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Doctoral Thesis

# Essays on Social Groups: Inequality, Influence and the Structure of Interactions

*Author:*
Tugce Cuhadaroglu

*Supervisor:*
Miguel Angel Ballester

International Doctorate in Economic Analysis
Departament d'Economia i d'História Económica

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# Contents

# Introduction

One of the main questions of economics has always been to understand and formalize the dynamic relation between what is individual and what is social. This dissertation includes two complementary perspectives to explore this major question.

In the first approach, which refers to the first chapter, we investigate how to evaluate the degree to which differences in individual characteristics result in differences in social outcomes; so to speak, we chase the 'individual' in 'social'. Focusing on non-income inequalities between social groups, we propose a new methodology to evaluate them. Up on providing the foundational analysis of this methodology, we explore the intimate link between inequality and segregation by means of it.

Unequal distribution of social groups across different levels of a quality-of-life variable is a frequently observed phenomenon. Since group-based differences in welfare outcomes are important determinants of social and economic well-being of societies, measuring the extent of these inequalities is of particular concern. However, when we go beyond income and consider other welfare variables such as educational attainment, occupational status, health or subjective well-being, we run short of well-developed inequality measurement techniques. Moreover, the existing techniques for evaluating income inequality are not necessarily applicable in the treatment of these non-income variables since these variables do not share a very important feature of income: they are not cardinal in nature. Rather they are defined over categories that can be ordered unambiguously. In Chapter 1, *My Group Beats Your Group: Evaluating Non-Income Inequalities*, we propose a new methodology, the Domination Index, to evaluate non-income inequalities between social groups without making use of restrictive cardinalisation assumptions. The Domination Index measures the discrepancy in group distributions in terms of the number of times a group beats another group in pairwise confrontations. Since beating is defined purely as an ordinal concept, no cardinalisation for different levels of the variable of interest is required. We provide an axiomatic approach and show that a set of desirable properties for a group inequality measure when the variable of interest is not cardinal but ordinal, characterize the Domination Index. Moreover, depending on our analysis, we explore the close relation between segregation and group inequalities, a topic that is widely discussed both from theoretical and empirical perspectives in a variety of contexts with different variables of interest. Segregation refers to the degree to which social groups are distributed differently from each other; it is the inequality in the distributions of social groups across neither measurable nor comparable categories. We make use of The Domination Index to formulate a unifying framework to approach these two related concepts. In a first result we show that for a very specific organization of the society the inequality between groups as measured by the Domination Index is equal to the segregation in that society as measured by a known segregation measure, the Gini Segregation Index. This particular organization is the one in which the importance of each category reflects how uneven the distribution of groups in that category is. Moreover, this organization is actually the one with a higher level of inequality than any other organization of the society, i.e., the order of the categories are such that the resulting inequality is the highest it can ever be for any other order, as established by a second result. The interpretation is clear: Segregation of social groups is actually the inequality between them for a hypothetical scenario, where the order of the categories are such that the inequality is at its peak.

The concept of a domination developed in this first chapter is related to the relative position of an individual in the society. Overall, the Domination Index presents an evaluation of the inequality in the society by aggregating these dominations created by all individuals. Departing from 'individual', we arrive to the 'social'. The remaining two chapters, on the other hand, can be seen as a chase for the 'social in 'individual. We consider an individual as a social agent and investigate the role of social interactions in individual decision making. The second chapter is concentrated around the identification problem of social influence and homophily. We suggest a methodology that exploits individual decision outcomes in order to assess the level of homophily and influence related to social interaction.

