Diagrams of fibrations and fibrewise cellularization

Carlos A. Giraldo Hernández

Thesis presented to the

Department of Mathematics of the

Universitat Autònoma de Barcelona

Director: Dr. Carles Broto Blanco

CERTIFICO que esta memoria ha sido realizada por Carlos A. Giraldo Hernández bajo la dirección del Dr. Carles Broto Blanco

Bellaterra, Octubre de 2012.

Dr. Carles Broto Blanco.

A Dios que me ha levantado cuantas veces he necesitado

A mi familia que me ha amado incondicionalmente

A mis amigos que han estado ahí para escucharme

CONTENTS

INTRODUCTION

Given a map $g : C \to D$ and a fibration $p : E \to B$ with fibers of the homotopy type of F, it is well known that we can always obtain a fibrewise localization of p under the localization functor L_g; although the same result is not true in general for the A-cellularization functor CW_A of pointed topological spaces. So our initial interest was to find conditions over the base space B of p and over its fibers, in order to determine when p admits a fiberwise cellularization, that is, a commutative diagram like the following

$$\begin{tikzcd}
CW_AF \arrow[r] \arrow[d, "c_F"'] & \overline{E} \arrow[r, "q"] \arrow[d, "b"'] & B \arrow[d, equal] \\
F \arrow[r] & E \arrow[r, "p"'] & B
\end{tikzcd}$$

where $c_F : CW_A \to F$ is the augmentation map for F, q is a fibration and b is a map that induce the A-cellularization of homotopy fibers of p. Another natural question around this situation is: if p admits a diagram like this, is it unique as it happens in the fibrewise localization of p?.

The second question suggested us that a previous step to solve this problem has to do with the homotopy classification of the above diagrams of fibrations in terms of its fiber, that is, we should be able to find a universal diagram of fibrations, with fiber $c_F : CW_AF \to F$, that allow us to obtain the above diagrams as pullbacks of the universal one, as it happens in the classical case. By following the classical way we can define $aut(c_F)$ as the topological submonoid of $aut(CW_AF) \times aut(F)$ that consists of pairs (α, β), such that $\beta c_F \simeq c_F\alpha$, where

$aut(CW_AF)$ and $aut(F)$ are the monoids of self homotopy equivalences of CW_AF and F, respectively.

The monoid $aut(c_F)$ acts on the right of the universal contractible space $Eaut(c_F)$, and on the left of the spaces CW_AF and F. Then we can consider the following Borel construction [2]:

$$
\begin{tikzcd}
CW_AF \arrow[r, "c_F"] \arrow[d] & F \arrow[d] \\
Eaut(c_F) \times_{aut(c_F)} CW_AF \arrow[r] \arrow[d] & Eaut(c_F) \times_{aut(c_F)} F \arrow[d] \\
Baut(c_F) \arrow[r, equal] & Baut(c_F)
\end{tikzcd}
$$

So by pulling-back this diagram with a map $B \to Baut(c_F)$, we obtain diagrams with fiber $c_F : CW_AF \to F$. Thus, the next step is to proof that this diagram is universal and classifies all the diagrams of fibrations with fiber c_F.

We considered that the proper context to get this is the simplicial one, furthermore, this classification problem is independent of the functor CW_A and is part of a general classification problem, that is, if $\mathbf{S}$ is the category of simplicial sets and $\mathcal{C}$ is a small category, what we want is to classify fibrations in the category $\mathbf{S}^{\mathcal{C}}$ of functors from $\mathcal{C}$ to $\mathbf{S}$ (or the category of $\mathcal{C}$-diagrams in $\mathbf{S}$). In this case we fix a $\mathcal{C}$-diagram $\mathbf{F}$ and consider fibrations $p : \mathbf{E} \to B$ with fiber of the homotopy type of $\mathbf{F}$ and base space the constant $\mathcal{C}$-diagram to the space B of $\mathbf{S}$.

The classical case that corresponds to the category $\mathcal{C}$ with one object and no non-identity morphisms was solved by Barrat [1] around 1958. In this situation to classify fibrations $p : E \to B$ with fiber F, the firs step is to replace p by a deformation retract of it which is a minimal fibration, and then to show that a minimal fibration is a fibre bundle. Next, each fiber bundle can be considered a twisted cartesian product of its base B and its fiber F, and each of them is determined by a twisting function $t : B \to aut(F)$. Finally they show that the equivalence classes of such twisted products is in one to one correspondence with the homotopy classes of maps from B to $\overline{W}aut(F)$.

What we do in this memory is to complete this classical sequence followed to classify fibrations in $\mathcal{S}$ in order to classify $\mathcal{C}$-diagrams of fibrations in $\mathbf{S}^{\mathcal{C}}$. In this case the classification theorem can be established as follows:

Theorem. *Let $\mathcal{C}$ be a small EI-category with a finite number of objects, and* $\mathbf{F}$ *and* $\mathbf{B}$*, C-diagrams, where* $\mathbf{B}$ *is the constant diagram to the connected simplicial set B. Then the set $[B, \overline{W}haut(\mathbf{F})]$ of homotopy classes of maps from B to $\overline{W}haut(F)$ is in bijective correspondence with the set of equivalence classes of $\mathcal{C}$-diagrams of fibrations with base space B and fibers with the homotopy type of* $\mathbf{F}$.

By using this result we are able to express the existence and uniqueness of the fibrewise cellularization, for a given fibration p, in terms of the obstructions to the existence of a lifting of certain map.

Below we summarize briefly the work done in this memory, and we refer the reader to each chapter for further details on a specific subject.

In the Chapter 0 we exposed the basic definitions and results about simplicial sets. Some proofs about fiber bundles and twisted cartesian products are included, since the comprehension of the combinatorial calculations involved to the level of spaces was fundamental to understand the behavior of the situation in which we have many spaces connected by arrows between them. Although some proofs are briefly commented in the literature, mainly in [21] and [1], we made the exercise to complete them in a detailed way, as for example, the Propositions 0.1.28 and 0.1.29.

We also introduce some basic language about model categories, simplicial model categories and cofibrantly generated model categories. By exploiting the cofibrantly generated model structure over $\mathbf{S}^{\mathcal{C}}$ we are able to generalize the classical concepts about minimal fibrations, fibre bundles, or structural group, to the case of $\mathcal{C}$-diagrams of fibrations over a constant base space. We finish the chapter by introducing the localization functor L_g and the cellularization functor CW_A.

The first chapter introduces the notion of free diagram, as is defined by Philip Hirschhorn in [16], and its properties are given in terms of adjoint pairs of functors. The free diagrams on the standard n-simplex $\triangle[n]$ will be the basic pieces through we will study the $\mathcal{C}$-diagrams, since as we have shown in Proposition 1.1.8, any diagram can be recover through a colimit process by these objects. The structure of cofibrantly generated model category over $\mathbf{S}^{\mathcal{C}}$ is also introduced and the $\mathcal{C}$-fibre bundles and $\mathcal{C}$-twisted cartesian products are introduced as a natural generalization of the classical ones in $\mathbf{S}$.

We also establish a classification result for $\mathcal{C}$-fiber bundles an hence for $\mathcal{C}$-twisted cartesian products in Theorem 1.4.12. Finally we describe the automorphism group $aut(\mathbf{F})$ by using free diagrams and also by taking the description given by Dwyer and Kan in [7]. With the first description given we are able to find an equivalent definition of $\mathcal{C}$-fiber bundle in terms of free diagrams, as it is shown in Appendix C.

The Chapter 2 starts with preliminary concepts and properties about the homotopy relation in $\mathbf{S}^{\mathcal{C}}$. In the second section we define the sub-homotopy relation between n-simplices, which is used to compare homotopycally different pieces of orbits of simplices. By using these relation in free cell complexes we are able to fit (homotopically) whole orbits of n-simplices into another orbits, thus the concept of free cell complex is quite important, since it is the one for which we can formulate the definition of minimal diagram.

At this point the Quillen's small object argument exposed in Appendix B is also important. This argument is the first additional steep added to the classical sequence followed to classify fibrations, since it allow us to obtain from a fibration p with fiber F an equivalent one whose total space is a free cell complex. For fibrations whose total space is a free cell complex the well known results about minimal fibrations are proved.

Joining the previous arguments with the ones given in the Chapter 1 we can establish a classification theorem in terms of the fiber F' obtained from the minimal fibrations constructed. The second step added to finish the classification consists in to connect the homotopy type of the monoid $aut(F)$ with the one of $aut(F')$, by using [7].

It is important to remark that in order to obtain a minimal fibrations some restrictions over the category $\mathcal{C}$ are necessary, it becomes clear in Proposition 2.2.6 and the Appendix A.

In Chapter 3 we back again to the fibrewise cellularization problem for a given fibration $p : E \to B$. As we already know, the fibration p is classified by a map $h : B \to Baut(F)$, then by our classification theorem it holds that p has a fiberwise cellularization, if in the following diagram h has a lift

$$\begin{tikzcd}
 & Baut(c_F) \arrow[d] \\
B \arrow[ur, dotted, "\widetilde{h}"] \arrow[r, "h"] & Baut(F)
\end{tikzcd}$$

It holds that the obstructions to the existence and uniqueness of this lifting lies in $H^{i+1}(B, \pi_i(L))$ and $H^i(B, \pi_i(L))$, respectively. So in the first part of this chapter we strive to characterize the homotopy fiber of $Baut(c_X) \to Baut(X)$, and in the last section we give some examples where it is possible to obtain a fiberwise cellularization.

With respect to the classification problem it is important to remark that recently, Blomgren and Chachólski [Martin Blomgren, Wojciech Chachólski. *On the Classification of Fibrations*, Preprint, 2012] have independently obtained a classification theorem for fibrations in general model categories $\mathcal{M}$ using different methods.

AGRADECIMIENTOS

Lo que en un principio surgió como un problema de clasificación de diagramas de apariencia sencilla y cuya solución estaba destinada a comprender ciertas celularizaciones, se fue convirtiendo en un proyecto de mayores dimensiones. A Carles Broto y a Ramón Flores quiero expresarles mi gratitud por permitirme ser parte de este proyecto y por los conocimientos compartidos durante las largas sesiones de trabajo.

El poder comprender lo que significa ser minimal en el contexto de los diagramas de conjuntos simpliciales fue posible gracias al paciente acompañamiento y a las rigurosas observaciones de Carles. *No solo tienes talento, también lo cultivas dedicando gran parte de tú tiempo a las matemáticas, has sido un maestro para mí.*

Ramón, muy entusiastamente, nos impulsó a estudiar el funtor de A-celularización. Su conocimiento en el área han fácilitado mi comprensión sobre el problema de celularizar fibra a fibra. Los encuentros de topología y otros escenarios han sido los espacios propicios para poder confluir. *Gracias por movilizarte y por dedicar tú tiempo a la discusión de estas cuestiones.*

Agradezco al Grupo de Topología Algebraica de Barcelona la formación que me ha dado durante estos últimos cuatro años a través del seminario de topología, del seminario junior y de los diferentes cursos y eventos que ha organizado. Con ellos me inicié en teoría de homotopía y he tenido la oportunidad de hacer topología algebraica. *Para mí son unos Gigantes, el listón a superar.*

Gracias al Departamento de Matemáticas de la Universidad Autónoma de

Barcelona por proporcionarme los medios necesarios para poder llevar a cabo mi investigación. En especial quiero agradecer al personal de secretaría del departamento por haber colaborado en mis solicitudes cuantas veces lo he necesitado. No saben lo importante que es esto, sobre todo cuando uno es extranjero.

Agradezco al Ministerio de Ciencia e Innovación de España por haberme concedido una beca FPI para realizar mis estudios de doctorado en Barcelona. Su soporte económico y el del grupo de Topología Algebraica de Barcelona me han permitido desplazarme y ampliar mi panorama académico.

I wish to thank the Centre for Simmetry and Deformation of the University of Copenhagen for their hospitality during the three months that I was there, specially to the professor Jesper Michael Moller and Sara Arklint. I'm also grateful to professor Wojciech Chacholski and Martin Blomgren for sharing their knowledge during the days that I spent in Stockholm.

A mis colegas del departamento les quedo muy agradecido por los amenos momentos de conversación que permiten distensionar la mente. A Alberto, con quien he compartido a la par esta aventura, a Wolfgang y Natàlia, quienes siempre estuvieron dipuestos a escucharme y a aclarar mis dudas, a Alex, por su buen rollo y por haberse tomado el tiempo de leer parte de esta memoria y hacer sugerencias, a Albert, que hizo que todo fuese más fácil en Copenhaguen, a Elena, quién muy pacientemente ha estado ahí para ayudarme, a Victor, por su hospitalidad y generosidad, a Simone, quien me ha soportado como compañero de despacho y a Isabel, que con su calidez humana hizo más sencillo mi acoplamiento con el grupo.

Agradezco a Laia y a David el haberme hecho parte de sus costumbres y por permitirme compartir con sus familias, a Paula por estar presente para escucharme y apoyarme y a mis compañeros de piso de Barcelona y Cerdanyola, con quienes he compartido momentos agradables. *Vuestro apoyo emocional ha sido invaluable.*

Agradezco a mis amigos de Colombia que me animaron a cruzar el océano para emprender esta travesía, ellos fueron quienes me proporcionaron ese impulso inicial. A mi familia, gracias por su confianza y comprensión, en especial a mis padres y a mi madrina, que siempre han estado presentes para apoyarme y para quererme. *Cuán importante ha sido el que crean en mí.*

Finalmente quiero agradecer a Deisy por ayudarme con mi inglés al escribir

esta memoria.

CHAPTER 0

PRELIMINARY NOTIONS ABOUT SPACES

In this chapter we introduce the category of Simplicial Sets $\mathbf{S}$, or the category of Spaces, and its properties as a closed model category. Some definitions and properties about simplicial sets are necessary to deal with diagrams of spaces, since many of them will be generalized, for example those concerning n-simplices, locally trivial maps, twisted cartesian products and fibre bundles among others.

In this chapter after introduce the category of simplicial sets, in which the concept of Kan fibration is central, we define what a fibre bundle is, and following the classical approach we show how the transformation elements for a given atlas of a bundle are obtained (basically we followed Barrat's paper [1] and the one of Curtis [4]). We proceed in this way because the classical reasonings admit a natural generalization in the case of $\mathcal{C}$-diagrams of fibre bundles, as for example the Proposition 0.1.19 that allow us to find regular atlasses. In a similar way the twisted cartesian products are studied.

The section 2 consists of a battery of definitions and main results about model categories. In particular we are interested in the different ways to obtain a structure of cofibrantly generated model categories for a given category $\mathfrak{M}$, since in the case of spaces it allow us to determine the form of trivial cofibrations in $\mathbf{S}^{\mathcal{C}}$, and hence the one of fibrations, by knowing how cofibrations and trivial cofibrations are in $\mathbf{S}$. Not less important are the results about simplicial model categories from which we can have the valuable covering homotopy property for fibrations.

Finally we introduce two homotopy idempotent functors: the localization functor L_f with respect to a map $f : A \to B$ in $\mathbf{S}$ between cofibrant spaces, and the cellularization functor CW_A with respect to a pointed and connected space A. The concepts of fibrewise localization and fibrewise cellularization are also introduced.

0.1 Simplicial Sets

Let Δ be the category whose objects are finite, non-empty, totally ordered sets $[n] = \{0, 1, ..., n\}$ and whose morphisms are the order preserving functions. If $\mathfrak{Sets}$ is the category of sets, the category of contravariant functors $\mathfrak{Sets}^{\Delta^{op}}$ from Δ to $\mathfrak{Sets}$ is called the category of *Simplicial Sets*, also known as the *Category of Spaces* . To shorten notation this category will be denoted by $\mathbf{S}$, and for a given simplicial set $X \in \mathbf{S}$, X_n will stand for the set $X[n]$ (its elements will be called *n-simplices*).

Notice that in Δ for every n and $i = 0, 1, ..., n$ there exists a unique injective morphism $\varepsilon_i : [n-1] \to [n]$ whose image misses i and a unique surjective map $\sigma_i : [n+1] \to [n]$ with two elements mapping to i. It is not difficult to show that every morphism $\beta : [n] \to [m]$ in Δ has a unique *epi-monic factorization* $\beta = \varepsilon\sigma$, where the monic ε can be uniquely expressed as a composition of maps $\varepsilon = \varepsilon_{k_s}...\varepsilon_{k_1}$ with $0 \le k_1 < ... < k_s \le m$ and the epi σ is uniquely a composition of maps $\sigma = \sigma_{j_1}...\sigma_{j_t}$ with $0 \le j_1 < ... < j_t \le n$ (see [23, Chapter 8] for further details of this factorization).

Exploiting this property we can prove that in order to define a simplicial set, it is necessary and sufficient to give a sequence of sets X_0, X_1,X_2,... together with operators $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ for $i = 0, 1, ..., n$, which satisfy the following identities

$$\begin{cases} d_i d_j = d_{j-1} d_i & \text{if } i < j \\ s_i s_j = s i_{j+1} s_i & \text{if } i \le j \end{cases} \qquad d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ id & \text{if } i = j, i = j+1 \\ s_j d_{i-1} & \text{if } i > j+1 \end{cases}$$

The operators d_i and s_i are called *faces* and *degeneracies* respectively, and the above identities are known as *simplicial identities.* A subset Y of a simplicial set X forms or is a *simplicial subset* of X if it is closed under these operators. A simplex $x \in X_n$ will be called *degenerate* if it is the image of some degeneracy map s_i and *non-degenerate* otherwise. The set of degenerate simplices of X will be denoted by $X^\flat$.

Definition 0.1.1 ([21]) *A simplicial map $f : X \to Y$ between the simplicial sets X and Y is a collection of set maps $f_n : X_n \to Y_n$, $n \geq 0$, which commutes with the face and degeneracy operators, that is, $f_n \circ d_i = d_i \circ f_{n+1}$ and $f_n \circ s_i = s_i \circ f_{n-1}$.*

As a consequence of the epi-monic factorization of morphisms in Δ, every composition of faces and degeneracies can be expressed as follows.

Lemma 0.1.2 ([15], *Lemma* **1)**
Every operator which is composition of $d_i's$ and $s_j's$ can be written uniquely in the form

$$s_{j_t}...s_{j_1}d_{k_1}...d_{k_s}$$

where $j_1 < ... < j_t$ and $k_1 < ... < k_s$ (the cases t=0 and s=0 are included).

Such operators are called *semisimplicial operators* and after re-expressing these by using the lemma, the difference $(t - s)$ is called the *height*.

Example 0.1.3 *The simplest examples of simplicial sets are the well known* standard n-simplices $\triangle[n]$. *For $[n] \in \Delta$, $\triangle[n]$ is the* represented functor $\Delta(-, [n]) : \Delta^{op} \to \mathfrak{Sets}$ *that takes an object $[p] \in \Delta$ to $\Delta([p], [n])$ and a morphism $\alpha : [q] \to [p]$ to the precomposition arrow $\alpha^* : \Delta([p], [n]) \to \Delta([q], [n])$. So the set of k-simplices of $\triangle[n]$ is $\Delta([k], [n])$ and its face and degeneracy maps d_i and s_i are given by precomposition in Δ with ε_i and σ_i, respectively.*

Identifying $\beta \in \Delta([k], [n])$ with its image $\beta([k])$, a k-simplex of $\triangle[n]$ can be seen as a sequence of integers $(a_0, ..., a_k)$ with $0 \leq a_0 \leq a_1 \leq ... \leq a_k \leq n$. Thus the semisimplicial operators are defined by $d_i(a_0, ..., a_k) = (a_0, ..., a_{i-1}, \widehat{a_i}, ..., a_k)$, and $s_i(a_0, ..., a_k) = (a_0, ..., a_i, a_i, a_{i+1}, ..., a_k)$, where $\widehat{a_i}$ means that we skip the i-th element. We let ι_n denote $(0, 1, 2, ..., n) \in \triangle[n]$.

For our purposes one alternative way to define $\triangle[n]$ is as follows: let ι_n be a symbol, then $\triangle[n]_k$ is the set of all elements $\varphi\iota_n$, where φ is any semisimplicial operator with domain n and height $k - n$; its faces and degeneracies are defined by $d_i(\varphi\iota_n) = (d_i\varphi)\iota_n$ and $s_i(\varphi\iota_n) = (s_i\varphi)\iota_n$. The complex $\triangle[n]$ contains subcomplexes $\dot{\triangle}[n]$ (*boundary of* $\triangle[n]$) and $\Lambda^k[n]$, $0 \leq k \leq n$ (k^{th}-*horn*), where $\dot{\triangle}[n]$ is the smallest subcomplex of $\triangle[n]$ containing the faces $d_i\iota_n$, $0 \leq i \leq n$, and the

k^{th}-horn ($n \geq 1$) is the subcomplex of $\triangle[n]$ generated by all faces $d_i\iota_n$, except for the face $d_k\iota_n$.

One version of Yoneda's Lemma [20, Chapter $III.2$] tells us that the set of maps $[n] \to [m]$ in Δ corresponds bijectively with the set of morphisms $\triangle[n] \to \triangle[m]$. Therefore the maps ε_i and σ_i in Δ correspond to maps $\varepsilon^i : \triangle[n-1] \to \triangle[n]$ and $\sigma^i : \triangle[n+1] \to \triangle[n]$ in $\mathbf{S}$, and these morphisms are defined explicitly by the formulas $\varepsilon^i(\varphi\iota_{n-1}) = \varphi d_i\iota_n$ and $\sigma^i(\varphi\iota_{n+1}) = \varphi s_i\iota_n$. For any simplicial set X a stronger version of Yoneda's Lemma also permits to give a natural bijective correpondence between X_n and the set of morphisms $\triangle[n] \to X$ in $\mathbf{S}$. Thus, for $x \in X_n$ its corresponding map $x : \triangle[n] \to X$ is called the *representing map for x* and is defined by $x(\varphi\iota_n) = \varphi x$.

Example 0.1.4 *Given two simplicial sets X and Y we can define with them other simplicial sets. The product $X \times Y$ is the simplicial set whose set of n-simplices is the cartesian product $X_n \times Y_n$, and for all $(x, y) \in (X \times Y)_n$ the faces and degeneracies are defined by $d_i(x, y) = (d_ix, d_iy)$ and $s_i(x, y) = (s_ix, s_iy)$, repectively. The* function complex *$hom(X, Y)$ has as set of n-simplices $hom(X, Y)_n$ the simplicial maps $Hom_{\mathbf{S}}(\triangle[n] \times X, Y)$. If $g \in Hom_{\mathbf{S}}(\triangle[n] \times X, Y)$, then the faces d_ig and the degeneracies s_ig are given by the following compositions $\triangle[n-1] \times X \xrightarrow{\varepsilon^i \times 1} \triangle[n] \times X \xrightarrow{g} Y$ and $\triangle[n+1] \times X \xrightarrow{\sigma^i \times 1} \triangle[n] \times X \xrightarrow{g} Y$, that is, $d_ig = g(\varepsilon^i \times 1)$ and $s_ig = g(\sigma^i \times 1)$ for $0 \leq i \leq n$.*

Sometimes it is convenient to replace a map $g : \triangle[n] \times X \to Y$ by the map $\widetilde{g} : \triangle[n] \times X \to \triangle[n] \times Y$ defined by $\widetilde{g}(t, x) = (t, g(t, x))$, where $t \in \triangle[n]$ and $x \in X$. Thus given $g \in hom(Z, Y)$ and $f \in hom(Y, X)$, $fg \in hom(Z, X)$ is defined by $\widetilde{fg} = \widetilde{f} \circ \widetilde{g}$. Under this operation the complex $hom(X, X)$ is a monoid and it operates on the left of X by $f.x = f(\iota_n, x)$, for $f \in hom(X, X)$ and $x \in X$ (see Definition 0.1.23). The group of automorphisms $aut(X)$ of X is the maximal subgroup $aut(X) \subseteq hom(X, X)$, so $f \in hom(X, X)$ belongs to $aut(X)$ if and only if $\widetilde{f}$ has an inverse.

From a homotopical viewpoint it is technically convenient to focus on certain simplicial sets that satisfies the extension condition.

Definition 0.1.5 ([21]) *A simplicial set X is said to satisfy the extension condition if for every colletion of $n+1$ n-simplices $x_0, x_1, ..., x_{k-1}, x_{k+1}, ..., x_{n+1}$ which* *satisfy* $d_i x_j = d_{j-1} x_i$, $i < j$, $i \neq k$, $j \neq k$, *there exists an* $(n+1)$*-simplex* x *such that* $d_i x = x_i$ *for* $i \neq k$.

Equivalently X satisfies the extension condition if for every simplicial map $f : \Lambda^k[n] \to X$ there is a map $\widehat{f} : \triangle[n] \to X$ such that $\widehat{f}|_{\Lambda^k[n]} = f$. If X satisfies the extension condition, it is called a *Kan complex* or *fibrant object* of $\mathbf{S}$. As in [21] the word 'complex' will always mean simplicial set.

If X is a fibrant simplicial set, it holds that $hom(Z, X)$ is fibrant as well [15], and if G is a simplicial group, that is, a simplicial set where every G_n is a group being its faces and degeneracies homomorphisms of groups, then G is fibrant [21].

Example 0.1.6 *If* G *is a simplicial group the simplicial set* $\overline{W}G$ *is defined by* $\overline{W}G_0 = \{*\}$ *and* $\overline{W}G_n = G_{n-1} \times G_{n-2} \times ... \times G_0$ *if* $n \geq 1$, *where the simplicial operators are given by* $s_0(*) = e_0$ *and* $d_i(g_0) = *$ *if* $i = 0, 1$. *If* $n > 1$, *then the face operators are defined by* $d_0(g_{n-1}, ..., g_0) = (g_{n-2}, ..., g_0)$, $d_i(g_{n-1}, ..., g_0) = (d_{i-1}g_{n-1}, d_{i-2}g_{n-2}, d_{i-3}g_{n-3}, ..., d_0 g_{n-i} g_{n-i-1}, g_{n-i-2}, ..., g_0)$, *and the degeneracies by* $s_0(g_{n-1}, ..., g_0) = (e_n, g_{n-1} ..., g_0)$. *The Kan complex* $\overline{W}G$ *is known as the bar construction of G and is called the classifying complex of G (it classifies principal fibrations with fiber G, see Theorem 0.1.32).*

Example 0.1.7 *If* X *is a topological space, it is possible to construct functorially an associated simplicial set* $Sing(X)$ *by letting the set of n-simplices* $Sing(X)_n$ *be the set of all continuous maps* $\triangle_n \to X$ *(where* $\triangle^n$ *is the topological standard n-simplex). If* $f \in Sing(X)_n$ *its faces and degeneracies are defined by* $(d_i f)(t_0, ..., t_{n-1}) = f(t_0, ..., t_{i-1}, 0, t_i, ..., t_{n-1})$ *and* $(s_i f)(t_0, ..., t_n) = f(t_0, ..., t_{i-1}, t_i + t_{i+1}, t_{i+2}, ..., t_n)$, *for* $1 \leq i \leq n$. *The Kan simplicial set* $Sing(X)$ *will be called the singular complex of* X *and the functor* $Sing : \mathfrak{Top} \to \mathbf{S}$ *will be known as the singular functor.*

Definition 0.1.8 ([21]) *Let* X *be a simplicial set. Two n-simplices* x *and* w *of* X *are said to be* homotopic $x \simeq w$, *if* $d_i x = d_i w$ *for* $0 \leq i \leq n$, *and if there exists an* $(n+1)$*-simplex* z *such that* $d_n(z) = x$, $d_{n+1} z = w$ *with* $d_i z = s_{n-1} d_i x = s_{n-1} w$ *for* $0 \leq i < n$. *The simplex* z *is said to be a homotopy between* x *and* w $(z : x \simeq w)$.

The homotopy relation may fail to be an equivalence relation in general. Consider the maps $\iota_0, \iota_1 : \triangle[0] \to \triangle[n]$, $n \geq 1$, which classify the vertices 0 and 1, respectively. The 1-simplex $(0, 1)$ is a homotopy between ι_0 and ι_1, but there is not a 1-simplex which could give a homotopy $\iota_1 \xrightarrow{\simeq} \iota_0$, since $0 \leq 1$. Although if X is a fibrant simplicial set the homotopy relation between the set of its n-simplices is an equivalence relation [14].

Definition 0.1.9 ([21]) *A Kan complex X is said to be minimal if $x \simeq y$ implies that $x = y$.*

Proposition 0.1.10 ([21])
Let x and w be simplices in a complex X. Suppose that both x and w are degenerate and that $d_i x = d_i w$ for all i, then $x = w$.

Definition 0.1.11 ([4]) *A simplicial map $p : X \to B$ is called a fibre map or a Kan fibration if for every commutative diagram*

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\ \triangle[n] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

the dotted arrow exists and makes the diagram commutative.

Definition 0.1.12 ([21]) *Let $p : X \to B$ be a simplicial map and let $x, w \in X$. x is said to be p-homotopic to w if a there is a homotopy $z : x \simeq w$, such that $pz = ps_n x$.*

To shorten the notation the p-homotopy relation between n-simplices x and w will be denoted by $x \simeq_p w$.

Definition 0.1.13 ([21]) *Let $f, g : X \to Y$ be simplicial maps. Then f is homotopic to g, $f \simeq g$, if there exist functions $h_i : X_n \to Y_{n+1}$, $0 \leq i \leq n$, which satisfy*

$$\begin{array}{ll} d_0 h_0 = f, \quad d_{n+1} h_n = g & d_i h_j = h_{j-1} d_i \quad if \quad i < j \\ s_i h_j = h_{j+1} s_i \quad if \quad i \leq j & d_{j+1} h_{j+1} = d_{j+1} h_j \\ s_i h_j = h_j s_{i-1} \quad if \quad i > j & d_i h_j = h_j d_{i-1} \quad if \quad i > j+1 \end{array}$$

Two complexes X and Y are said to be of the same homotopy type if there exists maps $f : X \to Y$ and $g : Y \to X$ which satisfy $gf \simeq 1_X$ and $fg \simeq 1_Y$. The homotopy relation between maps $f, g : X \to Y$ is not in general an equivalence relation, although if Y is a Kan complex it becomes an equivalence relation.

Lemma 0.1.14 ([21])
$\triangle[n]$ is contractible for all n.

Proof. Define $h_0(\iota_n) = s_0\iota_n$, $h_1(\iota_n) = s_0\iota_n$ and $h_i(\iota_n) = s_{i-1}...s_0d_1...d_{i-1}\iota_n$, $1 < i \leq n$. Then if $j : \triangle[n] \to (0) \subseteq \triangle[n]$ it holds that $h : 1 \simeq j$.

∎

0.1.1 Fibre bundles

Given an 0-simplex $v \in B$, we say that the *fiber of* $p : X \to B$ *over* v, $F_p(v)$, is the complex obtained from the following pullback

$$\begin{tikzcd} \triangle[0] \times_B X \arrow[r, dotted] \arrow[d, dotted] & X \arrow[d, "p"] \\ \triangle[0] \arrow[r, "v"'] \arrow[ur, phantom, "\lrcorner"] & B \end{tikzcd} \qquad F_p(v)=\triangle[0]\times_B X$$

where $\triangle[0]$ is the standard 0-simplex. Notice that $\triangle[0] \times_B X$ is the simplicial subset of B that in dimension n has only the simplex $s_0^n v$. If v is an n-simplex we define the fiber over v as $F_p(\varepsilon_0 v)$, where $\varepsilon_0 v = \widehat{d_0}d_1d_2...d_nv$ ($\widehat{d_0}$ means that we skip the face d_0).

Definition 0.1.15 ([11]) *A map $p : X \to B$ in* **S** *is said to be locally trivial, if for every n-simplex $v \in B$ there is an isomorphism $\alpha_p(v)$ from $\triangle[n] \times F_p(\varepsilon_0 v)$ to $\triangle[n] \times_B X$, such that the following diagram commutes*

$$\begin{tikzcd} \triangle[n] \times F_p(\varepsilon_0 v) \arrow[r, "\alpha_p(v)", "\cong"'] \arrow[d, "pr"'] & \triangle[n] \times_B X \arrow[r, dotted] \arrow[d, dotted, "pr"'] & X \arrow[d, "p"] \\ \triangle[n] \arrow[r, "-"'] & \triangle[n] \arrow[r, "v"'] \arrow[ur, phantom, "\lrcorner"] & B \end{tikzcd}$$

Lemma 0.1.16 ([11], *Proposition* 4.2.2**)**
Let $p : X \to B$ be a locally trivial map. If v and w are simplices in B with $w = \alpha v$ for some operator α, then the fibers over v and w are isomorphic.

Proof. Assume that $v \in X_{c,n}$ and $w \in X_{c,m}$. From the local triviality of p we obtain an isomorphism $\alpha_p(v) : \triangle[n] \times F_p(\varepsilon_0 v) \to \triangle[n] \times_B X$, and considering the 0-simplex $\varepsilon_0\alpha\iota_n$ of $\triangle[n]$ we have the following diagram of pullbacks

$$\begin{tikzcd}
\triangle[0] \times F_p(\varepsilon_0 v) \arrow[r] \arrow[d, "pr"'] & \triangle[n] \times F_p(\varepsilon_0 v) \arrow[r, "\alpha_p(v)", "\cong"'] \arrow[d, "pr"'] & \triangle[n] \times_B X \arrow[r, dotted] \arrow[d, dotted, "pr"'] & X \arrow[d, "p"] \\
\triangle[0] \arrow[r, "\varepsilon_0 \alpha \iota_n"'] & \triangle[n] \arrow[r, "-"'] & \triangle[n] \arrow[r, "v"'] & B
\end{tikzcd}$$

Note that the composite of the lower row is equal to $\varepsilon_0 w$, then there is an isomorphism $\alpha_p(\varepsilon_0 w) : \triangle[0] \times F_p(\varepsilon_0 w) \to \triangle[0] \times F_p(\varepsilon_0 v)$ since p is locally trivial.

$\blacksquare$

Definition 0.1.17 ([1]) *Let F be a simplicial set. A map $p : X \to B$ will be called a* fibre bundle *with fibre F if p is onto and for every n-simplex $v \in B$ there exists an isomorphism $\alpha_p(v) : \triangle[n] \times F \to \triangle[n] \times_B X$ such that the following diagram commutes.*

$$\begin{tikzcd}
\triangle[n] \times F \arrow[r, "\alpha_p(v)", "\cong"'] \arrow[d, "pr"'] & \triangle[n] \times_B X \arrow[r, dotted, "\hat{v}"] \arrow[d, dotted, "pr"'] & X \arrow[d, "p"] \\
\triangle[n] \arrow[r, "-"'] & \triangle[n] \arrow[r, "v"'] & B
\end{tikzcd}$$

The set of isomorphisms $\{\alpha_p(v)\}$ will be called an *atlas* of the bundle and if F is fibrant p will be called a *Kan fibre bundle.* Notice that given two atlases $\{\alpha_p(v)\}$, $\{\widetilde{\alpha}_p(v)\}$ of p, then $\alpha_p(v)^{-1}\widetilde{\alpha}_p(v) \in aut(F)_n$ and conversely if for every $v \in B_n$ we choose $\gamma(v) \in aut(F)_n$, then $\{\alpha_p(v)\gamma(v)\}$ is another atlas. If for every v we define $\beta_p(v) = \hat{v}\alpha_p(v)$, then the atlas $\{\alpha_p(v)\}$ and the set of maps $\{\beta_p(v)\}$ will determine each other.

