

TESI DOCTORAL

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Preface

The objective of this thesis is to study cages, constructions and properties of such families of graphs. For this, the study of graphs without short cycles plays a fundamental role in order to develop some knowledge on their structure, so we can later deal with the problems on cages. In this work we study structural properties such as the connectivity, diameter, and degree regularity of graphs without short cycles.

After what is considered to be the seminal paper of Graph Theory and Topology by Euler, on the well known Königsberg Bridge Problem (published in *Commentarii Academiae Scientiarum Petropolitanae* 8, 1741, pp. 128-140), the fundamental concepts of connection, paths and cycles were defined. The concept of distance arises naturally, and therefore that of diameter. In the framework of geometry, the ideas of symmetry and of regularity appear frequently.

Within the study of finite geometries all these concepts are related, for example, the incidence graphs of finite projective planes are bipartite regular graphs that do not contain cycles of length less than six and have diameter three.

It is natural to ask in general if there exists a graph that is regular of degree k (that is, all its vertices have degree k), not containing cycles of length less than a given length g , (i.e., of girth g). Such graphs with the minimum number of vertices are known as *cages*; they were introduced by Tutte [127] in 1947. In 1963, Erdős and Sachs [55] proved that $(k; g)$ -cages exist for any given values of k and g . Since then, large amount of research in cages has been devoted to their construction. For more information on this problem see the survey by Wong [128], or the survey by Holton and Sheeham [121], or the more recent one by Exoo and Jajcay [57].

Entire families of cages can be obtained from finite geometries, for example, the already mentioned graphs of incidence of projective planes of order q a prime power, are $(q + 1, 6)$ -cages. Also by using other incidence structures such as the generalized quadrangles or generalized hexagons, it can be obtained families of cages of girths 8 and 12.

Concerning the degree and the diameter, there is the concept of a Moore graph. A Moore

graph is a regular graph of degree k and diameter d whose number of vertices equals

$$1 + k \sum_{i=0}^{d-1} (k-1)^i.$$

An equivalent definition of a Moore graph is that it is a graph of diameter d with girth $2d + 1$. Moore graphs were named by Hoffman and Singleton [74] after Edward F. Moore, who posed the question of describing and classifying these graphs. As well as having the maximum possible number of vertices for a given combination of degree and diameter, Moore graphs have the minimum possible number of vertices for a regular graph with given degree and girth. That is, any Moore graph is a cage [54]. The formula for the number of vertices in a Moore graph can be generalized to allow a definition of Moore graphs with even girth (bipartite Moore graphs) as well as odd girth, and again these graphs are cages.

A basic structural property of a graph is its connectivity. Thus a classical problem is to find sufficient conditions for guaranteeing high connectivity of a graph. With this aim, conditions on the diameter, on the order, on the girth, and on the maximum and minimum degree have been given in the literature [15, 16, 49, 60, 59, 61, 58, 83, 101, 106].

In some sense, connectivity is a measure of the reliability of a network, therefore by refining the connectivity properties, we obtain more refined indices of reliability. Then two graphs with the same edge-connectivity λ may be considered to have different reliabilities, as a more refined index than the edge-connectivity, edge-superconnectivity is proposed in [39, 40]. Related to the superconnectivity of a graph, new parameters called *restricted connectivities* such as the parameter $\lambda'(G)$ were introduced in [56].

By relaxing the conditions that are imposed for the graphs to be cages, we can achieve more refined connectivity properties on these families and also we have an approach to structural properties of the family of graphs with more restrictions (i.e., the cages). Our aim, by studying such structural properties of cages is to get a deeper insight into their structure so we can attack the problem of their construction.

This thesis is organized as follows. The first chapter is a brief introduction to the subjects of this thesis together with basic definitions and examples. The second chapter is devoted to our contribution in solving the conjecture of Fu, Huang, and Rodger on cages [64]. Third chapter is about refined indexes of connectivity in relation with the girth, specifically studying restricted edge-connectivity of graphs with a given girth pair, and following with the edge superconnectivity of semiregular cages.

In the fourth chapter the problem of constructing small regular graphs with a given girth pair is addressed, also a question of Harary and Kovacs on the order of a $(k; g, h)$ -cage is answered in the affirmative for almost all cases. And some relations and results concerning the excess of a cage, are also presented. In the fifth chapter we deal with the problem of constructing a small regular graph with girth 7, we manage to construct a whole family of such

graphs, by using the incidence graph of a Generalized Quadrangle and some combinatorial properties of such graphs. The last chapter is devoted to presenting some conclusions and open problems arising from our study.

Chapter 1

Terminology and Introduction

1.1 Graph theory terminology

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For any $S \subset V(G)$, the subgraph *induced* by S is denoted by $G[S]$. For $u, v \in V(G)$, $d(u, v) = d_G(u, v)$ denotes the *distance* between u and v in G , that is, the length of a shortest (u, v) -path in G . For $S, F \subset V(G)$, $d(S, F) = d_G(S, F) = \min\{d(s, f) : s \in S, f \in F\}$ denotes the *distance* between S and F . For every $v \in V$ and every integer $t \geq 0$, $N_t(v) = \{w \in V(G) : d(w, v) = t\}$ denotes the *neighborhood* of v at distance t . If $S \subset V(G)$, then $N_t(S) = \{w \in V(G) : d(w, S) = t\}$. Observe that $N_0(S) = S$ for every subset S of vertices and, when $t = 1$, we put simply $N(v)$ and $N(S)$ instead of $N_1(v)$ and $N_1(S)$. The *degree* of a vertex v is $d(v) = |N(v)|$, whereas $\delta = \delta(G)$ is the *minimum degree* over all vertices of G , and Δ the *maximum degree*. A graph is called *k-regular* if all its vertices have the same degree k . The *diameter* $diam(G)$ is the maximum distance over all pairs of vertices in G and clearly $diam(G) < \infty$ if and only if G is connected. The graph G^C is called the *complement* of G and is defined as the graph with vertex set $V(G^C) = V(G)$ and edge set $E(G^C)$ such that $uv \in E(G)$ if and only if $uv \notin E(G^C)$. A subset $S \subset V(G)$ is an *independent set* of vertices if there is not any uv edge for all $u, v \in V(S)$.

Let G and H be two graphs. The Kronecker product of G and H , denoted as $G \otimes H$, is the graph with vertex set $V(G \otimes H) = V(G) \times V(H)$ and edge set $E(G \otimes H) = \{(u, v)(u', v') : uu' \in E(G) \text{ and } vv' \in E(H)\}$. Observe that $|V(G \otimes H)| = |V(G)| \cdot |V(H)|$; $|E(G \otimes H)| = 2 \cdot |E(G)| \cdot |E(H)|$ and, for every $(u, v) \in V(G \otimes H)$; its degree is $d_{G \otimes H}((u, v)) = d_G(u) \cdot d_H(v)$. This product (which is commutative and associative up to isomorphism) is variously known as *direct product* [42], *categorical product* [95], *tensor product* [47] and *graph conjunction* [35]. It is considered to be one of the most important of all graph products. Several applications and characteristics appear in [43, 47, 95, 112].

1.2 Connectivity and restricted connectivities

A graph G is called *connected* if every pair of vertices is joined by a path. If $S \subset V(G)$ and $G - S$ is not connected, then S is said to be a *cut set*. Certainly, every connected graph different from a complete graph has a cut set. A *component* of a graph G is a maximal connected subgraph of G . A (non-complete) connected graph is called *k -connected* if every cut set has cardinality at least k . The *connectivity* $\kappa = \kappa(G)$ of a (noncomplete) connected graph G is defined as the maximum integer k such that G is k -connected. The *minimum cut sets* are those having cardinality κ . The connectivity κ of a complete graph $K_{\delta+1}$ on $\delta + 1$ vertices is defined as $\kappa(K_{\delta+1}) = \delta$.

Instead of removing a subset $S \subset V(G)$ to disconnect G , a subset $W \subset E(G)$ may be taken. If $G - W$ is disconnected then W is an *edge-cut* of the graph G . Every connected graph on at least two vertices has an edge-cut. The *edge-connectivity* $\lambda = \lambda(G)$ of a graph G is the minimum cardinality of an edge-cut of G . The *minimum edge-cuts* are those having cardinality λ . If W is a minimum edge-cut of a connected graph G , then $G - W$ contains exactly two components. It is well known that, for every graph G , $\kappa \leq \lambda \leq \delta$ (Whitney's inequality) and when $\kappa = \delta$ ($\lambda = \delta$) we say that the graph is *maximally connected* (maximally edge-connected). Observe that the situation $\lambda < \delta$ is precisely the situation where no minimum edge-cut isolates a vertex.

As a more refined index than edge-connectivity *edge-superconnectivity* is proposed in [39, 40]. A subset of edges W is called *trivial* if it contains the set of edges incident with some vertex of the graph. Clearly, if $|W| \leq \delta - 1$, then W is nontrivial. A graph is said to be *edge-superconnected* if $\lambda = \delta$ and every minimum edge-cut is trivial. In Figure 1.1 the three edges in the middle form a non-trivial cutset, hence this graph is not edge-superconnected.

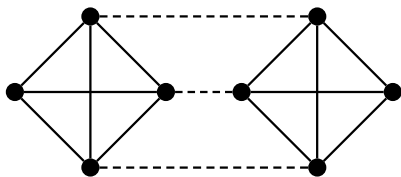


Figure 1.1: A non edge-superconnected graph

The *restricted edge-connectivity* $\lambda' = \lambda'(G)$ was introduced by Esfahanian and Hakimi [56] as the minimum cardinality over all *restricted edge-cuts* S , i.e., those for which there remain no isolated vertices in $G - S$. A restricted edge-cut S is called a *λ' -cut* if $|S| = \lambda'$. A connected graph G is called *λ' -connected* if λ' exists. Esfahanian and Hakimi [56] showed that

each connected graph G of order $n(G) \geq 4$ except a star, is λ' -connected and satisfies $\lambda' \leq \xi$, where $\xi = \xi(G)$ denotes the *minimum edge-degree* of G defined as $\xi(G) = \min\{d(u)+d(v)-2 : uv \in E(G)\}$. See Figure 1.2.

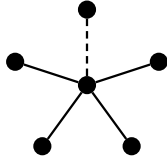


Figure 1.2: A star is not λ' -connected

Furthermore, a λ' -connected graph is said to be λ' -*optimal* if $\lambda' = \xi$. Recent results on this property are obtained in [17, 20, 70, 71, 116]. In Figure 1.3, the minimum edge degree of the graph on the left is $\xi = 3$, whereas for the graph on the right $\xi = 4$ and both have an edge cut with three edges, hence the first one is λ' -optimal and the other is not.

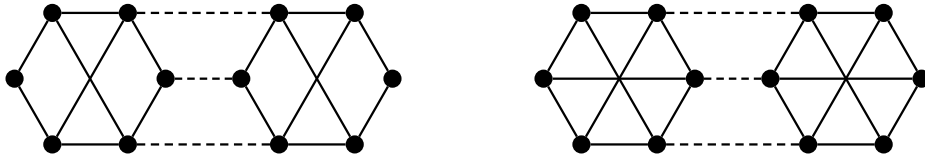


Figure 1.3: λ' -optimal vs. Non λ' -optimal

The parameter λ measures quantitatively the edge-connectivity of a graph (i.e. gives us information about the minimum number of edges needed to be removed in order to disconnect the graph).

When $\lambda = \delta$, where δ is the minimum degree, the parameter λ just says that the graph is maximally edge-connected, but it does not give any information about the edge-cut. However, the restricted edge-connectivity λ' gives us information on the structure of the edge-cuts of G .

The inequality $\lambda \leq \lambda'$ always holds, and when $\lambda' > \delta$ the graph is edge superconnected. Hellwig and Volkmann [72] provide a comprehensive survey of sufficient conditions for a graph to achieve lower bounds on λ and λ' .

1.3 Cages and generalized cages

1.3.1 Cages

The length of a shortest cycle in a graph G is called the *girth* of G . A k -regular graph with girth g is called a $(k; g)$ -*graph*. A $(k; g)$ -graph is said to be a $(k; g)$ -*cage* if it has the least possible number of vertices $n(k; g)$.

The cage problem asks for the construction of regular simple graphs with specified degree and girth and minimum order. This problem was first considered by Tutte [127]. In 1963, Erdős and Sachs [55] proved that $(k; g)$ -cages exist for any given values of k and g .

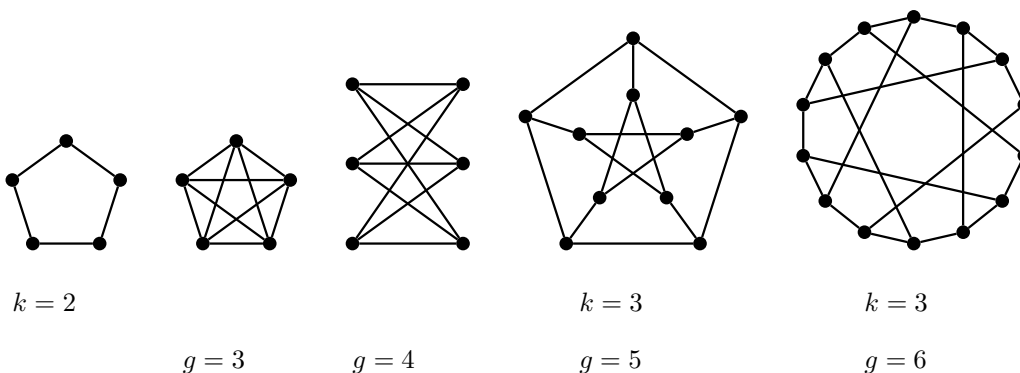


Figure 1.4: Some examples of cages

At about the same time, the study of Moore graphs, first proposed by Moore, was developed by Hoffman and Singleton [74]. Their study begins with the observation that a regular graph of degree k and diameter d has at most $1 + k + k(k - 1) + \dots + k(k - 1)^{d-1}$ vertices, and graphs that achieve this bound must have girth $g = 2d + 1$.

This observation can be turned around and make another regarding the order, n , of a regular graph with degree k and girth g . Such a graph is called an $(k; g)$ -graph. Counting the numbers of vertices in the distance partition with respect to a vertex when g is odd, and with respect to an edge when g is even, yields the lower bound $n_0(k; g)$ on the order of a $(k; g)$ -cage; it clearly depends on the parity of g :

For $k \geq 3$ and $g \geq 5$ the order $n(k; g)$ of a cage is bounded by

$$n(k; g) \geq n_0(k; g) = \begin{cases} 1 + k \sum_{i=0}^{(g-3)/2} (k-1)^i & g \text{ odd} \\ 2 \sum_{i=0}^{(g-2)/2} (k-1)^i & g \text{ even} \end{cases} \quad (1.1)$$

This bound is called the *Moore bound*. Graphs for which equality holds are called *Moore graphs*. The following theorem states for which parameters a Moore graphs does exist.

Theorem 1.3.1 [32, 51] *There exists a Moore graph of degree k and girth g if and only if*

(i) $k = 2$ and $g \geq 3$, *cycles*;

(ii) $g = 3$ and $k \geq 2$, *complete graphs*;

(iii) $g = 4$ and $k \geq 2$, *complete bipartite graphs*;

(iv) $g = 5$ and: $k = 2$, *the 5-cycle*;

$k = 3$, *the Petersen graph*;

$k = 7$, *the Hoffman-Singleton graph*;

and possibly $k = 57$;

(v) $g = 6, 8, 12$ and *there exists a symmetric generalized n -gon of order $k - 1$, $n = 3, 4, 6$.*

Regarding (v), the 4-gons of order $k - 1$ are called *generalized quadrangles* of order $k - 1$, and the 6-gons of order $k - 1$ are called *generalized hexagons* of order $k - 1$. All these objects are known to exist for all prime power values of $k - 1$ [33, 67, 124], and no example is known when $k - 1$ is not a prime power. As a particular case, finite projective planes are the generalized triangles (or 3-gons), they are known to exist whenever their order $k - 1 = q$ where q is a prime power. If q is not a prime power, $q \equiv 1, 2 \pmod{4}$, and q is not the sum of two integer squares, it has been proved that no finite projective plane exists [46].

A finite projective plane of order q has $q^2 + q + 1$ points and $q^2 + q + 1$ lines and satisfies the following properties:

- PP1: Any two points determine a line.
- PP2: Any two lines determine a point.
- PP3: Every point is incident with $q + 1$ lines.

- PP4: Every line is incident with $q + 1$ points.

Let us define Γ_q to be the incidence graph of a projective plane of order q ; it is defined as follows: Let $V(\Gamma_q) = (P, L)$, where P is the set of points and L is the set of lines of the projective plane. A point p and a line ℓ are adjacent in Γ_q if they are incident in the corresponding projective plane. Notice that Γ_q is regular of degree $q + 1$, has $2(q^2 + q + 1)$ vertices, diameter 3, and girth 6. Since the Moore bound for degree $q + 1$ and girth 6 is equal to the order of these graphs, the incidence graphs of projective planes are $(q + 1; 6)$ -cages.

For example, the $(3; 6)$ -cage Heawood graph is the incidence graph of the plane of order 2 known as the Fano plane (see Figure 1.5).

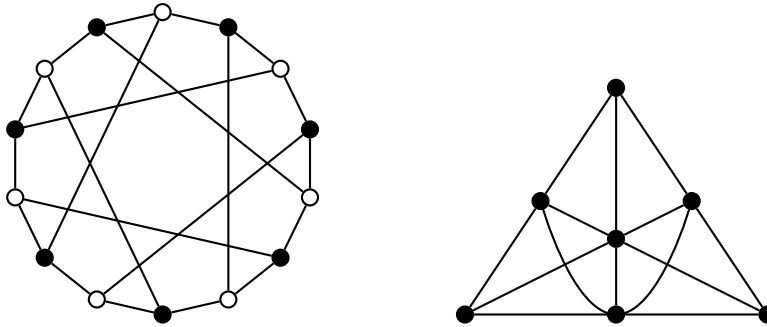


Figure 1.5: Heawood graph and Fano plane

In order to prove that a specific graph is a $(k; g)$ -cage, the non-existence of a smaller $(k; g)$ -graph has to be established. These lower bound proofs are in general very difficult and, consequently, in addition to the Moore graphs, very few cages are known.

Erdős and Sachs [55], Holton and Sheeham [121], and Fu, Huang and Rodger [64] independently proved the following monotonicity result concerning the girth of cages which turns out to be the foundation in exploring the connectivity of cages.

Theorem 1.3.2 [55, 64, 121] *If $k \geq 3$ and $3 \leq g_1 < g_2$ then $n(k; g_1) < n(k; g_2)$*

With this result Fu, Huang, and Rodger proved that $(k; g)$ -cages are 2-connected [64]. In addition they posed the following conjecture and proved it for $k = 3$.

Conjecture 1.3.1 ([64]) *Every $(k; g)$ -cage is k -connected.*

Another consequence of the theorem of monotonicity is the following upper bound for the diameter of a cage.

Theorem 1.3.3 [105] *The diameter of a $(k;g)$ -cage is at most g .*

1.3.2 Generalizations

One way to generalize the concept of cage is to allow graphs to have two girths. The *odd girth* (*even girth*) of a graph G is the length of a shortest odd (even) cycle in G . If there is no odd (even) cycle in G then the odd (even) girth of G is taken as ∞ . Let $g < h$, where g is the girth of a graph G , and h the length of a smallest cycle of different parity than g . Then (g, h) is called the *girth pair* of G .

Girth pairs were introduced by Harary and Kovács [62] in 1983. They developed a new generalization to the original question on cages, by relaxing the girth condition allowing the graphs to have two girths, the odd and the even.

A lower bound on the order of a regular graph with girth pair (g, h) , for odd g and even $h \geq g + 3$, can be found in [24].

Another generalization of a cage is to allow the graphs to have two degrees. A *semiregular graph* is a graph with degree set $\{k, k + 1\}$ and girth g . The concept of *semiregular cage*, $(k, k + 1; g)$ -cage, is defined analogous to that of a cage.

More generally, Chartrand, Gould and Kapoor [50] defined and proved the existence of $(D; g)$ -cages. In this case, the condition on the degrees is generalized by allowing the graphs to have a *degree set* D of different degrees. The concepts of $(D; g)$ -graph and $(D; g)$ -cage are defined analogous to those of $(k; g)$ -graph and $(k; g)$ -cage. As special cases, when $D = \{k\}$ we have the $(k; g)$ -cages, when $D = \{k, k + 1\}$ the semiregular cages, and when $D = \{k, m\}$ the bi-regular cages.

The most general definition of cage is when we allow the graphs to have a set of degrees $1 \leq k_1 < k_2 < \dots < k_t$ and the set of cycle lengths $3 \leq g_1 < g_2 < \dots < g_s < N$. Thus a $(k_1, k_2, \dots, k_t; g_1, g_2, \dots, g_s; N)$ -graph is a graph that contains vertices of degrees k_1, k_2, \dots, k_t but no other degrees and cycles of lengths g_1, g_2, \dots, g_s but no other cycles of length $< N$.

Chapter 2

Connectivity of cages: A conjecture of Fu, Huang, Rodger

Coming back to Conjecture 1.3.1: It is clearly true for $g = 3, 4$ because $(k; 3)$ -cages are the complete graphs and $(k; 4)$ -cages are the complete bipartite graphs.

Next, we summarize previous results concerning Conjecture 1.3.1.

Theorem 2.0.4 (i) [94] Every $(k; 6)$ -cage is k -connected.

(ii) [94] Every $(k; 8)$ -cage is k -connected.

(iii) [94] Every $(k; 5)$ -cage is k -connected when $k + 1$ is a prime power.

(iv) [94] Every $(k; g)$ -cage is 10-connected if $k \geq 10$ and $r + 2$ is a prime power.

(v) [11] Every $(k; g)$ -cage is k -connected if $g = 12$ or $g \in \{7, 11\}$ and $k - 1$ is a prime power.

(vi) [52, 78] Every $(k; g)$ -cage with $k \geq 3$ is 3-connected.

(vii) [93] Every $(k; g)$ -cage with $k \geq 4$ and $g \geq 10$ is 4-connected.

(viii) [85] Every $(k; g)$ -cage with $k \geq 3$ and odd girth $g \geq 7$ is at least $\sqrt{k + 1}$ -connected.

(ix) [84] Every $(k; g)$ -cage with $k \geq 3$ and even girth g is $(t + 1)$ -connected, where t is the largest integer such that $t^3 + 2t^2 \leq k$.

(x) [89] Every $(k; g)$ -cage with $k \geq 3$ and odd girth $g \geq 9$ is t -connected, where $(t - 1)^2 \leq k + \sqrt{k} - 2 < t^2$.

(xi) [89] Every $(k; g)$ -cage with $k \geq 4$ and odd girth $g \geq 10$ is $(t + 1)$ -connected, where t is the largest integer such that $\frac{t(t-1)^2}{4} + 1 + 2t(t - 1) \leq k$.

In this section we prove that all $(k; g)$ -cages are at least $k/2$ -connected for every odd girth $g \geq 7$ by means of a matrix technique which allows us to construct graphs without short cycles (Theorem 2.0.6). This lower bound on the vertex connectivity of cages is a new advance in proving the conjecture of Fu, Huang and Rodger which states that all $(k; g)$ -cages are k -connected. In order to prove Theorem 2.0.6, we must first state a theorem and recall Theorem 1.3.2 from the Introduction.

Theorem 2.0.5 *Let S be a cutset of a graph G and C a component of $G - S$. The following assertions hold:*

- (i) *([16, 60, 106]) If $|S| \leq \delta - 1$. There exists some vertex $x \in V(C)$ such that $d(x, S) \geq \lfloor (g - 1)/2 \rfloor$.*
- (ii) *[15] For every $x \in V(C)$ such that $d(x, S) \geq \lfloor (g - 1)/2 \rfloor$ it holds that $|N(x) \cap N_{\lfloor (g-1)/2 \rfloor}(S)| \geq \delta - |S|$.*

Theorem 2.0.6 *Let G be a $(k; g)$ -cage, for $k \geq 3$, $g \geq 7$ odd. Then G is $k/2$ -connected.*

Proof. Let G be a $(k; g)$ -cage. We know that G is 3-connected by Theorem 2.0.4, so the theorem is true for $r = 3, 4, 5, 6$. Suppose that G has vertex connectivity κ with $\kappa \leq k/2 - 1$.

Consider the set \mathcal{F} of all cut sets of G having cardinality κ . For every $F \in \mathcal{F}$, let C_F denote a smallest component of $G - F$. Take $S \in \mathcal{F}$ satisfying $|V(C_S)| \leq |V(C_F)|$ for every $F \in \mathcal{F}$. Then

$$|N(s) \cap V(C_S)| \geq 2, \text{ for all } s \in S.$$

Indeed, suppose $N(s) \cap V(C_S) = \{v\}$, for some $s \in S$. Then the set $F = \{v\} \cup (S - s)$ is a cutset belonging to \mathcal{F} and satisfying $|V(C_F)| < |V(C_S)|$, contradicting the definition of S .

From now on we will denote C_S simply by C .

By Theorem 2.0.5, there exists a vertex $u \in V(C)$, such that $d(u, S) \geq (g - 1)/2$; and from Theorem 2.0.5 (ii), we know that $|N(u) \cap N_{(g-1)/2}(S)| \geq k - |S| = k - \kappa \geq k - k/2 + 1 = k/2 + 1 \geq \kappa + 1$. Hence, let $U = \{u_1, \dots, u_\kappa, u_{\kappa+1}\} \subset N(u) \cap N_{(g-1)/2}(S)$ and consider the subgraph

$$G_1 = G[(V(C) - u - U) \cup S] - E[S], \quad (2.1)$$

where $E[S]$ denotes the edges joining vertices in the cutset S , (see Figure 2.1).

Let

$$\Omega = (N(u) - U) \cup \left(\bigcup_{i=1}^{\kappa+1} N(u_i) - u \right) \quad (2.2)$$

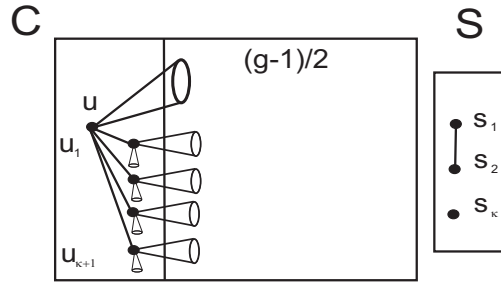


Figure 2.1: Subgraph G_1 .

Notice that the vertices in $\Omega \cup S$ satisfy:

$$\begin{aligned}
 d_{G_1}(w, v) &\geq g - 4, \text{ for every } w, v \in \Omega. \\
 d_{G_1}(s, t) &\geq 2, \text{ for every } s, t \in S. \\
 d_{G_1}(w, S) &\geq (g - 1)/2, \text{ for every } w \in \Omega \cap \mathcal{F}_C; \\
 d_{G_1}(w, S) &= (g - 3)/2, \text{ for every } w \in \Omega \setminus \mathcal{F}_C.
 \end{aligned}
 \tag{2.3}$$

As a consequence, every vertex in $G_1 - \Omega - S$ has degree k in G_1 , every vertex in Ω has degree $k - 1$ in G_1 , and every vertex $s \in S$ has degree $|N(s) \cap V(C)|$ in G_1 .

We will construct a k -regular graph with girth at least g by using two copies of the subgraph G_1 , as defined by (2.1). The order of the resulting graph will be

$$2|V(G_1)| = 2(|V(C)| - \kappa - 2 + |S|) = 2|V(C)| - 4 < |V(G)|,$$

the strict inequality due to C being a smallest component of $G - S$. Thus we will have constructed a $(k; g)$ -graph with fewer vertices than the number of vertices of the original graph G , and since G was assumed to be a $(k; g)$ -cage, we will obtain a contradiction (See Figure 2.2).

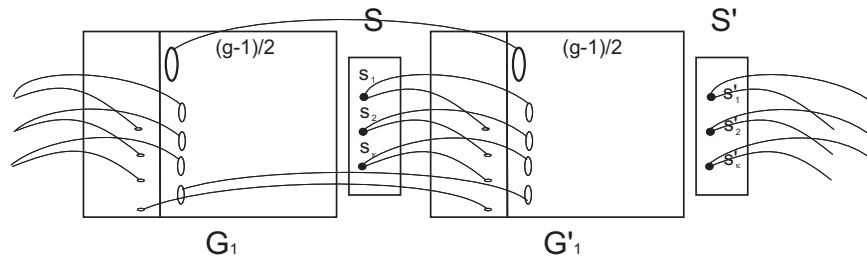


Figure 2.2: $(k; g)$ -graph with fewer vertices than G .

Let G'_1 be a copy of the subgraph G_1 . In G'_1 , the corresponding sets of interest will be denoted by U' , Ω' , C' , and S' . We must add the necessary edges between $\Omega \cup S$ and its copy

$\Omega' \cup S'$ in order to get regularity. The idea is that every vertex s in S will be matched with a vertex u'_i in U' and then connected to the appropriate neighbors of the vertex u'_i . Next, we will describe the construction.

Matrices for adding edges

In what follows we denote $S = \{s_1, \dots, s_\kappa\}$.

Remark 2.0.1 $|N(u_i) \cap N_{(g-3)/2}(s_j)| \leq 1$, for all $u_i \in U$.

Proof. This is clear because if $d_{G_1}(v, s_j) = d_{G_1}(v', s_j) = (g-3)/2$ for $v, v' \in N(u_i)$, $v \neq v'$, then the cycle $u_i, v, \dots, s_j, \dots, v', u_i$ has length less than g , which is a contradiction. ■

Let $u_i \in U$. If $d_{G_1}(v, S) = (g-3)/2$, for some $v \in N(u_i)$, let $m = \min\{j : d(v, s_j) = (g-3)/2\}$ and label $v := u_{i,m}$. The remaining labels are distributed arbitrarily among the neighbors v of u_i which clearly satisfy $d_{G_1}(v, S) \geq (g-1)/2$. Thus, $N(u_i) = \{u_{i,1}, u_{i,2}, \dots, u_{i,k} = u\}$, where the vertices $u_{i,j}$ for $\kappa+1 \leq j \leq k-1$ are arbitrarily chosen and eventually some j , $1 \leq j \leq \kappa$, has also been arbitrarily chosen. Remark 2.0.1 allows us to define a matrix which will play an important role in the proof of our main result.