Individuals sharing the same environment, such as members of the same household, colleagues from workplace, friends from school, tend to behave similarly in many occasions. As an immediate implication of being part of a society, we do influence each other's behaviors. However, performance of similar behavior does not necessarily imply the influence of one on the other, but performing similarly may have been the reason that a particular social relation is built at the first place. This refers to the identification problem of influence and homophily. In Chapter 2, *Choose What You Like or Like What You Choose? Identifying Influence and Homophily out of Individual Decisions*, we investigate the micro-foundations of this identification problem. We provide a novel framework that focuses on individual decision making in order to identify the social influence and homophily effects. Having a closer look to the decision making processes of individuals that interact, we investigate how they affect each other's behaviors. We propose simple and direct measures of homophily and influence by making use of individual preferences. However since in many occasions, preferences are not easily observed, we extend our analysis to the observables, decision outcomes. In order to infer the underlying preferences out of decision outcomes, we follow a foundational approach. We analyze the behavioral characteristics of individual decision making that includes interaction and finally we make use of the tools that are provided by revealed preference theory in order to uncover the underlying preferences of the individuals. Based on revealed preference analysis, we revisit our measurement techniques for homophily and influence.

In the third chapter, we carry on considering individuals that are in interaction. The subject matter of this chapter, on the other hand, is the structure of social interactions. We suggest to uncover the underlying structure of a social network by analyzing individual behavior patterns.

Social networks are known to affect the way individuals behave. From consumption habits, to voting behavior, from school achievement of teenagers to investment interests of business people, a variety of decisions that people make in everyday life have an intimate relation with their social interactions. However, quite often the structure of the social network is not clearly observable. In Chapter 3, *Tell Me Who You Are, I Tell You Who Your Friends Are: Understanding Social Networks Out of Individual Decisions*, we suggest to exploit individual behavior in order to understand the underlying social network structure. Our approach is based on the observation that socially connected individuals, influence each other's decisions especially at times of indecisiveness. It is quite common practice for all of us to refer to the opinions of people around if we do not have sufficient information or experience to compare several options that we are facing. We may use different communication tools such as asking for advice, gathering suggestions or we may observe people behaving in a particular way and adapt similar behavioral patterns. Departing from this observation, we consider individuals that are sharing a common social environment, in which the exact structure of the interactions are not known. We first present a decision model that allows these individuals to refer to their social contacts when they need it. Then we investigate the properties on their decision outcomes that will reveal out the specific way in which they are connected. Overall we characterize four different possible interaction structures by which individuals may be connected in a social network.

The relation between what is individual and what is social is clearly multi-layered and a quite complicated one. But it is also key to the understanding of many economic decisions. We believe that both perspectives analyzed in this dissertation presents humble contributions to this understanding.

# Chapter 1

# My Group Beats Your Group: Evaluating Non-Income Inequalities

## 1.1 Introduction

Inequalities between groups are important determinants of social and economic well-being of societies. For more than a century now, economists have been interested in evaluating the extent of inequalities in order to understand: (i) how they change, by comparing them across time; (ii) why they change and what they change, by comparing them across societies with different characteristics, revealing their relation to other social and economical phenomena.

Income or wealth disparities between social groups are well-known, well-documented and deeply analyzed inequalities. However it has been recognized long ago that comparing levels of income is not sufficient on its own to assess differences in individual well-being. Atkinson makes the pioneering move in departing from the classical approach of measuring inequality as the dispersion of levels of income, first bringing in the idea of social-welfare based income inequality measurement [7] and second incorporating the differences in individual needs to the assessment of income inequality [8]. Sen, in a series of papers and books, explores the need for going beyond income inequality and shifts the focus to many other variables such as longevity, survival, literacy, fertility, employment status, that influence individual well-being "but not captured by the simple statistics of incomes and commodity holdings" [77–80]. There are many other variables that jointly contribute to one's quality of life, and hence whose uneven distribution between social groups is of interest. However they lack the attention and well-developed theoretical approach that income received.