Definition 0.1.18 ([1], *Definition* 2.4**)** *Let $G \subseteq aut(F)$ be a simplicial subgroup. An atlas $\{\alpha_p(v)\}$ of a fiber bundle p all of whose transformation elements lie in G is called a G-*atlas*; two G-atlases $\{\alpha_p(v)\}$, $\{\widetilde{\alpha}_p(v)\}$ are said to be equivalent if there exists $\gamma(v) \in G$ such that $\widetilde{\alpha}_p(v) = \alpha_p(v)\gamma(v)$. A fiber bundle together with a given G-equivalence class of G-atlases will be called a G-*bundle*; thus any bundle with fibre F is an $aut(F)$-bundle.*

Given an atlas $\{\alpha_p(v)\}$ of p, in general it is not true that $\beta_p(s_i v) = s_i\beta_p(v)$, that is, the top square of the following diagram does not commute

$$\begin{tikzcd}
 & \triangle[n]\times F \arrow[r, "\alpha_p(v)", "\cong"'] & \triangle[n]\times_B X \arrow[r] \arrow[d, dotted] & X \arrow[d, "p"] \\
\triangle[n+1]\times F \arrow[ur, "\sigma^i\times 1"] \arrow[r, "\alpha_p(s_iv)", "\cong"'] & \triangle[n+1]\times_B X \arrow[ur, "\widetilde{\sigma}^i\times 1"] \arrow[urr] \arrow[d] & \triangle[n] \arrow[r, "v"] & B \\
 & \triangle[n+1] \arrow[ur, "\sigma^i"] \arrow[urr, "s_iv"'] & &
\end{tikzcd}$$

An atlas for which $\beta_p(s_iv) = s_i\beta_p(v)$ is called a normalized atlas. By redefining the atlas over the non-degenerate simplices $v \in B$, for all i and v, as follows

$$(*) \quad \beta'_p(v) := \beta_p(v) \quad \textit{and} \quad \beta'_p(s_iv) := s_i\beta_p(v)$$

we can replace β_p by the *normalized atlas* β'_p.

The same analysis can be done for faces operators, and again it is not true in general that $\beta_p(d_iv) = d_i\beta_p(v)$

$$\begin{tikzcd}
 & \triangle[n]\times F \arrow[r, "\alpha_p(v)", "\cong"'] & \triangle[n]\times_B X \arrow[r] \arrow[d, dotted] & X \arrow[d, "p"] \\
\triangle[n-1]\times F \arrow[ur, "\varepsilon^i\times 1"] \arrow[r, "\alpha_p(d_iv)", "\cong"'] & \triangle[n-1]\times_B X \arrow[ur, "\widetilde{\varepsilon}^i\times 1"] \arrow[urr] \arrow[d] & \triangle[n] \arrow[r, "v"] & B \\
 & \triangle[n-1] \arrow[ur, "\varepsilon^i"] \arrow[urr, "d_iv"'] & &
\end{tikzcd}$$

Notice that ε^i is injective and for every $v \in B_n$ the isomorphism $\alpha_p(v)$ sends $(\varepsilon^i \times 1)(\triangle[n-1] \times F)$ isomorphically to the part of $\triangle[n] \times_B X$ over $d_i\iota_n$. Since $\varepsilon^i \times 1$ is one-to-one and $\alpha_p(v)[(\varepsilon^i \times 1)(\triangle[n-1] \times F)] = Im(\widetilde{\varepsilon}^i \times 1)$ we can define the isomorphism $\theta_p(v) : d_i\alpha_p(v)[\triangle[n-1] \times F] \to \triangle[n-1] \times_B X$ by $\theta_p(v)(\varphi d_i\iota_n, z) = (\varphi\iota_{n-1}, z)$. Then for every $v \in B_n$ the following composition is an isomorphism

$$(\dagger) \quad \xi^i_p(v) := \alpha_p(d_iv)^{-1}\theta_p(v)d_i\alpha_p(v) \in aut(F)_{n-1}$$

We refer to $\{\xi^i_p(v)\}$ as the set of *transformation elements* associated to the atlas $\{\alpha_p(v)\}$. An atlas $\{\alpha_p(v)\}$ of a fibre bundle $p : X \to B$ is said to be *regular* if for every $v \in B_n$, $n \geq 1$, it holds that $\xi^i_p(v) = e_{n-1}$, if $i > 0$, where e_{n-1} is the identity element of $aut(F)_{n-1}$. Notice that $\xi^i_p(v) = e_{n-1}$ implies $\beta_p(d_iv) = d_i\beta_p(v)$ and viceversa, for $i > 0$.

Proposition 0.1.19 ([4], *Theorem* 6.6**)**
In every G-equivalence class of atlases there is one regular G-atlas.

Proof. In (†) we can omit the isomorphism $\theta_p(v)$ and keep in mind the relation $\xi_p^i(v) = \alpha_p(d_i v)^{-1} d_i \alpha_p(v)$. So let us specify a new atlas on the non-degenerate elements.

If $v \in B_0$, then let $\alpha'_p(v) = \alpha_p(v)$. Let now $v \in B_1$ be non-degenerate, since G is a simplicial group, it satisfies the extension condition. Therefore there exists $\gamma(v) \in G_1$ with $d_1\gamma(v) = \xi_p^1(v)$. We replace $\alpha_p(v)$ by $\alpha'_p(v) = \alpha_p(v)\gamma(v)^{-1}$:

$$d_1\alpha'_p(v) = d_1\alpha_p(v)d_1\gamma(v)^{-1} = \alpha_p(d_1v)\xi_p^1(v)\xi_p^1(v)^{-1} = \alpha_p(d_1v) = \alpha'_p(d_1v)$$

Next, suppose inductively that $\{\alpha_p(v)\}$ satisfies $\xi_p^i(v) = 1$ for $i > 0$ and for all v of dimension less or equal than $n-1$, and let $v \in B_n$ a non-degenerate simplex. From the induction hypothesis we have that

$$d_i\xi_p^j(v) = d_{j-1}\xi_p^i(v) \quad for \quad 0 < i < j$$

Hence there exists $\gamma(v) \in G_n$ with $d_i\gamma(v) = \xi_p^i(v)$ for $i > 0$, since G is a Kan complex. Replacing $\alpha_p(v)$ by $\alpha'_p(v) = \alpha_p(v)\gamma(v)^{-1}$, we have the following equalities for for $i > 0$

$$d_i\alpha'_p(v) = d_i\alpha_p(v)d_i\gamma(v)^{-1} = \alpha_p(d_iv)\xi_p^i(v)\xi_p^i(v)^{-1} = \alpha_p(d_iv) = \alpha'_p(d_iv)$$

This way, we have construct a new atlas $\{\alpha'_p(v)\}$ which is both regular and G-equivalent to $\{\alpha_p(v)\}$.

∎

Notice that, once a regular atlas $\{\alpha_p(v)\}$ of a bundle p is specified, the normalizing process takes care of the degenerate simplices. That is, if $v \in B$ is a non-degenerate simplex, then after normalizing (using the formula $(*)$) we have: $d_k\beta'_p(s_iv) = d_ks_i\beta_p(v) = s_*d_\diamond\beta_p(v) = s_*\beta_p(d_\diamond v) = \beta'_p(s_*d_\diamond v) = \beta'_p(d_ks_iv)$.

Definition 0.1.20 ([21]) *If $p : X \to B$ and $p' : X' \to B$ are G-bundles with fibre F, a map $h : X \to X'$ will be called a G-map if for every n-simplex $v \in B$ and any G-atlases $\{\alpha_p(v)\}$, $\{\alpha_{p'}(v)\}$ in the given G-equivalence classes of atlases, there exists $\gamma(v) \in G_n$ such that the following diagram commutes*

$$\begin{tikzcd}
 & \triangle[n]\times F \arrow[rr, "\alpha_{p'}(v)"] \arrow[ddl, dotted] & & \triangle[n]\times_B X' \arrow[rr, "\widehat{v}'"] \arrow[ddl, dotted] & & X' \arrow[ddl, "p'"] \\
\triangle[n]\times F \arrow[ur, "\gamma(v)"] \arrow[rr, "\alpha_p(v)"] \arrow[d] & & \triangle[n]\times_B X \arrow[rr, "\widehat{v}"] \arrow[d] & & X \arrow[ur, "h"] \arrow[d, "p"] & \\
\triangle[n] \arrow[rr, "_"'] & & \triangle[n] \arrow[rr, "v"'] & & B &
\end{tikzcd}
\qquad h\beta_p(v) = \beta_{p'}(v)\gamma(v)$$

*If h is an isomorphism we'll say that p and p' are G-*equivalent *. The concept of $aut(F)$-equivalence is identical with that of isomorphism of fibre bundles with a given fibre.*

Lemma 0.1.21 ([21])
Let $p : X \to B$ be a G-fibre bundle with fibre F. If F is fibrant, then p is a fibration.

0.1.2 Twisting Cartesian Products

A twisting cartesian product is a combinatorial model for fibrations based on the notion of a twisting function. They are constructed by starting with a base complex B and a fibre F and trying to 'deform' the simplicial product $B\times F$ to get some non-trivial fibred object $B\times_t F$. This deformation will involve the simplicial group of automorphisms $aut(F)$ of the fibre F and the resulting twisting function 't', going from the base B to $aut(F)$, will be forced by the simplicial identities to obey certain relations in order that $B\times_t F$ be a simplicial set.

Definition 0.1.22 ([21]) *Given a simplicial set B and a simplicial group G, a* twisting map *$t : B \to G$ is a collection of functions $\{t_n : B_n \to G_{n-1}\}_{n\geq 1}$ satisfying*

$$\begin{aligned}
d_i t_{n+1}(v) &= t_n(d_{i+1}v), \quad i > 0\\
s_i t_n(v) &= t_{n+1}(s_{i+1}v), \quad i \geq 0\\
d_0 t_{n+1}(v) &= [t_n(d_0 v)]^{-1} t_n(d_1 v)\\
t_{n+1}(s_0 v) &= e_n
\end{aligned}$$

where e_n is the identity of G_n.

Definition 0.1.23 ([21]) *A group complex G is said to operate (or to act) from the left on a complex F if there exists a simplicial map $\mu : G \times F \to F$ such that $\mu(e_n, z) = z$ and $\mu(g_1, \mu(g_2, z)) = \mu(g_1 g_2, z)$ for all $g_1, g_2 \in G_n$ and all $z \in F_n$. The simplex $\mu(g, z)$ will be denoted by $g.z$ or simply by gz.*

We will say that G operates *effectively* if $\mathrm{g}.z = z$ for all $z \in F_n$ implies that $\mathrm{g} = e_n$; the action is said to be *principal or free* if $\mathrm{g}.z = z$ for some $z \in F_n$ implies that $\mathrm{g} = e_n$. Note that an action $\mu : G \times F \to F$ induce a group homomorphism $\theta : G \to aut(F)$ by setting $\theta(\mathrm{g})(\varphi\iota_n, z) = \varphi\mathrm{g}.z$, for $\mathrm{g} \in G_n$ and any k-simplex $(\varphi\iota_n, z)$ in $\triangle[n] \times F$. Clearly there is an equality $\mathrm{g}.z = \theta(\mathrm{g}).z$. The homomorphism θ is a monomorphism if and only if G operates effectively over F; although, by replacing G by $G/ker(\theta)$ we can always assume that operations by groups are effective.

Definition 0.1.24 ([21]) *Let B and Y be simplicial sets, G a simplicial group acting on F by the left and $t : B \to G$ a twisting map. The* twisted cartesian product*, or TCP for short with base B, fibre F and group G is the simplicial set $B \times_t F$ defined by*

$$(B \times_t F)_n = B_n \times F_n$$

$$d_i(v, z) = (d_i v, d_i z), \quad i \geq 1$$

$$s_i(v, z) = (s_i v, s_i z), \quad i \geq 0$$

$$d_0(v, z) = (d_0 v, t_n(v) d_0 z)$$

for every $v \in B_n$ and every $z \in F_n$

If $F = G$, then $B \times_t G$ is called a *principal twisted cartesian product (PTCP for short*. The term TCP also stands for the projection map $pr : B \times_t F \to B$.

Proposition 0.1.25 ([21])
Let $pr : B \times_t F \to B$ be a TCP with group G. If F is a Kan complex, then pr is a Kan fibration.

Definition 0.1.26 ([21]) *Let $B \times_t F$ and $B' \times_l F$ be twisted cartesian products with group G. A simplicial map $h : B \times_t F \to B' \times_l F$ is said to be a map of TCP's if $h(v, z) = (g(v), \gamma(v).z)$, where $g : B \to B'$ is a simplicial map and $\gamma : B \to G$ is a dimension preserving function. If $B = B'$ and $g = id_B$ we say that t and l are equivalent twisting functions, $t \sim l$, or that $B \times_t F$ and $B' \times_l F$ are equivalent TCP's.*

Note that given equivalent $TCP's$, $B \times_t F$ and $B' \times_l F$, h is an isomorphism and the requirement that h be simplicial is equivalent to the following identities on γ for all $v \in B_n$

$$\begin{array}{l} l_n(v)d_0\gamma_n(v) = \gamma_{n-1}(d_0 v)t_n(v) \\ d_i\gamma_n(v) = \gamma_{n-1}(d_i v),\ i \geq 1 \\ s_i\gamma_n(v) = \gamma_{n+1}(s_i v),\ i \geq 0 \end{array}$$

Proposition 0.1.27 ([19], *Proposition* 4.1.5.36**)**
Let $pr : B \times_t F \to B$ be a TCP with fibre F and group G, then p is a G-bundle with fibre F.

Proof. In the following diagram

$$\begin{tikzcd} \triangle[n] \times F \arrow[r, "\alpha_t(v)"] \arrow[d] & \triangle[n] \times_{tov} F \arrow[r, "v\times 1"] \arrow[d] & B \times_t F \arrow[d] \\ \triangle[n] \arrow[r, equal] & \triangle[n] \arrow[r, "v"'] & B \end{tikzcd}$$

let $\alpha_t(b)$ be the map defined by $\alpha_t(v)(\varphi\iota_n, z) = (\varphi\iota_n, \psi_t^v(\varphi\iota_n).z)$, where the map $\psi_t^v : \triangle[n] \to G$ is a grading preserving and is defined by $\psi_t^v(\varphi\iota_n) = s't(d'v)$ if $\varphi = s'd_0d'$, and $\psi_t^v(\varphi\iota_n) = e$ if $\varphi = s'd'$ (that is, if φ does not contain the operator d_0), where e is the identity of G. Since ψ_t^v satisfies the above identities, it holds that $\alpha_t(b)$ is an isomorphism.

∎

The following Lemma says that the corresponding notion of equivalence for $TCP's$ with group G corresponds exactly to that of G-equivalence of bundles.

Proposition 0.1.28 ([1])
The $TCP's$, $B \times_t F$ and $B \times_l F$ with group G are equivalents if there is an isomorphism $h : B \times_t F \to B \times_l F$, such that for every $v \in B_n$ there exists $\gamma(v) \in G_n$ that makes the following diagram commutative

$$\begin{tikzcd}
& \triangle[n]\times F \arrow[rr, "\alpha_l(v)"] \arrow[ddl, dotted] & & \triangle[n]\times_{lov} F \arrow[rr, "v\times 1"] \arrow[ddl, dotted] & & B\times_l F \arrow[ddl] \\
\triangle[n]\times F \arrow[ur, "\gamma(v)"] \arrow[rr, "\alpha_t(v)"] \arrow[d] & & \triangle[n]\times_{tov} F \arrow[rr, "v\times 1"] \arrow[d] & & B\times_t F \arrow[ur, "h"] \arrow[d] & \\
\triangle[n] \arrow[rr] & & \triangle[n] \arrow[rr, "v"'] & & B &
\end{tikzcd}
\qquad h(v\times 1)\alpha_t(v) = (v\times 1)\alpha_l(v)\gamma(v)$$

Proof. Let us see that $\gamma : B \to G$ defined by the above diagram, which is dimension preserving, defines an equivalence of $TCP's$. Take an n-simplex (v, z) of the twisted cartesian product $B \times_t F$ and evaluate (ι_n, z) in the diagram: $h(v \times 1)\alpha_t(v)(\iota_n, z) = h(v \times 1)(\iota_n, z) = h(v, z)$ and $(v \times 1)\alpha_l(v)\gamma(v)(\iota_n, z) = (v \times 1)\alpha_l(v)(\iota_n, \gamma(v).z) = (v \times 1)(\iota_n, \gamma(v).z) = (v, \gamma(v).z)$, since the diagram commutes we have that $h(v, z) = (v, \gamma(v).z)$, then the isomorphism h and the function γ defines an equivalence of $TCP's$.

Now suppose that we have an equivalence between $B \times_t F$ and $B \times_l F$ given by a simplicial map h and a dimension preserving function $\gamma : B \to G$. Then let us see that for every $v \in B_n$ the simplex $\gamma(v)$ makes the above diagram commutative. Take a k-simplex $(\varphi\iota_n, z)$ of $\triangle[n] \times F$ and evaluate it in the above diagram: on one side $h(v \times 1)\alpha_t(v)(\varphi\iota_n, z) = h(v \times 1)(\varphi\iota_n, \psi_t^v(\varphi\iota_n).z) = h(\varphi\iota_n, \psi_t^v(\varphi\iota_n).z) = (\varphi v, \gamma(\varphi v).\psi_t^v(\varphi\iota_n).z)$ and on the other side we have the following sequence $(v \times 1)\alpha_l(v)\gamma(v)(\varphi\iota_n, z) = (v \times 1)\alpha_l(v)(\varphi\iota_n, \gamma(v)(\varphi\iota_n, z)) = (v \times 1)(\varphi\iota_n, \psi_l^v(\varphi\iota_n).\gamma(v).(\varphi\iota_n, z)) = (\varphi v, \psi_l^v(\varphi\iota_n).\varphi\gamma(v).z)$. Then we must show that $\gamma(\varphi v).\psi_t^v(\varphi\iota_n) = \psi_l^v(\varphi\iota_n).\varphi\gamma(v)$.

To check that, first let us suppose that $\varphi = s'd'$: applying the equalities given after definition 0.1.26 it holds $\gamma(s'd'v).\psi_t^v(s'd'\iota_n) = s'd'\gamma(v).e = e.s'd'\gamma(v) = \psi_l^v(s'd'\iota_n).s'd'\gamma(v)$. Now suppose that $\varphi = s'd_0d'$ and apply again the same equalities: $\gamma(s'd_0d'v).\psi_t^v(s'd_0d'\iota_n) = s'[\gamma(d_0d'v).t(d'v)] = s'[l(d'v)d_0\gamma(d'v)] = s'l(d'v).s'd_0d'\gamma(v) = \psi_l^v(s'd_0d'\iota_n).s'd_0d'\gamma(v)$.

∎

Proposition 0.1.29 ([4])
Let $p : X \to B$ a G-bundle with fibre F and a regular atlas $\{\alpha_p(v)\}$. Then the transformation elements $\{\xi_p^0(v)\}$ determine a twisting function $\xi_p^0 : B_n \to G_{n-1}$ and thereby $B \times_{\xi_p^0} F$ becomes a TCP with fibre F and group G. Furthermore, there is an isomorphism of fibre bundles $h : B \times_{\xi_p^0} F \to X$.

Proof. The map h is given by $h(v,z) = \beta_p(v)(\iota_n, z)$ for every n-simplex (v,z) in $B \times_{\xi_p^0} F$, then the G-equivalence is defined by the composition $\gamma(v) = \alpha_p(v)^{-1} h^* \alpha_{\xi_p^0}(v)$ of the following diagram, where $h^*(\varphi\iota_n, z) = (\varphi\iota_n, h(\varphi v, z))$ for every k-simplex $(\varphi\iota_n, z)$ in $\triangle[n] \times_{\xi_p^0 \circ v} F$

$$\begin{tikzcd}
& \triangle[n] \times F \arrow[rr, "\alpha_p(v)"] \arrow[ddl, dotted] & & \triangle[n] \times_B X \arrow[rr, "\widehat{v}"] \arrow[ddl, dotted] & & X \arrow[ddl, "p"] \\
\triangle[n] \times F \arrow[ur, "\gamma(v)"] \arrow[rr, "\alpha_{\xi_p^0}(v)"] \arrow[d] & & \triangle[n] \times_{\xi_p^0 \circ v} F \arrow[ur, "h^*"] \arrow[rr, "v\times 1"] \arrow[d] & & B \times_{\xi_p^0} F \arrow[ur, "h"] \arrow[d] & \\
\triangle[n] \arrow[rr, "_"'] & & \triangle[n] \arrow[rr, "v"'] & & B &
\end{tikzcd}$$

In the above G-equivalence let us see that $\gamma(v)$ is the identity map. Take a k-simplex $(\varphi\iota_n, z)$ in $\triangle[n] \times F$ and consider the cases $\varphi = s'd'$ and $\varphi = s'd_0d'$. If $\varphi = s'd'$ we have that $\alpha_p(v)^{-1} h^* \alpha_{\xi_p^0}(v)(\varphi\iota_n, z) = \alpha_p(v)^{-1} h^*(\varphi\iota_n, \psi^v_{\xi_p^0}(\varphi\iota_n).z) = \alpha_p(v)^{-1}(\varphi\iota_n, h(\varphi v, z)) = \alpha_p(v)^{-1}(\varphi\iota_n, \beta_p(\varphi v)(\iota_k, z))$, since $\{\alpha_p(v)\}$ is a normalized and regular atlas $\alpha_p(v)^{-1}(\varphi\iota_n, \beta_p(\varphi v)(\iota_k, z)) = \alpha_p(v)^{-1}(\varphi\iota_n, \varphi\beta_p(v)(\iota_k, z)) = \alpha_p(v)^{-1}(\varphi\iota_n, \beta_p(v)(\varphi\iota_n, z)) = (\varphi\iota_n, z)$.

If $\varphi = s'd_0d'$, then $\alpha_p(v)^{-1} h^* \alpha_{\xi_p^0}(v)(\varphi\iota_n, z) = \alpha_p(v)^{-1} h^*(\varphi\iota_n, \psi^v_{\xi_p^0}(\varphi\iota_n).z) = \alpha_p(v)^{-1}(\varphi\iota_n, h(\varphi v, \psi^v_{\xi_p^0}(\varphi\iota_n).z)) = \alpha_p(v)^{-1}(\varphi\iota_n, \beta_p(\varphi v)(\iota_k, \psi^v_{\xi_p^0}(\varphi\iota_n).z))$. From the last term of the equation consider the second factor $\beta_p(\varphi v)(\iota_k, \psi^v_{\xi_p^0}(\varphi\iota_n).z))$, so $\beta_p(\varphi v)(\iota_k, \psi^v_{\xi_p^0}(\varphi\iota_n).z)) = s'\beta_p(d_0d'v)(\iota_k, \psi^v_{\xi_p^0}(\varphi\iota_n).z))$ applying the degeneracies $s'\beta_p(d_0d'v)(\iota_k, \psi^v_{\xi_p^0}(\varphi\iota_n).z)) = \beta_p(d_0d'v)(\sigma' \times 1)(\iota_k, \psi^v_{\xi_p^0}(\varphi\iota_n).z))$, from (†) it holds that $\beta_p(d_0d'v)(\sigma' \times 1)(\iota_k, \psi^v_{\xi_p^0}(\varphi\iota_n).z) = d_0\beta_p(d'v)\xi_p^0(d'v)^{-1}(\sigma' \times 1)(\iota_k, \psi^v_{\xi_p^0}(\varphi\iota_n).z) = d_0\beta_p(d'v)s'\xi_p^0(d'v)^{-1}(\iota_k, \psi^v_{\xi_p^0}(\varphi\iota_n).z)) = d_0\beta_p(d'v)\psi^v_{\xi_p^0}(\varphi\iota_n)^{-1}(\iota_k, \psi^v_{\xi_p^0}(\varphi\iota_n).z))$ that is, $d_0\beta_p(d'v)\psi^v_{\xi_p^0}(\varphi\iota_n)^{-1}(\iota_k, \psi^v_{\xi_p^0}(\varphi\iota_n).z)) = d_0\beta_p(d'v)(s'\iota_m, z) = d_0d'\beta_p(v)(s'\iota_m, z) = \beta_p(v)(\varphi\iota_n, z)$. Joining again we have that $\alpha_p(v)^{-1}(\varphi\iota_n, \beta_p(\varphi v)(\iota_k, \psi^v_{\xi_p^0}(\varphi\iota_n).z)) = \alpha_p(v)^{-1}(\varphi\iota_n, \beta_p(v)(\varphi\iota_n, z)) = (\varphi\iota_n, z)$.

∎

Proposition 0.1.30 ([21], *Lemma* 20.2**)**
Let $p : X \to B$ and $p' : X' \to B$ G-bundles with fibre F. Then p and p' are G-equivalent if and only if t and t' are equivalent, where t and t' are any twisting functions of p and p'.

Proof. If p and p' are G-equivalent there exists and isomorphism $h : X \to X'$ such that $p'h = p$ and $h\beta_p(v) = \beta_{p'}(v)\gamma(v)$ for any $v \in B_n$, where $\gamma(v) \in G$ and $\{\alpha_p(v)\}$, $\{\alpha_{p'}(v)\}$ are any G-atlases of p and p' respectively. If t and t' are twisting functions defined by $\{\alpha_p(v)\}$ and $\{\alpha_{p'}(v)\}$, from Propositions 0.1.29 and 0.1.28 it holds that γ defines the equivalence between t and t'. Conversely if t and t' are equivalents, from proposition 0.1.29 we can define an isomorphism between X and X' by composing, and again from Proposition 0.1.29 and Proposition 0.1.28 it holds that p and p' are G-equivalents.

∎

Definition 0.1.31 ([21]) *If G is a complex group that acts on the right of a complex X and on the left of a complex F, then $X \times_G F$ is defined to be the quotient of $X \times F$ obtained by identifying $(x, g.z)$ with $(x.g, z)$ for all $x \in X$, $z \in F$ and $g \in G$.*

Notice that given a twisting cartesian product $B \times_t F$ there is an associated $PTCP$ by considering the left action of G over itself and using the same twisting function $t : B_n \to G_{n-1}$. Conversely given a principal twisted cartesian product $B \times_t G$ and a left action of G over a complex F the complex $(B \times_t G) \times_G F$ can be identified with $B \times_t F$ and is called the *twisting cartesian product with fibre F associated to $B \times_t G$.*

Proposition 0.1.32 ([1], *Theorem* 5.6**)**
Let G be a simplicial group and B any complex. Then the set of homotopy classes of maps $[B, \overline{W}G]$ from B to $\overline{W}G$ are in bijective correspondence with the equivalence classes of principal twisted cartesian products with base complex B and fibre G.

0.2 Model Categories

A structure of model category over a category $\mathcal{M}$ provides a suitable environment to do homotopy theory in $\mathcal{M}$. The notion of homotopy in $\mathcal{M}$ carry out the construction of a homotopy category $\mathfrak{Ho}(\mathcal{M})$, which is equivalent to the localization of $\mathcal{M}$ with respect to a class of morphisms in $\mathcal{M}$ called 'weak equivalences'. In the localized category, the class of weak equivalences will be considered to be isomorphisms when they are not by formally inverting them.

Although there is a foundational problem with inverting the weak equivalences, since the class of maps between two objects in the localized category may not be a set. In a model category besides weak equivalences there is additional structure (other classes of maps called cofibrations and fibrations) that allow one to get a precise control of the maps in the homotopy category.

Cofibrations and fibrations will enable us to do homotopy theory, because while many of the homotopy notions involved can be defined in terms of the weak equivalences, the verification of many of their properties requires the cofibrations and/or the fibrations.

Definition 0.2.1 ([16]) *A model category is a category $\mathcal{M}$ together with three classes of maps, called the weak equivalences, the cofibrations and the fibrations, satisfying the following five axioms:*

M1. *(Limit axiom) The category $\mathcal{M}$ is complete and cocomplete.*

M2. *(Two out of three axiom) If f and g are maps in $\mathcal{M}$ such that gf is defined and two of f, g and fg are weak equivalences, then so is the third.*

M3. *(Retract axiom) If f and g are maps in $\mathcal{M}$ such that f is a retract of g (in the category of maps of $\mathcal{M}$) and g is a weak equivalence, a cofibration or a fibration, then so is f.*

M4. *(Lifting axiom) Given the commutative solid arrow diagram in $\mathcal{M}$*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

the dotted arrow exists if either

1) *i is a cofibration and p a trivial fibration (i.e., a fibration that is also a weak equivalence), or*

2) *i is a trivial cofibration (i.e., a cofibration that is also a weak equivalence) and p is a fibration.*

M5. *(Factorization axiom) Every map $g \in \mathcal{M}$ has two functorial factorizations*

1) *$g = qi$, where i is a cofibration and q a trivial fibration, and*

2) *$g = pj$, where j is a trivial cofibration and p a fibration.*

Proposition 0.2.2 ([16], *Proposition* 7.17**)**
If S is a set and for every element s of S we have a model category $\mathcal{M}_s$, then the category $\prod_{s\in S}\mathcal{M}_s$ is a model category in which a map is a cofibration, a fibration or a weak equivalence if each of its components is, respectively, a cofibration, a fibration, or a weak equivalence.

Note that a model category always has initial and final objects. An object for which the unique map from the initial object is a cofibration is said to be *cofibrant* and an object for which the unique map to the final object is a fibration is said to be *fibrant* .

The main problem in verifying the model category axioms is in constructing the required functorial factorizations and this is overcome by a so-called small object argument which involves the idea of an infinitely long sequence of maps, and the composition of such sequence (see Apendix B). This argument also will be used to define cofibrantly generated model categories which requires the notion of transfinite composition and smallness.

Example 0.2.3 *The category $\mathfrak{Top}$ of compactly generated topological spaces can be given the structure of a model category by defining $f : X \to Y$ to be a weak equivalence if f is a weak homotopy equivalence, a cofibration if f is a retract of a map $X \to Y'$ in which Y' is obtained from X by attaching cells, and a fibration if f is a Serre fibration.*

With respect to this model category structure the homotopy category $\mathfrak{Ho}(\mathfrak{Top})$ (its localized category w.r.t. weak equivalences) is equivalent to the usual homotopy category of CW-complexes. Every object is fibrant, and the cofibrant objects are exactly the spaces which are retracts of generalized CW-complexes (where a *generalized CW-complex* is a space built up from cells, without the requirement that the cells be attached in order by dimension).

Identifying the elements $[n]$ of the category Δ with the vertices $v_0 = (1, 0, ..., 0)$, $v_1 = (0, 1, ..., 0)$,...,$v_n = (0, 0, ..., 1)$ of $\triangle^n$, a map $\alpha : [q] \to [p] \in \Delta$ sends the vertices of $\triangle^q$ to the vertices of $\triangle^p$ by the rule $\alpha(v_i) = v_{\alpha(i)}$, by extending linearly we obtain a map $\alpha_* : \triangle^q \to \triangle^p$. So to every simplicial set X we can associate functorially a space $|X|$ called the geometric realization of X, which is defined by $|X| = \underset{\Delta\downarrow X}{colim}\triangle^n$. The singular functor $Sing : \mathfrak{Top} \to \mathbf{S}$ defined in example 0.1.7 becomes a right adjoint of the geometric realization functor.

Example 0.2.4 *In the category* **S** *call a map* $f : X \to Y$ *a weak equivalence if* $|f|$ *is a weak homotopy equivalence between topological spaces, a cofibration if each morphism* $f_n : X_n \to Y_n$ $(n > 0)$ *is a monomorphism, and a fibration if* f *is a Kan fibration. With these choices* **S** *is a model category, and since the injections in* **S** *are the cofibrations all objects are cofibrant. Fibrant objects will be those that satisfy the extension condition (that is, the Kan complexes).*

Definition 0.2.5 ([16]) *If* $i : A \to B$ *and* $p : X \to Y$ *are maps for which the dotted arrow exists in every solid arrow diagram of the form*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

then

i) *i is said to have the* left lifting property *(LLP) with respect to p, and*

ii) *p is said to have the* right lifting property *(RLP) with respect to* i.

Axiom $(M4)$ of 0.2.1 says that cofibrations has the LLP with respect to trivial fibrations and that fibrations have the right lifting property with respect to trivial cofibrations.

Proposition 0.2.6 ([16], *Proposition* 7.2.3**)**
Let $\mathcal{M}$ *be a model category*

1) *The map* $i : A \to B$ *is a cofibration iff has the* LLP w.r.t. *all trivial fibrations.*

2) *The map* $i : A \to B$ *is a trivial cofibration iff has the* LLP w.r.t. *all fibrations.*

3) *The map* $p : X \to Y$ *is a fibration iff has the* LLP w.r.t. *all trivial cofibrations.*

4) *The map* $p : X \to Y$ *is a trivial fibration iff has the* LLP w.r.t. *all cofibrations.*

From the above proposition we can see that any two of the three distinguished classes of maps, weak equivalences, fibrations and cofibrations determine the third. Although these can also be described by using cofibrations, since weak equivalences are the maps that can be written as the composition of a trivial cofibration followed by a trivial fibration and trivial fibrations are the maps with the right lifting property with respect to all cofibrations. So the class of cofibrations and of trivial cofibrations enterely determine the model category structure of $\mathcal{M}$.

Example 0.2.7 *In* **S** *the model category structure can be described as follows:*

- *A map is a cofibration if it is a retract of a transfinite composition of pushouts of the maps $\dot{\triangle}[n] \to \triangle[n]$ for all $n \geq 0$, and it is a trivial fibration if it has the RLP w.r.t. the maps $\dot{\triangle}[n] \to \triangle[n]$ for all $n \geq 0$.*

- *A map is a trivial cofibration if it is a retract of a transfinite composition of pushouts of the maps $\Lambda^k[n] \to \triangle[n]$ for all $n \geq 1$, $0 \leq k \leq n$, and it is a fibration if it has the RLP w.r.t. the maps $\Lambda^k[n] \to \triangle[n]$ for all $n \geq 1$, $0 \leq k \leq n$.*

- *A map is a weak equivalence if it is the composition of a trivial cofibration followed by a trivial fibration.*

Definition 0.2.8 ([16], *Definition* 11.1.2**)** *A cofibrantly generated model category is a model category $\mathcal{M}$ such that*

1) *There exists a set I of maps (called a set of generating cofibrations) that permits the small object argument (see Definition B.0.27) and such that a map is a trivial fibration if and only if it has the RLP with respect to every element of I, and*

2) *There exists a set J of maps (called a set of trivial generating cofibrations) that permits the small object argument and such that a map is fibration if and only if it has the RLP with respect to every element of J*

Proposition 0.2.9 ([16], *Proposition* 11.1.10**)**
If S is a set and for every element s of S we have a cofibrantly generated model category $\mathcal{M}_s$ with generating cofibrations I_s and generating trivial cofibrations J_s, then the model category structure on $\prod\limits_{s\in S}\mathcal{M}_s$ of Proposition 0.2.2 is cofibrantly generated with generating cofibrations I and generating trivial cofibrations J, where

$$I = \bigcup_{s\in S}(I_s \times \bigcup_{t\neq s} 1_{\emptyset_t}) \ \text{ and } \ J = \bigcup_{s\in S}(J_s \times \bigcup_{t\neq s} 1_{\emptyset_t})$$

and where $1_{\emptyset_t}$ is the identity map of the initial object in $\mathcal{M}_t$.