Let $M = (a_{ij})$ be a matrix of order $(\kappa+1) \times \kappa$ defined as follows:

$$a_{ij} = \begin{cases} l & \text{if } d_{G_1}(u_{i,l}, s_j) = (g-3)/2; \\ 0 & \text{otherwise.} \end{cases} \quad (2.4)$$

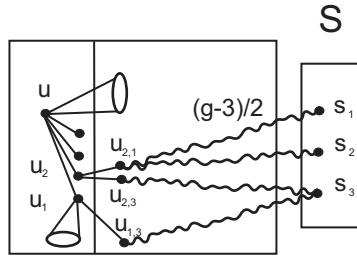
Note that $a_{ij} \leq j \leq \kappa$, and if $a_{ij} = l$ with $l < j$ and $a_{il} \neq 0$, then $a_{il} = l$. By way of example suppose that $M = (a_{ij})$ is such that $a_{ij} = j$. This means that $d_{G_1}(u_{i,t}, s_j) \geq (g-1)/2$, for all $t \neq j$ and $d_{G_1}(u_{i,j}, s_j) = (g-3)/2$.

For example, the following matrix is represented in Figure 2.3

$$\begin{pmatrix} 0 & 0 & 3 \\ 1 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Completing the degrees of the vertices s_i without creating short cycles

Let $\eta_j(M) = |\{i : a_{ij} \neq 0 \text{ for } a_{ij} \in M\}|$. When there is no possibility of confusion we may omit M and simply write η_j . Since for all $u_{i,j} \in N(u_i)$, $u_{h,j} \in N(u_h)$, $i \neq h$, $d_G(u_{i,j}, u_{h,j}) \geq 3$ because of the girth, we have $N_{(g-5)/2}(u_{i,j}) \cap N_{(g-5)/2}(u_{h,j}) \cap N(s_j) = \emptyset$

Figure 2.3: Example of a matrix $M = (a_{ij})$.

yielding $|N(s_j) \cap V(C)| \geq \eta_j$. Hence, for each vertex s_j , we need to add $k - |N(s_j) \cap V(C)| \leq k - \eta_j$ edges to complete its degree. Also, since $|N(s_j) \cap V(C)| \geq 2$ because of the minimality of C , we know that $k - |N(s_j) \cap V(C)| \leq k - 2$, even though $\eta_j = 0$.

Suppose that $s_l \in S$ has been matched with $u_h \in U$, and then some appropriate edges joining the vertex s_l with $u'_{h,t}$ and s'_l with $u_{h,t}$ have been added to the graph $G_1 \cup G'_1$ to complete the degree of the vertices s_l and s'_l . Let $\mathcal{L}_l^h \subseteq \{-1, -2, \dots, -(k-1)\}$ be such a set of labels, that for each $-t \in \mathcal{L}_l^h$, the edges $s_l u'_{h,t}$ and $s'_l u_{h,t}$ have been added to the graph $G_1 \cup G'_1$. Therefore $|\mathcal{L}_l^h| = k - |N(s_l) \cap V(C)|$. Let $\widehat{M} = (\widehat{a}_{ij})$ be the matrix obtained from $M = (a_{ij})$ as follows:

$$\widehat{a}_{ij} = \begin{cases} \{a_{ij}\} \cup \mathcal{L}_j^i & \text{if } s_j \in S \text{ is matched with } u_i \in U; \\ a_{ij} & \text{otherwise.} \end{cases}$$

Note that \widehat{M} has exactly one set-entry of the form $\{a_{ij}\} \cup \mathcal{L}_j^i$ in each column j and in each row i except for one row. By $\mathcal{L}_j^i(M)$ we will refer to the \mathcal{L}_j^i corresponding to the matrix \widehat{M} . In the following remark we establish which requirements should satisfy \widehat{M} to guarantee that after adding the edges indicated by the sets \mathcal{L}_j^i the resulting graph will have girth at least g .

Remark 2.0.2 *Given any matrix \widehat{M} suppose that if $h \in \widehat{a}_{ij}$ then $-h \notin \mathcal{L}_j^i$. Further suppose that \widehat{M} is free of any $\begin{pmatrix} -i & i \\ j & -j \end{pmatrix}$ submatrix, where $i, j \in \{1, 2, \dots, \kappa\} \cup \{-1, -2, \dots, -\kappa\}$, $i \neq j$, both with the same sign. Let \mathcal{W} be the corresponding new added edges. Then the graph $G_1 \cup G'_1 \cup \mathcal{W}$ has girth at least g and degrees $\{k-1, k\}$.*

Proof. Otherwise suppose that the graph $G_1 \cup G'_1 \cup \mathcal{W}$ has girth less than g . A cycle of length less than g must contain an even number of new edges joining G_1 with G'_1 , say $s_j u'_{i,t}$ and $s'_l u_{m,h}$, in such a way that $d_{G_1}(u_{m,h}, s_j) = (g-3)/2$ and $d_{G'_1}(u'_{i,t}, s'_l) = (g-3)/2$. Thus $-t \in \mathcal{L}_j^i$, $-h \in \mathcal{L}_l^m$ by the definition of the sets \mathcal{L}_j^i , and $a_{mj} = h$, $a_{il} = t$ by (2.4).

Therefore, if $j < l$ and $i < m$ then \widehat{M} contains the submatrix $\begin{pmatrix} -t & t \\ h & -h \end{pmatrix}$, contradicting the hypothesis of the remark. If $j = l$ then $t = a_{il} = a_{ij}$, implying that $t, -t \in \widehat{a}_{ij}$, contradicting the hypothesis. Further, if $i = m$ then $j = l$ because \mathcal{L}_j^i and \mathcal{L}_l^m are located neither two in the same column nor two in the same row. Hence we obtain again a contradiction. Therefore the graph $H = G_1 \cup G'_1 \cup \mathcal{W}$ has girth at least g and degrees $\{k-1, k\}$ because all vertex s_j, s'_j has degree k in H , and the vertices in $\Omega \cup \Omega'$ (see (2.2)) that have been not used to complete the degree of s_j and s'_j have degree $k-1$. ■

In what follows we will say that the matrix M is *solved* if we find \widehat{M} satisfying the conditions of Remark 2.0.2. The matrix \widehat{M} will be said to be a *solution* of M . Set $\mathcal{L}^* = \{-(\kappa+1), \dots, -(k-1)\}$.

Remark 2.0.3 *Let M be such that $\eta_j(M) = \kappa+1$ for all $j = 1, \dots, \kappa$. Then $|N(s_j) \cap V(C)| \geq \eta_j = \kappa+1$ meaning that for each vertex s_j we need to add $r - |N(s_j) \cap V(C)| \leq r - \kappa - 1$ edges to complete its degree. Hence $\mathcal{L}_j^i \subset \mathcal{L}^*$, $i, j = 1, \dots, \kappa$, and clearly \widehat{M} is a matrix satisfying Remark 2.0.2. Thus \widehat{M} is a solution for M .*

Remark 2.0.4 *Let $M = (a_{ij})$ be such that $a_{ij} = 0$ for all $i \neq i_1, i_2$, and suppose $i_1 \neq \kappa+1$. A solution for M is $\widehat{M} = (\widehat{a}_{ij})$ where for all $t \neq i_2$, $\widehat{a}_{tt} = \{a_{tt}\} \cup (\{-1, \dots, -\kappa\} \setminus \{-h\}) \cup \mathcal{L}^*$ with $h = a_{i_1 i_1}$ if $t = i_1$ and $a_{i_1 i_1} \neq 0$, or $h = t$ otherwise. If $i_2 \neq \kappa+1$ then $\widehat{a}_{\kappa+1, i_2} = (\{-1, \dots, -\kappa\} \setminus \{-i_2\}) \cup \mathcal{L}^*$; and $\widehat{a}_{ij} = a_{ij}$ otherwise.*

By way of example, suppose that $M = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, i.e., $\kappa = 3$ and $k \geq 7$. By Remark 2.0.4, a solution for M is

$$\widehat{M} = \begin{pmatrix} \{1\} \cup \{-2, -3\} \cup \mathcal{L}^* & 2 & 3 \\ 0 & \{-1, -3\} \cup \mathcal{L}^* & 2 \\ 0 & 2 & 3 \\ 0 & 0 & \{-1, -2\} \cup \mathcal{L}^* \end{pmatrix}$$

In the following remark we prove that any matrix $M_{(\kappa+1) \times \kappa}$ can be solved by solving another matrix $M'_{(\kappa+1) \times \kappa}$ which only differs from $M_{(\kappa+1) \times \kappa}$ in its zero entries.

Remark 2.0.5 *Let $M = (a_{ij})$ and suppose that $a_{i_0 j_0} \neq 0$. Let $M' = (a'_{ij})$ be such that*

$$a'_{ij} = \begin{cases} 0 & \text{if } i = i_0, j = j_0; \\ a_{ij} & \text{otherwise.} \end{cases}$$

and \widehat{M}' is a solution of M' . Then M can also be solved by using \widehat{M}' .

Proof. Let $\widehat{M}' = (\widehat{a'_{ij}})$ and denote by $\widehat{M} = (\widehat{a_{ij}})$ the solution for M we are looking for.

First suppose that $\widehat{a'_{i_0j_0}} \neq 0$. Since by hypothesis $a'_{i_0j_0} = 0$, it follows that $\widehat{a'_{i_0j_0}} = \mathcal{L}_{j_0}^{i_0}(M')$. In this case we define $\widehat{a_{ij}} = \widehat{a'_{ij}}$, for all $(i, j) \neq (i_0, j_0)$, and

$$\widehat{a_{i_0j_0}} = \begin{cases} \widehat{a'_{i_0j_0}} \setminus \{-a_{i_0j_0}\} & \text{if } -a_{i_0j_0} \in \widehat{a'_{i_0j_0}}; \\ \widehat{a'_{i_0j_0}} \setminus \{-t\} & \text{for some } -t \in \widehat{a'_{i_0j_0}} \text{ if } -a_{i_0j_0} \notin \widehat{a'_{i_0j_0}}. \end{cases}$$

Second, suppose $\widehat{a'_{i_0j_0}} = 0$. Then there exists $i_1 \neq i_0$ such that $\widehat{a'_{i_1j_0}} = \{a_{i_1j_0}\} \cup \mathcal{L}_{j_0}^{i_1}(M')$ and there exists $j_1 \neq j_0$ such that $\widehat{a'_{i_0j_1}} = \{a_{i_0j_1}\} \cup \mathcal{L}_{j_1}^{i_0}(M')$. In this case we define $\widehat{a_{ij}} = \widehat{a'_{ij}}$ for all $(i, j) \notin \{(i_0, j_0), (i_1, j_0)\}$, $\widehat{a_{i_0j_0}} = a_{i_0j_0}$ and

$$\widehat{a_{i_1j_0}} = \begin{cases} \widehat{a'_{i_1j_0}} \setminus \{-a_{i_1j_1}\} & \text{if } -a_{i_1j_1} \in \widehat{a'_{i_1j_0}}; \\ \widehat{a'_{i_1j_0}} \setminus \{-t\} & \text{for some } -t \in \widehat{a'_{i_1j_0}} \text{ if } -a_{i_1j_1} \notin \widehat{a'_{i_1j_0}}. \end{cases}$$

It is not difficult to check that \widehat{M} (the solution for M) satisfies the conditions of Remark 2.0.2. ■

Now we are ready to construct a graph $G_1 \cup G'_1 \cup W$ as that given by Remark 2.0.2. Let $M = (a_{ij})$ be the matrix given by (2.4). Let us consider the matrix $M^0 = (a_{ij}^0)$ such that $a_{ij}^0 = a_{ij}$, $i = 1, 2$, and $a_{ij}^0 = 0$ otherwise. To solve M^0 we use Remark 2.0.4. Then we solve the matrix M by replacing one zero entry $\widehat{a_{ij}^0} = 0$ from M^0 by the corresponding $a_{ij} \neq 0$ from M applying recursively Remark 2.0.5, until arriving to a matrix $M^p = M$, which is solved by $\widehat{M^{p-1}}$.

By way of example suppose that $M = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 0 & 0 \\ 0 & 2 & 3 \end{pmatrix}$, i.e., $\kappa = 3$ and $k \geq 7$. Then:

$$M^0 = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \widehat{M^0} = \begin{pmatrix} \{1\} \cup \{-2, -3\} \cup \mathcal{L}^* & 1 & 3 \\ 1 & 2 & 2 \\ 0 & 0 & \{-1, -2\} \cup \mathcal{L}^* \\ 0 & \{-1, -3\} \cup \mathcal{L}^* & 0 \end{pmatrix}$$

$$M^1 = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \widehat{M^1} = \begin{pmatrix} \{1\} \cup \{-2\} \cup \mathcal{L}^* & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 0 & \{-1, -2\} \cup \mathcal{L}^* \\ 0 & \{-1, -3\} \cup \mathcal{L}^* & 0 \end{pmatrix}$$

$$M^2 = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \widehat{M^2} = \begin{pmatrix} \{1\} \cup \{-2\} \cup \mathcal{L}^* & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 0 & \{-1, -2\} \cup \mathcal{L}^* \\ 0 & \{2\} \cup \{-1\} \cup \mathcal{L}^* & 0 \end{pmatrix}$$

$$M^3 = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 0 & 0 \\ 0 & 2 & 3 \end{pmatrix} \widehat{M^3} = \begin{pmatrix} \{1\} \cup \{-2\} \cup \mathcal{L}^* & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 0 & \{-1\} \cup \mathcal{L}^* \\ 0 & \{2\} \cup \{-1\} \cup \mathcal{L}^* & 3 \end{pmatrix}$$

In this way we construct a graph $H = G_1 \cup G'_1 \cup W$ of girth at least g and degrees $k - 1$, k . To finish the proof, note that the only vertices having degree $k - 1$ in H are the vertices in $(\Omega \cup \Omega') \setminus N_H(S \cup S')$. Observe that, for all $w \in \Omega$ and $v' \in \Omega'$ of degree $k - 1$ in H , there exists (in H) a path joining these two vertices, namely, $w \cdots s_j u'_{i,t} \cdots v'$ at distance

$$d_H(w, v') \geq d_{G_1}(w, s_j) + 1 + d_{G'_1}(u'_{i,t}, v') \geq (g - 3)/2 + 1 + g - 4 \geq g - 1,$$

due to (2.3) and because, by hypothesis, $g \geq 7$. Therefore we construct a $(k; g)$ -graph by adding to H a matching joining every vertex $\Omega \setminus N_H(S \cup S')$ with its corresponding in Ω' . This new $(k; g)$ -graph has fewer vertices than G which is a contradiction to the monotonicity Theorem.

Hence we conclude that $\kappa \geq k/2$. ■

Chapter 3

Restricted Connectivities

3.1 A condition on graphs with a girth pair to be λ' -optimal

We use the following notation introduced in [20]. Let U and F be vertex subsets of a graph G . Then $[U, F]$ stands for the set of edges $\{uf \in E : u \in U, f \in F\}$. If $U = \{u\}$ then we write simply $[u, F]$ instead of $[\{u\}, F]$.

Let G be a graph. We will denote an edge cut W by $[W_0, W_1]$, where H_i is a component of $G - W$ and $W_i \subset V(H_i)$ is the set of vertices of H_i which are incident with some edge in W . Let

$$\mu_i = \max\{d(x, W_i) : x \in V(H_i)\}, \quad i = 0, 1.$$

For any vertex $v \in V(H_i)$ and an edge cut $W = [W_0, W_1]$, we define the following sets:

$$S^-(v) = \begin{cases} \{z \in N(v) : d(z, W_i) = d(v, W_i) - 1\} & \text{if } v \notin W_i; \\ z \in W_{i+1} \cap N(v) & \text{if } v \in W_i. \end{cases}$$

$$S^+(v) = \{z \in N(v) : d(z, W_i) = d(v, W_i) + 1\}$$

$$S^=(v) = \{z \in N(v) : d(z, W_i) = d(v, W_i)\}.$$

The following result was the starting point for an extensive study of connectivity and refined measures of connectivity for graphs relating the diameter and the girth of a graph, guaranteeing maximal edge connectivity ($\lambda = \delta$).

Proposition 3.1.1 [106] *Let G be a graph with minimum degree $\delta \geq 2$ and girth g .*

Then $\lambda = \delta$ if $\text{diam}(G) \leq \begin{cases} g-2 & \text{if } g \text{ is even;} \\ g-1 & \text{if } g \text{ is odd.} \end{cases}$

Note that λ' -optimality implies edge superconnectivity and this implies maximal edge-connectivity.

Hence, it would be a stronger result to prove that a graph G is λ' -optimal under the assumptions of Proposition 3.1.1. This is what Balbuena, Garca-Vazquez and Marcote achieved in the next theorem.

Theorem 3.1.1 [20] *Let G be a graph with minimum degree $\delta \geq 2$ and girth g . Then G is λ' -optimal if $\text{diam}(G) \leq g - 2$.*

Figure 3.1 presents an example which shows that the hypothesis on the diameter in Theorem 3.1.1 is best possible.

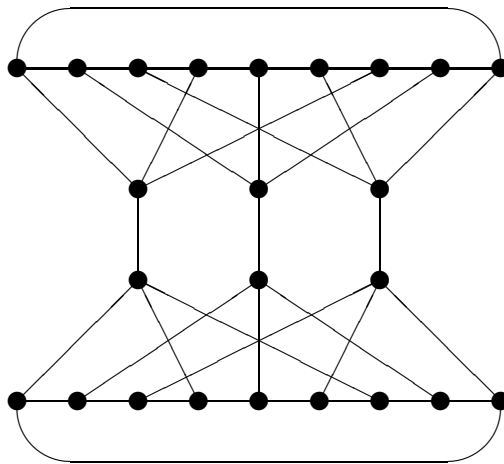


Figure 3.1: A non λ' -optimal graph with $\text{diam}(G) = 4$, $g = 5$ and $\lambda' = 3$

Another way of strengthening Proposition 3.1.1 is by considering graphs with a given girth pair (g, h) and proving maximal edge-connectivity for such graphs, as in the following theorem.

Theorem 3.1.2 [19] *Let G be a graph of minimum degree $\delta \geq 3$, girth pair (g, h) , odd g and even h with $g + 3 \leq h < \infty$. Then*

(i) $\lambda = \delta$ if $\text{diam}(G) \leq h - 3$;

- (ii) $\lambda = \delta$ if $\text{diam}(G) \leq h - 2$ and for all pairs of vertices at distance $d(u, v) = h - 2$ the induced subgraph $G[N_{(h-2)/2}(u) \cap N_{(h-2)/2}(v)]$ has edges;
- (iii) $\lambda = \delta$ if $\text{diam}(G) \leq h - 2$ and G is δ -regular with $\delta \geq 4$ even.

This is an improvement for it shows that maximal connectivity can be obtained by bounding the diameter not by the girth g of the graph but in this case by the even girth $h - 3$, which can be very very big even though g is small.

In this section we put together both generalizations thus obtaining λ' -optimality for graphs with a girth pair (g, h) . A key element for obtaining λ' -optimality in Theorem 3.1.1 is the following proposition.

Proposition 3.1.2 [20] *Let G be a λ' -connected graph with girth g and minimum degree $\delta \geq 2$. Let $[W_0, W_1]$ be a λ' -cut.*

Then if G is non λ' -optimal, there exists some vertex $u \in H_i$ such that $d(u, W_i) \geq \lceil (g - 3)/2 \rceil$.

Inspired by the aforementioned result, the following proposition shows that the even girth h is a suitable index in order to study how far away a vertex of a non λ' -optimal graph may be from a cutset.

Proposition 3.1.3 *Let G be a λ' -connected graph with girth pair (g, h) , odd g and even h such that $g + 3 \leq h < \infty$. Let $[W_0, W_1]$ be a λ' -cut.*

If G is non λ' -optimal then there exists a vertex $u \in H_i$ such that

- (i) $d(u, H_i) \geq 1$ if $\delta \geq 3$ and G has no triangle with all its vertices of degree 3;
- (ii) $d(u, H_i) \geq (h - 4)/2$ if $g \geq 5$ and $\delta \geq 4$ or $g = 3$ and $\delta \geq 5$.

Proof. Let G be a λ' -connected graph and $[W_0, W_1]$ be a λ' -cut. We will do the proof only for H_0 , the proof for H_1 is similar.

(i) Suppose that $\mu_0 = 0$. This implies that every vertex of H_0 is an end of some edge in $[W_0, W_1]$, that is, $H_0 = W_0$. Moreover, since G is λ' connected there exists an edge uv in C . Note that $(N(v) - u) \cup (N(u) - v) \subset W_0 \cup W_1$ and that $|(N(v) - u) \cap (N(u) - v)| \leq 1$, since $h \geq 6$. If $(N(v) - u) \cap (N(u) - v) = \emptyset$ then

$$\begin{aligned} \lambda'(G) &\geq |[v, W_1]| + |(N(v) - u) \cap W_0, W_1| + |[u, W_1]| + |(N(u) - v) \cap W_0, W_1| \\ &= d(v) + d(u) - 2 \geq \xi(G), \end{aligned}$$

which is a contradiction. Thus, $(N(v) - u) \cap (N(u) - v) = \{z\}$ yielding that u, v, z induce a triangle in G , i.e., $g = 3$.

Observe that the sets $(N(v) \cap W_0) \setminus \{u, z\}$, $(N(u) \cap W_0) \setminus \{v, z\}$, $(N(z) \cap W_0) \setminus \{u, v\}$ and $\{u, v, z\}$ are pairwise disjoint since G has no cycle of length four. Therefore,

$$\begin{aligned} \lambda'(G) &\geq |[v, W_1]| + |[(N(v) \cap W_0) \setminus \{u, z\}, W_1]| + |[u, W_1]| \\ &\quad + |[(N(u) \cap W_0) \setminus \{v, z\}, W_1]| + |[z, W_1]| + |[(N(z) \cap W_0) \setminus \{u, v\}, W_1]| \\ &\geq |N(v) \setminus \{u, z\}| + |N(u) \setminus \{v, z\}| + |N(z) \setminus \{u, v\}| \\ &\geq d(v) - 2 + d(u) - 2 + d(z) - 2 \\ &= d(v) + d(u) + d(z) - 6, \end{aligned}$$

which is greater than $\xi(G)$ because u, v or z must have degree at least four by hypothesis when $g = 3$, leading to a contradiction. Thus $\mu_0 \geq 1$ and item (i) holds.

(ii) Assume by way of contradiction that $1 \leq \mu_0 \leq (h-6)/2$ (i.e., $h \geq 8$). Let us choose a vertex $u \in N_{\mu_0}(W_0) \cap H_0$ such that $|S^-(u)| \leq |S^-(v)|$, for all $v \in N_{\mu_0}(W_0) \cap H_0$, and denote by $\delta_{N(u)} = \min\{d(v) : v \in N(u)\}$. Let us consider the sets $A = N_2(u) \cap N_{\mu_0}(W_0)$, $B = N_2(u) \cap N_{\mu_0-1}(W_0)$ and $D = N_2(u) \setminus (A \cup B)$. Note that $N_{\mu_0-1}(B) \cap W_0 = B$ if $\mu_0 = 1$ and $|N_{\mu_0-1}(B) \cap W_0| \geq |B|$ if $\mu_0 \geq 2$, otherwise an even cycle of length at most $2\mu_0 + 2 \leq h - 4$ would be created. Also observe that $|D| \geq 1$ and $D \subset W_1$ if $\mu_0 = 1$. Two cases need to be distinguished.

Case 1. Suppose that $|S^-(u)| \geq 2$. Therefore $|S^-(v)| \geq 2$, for every $v \in N_{\mu_0}(W_0) \cap H_0$, due to the way u has been chosen.

In particular, $|N_{\mu_0}(a) \cap W_0| \geq |S^-(a)| \geq 2$, for all $a \in A$, yielding $| (N_{\mu_0}(A) \setminus N_{\mu_0}(u)) \cap W_0 | \geq |A|$ for if not, an even cycle of length at most $2\mu_0 + 4 \leq h - 2$ appears. Moreover, for the same reason, $|[N_{\mu_0}(u) \cap W_0, W_1]| \geq |D|$. Hence,

$$\begin{aligned} \xi(G) - 1 &\geq |[W_0, W_1]| \\ &\geq |[N_{\mu_0}(u) \cap W_0, W_1]| + |[(N_{\mu_0}(A) \setminus N_{\mu_0}(u)) \cap W_0, W_1]| \\ &\geq |D| + |A| \\ &= (|N_2(u)| - |A| - |B|) + |A| \\ &\geq \begin{cases} \sum_{v \in N(u)} (d(v) - 1) - |B| & \text{if } g \geq 5; \\ \sum_{v \in N(u)} (d(v) - 2) - |B| & \text{if } g = 3. \end{cases} \end{aligned} \tag{3.1}$$

If $g = 3$ and $\delta \geq 5$, from (3.1) it follows that

$$d(u) + \delta_{N(u)} - 3 \geq \xi(G) - 1 \geq \sum_{v \in N(u)} (d(v) - 2) - |B| \geq d(u)(\delta_{N(u)} - 2) - |B|, \text{ yielding that}$$

$$|B| \geq d(u)(\delta_{N(u)} - 2) - d(u) - \delta_{N(u)} + 3 = (d(u) - 1)(\delta_{N(u)} - 3).$$

Taking into account that $(a - 1)(b - 3) \geq a + b - 2$, for $a, b \geq 5$ two integer numbers, we have

$$|B| \geq d(u) + \delta_{N(u)} - 2.$$

Hence, $\lambda'(G) \geq |W_0| \geq |N_{\mu_0-1}(B) \cap W_0| \geq |B| \geq d(u) + \delta_{N(u)} - 2 \geq \xi(G)$, against the fact that G is non λ' -optimal. If $g \geq 5$, by hypothesis $\delta \geq 4$. From (3.1) it follows that $|B| \geq (d(u) - 1)(\delta_{N(u)} - 2) + 1$. Thus, we arrive again to contradiction by applying $(a - 1)(b - 2) \geq a + b - 3$ for two integers $a, b \geq 4$.

Case 2. $|S^-(u)| = 1$.

Let us denote $S^-(u) = \{w\}$ and $A_w = N(w) \cap N_{\mu_0}(W_0)$, $A' = A \setminus A_w$, $B_w = N(w) \cap N_{\mu_0-1}(W_0)$ and $B' = B \setminus B_w$.

Let us prove the following claim.

$$\text{Claim 1. } |A'| + |B'| \geq \begin{cases} 2\xi(G) - 4 & \text{if } g = 3; \\ 2\xi(G) - 3 & \text{if } g \geq 5. \end{cases}$$

If $g \geq 5$, by hypothesis, $\delta \geq 4$. Then

$$(d(u) - 1)(\delta_{N(u)} - 1) \leq \sum_{v \in N(u)-w} (d(v) - 1) \leq |A'| + |B'|,$$

yielding that $|A'| + |B'| \geq 2\xi(G) - 3$ because $ab \geq 2(a + b) - 3$ holds for any two integers $a, b \geq 3$. Furthermore, for $g = 3$ and $\delta \geq 5$, as $h \geq 8$, we get

$$(d(u) - 1)(\delta_{N(u)} - 2) \leq \sum_{v \in N(u)-w} (d(v) - 2) \leq |A'| + |B'|.$$

Thus $|A'| + |B'| \geq 2\xi(G) - 4$ because $(a - 1)(b - 2) \geq 2(a + b - 2) - 4$ holds for any integers $a, b \geq 5$. \square

Note that the sets $N_{\mu_0}(A') \cap W_0$ and $N_{\mu_0-1}(w) \cap W_0$ are disjoint, otherwise an even cycle of length at most $2\mu_0 + 2 \leq h - 4$ would be created. Furthermore, by the same reason, $|[N_{\mu_0}(A') \cap W_0, W_1]| \geq |N_{\mu_0}(A') \cap W_0| \geq |A'|$ and $|[N_{\mu_0-1}(w) \cap W_0, W_1]| \geq |D|$. Thus

$$|A'| + |D| \leq |[N_{\mu_0}(A') \cap W_0, W_1]| + |[N_{\mu_0-1}(w) \cap W_0, W_1]| \leq \xi(G) - 1. \quad (3.2)$$

Also, since the sets $N_{\mu_0-1}(B') \cap W_0$ and $N_{\mu_0-1}(B_w) \cap W_0$ are pairwise disjoint, and $|N_{\mu_0-1}(B') \cap W_0| \geq |B'|$ and $|N_{\mu_0-1}(B_w) \cap W_0| \geq |B_w|$, hence

$$|B'| + |B_w| \leq |N_{\mu_0-1}(B') \cap W_0| + |N_{\mu_0-1}(B_w) \cap W_0| \leq \xi(G) - 1. \quad (3.3)$$

If $g \geq 5$, from (3.2), (3.3), and *Claim 1* it follows that $2\xi(G) - 2 \leq |A'| + 1 + |B'| \leq |A'| + |D| + |B'| + |B_w| \leq 2\xi(G) - 2$, yielding that all the above inequalities are equalities. That is,

$$\begin{aligned} |D| + |B_w| &= 1 \\ |A'| &= \xi(G) - 2 \\ |B'| &= \xi(G) - 1. \end{aligned} \tag{3.4}$$

If $g = 3$, from (3.2), (3.3), and *Claim 1* it follows that $2\xi(G) - 3 \leq |A'| + 1 + |B'| \leq |A'| + |D| + |B'| + |B_w| \leq 2\xi(G) - 2$, yielding that

$$\begin{aligned} 1 &\leq |D| + |B_w| \leq 2 \\ \xi(G) - 3 &\leq |A'| \leq \xi(G) - 2 \\ \xi(G) - 2 &\leq |B'| \leq \xi(G) - 1. \end{aligned} \tag{3.5}$$

Note that $A_w \cup B_w \cup D$ is a partition of $N(w) - u$, and $4 \leq |N(w) - u| = |A_w| + |B_w| + |D| = |A_w| + 2$ if $g = 3$ and $\delta \geq 5$, and $3 \leq |N(w) - u| = |A_w| + |B_w| + |D| = |A_w| + 1$. Thus $|A_w| \geq 2$.