Moreover for the treatment of these non-income variables such as education, health, occupational status or subjective well-being, we cannot generally apply the techniques developed for evaluating income inequality since these variables do not share a very important feature of income: they are not cardinal in nature. They are rather defined over categories that are not necessarily associated with cardinal values. However notice that although these categories do not convey any cardinal information, they are not completely unrelated either. In most of the cases, categories can be compared unambiguously. Everybody will agree that a college graduate's educational attainment is higher than a secondary school drop out though we would not know by how much it is higher. Or it will be safe to claim that an individual that selects the score 3 as answer to the question of "Taking all things together, how happy would you say you are, on a scale from 1 to 10 where [1] means you are very unhappy and [10] means you are very happy?" has selected a lower happiness score than an individual with a score of 9.[1] But this would not necessarily imply that the second individual is three times happier than the first one. Since these variables are vaguely measurable, the methods designed for measuring income inequality are essentially futile. Representing these categories by making use of specific cardinalisations requires further assumptions if not result in misevaluations. Let us give a closer look to a pair of specific examples:

**Example 1: Gender based occupational status inequality.** The following table summarizes the gender distribution across occupational hierarchy within the class of the Management, business and financial occupations in United States.[2]

**Table 1. Women Share of MBF Occupations, U.S., 2010**

| | Total employed | Percent women |
|---|---|---|
| MBF occupations | 20,938 | 43% |
| Chief executives | 1,505 | 25,5% |
| General and operations managers | 1,007 | 29,9% |
| Managers | 12,489 | 40,5% |
| Operations | 5,937 | 54,9% |

According to the US census data, in 2010, out of almost 21 million employees in management, business and financial occupations, 43% were women. Within this class, Chief

[1]This is the Question 42 of the Second European Quality of Life Survey, 2007-2008. Questions of the same sort are found in population surveys such as United States General Social Survey or Euro-Barometer Survey Series.

[2]Civilian noninstitutionalized population 16 years old and over. Annual average of monthly figures. Figures are in millions. *Source:* U.S. Census Bureau, The 2012 Statistical Abstract, The National Data Book, Labor Force, Employment and Earnings Section, Table 616. Employed Civilians by Occupation, Sex, Race, and Hispanic Origin: 2010.

executives are the ones with the highest status, followed by General and operations managers. Managers occupy the third position in the hierarchy and finally the last position is occupied by Operations employees. If the distributions of genders across these positions were completely equal, we would observe a women share of 43% in each position. However, the increasing women share going down the hierarchy signals an inequality in the distribution of genders. How do we treat this data? In order to assess gender inequality of occupational status consistently we need to take into account the hierarchy of positions. One could falsely argue that this hierarchy can be represented by the corresponding wage levels of the occupational statuses eliminating the need for going beyond wage inequality. However, as shown in different works [40, 64, 69] the average wage of a female dominated job do not correspond to women's occupational prestige for that status.

**Example 2: Racial disparities in educational attainment.** According to United States Census data in the year 1970, 43.2% of the White citizens and 27% of the Black citizens were high school graduates without a further degree, whereas 11,3% of the Whites and 4.4% of the Blacks were college graduates or more. These numbers would roughly imply that back in '70s, in all higher categories of educational attainment, White citizens had more representation than Black citizens in relative terms. However in the year 2010, 57.3% of the Whites and 64.4% of the Blacks had high school diploma, while 30.3% of the Whites and 19.8% of the Blacks had college diplomas or even higher degrees, which definitely points to a decrease in the discrepancy in higher levels of educational attainment between race groups. But by how much? Or only by looking at these categories can we say that the inequality of educational attainment between race groups has declined from 1970 to 2010? Justified answers to these questions require to compare the entire distributions of Blacks and Whites across all educational attainment categories and a method to evaluate the difference in these distributions.[3] Since the categories refer to the highest level of education attained, no obvious cardinal values are attached to them. In empirical works, this problem is often resolved by assigning the average number of years of schooling to educational attainment categories in order to make use of cardinal measures. Different countries, however, possess different educational cycles or countries make adjustments in their educational systems over time. Since cardinal measurement techniques are not robust to these changes, application of income inequality measures will cause misevaluations of inequality especially in cross-contry comparisons [71].