The category **S** is cofibrantly generated. The generating cofibrations are the inclusions $\dot{\triangle}[n] \to \triangle[n]$ for $n \geq 0$, and the generating trivial cofibrations are the inclusions $\Lambda^k[n] \to \triangle[n]$ for $n > 0$ and $0 \leq k \leq n$. The model category $\mathfrak{Top}$ is also cofibrantly generated, the generating cofibrations are the inclusions $|\dot{\triangle}[n]| \to |\triangle[n]|$ for $n \geq 0$, and the generating trivial cofibrations are the inclusions $|\triangle^k[n]| \to |\triangle[n]|$ for $n > 0$ and $0 \leq k \leq n$.

Definition 0.2.10 ([16]) *Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations I. A relative I-cell complex (see Definition B.0.24) will be called a relative cell complex, and an I-cell complex will be called a cell complex.*

Proposition 0.2.11 ([16], *Proposition* 11.2.1)
Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations I. Then the class of cofibrations of $\mathcal{M}$ equals the class of retracts of relative I-cell complexes, which equals the class of I-cofibrations (see Propositions B.0.25 and B.0.29).

Proof. This follows from Definition 0.2.8 and the first item of Proposition 0.2.6. ∎

In the context of cofibrantly generated model categories the small object argument is also useful to show that for every small category $\mathcal{C}$ the cofibrantly generated model structures over $\mathfrak{Top}$ and $\mathbf{S}$ induce cofibrantly generated model structures on the diagram categories $\mathfrak{Top}^{\mathcal{C}}$ and $\mathbf{S}^{\mathcal{C}}$, in which the weak equivalences and the fibrations are the objectwise ones. It is done by lifting the cofibrantly generated model category structure from $\mathbf{S}^{\mathcal{C}^o}$ to $\mathbf{S}^{\mathcal{C}}$ via a pair of Quillen functors (Theorem 0.2.14).

Quillen functors are pairs of adjoint functors that will provide a good notion of morphism between model categories, in the sense that each of them is compatible with one half of the model category structures. That means that the left adjoint preserves cofibrations and trivial cofibrations and the right adjoint preserves fibrations and trivial fibrations. Then there is a corresponding notion of equivalences between model categories (called Quillen equivalences) given by Quillen functors that induce equivalences of homotopy theories.

Definition 0.2.12 ([16]) *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories and let $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$ a pair of adjoint functors. We will say that*

1) *F is a left Quillen functor,*

2) *U is a right Quillen functor, and*

3) *(F, U) is a Quillen pair,*

if

1) *the left adjoint F preserves both cofibrations and trivial cofibration and*

2) *the right adjoint U preserves both fibrations and trivial fibration.*

For example, $|-|: \mathbf{S} \to \mathfrak{Top} : Sing$ are an adjoint Quillen pair that induce an equivalence between the homotopy categories $\mathfrak{Ho}(\mathbf{S})$ and $\mathfrak{Ho}(\mathfrak{Top})$ [22]. This equivalence gives us a powerful tool to study the homotopy properties of topological spaces, since the category of simplicial sets is a good category of algebraic and combinatorial models.

Theorem 0.2.13 ([6], *Proposition* 6.11)
For every small category $\mathcal{C}$ the geometric realization and the singular functor induce an adjoint pair of Quillen equivalences

$$|-|^{\mathcal{C}} : \mathbf{S}^{\mathcal{C}} \longleftrightarrow \mathfrak{Top}^{\mathcal{C}} : Sing^{\mathcal{C}}$$

Theorem 0.2.14 (D.M. Kan, [16] *Proposition* 11.3.2)
Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations I and generating trivial cofibrations J. Let $\mathcal{N}$ be a category that is closed under small limits and colimits, and let $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$ be a pair of adjoint functors. If we let $FI = \{Fu : u \in I\}$ and $FJ = \{Fv : v \in J\}$ and if

1) *both of the sets FI and FJ permit the small object argument, and*

2) *U takes relative FJ-cell complexes to weak equivalences,*

then there is a cofibrantly generated model category structure on $\mathcal{N}$ in which FI is a set of generating cofibrations, FJ is a set of generating trivial cofibrations, and the weak equivalences are the maps that U takes into weak equivalences in $\mathcal{M}$. Furthermore, with respect to this model category structure, (F, U) is a Quillen pair.

All the categories that we have mentioned in this section besides being cofibratly generated are simplicial categories too, that is categories endowed with a simplicial set 'function complex' for every pair of objects in the category. If the simplicial structure is compatible related with the closed model structure we will have a closed simplicial model category.

Definition 0.2.15 ([22]) *A simplicial category $\mathcal{M}$ is a category endowed with the following structure:*

i) *A functor $X, Y \to hom(X, Y)$ from $\mathcal{M}^{op} \times \mathcal{M}$ to $\mathbf{S}$. The simplicial set $hom(X, Y)$ is called the simplicial mapping space or the function complex from X to Y.*

ii) *for every three objects X, Y and Z in $\mathcal{M}$ a morphism of simplicial sets $hom(X,Y) \times hom(Y,Z) \to hom(X,Z)$, $(f,g) \mapsto g \circ f$ called the composite rule.*

iii) *for every two objects $X, Y \in \mathcal{M}$ an isomorphism $hom(X,Y)_0 \equiv hom_{\mathcal{M}}(X,Y)$ that commutes with the composite rule.*

This structure is subject to the following two conditions

1) *If $f \in hom(X,Y)_n$, $g \in hom(Y,Z)_n$ and $h \in hom(Z,W)_n$, then $(h \circ g) \circ f = h \circ (g \circ f)$.*

2) *If $u \in hom_{\mathcal{M}}(X,Y)$ and $f \in hom(Y,Z)_n$, then $f \circ s_0^n u = hom(u,Z)_n(f)$. Also $s_0^n u \circ g = hom(W,u)_n(g)$ if $g \in hom(W,X)_n$.*

Given two maps $p : X \to B$ and $g : A \to B$ in $\mathcal{M}$ the pullback of p along g is denoted by $p_g : A \times_B X \to A$ and it is a fibration whenever p so is. So if $i : A \to B$ and $p : X \to Y$ are maps in $\mathcal{M}$ the pullback of $i^* : hom(B,Y) \to hom(A,Y)$ and $p_* : hom(A,X) \to hom(A,Y)$ is denoted by $hom(A,X) \times_{hom(A,Y)} hom(B,Y)$. If $f : L \to K$ is a map in $\mathbf{S}$, the pushout of $1 \otimes f : A \otimes L \to A \otimes K$ and $i \otimes 1 : A \otimes L \to B \otimes L$ is denoted by $A \otimes K \coprod_{A \otimes L} B \otimes$.

Definition 0.2.16 ([16]) *A simplicial model category is a model category $\mathcal{M}$ that is also a simplicial category, such that the following two axioms hold:*

M6. *For every two objects X and Y of $\mathcal{M}$ and every simplicial set K there are objects $X \otimes K$ and Y^K of $\mathcal{M}$ such that there are isomorphisms of simplicial sets $hom(X \otimes K, Y) \equiv hom(K, hom(X,Y)) \equiv hom(X, Y^K)$ that are natural in X, Y and K.*

M7. *If $i : A \to B$ is a cofibration in $\mathcal{M}$ and $p : X \to Y$ is a fibration in $\mathcal{M}$, then the map of simplicial sets*

$$(i^*, p_*) : hom(B,X) \to hom(A,X) \times_{hom(A,Y)} hom(B,Y)$$

is a fibration that is a trivial fibration if either i or p is a weak equivalence.

The category of simplicial sets $\mathbf{S}$ is a simplicial model category, the complex function $hom(X,Y)$ defined in Example 0.1.4 is the simplicial set that in degree n is the set $hom_{\mathbf{S}}(X \times \triangle[n], Y)$, for $K \in \mathbf{S}$, let $X \otimes K$ be $X \times K$ and X^K

be $hom(K, X)$. The category $\mathfrak{Top}$ is also a simplicial model category by defining the complex function $hom(X, Y)$ in degree n, for $X, Y \in \mathfrak{Top}$, as the set of continuous maps from $X \times |\triangle[n]|$ to Y, for $K \in \mathbf{S}$, $X \times K$ by $X \times |K|$ and $X^K = hom_{\mathfrak{Top}}(|K|, X)$.

Example 0.2.17 *Let $\mathcal{C}$ be a small category and $\mathbf{S}^{\mathcal{C}}$ the category of functors from $\mathcal{C}$ to $\mathbf{S}$, or the category of $\mathcal{C}$-diagrams of simplicial sets . For every $\mathcal{C}$-diagram X, X_c will denote the evaluation $X(c)$ of X in c, for every $c \in \mathcal{C}$. $\mathbf{S}^{\mathcal{C}}$ is a simplicial model category, where its model structure is given by defining weak equivalences, fibrations and cofibrations in the following way for $f : X \to Y \in \mathbf{S}^{\mathcal{C}}$:*

- *the natural transformation f is a weak equivalence if for every $c \in \mathcal{C}$, $f_c : X_c \to Y_c$ is a weak equivalence in $\mathbf{S}$. Two objects will be called* weakly equivalents *if they can be connected by a finite string of weak equivalences.*
- *f is a fibration if for every $c \in \mathcal{C}$, $f_c : X_c \to Y_c$ is a fibration in $\mathbf{S}$. In particular an object $X \in \mathbf{S}^{\mathcal{C}}$ is fibrant if for every $c \in \mathcal{C}$, $X_c \in \mathbf{S}$ is fibrant.*
- *f is a cofibration if it has the L.L.P. w.r.t. the class of trivial fibrations.*

Given a diagram $X \in \mathbf{S}^{\mathcal{C}}$ and a simplicial set K a simplicial model structure over $\mathbf{S}^{\mathcal{C}}$ holds by defining $X \otimes K$ as the functor that makes correspond to $c \in \mathcal{C}$ the simplicial set $X_c \times K$, and if $c \to d \in \mathcal{C}$, $X \otimes K(f) = X_f \times 1$. For $X, Y \in \mathbf{S}^{\mathcal{C}}$ the function complex $hom(X, Y)$ is the simplicial set that in degree n has as simplices the set $hom_{\mathbf{S}^{\mathcal{C}}}(X \otimes \triangle[n], Y)$, and X^K is defined by $hom(K, X)$ taking K as the constant diagram.

To short notation, the diagram $X \otimes \triangle[n]$ will be denoted by $X \times \triangle[n]$ or equivalently by $\triangle[n] \times X$. Notice that $hom(X, Y)_n$ can be seen as the set formed by the following commutative diagrams of $\mathbf{S}^{\mathcal{C}}$

$$\begin{tikzcd} \triangle[n] \times X \arrow[rr] \arrow[dr, "pr"'] & & \triangle[n] \times Y \arrow[dl, "pr"] \\ & \triangle[n] & \end{tikzcd}$$

The monoid of self weak equivalences of a diagram X is the submonoid $we(X) \subseteq hom(X, X)$ consisting of the above commutative triangles (taking $X = Y$) where the horizontal arrow is a weak equivalence. The group of automorphisms of the $\mathcal{C}$-diagram X is the maximal subgroup $aut(X) \subseteq we(X)$, where the horizontal arrows are isomorphisms.

Given two maps $i : A \to B$ and $p : X \to Y$ in $\mathcal{M}$ and a map $f : L \to K$ in $\mathbf{S}$, the map of Definition 0.2.16 in $M7$. is called the *pullback corner map* of i and p, and the map $A \otimes K \coprod_{A \otimes L} B \otimes L \to B \otimes K$ is called *pushout corner map* of i and f.

Lemma 0.2.18 ([16], *Lemma* 9.3.6)
Let $\mathcal{M}$ be a simplicial model category. If $i : A \to B$ and $p : X \to Y$ are maps in $\mathcal{M}$ and $f : L \to K$ is a map of simplicial sets, then the following are equivalents:

1) *The dotted arrow exists in every solid arrow diagram of the form*

$$\begin{tikzcd} L \arrow[r] \arrow[d] & hom(B,X) \arrow[d] \\ K \arrow[r] \arrow[ur, dotted] & hom(A,X) \times_{hom(A,Y)} hom(B,Y) \end{tikzcd}$$

2) *The dotted arrow exists in every solid arrow diagram of the form*

$$\begin{tikzcd} A \otimes K \coprod_{A \otimes L} B \otimes L \arrow[r] \arrow[d] & X \arrow[d] \\ B \otimes K \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

0.3 Idempotent Functors

Localized model category structures originated in Bousfield's work on localization with respect to homology, given a homology theory h_*. Bousfield established a model category structure on the category of simplicial sets in which the weak equivalences where the maps that induced isomorphisms of all the homology groups. The problem that led to Bousfiled's model category structure was that of constructing a localization functor for a homology theory, that is, given a homology theory h_*, the problem was to define for each space X a local space $L_{h_*}X$ and a natural homology equivalence $X \to L_{h_*}X$.

Some years later, Bousfield and Dror Farjoun independently considered the notion of localizing spaces with respect to an arbitrary map $f : A \to B$ in $\mathbf{S}$ (or in $\mathfrak{Top}$). A map $g : X \to Y$ is defined to be a f-local equivalence if for every f-local space W the induced map $g^\sharp : hom(Y, W) \to hom(X, W)$ is a weak equivalence. An f-localization of a space is then an f-local space $L_f X$ together with an f-local equivalence $X \to L_f X$. Given any map f it is possible to construct a model category structure on the category of spaces in which the weak equivalences are the f-local equivalences, and in which an f-localization functor is a fibrant approximation functor for the f-local model category [16].

0.3.1 Localization

Definition 0.3.1 ([9]) *Let $f : A \to B$ be a map between cofibrant spaces. The simplicial set X is said to be f-local if X is fibrant and the map f induces a weak homotopy equivalence on function complexes*

$$hom(f, X) = f^\sharp : hom(B, X) \xrightarrow{\simeq} hom(A, X)$$

In what follows the word 'equivalence' stands for weak homotopy equivalence. One also defines this concept in the pointed category of spaces, so by using the fibration $hom_*(V, Z) \to hom(V, Z) \to Z$ over any connected space Z, and the Dold's theorem about fibrewise homotopy it holds that $hom(f, X)$ is an equivalence if and only if the map $hom_*(f, X)$ is an equivalence. So a connected space X is f-local in $\mathbf{S}$ if and only if it is f-local in $\mathbf{S}_*$.

If the map is simply $f : * \to A$ one refers to an f-local space X as an A-null space, this means that the natural map $hom(A, X) \xrightarrow{\simeq} Y$ is an equivalence, or equivalently that $hom_*(A, X) \simeq *$ is weakly contractible.

Definition 0.3.2 ([9]) *Let $f : A \to B$ be a map between cofibrant spaces. A map $g : X \to Y$ is called an f-local equivalence if for all f-local space W the induced map*

$$g^\sharp : hom(Y, W) \xrightarrow{\simeq} hom(X, W)$$

is an equivalence.

Definition 0.3.3 ([9]) *A functor $L : \mathbf{S} \to \mathbf{S}$ is called coaugmented if it comes with a natural transformation $l : Id \to L$. A coaugmented functor L is said to be homotopically idempotent if for every $X \in \mathbf{S}$ both natural maps $LX \rightrightarrows LLX$, namely both l_{LX} and $L(l_X)$, are weak equivalences and are homotopic to each other.*

Theorem 0.3.4 ([9], $Theorem\ A.3$**)**
For any map $f : A \to B$ in $\mathbf{S}$ there exists a functor $L_f : \mathbf{S} \to \mathbf{S}$, called the f-localization functor, which is coaugmented and homotopically idempotent. Any two of such functors are naturally weakly equivalent to each other. For any space X its coaugmentation $l_X : X \to L_f X$ is initial up to homotopy among all maps from X to f-local spaces, and it is final up to homotopy among all f-local equivalences $X \to Y$. Moreover L_f can be chosen to be a simplicial functor.

The coaugmentation map $l_X : X \to L_f X$ is determined up to equivalence by three basic properties:

- l_X is an f-local equivalence
- $L_f X$ is f-local, and
- $l_X : X \to L_f X$ is a weak equivalence when it is applied to an f-local space

For the map $* \to A$ or $A \to *$, $L_{*\to A} = L_{A\to *}$ is denoted by P_A and is called the *A-nullification functor*. For example, if $A = S^{n+1}$, then $P_{S^{n+1}}$ is the n-th Postnikov section X_n which can be characterized by $\Omega^{n+1} X_n \simeq *$.

Since L_f is a homotopy functor (that is, it preserves weak equivalences) it can be applied to fibre sequences $E \to B$ by applying L_f to each fibre. It is not difficult to get a functorial construction of a fibrewise application in the homotopy category, that is, a homotopy commutative diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[r, "p"] \arrow[d] & B \arrow[d, equal] \\ L_f F \arrow[r] & \overline{E} \arrow[r, "q"'] & B \end{tikzcd}$$

where q is a fibration. Although it is also possible to construct a rigid fibrewise localization that will have universal properties in the category of spaces over B similar to those of L_f in the category of spaces (see theorem 0.3.7).

Definition 0.3.5 ([16], $Definition\ 6.1.1$**)** *Let $f : A \to B$ be a map between cofibrant spaces. A fiberwise f-localization of a fibre map $p : E \to B$ is a factorization $E \xrightarrow{a} \overline{E} \xrightarrow{q} B$ of p such that:*

1) *q is a fibration*

2) *for every point $v \in B$ the induced map $hFib_p(v) \to hFib_q(v)$ of homotopy fibers is the f-localization of $hFib_p(v)$.*

Theorem 0.3.6 ([9], *Theorems F.3 and F.4***)**
Let $p : E \to B$ a fibre map. Then there is a commutative diagram over B

$$\begin{tikzcd} F \arrow[r] \arrow[d, "l_F"'] & E \arrow[r, "p"] \arrow[d, "a"'] & B \arrow[d, equal] \\ L_f F \arrow[r] & \overline{E} \arrow[r, "q"'] & B \end{tikzcd}$$

where q is a fibre map, F is the homotopy fiber over any component of B and $L_f F$ the corresponding homotopy fibre of q. Moreover, the map $a : E \to \overline{E}$ obtained by applying L_f fibrewise is an f-local equivalence.

Theorem 0.3.7 ([16], *Theorem 6.1.3***)**
Let $p : E \to B$ a fibre map. If $E \xrightarrow{a} \overline{E} \xrightarrow{q} B$ is a fiberwise f-localization of p, then for every solid arrow diagram

$$\begin{tikzcd} E \arrow[r, "a"] \arrow[rr, bend left, "\overline{a}"] \arrow[dr, "p"'] & \overline{E} \arrow[r, dotted, "k"] \arrow[d, "q"] & \overline{\overline{E}} \arrow[dl, "r"] \\ & B & \end{tikzcd}$$

in which $\overline{a} : E \to \overline{E}$ is another fiberwise f-localization of p, there exists a map $k : \overline{E} \to \overline{\overline{E}}$ unique up to simplicial homotopy in $E \downarrow \mathbf{S} \downarrow B$, such that, k is a weak equivalence and $ka = \overline{a}$.

0.3.2 Cellularization

For many applications it is often useful to approximate a given topological space by simpler ones. In the case of pointed simplicial sets an important example is the cellularization functor $CW_A : \mathbf{S}_* \to \mathbf{S}_*$, which associates to each pointed space a space built through the process of attaching copies of a fixed space A. This general concept of cellularization was developed systematically for the category of topological spaces and simplicial sets by Dror Farjoun [9] building upon the general foundational work on homotopy localization of Bousfield.

Definition 0.3.8 ([9]) *Let A be a cofibrant space. A map $g : X \to Y$ in $\mathbf{S}_*$ between fibrant spaces is called an A-equivalence if the map g induces a weak homotopy equivalence on function complexes*

$$hom(A, g) = g_\sharp : hom_*(A, X) \xrightarrow{\simeq} hom_*(A, Y)$$

Notice that in the unpointed case if $A \neq \emptyset$, then a one point space is a retract of A, and so every space X is retract of $hom(A, X)$. This implies that an A-local equivalence of unpointed spaces must be a weak equivalence, since g is a retract of $g_\sharp$. Thus, consider the notion of A-equivalence of unpointed spaces would be pointless.

Definition 0.3.9 ([24], *Definition* 5.1) *A connected simplicial set Z is called A-cellular if for any choice of the base point $z \in Z$ and for any map of pointed Kan complexes $g : X \to Y$, for which $g_\sharp$ is an A-equivalence, the map $g_\sharp : hom_*(Z, X) \to hom_*(Z, Y)$ is a weak equivalence.*

It can be proved that this is equivalent to saying that X can be built as an iterated pointed homotopy colimit of copies of A.

Definition 0.3.10 ([9]) *A functor $C : \mathbf{S}_* \to \mathbf{S}$ is called augmented if it comes with a natural transformation $c : Id \to C$. A augmented functor C is said to be homotopically idempotent if for every $X \in \mathbf{S}_*$ both natural maps $CCX \rightrightarrows CX$, namely both c_{CX} and $C(c_X)$, are weak equivalences and are homotopic to each other.*

In the construction of the functor CW_A (see the Chapter 2 of [9], numerals $E.2$ and $E.3$) the half smash $\widetilde{\Sigma^n} A$ is the basic building block, this is defined by

$$\widetilde{\Sigma^n} A = (S^n \times A) \cup (D^{n+1} \times \{x_0\}) \subset D^{n+1} \times A$$

with the base point $\{*\} \times \{*\}$.

If λ is the first limit ordinal bigger than the cardinality of the set of simplices of A and X is a pointed simplicial set, then a functor $C : \lambda \to \mathbf{S}_*$ and a pointed map $c : C \to X$ is constructed, where λ is the category whose objects are all ordinal numbers smaller than λ and for any two ordinal numbers $i \leq j$, there is only one arrow $i \to j$.

The space C_0 and the map $C_0 \xrightarrow{c_0} X$ are defined by

$$C_0 = \bigvee_{\substack{i \geq 0 \\ h \in hom_*(\widetilde{\Sigma}^i A, X)}} \widetilde{\Sigma}^i A \quad \text{and} \quad \bigvee_{\substack{i \geq 0 \\ h \in hom_*(\widetilde{\Sigma}^i A, X)}} h$$

Notice that any element $h : \widetilde{\Sigma}^i A \to X$ representing a homotopy class of $[A, X]_*$ in the component $h|\{*\} \times A : A \to X$ is null homotopic in that component if and only if h can be extended along the map $\widetilde{\Sigma}^i A \hookrightarrow D^{n+1} \times A$. So let k_0 the wedge of all the maps $g : \widetilde{\Sigma}^i A \to C_0$ with a given extension of $c_0 g$, the map $D_0 \to C_0$ is given by g. The space C_1 is defined as the pushout along the extension to $D^{n+1} \times A$

$$\begin{tikzcd}
\bigvee_{K_0} \widetilde{\Sigma}^i A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \bigvee_{K_0} D^{n+1} \times A \arrow[d] \\
C_0 \arrow[r] & C_1
\end{tikzcd}$$

Proceeding by induction it holds the following

$$\begin{tikzcd}
\bigvee_{K_0} \widetilde{\Sigma}^i A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \bigvee_{K_0} D^{i+1} \times A \arrow[dr] & \bigvee_{K_1} \widetilde{\Sigma}^i A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \bigvee_{K_1} D^{i+1} \times A \arrow[dr] & \\
C_0 \arrow[rr] & & C_1 \arrow[r] & \dots & {}
\end{tikzcd}$$

$$\begin{tikzcd}
\dots & \bigvee_{K_\beta} \widetilde{\Sigma}^i A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \bigvee_{K_\beta} D^{i+1} \times A & \dots \\
{} \arrow[r] & C_\beta & \dots & (\beta \leq \lambda)
\end{tikzcd}$$

With this construction $CW_A : \mathbf{S}_* \to \mathbf{S}_*$ is a functor defined as $CW_A(X) = \underset{\lambda}{colim} C$ and the augmentation $c_X : CW_A(X) \to X$ is the natural transformation defined as the map $c_X = \underset{\lambda}{colim}(p)$.

Proposition 0.3.11
If A is $\mathbb{Z}$-acyclic, then $CW_A X$ is $\mathbb{Z}$-acyclic

Proof. Since $\Sigma^i A$ is weakly equivalent to the homotopy cofiber of the map $A \to \widetilde{\Sigma}^i A$ we have the following exact sequence

$$1 \to \widetilde{H}_n(A) \to \widetilde{H}_n(\widetilde{\Sigma}^i A) \to \widetilde{H}_n(\Sigma^i A) \to 1$$

The space A is $\mathbb{Z}$-acyclic, that is, $\widetilde{H}_n(A) = 0$ for every n, therefore $\widetilde{H}_n(\Sigma^i A) \cong \widetilde{H}_{n-i}(A) \cong 0$, thus $\widetilde{H}_n(\widetilde{\Sigma}^i A) \cong 0$. We also know that $\widetilde{H}_n(\bigvee_\Lambda \widetilde{\Sigma}^i A) \cong \bigoplus_\Lambda \widetilde{H}_n(\widetilde{\Sigma}^i A)$ for every index set Λ, then $\widetilde{H}_n(\bigvee_\Lambda \widetilde{\Sigma}^i A) \cong 0$ and the contractibility of D^{n+1} implies that the homology group $\widetilde{H}_n(D^{i+1} \times A)$ is also trivial.

Applying the Mayer-Vietoris sequence to the first pushout of the above diagram used to define CW_A

$$... \to \widetilde{H}_n(\bigvee_{I_0} \widetilde{\Sigma}^i A) \bigoplus \widetilde{H}_n(\bigvee_{K_0} D^{n+1} \times A) \to \widetilde{H}_n(C_1) \to \widetilde{H}_{n-1}(\bigvee_{K_0} \widetilde{\Sigma}^i A) \to ...$$

we have that $\widetilde{H}_n(C_1) \cong 0$. Since it happens for every ordinal $\beta \leq \lambda$ it holds that $colim_{\beta \leq \lambda} \widetilde{H}_n(C_\beta) \cong 0$, and therefore $CW_A X$ is $\mathbb{Z}$-acyclic.

∎

In general we have that pointed homotopy colimits of acyclic spaces with respect to any generalized homology theory are again acyclic [9], [D.2.5].

Theorem 0.3.12 ([9], $2.E.8$**)**
The functor $CW_A : \mathbf{S}_* \to \mathbf{S}_*$ is a homotopy idempotent, augmented simplicial functor, with the aumentation map $c_X : CW_A X \to X$ final up to homotopy among all maps from A-cellular spaces to X, and it is initial up to homotopy among all A-equivalences $Y \to X$.

By computational reasons we will also work with another construction of CW_A given by Wojciech [24].

Theorem 0.3.13 ([24])
If A is a pointed and 0-connected space, the A-cellularization of X has the homotopy type of the homotopy fibre of the map $\eta : X \to P_{\Sigma A} C_{ev}$, where η is the composition of the inclusion $X \hookrightarrow C_{ev}$ into the homotopy cofibre of the evaluation map $ev : \bigvee_{[A,X]_*} A \to X$ with the coaugmentation $l_{C_{ev}} : C_{ev} \to P_{\Sigma A} C_{ev}$.

For every X in $\mathbf{S}_*$, the augmentation $c_X : CW_A X \to X$ is characterized up to equivalence by three properties:

- c_X is an A-cellular equivalence
- $CW_A X$ is A-cellular

- $hom_*(CW_A X, g)$ is a weak equivalence whenever g is an A-cellular equivalence

A well known example of an augmented functor is the universal cover $\widehat{X} \to X$. More generally for $X \in \mathbf{S}_*$, if $A = S^{n+1}$ then $CW_{S^{n+1}} X$ is the connected n-cover $X\langle n \rangle$ of X.

As in Section 0.3.1 the concept of fibrewise localization as its counterpart in the cellularization context.

Definition 0.3.14 *Let A be a pointed and connected space. A fiberwise cellularization of a fibration $p : E \to B$ is a fibration $q : \overline{E} \to B$ such that*

1) *there exists a map $b : \overline{E} \to E$ such that $pb = q$, and*

2) *for every point $v \in B$ the induced map $hFib_q(v) \to hFib_p(v)$ of homotopy fibers is the A-cellularization of $hFib_p(v)$.*

CHAPTER 1

DIAGRAMS OF SPACES

We are interested in the structure of $\mathbf{S}$ as a cofibrantly generated model category and how it passes to the category of functors $\mathbf{S}^{\mathcal{C}}$, from a small category $\mathcal{C}$ to $\mathbf{S}$ (or, category of $\mathcal{C}$-diagrams of spaces). This structure will be studied by using free diagrams [16] over the standard n-simplex $\triangle[n]$, which are $\mathcal{C}$-diagrams that lie in a full subcategory Γ of $\mathbf{S}^{\mathcal{C}}$. In the first section we show that these diagrams are the basic pieces through which we may cover a given $\mathcal{C}$-diagram X, by proving that Γ is a dense subcategory of $\mathbf{S}^{\mathcal{C}}$. Thus these objects will play the role of simplices in $\mathbf{S}^{\mathcal{C}}$ as it is done by the well known n-simplices in $\mathbf{S}$.

In section 2 the sets of cofibrations and trivial cofibrations in $\mathbf{S}^{\mathcal{C}}$ are explicitly describe and we closed the section by characterizing the free cell complexes, which play the central role in the construction of minimal $\mathcal{C}$-diagrams. As we will see in Chapter 2 the notion of base for a free cell complex will allow us to identify homotopically zig-zags of simplices.

By understanding the automorphisms group $aut(F)$, for a given $\mathcal{C}$-diagram F we are able to generalize the notions of fibre bundle and twisted cartesian products to the context of $\mathcal{C}$-diagrams (it is done in sections 3 and 4). Generalize the properties concerning to fiber bundles is quite natural and technically it is not so difficult; although less obvious is the equivalent definition of $\mathcal{C}$-fiber bundle given in the appendix C, since it allow us to express the local triviality property of bundles in terms of its n-$\mathcal{C}$-simplices, which are not necessarily constant diagrams.

Given a twisting function $t : B_n \to aut(F)_{n-1}$ it is easy to define a $\mathcal{C}$-diagram of twisted cartesian products, in this situation the technical difficulty is to obtain the reciprocal way, as it is shown in Proposition 1.4.3. After showing this we proof that any $\mathcal{C}$-fiber bundle over B is a $\mathcal{C}$-TCP and viceversa, as it happens in the simplicial case, thus we are able to classify both of them in terms of the set of homotopy classes of maps $[B, \overline{W}aut(F)]$ from B to $\overline{W}aut(F)$.

In section 5 we characterize the function complex $hom(F, F)$.

1.1 Free objects in the category of diagrams of simplicial sets

Let $\mathcal{C}$ and $\mathcal{M}$ be categories. If $\mathcal{C}$ is a small category, then the category $\mathcal{M}^{\mathcal{C}}$ of $\mathcal{C}$-diagrams in $\mathcal{M}$ is the category whose class of objects is the class of functors from $\mathcal{C}$ to $\mathcal{M}$ and whose set of morphisms between two functors are the natural transformations between them. If $F : \mathcal{N} \to \mathcal{M}$ is a functor and $m \in \mathcal{M}$, the category $F \downarrow m$ of *objects F-over m* has as objects the arrows $\alpha : F(n) \to m$ in $\mathcal{M}$ and as morphisms between $\alpha : F(n) \to m$ and $\beta : F(n') \to m$ the maps $g : n \to n'$ in $\mathcal{N}$, such that $\beta F(g) = \alpha$. This construction comes with a natural projection functor θ_m defined by $\theta_m(g : \alpha \to \beta) = g : n \to n'$.

Definition 1.1.1 ([20]) *A functor $F : \mathcal{N} \to \mathcal{M}$ is* dense *if for each $m \in \mathcal{M}$, $m = \underset{F\downarrow m}{colim} F\theta_m$.*

In the sequel for a given category $\mathcal{M}$, the set $Mor_{\mathcal{M}}(m, m')$ will be denoted by $\mathcal{M}(m, m')$ or $[m, m']$ if the context category is clear.

Lemma 1.1.2 ([20])
A functor $F : \mathcal{N} \to \mathcal{M}$ is dense iff the functor $\mathcal{M} \to \mathfrak{Sets}^{\mathcal{N}^{op}}$ defined by the assignation rule $m \mapsto \mathcal{M}(F(\_), m)$ is faithful and full.

Example 1.1.3 *In the category Δ every morphism $\beta : [n] \to [m]$ defines a natural transformation $\beta_* : \triangle[n] \to \triangle[m]$ given pointwise by postcomposition with β (see Example 0.1.3), then we can define a functor $\triangle[-] : \Delta \to \mathfrak{Sets}^{\Delta^{op}}$, where $\triangle[-]([n]) = \triangle[n]$. It is an immediate consequence from Yoneda's Lemma [20] that the functor $\mathfrak{Sets}^{\Delta^{op}} \to \mathfrak{Sets}^{\Delta^{op}}$ defined by $X \mapsto \mathbf{S}(\triangle[-], X)$ is full and faithfull, then by Lemma 1.1.2 every object $X \in \mathbf{S}$ can be expressed as a colimit. The category $\triangle[-] \downarrow X$ or for short $\Delta \downarrow X$ will be called the simplex category of X, so $X = \underset{\Delta\downarrow X}{colim}(\triangle[n] \to X)$.*

Viewing a simplex $x \in X_n$ as a map $x : \triangle[n] \to X$ between simplicial sets becomes a good tool to understand some properties of maps in $\mathcal{S}$, specially when we are dealing with lifting properties since purely combinatorial simplicial proofs can be quite technical. So the spirit of this section in what follows is to define the objects that will allow us to see a diagram $X \in \mathbf{S}^{\mathcal{C}}$ in terms of blocks (as it happens for simplicial sets with their representing maps), through which we could recover X by a colimit process. That is why we introduce the notion of free object.

Definition 1.1.4 ([16], *Definition* 11.5.25) *Let $\mathcal{M}$ be a cocomplete category, C a small category, fix $c \in \mathcal{C}$, take $X \in \mathcal{M}$ and denote the set $Mor_{\mathcal{C}}(c, a)$ by $[c, a]$. The functor $X \otimes [c, -] : \mathcal{C} \to \mathcal{M}$ defined by*

$$X \otimes [c,-](f : a \to b) = \coprod_{[c,a]} 1_X : \coprod_{[c,a]} X \to \coprod_{[c,b]} X$$

for every arrow $f : a \to b \in \mathcal{C}$, is called the free diagram on X generated at c .

