-D}\epsilon_G[A^\mu]. \quad (1.11)$$

Again it is assumed that each term contributes positively to the total energy. The extremum condition of the rescaled energy functional in $\lambda = 1$ gives the constraint

$$(n - D)\epsilon_S - D\epsilon_U + (4 - D)\epsilon_G = 0, \quad (1.12)$$

to be satisfied. For having a static soliton one needs a zero exponent against rescalings in all terms contributing to the energy functional [5] or a balance between positive and negative exponents, otherwise there are not finite-energy static solutions, expecting the vacuum. As previously seen, such a balance depends critically on space dimensionality. Many soliton solutions show this structure. A short list includes:

- (1+1) dimensions: Scalar ϵ_S + potential ϵ_U terms → kinks.
- (2+1) dimensions:
  - Scalar ϵ_S + potential ϵ_U + gauge ϵ_G terms → → Nielsen-Olesen vortex.
  - Only a scalar ϵ_S term → O(3) sigma model lump.
- (3+1) dimensions: Scalar ϵ_S + potential ϵ_U + gauge ϵ_G terms → → 't Hooft-Polyakov monopole.
- (4+1) dimensions: A pure gauge term ϵ_G in an Euclidean space → Instanton.

• To consider time-dependent but non-dispersive solutions → Q-Balls.

• To include higher powers of the derivatives of the scalar field → Skyrme and Faddeev-Niemi models. Here, aside from standard kinetic terms a quartic term is included.

• To consider more complicated models defined by lagrangians which are general functions of the standard kinetic terms (both for scalar and gauge field theories), in such a way that the pertinent symmetries be maintained.

[5] For instance, if the energy density consists only on scalar fields, then one is forced to fix $n = D$.

Some comments are in order. First, Derrick's theorem shows a striking feature: the requirement of stability against rescalings represents a sort of "virial" theorem in the sense that the extremum condition usually provides identities relating the different contributions to the energy functional. If we take, for example, the simplest case, the scalar field theory with lagrangian density (1.1) and energy (1.6), the extremum condition gives, as we have seen, a relation between both contributions, namely, Eq.(1.9). For $D = 1$ one trivially gets $\epsilon_S = \epsilon_U$. This implies that scalar and potential terms contribute equally to the total energy of the static solitons. Another example showing this feature is the three-dimensional Skyrme model.

Second, Derrick's argument only involves a particular case within the set of all possible perturbations applicable upon the soliton, which could be unstable against other perturbations or singularities of the field. That is to say, Derrick's theorem only provides *necessary* conditions for the existence and stability of static solitons, but not *sufficient* ones. Moreover, for the case of gauge field theories aside from Derrick's theorem other non-existence theorems have been established which, consequently, must also be circumvented in order to obtain soliton solutions. We shall tackle these problems in the next chapter.

Chapter 2

Generalized field theories

In this chapter we focus our attention upon a class of relativistic field theories defined as generalizations of well-known models, such as the non-linear sigma model, Maxwell electrodynamics or non-abelian gauge field theories. The latter cannot possess stable, static, finite-energy soliton solutions in a four dimensional flat space-time (without the inclusion of additional terms) due to the aforementioned Derrick's theorem and other non-existence theorems [46, 47, 48, 49], while the former, when some conditions are fulfilled, *do* possess. We shall call this class of theories, defined for various kinds of fields, *generalized field theories*.

2.1 Effective field theories

Effective field theories (EFTs) have a long history (see [67, 98, 99] for reviews). The first developments of the subject date back to the work of Euler and Heisenberg in 1936, where quantum corrections to the Maxwell electrodynamics were computed [68, 100].

EFTs have become a common tool for the analysis of physical problems which involve two (or more) widely separated energy scales. To understand how EFTs work, it is useful to adopt the and approach based on the integration of the high-energy degrees of freedom. Let be a field theory which contains "heavy" particles Φ of mass M and "light" particles ϕ of mass m, such that $m << M$. Let us consider now a physical process which typical energy scale Λ is much below the heavy mass scale so real heavy particles cannot be generated. Following this scheme the lagrangian of the EFT can be written as

$$L(\Phi, \phi) = L(\phi) + L_{\Phi}(\Phi, \phi), \tag{2.1}$$

where the contribution of the light fields is isolated in the part $L(\phi)$, while the heavy fields and the interaction between light and heavy fields is contained in $L_{\Phi}(\phi, \Phi)$. Next, we must consider the generating functional of the full theory

$$Z[\phi] = \int D\Phi D\phi e^{iS(\Phi,\phi)} = \int D\phi e^{iS(\phi)} \int D\Phi e^{iS_\Phi(\Phi,\phi)}. \tag{2.2}$$

Then all heavy modes are integrated out (since they live on the scale $M >> \Lambda$) in (2.2) and one is left with a theory for small momentum (or large distance) modes of the light fields, defined by an effective lagrangian obtained in a perturbative way and having infinitely many terms. The physics on the scale M is now encoded into the parameters of this low-energy lagrangian.

A more useful approach (for practical calculations) to construct an EFT amounts to identify the dynamical degrees of freedom (the light fields), satisfying the pertinent symmetries of the underlying theory at the scale of reference. Then the most general lagrangian (including the light degrees of freedom) consistent with these symmetries must be written down. This low-energy theory can now describe accurately the predictions of the underlying theory at this scale simply by including sufficiently many operators in the effective lagrangian. This lagrangian can be written in terms of an expansion

$$L_{eff}(\phi) = L_0(\phi) + \sum_n c_n O_n, \tag{2.3}$$

where we have isolated the renormalizable part (L_0) of L_{eff}, and O_n are operators of dimension n built with the light fields, while the information of the heavy degrees of freedom is contained in the couplings c_n. The numerical value of these coefficients c_n could be obtained from the "fundamental" theory if it were available and manageable or, alternatively, could be fitted from the experiments. Thus, this quasi-classical theory, which contains new non-linear self-couplings, accounts, at a classical phenomenological level, for quantum effects and interactions with the removed heavy-mode sector.

2.1.1 Skyrme model

For a long time the possibility that solitons may describe particles has attracted lots of attention. An early attempt at this regard was the Skyrme model [8, 9, 97], developed in the sixties as a model for the analysis of hadron physics, which only involves meson fields (pions) and where baryons arise as topological solitons. A convenient parameterization of these degrees of freedom amounts to define an unitary matrix $U = \exp(i\tau^a\pi^a/f_\pi)$, π^a being the pion fields, τ^a the Pauli matrices and $f_\pi \simeq 93$MeV the pion decay constant. Written in this language the lagrangian reads

$$L_{SK} = -\frac{f_\pi^2}{4} Tr\left(L_\mu L^\mu\right) + \frac{1}{32e^2} Tr\left[L_\mu, L_\nu\right]^2 = \frac{f_\pi^2}{4} tr\left(\partial_\mu U \partial^\mu U^+\right) + \frac{1}{32e^2} tr\left(\left[U^+\partial_\mu U, U^+\partial_\nu U\right]^2\right), \tag{2.4}$$

where $L_\mu = U^+ \partial_\mu U$ and e is a constant. The first term in Eq.(2.4) corresponds to the non-linear sigma model, which does not support static, finite-energy solutions because of Derrick's theorem [1]. As previously mentioned this can be cured with the addition of higher-order terms in derivatives. Indeed this is what Skyrme did through the inclusion of a quartic term in the derivatives (the second term in the lagrangian (2.4)) in order to balance this scaling, and thus allow for topological finite-energy soliton solutions.

The interest in the Skyrme model lies in the fact that gives a phenomenological description ($\sim 30\%$ of error for single-baryon properties [97]) of nucleons as solitons of the pion field. This implies that the Skyrme model describes *fermionic* states although their fundamental fields are *bosons*. This is a (3+1) dimensional version of the phenomenon already seen in the (1+1) dimensional Sine-Gordon system. Indeed, the identification of baryons with solitons arising in an effective boson field theory is related to the limit of large number of colors N_C of QCD. In the 70's t'Hooft [10] showed that if QCD is generalized from $SU(3)_C$ to $SU(N_C)$ with $N_C \to \infty$ then $1/N_C$ can be taken as an expansion parameter for the low-energy regimen. Planar diagrams are dominant in this limit and QCD is reduced to a field theory of weakly coupled (i.e. with coupling constant $g \sim 1/\sqrt{N_C}$) mesons and glueballs. Indeed, Witten [12] was able to show that baryons emerge in this theory as topological soliton solutions.

However, the quartic term in (2.4) is rather arbitrary since it has been introduced "ad hoc" only for evading the hypothesis of Derrick's theorem, and there are no real physical basis to fix this particular choice. Clearly any term of degree fourth or higher in derivatives would do equally well at this respect, which shows that the need to fix a specific choice seems rather unnatural.

2.1.2 Scalar field models with generalized dynamics

To get around this ambiguity as well as to circumvent Derrick's theorem, Deser, Duff and Isham [20] proposed a modification of the non-linear sigma model, by defining a new lagrangian as a function of the previous one, whilst maintaining the original symmetries. This lagrangian reads (avoiding constants by simplicity)

$$L_{DDI} = (-L_{NLSM})^{3/2} = (TrL_\mu L^\mu)^{3/2} . \tag{2.5}$$

The motivation of this choice is given by the fact that the power (3/2) is just the necessary one required to balance the rescaling of the standard kinetic term in (3+1) dimensions.

Classical field theories with non-canonical kinetic terms are not unfamiliar in modern theoretical physics, where they find several applications, in particular, within the context of soliton physics. Following the fractional power idea of the example (2.5) other models

[1] Two spatial derivatives in the lagrangian do not rescale appropriately in three-space dimensions.

have been studied. Two particular lagrangians, which are parameterized by a three-component unit vector field $\widehat{n} : R^3 \rightarrow S^2$, $|\widehat{n}| = 1$, have attracted special attention

$$L_2 \equiv (\partial_\mu \widehat{n})^2 \;\; ; \;\; L_4 \equiv [\widehat{n}\,(\partial_\mu \widehat{n} \times \partial_\nu \widehat{n})]^2 \,. \tag{2.6}$$

Making use of these lagrangians several theories have been proposed. For example a model which finds applications both in field theory and condensed matter physics is the well-known Skyrme-Faddeev-Niemi model [101, 26] with lagrangian $L_{SFN} = L_2 - \lambda L_4$, λ being a constant. On the other hand, it is also possible to select lagrangians defined only in terms of one of the previous lagrangians, either L_2 or L_4. First, Nicole introduced the lagrangian [22] (see also [23])

$$L_{NI} = (L_2)^{3/2} \,, \tag{2.7}$$

showing that toroidal solitons are allowed to exist within this model. Another lagrangian is the Aratyn-Ferreira-Zimmerman (AFZ) model [102, 103] defined as

$$L_{AFZ} = -\,(L_4)^{3/4} \,, \tag{2.8}$$

which contains infinitely many toroidal soliton solutions. Note that these theories L_{DDI}, L_{NI}, L_{AFZ} are scale-invariant under Derrick's transformations (1.7), which is a trivial consequence of the way these lagrangians evade Derrick's theorem - the rational powers included are just the necessary ones in order to balance Derrick's scaling in lagrangians composed of a single term -. It has been argued that such scale-invariant models could be unstable due to zero modes associated with changes in the scale of the soliton, which might give rise to soliton collapse in a finite time [78, 104].

For both circumventing Derrick's theorem and breaking the above scale invariance in three-space dimensions it is necessary to go beyond the aforementioned single-term models. During the last two decades lots of attention has been paid to scalar field models with generalized dynamics. The terms *"k-fields"* or *"k-essence"*, where k stands for "kinetic", have been coined for referring to these extensions. They are scalar field theories defined by generalized actions of the canonical kinetic term, which can have unusual physical properties. K-essence models are specially interesting in Cosmology. Here the idea was proposed at first in the context of inflation (k-inflation [64, 105]) but it was soon noted that it can serve for modelling both dark energy and dark matter [65, 66].

Concerning solitons, the search for scalar models supporting soliton solutions, by considering lagrangians with generalized dynamics, circumventing in this way the constraints of Derrick's theorem, was partially treated in [70]. K-fields have also been used in investigations of defect structures of topological nature. Topologically non-trivial configurations with symmetry-breaking potential terms, but with the standard canonical kinetic term being replaced by a k-field have been investigated in [106, 107]. K-vortices [108] and compactons [109] (i.e. solitons with compact support, a class of topological defects approaching the vacuum at a finite distance) have been considered as well.

2.2 Non-linear electrodynamics

Let us now focus our attention upon the non-linear generalizations of Maxwell's electrodynamics. Classical electrodynamics with charged massive point particles has two restrictive features. First, the self-energy of the field of a point particle is infinite. Second, a Lorentz force-law supplementing Maxwell's equations must be postulated to describe interactions between point particles and the electromagnetic field. Since the discovery of the electron, physicists have tried to develop non-linear extensions of Maxwell's electrodynamics, namely, non-linear electrodynamics (NED), representing finite-energy charged particles.

2.2.1 Born-Infeld theory

In 1912 G. Mie [1] introduced a "maximal field" E_{max} (by analogy with the notion of maximum velocity c in Relativity) in the framework of a NED, in such a way that the electric field of a point-like charge does not diverge at the origin, remaining bounded everywhere

$$L = \sqrt{1 - \frac{\vec{E}^2}{\vec{E}^2_{max}}}. \tag{2.9}$$

Although within this model the electron is represented by a solution of finite-energy it is obvious that such an extension cannot be regarded as physically acceptable from the point of view of relativistic field theories since it is not covariant under Lorentz transformations.

Looking for physically acceptable models Born [2] extended Mie's idea by considering a Lorentz and gauge-invariant NED (now including magnetic fields in the lagrangian definition) supporting finite-energy electrostatic field solutions. Finally, Born and Infeld [3] considered a more general NED including two quadratic, Lorentz and gauge invariants of the field, giving rise to the well-known *Born-Infeld electrodynamics*. In the classical standard picture infinities come from the fact that elementary particles are regarded as mathematical points and then physical quantities, as the energy of the associated field, diverge there. In BI theory the electric field reduces to Coulomb's field at large distances, but differ near the origin, where it attains its maximal value. The model defines a characteristic length $r_o = (q/E_{max})^{1/2}$, which sets the scale of distance from the center where the non-linear effects become relevant. Born and Infeld were motivated by the hope of finding sourceless, regular, finite-energy solutions to represent elementary particles. However they did not completely succeed in this regard since point-like electrostatic solutions of BI theory still carry singularities in the *electric displacement* field at the origin and, as a consequence, there is a Dirac-delta function [2], which can be interpreted as a source (charge) at this point. Nevertheless, the model allows an alternative interpretation of the charge of these solutions as a continuous distribution in space. On the other hand, the main features ascribed to solitons, such as the finite-energy character and the stability (in the usual weak sense, see chapter 5) are fulfilled by the electrostatic solutions of this theory.

BI model is a very special theory among the NEDs. It is instructive to recall some of its features, before studying the general NED case, which shall be discussed in greater depth in chapter 4. The action reads [3]

$$\begin{aligned} L_{BI} &= \beta^2 \left(\sqrt{-\det(g_{\mu\nu})} - \sqrt{\det(g_{\mu\nu} + \beta^{-1} F_{\mu\nu})} \right) = \\ &= \beta^2 \left(1 - \sqrt{1 + \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{16\beta^4} \left(F_{\mu\nu} F^{*\mu\nu} \right)^2} \right) = \\ &= \beta^2 \left(1 - \sqrt{1 + \frac{1}{\beta^2} \left(\vec{B}^2 - \vec{E}^2 \right) - \frac{1}{\beta^4} \left(\vec{E} \cdot \vec{B} \right)^2} \right). \end{aligned} \tag{2.10}$$

Among the properties of this theory let us mention

1. Maxwell action is recovered in the $\beta \to \infty$ limit:

$$L_{BI} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + O(1/\beta^2) = \frac{1}{2} (\vec{E}^2 - \vec{B}^2) + O(1/\beta^2) \tag{2.11}$$

2. For purely electric configurations ($\vec{B} = 0$)

$$L_{BI} \mid_{\vec{B}=0} = \beta^2 \left(1 - \sqrt{1 - \beta^{-2} \vec{E}^2} \right), \tag{2.12}$$

there exist an upper bound for the electric field given by $\vec{E}(r) = \beta$, which is an analogue of Mie's maximal field. Indeed, a key point in BI electrodynamics (as opposed to Maxwell electrodynamics) is the difference between the electric field $\vec{E}$ and the "electric displacement" vector $\vec{D}$. Here singularities are associated to $\vec{D}$ rather than $\vec{E}$ since the former fulfils the relation

[2] Due to this feature the term *BIon* is frequently used in some contexts for referring to these configurations [110].

[3] Note that the first equality is obtained for the case of Minkowski space $g_{\mu\nu} = \eta_{\mu\nu}$.

$$\vec{\nabla} \cdot \vec{D} = 4\pi q \delta^3(\vec{r}), \tag{2.13}$$

and, moreover, electric and displacement fields are related by the equality

$$\vec{D} = \frac{\delta L_{BI}}{\delta \vec{E}}\bigg|_{\vec{B}=0} = \frac{\vec{E}}{\sqrt{1 - \frac{1}{\beta^2}\vec{E}^2}} \Rightarrow \vec{E} = \frac{\vec{D}}{\sqrt{1 + \frac{1}{\beta^2}\vec{D}^2}}; \tag{2.14}$$

so although the vector $\vec{D}$ diverges at the origin (since $\vec{D} = \frac{q}{r^2}\hat{r}$, which is identical to the Maxwell solution) the electric field $\vec{E} = \frac{q}{\sqrt{r^4 + \frac{q^2}{\beta^2}}}\hat{r}$ is everywhere finite and so is the energy, as can be easily checked:

$$\epsilon = \int d^3\vec{r}(\vec{E} \cdot \vec{D} - L) \sim \int_0^\infty dr \left(\sqrt{q^2 + r^4} - r^2\right) < \infty \tag{2.15}$$

3. Finally, BI theory has a set of good physical properties concerning wave propagation, such as the absence of birefringence and shock waves, belonging to the class of theories called "completely exceptional" [111, 112] (see section 4.3). It also possess some duality symmetries such as electric-magnetic duality (see appendix B).

Let us also recall that in recent decades there has been a renewed interest on BI theory and its non-abelian extensions as well as other types of NEDs since some of them arise in the low-energy limits of string theory and in the physics of D-Branes [30, 31, 110, 113, 114].

2.2.2 Euler-Heisenberg effective lagrangian

We shall now discuss some aspects concerning another physically meaningful example of NED with applications in theoretical physics, which is found in a different context.

In classical electrodynamics, described by Maxwell linear field equations, photons do not "fell" the presence of other photons. This is not so in QED, where the interaction between photons and electron-positron virtual pairs in vacuum gives rise to measurable effects. For energies much below the electron mass ($\epsilon << m_e$), these effects can be described through effective lagrangians [67], where the dominant term corresponds to the Maxwell lagrangian $L_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, describing free photons. In the scheme (2.3) the subsequent terms O_n correspond to the coupling of photons to virtual electronic loops in vacuum. These operators O_n must respect the symmetries of the underlying theory (namely, Lorentz, gauge and parity invariance). Thus, they must be built with terms invariant under the above symmetries and constructed from the field strength tensor, its dual, and (in general) their derivatives.

In determining the first terms of this expansion, Euler and Heisenberg [68] (see also [100]) wrote down the four-vertex contribution (interaction of four photons via coupling to an electronic loop in vacuum) for "slowly varying" fields, which reads

$$L_{EH} = L_0 + \frac{\alpha^2}{90m_e^4}\left[(F_{\mu\nu}F^{\mu\nu})^2 + \frac{7}{4}\left(F_{\mu\nu}F^{*\mu\nu}\right)^2\right]. \tag{2.16}$$

Higher-order corrections in the field invariants, corresponding to the coupling between more photons, can be added to (2.16) (see e.g. [69]). We shall see in section 6.1.1 how the Euler-Heisenberg lagrangian and the sequence of higher-order correction lagrangians behave from the point of view of particle-like solutions.

2.3 Generalized non-abelian gauge field theories

We shall consider now the extension of non-linear electrodynamics to the case of non-abelian gauge fields. In this case the study of generalized actions for the soliton problem raises some mathematical subtleties which we are going to discuss briefly in this section. Obviously the whole subject of non-abelian solitons is too vast to be fully discussed here and we refer the reader to the review of Ref.[52] for more details.

2.3.1 Non-existence theorems

We have seen that the existence of soliton solutions in field theories is severely restricted by Derrick's theorem, widely discussed in previous sections. But concerning gauge field theories supplementary non-existence theorems have been established, forbidding the existence of finite-energy, stable field configurations made up only of gauge fields. In order to understand how generalized gauge field models circumvent these restrictions let us sketch the arguments underlying these theorems.

First of all, Deser [46] observed that *"There do not exist non-trivial static solutions of finite-energy in Yang-Mills theory, excepting in the (4+1) dimensional case"*.

Subsequently Coleman [47] established that *"There are not static, finite-energy solutions for a traceless gauge field theory, excepting the vacuum"*. Let us consider, by simplicity, the (3+1) dimensional case. It is assumed that the field approaches zero at infinity fast enough for the total energy to be finite. The requirement of traceless energy-momentum tensor $T^\mu_\mu = 0$, which is a consequence of the scale invariance of the theory [4], implies the existence of a "dilational" charge defined as

[4]As a consequence of this scale invariance an associated current can be defined as $j_\nu = x^\mu T_{\mu\nu}$, from which the charge (2.17) defined below is derived. The tracelessness of the energy-momentum tensor follows immediately since $\partial_\nu j^\nu = T^\nu_\nu$=0.

$$Q_D = \int d^3x^\mu T_{\mu 0} = \int d^3x(x_0 T_{00} - \sum_i x_i T_{i0}) = t\epsilon - \int d^3x \sum_i x_i T_{i0}, \qquad (2.17)$$

which is conserved under the above assumptions, as can be easily seen

$$\begin{aligned} \partial_0 Q_D &= \int d^3x(T_{00} - \sum_i x_i(\partial_0 T_{i0})) = \int d^3x(T_{00} + \sum_i x_i \partial_j T_{ij}) = \\ &= \int d^3x(T_{00} + \sum_{i,j}(\partial_j(x_i T_{ij}) - \sum_i T_{ii}) = \int d^3x \sum_{i,j} \partial_j(x_i T_{ij}) = 0, \end{aligned} \qquad (2.18)$$

Now, since for static solutions T_{i0} does not depend on the time coordinate then it follows immediately that $\partial_0 Q_D = \int d^3x T_{00} = \epsilon = 0$, that is, the vacuum solution.

Other theorems rule out the existence of static, finite-energy configurations which hold themselves for a long time before radiating away their energy to infinity [48], or spatially localized solutions but with a periodic time dependence [49].

The above non-existence theorems can be connected with physical grounds as follows: as mentioned, the scale invariance of the theory implies the tracelessness of the energy-momentum tensor $T_\mu^\mu = T_{00} - \sum_{i=1}^3 T_{ii} = 0$, which can be physically understood in the sense that the total stresses in an extended object must balance. Since the energy density is positive definite by construction ($T_{00} > 0$) this implies that the sum of the principal pressures $\sum_{i=1}^3 T_{ii}$ must be everywhere positive, i.e. Yang-Mills "matter" is purely repulsive and a force balance within the localized, static Yang-Mills configurations cannot be reached.

Although the standard Yang-Mills theory does not admit classical particle-like solutions, in Yang-Mills systems coupled to gravity both attractive and repulsive forces are present so the existence of such solutions is not excluded. *But* this does not guarantee their existence. Thus it was a surprise when such solutions were numerically found by Bartnik and McKinnon [50]. Here scale invariance is broken down by gravity. However, it was next shown that Bartnik-McKinnon particles are unstable [115]. Anyhow, as this was actually the first example of self-gravitating particle-like configurations a lot of interest towards this kind of systems has been triggered.

Despite this discovery it remained the question if it is possible for generalized gauge field theories (modifications of the standard Yang-Mills lagrangian) to admit these solutions in *flat* space. For obtaining particle-like solutions the scale invariance should be broken. A natural way to achieve this goal would be to consider gauge field theories spontaneously broken down by scalar fields, which gives rise to configurations such as monopoles and sphalerons. In this case the contribution of the Higgs field provides the pure attraction needed for making a force balance possible. However, another option is given by field theories inspired by BI generalizations of classical electrodynamics. Indeed a suitable extension of these theories to the non-abelian fields will remove the obstruction of the non-existence theorems since they break the scale invariance of the classical Yang-Mills theory [5]. However, it was soon realized that the extension of generalized *abelian* actions to the *non-abelian* case raises a new problem.

2.3.2 Trace prescriptions

This problem is concerned with the trace definition, due to the existence of an ambiguity about the way the trace of the non-abelian action is specified over the gauge group generators. Formally several possibilities can be envisaged. Taking BI theory as an example, an *ordinary* trace is usually considered, which leads to a simple and closed form for the action [116]. On the other hand, another trace definition is favored by superstring theory, the *symmetrized* trace [33], but the explicit lagrangian with such a trace is known only as a perturbative series [117]. As a consequence of this ambiguity, in the general case, if $F_{\mu\nu} = \sum_a F^a_{\mu\nu} T^a$ (T^a being the gauge group generators) is the field strength of the non-abelian group, there is not a direct relation between the "determinant" form of the BI action (the first line of (2.10)) and the "square root" form given by the second line of (2.10). In the ordinary prescription of the trace, this square-root form formulated for gauge fields (i.e. simply by replacing $F_{\mu\nu} \to F^a_{\mu\nu}$ and summing over a)

$$L_{NBI} = \beta^2 Tr\left(1 - \sqrt{1 + \frac{1}{2\beta^2}\sum_a F^a_{\mu\nu}F^{\mu\nu a} - \frac{1}{16\beta^2}\left(\sum_a F^a_{\mu\nu}F^{*\mu\nu a}\right)^2}\right) \tag{2.19}$$

can be taken as a starting point for the non-abelian generalization of BI abelian actions.

Despite these difficulties, non-abelian Born-Infeld (NBI) actions have been widely studied over the last decade both in flat and in curved space using different trace prescriptions. In particular, classical glueballs for a $SU(2)$ NBI in flat space, within the monopole ansantz $A^a_0 = 0$, $A^a_i = \epsilon_{aij}\frac{x^j}{r^2}(1-\omega(r))$, were found by Gal'tsov and Kerner [29]. These solutions show a remarkable similarity with Bartnik-McKinnon particles, despite the fact that the former arise in flat space while the latter live in curved space. For models already containing particle-like solutions in flat space it is interesting to study how gravity affects them [118, 119]. The inclusion of Higgs fields allows for other field configurations, such as monopoles and dyons, to exist both in flat and in curved space [117, 120, 121, 122]. Finally, other trace prescriptions make use of results coming from non-commutative geometry [123].

[5]Note that removing scale invariance, which is a necessary condition for the existence of classical particle-like solutions, by no means is a sufficient one. Each particular case must be analyzed in looking for such solutions.

Chapter 3

Scalar solitons

In this chapter we define models for scalar fields where the lagrangian densities are chosen as *general* functions of the D'Alembert kinetic lagrangian, circumventing in this way the constraints of Derrick's theorem [21]. Many general results concerning the properties of these families of field theories and their associated solitons solutions can be obtained without specifying the particular form of the lagrangian function. A partial study on this subject was considered in [70]. Here we shall extended those results and perform a broad analysis of this class of theories.

We shall first introduce the lagrangian of the models. We write down the field equations and compute the energy-momentum tensor. Then we shall establish the necessary conditions to be imposed on our models ir order to obtain physically admissible theories supporting unique, static, weakly-stable, finite-energy, spherically symmetric non-topological soliton solutions. Finally we shall consider the extension of these methods to generalized multiscalar models.