Note that $|A_w \cap N(u)| \leq 1$ because G has no cycles of length 4. Thus there exists a vertex $a_w \in A_w \setminus N(u)$ such that there is $z \in S^=(a_w)$ with $z \notin N(u) \cup \{u\}$. As the sets $N_{\mu_0-1}(B') \cap W_0$ and $N_{\mu_0}(z) \cap W_0$ are disjoint, then by (3.4) and (3.5), we have

$$\xi(G) = |B'| + 1 \leq |N_{\mu_0-1}(B') \cap W_0| + |N_{\mu_0}(z) \cap W_0| \leq \xi(G) - 1,$$

which is a contradiction. Hence $S^=(a_w) = \emptyset$, and $|S^-(a_w)| \geq 4$. This implies, as $N_{\mu_0-1}(S^-(a_w)) \cap W_0$ and $N_{\mu_0}(A') \cap W_0$ are disjoint, and applying (3.4) and (3.5), that

$$\xi(G) \leq |A'| + 4 \leq |A'| + |S^-(a_w)| \leq |N_{\mu_0}(A') \cap W_0| + |N_{\mu_0-1}(S^-(a_w)) \cap W_0| \leq \xi(G) - 1$$

a contradiction, which finishes the proof. \blacksquare

As a consequence of Proposition 3.1.3, the following theorem provides a sufficient condition for a graph with girth pair (g, h) to be λ' -optimal.

Theorem 3.1.3 *Let G be a λ' -connected graph of minimum degree δ and girth pair (g, h) , odd g and even h with $g + 3 \leq h < \infty$. Then G is λ' -optimal if the diameter is*

- (i) $\text{diam}(G) \leq 2$ if $g \geq 5$ or $g = 3$, $\delta \geq 3$ and G has no triangle with all its vertices of degree 3;
- (ii) $\text{diam}(G) \leq (h - 4)/2$ if $g \geq 5$ and $\delta \geq 4$ or $g = 3$ and $\delta \geq 5$.

Proof. We will only do the proof of (ii), the proof of (i) is analogous. Suppose that G is non λ' -optimal and consider a λ' -cut $[W_0, W_1]$. By Proposition 3.1.3, there exists a vertex $u \in H_0$ such that $d(u, W_0) \geq (h - 4)/2$ and there exists a vertex $v \in V(W_1)$ such that $d(W_1, v) \geq (h - 4)/2$. Hence $diam(G) \geq d(u, W_0) + 1 + d(W_1, v) \geq h - 3$, against the hypothesis $diam(G) \leq h - 4$. ■

3.1.1 Polarity graphs

Related to the diameter and girth there is a very nice characterization of the finite simple graphs with diameter two and no 4-cycles, due to Bondy, Erdős and Fajtlowicz [41]. They showed that every such graph falls into one of three well defined classes; it is either a Moore graph of diameter two, a polarity graph, or a graph that contains a vertex adjacent to all the other vertices. Next, we present such characterization, and later we show that polarity graphs are λ' -optimal as a corollary of Theorem 3.1.3.

Definition 3.1.1 *Let \mathcal{P} be a finite projective plane, and let π be a polarity of \mathcal{P} , that is, a one-to-one mapping of points onto lines such that $p' \in \pi(p)$ whenever $p \in \pi(p')$. The polarity graph $G(\mathcal{P}, \pi)$ is the graph whose vertex set is the set of points of \mathcal{P} and whose edge set is $\{pp' : p \in \pi(p')\}$.*

Theorem 3.1.4 [41] *Let G be a graph with diameter two no 4-circuit and order n . Then one of the following is true:*

(i) $\Delta(G) = n - 1$;

(ii) G is a Moore graph;

(iii) G is a polarity graph.

Proof. There are three cases.

(i) Suppose that G^C is disconnected. Since G does not contain 4-cycles, at most one component of G^C has two or more vertices. Hence $\Delta = n - 1$, that is, there is vertex in G adjacent to all the rest vertices.

(ii) Suppose that G^C is connected, and no vertex of degree Δ lies on a triangle. Let us prove that if $xy \notin E(G)$ and $d(x) = \Delta$, then $d(y) = \Delta$. As $diam(G) = 2$, there exists a vertex z which is the only common neighbor of x and y . Thus for each $x_i \in N(x) - z$, there is a unique vertex y_i adjacent to x_i and y . As x lies in no triangle, then $y_i \neq z$; moreover, $y_i \neq y_j$

and $x_i \neq x_j$ because G contains no 4-cycles. Therefore y has at least as many neighbors as x , and $d(y) = \Delta$. Since G is connected, it must be δ -regular, and it must contain a cycle, otherwise it would be a tree of diameter two (i.e., a star), contradicting that G^C is connected. As G has diameter two it must have girth five, hence G is a Moore graph.

(iii) Suppose that G^C is connected, and some vertex of degree Δ lies on a triangle. First we will prove that if xy is an edge belonging to a triangle xyz then $d(x) = d(y) = \delta$. Denote the sets of neighbors of x, y and z not belonging to the triangle, by X, Y and Z respectively. For each $x_i \in X$ and $y_j \in Y$, there is a unique common neighbor w_{ij} , hence $d(x) = d(y) = r$ and as X, Y and Z are pairwise disjoint since G has no 4-cycles, then

$$|X| = |Y| = |Z| = r - 2.$$

Let W the vertices in $G \setminus X \cup Y \cup Z$. Each w is adjacent to exactly one vertex of X and exactly one vertex of Y . Hence,

$$|W| = |X||Y| = (r - 2)^2$$

and so

$$n = (r - 2)^2 + 3(r - 2) + 3 = r^2 + r + 1.$$

But as r is independent of the triangle chosen, and by hypothesis some vertex of degree Δ lies on a triangle, we have $r = \Delta$. Thus

$$n = (\Delta - 2)^2 + 3(\Delta - 2) + 3 = \Delta^2 + \Delta + 1. \quad (3.6)$$

Now, let us prove that if xy is an edge not belonging to a triangle then $\{d(x), d(y)\} = \{\Delta, \Delta - 1\}$. Suppose that the edge xy lies in no triangle. Denote the neighbors of x and y not on this edge as X and Y respectively, and let W be the set of remaining vertices of G . As X and Y are disjoint, $|X| = d(x) - 1$ and $|Y| = d(y) - 1$. Arguing as before, we obtain

$$|W| = |X||Y| = (d(x) - 1)(d(y) - 1)$$

and

$$n = (d(x) - 1)(d(y) - 1) + d(x) + d(y) = d(x)d(y) + 1.$$

Together with (3.6), we get $d(x)d(y) = \Delta^2 - \Delta$, concluding that $\{d(x), d(y)\} = \{\Delta, \Delta - 1\}$. From these two properties, each vertex has degree Δ or $\Delta - 1$.

Now, we define a projective plane \mathcal{P} from G as follows: The points of \mathcal{P} are the vertices of G and the lines of \mathcal{P} are the sets $L(v)$, for each $v \in V(G)$, defined by:

$$L(v) = \begin{cases} N(v) & \text{if } d(v) = \Delta, \\ N(v) \cup \{v\} & \text{if } d(v) = \Delta - 1. \end{cases}$$

As G has no 4-cycles then for every $u, v \in V(G)$ such that $uv \notin E(G)$ it holds that $|N(u) \cap N(v)| \leq 1$, and since G has diameter 2, it follows that $|N(u) \cap N(v)| \geq 1$, yielding $|N(u) \cap N(v)| = 1$.

$|N(v)| = 1$. If $uv \in E(G)$, we have two cases: When uv belongs to no triangle then $d(u) = \Delta$, $d(v) = \Delta - 1$ and by definition $L(u) \cap L(v) = v$. If uv belongs to a triangle uvw then $L(u) \cap L(v) = w$. Hence, in any case, every two lines $L(u)$ and $L(v)$ determine a point and every two points u and v determine a line, and every point belongs to Δ lines, concluding that \mathcal{P} is a projective plane. Moreover, the mapping defined by $\pi(v) = L(v)$, for each $v \in V(G)$, is a polarity of \mathcal{P} , and $G(\mathcal{P}, \pi)$. Hence, G is a polarity graph. ■

Note that polarity graphs are a family of graphs with girth pair (g, h) , for $g = 3$, $h = 6 \geq g + 3$ and diameter 2. Therefore they satisfy the hypothesis of Theorem 3.1.3, and, as a consequence of Theorem 3.1.3, we obtain the following.

Theorem 3.1.5 *Polarity graphs are λ' -optimal.*

Proof. It is not difficult to check for $q = 2$ that the corresponding graph on seven vertices is λ' -optimal, see Figure 3.2. Furthermore, by Theorem 3.1.3, polarity graphs are λ' -optimal for $q \geq 4$. Finally, the polarity graph for $q = 3$ has diameter 2 and there is no triangle with all its vertices of degree 3. Hence, by Theorem 3.1.3 (i), this graph is also λ' -optimal. ■

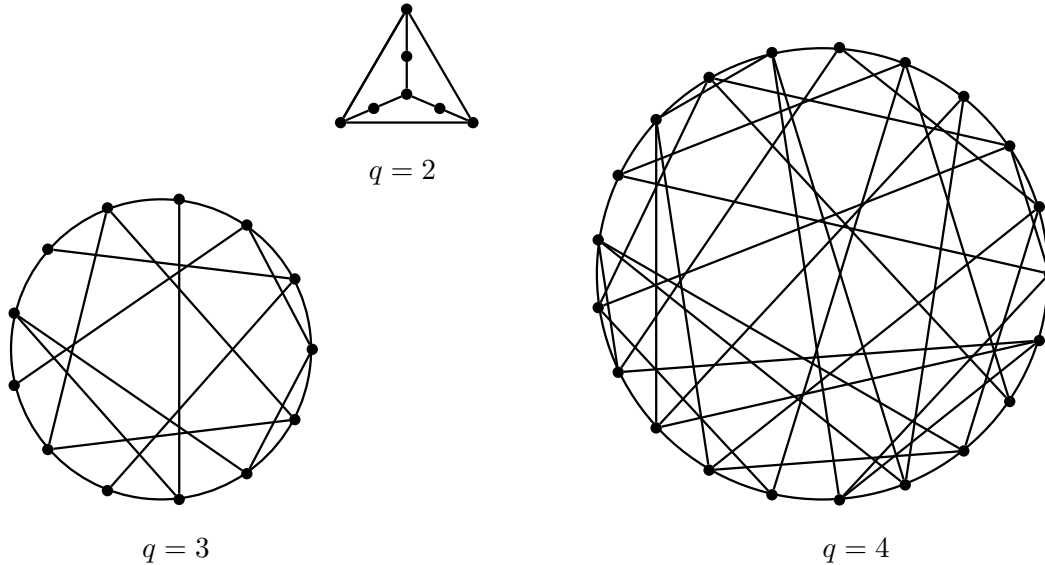


Figure 3.2: Polarity graphs for $q = 2, 3, 4$.

Note that if we define a new graph G' by adding a loop to every vertex of degree Δ of a polarity graph G , and apply the Kronecker product $G' \otimes K_2$, we obtain a Δ regular graph, of girth 6 and order $2(\Delta^2 + \Delta + 1)$, that is, the incidence graph of a projective plane. In [2], Abreu, Balbuena and Labbate present a method for explicitly obtaining the adjacency matrices of polarity graphs from the incidence matrices of projective planes.

3.2 Edge-superconnectivity of semiregular cages

By Whitney's inequality, Conjecture 1.3.1 implies that the edge-connectivity of $(k; g)$ -cages is k as well. Although Conjecture 1.3.1 is still open, for the edge connectivity the corresponding conjecture has already been settled.

Wang, Xu, and Wang [110] obtained the following result for odd girth.

Theorem 3.2.1 [110] *Let G be a $(k; g)$ -cage, where $k \geq 3$ and g is odd. Then G is k -edge connected.*

Together with the corresponding result for even girth due to Lin, Miller and Rodger [87], the problem of the edge-connectivity of cages is solved.

Theorem 3.2.2 [87] *A $(k; g)$ -cage is k -edge connected if g is even.*

These last two results on the edge connectivity of cages were extended in [86, 92], where the edge-superconnectivity of cages was established.

Theorem 3.2.3 [92] *Let G be a $(k; g)$ -cage with odd girth g and $k \geq 3$. Then G is edge-superconnected.*

Theorem 3.2.4 [86] *All $(k; g)$ -cages are edge-superconnected if g is even.*

Concerning the study of connectivity for semiregular cages, it has been proved in [22] that they are maximally edge-connected.

Corollary 3.2.1 [22] *Every $(k, k + 1; g)$ -cage with $r \geq 2$ is maximally edge-connected.*

Hence it is natural to follow the study of connectivity for such families of graphs with the study the edge-superconnectivity of semiregular cages. That is what we do in this section: We prove that semiregular cages of odd girth are edge-superconnected.

With this aim we need the following two generalizations of Theorems 1.3.2 and 1.3.3 on cages.

Theorem 3.2.5 [117] *Let g_1, g_2 be two integers such that $3 \leq g_1 < g_2$. Then $n(k, k + 1; g_1) < n(k, k + 1; g_2)$.*

Theorem 3.2.6 [28] *The diameter of a $(k, k + 1; g)$ -cage is at most g .*

In what follows, let X_0, X_1 be two subsets of $V(G)$ such that $|X_0| = |X_1|$. Let \mathcal{B}_Γ be the bipartite graph with bipartition (X_0, X_1) and $E(\mathcal{B}_\Gamma) = \{u_i v_j : u_i \in X_0, v_j \in X_1, d_\Gamma(u_i, v_j) \geq g - 1\}$, where Γ is a certain subgraph of G . Note that if a graph G contains two vertices u, v at distance at least $g - 1$, the edge uv can be added to G without creating cycles of length less than g . Moreover, if G contains two sets of vertices of the same cardinality and the graph \mathcal{B}_Γ contains a matching, this means that a matching can be added to G without creating cycles of length less than g .

In order to study the edge-superconnectivity of a graph in terms of its diameter and its girth, in the same vein as in Proposition 3.1.2, the following result was established.

Proposition 3.2.1 [16, 76] *Let G be a graph and W be a minimum nontrivial edge-cut. Then there exists some vertex $x_i \in V(H_i)$ such that $d(x_i, W_i) \geq \lfloor (g - 1)/2 \rfloor$, if $|W_i| \leq \delta - 1$.*

When W is nontrivial and $|W| \leq \xi - 1$, it follows from Proposition 3.1.2 that $\mu_i \geq \lfloor (g - 3)/2 \rfloor$. In the case when g is odd and $\mu_i = (g - 3)/2$, we obtain more structure in G as shown in the next lemma.

Lemma 3.2.1 *Let G be a graph with minimum degree $\delta \geq 3$ and odd girth $g \geq 5$. Let W be a minimum nontrivial edge-cut with cardinality $|W| \leq \delta$.*

If $\mu_i = (g - 3)/2$ the following statements hold:

- (i) $|W_i| = |W| = \delta$, i.e., every $a \in W_i$ is incident to one unique edge of W .
- (ii) Every vertex $z \in V(H_i)$ such that $d(z, W_i) = \mu_i$ has degree $d(z) = \delta$.
- (iii) For every $a \in W_i$ there exists a vertex $x \in V(H_i)$ such that $d(x, W_i) = d(x, a) = \mu_i$ and $N_{(g-3)/2}(x) \cap W_i = \{a\}$. Further, $N(x)$ and W_i can be ordered as $\{u_1, u_2, \dots, u_\delta\}$ and $\{a = a_1, a_2, \dots, a_\delta\}$, respectively, so that $N_{(g-5)/2}(u_1) \cap W_i = \{a_1\}$ and $N_{(g-3)/2}(u_k) \cap W_i = \{a_k\}$, for every $k > 1$.

Proof. (i) Since $\mu_i = (g - 3)/2$ we have $d(x, W_i) \leq \mu_i = (g - 3)/2 < (g - 1)/2$, for all $x \in V(H_i)$. Hence, from Proposition 3.2.1, it follows that $|W_i| \geq \delta$, yielding that $|W_i| = \delta$ because $|W_i| \leq |W| \leq \delta$. Observe that $\delta = |W_i| = |W|$ means that $|N(a) \cap W_{i+1}| = 1$ for each vertex $a \in W_i$ (taking the sum of subindexes mod 2).

- (ii) First observe that $\mu_i = (g - 3)/2 \geq 1$ since $g \geq 5$.

Let z be a vertex of H_i such that $d(z, W_i) = \mu_i = (g-3)/2$. Then we have

$$\begin{aligned}
N(z) &= S^=(z) \cup S^-(z); \\
|N_{(g-3)/2}(S^=(z)) \cap W_i| &\geq |S^=(z)|; \\
|N_{(g-5)/2}(S^-(z)) \cap W_i| &\geq |S^-(z)|; \\
N_{(g-3)/2}(S^=(z)) \cap N_{(g-5)/2}(S^-(z)) &= \emptyset,
\end{aligned} \tag{3.7}$$

because otherwise cycles of length less than the girth g appear. Since

$$\begin{aligned}
\delta \leq d(z) &= |S^=(z)| + |S^-(z)| \\
&\leq |N_{(g-3)/2}(S^=(z)) \cap W_i| + |N_{(g-5)/2}(S^-(z)) \cap W_i| \\
&\leq |W_i| = \delta
\end{aligned}$$

it follows that $\delta = d(z)$. Therefore item (ii) holds.

(iii) First let us prove that there exists an edge zz' such that $d(z, W_i) = d(z', W_i) = (g-3)/2$. Otherwise, $S^=(z) = \emptyset$ for all z with $d(z, W_i) = (g-3)/2$. This implies that for all $u \in N(z)$, $u \in S^-(z)$ and $S^=(S^+(u)) = \emptyset$. Further, $|N_{(g-5)/2}(u) \cap W_i| = 1$ for all $u \in N(z)$, because $\delta = |W_i| = \sum_{u \in N(z)} |N_{(g-5)/2}(u) \cap W_i| \geq \delta$. Hence $|S^-(u)| = 1$, and so $|S^+(u)| + |S^-(u)| = d(u) - 1 \geq \delta - 1 \geq 2$. Suppose that $|S^-(u)| \geq 1$ for some $u \in N(z)$, then as $N_{(g-3)/2}(z) \cap W_i$ and $N_{(g-5)/2}(S^-(u)) \cap W_i$ are two vertex disjoint sets we have $|W| \geq |N_{(g-3)/2}(z) \cap W_i| + |N_{(g-5)/2}(S^-(u)) \cap W_i| \geq \delta + 1$ which is a contradiction because $|W| = \delta$. Then we must assume that for all $u \in N(z)$, $|S^+(u)| = d(u) - 1 \geq \delta - 1 \geq 2$. Let $t \in S^+(u) - z$, according to our first assumption $S^-(t) = \emptyset$ meaning that $N(t) = S^-(t)$. Since t has the same behavior as z we have $W_i = N_{(g-3)/2}(S^-(z)) = N_{(g-3)/2}(S^-(t))$, and as $2 < \delta \leq d(z) = d(t)$, there exist cycles through $\{z, u, t, w\}$ for some $w \in W_i$ of length less than g which is a contradiction.

Hence we may assume that there exists an edge zz' such that $d(z, W_i) = d(z', W_i) = (g-3)/2$. Since $N_{(g-5)/2}(S^-(z)) \cap W_i$, $N_{(g-5)/2}(S^-(z')) \cap W_i$ and $N_{(g-3)/2}(S^=(z') - z) \cap W_i$ are three pairwise disjoint sets because $g \geq 5$, and taking into account (3.7) we have

$$\begin{aligned}
\delta = |W| &\geq |N_{(g-5)/2}(S^-(z)) \cap W_i| + |N_{(g-5)/2}(S^-(z')) \cap W_i| + |N_{(g-3)/2}(S^=(z') - z) \cap W_i| \\
&\geq |S^-(z)| + |S^-(z')| + |S^=(z') - z| \\
&= d(z) - 1 + |S^-(z)| \geq \delta.
\end{aligned}$$

Therefore, all inequalities become equalities, i.e. $|S^-(z)| = 1 = |N_{(g-5)/2}(S^-(z)) \cap W_i|$. So $S^-(z) = \{z_1\}$ and $N(z) - z_1 = S^=(z)$ yielding to a partition of W_i :

$$W_i = (N_{(g-5)/2}(z_1) \cap W_i) \cup (\cup_{z' \in N(z) - z_1} N_{(g-3)/2}(z') \cap W_i),$$

because for all $z' \in N(z) - z_1$ the sets $N_{(g-3)/2}(z') \cap W_i$ and the set $N_{(g-5)/2}(z_1) \cap W_i$ are mutually disjoint. Thus, $|N_{(g-3)/2}(z') \cap W_i| = 1$ for all $z' \in N(z) - z_1$. Therefore, for every

vertex $a \in W_i$ there exists a vertex $x \in (N(z) - z_1) \cup \{z\}$ such that $d(x, W_i) = d(x, a) = (g-3)/2$ and $N_{(g-3)/2}(x) \cap W_i = \{a\}$. Furthermore, since every vertex $z' \in N(z) - z_1$ has the same behavior as z , $N(x)$ can be ordered as $\{u_1, u_2, \dots, u_\delta\}$, and W_i can be ordered as $\{a_1, a_2, \dots, a_\delta\}$, where $a_1 = a$, so that $N_{(g-5)/2}(u_1) \cap W_i = \{a_1\}$ and $N_{(g-3)/2}(u_k) \cap W_i = \{a_k\}$ for every $k > 1$, which finishes the proof. ■

Semiregular cages are known to be maximally edge-connected [22]. Now, we are ready to prove that semiregular cages with odd girth are edge-superconnected. As will be seen, the following theorem due to Hall is a key point for this study.

Theorem 3.2.7 [68] *A bipartite graph with bipartition (X_0, X_1) has a matching which covers every vertex in X_0 if and only if*

$$|N(S)| \geq |S| \text{ for all } S \subset X_0.$$

Using Hall's Theorem, Jiang [77] proved the following result.

Lemma 3.2.2 [77] *Let G be a bipartite graph with bipartition (X_0, X_1) , where $|X_0| = |X_1| = r$. If G contains at least $r^2 - r + 1$ edges then G contains a matching.*

The following lemma is a stronger version of Lemma 3.2.2, which is also proved using Hall's Theorem.

Lemma 3.2.3 *Let \mathcal{B} be a bipartite graph with bipartition (X_0, X_1) and $|X_0| = |X_1| = r$. If $\delta(\mathcal{B}) \geq 1$ and $|E(\mathcal{B})| \geq r^2 - r$ then \mathcal{B} contains a perfect matching.*

Proof. Let $\mathcal{B} = (X_0, X_1)$ be a bipartite graph with $|X_0| = |X_1| = r$, $\delta(\mathcal{B}) \geq 1$ and $|E(\mathcal{B})| \geq r^2 - r$. We shall show that for a subset $S \subset X_0$, $|N(S)| \geq |S|$. Notice that if $|S| = 1$ then $|N(S)| \geq 1 = |S|$ because $\delta(\mathcal{B}) \geq 1$; and if $S = X_0$, $N(S) = X_1$ because $\delta(\mathcal{B}) \geq 1$ implies that each vertex $u \in X_1$ must have a neighbor in S , hence $|S| = |N(S)|$.

Therefore if we assume that $1 \leq |N(S)| < |S| = t \leq r - 1$ then the number of edges in \mathcal{B} is at most

$$|E(\mathcal{B})| = |[S, N(S)]| + |[X_0 \setminus S, X_1]| \leq t(t-1) + (r-t)r,$$

and, by hypothesis, $|E(\mathcal{B})| \geq r^2 - r$. Thus $r^2 - r \leq t(t-1) + (r-t)r$, yielding $0 \leq (t-r)(t-1)$, which is an absurdity because $1 < t < r$. Therefore $|N(S)| \geq |S|$, for all $S \subset X_0$, and by Hall's Theorem 3.2.7 the lemma follows. ■

Theorem 3.2.8 *Let G be a $(k, k + 1; g)$ -cage with odd girth $g \geq 5$, and $k \geq 3$. Then G is edge-superconnected.*

Proof. Let us assume that G is a $(k, k + 1; g)$ -cage that is not edge-superconnected, and we will arrive at a contradiction. To this end, let us take a minimum nontrivial edge-cut $W = [W_0, W_1]$ such that $|W| \leq \delta$, $G - W = H_0 \cup H_1$, and $W_i \subset V(H_i)$ for $i = 0, 1$. From Proposition 3.1.2 it follows that $\mu_i = \max\{d(x, W_i) : x \in V(H_i)\} \geq (g - 3)/2$, $i = 0, 1$. Let $x_i \in V(H_i) \cap N_{\mu_i}(W_i)$. As G is a $(k, k + 1; g)$ -cage, the diameter is at most $\text{diam}(G) \leq g$ by Theorem 3.2.6, so we get the following chain of inequalities:

$$g \geq \text{diam}(G) \geq d(x_0, x_1) \geq d(x_0, W_0) + 1 + d(x_1, W_1) = \mu_0 + 1 + \mu_1 \geq g - 2.$$

If we assume henceforth $\mu_0 \leq \mu_1$ (without loss of generality), then we have the cases summarized in the following table:

μ_0	\leq	μ_1
$(g - 3)/2$		$(g + 1)/2$
		$(g - 1)/2$
		$(g - 3)/2$
$(g - 1)/2$		$(g - 1)/2$

We begin with the study of *Case (a)*: $\mu_0 = (g - 3)/2$.

From Lemma 3.2.1 (i), $|W_0| = k = |W|$, so that each vertex of W_0 is incident to one unique edge of W yielding that every vertex $a \in W_0$ has $d_{H_0}(a) \in \{k - 1, k\}$. Also by Lemma 3.2.1 (ii), every vertex $x \in N_{(g-3)/2} \cap V(H_0)$ has $d(x) = k$. And by Lemma 3.2.1 (iii), for every $a \in W_0$, there exists a vertex $x_0 \in N_{(g-3)/2} \cap V(H_0)$ such that $N(x_0) = \{u_1, u_2, \dots, u_d\}$ and $W_0 = \{a_1, a_2, \dots, a_d\}$, where $a_1 = a$, in such a way that $d(u_1, a_1) = d(u_1, W_0) = (g - 5)/2$, $d(u_j, W_0) = d(u_j, a_j) = (g - 3)/2$, and by (ii), $(u_j) = d$ for every $j \geq 2$. This implies that $d_{G-x_0}(u_1, a_j) \geq (g - 1)/2$, for all $j \geq 2$, because the shortest (u_1, a_j) -path in $G - x_0$, the shortest (u_j, a_j) -path in G , and the path $u_j x_0 u_1$ in G of length two, form a closed walk containing a cycle. Reasoning analogously, $d_{G-x_0}(u_j, a_1) \geq (g + 1)/2$, for all $j \geq 2$, and $d_{G-x_0}(u_j, a_i) \geq (g - 1)/2$, for $j \neq i$ and $j, i \in \{2, \dots, k\}$. Furthermore, $[N_{(g-3)/2}(x_0) \cap W_0, W_1] = \{a_1 b_1\}$, for some $b_1 \in W_1$.

Case (a.1): $\mu_1 = (g + 1)/2$.

Let $x_1 \in V(H_1)$ be any vertex such that $d(W_1, x_1) = (g + 1)/2$. Let $X_0 = \{u_2, \dots, u_k\} \cup \{x_0\}$ and $X_1 = \{v_1, v_2, \dots, v_k\} \subseteq N(x_1)$. As $d(u_i, W_0) = (g - 3)/2$, for $i \geq 2$ and $d_{G-x_1}(N(x_1), W_1) \geq (g - 1)/2$, then $d_{G-x_1}(X_0, X_1) \geq g - 1$, so $|E(\mathcal{B}_\Gamma)| = k^2$, where $\Gamma = G - x_1$. Clearly, \mathcal{B}_Γ is a complete bipartite graph, so there is a perfect matching M which

covers every vertex in X_0 and if $d(x_1) = k$, also covers $N(x_1)$. Hence, in this case the graph $G^* = (G - \{x_1\} - \{x_0u_k\}) \cup M$ has girth at least g and the vertices u_2, \dots, u_{k-1} have degree $k+1$ in G^* as they had degree k in G ; for the same reason x_0 and u_k have degree k in G^* . The remaining vertices have the same degree they had in G . As G^* is a $(k, k+1; g^*)$ -graph with girth $g^* \geq g$ and $|V(G^*)| < |V(G)|$, we get a contradiction to the monotonicity Theorem 1.3.2. If $d(x_1) = k+1$, since $d_{G^*}(u_d, v_{k+1}) \geq g-1$, where $v_{k+1} \in N(x_1) \setminus X_1$, we can add the new edge u_kv_{k+1} to G^* without decreasing the girth. Then $G^* \cup \{u_kv_{k+1}\}$ give us again a contradiction.

Case (a.2): $\mu_1 = (g-3)/2$.

By Lemma 3.2.1, given $b_1 \in W_1$ we can take $x_1 \in V(H_1) \cap N_{(g-3)/2}(W_1)$ of $d(x_1) = k$ such that $N(x_1) = \{v_1, v_2, \dots, v_k\}$, $W_1 = \{b_1, b_2, \dots, b_k\}$ and each vertex of W_1 is incident to one unique edge of W , hence $W = \{a_1b_1, a_2b_2, \dots, a_kb_k\}$. Also, $d(b_1, v_1) = d(W_1, v_1) = (g-5)/2$, and $d(W_1, v_j) = d(b_j, v_j) = (g-3)/2$, for every $j \geq 2$, and besides $d(v_j) = k$. Then $d(x_0, x_1) = d(x_0, a_1) + 1 + d(b_1, x_1) = g-2$. Now let $X_0 = N(x_0)$, $X_1 = N(x_1)$ and $\Gamma = G - \{x_0, x_1\}$. We have

$$\begin{aligned} d_\Gamma(u_1, N(x_1) - v_1) &= \\ &= \min\{d_\Gamma(u_1, a_1) + 1 + d_\Gamma(b_1, N(x_1) - v_1); d_\Gamma(u_1, a_j) + 1 + d_\Gamma(b_j, N(x_1) - v_1), j \geq 2\} \\ &\geq \min\left\{\frac{g-5}{2} + 1 + \frac{g+1}{2}; \frac{g-1}{2} + 1 + \frac{g-3}{2}\right\} = g-1, \end{aligned}$$

since $d_\Gamma(b_1, v_j) \geq (g+1)/2$, for all $j \geq 2$, because the shortest (b_1, v_j) -path in Γ , the shortest (b_1, v_1) -path in Γ , and the path $v_jx_1v_1$ in G of length two, form a closed walk containing a cycle. Reasoning in the same way, it follows for all $j \geq 2$ that

$$\begin{aligned} d_\Gamma(u_j, N(x_1) - v_j) &= \\ &= \min\{d_\Gamma(u_j, a_j) + 1 + d_\Gamma(b_j, N(x_1) - v_j); d_\Gamma(u_j, a_h) + 1 + d_\Gamma(b_h, N(x_1) - v_j) : h \neq j\} \\ &\geq \min \left\{ \begin{array}{l} \left\{ \frac{g-3}{2} + 1 + \frac{g-1}{2}; \frac{g-1}{2} + 1 + \frac{g-3}{2} \right\} \text{ if } h \geq 2, h \neq j \\ \left\{ \frac{g-3}{2} + 1 + \frac{g-1}{2}; \frac{g+1}{2} + 1 + \frac{g-5}{2} \right\} \text{ if } h = 1 \end{array} \right\} = g-1. \end{aligned}$$

Similarly, $d_\Gamma(N(x_0) - u_1, v_1) \geq g-1$ and $d_\Gamma(N(x_0) - u_j, v_j) \geq g-1$ for all $j \geq 2$. Hence the bipartite graph $\mathcal{B}_\Gamma = X_0, X_1$ has $|E(\mathcal{B}_\Gamma)| = k^2 - k$ and $d_{\mathcal{B}_\Gamma}(w) \geq 1$ for all $w \in X_0 \cup X_1$. From Lemma 3.2.3, there is a perfect matching M between $X_0 = N(x_0)$ and $X_1 = N(x_1)$. Hence $G^* = (G - \{x_0, x_1\}) \cup M$ is a $(k, k+1; g^*)$ -graph (because every vertex in G^* has the same degree it had in G and the removed vertices x_0, x_1 had degree k , as well as the vertices u_i, v_j , for every $i, j \geq 2$), with $g^* \geq g$ and $|V(G^*)| \leq |V(G)|$, which contradicts the monotonicity Theorem 3.2.5, and we are done.