For example if $\mathcal{C} = \Delta^{op}$ and $\{*\}$ is the singleton set, the free diagram on $\{*\}$ generated at $[n]$ is the standard n-simplex $\triangle[n]$. Let $\mathcal{M} = \mathbf{S}$, $\triangle[n] \in \mathbf{S}$ and $c \in \mathcal{C}$, since the free diagram $\triangle[n] \otimes [c, -]$ on $\triangle[n]$ generated at c will appear repeatedly in what follows we will denote it by δ^c_n, and the free diagrams $\dot{\triangle}[n] \otimes [c, -]$, $\triangle^k[n] \otimes [c, -]$ by $\dot{\delta}^c_n$ and $\delta^c_{n,k}$, respectively. For a $\mathcal{C}$-diagram X a map $\delta^c_n \to X$ will be called an *n-$\mathcal{C}$-simplex of X at c*

Proposition 1.1.5 ([16], *Proposition* 11.5.26)
Let $\mathcal{C}$ be a small category and $c \in \mathcal{C}$. Then the functor $(-) \otimes [c, -] : \mathbf{S} \to \mathbf{S}^{\mathcal{C}}$ is left adjoint to the functor $ev_c : \mathbf{S}^{\mathcal{C}} \to \mathbf{S}$ that evaluates at c, i.e., for every object $Y \in \mathbf{S}$ and every diagram $X \in \mathbf{S}^{\mathcal{C}}$ there's a natural isomorphism $\mathbf{S}(Y, X_c) \cong \mathbf{S}^{\mathcal{C}}(Y \otimes [c, -], X)$, where $X_c = ev_c(X)$.

Sketch. The proof of this lemma is given in [16, proposition 11.5.26] for any cocomplete category $\mathcal{M}$, but for the sake of completeness of these notes we make the sketch in the simplicial case by considering the simplicial set $\triangle[n]$.

If $X \in \mathbf{S}^{\mathcal{C}}$ and w is the representing map for a given n-simplex $w \in X_c$, then the bijection $\psi^c_n : \mathbf{S}(\triangle[n], X_c) \to \mathbf{S}^{\mathcal{C}}(\delta^c_n, X)$ is given by

$$w : \triangle[n] \to X_c \qquad \Rightarrow \qquad \begin{tikzcd} \triangle[n] \arrow[r, "i_g^{[c,d]}"] \arrow[dr, "w"'] & \coprod_{[c,d]} \triangle[n] \arrow[r, dotted, "\psi^c_n(w)_d"] & X_d \\ & X_c \arrow[ur, "X_g"'] & \end{tikzcd}$$

(1)

Where for every $g \in [c,d]$, $i_g^{[c,d]}$ symbolize the inclusion map from $\triangle[n]_g = \triangle[n]$ into the coproduct $\coprod_{[c,d]} \triangle[n] = \delta_n^c(d)$, and for every $d \in \mathcal{C}$ the map $\psi_n^c(w)_d$ is the one given by the universal property of the coproduct. If $\eta : \coprod_{[c,-]} \triangle[n] \to X \in \mathbf{S}^{\mathcal{C}}(\delta_n^c, X)$, then $(\psi_n^c)^{-1}(\eta)$ is given by the composite $\triangle[n] \xrightarrow{i_{1_c}^{[c,c]}} \coprod_{[c,c]} \triangle[n] \xrightarrow{\eta_c} X_c$

∎

Definition 1.1.6 *Let Γ be the subcategory of $\mathbf{S}^{\mathcal{C}}$ whose set of objects is formed by the diagrams δ_n^c, for every $c \in \mathcal{C}$ and $n \geq 0$, and whose morphisms is given by the composition of the following ones*

$$\begin{array}{lll} \sigma^i : \delta_{n+1}^c \to \delta_n^c & , & 0 \leq i \leq n \\ \varepsilon^i : \delta_n^c \to \delta_{n+1}^c & , & 0 \leq i \leq n+1 \\ j_f : \delta_n^b \to \delta_n^a & \text{for} & f : a \to b \in \mathcal{C} \end{array}$$

where $\sigma^i = \coprod_{[c,-]} \sigma^i$ and $\varepsilon^i = \coprod_{[c,-]} \varepsilon^i$. Remember that $\sigma^i : \triangle[n+1] \to \triangle[n]$ and $\varepsilon^i : \triangle[n] \to \triangle[n+1]$ are the induced arrows by $\sigma_i : [n+1] \to [n]$ and the map $\varepsilon_i : [n] \to [n+1]$ in Δ, respectively (see Section 0.1). Given a morphism $f : a \to b \in \mathcal{C}$, the map $j_{f,c} : \coprod_{[b,c]} \triangle[n] \to \coprod_{[a,c]} \triangle[n]$ is defined as follows, for every $c \in \mathcal{C}$: if $z \in \triangle[n]_h$ for a given $h \in [b,c]$, then $j_{f,c}(z) = z \in \triangle[n]_{hf}$.

Proposition 1.1.7
Γ is a full subcategory of $\mathbf{S}^{\mathcal{C}}$.

Proof. Take a morphism $\eta : \delta_n^c \to \delta_m^d$ and apply the Proposition 1.1.5 with $X = \delta_m^d$ in the diagram (1). Then $\eta = \psi_n^c(w^g)$, where w^g is the representing map $\triangle[n] \xrightarrow{w} \triangle[m] \xrightarrow{i_g^{[d,c]}} \coprod_{[d,c]} \triangle[m]$ of some simplex $w \in \coprod_{[d,c]} \triangle[m]$. Therefore $\eta_e = \coprod_{[c,e]} w : \coprod_{h \in [c,e]} \triangle[n]_h \to \coprod_{h \in [c,e]} \triangle[m]_{hg} \subseteq \coprod_{[d,e]} \triangle[m]$, for every $e \in \mathcal{C}$. According to the Lemma 0.1.2 the n-simplex w can be expressed as $s_{j_t}...s_{j_1} d_{k_1}...d_{k_s} \iota_m$; hence for every $h \in [c,e]$, w has a factorization:

$$\triangle[n]_h = \triangle[n]_{hg} \xrightarrow{\sigma^{j_t}} \triangle[n-1] \xrightarrow{\sigma^{j_{t-1}}} ... \xrightarrow{\varepsilon^{k_{s-1}}} \triangle[m-1] \xrightarrow{\varepsilon^{k_s}} \triangle[m]_{hg}$$

that is, $\eta_e = \varepsilon^{k_s}...\varepsilon^{k_1}\sigma^{j_1}...\sigma^{j_t} j_{g,e}$.

∎

Proposition 1.1.8
The inclusion functor $i : \Gamma \hookrightarrow \mathbf{S}^{\mathcal{C}}$ is dense.

Proof. We have to check that $\mathbf{S}^{\mathcal{C}}(i(-),-) : \mathbf{S}^{\mathcal{C}} \to \mathfrak{Sets}^{\Gamma^{op}}$ is a faithful and full functor (see Lemma 1.1.2).

i) $\mathbf{S}^{\mathcal{C}}(i(-),-)$ is faithful: Suppose that $\eta, \rho : X \to Y \in \mathbf{S}^{\mathcal{C}}$ are natural transformations, such that their images η_* and ρ_* by $\mathbf{S}^{\mathcal{C}}(i(-),-)$ are equal, that is, $\mathbf{S}^{\mathcal{C}}(i(-),\eta) = \mathbf{S}^{\mathcal{C}}(i(-),\rho)$. Let c be an object of $\mathcal{C}$, $w \in \mathbf{S}(\triangle[n], X_c)$ and $\psi_n^c(w) : \delta_n^c \to X$ its corresponding natural transformation (diagram (1)).

Since $\eta_* = \rho_*$ we have that $[\eta_* \psi_n^c(w)]_d = [\rho_* \psi_n^c(w)]_d$ for every $d \in \mathcal{C}$, in particular if $c \in \mathcal{C}$ and $1_c \in [c,c]$ the equality $\eta_c \psi_n^c(w)_c i_{1_c}^{[c,c]} = \rho_c \psi_n^c(w)_c i_{1_c}^{[c,c]}$ holds. Then $\eta_c(w) = \rho_c(w)$, since (1) commutes for every $d \in \mathcal{C}$ and every $g \in [c,d]$.

ii) $\mathbf{S}^{\mathcal{C}}(i(-),-)$ is full: Take a map $\tau : \mathbf{S}^{\mathcal{C}}(i(\_),X) \to \mathbf{S}^{\mathcal{C}}(i(\_),Y) \in \mathfrak{Sets}^{\Gamma^{op}}$ and define the map $\rho_n^c : X_{c,n} \to Y_{c,n}$ by $\rho_n^c := (\phi_n^c)^{-1} \tau_n^c \psi_n^c$ for every $c \in \mathcal{C}$ and every $n \geq 0$, where ψ and ϕ are the bijections given in (1) for X and Y. Then let us show that ρ is a natural transformation from X into Y and that $\mathbf{S}^{\mathcal{C}}(i(-),\rho) = \tau$. To see that ρ is natural it is enough to check that the following square commutes for every arrow $g : c \to d$ in $\mathcal{C}$.

$$\begin{tikzcd} \mathbf{S}(\triangle[n], X_c) \arrow[r, "X_{g,*}"] \arrow[d, "\psi_n^c"'] & \mathbf{S}(\triangle[n], X_d) \arrow[d, "\psi_n^d"] \\ \mathbf{S}^{\mathcal{C}}(\delta_n^c, X) \arrow[r, "j^*_{g,n}"] & \mathbf{S}^{\mathcal{C}}(\delta_n^d, X) \end{tikzcd}$$

Note that for every n-simplex w in $\mathbf{S}(\triangle[n], X_c)$ the down diagram commutes

$$\begin{tikzcd} \triangle[n] \arrow[r, "i_f^{[d,e]}"] \arrow[drr, "w"] \arrow[ddrr, "X_g(w)"'] & \coprod_{f\in[d,e]} \triangle[n]_f \arrow[r, "j_{g,n}"] & \coprod_{f\in[d,e]} \triangle[n]_{fg} \subseteq \coprod_{[c,e]} \triangle[n] \arrow[r, "\psi_c^n(w)_e"] & X_e \\ & & X_c \arrow[ur, "X_{fg}"] \arrow[d, "X_g"] & \\ & & X_d \arrow[uur, "X_f"'] & \end{tikzcd}$$

Then $\psi_n^c(w)_e j_{g,n} = \psi_n^d(X_g(w))_e$ for every $e \in \mathcal{C}$ and every $f : d \to e \in \mathcal{C}$, and therefore $j_{g,n}^* \psi_n^c(w) = \psi_n^c(w) j_{g,n} = \psi_n^d(X_g(w)) = \psi_n^d X_{g,*}(w)$. It only remains to show that $\mathbf{S}^{\mathcal{C}}(i(-),\rho) = \tau$, that is, $\tau_n^c = \phi_n^{c,-1}\tau_n^c\psi_n^c$ for every $\delta_n^c \in \Gamma$, but this is a consequence from commutativity of the following diagram

$$\begin{tikzcd}
\mathbf{S}(\triangle[n], X_c) \arrow[d, "\phi_n^{c,-1}\tau_n^c\psi_n^c"] & \mathbf{S}^{\mathcal{C}}(\delta_n^c, X) \arrow[l, "\psi_n^{c,-1}"'] \arrow[d, "\tau_n^c"] \\
\mathbf{S}(\triangle[n], Y_c) & \mathbf{S}^{\mathcal{C}}(\delta_n^c, Y) \arrow[l, "\phi_n^{c,-1}"]
\end{tikzcd}$$

∎

The density of the functor $i : \Gamma \hookrightarrow \mathbf{S}^{\mathcal{C}}$ says us that every $\mathcal{C}$-diagram X can be expressed as a colimit of its n-$\mathcal{C}$-simplices, that is, $X = \underset{\Gamma\downarrow X}{colim}\delta_n$.

Corollary 1.1.9
$\mathbf{S}^{\mathcal{C}}(i(-),-) : \mathbf{S}^{\mathcal{C}} \to \mathfrak{Sets}^{\Gamma^{op}}$ *is an equivalence of categories.*

Proof. From Proposition 1.1.8 we have that $\mathbf{S}^{\mathcal{C}}(i(-),-)$ is faithful and full, then it only remains to show that $\mathbf{S}^{\mathcal{C}}(i(-),-)$ is representative, that is, we have to show that for every $A \in \mathfrak{Sets}^{\Gamma^{op}}$ there exists a $\mathcal{C}$-diagram $\widehat{A}$, such that $\mathbf{S}^{\mathcal{C}}(i(-),\widehat{A})$ is isomorphic to A.

Take $A \in \mathfrak{Sets}^{\Gamma^{op}}$ and define $\widehat{A} \in \mathbf{S}^{\mathcal{C}}$ by $\widehat{A}_{c,n} := A(\delta_n^c)$, $\widehat{A}(g)_n := A(j_{g,n})$, $s_i := A(\sigma^i)$ and $d_i := A(\varepsilon^i)$, for every $c \in \mathcal{C}$ and every $g : c \to d \in \mathcal{C}$. Then $\mathbf{S}^{\mathcal{C}}(i(-),\widehat{A}) \cong A$, since $\mathbf{S}^{\mathcal{C}}(\delta_n^c,\widehat{A}) \cong \mathbf{S}(\triangle[n],\widehat{A}_c) \cong \widehat{A}_{c,n} = A(\delta_n^c)$ for every $\delta_n^c \in \Gamma$.
∎

The notion of free diagram generated at $c \in \mathcal{C}$ (Definition 1.1.4) can be extended to the whole category $\mathcal{C}$ by considering the discrete category $\mathcal{C}^o$ of $\mathcal{C}$, whose objects are the same as $\mathcal{C}$ and with no non-identity morphisms.

Definition 1.1.10 ([16], *Definition* 11.5.27**)** *If $\mathcal{C}$ is a small category, $\mathcal{M}$ is a cocomplete category and $\mathbf{X} \in \mathcal{M}^{\mathcal{C}^o}$, then the free $\mathcal{C}$-diagram in $\mathcal{M}$ generated by $\mathbf{X}$ is the functor $\coprod_{c\in\mathcal{C}}((-)_c \otimes [c,\_\,]) : \mathcal{M}^{\mathcal{C}^o} \to \mathcal{M}^{\mathcal{C}}$, which is defined over objects by $\coprod_{c\in\mathcal{C}}(-_c \otimes [c,\_\,])(\mathbf{X}) = \coprod_{c\in\mathcal{C}}(\mathbf{X}_c \otimes [c,\_\,])$ for every $X \in \mathcal{M}^{\mathcal{C}^o}$, and over arrows by $\coprod_{c\in\mathcal{C}}((-)_c \otimes [c,\_\,])(\eta) = \coprod_{c\in\mathcal{C}}(\eta_c \otimes [c,\_\,])$ for every $\eta : \mathbf{X} \to \mathbf{Y}$ in $\mathcal{M}^{\mathcal{C}^o}$.*

Proposition 1.1.11 ([16], *Theorem* 11.5.28**)**
Under the above assumptions, if $U : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}^{\mathcal{C}^o}$ is the forgetful functor, then $\coprod_{c\in\mathcal{C}}((-)_c \otimes [c,\_\,])$ is left adjoint to U, that is, If $\mathbf{Y} \in \mathcal{M}^{\mathcal{C}^o}$ and $X \in \mathcal{M}^{\mathcal{C}}$, then there's a natural isomorphism

$$\mathcal{M}^{\mathcal{C}}(\coprod_{c\in\mathcal{C}}(\mathbf{Y}_c \otimes [c,\_\,]), X) \cong \mathcal{M}^{\mathcal{C}^o}(\mathbf{Y}, U(X))$$

Notice that $\mathcal{M}^{\mathcal{C}^o}(\mathbf{Y}, U(X)) = \prod_{c\in\mathcal{C}} \mathcal{M}(\mathbf{Y}_c, X_c)$, since in $\mathcal{M}^{\mathcal{C}^o}$ the only arrows are the identity morphisms.

1.2 Cofibrantly generated model structure over $\mathbf{S}^{\mathcal{C}}$

In the Section 0.3 we saw that $\mathbf{S}$ is a simplicial model category and we showed how this structure induced a model structure over the category of $\mathcal{C}$-diagrams of simplicial sets (see Example 0.2.17); although we also saw that $\mathbf{S}$ has a cofibrantly generated model structure. In this section we will see how the cofibrantly generated structure of $\mathbf{S}$ can be lifted to $\mathbf{S}^{\mathcal{C}}$ via a pair of adjoint functors, in particular we are interested in determining explicitly how the cofibrations and trivial cofibrations are in $\mathbf{S}^{\mathcal{C}}$.

In the previous section, the free diagrams were introduced, since the cofibrant objects in this model category will be the free cell complexes and their retracts. The relative free cell complexes will be the analogues for diagrams of topological spaces of relative cell complexes for topological spaces.

Definition 1.2.1 ([16]) *Let $\mathcal{C}$ be a small category and let c be an object of $\mathcal{C}$. If $\mathcal{M}$ is a model category and I is a set of maps in $\mathcal{M}$, then a free I-cell generated at c in $\mathcal{M}^{\mathcal{C}}$ is a map of the form*

$$\coprod_{[c,-]} A \to \coprod_{[c,-]} B$$

where $A \to B$ is an element of I.

A free cell generated at c in $\mathfrak{Top}^{\mathcal{C}}$ is a map of the form $\coprod_{[c,-]} |\dot{\triangle}[n]| \to \coprod_{[c,-]} |\triangle[n]|$ and a free cell generated at c in $\mathbf{S}^{\mathcal{C}}$ has the form $\dot{\delta}^c_n \to \delta^c_n$.

Definition 1.2.2 ([16]) *If $\mathcal{M}$ is a cofibrantly generated model category and $\mathcal{C}$ is a small category, then a relative free cell complex in $\mathcal{M}^{\mathcal{C}}$ is a map that is a transfinite composition (see Appendix B) of pushouts of free cells, and a free cell* *complex in $\mathcal{M}^{\mathcal{C}}$ is a diagram X such that the map from the initial object of $\mathcal{M}^{\mathcal{C}}$ to X is a relative free cell complex.*

In the following theorem we can see that the relative free cell complexes and their retracts will be the cofibrations in the model category of $\mathcal{C}$-diagrams in a cofibrantly generated model category $\mathcal{M}$.

Theorem 1.2.3 ([16], *Theorem* 11.6.1**)**
If $\mathcal{C}$ is a small category and $\mathcal{M}$ is a cofibrantly generated model category with generating cofibrations I and generating trivial cofibrations J, then the category $\mathbf{M}^{\mathcal{C}}$ of $\mathcal{C}$-diagrams in $\mathcal{M}$ is a cofibrantly generated model category with generating cofibrations $I_{\mathcal{C}} = \bigcup\limits_{c\in\mathcal{C}} I_c$ where

$$I_c = \{\, X_i \otimes [c,_-\,] \xrightarrow{f_i\otimes[c,_-]} Y_i \otimes [c,_-\,] \ : f_i : X_i \to Y_i \in I\}$$

and generating trivial cofibrations $J_{\mathcal{C}} = \bigcup\limits_{c\in\mathcal{C}} J_c$ where

$$J_c = \{\, X_j \otimes [c,_-\,] \xrightarrow{f_j\otimes[c,_-]} Y_j \otimes [c,_-\,] \ : f_j : X_j \to Y_j \in J\}$$

In this model category structure, a map $X \to Y \in \mathbf{S}^{\mathcal{C}}$ is

- *a weak equivalence if $X_c \to Y_c$ is a weak equivalence in $\mathcal{M}$ for every $c \in \mathcal{C}$,*
- *a fibration if $X_c \to Y_c$ is a fibration in $\mathcal{M}$ for every $c \in \mathcal{C}$, and*
- *a cofibration if it is a retract of a transfinite composition of pushouts of elements of $I_{\mathcal{C}}$.*

For our purposes we will make a sketch of the proof for $\mathcal{M} = \mathbf{S}$ (the detailed proof can be found in [16, Theorem 11.6.1]).

Sketch. If $\mathcal{C}^o$ is the discrete category with objects equal to the objects of $\mathcal{C}$, then $\mathbf{S}^{\mathcal{C}^o} = \prod\limits_{c\in Obj(\mathcal{C})} \mathbf{S}$ is a cofibrantly generated model category, by Proposition 0.2.9, with the sets of generating cofibrations I_Π and generating trivial cofibrations J_Π given by

$$I_\Pi = \bigcup_{c\in Obj(\mathcal{C})} (I_c \times \prod_{t\neq c} 1_{\emptyset_t})$$

$$J_\Pi = \bigcup_{c\in Obj(\mathcal{C})} (J_c \times \prod_{t\neq c} 1_{\emptyset_t})$$

where $1_{\emptyset_t}$ is the identity map of the initial object of $\mathbf{S}$, and $I_c = I$, $J_c = J$ for every $c \in \mathcal{C}$. The adjoint pair $\coprod_{c\in\mathcal{C}}((-)_c \otimes [c,_-]) : \mathbf{S}^{\mathcal{C}^{disc}} \rightleftarrows \mathbf{S}^{\mathcal{C}} : U$ of Proposition 1.1.11 satisfy the conditions of Theorem 0.2.14, and therefore there exists a cofibrantly generated model structure over $\mathbf{S}^{\mathcal{C}}$ in which $\coprod_{c\in\mathcal{C}}((-)_c \otimes [c,_-])(I_\Pi)$ is a set of generating cofibrations, $\coprod_{c\in\mathcal{C}}((-)_c \otimes [c,_-])(J_\Pi)$ is a set of generating trivial cofibrations, and the weak equivalences are the maps that U takes into weak equivalences in $\mathcal{M}$. It also holds that with respect to this model category structure $(\coprod_{c\in\mathcal{C}}((-)_c \otimes [c,_-]), U)$ is a Quillen pair.

∎

Remember that in $\mathbf{S}$ the set of generating cofibrations is formed by the inclusions $\dot{\triangle}[n] \hookrightarrow \triangle[n]$ for $n \geq 0$, and the generating trivial cofibrations are the inclusions $\triangle^k[n] \hookrightarrow \triangle[n]$ for $n > 0$ and $0 \leq k \leq n$. So from the above sketch we see the set of generating cofibrations in $\mathbf{S}^{\mathcal{C}}$ is formed by the inclusions $\dot{\delta}^c_n \hookrightarrow \delta^c_n$ for every $c \in \mathcal{C}$, and every $n \geq 0$, and the set of generating trivial cofibrations is given by the inclusions $\delta^c_{n,k} \hookrightarrow \delta^c_n$ for every $c \in \mathcal{C}$, $n > 0$ and $0 \leq k \leq n$.

It also holds that $X \to Y \in \mathbf{S}^{\mathcal{C}}$ is a fibration if $X_c \to Y_c$ is a fibration in $\mathbf{S}$ for every $c \in \mathcal{C}$, or equivalently if it has the *RLP* with respect to the set of generating trivial cofibrations, that is, if for every commutative diagram as the following, the dotted arrow exists

$$\begin{tikzcd} \delta^c_{n,k} \arrow[r] \arrow[d] & X \arrow[d] \\ \delta^c_n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

Note that with this cofibrantly generated structure over $\mathbf{S}^{\mathcal{C}}$ their relative free cell complexes are the relative cell complexes and their free cell complexes are the cell complexes (see Definitions 0.2.10 and 1.2.2). Then from Proposition 0.2.11 it holds that the class of cofibrations equals the class of retracts of relative free cell complexes, and therefore the class of I-injectives corresponds to the class of trivial fibrations (see item 4 of Proposition 0.2.6).

The following proposition characterizes those diagrams of simplicial sets that are free cell complexes. This characterization will be central to define and construct minimal models of $\mathcal{C}$-diagrams.

Proposition 1.2.4 ([16], *Proposition* 14.8.1**)**
If $\mathcal{C}$ is a small category and X is a $\mathcal{C}$-diagram of simplicial sets, then X is a free cell complex if and only if there is a sequence $\{\Sigma_0, \Sigma_1, \Sigma_2, ...\}$ of $\mathcal{C}^o$-diagrams of sets such that

a) *For every $n \geq 0$ and c an object of $\mathcal{C}$, the set $\Sigma_{c,n} := \Sigma_n(c)$ is a subset of $X_{c,n}$.*

b) *For $0 \leq i \leq n$ and c an object of $\mathcal{C}$, we have $s_i(\Sigma_{c,n}) \subseteq \Sigma_{c,(n+1)}$, that is Σ is closed under degeneracies.*

c) *If $n \geq 0$, d is an object of $\mathcal{C}$ and z is an n-simplex of X_c, then there exists an object c in $\mathcal{C}$, an element $w \in \Sigma_{c,n}$ and a map $f : c \to d \in \mathcal{C}$ such that $X_f(w) = z$, and such a triple is unique.*

The sequence $\{\Sigma_0, \Sigma_1, \Sigma_2, ...\}$ of the proposition will be called a *basis* for X, and an element of $\Sigma_{c,n}$ will be called a *generator* of the free cell complex X. We will let $\Sigma^\flat \subseteq \Sigma$ be the subset of degenerate simplices, and we will call an element of $\Sigma^\flat$ a *degenerate generator* of X. An element of $\Sigma - \Sigma^\flat$ will be called a *nondegenerate generator*.

1.3 $\mathcal{C}$-Fiber bundles

Our aim in this section is to extend the definition of fiber bundle given in Section 0.1.1 to the context of $\mathcal{C}$-diagramas of simplicial sets. In particular we are interested in maps $p : X \to B$ of $\mathbf{S}^{\mathcal{C}}$, where B is a constant diagram to the simplicial set B; for p the concepts about local triviality, atlases and structural group are reformulated in such way that we can recover again the theory of twisted cartesian products.

Definition 1.3.1 *Let F be a $\mathcal{C}$-diagram and B a simplical set. A map $p : X \to B$ in $\mathbf{S}^{\mathcal{C}}$, where B is the constant diagram, will be called a $\mathcal{C}$-* fibre bundle *with fibre F if p is an epimorphism and for every n-simplex $v \in B$ there exists an isomorphism $\alpha_p(v) : \triangle[n] \times F \to \triangle[n] \times_B X$ such that the following diagram commutes.*

$$\begin{tikzcd}
\triangle[n] \times F \arrow[r, "\alpha_p(v)", "\cong"'] \arrow[d, "pr"'] & \triangle[n] \times_B X \arrow[r, dotted, "\hat{v}"] \arrow[d, dotted, "pr"'] \arrow[dr, phantom, "\lrcorner", very near end] & X \arrow[d, "p"] \\
\triangle[n] \arrow[r, "-"'] & \triangle[n] \arrow[r, "v"'] & B
\end{tikzcd}$$

The set of isomorphisms $\{\alpha_p(v)\}$ will be called a $\mathcal{C}$*-atlas* of the $\mathcal{C}$-bundle and if F is fibrant p will be called a *Kan* $\mathcal{C}$*-fibre bundle* or for short a $\mathcal{C}_K$-bundle . Given two atlases $\{\alpha_p(v)\}$ and $\{\widetilde{\alpha}_p(v)\}$ of p, $\alpha_p(v)^{-1}\widetilde{\alpha}_p(v) \in aut(F)_n$ and conversely if for every $v \in B_n$ we choose $(\gamma(v)) \in aut(F)_n$, then $\{\alpha_p(v)\gamma(v)\}$ is another atlas. If for every v we define $\beta_p(v)$ as the composition $\hat{v}\alpha_p(v) : \triangle[n] \times F \to \triangle[n] \times_B X \to X$, then the atlas $\{\alpha_p(v)\}$ and the set of maps $\{\beta_p(v)\}$ determine each other.

Given an arrow $f : c \to d$ in $\mathcal{C}$ a $\mathcal{C}$-fibre bundle $p : X \to B$ with fibre F looks locally as follows

$$(2) \qquad \begin{tikzcd}
\triangle[n] \times F_c \arrow[r, "\alpha_{p_c}(v)", "\cong"'] \arrow[d, "1\times F_f"'] & \triangle[n] \times X_c \arrow[r, "\widehat{v}_c"] \arrow[d, "X_f^*"] & X_c \arrow[d, "X_f"] \\
\triangle[n] \times F_d \arrow[r, "\alpha_{p_d}(v)", "\cong"'] \arrow[d] & \triangle[n] \times X_d \arrow[r, "\widehat{v}_d"] \arrow[d] & X_d \arrow[d, "p_d"] \\
\triangle[n] \arrow[r, equal] & \triangle[n] \arrow[r, "v"'] & B
\end{tikzcd}$$

So for every $c \in \mathcal{C}$, p_c is a fibre bundle with fibre F_c and atlas $\{\alpha_{p_c}(v)\}$. Like in the classic theory for fibre bundles we could ask if given a $\mathcal{C}$-atlas $\{\alpha_p(v)\}$ of p it is normal and regular, that is, if $\{\alpha_{p_c}(v)\}$ is normal and regular for every $c \in \mathcal{C}$.

To get a normal $\mathcal{C}$-atlas we can redefine $\{\alpha_{p_c}(v)\}$ for every $c \in \mathcal{C}$ over the non-degenerate simplices $v \in B$, as we did in Section 0.1.1 with the formula $(*)$. After doing that we just have to verify that the commutativity of the diagram (2) is preserved, that is, we have to see if the frontal square of the following diagram commutes for every arrow $f : c \to d \in \mathcal{C}$:

$$\begin{tikzcd}
 & \triangle[n] \times F_c \arrow[rr, dotted, "\beta'_{p_c}(v)"] \arrow[d, dotted, "1\times F_f"] & & X_c \arrow[d, "X_f"] \\
\triangle[n+1] \times F_c \arrow[ur, dotted, "\sigma^i\times 1"] \arrow[rrru, "\beta'_{p_c}(s_iv)"'] \arrow[d, "1\times F_f"'] & \triangle[n] \times F_d \arrow[rr, dotted, "\beta'_{p_d}(v)"] & & X_d \\
\triangle[n+1] \times F_d \arrow[ur, dotted, "\sigma^i\times 1"] \arrow[rrru, "\beta'_{p_d}(s_iv)"'] & & &
\end{tikzcd}$$

Given a non-degenerate simplex $v \in B$ we define $\beta'_{p_c}(v) := \beta_{p_c}(v)$ and $\beta'_{p_c}(s_iv) := s_i\beta_{p_c}(v)$ for every $c \in \mathcal{C}$. So the following sequence of equalities holds $X_f\beta'_{p_c}(s_iv) = X_f s_i\beta_{p_c}(v) = X_f\beta_{p_c}(v)(\sigma^i \times 1) = \beta_{p_d}(v)(1\times F_f)(\sigma^i \times 1) = \beta_{p_d}(v)(\sigma^i \times 1)(1 \times F_f) =$

$s_i\beta_{p_d}(v)(1 \times F_f) = \beta'_{p_d}(s_i v)(1 \times F_f)$.

Note that it is also possible to define a set of transformation elements $\{\xi^i_p(v)\}$ for a given $\mathcal{C}$-bundle p by doing a parallel treatment over the faces $d_i v$ of a given n-simplex $v \in B$, as it was done in Section 0.1.1. In the present situation $\xi^i_p(v)$ is defined over every $c \in \mathcal{C}$ by $\xi^i_{p_c}(v) := \alpha_{p_c}(d_i v)^{-1}\theta_{p_c}(v)d_i\alpha_{p_c}(v)$ and it is easy to see that $\xi^i_p(v) \in aut(F)_{n-1}$, since (2) is commutative. Following the same reasoning of Proposition 0.1.19 we can obtain a regular atlas $\{\alpha'_p(v)\}$ of p, and as in the normalization process we must prove that the new atlas makes that (2) commutes, that is, if $X_f\beta'_{p_c}(v) = \beta'_{p_d}(1 \times F_f)$ holds for every $f : c \to d \in \mathcal{C}$.

In the induction process of Proposition 0.1.19 we can see that $\alpha_p(v)$ is redefined by $\alpha'_p(v) = \alpha_p(v)\gamma(v)^{-1}$ for every non-degenerate n-simplex v in B, where $\gamma(v) \in aut(F)_n$. Since $\gamma(v) \in aut(F)_n$, for every $f : c \to d \in \mathcal{C}$ it holds that $(1 \times F_f)\gamma_c(v) = \gamma_d(v)(1 \times F_f)$, then it follows that $X_f\beta'_{p_c}(v) = X_f\widehat{v}_c\alpha'_{p_c}(v) = X_f\widehat{v}_c\alpha_{p_c}(v)\gamma_c(v)^{-1} = \widehat{v}_d\alpha_{p_d}(v)(1 \times F_f)\gamma_c(v)^{-1} = \widehat{v}_d\alpha'_{p_d}(v)\gamma_d(v)(1 \times F_f)\gamma_c(v)^{-1} = \beta'_{p_d}(v)\gamma_d(v)\gamma_d(v)^{-1}(1 \times F_f) = \beta'_{p_d}(v)(1 \times F_f)$.

From now on, given any $\mathcal{C}$-bundle with group G we may suppose that we are dealing with regular and normalized atlases. Notice that the concept of G-map and G-equivalence of Definition 0.1.20 is similar in the context of $\mathcal{C}$-bundles.

Definition 1.3.2 *If $p : X \to B$ and $p' : X' \to B$ are $\mathcal{C}$-bundles with fibre F and group G, a G-map from p to p' is a map of $\mathcal{C}$-diagrams $h : X \to X'$ such that for every n-simplex $v \in B$ and any G-atlases $\{\alpha_p(v)\}$, $\{\alpha_{p'}(v)\}$ in the given G-equivalence classes of atlases, there exists $\gamma(v) \in G_n$ such that the following diagram commutes*

$$\begin{tikzcd}
 & \triangle[n] \times F \arrow[r, "\alpha_{p'}(v)"] \arrow[ddl, dotted] & \triangle[n] \times_B X' \arrow[r, "\widehat{v}'"] \arrow[ddl, dotted] & X' \arrow[ddl, "p'"] \\
\triangle[n] \times F \arrow[ur, "\gamma(v)"] \arrow[r, "\alpha_p(v)"] \arrow[d] & \triangle[n] \times_B X \arrow[r, "\widehat{v}"] \arrow[d] & X \arrow[ur, "h"] \arrow[d, "p"] & \\
\triangle[n] \arrow[r, "_"'] & \triangle[n] \arrow[r, "v"'] & B &
\end{tikzcd} \qquad h\beta_p(v) = \beta_{p'}(v)\gamma(v)$$

If h is a natural isomorphism we'll say that p and p' are G-equivalent .

Lemma 1.3.3
If $p : X \to B$ is a $\mathcal{C}_K$-bundle with fibre F, then p is a fibration.