3.1 The models

We begin with lagrangian densities for scalar field *potentials* $\phi(x)$, defined in a four dimensional Minkowski space-time as

$$L = f(\partial_\mu \phi \cdot \partial^\mu \phi), \tag{3.1}$$

where $f(X)$ is a given *continuous and derivable* function in the domain of definition (Ω) which is assumed to be open, connected and including the vacuum ($X \equiv \partial_\mu \phi \cdot \partial^\mu \phi = 0$). For future purposes, we also require $f(X)$ to be monotonically increasing (more precisely, $df/dX > 0, \forall X \neq 0 \in \Omega$ and $df/dX \geq 0$ for $X = 0$) and df/dX to be continuous for $X < 0 (X \in \Omega)$. In absence of a coupling to external sources, the associated field equations take the form of a local conservation law

$$\partial_\mu J^\mu = 0, \tag{3.2}$$

where the conserved current J^μ is

$$J^\mu = \overset{\bullet}{f}(X)\partial^\mu \phi, \tag{3.3}$$

with $\overset{\bullet}{f}(X) = df/dX$. In these models, the D'Alembert linear wave equation corresponds to [1] $f(X) = X/2$. For the SSS solutions $\phi(r)$, Eq.(3.2) has the first-integral

$$r^2 \phi' \overset{\bullet}{f}(-\phi'^2) = \Lambda, \tag{3.4}$$

where $\phi' = d\phi/dr$, and Λ is the integration constant. This is an algebraic equation which allows, in principle, the determination of the field strength as a function of r and Λ. The positivity of $\overset{\bullet}{f}(X)$ requires both Λ and $\phi'(r)$ to be simultaneously either positive or negative. We can then consider the positive-sign case only without loss of generality. Strictly speaking Eq.(3.4) determines the field $\phi'(r)$ only for $r > 0$. If we replace the solutions of Eq.(3.4) in Eq.(3.2) we do not obtain zero, but a Dirac δ distribution of weight $4\pi\Lambda$. We can then identify this parameter with the central scalar charge source of the (at rest) SSS solution, in analogy with the point-like charges in the Maxwell theory. Alternatively, in some cases as, for instance, the non-linear electromagnetism of Born-Infeld, this charge may be interpreted as a continuous charge density distribution in space. For non-linear electromagnetic models the continuous interpretation of the charge is rather natural, owing to the conservation of the electric charge as a consequence of the field equations, but this is not so for the scalar models [124]. Nevertheless, following the analogy with the electromagnetic case, we can define for the models (3.1) the total scalar charge associated with a given *static* asymptotically vanishing field solution $\phi(\vec{r})$ as

$$\frac{1}{4\pi}\int d_3\vec{r}\left(\vec{\nabla}\cdot\left[\overset{\bullet}{f}(X)\vec{\nabla}(\phi)\right]\right) = \frac{1}{4\pi}\int_{S_\infty} \overset{\bullet}{f}(X)\vec{\nabla}(\phi)\cdot d\vec{\sigma}, \tag{3.5}$$

which, owing to the field equation (3.2), vanishes for everywhere regular solutions. For the SSS solutions of (3.4) we have $\overset{\bullet}{f}(X)\vec{\nabla}(\phi) = \Lambda\frac{\vec{r}}{r^3}$ and the total charge equals Λ. We can then define the spatial charge-density distribution as

$$\sigma(r) = (1/4\pi)\overset{\bullet}{f}(0)\vec{\nabla}^2\phi, \tag{3.6}$$

[1] As in this linear case, plane waves of the form $\phi = \phi(k_\mu \cdot x^\mu)$, with $k^2 = k_\mu k^\mu = 0$, are solutions of Eq.(3.2) but superpositions of such waves are not, in general.

which gives for the total scalar charge of the SSS solutions

$$\frac{1}{4\pi}\int d_3\vec{r}\overset{\bullet}{f}(0)\vec{\nabla}^2\phi = \lim_{r\to\infty}\overset{\bullet}{f}(0)r^2\phi^{'}(r) = \lim_{r\to\infty}\frac{\Lambda\overset{\bullet}{f}(0)}{\overset{\bullet}{f}(X)}. \tag{3.7}$$

Clearly, this interpretation is only possible if $\overset{\bullet}{f}(0)$ is finite, or equivalently, if the function $r^2\phi^{'}(r)$ goes to a constant as $r \to \infty$ (asymptotically coulombian fields). This function must also vanish as $r \to 0$. As we shall see at once, this latter condition is fulfilled for all the models with finite-energy SSS solutions, but the former defines a sub-class of those models (see case B-2 below in this chapter) to which the scalar version of the Born-Infeld model belongs.

Once the form of $f(X)$ is fixed, equation (3.4) gives the expression of the field $\phi^{'}(r) \equiv \phi^{'}(r,\Lambda)$ in implicit form and allows the determination of the potential $\phi(r)$ (up to an additive arbitrary constant) through a quadrature. We also note that Eq.(3.4) implies that $\phi^{'}(r)$, if single-branched [2], is necessarily a monotonic function of r. Moreover, the solution depends on r and Λ through the ratio $r/\sqrt{\Lambda}$. This is a straightforward consequence of the invariance of the solutions of the field equations (3.2) under space-time scale transformations. Indeed, if $\phi(\vec{r},t)$ is a solution of this equation, so is the modified function

$$\varphi(\vec{r},t,\lambda) = \lambda^{-1}\phi(\lambda\vec{r},\lambda t), \tag{3.8}$$

where λ is a positive constant. This is a symmetry of the solutions of the field equations without sources, but is not an invariance of the action [125] (note that this scale transformation is *not* the same as that given by Derrick's theorem, defined by Eq.(1.7)).

The canonical energy-momentum tensor associated to the lagrangian (3.1) is

$$T_{\mu\nu} = 2\overset{\bullet}{f}(X)\partial_\mu\phi\partial_\nu\phi - f(X)\eta_{\mu\nu}, \tag{3.9}$$

The trace of the energy-momentum tensor (3.9) reads

$$T^\mu_\mu = 2\left(X\overset{\bullet}{f}(X) - f(X)\right), \tag{3.10}$$

which shows that the only traceless model of this class of theories is, precisely, D'Alembert model $f(X) \sim X$.

---

[2] In some cases Eq.(3.4) can lead to discontinuities or several branches for the function $\phi^{'}(r)$. We shall regard such cases as "unphysical" and rule them out from this analysis, considering only models for which the fields of the SSS solutions are (for $r > 0$) continuous, single-branched functions defined everywhere. We shall establish at the end of this chapter that the corresponding admissibility condition for the lagrangian densities is the strict monotonicity of $f(X)$.

The energy density in the SSS case becomes

$$\rho = T^{00} = -f(-\phi^{'2}), \tag{3.11}$$

whereas the total energy is

$$\epsilon(\Lambda) = -4\pi \int_0^\infty r^2 f(-\phi^{'2}(r,\Lambda))dr = \Lambda^{3/2}\epsilon(\Lambda = 1), \tag{3.12}$$

where the last equality is a consequence of the aforementioned scale invariance [3]. Using the first-integral (3.4) and integrating by parts we can obtain the following useful expression for the energy in this SSS case

$$\epsilon(\Lambda) = \frac{8\pi\Lambda}{3}\left\{[\phi(r,\Lambda)]|_0^\infty - \left[r\phi^{'}(r,\Lambda) + \frac{r^3}{2\Lambda}f(-\phi^{'2}(r,\Lambda))\right]|_0^\infty\right\}. \tag{3.13}$$

As we shall see, if the energy of SSS solutions is finite the second bracket in the r.h.s. vanishes and this expression reduces to

$$\epsilon(\Lambda) = \frac{8\pi\Lambda}{3}\left[\phi(\infty,\Lambda) - \phi(0,\Lambda)\right], \tag{3.14}$$

which shows that the potential $\phi(r)$ must be a bounded function of r, defined up to an arbitrary constant. Note also that Eq.(3.14) has the form of the potential energy of a point-like scalar charge of value 2Λ placed at infinity in the soliton field.

There is another way round to obtain the expression (3.14). Indeed, by performing the usual rescaling of Derrick's theorem, namely Eq.(1.7), the energy rescales as

$$\begin{aligned}\epsilon_\lambda(\phi^{'}(r),\Lambda) = \epsilon(\phi^{'}(\lambda r),\Lambda) &= -4\pi \int_0^\infty r^2 f(-\phi^{'2}_{(r)}(\lambda r,\Lambda))dr = \\ &= -4\pi \int_0^\infty r^2 f(-\lambda^2\phi^{'2}_{(\lambda r)}(\lambda r,\Lambda))dr,\end{aligned} \tag{3.15}$$

and can be easily checked that the condition of extremum of the energy against these rescalings $\left(\frac{d\epsilon_\lambda(\phi^{'}(r),\Lambda)}{d\lambda}|_{\lambda=1} = 0\right)$ leads automatically to the relation (3.14) when the condition of finiteness of the energy is assumed. Although Derrick's theorem is only a *necessary* condition for stability, this signals the presence of a connection between stability and finite-energy condition of the SSS solutions considered here. This connection will be precisely established in chapter 5.

[3]This kind of relation is completely general; in fact, for a theory in D dimensions one would have a scaling of energies given by $\varepsilon(\Lambda) = \Lambda^{\frac{D}{D-1}}\varepsilon(\Lambda = 1)$.

Going beyond the results of reference [70], we shall determine general conditions to be imposed on the functions $f(X)$ in order to obtain physically consistent field theories such that the SSS solutions of the corresponding field equations be stable and the associated energy (3.12) be finite (non-topological solitons). We first summarize some criteria of physical consistency adopted for the purposes of the present analysis (defining what we shall call *"admissible"* field theories) and obtain the associated restrictions on the lagrangian densities. Next we will obtain the conditions for such admissible models to support SSS soliton solutions.

3.2 Conditions on the energy functional

If any of these models are to be used for descriptions of quantum physical systems, the possibility of their quantization becomes important. This implies supplementary conditions to be satisfied by the lagrangian densities, as necessary for any quantum extension. Thus, aside from the above mentioned continuity and derivability conditions, we must require the function $f(X)$ to be defined everywhere ($\Omega \equiv \Re$), in order to allow for the proper definition of the associated path integral. Here we shall call such models *class-1* field theories. These conditions exclude models such as the scalar version of the Born-Infeld Electrodynamics [3]. In such cases it is always possible to generalize the model, by continuing the lagrangian density function to the undefined regions through some prescription which must preserve the classical dynamical content of the initial model. Then, the quantum behaviour of the extended model would depend on this prescription. But this procedure will necessarily enlarge the space of solutions of the classical theory. As we shall show below, these extensions introduce new branches for the SSS solutions which become spurious at the classical level. We shall exclude such extended models from the present analysis since they are non-admissible according to our physical criteria.

Alternatively, as previously discussed, we can consider these scalar models (and their generalized versions proposed below) as effective classical lagrangians of more "fundamental" theories (see section 2.1), including integrated high-energy and quantum effects through new non-linear couplings. In these cases we can relax the everywhere definiteness conditions of the admissible lagrangians and require their regularity only within a restricted domain of definition ($\Omega \subset \Re$), which is assumed to be open, connected and including the vacuum ($0 \in \Omega$). We shall call these models *class-2* field theories. Using this criterion, models such as the Born-Infeld one become admissible field theories belonging to this class. Although these models are essentially classical, quantum corrections to their particle-like solutions can be obtained by quantizing the field of small fluctuations around these ground states. Such fluctuations obey Euler-Lagrange linear field equations of admissible lagrangian densities defined everywhere (see Ref. [76] and chapter 5, Eqs.(5.12) and (5.13)).

Obviously, a second condition to be imposed for admissibility in all cases concerns the positive definite character of the energy, which is required to hold in the entire domain of definition (Ω) of the lagrangian density. The expression for the energy density in terms of $f(X)$ is

$$\rho = T^{00} = 2 \overset{\bullet}{f}(X) \left(\frac{\partial \phi}{\partial t}\right)^2 - f(X) = 2X \overset{\bullet}{f}(X) - f(X) + 2 \overset{\bullet}{f}(X) \left(\vec{\nabla}\phi\right)^2 . \tag{3.16}$$

For the D'Alembert lagrangian we have $f(X) = X/2$, and the energy density reduces to

$$\rho = \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial t}\right)^2 + \left(\vec{\nabla}\phi\right)^2\right], \tag{3.17}$$

which is positive for any non-constant function $\phi(t, \vec{r})$ and vanishes in vacuum. In order to obtain a similar behaviour in the general case (3.16) (requiring also the energy density to vanish in vacuum) we are lead to the necessary conditions

$$\begin{aligned} f(0) &= 0 \\ \overset{\bullet}{f}(X) &\geq 0 \;\; \forall X. \end{aligned} \tag{3.18}$$

For minimal sufficient conditions on $f(X)$ let us analyze separately the cases $X < 0$ and $X > 0$. On the one hand for $X < 0$ the term $(\partial_t \phi)^2$ may vanish and the positivity of the energy density (3.16) requires

$$f(X) < 0 \;\; ; \;\; \forall X < 0. \tag{3.19}$$

On the other hand, for $X > 0$ the term $(\vec{\nabla}\phi)^2$ may vanish and the positivity of the energy requires

$$\rho(X) \geq 2X \overset{\bullet}{f}(X) - f(X) = X f(X) \frac{d}{dX}\left[\ln\left(\frac{f^2(X)}{X}\right)\right] \geq 0. \tag{3.20}$$

This equation, together with the conditions $f(0) = 0$ and $\overset{\bullet}{f}(X) \geq 0$ $(\forall X)$ lead to

$$\frac{d}{dX}\left[\ln\left(\frac{f^2(X)}{X}\right)\right] \geq 0, \tag{3.21}$$

or, equivalently, the function $\frac{f(X)}{\sqrt{X}}$ (and hence $f(X)$ itself) must be a positive monotonically increasing function of X for $X > 0$. Equation (3.21) with the boundary condition

$f(0) = 0$ fix the behaviour of $f(X)$ around $X = 0$ as

$$f(X \to 0^+) \sim X^{1+\alpha}, \tag{3.22}$$

with $\alpha > -1/2$, and the energy density behaves there as [4]

$$\rho(X \to 0^+) \geq (1 + 2\alpha)X^{1+\alpha}. \tag{3.23}$$

3.3 Conditions for finite-energy SSS solutions

The convergence of the integral of energy (3.12) for the SSS solutions is governed by the behaviour of the integrand near the limits $r \to \infty$ and $r \to 0$. This imposes supplementary conditions on the form of the function $f(X)$ around the values of $X(r) = -\phi'^2(r)$ in these limits. Let us assume a power law expression for the field around these regions [5] ($\phi'(r) \sim r^q$ as $r \to \infty$ or as $r \sim 0$). From the first-integral (3.4) we obtain the relation

$$df/dr = -\frac{2\Lambda\phi''(r)}{r^2} \sim -2\Lambda q r^{q-3}, \tag{3.24}$$

and in the limits of integration $f(r)$ behaves as

$$f(r) \sim \frac{2\Lambda q}{2-q} r^{q-2} + D, \tag{3.25}$$

if $q \neq 2$, or as

$$f(r) \sim -4\Lambda \ln(r) + D, \tag{3.26}$$

if $q = 2$. The integration constants D in these expressions are easily related to the values of X and $f(X)$ on the limits of the integral of energy ϵ. Around each of these limits this integral takes the form

[4]For values of α in Eqs.(3.22) and (3.23) which lie in the interval $-1 < \alpha < -1/2$ the condition $f(0) = 0$ is fulfilled, but $\overset{\bullet}{f}(X)$ diverges at $X = 0$ in such a way that the energy density becomes necessarily negative in the neighbourhood of $X = 0$. For $\alpha \leq -1$, $f(X)$ diverges in vacuum. The limit case $\alpha = -1/2$ is singular.

[5]Although this assumption excludes some transcendent behaviours such as the asymptotic exponential damping, our conclusions will remain valid for models exhibiting these "short-ranged" SSS solutions. In fact such models are included in the case B-3 below. Note that damped *oscillatory* behaviour at infinity is excluded by the monotonicity of the SSS field solutions.

$$-4\pi \int dr \left[\frac{2\Lambda q}{q-2} r^q + Dr^2 \right], \tag{3.27}$$

for $q \neq 2$ and

$$-4\pi \int dr \left[4\Lambda r^2 \ln(r) + Dr^2 \right], \tag{3.28}$$

for $q = 2$. Let us analyze separately the convergence of the energy integral around $r \sim 0$ (case A) and in the asymptotic limit $r \to \infty$ (case B).

3.3.1 Convergence at the origin

In case A the convergence of (3.27) requires $q > -1$ and we can distinguish three sub-cases:

• **A-1)** If $-1 < q < 0$ the field $\phi'(r)$ diverges at $r \to 0$ but the integral of energy converges there and the potential $\phi(r)$ is finite at the origin. Then, as r approaches zero, $X \to -\infty$ and $f(X)$ and $\overset{\bullet}{f}(X)$ diverge as

$$f(X) \sim -(-X)^{\frac{q-2}{2q}}, \quad \overset{\bullet}{f}(X) \sim (-X)^{-\frac{q+2}{2q}} \tag{3.29}$$

(see figure 3.1). Such solutions can be stable and finite-energy SSS fields (depending on their large-r behaviour) and, in this sense, they might be considered as genuine non-topological solitons.

• **A-2)** If $q = 0$ the field $\phi'(r)$ goes to a constant value at the origin ($\phi'(0) = C$, which corresponds to $X + C^2 = 0$) and can be written around this point as

$$\phi'(r) \sim C - \theta r^{\sigma}, \tag{3.30}$$

where C, θ and σ are positive constants [6]. Then $\overset{\bullet}{f}(X)$ diverges there as

$$\overset{\bullet}{f}(X) \sim (X + C^2)^{-\frac{2}{\sigma}}. \tag{3.31}$$

Consequently, for $\sigma \neq 2, f(X)$ behaves around $X = -C^2$ as

$$f(X) \sim \frac{(X + C^2)^{1-\frac{2}{\sigma}}}{\left(1 - \frac{2}{\sigma}\right)} + \Delta, \tag{3.32}$$

where Δ is a constant. For $\sigma = 2$ this behaviour becomes

[6] Note that the scalar version of the Born-Infeld model is an example which belongs to this case, corresponding to $\sigma = 4$.

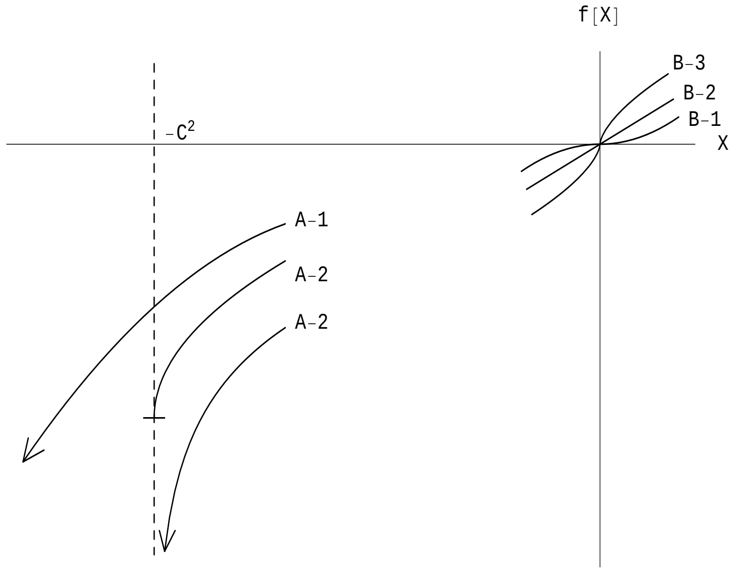

Figure 3.1: Different possible behaviours of the admissible lagrangians supporting finite-energy SSS solutions. Note the existence of two A-2 cases, both ending at a finite value $X = -C^2$ (corresponding to the maximum value of the field strength) but with one of them ending at a finite value of $f(X)$ there while the other one exhibits a vertical asymptote. A-1 branch is unbounded in $X < 0$.

$$f(X) \sim \ln(X + C^2). \tag{3.33}$$

We see that when $\sigma \leq 2$ these lagrangians diverge at $X = -C^2$ (figure 3.1). However they can be accepted as admissible class-2 field theories if this point is located on the frontier of the (open) domain of definition where the lagrangian density must be regular (see the third example of chapter 6). Consequently the set $X \leq -C^2$ must be excluded from the domain of definition (Ω) in this case. This requires σ to be an irrational number or four times the irreducible ratio between any natural and an odd natural. When $\sigma > 2$ these values also exclude the same region, leading again to admissible class-2 field theories. However, in this case the lagrangians are finite in $X = -C^2$ (even if $\dot{f}(X)$ diverges there). Then, for rational values of $\sigma > 2$ which are irreducible ratios of an odd natural and any natural [7], the lagrangians are defined and continuous for any X around $X = -C^2$, exhibiting a vertical-slope inflexion in this point. If suitably extended to all X, they lead to class-1 field theories which satisfy the everywhere positive definiteness condition of the energy. Nevertheless they violate the requirement of continuity of $\dot{f}(X)$ for $X < 0 \in \Omega$ and this leads to associated point-like solutions exhibiting several branches. Indeed, as we shall see at once, this requirement for admissibility is introduced

[7] Other values of σ, for which the lagrangian is also defined for $X < -C^2$, lead to negative energy densities there and consequently must be excluded.

because it endorses the single-branched character of the SSS solutions. Consequently, in this case A-2 all admissible lagrangians must remain undefined for $X \leq -C^2$ and thus belong to class-2 field theories. This implies (as happens in the Born-Infeld model [3]) the existence of a maximum value of the field strength ($\phi^{'}(r) < C$).

As a function of r, the energy density behaves around the center as

$$r^2 f(r) \sim r^\sigma - \Delta r^2, \tag{3.34}$$

for $\sigma \neq 2$ and as

$$r^2 f(r) \sim 2r^2 \ln(r) + r^2 \ln(2C\theta), \tag{3.35}$$

for $\sigma = 2$. As expected, the energy integral converges there in both cases.

- **<u>A-3</u>)** The case $q > 0$ must be discarded. Indeed, in this case $\overset{\bullet}{f}(X)$ behaves as

$$\overset{\bullet}{f}(X) \sim X^{-\frac{q+2}{2q}}, \tag{3.36}$$

around $X = 0$. Consequently, $f(X)$ is singular in vacuum for $0 < q \leq 2$. For $q > 2$ the energy density for $X \to 0^+$ behaves as

$$\rho(X) \sim -\frac{2}{q} X^{\frac{q-2}{2q}}, \tag{3.37}$$

and becomes negative around the vacuum (see also Eqs.(3.22), (3.23) and the footnote there).

3.3.2 Convergence at infinity

In **<u>case B</u>** the convergence of (3.27) in the $r \to \infty$ limit requires $q < -1$ (in this case $\phi^{'}(r \to \infty) = 0$ and the integration constant D in (3.27) vanishes). Then the behaviour of $f(X)$ around $X = 0$ must be [8]

$$f(X) \sim X^{\frac{p+2}{2p}}. \tag{3.38}$$

Now the existence of the lagrangian on both sides around $X = 0$ becomes crucial for the consistency of the theory and this imposes supplementary restrictions on the possible values of the parameter p. Indeed, the exponent in Eq.(3.38) must be the ratio of two

[8] For the sake of clarity we use here the parameter $p = -q$ in the exponent, in terms of which $\phi(r \to \infty) \sim \frac{1}{r^p}$ with $p > 1$.

odd naturals [9] and this restricts the possible values of p to a sub-class of the rational numbers. Let us analyze separately three possibilities:

• **B-1)** Consider first the case $1 < p < 2$. We define P and Q as two positive odd natural numbers in such a way that $P < Q$ and the ratio $\Sigma = P/Q$ be irreducible. Then the admissible values of the exponent in (3.38) are given by

$$\frac{p+2}{2p} = \frac{3}{2+\Sigma}, \tag{3.39}$$

and the corresponding admissible values of p can be written as

$$p = \frac{4+2\Sigma}{4-\Sigma}. \tag{3.40}$$

Now $\overset{\bullet}{f}(0) = 0$ and the slope of the lagrangian vanishes in vacuum (see figure 3.1).

• **B-2)** For $p = 2$ the lagrangian behaves around $X = 0$ as the D'Alembert lagrangian

$$f(X \to 0^{\pm}) \sim X, \tag{3.41}$$

and the soliton field becomes asymptotically Coulombian (see figure 3.1).

• **B-3)** For $p > 2$ the behaviour of the lagrangian is also given by Eq.(3.38) but now the admissible values of the exponent are constrained by

$$\frac{p+2}{2p} = \frac{1}{1+\Sigma}, \tag{3.42}$$

where $\Sigma = P/Q$ must be the irreducible ratio between an even natural P and an odd natural Q such that $Q > P$. The corresponding admissible values of p are

$$p = 2\frac{1+\Sigma}{1-\Sigma}. \tag{3.43}$$

As easily seen, the slope of the lagrangian diverges at $X = 0$ in this case (see figure 3.1), but the energy density remains positive definite there.

We conclude that the set of admissible models exhibiting finite-energy SSS solutions can be classified into six families which are the combinations of the cases A-1 or A-2, governing the central field behaviour and the cases B-1, B-2 or B-3, determining the asymptotic field behaviour. Moreover, any given scalar, monotonically decreasing, SSS function $\phi'(r)$, which satisfies boundary conditions of A-type at the center and of B-type asymptotically, is a finite-energy SSS solution of a particular lagrangian model belonging

[9] If this exponent is the irreducible ratio between an even and an odd natural numbers the lagrangian is well defined on both sides of $X = 0$, but the energy density becomes negative for $X < 0$.

to one of these families. The explicit form of the corresponding lagrangian density can be found by integrating Eq.(3.4) with respect to the variable $X = -\phi'^2(r)$ with the corresponding boundary conditions (see section 5.2).

3.4 Conditions for stability

For the sake of completeness of this chapter, we describe here the main steps in the analysis of the linear stability of the scalar SSS soliton solutions (the detailed calculations are given in chapter 5). The linear stability of these solutions requires their energy (3.12) to be a local minimum against charge-preserving perturbations. We consider finite-energy SSS solutions $\phi(r)$ and small static perturbations $\delta\phi(\vec{r})$, finite and regular everywhere and vanishing (as well as their radial derivatives) as $r \to \infty$, in such a way that the scalar charge of the perturbed fields remains unchanged at the first order in the perturbations. For the static solutions of the field equations (3.2) the first variation of the energy (3.16) vanishes, while the second variation is positive if and only if the condition [10]

$$\overset{\bullet}{f}(X) + 2X \overset{\bullet\bullet}{f}(X) > 0, \tag{3.44}$$

is satisfied in all the range of values of X covered by the solution ($X = -\phi'^2(r), 0 \leq r < \infty$). As we shall see this requirement is always fulfilled by the finite-energy SSS solutions of the *admissible* models defined in this chapter, proving their linear stability. A detailed spectral analysis of the small oscillations around these SSS finite-energy solutions has also been performed. It leads, for admissible models, to discrete spectra of eigenvalues and normalizable orthogonal eigenfunctions in Hilbert spaces, whose scalar products are built as three-dimensional integrals of their products with the functions $f(X(r))$ as kernels. In their temporal evolution the perturbations to the soliton solutions remain bounded in this norm, confirming the stability (see chapter 5).

3.5 Conditions for uniqueness

We return now to Eq.(3.4), which defines the SSS field solutions $\phi'(r)$, and analyze the conditions under which they are single-branched. As already mentioned, owing to the positivity of $\overset{\bullet}{f}(X)$ both Λ and $\phi'(r)$ must be simultaneously either positive or negative and then we can analyze only the case $\phi'(r) > 0$ without loss of generality. Let us write (3.4) under the form

[10] See Eq. (5.7) and the footnote there.