Inequalities of health or subjective well-being are other examples of non-income variables that face the same difficulty of treatment. The data on health and subjective well-being are collected via nation-wide surveys held by the health or statistics authorities of the

[3] Developed by UNESCO, the International Standard Classification of Education (ISCED) provides an internationally harmonized classification system for educational attainment.

countries. For practical purposes these variables are either defined over ordered categories such as "poor, fair, good, excellent" or over a cardinal scale such as "1,2,3,4", where 1 corresponding to "poor", "2" to "fair" and so on.[4] Allison and Foster show that application of cardinal measures of inequality over these categories results in incomparable levels of inequalities for different societies since these techniques are sensitive to scale changes [1].

As shown by the previous examples, measuring the extent of inequality in occupational status, educational attainment, health or subjective well-being is subject to restrictive assumptions or misevaluations caused by specific cardinalisations. There exists, therefore, a need for going beyond measurement of income inequality techniques and developing justified measurement methodologies for the evaluation of these non-income, social inequalities.

In this study we suggest a methodology to evaluate social inequalities: the Domination Index. Given a society with ordered categories and two social groups, the Domination Index follows a very natural logic to compare the distributions of social groups over categories: It basically counts the number of times a group beats the other group in pairwise confrontations. Consider a pair of individuals where each of them is a member of a different group, say Women and Men. The woman beats, dominates the man if she is in a better category than him. That is what we define as a domination. Then, the total number of dominations by the group Women is the total number of times that a woman beats a man. The Domination Index evaluates inequality in terms of the difference in the number of dominations. It actually is equal to the absolute average difference in the number of dominations by groups.

On top of its conceptual simplicity, the Domination Index has several appealing properties. First of all, it has a very intuitive interpretation. Since it compares the average number of dominations, it actually gives out the ex-ante probability advantage of a group over the other. In other words, the Domination Index gives out the extra probability that on a random selection of a pair of individuals from different groups, the member of one group occupies a better category than the member of the other group.[5] Second, it is efficient. It makes use of all the information available regarding the distributions of the social groups, and only of this information without going for further assumptions. Third, it is easy-to-use. Large samples of populations or long lists of categories do not create computational complexities. Fourth, it is well-founded. Our axiomatic analysis shows that it satisfies a set of reasonable properties. Moreover, it represents the only

[4] National Medical Expenditure Survey and National Health Interview Survey of United States, General Household Survey of United Kingdom, Swiss Health Survey and Survey on Health and Retirement in Europe make use of ordinal health categories.

[5] The Domination Index is closely related to Mann-Whitney's Statistic U and the Net Difference Index [62]. More on this can be found in the following review of literature.

family that satisfies these properties. These characterizing properties are variations of classical notions such as a symmetry property that requires equal treatment to social groups; a monotonicity property that controls the change in inequality for very specific changes in the society and finally a decomposability and an additivity property that allow to concentrate in different parts of the society and express the overall inequality as an aggregation of the inequalities in these parts. For instance, in order to understand the specific structure of the inequality, one may want to focus on upper and lower parts of the society separately, where the upper part consists of the better positions and the lower part consists of the worse ones. Decomposability ensures that the inequality in the entire society can be expressed in terms of the inequalities in upper and lower parts. On the other hand, with the same purpose, one may want to identify the contributions of different sections of the social groups to the overall inequality. An additivity property ensures that the overall inequality can be expressed in terms of the inequalities between different sections of the social groups. In a first theorem, we show that these properties yield us the Domination Index up to a positive scalar transformation.