Proof. The map $p_c : X_c \to B$ is a fibre bundle with fibre F_c for every $c \in \mathcal{C}$, and since F is fibrant by Lemma 0.1.21 p_c is a fibration for every $c \in \mathcal{C}$.
∎

1.4 $\mathcal{C}$-Twisted cartesian products

According to the example 0.2.17 of Section 0.3, if F is a $\mathcal{C}$-diagram the group $aut(F)$ of automorphisms of F consists in dimension n of tuples $(\varepsilon_c)_{c\in\mathcal{C}}$ that satisfies:

1. $\varepsilon_c \in aut(F_c)_n$, for every $c \in \mathcal{C}$
2. the following diagram commutes, for every morphism $f : c \to d$ in $\mathcal{C}$

$$\begin{tikzcd} \triangle[n] \times F_c \arrow[r, "1\times F_f"] \arrow[d, "\varepsilon_c"'] & \triangle[n] \times F_d \arrow[d, "\varepsilon_d"] \\ F_c \arrow[r, "F_f"'] & F_d \end{tikzcd}$$

   or equivalently the following one

$$\begin{tikzcd} \triangle[n] \times F_c \arrow[rr, "1\times F_f"] \arrow[d, "\tilde{\varepsilon}_c"'] \arrow[ddr, dotted] & & \triangle[n] \times F_d \arrow[d, "\tilde{\varepsilon}_d"] \arrow[ddl, dotted] \\ \triangle[n] \times F_c \arrow[rr, "1\times F_f"] \arrow[dr] & & \triangle[n] \times F_d \arrow[dl] \\ & \triangle[n] & \end{tikzcd}$$

   where $\tilde{\varepsilon}_c(\tau, z) = (\tau, \varepsilon_c(\tau, z))$, for every k-simplex (τ, z) in $\triangle[n] \times F_c$ and every $c \in \mathcal{C}$.

Since many of the properties of $aut(F)$ are a consequence of commutativity of the above square, we will see how it can be expressed in combinatorial terms, that is, consider $(\varphi\iota_n, z) \in (\triangle[n] \times F_c)_k$ and evaluate this simplex in the diagram: $\varepsilon_d(1 \times F_f)(\varphi\iota_n, z) = \varepsilon_d(\varphi\iota_n, F_f(z)) = \varphi\varepsilon_d(\iota_k, F_f(z))$ and

$F_f\varepsilon_c(\varphi\iota_n, z) = F_f[\varphi\varepsilon_c(\iota_k, z)]$, therefore $\varphi\varepsilon_d(\iota_k, F_f(z)) = F_f[\varphi\varepsilon_c(\iota_k, z)]$ or equivalently

$$(\ddagger)\ \varphi\varepsilon_d.F_f(z) = F_f[\varphi\varepsilon_c.z]$$

Given a subgroup G of $aut(F)$ the subgroup of $aut(F_c)$ whose set of n-simplices consists of the elements ε_c such that $(\varepsilon_c) \in G_n$ will be denoted by G_c, and in what follows (ε_c) will denote the tuple $(\varepsilon_c)_{c\in\mathcal{C}}$. The following definition will be justified by Proposition 1.4.2 and Proposition 1.4.3.

Definition 1.4.1 *Let B be a simplicial set, F a $\mathcal{C}$-diagram, G a simplicial group acting effectively on the left of F and $t : B \to G$ a twisting map. Then the $\mathcal{C}$-Twisted cartesian product with base B, fibre F and group G is the $\mathcal{C}$-diagram $B \times_t F$ defined by $B \times_t F(c) = B \times_{t_c} F_c$ for every $c \in \mathcal{C}$, and by $B \times_t F(c)(f) = 1 \times F_f$ for every morphism $f \in \mathcal{C}$. The term $\mathcal{C}$-TCP will also refer to the projection map $pr : B \times_t F \to B$.*

In the definition by an effective left action of G over F we mean an injective group homomorphism $\theta : G \to aut(F)$. Note that θ is given by a tuple (θ_c) of group homomorphisms $\theta_c : G \to aut(F_c)$ such that $(\theta_c(\mathrm{g})) \in aut(F)$, or equivalently by a left action of G over F_c for every $c \in \mathcal{C}$, such that $\varphi\mathrm{g}.F_f(z) = F_f[\varphi\mathrm{g}.z]$ for every arrow $f : c \to d \in \mathcal{C}$, every k-simplex z of F_c and every semisimplicial operator φ of length $(n-k)$. Since θ is injective we can assume that G is a subgroup of $aut(F)$.

Notice that a $\mathcal{C}$-twisting map $t = \{t_n : B_n \to G_{n-1}\}_{n\geq 1}$ has the following form: If $v \in B_n$, $t(v) = (t(v)_c)$. Then for every $c \in \mathcal{C}$ we can obtain a map $t_c : B_n \to G_{c,n-1}$, by defining $t_c(v) := t(v)_c$, since t is a twisting map it is easy to see that t_c is also a twisting map. Then $B \times_{t_c} F_c$ is a TCP with fibre F_c and group G_c, for every $c \in \mathcal{C}$.

Proposition 1.4.2
Let B be a simplicial set, F a $\mathcal{C}$-diagram, $G \leq aut(F)$ and $t : B \to G$ a twisting map. Then for every arrow $f : c \to d \in \mathcal{C}$ the map $1 \times F_f : B \times_{t_c} F_c \to B \times_{t_d} F_d$ is a simplicial map.

Proof. To show that $1\times F_f$ is simplicial is enough with checking the commutativity of $1 \times F_f$ with the face operator d_0. Let us take an n-simplex (v, z) of $B \times_{t_c} F_c$: $d_0(1 \times F_f)(v, z) = d_0(v, F_f(z)) = (d_0v, t_d(v).d_0F_f(z)) = (d_0v, t_d(v).F_f(d_0z))$, since

$t(v) \in G_{n-1}$ from (‡) it holds that $(d_0b, t_d(v).F_f(d_0z)) = (d_0b, F_f[t_c(v).d_0z]) = (1 \times F_f)(d_0v, t_c(v).d_0z) = (1 \times F_f)d_0(v, z)$.

∎

The above proposition says that from every twisting map $t : B \to G$, where $G \leq aut(F)$, we can obtain a well behaved $\mathcal{C}$-diagram of $TCP's$. Although it is also possible to show that every $\mathcal{C}$-diagram of well behaved $TCP's$ comes from a $\mathcal{C}$-twisting map, that is, if for a given diagram $F \in \mathbf{S}^{\mathcal{C}}$ and a given simplicial set B there is a $\mathcal{C}$-diagram X such that for every $f : c \to d$ in $\mathcal{C}$, $X(a \xrightarrow{f} b) = B \times_{t_c} F_c \xrightarrow{1\times F_f} B \times_{t_d} F_d$, then it is fully characterized by a $\mathcal{C}$-twisting map $t : B \to G$ for some $G \leq aut(F)$.

Proposition 1.4.3
Let F be a $\mathcal{C}$-diagram and B a simplicial set. If X is a $\mathcal{C}$-diagram such that for every $f : c \to d$ in $\mathcal{C}$, $X(c \xrightarrow{f} d) = B \times_{t_c} F_c \xrightarrow{1\times F_f} B \times_{t_d} F_d$, then $t = (t_c)$ is a twisting map.

Proof. To prove that $t : B \to aut(F)$ defined by $t = (t_c)$ is a twisting map we must show that $t(v) \in aut(F)_{n-1}$ is well defined for every $v \in B_n$ (that is, (t_c) satisfies the relation (‡)) and that t is a twisting map. Since $1 \times F_f$ is a simplicial map it commutes with the face operator d_0, then from a straightforward calculus we can see that

$$(\diamond) \quad t_d(v).F_f(z) = F_f[t_c(v).z]$$

for every $f : c \to d$ in $\mathcal{C}$, $v \in B_n$ and $z \in F_{c,(n-1)}$. Using this relation we will show that $\varphi t_d(v).F_f(z) = F_f[\varphi t_c(v).z]$ by considering the following two cases:

i) $\varphi = d_0$. Check the equation $d_0t_d(b).F_f(z) = F_f[d_0t_c(b).z]$ is equivalent to check that $t_d(d_0b)^{-1}.t_d(d_1b).F_f(z) = F_f[t_c(d_0b)^{-1}.t_c(d_1b).z]$, or equivalently that $t_d(d_1b).F_f(z) = t_d(d_0b).F_f[t_c(d_0b)^{-1}.t_c(d_1b).z]$, since the twisting maps (t_c) satisfy the relations with the face operator d_0 given in Definition 0.1.22. Starting with the right part of the last equation we obtain the following equality $t_d(d_0b).F_f[t_c(d_0b)^{-1}.t_c(d_1b).z] = F_f[t_c(d_0b).t_c(d_0b)^{-1}.t_c(d_1b).z]$ by using $(\diamond)$, and again by the equation $(\diamond)$ it holds that $F_f[t_c(d_1b).z] = t_d(d_1b).F_f(z)$.

ii) $\varphi = s'd_0d'$. Every step of the following sequence holds by using the equations given in Definition 0.1.22 for $TCP's$: $\varphi t_d(v).F_f(z) = s'd_0d't_d(v).F_f(z) =$

$d_0s''d't_d(v).F_f(z) = d_0t_d(s'''d''v).F_f(z)$, by using the equality of the part i) we have that $d_0t_d(s'''d''v).F_f(z) = F_f[d_0t_c(s'''d''v).z]$, and using again the relations of Definition 0.1.22 we obtain the following equalities $F_f[d_0t_c(s'''d''v).z] = F_f[d_0s''d't_c(v).z] = F_f[s'd_0d't_c(v).z] = F_f[\varphi t_c(b).z]$.

Therefore $t : B \to aut(F)$ is well defined. Since t_c is a twisting map for every $c \in \mathcal{C}$, it is easy to check that t is a twisting map.

∎

Lemma 1.4.4
Let $pr : B \times_t F \to B$ a $\mathcal{C}$-TCP. If F is fibrant, then pr is a fibration.

Definition 1.4.5 *Let $B \times_t F$ and $B' \times_l F$ be $\mathcal{C}$-$TCP's$ with fibre F and group G. A morphism of $\mathcal{C}$-diagrams $h : B \times_t F \to B' \times_l F$ is said to be a map of $\mathcal{C}-TCP's$ if $h_c(v,z) = (g(v), \gamma(v)_c.z)$ for every $c \in \mathcal{C}$, $v \in B_n$ and every $z \in F_{c,n}$, where $g : B \to B'$ is a simplicial map and $\gamma : B \to G$ is a dimension preserving function. If $B = B'$ and $g = id_B$ we say that t and l are twisting maps equivalents, $t \sim_{\mathcal{C}} l$, or that $B \times_t F$ and $B' \times_l F$ are $\mathcal{C}$-$TCP's$ equivalents.*

Remark 1.4.6 *If we want a map between $\mathcal{C}$-$TCP's$ it is enough with defining a map of $\mathcal{C}$-diagrams $h : B \times_t F \to B' \times_l F$, such that h_c is a map of $TCP's$ for every $c \in \mathcal{C}$. That is if $h_c(v,z) = (g(v), \gamma_c(v).z)$ for a given n-simplex (v,z) of $B \times_{t_c} F_c$, then we can define $\gamma : B \to G$ by $\gamma(v) = (\gamma_c(v))$, and we just have to see that $(\gamma_c(v)) \in G$.*

To check that $(\gamma_c(v)) \in G$ let us remember that given an n-simplex (v,z) of $B \times_{t_c} F_c$ the following equalities hold for every $c \in \mathcal{C}$ and every arrow $f : c \to d$ in $\mathcal{C}$:

$$\begin{array}{ll} (i) & t_d(v).F_f(z) = F_f[t_c(v).z] \\ (ii) & \gamma_d(v).F_f(z) = F_f[\gamma_c(v).z] \\ (iii) & l_c(v)d_0\gamma_c(v) = \gamma_c(d_0v)t_c(v) \end{array}$$

The first one holds since $1 \times F_f : B \times_{t_c} F_c \to B \times_{t_d} F_d$ is a simplicial map (see Proposition 1.4.3), the second one holds since h is a map of $\mathcal{C}$-diagrams and the third one holds because h_c is a map of $TCP's$ (see the equalities given after Definition 0.1.26. So to see that $(\gamma_c(v)) \in G$ we have to verify that $\gamma_c(v))$ satisfies the equality (‡), that is, $\varphi\gamma_d(v).F_f(z) = F_f[\varphi\gamma_c(v).z]$.

We are going to consider first the case in which $\varphi = d_0$: $d_0\gamma_d(v).F_f(z) =^{(iii)} l_d(v)^{-1}.\gamma_d(d_0v).t_d(v).F_f(z) =^{(i)} l_d(v)^{-1}.\gamma_d(d_0v).F_f[t_c(v).z]$ using (ii) it holds that

$l_d(v)^{-1}.\gamma_d(d_0v).F_f[t_c(v).z] = l_d(b)^{-1}.F_f[\gamma_c(d_0v).t_c(v).z]$, applying (iii) we have that $l_d(b)^{-1}.F_f[\gamma_c(d_0v).t_c(v).z] = l_d(v)^{-1}.F_f[l_c(v).d_0\gamma_c(v).z]$, and from the equation (i), $l_d(v)^{-1}.F_f[l_c(v).d_0\gamma_c(v).z] = l_d(v)^{-1}l_d(v)F_f[d_0\gamma_c(v).z] = F_f[d_0\gamma_c(v).z]$. Therefore: $d_0\gamma_d(b).F_f(z) = F_f[d_0\gamma_c(b).z]$

Now let us suppose that $\varphi = s'd_0d'$: $\varphi\gamma_d(v).F_f(z) = s'd_0d'\gamma_d(v).F_f(z) = d_0s''d'\gamma_d(v).F_f(z) = d_0\gamma_d(s''d'v).F_f(z)$, by the last equation of the above paragraph we have that $d_0\gamma_d(s''d'v).F_f(z) = F_f[d_0\gamma_c(v)(s''d'v).z] = F_f[d_0s''d'\gamma_c(v).z] = F_f[s'd_0d'\gamma_c(v).z] = F_f[\varphi\gamma_c(v).z]$.

Proposition 1.4.7
Let $pr : B \times_t F \to B$ a $\mathcal{C}$-twisted cartesian product with base B, fibre F and group G. Then pr is a $\mathcal{C}$-bundle with fibre F and group G.

Proof. Given $v \in B_n$ let us define the map $\alpha_t(v) : \triangle[n] \times F \to \triangle[n] \times_{t\circ v} F$ by $\alpha_{t_c}(v)(\varphi\iota_n, z) = (\varphi\iota_n, \psi_t^v(\varphi\iota_n).z)$, for every $c \in \mathcal{C}$, where $\psi_t^v(\varphi\iota_n) = s't(d'v)$ if $\varphi = s'd_0d'$, and $\psi_t^v(\varphi\iota_n) = e$ if $\varphi = s'd'$. From Proposition 0.1.27 it holds that $\alpha_{t_c}(v)$ is an isomorphism for every $c \in \mathcal{C}$, so it remains to show that for every $f : c \to d \in \mathcal{C}$ the following square is commutative

$$\begin{tikzcd}
\triangle[n] \times F_c \arrow[r, "\alpha_{t_c}(v)"] \arrow[d, "1\times F_f"'] & \triangle[n] \times_{t_c\circ v} F_c \arrow[d, "F_f^*"] \\
\triangle[n] \times F_d \arrow[r, "\alpha_{t_d}(v)"'] & \triangle[n] \times_{t_d\circ v} F_d
\end{tikzcd}$$

where $F_f^*(\tau, z) = (\tau, F_f(z))$.

Take a simplex $(\varphi\iota_n, z) \in \triangle[n] \times F_c$ and consider its evaluations in the above square: $F_f^*\alpha_{t_c}(v)(\varphi\iota_n, z) = F_f^*(\varphi\iota_n, \psi_{t_c}^v(\varphi\iota_n).z) = (\varphi\iota_n, F_f[\psi_{t_c}^v(\varphi\iota_n).z])$ and $\alpha_{t_d}(v)(1 \times F_f)(\varphi\iota_n, z) = \alpha_{t_d}(v)(\varphi\iota_n, F_f(z)) = (\varphi\iota_n, \psi_{t_d}^v(\varphi\iota_n).F_f(z))$, then the above diagram commutes if $F_f[\psi_{t_c}^v(\varphi\iota_n).z] = \psi_{t_d}^v(\varphi\iota_n).F_f(z)$. To show that consider the following two casses:

i) $\varphi = s'd'$. $F_f[\psi_{t_c}^v(\varphi\iota_n).z] = F_f[e.z] = F_f(z) = e.F_f(z) = \psi_{t_d}^v(\varphi\iota_n).F_f(z)$.

ii) $\varphi = s'd_0d'$. $F_f[\psi_{t_c}^v(\varphi\iota_n).z] = F_f[s't_c(d'v).z] = F_f[t_c(s''d'v).z]$, since (t_c) satisfies the equation (‡) (because $t(v) \in aut(F)$) it holds that $F_f[t_c(s''d'v).z] = t_d(s''d'v).F_f(z) = s't_d(d'v).F_f(z) = \psi_{t_d}^v(\varphi\iota_n).F_f(z)$.

∎

Proposition 1.4.8
Let $p : X \to B$ a $\mathcal{C}$-fibre bundle with fibre F and a regular G-atlas $\{\alpha_p(v)\}$. Then the transformation elements $\{\xi_p^0(v)\}$ define a $\mathcal{C}$-twisting map $\xi_p^0 : B_n \to G_{n-1}$ and thereby $B \times_{\xi_p^0} F$ becomes a $\mathcal{C}$-TCP with fibre F and group G. Furthermore there is an isomorphism $h : B \times_{\xi_p^0} F \to X$ of $\mathcal{C}$-fibre bundles with group G.

Proof. First notice that $\xi_p^0(v) \in G_{n-1}$ for every $v \in B_n$, since p is a G-bundle. By Proposition 0.1.29 it holds that $\xi_{p_c}^0 : B_n \to G_{c,n-1}$ defined by $\xi_{p_c}^0(v) = (\xi_p^0(v))_c$ is a twisting map for every $c \in \mathcal{C}$, then ξ_p^0 is a twisting map.

The map h is defined by $h_c(v, z) = \beta_{p_c}(v)(\iota_n, z)$, for every $c \in \mathcal{C}$, where (v, z) is an n-simplex of $B \times_{\xi_{p_c}^0} F_c$. Applying the Proposition 0.1.29 it holds that h_c is an isomorphism for every $c \in \mathcal{C}$, then it remains to show that for every $f : c \to d \in \mathcal{C}$ the following square commutes

$$\begin{tikzcd} B \times_{\xi_{p_c}^0} F_c \arrow[r, "h_c"] \arrow[d, "1\times F_f"'] & X_c \arrow[d, "X_f"] \\ B \times_{\xi_{p_d}^0} F_d \arrow[r, "h_d"'] & X_d \end{tikzcd}$$

Take an n-simplex (v, z) in $B \times_{\xi_{p_c}^0} F_c$ and evaluate it in the above square, that is, $X_f h_c(v, z) = X_f \beta_{p_c}(v)(\iota_n, z)$ and $h_d(1 \times F_f)(v, z) = h_d(v, F_f(z)) = \beta_{p_d}(v)(\iota_n, F_f(z))$. Since p is a $\mathcal{C}$-fibre bundle it looks locally as the diagram (2), which is commutative, then $X_f \beta_{p_c}(v)(\iota_n, z) = \beta_{p_d}(v)(1 \times F_f)(\iota_n, z)$, where $\beta_{p_d}(v)(1 \times F_f)(\iota_n, z) = \beta_{p_d}(v)(\iota_n, F_f(z))$.

∎

As in Proposition 0.1.28 of Section 0.1.2 the following lemma says that the corresponding notion of equivalence for $\mathcal{C}$-$TCP's$ with group G corresponds exactly to that of G-equivalence of $\mathcal{C}$-bundles.

Proposition 1.4.9
The $\mathcal{C}$-$TCP's$, $B \times_t F$ and $B \times_l F$ with group G are equivalents if there exists an isomorphism $h : B \times_t F \to B \times_l F$, such that for every $v \in B_n$ there exists $\gamma(v) \in G$ that makes the following diagram commutative

$$
\begin{tikzcd}
& \triangle[n]\times F \arrow[rr, "\alpha_l(v)"] \arrow[ddl, dotted] & & \triangle[n]\times_{l\circ v} F \arrow[rrr, "v\times 1"] \arrow[ddl, dotted] & & & B\times_l F \arrow[ddll] \\
\triangle[n]\times F \arrow[ur, "\gamma(v)"] \arrow[rr, "\alpha_t(v)"] \arrow[d] & & \triangle[n]\times_{t\circ v} F \arrow[rr, "v\times 1"] \arrow[d] & & B\times_t F \arrow[urr, "h"] \arrow[d] & & \\
\triangle[n] \arrow[rr] & & \triangle[n] \arrow[rr, "v"'] & & B & &
\end{tikzcd}
\qquad h(v\times 1)\alpha_t(v) = (v\times 1)\alpha_l(v)\gamma(v)
$$

From the analogue of Proposition 0.1.30 in the context of $\mathcal{C}$-bundles and $\mathcal{C}$-$TCP's$ we can formulate the following one.

Proposition 1.4.10
The equivalence classes of $\mathcal{C}$-bundles with fibre F and group G are in bijective correspondence with the equivalence classes of $\mathcal{C}$-$TCP's$ with fibre F and group G.

The next result is a straightforward consequence of definitions 0.1.26 and 1.4.5.

Proposition 1.4.11
Let F be a $\mathcal{C}$-diagram and G a subgroup of $aut(F)$. Then the equivalence classes of principal twisted cartesian products with group G are in bijective correspondence with the equivalence classes of $\mathcal{C}$-$TCP's$ with fibre F and group G.

From Propositions 1.4.10, 1.4.11 and Theorem 0.1.32 we have that any $\mathcal{C}$-fibre bundle with fibre F, group G and base diagram B can be obtained from a map with domain B and codomain $\overline{W}G$, and conversely that any map from B into $\overline{W}G$ defines a $\mathcal{C}$-fibre bundles with fibre F and group G.

Theorem 1.4.12
Let F be a $\mathcal{C}$-diagram, G a subgroup of $aut(F)$ and B a connected simplicial set. Then the set of homotopy classes of maps $[B, \overline{W}G]$ from B to $\overline{W}G$ are in bijective correspondence with the set of equivalence classes of $\mathcal{C}$-fibre bundles with fibre F and group G.

1.5 aut(F)

Given two $\mathcal{C}$-diagrams X and Y let us define a new $\mathcal{C}$-diagram $\underline{Hom}(X, Y)$ in $\mathfrak{Sets}^{\Gamma^{op}}$ (remember that $\mathbf{S}^{\mathcal{C}}$ and $\mathfrak{Sets}^{\Gamma^{op}}$ are equivalents as categories, see Corollary 1.1.9), for the arrow $\alpha : \delta^b_n \to \delta^e_k$ in Γ define the functor as follows

$\underline{Hom}(X,Y)(\alpha : \delta_n^b \to \delta_k^e) = (\alpha \times 1)^\sharp : \mathbf{S}^{\mathcal{C}}(\delta_k^e \times X, Y) \to \mathbf{S}^{\mathcal{C}}(\delta_n^b \times X, Y)$. Now for every n consider the restriction of $\underline{Hom}(X,Y)$ to the full subcategory Γ_n of Γ, whose set of objects is given by $\{\delta_n^c : c \in \mathcal{C}\}$ and denote the restriction by $\underline{Hom}(X,Y)_n : \Gamma_n^{op} \to \mathfrak{Sets}$.

For every n take the inverse limit $\underline{hom}(X,Y)_n = \underset{\leftarrow}{lim}\underline{Hom}(X,Y)_n$. Notice that $\underline{hom}(X,Y)_n$ is the set formed by the set of tuples $(\epsilon_c)_{c\in\mathcal{C}}$ such that for every arrow $g : b \to c$ in $\mathcal{C}$ the following diagram commutates

$$\begin{tikzcd}
 & & \delta_n^b \times X \arrow[rr, "\epsilon_b"] \arrow[dr, dotted] & & \delta_n^b \times Y \arrow[dl] \\
\delta_n^c \times X \arrow[urr, "j_g \times 1"] \arrow[rr, "\epsilon_c"] \arrow[dr] & & \delta_n^c \times Y \arrow[urr, "j_g \times 1"] \arrow[dl] & \delta_n^b & \\
 & \delta_n^c \arrow[urr, "j_g"] & & &
\end{tikzcd}$$

So we can define the simplicial set $\underline{hom}(X,Y)$, whose set of n-simplices is given by $\underline{hom}(X,Y)_n$ and with operators faces and degeneracies given by composing with $1 \times \varepsilon^i$ and $1 \times \sigma^i$, respectively. If F is a $\mathcal{C}$-diagram, then $\underline{we}(F)$ will be the simplicial set whose set of n-simplices is formed by the above commutative diagrams where the horizontal arrows are weak equivalences (in that case $X = Y = F$), and $\underline{aut}(F)$ is the simplicial subset of $\underline{we}(F)$ in which the horizontal arrows are isomorphisms for every $c \in \mathcal{C}$.

Proposition 1.5.1
Let X and Y be $\mathcal{C}$-diagrams. Then the simplicial sets $hom(X,Y)$ and $\underline{hom}(X,Y)$ are isomorphic.

Proof. Notice that for every $a \in \mathcal{C}$ the commutativity of the triangles in the above diagram tell us that every arrow ϵ_c of one tuple (ϵ_c), evaluated in a, has the form $\epsilon_c(a) = (\epsilon_{c,g}^a)_{g\in[c,a]}$, where $\epsilon_{c,g}^a : \triangle[n]_g \times X_a \to \triangle[n]_g \times Y_a$. Then let us see that for every $c \in \mathcal{C}$, $\epsilon_{c,1_c}^c = \epsilon_{b,g}^c = \epsilon_{c,f}^c$ for every $g \in [b,c]$ and every $f \in [c,c]$.

If we take an arrow $g \in [b,c]$ and consider the composite $b \xrightarrow{g} c \xrightarrow{1_c} c$, the following diagram holds

$$\begin{tikzcd}
\triangle[n]_g \times X_c \arrow[r, "\epsilon^c_{b,g}"] & \triangle[n]_g \times Y_c \\
\triangle[n]_{1_c} \times X_c \arrow[u, "\shortmid"'] \arrow[r, "\epsilon^c_{c,1_c}"] & \triangle[n]_{1_c} \times Y_c \arrow[u, "\shortmid"']
\end{tikzcd}$$

That is, $\epsilon^c_{c,1_c} = \epsilon^c_{b,g}$ for every $g \in [b,c]$. Now take an arrow $f \in [c,c]$ and an arrow $h \in [b,c]$, as before we can conclude that $\epsilon^c_{c,f} = \epsilon^c_{b,fh}$. Since $fh \in [b,c]$ by the first part holds that $\epsilon^c_{c,1_c} = \epsilon^c_{c,f}$. Thus every tuple $(\epsilon_c) \in \underline{hom}(X,Y)$ defines a unique natural transformation $\varepsilon \in hom(X,Y)$ given by $\varepsilon_c = \epsilon^c_{c,1_c}$, for every $c \in \mathcal{C}$.
∎

If $\mathfrak{Cat}$ stands for the category of all small categories and $H : \mathcal{C} \to \mathfrak{Cat}$ is a functor, then the Grotendieck construction $Gr_{\mathcal{C}}H$ is the category whose set of objects are the pairs (c,x), where $c \in \mathcal{C}$ and $x \in H(c)$, and a morphism $(\alpha,h) : (a,x) \to (b,y)$ is a pair (α,h) consisting of a morphism $\alpha : a \to b$ in $\mathcal{C}$ and a morphism $h : H(\alpha)(x) \to y$ in $H(b)$. The composite of $(\alpha,h) : (a,x) \to (b,y)$ and $(\beta,g) : (b,y) \to (c,z)$ is defined to be $(\beta\alpha, gH(\beta)(h))$.

For a given $c \in \mathcal{C}$ consider the *under category* $c \downarrow \mathcal{C}$ whose set of objects consists of arrows $f : c \to d$ in $\mathcal{C}$, and whose set of morphisms between $f : c \to d$ and $g : c \to e$ is the set of morphisms $h \in \mathcal{C}$ such that $hf = g$. Let us define the functor $H : \mathcal{C}^{op} \to \mathfrak{Cat}$ by $H(c) = c \downarrow \mathcal{C}$, for which $H(f^{op}) = (f^\sharp)$ for $f^{op} : c \to d \in \mathcal{C}^{op}$. It is easy to see that the Grotendieck construction $Gr_{\mathcal{C}^{op}}H$ for H is the *twisted arrow category* $a\mathcal{C}$ of $\mathcal{C}$, that is, the category which has as objects the arrows $f : a \to b$ in $\mathcal{C}$, and as maps between $f : a \to b$ and $g : c \to d$ the pairs of arrows (α,β) of $\mathcal{C} \times \mathcal{C}$ that makes the following square commutative

$$\begin{tikzcd}
a \arrow[r, "f"] & b \arrow[d, "\beta"] \\
c \arrow[u, "\alpha"] \arrow[r, "g"'] & d
\end{tikzcd}$$

If X and Y are $\mathcal{C}$-diagrams let us consider the functor $hom_a(X,Y) : a\mathcal{C} \to \mathbf{S}$ defined over objects by $hom_a(X,Y)(a \to b) = hom(X_a, Y_b)$ and over arrows by $hom_a(X,Y)(\alpha,\beta)(h) = Y_\beta h(1 \times X_\alpha)$, for every $h \in hom(X_a, Y_b)$ (see the above square). It is not difficult to see that $hom(X,Y)$ is isomorphic to $\underleftarrow{lim} hom_a(X,Y)$.

Proposition 1.5.2 ([7])
Let X and Y be $\mathcal{C}$-diagrams. If X is cofibrant and Y is fibrant, then the natural map

$$hom(X,Y) \to \underset{a\mathcal{C}}{holim} hom_a(X,Y)$$

is a weak equivalence.

For a given $\mathcal{C}$-diagram F, in what follows $haut(F)$ will denote the components of $\underset{a\mathcal{C}}{holim} hom_a(F,F)$ that are weakly equivalents to $aut(F)$.

Example 1.5.3 *If $\mathcal{C}$ consists only of the arrow $f : a \to b$, then $haut(F)$ is given by the homotopy pullback of the following diagram*

$$\begin{tikzcd} haut(F) \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner"] & F_b \arrow[d, "F_f^\sharp"] \\ aut(F_a) \arrow[r, "F_{f,\sharp}"'] & hom(F_a, F_b) \end{tikzcd}$$

and $aut(F)$ corresponds to the strict pullback, although if the diagram F is fibrant and cofibrant, then $haut(F) \simeq aut(F)$. Given a group G and denoting by $\mathfrak{B}G = \{o_G\}$ the category determined by G, it holds that $haut(F) = aut(F_{o_G})^{hG}$ and $aut(F) = aut(F_{o_G})^G$.

CHAPTER 2

MINIMAL MODELS IN $\mathbf{S}^{\mathcal{C}}$

In the category of simplicial sets the problem of classification of spaces up to homotopy equivalence admits an attractive formulation in terms of the theory of minimal Kan complexes (or more generally minimal Kan fibrations). Every Kan complex X has a strong deformation retract X', which is a minimal Kan complex and a map $X \to Y$ between minimal Kan complexes is a homotopy equivalence if and only if it is an isomorphism. So the classification of Kan complexes up to homotopy equivalence is equivalent to the classification of minimal Kan complexes up to isomorphism.

In this chapter our aim is to generalize the theory of minimal complexes in the setting of $\mathcal{C}$-diagrams of spaces, on one hand by defining a convenient homotopy relation over the set of n-$\mathcal{C}$-simplices of a $\mathcal{C}$-diagram X, and on the other hand by using the theory of free cell complexes exposed in the previous chapter.

We start the section 1 by defining the homotopy relation in $\mathbf{S}^{\mathcal{C}}$ and the fibrewise homotopy relation, which is the one used in this memory to classify fibrations. In the section 2 we define the sub-homotopy relation between n-$\mathcal{C}$-simplices, which is used to identify pieces of zig-zag's of n-simplices, by using the well known homotopy relation between them. The usefulness of this relation becomes clear by working with free cell complexes, since the generators permit us to identify the orbits of the diagram that are in excess (in homotopy).

Taking $\mathcal{C}$ as a small EI-category with a finite number of objects we can generalize the classical results concerning to minimal simplicial sets and minimal fibrations to the category of $\mathcal{C}$-diagrams of spaces. It is important to remark that it can also be done in some categories with non necessarily a finite number of objects, and it depends on the existence of certain maximal orbits (see Appendix A). We finish the chapter with the classification Theorem of fibrations in $\mathbf{S}^{\mathcal{C}}$ up to fibrewise homotopy, by using the results of Chapter 1 and the first two sections of the present chapter too.

2.1 Homotopy in $\mathbf{S}^{\mathcal{C}}$

Let $\triangle[1]$ be the standard 1-simplex in $\mathbf{S}$. If X belongs to $\mathbf{S}^{\mathcal{C}}$ the $\mathcal{C}$-diagram $X \times \triangle[1]$ is defined by $X \times \triangle[1](a) = X_a \times \triangle[1]$ for $a \in \mathcal{C}$, and over arrows by $X \times \triangle[1](f) = X_f \times 1$, for any morphism $f : a \to b \in \mathcal{C}$. Thus if, X and Y are diagrams in $\mathbf{S}^{\mathcal{C}}$ a *homotopy* from X to Y will be a natural transformation $H : X \times \triangle[1] \to Y \in \mathbf{S}^{\mathcal{C}}$.

Definition 2.1.1 *Let* 0 *stand for the vertex* $(0) \in \triangle[1]_0$ *and for any of its degeneracies* $s_0^n(0)$ *and* 1 *for the vertex* (1) *and its degeneracies* $s_0^n(1)$*. Two maps* $f, g : X \to Y \in \mathbf{S}^{\mathcal{C}}$ *are said to be homotopic (*$f \simeq g$*) if there exists a map* $H : X \times \triangle[1] \to Y \in \mathbf{S}^{\mathcal{C}}$*, such that for every* $c \in \mathcal{C}$ *and every simplex* $x \in X_{c,n}$*,* $H_c(x, 0) = f_c(x)$ *and* $H_c(x, 1) = g_c(x)$*.*

Proposition 2.1.2 ([22], *II.2 Proposition* 2.5**)**
If X *is cofibrant and* Y *is fibrant, then the homotopy relation between maps from* X *to* Y *is an equivalence relation.*

Definition 2.1.3 *Let* D *be a subdiagram of* X*,* $f, g : X \to Y$ *and* $p : Y \to Z$ *arrows in* $\mathbf{S}^{\mathcal{C}}$*, and* H *a homotopy from* f *to* g*. We will say that*

1) *f is homotopic to g relative to D , $f \simeq g$ $(relD)$, if $f|_D = g|_D$ and there is a homotopy H from f to g such that the following diagram commutes*

$$\begin{tikzcd} D \times \triangle[1] \arrow[r, "pr"] \arrow[d, hook] & D \arrow[d, "f|D"] \\ X \times \triangle[1] \arrow[r, "H"'] & X \end{tikzcd}$$

2) *f is fibrewise homotopic to g (or f is homotopic to g over Z), $f \simeq_Z g$, if $pf = pg$ and there is a homotopy H from f to g such that the diagram*

$$\begin{tikzcd} X \times \triangle[1] \arrow[r, "H"] \arrow[d, "pr"] & Y \arrow[d, "p"] \\ X \arrow[r, "pf"'] & Z \end{tikzcd}$$

commutes

3) *f is homotopic to g over Z relative to D, $f \simeq_Z g$, $rel(D)$, if $f|_D = g|_D$, $pf = pg$ and there is a homotopy H from f to g such that the following diagram commutes*

$$\begin{tikzcd} D \times \triangle[1] \arrow[r, "pr"] \arrow[d, hook] & D \arrow[d, "f|_D"] \\ X \times \triangle[1] \arrow[r, "H"] \arrow[d, "pr"'] & Y \arrow[d, "p"] \\ X \arrow[r, "pf"'] & Z \end{tikzcd}$$

The following proposition is a consequence of Proposition 6.2 of [18]

Proposition 2.1.4
Let $p : Z \to Y$ be a fibration between $\mathcal{C}$-diagrams. If the inclusion $D \hookrightarrow X$ is a cofibration, then homotopy $rel(D)$ over Z is an equivalence relation on $\mathbf{S}^{\mathcal{C}}(X, Z)$.