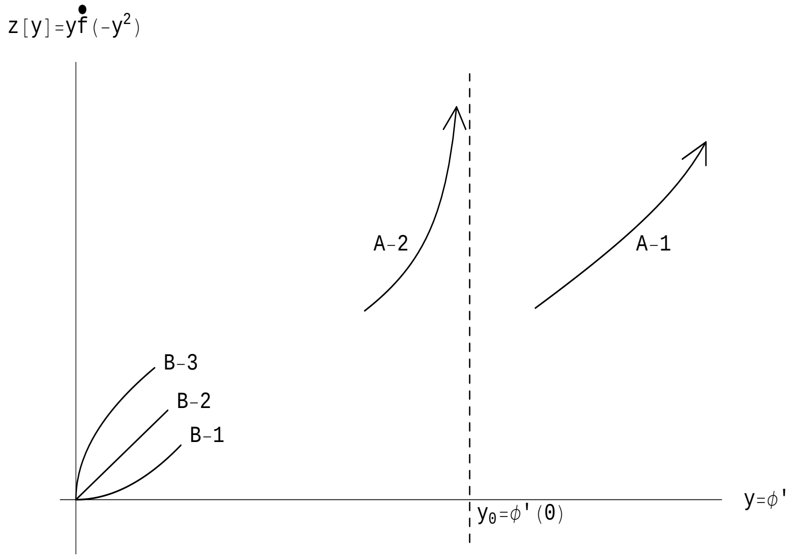

Figure 3.2: Characterization in the y-$z(y) = y \overset{\bullet}{f}(-y^2)$ plane of the admissible models supporting finite-energy SSS solutions. Observe that all curves start at $y = 0, z = 0$ ($y = \phi'(r \to \infty) = 0$, cases B-1, B-2 and B-3) and grow monotonically without limit. This guarantees the existence of a single cutting point with the $z = constant$ lines and thus a single-valued solution (compare to figure 3.1)

$$z(y) = y \overset{\bullet}{f}(-y^2) = \frac{\Lambda}{r^2}, \tag{3.45}$$

where we have introduced the variable $y = \phi'(r) > 0$. The field strength function is given by the values of y obtained by cutting the curve $z(y)$ with horizontal lines corresponding to the different values of r. Then, for the field $\phi'(r)$ to be defined in all space, z must range from 0 to ∞ and the uniqueness of the solution requires a single cut point on every $z = constant$ line. This restricts $z(y)$ to be a continuous monotonic function. As a consequence, the requirements of continuity (for $X < 0$) and strict positivity (for any $X \neq 0$) imposed on $\overset{\bullet}{f}(X)$ at the beginning of this chapter, when establishing the admissibility conditions, are mandatory for the proper definition of the SSS solutions. Indeed, a glance at the form of the function $z(y)$ reveals that the existence of a jump in the function $\overset{\bullet}{f}(X)$ at a finite value $X < 0$ would lead to SSS solutions which are double-branched or undefined in some range of values of r. Moreover, if $\overset{\bullet}{f}(X)$ vanishes for a value $X_0 < 0$ (horizontal-slope inflexion point for $f(X)$), then $z(y)$ vanishes at $y = (-X_0)^{1/2}$ reaching a minimum there and the SSS solutions become necessarily multiple-valued (the vanishing of $\overset{\bullet}{f}(X)$ for a positive value $X_0 > 0$ is discarded by the energy-positivity condition (3.21)).

For soliton solutions the finiteness of the energy requires the origin to be a point of the curve $z(y)$ (see figure 3.2) and the large-z behaviour is determined by the behaviour of the field at the center of the soliton. Then, in the cases where the field strength diverges as $r \to 0$ (case A-1 above) the unicity of the solution requires the curve $z(y)$ to start at the origin and grow *monotonically* without limit as $y \to \infty$. When the field is finite at $r = 0$ (case A-2) the curve $z(y)$ must increase *monotonically* from the origin and diverge at $y_0 = \phi'(0)$, showing a vertical asymptote there. In this case $\overset{\bullet}{f}(-y^2)$ diverges at $y = y_0$ and the unicity condition requires the lagrangian function to remain undefined for $y > y_0$, for the function $z(y)$ to exhibit an unique growing branch. Thus in all A-2 cases the set $X < -y_0^2$ must be excluded from the domain of definition (Ω) and then, the associated models must be necessarily class-2 field theories [11]. The monotonicity condition for the unique branch of $z(y)$ in the admissible models with finite-energy solutions takes the form

$$\frac{dz}{dy} = \overset{\bullet}{f}(-y^2) - 2y^2 \overset{\bullet\bullet}{f}(-y^2) \geq 0, \tag{3.46}$$

for any $y \geq 0$. This requirement coincides with the condition (3.44) for stability which, as already mentioned, is fulfilled by all these admissible models. To summarize, we conclude that all *finite-energy* SSS solutions of *admissible* scalar field theories considered in this chapter are *single-branched, stable and defined everywhere.*

To close this chapter let us give an additional expression for the energy of the SSS solutions in terms of the function $z(y)$ which, as we shall see in the examples of chapter 6, will be useful in the explicit calculation of the soliton energy. This expression can be obtained by taking into account Eqs.(3.4), (3.13) and (3.45) (or rewriting Eq.(**??**) for the variable $z(y)$) and reads

$$\epsilon = \frac{4\pi}{3}\Lambda^{3/2}\left\{ \left.\frac{f(-y^2)}{z(y)^{3/2}}\right|_{y(r=0)}^{y(r\to\infty)} - 2\int_{y(r=0)}^{y(r\to\infty)} \frac{dy}{\sqrt{z(y)}} \right\}. \tag{3.47}$$

As can be seen from the preceding analysis, the first term in this formula vanishes for admissible models with soliton solutions. Conversely, the conditions to be imposed on the lagrangian densities of admissible models for supporting finite-energy SSS solutions could have been directly obtained from the requirement of cancellation of the first term in (3.47). Once the expression of the lagrangian density is known, the second term gives the soliton energy directly through a quadrature.

[11] It can be shown from the analysis of the possible continuations of $z(y)$ for $y > y_0$ that the new branches of SSS field solutions are pathological (non-defined everywhere, unstable, or both).

3.6 The multicomponent scalar field

We shall extend the results of previous sections to the case of a set of scalar fields $\phi_i(x)(i = 1 \to N)$. In many cases the covariant lagrangians including N scalar fields and their first-order derivatives are constrained by conditions which allow to implement some internal symmetries. Such conditions manifest themselves in the structure of the manifold where the field takes its values. A well-known example is the non-linear sigma model where this manifold is a Riemann space implementing the chiral $SU(2) \times SU(2)$ symmetry [76, 46]. Here we shall restrict ourselves to the case where the field manifold is the N-dimensional Euclidean space and the $SO(N)$ invariant lagrangian density depends only on derivative terms

$$L(\phi_i, \partial_\mu \phi_i) = f\left(\sum_{i=1}^{N} \partial_\mu \phi_i \partial^\mu \phi_i\right), \tag{3.48}$$

where, as in the one-component case, $f(X)$ is a given *continuous, derivable (C^1 for $X < 0$) and monotonically increasing* ($\frac{df}{dX} > 0, \forall X \neq 0; \frac{df}{dX} \geq 0, X = 0$) function defined in a open and connected domain ($\Omega \subseteq \Re$) which includes the vacuum ($(X = 0) \in \Omega$). Besides the fact that these models are the natural generalizations of the scalar field theories studied so far, there is another motivation for their analysis. Indeed, as we shall see in the following chapters, a class of soliton solutions arising in generalized gauge field theories of some compact semisimple Lie group of dimension N reduce to multiscalar (N components) solitons of some of the models (3.1).

The field equations associated to the lagrangians (3.1) take the form of N local conservation laws

$$\partial_\mu J_i^\mu = 0, \tag{3.49}$$

where the conserved currents J_i^μ are

$$J_i^\mu = \overset{\bullet}{f}(X) \partial^\mu \phi_i, \tag{3.50}$$

with $X = \sum_{i=1}^{N} \partial_\alpha \phi_i \cdot \partial^\alpha \phi_i$. The canonical energy-momentum tensor is

$$T_{\mu\nu} = 2\overset{\bullet}{f}(X) \sum_{i=1}^{N} \partial_\mu \phi_i \partial_\nu \phi_i - f(X)\eta_{\mu\nu}, \tag{3.51}$$

and the corresponding energy density

$$\rho(x) = 2\overset{\bullet}{f}(X) \sum_{i=1}^{N} \left(\frac{d\phi_i}{dt}\right)^2 - f(X), \tag{3.52}$$

is positive definite under the same conditions as for the function $f(X)$ defined in the one-component case.

For the SSS solutions $\phi_i(r)$, equations (3.49) have N first-integral field equations of the form

$$r^2\phi_i' \overset{\bullet}{f} \left(-\sum_{j=1}^{N} \phi_j'^2\right) = \Lambda_i, \tag{3.53}$$

where $\phi_i' = d\phi_i/dr$ and Λ_i are the integration constants. Now the signs of every component of the scalar field and of the corresponding integration constant are the same, but may differ for different components. In order to solve the system (3.53) let us introduce the functions $X_i(r) = -\phi_i'^2(r)$, in such a way that $X(r) = \sum_{i=1}^{N} X_i(r)$. By squaring and adding Eqs.(3.53) we obtain

$$r^4 X \overset{\bullet}{f}^2 (X) = -\sum_{i=1}^{N} \Lambda_i^2, \tag{3.54}$$

or, equivalently,

$$r^2\sqrt{-X} \overset{\bullet}{f} (X) = \Lambda, \tag{3.55}$$

where

$$\Lambda = \sqrt{\sum_{i=1}^{N} \Lambda_i^2}, \tag{3.56}$$

Equation (3.55) has the same form as the first-integral of the one-component scalar case (3.4). Consequently, if the function $f(X)$ is the same in both cases, we can associate to any SSS solution of the one-component case, of the form $\phi'(r, \Lambda)$, a set of sequences of N functions which are SSS solutions of the multicomponent scalar equations. Such functions take the form

$$\phi_i'(r, \Lambda_j) = \frac{\Lambda_i}{\Lambda}\phi'(r, \Lambda), \tag{3.57}$$

and, owing to Eq.(3.56), there is a one-to-one correspondence between such sequences and the points of the sphere of radius Λ in the N-dimensional Euclidean space ($\Re^N$). Obviously, this is a straightforward consequence of the invariance of the lagrangian (3.48) under rotations in the internal space. The constants Λ_i can now be identified as the "source point-charges" associated to the different components of the SSS field, namely

$$\Lambda_i = \frac{1}{4\pi} \int d^3x \vec{\nabla} \cdot \left(\overset{\bullet}{f} (X) \vec{\nabla} \phi_i \right), \tag{3.58}$$

and Λ as the mean-square scalar charge. The field potentials obtained by integration of (3.57) read

$$\phi_i(r, \Lambda_j) = \frac{\Lambda_i}{\Lambda} \phi(r, \Lambda) + \Delta_i, \tag{3.59}$$

where Δ_i are integration constants.

If we consider now the energy associated to these SSS solutions we find from (3.52)

$$\epsilon = -4\pi \int_0^\infty r^2 f \left(\sum_{i=1}^{N} X_i(r) \right) dr = -4\pi \int_0^\infty r^2 f\left(X(r)\right) dr. \tag{3.60}$$

This is the energy of the *one-component* SSS solution corresponding to the integration constant (3.56). Thus the set of SSS solutions of the multicomponent scalar field associated to the sphere of radius Λ in $\Re^N$ is degenerate in energy. Moreover, the search of conditions to be imposed on $f(X)$ for the existence of finite-energy SSS solutions (as well as the admissibility constraints) in the multicomponent case reduces to the analysis performed for the one-component case.

Concerning the conditions for stability of the solutions (3.59), the analysis of the one-component case can be straightforwardly generalized to the present situation (see chapter 5 for details). The final conclusion is that the multicomponent soliton solutions of admissible models are also linearly stable against charge-preserving perturbations. This charge preservation prevents a soliton from evolving spontaneously towards another equal-energy soliton state in the sphere of radius Λ.

Chapter 4

Abelian and non-abelian gauge solitons

In this chapter we shall extend the analysis developed in the former chapter to generalized gauge field theories of compact semi-simple Lie groups. In this case the dependence of the lagrangian on the field invariants is a natural consequence of the gauge principle.

To start with, we study the simpler case of generalized electromagnetic ($U(1)$-invariant) fields. After defining the models we impose conditions for admissibility in this case and solve the field equations for electrostatic finite-energy central field solutions. We then establish a correspondence between families of admissible generalized gauge-invariant theories supporting electrostatic, finite-energy solutions and a family of admissible (multi-) scalar field theories supporting similar static, finite-energy solutions. We discuss briefly the new conditions for stability to be imposed on the lagrangian densities, leaving the full analysis for chapter 5. We then proceed further to the case of non-abelian gauge fields, following a similar procedure as in the abelian case.

4.1 Generalized abelian field theories

We define lagrangian densities for *generalized electromagnetic* fields defined as arbitrary functions of the two quadratic field invariants, built from the Maxwell tensor and its dual. Following the conventions of Ref.[126], these tensors are defined as

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ F^*_{\mu\nu} &= \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}, \end{aligned} \tag{4.1}$$

where $\varepsilon^{0123} = -\varepsilon_{0123} = 1$. The electric and magnetic fields are defined as $E^i = -F^{0i}$ and $H^i = -\frac{1}{2}\varepsilon^{ijk}F_{jk}$, respectively. We introduce the two standard first-order gauge field invariants, X and Y, defined as

$$\begin{aligned} X &= -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \vec{E}^2 - \vec{H}^2 \\ Y &= -\frac{1}{2}F_{\mu\nu}F^{*\mu\nu} = 2\vec{E}\cdot\vec{H}. \end{aligned} \tag{4.2}$$

We define the general form of the lagrangian density of our models as

$$L = \varphi(X, Y), \tag{4.3}$$

where φ is a given *continuous and derivable* function on its domain of definition (Ω) of the $X - Y$ plane ($\Re^2$). As in the scalar case we assume Ω to be open and connected and including the vacuum ($(X = 0, Y = 0) \in \Omega$). As a minimal extension of the assumptions of chapter 3 concerning the regularity properties of the scalar lagrangian functions, which is necessary for future purposes (see Eq.(4.21) below), we shall assume $\varphi(X, Y)$ to be of class C^1 on the line $(X > 0, Y = 0)\bigcap\Omega$ and $\partial\varphi/\partial X$ to be strictly positive there. By generalizing the definitions of chapter 3 we shall call "class-1 field theories" the models defined an regular everywhere ($\Omega \equiv \Re^2$) and "class-2 field theories" those with $\Omega \subset \Re^2$. We also require $\varphi(X, Y)$ to be symmetric in the second argument, in order to implement parity invariance

$$\varphi(X, Y) = \varphi(X, -Y). \tag{4.4}$$

This implies that the odd partial derivatives of $\varphi(X, Y)$ with respect to Y must vanish on $Y = 0$. In our notation the Maxwell lagrangian density corresponds to $\varphi(X, Y) = \frac{X}{8\pi}$ while the Born-Infeld electrodynamics is given by the lagrangian density

$$L_{BI} = \varphi_{BI}(X, Y) = \frac{1 - \sqrt{1 - \mu^2 X - \frac{\mu^4}{4}Y^2}}{4\pi\mu^2}, \tag{4.5}$$

where $\frac{1}{\mu}$ is the maximum field strength, attained at the center of the solution. The lagrangian (4.5) reduces to the Maxwell one in the limit $\mu \to 0$.

4.1.1 Conditions on the energy functional

The symmetric (gauge-invariant) energy-momentum tensor obtained from the lagrangian density (4.3) takes the form

$$T^s_{\mu\nu} = 2\left(\frac{\partial\varphi}{\partial X}F_{\mu\alpha}F^{\alpha}_{\nu} + \frac{\partial\varphi}{\partial Y}F_{\mu\alpha}F^{*\alpha}_{\nu}\right) - \varphi\eta_{\mu\nu} = 2\frac{\partial\varphi}{\partial X}F_{\mu\alpha}F^{\alpha}_{\nu} + \left(Y\frac{\partial\varphi}{\partial Y} - \varphi\right)\eta_{\mu\nu}, \quad (4.6)$$

and the associated energy density is

$$\begin{aligned} \rho^s = T^s_{00} &= 2\frac{\partial\varphi}{\partial X}\vec{E}^2 + 2\frac{\partial\varphi}{\partial Y}\vec{E}\cdot\vec{H} - \varphi(X,Y) = \\ &= 2X\frac{\partial\varphi}{\partial X} - \varphi(X,Y) + Y\frac{\partial\varphi}{\partial Y} + 2\frac{\partial\varphi}{\partial X}\vec{H}^2. \end{aligned} \quad (4.7)$$

We assume the symmetric energy-momentum tensor (4.6) to give the correct space-time energy density distribution. Let us analyze the conditions for the positivity of the energy density of any field configuration. The inspection of Eq.(4.7), together with the requirement of vanishing of the vacuum energy, lead to the set of *necessary* conditions

$$\varphi(0,0) = 0 \ \ ; \ \ \varphi(X,0) < 0 \ \forall(X < 0, Y = 0 \in \Omega) \ \ ; \ \ \frac{\partial\varphi}{\partial X} > 0 \ \forall(X,Y) \in \Omega, \quad (4.8)$$

to be satisfied by the lagrangian densities. However, it is possible to obtain a minimal set of *necessary and sufficient* conditions of admissibility for a satisfactory energetic behaviour. Solving Eqs.(4.1) for the fields we obtain

$$\begin{aligned} E^2 &= \frac{1}{2}\left(\sqrt{X^2 + \frac{Y^2}{\cos^2(\vartheta)}} + X\right) \geq \frac{1}{2}\left(\sqrt{X^2+Y^2} + X\right) \\ H^2 &= \frac{1}{2}\left(\sqrt{X^2 + \frac{Y^2}{\cos^2(\vartheta)}} - X\right) \geq \frac{1}{2}\left(\sqrt{X^2+Y^2} - X\right), \end{aligned} \quad (4.9)$$

where ϑ is the angle between $\vec{E}$ and $\vec{H}$. Note that the equality in the above expression is reached for the special case of parallel $\vec{E}$ and $\vec{H}$ vectors. From these expressions the energy density can be written as

$$\rho^s = \frac{\partial\varphi}{\partial X}\left(\sqrt{X^2 + \frac{Y^2}{\cos^2(\vartheta)}} + X\right) + Y\frac{\partial\varphi}{\partial Y} - \varphi(X,Y). \quad (4.10)$$

Consequently, the requirement of the positive definite character of the energy leads to the minimal *necessary and sufficient* condition

$$\rho^s \geq \left(\sqrt{X^2+Y^2}+X\right)\frac{\partial\varphi}{\partial X}+Y\frac{\partial\varphi}{\partial Y}-\varphi(X,Y)\geq 0, \tag{4.11}$$

to be satisfied in the entire domain of definition (Ω). Generalizing the criteria of chapter 3, we only regard as "admissible" those models whose lagrangian densities satisfy the condition (4.11), aside from the vanishing of the vacuum energy and the regularity and parity-invariance conditions stated above [1]. Consequently the admissible lagrangians must be solutions of the first-order linear inhomogeneous partial differential equation

$$\left(\sqrt{X^2+Y^2}+X\right)\frac{\partial\varphi}{\partial X}+Y\frac{\partial\varphi}{\partial Y}-\varphi(X,Y)=\Psi(X,Y), \tag{4.12}$$

where $\Psi(X,Y)$ is any function being positive definite in Ω and vanishing in vacuum ($\Psi(0,0)=0$). Such solutions must also satisfy the regularity and parity-invariance requirements, as boundary conditions.

Let us also consider the trace of the symmetric energy-momentum tensor

$$\begin{aligned} T^s &= 4\left[\frac{\partial\varphi}{\partial X}(\vec{E}^2-\vec{H}^2)+2\frac{\partial\varphi}{\partial Y}\vec{E}\cdot\vec{H}-\varphi(X,Y)\right]= \\ &= 4\left[X\frac{\partial\varphi}{\partial X}+Y\frac{\partial\varphi}{\partial Y}-\varphi(X,Y)\right]. \end{aligned} \tag{4.13}$$

From the last expression we see that the sub-class of models with traceless symmetric energy-momentum tensors is given by the lagrangian densities $\varphi(X,Y)$ which are solutions of the first-order linear homogeneous partial differential equation

$$X\frac{\partial\varphi}{\partial X}+Y\frac{\partial\varphi}{\partial Y}-\varphi(X,Y)=0. \tag{4.14}$$

The general solution of this equation is the family of all conic surfaces in the (X,Y,φ)-space having the origin as a vertex. Clearly the set of planes of the form $\varphi=aX+bY$ (a and b being constants) are particular solutions of this equation (which violate parity invariance if $b\neq 0$). The simple case $b=0$ with $a=\frac{1}{8\pi}$ corresponds to the Maxwell theory.

4.1.2 Field equations

The field equations are obtained by extremizing the action corresponding to the lagrangian (4.3) with respect to A_μ, which yields

[1] Obviously, excepting the vanishing of the vacuum energy, the remaining conditions in (4.8) are consequences of (4.11).

$$\partial_\mu \left(\frac{\partial \varphi}{\partial X} F^{\mu\nu} + \frac{\partial \varphi}{\partial Y} F^{*\mu\nu} \right) = 0, \tag{4.15}$$

to which we add the Bianchi identities

$$\partial_\mu F^{*\mu\nu} = 0. \tag{4.16}$$

In terms of the fields these equations (4.15) and (4.16) read

$$\begin{aligned} \vec{\nabla} \cdot \left(\frac{\partial \varphi}{\partial X} \vec{E} + \frac{\partial \varphi}{\partial Y} \vec{H} \right) &= 0 \\ -\frac{\partial}{\partial t} \left(\frac{\partial \varphi}{\partial X} \vec{E} + \frac{\partial \varphi}{\partial Y} \vec{H} \right) &+ \vec{\nabla} \times \left(\frac{\partial \varphi}{\partial X} \vec{H} - \frac{\partial \varphi}{\partial Y} \vec{E} \right) = 0, \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{H}}{\partial t} \\ \vec{\nabla} \cdot \vec{H} &= 0. \end{aligned} \tag{4.18}$$

If we define now a tensor $P^{\mu\nu} = \frac{\partial \varphi}{\partial X} F^{\mu\nu} + \frac{\partial \varphi}{\partial Y} F^{*\mu\nu}$ and introduce *electric displacement* $D^i = -P^{0i} = \frac{\partial \varphi}{\partial X} E^i + \frac{\partial \varphi}{\partial Y} H^i$ and *magnetic intensity* $B^i = -\frac{1}{2}\varepsilon^{ijk} P_{jk} = \frac{\partial \varphi}{\partial X} H^i - \frac{\partial \varphi}{\partial Y} E^i$ vectors, it is immediately seen that the first group of field equations (4.17) take the form

$$\begin{aligned} \vec{\nabla} \cdot \vec{D} &= 0 \\ \vec{\nabla} \times \vec{B} - \frac{\partial \vec{D}}{\partial t} &= 0, \end{aligned} \tag{4.19}$$

which takes exactly the same form as in the usual Maxwell theory but with the "basic" electric and magnetic fields $\vec{E}, \vec{H}$ being replaced by their generalized versions $\vec{D}, \vec{B}$ [2]. Obviously in Maxwell theory $\vec{D} = \vec{E}$ and $\vec{B} = \vec{H}$ and we recover the usual Maxwell field equations. As we shall see below (Eq.(4.24)) the fact that the Dirac-delta singularities at the center of the soliton be related to the vector $\vec{D} = \frac{q}{r^2}\widehat{e}_r$ rather than $\vec{E}$, is a key difference between Maxwell and generalized electrodynamic theories supporting soliton solutions, leading in the latter case to an integrable behaviour of the corresponding $\vec{E}$

[2] Note that we have reversed the notation with respect to the more conventional one found in the literature, where the $\vec{B}$ vector is taken as the basic magnetic field and $\vec{H}$ as the generalized one.

field [3].

For electrostatic fields we have $Y = 0$ and in the ESS case these equations can be written in terms of the electrostatic potential $A^0(r)$ ($\vec{A} = 0, \vec{E}(r) = -\vec{\nabla}A^0(r)$) in such a way that the first group of equations in (4.17) leads to the first-integral [4]

$$r^2 \frac{dA^0}{dr} \frac{\partial \varphi}{\partial X}(X, Y = 0) = q, \tag{4.20}$$

where now $X = \left(\frac{dA^0}{dr}\right)^2$ and q is an integration constant. Using the identification $\phi(r, \Lambda) \equiv A^0(r, q = \Lambda)$ this equation coincides with the first-integral (3.4) for the SSS solutions of a scalar field model with a lagrangian density defined by

$$L_{scalar} = f(\partial_\mu \phi \cdot \partial^\mu \phi) \equiv f(X) = -\varphi(-X, Y = 0), \tag{4.21}$$

which leads to

$$\overset{\bullet}{f}(X) = \frac{\partial \varphi}{\partial X}(-X, Y = 0). \tag{4.22}$$

Conversely we can associate to each scalar model defined by a lagrangian density $f(X)$, a family of electromagnetic field models defined by lagrangian densities $\varphi(X, Y)$ satisfying Eq.(4.21) as well as the admissibility (4.11) and stability constraints (see Eq.(4.32) below). The ESS field solutions of all electromagnetic generalizations ($|E(\vec{r}, q)|$) have the same form, as functions of r, as the SSS field solutions ($\phi'(r, \Lambda)$) of the original scalar model, q and Λ being the integration constants, which should be identified as the electric and scalar point-like charges associated to the solution, respectively. Indeed, in the generalized electromagnetic case the definition of the electric charge associated to a given field is

$$\frac{1}{4\pi} \int d^3\vec{r} \vec{\nabla} \cdot \left(\frac{\partial \varphi}{\partial X} \vec{E} + \frac{\partial \varphi}{\partial Y} \vec{H} \right), \tag{4.23}$$

which now is conserved as a consequence of the field equations. Substituting in this equation the ESS field coming from the solution of (4.20) we obtain q as the value of its total electric charge

$$\frac{1}{4\pi} \vec{\nabla} \cdot \left[\frac{\partial \varphi}{\partial X} \vec{E} \right] = q\delta_3(\vec{r}). \tag{4.24}$$

[3] Note that the field $\vec{E}$ can also diverge at the center of the soliton (case A-1 of chapter 3), but the divergences are now ameliorated enough for the total energy to be finite.

[4] The remaining three equations in (4.17) are identically satisfied for arbitrary electrostatic fields, owing to Eq.(4.4). On the other hand, the set of equations (4.18) are trivially satisfied by the ESS solutions.

In terms of the $\vec{D}$ vector the charge (4.23) takes the form

$$q = \frac{1}{4\pi}\int d^3r \vec{\nabla}\cdot\vec{D}. \tag{4.25}$$

As pointed out in Ref.[3] for the Born-Infeld model this charge can be interpreted as a source point-like charge at the center of the ESS solution or, alternatively, as a continuous charge-density distribution associated with the field and given by

$$\frac{1}{4\pi}\frac{\partial\varphi}{\partial X}(X=0, Y=0)\vec{\nabla}\cdot\vec{E}. \tag{4.26}$$

This interpretation, as already discussed for scalar models, requires the function $r^2E(r)$ to vanish at the origin (this condition is always fulfilled for the finite-energy ESS solutions) and the field $E(r)$ to be asymptotically coulombian (B-2 case models).

4.1.3 Energy

In calculating the total energy of these electrostatic central fields from the energy density (4.7) we are lead to

$$\begin{aligned}\epsilon_e(q) &= 8\pi\int_0^\infty r^2\frac{\partial\varphi}{\partial X}\left[X=\vec{E}^2(r,q), Y=0\right]\vec{E}^2(r,q)dr - \\ &- 4\pi\int_0^\infty r^2\varphi\left[X=\vec{E}^2(r,q), Y=0\right]dr,\end{aligned} \tag{4.27}$$

(the index **e** stands for electric field). The energy associated with the corresponding SSS scalar field solutions, obtained from Eqs.(3.12) and (4.21) reads

$$\epsilon_s(\Lambda) = 4\pi\int_0^\infty r^2\varphi\left[\vec{E}^2(r,\Lambda), 0\right]dr, \tag{4.28}$$

(the index **s** stands for scalar field). If the total energy (4.28) associated to the scalar field is finite, so is $\epsilon_e(q)$. Indeed, using Eq.(4.20) this energy becomes

$$\epsilon_e(q) = 8\pi q\left[A^0(\infty,q) - A^0(0,q)\right] - \epsilon_s(q), \tag{4.29}$$

which, owing to (3.14), is related with the scalar soliton energy through

$$\epsilon_{ef}(q) = 2\epsilon_{sf}(q), \tag{4.30}$$

and must be also finite. Thus the energy of a ESS solution is twofold the energy of the corresponding SSS soliton when the integration constants take the same value ($\Lambda = q$).