The Domination Index is also instrumental to understand the connection of social inequalities with a related problem: segregation. Segregation is defined as the inequality in the distribution of groups over neither measurable nor comparable categories. The relation of between-group inequalities to segregation has been discussed in different literatures from both theoretical and empirical perspectives. Segregation simply captures the nominal difference of distributions without any regard to how relatively good or bad the distribution is. Inequality on the other hand involves an evaluation of the distributions. The difference of the distributions is assessed taking into account how beneficiary they are for the corresponding groups. Consider an imaginary building with the residents being from two different groups. A scenario such that one of the groups is occupying all the nicer flats with the view at the higher floors of the building, whereas all of the members of the other group living downstairs facing the facade of the building across the street will be maximally and as equally segregated as the scenario where all members of the first group are living in the odd numbered floors and all members of the second group are living in the even numbered floors, hence two groups are never sharing the same floor. However an inequality measure will label the first scenario more unequal than the second one. This certainly does not imply the dominance of one concept over the other but simply demonstrates that although closely related their focuses are different.

Clarifying the theoretical link between segregation and inequality, the Domination Index helps to understand the structure of the relation between these two concepts. We start by showing that for some societies the inequality between social groups measured by the Domination Index coincides with the level of segregation measured by a well-known segregation measure, the Gini Segregation Index. In other words, for a particular organization of the society, segregation is equal to the level of inequality between groups. This particular organization is the one in which the importance of each category reflects how uneven the distribution of groups in that category is. In other words, the order relation of the categories is in line with the relative distribution of groups across categories. If from the best to the worst category the ratio between the members of groups is always decreasing or increasing, i.e., if the ratio of the number of members of a group to the one of the other group is the highest in the best position, the second highest in the second best position and so on, then segregation in this society according to the Gini Segregation Index is equal to the inequality measured by the Domination Index. We then show that this organization is actually the one that results in the maximum possible level of inequality for that society. Hence, level of segregation in general gives an upper bound for the level of inequality. These observations not only provide a theoretical contribution to the debate on the relation of segregation to inequality but also gives out the characterization of the Gini Segregation index as a by-product. We show that variants of the properties that characterize the Domination Index do characterize the Gini Segregation Index. As a second by-product, we consider an extension of our methodology to assess inequalities under incomplete information about the ordering of categories. We exploit the relation between segregation and inequality to provide a way to measure inequalities between groups when the categories are not completely ordered.

The organization of the paper is as follows: First in a subsection we present a review of related literature. Then, the following section introduces the basic set up and the Domination Index. We provide a set of properties and the foundational analysis of the Domination Index. In the third section, we explore the link between segregation and inequality with the help of the Domination Index. The fourth section is an extension of our model to incomplete information. The proofs of the theorems in general are left to an appendix.

### 1.1.1 Related Literature

Although works discussing evaluation of social inequalities have not developed in a comprehensive and systematic way, there are various related literatures that we refer to. The most deeply analyzed and well-developed is, not surprisingly, the literature of between-group income inequality. A major part of this literature analyzes the decomposition of income inequality to its within-group and between-group components. The measures that allow the overall inequality to be expressed as the sum of between-group and within-group inequalities are qualified as additively decomposable measures [27, 28, 82, 83].

For this class of measures the between-group component of income inequality is simply found by assuming that each member of a social group receives that group's mean income. Then, comparison between-groups essentially becomes a comparison of group means. Bourguignon (1979) characterizes the family of decomposable functions that also satisfy other desirable properties and he shows that only two functions serve to this purpose: One of them is Theil's entropy measure [85] and the other is the mean logarithmic deviation, which is closely related to the Theil measure. Lasso de la Vega, Urrutia and Volij recently provide a characterization of the Theil measure by only making use of ordinal axioms [60]. Other methods based on comparison of representative levels of income of groups instead of mean income [16] or comparison of the observed between-group inequality with the maximum inequality that could occur [39] have been proposed as well.