Case (a.3): $\mu_1 = (g-1)/2$. In this case we distinguish two other possible subcases.

Case (a.3.1): Suppose that there exists $x_1 \in V(H_1) \cap N_{(g-1)/2}(W_1)$ such that $d(b, v) \leq (g-1)/2$ for all $b \in W_1$ and for all $v \in N(x_1)$.

Then, every $b \in W_1$ has $d_{H_1}(b) = d(x_1) \in \{k, k+1\}$ because $|N_{(g-3)/2}(z) \cap N(x_1)| \leq 1$, for all $z \in N(b)$ (otherwise cycles of length less than g appears). Hence $d(x_1) = k$ and $d(b) = k+1$, for all $b \in W_1$. Thus $N(x_1) = \{v_1, \dots, v_k\}$ and W is a matching, i.e., $W = \{a_1b_1, \dots, a_kb_k\}$. Therefore the subgraph H_1 gives a contradiction unless H_1 is k -regular. In this case let us consider the graph $\tilde{G} = (G - x_1 - W) \cup \{a_1v_1, \dots, a_kv_k\}$ which clearly has girth at least g . Moreover, $d_{\tilde{G}}(b_i) = d(b_i) - 1 = k$ and every vertex different from b_i has the same degree it had in G . Thus we may suppose that \tilde{G} is k -regular because otherwise \tilde{G} would be a $(k, k+1; g^*)$ -graph with girth $g^* \geq g$ and smaller than G , a contradiction. Also, we may assume that $d_{H_1}(b_1, v_1) = (g-3)/2$ and $d_{H_1}(b_1, N(x_1) - v_1) = (g-1)/2$. Thus we have

$$\begin{aligned} d_{\tilde{G}}(b_1, u_2) &\geq \min\{d_{H_1}(b_1, v_2) + d_{\tilde{G}}(v_2, a_2) + d_{H_0}(a_2, u_2); d_{H_1}(b_1, v_1) + d_{\tilde{G}}(v_1, a_1) + d_{H_0}(a_1, u_2)\} \\ &\geq \min\left\{\frac{g-1}{2} + 1 + \frac{g-3}{2}; \frac{g-3}{2} + 1 + \frac{g+1}{2}\right\} \\ &= g-1, \end{aligned}$$

which implies that we can add to \tilde{G} the edge b_1u_2 to obtain a graph without decreasing the girth g . As this new graph is smaller than G and has degrees $\{k, k+1\}$ we get a contradiction to the monotonicity Theorem 3.2.5, and we are done.

Case (a.3.2): Suppose that for all $z \in V(H_1) \cap N_{(g-1)/2}(W_1)$ there exist $v \in N(x_1)$ and $b \in W_1$ such that $d(b, v) \geq (g+1)/2$.

Let $x_1 \in V(H_1) \cap N_{(g-1)/2}(W_1)$, $v_1 \in N(x_1)$ and $b^* \in W_1$ such that $d(b^*, v_1) \geq (g+1)/2$. By Lemma 3.2.1, there exists a unique edge $a^*b^* \in W$ to which vertex $a^* \in W_0$ is incident and there exists a vertex $x^* \in V(H_0)$ of $d(x^*) = k$ such that $d(x^*, W_0) = d(x^*, a^*) = (g-3)/2$ and $N_{(g-3)/2}(x^*) \cap W_0 = \{a^*\}$. Further, $N(x^*)$ can be ordered as $\{z_1, z_2, \dots, z_k\}$, and W_0 can be ordered as $\{a_1, a_2, \dots, a_d\}$, where $a_1 = a^*$, so that $N_{(g-5)/2}(z_1) \cap W_i = \{a_1\}$, $N_{(g-3)/2}(z_j) \cap W_i = \{a_j\}$ and $d(z_j) = k$ for every $j > 1$. Furthermore, $[N_{(g-3)/2}(x^*) \cap W_0, W_1] = \{a_1b^*\}$

Let $\Gamma = G - \{x^*, x_1\}$. We obtain:

$$\begin{aligned} d_{\Gamma}(z_1, v_1) &= \min\{d_{\Gamma}(z_1, a_1) + 1 + d_{\Gamma}(b^*, v_1); d_{\Gamma}(z_1, a_j) + 1 + d_{\Gamma}(b', v_1) : j \geq 2 \text{ and } a_jb' \in W\} \\ &\geq \min\left\{\frac{g-5}{2} + 1 + \frac{g+1}{2}; \frac{g-1}{2} + 1 + \frac{g-3}{2}\right\} = g-1. \end{aligned}$$

Moreover, $d_{H_0}(z_j, W_0) = (g-3)/2$ for all $z_j \in N(x^*) - z_1$ and there exists a unique vertex say $b_j \in W_1$ for which $a_jb_j \in W$. As $|N_{(g-3)/2}(b) \cap N(x_1)| \leq 1$, for each $b \in W_1$ (otherwise cycles of length less than g appears) we denote by v_j the vertex in $N(x_1) - v_1$ such that $d(b_j, v_j) = (g-3)/2$, if any. Thus we obtain:

$$d_{\Gamma}(z_j, N(x_1) \setminus \{v_1, v_j\}) = d(z_j, a_j) + 1 + d(b_j, N(x_1) \setminus \{v_1, v_j\}) \geq \frac{g-3}{2} + 1 + \frac{g-1}{2} = g-1.$$

Let us consider $X_0 = N(x^*) - z_1$ and $X_1 \subseteq N(x_1) - v_1$, with $|X_1| = d - 1$. It is clear that $|d_{\mathcal{B}_\Gamma}(z_j)| \geq d - 2 \geq 1$, for all $z_j \in N(x^*) - u_1$ giving that $|E(\mathcal{B}_\Gamma)| \geq (k - 2)|X_0| = (k - 2)(k - 1) = (k - 1)^2 - (k - 1)$.

First, suppose that $|d_{\mathcal{B}_\Gamma}(v)| \geq 1$ for all $v \in N(x_1) - v_1$. From Lemma 3.2.3, there is a matching M which covers every vertex in $N(x^*) - z_1$ and every vertex in $N(x_1) - v_1$ if $d(x_1) = k$. In this case $G^* = (G - \{x^*, x_1\}) \cup M \cup \{z_1 v_1\}$ is a graph with girth $g^* \geq g$ and smaller than G whose vertices have the same degree they had in G , thus G^* is a $(k, k + 1; g^*)$ -graph and we are done. Thus suppose that $d(x_1) = k + 1$ and that after adding the matching $M \cup \{z_1 v_1\}$ to $G - \{x^*, x_1\}$ it remains the vertex $v_{k+1} \in (N(x_1) - v_1) \setminus X_1$ of degree $k - 1$. By Lemma 3.2.1, every z_j , $j > 1$, has degree k in G , and we have proved that $d(z_j, N(x_1) \setminus \{v_1, v_j\}) \geq g - 1$. Then we add one extra edge $z_j v_{k+1}$ to G^* obtaining a new $(k, k + 1; g^*)$ -graph with $g^* \geq g$ and smaller than G , a contradiction to the monotonicity Theorem 3.2.5, so we are done.

Therefore we must suppose that there exists $v_2 \in N(x_1) - v_1$ such that $|d_{\mathcal{B}_\Gamma}(v_2)| = 0$. This implies that $d(v_2, b) = (g - 3)/2$ for all $b \in W_1 - b^*$, and hence $d(v, W_1 - b^*) = (g - 1)/2$, for all $v \in N(x_1) - v_2$, because of the girth. If $d(v_2, b^*) \geq (g + 1)/2$ then $d_\Gamma(z_1, v_2) \geq g - 1$, $d_\Gamma(z_k, N(x_1) - v_2) = g - 1$, for all $j \geq 2$; thus we consider the set $X_1 \subseteq N(x_1) - v_2$ with $|X_1| = k - 1$. It is clear that $|d_{\mathcal{B}_\Gamma}(w)| \geq k - 1$, for all $w \in X_0 \cup X_1$. By using Lemma 3.2.3 and reasoning as before we get a contradiction. Therefore we must suppose that $d(v_2, b^*) \leq (g - 1)/2$. Since $N(x_1) - v_2 \subseteq N_{(g-1)/2}(W_1) \cap V(H_1)$, we have by hypothesis that for all $v \in N(x_1) - v_2$ there exists $\tilde{v}_1 \in N(v)$ and $\tilde{b} \in W_1$ such that $d(\tilde{b}, \tilde{v}_1) \geq (g + 1)/2$. As the behavior of any $v \in N(x_1) - v_2$ is the same as that of a vertex x_1 , then reasoning as before we get a contradiction unless, for all $v \in N(x_1) - v_2$, there exists $\tilde{v}_2 \in N(v) - \tilde{v}_1$ such that $|d_{\mathcal{B}_\Gamma}(\tilde{v}_2)| = 0$ satisfying that $d(\tilde{v}_2, b) = (g - 3)/2$, for all $b \in W_1 - \tilde{b}$ and $d(\tilde{v}_2, \tilde{b}) \leq (g - 1)/2$. Therefore we conclude that every vertex $b \in W_1$ has $d_{H_1}(b) = d(x_1) \in \{k, k + 1\}$. Now considering the same graph as in Subcase (a.3.1), we get a contradiction.

Case (b): $\mu_0 = \mu_1 = (g - 1)/2$. Let $x_i \in V(H_i)$ such that $d(x_i, W_i) = \mu_i$, for $i = 0, 1$.

First of all note that there must exist a vertex in $N(x_0)$ of degree k , otherwise $G - x_0$ would be either a $\{k, k + 1\}$ -graph or a k -regular graph. In the former case we get a contradiction because $G - x_0$ is smaller than G and has girth at least g . In the latter case we consider the graph $(G - x_0) \cup \{u_i x_1\}$ with $u_i \in N(x_0)$, which gives again a contradiction. Similarly, note that there must exist a vertex in $N(x_1)$ of degree k .

Suppose that $d(x_0) = d(x_1) = r \in \{d, d + 1\}$. Let $X_0 = N(x_0)$, $X_1 = N(x_1)$ and $\Gamma = G - \{x_0, x_1\}$. Define $A = \{u_i v_j : u_i \in X_0, v_j \in X_1, d_\Gamma(u_i, v_j) \leq g - 2\}$ and observe that $E(\mathcal{B}_\Gamma) = \{uv : u \in X_0, v \in X_1\} \setminus A$. Note that every (u_i, v_j) -path in G goes through an edge of W . Therefore every edge in W gives rise to at most one element in A , otherwise G would contain a cycle of length at most $2(g - 3)/2 + 2 = g - 1$. Hence $|A| \leq |W| \leq k$ and $|E(\mathcal{B}_\Gamma)| = |K_{r,r}| - |A| \geq r^2 - k$.

If $r = k + 1$ then $|E(\mathcal{B}_\Gamma)| \geq (k + 1)^2 - k = k^2 + k + 1$, and by Lemma 3.2.2, the graph \mathcal{B}_Γ contains a perfect matching M . Therefore the graph $G' = G - \{x_0, x_1\} \cup M$ has fewer vertices than G and girth at least g , producing a contradiction unless G' is regular of degree k . In this case we consider the graph $G'' = G' \cup \{uv\}$, where $u \in N(x_0)$, such that $d(u, W_0) = (g - 1)/2$ (which must exist because $d(x_0) = k + 1$ and $|W_1| \leq k$) and $v \in N(x_1)$ such that $uv \notin M$. As G'' is a $(k, k + 1; g)$ -graph with fewer vertices than G and girth g a contradiction is again obtained.

Suppose $r = k$. If $d_{\mathcal{B}_\Gamma}(z) \geq 1$, for all $z \in \mathcal{B}_\Gamma$ then, by Lemma 3.2.3, there exists a matching M between X_0 and X_1 ; reasoning as before we obtain again a contradiction. Hence, we may assume that $d_{\mathcal{B}_\Gamma}(u_1) = 0$, for some $u_1 \in X_0$. This implies that $d_\Gamma(u_1, v_j) = g - 2$, for all $v_j \in N(x_1)$, or equivalently $d_\Gamma(v_j, W_1) = (g - 3)/2$, for all $v_j \in N(x_1)$. From this, and because the girth $g \geq 5$, we get that $|W_1| \geq |N(x_1)| = k$, yielding $|W_1| = k$ (since $k = |W| \geq |W_1|$), and also that $N_{(g-3)/2}(v_j) \cap W_1 = \{b_j\}$ for all $v_j \in N(x_1)$. That is, $|N(b_j) \cap W_0| = 1$ for every $b_j \in W_1$. Also we have $N_{(g-1)/2}(u_1) \cap W_1 = W_1$, hence $N_{(g-3)/2}(u_1) \cap W_0 = W_0$. Thus $d(u_i, W_0) = (g - 1)/2$, for $i \geq 2$.

Let $u_l \in N(x_0)$, $l \geq 2$, define $\Gamma_l = G - \{u_l, x_1\}$ and consider the sets

$$X_l = \begin{cases} N(u_l) & \text{if } d(u_l) = k; \\ N(u_l) - x_0 & \text{if } d(u_l) = k + 1; \end{cases}$$

$$X_1 = N(x_1);$$

$$A_l = \{z_i v_j : z_i \in X_l, v_j \in X_1, d_{\Gamma_l}(z_i, v_j) \leq g - 2\}.$$

Let $\mathcal{B}_{\Gamma_l} = K_{|X_l|, |X_1|} - A_l$.

If $d_{\mathcal{B}_{\Gamma_l}}(z) \geq 1$, for all $z \in X_l$, we get a matching M between X_l and $N(x_1)$ by Lemma 3.2.3; if $d(u_l) = k$ the graph $\Gamma_l \cup M$ yields a contradiction; if $d(u_l) = k + 1$ the graph $\Gamma_l \cup M \cup \{x_0 v_j\}$, where v_j is a vertex of $N(x_1)$ with degree k , yields again a contradiction. Therefore we can suppose that for every $u_l \in N(x_0) - u_1$ there exists $\tilde{z}_l \in N(u_l)$ such that $d_{\Gamma_l}(\tilde{z}_l, v_j) = g - 2$ for all $v_j \in N(x_1)$. Hence, $N_{(g-3)/2}(\tilde{z}_l) \cap W_0 = W_0$, that is, $d_{\Gamma_l}(\tilde{z}_l, a_j) = (g - 3)/2$, for each $a_j \in W_0$, yielding $d_{H_0}(a_j) = k$, thus $d(a_j) = k + 1$ and $[W_0, W_1]$ is a matching (recall that $|N(b_j) \cap W_0| = 1$ for every $b_j \in W_1$). We can now use the same graph $\tilde{G} = (G - \{x_0\} - W) \cup \{b_1 u_1, \dots, b_k u_k\}$ as we used in Case (a.3.2), arriving again at a contradiction.

The only remaining case is when x_0 and x_1 have different degrees. Let us suppose $d(x_0) = k$ and $d(x_1) = k + 1$. As $d(x_1) = k + 1 > |W_1|$, there exists, say $v_{k+1} \in N(x_1)$, such that $d(v_{k+1}, W_1) = (g - 1)/2$. We proceed as before, with the sets $X_0 = N(x_0)$ and $X_1 = N(x_1) - v_{k+1}$, finding a graph G' with fewer vertices and the same girth and degrees as G , except for the vertex v_{k+1} . Recall that there must exist a vertex $y \in N(x_0)$ such that $d(y) = k$, then we construct the graph $G^* = G' \cup \{y v_{k+1}\}$, which is a new $\{k, k + 1\}$ -graph

with girth g , arriving at a contradiction. This concludes the proof of the theorem. ■

Chapter 4

Constructions of Small Regular Graphs with a Given Girth Pair

There are many cages that have been obtained by a particular construction, for example, the $(7; 6)$ -cage by O’Keefe and Wong [97], or the famous Hoffman-Singleton graph which is the $(7; 5)$ -cage [74]. As more general constructions that obtain entire families of regular graphs with arbitrarily large girth, there are the construction of Sachs [103], the trivalent sextet, hexagon and triplet graphs [37, 73], as well as the higher degree constructions of Lubotzky, Phillips and Sarnak [90], and Lazebnik, Ustimenko and Woldar [82]. By constructing such families, upper bounds for the order of $(k; g)$ -cages are obtained.

The problem of obtaining similar upper bounds for families of graphs with a given girth pair (g, h) is addressed in this chapter. We present a construction for graphs with a girth pair (g, h) , defined in such a way that the order of a $(k; g, h)$ -cage is bounded by the order of a $(k; h)$ -cage, more precisely, we prove that $n(k; g, h) < n(k; h)$ for all $(k; g, h)$ -cages when g is an odd girth, and also for h sufficiently large and even girth g ; in both cases under the assumption that $(k; g)$ -cages are bipartite for g even. This result answers in the affirmative a conjecture by Harary and Kovács.

We would like to emphasize that every known $(k; g)$ -cage with even girth g is bipartite, furthermore it is conjectured that all cages with even girth are bipartite [126, 128]. Hence, the requirement of the existence of a bipartite $(k; g)$ -cage for even g is natural.

We begin first section by presenting the bipartition theorem by Biggs and Ito [38] related to this conjecture. We rewrite it here with our notation. Second section contains the bounds for cages with a given girth pair, and in third section we obtain more specific bounds, by studying the cases when the cages reach the Moore bound (i.e., $g = 6, 8, 12$). Also in this case, we obtained a result concerning the bipartition theorem of Biggs and Ito [38], for $g = 6$.

For $uv \in E(G)$ and $0 \leq l \leq g/2 - 1$, let us denote the sets

$$B_{uv}^l = \{x \in V(G) : d(x, u) = l \text{ and } d(x, v) = l + 1\} \text{ and } \overline{B}_{uv}^l = \bigcup_{i=0}^l B_{uv}^i.$$

Observe that $B_{uv}^0 = \{u\} = \overline{B}_{uv}^0$ and $B_{uv}^1 = N(u) - v$ while $\overline{B}_{uv}^1 = (N(u) - v) \cup \{u\}$. Moreover, note that $B_{uv}^l \neq B_{vu}^l$ and $\overline{B}_{uv}^l \neq \overline{B}_{vu}^l$.

Denote $T_{uv}^l = G[\overline{B}_{uv}^l \cup \overline{B}_{vu}^l]$ and observe that if $l \leq g/2 - 2$, where g is the girth of G , then T_{uv}^l is the tree rooted in the edge uv of depth l . When $l = g/2 - 1$ the subgraph T_{uv}^l may not be a tree, it can contain edges between vertices in B_{uv}^l and vertices in B_{vu}^l .

We will denote the set of cycles in G by $\mathcal{C}(G) = \{\alpha : \alpha \text{ is a cycle in } G\}$.

Let G be a $(k; g)$ -cage of even girth $g = 2r$. The *excess* e of G with respect to an edge $uv \in E(G)$ is the cardinality of the following set:

$$X = \{x \in V(G) : d(x, uv) \geq r\} = V(G) \setminus T_{uv}^{r-1}.$$

4.1 Excess

The values of k and g for which $(k; g)$ -cages can reach the Moore bound $n_0(k; g)$ are few and well known. In particular, for g even: $g = 6, 8, 12$ and $k = q + 1$ with q a prime power and for g odd: $g = 5$ and $k = 3, 7, 57$. It is natural to investigate what happens when the number of additional vertices is small. This number $n(k; g) - n_0(k; g)$ is known as the excess and it was defined by Biggs and Ito [38]. In that paper the authors obtain interesting results, some by algebraic methods and other by combinatorial means. In this section we present the combinatorial results with their proofs and a theorem obtained by algebraic means that states that there are no $(k; g)$ -cages with even girth $g \geq 8$ and excess $e = 2$.

Lemma 4.1.1 [38] *Let G be a $(k; g)$ -cage of girth $g = 2r$, let $xy \in E(G)$ be contained in $B_{uv}^{r-1} \cup X$. Then*

$$|N(x) \cap B_{vu}^{r-1}| + |N(y) \cap B_{vu}^{r-1}| \leq k - 1.$$

The same result holds interchanging B_{uv}^{r-1} , for B_{vu}^{r-1} .

Proof. Note that the $(k - 1)^r - 1$ vertices in B_{vu}^{r-1} are partitioned into the subsets $B_{v_i v}^{r-2}$ for each $v_i \in N(v) - u$, which are $k - 1$ subsets of cardinality $(k - 1)^r - 2$. As $N(x) \cap B_{vu}^{r-1}$ and $N(y) \cap B_{vu}^{r-1}$ are disjoint, otherwise G would have triangles, their union has cardinality $|N(x) \cap B_{vu}^{r-1}| + |N(y) \cap B_{vu}^{r-1}|$. If its sum is k or greater, by the pigeon-hole principle one of the $B_{v_i v}^{r-2}$ must contain two vertices x' and y' belonging to $(N(x) \cap B_{vu}^{r-1}) \cup (N(y) \cap B_{vu}^{r-1})$,

hence forming a cycle $xx' \cdots v_i \cdots y'yx$ of length $2(r-2) + 3 = 2r - 1 < g$, arriving at a contradiction. ■

Lemma 4.1.2 [38] *Let G be a $(k; g)$ -cage of girth $g = 2r$, let $xy \in E(G)$, with $x \in X, y \in B_{uv}^{r-1}$. Then*

$$|N(y) \cap X| \geq |N(x) \cap B_{vu}^{r-1}|.$$

The same result holds interchanging B_{uv}^{r-1} for B_{vu}^{r-1} .

Proof. Note that the vertex y has one neighbor in $y \in B_{uv}^{r-2}$ and the rest in $B_{vu}^{r-1} \cup X$. Hence $|N(y) \cap X| = k - 1 - |N(y) \cap B_{vu}^{r-1}|$ which is greater than or equal to $|N(x) \cap B_{vu}^{r-1}|$ by Lemma 4.1.1. ■

With these previous lemmas we are now ready to prove the bipartition theorem.

Theorem 4.1.1 [38] *Let G be a $(k; g)$ -cage of girth $g = 2r \leq 6$ and excess e . If $e \leq k - 2$ then G is bipartite and its diameter is $r + 1$.*

Proof. If a vertex $z \in X$ has all its neighbors in X then $|X| \leq k + 1$, contradicting the hypothesis. Hence every vertex in X must have a neighbor in T_{uv}^{r-1} . Let us suppose that $z \in X$ is adjacent to $x \in B_{vu}^{r-1}$ and to $y \in B_{uv}^{r-1}$. Then the sets $N(z) \cap X, N(x) \cap X - \{z\}, N(y) \cap X - \{z\}$ and $\{z\}$ are disjoint, therefore:

$$\begin{aligned} e = |X| &\geq |N(z) \cap X| + |N(x) \cap X| - 1 + |N(y) \cap X| - 1 + 1 \\ &= |N(z) \cap X| + |N(x) \cap X| + |N(y) \cap X| - 1 \\ &\geq |N(z) \cap X| + |N(z) \cap B_{vu}^{r-1}| + |N(z) \cap B_{uv}^{r-1}| \\ &\geq |N(z)| - 1 = k - 1. \end{aligned}$$

by Lemma 4.1.2, arriving at a contradiction. Hence every vertex in X has neighbors in B_{vu}^{r-1} or B_{uv}^{r-1} , but not in both. Define a partition $X = X_u \cup X_v$, the subsets of X whose vertices are at distance r from u and v , respectively. Suppose that X_u contains two adjacent vertices xy , by definition of X_u there are vertices x' and y' both in B_{uv}^{r-1} such that $xx' \in E(G)$ and $yy' \in E(G)$. The sets $(N(x) \cap X) - \{y\}, (N(y) \cap X) - \{x\}, \{x\}, \{y\}$ are disjoint, so

$$\begin{aligned} e = |X| &\geq |N(x) \cap X| - 1 + |N(y) \cap X| - 1 + 2 \\ &= |N(x) \cap X| + |N(y) \cap X|. \end{aligned}$$

Thus we have

$$\begin{aligned} e &\geq k - |N(x) \cap B_{uv}^{r-1}| + k - |N(y) \cap B_{uv}^{r-1}| \\ &= 2k - |N(x) \cap B_{uv}^{r-1}| + |N(y) \cap B_{uv}^{r-1}| \\ &\geq k + 1, \end{aligned}$$

by Lemma 4.1.1, contradicting the hypothesis $e \leq k - 2$, therefore X_u is an *independent set* (and similarly X_v). Hence, G is bipartite and of diameter $r + 1$. ■

By means of spectral techniques Biggs and Itto also obtained the following theorem.

Theorem 4.1.2 [38] *There is no regular graph with girth $2r \geq 8$ and excess $e = 2$.*

4.2 Constructions of girth pair graphs

In [62], Harary and Kóvacs introduced the concept of $(k; g, h)$ -cages and proved their existence for $3 \leq g < h$, obtaining the bound $n(k; g, h) \leq 2n(k; h)$, they also showed that in it general is not the best.

Proposition 4.2.1 [62] *If $k \geq 3$ and $g \geq 4$ then $n(k; h - 1, h) \leq n(k; h)$.*

As a consequence they stated the following conjecture.

Conjecture 4.2.1 [62] *$n(k; g, h) \leq n(k; h)$, for all $k, g \geq 3$.*

Later Xu, Wang, Wang proved that the inequality in Proposition 4.2.1 is strict.

Theorem 4.2.1 [114] *$n(k; h - 1, h) < n(k; h)$, where $k \geq 3, h \geq 4$.*

Regarding the 3-regular cages of girth pair $(4; h)$, Kovács proved the following:

Theorem 4.2.2 [80] *Let h be an odd integer with $h \geq 5$. Then the Möbius ladder of order $2(h - 1)$ is the unique minimal $(3; 4, h)$ -graph (i.e., is the $(3; 4, h)$ -cage).*

Campbell [48] studied the size of smallest cubic graphs with girth pair $(6, b)$ and constructed the cages for the exact values $(3; 6, 7)$, $(3; 6, 9)$ and $(3; 6, 11)$.

In order to study the cycles in cages, the following lemma is a useful consequence of Theorem 1.3.2.

Lemma 4.2.1 ([77]) *Let G be a $(k; g)$ -cage with $k \geq 3$ and girth $g \geq 4$. Then every edge of G lies on at least $k - 1$ cycles of length at most $g + 1$.*

4.2.1 Constructions for g odd

Lemma 4.2.2 *Let G be a $(k; h)$ -cage with $k \geq 3$ and even girth $h \geq 6$. Then G contains a girdle β such that $V(\beta) \cap B_{uv}^{h/4-1} = \emptyset$ or $V(\beta) \cap B_{vu}^{h/4-1} = \emptyset$.*

Proof. Let uv be an edge of a girdle α of G , take the subgraph T_{uv}^ℓ for $\ell = h/2 - 1$. There exists an edge $u_\ell v_\ell \in E(\alpha)$, where $u_\ell \in B_{uv}^\ell$ and $v_\ell \in B_{vu}^\ell$. From Lemma 4.2.1, it follows that there is another girdle β of G such that $u_\ell v_\ell \in E(\beta)$. If $V(\beta) \cap B_{uv}^{h/4-1} \neq \emptyset$ and $V(\beta) \cap B_{vu}^{h/4-1} \neq \emptyset$ then $|E(\beta)| \geq 4(\ell - (h/4 - 1)) + 2 = 4(h/4) + 2 > h$, which is a contradiction. Therefore, either $V(\beta) \cap B_{uv}^{h/4-1} = \emptyset$ or $V(\beta) \cap B_{vu}^{h/4-1} = \emptyset$. ■

Theorem 4.2.3 *Let $h \geq 6$ even and $k \geq 3$. Suppose that there is a bipartite $(k; h)$ -cage. If $g \geq 5$ is an odd number such that $h/2 + 1 \leq g < h$ then*

$$n(k; g, h) \leq n(k; h) - 2 \sum_{i=0}^{(h-g-3)/2} (k-1)^i - (k-1)^{(h-g-1)/2}.$$

Proof. Let H be a bipartite $(k; h)$ -cage with $n(k; h)$ vertices. Take $uv \in E(H)$, the subgraph T_{uv}^ℓ for $\ell = h/2 - 1$, and the girdles α and β as in Lemma 4.2.2. From this lemma we may suppose $V(\beta) \cap B_{vu}^{h/4-1} = \emptyset$. Let $z_l \in V(\alpha) \cap B_{vu}^l$ and $w_l \in V(\alpha) \cap B_{uv}^l$, for $0 \leq l \leq \ell = h/2 - 1$. Hence $v = z_0$, $u = w_0$ and $\alpha = w_0 w_1 \cdots w_\ell z_\ell z_{\ell-1} \cdots z_0 w_0$.

Let $\ell_v = (h - g + 1)/2$ and $\ell_u = (h - g - 1)/2$. Notice that $(\ell - \ell_u) + (\ell - \ell_v) + 2 = g$. Let us consider the graph $G_0 = H - (\overline{B}_{uv}^{\ell_u-1} \cup \overline{B}_{vu}^{\ell_v-1})$ and observe that every vertex of G_0 has degree k except for the ones in $B_{uv}^{\ell_u} \cup B_{vu}^{\ell_v}$ of degree $k - 1$.