Equivalently, Eq.(3.12), which gives the scaling of energies, leads to the following relation between the integration constants (the electric (q) and the scalar (Λ) charges) of an electrostatic soliton and the associated scalar soliton of equal-energy

$$q = (2)^{2/3}\Lambda. \tag{4.31}$$

We then conclude that the ESS solutions for the families of electromagnetic models which generalize (through Eq.(4.21)) the different classes of scalar models with soliton solutions, have the same functional forms as the corresponding SSS scalar solutions and are also of finite-energy. Moreover, the classification of the admissible models with soliton solutions in the scalar case according to the central and asymptotic behaviours of the fields, immediately induces, through Eq.(4.21), a similar classification of the finite-energy ESS solutions in the electromagnetic case.

Let us point out an immediate consequence of this analysis (which is a corollary of the non-existence theorems established in Refs. [46, 47]): *there are not ESS soliton solutions for admissible generalized electromagnetic field theories with traceless energy-momentum tensor*. Indeed, as mentioned above the lagrangian densities of such theories (see Eq.(4.13)) are given by conic surfaces in the (X, Y, φ) space and the associated scalar field lagrangian densities $f(X) = -\varphi(-X, Y = 0)$ are straight lines in the (X, f) plane for $X < 0$. Consequently the associated SSS solutions are *coulombian* in form as well as energy-divergent, and so are the ESS solutions of these generalized electromagnetic models.

4.1.4 Stability

Although finite-energy SSS solutions of admissible scalar models are always linearly stable against charge-preserving perturbations, this is not so for the finite-energy ESS solutions of admissible generalized electromagnetic theories. Indeed, the analysis of the linear stability of the electrostatic solitons leads to a generalization of the criteria obtained in the scalar case (see chapter 5 for details). As a result of this analysis, the electrostatic finite-energy central field solutions of admissible generalized electromagnetic field models, with their lagrangian densities $\varphi(X, Y)$ satisfying the supplementary condition

$$\frac{\partial\varphi}{\partial X} - 2X\frac{\partial^2\varphi}{\partial Y^2} > 0, \tag{4.32}$$

in the entire domain of existence of the ESS solutions in the plane $(X, Y = 0)$, can be shown to be local minima of the energy functional against small charge-preserving perturbations. Consequently, the *admissibility* and *finite-energy* conditions of the ESS solutions, aside from Eq.(4.32), are *necessary and sufficient conditions* for linear static stability. Moreover, the linear analysis of the dynamics of the small perturbations of the soliton solutions performed for the scalar models can be generalized to the associated families of electromagnetic models which satisfy (4.32) (see chapter 5).

Finally, the conditions for univoque and everywhere defined ESS solutions are straightforwardly deduced from those of the scalar case and Eq.(4.21).

4.2 Generalized non-abelian field theories

The results obtained for generalized electromagnetic fields can be extended to non-abelian generalized gauge field theories of compact semi-simple Lie groups of dimension N. Let then G be a gauge group with generators $T_a, a = 1 \dots N$, satisfying the Lie algebra $[T_a, T_b] = iC_{abc}T_c$. As usual, in this case the tensor field strength components in the algebra and their duals are defined from the gauge fields $A_{a\mu}$ and the structure constants C_{abc} as

$$\begin{aligned} F_{a\mu\nu} &= \partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} - g \sum_{bc} C_{abc} A_{b\mu} A_{c\nu} \\ F^*_{a\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} F_a^{\alpha\beta}, \end{aligned} \tag{4.33}$$

whose components define the fields $\vec{E}_a$, $\vec{H}_a$ in the usual form, i.e.

$$\begin{aligned} \vec{E}_a &= -\frac{\partial \vec{A}_a}{\partial t} - \vec{\nabla} A_a^0 - g \sum_{bc} C_{abc} \vec{A}_b A_c^0 \\ \vec{H}_a &= \vec{\nabla} \times \vec{A}_a + \frac{g}{2} \sum_{bc} C_{abc} \vec{A}^b \times \vec{A}^c. \end{aligned} \tag{4.34}$$

In order to introduce the lagrangian densities governing the generalized dynamics of these fields we must define pertinent field invariants. However there is now an ambiguity in the calculation of the traces over the group indices, leading to different possibilities in the definition of these invariants (see section 2.3.2). Although at this regard different prescriptions have been introduced, mainly in the context of string theory, where BI-like actions arise as a low-energy effective field limit [30, 31, 32, 33, 34, 116, 123, 127, 128, 129], here we shall restrict our analysis to the case of actions built from the two simplest first-order field invariants, defined from the ordinary prescription for the calculation of the traces as

$$
\begin{aligned}
X &= -\frac{1}{2}\sum_a (F_{a\mu\nu}F_a^{\mu\nu}) = \sum_a \left(\vec{E}_a^2 - \vec{H}_a^2\right) \\
Y &= -\frac{1}{2}\sum_a (F_{a\mu\nu}F_a^{*\mu\nu}) = 2\sum_a \left(\vec{E}_a \cdot \vec{H}_a\right),
\end{aligned} \tag{4.35}
$$

where $1 \leq a \leq N$. The generalized lagrangian density is now assumed to be a given function $\varphi(X,Y)$ which (again for parity invariance) must be symmetric in the second argument ($\varphi(X,Y) = \varphi(X,-Y)$) and satisfy the same admissibility constraints of definition, continuity and derivability (as well as the distinction between class-1 and class-2 field theories) as in the electromagnetic case.

The associated field equations read now

$$
\sum_c D_{ac\mu}\left[\frac{\partial\varphi}{\partial X}F_c^{\mu\nu} + \frac{\partial\varphi}{\partial Y}F_c^{*\mu\nu}\right] = 0, \tag{4.36}
$$

where $D_{ac\mu} \equiv \delta_{ac}\partial_\mu - g\sum_b C_{abc}A_{b\mu}$ is the gauge-covariant derivative. In terms of the fields these equations reads

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{D}^a &+ g\sum_{bc} C_{abc}\vec{A}^b\cdot\vec{D}^a = 0 \\
-\frac{\partial}{\partial t}\vec{D}^a &+ \vec{\nabla}\times\vec{B}^a + g\sum_{bc} C_{abc}\left(A_0^b\vec{D}^c + \vec{A}^b\times\vec{B}^c\right) = 0,
\end{aligned} \tag{4.37}
$$

where we have introduced the vectors

$$
\begin{aligned}
\vec{D}^a &= \left(\frac{\partial\varphi}{\partial X}\vec{E}^a + \frac{\partial\varphi}{\partial Y}\vec{H}^a\right) \\
\vec{B}^a &= \left(\frac{\partial\varphi}{\partial X}\vec{H}^a - \frac{\partial\varphi}{\partial Y}\vec{E}^a\right),
\end{aligned} \tag{4.38}
$$

This defines a transformation between the solutions of a *basic* gauge field theory and a *generalized* one, generated by the lagrangian function $\varphi(X,Y)$, through the equations

$$
\begin{aligned}
\frac{\partial\varphi}{\partial\vec{E}^a} &= 2\vec{D}^a \\
\frac{\partial\varphi}{\partial\vec{H}^a} &= -2\vec{B}^a,
\end{aligned} \tag{4.39}
$$

where the fields $\vec{E}^a$, $\vec{H}^a$ obey to the field equations of the *basic* theory while $\vec{D}^a$, $\vec{B}^a$ are solutions of the *generalized* theory.

The symmetric energy-momentum tensor is

$$T^s_{\mu\nu} = 2\sum_a F_{a\mu\alpha}\left(\frac{\partial\varphi}{\partial X}F^{\alpha}_{a\nu} + \frac{\partial\varphi}{\partial Y}F^{*\alpha}_{a\nu}\right) - \varphi\eta_{\mu\nu}, \tag{4.40}$$

and the energy density takes the form

$$\rho^s = T^s_{00} = 2\frac{\partial\varphi}{\partial X}\sum_a \vec{E}_a^2 + 2\frac{\partial\varphi}{\partial Y}\sum_a \vec{E}_a\cdot\vec{H}_a - \varphi(X,Y). \tag{4.41}$$

The admissibility conditions to be imposed on the lagrangian density, for the energy functional to be positive definite and vanishing in vacuum, take the same form as in the electromagnetic case (see Eqs.(4.8) and (4.11)). Also the trace of the energy-momentum tensor has the same expression (see the last member of Eq.(4.13)), and vanishes under the same conditions (4.14). The subclass of models with non-vanishing trace energy-momentum tensor (those which violate condition (4.14)) breaks the scale invariance and thus circumvents the non-existence theorems [46, 47, 48, 49] allowing for particle-like solutions (see section 2.3.1).

4.2.1 ESS solutions

Let us consider the ESS solutions of these models. We consider fields of the form

$$\vec{E}_a(\vec{r}) = -\vec{\nabla}\left(A^0_a(r)\right) = A'^0_a(r)\frac{\vec{r}}{r} \quad ; \quad \vec{H}_a = 0, \tag{4.42}$$

where the functions $A^0_a(r)$ are the time-like components of the gauge potential in the Lorentz gauge ($\vec{A}_a = 0$) and $A'^0_a = \frac{dA^0_a}{dr}$. When replaced in the field equations (4.36) we are lead to

$$\begin{aligned}
\vec{\nabla}\cdot\left(\frac{\partial\varphi}{\partial X}\vec{E}_a\right) &= -\vec{\nabla}\cdot\left(\frac{\partial\varphi}{\partial X}\vec{\nabla}A^0_a(r)\right) = -\vec{\nabla}\cdot\left(\frac{\partial\varphi}{\partial X}A'^0_a(r)\frac{\vec{r}}{r}\right) = 0\\
\frac{\partial\varphi}{\partial X}\sum_{bc} C_{abc}A^0_b(r)\vec{E}_c &= -\frac{\partial\varphi}{\partial X}\sum_{bc} C_{abc}A^0_b(r)\vec{\nabla}A^0_c(r) =\\
&= -\frac{\partial\varphi}{\partial X}\sum_{bc} C_{abc}A^0_b(r)A'^0_c(r)\frac{\vec{r}}{r} = 0,
\end{aligned}\tag{4.43}$$

where $X = \sum_a \vec{E}_a^2$. The first group of equations has a set of first-integrals of the form

$$r^2 \frac{\partial \varphi}{\partial X} A_a^{\prime 0}(r) = Q_a, \tag{4.44}$$

where the Q_a are integration constants which will be identified below as "source color charges". With the identification $\phi_a(r) \equiv A_a^0(r)$ and $\Lambda_a \equiv Q_a$, these equations coincide with the field equations (3.11) for a multicomponent SSS scalar field theory whose lagrangian density is given by

$$L = f\left(\sum_a \partial_\mu \phi_a \cdot \partial^\mu \phi_a\right) \equiv f(X) = -\varphi(-X, Y = 0). \tag{4.45}$$

Thus the solutions of equations (4.44) are obtained from Eq.(3.57) as

$$|\vec{E}_a(r, Q_b)| = A_a^{\prime 0}(r, Q_b) = \frac{Q_a}{Q}\phi^{'}(r, Q) \quad ; \quad \vec{H}_a = 0, \tag{4.46}$$

where

$$Q = \sqrt{\sum_a Q_a^2}, \tag{4.47}$$

is the mean-square color charge and $\phi^{'}(r, Q)$ is the SSS solution of the associated one-component scalar model defined by a lagrangian density of the form (4.45). These equations can be integrated once, leading to

$$A_a^0(r, Q_b) = \frac{Q_a}{Q}\phi(r, Q) + \chi_a, \tag{4.48}$$

where χ_a are integration constants. These functions must also satisfy the second set of equations (4.43). Owing to the antisymmetry of the structure constants these equations lead to the supplementary restriction

$$\chi_a = \frac{Q_a}{Q}\chi, \tag{4.49}$$

and the final solution of (4.43) is

$$A_a^0(r, Q_b) = \frac{Q_a}{Q}\left(\phi(r, Q) + \chi\right), \tag{4.50}$$

where χ is an arbitrary constant, being now the same for all components of the potential. In terms of the fields the final solution is still given by Eq.(4.46). The integration constants Q_a in this solution must be interpreted as source point-like color charges associated with the different components of the gauge field. Indeed, color charges are defined in general as

$$Q_a = \frac{1}{4\pi}\int d_3\vec{r}\vec{\nabla}\cdot\left(\frac{\partial\varphi}{\partial X}\vec{E}_a + \frac{\partial\varphi}{\partial Y}\vec{H}_a\right), \tag{4.51}$$

which, owing to the field equations (4.36), include now external source charges and charges carried by the field itself. The latter ones come from the integration of the term

$$g\sum_{bc} C_{abc}A_{b\mu}\left[\frac{\partial\varphi}{\partial X}F_c^{\mu\nu} + \frac{\partial\varphi}{\partial Y}F_c^{*\mu\nu}\right], \tag{4.52}$$

which vanishes for the ESS solutions. On the other hand the former are associated to Dirac distributions of weight $4\pi Q_a$, as can be easily seen from the substitution of Eqs.(4.44) in the first set of Eqs.(4.43).

The calculation of the energy associated to these solutions proceeds in the same way as in the electromagnetic case. The integration of Eq.(4.41) gives

$$\begin{aligned}\epsilon_{gf}(Q_a) &= 8\pi\int_0^\infty r^2\frac{\partial\varphi}{\partial X}\left[X = \sum_b \vec{E}_b^2(r,Q_a), Y=0\right]\sum_a \vec{E}_a^2(r,Q_a)dr \\ &- 4\pi\int_0^\infty r^2\varphi\left[X = \sum_a \vec{E}_a^2(r,Q_a), Y=0\right]dr,\end{aligned} \tag{4.53}$$

where the index **gf** stands for gauge field. From this expression it is now straightforward to show that, as in the corresponding multiscalar case, this energy is finite if the energy of the associated scalar solitons is finite, and depends on the charges only through the constant Q, having the same kind of degeneration on spheres of radius Q in the N-dimensional color-charge space. The relation between the finite energies of the solitons with equal mean-square charges in the gauge models and in the associated multiscalar models is given by the same equation (4.30) relating the finite energies in the cases of the abelian models and their associated scalar models.

Concerning the stability of the gauge field solitons we shall show in the next chapter that the finite-energy ESS solutions of admissible generalized non-abelian models are linearly stable if (and only if) the lagrangian density functions satisfy the same criterion (4.32) obtained in the abelian case.

4.3 Discussion

Let us summarize the main conclusions of this chapter: The set of generalized gauge field theories of compact semi-simple Lie groups, whose lagrangian densities are functions $\varphi(X,Y)$ of the field invariants (4.35), satisfying the *admissibility conditions* and the stability criterion (4.32), and supporting finite-energy ESS non-topological soliton solutions, can be split in equivalence classes. Two models belong to the same class if their respective lagrangian densities satisfy the condition

$$\varphi_1(X,0) = \varphi_2(X,0). \tag{4.54}$$

The forms of the ESS soliton solutions and their energies coincide for all the models belonging to the same class. There is a one-to-one correspondence, given by Eq.(4.21), between the set of these classes and the set of *admissible* scalar field models defined by Eq.(3.1) and supporting finite-energy SSS non-topological solitons. The form and energies of the gauge solitons are obtained from those of the corresponding scalar solitons through Eqs.(4.46), (4.48) and (4.30). The analysis and classification of scalar solitons performed in chapter 3 can be immediately generalized to the gauge solitons through this correspondence. Furthermore, the explicit examples supporting scalar solitons, which shall be introduced in chapter 6, can also be extended to the gauge field case simply by including the Y invariant in such a way that the admissibility (4.11) and stability (4.32) constraints be fulfilled by the extended models.

Let us conclude this chapter with some comments concerning an important question which has not been addressed here. It refers to the analysis of propagation of wave-like solutions of these models. As can be easily seen, all these theories exhibit plane wave solutions propagating with the speed of light. But, owing to the non-linear self-coupling, they also support other radiative solutions propagating with more complex dispersion relations. In most cases such waves evolve towards spatially-singular configurations. Roughly speaking, the wave fronts travelling with velocities which are dependent on the values of the fields at every point tend to cumulate, generating discontinuities and shocks after a critical time. Regularly evolving wave solutions of a system of field equations are called *exceptional*. If all the wave solutions of a given system are exceptional, the system is called *completely exceptional* [130]. A detailed analysis of the problem of wave propagation for generalized electromagnetic field models was performed by G. Boillat [111, 112], who established the complete exceptionality of the Born-Infeld electrodynamics. Moreover, BI is the only admissible generalized electromagnetic field theory (with asymptotically coulombian elementary solutions) exhibiting this property [5]. Nevertheless, the Boillat analysis considers only models which satisfy the condition

[5]There is another lagrangian belonging to this class of completely exceptional theories, given by $\varphi(X,Y) = X/Y$. However, this is not an admissible model according to the criteria established in section 4.1.

$$\frac{\partial \varphi}{\partial X}(X = 0, Y = 0) = 1, \tag{4.55}$$

(case B-2) and, consequently, excludes the models belonging to B-1 and B-3 cases. It would be interesting to perform a similar analysis for these cases. However, when one considers the extensions of BI electrodynamics to the non-abelian case or in the Kaluza-Klein context, this exceptionality character is lost [131, 132].

Chapter 5

Stability analysis

We turn now to analyze the stability of the finite-energy solutions of the different models introduced so far. Aside from the fact that stability is a main feature of a soliton, there is another motivation for such an analysis. If soliton solutions are to be useful for modelling particle structure, the analysis of small fluctuations around them is the first step towards a quantization of the modes of the soliton field (see chapter 7 for future perspectives at this regard).

We first discuss the different criteria of stability. We then consider both the static and dynamic stability for each class of problems. In all cases we shall find necessary and sufficient conditions to be imposed on the lagrangian densities of the admissible models for the linear stability of the associated soliton solutions.

5.1 Stability criteria

In the literature there are two main definitions of stability:

- **Strong**: We define the strong stability as the ability of a soliton to maintain its identity under any perturbation or in closed many-soliton configurations [5, 133].
- **Weak**: Identified with usual *linear* stability, i.e. with stability under small perturbations.

Rigorous analysis of stability in the strong sense has been performed for a few field theories in one-space dimension which exhibit conserved discrete topological charges associated with the soliton solutions. In three-space dimensions similar topological conservation laws are responsible for the stability of the 't Hooft-Polyakov monopole solution [6, 7] or the chiral soliton solution of Deser et al. [20]. But satisfactory *general methods* for the analysis of interactions between non-topological solitons and strong external fields in three space dimensions are still lacking and only numerical analysis of the evolution of the solutions can give some insight on this issue for most models. In our context, a tentative approach to this question has been developed by Chernitsky for the Born-Infeld model [134]. It is based on the use of the discontinuity of the field strength at the center of *static* B-I solitons as a marker of the presence and location of the *dynamic* soliton evolving in interaction with strong external fields, or in many-soliton configurations. Since all SSS soliton solutions of the models considered here exhibit similar central field singularities, this procedure might be extended to these cases, but such an extension lies beyond the scope of the present work.

In the case of interactions between solitons and weak external fields (or for widely separated soliton configurations) linear stability *ensures* the identity preservation of the solitons and becomes a basic condition for the consistency of the *low-energy* analysis. The results of this analysis may be interpreted in terms of particle-field (or particle-particle) force laws and describe the radiative behaviour in these processes [134].

Anyway, to perform an analysis of stability in the strong sense, the function $f(X)$ or $\varphi(X,Y)$ should be explicitly fixed before studying the stability of the solutions of a particular theory. However, we shall show that the analysis of stability in the weak sense can be performed without fixing the particular form of $f(X)$ or $\varphi(X,Y)$, since the admissibility criteria, adopted in chapters 3 and 4, together with the finite-energy condition, allow to determine the conditions that must be fulfilled by these models in order for them to support weakly-stable SSS or ESS solutions.

5.2 Stability of one-component scalar solitons

5.2.1 Static stability

We shall begin with the study of the *static* linear stability for the soliton solutions of the scalar models of chapter 3, by analyzing the behaviour of their energy, which must be a minimum against appropriate small perturbations. As emphasized in Ref.[135], this criterion is a sufficient condition for stability, but is not a necessary one. Here we shall not consider the problem of stability of solutions which do not correspond to minima of the energy, a complicated task which deserves a study in itself.

We shall show that, for these models, the conditions of admissibility guarantee this kind of stability for all the finite-energy SSS solutions. We start with the SSS potential $\phi(r)$ and introduce a set of small static perturbations $\delta\phi(\vec{r})$, finite and regular (as well as their first order spatial derivatives) everywhere. We also require the perturbation to leave unchanged the scalar charge associated to the solution. Let us emphasize that this condition is essential for the energy of the soliton to be a minimum. Indeed, without such a condition the perturbation does not necessarily lead to an increase of the energy of the soliton (as can be easily seen by differentiating Eq.(3.12) with respect to small variations of Λ) and the perturbed soliton might evolve towards less energetic states. To first order in the perturbations the modification of this scalar charge, obtained from Eq.(3.5), reads

$$\Delta\Lambda = \frac{1}{4\pi}\int d_3\vec{r}\vec{\nabla}\cdot\left[\overset{\bullet}{f}(X_0)(\vec{\nabla}\delta\phi) - 2\overset{\bullet\bullet}{f}(X_0)\left(\vec{\nabla}\phi\cdot\vec{\nabla}\delta\phi\right)\vec{\nabla}\phi\right] = 0, \tag{5.1}$$

where now $X_0 = -(\vec{\nabla}(\phi))^2 = -\phi'^2(r)$. The condition $\Delta\Lambda = 0$ imposes restrictions on the behaviour of the admissible perturbations at $r = 0$ and as $r \to \infty$. In particular, $\delta\phi(\vec{r})$ must satisfy

$$\lim_{r\to\infty}\frac{\delta\phi(\vec{r})}{\phi(\infty)-\phi(r)} = 0. \tag{5.2}$$

In this manner the perturbed fields remain inside the space of functions defined by the prescribed boundary conditions (on S_∞ in this case) which determine uniquely the solution associated to a given value of the charge. At the center of the soliton $\delta\phi(\vec{r})$ must be regular (see Eq.(5.28) and the analysis of the dynamic stability below).

The first-order perturbation of the energy, calculated by expanding (3.12) becomes

$$\begin{aligned}\Delta_1\epsilon &= 2\int d_3\vec{r}\,\overset{\bullet}{f}(X_0)\vec{\nabla}(\phi)\cdot\vec{\nabla}(\delta\phi) = \\ &= 2\int d_3\vec{r}\vec{\nabla}\cdot\left(\overset{\bullet}{f}(X_0)\delta\phi\vec{\nabla}(\phi)\right) - 2\int d_3\vec{r}\delta\phi\vec{\nabla}\cdot\left(\overset{\bullet}{f}(X_0)\vec{\nabla}(\phi)\right),\end{aligned} \tag{5.3}$$

where a partial integration has been performed. Owing to Eq.(3.4) and the assumed asymptotic behaviour of the perturbation $\delta\phi(\vec{r})$, the two integrals in the last equation converge and cancel each other so that the first variation of the energy vanishes. This is the necessary condition for the energy of the soliton to be an extremum. The second variation reads

$$\Delta_2\epsilon = \int d_3\vec{r}\,\overset{\bullet}{f}(X_0)(\vec{\nabla}\delta\phi)^2 - 2\int d_3\vec{r}\,\overset{\bullet\bullet}{f}(X_0)\left(\vec{\nabla}\phi\cdot\vec{\nabla}\delta\phi\right)^2, \tag{5.4}$$

where, owing to the boundary behaviours of the perturbation and the SSS field itself, both integrals are also convergent. From the arbitrariness of $\delta\phi$, the positivity of $\overset{\bullet}{f}(X)$ and the minimum condition of the energy $\Delta_2\epsilon > 0$, we see that static stability is reached *if* the requirement

$$\overset{\bullet\bullet}{f}(X) < 0, \tag{5.5}$$

is fulfilled in all the range of values of $X = X_0 = -\phi'^2(r)$ covered by the solution. However, if we rewrite Eq.(5.4) as

$$\begin{aligned}\Delta_2\epsilon &= \int d_3\vec{r}\left[\overset{\bullet}{f}(X_0) + 2X_0\overset{\bullet\bullet}{f}(X_0)\right]\left(\frac{\partial\delta\phi}{\partial r}\right)^2 \\ &+ \int d_3\vec{r}\,\overset{\bullet}{f}(X_0)\left[\frac{1}{r^2}\left(\frac{\partial\delta\phi}{\partial\theta}\right)^2 + \frac{1}{r^2\cos^2(\theta)}\left(\frac{\partial\delta\phi}{\partial\varphi}\right)^2\right],\end{aligned} \tag{5.6}$$

we are lead to the less restrictive *static stability criterion* [1]

$$\overset{\bullet}{f}(X_0) + 2X_0\overset{\bullet\bullet}{f}(X_0) \geq 0, \tag{5.7}$$

which is a necessary and sufficient condition for linear stability, as opposed to Eq.(5.5) which is only a sufficient one. This criterion is always fulfilled for admissible models with finite-energy SSS solutions. Indeed, by deriving the first-integral equation (3.4) with respect to r we obtain

$$\overset{\bullet}{f}(X_0) - 2\phi'^2(r)\overset{\bullet\bullet}{f}(X_0) = -\frac{2\Lambda}{r^3\phi''(r)}, \tag{5.8}$$

which, owing to the monotonicity of $\phi'(r)$, is positive in all the range of values of X_0 covered by the solution. We conclude that **the finite-energy SSS solutions of admissible scalar models are statically stable**.

Against perturbations which modify the charge the solitons are unstable, but these instabilities are blocked if charge conservation is implicit in the model (as in the case of generalized gauge field theories considered below) or if it is a consequence of the nature of the external sources.

5.2.2 Dynamic stability

Let us consider now the *dynamic stability* of the SSS solutions. The initial perturbation defines the following Cauchy conditions

$$\Phi(\vec{r}, t=0) = \phi(r) + \delta\phi(\vec{r}) \quad ; \quad \frac{\partial\Phi}{\partial t}(\vec{r}, t=0) = 0, \tag{5.9}$$

[1] The only model for which (5.7) vanishes everywhere is singular and corresponds to the lagrangian $f(X) = \lambda\sqrt{X}$.

for a dynamical problem determining the temporal evolution of the perturbed field $\Phi(\vec{r},t)$, which is governed by the hyperbolic field equations (3.2). At the first order, the evolution of the perturbation

$$\delta\phi(\vec{r},t) = \Phi(\vec{r},t) - \phi(r), \tag{5.10}$$

is given by the linearized scalar field equation

$$\partial_\mu \left(\overset{\bullet}{f}(X_0)\partial^\mu \delta\phi + 2\overset{\bullet\bullet}{f}(X_0)\partial_\nu\phi\partial^\nu\delta\phi\partial^\mu\phi \right) = 0, \tag{5.11}$$

which can be rewritten as

$$\frac{\partial}{\partial t}\left(\overset{\bullet}{f}(X_0)\frac{\partial\delta\phi}{\partial t} \right) - \vec{\nabla}\cdot\left[\overset{\bullet}{f}(X_0)\vec{\nabla}(\delta\phi) - 2\overset{\bullet\bullet}{f}(X_0)\left(\vec{\nabla}\phi\cdot\vec{\nabla}\delta\phi \right)\vec{\nabla}\phi \right] = 0. \tag{5.12}$$

This is the Euler-Lagrange equation associated with the lagrangian density

$$L = \frac{1}{2}\left[\overset{\bullet}{f}(X_0)\partial_\mu\delta\phi\cdot\partial^\mu\delta\phi - 2\overset{\bullet\bullet}{f}(X_0)\left(\vec{\nabla}\phi\cdot\vec{\nabla}\delta\phi \right)^2 \right], \tag{5.13}$$

which is defined everywhere. Equation (5.12) has the form of a local conservation law for a charge density $\eta = \overset{\bullet}{f}(X_0)\frac{\partial\delta\phi}{\partial t}$ which, in integral form, becomes

$$\frac{d}{dt}\int d_3\vec{r}\,\overset{\bullet}{f}(X_0)\frac{\partial\delta\phi}{\partial t} = \int d_3\vec{r}\vec{\nabla}\cdot\left[\overset{\bullet}{f}(X_0)\vec{\nabla}(\delta\phi) - 2\overset{\bullet\bullet}{f}(X_0)\left(\vec{\nabla}\phi\cdot\vec{\nabla}\delta\phi \right)\vec{\nabla}\phi \right] = 0. \tag{5.14}$$

The r.h.s. of this equation is proportional to the first-order perturbation of the scalar charge of the soliton (see Eq.(5.1)) which, consequently, remains conserved as time evolves. Moreover, for solutions satisfying the initial conditions (5.9) the quantity

$$\int d_3\vec{r}\,\overset{\bullet}{f}(X_0)\delta\phi(\vec{r},t), \tag{5.15}$$

remains constant in time.