A second major branch of group inequalities literature corresponds to segregation theories. The very first paper on this issue focuses on the residential segregation of race groups [56]. Research on school segregation by ethno-race groups [38, 43] developed parallel to the research on residential segregation [33, 68, 88] as well as occupational segregation by gender and race [26, 32, 72]. Most of the literature on segregation is based on development and application of indices, that are generally adaptations of measures of income inequality. Axiomatic characterizations of indices that are relevant for all questions of segregation are provided in [36, 50–52]. Research on the measurement of social inequalities is far from forming a well-developed, systematic literature but rather different pieces can be found as parts of different literatures. In a statistics spin off paper, Lieberson proposes the Net Difference Index to examine situations where two populations are to be compared with respect to a completely ordered characteristic such as age or years of schooling. Net Difference Index is based on Mann-Whitney's U Statistics (1947), which gives a non-parametric rank test that is used to determine if two samples are from the same population. The Statistics U is simply the number of times the observations from one sample precede the observations from the other sample when all of the observations are ordered into a single ranked series. The probability distribution tables of U are provided for testing the null hypothesis that two samples share the same distribution. The Statistics U is different from well-known Wilcoxon rank-sum statistics in that U allows for different sample sizes. Another rank-based statistics Somer's D, which is essentially a measure of association for ordinal variables, is used by several sociologists in the measurement of gender-based inequality of occupational status [15, 84].

Hutchens studies the question of gender-based occupational status inequality when the occupational status is determined by a prestige score [53]. He provides a set of desirable properties both for cardinal and ordinal variables of prestige. Reardon discusses the inequality of an ordinal variable such as education or occupational status between social groups and proposes to measure it in terms of the distances of the distributions of groups to a completely polarized distribution [76]. He proposes desirable properties for a measure of this sort, and then introduces four different functions that satisfy those properties without going for further axiomatic analysis.

Allison and Foster discuss the measurement of health inequality using self reported health status data, which is based on ordinal categories attached to a scale [1]. They argue that traditional measures are not applicable since they are not order preserving to scale changes and propose a partial ordering of health inequality that is invariant to scale changes. Based on Allison and Foster methodology, Naga and Yalcin propose a parametric family of indices that satisfy a basic normalization axiom [73]. Dutta and Foster apply the same methodology to measure the inequality of happiness in US by using self reported subjective well-being data [35]. They further use additive decomposition techniques to measure the group inequality of happiness between races, genders and regions. Kobus and Milos provides a characterization of a decomposable family of indices that respect Allison and Foster partial ordering [58]. Kobus also proposes an extension of this ordering to evaluate multi-dimensional inequalities [57]. In the measurement of inequality in educational attainment, although the data is collected over educational categories, the average number of years of schooling is assigned as a cardinal value to corresponding categories [10–12, 86]. This cardinalisation allows to use common inequality measures such as Gini coefficient and Theil indices while keeping the aforementioned problems of this procedure unsolved. Quite recently, Herrero and Villar propose a methodology to compare the educational achievements of different groups and provide different applications of this methodology in order to evaluate inequality of opportunities in education and health [46, 47]. Their methodolgy shares a similar statistical reference with the one proposed in this paper.

## 1.2 The Domination Index

A society is composed of individuals from different social groups distributed across ordered positions. We restrict our analysis to two social groups, namely Women and Men.[6] Formally, a society is a pair of elements $(S, L_{\mathcal{I}})$, where $\mathcal{I}$ denotes a finite set of $I$ positions, $S$ is a society matrix that shows the distribution of Women and Men over $I$ positions and $L$ is the order relation over $\mathcal{I}$. We assume that $L$ is an exogenous total order (a complete, transitive and asymmetric binary relation), where for every $i, j$ in $\mathcal{I}$, $iLj$ is interpreted as social position $i$ is better than position $j$. A society matrix, $S = (S_W, S_M)$ is a positive real matrix of dimension $I \times 2$ with the first column, $S_W$

[6] An extension to multigroup case is immediate though, as suggested in the concluding remarks.