The following definition corresponds to the Definition 13.4.1 of [16]

Definition 2.1.5 *Let X be an object in $\mathbf{S}^{\mathcal{C}}$, $g : Z \to X$ a map between $\mathcal{C}$-diagrams and $*$ the terminal object in $\mathbf{S}^{\mathcal{C}}$. By a point in X we will mean a map $x : * \to X$ and by the fiber of g over x the pullback of the diagram $* \xrightarrow{x} X \longleftarrow Z$*

A map $f : X \to Y \in \mathbf{S}^{\mathcal{C}}$ is called a *homotopy equivalence* if there is a map $g : Y \to X$ such that gf is homotopic to 1_X and fg is homotopic to 1_Y. If $p : Y \to Z$ is a map $\mathbf{S}^{\mathcal{C}}$, f is said to be a *fibrewise homotopy equivalence* if there is a map $g : Y \to X$ such that $gf \simeq_Z 1_X$ and $fg \simeq_Z 1_Y$; in particular we have the notion of *homotopy retract* and *strong homotopy retract*. A diagram is called *contractible* to the point $x : * \to X$ if there is either a homotopy from 1_X to the constant map $X \to * \to X$, or a homotopy from $X \to * \to X$ to 1_X.

In $\mathbf{S}^{\mathcal{C}}$ we can define a *cocylinder* $hom(\triangle[1],-)$ on $\mathbf{S}^{\mathcal{C}}$ (see Definition 3.1 of [18]) in the following way: If X is a $\mathcal{C}$-diagram, then the $\mathcal{C}$-diagram $hom(\triangle[1],X)$ is defined by $hom(\triangle[1],X)(a) := \triangle[1]\times X_a$ for every $a\in\mathcal{C}$, and if $f: a\to b\in\mathcal{C}$, $hom(\triangle[1],X)(f) := X_{f,\sharp}$. Thus, as a consequence of the Theorem $I.6.3$ of [18] the following result holds in $\mathbf{S}^{\mathcal{C}}$.

Proposition 2.1.6 (Dold's Theorem)
Let $p: X\to B$ and $p': X'\to B$ be fibrations in $\mathbf{S}^{\mathcal{C}}$ and $g: X\to X'$ a homotopy equivalence such that $p'g = g$, then g is a homotopy equivalence over B.

The following result is an immediate consequence of Proposition 13.4.7 of [16] in the category $\mathbf{S}^{\mathcal{C}}$.

Proposition 2.1.7
Let $g: Z\to X$ be a fibration in $\mathbf{S}^{\mathcal{C}}$. If $x: *\to X$ and $z: *\to X$ are homotopic points in X, then the fiber of g over x is weakly equivalent to the fiber of g over z.

The proof of the following Lemma is analogous to Lemma 10.6 of [14] by considering the structure of closed simplicial model category over $\mathbf{S}^{\mathcal{C}}$. The lifting properties of the fibrations involved in this proof can be obtained for $\mathbf{S}^{\mathcal{C}}$ by using the Axioms $(M7)$ and $(M6)$ of Definition 9.1.6 in [16].

Lemma 2.1.8
Let $p: X\to B$ and $g: A\to B$ be maps in $\mathbf{S}^{\mathcal{C}}$. If $f_0, f_1: A\to B$ are homotopic maps and p is a fibration, then p_{f_0} and p_{f_1} are fibrewise homotopy equivalents.

The following result which is a particular case of Lemma 0.2.18 shows how the combinatorial structure of $\mathbf{S}$ is reflected over $\mathbf{S}^{\mathcal{C}}$. It has to do with the theory of anodyne extensions of Gabriel-Zisman [12], which encodes the difficult calculations based on the standard subdivision of a prism in $\mathfrak{Top}$, by means of combinatorial manipulations in simplicial sets. This theory suppresses or engulfs most of the old combinatorial arguments. We will prove the Lemma 2.1.9 by using the theory of anodyne extensions and Theorems 0.2.14, 1.2.3.

Lemma 2.1.9
If $p: X\to B$ is a fibration in $\mathbf{S}^{\mathcal{C}}$, then in every commutative square as the following, the dotted arrow exists

$$\begin{tikzcd}
\dot{\delta}^c_n\times\triangle[m]\cup\delta^c_n\times\Lambda^k[m] \arrow[r] \arrow[d, hook] & X \arrow[d] \\
\delta^c_n\times\triangle[m] \arrow[r] \arrow[ur, dotted] & B
\end{tikzcd}$$

Proof. In the category of simplicial sets $\Lambda^k[n] \hookrightarrow \triangle[n]$ is an anodyne extension and $\dot{\triangle}[n]$ is a subsimplicial set of $\triangle[n]$, so the inclusion

$$[(\dot{\triangle}[n] \times \triangle[m]) \cup (\triangle[n] \times \Lambda^k[m])] \hookrightarrow [\triangle[n] \times \triangle[m]]$$

is an anodyne extension [11, Proposition 4.5.8] and hence a trivial cofibration.

Applying the functor $\coprod_{c\in\mathcal{C}}(-_c \otimes [c,\_\,])$ given in the sketch of Theorem 1.2.3 to this trivial cofibration we obtain the following map

$$(4) \qquad \coprod_{[c,-]} [(\dot{\triangle}[n] \times \triangle[m]) \cup (\triangle[n] \times \Lambda^k[m])] \hookrightarrow \coprod_{[c,-]} [\triangle[n] \times \triangle[m]]$$

Which is a trivial cofibration, since $\coprod_{c\in\mathcal{C}}(-_c \otimes [c,\_\,])$ is a left Quillen functor (see Definition 0.2.12 and Theorem 0.2.14).

Now note that the followings equalities hold:

$$\coprod_{[c,-]}[(\dot{\triangle}[n] \times \triangle[m]) \cup (\triangle[n] \times \Lambda^k[m])] = \coprod_{[c,-]}(\dot{\triangle}[n] \times \triangle[m]) \cup \coprod_{[c,-]}(\triangle[n] \times \Lambda^k[m])$$
$$= (\coprod_{[c,-]}\dot{\triangle}[n]) \times \triangle[m] \cup (\coprod_{[c,-]}\triangle[n]) \times \Lambda^k[m] = (\dot{\delta}^c_n \times \triangle[m]) \cup (\delta^c_n \times \Lambda^k[n])$$

and $\coprod_{[c,-]}\triangle[n] \times \triangle[m] = \delta^c_n \times \triangle[m]$. Therefore (4) is the inclusion morphism $\dot{\delta}^c_n \times \triangle[m] \cup \delta^c_n \times \Lambda^k[n] \hookrightarrow \delta^c_n \times \triangle[m]$. Hence the doted arrow of the above square exists, since $X \to Y$ is a fibration (Definition 0.2.8(2)).

∎

In Proposition 1.1.5 we showed that $\mathbf{S}^{\mathcal{C}}(\delta^c_n, X) \cong \mathbf{S}(\triangle[n], X_c)$ for every diagram X and every $c \in \mathcal{C}$, and since $\mathbf{S}(\triangle[n], X_c) \cong X_{c,n}$, each n-$\mathcal{C}$-simplex $\delta^c_n \to X \in \mathbf{S}^{\mathcal{C}}(\delta^c_n, X)$ will be denoted by its correspondient simplex $x^c \in X_{c,n}$ by adding to x the upper index c. Then by abuse of notation we won't make any distinction between x as n-simplex or x^c as n-$\mathcal{C}$-simplex.

Note that $X \in \mathbf{S}^{\mathcal{C}}$ can be seen as a simplicial set by defining the set of n-simplices as $X_n = \{x^c \in X_{c,n} : c \in Obj(\mathcal{C})\}$ and faces and degeneracies for a given $x^c \in X_{c,n}$ by $d_i x^c = x^c \circ \varepsilon^i$ and $s_i x^c = x^c \circ \sigma^i$, where ε^i and σ^i belongs to Γ (Definition 1.1.6). Therefore combinatorial definitions about homotopy of simplices in the category $\mathbf{S}$ have an immediate translation to $\mathbf{S}^{\mathcal{C}}$ when we deal with the homotopy relation between n-$\mathcal{C}$-simplices. As in simplicial sets $X^\flat$, will denote the set of degenerate n-$\mathcal{C}$-simplices of X.

2.2 Minimal $\mathcal{C}$-Diagrams

Definition 2.2.1 *Let X be a $\mathcal{C}$-diagram and $x^d, w^c \in X$ two n-$\mathcal{C}$-simplices.*

- *the simplex x^d is said to be sub-homotopic to w^c, $x^d \rightsquigarrow w^c$, if there exists an arrow $f : c \to d \in \mathcal{C}$ such that $X_f(w^c) \simeq x^d$ $rel(\dot{\delta}^d_n)$. If $c = d$ and f is the identity map, then x and w are said to be homotopic, $x^c \simeq w^c$.*

- *given a map $p : X \to B$, x^d is said to be* sub-p-homotopic *to w^c ,$x^d \rightsquigarrow_p w^c$, if $X_f(w^c) \simeq_B x^d$ $rel(\dot{\delta}^d_n)$, for some arrow $f : c \to d \in \mathcal{C}$. If $c = d$ and f is the identity map, then x and w are said to be p-homotopic, $x^c \simeq_p w^c$.*

Given a fibration $p : X \to B$ in $\mathbf{S}^{\mathcal{C}}$ we can suppose that X is a free cell complex. That is, we can factorize the map $\emptyset \to X$ as $\emptyset \hookrightarrow X' \twoheadrightarrow X$ by applying the small object argument (see Proposition B.0.28), where $\emptyset \to X$ is a I-cell complex and $X' \twoheadrightarrow X$ is an I-injective. But in Section 1.2 we conclude that the class of I-cell complexes equals the class free cell complexes and the class of I-injectives equals the class of trivial fibrations (see Proposition 1.2.3).

Remark 2.2.2 *Given a free cell complex $X \in \mathbf{S}^{\mathcal{C}}$, a basis Σ for X may not be unique (see Proposition 1.2.4), then when we use the notation Σ, we are under the assumption of having fixed one of them. If X and Y are free cell complex and we have to specify bases for both of them, we will denote the chosen ones by $\Sigma(X)$ and $\Sigma(Y)$, respectively. Although a basis is not unique, according with the item c) of Proposition 1.2.4 the following definitions do not depend of the chosen basis.*

Definition 2.2.3 *Let X be a $\mathcal{C}$-diagram. Two n-simplices $w \in X_{c,n}$ and $x \in X_{d,n}$ are said to be joined , $w \to x$, if there exists a map $f : c \to d$ such that $X_f(w) = x$. A zig-zag from w to x is a finite sequence $\{z_k\}_{k=1}^{k=n}$ of simplices of X, such that $w \to z_1 \leftarrow z_2 \leftarrow z_3 \to ... \to z_n \leftarrow x$.*

If X is a free cell complex we can see that every zig-zag of simplices has at most one element of Σ. Thus, if x^d and w^c are non equal n-$\mathcal{C}$-simplices in Σ such that $x^d \rightsquigarrow w^c$ it holds that the orbit generated by x can be identified in homotopy with one part of the orbit generated by w.

Definition 2.2.4 *Let X be a free cell complex. A fibration $X \to B$ in $\mathbf{S}^{\mathcal{C}}$ is said to be minimal if every pair of non-equal generators of X are not sub-p-homotopic.*

In what follows we will consider the category $\mathcal{C}$ as in Appendix A, that is, a small EI-category with a finite number of objects (maybe with an infinite number of objects, see 3 in Example A.0.17 of Appendix A).

Proposition 2.2.5
Let X be a free cell complex and $p : X \to B$ a fibration. Then p is minimal if and only if no proper fibration $q : D \to B$ of p (that is, $D \subsetneq X$ and $q = p|_D$) is a strong deformation retract of p over B.

Proof. Take p minimal. Let us suppose that there exists a fibration $q : D \to B$ such that D is a subdiagram strictly contained in X, $q = p|_D$ and q is a strong deformation retract of p over B. Let $i : D \hookrightarrow X$ the inclusion map and $h : X \times \triangle[1] \to X$ the homotopy from 1_X to ir, where $r : X \to D$ is the respective retraction.

- Since $D \subsetneq X$ it is possible to choose one $\mathcal{C}$-simplex u^e of mínimum dimension ,say n, such that $u \in X_{e,n} - D_{e,n}$. Since X is a free cell complex there exists a generator $\mathbf{z}^d$ of X and a map $f : d \to e$ such that $X_f(\mathbf{z}) = u$ (see Proposition 1.2.4). If $z \in D_{d,n}$ it must hold that $u \in D_{e,n}$ since D is a subdiagram of X, therefore $\mathbf{z} \notin D_{d,n}$. Then all faces $d_k\mathbf{z}$ of $\mathbf{z}$ belong to $D_{d,n-1}$, for $0 \le k \le n$, hence we can consider the following diagram

$$\begin{tikzcd}
\dot{\delta}^d_n \times \triangle[1] \arrow[r, "\dot{\mathbf{z}}\times 1"] \arrow[d, hook] & D \times \triangle[1] \arrow[r] \arrow[d, hook] & D \arrow[d, hook, "i"] \arrow[dd, bend left, "q"] \\
\delta^d_n \times \triangle[1] \arrow[r, "\mathbf{z}\times 1"] \arrow[d] & X \times \triangle[1] \arrow[r, "h"] \arrow[d] & X \arrow[d, "p"] \\
\delta^d_n \arrow[r, "\mathbf{z}"'] & X \arrow[r, "p"'] & B
\end{tikzcd}$$

  That is, $\mathbf{z} \simeq_p r_d(\mathbf{z})$. If $r_d(\mathbf{z})$ is generator of X it must hold that $\mathbf{z} = r_d(\mathbf{z})$, since p is minimal, and therefore $u \in D_{e,n}$ which is not possible. Hence there exists a generator $\mathbf{w}$ of X and a map $g : c \to d$ such that $X_g(\mathbf{w}) = r_d(\mathbf{z})$, then $\mathbf{z}^d \rightsquigarrow_p \mathbf{w}^c$. Since p is minimal it must hold that $c = d$ and $\mathbf{z} = \mathbf{w}$, therefore $u \in D_{e,n}$ which is a contradiction since $u \notin D_{d,n}$.

- The reciprocal way of the theorem is consequence of Proposition 2.2.6.

$\blacksquare$

Since $\Lambda^k[n]$ is the subcomplex of $\triangle[n]$ generated by all the faces $d_i\iota_n$ of $\triangle[n]$, except by the face $d_k\iota_n$, we will denote a given map $f : \Lambda^k[n] \to X$ by $(x_0, , ..., x_{k-1}, -, x_{k+1}, ..., x_n)$, where $x_i = f(d_i\iota_n)$. The complex $\triangle[1]$ can be seen as a subcomplex of $\triangle[2]$ by defining the three possible inclusion maps $\triangle[1] \hookrightarrow \triangle[2]$ by $\iota_1 \mapsto d_i\iota_2$, where i is equal to $0, 1$ or 2. Thus, given two maps $f : Z \times \triangle[1] \to X$ and $g : Z \times \triangle[1] \to X$, the map $(f, g, -) : Z \times \Lambda^2[2] \to X$ will denote the one defined by $(f, g, -)(z, \varphi d_0\iota_2) = f_c(z, \varphi\iota_1)$ and $(f, g, -)(z, \varphi d_1\iota_2) = g_c(z, \varphi\iota_1)$, for every $c \in \mathcal{C}$ and every $z \in Z_c$ (similar for $(f, -, g)$ and $(-, f, g)$, whose domains are $Z \times \Lambda^1[2]$ and $Z \times \Lambda^0[2]$, respectively).

Proposition 2.2.6
Let $p : X \to B$ a fibration for which X is a free cell complex. Then p has a strong fibrewise deformation retract $q : \hat{X} \to B$ which is a minimal fibration.

Proof. Consider a basis Σ of X. Let Σ' the set formed by taking from Σ a *minimal number* of elements such that:

For every $\mathbf{x}^d \in \Sigma$ there exists some $\mathbf{w}^c \in \Sigma'$ such that $\mathbf{x}^d \rightsquigarrow_p \mathbf{w}^c$

and such that we choose the degenerate ones whenever it is possible. The set Σ' exists, since $\mathcal{C}$ is a small EI-category with a finite number of objects (see Appendix A, Example A.0.17).

Notice that if $\mathbf{x}^d$ and $\mathbf{w}^c$ are degenerate simplices in Σ, such that $\mathbf{x}^d \rightsquigarrow_p \mathbf{w}^c$, then they must be equal. Since $\mathbf{x}^d \rightsquigarrow_p \mathbf{w}^c$, there exists a map $f : c \to d$ in $\mathcal{C}$ such that $X_f(\mathbf{w}^c) \simeq_p \mathbf{x}^d$, and hence $X_f(\mathbf{w}^c) = \mathbf{x}^d$ (see Proposition 0.1.10). But $\mathbf{w}^c$ and $\mathbf{x}^d$ are generators of X, then $\mathbf{x}^d = \mathbf{w}^c$. Thus, in the above choice we can always take the degenerate one.

Claim: Σ' is closed under degeneracy operators. Note that given $\mathbf{x}^d \in \Sigma'_n$ its degeneracies $s_k\mathbf{x}$ belongs to Σ_n, for $0 \leq k \leq n$, since Σ is closed under degeneracy operators (Proposition 1.2.4). Then there exists a simplex $\mathbf{w}^c \in \Sigma'_{n+1}$ such that $s_k\mathbf{x}^d \rightsquigarrow_p \mathbf{w}^c$. If $\mathbf{w}^c$ is degenerate, then $s_k\mathbf{x}^d = \mathbf{w}^c$, otherwise we must consider the following cases:

- $c = d$. It is easy to see that $\mathbf{w}^c \rightsquigarrow_p s_k\mathbf{x}^d$, since every endomorphism in $\mathcal{C}$ has a left inverse. Therefore $s_k\mathbf{x}^d \in \Sigma'_{n+1}$, since in our choice we prefer the degenerate ones.

- $c \neq d$. Since $s_k\mathbf{x}^d \rightsquigarrow_p \mathbf{w}^c$, there exists a map $f : c \to d$ such that $X_f(\mathbf{w}^c) \simeq_p s_k\mathbf{x}^d$ and hence $d_k X_f(\mathbf{w}^c) = \mathbf{x}^d$, so $X_f(d_k\mathbf{w}^c) = \mathbf{x}^d$. Then by item *c*) of Proposition 1.2.4 there exists a map $g : d \to c$ such that $X_g(\mathbf{x}^d) = d_k\mathbf{w}^c$.

  In the category $\mathcal{C}$ every endomorphism has a left inverse, then there exists $\alpha : c \to c$ such that $\alpha g f = 1_c$. We have that $X_f(\mathbf{w}^c) \simeq_p s_k\mathbf{x}^d$, then $X_{gf}(\mathbf{w}^c) \simeq_p X_g(s_k\mathbf{x}^d)$, and hence $\mathbf{w}^c \simeq_p X_{\alpha g}(s_k\mathbf{x}^d)$. That is, it holds that $\mathbf{w}^c \rightsquigarrow_p s_k\mathbf{x}^d$, and therefore $s_k\mathbf{x}^d \in \Sigma'_{n+1}$.

Then if $\mathbf{x}^d \in \Sigma'$, it holds that $s_k\mathbf{x}^d \in \Sigma'_{n+1}$ as we wanted to show.

Now we define $\mathfrak{A}$ as follows:

$$\mathfrak{A} = \{\widetilde{X} \in \mathbf{S}^{\mathcal{C}} : \widetilde{X} \leq X, \text{which is free cell complex, such that } \ \Sigma(\widetilde{X}) \subseteq \Sigma'\}$$

Let us see that $\mathfrak{A}$ admits maximal elements:

Since Σ' is closed under degeneracy operators, $\langle \Sigma'_0 \rangle$ the smallest subdiagram of X containing Σ'_0 is a free cell complex whose basis is contained in Σ', then $\langle \Sigma'_0 \rangle \in \mathfrak{A}$ and therefore $\mathfrak{A} \neq \emptyset$. If we take a chain of $\mathfrak{A}$ its union is an upper bound of the chain that belongs to $\mathfrak{A}$, then by Zorn's Lemma we can find a free subdiagram $\widehat{X}$ of X which is maximal with respect to the property of having all its generators in Σ'.

Fix a maximal element $\widehat{X}$ of $\mathfrak{A}$. Then any simplex of X, whose faces are in $\widehat{X}$ and whose generator belongs to Σ', belongs to $\widehat{X}$:

Take some simplex $x^d \in X_{d,n}$ such that $d_k x^d \in \widehat{X}$ for $0 \leq k \leq n$, and for which there exists $\mathbf{w}^c \in \Sigma'$ and a map $f : c \to d \in \mathcal{C}$ such that $X_f(\mathbf{w}^c) = x^d$. Since $d_k x^d \in \widehat{X}$, there exists $\mathbf{z}^b \in \Sigma(\widehat{X})$ (and therefore in $\widehat{X}$) and a map $g : b \to d \in \mathcal{C}$ such that $X_g(\mathbf{z}^b) = d_k x^d$, which means that $d_k\mathbf{w}^c$ and $\mathbf{z}^b$ are joined by a zig-zag ($d_k\mathbf{w}^c \xrightarrow{X_f} d_k x^d \xleftarrow{X_g} \mathbf{z}^b$), since $X_f(d_k\mathbf{w}^c) = d_k x^d$. Then by item *c*) of Proposition 1.2.4 there exists a map $h : b \to c \in \mathcal{C}$, such that $X_h(\mathbf{z}^b) = d_k\mathbf{w}^c$ and since $\mathbf{z}^b \in \widehat{X}$ it holds that $d_k\mathbf{w}^c \in \widehat{X}$. Then the subdiagram $\widehat{X} \cup \langle \mathbf{w}^c \rangle$ of X generated by $\widehat{X}$ and $\mathbf{w}^c$ belongs to $\mathfrak{A}$, but $\widehat{X}$ is maximal with respect to the property of having all its generators in Σ', therefore $\widehat{X} \cup \langle \mathbf{w}^c \rangle = \widehat{X}$. Thus, $x \in \widehat{X}$, since $X_f(\mathbf{w}^c) = x^d$.

Let $\mathfrak{B}$ be the set that consists of all the pairs (Y, H), where Y is a subdiagram of X containing $\widehat{X}$ and $H : Y \times \triangle[1] \to X$ is a homotopy such that H maps $Y \times \{0\}$ into $\widehat{X}$, is the inclusion on $Y \times \{1\}$, H is constant on $\widehat{X} \times \triangle[1]$ and pH is the constant homotopy restricted to Y.

The set $\mathfrak{B}$ has maximal elements:

We can define a partial order '$\preceq$' over $\mathfrak{B}$ as follows: $(Y, H) \preceq (Y', H')$ if $Y \subset Y'$ and $H'|_Y = H$. The diagram $\widehat{X}$ with the constant homotopy belongs to $\mathfrak{B}$, then $\mathfrak{B} \neq \emptyset$. Moreover, if we take a chain of $\mathfrak{B}$ its colimit will be an upper bound of the chain in $\mathfrak{B}$, then applying the Zorn's lemma we get a maximal element $(\widehat{Y}, H)$ in $\mathfrak{B}$.

Take a maximal diagram $\widehat{Y}$ of $\mathfrak{B}$, and let us see that $\widehat{Y} = X$:

Assume that $\widehat{Y} \neq X$ and choose a $\mathcal{C}$-simplex $z_1^d \in X$ of lowest dimension such that $z_1^d \in X - \widehat{Y}$, say $z_1^d : \delta_n^d :\to X$. It is non degenerate (otherwise it would belong to $\widehat{Y}$) and its faces belong to $\widehat{Y}$. So we want to extend H to $\widehat{Y} \cup \langle z_1^d \rangle$ the subdiagram of X generated by $\widehat{Y} \cup \{z_1^d\}$. First consider the following commutative diagram

$$\begin{tikzcd}
\dot{\delta}^d_n \times \triangle[1] \arrow[rr, "\dot{z}_1{}^d \times 1"] \arrow[dd, hook] \arrow[drr, hook] & & \widehat{Y} \times \triangle[1] \arrow[rr, "H"] & & X \arrow[dd, "p"] \\
& & \dot{\delta}^d_n \times \triangle[1] \cup \delta^d_n \times \Lambda^1[1] \arrow[dll, hook] \arrow[urr] & & \\
\delta^d_n \times \triangle[1] \arrow[rr, "pr"'] \arrow[uurrrr, dotted, "H'"] & & \delta^d_n \arrow[rr, "pz_1^d"'] & & B
\end{tikzcd}$$

where the homotopy H' is obtained from Lemma 2.1.9, and is such that $H'_1 = z_1^d$. For $H'_0 = z^d$ there exists $\mathbf{w}^c \in \Sigma$ and a map $f : c \to d$ such that $X_f(\mathbf{w}^c) = z$ (by Proposition 1.2.4), then there exists $\mathbf{m}^b \in \Sigma' \subset X'$ such that $\mathbf{w}^c \rightsquigarrow_p \mathbf{m}^b$, that is, $X_g(\mathbf{m}^b) \simeq_p \mathbf{w}^c$ for some arrow $g : b \to c \in \mathcal{C}$. From the above diagram we can see that z^d has its faces in $\widehat{X}$ and then $X_{fg}(\mathbf{m}^b) := x^d$ as well, since $z \simeq_p X_{fg}(\mathbf{m}^b)$. Therefore $x \in \widehat{X}$, since x has its faces in $\widehat{X}$ and its generator $\mathbf{m}^b$ belongs to Σ'.

If $r : \widehat{Y} \to \widehat{X}$ is the retraction from $\widehat{Y}$ to $\widehat{X}$ let us extend it to $r' : \widehat{Y} \cup \langle z_1^d \rangle \to \widehat{X}$ by defining $r'(z_1^d) = x^d$. Then using H' and r' we can define one homotopy

$\widehat{H} : \delta_n^d \times \triangle[1] \to X$ such that $\widehat{H}$ is z_1^d on $\delta_n^d \times \{1\}$, $p\widehat{H}$ is the constant homotopy to pz_1^d, $\widehat{H}$ extends H on $\dot{\delta}_n^d \times \triangle[1]$ and $\widehat{H}(\delta_n^d \times \{0\})$ belongs to $\widehat{X}$; that is, we have an extension of H to $\widehat{Y} \cup \langle z_1^d \rangle \times \triangle[1]$. Then the pair $(\widehat{Y} \cup \langle z_1^d \rangle, \widehat{H})$ is a contradiction with the maximality of $(\widehat{Y}, H)$ and therefore $\widehat{Y} = X$.

The restriction $q = p|_{\widehat{X}}$ is a minimal fibration:

Since $p|_{\widehat{X}}$ is a retraction of p, it is a fibration (retraction axiom for model categories, see Definition 0.2.1). By construction $\widehat{X}$ is a free cell complex and if $\mathbf{w}$ is a generators of $\widehat{X}$, then it is also a generator of X that belongs to Σ', so none of such simplices can be sub-p-homotopic, otherwise the minimality condition that satisfies Σ' would be contradicted. Therefore q is a minimal fibration.

∎

Lemma 2.2.7
Let $p : X \to B$ be a map of diagrams, where X is a free cell complex and $h : A \to B$ a map with domain the constant diagram A. Then the pullback diagram $A \times_B X$ is a free cell complex.

Proof. Let Σ be a base for X. Let us show that $\Sigma^{\sqcap} = \{(u,x) \in A \times_B X : x \in \Sigma\}$ is a base for $A \times_B X$ (see Proposition 1.2.4). First take a pair of n-simplices $(u,x) \in \Sigma^{\sqcap}$ and show that $\Sigma^{\sqcap}$ is closed under degeneracies. Since $(u,x) \in A \times_B X$ it holds that $h(u) = p(x)$, hence for every degeneracy operator s_k it holds that $h(s_k u) = p(s_k x)$, for $0 \leq k \leq n$. Σ is closed under degeneracies, that is, $s_k x \in \Sigma$, therefore $s_k(u,x) = (s_k u, s_k x) \in \Sigma^{\sqcap}$.

If $(v,z)^d \in \Sigma^{\sqcap}$ there exists a unique map $\alpha : c \to d \in \mathcal{C}$ and a unique $x^c \in \Sigma$ such that $X_\alpha(x) = z$, since Σ is a base for X. For the pair $(v,x)^c$ let us calculate $p_c(x)$: $p_c(x) = p_d X_\alpha(x) = p_d(z) = h(v)$, then $(v,x)^c \in \Sigma^{\sqcap}$ and $A \times_B X(\alpha)(v,x) = X_\alpha^*(v,x) = (v, X_\alpha(x)) = (v,z)$. If there exists an arrow $\beta : b \to d \in \mathcal{C}$ and an n-simplex $(w,y) \in \Sigma^{\sqcap}$ such that $X_\beta^*(w,y) = (v,z)$ it holds that $(w, X_\beta(y)) = (v,z) = (v, X_\alpha(x))$, then $w = v$ and since Σ is a fundamental domain it must hold that $\beta = \alpha$ and $y = z$.

∎

Lemma 2.2.8
Let $p : X \to B$ a minimal fibration and $h : A \to B$ a map in $\mathbf{S}^{\mathcal{C}}$, where A and B are constant diagrams. Then the fibration $pr : A \times_B X \to A$ is minimal.

Proof. If Σ is a base for X, then from Lemma 2.2.7 it holds that $A \times_B X$ is a free cell complex with base $\Sigma^{\sqcap} = \{(u, x) \in A \times_B X : x \in \Sigma\}$. Take two generators $(v, z)^d$ and $(w, x)^c$ of $A \times_B X$ and suppose that $(v, z)^d \rightsquigarrow_{pr} (w, x)^c$. Then there exists a map $\alpha : c \to d \in \mathcal{C}$ such that $X^*_\alpha(w, x)^c \simeq_{pr} (v, z)^d$, so $w = v$ since pr-homotopic simplices are in the same fiber. Applying the projection map $pr : A \times_B X \to X$ we have that $x^c \simeq_p z^d$, but from minimality of p it must hold that $c = d$ and $x = z$, that is, $(v, z) = (w, x)$.

∎

The following proof is an extension of sublemma 10.5 from [14].

Proposition 2.2.9
Let $p : X \to B$ and $q : Z \to B$ be fibrations in $\mathbf{S}^{\mathcal{C}}$ and $f, g : Z \to X$ fibre-wise homotopic maps, where g is an isomorphism and p minimal. Then f is an isomorphism.

Proof. We will show that f is an isomorphim by induction over the dimension of the $\mathcal{C}$-simplices. Suppose that f is an isomorphism on k-simplices for all $k < n$, and take $h : Z \times \triangle[1] \to X$ the homotopy from g to f.

1) f is surjective:

   Let Σ be base for X. Then to prove that f is surjective it is enough with showing that f is surjective over Σ. Given $\mathbf{x}^d$ a generator of X we have that for every face $d_i\mathbf{x}^d$ of $\mathbf{x}^d$ there exists a unique $(n-1)$-simplex z_i^d such that $f(z_i) = d_i\mathbf{x}^d$ (by the induction hypothesis), that is, the composite $\dot{\delta}^d_n \xrightarrow{(z_0,\ldots,z_n)} Z \xrightarrow{f} X$ is equal to $(d_0\mathbf{x}^d, \ldots, d_n\mathbf{x}^d)$. Then we obtain the following commutative diagram

$$\begin{tikzcd} \dot{\delta}^d_n \times \triangle[1] \cup \delta^d_n \times \Lambda^1[1] \arrow[rr, "(h|_{(z_0,\ldots,z_n)}{,}\mathbf{x}^d)"] \arrow[d, hook] & & X \arrow[d, "p"] \\ \delta^d_n \times \triangle[1] \arrow[r, "pr"'] & \delta^d_n \arrow[r, "p\mathbf{x}^d"'] & B \end{tikzcd}$$

   Since p is a fibration, there is a lift $G : \delta^d_n \times \triangle[1] \to X$ (see Lemma 2.1.9). Then there is an n-$\mathcal{C}$-simplex $w^d : \delta^d_n \times \triangle[0] \xrightarrow{1\times\varepsilon^1} \delta^d_n \times \triangle[1] \xrightarrow{G} X$ homotopic to $\mathbf{x}^d$ and such that $g(z_i) = d_i w$. By surjectivity of g there exists a $\mathcal{C}$-simplex $z^d \in Z$, such that $g(z) = w$. Then $g(z_i) = g(d_i z)$, which implies that $z_i = d_i z$ since g is injective. Therefore there is a commutative diagram

$$\begin{tikzcd} \dot{\delta}^d_n \times \triangle[2] \cup \delta^d_n \times \Lambda^0[2] \arrow[rr, "(s_1 h|_{(z_0,\dots,z_n)},(-,G,h_z))"] \arrow[d, hook] & & X \arrow[d, "p"] \\ \delta^d_n \times \triangle[2] \arrow[r, "pr"'] & \delta^d_n \arrow[r, "p\mathbf{x}^d"'] & B \end{tikzcd}$$

where $h_z = h(z^d \times 1)$; by Lemma 2.1.9 there is a lift F from $\delta^d_n \times \triangle[2]$ to X. Notice that the composite $\delta^d_n \times \triangle[1] \xrightarrow{1\times\varepsilon^0} \delta^d_n \times \triangle[2] \xrightarrow{F} X$ is a sub-p-homotopy from $\mathbf{x}^d$ to $f(z)^d$.

If $f(z)^d \in \Sigma$ it must hold that $f(z)^d = \mathbf{x}^d$, since p is minimal. But if $f(z)^d \notin \Sigma$ by Proposition 1.2.4 there exists $\mathbf{m}^c \in \Sigma$ and a map $\alpha : c \to d$ such that $X_\alpha(\mathbf{m}^c) = f(z^d)$; therefore $\mathbf{x}^d \rightsquigarrow_p \mathbf{m}^c$, but p is a minimal fibration and any two different generators can not be sub-p-homotopic, therefore $c = d$ and $\mathbf{x} = \mathbf{m}$. Since $\mathcal{C}$ is an EI-category there exists $\alpha^{-1} : d \to d$ such that $\alpha^{-1}\alpha = 1_d$ and therefore $\mathbf{x}^d = X_{\alpha^{-1}} f(z^d) = f[Z_{\alpha^{-1}}(z^d)]$.