For each $s, s' \in B_{vu}^{\ell_v}$ their distance $d_H(s, s') \leq 2\ell_v$, which yields $d_{G_0}(s, s') \geq h - 2\ell_v = h - 2((h - g + 1)/2) = g - 1$; and for $t, t' \in B_{uv}^{\ell_u}$ their distance $d_{G_0}(t, t') \geq h - 2\ell_u = h - 2((h - g - 1)/2) \geq g + 1$. By a similar argument, the distance from $s \in B_{vu}^{\ell_v}$ to $t \in B_{uv}^{\ell_u}$ in G_0 is $h - (\ell_u + \ell_v + 1) \geq h - (h + g + 1) = g - 1$. Notice that all these distances are even because H is bipartite.

Let us define a graph G whose vertex set is $V(G) = V(G_0)$ and for k even its edge set is $E(G) = E(G_0) \cup M_u \cup M_v \cup \{w_{\ell_u} z_{\ell_v}\}$, where M_u is any matching connecting the vertices of $B_{uv}^{\ell_u} \setminus \{w_{\ell_u}\}$ and M_v is any matching connecting the vertices of $B_{vu}^{\ell_v} \setminus \{z_{\ell_v}\}$.

For k odd its edge set is $E(G) = E(G_0) \cup M_u \cup M_v \cup \{w_{\ell_u} z_{\ell_v}, w^* z^*\}$, where $w^* \in B_{uv}^{\ell_u} \setminus \{w_{\ell_u}\}$, M_u is a matching joining the vertices in $B_{uv}^{\ell_u} \setminus \{w_{\ell_u}, w^*\}$, $z^* \in B_{vu}^{\ell_v} \setminus \{z_{\ell_v}\}$, and M_v is a matching joining the vertices in $B_{vu}^{\ell_v} \setminus \{z_{\ell_v}, z^*\}$.

Therefore in both cases G is k -regular. The cycle $w_{\ell_u}w_{\ell_u+1}\cdots w_{\ell}z_{\ell}z_{\ell-1}\cdots z_{\ell_v}w_{\ell_u}$ has odd length $(\ell - \ell_u) + (\ell - \ell_v) + 2 = g$, the even cycle β is still contained in G because by hypothesis $g \geq h/2 + 1$ which yields $\ell_v \geq h/4$. And since any new even cycle must have at least two new edges, it follows that it must have length at least $2(g - 1) + 2 \geq h$. Therefore the girth pair of G is (g, h) and $|V(G)| = |V(G_0)| = n(k, h) - 2 \sum_{i=0}^{(h-g-3)/2} (k-1)^i - (k-1)^{(h-g-1)/2}$. Hence the result holds. ■

Theorem 4.2.4 *Let $h \geq 6$ even and $k \geq 3$. Suppose that there is a bipartite $(k; h)$ -cage. If g is an odd number such that $g < h$ then $n(k; g, h) < n(k; h)$.*

Proof. We may assume that $g \leq h/2$, otherwise, by Theorem 4.2.3, the conclusion holds. Let H be a bipartite $(k; h)$ -cage with $n(k; h)$ vertices. Take $uv \in E(H)$, the subgraph T_{uv}^{ℓ} for $\ell = h/2 - 1$, and the girdle $\alpha = w_0w_1\cdots w_{\ell}z_{\ell}z_{\ell-1}\cdots z_0w_0$ as in Theorem 4.2.3, that is, $u = w_0$ and $v = z_0$.

From Lemma 4.2.1, since $k \geq 3$, there is at least one edge xx' with $x \in B_{vu}^{\ell} \setminus \{z_{\ell}\}$ and $x' \in B_{uv}^{\ell} \setminus \{w_{\ell}\}$ such that $\beta = v \cdots xx' \cdots uv$ is a cycle of H of length h . Let $v_{\ell} \in B_{vu}^{\ell} \setminus \{x\}$ be such that $d_H(v_{\ell}, w_{(g-1)/2}) > \ell - (g-1)/2$, and let $v_{\ell-1}$ be a neighbor of v_{ℓ} that belongs to $B_{vu}^{\ell-1}$.

For k even consider the graph with vertex set $V(G) = V(H) \setminus \{v_{\ell}\}$ and edge set $E(G) = E(H - v_{\ell}) - \{w_{(g-1)/2}w_{(g+1)/2}, z_{(g-3)/2}z_{(g-1)/2}\} \cup (M_t \cup \{w_{(g-1)/2}z_{(g-3)/2}, w_{(g+1)/2}z_{(g-1)/2}\})$, where M_t is a matching between the vertices in $N(v_{\ell})$.

For k odd consider the graph with vertex set $V(G) = V(H) - \{v_{\ell-1}, v_{\ell}\}$ and edge set $E(G) = E(H - v_{\ell-1}v_{\ell}) - \{w_{(g-1)/2}w_{(g+1)/2}, z_{(g-3)/2}z_{(g-1)/2}\} \cup (M_s \cup M_t \cup \{w_{(g-1)/2}z_{(g-3)/2}, w_{(g+1)/2}z_{(g-1)/2}\})$, where M_s is a matching between the vertices in $N(v_{\ell-1}) - v_{\ell}$ and M_t is a matching between the vertices in $T = N(v_{\ell}) - v_{\ell-1}$.

In both cases G is a k -regular graph. Let us show that the girth pair of G is (g, h) . Suppose that k is odd. For each $s, s' \in N(v_{\ell-1}) - v_{\ell}$ or for $t, t' \in N(v_{\ell})$, their distance in $H - \{v_{\ell-1}, v_{\ell}\}$ is at least $h - 2$; and the distance from s to t is at least $h - 3$. Therefore a cycle of G having one or two such edges has length at least $h - 1 > g$. A cycle having the edge $w_{(g-1)/2}z_{(g-3)/2}$ and an edge from M_s must have length greater than $h/2 - 1 - (g-1)/2 + (g-1)/2 + 2 + h/2 - 3 + 1 = h - 1$. If the cycle has some edge in M_t its length must be greater than $h/2 - 1 - (g-1)/2 + h/2 - 1 + 2 + (g-3)/2 = h - 1$. A cycle having the edge $w_{(g+1)/2}z_{(g-1)/2}$ and an edge from M_s or from M_t must have length at least $2(h/2 + 1) = h + 2$.

Finally note that $w_0 \cdots w_{(g-1)/2}z_{(g-3)/2} \cdots z_0w_0$ is an odd cycle of length g , the cycle $w_{(g+1)/2} \cdots w_{\ell}z_{\ell} \cdots z_{(g-1)/2}w_{(g+1)/2}$ has odd length $h - g \geq g$ because $g \leq h/2$. Since the

cycle β of length h remains in G , the girth pair of G is (g, h) and the result holds. For k even the reasoning is similar. ■

In this way we have proved that Conjecture 4.2.1 holds for g odd and all values of h . When the girth g is even we could not settle the conjecture completely but asymptotically, in such a way that all but few remaining small cases are solved. That is what we present in the following subsection.

4.2.2 Constructions for g even and h odd large enough

In [48] the exact values $n(3; 6, 7) = 18$, $n(3; 6, 9) = 24$ and $n(3; 6, 11) = 28$ are determined. Also, it is proved that $n(3; 6, h) \leq \frac{1}{3}(10h + 2k)$ where $0 \leq k \leq 2$ and $h \equiv k \pmod{3}$. Hence in the following corollary we point out that Conjecture 4.2.1 holds for every cubic cage with girth pair $(6, h)$.

Corollary 4.2.1 $n(3; 6, h) < n(3; h)$.

Proof. In the survey [57] we can check that $n(3; 7) = 24$, $n(3; 9) = 58$ and $n(3; 11) = 112$. Then from the exact values shown in [48], the result holds for $h = 7, 9, 11$. Moreover, for $h \geq 13$, both the upper bound given in [48] and the Moore bound (1.1) imply $n(3; 6, h) \leq \frac{1}{3}(10h + 2k) < n_0(3, h) = 2(2^h - 1) \leq n(3; h)$. ■

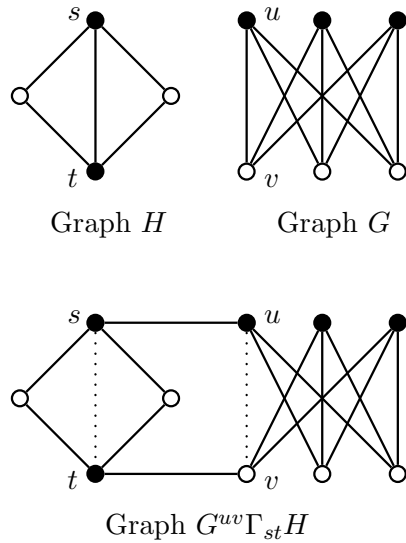
In our study of Conjecture 4.2.1 for every $(k; g, h)$ -cage when g is even and h odd we must introduce a construction that we will use later for breaking short odd cycles while preserving the regularity and the even girth.

Definition 4.2.1 Let G, H be two vertex-disjoint graphs, $uv \in E(G)$ and $st \in E(H)$. We will define a new graph $G^{uv}\Gamma_{st}H$, that we will call the insertion of (G, uv) into (H, st) by letting:

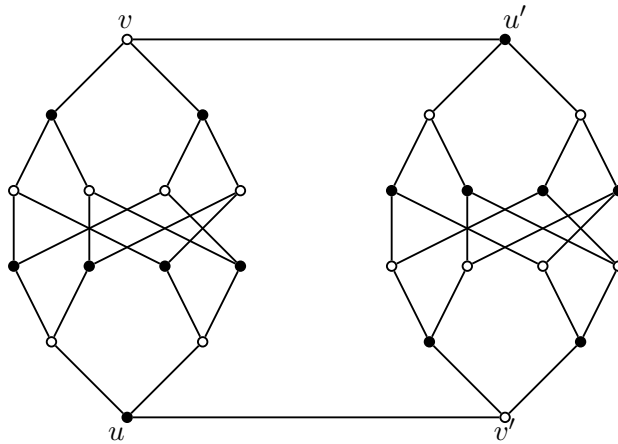
- $V(G^{uv}\Gamma_{st}H) = V(G) \cup V(H)$
- $E(G^{uv}\Gamma_{st}H) = (E(G) \setminus \{uv\}) \cup (E(H) \setminus \{st\}) \cup \{us, vt\}$.

See Figure 4.1, for an example illustrating this definition.

The well known Hajs construction [?], related to coloring and k -chromatic critical graphs, may be obtained by applying the insertion $G^{uv}\Gamma_{st}H$, and contracting the edges vt or us . Observe that if G and H are k -regular and bipartite then $G^{uv}\Gamma_{st}H$ is k -regular and bipartite.

Figure 4.1: The insertion $G^{uv} \Gamma_{st} H$

The first basic result we obtained with respect to g even and h odd is the following theorem; it is also useful as an introduction to the techniques we will use later. Notice that in all cases we are going to insert a graph (G, uv) into a copy $(G', u'v')$. By way of example let us consider the $(3; 6)$ -cage G or Heawood graph and G' a vertex disjoint copy of G . The graph $G^{uv} \Gamma_{v'u'} G'$ is depicted in Figure 4.2.

Figure 4.2: The graph $G^{uv} \Gamma_{v'u'} G'$.

Theorem 4.2.5 *Let $k \geq 3$ and $g \geq 6$ even. Then $n(k; g, 2g - 1) \leq 2n(k; g)$ provided that there is a bipartite $(k; g)$ -cage.*

Proof. Let G be a bipartite $(k; g)$ -cage with $k \geq 3$ and $g \geq 6$. Let $uv \in E(G)$ be an edge belonging to a girdle α of G , consider the insertion $G^{uv}\Gamma_{v'u'}G'$, where G' is a vertex disjoint copy of G and denote by $x' \in V(G')$ the copy of the vertex $x \in V(G)$. Observe that there is a natural 2-coloring (bipartition) of the vertices of $G^{uv}\Gamma_{v'u'}G'$, (the one inherited by the colors of u and v in G). Let $N_G(v) - u = \{v_1, v_2, \dots, v_{k-1}\}$, and $w \in (N_G(v_1) - v) \cap V(\alpha)$. Let us construct a new graph H from $G^{uv}\Gamma_{v'u'}G'$ as follows:

- Delete the edges vv_2 and v_1w and add the edges vw , v_1v_2 .
- Delete the edges v_1x (in G) and v'_1x' (in G') and add the edges v_1x' , v'_1x , for all $x \in N(v_1) \setminus \{v, w\}$.
- If $k \geq 4$, delete the edges v_iz (in G) and v'_iz' (in G') and add the edges v'_iz , v_iz' , for all $z \in N(v_i) \setminus \{v\}$ and $i = 3, \dots, k-1$.

It is straightforward to check that the resulting graph is k -regular. Also observe that the only monochromatic edges of H are vw and v_1v_2 . To prove the theorem we need to show that the girth pair of H is $(g, 2g-1)$.

Let C be a cycle of H having new edges. If $V(C) \subset V(G)$ then the path wv_1v_2 is contained in C which yields $|C| \geq 3 + d_{G-\{vv_2, v_1w\}}(w, v_2) \geq g$ because $d_{G-\{vv_2, v_1w\}}(w, v_2) \geq g-3$. Since $G - \{vv_2, v_1w\}$ is bipartite, $d_{G-\{vv_2, v_1w\}}(w, v_2)$ is odd because it has the same parity as $g-3$. Hence $|C| \geq g$ and is even.

Observe that if $V(C) \cap V(G') \neq \emptyset$, then $V(C) \cap V(G) \neq \emptyset$. Suppose that both $vw, v_1v_2 \notin E(C)$. If $v_1x' \in E(C)$ then C must go through $v_1v'u'$ or through $v_1vv_iz' \in N(v'_i) \setminus \{v'\}$ for $i \in \{3, \dots, k-1\}$. In the first case every $u'x'$ -path has odd length at least $g-3$, in the second case every $z'x'$ -path has even length at least $g-4$. Therefore C has even length at least g .

If C contains only one monochromatic edge, either vw or v_1v_2 , then C must be an odd cycle. If $vw \in E(C)$ then $C = vws \cdots uv' \cdots u'v$ ($s \neq v_1$) has length at least $2g-1$ and if $ws \in E(\alpha)$ then C is a $(2g-1)$ -cycle. If $v_1v_2 \in E(C)$ then $C = v_2v_1x' \cdots v'_iv'u \cdots v_2$ ($i \geq 2$) has length at least $2g-1$ because the $x'v'_i$ -path has length at least $g-3$ and the uv_i -path has length at least $g-2$.

Finally, let us prove that H contains a cycle of length g . From Lemma 4.2.1, it follows that G contains $k-2$ cycles of length g through the edge uv different from $\alpha = uvv_1ws \cdots u$. Let C_0 be one of such cycles; note that C_0 either goes through $\hat{x} \in N(v_1)$, $\hat{y} \in N(v_2)$, or $\hat{z} \in N(v_i)$ for $i \geq 3$. In the first case H contains the cycle $\hat{x}v'_1v'u \cdots \hat{x}$, in the second it contains the cycle $svvv_1v_2\hat{y} \cdots s$, in the latter it contains the cycle $\hat{z}v'_iv'u \cdots \hat{z}$, all three cycles of length g .

Therefore H has girth pair $(g, 2g-1)$. ■

To prove the corresponding result for $f(k; g, g + r)$ and $1 \leq r \leq g - 3$ we will use the following remark.

Remark 4.2.1 *Let G, H be graphs with girths g, h , respectively, such that $g \leq h$, and let $G^{uv}\Gamma_{st}H$ be the insertion of (G, uv) into (H, st) . Then the set of cycles in $G^{uv}\Gamma_{st}H$ is:*

$$\mathcal{C}(G^{uv}\Gamma_{st}H) = (\mathcal{C}(G) \setminus \{\alpha \in \mathcal{C}(G) : uv \in E(\alpha)\}) \cup (\mathcal{C}(H) \setminus \{\beta \in \mathcal{C}(H) : st \in E(\beta)\}) \cup \{\gamma = P_1vtP_2su : P_1 \text{ is a } uv\text{-path in } G - uv \text{ and } P_2 \text{ is a } ts\text{-path in } H - st\}.$$

This means that if there were cycles of lengths c_1 and c_2 in graphs G and H that used the edges uv and st , respectively, they are removed in the new graph $G^{uv}\Gamma_{st}H$ and new cycles of length $c_1 + c_2$ are created.

Theorem 4.2.6 *Let $k \geq 3$, g, g' even such that $6 \leq g$ and r an odd number such that $1 \leq r \leq g - 3$. Then $n(k; g, g + r) \leq 4n(k; g)$, provided that there is a bipartite $(k; g)$ -cage.*

Proof. Let G be a bipartite $(k; g)$ -cage with $k \geq 3$ and $g \geq 6$. Let $uv \in E(G)$ be an edge belonging to a girdle α of G , consider the insertion $G^{uv}\Gamma_{v'u'}G'$, where G' is a vertex disjoint copy of G and denote by $x' \in V(G')$ the copy of the vertex $x \in V(G)$. Let $N_G(v) - u = \{v_1, v_2, \dots, v_{k-1}\}$ and $N_G(u) - v = \{u_1, u_2, \dots, u_{k-1}\}$.

Let $z_l \in V(\alpha) \cap B_{vu}^l$ and $w_l \in V(\alpha) \cap B_{uv}^l$ for $0 \leq l \leq g/2 - 1$. Hence $v = z_0$, $u = w_0$ and $\alpha = w_0w_1 \cdots w_{g/2-1}z_{g/2-1}z_{g/2-2} \cdots z_0w_0$. Suppose without loss of generality that $v_2 = z_1$ and $u_1 = w_1$.

For each odd number r between 1 and $g - 3$ let $\ell = \frac{r+1}{2}$, note that $1 \leq \ell \leq g/2 - 1$. We will construct a $(k; g, g + r)$ -graph H from $G^{uv}\Gamma_{v'u'}G'$ as follows.

First we construct a graph H_0 from $G^{uv}\Gamma_{v'u'}G'$ in the following way:

- Delete the vertices v, u' and the edges $uu_2, v'v'_2$.
- Add the edges $\{v_iu'_i : i > 2\} \cup \{v_1v'\} \cup \{uu'_1\}$.

Second, we get the graph H from the graph H_0 by performing the following operations depending on ℓ :

- For $\ell = g/2 - 1$, add the edges $z_1z'_1 = v_2v'_2, u_2u'_2$.
- If $\ell = g/2 - 2$, denote $N_G(v_2) - v = \{v_{21}, v_{22}, \dots, v_{2(k-1)}\}$. Suppose without loss of generality that $v_{21} = z_2$ and recall that $u_1 = w_1$. Delete the vertices v_2, v'_2 and add the edges $\{u_2v'_{22}\} \cup \{v_{22}u'_2\} \cup \{v_{2i}v'_{2i} : i \neq 2\}$.

- When $\ell \leq g/2 - 3$, delete the edge $z_{g/2-\ell-1}z_{g/2-\ell}$ and add the edges $z_1z'_1, u_2z'_{g/2-\ell-1}, z_{g/2-\ell-1}u'_2, z_{g/2-\ell}z'_{g/2-\ell}$.

The resulting graph H is k -regular and observe that the only monochromatic edges of H are $\{z_1z'_1, u_2u'_2\}$ for $\ell = g/2 - 1$; $\{z_1z'_1, u_2v'_{22}, v_{22}u'_2\} \cup \{v_{2i}v'_{2i} : i > 2\}$ for $\ell = g/2 - 2$; $\{z_1z'_1, z_{g/2-\ell}z'_{g/2-\ell}\}$ for $\ell \leq g/2 - 3$, and possibly $\{u_2z'_{g/2-\ell-1}, z_{g/2-\ell-1}u'_2\}$ depending on the parity of ℓ for $\ell \leq g/2 - 3$.

We will show that the girth pair of H is $(g', g + r)$ for $g' \geq g$ even and the last step will be to guarantee the existence of cycles of length g in our graphs.

Let C be a cycle of H . If C contains only the edge v_1v' or uu'_1 its length must be even and greater than g , because such a cycle should use the uv' edge in both cases together with an v_1u -path or a $v'u'_1$ -path (respectively). If C contains only the edge $v_iu'_i$ its length must be even and greater than $2(g/2 - 1) + 2(g/2) \geq 2g - 2$.

Suppose that C contains only one monochromatic edge, observe that C must have odd length. If C contains the edge $z_{g/2-\ell}z'_{g/2-\ell}$ then it contains the $z_{g/2-1}z_{g/2-\ell}$ -path of length $\ell - 1$ and

$C = w_0w_1 \cdots w_{g/2-1}z_{g/2-1} \cdots z_{g/2-\ell}z'_{g/2-\ell} \cdots z'_{g/2-1}w'_{g/2-1} \cdots w'_1w_0$ has odd length $g + 2\ell - 1 = g + r$ including the case when $\ell = 1$. Similarly if C contains any other monochromatic edge its length is odd and it must be greater than $g + 2\ell - 1 \geq g + r$ by construction.

Suppose that C has two monochromatic edges then it has even length and we have the following cases:

For $\{z_1z'_1, u_2z'_{g/2-\ell-1}\} \subset E(C)$, the length of C is at least $2(g/2 - 1) + 2 = g$; observe that when $\ell = g/2 - 1$ we have $z_1z'_1, u_2u'_2$.

For $v_{2i}v'_{2i} \in E(C)$ together with $u_2v'_{22} \in E(C)$ or $v_{22}u'_2 \in E(C)$, we have that $|C| \geq g/2 - 1 + 3(g/2 - 2) + 1 > 2g - 2$, similarly if we have $v_{2i}v'_{2i} \in E(C)$ together with $v_{2j}v'_{2j} \in E(C)$, the length of C is at least $4(g/2 - 2) + 2 = 2g - 6$.

For the case when $\ell \leq g/2 - 3$, if C contains the monochromatic edge $z_1z'_1$ and the edge $u_2z'_{g/2-\ell-1}$ its length is odd at least $2(g/2 - 1) + (g/2 - 1 - \ell) \geq g$ but it may not be greater than $g + r$. Hence, in this case we must apply the insertion of G on both $u_2z'_{g/2-\ell-1}, z_{g/2-\ell-1}u'_2$ edges, that is $G^{uv}\Gamma_{u_2z'_{g/2-\ell-1}}H = H^1$ and $G^{uv}\Gamma_{z_{g/2-\ell-1}u'_2}H^1 = H^2$. In such a way we obtain a graph with $4n(k; g)$ vertices, nevertheless its odd girth is $g + r$ by Remark 4.2.1. Note that the insertion of G creates cycles of length g by Remark 4.2.1 and the observation that not all girdles in G use the uv edge, hence the girth pair of H^2 is $(g, g + r)$.

Finally, for the cases $g/2 - 2 \leq \ell \leq g/2 - 1$, we have proved that its girth pair is $\{g', g+r\}$, for even $g' \geq g$, therefore applying the insertion of G into any monochromatic edge different from $z_{g/2-\ell}z'_{g/2-\ell}$ we obtain a graph H^2 with girth pair $(g, g+r)$ and less than $4n(k; g)$ vertices, which finishes the proof. ■

Lemma 4.2.3 *Let $k \geq 3$, $g \geq 6$ even and suppose that there is a bipartite $(k; g)$ -cage. Then $n(k; g, mg+r) \leq 4n(k; g) + k(m-1)n(k; g)$, for $m \geq 1$ and r any odd number such that $1 \leq r \leq g-1$. In particular when $r = g-1$, from Theorem 4.2.5, we have $n(k; g, (m+1)g-1) \leq 2mn(k; g)$.*

Proof. First suppose that $r = g-1$. From Theorem 4.2.5, the lemma holds for $m = 1$. Moreover, let us recall that the graph H constructed in Theorem 4.2.5 has exactly two monochromatic edges vw and v_1v_2 . Let $H_1 = G^{uv}\Gamma_{vw}H$, and $H_2 = G^{uv}\Gamma_{v_1v_2}H_1$, that is, H_1 and H_2 are obtained by inserting G in both monochromatic edges vw and v_1v_2 . Notice that, by applying insertion, we get an even girth g in H_2 and its odd girth is $3g-1$ from Remark 4.2.1. Also notice that H_2 has only the two new monochromatic edges vw and uv_1 . Therefore H_2 is a $(k; g, 3g-1)$ -graph with at most $4n(k; g)$ vertices and exactly two monochromatic edges. Therefore, by applying the same argument inductively, we obtain that $n(k; g, (m+1)g-1) \leq 2mn(k; g)$, for every $m \geq 1$ as desired.

For r such that $1 \leq r \leq g-3$, the corresponding graph for $m = 1$ of Theorem 4.2.6, has at most k monochromatic edges, $k-2$ of the form xx' and in some case (for $\ell = g/2 - 2$) it has 2 more of the form $u_2v'_{22}$ and $v_{22}u'_2$. Let us label those k vertices as x_1, x_2, \dots, x_k . Let $H_1 = G^{uv}\Gamma_{x_1x'_1}H^2$, where H^2 is the $(k; g, g+r)$ -graph constructed in the proof of Theorem 4.2.6, $H_2 = G^{uv}\Gamma_{x_2x'_2}H_1$, until $H_k = G^{uv}\Gamma_{x_{k-1}x'_{k-1}}H_{k-1}$. By the same argument as above we have H_k is a $(k; g, 2g+r)$ -graph with at most $kn(k; g) + 4n(k; g)$ vertices. Again, inductively we obtain that $n(k; g, mg+r) \leq (k-1)mn(k; g) + 4n(k; g)$ for every $m \geq 1$, finishing the proof. ■

As a consequence we obtain the following theorem.

Theorem 4.2.7 *Suppose there is a bipartite $(k; g)$ -cage with degree $k \geq 3$ and even girth $g \geq 6$. Then $n(k; g, h) < n(k, h)$, for h sufficiently large.*

Proof. Any h can be expressed as $h = mg+r$, for some $m \in \mathbb{N}$ and $1 \leq r \leq g-1$. Given that $1 \leq r \leq g-1$ then $h = mg+r \leq (m+1)g-1$, for any $r \leq g-1$, and from Lemma 4.2.3, we know $n(k; g, mg+r) \leq 4n(k; g) + k(m-1)n(k; g)$. Therefore for any $h = mg+r$, we have $n(k; g, h) \leq 4n(k; g) + kmn(k; g)$, so the upper bound for $n(k; g, h)$ obtained in Lemma 4.2.3 is linear on h .

On the other hand, since the Moore bound equals $n_0(k; h) = 1 + k \sum_{i=0}^{(h-3)/2} (k-1)^i > k(k-1)^{(h-3)/2}$, it grows exponentially on h and it is a lower bound for $n(k; h)$. Thus, we obtain $n(k; g, h) < n(k; h)$, for h sufficiently large. ■

So, we have proved Conjecture 4.2.1 for even girth g in general but asymptotically. For specific values of k and g , the Conjecture 4.2.1 can be completely settled, as we will show in the following section.

4.2.3 Particular cases, small excess, and an exact value

As $n(k; g, h)$ grows linearly and $n(k; h)$ grows exponentially on h , it is expected that $n(k; g, h) < n(k; h)$, for not very large h . It is possible that some of the remaining cases can be treated separately as we will see next. Notice that for $1 \leq r \leq g-1$ the inequality $n(k; g, g+r) \leq 4n(k; g)$ is obtained in Theorem 4.2.6. Therefore, it is an interesting question for which cases the strict inequality $4n(k; g) < n(k; g+r)$ holds. The following corollary answers this question when the $(k; g)$ -cage achieves the Moore bound. It is well known that cages with even girth g reach the Moore bound $n_0(k; g)$ only when $g = 6, 8, 12$ and $k = q+1$, where q is a prime power, and such cages are bipartite.

Corollary 4.2.2 *For every girth $g = 6, 8, 12$ and every prime power q ,*

$$(i) \ n(q+1; g, g+1) < n(q+1; g+1), \text{ for } q \geq 7;$$

$$(ii) \ n(q+1; g, g+r) < n(q+1; g+r), \text{ for every odd number such that } 3 \leq r \leq g-1.$$

Proof. Let us recall that the Moore bound for g even and $g+r$ odd is respectively:

$$n_0(q+1; g) = 2 \sum_{i=0}^{(g-2)/2} q^i \text{ and } n_0(q+1; g+r) = 1 + (q+1) \sum_{i=0}^{(g+r-3)/2} q^i.$$

Hence, for $r = 1$ and $q \geq 7$, we get $4n_0(q+1; g) < n_0(q+1; g+1) \leq n(q+1; g+1)$ and (i) holds. If $r \geq 3$, we get $4n_0(q+1; g) < n_0(q+1; g+r) \leq n(q+1; g+r)$, for every q , and so (ii) holds. ■

From Lemma 4.2.3, we know that $n(k; g, mg+r) \leq 4n(k; g) + k(m-1)n(k; g) = k(m+3)n(k; g)$, for every $m \geq 1$, together with a similar argument as in Corollary 4.2.2, we obtain the following corollary.

Corollary 4.2.3 *For every prime power q , girth $g = 6, 8, 12$, all $m \geq 2$, every odd r such that $1 \leq r \leq g - 1$ and $h = mg + r$, $h > g + 1$, the following inequalities hold:*

$$\begin{aligned} n(q+1; g, h) = n(q+1; g, mg+r) &\leq (q+1)(m+3)n_0(q+1; g) \\ &< n_0(q+1; mg+r) \leq n(q+1; mg+r) \\ &= n(q+1; h). \end{aligned}$$

We conclude that Conjecture 4.2.1 holds for every prime power q , girths $g = 6, 8, 12$ and $h > g + 1$, if $h = g + 1$ it holds for every prime power $q \geq 7$.

Therefore the only remaining cases are the $(3; 6, 7)$, $(3; 8, 9)$, $(3; 12, 13)$, $(5; 6, 7)$, $(5; 8, 9)$, $(5; 12, 13)$, $(6; 6, 7)$, $(6; 8, 9)$, $(6; 12, 13)$ -cages. The $(3; 6, 7)$ -cage on 18 vertices is constructed in [48].

Another interesting consequence is for graphs with small excess. Let us recall that if a graph G is a $(k; g)$ -cage of even girth g and excess $e \leq k - 2$, from Theorem 4.1.1, it must be bipartite. Then the hypothesis for Lemma 4.2.3 is fulfilled. Hence $n(k; g, mg + r) \leq 4n(k; g) + k(m - 1)n(k; g)$. Thus, as in Corollary 4.2.2, we obtain:

Corollary 4.2.4 *Let G be $(k; g)$ -cage of even girth g , degree $k \geq 3$ and excess $e \leq k - 2$. It follows that:*

- (i) $n(k+1; g, g+1) < n(k+1; g+1)$, for $k \geq 8$;
- (ii) $n(k+1; g, g+r) < n(k+1; g+r)$, for every odd number such that $2 \leq r \leq g-1$.