Equation (5.12), together with the conditions (5.9) and (5.2), outline a spectral problem to which we can apply standard methods. We look for solutions separating time and spatial variables under the form

$$\delta\phi(\vec{r},t,\Gamma) = T(t,\Gamma)\psi(\vec{r},\Gamma), \tag{5.16}$$

where Γ is the separation constant and the eigenfunction $\psi(\vec{r},\Gamma)$ is assumed to satisfy the boundary condition (5.2). Replacing this expression in (5.12) and using the second of the initial conditions (5.9) we are lead to

$$T(t,\Gamma) = \cos(\sqrt{\Gamma}t), \tag{5.17}$$

and

$$\Gamma \overset{\bullet}{f}(X_0)\psi = -\vec{\nabla}\cdot\left[\overset{\bullet}{f}(X_0)\vec{\nabla}(\psi) - 2\overset{\bullet\bullet}{f}(X_0)\phi'^2\left(\frac{\vec{r}}{r}\cdot\vec{\nabla}\psi\right)\frac{\vec{r}}{r}\right]. \tag{5.18}$$

The sign of the eigenvalue Γ is crucial for stability. Multiplying this equation by $\psi(\vec{r},\Gamma)$ and integrating over all space we are lead (after an integration by parts of the right-hand-side) to

$$\Gamma\int d_3\vec{r}\,\overset{\bullet}{f}(X_0)\psi^2 = \int d_3\vec{r}\left[\overset{\bullet}{f}(X_0)(\vec{\nabla}\psi)^2 - 2\overset{\bullet\bullet}{f}(X_0)\phi'^2\left(\frac{\partial\psi}{\partial r}\right)^2\right]. \tag{5.19}$$

Owing to Eq.(5.8), together with the admissibility and boundary conditions, both the r.h.s. of this equation and the coefficient of Γ are finite and positive. Then so is for Γ, and the evolution is oscillatory and bounded in time [2]. Moreover, if we consider two different eigenvalues ($\Gamma_i, i=(1,2)$) and their associated eigenfunctions ($\psi_i = \psi(\vec{r},\Gamma_i), i=(1,2)$) equation (5.18) leads to

$$(\Gamma_2-\Gamma_1)\int d_3\vec{r}\,\overset{\bullet}{f}(X_0)\psi_1\psi_2 = \int d_3\vec{r}\left(\psi_2\vec{\nabla}\cdot\vec{\Sigma_1} - \psi_1\vec{\nabla}\cdot\vec{\Sigma_2}\right), \tag{5.20}$$

where

$$\vec{\Sigma_i} = \overset{\bullet}{f}(X_0)\vec{\nabla}\psi_i - 2\overset{\bullet\bullet}{f}(X_0)\left(\vec{\nabla}\phi\cdot\vec{\nabla}\psi_i\right)\vec{\nabla}\phi. \tag{5.21}$$

After a partial integration and making use of the boundary conditions we see that the right-hand-side of (5.20) vanishes and thus we are lead to the orthogonality relation

$$\int d_3\vec{r}\,\overset{\bullet}{f}(X_0)\psi_1\psi_2 = 0. \tag{5.22}$$

These results outline a Sturm-Liouville problem for each admissible scalar model of the form (3.1) supporting finite-energy SSS solutions and lead to the following conclu-

[2]Note that the r.h.s. of (5.19) coincides with the second variation of the energy associated to the eigenfunction $\psi(\vec{r},\Gamma)$ and has the same sign as Γ. This establishes a strict correspondence between static and dynamic stabilities of the SSS solutions.

sions [136]:

1. The analysis of the dynamics of the small oscillations around these solutions leads in all cases to discrete spectra of eigenvalues Γ_i,

2. The associated eigenfunctions are orthogonal and finite-norm with respect to the scalar product

$$< \psi_i, \psi_j >= \int d_3\vec{r} \overset{\bullet}{f} (X_0)\psi_i(\vec{r})\psi_j(\vec{r}) = \frac{\Delta_2 \epsilon_i}{\Gamma_i}\delta_{ij}, \tag{5.23}$$

defined with the kernel $\overset{\bullet}{f} (X_0) > 0$. Such functions generate a complete Hilbert space in which any perturbation can be expanded.

On the other hand, we can now separate the spatial eigenfunctions in radial and angular coordinates as

$$\psi(r, \vartheta, \varphi, \Gamma, l) = R(r, \Gamma, l)\, Y_l(\vartheta, \varphi), \tag{5.24}$$

where the angular components are the usual spherical harmonics satisfying

$$\sin\vartheta \frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial Y_l}{\partial\vartheta}\right) + \frac{\partial^2 Y_l}{\partial\varphi^2} + l(l+1)\sin^2\vartheta\, Y_l = 0. \tag{5.25}$$

The radial components obey to the equation

$$\frac{d}{dr}\left(\frac{1}{r\phi''}\frac{dR}{dr}\right) + \frac{l(l+1) - \Gamma r^2}{2r^2\phi'}R = 0, \tag{5.26}$$

which is obtained from (5.18), (5.24) and using the first-integral (3.4). These equations have the standard Sturm-Liouville form [136]

$$\begin{aligned} Ly &+ \lambda\rho(x)y = 0 \\ Ly &= \frac{d}{dx}\left[k(x)\frac{dy}{dx}\right] - q(x)y \quad \text{with } k(x) > 0, \rho(x) > 0. \end{aligned} \tag{5.27}$$

The asymptotic behaviour of $R(r)$ is obtained from the asymptotic form of the admissible solitons ($\phi'(r \to \infty) \sim 1/r^p; p > 1$) through

$$\frac{d^2R}{dr^2} + \frac{p}{r}\frac{dR}{dr} - p\frac{(l(l+1) - \Gamma r^2)}{2r^2}R = 0, \tag{5.28}$$

which is a Lommel equation and can be solved in terms of Bessel functions [137]. For large r we can assume for the solution the asymptotic form

$$R(r \to \infty) \sim \frac{\varrho(r)}{r^q}, \tag{5.29}$$

where $\varrho(r)$ is a bounded function and, owing to the boundary condition (5.2), q is restricted to be $q > p > 1$. By neglecting the higher-order terms in $1/r$ in the resulting equation for ρ we are lead to

$$\varrho'' + \frac{p\Gamma}{2}\varrho = 0. \tag{5.30}$$

Owing to the positivity of Γ the solution of this equation is oscillatory and the asymptotic behaviour of the eigenfunctions is given by [3]

$$R(r \to \infty) \sim \frac{\cos\left(\sqrt{\frac{p\Gamma}{2}}r + \chi\right)}{r^q}, \tag{5.31}$$

where χ is a constant phase. This asymptotic form of the eigenfunctions makes the integral of the second variation of the energy (5.6) to converge in the $r \to \infty$ limit.

To determine the behaviour of $R(r)$ around the center of the soliton we must consider separately cases A-1 and A-2. Let us assume in both cases a form

$$R(r \to 0) \sim \alpha - \beta r^q. \tag{5.32}$$

In case A-1 ($\phi'(r \to 0) \sim 1/r^p, 0 < p < 1$) equation (5.26) becomes

$$2\beta q \frac{p+q-1}{p} r^q + \left(l(l+1) - \Gamma r^2\right)(\alpha - \beta r^q) \approx 0. \tag{5.33}$$

For the first surface spherical harmonic ($l = 0$) we are lead to [138]

$$\alpha = 0 \quad ; \quad q = 1 - p, \tag{5.34}$$

or

$$\frac{\beta}{\alpha} = \frac{\Gamma p}{4(p+1)} \quad ; \quad q = 2. \tag{5.35}$$

For $l \neq 0$ we obtain

[3] The value of q as a function of p and l can be explicitly obtained from recursion formulae methods [138] applied to Eq.(5.28), which also allow for the approximate determination of the eigenvalue spectrum of Γ.

$$\alpha = 0 \;\; ; \;\; q = \frac{1 - p + \sqrt{(1-p)^2 + 2l(l+1)p}}{2}. \tag{5.36}$$

In <u>case A-2</u> ($\phi^{'}(r \to 0) \sim a - br^{\sigma}, \sigma > 0$) equation (5.26) becomes

$$2\beta q \frac{q - \sigma - 1}{b\sigma} r^{q-\sigma} + \frac{l(l+1) - \Gamma r^2}{a} (\alpha - \beta r^q) \approx 0. \tag{5.37}$$

If $l = 0$ we must have

$$\frac{\beta}{\alpha} = \frac{b}{2a}\frac{\Gamma\sigma}{\sigma + 2} \;\; ; \;\; q = \sigma + 2, \tag{5.38}$$

or

$$\alpha = \;\; ; \;\; q = \sigma + 1. \tag{5.39}$$

For $l \neq 0$

$$\frac{\beta}{\alpha} = \frac{b}{2a} l(l+1) \;\; ; \;\; q = \sigma, \tag{5.40}$$

or

$$\alpha = 0 \;\; ; \;\; q = \sigma + 1. \tag{5.41}$$

In all these cases the integral of the second order variation of the energy (5.6) can be shown to converge in the limit $r \to 0$.

We then conclude that all finite-energy SSS solutions of admissible scalar models, which are *statically* stable, are also *dynamically* stable.

5.3 Stability of multicomponent scalar solitons

The analysis of stability in the scalar case can be extended to the multicomponent scalar fields. Now we have N integration constants Λ_i and a degeneration in energy of the SSS solutions on the sphere of radius $\Lambda = \sqrt{\sum_{i=1}^{N} \Lambda_i^2}$ in $\Re^N$. Obviously the variations of the energy vanish for perturbations which remain inside this sphere obtained by modifying the constants Λ_i in equation (3.57) in such a way that the "total mean-square charge" Λ remains unchanged. The asymptotic boundary conditions satisfied by the fields $\phi_i^{'}(r, \Lambda_j)$ (obtained from the asymptotic behaviour of the associated one-component scalar field solution $\phi^{'}(r, \Lambda)$ through Eq.(3.57)) are modified by these perturbations and the associated charges (defined from Eq.(3.58)) as Λ_i) are modified. Consequently, charge conservation condition blocks such perturbations and prevents a soliton from evolving spontaneously towards another equal-energy configuration in the sphere.

For general perturbations $\delta\phi_i(\vec{r})$ the first-order modifications of the scalar charges take the form

$$\Delta\Lambda_i = \frac{1}{4\pi}\int d_3\vec{r}\vec{\nabla}\cdot\left[\overset{\bullet}{f}(X_0)(\vec{\nabla}\delta\phi_i) - 2\overset{\bullet\bullet}{f}(X_0)\sum_{j=1}^{N}\left(\vec{\nabla}\phi_j\cdot\vec{\nabla}\delta\phi_j\right)\vec{\nabla}\phi_i,\right], \quad (5.42)$$

with $X = -\sum_{i=1}^{N}\phi_i'^2$. The requirement of charge conservation ($\Delta\Lambda_i = 0$) imposes boundary conditions on the perturbing fields which, as in the one-component case, must vanish asymptotically faster than the SSS fields themselves.

The first-order variation of the energy functional is obtained from the integral of Eq.(3.52) and reads

$$\Delta_1\epsilon = 2\sum_{i=1}^{N}\frac{\Lambda_i}{\Lambda}\left(\int d_3\vec{r}\nabla\cdot\left[\overset{\bullet}{f}(X_0)\delta\phi_i\vec{\nabla}\phi\right] - \int d_3\vec{r}\delta\phi_i\vec{\nabla}\cdot\left[\overset{\bullet}{f}(X_0)\vec{\nabla}(\phi)\right]\right), \quad (5.43)$$

where Eq.(3.59) has been used and a partial integration has been performed. Each term of this sum has the form of Eq.(5.3) and vanishes because of the same reasons. Thus the first variation of the energy vanishes, which is an extremum condition. The second variation of the energy functional takes the form

$$\Delta_2\epsilon = \int d_3\vec{r}\left[\overset{\bullet}{f}(X_0)\sum_{i=1}^{N}\left(\vec{\nabla}\delta\phi_i\right)^2 - 2\phi'^2\overset{\bullet\bullet}{f}(X_0)\left(\sum_{i=1}^{N}\frac{\Lambda_i}{\Lambda}\frac{\partial\delta\phi_i}{\partial r}\right)^2\right], \quad (5.44)$$

and can be written as

$$\begin{aligned}\Delta_2\epsilon &= \int d_3\vec{r}\left[\overset{\bullet}{f}(X_0) + 2X_0\overset{\bullet\bullet}{f}(X_0)\cos^2(\Omega)\right]\sum_{i=1}^{N}\left(\frac{\partial\delta\phi_i}{\partial r}\right)^2 + \quad (5.45)\\ &+ \int d_3\vec{r}\overset{\bullet}{f}(X_0)\sum_{i=1}^{N}\left[\frac{1}{r^2}\left(\frac{\partial\delta\phi_i}{\partial\theta}\right)^2 + \frac{1}{r^2\cos^2(\theta)}\left(\frac{\partial\delta\phi_i}{\partial\varphi}\right)^2\right],\end{aligned}$$

in terms of the angle $\Omega(\vec{r})$ in the internal space between the vector formed by the radial derivatives of the components of the perturbing fields and the direction $\frac{\Lambda_i}{\Lambda}$ defined by the SSS solution . The second integral in this equation is always positive. If $\ddot{f}(X_0)$ is negative the first integral is also positive. Otherwise we have

$$\dot{f}(X_0) + 2X_0\,\ddot{f}(X_0)\cos^2(\Omega) \geq \dot{f}(X_0) + 2X_0\,\ddot{f}(X_0) \geq 0, \tag{5.46}$$

and, owing to Eq.(5.8), this integral is always positive for admissible many-components scalar models with finite-energy SSS solutions. Consequently, all these solutions are statically stable. Moreover, the analysis of the dynamical evolution of small perturbations performed for scalar solitons can be straightforwardly generalized to this multicomponent case. Such an analysis proves the dynamical stability of these solitons.

5.4 Stability of generalized electromagnetic solitons

We turn now to analyze the linear stability of generalized gauge field solitons. Methods applied for scalar solitons in looking for the conditions for these solutions to be weakly stable can be extended to generalized electromagnetic solitons. In this case, due to gauge transformations, the analysis is quite more involved. Anyway, as we shall see, following a similar procedure as in the scalar case we will obtain *supplementary* conditions to be fulfilled by the admissible lagrangian densities in order for their soliton solutions to be stable against small charge-preserving perturbations.

5.4.1 Static stability

A similar analysis of static stability as in the scalar case can be performed for generalized abelian gauge fields. We consider a finite-energy ESS solution [4] of the field equations (4.15) ($\vec{E}_0(r), \vec{H}_0 = 0$) and introduce a small perturbing field

$$\begin{aligned} \vec{E}_1(\vec{r}) &= -\frac{\partial A_0(\vec{r})}{\partial t} - \vec{\nabla}\vec{A}_1(\vec{r}) \\ \vec{H}_1(\vec{r}) &= \vec{\nabla} \times \vec{A}_1(\vec{r}), \end{aligned} \tag{5.47}$$

which does not modify the total electric charge of the soliton. The first-order modification of the charge density is obtained by perturbing the first of the field equations (4.17), which leads to

[4] To avoid difficulties related to the gauge determination we work directly with the fields. When the use of the potentials becomes necessary in some step of the calculation we will fix the gauge through appropriate conditions.

$$\vec{\nabla}\cdot\vec{\sigma}=0, \tag{5.48}$$

where

$$\vec{\sigma}=\frac{\partial\varphi}{\partial X_0}\vec{E}_1+2\frac{\partial^2\varphi}{\partial X_0^2}(\vec{E}_0\cdot\vec{E}_1)\vec{E}_0, \tag{5.49}$$

and the index 0 in the derivatives means that they are calculated for the unperturbed solution (note that, owing to the parity invariance, the odd partial derivatives of φ with respect to Y vanish in $Y=0$). From the integration of (5.48), which leads to the requirement of conservation of the electric charge

$$\delta q=\frac{1}{4\pi}\int d^3\vec{r}\vec{\nabla}\cdot\vec{\sigma}=0, \tag{5.50}$$

we see that $\vec{\sigma}$ must vanish asymptotically faster than r^{-2}. Then, the regular perturbing field $|\vec{E}_1(\vec{r})|$ must be damped faster than $E_0(r)$ itself. This boundary condition is similar to the one introduced for scalar models but, owing to the electric charge conservation implicit in the field equations, the physical meaning becomes here more transparent.

Let us now consider the variations of the energy functional under such charge-preserving perturbations. The first-order variation is obtained from the integration of Eq.(4.7) and reads

$$\Delta_1\epsilon=-2\int d_3\vec{r}\vec{\nabla}\cdot\left[A^0\vec{\sigma}\right]+2\int d_3\vec{r}A^0\vec{\nabla}\cdot\vec{\sigma}, \tag{5.51}$$

where we have introduced the time-like component of the four-vector potential for the solution ($\vec{E}_0=-\vec{\nabla}A^0, \vec{A}=0$). This expression vanishes due to the boundary conditions and the linearized field equation (5.48). This is an extremum condition.

In calculating the second variation of the energy let us expand the first of the field equations (4.17) to the second-order. We are lead to

$$\vec{\nabla}\cdot(\vec{\sigma}+\vec{\eta})=0, \tag{5.52}$$

where now the term

$$\begin{aligned}\vec{\eta} &= 2\frac{\partial^2\varphi}{\partial X_0^2}(\vec{E}_0\cdot\vec{E}_1)\vec{E}_1+\frac{\partial^2\varphi}{\partial X_0^2}(\vec{E}_1^2-\vec{H}_1^2)\vec{E}_0+2\frac{\partial^3\varphi}{\partial X_0^3}(\vec{E}_0\cdot\vec{E}_1)^2\vec{E}_0+\\ &+ 2\frac{\partial^3\varphi}{\partial X_0\partial Y_0^2}(\vec{E}_0\cdot\vec{H}_1)^2\vec{E}_0+2\frac{\partial^2\varphi}{\partial Y_0^2}(\vec{E}_0\cdot\vec{H}_1)\vec{H}_1,\end{aligned} \tag{5.53}$$

includes the second-order corrections. Using this equation the second variation of the energy, obtained from the integration of Eq.(4.7), becomes

$$\begin{aligned}\Delta_2\epsilon &= \int d_3\vec{r}\left[\frac{\partial\varphi}{\partial X_0}\vec{E}_1^2 + 2\frac{\partial^2\varphi}{\partial X_0^2}(\vec{E}_0\cdot\vec{E}_1)^2\right] + \int d_3\vec{r}\left[\frac{\partial\varphi}{\partial X_0}\vec{H}_1^2 - 2\frac{\partial^2\varphi}{\partial Y_0^2}(\vec{E}_0\cdot\vec{H}_1)^2\right] - \\ &- 2\int d_3\vec{r}\vec{\nabla}\cdot\left[A^0\vec{\eta}\right]. \end{aligned} \tag{5.54}$$

The last integral in the r.h.s. of this equation vanishes, owing to the boundary conditions. The first term is positive if $\frac{\partial^2\varphi}{\partial X_0^2} \geq 0$; on the other hand, if $\frac{\partial^2\varphi}{\partial X_0^2} < 0$ the integrand of this term can be written as

$$(\vec{E}_1)^2\left(\frac{\partial\varphi}{\partial X_0} + 2\frac{\partial^2\varphi}{\partial X_0^2}(\vec{E}_0^2\cos^2(\theta))\right) \geq (\vec{E}_1)^2\left(\frac{\partial\varphi}{\partial X_0} + 2\frac{\partial^2\varphi}{\partial X_0^2}\vec{E}_0^2\right), \tag{5.55}$$

where θ is the angle between $\vec{E}_0$ and $\vec{E}_1$. By deriving Eq.(4.20) with respect to r and taking into account the monotonicity of $E_0(r)$ we see that the first term in (5.54) is always positive. Concerning the second term of (5.54) it is positive if $\frac{\partial^2\varphi}{\partial Y_0^2} \leq 0$ while if $\frac{\partial^2\varphi}{\partial Y_0^2} > 0$ we can write its integrand as

$$(\vec{H}_1)^2\left(\frac{\partial\varphi}{\partial X_0} - 2\frac{\partial^2\varphi}{\partial Y_0^2}(\vec{E}_0^2\cos^2(\theta))\right) \geq (\vec{H}_1)^2\left(\frac{\partial\varphi}{\partial X_0} - 2\frac{\partial^2\varphi}{\partial Y_0^2}(\vec{E}_0^2)\right). \tag{5.56}$$

From the arbitrariness of the perturbing fields, the positivity of this term and, finally, the positivity of the second variation of the energy requires the condition

$$\frac{\partial\varphi}{\partial X} \geq 2X\frac{\partial^2\varphi}{\partial Y^2}, \tag{5.57}$$

to be fulfilled in the range of values of X ($Y = 0$) where the ESS solutions are defined. This is a *necessary and sufficient condition of static stability* to be satisfied by the lagrangian densities of *admissible* models supporting finite-energy ESS solutions. This stability criterion goes beyond the widely used Derrick's *necessary conditions* [21].

Let us check, using this criterion, the linear stability of the electrostatic finite-energy solutions of the BI model. From Eqs. (4.5) and (5.57) we immediately obtain

$$\frac{\partial\varphi}{\partial X_0} - 2X_0\frac{\partial^2\varphi}{\partial Y_0^2} = \frac{1}{8\pi}\left(1 - \mu^2 X_0\right)^{1/2} > 0, \tag{5.58}$$

and the stability condition (5.57) is fulfilled since the ESS field is bounded everywhere

$(X_0 < 1/\mu^2)$.

5.4.2 Dynamic stability

Let us now analyze the dynamical evolution of the small perturbations of the ESS solitons. The system of linearized field equations, obtained by expanding (4.17) to first order, is formed by Eq.(5.48) aside from a vector equation:

$$\begin{aligned} \vec{\nabla} \cdot \vec{\sigma} = \vec{\nabla} \cdot (\mathbf{\Sigma} \cdot \vec{E}_1) &= 0 \\ \frac{\partial \vec{\sigma}}{\partial t} - \vec{\nabla} \times (\mathbf{\Omega} \cdot \vec{H}_1) &= \frac{\partial}{\partial t}(\mathbf{\Sigma} \cdot \vec{E}_1) - \vec{\nabla} \times (\mathbf{\Omega} \cdot \vec{H}_1) = 0, \end{aligned} \tag{5.59}$$

where now we have introduced the symmetric tensors

$$\begin{aligned} \mathbf{\Sigma} &= \frac{\partial \varphi}{\partial X_0} \mathbb{I}_3 + 2 \frac{\partial^2 \varphi}{\partial X_0^2} (\vec{E}_0 \otimes \vec{E}_0) \\ \mathbf{\Omega} &= \frac{\partial \varphi}{\partial X_0} \mathbb{I}_3 - 2 \frac{\partial^2 \varphi}{\partial Y_0^2} (\vec{E}_0 \otimes \vec{E}_0), \end{aligned} \tag{5.60}$$

which will be useful in simplifying the notations in the sequel. The perturbing fields must also satisfy the first set of Maxwell equations. Expanding (4.18) up to the first order we obtain

$$\begin{aligned} \vec{\nabla} \times \vec{E}_1 &= -\frac{\partial \vec{H}_1}{\partial t} \\ \vec{\nabla} \cdot \vec{H}_1 &= 0. \end{aligned} \tag{5.61}$$

Let us look for solutions of these equations which are products of functions of time and space variables for both electric and magnetic fields, of the form

$$\begin{aligned} \vec{E}_1(t, \vec{r}) &= T_e(t) \cdot \vec{e}(\vec{r}) \\ \vec{H}_1(t, \vec{r}) &= T_h(t) \cdot \vec{h}(\vec{r}). \end{aligned} \tag{5.62}$$

In this way, by substituting (5.62) into the second of Eqs.(5.59) and the first of (5.61), we are lead to the first-order equations for the time variables

$$\begin{aligned} \dot{T}_e &= \lambda T_h \\ \dot{T}_h &= -\mu T_e, \end{aligned} \tag{5.63}$$

where λ and μ are separation constants. By deriving these equations we are lead to the system

$$\begin{aligned} \ddot{T}_e(t) + \Gamma T_e(t) &= 0 \\ \ddot{T}_h(t) + \Gamma T_h(t) &= 0, \end{aligned} \tag{5.64}$$

with $\Gamma = \lambda \cdot \mu$. The identification $\lambda = \mu = \sqrt{\Gamma}$ can be introduced without loss of generality, as a consequence of the first-order equations and *the positivity of* Γ, which will be established below. The eigenvalue Γ being positive, the solutions (normalized to unity) take the form

$$\begin{aligned} T_e(t) &= \cos(\sqrt{\Gamma}t + \delta) \\ T_h(t) &= \sin(\sqrt{\Gamma}t + \delta), \end{aligned} \tag{5.65}$$

where δ is the same constant phase for both solutions. In this case the eigenfunctions remain bounded as time evolves and the soliton is dynamically stable. The field equations for the spatial components are

$$\begin{aligned} \vec{\nabla} \cdot (\mathbf{\Sigma} \cdot \vec{e}) &= 0 \\ \vec{\nabla} \times (\mathbf{\Omega} \cdot \vec{h}) &= \sqrt{\Gamma}\mathbf{\Sigma} \cdot \vec{e}, \end{aligned} \tag{5.66}$$

where we have used the definitions (5.60). Note that the first of Eqs.(5.66) is an immediate consequence of the second one. Moreover, the first set of Maxwell equations leads to

$$\begin{aligned} \vec{\nabla} \times \vec{e} &= \sqrt{\Gamma}\vec{h} \\ \vec{\nabla} \cdot \vec{h} &= 0. \end{aligned} \tag{5.67}$$

where again the second equation is a trivial consequence of the first one. We shall now introduce a four-potential A^μ for the perturbing fields, defined in the Hamilton gauge

($A^0 = 0$) in such a way that [5]

$$\begin{aligned} \vec{E}_1 &= -\frac{\partial \vec{A}}{\partial t} \\ \vec{H}_1 &= \vec{\nabla} \times \vec{A}. \end{aligned} \tag{5.68}$$

This vector potential is determined up to the gradient of an arbitrary time-independent scalar field. In terms of this vector potential, the first set of field equations (5.61) are identically satisfied while the second set (5.59) becomes

$$\begin{aligned} \frac{\partial}{\partial t} &\left(\vec{\nabla} \cdot (\mathbf{\Sigma} \cdot \vec{A})\right) = 0, \\ \frac{\partial^2}{\partial t^2} &(\mathbf{\Sigma} \cdot \vec{A}) + \vec{\nabla} \times \left(\mathbf{\Omega} \cdot (\vec{\nabla} \times \vec{A})\right) = 0, \end{aligned} \tag{5.69}$$

If we write equations (5.68) for the separated functions (5.62), by integrating in time the first one and using (5.63), we obtain the general form of the vector potential *for these functions* as

$$\vec{A}(t, \vec{r}) = T_a(t)\vec{a}(\vec{r}) + \vec{\nabla}\phi(\vec{r}), \tag{5.70}$$

where $\phi(\vec{r})$ is a time-independent function and

$$T_a(t) = \frac{T_h(t)}{\sqrt{\Gamma}} \quad ; \quad \vec{a}(\vec{r}) = \vec{e}(\vec{r}). \tag{5.71}$$

This vector potential, determined in Eq.(5.70) up to the gradient of a time-independent scalar field, becomes univocally fixed by requiring its form to separate in time and space variables, taking the form of the first term in the r.h.s of Eq.(5.70). In terms of this potential, using the definitions (5.60), the field equations for the spatial part of the perturbation reduce to the unique vector equation

$$\vec{\nabla} \times \left(\mathbf{\Omega} \cdot (\vec{\nabla} \times \vec{a})\right) = \Gamma \mathbf{\Sigma} \cdot \vec{a}, \tag{5.72}$$

which outlines the *eigenvalue problem* for the linear oscillations in this electromagnetic case.