2) f is injective:

i) First let us show that if z^c and w^c are two n-$\mathcal{C}$-simplices from Z, such that $fz^c = fw^c$, then we can construct a p-homotopy from $g(z)$ to $g(w)$.

If $fz^c = fw^c$ we have that $f(d_i z^c) = f(d_i w^c)$ and inductively it holds that $d_i z^c = d_i w^c$, for $0 \le i \le n$. Then the composites

$$\dot{\delta}^c_n \times \triangle[1] \xhookrightarrow{i\times 1} \delta^c_n \times \triangle[1] \underset{w^c\times 1}{\overset{z^c\times 1}{\rightrightarrows}} Z \times \triangle[1] \xrightarrow{h} X$$

are both equal to a map l. Taking $h_z = h(z^c \times 1)$ and $h_w = h(w^c \times 1)$ we can construct the following commutative diagram

$$\begin{tikzcd} \delta^c_n \times \Lambda^2[2] \cup \dot{\delta}^c_n \times \triangle[2] \arrow[rr, "((h_z,h_w,-),s_0 l)"] \arrow[d, hook] & & X \arrow[d, "p"] \\ \delta^c_n \times \triangle[2] \arrow[r, "pr"'] & \delta^c_n \arrow[r, "pz^c=pw^c"'] & B \end{tikzcd}$$

By Lemma 2.1.9 there exists a lift $H : \delta_n^c \times \triangle[2] \to X$ that makes the above diagram commutative, since p is a fibration. The composite $H(1 \times \varepsilon^2)$ gives us a p-homotopy from gz^c to gw^c.

ii) Let us show that given two n-simplices of Z whose images by f are equal, must be equal. Since $g : Z \to X$ is an isomorphism we can assume that Z is a free cell complex, for which there is a bijection $\Sigma(Z) \cong \Sigma(X)$ between bases, given by g.

Take x^e and y^e n-$\mathcal{C}$-simplices of Z such that $f(x) = f(y)$. By Proposition 1.2.4 there exists $\mathbf{x_1}^d \in \Sigma(Z)$ and a map $\alpha : d \to e$ in $\mathcal{C}$ such that $Z_\alpha(\mathbf{x_1}^d) = x$, and it holds for y^e as well, say $Z_\beta(\mathbf{y_1}^c) = y^e$. Since $X_\alpha f(\mathbf{x_1}^d) = f(x)$ and $X_\beta f(\mathbf{y_1}^c) = f(x)$, there exists $\mathbf{m}^b \in \Sigma(X)$ and maps $\theta : b \to d$, $\tau : b \to c$ in $\mathcal{C}$ such that $X_\theta(\mathbf{m}^b) = f(\mathbf{x_1}^d)$ and $X_\tau(\mathbf{m}^b) = f(\mathbf{y_1}^c)$.

We showed in 1) that f is surjective, then there exists $n^b \in Z_{b,n}$ such that $f(n) = \mathbf{m}^b$, so for $g(n^b)$ there exists $\mathbf{z}^a \in \Sigma(X)$ and a map $\varepsilon : a \to b$ in $\mathcal{C}$ such that $X_\varepsilon(\mathbf{z}^s) = g(n)$. Note that $fZ_\theta(n) = f(\mathbf{x_1}^d)$ and $fZ_\tau(n) = f(\mathbf{y_1}^c)$, then by item $i)$ it holds that $g(\mathbf{x_1}^d) \simeq_p gZ_\theta(n)$ and $g(\mathbf{y_1}^c) \simeq_p gZ_\tau(n)$, but $X_\theta X_\varepsilon(\mathbf{z}^a) = X_\theta g(n) = gZ_\theta(n)$ and $X_\tau X_\varepsilon(\mathbf{z}^a) = X_\tau g(n) = gZ_\tau(n)$, therefore $g(\mathbf{x_1}^d) \rightsquigarrow_p \mathbf{z}^a$ and $g(\mathbf{y_1}^c) \rightsquigarrow_p \mathbf{z}^a$. Since $g(\mathbf{x_1}^d), g(\mathbf{y_1}^c) \in \Sigma(X)$ and p is minimal it must hold that $a = c = d$ and $\mathbf{z} = g(\mathbf{x_1}) = g(\mathbf{y_1})$. That is, $a = c = d$ and $\mathbf{x_1} = \mathbf{y_1}$, since g is injective.

Since $\mathbf{m}^b \in \Sigma(X)$ and $X_\theta(\mathbf{m}^b) = f(\mathbf{x_1}) = X_\tau(\mathbf{m}^b)$, then $\theta = \tau$; analogously it must hold that $\alpha\theta = \beta\theta$ (see Proposition 1.2.4). Then $\alpha\theta\varepsilon = \beta\theta\varepsilon$, so $\alpha = \beta$, since $\theta\varepsilon$ is invertible ($\theta\varepsilon$ is an endomorphism in the EI-category $\mathcal{C}$). Since $Z_\alpha(\mathbf{x_1}) = x$, $Z_\beta(\mathbf{y_1}) = y$ and $\mathbf{x_1} = \mathbf{y_1}$ we have that $x = y$.

∎

Corollary 2.2.10
Let $p : X \to B$ be a minimal fibration in $\mathbf{S}^{\mathcal{C}}$ and $f, g : A \to B$ homotopic maps, where A and B are constant diagrams. Then the fibrations p_f and p_g from $A \times_B X$ to A are isomorphic.

Proof. It follows from Lemmas 2.1.8, 2.2.8 and Proposition 2.2.9.

∎

Corollary 2.2.11
Let $p : X \to B$ be a minimal fibration, where B is a constant diagram to the connected simplicial set B. Suppose that F is the fibre of p over a base point $*$ of B. Then p is a $\mathcal{C}$-fibre bundle with fibre F.

Proof. First let us show that p is locally trivial. Take an n-simplex v in B and make the composite of its representing map $v : \triangle[n] \to B$ with the maps $1, j : \triangle[n] \to \triangle[n]$, where 1 is the identity map and j is the composite of the map $\triangle[0] \to \triangle[n]$ that picks out the vertex (0) of $\triangle[n]$ with the map $\triangle[n] \to \triangle[0]$. By considering the pullback of p along this composites, the following diagram holds

$$\begin{tikzcd}
\triangle[n] \times F_p(\varepsilon_0 v) \arrow[r, "h"] \arrow[dr] & \triangle[n] \times_B X \arrow[r, "1"] \arrow[d] & \triangle[n] \times_B X \arrow[r] \arrow[d] \arrow[dr, phantom, very near end, "\lrcorner"] & X \arrow[d, "p"] \\
& \triangle[n] \arrow[r, shift left, "1"] \arrow[r, shift right, "j"'] & \triangle[n] \arrow[r, "v"'] & B
\end{tikzcd}$$

since 1 and j are homotopic maps (see Lemma 0.1.14). The induced pullbacks over $\triangle[n]$ are minimal fibrations since p is minimal (see Lemma 2.2.8), then by Corollary 2.2.10 it holds that h is an isomorphism. The Lemma 0.1.16 of Section 0.1 admits a natural generalization in the case of $\mathcal{C}$-diagrams, so $F_p(\varepsilon_0 v) \cong F$, since B is connected.

∎

2.3 Classification of $\mathcal{C}$-Fibrations

Consider two $\mathcal{C}$-diagrams F and B, where B is the constant diagram to the connected simplicial set B. The objective of this section is to classify fibrations $p : X \to B$ in $\mathbf{S}^{\mathcal{C}}$ whose fiber is homotopically equivalent to F. The relation used to classify them is that of fiberwise homotopy equivalence (See Definition 2.1.3 part (2) and Proposition 2.1.6).

Theorem 2.3.1
Let $\mathcal{C}$ be a small EI-category with a finite number of objects and F, B, C-diagrams, where B is the constant diagram to the connected simplicial set B.

Then the set $[B, \overline{W}haut(F)]$ *of homotopy classes of maps from* B *to* $\overline{W}haut(F)$ *is in bijective correspondence with the set of equivalence classes of* $\mathcal{C}$*-diagrams of fibrations with base space* B *and fibers with the homotopy type of* F*.*

Proof. In view of Proposition B.0.28 every fibration p is equivalent to a fibration p' whose total space is a free cell complex. By Proposition 2.2.6 the fibration p' is equivalent to a minimal one p'', and from Proposition 2.2.9 it holds that any two minimal fibrations which are equivalents are isomorphic. So if F'' is the fiber of p'', then p determines a unique element in $\pi_0 hom(B, \overline{W}aut(F''))$, since any minimal fibration is a fiber bundle (See Corollary 2.2.11 and Theorem 1.4.12).

From Proposition 1.5.2 we have that $aut(F'') \simeq haut(F'')$, and since $F'' \simeq F$ it holds that $aut(F'') \simeq haut(F)$. Then $\overline{W}aut(F'') \simeq \overline{W}haut(F)$, so p determines a unique element in $\pi_0 hom(B, \overline{W}haut(F))$.

∎

CHAPTER 3

APPLICATIONS

We present some applications of the classification theorem 2.3.1 for $\mathcal{C}$-diagrams of fibrations over a connected base space B, when the category $\mathcal{C}$ has the shape $\{\bullet \to \bullet\}$. Some classical results can be deduced from it, as for example, those related with the fiberwise localization of a fibration p and the one related with the homotopy type of certain Gauge group. We also use it to state the existence and uniqueness of the fiberwise cellularization for a fibration p in terms of the obstructions to the existence and uniqueness of a certain lifting.

In what follows we will use either topological techniques, or simplicial techniques to proof some facts. Thus, given a monoid M, BM will denote its classifying space, and if F is a topological space $aut(F)$ will denote the topological monoid of self-homotopy equivalences of F. Since in the simplicial context we use to work with minimal simplicial sets F, then $aut(F)$ will denote the simplicial monoid of sel-fhomotopy equivalences of F, or the simplicial group of automorphisms of F.

3.1 Fiberwise Localization

Given a cofibration $f : A \to B$ in $\mathbf{S}$ and a fibre map $p : E \to B$ with fibre F, we reprove the Theorems 0.3.6 and 0.3.7 of Section 0.3.1. To do this consider the cofibrant f-localization $l_F : F \to L_f F$ of F, and then all the $\mathcal{C}$-diagrams of fibrations over the connected space B, where $\mathcal{C} = \{\bullet \to \bullet\}$, whose fibers are homotopy equivalent to l_F.

In this case we can consider the following pullback diagram given in the Example 1.5.3

$$\begin{tikzcd}
aut(l_X) \arrow[r, dotted, "s_{L_fF}"] \arrow[d, dotted, "s_F"'] \arrow[dr, phantom, "\lrcorner"] & aut(L_fF)_{\Phi''} \arrow[d, "(l_F)^\sharp"] \\
aut(F) \arrow[r, "(l_F)_\sharp"'] & Map(F, L_fF)_\Phi
\end{tikzcd}$$

where $Map(F, L_fF)_\Phi$ are the components of maps in $Map(F, L_fF)$ that factorize well, that is, g belongs to $hom(F, L_fF)_\Phi$ if there exists self-homotopy equivalences $\alpha \in aut(F)$ and $\beta \in aut(L_fF)$, such that $g \simeq l_F\alpha \simeq \beta l_F$. $aut(L_fF)_{\Phi''}$ are the components of $aut(L_fF)$ that are in correspondence with the ones of $hom(F, L_fF)_\Phi$, and since L_f is a coaugmented functor all the components of the monoid $aut(F)$ are in correspondence with the components of $Map(F, L_fF)_\Phi$.

Since l_F is an f-local equivalence and L_fF is an f-local space, we have a weak equivalence $Map(L_fF, L_fF) \simeq Map(F, L_fF)$ (See section 0.3.1), and therefore $aut(L_fF)_{\Phi'}$ and $Map(F, L_fF)_\Phi$ are weakly equivalent. Since the above diagram is a pullback it holds that $aut(l_X) \simeq aut(F)$. So applying the classifying space functor to $s_F : aut(l_F) \to aut(F)$, which is a map of topological monoids, we have that

$$Baut(l_F) \xrightarrow[\simeq]{Bs_F} Baut(F)$$

If p is classified by a map $h : B \to Baut(X)$, then there is a unique lifting $\widetilde{h} : B \to Baut(l_F)$ of h up to homotopy. Therefore (E, p, B) admits up to homotopy a unique fiberwise f-localization

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "l_F"'] & E \arrow[r, "p"] \arrow[d, "a"'] & B \arrow[d, equal] \\
L_fF \arrow[r] & \overline{E} \arrow[r, "q"'] & B
\end{tikzcd}$$

where a is an f-local equivalence (by Example $D.8$ of [9]).

3.2 Fibrewise cellularization

We are interested in the study of the existence of fiberwise cellularizations (see Definition 0.3.14) for a given fibre sequence $F \to E \to B$. A fibration together with a fibrewise cellularization form a diagram

$$\begin{tikzcd} CW_AF \arrow[r, "c_F"] \arrow[d] & F \arrow[d] \\ \overline{E} \arrow[r, "b"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

where the homotopy fiber of the diagram is determined by the augmentation map $c_F : CW_AF \to F$.

To deal with these diagrams we replace in the diagram of the Example 1.5.3 the map F_f by the fibration $c_X : CW_AF \to F$

$$\begin{tikzcd} aut(c_F) \arrow[r, dashed, "s_F"] \arrow[d, dotted, "s_{CW_AF}"'] & aut(F) \arrow[d, "(c_F)^\sharp"] \\ aut(CW_AF) \arrow[r, "(c_F)_\sharp"'] & Map(CW_AF, F) \end{tikzcd}$$

and as in the previous section we apply the classifying space functor to the upper row

$$L \longrightarrow Baut(c_F) \xrightarrow{Bs_F} Baut(F)$$

in which the space L will denote the homotopy fibre of Bs_F. Since $p : E \to B$ is classified by a map $h : B \to Baut(F)$, p has a fiberwise cellularization if h has a lift

$$\begin{tikzcd} & Baut(c_F) \arrow[d, "Bs_F"] \\ B \arrow[ur, dotted, "\widetilde{h}"] \arrow[r, "h"] & Baut(F) \end{tikzcd}$$

Thus, the obstructions to the existence and uniqueness of this lifting lies in $H^{i+1}(B, \pi_i(L))$ and $H^i(B, \pi_i(L))$ respectively. So we will characterize the fiber

L of the fibration $Bs_X : Baut(c_X) \to Baut(X)$ in order to determine these cohomology groups.

3.2.1 Homotopy Fibers

Given a fibration $f : X \to Y$ we are interested in determining the homotopy groups of the homotopy fiber of $Bs_Y : Baut(f) \to Baut(Y)$, which is obtained from the pullback diagram

$$\begin{tikzcd} aut(f) \arrow[r, dotted, "s_Y"] \arrow[d, dotted, "s_X"'] & aut(Y) \arrow[d, "f^\sharp"] \\ aut(X) \arrow[r, "f_\sharp"'] & Map(X,Y) \end{tikzcd}$$

Theorem 3.2.1
Let $X \xrightarrow{f} Y \xrightarrow{p} Z$ be a fibre sequence with the space Z pointed, 0-connected and ΣX-null (that is, $Map_*(\Sigma X, Z) \simeq *$). Then $\pi_i L \cong \pi_i Z$ for $i \geq 1$, where L is the fiber of the fibration $Baut(f) \to Baut(Y)$.

Proof. By considering the fibre sequence $X \xrightarrow{f} Y \xrightarrow{p} Z$ we can enlarge the above diagram as follows

$$(I) \qquad \begin{tikzcd} aut(f) \arrow[r, dotted, "s_Y"] \arrow[d, dotted, "s_X"'] & aut(Y)_{\phi''} \arrow[d, "f^\sharp"] & \\ aut(X)_{\phi'} \arrow[r, "f_\sharp"'] & Map(X,Y)_\phi \arrow[r, "p'_\sharp"'] & Map(X,Z)_c \end{tikzcd}$$

where $Map(X,Z)_c$ is the component of the constant map in $Map(X,Z)$. Notice that given $g \in Map(X,Y)_\phi$, it can be written as $g \simeq f\alpha \simeq \beta f$ where $(\alpha, \beta) \in aut(X) \times aut(Y)$, hence $p_\sharp(g) = pg$ is nullhomotopic, since pf is nullhomotopic ($pg \simeq pf\alpha$). Therefore $p'_\sharp$ is the restriction of $p_\sharp : Map(X,Y) \to Map(X,Z)$ to $Map(X,Y)_\phi$.

Taking loops in the lower row of the above diagram we obtain the sequence

$$\Omega Map(X,Z)_c \longrightarrow aut(X)_{\phi'} \xrightarrow[f_\sharp]{} Map(X,Y)_\phi \xrightarrow[p_\sharp]{} Map(X,Z)_c$$

Since the square (I) is a homotopy pull-back, we can construct a homotopy equivalence from $\Omega Map(X,Z)_c$ to the homotopy fibre of s_Y, and hence the following is a fibre sequence

$$\Omega Map(X,Z)_c \longrightarrow aut(f) \xrightarrow{s_Y} aut(Y)$$

From $L \to Baut(f) \to Baut(Y)$ we can consider the fibre sequence

$$\Omega L \longrightarrow aut(f) \xrightarrow{S_Y} aut(Y) \longrightarrow L \longrightarrow Baut(f) \xrightarrow{Bs_Y} Baut(Y)$$

and therefore $\Omega L \simeq \Omega Map(X,Z)_c$, although not necessarily as loop spaces.

By hypothesis $Map_*(\Sigma X, Z) \simeq *$, that is, $\pi_i Map_*(\Sigma X, Z) = 0$, so the homotopy groups $\pi_{i+1} Map_*(X,Z)$ are equal to zero for $i \geq 0$ (by the adjunction property of mapping spaces).

Applying the long exact sequence in homotopy groups for the following fibre sequence

$$Map_*(X,Z)_{\tilde{c}} \longrightarrow Map(X,Z)_c \xrightarrow{ev} Z$$

we have that $\pi_i Map(X,Z)_c \cong \pi_i Z$ for $i \geq 0$, since Z and $Map(X,Z)_c$ are 0-connected (we have chosen only one component), and $\pi_i Map_*(X,Z) = 0$ for $i \geq 1$. Since $\Omega L \simeq \Omega Map(X,Z)_c$, it holds that $\pi_i L \cong \pi_i Z$ for $i \geq 1$.

∎

The following result is a straightforward consequence of this Theorem, by using the sequence $CW_A F \to F \to P_{\Sigma A} C_{ev}$, given by Chachólski in the Theorem 0.3.13.

Corollary 3.2.2
Let $p : E \to B$ a fibration with fiber F. Then the obstructions to the existence and uniqueness of a fiberwise cellularization for p lies in $H^{i+1}(B, \pi_i(P_{\Sigma A} C_{ev}))$ and $H^i(B, \pi_i(P_{\Sigma A} C_{ev}))$, respectively.

The following fibration was suggested by Chachólski

Theorem 3.2.3
Let (X, f, Y) be a fibration with fiber F, classified by a map $cl : Y \to Baut(F)$. Then the following is a fibre sequence

$$Map(Y, Baut(F))_{cl} \hookrightarrow Baut(f) \xrightarrow{Bs_Y} Baut(Y)$$

In order to proof this theorem let us first consider the following result

Theorem 3.2.4 ([5], *Theorem* 1.2(*ii*)**)**
Let $p : E \to B$ a principal fibration with group G, classified by the map cl. If $aut_G(p)$ is the simplicial group which has as set of n-simplices the commutative diagrams

$$\begin{tikzcd} \triangle[n] \times E \arrow[rr] \arrow[dr, "1\times p"'] & & \triangle[n] \times E \arrow[dl, "1\times p"] \\ & \triangle[n] \times B & \end{tikzcd}$$

in which the horizontal map is compatible with the action of G (an therefore an isomorphism). Then $\overline{W}aut_G(p)$ has the homotopy type of $hom(B, \overline{W}G)_{cl}$, that is, it has the homotopy type of the connected component of $hom(B, \overline{W}G)$ which corresponds to cl.

Proof. Note that $aut_G(p)$ acts from the left of E and from the right of $Waut_G(p) = \overline{W}aut_G(p) \times_\tau aut_G(p)$. Since $(1 \times p) : Waut_G(p) \times_{aut_G(p)} E \to \overline{W}aut_G(p) \times B$ is principal and with group G, it is classified by a map $g : \overline{W}aut_G(p) \times B \to \overline{W}G$, so by using the adjointness property of function complexes we obtain a map $\hat{g} : \overline{W}aut_G(p) \to hom(B, \overline{W}G)_{cl}$, since $(1 \times p)$ restricted to every vertex of $Waut_G(p)$ is the fibration p. To show that $\hat{g}$ is an homotopy equivalence let us see that $\overline{W}aut_G(p)$ and $hom(B, \overline{W}G)_{cl}$ classifies the same principal fibrations.

Notice that at every map $h : Y \to hom(B, \overline{W}G)_{cl}$ corresponds to a map $\hat{h} : Y \times B \to \overline{W}G$, such that its restriction $\hat{h}|_y : B \to \overline{W}G$ to the first component over any vertex y of Y is homotopic to cl (that is, $\hat{h}|_y \simeq cl$). So by pulling-back $\hat{h}$ with the map $\pi : WG \to \overline{W}G$ we obtain up to homotopy a unique twisting cartesian product of the form $Y \times_t E \to Y \times B$.

From proposition 1.4.11 it holds that $\overline{W}aut(p)$ classifies the following diagrams of fibrations

$$\begin{tikzcd}
E \arrow[rr, "p"] \arrow[d] & & B \arrow[d] \\
Y \times_{t_1} E \arrow[rr, "1\times p"] \arrow[dr] & & Y \times_{t_2} B \arrow[dl] \\
& Y &
\end{tikzcd}$$

Since a twisting map $t : Y_n \to aut_G(p)_{n-1}$ has the form (t_1, id), where $t_1(y) : \triangle[n] \times E \to \triangle[n] \times E$ is equivariant, for any $y \in Y_n$, it hods that $\overline{W}aut_G(p)$ classifies principal fibrations $Y \times_t E \to Y \times B$.

∎

The group $aut_G(p)$ is known as the Gauge group of p.

Theorem 3.2.5 ([5], *Theorem* 1.4(*ii*)**)**
Let F be a minimal simplicial set, $f : X \to Y$ a fibration with all the fibres of the homotopy type of F, and let $aut_{id}(f)$ be the simplicial monoid which has as set of n-simplices the commutative diagrams

$$\begin{tikzcd}
\triangle[n] \times X \arrow[rr] \arrow[dr, "1\times f"'] & & \triangle[n] \times X \arrow[dl, "1\times f"] \\
& \triangle[n] \times Y &
\end{tikzcd}$$

in which the horizontal map is a self-homotopy equivalence of X. If f is classified by a map $cl : Y \to \overline{W}aut(F)$, then $\overline{W}aut_{id}(f) \simeq hom(Y, \overline{W}aut(F))_{cl}$.

Proof. This Theorem can be reduced to Theorem 3.2.4.

∎

Proof. [Theorem 3.2.3] Consider the fibration $F_{id_Y}(s_Y) \hookrightarrow aut(f) \xrightarrow{s_Y} aut(Y)$, where $F_{id_Y}(s_Y)$ is the fiber of s_Y over the identity map id_Y. By Applying $\overline{W}$ to s_Y, we obtain the fibre sequence $\overline{W}F_{id_Y}(s_Y) \hookrightarrow \overline{W}aut(f) \to \overline{W}aut(Y)$.

Let $f' : X' \to Y$ a minimal subfibration of f (which is a strong deformation retract of f). Notice that for $s'_Y : aut(f') \to aut(Y)$, its fiber $F_{id_Y}(s'_Y)$ consists of the following commutative triangles

$$\begin{tikzcd} \triangle[n] \times X' \arrow[rr, "\alpha"] \arrow[dr, "1\times f'"'] & & \triangle[n] \times X' \arrow[dl, "1\times f'"] \\ & \triangle[n] \times Y & \end{tikzcd}$$

where α is an automorphism of X'. Thus, by Theorem 3.2.5 it holds that $\overline{W}F_{id_Y}(s'_Y) \simeq hom(Y, \overline{W}aut(F))_{cl}$, and since $\overline{W}F_{id_Y}(s'_Y) \simeq \overline{W}F_{id_Y}(s_Y)$ it holds that $\overline{W}F_{id_Y}(s_Y) \simeq hom(Y, \overline{W}aut(F))_{cl}$.

∎

Proposition 3.2.6
Let $X \xrightarrow{f} Y \xrightarrow{p} Z$ *be a fibre sequence. If the map* $cl : Y \to Baut(\Omega Z)$ *classifies* f*, then* $Map(Y, Baut(\Omega Z))_{cl} \simeq Map(X, Y)_c$*, where* $Map(X, Y)_c$ *is the component of the constant map in* $Map(X, Y)$*.*

Proof. Taking loops in the fibre sequence of Theorem 3.2.3 and using the diagram (I) given in the proof of Theorem 3.2.1, we can consider the following diagram

$$\begin{tikzcd} aut(f) \arrow[r, dotted, "s_Y"] \arrow[d, dotted, "s_X"'] \arrow[dr, phantom, "I"] & aut(Y)_{\phi''} \arrow[r, "cl_\sharp"] \arrow[d, "f^\sharp"] & Map(Y, Baut(\Omega Z))_{cl} \\ aut(X)_{\phi'} \arrow[r, "f_\sharp"'] & Map(X, Y)_\phi \arrow[r, "p'_\sharp"'] & Map(X, Z)_c \end{tikzcd}$$

Let us see that the map $cl_\sharp$ is well defined, that is, given $\beta \in aut(Y)_{\phi''}$ we must see that the induced fibration $\widehat{f} : \beta^*(X) \to Y$ given by he pullback of f along β is fiberwise homotopic to f. To check this consider the following sequence of homotopy commutative diagrams

$$\begin{tikzcd} \beta^*(X) \arrow[rr, "\widehat{\beta}"] \arrow[dr, "\widehat{f}"'] & & X \arrow[dl, "\beta^- f"] \\ & Y & \end{tikzcd} \quad \begin{tikzcd} \beta^*(X) \arrow[r, "\simeq"] \arrow[dr, "\widehat{f}"'] & X \arrow[r, "\simeq"] \arrow[d, "f\alpha^-"] & P_{f\alpha^-} \arrow[dl, "q"] \\ & Y & \end{tikzcd} \quad \begin{tikzcd} P_{f\alpha^-} \arrow[r, "\simeq"] \arrow[dr, "q"'] & X \arrow[r, "\simeq", "\alpha^-"'] & X \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

in the first triangle we compose f with the homotopy inverse β^- of β and since $\beta f \simeq f\alpha^-$ we can pass to the second triangle, where q is the fibration that factorize $f\alpha^-$. By using the third triangle we can conclude that $\widehat{f}$ is fiberwise homotopic to f.

Now consider the following pullback diagram

$$\begin{tikzcd}
\Omega Z \arrow[d] & & \Omega Z \arrow[d] \\
aut(Y)_{\phi''} \times X \arrow[rr, "p_\sharp f^\sharp \times 1"] \arrow[d, "id \times f"] & & Map(X,Z)_c \times X \arrow[d, "1 \times f"] \\
aut(Y)_{\phi''} \times Y \arrow[rr, "p_\sharp f^\sharp \times 1"'] & & Map(X,Z)_c \times Y
\end{tikzcd}$$

Notice that these two fibrations are classified by maps θ and η that make the following diagram homotopy commutative, because the above square is a pullback

$$\begin{tikzcd}
aut(Y)_{\phi''} \times Y \arrow[r, "p_\sharp f^\sharp \times 1"] \arrow[dr, "\theta"'] & MapX,Y)_c \times Y \arrow[d, "\eta"] \\
 & Baut(\Omega Z)
\end{tikzcd}$$

By using the adjunction property of mapping spaces we can obtain a homotopy commutative diagram

$$\begin{tikzcd}
aut(Y)_{\phi''} \arrow[r, "p_\sharp f^\sharp"] \arrow[dr, "\widetilde{\theta}"'] & Map(X,Y)_c \arrow[d, "\widetilde{\eta}"] \\
 & Map(Y, Baut(\Omega Z))
\end{tikzcd}$$

Since the restriction of the fibration $(1 \times f)$ and $(id \times f)$ over any vertex of $Map(X,Y)_c$ and $aut(Y)_{\phi''}$ respectively, are fibre wise homotopic to f, then the codomain of $\widetilde{\theta}$ and $\widetilde{\eta}$ is $Map(Y, Baut(\Omega Z))_{cl}$. Thus the map $\widetilde{\theta}(\beta)$ is homotopic to cl for any $\beta \in aut(Y)_{\phi''}$, but $cl\beta \simeq cl$ for any $\beta \in aut(Y)_{\phi''}$, and hence $\widetilde{\theta} \simeq cl_\sharp$.

Now consider the following diagram

$$\begin{tikzcd}
(II) & aut(X)_{\phi'} \times_{f_\sharp} P_{f^\sharp} \arrow[rr, "\pi"] \arrow[ddl, dotted] & & P_{f^\sharp} \arrow[ddl, "q"] & \\
aut(X)_{\phi'} \times_{f_\sharp} aut(Y)_{\phi''} \arrow[r] \arrow[ur, "1\times\upsilon"] \arrow[d] & & aut(Y)_{\phi''} \arrow[ur, "\upsilon"] \arrow[rr, "cl_\sharp"] \arrow[d, "f^\sharp"'] & & Map(Y, Baut(\Omega Z))_{cl} \\
aut(X)_{\phi'} \arrow[rr, "f_\sharp"'] & & Map(X,Y)_\phi \arrow[rr, "p_\sharp"'] & & Map(X,Y)_c \arrow[u, "\widetilde{\eta}"]
\end{tikzcd}$$

where $aut(X)_{\phi'} \times_{f_\sharp} aut(BF)_{\phi''} = aut(f)$ and $P_{f^\sharp}$ is the path space of the map $f^\sharp$, that is, $P_{f^\sharp} = \{(\beta, \theta) \in aut(Y)_{\phi''} \times hom(X,Y)^I_\phi : f^\sharp(\beta) = \theta(0)\}$. The map q is the fibration that factorizes $f^\sharp$ and υ is the correspondent homotopy equivalence. $aut(X)_{\phi'} \times_{f_\sharp} P_{f^\sharp}$ denotes the pullback of $. \xrightarrow{f_\sharp} . \xleftarrow{q} .$.

From the above diagram take the following one

$$\begin{tikzcd}
aut(X)_{\phi'} \times_{f_\sharp} aut(Y)_{\phi''} \arrow[r] \arrow[d, "1\times\upsilon"] & aut(Y)_{\phi'} \arrow[r, "cl_\sharp"] \arrow[d, "\upsilon"'] & Map(Y, Baut(\Omega Z))_{cl} \\
aut(X)_{\phi'} \times_{f_\sharp} P_{f^\sharp} \arrow[r, "\pi"'] & P_{f^\sharp} \arrow[r, "p_\sharp q"'] & Map(X,Y)_c \arrow[u, "\widetilde{\eta}"]
\end{tikzcd}$$

Since $p_\sharp$ and q are fibrations we have that $Fib(p_\sharp q) \simeq aut(X)_{\phi'} \times_{f_\sharp} P_{f^\sharp}$ and by Theorem 3.2.3 the upper row is a fibre sequence by . Note that the upper square of diagram II is a pullback square, and hence $(1 \times \upsilon)$ is an homotopy equivalence since υ is so too. Taking the long exact sequences of homotopy groups for these two fibrations we have that $Map(Y, Baut(\Omega Z))_{cl} \simeq Map(X,Y)_c$.

∎

As an illustration of this Proposition let us consider the following examples

Example 3.2.7 *Consider the fibre sequence* $K(\mathbb{Z},1) \longrightarrow X \xrightarrow{f} K(\mathbb{Z},2)$ *where* $X \simeq *$. *Since* f *is classified by a map* $p : K(\mathbb{Z},2) \to BK(\mathbb{Z},1)$ *it holds that*

$$Map(X, BK(\mathbb{Z},1))_c \simeq Map(*, K(\mathbb{Z},2))_c \simeq K(\mathbb{Z},2)$$

but f *is also classified by a map* $cl : K(\mathbb{Z},2) \to BweK(\mathbb{Z},1)$ *and in that case*

$$Map(K(\mathbb{Z},2), BweK(\mathbb{Z},1))_{cl} \simeq Map(K(\mathbb{Z},2), BK(\mathbb{Z},1))_{cl}$$

$$\simeq Map(K(\mathbb{Z},2),K(\mathbb{Z},2))_{cl} \simeq K(\mathbb{Z},2)$$

Example 3.2.8 *Let us take the principal fibration* $G \hookrightarrow EG \xrightarrow{f} BG$ *where* G *is a finite group. If* f *is classified by* $p : BG \to BG$*, then*

$$Map(EG,BG)_c \simeq Map(*,BG)_c \simeq BG$$

Remember the inclusion map $G \overset{i}{\hookrightarrow} \Sigma_G$ *, where* Σ_G *is the symmetric group of* $|G|$ *letters.* i *gives rise a group homomorphism* $C_{\Sigma_G}(i(G)) \times G \to \Sigma_G$ *for which we can apply the classifying space functor* $BC_{\Sigma_G}(i(G)) \times BG \to B\Sigma_G$ *and take the adjoint map* $BC_{\Sigma_G}(G) \to Map(BG, B\Sigma_G)_i$*, which is a homotopy equivalence. Since* $BweG \simeq B\Sigma_G$ *and* $Map(BG,\Sigma_G)_h \simeq Map(BG,B\Sigma_G)_i$*, where* $h \in Rep(G,\Sigma_G)$ *we have that*

$$Map(BG,BweG)_{cl} \simeq Map(BG,B\Sigma_G)_i$$

where $cl : BG \to Bwe(G)$ *is the classifying map of* f*. Then*

$$Map(BG,BweG)_{cl} \simeq BC_{\Sigma_G}(G)$$

but $C_{\Sigma_G}(G) \cong G' \leq \Sigma_G$ *with* $G \cong G'$*, therefore*

$$Map(BG,BweG)_{cl} \simeq BC_{\Sigma_G}(G) \simeq BG$$

3.2.2 Examples

As we saw in Section 0.3.1 given a cofibration $f : A \to B$ and a fibration $E \to B$ it is always possible to obtain a fibrewise L_f-localization of g, but it is not true in general for the cellularization functor CW_A. For example, for $A = B\mathbb{Z}/2$ consider the fibre sequence $S^2 \hookrightarrow \mathbb{R}P^2 \to B\mathbb{Z}/2$ and suppose that it has a fibrewise cellularization

$$\begin{tikzcd} S^2 \arrow[r] & \mathbb{R}P^2 \arrow[r] & B\mathbb{Z}/2 \arrow[d, equal] \\ CW_{B\mathbb{Z}/2}S^2 \arrow[u] \arrow[r] & \overline{E} \arrow[u] \arrow[r] & B\mathbb{Z}/2 \end{tikzcd}$$

The sphere is a finite CW-complex and hence $hom_*(B\mathbb{Z}/2,S^2) \simeq *$. So it holds that $CW_{B\mathbb{Z}/2}S^2 \simeq *$, and therefore $\overline{E} \simeq B\mathbb{Z}/2$; but $hom_*(B\mathbb{Z}/2,\mathbb{R}P^2) \simeq *$, since $\mathbb{R}P^2$ is also a finite CW-complex. Thus, there is not nontrivial maps from

$B\mathbb{Z}/2$ to $\mathbb{R}P^2$ and hence the above diagram cannot be commutative.