Proof. From the Moore bound and the excess of G , the following equalities hold: $n(k+1; g) = 2 \sum_{i=0}^{(g-2)/2} k^i + k - 2$ and $n_0(k+1; g+r) = 1 + (k+1) \sum_{i=0}^{(g+r-3)/2} k^i$.

Hence, for $r = 1$ and $k \geq 8$, we get $4(2 \sum_{i=0}^{(g-2)/2} k^i + k - 2) < n_0(k+1; g+1) \leq n(k+1; g+1)$ and (i) holds.

If $r \geq 3$, by subtracting

$$n_0(k+1; g+3) - 4n(k+1; g) = (1 + (k+1) \sum_{i=0}^{g/2} k^i) - 4(2 \sum_{i=0}^{(g-2)/2} k^i + k - 2)$$

since $k \geq 3$ and $(k+1)k^{g/2} - 4(k-2) \geq 0$, then $n_0(k+1; g+r) - 4n(k+1; g) > 0$, concluding that $4n(k+1; g) < n_0(k+1; g+r) \leq n(k+1; g+r)$, for every $k \geq 3$, and (ii) holds. ■

In general, by a counting argument

$$(k+1)(m+3)n(k+1;g) = (k+1)(m+3)\left(2 \sum_{i=0}^{(g-2)/2} k^i + k - 2\right),$$

if we subtract it from $n_0(k+1;mg+r) = 1 + (k+1) \sum_{i=0}^{(mg+r-3)/2} k^i$ we get

$$1 + (k+1) \sum_{i=0}^{(mg+r-3)/2} k^i - (k+1)(m+3)\left(2 \sum_{i=0}^{(g-2)/2} k^i + k - 2\right),$$

it is positive if

$$\sum_{i=0}^{(mg+r-3)/2} k^i - (m+3)\left(2 \sum_{i=0}^{(g-2)/2} k^i + k - 2\right) \quad (4.1)$$

is positive.

Note that $k^{(mg+r-3)/2} \geq k^{(mg-2)/2}$ and $k^{(mg-2-j)/2} = (k^{(g-2-j)/2})^m (k^{(2+j)/2})^{m-1}$. Hence, we rewrite the last $(g-2)/2$ terms in the subtraction in the following way considering j even, $\sum_{j=0}^{g-2} (k^{(g-2-j)/2})^m (k^{(2+j)/2})^{m-1} - 2(m-3)(k^{(g-2-j)/2})$.

Therefore the Equation 4.1 would become:

$$1 + (k+1) \sum_{i=0}^{((m-1)g+r)/2} k^i + (k+1) \sum_{j=0}^{g-2} (k^{(g-2-j)/2})^m (k^{(2+j)/2})^{m-1} - 2(m-3)(k^{(g-2-j)/2}).$$

Since this sum has only positive terms, its positive, therefore $(k+1)(m+3)n(k+1;g) < n_0(k+1;mg+r)$ as desired. yielding

Corollary 4.2.5 *For every $(k;g)$ -cage of even girth g , degree $k \geq 3$ and excess $e \leq k-2$, any numbers $m \geq 2$, and odd r such that $1 \leq r \leq g-1$ and $h = mg+r$. The following inequalities hold:*

$$\begin{aligned} n(k+1;g,h) = n(k+1;g,mg+r) &\leq (k+1)(m+3)n(k+1;g) \\ &< n_0(k+1;mg+r) \leq n(k+1;mg+r) \\ &= n(k+1;h). \end{aligned}$$

That is, Conjecture 4.2.1 holds for graphs with girth pair (g,h) with $h \geq g+3$ and such that the corresponding $(k;g)$ -cage has excess at most $k-2$.

Concerning girth pair cages whose girth is odd we found an exact value using the Kronecker product together with the following proposition that states certain relevant characteristics of Kronecker-product graphs.

Proposition 4.2.2 *Let G be a connected graph with girth g . Then*

(i) [112] $G \otimes K_2$ is a bipartite graph. Furthermore, $G \otimes K_2$ is disconnected if and only if G is bipartite.

(ii) [1] For every $(u, i), (v, j) \in V(G \otimes K_2)$, $d_{G \otimes K_2}((u, i), (v, j)) \geq d_G(u, v)$.

(iii) [121] For every $u \in V(G)$, $d_{G \otimes K_2}((u, i), (u, j)) \geq g$ for $i \neq j$.

(iv) [121, 105] Let G be a graph with odd girth g . Then $g(G \otimes K_2) \geq g + 1$.

Theorem 4.2.8 $n(3; 5, 8) = 18$.

Proof. Regarding $(k; g, g+3)$ -graphs with $g \geq 5$ odd, in [?] it was proved that $n(k; g, g+3) > k + k(k-1)^{(g-1)/2}$, yielding $n(3; 5, 8) \geq 16$. If there was a $(3; 5, 8)$ -graph G on 16 vertices, then the Kronecker product $G \times K_2$ would be a $(3, 8)$ -graph on 32 vertices. Biggs and Ito [38] proved that this graph does not exist, see Theorem 4.1.2. Therefore $n(3; 5, 8) \geq 18$ because a cubic graph must have even order. Figure 4.4 depicts a $(3; 5, 8)$ -graph on 18 vertices which implies that $n(3; 5, 8) \leq 18$. To check that this graph has girth pair $(5, 8)$, it is enough to check that the the Kronecker product of this graph by K_2 is a graph on 36 vertices with girth 8. ■

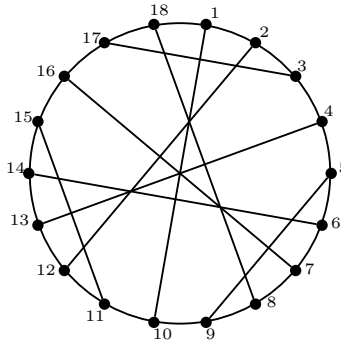


Figure 4.3: A $(3; 5, 8)$ -graph of 18 vertices.

4.3 Excess and bounds for girth pair graphs when $g = 6, 8, 12$

As the known cages of girth $g = 6, 8, 12$ reach the Moore bound the corresponding bounds for girth pair graphs can be improved. Hence, in this section we present lower and upper bounds on the order of the corresponding $(k; g, h)$ -cages. Also we obtained a result concerning the bipartition of $(k; 6)$ -cages.

In the following theorem we gather the more recent upper bounds on the order of $(k; g)$ -cages for $g = 6, 8, 12$ and every $k \leq q$ where q is a prime power.

Theorem 4.3.1 *Let $q \geq 2$ be a prime power and $g = 6, 8, 12$.*

- (i) [3, 13] $n(q; 6) \leq 2(q^2 - 1)$;
- (ii) [13] $n(q - 1; 6) \leq 2(q^2 - q - 2)$;
- (iii) [7] $n(k; 6) \leq 2(qk - 2)$ for all $k \leq q - 1$;
- (iv) [66] $n(k; 6) \leq 2(kq - (q - k)(\sqrt{q} + 1) - \sqrt{q})$ for all $k \leq q$ and q is a square;
- (iv) [14] $n(k; 8) \leq 2q(qk - 1)$ for all $k \leq q$;
- (v) [66] $n(q; 8) \leq 2q(q^2 - 2)$ if q is a square;
- (vi) [8] $n(k; 12) \leq 2kq^2(q^2 - 1)$ for all $k \leq q$.

4.3.1 Lower bounds and excess

As an immediate consequence from Theorem 4.1.1 we can write the following corollary.

Corollary 4.3.1 *The order of every $(k; g, h)$ -graph with $k \geq 3$, $g \geq 6$ even is at least $n(k; g, h) \geq n_0(k, g) + k - 1$.*

By (1.1) we have $n_0(k, 6) = 2(k^2 - k + 1)$. Then, for the particular case when $k = 3$ and $g = 6$, Corollary 4.3.1 implies that $n(3; 6, h) \geq 16$. The following result which is an improvement of Corollary 4.3.1, for $k = 3$ and $g = 6$, can be found in [48].

Theorem 4.3.2 [48] *The order of every $(3; 6, h)$ -graph is at least $n(3; 6, h) \geq (7h + 1)/3$ for all $h \geq 7$.*

In order to improve Corollary 4.3.1 for $g = 6$ and to extend Theorem 4.3.2 for any degree $k \geq 3$. We proved the following two lemmas.

Lemma 4.3.1 *Let G be a $(k; g, h)$ -graph with $k \geq 3$, $g \geq 6$ even and γ be an h -cycle of G . Then every vertex of $G - \gamma$ is joined to at most one vertex of γ .*

Proof. Note that γ is an induced subgraph of G since γ has no chord, otherwise an odd h' -cycle with $h' < h$ results in G which is a contradiction. If some vertex z of $G - \gamma$ is joined to $u, v \in V(\gamma)$ and $d_\gamma(u, v) = \ell$, then G contains two cycles, one of length $\ell + 2$ and another of length $h - \ell + 2$. If ℓ is even, $\ell + 2 \geq g$ and $h - \ell + 2 \geq h$ must hold. Consequently, $\ell \leq 2$, implying that $\ell + 2 \leq 4$ which is a contradiction because $\ell + 2 \geq g \geq 6$. Therefore ℓ is odd, $\ell + 2 \geq h$ and $h - \ell + 2 \geq g$ must hold. Then, from these two inequalities we obtain $h - \ell + 2 \leq h - (h - 2) + 2 = 4$ which is again a contradiction. ■

Lemma 4.3.2 *Let G be a $(k; g, h)$ -graph with $k \geq 3$, $g \geq 6$ even and $h \geq g + 1$ odd. Let γ be an h -cycle of G and w any vertex in $N(\gamma) \setminus V(\gamma)$. If $g = 6$, w is adjacent to at most one vertex in $N(\gamma) \setminus V(\gamma)$; and if $g \geq 8$, w is adjacent to no vertex in $N(\gamma) \setminus V(\gamma)$.*

Proof. We reason by contradiction assuming that there are $x, y, z \in N(\gamma) \setminus V(\gamma)$ such that $x, z \in N(y)$. Let $u_x, u_y, u_z \in V(\gamma)$ be such that $u_x x, u_y y, u_z z \in E(G)$ and suppose that $d_\gamma(u_x, u_y) = \ell_1$, $d_\gamma(u_y, u_z) = \ell_2$ and $d_\gamma(u_x, u_z) = \ell_1 + \ell_2$. Observe that the $u_x u_y$ -path of length ℓ_1 together with the path $u_x x y u_y$ form a cycle of length $\ell_1 + 3$. Therefore $\ell_1 + 3 \geq g$ if ℓ_1 is odd or $\ell_1 + 3 \geq h$ if ℓ_1 is even. In either case we have $\ell_1 \geq 3$ and analogously $\ell_2 \geq 3$. If $\ell_1 + \ell_2$ even, then $h - (\ell_1 + \ell_2) + 4 \geq h$ yielding $\ell_1 + \ell_2 \leq 4$ which is a contradiction. Therefore $\ell_1 + \ell_2$ is odd, which implies that $h - (\ell_1 + \ell_2) + 4 \geq g$. Moreover we can assume that ℓ_1 is odd and ℓ_2 is even so that $\ell_1 \geq h - 3$ and $\ell_2 \geq g - 3$. Thus $g \leq h - (\ell_1 + \ell_2) + 4 \leq h - (h + g - 6) + 4 = 10 - g$, yielding $g \leq 5$ which is a contradiction.

Finally, suppose that $x, y \in N(\gamma) \setminus V(\gamma)$ such that $x \in N(y)$. Let $u_x, u_y \in V(\gamma)$ be such that $u_x x, u_y y \in E(G)$ and suppose that $d_\gamma(u_x, u_y) = \ell$. As above we have $\ell + 3 \geq g$ and $h - \ell + 3 \geq h$ if ℓ is odd; or $\ell + 3 \geq h$ and $h - \ell + 3 \geq g$ if ℓ is even. In either case we conclude that this is only possible if $g = 6$. ■

Let $S_{h,k}$ denote the graph obtained from a cycle of length h attaching to each vertex $k - 2$ pendant edges. By Lemma 4.3.1 and Lemma 4.3.2, an h -cycle γ of G determines an $S_{h,k}$ and every vertex of $S_{h,k}$ not on the cycle γ is joined to at most two vertices of $S_{h,k}$ if $g = 6$ or is joined to at most one vertex if $g \geq 8$. We use these facts in the following theorem.

Theorem 4.3.3 *Let G be a $(k; g, h)$ -graph with $k \geq 3$, $g \geq 6$ even and $h \geq g + 1$ odd. Then*

$$n(k; g, h) \geq \begin{cases} \max\{h(k-1) + 2(k-2)^2, (2k-5)h + \lceil 4h/k \rceil\} & \text{if } g = 6; \\ \max\{h(k-1) + (g-4)(k-2)(k-1), (2k-4)h + \lceil 2h/k \rceil\} & \text{if } g \geq 8. \end{cases}$$

Proof. Let G be a $(k; g, h)$ -graph and $\gamma = u_1 u_2 \cdots u_h$ an h -cycle. By Lemma 4.3.1, G contains a subgraph $S_{h,k}$ consisting of the h -cycle γ and $k - 2$ pendant edges $u_i z_{i,j}$ attached

to each vertex u_i of γ . Then $|V(S_{h,k})| = h + h(k-2) = hk - h$, and by Lemma 4.3.2, $|N_G(z_{i,j}) \cap V(S_{h,k})| \leq 2$ if $g = 6$ and $|N_G(z_{i,j}) \cap V(S_{h,k})| = 1$ if $g \geq 8$ for all $i = 1, \dots, 8$ and $j = 1, \dots, k-2$. Moreover, since $g \geq 6$, $N_G(z_{i,j}) \cap N_G(z_{i+t,s}) = \emptyset$, $i = 1, \dots, h$, $t = 0, 1, \dots, g-4$, the sum of subindex taken modulo h , and $j, s = 1, \dots, k-2$. Let $\partial(S_{h,k}) = N(S_{h,k}) \setminus V(S_{h,k})$. Then

$$|\partial(S_{h,k})| \geq \sum_{i=1}^{g-4} \sum_{j=1}^{k-2} |N_G(z_{i,j}) \setminus V(S_{h,k})| \geq \begin{cases} 2(k-2)^2 & \text{if } g = 6; \\ (g-4)(k-2)(k-1) & \text{if } g \geq 8. \end{cases}$$

Therefore

$$\begin{aligned} |V(G)| &\geq |V(S_{h,k})| + |\partial(S_{h,k})| \geq h(k-1) + 2(k-2)^2 && \text{if } g = 6; \\ &\geq h(k-1) + (g-4)(k-2)(k-1) && \text{if } g \geq 8. \end{aligned} \quad (4.2)$$

On the other hand, by Lemma 4.3.1, every vertex of $S_{h,k}$ not lying on γ has degree at most 2 in $S_{h,k}$ if $g = 6$ or degree 1 if $g \geq 8$. Since G is k -regular $S_{h,k}$ must receive at least $h(k-2)^2$ edges from vertices external to $S_{h,k}$ if $g = 6$ or $h(k-2)(k-1)$ edges if $g \geq 8$. Thus, if $g = 6$, there must be at least $\lceil (k-2)^2 h/k \rceil$ external vertices to $S_{h,k}$, yielding at least $hk - h + \lceil (k-2)^2 h/k \rceil = (2k-5)h + \lceil 4h/k \rceil$ vertices in G . And if $g \geq 8$, there must be at least $\lceil (k-2)(k-1)h/k \rceil$ external vertices to $S_{h,k}$, yielding at least $hk - h + \lceil (k-2)(k-1)h/k \rceil = (2k-4)h + \lceil 2h/k \rceil$ vertices in G . Hence combining these results with (4.2) the theorem holds. ■

The following result is immediate from Theorem 4.3.3 and it is an extension of Theorem 4.3.2, for any degree $k \geq 3$ and an improvement of Corollary 4.3.1, for $g = 6$ and $h \geq 9$.

Corollary 4.3.2 *The order of every $(k; 6, h)$ -graph with $k \geq 3$, and $h \geq 7$ is at least*

$$n(k; 6, h) \geq \begin{cases} (2k-5)h + \lceil 4h/k \rceil & \text{if } h \geq 2k+1; \\ h(k-1) + 2(k-2)^2 & \text{if } h \leq 2k-1. \end{cases}$$

Note that Corollary 4.3.2 gives the same bound as Corollary 4.3.1, for $g = 6$ and $h = 7$, and improves it for $h \geq 9$.

The following theorem implies that graphs with larger excess than that of Theorem 4.1.1 are also bipartite, when $g = 6$, and they do not contain odd cycles of length at most $2k-1$.

Theorem 4.3.4 *Every $(k, 6)$ -graph with $k \geq 3$ free of odd cycles of length at most $2k-1$ and order at most $n_0(k, 6) + 2k^2 - 6k + 1$ must be bipartite.*

Proof. By Corollary 4.3.2, the minimum possible order of a $(k, 6)$ -graph with odd girth $h \geq 2k+1$ is at least $(2k-5)h + \lceil 4h/k \rceil \geq (2k-5)(2k+1) + \lceil 4(2k+1)/k \rceil = 4k^2 - 8k + 4$. By

(1.1) we have $n_0(k, 6) = 2(k^2 - k + 1)$. Therefore a $(k, 6)$ -graph free of odd cycles of length at most $2k + 1$ and order at most $n_0(k, 6) + 2k^2 - 6k + 1$ can not have odd cycles. Hence this graph is bipartite. ■

With respect to the conjecture that cages with even girth are bipartite (see [126, 128]). As a consequence of both Theorem 4.3.1 and Theorem 4.3.4, we can establish the following result which is a contribution to this problem for cages of girth $g = 6$.

Theorem 4.3.5 *Every $(k, 6)$ -cage is bipartite if it is free of odd cycles of length at most $2k - 1$.*

Proof. For $k = 3$ the only $(3, 6)$ -cage is Heawood's graph which is bipartite. Hence, we may suppose $k \geq 4$. First, suppose $k = p$ where $p \geq 4$ is a prime number. By Theorem 4.3.1 we know that $n(p, 6) \leq 2(p^2 - 1)$. Since $2(p^2 - 1) = n_0(p, 6) + 2p - 4$ and $2p - 4 \leq 2p^2 - 6p + 1$ for all $p \geq 4$ it follows from Theorem 4.3.4 that every $(p, 6)$ -cage is bipartite if it is free of odd cycles of length at most $2p - 1$. Now assume that $k \leq p - 1$. By Theorem 4.3.1, we know that $n(k, 6) \leq 2(pk - 2)$. Since $2(pk - 2) = n_0(p, 6) + 2k(p - k + 1) - 6$ and $2k(p - k + 1) - 6 \leq 2k^2 - 6k + 1$ holds when $p - k \leq k - 4$, the result will be true by Theorem 4.3.4 if the $(k, 6)$ -cage is free of odd cycles of length at most $2k - 1$, for all $k \geq (p + 4)/2$. Finally, by Bertrand's postulate it always exists a prime number p such that $k \leq p < 2k - 3$. Therefore $(p + 3)/2 < k$ as desired, concluding the proof. ■

The following result is also immediate from Theorem 4.3.3.

Corollary 4.3.3 *The order of every $(k; g, h)$ -graph with $k \geq 3$, $g \geq 8$ even and $h \geq g + 1$ odd is at least*

$$n(k; g, h) \geq \begin{cases} (2k - 4)h + \lceil 2h/k \rceil & \text{if } h \geq k(g - 4); \\ h(k - 1) + (g - 4)(k - 2)(k - 1) & \text{if } h < k(g - 4). \end{cases}$$

In the following corollaries we observe that Corollary 4.3.3 is an improvement of Corollary 4.3.1 for $g = 8, 12$ whenever h is large enough in terms of k .

Corollary 4.3.4 *If $h \geq k^2 + 3$ odd, then $n(k; 8, h) \geq n_0(k; 8, h) + 2k - 6$.*

Corollary 4.3.5 *If $h \geq k^4$ odd, then $n(k; 12, h) \geq n_0(k; 12, h) + 4k^4 - 12k^3 + 12k^2 - 6k + 1$.*

4.3.2 Upper bounds

In the following theorem we establish an upper bound on the order of every $(q+1; 6, 7)$ -graph.

Theorem 4.3.6 *Let $q \geq 2$ be a prime power. Then*

$$n(q+1; 6, 7) \leq 2n_0(q+1, 6) - 4q - 2 = 4q^2 + 2.$$

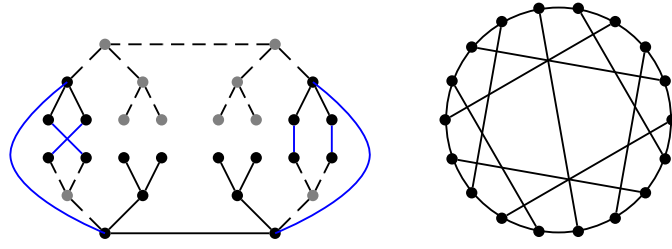
Proof. Let Γ_q be a $(q+1, 6)$ -cage on $n_0(q+1, 6)$ vertices. Since Γ_q is a Moore cage it follows that Γ_q is the incidence graph of a projective plane of order q . Let $V(\Gamma_q) = (P, L)$ where P is the set of points and L is the set of lines of the projective plane. A point p and a line ℓ are adjacent in Γ_q if they are incident in the corresponding projective plane. Let us take $p^* \in P$ and $\ell^* \in L$ such that $p^*\ell^* \notin E(\Gamma_q)$. Let $Z = \Gamma_q - (N_{\Gamma_q}[p^*] \cup N_{\Gamma_q}[\ell^*])$. Every vertex z of Z has degree q because if $z \in P$, then there exists a unique line $\hat{\ell} \in L$ such that $p^*, z \in N_{\Gamma_q}(\hat{\ell})$; hence $\hat{\ell} \in N_{\Gamma_q}(p^*)$. And if $z \in L$, there exists a unique point $\hat{p} \in P$ such that $z, \ell^* \in N_{\Gamma_q}(\hat{p})$; hence $\hat{p} \in N_{\Gamma_q}(\ell^*)$.

Let $\ell_0 \in N_{\Gamma_q}(p^*)$ and denote by $P_0 = N_{\Gamma_q}(\ell_0) \cap V(Z)$. Let $\{p_0\} = N_{\Gamma_q}(\ell_0) \cap N_{\Gamma_q}(\ell^*)$ and denote by $L_0 = N_{\Gamma_q}(p_0) \cap V(Z)$. Note that $|P_0| = |L_0| = q - 1$ and that $V(Z)$ can be partitioned as $P_0 \cup L_0 \cup N_Z(P_0) \cup N_Z(L_0)$.

Let Γ'_q be a vertex disjoint copy of Γ_q and denote by $z' \in V(\Gamma'_q)$ the copy of the vertex $z \in V(\Gamma_q)$. Let us define $Z' = \Gamma'_q - (P'_0 \cup L'_0)$ and observe that every vertex of Z' has degree $q+1$ except the neighbors of each vertex in $P'_0 \cup L'_0$. More precisely, the vertices ℓ'_0 and p'_0 have degree 2 in Z' , any vertex of $N_{Z'}(P'_0 \cup L'_0)$ has degree q and any other vertex of Z' has degree $q+1$. Let $U = Z \cup Z'$. We construct a new graph G from U as follows:

- Add the edges $\ell'_0 p$ for all $p \in P_0$, and add the edges $p'_0 \ell$ for all $\ell \in L_0$.
- Add the edges aa' for all $a \in N_Z(L_0)$.
- For all $p \in P_0$ connect the vertices of $N_Z(p)$ with its copies in Z' by a perfect matching M such that if $bc' \in M$ then $b \neq c$.

By way of example the resulting graph G for $q = 2$ is depicted on the right of Figure 4.4. On the left side in Figure 4.4, two spanning trees of a $(3, 6)$ -cage are depicted. The eliminated vertices are incident with dashed lines and are in gray color while the added edges are in blue color. We can check that G is $(q+1)$ -regular and has cycles of length 6 since Z and Z' have cycles of length 6. For example there are 6-cycles in Z' through the path $\ell'^* p'_0 \ell'_0 p'^*$. To prove the theorem we need to show that the girth pair of G is $(6, 7)$. Let C be a cycle of G having new edges. By construction C must have at least two new edges, i.e., $V(C) \cap V(Z') \neq \emptyset$ and $V(C) \cap V(Z) \neq \emptyset$.

Figure 4.4: A $(3; 6, 7)$ -cage on 18 vertices.

If two edges $\ell'_0 p, y'_0 z \in E(C)$ with $p, z \in P_0$, then C has even length at least 6 because since $p, z \in N_{\Gamma_q}(\ell_0) \cap V(Z)$ it follows that $d_Z(p, z) \geq 4$. The same occurs if two edges $p'_0 \ell, p'_0 z \in E(C)$ with $\ell, z \in L_0$.

If two edges $\ell'_0 p, aa' \in E(C)$ with $p \in P_0$ and $a \in N_Z(L_0)$, then C has odd length at least 7 because $d_Z(a, p) = 2$ and $d_{Z'}(a', \ell'_0) = 3$. The same occurs if two edges $p'_0 \ell, bc' \in E(C)$ with $\ell \in L_0$ and $b, c \in P_0$.

If two edges $\ell'_0 p, bc' \in E(C)$ with $p \in P_0$ and $b, c \in N_Z(P_0)$, then C has odd length at least 7 because $d_Z(b, p) \geq 1$ and $d_{Z'}(c', \ell'_0) = 4$ because both c'_0 and ℓ'_0 were adjacent to a vertex from P'_0 that has been eliminated from Γ'_q to obtain Z' . The same occurs if two edges $p'_0 \ell, aa' \in E(C)$ with $\ell \in L_0$ and $a \in N_Z(L_0)$.

Finally, if $aa', bc' \in E(C)$ with $a \in N_Z(L_0)$ and $b, c \in N_Z(P_0)$, then $|E(C)| \geq d_Z(a, b) + d_{Z'}(a', c') + 2 \geq 6$ because note that if $ab \in E(\Gamma_q)$ then $d_Z(a, c) = 3$ since $b, c \in N(p)$ for some $p \in P_0$.

Therefore G is a $(q+1; 6, 7)$ -graph of order $2n_0(q+1, g) - |N_{\Gamma_q}[p^*] \cup N_{\Gamma_q}[\ell^*]| - |P_0 \cup L_0| = 2n_0(q+1, g) - 2(q+2) - 2(q-1)$, so the theorem holds. ■

By (1.1) we have $n_0(k, 6) = 2(k^2 - k + 1)$. Hence, by Theorem 4.3.6, we can observe that the strict inequality $n(q+1; 6, 7) < n(q+1, 7)$ is fulfilled as proved in [114]. Moreover, as a consequence of Corollary 4.3.1 and Theorem 4.3.6 we obtain the following corollary.

Corollary 4.3.6 *Let $q \geq 2$ be a prime power. Then $2q^2 + 3q + 2 \leq n(q+1; 6, 7) \leq 4q^2 + 2$.*

For $q = 2$ the above corollary together with Theorem 4.3.2 gives in particular that $n(3; 6, 7) = 18$. Therefore we conclude that a $(3; 6, 7)$ -cage can be constructed considering two copies of a $(3, 6)$ -cage and following the proof of Theorem 4.3.6, see Figure 4.4.

Theorem 4.3.7 *Let $q \geq 2$ be a prime power and $g = 6, 8, 12$. Let ℓ be a positive integer such that $1 \leq \ell \leq \lfloor g/4 \rfloor$. Then*

$$n(q+1; g, 2g-4\ell+1) \leq 2n_0(q+1, g) - 4(q^\ell - 1)/(q-1).$$

Proof. Let Γ_q be a $(q+1, g)$ -cage with $g = 6, 8, 12$ and $n_0(q+1, g)$ vertices. Let $uv \in E(\Gamma_q)$ and consider a tree $T_{\ell-1}$ rooted in uv of degree $q+1$ and of depth $\ell-1$ for $1 \leq \ell \leq \lfloor g/4 \rfloor$ (if $\ell = 1$ the tree $T_{\ell-1}$ is the edge uv). Let $Z = \Gamma_q - V(T_{\ell-1})$, Z' a vertex disjoint copy of Z and denote by $z' \in V(Z')$ the copy of the vertex $z \in V(Z)$. Let $U = Z \cup Z'$ and denote by Ω the vertices of Z that were adjacent to vertices in $T_{\ell-1}$ in Γ_q . Let us consider the partition $\Omega = \Omega^u \cup \Omega^v$ where $\Omega^u = \{w \in \Omega : d_{\Gamma_q}(w, u) = \ell, d_{\Gamma_q}(w, v) = \ell + 1\}$ and $\Omega^v = \{w \in \Omega : d_{\Gamma_q}(w, v) = \ell, d_{\Gamma_q}(w, u) = \ell + 1\}$. Since $\ell \leq \lfloor g/4 \rfloor$ it follows that every vertex of $\Omega \cup \Omega'$ has degree q and any other vertex of U has degree $q+1$. We construct a new graph G from U by adding a matching connecting the vertices of Ω with the vertices of Ω' as follows:

- Choose a vertex $x_1 \in \Omega^u$ and a vertex $y_1 \in \Omega^v$. Add the edges $x_1x'_1, y_1y'_1$ and match arbitrarily each $x \in \Omega^u - x_1$ with one $y \in \Omega^v - y_1$, then add the edges xy', yx' .

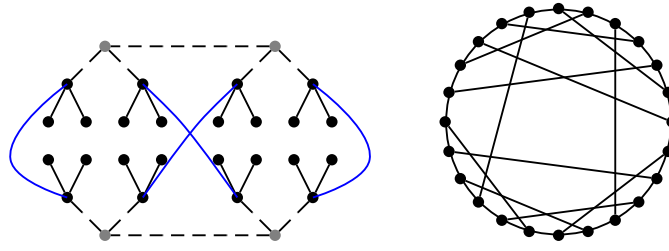
By way of example the resulting graph G for $q = 2$ is depicted on the right of Figure 4.5. On the left side in Figure 4.5, two spanning trees of a $(3, 6)$ -cage are depicted. The eliminated vertices are incident with dashed lines and are in gray color while the added edges are in blue color. Note that G is $(q+1)$ -regular and has cycles of length g since Γ_q is a Moore cage. Also observe that the monochromatic edges of G are $x_1x'_1, y_1y'_1$ and denote by $S = \{xy', yx' : x \in \Omega^u - x_1, y \in \Omega^v - y_1\}$ the set of new heterochromatic edges. To prove the theorem we need to show that the girth pair of G is $(g, 2g - 4\ell + 1)$. Let C be a cycle of G having new edges. By construction C must have at least two new edges, i.e., $V(C) \cap V(Z') \neq \emptyset$ and $V(C) \cap V(Z) \neq \emptyset$.