As in the scalar case, the standard analysis of this problem can be performed for the ESS solitons of admissible generalized electromagnetic field models without any reference to the explicit form of the lagrangian density. In this way we shall show that two given

[5]This gauge-fixing condition is allowed by the gauge invariance of the equations for the perturbing fields (5.59)-(5.61) which are independent from any gauge choice for the unperturbed fields.

eigenfunctions $\vec{a}_1(\vec{r})$ and $\vec{a}_2(\vec{r})$ associated to the eigenvalues Γ_1 and Γ_2, respectively, are orthogonal and finite-norm with respect to the scalar product

$$< \vec{a}_1 \cdot \vec{a}_2 >= \zeta \int d_3\vec{r}(\vec{a}_1 \cdot \mathbf{\Sigma} \cdot \vec{a}_2), \tag{5.73}$$

where ζ is a normalizing factor. Indeed, multiplying Eq.(5.72) (defined for a given eigenfunction $\vec{a}_1$) by an eigenfunction $\vec{a}_2$, integrating in space, using the identity

$$\vec{a}_2 \cdot \vec{\nabla} \times \left(\mathbf{\Omega} \cdot (\vec{\nabla} \times \vec{a}_1)\right) = \vec{\nabla} \cdot \left[\vec{a}_2 \times \left(\mathbf{\Omega} \cdot (\vec{\nabla} \times \vec{a}_1)\right)\right] + (\vec{\nabla} \times \vec{a}_1) \cdot \mathbf{\Omega} \cdot (\vec{\nabla} \times \vec{a}_2), \tag{5.74}$$

and taking into account the symmetry of the tensors $\mathbf{\Sigma}$ and $\mathbf{\Omega}$, we obtain the equation

$$\int d_3\vec{r}\vec{\nabla} \cdot \left[\vec{a}_2 \times \left(\mathbf{\Omega} \cdot (\vec{\nabla} \times \vec{a}_1)\right)\right] + \int d_3\vec{r}(\vec{\nabla} \times \vec{a}_1) \cdot \mathbf{\Omega} \cdot (\vec{\nabla} \times \vec{a}_2) = \Gamma_1 \int d_3\vec{r}(\vec{a}_1 \cdot \mathbf{\Sigma} \cdot \vec{a}_2). \tag{5.75}$$

Owing to the boundary conditions the first integral in the l.h.s of this equation vanishes while the second one converges. By permuting the indices and subtracting we finally obtain

$$(\Gamma_1 - \Gamma_2) \int d_3\vec{r}(\vec{a}_1 \cdot \mathbf{\Sigma} \cdot \vec{a}_2) = (\Gamma_1 - \Gamma_2) < \vec{a}_1 \cdot \vec{a}_2 >= 0. \tag{5.76}$$

We see that the eigenfunctions associated to different eigenvalues are orthogonal with respect to the scalar product (5.73). Moreover, if $\vec{a}_1 = \vec{a}_2$ we obtain from (5.75)

$$\Gamma \int d_3\vec{r}(\vec{a} \cdot \mathbf{\Sigma} \cdot \vec{a}) = \int d_3\vec{r}(\vec{\nabla} \times \vec{a}) \cdot \mathbf{\Omega} \cdot (\vec{\nabla} \times \vec{a}), \tag{5.77}$$

and both integrals converge. The integrand in the l.h.s. of this equation can be written as

$$(\vec{a} \cdot \mathbf{\Sigma} \cdot \vec{a}) = \frac{\partial \varphi}{\partial X_0}\left((\vec{a})^2 - \left(\vec{a} \cdot \frac{\vec{r}}{r}\right)^2\right) + \left(\frac{\partial \varphi}{\partial X_0} + 2\vec{E}_0^2 \frac{\partial^2 \varphi}{\partial X_0^2}\right)\left(\vec{a} \cdot \frac{\vec{r}}{r}\right)^2, \tag{5.78}$$

and is positive for ESS soliton solutions of any admissible electromagnetic model. The integrand of the r.h.s. of (5.77) takes the form

$$
\begin{aligned}
(\vec{\nabla} \times \vec{a}) \cdot \mathbf{\Omega} \cdot (\vec{\nabla} \times \vec{a}) &= \frac{\partial \varphi}{\partial X_0} \left((\vec{\nabla} \times \vec{a})^2 - \left(\frac{\vec{r}}{r} \cdot \vec{\nabla} \times \vec{a} \right)^2 \right) \\
&+ \left(\frac{\partial \varphi}{\partial X_0} - 2\vec{E}_0^2 \frac{\partial^2 \varphi}{\partial Y_0^2} \right) \left(\frac{\vec{r}}{r} \cdot \vec{\nabla} \times \vec{a} \right)^2 . \qquad (5.79)
\end{aligned}
$$

This term is positive *if (and only if)* the condition for static stability (5.57) is fulfilled. Under this condition the eigenvalues Γ are well defined and positive and, consequently, the behaviour of any initially bounded perturbation remains bounded as time evolves. We conclude that the statically stable ESS solitons of admissible electromagnetic models which satisfy (5.57) are also *dynamically stable*. Moreover, the spectrum of eigenvalues is discrete and the eigenfunctions generate the functional space of the *physical vector potentials*, which can be written as

$$
\vec{A}(t, \vec{r}) = \sum_n C_n \sin(\sqrt{\Gamma_n} t + \delta_n) \vec{a}_n(\vec{r}) \qquad (5.80)
$$

and are in a one-to-one correspondence with the physical perturbed states of the soliton. Indeed, any charge-preserving perturbation of the soliton field is described, in the Hamilton gauge, by vector potentials which can be obtained from one (and only one) of the form (5.80) by the addition of gradients of time-independent scalar functions.

The analysis of the spatial structure of the eigenfunctions and physical perturbations can now be performed by separating in radial and angular parts the components of the vector potentials $\vec{a}_n(\vec{r})$ in the natural basis of the polar coordinate system. This procedure, which is standard in spherically-symmetric physical problems [139, 140], will determine the asymptotic and central-behaviour of the perturbing fields, as in the scalar case already considered. We shall leave this study for future developments.

5.5 Stability of generalized non-abelian gauge solitons

Owing to the essential self-interactions involving the field potentials, the treatment of the static and dynamic stability of the solitons for generalized non-abelian gauge models is more involved than in the abelian case. A detailed study of the stability of some extended static finite-energy solutions for the standard Yang-Mills model has been performed in Ref. [135], where similar difficulties arise. Most methods of that work can be generalized to the present situation and we shall follow this way in analyzing the stability behaviour of the ESS solutions (4.46).

Consider a finite-energy ESS solution of the field equations (4.36) of the form (4.46) and introduce small regular perturbing fields through the definitions

$$\begin{aligned} A_a^0(r) &\rightarrow A_a^0 + \delta A_a^0(\vec{r}) \\ \vec{A}_a = 0 &\rightarrow \delta \vec{A}_a(\vec{r}) \\ \vec{E}_a(r) = -\vec{\nabla} A_a^0(r) &\rightarrow \vec{E}_a + \delta \vec{E}_a(\vec{r}) \quad (5.81) \\ \vec{H}_a = 0 &\rightarrow \delta \vec{H}_a(\vec{r}). \quad (5.82) \end{aligned}$$

To first order these fields are related through

$$\begin{aligned} \delta \vec{E}_a(\vec{r}) &= -\vec{\nabla} \delta A_a^0 - \frac{\partial \delta \vec{A}_a}{\partial t} - g \sum_{bc} C_{abc} \delta \vec{A}_b A_c^0 \\ \delta \vec{H}_a(\vec{r}) &= \vec{\nabla} \times \delta \vec{A}_a, \end{aligned} \quad (5.83)$$

and they are assumed to leave invariant the color charges Q_a associated to the unperturbed solution. To first-order the modifications of the charge densities are obtained from the perturbation of the time-components of the field equations (4.36) (the generalized Gauss laws) and read

$$\vec{\nabla} \cdot \vec{\sigma}_a = -g \sum_{bc} C_{abc} \frac{\partial \varphi}{\partial X_0} \delta \vec{A}_b \cdot \vec{E}_c, \quad (5.84)$$

where

$$\vec{\sigma}_a = \frac{\partial \varphi}{\partial X_0} \delta \vec{E}_a + 2 \frac{\partial^2 \varphi}{\partial X_0^2} \left(\sum_p \vec{E}_p \cdot \delta \vec{E}_p \right) \vec{E}_a, \quad (5.85)$$

and the modifications of the total charges read

$$\Delta Q_a = \frac{1}{4\pi} \int d_3 \vec{r} \vec{\nabla} \cdot \vec{\sigma}_a = -g \sum_{bc} C_{abc} \int d_3 \vec{r} \frac{\partial \varphi}{\partial X_0} \delta \vec{A}_b \cdot \vec{E}_c = 0. \quad (5.86)$$

The first-order perturbations of the vector equations, given by the spatial components of Eqs.(4.36) (the generalized Ampère laws) read

$$-\frac{\partial}{\partial t} \vec{\sigma}_a + \vec{\nabla} \times \vec{\omega}_a = -g \sum_{bc} C_{abc} \frac{\partial \varphi}{\partial X_0} \left[\delta A_b^0 \vec{E}_c + A_b^0 \delta \vec{E}_c \right], \quad (5.87)$$

where

$$\vec{\omega}_a = \frac{\partial\varphi}{\partial X_0}\delta\vec{H}_a - 2\frac{\partial^2\varphi}{\partial Y_0^2}\left(\sum_p \vec{E}_p\cdot\delta\vec{E}_p\right)\vec{E}_a. \tag{5.88}$$

The r.h.s. in Eq.(5.84) is the color-charge density carried by the perturbations and its spatial integral must vanish, according to our initial assumptions. This requirement restricts the asymptotic behaviour of $\vec{\sigma}_a$ and, as for the other field models already considered, leads to boundary conditions to be satisfied by the perturbing fields. Moreover, similarly to the multi-component scalar case, the color charges of the unperturbed solution (Q_a) fix a direction in the color-charge space (called in Ref. [135] "electromagnetic" direction, while the orthogonal directions are termed "charged"). Owing to the first-integral equation (4.44), the potentials A_a^0 and the fields $\vec{E}_a$ of the ESS solutions lie in this direction. For the perturbing fields to remain purely *electromagnetic* the associated charge densities that are induced by them must vanish and, owing to Eq.(5.84), $\delta\vec{A}_b$ must also lie in this direction. In what follows, we shall prove the stability of the finite-energy ESS solutions against this kind of non-charged perturbations.

Now let us analyze the variations of the energy functional. The first variation is obtained by perturbing the spatial integral of (4.41) (this is a gauge-invariant quantity, as well as its variations) around the ESS solutions, which reads

$$\Delta_1\epsilon = -2\int d_3\vec{r}\sum_a \vec{\nabla}\cdot\left[A_a^0\vec{\sigma}_a\right] + 2\int d_3\vec{r}\sum_a A_a^0\vec{\nabla}\cdot\vec{\sigma}_a. \tag{5.89}$$

The first integral in this expression vanishes, owing to the boundary conditions. Using Eq.(5.84) the variation becomes

$$\Delta_1\epsilon = -2g\int d_3\vec{r}\sum_{abc} C_{abc}\frac{\partial\varphi}{\partial X_0}A_a^0\delta\vec{A}_b\cdot\vec{E}_c. \tag{5.90}$$

As expected this expression vanishes, owing to the parallelism of A_a^0 and $\vec{E}_c$ in the color space (see Eqs.(4.46) and (4.50)) and the antisymmetry of the structure constants.

In obtaining the second variation of the energy functional we follow the same steps as in the abelian case. First we expand the generalized Gauss law to the second order. After the cancellation of the first-order terms we are lead to

$$\vec{\nabla}\cdot\vec{\omega}_a = -g\sum_{b,c} C_{abc}\left[\frac{\partial\varphi}{\partial X_0}\delta\vec{A}_b\cdot\delta\vec{E}_c + 2\frac{\partial^2\varphi}{\partial X_0^2}\left(\sum_p \vec{E}_p\cdot\delta\vec{E}_p\right)\delta\vec{A}_b\cdot\vec{E}_c\right], \tag{5.91}$$

where now

$$
\begin{aligned}
\vec{\omega}_a &= 2\frac{\partial^2\varphi}{\partial X_0^2}\left(\sum_p \vec{E}_p\cdot\delta\vec{E}_p\right)\delta\vec{E}_a + \frac{\partial^2\varphi}{\partial X_0^2}\sum_p\left(\delta\vec{E}_p^2-\delta\vec{H}_p^2\right)\vec{E}_a + \\
&+ 2\frac{\partial^3\varphi}{\partial X_0^3}\left(\sum_p \vec{E}_p\cdot\delta\vec{E}_p\right)^2\vec{E}_a + 2\frac{\partial^3\varphi}{\partial X_0\partial Y_0^2}\left(\sum_p \vec{E}_p\cdot\delta\vec{H}_p\right)^2\vec{E}_a + \\
&+ 2\frac{\partial^2\varphi}{\partial Y_0^2}\left(\sum_p \vec{E}_p\cdot\delta\vec{H}_p\right)\delta\vec{H}_a.
\end{aligned} \tag{5.92}
$$

By expanding the integral of (4.41) up to second order and using Eqs.(5.84) and (5.91) the second variation of the energy becomes

$$
\begin{aligned}
\Delta_2\epsilon &= \int d_3\vec{r}\left[\frac{\partial\varphi}{\partial X_0}\sum_a\delta\vec{E}_a^2 + 2\frac{\partial^2\varphi}{\partial X_0^2}\left(\sum_a\vec{E}_a\cdot\delta\vec{E}_a\right)^2\right] + \\
&+ \int d_3\vec{r}\left[\frac{\partial\varphi}{\partial X_0}\sum_a\delta\vec{H}_a^2 - 2\frac{\partial^2\varphi}{\partial Y_0^2}\left(\sum_a\vec{E}_a\cdot\delta\vec{H}_a\right)^2\right] - \\
&- 2\int d^3\vec{r}\vec{\nabla}\cdot(A_0^a\cdot\vec{\omega}^a) - 2g\int d_3\vec{r}\frac{\partial\varphi}{\partial X_0}\sum_{a,b,c}C_{abc}A_a^0\delta\vec{A}_b\cdot\delta\vec{E}_c.
\end{aligned} \tag{5.93}
$$

Once again, the divergence term in this expression vanishes owing to the boundary conditions. The integrand of the last term is the scalar product in color space between the potential A_a^0 of the unperturbed field and the first component of the second-order perturbation of the color-charge density in the r.h.s of Eq.(5.91). For *electromagnetic* perturbations this component must satisfy the condition

$$
\frac{\partial\varphi}{\partial X_0}\sum_{a,b,c}C_{abc}A_a^0\delta\vec{A}_b\cdot\delta\vec{E}_c = 0, \tag{5.94}
$$

(note that the remaining component in the r.h.s. of Eq.(5.91) lies already in the *electromagnetic* direction). Consequently, the last term in Eq.(5.93) must vanish and the second variation of the energy takes a form similar to that of the abelian case. We can now determine the conditions for stability of the finite-energy ESS solutions in this non-abelian case through a similar argumentation. As easily seen stability requires the lagrangian-density function $\varphi(X, Y)$ to satisfy the condition

$$\frac{\partial \varphi}{\partial X} \geq 2X \frac{\partial^2 \varphi}{\partial Y^2}, \tag{5.95}$$

in all the range of values of the gauge invariants $(X, Y = 0)$ defined by the solution. This condition is formally the same as in the abelian case and is also *necessary and sufficient* for the stability of the solitons against *electromagnetic* perturbations. Obviously, it is a gauge-invariant criterion.

The analysis of the **dynamical** stability of non-abelian solitons should now be performed starting with Eqs.(5.84) and (5.87) and following similar steps as in the abelian case. But the presence of the antisymmetric structure constants and the symplectic character of the eigenvalue problem require new qualitative procedures and longer calculations. This issue will be approached elsewhere.

Chapter 6

Some specific models

As illustrative examples of the considerations of the previous chapters we shall introduce and discuss in this chapter four families of admissible models representatives of the different classes analyzed so far. Some of these models should be viewed as approaches to the physical problems which we are addressing from the methods developed here (see chapter 7 for future perspectives). For each model we shall begin with lagrangian densities defined for *scalar* fields, which support stable, finite-energy SSS solutions for several values of the parameters. In some cases we also discuss how the Y-invariant can be included in order for the extended *gauge* models to be admissible and support stable, finite-energy ESS solutions. Obviously, for each admissible scalar model supporting soliton solution there exist a class of equivalence of associated admissible gauge field models supporting similar soliton solutions (see Eq.(4.54)).

6.1 Potential corrections to the D'Alembert Lagrangian

The first example is given by the two-parameter family of field theories defined by lagrangian densities of the form

$$f(X) = \frac{X}{2} + \lambda X^a, \tag{6.1}$$

where λ is a positive constant which gives the intensity of the self-coupling. When $\lambda = 0$, $f(X)$ reduces to the usual D'Alembert lagrangian density (see figure 6.1). The values of the exponent a are restricted to be irreducible ratios of two *odd* natural numbers (we consider 1 as odd) $a = P/Q$ such that

$$P > \frac{3}{2}Q \left(\Rightarrow a > \frac{3}{2}\right). \tag{6.2}$$

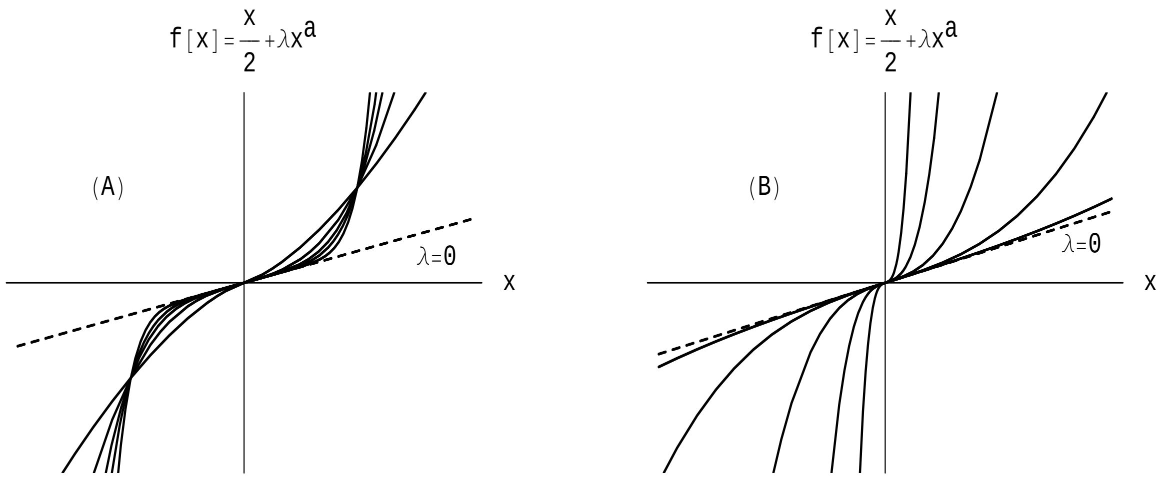

Figure 6.1: Functional form of the lagrangian densities for the family of models (3.1) (A) for a fixed value of the parameter $\lambda(=1)$ and several values of $a(=5/3, 3, 13/3, 17/3, 25/3)$ (B) for a fixed value of the exponent $a(=3)$ and several values of $\lambda(=0.01, 0.1, 1, 10, 100)$. The dashed line corresponds in both cases to the D'Alembert lagrangian ($\lambda = 0$).

As easily verified these restrictions allow $f(X)$ to be defined everywhere and the associated energy density to be positive definite and vanishing in vacuum (admissible class-1 field theories). The condition $a > 3/2$ is imposed to ensure the convergence of the integral of energy at $r = 0$, where the SSS field strengths diverge.

This family of models can be extended to include rational values of the exponent $a = P/Q > 3/2$ with P being an even natural number and Q an odd one. Such models are admissible for $X < 0$ if we replace $\lambda > 0$ by $-\lambda$ in Eq.(6.1). *But*, for complete admissibility, this function should be matched for $X > 0$ with another function satisfying the condition (3.21). For example, the lagrangian obtained by replacing λ by $\lambda \cdot sign(X)$ in (6.1). Since the structure and energy of the SSS solutions are determined by the form of the lagrangian density for $X < 0$, the SSS solutions of these models are also solitons.

The following considerations are valid for the extended family. The form of the SSS solutions is obtained from the equation (3.46), which now reads

$$z(y) \equiv \frac{y}{2} + (-1)^{P-1} \lambda a y^{(2a-1)} = \frac{\Lambda}{r^2}, \tag{6.3}$$

with $y(r, \Lambda) \equiv \phi^{'}(r, \Lambda)$. The function $z(y)$ shows an unique growing branch for every value of the scalar charge $\Lambda > 0$ and, consequently, there is an unique SSS solution of Eq.(6.3), which vanishes as $\phi^{'} \sim r^{-2}$ when $r \to \infty$ (case B-2 above, asymptotically coulombian) and diverges as $\phi^{'} \sim r^{-2/(2a-1)}$ when $r \to 0$. Thus the argument (X) of the lagrangian ranges from zero to $-\infty$ in this interval and, as expected, the stability condition (3.44), which now reads

$$\overset{\bullet}{f}(X) + 2X\,\overset{\bullet\bullet}{f}(X) = \frac{1}{2} + (-1)^{P-1}\lambda a(2a-1)X^{a-1} > 0, \quad \forall X < 0, \tag{6.4}$$

is fulfilled there.

The energy of the soliton, as a function of the model parameters, can be explicitly obtained from the integral term of Eq.(3.47). The final result is

$$\epsilon = \frac{4\sqrt{2}\pi}{3}\frac{\Lambda^{3/2}}{(a-1)(2a\lambda)^{\frac{1}{4(a-1)}}}B\left(\frac{1}{4(a-1)}, \frac{2a-3}{4(a-1)}\right), \tag{6.5}$$

where $B(x,y)$ is the Euler integral of first kind

$$B(x,y) = \int_0^1 dt t^{x-1}(1-t)^{y-1}, \quad \mathrm{Re}(x) > 0, \quad \mathrm{Re}(y) > 0. \tag{6.6}$$

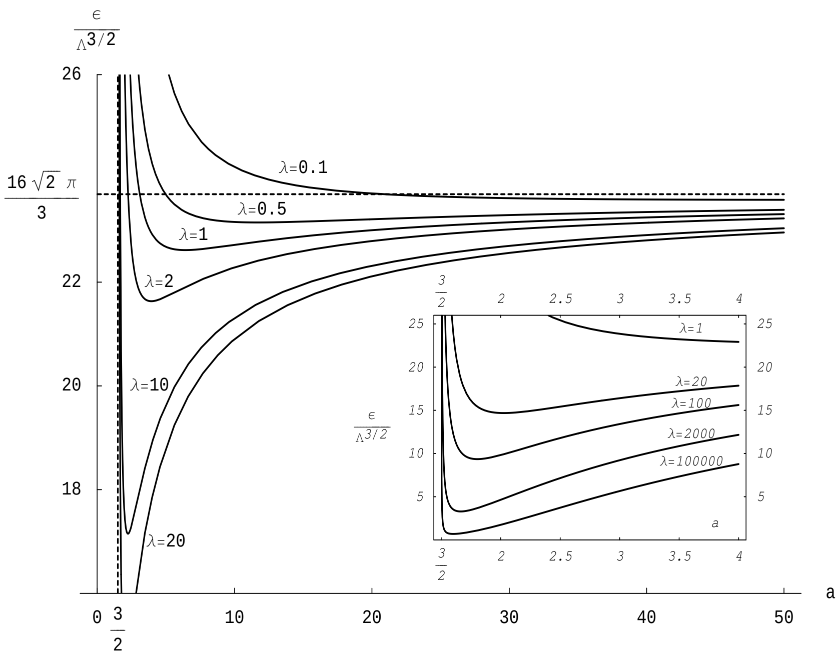

Figure 6.2: Energy of the solitons of the family (6.1) as a function of the exponent a with the coupling constant λ as parameter. The dashed line correspond to the asymptotic limit of the energy for $a \to \infty$. The small plot shows the behaviour of the energy for strong self-couplings. The energy is never zero for any $\lambda < \infty$.

In figure 6.2 we have plotted this energy as a function of the exponent a, with the coupling constant λ as a parameter. We see that the energy diverges, for any value of λ, as the exponent a approaches the value $3/2$. This energy is strongly reduced in the region of values of $a \gtrsim 3/2$ as the coupling constant λ increases, reaching minima which vanish as $\lambda \to \infty$ in this region. When the exponent a increases the energies of the solitons become less dependent on λ and approach asymptotically the value $\epsilon/\Lambda^{3/2} = 16\pi\sqrt{2}/3$ as $a \to \infty$. An interesting feature of these models is the existence of soliton solutions for any $a > 3/2$, no matter how small the coupling parameter λ may be. This implies that any small correction of this kind to the "bare" D'Alembert lagrangian leads to the possibility of excitation of soliton modes. For a fixed value of the exponent, the masses of such modes increase as the intensity of the coupling is reduced. This behaviour is similar to the one encountered in one-space dimensional models supporting topological soliton solutions [76, 141].

The preceding analysis can be generalized to the case of theories whose lagrangian densities for $X < 0$ take the form [1]

$$f_N(X) = \frac{X}{2} + \sum_{n=1}^{N} (-1)^{P_n - 1} \lambda_n X^{a_n}, \tag{6.7}$$

where, for admissibility, the λ_n are constrained to form a finite sequence of positive constants and the exponents $a_n = P_n/Q_n$ to form an increasing sequence of rational numbers, built as irreducible ratios of odd natural numbers or of even and odd naturals, such that $a_n > 1$ for $n < N$ and $a_N > 3/2$. In these models the energy density is positive definite for $X < 0$ whereas the SSS field solutions diverge at the origin as $\phi'(r \to 0) \sim r^{-2/(2a_N - 1)}$ and vanish asymptotically as $\phi'(r \to \infty) \sim r^{-2}$. The associated energies

$$\epsilon_N(\Lambda) = \frac{8\pi}{3}\sqrt{2}\Lambda^{3/2} \int_0^\infty \frac{dy}{y^{1/2}\sqrt{1 + 2\sum_{n=1}^{N} \lambda_n a_n y^{2(a_n - 1)}}}, \tag{6.8}$$

are finite and the corresponding solitons are stable.

A particular case of Eq.(6.7) is obtained when the exponents are a finite sequence of consecutive naturals $a_n = n(n > 1)$. Moreover, let us assume that we take the limit $N \to \infty$ and that the infinite sequence of $\lambda_n > 0$ converges to zero, in such a way that the series

$$f(X) = \frac{X}{2} + \sum_{n=2}^{\infty} (-1)^{n-1} \lambda_n X^n \tag{6.9}$$

be convergent in an interval including $\forall X < 0$ (case A-1), or in an interval including

[1] Obviously, for $X > 0$ they must be extended in agreement with the requirements of chapter 3.

the range $0 > X > -C^2$ but excluding the values $X < -C^2$ (case A-2). We can then extend the family (6.7) to a large class of analytic functions. Such functions must satisfy the conditions established in section 3.3 for the integrals of energy associated to the SSS solutions of such models to be finite. If we assume the positivity of the coefficients λ_n in Eq.(6.9) we can explicitly check for these models the fulfillment of the condition of stability (3.44), which now takes the form

$$\overset{\bullet}{f}(X) + 2X \overset{\bullet\bullet}{f}(X) = \frac{1}{2} + \sum_{n=1}^{\infty} \lambda_n (2n+1)(4n+1) X^{2n} > 0 \tag{6.10}$$

and holds in the entire domain of definition of the soliton. Thus the requirement of convergence of the series (6.9) for any X, with the assumed restrictions $\lambda_n > 0$ $(\forall n)$ leads to class-1 field theories supporting SSS solitons. As an example of this let us mention the analytic function $f(X) = \frac{1}{2} sh(X)$, which diverges when $X \to -\infty$ faster than X^γ with $\gamma > 3/2$ (as required by Eq.(3.29) in the A-1 case) and behaves like $X/2$ around $X = 0$ (case B-2).