Given a finite group G and a fibration $BG \hookrightarrow E \xrightarrow{g} B$, we may assume some conditions over G and the base space B in order to known when it is possible to obtain a $B\mathbb{Z}/p$-fibrewise cellularization for g. Before that let us remember some results related with $CW_A BG$ and the p-completion $X^\wedge_p$ of a space X.

Definition 3.2.9 ([10]) *Let G be a finite group and H a subgroup of some Sylow p-subgroup S of G. Then H is strongly closed in G if whenever $h \in H$ and $g \in G$ are such that $ghg^{-1} \in S$, then $ghg^{-1} \in H$.*

Remember that given two elements x and y in a subgroup H of G, they are said to be fused in G if they are conjugate in G, but not in H. If S is a Sylow p-subgroup S of G, let us denote the smallest strongly closed subgroup of S containing all elements of order p in S by $Cl(S)$.

Definition 3.2.10 ([10]) *Let G be a finite group and S a Sylow psubgroup. The normalizer $N_G(S)$ of the Sylow p-subgroup controls fusion if, whenever $P < G$ is a p-subgroup and $gPg^{-1} < N_G(S)$, we have $g = hc$, with $h \in N_G(S)$ and $c \in C_G(P)$.*

The following theorem is a consequence of the Theorem 5.6 of [10]

Proposition 3.2.11
Let G be a finite group generated by elements of order p and S a Sylow p-subgroup of G. If G is not a p-group, $N_G(S)$ the normalizer of S in G controls the fusion in G and $S = Cl(S)$, it holds that $CW_{B\mathbb{Z}/p}BG$ fits in a fibration

$$CW_{B\mathbb{Z}/p}BG \hookrightarrow BG \longrightarrow \prod_{q\neq p} BG^\wedge_q$$

where the product is taken over all primes q dividing the order of G.

It is important to remark that in the above fibration $\prod_{q\neq p} BG^\wedge_q \simeq P_{\Sigma B\mathbb{Z}_p} C_{ev}$, where C_{ev} is the cofiber given in Theorem 0.3.13.

Theorem 3.2.12 ([3], *Proposition $VII.4.3(i)$***)**
Let X be a pointed connected space such that $\pi_i(X)$ is finite for every $i \geq 1$, then $\pi_i(X^\wedge_p)$ is a finite p-group for all i and prime p.

Theorem 3.2.13
Let G be a finite group generated by elements of order p and S a Sylow p-subgroup of G. If G is not a p-group, $N_G(S)$ the normalizer of S in G controls

the fusion in G and $S = Cl(S)$ and B is $B\mathbb{Z}/p$-cellular, then any fibre sequence $BG \hookrightarrow E \xrightarrow{g} B$ admits a unique $B\mathbb{Z}/p$-fibrewise cellularization.

Proof. By Theorem 3.2.11 $CW_{\mathbb{Z}/p}BG$ is the fibre of $BG \to \prod_{q\neq p} BG^{\wedge}_q$, thus by Corollary 3.2.2 the obstruction to the existence of a fibrewise-cellularization of g lie in the cohomology groups

$$H^{i+1}(B, \pi_i(\prod_{q\neq p} BG^{\wedge}_q))$$

Taking $|G| = p^r q_1 ... q_s$ we have that

$$H^{i+1}(B, \pi_i(\prod_{q\neq p} BG^{\wedge}_q)) = H^{i+1}(B, \prod_{1\leq k\leq s} \pi_i(BG^{\wedge}_{q_k}))$$

The homotopy groups of $BG^{\wedge}_{q_k}$ are q_k groups (Theorem 3.2.12),then these cohomology groups can be expressed as a finite product of cohomology groups $H^{i+1}(B, \mathbb{Z}/q_k^{n_k})$ with coefficients in $\mathbb{Z}/q_k^{n_k}$. We also know that B is $B\mathbb{Z}/p$-cellular (that is, $B \simeq CW_{B\mathbb{Z}/p}B$), then it is constructed by assembling several copies of $B\mathbb{Z}/p$ together along some hocolimit scheme (see Section 0.3.2). Therefore we can study $H^{i+1}(B, \mathbb{Z}/q_k^{n_k})$ in terms of the simpler cohomology groups $H^{i+1}(B\mathbb{Z}/p, \mathbb{Z}/q_k^{n_k})$, or what is the same the groups $H^{i+1}(\mathbb{Z}/p, \mathbb{Z}/q_k^{n_k})$.

Since $H^{i+1}(\mathbb{Z}/p, \mathbb{Z}/q_k^{n_k}) \cong 0$, it holds that $\prod_{1\leq k\leq s} H^{i+1}(B, \pi_i(BG^{\wedge}_{q_k})) \cong 0$ and $\prod_{1\leq k\leq s} H^{i}(B, \pi_i(BG^{\wedge}_{q_k})) \cong 0$, therefore g admits a unique $CW_{B\mathbb{Z}/p}$-fibrewise cellularization.

∎

In the following proposition $L_{\mathbb{Z}[\frac{1}{p}]}$ denotes the homological localization with respect to the homology theory with coefficients in $\mathbb{Z}[\frac{1}{p}]$.

Theorem 3.2.14 ([13])
Let X be an 1-connected infinite loop space of finite type, p a prime number and r a positive integer. If $\pi_2 X$ is a torsion group, then $P_{\Sigma B\mathbb{Z}/p^r} C_{ev} \simeq L_{\mathbb{Z}[\frac{1}{p}]}X$.

Theorem 3.2.15
Let X be an 1-connected infinite loop space of finite type, p a prime number and r a positive integer. If $\pi_2 X$ is a torsion group and B a $B\mathbb{Z}/p^r$-cellular space, then any fibre sequence $E \xrightarrow{g} B$ with fiber X admits a unique $B\mathbb{Z}/p^r$-fibrewise-cellularization.

Proof. If $\pi_i(X)$ is a q-torsion group, then $\pi_i(X) \otimes \mathbb{Z}[\frac{1}{p}] \cong \pi_i(X)$ and if $\pi_i(X)$ is a p-torsion group it holds that $\pi_i(X) \otimes \mathbb{Z}[\frac{1}{p}] \cong 0$. Since B is $B\mathbb{Z}/p^r$-cellular space we can proceed as in proof of Proposition 3.2.13.

∎

In Section 0.3.1 we saw that given a fibration $p : E \to B$ for which its fibrewise localization is the factorization $E \xrightarrow{a} \overline{E} \xrightarrow{q} B$, it holds that a is a local equivalence. In the case of fibrewise cellularization it might happen that the map between the total spaces is not an A-equivalence, for example, if $A = B\mathbb{Z}/2$ the following diagram is a fibrewise cellularization of the fiber sequence $B\mathbb{Z}/3 \hookrightarrow B\Sigma_3 \to B\mathbb{Z}/2$

$$\begin{tikzcd} B\mathbb{Z}/3 \arrow[r] & B\Sigma_3 \arrow[r] & B\mathbb{Z}/2 \\ * \arrow[u] \arrow[r] & B\mathbb{Z}/2 \arrow[u, "b"] \arrow[r] & B\mathbb{Z}/2 \arrow[u, equal] \end{tikzcd}$$

we have that $Map_*(B\mathbb{Z}/2, B\mathbb{Z}/2) \simeq \mathbb{Z}/2$, while $Map_*(B\mathbb{Z}/2, B\Sigma_3)$ consists of three morphisms. Therefore b is not an $B\mathbb{Z}/3$-equivalence.

APPENDIX A

PREORDERED SETS

Let A be a set and '$\rightsquigarrow$' a preorder relation over A, that is, a reflexive and transitive relation. Consider the following subset A' from A

1. A' is the set formed by taking the minimum number of elements of A such that: for all $x \in A$ there exists $w \in A'$ with $x \rightsquigarrow w$

A natural question around this situation is: Does A' always exist?.

Example A.0.16 *Consider the following infinite sequence of sets connected by injective maps* $\ldots \longrightarrow A_n \xrightarrow{f_n} \ldots \longrightarrow A_1 \xrightarrow{f_1} A_0$ *and define '$\rightsquigarrow$' over* $\bigcup_{i\geq 0} A_i$ *for* $a \in A_j$ *and* $a' \in A_k$ *as follows*

$$a \rightsquigarrow a' \textit{ if there exists a map } f : A_k \to A_j \textit{ such that } f(a') = a$$

In this case $A' = \emptyset$*, since in the set* $\bigcup_{i\geq 0} A_i$ *we can consider infinite chains* $a_{i_1} \rightsquigarrow a_{i_2} \rightsquigarrow \ldots \rightsquigarrow a_{i_k} \rightsquigarrow \ldots$ *of non-equal elements (here* $\rightsquigarrow$ *is in particular an order relation).*

To solve our question we will reformulate it by defining an order relation over A, that is, consider the preorder set $(A, \rightsquigarrow)$ and define the following relation:

a. $x \sim w$ if $x \rightsquigarrow w$ and $w \rightsquigarrow x$

Note that '$\sim$' is and equivalence relation over A. Now in $A/\sim$ consider the relation:

b. given $[x], [w] \in A/\sim$, we say that $[x] \leq [w]$ if $x \rightsquigarrow w$

With the above relations $(A/\sim, \leq)$ is an ordered set. So the existence of A' depends on the existence of maximal elements in $A/\sim$, that is, A' will be the set formed by taking from every maximal class in $(A/\sim, \leq)$ one representant.

Let us consider the following situation:

Take a functor $X : \mathcal{C} \to \mathfrak{Sets}$ between one category $\mathcal{C}$ and the category of sets. Suppose that for every $c \in \mathcal{C}$ there is one equivalence relation '$\sim_c'$ defined over X_c, such that if $x \sim_c w$ and $f : c \to d$ is an arrow in $\mathcal{C}$ then $X_f(x) \sim_d X_f(w)$. Consider the set $A = \bigcup_{c\in\mathcal{C}} X_c$; if x is an element $\in X_d$ denote this element by x^d. With this notation over the set A let us define the following relation:

- given $x^d, w^c \in A$, we'll say that $x^d \rightsquigarrow w^c$ if there exists an arrow $f : c \to d$ such that $X_f(w) \sim_d x$.

Since $(A, \rightsquigarrow)$ is a preordered set we can also start to seek the set A' described in (**1**). To do that we will consider the relations given in (**a**) and (**b**) over A as before, although additionally we will need to impose some conditions over the category $\mathcal{C}$ to find some positive examples where A' exists.

Example A.0.17 *Let $\mathcal{C}$ be a small EI-category (that is, a category where all its endomorphisms are isomorphisms).*

1) Take $\mathcal{C}$ in such a way that $Obj(\mathcal{C}) < +\propto$. Consider a chain $\{[x_i^{c_i}]\}_{i\in I}$ in $A/\sim$ and suppose that there is a subsequence $[x_i^{c_i}] \leq [x_k^{c_k}] \leq [x_j^{c_j}]$ of the chain, with $c_i = c_j = c$. Then there are arrows $c \xrightarrow{f} c_k \xrightarrow{h} c$ in $\mathcal{C}$, such that $X_f(x_j)^{c_k} \sim_{c_k} x_k^{c_k}$ and $X_h(x_k)^c \sim_c x_i^c$. Notice that $x_i^c \sim_c X_{hf}(x_j)^c$ and since $\mathcal{C}$ is an EI-category there exists an arrow $\alpha : c \to c \in \mathcal{C}$ such that $\alpha hf = 1_c$, therefore applying α we have that $X_\alpha(x_i)^c \simeq_p x_j^c$ and then $[x_i^{c_i}] = [x_j^{c_j}]$.

This means that any chain in $(A/\sim, \leq)$ is finite and hence it has an upper bound. Then $A' \neq \emptyset$.

2) Let G be a group (may be infinite) and let $\mathcal{C}$ be the category with only one object and morphisms given by the elements of G. Note that '$\rightsquigarrow$' is already *an equivalence relation, since arrows in $\mathcal{C}$ are invertibles and have the same domain and codomain. Moreover every chain in $A/\rightsquigarrow$ has length* 1 *and therefore A' is formed by taking from every class in $A/\rightsquigarrow$ one representant (indeed this example is a particular case of* 1)*).*

3) *Let $\mathcal{C}$ be a small category for which there is a map $n : Obj(\mathcal{C}) \to \mathbb{Z}_{\geq 0}$ such that for every morphism $f : c \to d$ of $\mathcal{C}$ with $n_c \neq n_d$ it holds that $n_c < n_d$. If $c \in \mathcal{C}$ the number $n(c) = n_c$ will be called the* degree *of c.*

   Take a chain $\{[x_i^{n_{c_i}}]\}_{i \in I}$ in $A/\sim$ and suppose that there is a subsequence $[x_i^{n_{c_i}}] \leq [x_j^{n_{c_j}}]$ of the chain, with $n_{c_i} = n_{c_j} = n_c$. As in example 1) *we can conclude that $[x_i^{n_{c_i}}] = [x_j^{n_{c_j}}]$ and therefore we consider only chains where $n_{c_i} \neq n_{c_j}$, whenever $i \neq j$. So if $[x_i^{n_{c_i}}] \leq [x_j^{n_{c_j}}]$ there must be a map $f : c_j \to c_i$, such that $X_f(c_j) \sim_{c_i} x_i$ and since $n_c \neq n_d$ it holds that $n_{c_j} < n_{c_i}$. Therefore $[x_i^{n_{c_i}}] \leq [x_\tau^{n_{c_\tau}}]$, for every $i \in I$, where c_τ is such that $n_{c_\tau} = min\{n_{c_i}\}_{i \in I}$. Thus by Zorn's Lemma it holds that $A' \neq \emptyset$.*

   The category $\mathcal{C}$ will be called direct category *if every endomorphism is trivial.*

APPENDIX B

THE SMALL OBJECT ARGUMENT

The small object argument due to Quillen [22] is a tool that permits us to construct functorial factorizations with lifting properties. In order to develop this argument some results about transfinite compositions in categories are needed. Before introducing this machinery we will start with the exposition given by Dwyer and Spalinski in [8], which follows a sequence of very understandable reasonings that captures the essence of such argument.

Assume that $\mathcal{M}$ is a cocomplete category. Given a functor $X : \mathbb{Z}^+ \to \mathcal{M}$ and an object A in $\mathcal{M}$, the morphisms $i_n : X_n \to \underset{\to}{colim} X_k$ induce natural maps $i_{n,\sharp} : \mathcal{M}(A, X_n) \to \mathcal{M}(A, \underset{\to}{colim} X_k)$ defined by $i_{n,\sharp}(f) = i_n f$, which give the canonical map $\psi : \underset{\to}{colim} \mathcal{M}(A, X_n) \to \mathcal{M}(A, \underset{\to}{colim} X_k)$.

Definition B.0.18 ([8]) *An object A in $\mathcal{M}$ is said to be sequentially small if for every functor $X : \mathbb{Z}^+ \to \mathcal{M}$ the canonical map $\psi : \underset{\to}{colim} \mathcal{M}(A, X_n) \to \mathcal{M}(A, \underset{\to}{colim} X_k)$ is a bijection.*

Note that this bijection allows us to factorize any map g in $\mathcal{M}(A, \underset{\to}{colim} X_k)$ as $i_n g' = g$, where $g' : A \to X_n$ for some $n \in \mathbb{Z}^+$.

Take a set of maps $\mathcal{F} = \{f_i : A_i \to Bi\}_{i \in I}$ in $\mathcal{M}$ and suppose that $p : X \to Y$ is a map in $\mathcal{M}$ for which we want to factor p as a composite $X \to X' \to Y$ in such a way that the map $X' \to Y$ has the RLP (see Definition 0.2.5) with respect to all of the maps in $\mathcal{F}$. We can proceed as follows: for each $i \in I$ consider the set $S(1)$ formed by all the pairs of maps (g,h), such that the following diagram commutes

$$\begin{tikzcd} A_i \arrow[r, "g"] \arrow[d, "f_i"'] & X \arrow[d, "p"] \\ B_i \arrow[r, "h"'] & Y \end{tikzcd}$$

The *Gluing construction* $G^1(\mathcal{F},p)$ is defined by the following pushout diagram

$$\begin{tikzcd} \coprod\limits_{i\in I}\ \coprod\limits_{(g,h)\in S(1)} A_i \arrow[rr, "+_i+_{(g,h)}g"] \arrow[d, "\amalg f_i"'] & & X \arrow[d, "j_1"] \\ \coprod\limits_{i\in I}\ \coprod\limits_{(g,h)\in S(1)} B_i \arrow[rr, "\widehat{j}_1"'] & & G^1(\mathcal{F},p) \end{tikzcd}$$

By the universal property of colimits there is a map $p_1 : G^1(\mathcal{F},p) \to Y$ such that $p_1 j_1 = p$. Now repeat the process inductively for $n > 1$, that is, for each $i \in I$ consider the set $S(n-1)$ which contains all the pairs of maps (g,h), such that the following diagram commutes

$$\begin{tikzcd} A_i \arrow[r, "g"] \arrow[d, "f_i"'] & G^{n-1}(\mathcal{F},p) \arrow[d, "p_{n-1}"] \\ B_i \arrow[r, "h"'] & Y \end{tikzcd}$$

and as before define the map $p_n : G^n(\mathcal{F},p) \to Y$. What results is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "j_1"] \arrow[d, "p"'] & G^1(\mathcal{F},p) \arrow[r, "j_2"] \arrow[d, "p_1"'] & G^2(\mathcal{F},p) \arrow[r, "j_3"] \arrow[d, "p_2"'] & \ldots \arrow[r, "j_k"] & G^n(\mathcal{F},p) \arrow[r] \arrow[d, "p_n"'] & \ldots \\ Y \arrow[r, "-"'] & Y \arrow[r, "-"'] & Y \arrow[r, "-"'] & \ldots \arrow[r, "-"'] & Y \arrow[r, "-"'] & \ldots \end{tikzcd}$$

Let $G^\infty(\mathcal{F},p)$ the colimit of the upper row of the above diagram. There are natural maps $j_\infty : X \to G^\infty(\mathcal{F},p)$ and $p_\infty : G^\infty(\mathcal{F},p) \to Y$ such that $p_\infty j_\infty = p$.

Proposition B.0.19 ([8], *Proposition* 7.17**)**
In the above situation, suppose that for every $i \in I$ the object A_i of $\mathcal{M}$ is sequentially small. Then the map p_∞ has the RLP with respect to each of the maps in the family $\mathcal{F}$.

Proof. Consider the following commutative diagram

$$
\begin{tikzcd}
A_i \arrow[r, "g"] \arrow[d, "f_i"'] & G^\infty(\mathcal{F}, p) \arrow[d, "p_\infty"] \\
B_i \arrow[r, "h"'] & Y
\end{tikzcd}
$$

Since A_i is sequentially small, there exists an integer n such that the map g is the composite of a map $g' : A_i \to G^n(\mathcal{F}, p)$ with the map $G^n(\mathcal{F}, p) \to G^\infty(\mathcal{F}, p)$. Therefore the above commutative diagram can be enlarged as follows

$$
\begin{tikzcd}
 & A_i \arrow[r, dotted, "f_i"] \arrow[d, dotted, "\iota_{A_i}"] & B_i \arrow[d, dotted, "\iota_{B_i}"] & \\
 & \coprod_I \coprod_{S(n)} A_i \arrow[r, dotted, "\amalg f_i"] \arrow[d, dotted, "+_i +_{S(k)} g"] \arrow[dr, phantom, "\ulcorner", very near start] & \coprod_I \coprod_{S(n)} B_i \arrow[d, dotted, "\widehat{j}_{n+1}"] & \\
A_i \arrow[r, "g'"] \arrow[d, "f_i"'] & G^n(\mathcal{F}, p) \arrow[r, "j_{n+1}"] \arrow[d, "p_n"'] & G^{n+1}(\mathcal{F}, p) \arrow[r, "l"] \arrow[d, "p_{n+1}"'] & G^\infty(\mathcal{F}, p) \arrow[d, "p_\infty"'] \\
B_i \arrow[r, "h"'] \arrow[urr, dotted, "\widehat{j}_{n+1}\iota_{B_i}"'] & Y \arrow[r, "-"'] & Y \arrow[r, "-"'] & Y
\end{tikzcd}
$$

in which the composite of $lj_{n+1}g'$ is g. However the pair (g', h) contributes itself as an index in the construction of $G^{n+1}(\mathcal{F}, p)$ from $G^n(\mathcal{F}, p)$. Then by construction there exists a map $B_i \to G^{n+1}(\mathcal{F}, p)$ which makes the appropriate diagram commutative (that is, the map $\widehat{j}_{n+1}\iota_{B_i}$). Composing with the arrow $l : G^{n+1}(\mathcal{F}, p) \to G^\infty(\mathcal{F}, p)$ gives a lifting in the original square.

∎

It is possible to generalize the definition of smallness by considering sequences indexed by an ordinal, rather than just positive integers. Recall that an ordinal is the well-ordered set of all smaller ordinals. Every ordinal λ has a successor ordinal $\lambda + 1$ and an ordinal is said to be a *limit ordinal* if it is neither zero nor a successor ordinal.

Definition B.0.20 ([17], *Definition* 2.1.1) *Let $\mathcal{M}$ be a category closed under small colimits. If λ is an ordinal, then a λ-sequence in $\mathcal{M}$ is a functor $X : \lambda \to \mathcal{M}$, that is, a diagram in $\mathcal{M}$*

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \ldots \longrightarrow X_\beta \longrightarrow \ldots \qquad \beta < \lambda$$

such that for every limit ordinal $\gamma < \lambda$ the induced map $\underset{\beta<\lambda}{colim} X_\beta \to X_\gamma$ is an isomorphism.

We refer to the map $X_0 \to \underset{\beta<\lambda}{colim} X_\beta$ as the composition of the λ-sequence, though the composition is not unique, but only unique up to isomorphism under X, since the colimit is not unique. If $\mathcal{N}$ is a collection of morphisms of $\mathcal{M}$ and every map $X_\beta \to X_{\beta+1}$ for $\beta + 1 < \lambda$ is in $\mathcal{N}$, we refer to the composition $X_0 \to \underset{\beta<\lambda}{colim} X_\beta$ as a *transfinite composition* of maps of $\mathcal{N}$.

Definition B.0.21 ([16]) *A cardinal γ is regular if, whenever A is a set whose cardinal is less than γ and for every $a \in A$ there exists a set S_a whose cardinal is less than γ, the cardinal of the set $\underset{a\in A}{\cup} S_a$ is less than γ.*

Definition B.0.22 ([16]) *Let $\mathcal{M}$ be a cocomplete category and $\mathcal{N}$ a subcategory of $\mathcal{M}$.*

1) If κ is a cardinal, then an object W in $\mathcal{M}$ is κ-small relative to $\mathcal{N}$ if, for every regular cardinal $\lambda \geq \kappa$ and every λ-sequence

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \dots \longrightarrow X_\beta \longrightarrow \dots \qquad \beta < \lambda$$

in $\mathcal{M}$ such that the map $X_\beta \to X_{\beta+1}$ is in $\mathcal{N}$ for every ordinal β such that $\beta + 1 < \lambda$, the map of sets $\underset{\beta<\lambda}{colim} \mathcal{M}(W, X_\beta) \to \mathcal{M}(W, \underset{\beta<\lambda}{colim} X_\beta)$ is an isomorphism.

2) An object is small relative to $\mathcal{N}$ if it is κ-small relative to $\mathcal{N}$ for some cardinal κ, and it is small if it is small relative to $\mathcal{M}$.

If X is an object in $\mathbf{S}_*$ and κ is the first infinite cardinal greater than the cardinal of the set of nondegenerate simplices of X, then X is κ-small relative to the subcategory of inclusions. Thus, every simplicial set is small relative to the subcategory of inclusions.

Definition B.0.23 ([16], *Definition* 10.5.2**)** *Let $\mathcal{M}$ be a category and I a set of maps in $\mathcal{M}$.*

1) *The subcategory of I-injectives is the subcategory of maps that have the right lifting property (see Definition 0.2.5) with respect to every element of I. element*

2) *The subcategory of I-cofibrations is the subcategory of maps that have the left lifting property with respect to every I-injective. An object is I-cofibrant if the map to it from the initial object of $\mathcal{M}$ is an I-cofibration.*

For example if I is the set of inclusions $\dot{\triangle}[n] \to \triangle[n]$ in $\mathbf{S}$, then the injectives are the trivial fibrations and the I-cofibrations are the inclusions of simplicial sets (see Example 0.2.7). If J is the set of inclusions $\Lambda^k[n] \to \triangle[n]$ in $\mathbf{S}$, then the J-injectives are the Kan fibrations, and the J-cofibrations are the trivial cofibrations (see Proposition 0.2.6).

Definition B.0.24 ([16]) *Let $\mathcal{M}$ be a cocomplete category and I a set of maps in $\mathcal{M}$. The subcategory of relative I-cell complexes is the subcategory of maps that can be constructed as a transfinite composition of pushouts of elements of I. That is, if $f : A \to B$ is a relative I-cell complex, then there is an ordinal λ and a λ-sequence $X : \lambda \to \mathcal{M}$ such that f is the composition of X and such that, for every β for which $\beta + 1 < \lambda$, there is a pushout square as follows*

$$\begin{tikzcd} \coprod_{j\in J} C_j \arrow[r, "\coprod_{j\in J} g_j"] \arrow[d] & \coprod_{j\in J} D_j \arrow[d] \\ X_\beta \arrow[r] & X_{\beta+1} \end{tikzcd}$$

where $g_j \in I$ for every $j \in J$. An object is an I-cell complex if the map to it from the initial object of $\mathcal{M}$ is a relative I-cell complex.

Note that the identity map at A is the transfinite composition of the trivial 1-sequence A, so identity maps are relative I-cell complexes. If $f : A \to B$ is an isomorphism, then f is also the composition of the 1-sequence A, so f is a relative I-cell complex.

Proposition B.0.25 ([16], *Proposition* 10.5.10**)**
Let $\mathcal{M}$ be a cocomplete category and I a set of maps in $\mathcal{M}$. Then every relative I-cell complex is an I-cofibration.

Proposition B.0.26 ([16], *Proposition* 10.5.11**)**
Let $\mathcal{M}$ be a cocomplete category and I a set of maps in $\mathcal{M}$. Then a retract of a relative I-cell complex is an I-cofibration.

The reason for considering the theory of transfinite compositions and relative I-cell complexes is the small object argument, due to Quillen [22], though in fact we could get away with countable compositions in the examples that we have considered in this memory, we must not restrict ourselves to categories where countable composition will suffice. Furthermore, the localization process (the one discussed in [16]) will almost always require transfinite compositions. Using transfinite composition just means replacing ordinary induction arguments with transfinite induction arguments.

Definition B.0.27 ([16]) *If $\mathcal{M}$ is a category and I is a set of maps in $\mathcal{M}$, then we say that I permits the small object argument if the domains of the elements of I are small relative to I.*

Proposition B.0.28 (The small object argument, [16] *Proposition* 10.5.16**)** *Let $\mathcal{M}$ be a cocomplete category and I a set of maps in $\mathcal{M}$ that permits the small object argument, then there is a functorial factorization of every map in $\mathcal{M}$ into a relative I-cell complex followed by an I-injective.*

Proposition B.0.29 ([16])
Let $\mathcal{M}$ be a cocomplete category, I a set of maps in $\mathcal{M}$ that permits the small object argument and $i : A \to B$ an I-cofibration, then i is a retract of a relative I-cell complex.

Proof. We apply the factorization Proposition B.0.28 to factorize i as a composition $A \xrightarrow{j} A' \xrightarrow{p} B$, where j is a relative I-cell complex and p and I-njective. Since i is an I-cofibration the doted arrow in the following diagrams exists

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & A' \arrow[d, "p"] \\ B \arrow[r, "-"'] \arrow[ur, dotted, "g"] & B \end{tikzcd}$$

therefore the pair $(1_A, g)$ is a retraction from i to j.

∎

APPENDIX C

EQUIVALENT DEFINITION OF $\mathcal{C}$-FIBRE BUNDLE

As in Section 1.3 we will study the maps $p : X \to B \in \mathbf{S}^{\mathcal{C}}$, where B is the constant diagram to the simplicial set $B \in \mathbf{S}$.

Definition C.0.30 *Let F be a $\mathcal{C}$-diagram. A map $p : X \to B$ is said to be a $\mathcal{C}$-fibre bundle with fibre F if*

- *p is onto.*

- *For every n-$\mathcal{C}$-simplex v^c in B there is an isomorphism $\alpha_p(v^c)$ from $\delta_n^c \times F$ into $\delta_n^c \times_B X$, such that the following diagram commutates*

$$\begin{tikzcd}
\delta_n^c \times F \arrow[r, "\alpha_p(v^c)", "\cong"'] \arrow[d, "pr"'] & \delta_n^c \times_B X \arrow[r, dotted] \arrow[d, dotted, "pr"'] \arrow[dr, phantom, "\lrcorner", very near end] & X \arrow[d, "p"] \\
\delta_n^c \arrow[r, "-"'] & \delta_n^c \arrow[r, "v^c"'] & B
\end{tikzcd}$$

- *There exists a set of isomorphisms $\{\alpha_p(v^c)\}$ such that if $f : a \to b$ is an arrow in $\mathcal{C}$, then the following diagram commutes*

$$
\begin{tikzcd}
& \delta^a_n \times F \arrow[r, "\alpha_p(v^a)"] \arrow[dd, dotted] & \delta^a_n \times_B X \arrow[r] \arrow[dd, dotted] & X \arrow[dd, "p"] \\
\delta^b_n \times F \arrow[ur, "j_f \times 1"] \arrow[r, "\alpha_p(v^b)"] \arrow[dd] & \delta^b_n \times_B X \arrow[ur, "\widetilde{j}_f \times 1"] \arrow[urr] \arrow[dd, bend left] & & \\
& \delta^a_n \arrow[r, dotted, "1"] & \delta^a_n \arrow[r, "v^a"] & B \\
\delta^b_n \arrow[ur, "j_f"] \arrow[r, "1"] & \delta^b_n \arrow[ur, "j_f"] \arrow[urr, "v^b"'] & &
\end{tikzcd}
\tag{1}
$$

The set of isomorphisms $\{\alpha(v^c)\}$ is called an *atlas* of the $\mathcal{C}$-bundle. If F is fibrant, p will be called *$\mathcal{C}_K$-fibre bundle*. Notice that given two atlases $\{\alpha_p(v^c)\}$, $\{\widetilde{\alpha}_p(v^c)\}$ of p, then $\alpha_p(v^c)^{-1}\widetilde{\alpha}_p(v^c) \in \underline{aut}(F)_n$ (see Section 1.5) and conversely if for every $v \in B_n$ we choose $(\gamma(v^c)) \in \underline{aut}(F)_n$, then $\{\alpha_p(v^c)\gamma(v^c)\}$ is another atlas.

Proposition C.0.31
Let $p : X \to B$ be a $\mathcal{C}_K$-bundle with fibre F. Then p is a fibration.

As in Section 0.1.1 it is possible to define normalized and regular atlases in such a way that we can associate ceratin transformation elements to a given atlas. From Proposition 1.5.1 it holds that Definition 1.3.1 and Definition C.0.30 are equivalents.

BIBLIOGRAPHY

[1] V. Barrat, V. Gugenheim and J. Moore, *On Semisimplicial Fibre Bundles*, Amer. J. Math., **81** (1959), 639 – 657.

[2] A. Borel, *Seminar on Transformation Groups*, **46**, Princeton University Press, (1960).

[3] A. Bousfield and D. Kan, *Homotopy Limits, Completions and Localizations*, **304**, Springer-Verlag, (1972).

[4] E. Curtis, *Simplicial Homotopy Theory*, **6**, Advances in Math., (1971).

[5] F. Dror and W. Dwyer, *Automorphisms of Fibrations*, Proccedings A.M.S., **80** n.3 (1980), 491 – 494.

[6] W. Dwyer, P. Hirschhorn and D. Kan, *Model Categories and More General Abstract Homotopy Theory*, Preprint, (1997).

[7] W. Dwyer and D. Kan, *Function Complexes for Diagrams of Simplicial Sets*, Nederl. Akad.Wetensch. Indag. Math., **86** n.2 (1983), 139 – 147.

[8] W. Dwyer and J. Spalinski, *Homotopy Theories and Model Categories*, Elseiver Science, (1995).

[9] D. Farjoun, *Cellular Spaces, Null Spaces and Homotopy Colimits*, **1662**, Springer-Verlag, (1996).

[10] R. Flores and J. Scherer, *Cellularization of Classifying Spaces and Fusion Properties of finite groups*, J. London Math. Soc., (2007), 1 – 16.

[11] P. Fritsch, *Cellular Structures in Topology*, Cambridge University Press, (1990).

[12] P. Gabriel and M. Zisman, *Calculus of Fractions and Homotopy Theory*, Springer-Verlag, (1967).

[13] A. Gavira and N. Castellana, *Private communication*, (2012), .

[14] J. J. Goerss P., *Simplicial Homotopy Theory*, Birkhäuser Verlag, (1999).

[15] V. Gugenheim, *On Supercomplexes*, Trans. A.M.S., **85** n.1 (1957), 35 – 51.

[16] P. Hirschhorn, *Model Categories and Their Localizations*, **99**, American Mathematical Society, (2003).

[17] M. Hovey, *Model Categories*, **63**, American Mathematical Society, (1999).

[18] K. Kamps and T. Porter, *Abstract Homotopy and Simple Homotopy Theory*, World Scientific, (1997).

[19] R. Lewis, *Stacks and Formal Maps of Crossed Modules*, PhD. Thesis, University of Wales, Bangor, (2007).

[20] S. Mac Lane, *Categories for the Working Mathematician*, **5**, Springer-Verlag, (1971).

[21] J. May, *Simplicial Objects in Algebraic Topology*, Chicago Press, (1967).

[22] D. Quillen, *Homotopical Algebra*, reprinted by Springer-Verlag, (1967).

[23] C. Weibel, *An introduction to homological Algebra*, **38**, Cambridge University Press, (1994).

[24] C. Wojciech, *On the functors CW_A and P_A*, Duke Math,Journal, **84** n.3 (1996), 599 – 631.

INDEX