Suppose that C has no monochromatic edge and let $a_1b_1, a_2b_2 \in E(C) \cap S$. Then C has even length at least $|E(C)| \geq d_G(a_1, a_2) + d_G(b_1, b_2) \geq 2(g - 2\ell - 1) + 2 = 2g - 4\ell \geq g$ because $\ell \leq \lfloor g/4 \rfloor$.

Suppose C contains exactly one monochromatic edge. If $x_1x'_1 \in E(C)$ and $ab \in E(C) \cap S$, then C has odd length at least $|E(C)| \geq d_G(x_1, a) + d_G(x'_1, b) \geq (g - 2\ell - 1) + (g - 2\ell) + 2 = 2g - 4\ell + 1$. The same occurs if $y_1y'_1 \in E(C)$.

Finally if C contains both monochromatic edges, $x_1x'_1$ and $y_1y'_1$, then C has even length at least $|E(C)| \geq 2(g - 2\ell - 1) + 2 = 2g - 4\ell \geq g$.

Moreover G has cycles of length exactly $2g - 4\ell + 1$. Indeed, since Γ_q is a Moore cage, there exists in G a x_1x -path of length exactly $g - 2\ell$ for $x \in \Omega^u - x_1$; and there exists in G a x'_1y' -path of length exactly $g - 2\ell - 1$ for $y' \in \Omega'^v - y'_1$. Both paths together with the edges $x_1x'_1$ and xy' form a cycle of length $2g - 4\ell + 1$. Therefore G is a $(q+1; g, 2g - 4\ell + 1)$ -graph of order $2n_0(q+1, g) - 2|V(T_{\ell-1})| = 2n_0(q+1, g) - 4 \sum_{i=0}^{\ell-1} q^i$, so the theorem holds. ■

Figure 4.5: A $(3; 6, 9)$ -cage on 24 vertices.

As a consequence of Corollary 4.3.2 and Theorem 4.3.7 we obtain the following corollary for $g = 6$.

Corollary 4.3.7 $22 \leq n(3; 6, 9) \leq 24$ and $36 \leq n(4; 6, 9) \leq 48$. If $q \geq 4$ is a prime power, then $2q^2 + 5q + 2 \leq n(q + 1; 6, 9) \leq 4q^2 + 4q$.

The exact value $n(3; 6, 9) = 24$ is proved in [48], and a $(3; 6, 9)$ -cage can be constructed from the proof of Theorem 4.3.7, see Figure 4.5. Also as a consequence of Corollary 4.3.1 and Theorem 4.3.7 we obtain the following corollary for $g = 8, 12$.

Corollary 4.3.8 Let $q \geq 2$ be a prime power. Then

- (i) $n_0(q + 1, 8) + q \leq n(q + 1; 8, 9) \leq 4q^2(q + 1)$;
- (ii) $n_0(q + 1, 8) + q \leq n(q + 1; 8, 13) \leq 2n_0(q + 1, 8) - 4$;
- (iii) $n_0(q + 1, 12) + q \leq n(q + 1; 12, 13) \leq 4q^3(q^2 + q + 1)$;
- (iv) $n_0(q + 1, 12) + q \leq n(q + 1; 12, 17) \leq 2n_0(q + 1, 8) - 4(q + 1)$;
- (v) $n_0(q + 1, 12) + q \leq n(q + 1; 12, 21) \leq 2n_0(q + 1, 8) - 4$.

Theorem 4.3.8 Let $q \geq 2$ be a prime power and $g = 6, 8, 12$. Let ℓ be a positive integer such that $1 \leq \ell \leq \lfloor (g - 2)/4 \rfloor$. Then

$$n(q + 1; g, 2g - 4\ell - 1) \leq 2n_0(q + 1, g) - 4(q^\ell - 1)/(q - 1) - 2.$$

Proof. Let Γ_q be a $(q + 1, g)$ -cage with $g = 6, 8, 12$ and $n_0(q + 1, g)$ vertices. Let $uv \in E(\Gamma_q)$ and consider a tree $T_{\ell-1}$ rooted in uv of degree $q + 1$ and of depth $\ell - 1$ for $1 \leq \ell \leq \lfloor (g - 2)/4 \rfloor$. Let \hat{v} be a vertex in Γ_q such that $d_{\Gamma_q}(v, \hat{v}) = \ell$ and let $\hat{t} \in V(T_{\ell-1})$ such that $\hat{t}\hat{v} \in E(\Gamma_q)$. Let $Z = \Gamma_q - (V(T_{\ell-1}) \cup \{\hat{v}\})$, Z' a vertex disjoint copy of Z and denote by $z' \in V(Z')$ the copy

of the vertex $z \in V(Z)$. Let $U = Z \cup Z'$ and denote by Ω the vertices of Z that were adjacent to vertices in $T_{\ell-1}$ in Γ_q except \hat{v} , and A the vertices of Z that were adjacent to \hat{v} . Let us consider the partition $\Omega = \Omega^u \cup \Omega^v$ where $\Omega^u = \{w \in \Omega : d_{\Gamma_q}(w, u) = \ell, d_{\Gamma_q}(w, v) = \ell + 1\}$ and $\Omega^v = \{w \in \Omega : d_{\Gamma_q}(w, v) = \ell, d_{\Gamma_q}(w, u) = \ell + 1\}$. Clearly every vertex of $(\Omega \cup A) \cup (\Omega' \cup A')$ has degree q and any other vertex of U has degree $q + 1$. We construct a new graph G from U by adding a matching connecting the vertices of $\Omega \cup A$ with the vertices of $\Omega' \cup A'$ as follows:

- Choose a vertex $a_1 \in A$ and a vertex $b_1 \in \Omega^v$ such that $b_1 \hat{t} \notin E(\Gamma_q)$. Add the edges $a_1 b'_1, b_1 a'_1$ and the edges xx' for all $x \in (\Omega - b_1) \cup (A - a_1)$.

Observe that G is $(q + 1)$ -regular and has cycles of length g because Γ_q is a Moore cage. Also observe that the monochromatic edges of G are $a_1 b'_1, b_1 a'_1$. To prove the theorem we need to show that the girth pair of G is $(g, 2g - 4\ell - 1)$. Let C be a cycle of G having new edges. By construction C must have at least two new edges, i.e., $V(C) \cap V(Z') \neq \emptyset$ and $V(C) \cap V(Z) \neq \emptyset$.

Suppose C contains exactly one monochromatic edge. If $a_1 b'_1 \in E(C)$ and $xx' \in E(C)$, then C has odd length at least $|E(C)| \geq d_G(a_1, x) + d_G(b'_1, x') \geq (g - 2\ell - 2) + (g - 2\ell - 1) + 2 = 2g - 4\ell - 1$ because a_1 is at distance at least $g - 2\ell - 2$ from every $x \in \Omega \cup (A - a_1)$ and b_1 is at distance at least $g - 2\ell - 1$ from every $y \in (\Omega - b_1) \cup A$. The same occurs if $b_1 a'_1 \in E(C)$.

If C contains both monochromatic edges, $a_1 b'_1$ and $b_1 a'_1$, then C has even length at least $|E(C)| \geq 2(g - 2\ell - 1) + 2 = 2g - 2\ell \geq g$.

Finally, suppose that C has no monochromatic edge and let $xx', yy' \in E(C)$. If $x, y \in \Omega^u$, then C has even length at least $|E(C)| \geq 2(g - 2\ell) + 2 = 2g - 4\ell + 2 \geq g$. If $x \in \Omega^u$ and $y \in \Omega^v$, then C has even length at least $|E(C)| \geq 2(g - 2\ell - 1) + 2 = 2g - 4\ell \geq g$. If $x \in \Omega^u$ and $y \in A - a_1$, then C has even length at least $|E(C)| \geq 2(g - 2\ell - 2) + 2 = 2g - 4\ell - 2 \geq g$ because $\ell \leq \lfloor (g - 2)/4 \rfloor$.

Let us show that G has cycles of length exactly $2g - 4\ell - 1$. Since Γ_q is a Moore cage, there exists in G an $a_1 x$ -path of length exactly $g - 2\ell - 2$ and there exists in G a $b'_1 x'$ -path of length exactly $g - 2\ell - 1$ for $x \in \Omega^u$. Both paths together with the edges $a_1 b'_1$ and xx' form a cycle of length $2g - 4\ell - 1$. Therefore G is a $(q + 1; g, 2g - 4\ell - 1)$ -graph of order $2n_0(q + 1, g) - 2(|V(T_{\ell-1})| + 1) = 2n_0(q + 1, g) - 4 \sum_{i=0}^{\ell-1} q^i - 2$, so the theorem holds. ■

Also as a consequence of Corollary 4.3.1 and Theorem 4.3.8 we obtain the following corollary for $g = 8, 12$.

Corollary 4.3.9 *Let $q \geq 2$ be a prime power. Then*

- (i) $n_0(q+1, 8) + q \leq n(q+1; 8, 11) \leq 2n_0(q+1, 8) - 6;$
- (ii) $n_0(q+1, 12) + q \leq n(q+1; 12, 15) \leq 2n_0(q+1, 8) - 4q - 6;$
- (v) $n_0(q+1, 12) + q \leq n(q+1; 12, 19) \leq 2n_0(q+1, 12) - 6.$

Theorem 4.3.9 *Let $q \geq 2$ be a prime power and $g = 6, 8, 12$. Then*

$$n(q+1; g, 2g-1) \leq 2n_0(q+1, g).$$

Proof. Let Γ_q be a $(q+1, g)$ -cage with $g = 6, 8, 12$ and $n_0(q+1, g)$ vertices. Let $uv \in E(\Gamma_q)$ be an edge belonging to a girdle α of Γ_q , and consider the graph $X = \Gamma_q - uv$. Let X' be a vertex disjoint copy of X and denote by $x' \in V(X')$ the copy of the vertex $x \in V(X)$. Let $Y = X \cup X' + \{uv', vu'\}$. Observe that $d_Y(u, v) = d_Y(u', v') = g - 1$. Let $N_Y(v) - u' = \{v_1, v_2, \dots, v_q\}$ be such that $v_1 \in V(\alpha)$ and let $w \in (N_Y(v_1) - v) \cap V(\alpha)$. We construct a new graph G from Y as follows:

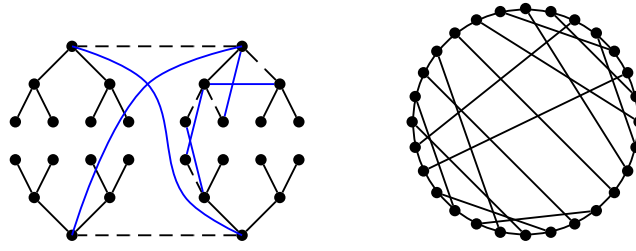
- Delete the edges vv_2 and v_1w and add the edges vw, v_1v_2 .
- Delete the edges v_1x (in X) and v'_1x' (in X') and add the edges v_1x', v'_1x for all $x \in N_Y(v_1) \setminus \{v, w\}$.
- If $q \geq 3$, delete the edges v_iz (in X) and v'_iz' (in X') and add the edges v'_iz, v_iz' for all $z \in N_Y(v_i) \setminus \{v\}$ and $i = 3, \dots, q$.

Observe that the resulting graph G is $(q+1)$ -regular and has cycles of length g since all the cycles of length g of Γ_q through v_2y where $y \in N_G(v_2) \setminus \{v_1\}$ remain in G . Also observe that the only monochromatic edges of G are vw and v_1v_2 . To prove the theorem we need to show that the girth pair of G is $(g, 2g-1)$.

Let C be a cycle of G having new edges. If $V(C) \subset V(X)$ then the path wvv_1v_2 is contained in C which yields C has length $|E(C)| = 3 + \ell$ where ℓ is the length of a wv_2 -path in $X - \{vv_2, v_1w\}$, so $\ell \geq g - 3$. Since $X - \{vv_2, v_1w\}$ is bipartite, ℓ is odd because it has the same parity as $g - 3$. Hence $|E(C)| = 3 + \ell \geq g$ is even.

Observe that if $V(C) \cap V(X') \neq \emptyset$, then $V(C) \cap V(X) \neq \emptyset$. Suppose that both $vw, v_1v_2 \notin E(C)$. If $v_1x' \in E(C)$, then C must go through v_1vu' or through $v_1vv_iz' \in N(v'_i) \setminus \{v'\}$ for $i \in \{3, \dots, q\}$. In the first case every $u'x'$ -path has odd length at least $g - 3$, in the second case every $z'x'$ -path has even length at least $g - 4$. Therefore C has even length at least g .

If C contains only one monochromatic edge, either vw or v_1v_2 , then C must be an odd cycle. If $vw \in E(C)$, then $C = vws \cdots uv' \cdots u'v$ ($s \neq v_1$) has length at least $2g - 1$ and if

Figure 4.6: A $(3; 6, 11)$ -cage on 28 vertices.

$ws \in E(\alpha)$ then C is a $(2g - 1)$ -cycle. If $v_1v_2 \in E(C)$, then $C = v_2v_1x' \cdots v'_i v'u \cdots v_2$ ($i \geq 2$) has length at least $2g - 1$ because the $x'v'_i$ -path has length at least $g - 3$ and the uv_i -path has length at least $g - 2$.

Therefore G is a $(q + 1; g, 2g - 1)$ -graph of order $2n_0(q + 1, g)$ and the theorem holds. ■

As a consequence of Corollary 4.3.2 and Theorem 4.3.9 we obtain the following corollary for $g = 6$.

Corollary 4.3.10 $26 \leq n(3; 6, 11) \leq 28$, $44 \leq n(4; 6, 11) \leq 52$ and $64 \leq n(5; 6, 11) \leq 124$. Let $q \geq 5$ be a prime power. Then $2q^2 + 7q + 2 \leq n(q + 1; 6, 11) \leq 4(q^2 + q + 1)$.

The exact value $n(3; 6, 11) = 28$ is shown in [?], and a $(3; 6, 11)$ -cage can be constructed from the proof of Theorem 4.3.7, see Figure 4.6.

Also as a consequence of Corollary 4.3.1 and Theorem 4.3.7 we can write the following corollary for $g = 8, 12$.

Corollary 4.3.11 Let $q \geq 2$ be a prime power. Then

- (i) $n_0(q + 1, 8) + q \leq n(q + 1; 8, 15) \leq 2n_0(q + 1, 8)$;
- (ii) $n_0(q + 1, 12) + q \leq n(q + 1; 8, 23) \leq 2n_0(q + 1, 11)$.

Chapter 5

Small Graphs of Girth 7 from Generalized Quadrangles of order q

Recalling Theorem 1.3.1, there are entire families of girth 6, 8, 12 cages that reach the Moore bound, when q is a prime power. From these families, it may be obtained graphs of girth 6, 8, 12 and degrees $k \neq q$, and also graphs of girth 5 (cf. [3, 6, 7, 8, 10, 11, 12, 13, 14, 34, 44, 45, 66, 79, 98, 126]).

In Table 5.1, we summarize the best known upper bounds, obtained in most cases from the aforementioned constructions, for degrees up to 20 and girths up to 16, (cf. [57]).

The main objective of this chapter is to give an explicit construction of small $(q + 1; 7)$ -graphs, obtained from such families of graphs.

It is well known [100, 91] that $Q(4, q)$ and $W(3, q)$ are the only two classical generalized quadrangles with parameters $s = t = q$. The generalized quadrangle $W(3, q)$ is the dual generalized of $Q(4, q)$, and they are selfdual for q even.

In 1966 Benson [34] constructed $(q + 1; 8)$ -cages from the generalized quadrangle $Q(4, q)$. He defined the point/line incidence graph Γ_q of $Q(4, q)$ which is a $(q + 1)$ -regular graph of girth 8 with $n_0(q + 1; 8)$ vertices. Hence, Γ_q is a $(q + 1; 8)$ -cage. Note that, Γ_q is isomorphic to the point/line incidence graph of $W(3, q)$.

Next we present the definition of generalized quadrangle for the sake of completeness. A *generalized quadrangle* is an incidence structure Let $Q = (P, L, I)$, where P and L denote respectively the sets of points and lines of Q , and for which I is a symmetric point-line incidence relation satisfying the following axioms:

- Each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with

at most one line.

- Each line is incident with $1 + s$ points ($s \geq 1$) and two distinct lines are incident with at most one point.
- For any pair $(p, \ell) \notin I$ there is a unique pair $(y, \ell') \in P \times L$ for which $(p, \ell') \in I$ and $(y, \ell) \in I$.

The integers s and t are the parameters of Q and is said to have order (s, t) ; if $s = t$, Q is said to have order s .

k/g	5	6	7	8	9	10	11	12	13	14	15	16
3	10	14	24	30	58	70	112	126	272	384	620	960
4	19	26	67	80	275	384		728				
5	30	42	152	170		1296	2688	2730				
6	40	62	294	312				7812				
7	50	90		672				32928				
8	80	114		800				39216				
9	96	146	1152	1170			74752	74898				
10	126	182		1640				132860				
11	156	240		2618				319440				
12	203	266		2928				354312				
13	240	336		4342				738192				
14	288	366		4760				804468				
15	312	462		7648				1957376				
16	336	504		8092				2088960				
17	448	546		8738				2236962				
18	480	614		10440				3017196				
19	512		720	13642				4938480				
20	576		762	14480				5227320				

Table 5.1: Summary of upper bounds for $n(k, g)$.

The *graph of incidence* Γ_q of a generalized quadrangle $Q = (P, L, I)$ is the graph whose vertex set is $V(\Gamma_q) = P \cup L$ and its edge set is $E(\Gamma_q) = \{uv : (u, v) \in I\}$.

An example of a graph of incidence of a generalized quadrangle is depicted in Figure 5.1

For any generalized quadrangle Q of order q and every point x of Q , let x^\perp denote the set of all points collinear with x . Note that in the incidence graph $x^\perp = N_2(x)$, with an abuse of notation supposing that $x \in \Gamma_q$ corresponds to the point $x \in Q$.

If X is a nonempty set of vertices of Q , then we define $X^\perp := \bigcap_{x \in X} x^\perp$. The *span* of the pair (x, y) is $sp(x, y) = \{x, y\}^{\perp\perp} = \{u \in P : u \in z^\perp \forall z \in x^\perp \cap y^\perp\}$, where P denotes the

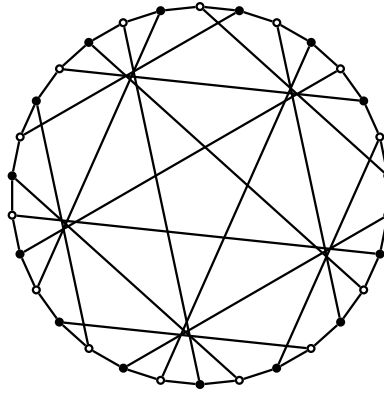


Figure 5.1: The Tutte-Coxeter graph, $(3,8)$ -cage, is the graph of incidence of $W(2)$.

set of points in Q . If x and y are not collinear, then $\{x, y\}^{\perp\perp}$ is also called the *hyperbolic line* through x and y . If the *hyperbolic line* through two noncollinear points x and y contains precisely $t + 1$ points, then the pair (x, y) is called *regular*. A point x is called *regular* if the pair (x, y) is regular for every point y not collinear with x . It is important to recall that the concept of regular also exists for a graph to avoid confusion. Hence we will emphasize when regular refers to a point or a graph.

Remark 5.0.1 [100] *Every point in $W(q)$ is regular.*

There are several equivalent coordinatizations of these generalized quadrangles (cf. [99], [108], [109], see also [91]) each giving a labeling for the graph Γ_q . Now we present a further labeling of Γ_q , equivalent to previous ones (cf. [4]), which will be central for our constructions.

Definition 5.0.1 *Let \mathbb{F}_q be a finite field with $q \geq 2$ a prime power. Let $\Gamma_q = \Gamma_q[V_0, V_1]$ be a bipartite graph with vertex sets $V_r = \{(a, b, c)_r, (q, q, a)_r : a \in \mathbb{F}_q \cup \{q\}, b, c \in \mathbb{F}_q\}$, $r = 0, 1$, and edge set defined as follows:*

For all $a \in \mathbb{F}_q \cup \{q\}$ and for all $b, c \in \mathbb{F}_q$:

$$N_{\Gamma_q}((a, b, c)_1) = \begin{cases} \{(x, ax + b, a^2x + 2ab + c)_0 : x \in \mathbb{F}_q\} \cup \{(q, a, c)_0\} & \text{if } a \in \mathbb{F}_q; \\ \{(c, b, x)_0 : x \in \mathbb{F}_q\} \cup \{(q, q, c)_0\} & \text{if } a = q. \end{cases}$$

$$N_{\Gamma_q}((q, q, a)_1) = \{(q, a, x)_0 : x \in \mathbb{F}_q\} \cup \{(q, q, q)_0\}.$$

Note that, in the labeling introduced in Definition 5.0.1, the second q in $\mathbb{F}_q \cup \{q\}$, usually denoted by ∞ , is meant to be just a symbol and no operations will be performed with it.

To finish, we define a *Latin square* as an $n \times n$ array filled with n different symbols, each occurring exactly once in each row and exactly once in each column.

5.1 Constructions of small $(q + 1; 7)$ -graphs for an even prime power q

In this section we will consider a $(q + 1; 8)$ -cage Γ_q with $q + 1 \geq 5$ an odd integer, since the only known $(q + 1; 8)$ -cages are obtained as the incidence graph of a Generalized Quadrangles, we let $q \geq 4$ a power of two.

Let $x \in V(\Gamma_q)$ and let $N(x) = \{x_0, \dots, x_q\}$, label $N(x_i) = \{x_{i0}, x_{i1}, \dots, x_{iq} = x\}$, for all $i \in \{0, \dots, q\}$, in the following way. Take x_{0j} and x_{1j} arbitrarily for $j = 0, \dots, q - 1$ and let $N_2(x_{0j}) \cap N_2(x_{1j}) - x = W_j$, note that $|W_j| = q$. Let $x_{ij} = (\bigcap_{w \in W_j} N_2(w)) \cap N(x_i)$, these vertices exist and are uniquely labeled since the generalized quadrangle $W(q)$ is regular.

$$\text{Let } H = x \cup N(x) \cup \{x_{q-1}, x_q\} \cup \bigcup_0^{q-2} N(x_i) \subset V(\Gamma_q).$$

We will delete the set H of vertices of Γ_q and add matchings M_Z between the remaining neighbors of such vertices in order to obtain a small regular graph of girth 7. In order to define the sets M_Z , we denote $X_i = N(x_i) \setminus \{x\}$ and $X_{ij} = N(x_{ij}) \setminus \{x_i\}$, for $i \in \{0, \dots, q\}$ and $j \in \{0, \dots, q - 1\}$.

Let \mathcal{Z} be the family of all $X_{q-1}X_q, X_{ij}$ for $i \in \{0, \dots, q - 2\}$ and $j \in \{0, \dots, q - 1\}$. For each $Z \in \mathcal{Z}$, M_Z will denote a perfect matching of $V(Z)$, which will eventually be added to Γ_q .

Definition 5.1.1 *Let Γ_q be a $(q + 1; 8)$ -cage, with odd degree $q + 1 \geq 5$.*

$$\text{Let } G_1 \text{ be the graph with: } V(G_1) := V(\Gamma_q - H) \text{ and } E(G_1) := E(\Gamma_q - H) \cup \bigcup_{Z \in \mathcal{Z}} M_Z.$$

Observe that the graph G_1 has order $|V(\Gamma_q)| - (q^2 + 2)$ and all its vertices have degree $q + 1$.

Next proposition states a condition for the graph G_1 to have girth 7, for this it is useful to state the following remark.

Remark 5.1.1 *Let $u, v \in V(\Gamma_q)$ a graph of girth 8, such that there is a uv -path P of length $t < 8$. Then every uv -path P' such that $E(P) \cap E(P') = \emptyset$ has length $|E(P')| \geq 8 - t$.*

Proposition 5.1.1 *Let Γ_q be a $(q+1, 8)$ -cage, with odd degree $q+1 \geq 5$ and G_1 as in Definition 5.1.1. Then G_1 has girth 7 if given $u_1v_1 \in M_{X_{ij}}$ and $u_2, v_2 \in X_{kl}$ such that $d(u_1, u_2) = 2$ and $d(v_1, v_2) = 2$, it holds $u_2v_2 \notin M_{X_{kl}}$, for $i \neq k \in \{0, \dots, q-2\}$ and $j, l \in \{0, \dots, q-1\}$.*

Proof. Let us consider the distances (in $\Gamma_q - H$) between the elements in the sets $Z \in \mathcal{Z}$. There are five possible cases:

- (1) Two vertices in the same set $u, v \in Z$ have a common neighbor w in Γ_q , therefore $d_{\Gamma_q-H}(u, v) \geq 6$.
- (2) If $u \in X_{q-1}$ and $v \in X_q$, then $d_{\Gamma_q-H}(u, v) \geq 4$, since x_{q-1}, x_q have x as a common neighbor in Γ_q .
- (3) If $u \in X_i$ for $i \in \{q-1, q\}$ and $v \in X_{kj}$ for $k \in \{0, \dots, q-2\}$ and $j \in \{0, \dots, q-1\}$ then $d_{\Gamma_q}(u, x_i) = 1$, $d_{\Gamma_q}(v, x_k) = 2$, and x_i, x_k have a common neighbor $x \in V(\Gamma_q)$, hence there is a uv -path of length 5 in Γ_q , concluding from Remark 5.1.1 that $d_{\Gamma_q}(u, v) \geq 3$.
- (4) If $u \in X_{ij}$ and $v \in X_{ik}$ for $i \in \{0, \dots, q-2\}$ and $j, k \in \{0, \dots, q-1\}$, then $ux_{ij}x_ix_{ik}v$ is a path of length 4 and from Remark 5.1.1 $d_{\Gamma_q-H}(u, v) \geq 4$.
- (5) If $u \in X_{ij}$ and $v \in X_{lk}$ for $i \neq l, i, l \in \{0, \dots, q-2\}$ and $j, k \in \{0, \dots, q-1\}$, then it is possible that there exist $w \in \Gamma_q - H$ such that $u, v \in N(w)$, that is $d_{\Gamma_q-H}(u, v) \geq 2$.

Let us consider C a shortest cycle in G_1 . If $E(C) \subset E(\Gamma_q - H)$ then $|C| \geq 8$. Suppose C contains edges in $M = \bigcup_{Z \in \mathcal{Z}} M_Z$. If C contains exactly one such edge, then by (1) $|C| \geq 7$. If C contains exactly two edges $e_1, e_2 \in M$, the following cases arise.

- If both e_1, e_2 lie in the same M_Z then by (1) $|C| \geq 14 > 7$.
- If $e_1 \in M_{X_{q-1}}$ and $e_2 \in M_{X_q}$ then by (2) $|C| \geq 10 > 7$.
- If $e_1 \in M_{X_i}$ and $e_2 \in M_{X_{kj}}$ then by (3) $|C| \geq 8 > 7$.
- If $e_1 \in M_{X_{ij}}$ and $e_2 \in M_{X_{ik}}$ then by (4) $|C| \geq 10 > 7$.
- If $e_1 \in M_{X_{ij}}$ and $e_2 \in M_{X_{lk}}$, for $i \neq l$, by hypothesis $|C| \geq 7$.

If C contains at least three edges of M , since $d(u, v) \geq 2$ for all $u, v \in \{X_{q-1}, X_q, X_{ij}\}$ with $i \in \{0, \dots, q-2\}$ and $j \in \{0, \dots, q-1\}$, $|C| \geq 9 > 7$.

Hence G_1 has girth 7 and we have finished the proof. ■

The following lemma gives sufficient conditions to define the matchings $M_{X_{ij}}$ for the sets X_{ij} , for $i \in \{0, \dots, q-2\}$ and $j \in \{0, \dots, q-1\}$, in order to fulfill the condition from Proposition 5.1.1.

Lemma 5.1.1 *There exist $q^2 - q$ matchings $M_{X_{ij}}$, for each $i \in \{0, \dots, q-2\}$ and $j \in \{0, \dots, q-1\}$ with the following property:*

Given $u_1v_1 \in M_{X_{ij}}$ and $u_2, v_2 \in X_{kj}$ such that $d(u_1, u_2) = 2$ and $d(v_1, v_2) = 2$ then $u_2v_2 \notin M_{X_{kj}}$.

Proof. By definition $\bigcap_{i=0}^{q-2} N(X_{ij}) = W_j$. Let $W_j = \{w_{j1}, \dots, w_{jq}\}$. Note that every vertex w_{jh} is adjacent to exactly one vertex in $N(X_{ij})$ that we will denote as x_{ijh} , for each $i \in \{0, \dots, q-2\}$ and $j \in \{0, \dots, q-1\}$.

Observe that x_{ijh} is well defined, because if x_{ijh} had two neighbors $w_h, w_{h'} \in \bigcap_{i=0}^{q-2} N(X_{ij})$, Γ_q would contain the cycle $x_{ijh}w_{jh'}x_{i'jh'}x_{i'jh}x_{ijh}w_{jh}$ of length 6.

Therefore, take the complete graph K_q label its vertices as $h = 1, \dots, q$. We know that it has a 1-factorization with $q-1$ factors F_1, \dots, F_{q-1} . For each $i = 0, \dots, q-2$, let $x_{ijh}x_{ijh'} \in M_{X_{ij}}$ if and only if $hh' \in F_i$.

To prove that the matchings $M_{X_{ij}}$ defined in this way fulfill the desired property suppose that $x_{ijh}x_{ijh'} \in M_{X_{ij}}$ and $x_{i'jh}x_{i'jh'} \in M_{X_{i'j}}$ for $i' \neq i$, then F_i and $F_{i'}$ would have the edge hh' in common contradicting that they are a factorization.