If we relax the requirement of positivity of the coefficients λ_n, whenever the series (6.9) converges in some restricted interval $X > -C^2$ and the sum remains a monotonically increasing function of X there, we are lead to admissible class-2 field theories exhibiting SSS soliton solutions. An example of this case is the lagrangian density $f(X) = \frac{1}{2} tg(X)$ restricted to the interval $-\frac{\pi}{2} < X < \frac{\pi}{2}$. This model supports finite-energy stable SSS solutions belonging to cases A-2 and B-2.

Let us consider the case $\lambda_n \geq 0, \forall n$. The partial sums in (6.9) give rise to an infinite sequence of admissible lagrangian models of the form (6.7) with natural exponents, all of them supporting SSS soliton solutions belonging to cases A-1 and B-2. The explicit forms $\phi_N(r, \Lambda)$ of these solutions can be obtained by solving the equation (3.45) for each lagrangian in the sequence. These equations take the form

$$\phi_N^{'} \left(\frac{1}{2} + \sum_{n=2}^{N} n \lambda_n (\phi_N^{'})^{2(n-1)} \right) = \frac{\Lambda}{r^2}. \tag{6.11}$$

If the series (6.9) converges in $X < 0$, defining an analytic lagrangian density function there, the sequence of SSS soliton solutions of the partial-sum models in the expansion, corresponding to the same value of the scalar charge Λ for all N, must converge to the SSS soliton solution (with the same charge) associated to this lagrangian density $(\phi_{N \to \infty}(r, \Lambda) \to \phi(r, \Lambda))$. This can be directly established from Eq.(6.11), which defines the forms of the SSS solutions. Then the limit solution can be written as a functional series expansion in terms of the members of the sequence as

$$\phi(r,\Lambda)=\phi_1(r,\Lambda)+\sum_{N=2}^{\infty}\delta_N(r,\Lambda), \tag{6.12}$$

where $\delta_N(r,\Lambda)=\phi_N(r,\Lambda)-\phi_{N-1}(r,\Lambda)$. Using Eqs.(3.13) and (3.14) it is easy to show that the sequence of energies of the equal-charge solitons associated to the partial-sum lagrangian densities converges towards the energy of the equal-charge soliton associated to the full series lagrangian (6.9), which can be written as the series expansion:

$$\epsilon(\Lambda)=\epsilon_1(\Lambda)+\sum_{N=2}^{\infty}\Delta_N\epsilon(\Lambda), \tag{6.13}$$

where $\Delta_N\epsilon(\Lambda)=\epsilon_N(\Lambda)-\epsilon_{N-1}(\Lambda)$ is the difference between the energies of two consecutive solitons in the sequence. The first two terms of this expansion are energy-divergent. The first one corresponds to the divergent self-energy of the Coulomb field whereas the first correction $\Delta_2\epsilon(\Lambda)$ cancels this divergence and "renormalizes" the self-energy to a finite value. The subsequent terms are all finite and the series converges towards the energy of the limit soliton. The energy associated with the soliton of order N in the sequence can be obtained making use of the expression (3.47) and reads

$$\epsilon_N(\Lambda)=\frac{8\pi}{3}\sqrt{2}\Lambda^{3/2}\int_0^{\infty}\frac{dy}{y^{1/2}\sqrt{1+2\sum_{n=2}^{N}n\lambda_n y^{2(n-1)}}}, \tag{6.14}$$

which can be numerically calculated once the coefficients are fixed.

To illustrate this procedure let us consider the above mentioned analytic lagrangian $f(X)=\frac{1}{2}sh(X)$. The energy associated to the soliton solutions of this model, obtained from Eq.(3.47), is $\epsilon(\Lambda)\simeq 28.5607\Lambda^{3/2}$. The partial sums of the McLaurin expansion of this lagrangian function are admissible models supporting a sequence of SSS soliton solutions. Their energies, obtained from Eq.(6.14), are plotted in figure 6.3 as functions of $i=\frac{N-1}{2}$, for the same value of the scalar charge Λ of each solution ($N=2i+1$ being the exponents of the surviving terms in the expansion which, in this case, are the sequence of odd naturals). Obviously the energy of the first-order term ($N=1, i=0$), which corresponds to the Coulomb field, diverges but, as expected, the first correction already "renormalizes" this coulombian divergent energy and the subsequent orders reduce the (now finite) energy, which approaches asymptotically the energy of the soliton of the exact model as i increases. The convergence in this example is related to the analytic character of the sum (6.9) but the "renormalization" of the divergent self-energy is due to the first correction to the pure D'Alembert lagrangian and would arise even if the series were not convergent.

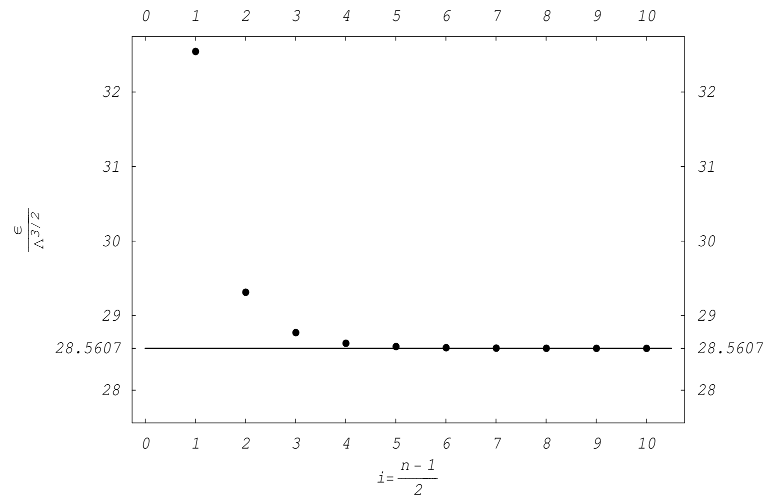

Figure 6.3: Energy of the sequence of SSS soliton solutions (with the same scalar charge, Λ) of the partial-sum lagrangian models obtained from the McLaurin expansion of the lagrangian density $f(X) = \frac{1}{2}sh(X)$, as functions of the integer parameter $i = \frac{N-1}{2}$, N being the odd exponents of the surviving terms in the expansion. As N increases these energies approach asymptotically the value $\epsilon(\Lambda) \simeq 28.5607\Lambda^{3/2}$, corresponding to the soliton energy of the exact theory.

6.1.1 Effective gauge-invariant lagrangian models

These results can be useful in the analysis of particle-like solutions in effective models of gauge-invariant interactions. Indeed, effective lagrangians arise frequently in perturbative schemas which lead to polynomial expressions in certain field invariants. For example, in the case of QED the perturbative expansion of the photon effective action, which is obtained by integrating out the high-energy degrees of freedom of the electron sector, defines a sequence of lagrangians which take this polynomial form in the field invariants (Euler-Heisenberg lagrangians [68] and the higher-order corrections [69]). On the other hand, as we have established in the preceding chapters, the solution of the electrostatic spherically symmetric problem for a generalized gauge-invariant lagrangian model can be reduced to that of an associated scalar field model, whose lagrangian density is univocally defined from the gauge-invariant one. If the sequence of gauge-invariant effective lagrangians are of polynomial forms in the field invariants, the associated scalar lagrangians are also polynomials in the kinetic term, taking the form of partial sums of a series as (6.9). In this way we have established that the sequence of effective lagrangians describing the low-energy photon-photon interaction in QED support electrostatic point-like finite-energy solutions [75]. Let us illustrate this result.

We can then proceed to extend the above scalar family (6.9) to the gauge field case. The most general lagrangian belonging to this class of theories and supporting ESS soliton solutions would be a series of products of powers of the X-invariant and even powers of the Y-invariant (in order for the parity requirement (4.4) to be fulfilled). This leads (for positive values of X) to lagrangians of the form

$$\varphi(X,Y) = \sum_{n,m=0}^{\infty} \alpha_{n,m} X^n Y^{2m}, \tag{6.15}$$

where the coefficients $\alpha_{n,m}$ are positive constants and the subindexes label the exponents of each field invariant for the corresponding term (note that the first coefficient in the expansion must take the value $\alpha_{0,0} = 0$ for vanishing vacuum energy). In our notation the second term in this expansion must be set to $\alpha_{1,0} = 1/2$ to recover Maxwell/Yang-Mills theory in the weak-field limit. Then the family reads

$$\varphi(X,Y) = \frac{X}{2} + \sum_{m=1}^{\infty}(\alpha_{0,m} + \alpha_{1,m}X)Y^{2m} + \sum_{n=2,m=0}^{\infty} \alpha_{n,m} X^n Y^{2m}, \tag{6.16}$$

for $X > 0$. As results from the preceding analysis, each partial-sum lagrangian in the series (6.16) corresponds to an admissible field theory supporting finite-energy ESS solutions, which energy can be calculated using formula (6.14) and the energy relation (4.30).

The stability requirement (4.32) reads for this family

$$\frac{\partial \varphi}{\partial X_0} - 2X_0 \frac{\partial^2 \varphi}{\partial Y_0^2} = \frac{1}{2} + \sum_{n=2}^{\infty} (n\alpha_{n,0} - 4\alpha_{n-2,1})\, X^{n-1} > 0, \forall X > 0. \tag{6.17}$$

A sufficient condition for this requirement to be satisfied is

$$\alpha_{n,0} > \frac{4}{n}\alpha_{n-2,1} \;\; ; \;\; \forall n. \tag{6.18}$$

However, if we think of the lagrangian (6.16) as an effective lagrangian coming from a perturbative expansion, obtained when some high-energy degrees of freedom are integrated out in the path-integral of the original action, then we can assume the coefficients $\alpha_{n,m}$ to be small and decreasing for each order. In fact, this is the case for the Euler-Heisenberg effective lagrangian (see section 2.2.2) and the sequence of higher-order corrections, where the terms in the expansion (6.16) come from the coupling of photons to virtual electron loops in the vacuum [68, 69]. With these assumptions, even though the condition (6.18) is not fulfilled to all orders, a less restrictive sufficient condition for stability demands that the requirement (6.17) be fulfilled beyond a certain order n_i in the expansion and for all the subsequent higher-order corrections, i.e.

$$\frac{1}{2} + \sum_{n=2}^{n_j \geq n_i} \left(n\alpha_{n,0} - 4\alpha_{n-2,1}\right) X^{n-1} > 0, \forall X > 0, \tag{6.19}$$

where the smallness of the coefficients $\alpha_{n,0}, \alpha_{n-2,1}$ guarantees that the 1/2 term will dominate over the remaining terms in the series (6.19) for small values of X, in such a way that the partial sums beyond the order n_i are positive everywhere. Moreover, we arrive at the conclusion that, if Eq.(6.18) is fulfilled to all orders, then each partial-sum lagrangian in the expansion (6.16) corresponds to an admissible model supporting finite-energy and stable ESS soliton solutions. On the other hand, if the condition (6.18) is violated but (6.19) remains fulfilled for a certain n_i and for all $n_j > n_i$, then each partial-sum lagrangian in the series (6.16) supports finite-energy ESS solutions which are unstable for orders $n < n_i$, but become stable for $n \geq n_i$. We notice that the EH lagrangian (2.16) does not fulfill the stability conditions but the higher-order terms in the effective lagrangian expansion [69] could do it.

Let us give a *tentative* physical interpretation of these results. The non-linear terms in these effective lagrangians describe, at a classical level, a self-interaction of the gauge field mediated by the Dirac vacuum. The point-like solution of the "bare" Maxwell lagrangian is the Coulomb field, which has a divergent self-energy. The first non-linear correction term of the effective lagrangian (Euler-Heinsenberg) incorporates polarization effects of the vacuum on the classical field of the point charge, calculated to lowest order in a perturbative expansion. These screening effects "renormalize" the charge field, which becomes finite-energy. The subsequent corrections in the expansion describe higher-order approximations to the behaviour of the screening, but the finite-energy character of the screened fields is preserved to all orders. Unfortunately, the validity of this effective approach is limited to energies much lower than the electron mass [67] and is not accurate to describe the strong fields arising near the center of the particle-like solutions. Consequently, this tentative interpretation can not be maintained only on these grounds. For a more rigorous investigation this question should be considered starting from a different effective approach incorporating the vacuum polarization effects in presence of the strong fields of point-like charges. The analysis of this approach is currently in progress (see also the comments on this point in chapter 7).

6.2 BI-like models

The second example is a two-parameter family of field theories defined by lagrangian densities of the form (see figure 6.4)

$$f(X) = \frac{(1+\mu^2 X)^\alpha - 1}{2\alpha\mu^2}, \tag{6.20}$$

where μ is a real constant. The admissibility conditions require the values of the

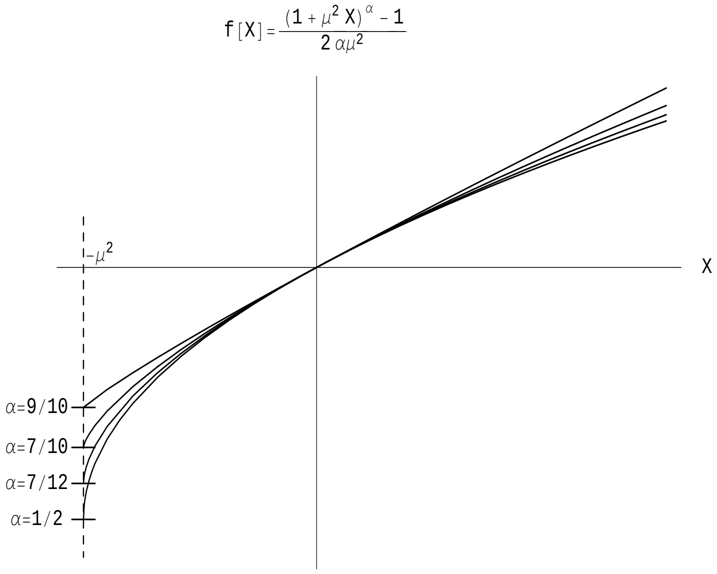

Figure 6.4: Functional form for the lagrangian densities (6.20). The dashed line ($\alpha = 1$) corresponds to the D'Alembert lagrangian while the case $\alpha = 1/2$ is associated to Born-Infeld theory.

parameter α to be restricted to the range $1/2 \leq \alpha < 1$. Indeed, if $1 \leq \alpha \leq 3/2$ the energy of the SSS solutions diverges around $r = 0$. On the other hand if $\alpha > 3/2$ the solution $\phi^{'}(r)$ is multi-valued and the different branches are either unstable or energy-divergent. We also discard the models with $0 < \alpha < 1/2$, since the energy density in this case is not positive definite for $X > 0$. Moreover, if α is a rational number built as the irreducible ratio of an even natural and an odd natural numbers, the function $f(X)$ is defined everywhere, but $\overset{\bullet}{f}(X)$ changes sign in $X = -1/\mu^2$ and the energy becomes negative for large negative values of X.

Finally, if α is the irreducible ratio of two odd naturals we are lead to models which exhibit multi-branched SSS solutions. In fact, the function $z(y)$ in (3.46) has now two separated branches. The field associated to the first branch ranges in the interval $0 \leq \phi^{'}(r) < 1/\mu$ and satisfies the condition (3.44) there, leading to a stable and finite-energy SSS solution, finite and defined everywhere (these branches fall inside the cases A-2 and B-2, with $\phi^{'}(0) = 1/\mu$ and coulombian asymptotic behaviour). The remaining branch of $z(y)$ ranges in the interval $1/\mu < y < \infty$ and exhibits a minimum at $y = (\mu\sqrt{2\alpha - 1})^{-1}$. Consequently there are two additional solutions $\phi^{'}(r)$ defined only inside the interval $0 \leq r \leq \sqrt{2\mu\Lambda}(2 - 2\alpha)^{(1-\alpha)/2}(2\alpha - 1)^{(2\alpha-1)/4}$ (see figure 6.5).

Consequently, we must exclude the models with these values of the parameter α and restrict the family to the lagrangian densities which result from *irrational values of* α *or rational values which are irreducible ratios of an odd and an even natural* (always

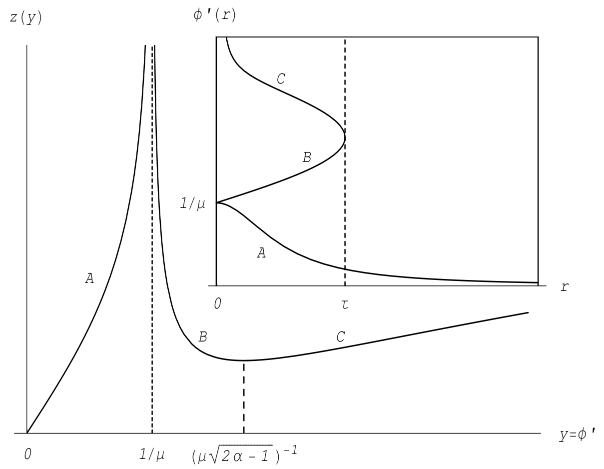

Figure 6.5: Multiple-branched SSS solutions. In the figure of the small plot the parameter $\tau = \sqrt{2\mu\Lambda}(2-2\alpha)^{\frac{1-\alpha}{2}}(2\alpha-1)^{\frac{2\alpha-1}{4}}$ has been used.

within the range $1/2 \leq \alpha < 1$). In these cases the lagrangian densities are defined only for $X > -1/\mu^2$ and behave as $X/2$ around $X = 0$, corresponding to class-2 field theories. There are now unique, stable and finite-energy SSS solutions for each model, which are defined everywhere and fall inside the cases A-2 and B-2 (with the maximum field strength $\phi^{'}(0) = 1/\mu$, and coulombian behaviour at infinity). In the limits $\mu \to 0$ or $\alpha \to 1$ Eq.(6.20) reduces to the D'Alembert lagrangian density. The scalar Born-Infeld model is a member of this family, corresponding to the frontier value $\alpha = 1/2$.

In calculating the energy of these soliton solutions as a function of the model parameters we evaluate the integral in (3.47), as in the previous example. The final expression is

$$\epsilon = \frac{4\sqrt{2}\pi}{3}\frac{\Lambda^{3/2}}{|\mu|^{1/2}}B\left(\frac{1}{2},\frac{3-\alpha}{2}\right). \qquad (6.21)$$

We see that, as function of μ the soliton energy behaves like $\epsilon \sim \frac{1}{\sqrt{\mu}}$ and diverges as $\mu \to 0$ (D'Alembert limit), whereas it vanishes in the strong-coupling limit $\mu \to \infty$. This energy is not very sensitive to the exponent α in the range of admissible values and the behaviour for the whole family is similar to that of the scalar Born-Infeld model.

This family can be extended to the gauge field case in a straightforward way. A natural choice would be to introduce the Y-invariant for the whole BI-like family in the same way as in the particular Born-Infeld lagrangian (4.5) and define a generalized family

$$\varphi(X,Y) = \frac{1 - \left(1 - \mu^2 X - \frac{\mu^4}{4} Y^2\right)^\alpha}{2\mu^2\alpha}. \tag{6.22}$$

It can be straightforwardly seen that such a family of lagrangians fulfills the admissibility constraints (4.8), (4.11) and supports finite-energy ESS solutions. Moreover, the condition of stability (4.32) reads now

$$\frac{\partial\varphi}{\partial X_0} - 2X_0 \frac{\partial^2\varphi}{\partial Y_0^2} = \frac{1}{2}\left(1 - \mu^2 X_0\right)^\alpha > 0, \tag{6.23}$$

and is satisfied by the whole family since the ESS field is bounded everywhere ($X_0 < 1/\mu^2$).

6.3 A three-parameter family

The third example is the three-parameter family of models defined by the lagrangian densities

$$f(X) = \frac{1}{2}\frac{X^\alpha}{(1+\mu^2 X)^\beta}, \tag{6.24}$$

where α is chosen as the irreducible ratio of two positive odd naturals (in order for the lagrangian to be well defined on both sides of $X = 0$). The exponent β must be chosen as a positive irrational number or as the irreducible ratio of an odd and an even natural numbers. In this way the lagrangian is defined only for $X > -1/\mu^2$, thus avoiding a non-positive definite character of the energy as well as a singularity inside the domain of definition. We emphasize that, as mentioned in chapter 3, we regard as acceptable singularities of the lagrangian density only those lying on the boundary of the (open and connected) domain of definition. In fact, one of the motivations in introducing this example is to show how models with such a kind of singularities can also lead to physically reasonable results.

These restrictions lead to a family of class-2 field theories belonging to the A-2 case, being examples of the sub-cases whose lagrangians diverge at $X \to -1/\mu^2$ in the boundary of the domain of definition. The behaviours of the lagrangians around $X = 0$ ($r \to \infty$ for the SSS solutions) belong respectively to the cases B-1 ($\alpha > 1$), B-2 ($\alpha = 1$) or B-3 ($\alpha < 1$), corresponding to asymptotic dampings of the soliton field strengths which are slower than coulombian, coulombian or faster than coulombian, respectively (see figure 6.6).

We also impose the condition $\alpha > \beta + 1/2$, necessary to ensure the positivity of the energy density for any $X \in \Omega$. Moreover, the convergence of the integral of energy for

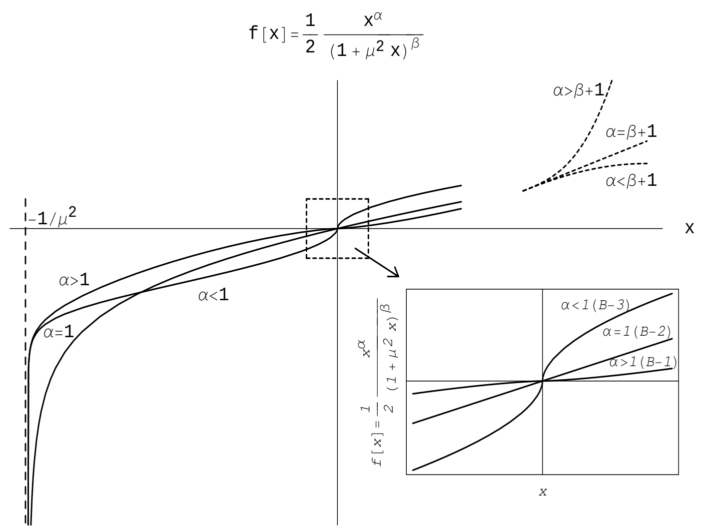

Figure 6.6: Form of the lagrangian functions corresponding to models of the family (6.24) for three sets of values of the parameter $\alpha \left(\lesseqgtr 1 \right)$. All lagrangians diverge at $X \to -1/\mu^2$, which corresponds to the maximum value of the field strength. At the point $X = 0$, which determines the asymptotic behaviour of the solitons, the three sets of values of α give the three different behaviours: case B-1 ($\alpha > 0$), case B-2 ($\alpha = 0$) and case B-3 ($\alpha < 0$). The dashed lines show the behaviour of the lagrangian function at large positive values of X for the admissible models corresponding to different relations among the parameters.

the SSS solutions as $r \to \infty$ requires $\alpha < 3/2$, as can be easily verified from the analysis of the field equation (3.4) and the integral of energy in this limit. Let us summarize in the following equations the restrictions imposed on the parameters of the models (6.24) in order to obtain admissible models with soliton solutions:

$$\frac{1}{2} < \alpha \equiv \frac{odd}{odd} < \frac{3}{2} \quad ; \quad \beta \equiv \frac{odd}{even} \text{ or irrational} \quad ; \quad 0 < \beta < \alpha - \frac{1}{2} < 1. \tag{6.25}$$

where the terms "odd" and "even" are implicitly understood to apply for natural numbers. The D'Alembert lagrangian is a limit member of this family obtained as $\alpha \to 1$ and $\beta \to 0$ or as $\alpha \to 1$ and $\mu \to 0$.

As results from the analysis of the A-2 cases, near the center the SSS solutions behave as

$$\phi'(r \to 0) \sim \frac{1}{\mu} - \lambda r^\sigma, \tag{6.26}$$

where the exponent σ is given by

$$1 < \sigma = \frac{2}{1+\beta} < 2, \tag{6.27}$$

and λ is a positive constant, which is the solution of the equation

$$2\alpha\mu\lambda + \beta = \Lambda\mu^{(\alpha+\beta)}(2\lambda)^{\beta+1}. \tag{6.28}$$

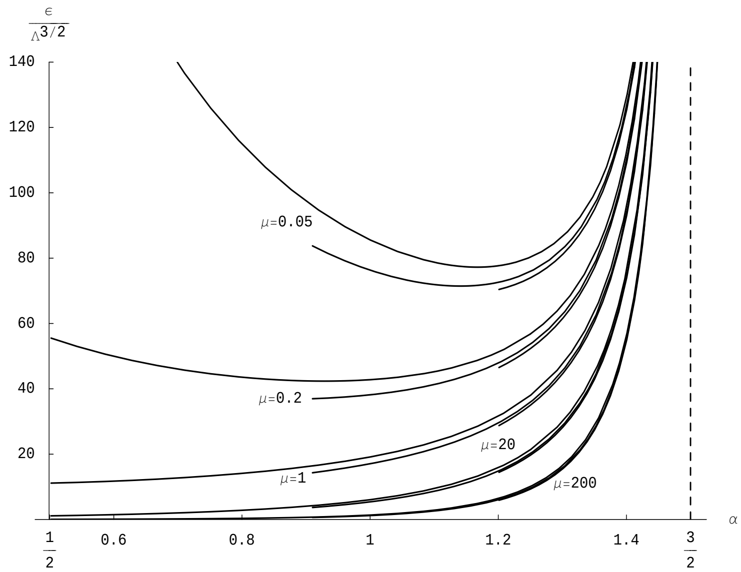

Figure 6.7: Behaviour of the energy as a function α for five values of μ and three of $\beta(0.01, 0.4, 0.7)$. Note that the lower branches for each value of μ do not cover all the range $1/2 < \alpha < 3/2$ since the constraint $\alpha > \beta + 1/2$ must be always fulfilled for admissibility.

As in the preceding examples, the energy of the soliton solutions can be explicitly obtained from (3.47). The final expression is

$$\epsilon = \frac{4\sqrt{2}\pi}{3}\frac{\Lambda^{3/2}}{\alpha^{1/2}}|\mu|^{\frac{2\alpha-3}{2}} B\left(\frac{3-2\alpha}{4}, \frac{\beta+3}{2}\right) F_1^2\left(\frac{1}{2}, \frac{3-2\alpha}{4}, \frac{9-2(\alpha-\beta)}{4}, \frac{\alpha-\beta}{\alpha}\right), \tag{6.29}$$

where $B(x, y)$ is again the Euler beta-function and $F_1^2(a, b, c, z)$ is the hypergeometric function defined as

$$F_1^2 = F(a,b,c,z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt, \ \mathrm{Re}(b) > 0, \ \mathrm{Re}(c) > 0. \tag{6.30}$$

In figure 6.7 we have plotted the energy as a function of α with μ and β as parameters. For a given value of μ the energy is rather insensitive to the parameter β. As α approaches the limit $3/2$ the energy diverges for all values of μ and β. As a function of μ the energy decreases as the power $1/\mu^{(\frac{3}{2}-\alpha)}$.

6.4 Short-ranged solutions

As a fourth example let us look for a family of models whose SSS solutions are exponentially damped for large r. As already mentioned, these kinds of theories allow to describe short-range interactions through the exchange of self-coupled scalar fields. Their generalizations to the case of gauge fields, which was performed in chapter 4, lead to effective non-linear lagrangians which also describe short-range interactions and preserve the explicit gauge-invariance. From this point of view such models provide alternatives to the usual symmetry breaking mechanism in the description of weak interactions.