describing the number of women in each position and the second column, $S_M$ denoting the number of men. We denote by $S_{iw}$ and $S_{im}$, the $i1^{st}$ and the $i2^{nd}$ elements of the matrix, the number of women and men in position $i$ respectively, whereas $S_w$ and $S_m$ stand for total number of women and men, i.e.; $\sum_i S_{iw} = S_w$ and $\sum_i S_{im} = S_m$. Small letters denote the proportions of individuals, i.e.; $s_{iw}$ denotes proportion of women in position $i$ to total number of women in society and $s_{im}$ denotes as of men. We consider $S_{iw}$ and $S_{im}$ to be nonnegative real numbers.[7] We denote with $C$ the space of all societies., i.e.; $C = \cup_{\mathcal{I}}(\mathbb{R}_+^{I\times 2} \times L^I)$, where $\mathbb{R}_+^{I\times 2}$ is the space of $I \times 2$ nonnegative real matrices and $L^I$ stands for the space of total orders over $\mathcal{I}$.

We define social inequality as the inequality in the distributions of women and men across ordered positions. Then, a social inequality measure is a non-zero continuous function $H : C \to \mathbb{R}_+$ that attaches to each possible society $(S, L_{\mathcal{I}})$, a nonnegative real number that shows the amount of social inequality.

The Domination Index, $D$ measures social inequality in terms of the number of times a group beats the other group in pairwise confrontations. Let us define **a domination** by a group as having a member in a better position than a counter-group member. Consider a woman in position $i$ in a society $(S, L_{\mathcal{I}})$. Her position is better than all the men that are in worse positions than $i$, thus she creates $\sum_{j:iLj} S_{jm}$ dominations in total. Then, the total number of dominations by women is equal to $\sum_i(S_{iw}\sum_{j:iLj} S_{jm})$, where total number of dominations by men is $\sum_i(S_{im}\sum_{j:iLj} S_{jw})$. The absolute difference in average number of dominations by women and men gives us the Domination Index:

$$D(S, L_{\mathcal{I}}) = \left|\frac{\sum_i(S_{iw}\sum_{j:iLj} S_{jm} - S_{im}\sum_{j:iLj} S_{jw})}{S_w S_m}\right| = \left|\sum_i(s_{iw}\sum_{j:iLj} s_{jm} - s_{im}\sum_{j:iLj} s_{jw})\right|$$

In a more compact form, it can equivalently be expressed as follows:

$$D(S, L_{\mathcal{I}}) = \left|\sum_i\sum_j c_{ij}s_{iw}s_{jm}\right| \text{ where } c_{ij} = \begin{cases} 1 & \text{if } iLj \\ 0 & \text{if } i = j \\ -1 & \text{if } jLi \end{cases}$$

This compact form notation highlights what $D$ measures in essence. $D$ actually gives out the ex-ante probability advantage between groups. Given a random pair of a woman and a man, the difference in probabilities of one individual beating the other is the ex-ante probability advantage of one group over the other, as shown in an immediate lemma:

**Lemma 1.1.** $D(S, L_{\mathcal{I}}) = |Pr(\textit{Women beating Men}) - Pr(\textit{Men beating Women})|$

[7] This choice not only ensures the generalization of our results but also is the convention in group-inequalities literatures. As noted in Hutchens (2001), in some empirical applications part-time employees are treated as fractional employees.

Since at the essence of social inequalities lies the idea of having an advantageous or a disadvantageous position just because being a member of a social group, $D$ does well in capturing this ex-ante probability advantage in evaluating social inequalities.

$D$ takes values between 0 and 1, 0 being complete equality and 1 being maximum inequality. The main attraction of $D$ depends on its simple structure and intuitive interpretation. It is very convenient and easy to apply to compare two distributions over ordered categories without making further cardinalisation assumptions. It is an efficient measure in the sense that it makes use of all the available information. Number of dominations by groups is the only relevant information of this setting and $D$ evaluates social inequality in terms of it. Although being easy to use, efficient and intuitive is important for an inequality measure for practical purposes, it is never sufficient unless supported by the properties that summarize the behavior of the function. To understand how $D$ behaves, we now introduce a set of properties that are not only satisfied by $D$ but are also 'reasonable' properties for any social inequality measure $H$.