Therefore, there exist $q^2 - q$ matchings $M_{X_{ij}}$ with the desired property. ■

To finish, notice that for $u_1v_1 \in M_{X_{ij}}$ and $u_2, v_2 \in X_{i'j'}$ with $j \neq j'$ and possibly $i = i'$, the distances $d(u_1, u_2)$ and $d(v_1, v_2)$ are at least 4. Then, counting the number of vertices of G_1 and using the Proposition 5.1.1 we have the following theorem.

Theorem 5.1.1 *Let $q \geq 4$ be a power of two. Then there is a $(q+1)$ -regular graph of girth 7 and order $2q^3 + q^2 + 2q$.*

5.2 Constructions of small $(q + 1; 7)$ -graphs for and odd prime power q .

In this section we will consider cages of even degree, that Γ_q is a $(q + 1, 8)$ -cage with q an odd prime power. We proceed as before, but as will be evident from the proofs, the result is not as good as in the previous section.

We will delete a set H of vertices of Γ_q and add matchings M_Z between the remaining neighbors of such vertices in order to obtain a small regular graph of girth 7. The sets H and M_Z are defined as follows.

Let $V = \{x, y\} \cup \{s_0, \dots, s_q\}$ be the vertices of $K_{2,q+1}$.

Let $\widehat{K_{2,q+1}}$ be the graph obtained subdividing each edge of $K_{2,q+1}$.

Let Γ_q be a graph containing a copy of $\widehat{K_{2,q+1}}$ as a subgraph and label its vertices as $H' = \{x, y, s_0, \dots, s_q\} \cup N(x) \cup N(y)$ where $N(x) = \{x_0, \dots, x_q\}$ and $N(y) = \{y_0, \dots, y_q\}$. Note that $N(x_i) \cap N(y_i) = s_i$ for $i = 0, \dots, q$. Define:

$$\begin{aligned} H &= \{x, y, s_3, s_4, \dots, s_q\} \cup N(x) \cup N(y) \subset V(\Gamma_q); \\ X_i &= N(x_i) \cap V(\Gamma_q - H), \quad i = 0, \dots, q; \\ Y_i &= N(y_i) \cap V(\Gamma_q - H), \quad i = 0, \dots, q; \\ S_i &= N(s_i) \cap V(\Gamma_q - H), \quad i = 3, \dots, q. \end{aligned}$$

See Figure 5.2. Notice that the vertices of $\Gamma_q - H$ have degrees $q - 1, q$ and $q + 1$. The vertices s_0, s_1, s_2 of degree $q - 1$, those in $X_i \cup Y_i \cup S_i$ of degree q and all the remaining vertices of $\Gamma_q - H$ have degree $q + 1$. Therefore, in order to complete the degrees to such vertices its necessary to add edges to $\Gamma_q - H$, we define such edges next.

Let \mathcal{Z} be the family of all X_i, Y_i, S_i . For each $Z \in \mathcal{Z}$, M_Z will denote a perfect matching of $V(Z)$, which will eventually be added to Γ_q .

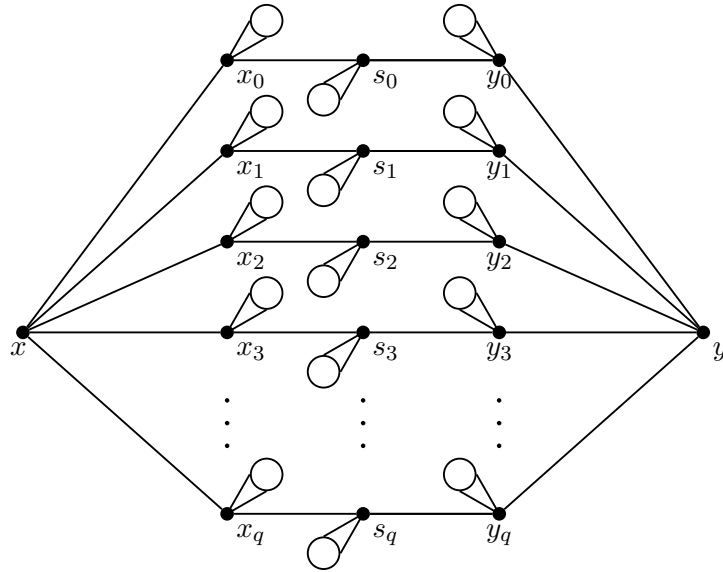
Definition 5.2.1 *Let Γ_q be a $(q + 1, 8)$ -cage, with even degree $q + 1 \geq 6$.*

- Let G_1 be the graph with: $V(G_1) := V(\Gamma_q - H)$ and $E(G_1) := E(\Gamma_q - H) \cup \bigcup_{Z \in \mathcal{Z}} M_Z$.

- Define G_2 as $V(G_2) := V(G_1)$ and

$$E(G_2) := (E(G_1) \setminus \{u_0v_0, u_1v_1, u_2v_2\}) \cup \{s_0u_0, s_0v_0, s_1u_1, s_1v_1, s_2u_2, s_2v_2\},$$

where $s_i \in H' - H$, the deleted edges $u_i v_i$ belong to M_{X_i} in G_1 and they are replaced by the paths of length two $u_i s_i v_i$, $i = 0, 1, 2$.

Figure 5.2: Sets H , X_i , Y_i and S_i .

By an immediate counting argument we know that the graph G_1 has order $|V(\Gamma_q)| - 3(q+1) + 1$, and observe that all vertices in G_1 have degree $q+1$ except for s_0, s_1, s_2 which remain of degree $q-1$. Hence, by the definition of $E(G_2)$, all vertices in G_2 are left with degree $q+1$.

Proposition 5.2.1 *Let Γ_q be a $(q+1, 8)$ -cage, with even degree $q \geq 5$ and G_1, G_2 be as in Definition 5.2.1.*

- (i) G_1 has girth 7 if the matchings M_{S_i}, M_{X_i} and M_{Y_i} have the following properties:
- (a) Given $u_1 v_1 \in M_{S_i}$ and $u_2, v_2 \in S_j$ such that $d(u_1, u_2) = 2$ and $d(v_1, v_2) = 2$, it holds that $u_2 v_2 \notin M_{S_j}$.
 - (b) Given $u_1 v_1 \in M_{X_i}$ and $u_2, v_2 \in Y_j$ such that $d(u_1, u_2) = 2$ and $d(v_1, v_2) = 2$, it holds that $u_2 v_2 \notin M_{Y_j}$.
- (ii) If conditions (a) and (b) hold then the graph G_2 also has girth 7.

Proof. To prove (i) let us consider the distances (in $\Gamma_q - H$) between the elements in the sets $Z \in \mathcal{Z}$. There are six possible cases:

- (1) Two vertices in the same set $u, v \in Z$ have a common neighbor w in Γ_q , therefore $d_{\Gamma_q - H}(u, v) \geq 6$.

- (2) If $u \in X_i$ and $v \in X_j$ then $d_{\Gamma_q-H}(u, v) \geq 4$, given that x_i, x_j have x as a common neighbor in Γ_q .
- (3) If $u \in Y_i$ and $v \in Y_j$ then $d_{\Gamma_q-H}(u, v) \geq 4$, as before.
- (4) If $u \in S_i$ and $v \in S_j$ then it is possible that there exist $w \in \Gamma_q - H$ such that $u, v \in N(w)$, that is, $d_{\Gamma_q-H}(u, v) \geq 2$.
- (5) If $u \in S_i$ and $v \in X_j \cup Y_j$ then $d_{\Gamma_q-H}(u, v) \geq 3$, since $s_i \in N(x_i) \cap N(y_i)$.
- (6) If $u \in X_i$ and $v \in Y_j$ then $d_{\Gamma_q-H}(u, v) \geq 2$.

Let us consider C a shortest cycle in G_1 . If $E(C) \subset E(\Gamma_q - H)$ then $|C| \geq 8$. Suppose C contains edges in $M = \bigcup_{Z \in \mathcal{Z}} M_Z$. If C contains exactly one such edge, then by (1) $|C| \geq 7$. If C contains exactly two edges $e_1, e_2 \in M$, the following cases arise:

- If both e_1, e_2 lie in the same M_Z , then by (1) $|C| \geq 14 > 7$.
- If $e_1 \in M_{X_i}$ and $e_2 \in M_{X_j}$ for $i \neq j$, by (2) $|C| \geq 10 > 7$.
- If $e_1 \in M_{Y_i}$ and $e_2 \in M_{Y_j}$ for $i \neq j$, by (3) $|C| \geq 10 > 7$.
- If $e_1 \in M_{S_i}$ and $e_2 \in M_{X_j} \cup M_{Y_j}$, by (5) $|C| \geq 8 > 7$.
- If $e_1 \in M_{S_i}$ and $e_2 \in M_{S_j}$ for $i \neq j$, by the first hypothesis in item (i)(b) $|C| \geq 7$.
- If $e_1 \in M_{X_i}$ and $e_2 \in M_{Y_j}$, by the second hypothesis in item (i)(b) $|C| \geq 7$.

If C contains at least three edges of M , since $d(u, v) \geq 2$ for all $u, v \in \{X_i \cup Y_i\}_{i=1}^k \cup \{S_i\}_{i=4}^k$, $|C| \geq 9 > 7$.

Hence G_1 has girth 7, concluding the proof of (i).

To prove (ii), let C be a shortest cycle in G_2 . If $E(C) \subset E(\Gamma_q - H) \cup M$ then $|C| \geq 7$.

- If C contains exactly one edge $s_i u_i$ or $s_i v_i$ then $|C| \geq 7$ since $d_{\Gamma_q}(s_i, u_i) = d_{\Gamma_q \Gamma_q 1}(s_i, v_i) = 2$ which implies $d_{G_1}(s_i, u_i) \geq 6$ and $d_{G_1}(s_i, v_i) \geq 6$.
- If C contains a path $u_i s_i v_i$ then $(C \setminus u_i s_i v_i) \cup u_i v_i$ is a cycle in G_1 with one less vertex than C , therefore $|C| \geq 8$.
- If C contains two edges $s_i u_i, s_j u_j$, for $i \neq j$. Their distances $d_{G_1}(s_i, u_j) \geq 4$, $d_{G_1}(s_i, s_j) \geq 4$, and $d_{G_1}(u_i, u_j) \geq 4$, therefore in any case C has length greater than 7 concluding the proof.

■

The following lemma gives sufficient conditions to define the matchings M_{S_i} for the sets S_i , in order that they fulfill condition (a) from Proposition 5.2.1 (i). Notice that in the incidence graph of a generalized quadrangle $\{x, y\}^{\perp\perp} = \bigcap_{s \in N_2(x) \cap N_2(y)} N_2(s)$, thus Remark 5.0.1 implies that $|\bigcap_{i=0}^q N(S_i)| = q-1$, recalling that $\{s_i\}_{i=0}^q = N_2(x) \cap N_2(y)$. Since $|\bigcap_{i=0}^q N(S_i)|$ is contained in $|\bigcap_{i=3}^q N(S_i)|$, and $|\bigcap_{i=3}^q N(S_i)| \leq |S_i| = q-1$ then the condition for the following lemma holds.

Lemma 5.2.1 *If $|\bigcap_{i=3}^q N(S_i)| = q-1$ then there exist matchings M_{S_i} , for $i = 3, \dots, q$, such that:*

- *Given $u_1 v_1 \in M_{S_i}$ and $u_2, v_2 \in S_j$ such that $d(u_1, u_2) = 2$ and $d(v_1, v_2) = 2$, it holds that $u_2 v_2 \notin M_{S_j}$.*

Proof. Let us suppose that $\bigcap_{i=3}^q N(S_i) = \{w_1, \dots, w_{q-1}\}$, and since S_i has $q-1$ vertices, every vertex w_j is adjacent to exactly one vertex in $s_{ij} \in S_i$.

Observe that s_{ij} is well defined, because if s_{ij} had two neighbors $w_j, w_{j'} \in \bigcap_{i=1}^{q+1} N(S_i)$, Γ_q would contain the cycle $(s_{ij} w_j s_{kj} s_k s_{kj'} w_{j'})$ of length 6.

Therefore, take the complete graph K_{q-1} , label its vertices as $j = 1, \dots, q-1$. We know that it has a 1-factorization with $q-2$ factors F_1, \dots, F_{q-2} . For each $i = 3, \dots, q+1$, let $s_{ij} s_{il} \in M_{S_i}$ if and only if $jl \in F_{i-3}$.

To prove that the matchings M_{S_i} defined in this way fulfill the desired property suppose that $s_{ij} s_{il} \in M_{S_i}$ and $s_{i'j} s_{i'l} \in M_{S_{i'}}$ for $i' \neq i$. Then F_i and $F_{i'}$ would have the edge jl in common contradicting that they were a factorization. ■

So far, the steps of our construction have been independent from the coordinatization of the chosen $(q+1, 8)$ -cage, however, in order to define M_{X_i} and M_{Y_i} satisfying condition (b) of Lemma 5.2.1, we need to fix all the elements chosen so far.

We will distinguish two cases, when q is a prime or when q is a prime power.

Choose $x = (q, q, q)_1$, $y = (0, 0, 0)_1$.

When q is a prime then $x_i = (q, q, i)_0$, $y_i = (i, 0, 0)_0$ for $i = 0, \dots, q$.

Therefore, $N(x_i) = \{(q, t, i)_1 : t = 0, \dots, q-1\} \cup x$ and $N(x_q) = \{(q, q, t)_1 : t = 0, \dots, q-1\} \cup x$; $N(y_i) = \{(t, -it, i+t^2)_1 : t = 0, \dots, q-2\} \cup (q, 0, i)_1$ and $N(y_q) = \{(0, t, 0)_1 : t = 0, \dots, q-1\} \cup (q, q, 0)_1$.

Thus, the corresponding vertices s_i are: $s_i = (q, 0, i)_1$ for $i = 0, \dots, q-1$ and $s_q = (q, q, 0)_1$; $N(s_i) = \{(i, 0, t)_0 : t = 1, \dots, q-1, i = 0, \dots, q\} \cup \{x_i, y_i\}$. Hence, $S_i = \{(i, 0, t)_0 : t = 1, \dots, q-1, i = 0, \dots, q\}$.

Then $N(S_i) = \{(a, b, c)_1 : b = -ia, c = t + a^2i, i = 0, \dots, q-1\}$, and $N(S_q) = \{(q, 0, t)_1 : t = 0, \dots, q-1\}$.

Solving the equations we obtain $N(S_i) \cap N(S_j) = \{(0, 0, t)_1 : t = 0, \dots, q-1\}$, moreover $N(i, 0, t)_0 \cap N(j, 0, t)_0 = (0, 0, t)_1$, for each $j \neq i$ and $t = 0, \dots, q-1$, or equivalently, $N(0, 0, t)_1 = \{(x, 0, t)_0 : t = 0, \dots, q-1, x = 0, \dots, q\}$. Hence the sets S_i satisfy the hypothesis of Lemma 5.2.1, yielding that there exist the matchings M_{S_i} with the desired property.

Notice that the sets X_i and Y_i are naturally defined as the sets $X_i = \{(q, t, i)_1 : t = 1, \dots, q-1, i = 0, \dots, q-1\}$, $X_0 = \{(q, t, 0)_1 : t = 1, \dots, q-1\}$ and $X_q = \{(q, q, t)_1 : t = 1, \dots, q-1\}$. The sets $Y_i = \{(t, -it, it^2)_1 : t = 1, \dots, q-1, i = 0, \dots, q-1\}$, and $Y_q = \{(0, t, 0)_1 : t = 1, \dots, q-1\}$.

In this way we have defined all the sets in Lemma 5.2.1, and from Lemma 5.2.1 we know that the matchings M_{S_i} have the property that:

- If $u_1v_1 \in M_{S_i}$ and $u_2, v_2 \in S_j$ are such that $d(u_1, u_2) = 2$ and $d(v_1, v_2) = 2$ then $u_2v_2 \notin M_{S_j}$.

It remains to define the matchings M_{X_i} and M_{Y_i} and prove they have property (b) from Proposition 5.2.1 (i).

For this we must analyze the intersection of the second neighborhood of an X_j with an Y_i , $N_2(X_j) \cap Y_i$. For each $w \in Y_i$, we know there is exactly one $z \in X_q$ such that $w \in N_2(z)$.

This allows us to define the following sets of latin squares: For each j , let the coordinate il of the j -th latin square to have the symbol s_{ilj} if there is a $w_{ilj} = (a, b, c)_1$ such that

$$w_{ilj} \in N((i, 0, 0)_0) \cap N_2((q, \ell, j)_1) \cap N_2((q, q, s_{ilj})_1),$$

where $(i, 0, 0)_0 = y_i$, $(q, \ell, j)_1 \in X_j$ and $(q, q, s_{ilj})_1 \in X_q$.

Since $N((i, 0, 0)_0) = \{(t, -it, i+t^2)_1 : t = 0, \dots, q-2\} \cup (q, 0, i)_1$, then $a = t$, $b = -it$, and $c = i + t^2$.

Observe that $w_{i\ell j} \in N_2((q, \ell, j)_1)$ is equivalent to $(j, \ell, t)_0 \in N((a, b, c)_1)$, since $N((q, \ell, j)_1) = \{(j, \ell, t)_0 : t = 0, \dots, q-1\} \cup \{(q, q, j)_0\}$. Hence, $aj + b = \ell$.

And $w_{i\ell j} \in N_2((q, q, s_{i\ell j})_1)$ implies $a = s_{i\ell j}$.

Therefore we obtain the following equation for $s_{i\ell j}$.

$$s_{i\ell j}(j - i) = \ell$$

Notice that this equation is undefined for $j = i$, otherwise it would mean that y_i has a neighbor at distance 3 from x_j and this would imply the existence of a cycle of length 6 in Γ_q .

Also from the equation we deduce that $-s_{i\ell j} = s_{i-\ell j}$, and $s_{i+1\ell j+1} = s_{i\ell j}$. This means that the $i+1$ -th row of the $j+1$ -th latin square is equal to the i -th row of the j -th latin square, hence all the set of latin squares have the same rows. This also implies that if we put an edge between two vertices on Y_i , $(s_{i\ell j}, -is_{i\ell j}, is_{i\ell j}^2)_1$ and $(-s_{i\ell j}, is_{i\ell j}, is_{i\ell j}^2)_1$, it will have at distance two in X_j only the vertices $(q, \ell, i)_1$ and $(q, -\ell, i)_1$.

Therefore, the matchings $M_{X_i} = \{(q, \ell, i)_1(q, -(\ell+2), i)_1 : i = 0, \dots, q-1, \ell = 1, \dots, q-3\} \cup \{(q, -2, i)_1(q, -1, i)_1 : i = 0, \dots, q-1\}$, $M_{X_q} = \{(q, q, \ell)_1(q, q, -(\ell+2))_1 : \ell = 1, \dots, q-3\} \cup \{(q, q, -2)_1(q, q, -1)_1\}$, and $M_{Y_i} = \{(t, -it, it^2)_1(-t, it, it^2)_1 : i = 0, \dots, q-1, t = 1, \dots, q-1\}$, have the property (b) from Proposition 5.2.1 (i).

When q is a prime power, let α a primitive root of unity in $GF(q)$. Then, $x_i = (q, q, \alpha^{i-1})_0$, $y_i = (\alpha^{i-1}, 0, 0)_0$ for $i = 1, \dots, q-1$, $x_0 = (q, q, 0)_0$, and $y_0 = (0, 0, 0)_0$. Moreover, $x_q = (q, q, q)_0$ and $y_q = (q, 0, 0)_0$.

Therefore, $N(x_i) = \{(q, \alpha^t, \alpha^{i-1})_1 : t = 0, \dots, q-2\} \cup (q, 0, \alpha^{i-1})_1 \cup x$ and $N(x_0) = \{(q, \alpha^t, 0)_1 : t = 0, \dots, q-2\} \cup (q, 0, 0)_1 \cup x$; $N(y_i) = \{(\alpha^t, -\alpha^{i-1+t}, \alpha^{i-1+2t})_1 : t = 0, \dots, q-2\} \cup (q, 0, \alpha^{i-1})_1$ and $N(y_0) = \{(\alpha^t, 0, 0)_1 : t = 0, \dots, q-2\} \cup (q, 0, 0)_1$; $N(x_q) = \{(q, q, \alpha^t)_1 : s = 0, \dots, q-2\} \cup (q, q, 0)_1 \cup x$; and $N(y_q) = \{(0, \alpha^t, 0)_1 : t = 0, \dots, q-2\} \cup (q, q, 0)_1 \cup y$.

Thus, the corresponding vertices s_i are: $s_i = (q, 0, \alpha^{i-1})_1$, for $i = 1, \dots, q-1$, $s_0 = (q, 0, 0)_1$ and $s_q = (q, q, 0)_1$; $N(s_i) = \{(\alpha^{i-1}, 0, \alpha^t)_0 : t = 0, \dots, q-2, i = 1, \dots, q-1\} \cup \{x_i, y_i\}$, and $N(s_0) = \{(0, 0, \alpha^t)_0 : t = 0, \dots, q-2\} \cup \{x_0, y_0\}$. Hence $S_i = \{(\alpha^{i-1}, 0, \alpha^t)_0 : t = 0, \dots, q-2, i = 0, \dots, q\}$ and $S_0 = \{(0, 0, \alpha^t)_0 : t = 0, \dots, q-2\}$.

Then $N(S_i) = \{(a, b, c)_1 : b = -\alpha^{i-1}a, c = \alpha^t + a^2\alpha^{i-1}, i = 1, \dots, q-1\}$, $N(S_0) = \{(a, b, c)_1 : b = 0, c = \alpha^t\}$ and $N(S_q) = \{(q, 0, \alpha^t)_1 : t = 0, \dots, q-2\} \cup (q, 0, 0)_1$.

Solving the equations we obtain $N(S_i) \cap N(S_j) = \{(0, 0, \alpha^t)_1 : t = 0, \dots, q-2\}$. Moreover, $N(\alpha^{i-1}, 0, \alpha^t)_0 \cap N(\alpha^{j-1}, 0, \alpha^t)_0 = (0, 0, \alpha^t)_1$, for each $j \neq i$ and $t = 0, \dots, q-2$, or

equivalently, $N(0, 0, \alpha^t)_1 = \{(\alpha^x, 0, \alpha^t)_0 : x = 0, \dots, q-2\} \cup (0, 0, \alpha^t)_0 \cup (q, 0, \alpha^t)_0$, for each $t = 0, \dots, q-2$. Hence the sets S_i satisfy the hypothesis of Lemma 5.2.1 yielding that there exist the matchings M_{S_i} with the desired property.

Notice that the sets X_i and Y_i are naturally defined as the sets $X_i = \{(q, \alpha^t, \alpha^{i-1})_1 : t = 0, \dots, q-2, i = 1, \dots, q-1\}$, $X_0 = \{(q, \alpha^t, 0)_1 : t = 0, \dots, q-2\}$ and $X_q = \{(q, q, \alpha^t)_1 : t = 0, \dots, q-2\}$. The sets

$Y_i = \{(\alpha^t, -\alpha^{i-1+t}, \alpha^{i-1+2t})_1 : t = 0, \dots, q-2\}$, $Y_0 = \{(\alpha^t, 0, 0)_1 : t = 0, \dots, q-2\}$ and $Y_q = \{(0, \alpha^t, 0)_1 : t = 0, \dots, q-2\}$.

In order to define the matchings M_{X_i} and M_{Y_i} and prove that they have the property (b) from Proposition 5.2.1 (i), we proceed as before, by defining the sets of latin squares:

For each j , let the coordinate $i\ell$ of the j -th latin square to have the symbol $s_{i\ell j} \in \{0, \dots, q-2\}$ if there is a $w_{i\ell j} = (a, b, c)_1$ such that

$$w_{i\ell j} \in N((\alpha^{i-1}, 0, 0)_0) \cap N_2((q, \alpha^\ell, \alpha^{j-1})_1) \cap N_2((q, q, \alpha^{s_{i\ell j}})_1) \text{ for } i, j \geq 1,$$

where $(\alpha^{i-1}, 0, 0)_0 = y_i$, $(q, \alpha^\ell, \alpha^{j-1})_1 \in X_j$ and $(q, q, \alpha^{s_{i\ell j}})_1 \in X_q$.

Since $N((\alpha^{i-1}, 0, 0)_0) = \{(\alpha^t, -\alpha^{i-1+t}, \alpha^{i-1+2t})_1 : t = 0, \dots, q-2\} \cup (q, 0, i)_1$, then $a = \alpha^t$, $b = -\alpha^{i-1+t}$, and $c = \alpha^{i-1+2t}$.

Also $w_{i\ell j} \in N_2((q, \alpha^\ell, \alpha^{j-1})_1)$ is equivalent to $(\alpha^{j-1}, \alpha^\ell, \alpha^t)_0 \in N((a, b, c)_1)$, since $N((q, \alpha^\ell, \alpha^{j-1})_1) = \{(\alpha^{j-1}, \alpha^\ell, \alpha^t)_0 : t = 0, \dots, q-2\}$. Hence $a\alpha^{j-1} + b = \alpha^\ell$.

And $w_{i\ell j} \in N_2((q, q, \alpha^{s_{i\ell j}})_1)$ implies $a = \alpha^{s_{i\ell j}}$.

Therefore we obtain the following equation for $s_{i\ell j}$.

$$\alpha^{s_{i\ell j}}(\alpha^{j-1} - \alpha^{i-1}) = \alpha^\ell$$

Notice that this equation is undefined for $j = i$, otherwise it would mean that y_i has a neighbor at distance 3 from x_j and this would imply the existence of a cycle of length 6 in Γ_q .

For $i = 0$, we obtain the equation $\alpha^{s_{0\ell j}}(\alpha^{j-1}) = \alpha^\ell$, and for $j = 0$, we obtain $\alpha^{s_{i\ell 0}}(-\alpha^{i-1}) = \alpha^\ell$. From the equation we obtain that $s_{i\ell+1j} = s_{i\ell j} + 1$, and each latin square is the sum table of the cyclic group \mathbb{Z}_{q-1} with the rows permuted.

Multiplying by α the equation $\alpha^{s_{i\ell-1j}}(\alpha^{j-1} - \alpha^{i-1}) = \alpha^{\ell-1}$, we obtain that $s_{i+1\ell j+1} = s_{i\ell-1j}$. This implies that the row $i+1$ of the $j+1$ -th latin square is equal to the row i of the

j -th latin square subtracting 1 to each symbol (i.e., $s_{i+1\ell j+1} + 1 = s_{i\ell j}$). That is, all the set of latin squares have the same rows but in a different order.

This also implies that if we put an edge between two vertices on Y_i , $(\alpha^{s_{i\ell j}}, -\alpha^{i-1+s_{i\ell j}}, \alpha^{i-1+2s_{i\ell j}})_1$ and $(\alpha^{s_{i\ell j}+1}, -\alpha^{i-1+(s_{i\ell j}+1)}, \alpha^{i-1+2(s_{i\ell j}+1)})_1$, it will have at distance two in X_j only the vertices, $(q, \alpha^\ell, i)_1$ and $(q, \alpha^{\ell+1}, i)_1$ and the other way around.

Therefore, the matchings $M_{X_i} = \{(q, \alpha^{2\ell}, i)_1 (q, \alpha^{2\ell+1}, i)_1 : i = 0, \dots, q-1, \ell = 1, \dots, (q-1)/2\}$, $M_{X_q} = \{(q, q, \alpha^{2\ell})_1 (q, q, \alpha^{2\ell+1})_1 : \ell = 1, \dots, (q-1)/2\}$, and $M_{Y_i} = \{(\alpha^{2t}, -\alpha^{i-1+2t}, \alpha^{i-1+4t})_1 (\alpha^{2t+3}, -\alpha^{i-1+(2t+3)}, \alpha^{i-1+2(2t+3)})_1 : i = 0, \dots, q-1, t = 1, \dots, (q-1)/2\}$ have the property (b) from Proposition 5.2.1 (i), proving the theorem for q prime power.

Theorem 5.2.1 *Let $q \geq 5$ be a prime power. Then there is a $q+1$ -regular graph of girth 7 and order $2q^3 + 2q^2 - q + 1$.*

Proof. Finally, by applying Lemma 5.2.1(ii), we obtain a $q+1$ -regular graph of girth 7 with $2(q^3 + q^2 + q + 1) - (q - 3 + 2(q + 2)) = 2q^3 + 2q^2 - q + 1$ vertices. ■

Chapter 6

Conclusions and Open Problems

This chapter is devoted to state open problems and conclusions obtained from the thesis.

We have been studying cages and properties such as connectivity and restricted connectivities. We gave a relevant contribution on solving the conjecture of Fu, Huang and Rodger obtaining that cages are $k/2$ -connected, (cf. [29, 104]).

We studied graphs with a given girth pair and, by imposing a condition on the diameter in relation to the girth pair of a graph, we obtained λ' -optimality, as a corollary we proved the λ' -optimality of polarity graphs, (cf. [21]). Also, we obtained a result proving the edge superconnectivity of semiregular cages, it is contained in [23]. Based on these studies it was possible to develop a deeper study of cages structure.

Thus, obtaining constructions for girth pair cages that prove a bound conjectured by Harary and Kovcs, relating the order of girth pair cages with the one for cages, (cf. [30]). Also, by studying the excess of graphs, we gave a contribution in the sense of the work of Biggs and Ito, relating the bipartition of girth 6 cages with their orders, (cf. [31]). Finally, we present a construction of an entire family of girth 7 cages that arises from some combinatorial properties of the incidence graphs of generalized quadrangles of order (q, q) , (cf. [5]).

Next, we present some possible lines of research to follow in the future:

6.0.1 Connectivity

- To extend Theorem 2.0.6 to cages with even girth.
- To study conditions for a graph with diameter $g - 1$ to be λ' -optimal.
- To extend Conjecture 1.3.1 for $(D; g)$ -cages.

- To prove that $(D; g)$ -cages are superconnected.
- To improve theorem of monotonicity (Theorem 1.3.2) for every degree set D .

6.0.2 Constructions

- To continue with the study of matrices of incidence in order to get geometric graphs, like in [2] and [13].
- To study the constructions of Lazebnik, Ustimenko and Woldar [82], in order to find new upper bounds for $n(k; g)$.
- To generalize constructions for girth 7 in order to obtain new families of larger odd girth.

6.0.3 Girth pair

- To prove that small k -regular graphs with girth pair (g, h) are 2-connected (Conjecture 4 in [62]).
- To continue with the study of the excess in graphs with girth pair.
- To construct the smallest $(r; 4, 5)$ -graphs for all integers $s > 1$ and r odd, the case unsolved by Harary and Kovács [62].

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