In obtaining these models we shall proceed backwards, looking for lagrangians whose associated field equations have *prescribed* SSS solutions. In this way we shall look for a family of lagrangian density functions of the form (3.1) whose associated SSS field solutions have the simple exponentially damped form

$$\phi^{'}(r,\Lambda) = A \exp\left(-\sigma \frac{r^n}{\Lambda^{n/2}}\right), \tag{6.31}$$

where A, σ and n are positive constants determining the different models within this family [2]. More complex choices of exponentially damped SSS fields (as, for example, $\phi^{'}(r) = a(r)\exp\left(-\sigma\frac{r^n}{\Lambda^{n/2}}\right)$, where $a(r)$ is assumed to be a bounded function) may be analyzed in a similar way, but in the present example the calculations can be performed in terms of elementary functions. With this choice the SSS field will be a soliton, but the method works also in obtaining models with exponentially-damped SSS solutions which are energy-divergent.

The constant Λ in (6.31) is the integration constant of the first-integral (3.4) of the field equation (whose solution is required to be (6.31)) and parameterizes all SSS solutions of a given model. It is explicitly introduced in (6.31) by implementing the

[2]The constant A is a parameter of the model and not an integration constant of the solutions. It plays the role of the maximum field strength and is shared by all SSS solutions of a given model, but differs for the various models in the family.

scale law (3.8). These fields belong to the cases A-2 (finite field strength at the center) and B-3 (asymptotic damping faster than coulombian), and the corresponding models are class-2 field theories.

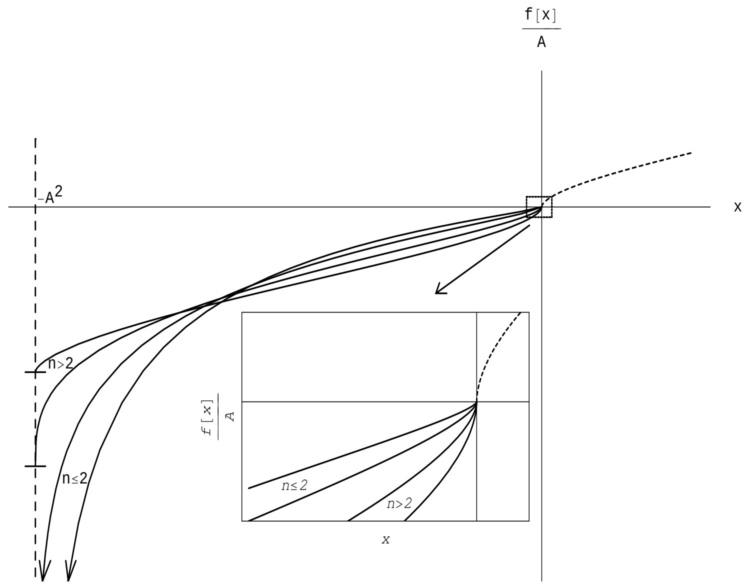

Figure 6.8: Functional form of the lagrangian densities for the family (6.33), exhibiting short-ranged soliton solutions. The dashed line indicates possible continuations of the lagrangian density for $X > 0$.

By eliminating r between (6.31) and the first-integral (3.4) we obtain the form of the first derivative of the lagrangian density

$$\overset{\bullet}{f}(X) = \frac{\Lambda}{r^2\phi'(r)} = \frac{(2\sigma)^{2/n}}{\sqrt{-X}\ln^{2/n}\left(\frac{-A^2}{X}\right)}, \tag{6.32}$$

which holds in the interval $-A^2 < X < 0$ (where the SSS solution (6.31) is defined) and diverges at the boundaries. The lagrangian density in this interval is

$$f(X) = 2\sigma^{2/n}\int_{\sqrt{-X}}^{0}\frac{dy}{\ln^{2/n}(A/y)} = -2A\sigma^{2/n}\int_{\ln(A/\sqrt{-X})}^{\infty}\frac{e^{-z}}{z^{2/n}}dz. \tag{6.33}$$

As easily verified, $f(0) = 0$ for any set of positive values of the parameters. In the lower boundary of the interval we have $f(X = -A^2) = -2A\sigma^{2/n}\Gamma\left(1 - \frac{2}{n}\right)$ for $n > 2$ (with $\Gamma(t) = \int_0^\infty z^{t-1}e^{-z}dz, t > 0$ being the usual Euler gamma function) and $f(X \to -A^2) \to -\infty$ for $n \leq 2$ (see figure 6.8). This expression of the lagrangian density could be continued to the region $X > 0$ by matching (6.33) to any function satisfying the admissibility conditions there, but such continuations do not affect the structure of the solitons, which is completely determined by the part (6.33) of the lagrangian density [3].

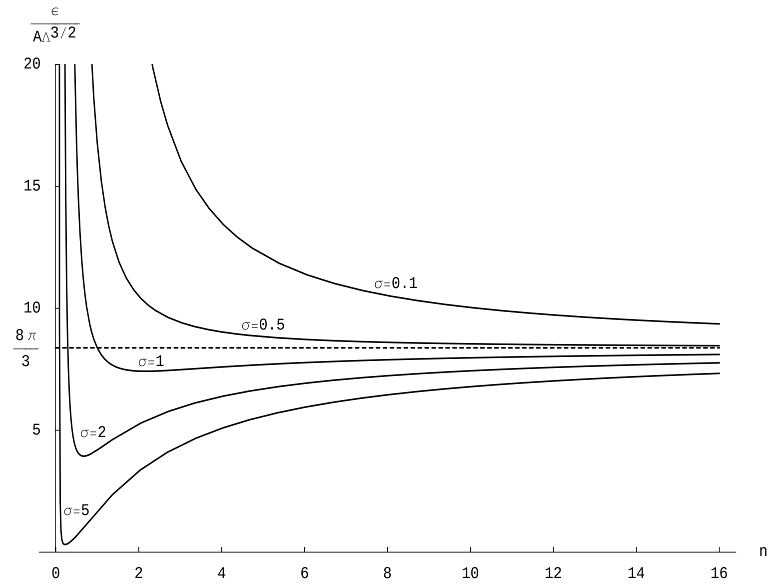

Figure 6.9: Energy for short-ranged solitons (6.31) of unit maximum-field strength as a function of the exponent n, for several values of the parameter σ (note that we have plotted $\frac{\epsilon}{A\Lambda^{3/2}}$ instead of $\frac{\epsilon}{\Lambda^{3/2}}$ as in the previous examples).

We calculate now the energy of these soliton solutions starting by convenience from Eq.(3.12) (although formula (3.47) would also work). After a partial integration we obtain

$$\epsilon = \frac{4\pi}{3}\left[-r^3 f\left(-\phi'^2(r)\right)|_0^\infty + \int_0^\infty r^3 \frac{df}{dr} dr\right]. \tag{6.34}$$

The integrated part in this equation can be shown to vanish for the prescribed solutions. In calculating the integral in the second term we use the first-integral field equation (3.4) and the expression of the field (6.31), which leads to

$$\frac{df}{dr} = -2 \overset{\bullet}{f}(X)\phi'(r)\phi''(r) = \frac{-2\Lambda}{r^2}\phi''(r) = 2\Lambda^{(2-n)/2} A n \sigma r^{n-3} \exp\left(-\sigma \frac{r^n}{\Lambda^{n/2}}\right), \tag{6.35}$$

and the final expression for the energy of the solitons reads

[3]Nevertheless, the requirements of positivity of the energy and vanishing vacuum energy of the complete lagrangians are still necessary for the stability of the solitons (see chapter 5).

$$\epsilon = \frac{8\pi A}{3\sigma^{1/n}} \Lambda^{3/2} \Gamma \left(\frac{1+n}{n} \right) , \tag{6.36}$$

which is proportional to the maximum amplitude of the field strength and decreases as the range of the field is reduced. In figure 6.9 we have plotted the energy of the unit maximum-field-strength as a function of n for several values of the constant σ. As we see the energy diverges as $n \to 0$ and becomes less dependent on the constant σ for large values of n, approaching asymptotically a fixed value given by $\frac{\epsilon}{A\Lambda^{3/2}} = \frac{8\pi}{3}$.

Chapter 7

Conclusions and perspectives

In this thesis we have solved the problem of characterizing a large class of physically consistent relativistic lagrangian field theories in three-space dimensions, supporting unique, static spherically symmetric finite-energy non-topological soliton solutions. The fields concerned were one and many-components scalar fields (whose lagrangian densities depend on the kinetic term alone) and generalized gauge fields of compact semi-simple Lie groups (whose lagrangian densities depend on the two standard first-order field invariants). This characterization is exhaustive and leads to the classification of such models into six types, according to the central and asymptotic behaviours of the soliton fields. We have performed a broad analysis of the linear stability of the solutions, obtaining necessary and sufficient stability conditions which go beyond the usual Derrick criterion. We also have carried out a general spectral analysis of the linear perturbations around the soliton solutions, confirming their dynamical stability and setting grounds for their quantum extensions.

All these results allow the explicit determination of a large number of examples of such a class of lagrangians, providing a wide panoply of tools for the analysis of diverse physical problems, such as those discussed in the introduction and in chapters 1 and 2. Among these problems let us outline several of particular interest, which we are addressing from the methods developed here. At this regard, the families of models introduced in this thesis, belonging to the different classes of solitons, should be seen as a first step on the approaches to tackle such problems.

Effective field theories supporting ESS soliton solutions in the framework of QED.

As already discussed in section 6.1.1, the photon-photon interaction mediated by the QED vacuum can be classically described in terms of effective lagrangians which are polynomial expressions in the gauge invariants X and Y and can be obtained in a perturbative procedure [67], where the lowest order is the well-known Euler-Heisenberg lagrangian [68]

$$\varphi(X, Y) \sim \eta X + \xi(4X^2 + 7Y^2), \tag{7.1}$$

(η and ξ being positive constants). The sequence of these lagrangians exhibits finite-energy ESS solutions and suggests an interpretation in terms of the screening effects of the vacuum on the field of point-like charges. Unfortunately, as already mentioned, the perturbative expansion involved in this procedure is a low-energy (or a low-intensity field) approximation and is not accurate to describe the strong fields present near the center of the ESS solutions. It is thus necessary to explore this issue with other effective lagrangians, obtained from the perturbative renormalization of the self-energy of point-like fields. The analysis of this problem is in progress [75, 142].

Short-ranged solutions without any symmetry breaking mechanism.

Scalar field models as that of the example treated in section 6.4, which belong to the case B-3, exhibiting short-ranged SSS solutions (solitons or not), can be extended to generalized gauge field models supporting similar ESS solutions. This behaviour, which is related to the form of the lagrangian density around the vacuum, may arise in effective actions for some fundamental forces. If we assume the effective dynamics of the non-abelian gauge fields in electroweak interactions to be described by this kind of lagrangians, the short range of these forces could be explained in terms of the non-linear self-couplings among these fields, coming from the integration of some higher energy degrees of freedom of a more fundamental theory. In this case the appeal to any symmetry breaking or Higgs mechanism should become superfluous. In our sense, this alternative deserves to be thoroughly explored.

Soliton-based phenomenological approaches to hadron internal structure and hadronic interactions.

A new approach to the phenomenological description of the hadronic structure can be envisaged, using the results of chapter 5 on the spectral analysis of the excitations of the (multi-) scalar or generalized gauge-invariant solitons. In the phenomenological descriptions based on the Skyrme model the hadron arises as a topological soliton of a non-linear field theory. Other models, which are believed to give an effective low-energy approach to the non-perturbative regime of QCD (as the Friedberg-Lee model [16, 17, 18, 19] and related theories), describe hadrons as confined states of quarks in non-topological solitonic bags of non-linear phenomenological fields. As an alternative to these viewpoints, we are considering a generalized gauge-invariant lagrangian model for gauge fields ("gluons") coupled to a quark-like fermionic sector and implementing properly chosen symmetries. Such a phenomenological model may be interpreted as an effective lagrangian for QCD or, alternatively, as a field-theoretical low-energy limit of string theory. The classical generalized gauge-field lagrangian can exhibit soliton solutions in absence of other fields. If such solutions are minima of the functional of energy of the full action, their small perturbations will involve fermionic and bosonic modes. The quantization of these modes leads to "quasi-quarks" and "quasi-gluons" as quantum excitations of the soliton field. This quantum extension becomes a model for the hadron containing these particles. In this picture the confinement would be a consequence of the fact that quarks are quasi-particles associated to these quantum excitations and (as the phonons in a solid) they cannot exist outside the hadron.

Self-gravitating configurations for scalar and gauge field models.

Another domain where the results of this work could be useful concerns the search for self-gravitating (scalar and gauge) field configurations in General Relativity [39, 40, 41, 51, 52]. The classification of the lagrangian field theories considered here, supporting non-topological solitons in flat space, can be extended to the static spherically symmetric solutions of the Einstein equations resulting from the coupling of these fields to gravitation. Indeed, we have verified (see appendix A) that these equations have first-integrals which have the same form as (or can be closely related to) the ones obtained from the corresponding field theories in flat space (of the generic form of Eqs.(3.4) or (4.20)). This result opens the possibility of generalizing to the gravitational case many of the methods and results obtained here [74]. We will continue to address this topic in future work.

Other ansantzs?.

It would be also interesting to study the soliton solutions of generalized non-abelian gauge field theories with other ansantzs than the ESS one. As mentioned in chapter 2, such a kind of solutions have been already found for the SU(2) non-abelian BI theory within the "monopole ansantz" $A_0^a = 0, A_i^a = \epsilon_{aij}\frac{x^j}{r^2}(1-\omega(r))$ [29] (see section 2.3.2). For general SU(2) non-abelian gauge field theories in this ansantz the field equations (4.15) read, in our notation, after proper normalization of the action

$$\left(\frac{\partial\varphi}{\partial X}\omega'\right)' = \frac{\partial\varphi}{\partial X}\frac{\omega(\omega^2-1)}{r^2}, \tag{7.2}$$

and new conditions for stable and finite-energy non-abelian solutions must be found. This issue should be a theme for a future investigation.

Appendix A

Self-gravitating ESS solitons

Let us consider some aspects of the problem of ESS soliton solutions in General Relativity, which we are currently addressing with the methods developed here. The general form of the action for a generalized electromagnetic theory coupled to gravity is given by

$$S = S_G + S_M = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R - \varphi(X,Y)\right], \tag{A.1}$$

where R is the scalar of curvature and g the determinant of the metric tensor $g_{\mu\nu}$. The field equations of the models (A.1) take the form

$$\nabla_\mu\left(\frac{\partial\varphi}{\partial X}F^{\mu\nu} + \frac{\partial\varphi}{\partial Y}F^{*\mu\nu}\right) = 0. \tag{A.2}$$

The gauge-invariant symmetric energy-momentum tensor reads

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = 2\left(\frac{\partial\varphi}{\partial X}F_{\mu\alpha}F^{\alpha}_{\nu} - \frac{\partial\varphi}{\partial Y}F_{\mu\alpha}F^{*\alpha}_{\nu}\right) - g_{\mu\nu}\varphi(X,Y). \tag{A.3}$$

We are interested in static, spherically symmetric solutions of Einstein-generalized electromagnetic theories. The metric for such a configuration can be cast into the form

$$ds^2 = e^{\nu(r)}dt^2 - e^{\mu(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{A.4}$$

where $\nu(r)$ and $\mu(r)$ are functions to be determined. For ESS solutions in this metric the field equations (A.2) can be written in terms of the electrostatic potential $A_0(r)$ as

$$\frac{d}{dr}\left(e^{-\left(\frac{\nu+\mu}{2}\right)}A_0'(r)\frac{\partial\varphi}{\partial X}(X, Y=0)\right) = 0. \tag{A.5}$$

The components of energy-momentum tensor in this case are given by

$$T_0^0 = T_1^1 = 2\frac{\partial\varphi}{\partial X}e^{-(\nu+\mu)}A_0'^2 - \varphi(X) \quad ; \quad T_2^2 = T_3^3 = -\varphi(X), \tag{A.6}$$

and the three independent Einstein equations may be written in the form

$$\begin{aligned} G_0^0 &= T_0^0 \to e^{-\mu}\left(-\frac{\mu'}{r} + \frac{1}{r^2}(1-e^{\mu})\right) = 2\frac{\partial\varphi}{\partial X}e^{-(\nu+\mu)}A_0'^2 - \varphi(X) \quad (I) \\ G_1^1 &= T_1^1 \to e^{-\mu}\left(\frac{\nu'}{r} + \frac{1}{r^2}(1-e^{\mu})\right) = 2\frac{\partial\varphi}{\partial X}e^{-(\nu+\mu)}A_0'^2 - \varphi(X) \quad (II) \\ G_2^2 &= T_2^2 \to e^{-\mu}\left(-\frac{\nu''}{2} + \frac{\nu'}{4}(\mu'-\nu') + \frac{1}{2r}(\mu'-\nu')\right) = \varphi(X) \quad (III), \end{aligned} \tag{A.7}$$

Due to the symmetry of the energy-momentum tensor $T_0^0 = T_1^1$, Einstein equations *(I)* and *(II)* combine to give $\mu' + \nu' = 0 \to \mu + \nu = \lambda$, where the constant of integration λ can be put equal to zero by a redefinition of the time coordinate. Then, without loss of generality we can write

$$ds^2 = g(r)dt^2 - \frac{dr^2}{g(r)} - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{A.8}$$

where we have defined $g(r) = e^{\nu(r)}$ [1]. With this form of the metric, the field equations (A.5) have a first-integral which is precisely given by the same equation obtained for the flat space case, namely, Eq.(4.20) and thus has no explicit dependence on the metric. When the Einstein equations (A.7) are satisfied, the field equations are a consequence of the Bianchi identities $\nabla_\mu G^{\mu\nu} = 0$. However, the first integral (4.20) is very useful for our purposes, as we shall see later. Thus, the remaining equations we have to solve are

$$\begin{aligned} \frac{d}{dr}(r(g(r)-1)) &= 2r^2\frac{\partial\varphi}{\partial X}A_0'^2 - r^2\varphi(X) \\ \frac{d^2}{dr^2}(rg(r)) &= -2r\varphi(X), \end{aligned} \tag{A.9}$$

where we have done some manipulations starting from (A.7). Integrating any of these equations, using the first integral (4.20), and comparing with Eq.(A.6) we arrive to the expression for the metric

[1] See [143] for generalizations of Birkhoff's theorem, applied to theories whose energy-momentum tensor satisfies $T_0^0 = T_1^1$, as in the present case.

$$g(r) = 1 - \frac{2M}{r} + \frac{1}{r}\int_r^\infty \tilde{r}^2 T_0^0(\tilde{r}) d\tilde{r}. \tag{A.10}$$

According to this, we define an "effective mass" $M(r) = M - \epsilon(r)$ where $\epsilon(r) = \frac{1}{2}\int_{\tilde{r}}^\infty \tilde{r}^2 T_0^0(\tilde{r})$ which allow us to write

$$g(r) = 1 - \frac{2M(r)}{r}, \tag{A.11}$$

so M is the gravitational mass given by $M = \lim_{r\to\infty} M(r)$ and satisfies $M'(r) = \epsilon'(r) = \frac{1}{2}r^2 T_0^0$. Some comments are in order. First, the quantity $\epsilon(r)$ that we have defined is just the expression of the energy density (integrated from r to ∞) for a central electrostatic field for a NED in *Minkowski* space. Second, the metric is reduced to well-known solutions in some particular choices of the function $\varphi(X,Y)$. For example, for the Maxwell case we have $\varphi(X,Y) = X$ and $M(r) = M - \frac{q^2}{2r}$ giving rise to the well-known metric $g(r)$ for the Reissner-Nördstrom solution $g(r) = 1 - \frac{2M}{r} + \frac{q^2}{r^2}$. For Born-Infeld electrodynamics [3], choosing the BI Lagrangian as $\varphi(X, Y=0) = 2\beta^2\left(1 - \sqrt{1 - \frac{X}{\beta^2}}\right)$ and using the expression of the BI electric field $E(r) = \frac{q}{\sqrt{r^4 + q^2/\beta^2}}$ the metric takes the form [41, 144, 145]

$$g(r) = 1 - \frac{2M(r)}{r} \;;\; M(r) = M - \beta\int_r^\infty \left(\sqrt{\beta^2\tilde{r}^4 + q^2} - \beta\tilde{r}^2\right) d\tilde{r}, \tag{A.12}$$

The expression (A.11) can be expanded to give

$$g(r) = 1 - \frac{2M}{r} + \frac{1}{3}\left[L(A_0'^2(r))r^2 - 2q\left(A_0'(r) - \frac{2}{r}\int_r^\infty A_0'(\tilde{r})d\tilde{r}\right)\right], \tag{A.13}$$

which can be useful in some calculations. For example, for BI electrodynamics we obtain the black hole solution found by Garcia-Salazar-Plebañski [39].

$$g(r) = 1 - \frac{2M}{r} + \frac{2\beta^2 r^2}{3}\left(1 - \sqrt{1 + \frac{q^2}{\beta^2 r^4}}\right) + \frac{4q^2}{3r}\int_r^\infty \frac{d\tilde{r}}{\sqrt{\tilde{r}^4 + q^2/\beta^2}}. \tag{A.14}$$

From the expression (A.11) we see that for a particular model the horizons are given by the roots of the equation $r = 2M(r)$ and the number of horizons depends on the sign of the quantity $M(0) = M - \epsilon$ where $\epsilon = \epsilon(0)$ is the total energy of the electromagnetic field in Minkowski space and $M(0)$ may be interpreted as the "binding energy". For $M(0) > 0$

there is exactly one non-degenerate horizon. The case $M(0) < 0$ has a similar behavior than Reissner-Nördstrom solution, with either having zero horizons, one degenerate horizon or two non-degenerate horizons, depending of the values of the parameters of the model. Finally, if $M(0) = 0$ the metric can have one non-degenerate horizon or none (again depending on the parameters of the model). In fact, the requirement that there be no infinities in the metric $g(r)$ forces this identification of the gravitational mass M with the electromagnetic mass ϵ.

Since we can write the metric (A.10) as

$$g(r) = 1 - \frac{2}{r}(M - \epsilon) - \frac{1}{r}\int_0^r \tilde{r}^2 T_0^0(\tilde{r}) d\tilde{r}, \tag{A.15}$$

for the case $M(0) = 0$ ($M = \epsilon$) we have

$$g(r) = 1 - \frac{1}{r}\int_0^r \tilde{r}^2 T_0^0(\tilde{r}) d\tilde{r}, \tag{A.16}$$

which has for large r the Schwarzschild asymptotic behaviour $g(r) = 1 - \frac{2m}{r}$ where $m = \frac{1}{2}\int_0^\infty r^2 T_0^0(r)$. Since $T_0^0(r)$ coincides with the expression of the energy-momentum tensor in flat space, the functions $\varphi(X, Y)$ supporting finite-energy ESS solutions generate non-divergent metric functions (within the identification $M = \epsilon$) as well as finite energies ϵ. However, although one is able to obtain a everywhere non-singular metric function, the curvature invariants blow up at the center of the solutions as a consequence of a theorem [43, 146], which establishes the non-existence of electrically charged solutions with Maxwell weak-field asymptotic limit having a regular center [2].

On the other hand, when the identification $M = \epsilon$ is made, the expression (A.13) is replaced by

$$g(r) = 1 + \frac{1}{3}\left[L(A_0'^2(r))r^2 - 2q\left(A_0'(r) + \frac{2\int_0^r A_0'(\tilde{r})d\tilde{r}}{r}\right)\right] \tag{A.17}$$

For BI electrodynamics this expression gives

$$g(r) = 1 + \frac{2\beta^2 r^2}{3}\left(1 - \sqrt{1 + \frac{q^2}{\beta^2 r^4}}\right) - \frac{4q^2}{3r}\int_0^r \frac{d\tilde{r}}{\sqrt{\tilde{r}^4 + q^2/\beta^2}} \tag{A.18}$$

which is the particle-like solution found by Demianski [40].

[2]Although the theorem *might* be circumvented if one considers models whose associated electromagnetic fields do not have the Maxwell weak-field limit, such as the cases B-1 and B-3 considered in chapter 3.

Appendix B

Extended electric-magnetic duality

This appendix is concerned with *electric-magnetic duality rotations*. As is well-known, in absence of external sources Maxwell equations are invariant under SO(2) rotations of electric and magnetic fields into each other. This duality transformation can be written under the form

$$F_{\mu\nu} \to \cos(\alpha)F_{\mu\nu} + \sin(\alpha)F^*_{\mu\nu}, \tag{B.1}$$

which in terms of the fields reads

$$\begin{aligned} \vec{E} &\to \cos(\alpha)\vec{E} - \sin(\alpha)\vec{H} \\ \vec{H} &\to \sin(\alpha)\vec{E} + \cos(\alpha)\vec{H}. \end{aligned} \tag{B.2}$$

However, as soon as one introduces generalized models through the function $\varphi(X, Y)$ this duality invariance property is lost. Nevertheless, it is possible to define generalized "electric-magnetic" dualities for the extended models. For example, the covariant generalization of invariance (B.2) was analyzed in [145] and amounts to define the transformation

$$\begin{aligned} F_{\mu\nu} &\to \cos(\alpha)F_{\mu\nu} - \sin(\alpha)P^*_{\mu\nu} \\ P_{\mu\nu} &\to \sin(\alpha)F^*_{\mu\nu} + \cos(\alpha)P_{\mu\nu}, \end{aligned} \tag{B.3}$$

(see section 4.1.2 for the definition of $P_{\mu\nu}$). In determining the conditions for a generalized electromagnetic field theory to be invariant under these extended duality rotations it was shown in [145] that the following equation

$$F_{\mu\nu}F^{*\mu\nu} = P_{\mu\nu}P^{\mu\nu}, \tag{B.4}$$

which, written in terms of our field invariants, reads

$$\frac{Y}{4} = \left(\frac{\partial\varphi}{\partial X}\right)^2 Y - \left(\frac{\partial\varphi}{\partial Y}\right)^2 Y - 2\left(\frac{\partial\varphi}{\partial X}\right)\left(\frac{\partial\varphi}{\partial Y}\right) X, \tag{B.5}$$

must be fulfilled by the lagrangian densities which have the Maxwell weak-field limit. It can be easily checked that BI theory (4.5) satisfies the above equation and thus it is invariant under this extended duality. It would be also interesting to determine other lagrangians, belonging to the class of soliton-supporting theories considered here, which are duality-invariant. We shall leave this issue for future developments.

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Abstract

In this thesis we perform a general analysis of the dynamic structure of two classes of relativistic lagrangian field theories exhibiting static spherically symmetric non-topological soliton solutions. The analysis is concerned with (multi-) scalar fields and generalized gauge fields of compact semi-simple Lie groups. The lagrangian densities governing the dynamics of the (multi-) scalar fields are assumed to be general functions of the kinetic terms, whereas the gauge-invariant lagrangians are general functions of the field invariants. These functions are constrained by requirements of regularity, positivity of the energy and vanishing of the vacuum energy, defining what we call "admissible" models. In the scalar case we establish the general conditions which determine exhaustively the families of admissible lagrangian models supporting this kind of finite-energy solutions. Next, we analyze the gauge field case, where we add the requirement of parity invariance to the admissibility constraints. We determine the general conditions defining the families of admissible gauge-invariant models exhibiting finite-energy electrostatic spherically symmetric solutions. We then establish a correspondence between any admissible soliton-supporting (multi-) scalar model and a family of admissible generalized gauge models supporting finite-energy electrostatic point-like solutions. Conversely, for each admissible soliton-supporting gauge-invariant model there is an associated *unique* admissible (multi-) scalar model with soliton solutions. From the variational analysis of the energy functional, we show that the admissibility constraints and the finiteness of the energy of the scalar solitons are necessary and sufficient conditions for their linear static stability against small charge-preserving perturbations. Furthermore we perform a general spectral analysis of the dynamic evolution of the small perturbations around the statically stable solitons, establishing their dynamic stability. In the gauge field case the variational analysis of the energy functional leads now to supplementary restrictions to be imposed on the lagrangian densities in order to ensure the linear stability of the solitons. We finally analyze some explicit examples of these different families, which are defined by the asymptotic and central behaviour of the fields of the corresponding particle-like solutions.