A first standard property is a symmetry property, that ensures equal treatment to groups. It simply requires that exchanging the distributions of groups should not change the amount of social inequality. Formally;

**Symmetry for Groups (SYM):** Consider two societies $(S, L_{\mathcal{I}})$ and $(S', L_{\mathcal{I}})$ with $S_W = S'_M$ and $S_M = S'_W$. Then $H(S, L_{\mathcal{I}}) = H(S', L_{\mathcal{I}})$.

The second property is about the relative character of the index. Inequality measures are usually differentiated according to their absolute or relative characters. For relative inequality measures what matters are the relative amount of individuals whereas absolute inequality measures do take into account absolute amounts. $D$ is a relative inequality measure as ensured by the following property:

**Scale Invariance (INV):** Given $(S, L_{\mathcal{I}})$ and any $\alpha, \beta \in \mathbb{R}_{++}$, consider $(S', L_{\mathcal{I}})$ such that for all $i$, $S'_{iw} = \alpha S_{iw}$ and $S'_{im} = \beta S_{im}$. Then, $H(S, L_{\mathcal{I}}) = H(S', L_{\mathcal{I}})$.

For scale invariant functions what matters is the proportion of individuals in each position, not the absolute amounts. Especially for cross society analysis this is an important property, since otherwise larger populations would always imply higher inequality.[8]

Next we will introduce a monotonicity property that defines the behavior of the function for certain changes in the distributions. For some distributions of the society there exist some changes that clearly do not increase or do not decrease social inequality. For instance consider the following simple example of a society $(S, L_{\mathcal{I}})$ with 3 positions

[8] The absolute version of the index, i.e., $|\sum_i (S_{iw} \sum_{j:iLj} S_{jm} - S_{im} \sum_{j:iLj} S_{jw})|$ is actually characterized by similar properties but INV. The characterization result replaces DEC and SAD properties, that would be introduced soon, with non-weighted versions of them and it is available upon request.

where all men occupy the best position and all women are grouped in the worst position: $S = \left(\begin{smallmatrix} 0 & 100 \\ 0 & 0 \\ 100 & 0 \end{smallmatrix}\right)$ and $1L2L3$. This is a society in which women and men are distributed in a maximum unequal way possible. Now consider addition of one women to the best position and one men to the worst position. The resulting society will be of the form: $S = \left(\begin{smallmatrix} 1 & 100 \\ 0 & 0 \\ 100 & 1 \end{smallmatrix}\right)$. One would not expect from a reasonable relative inequality measure to identify the resulting society with higher social inequality than the initial one. $D$ evaluates the second society as less unequal than the first one. Now, consider exactly the opposite society $(S', L_{\mathcal{I}})$ such that $S' = \left(\begin{smallmatrix} 100 & 0 \\ 0 & 0 \\ 0 & 100 \end{smallmatrix}\right)$ again with $1L2L3$. The same addition of one women to the best position and one man to the worst is now not an inequality decreasing change since it results in the society $S' = \left(\begin{smallmatrix} 101 & 0 \\ 0 & 0 \\ 0 & 101 \end{smallmatrix}\right)$. For any relative social inequality measure, inequality is still at its maximum. The monotonicity property will ensure that this particular addition will not increase inequality for societies like $(S, L_{\mathcal{I}})$ and it will not decrease inequality for societies like $(S', L_{\mathcal{I}})$.

We define a **women improving addition** to $(S, L_{\mathcal{I}})$, as a slight increase in the number of women in the best position and in the number of men in the worst position in $S$. Formally, a women improving addition to $(S, L_{\mathcal{I}})$ is the addition of an $\varepsilon_W$ matrix of dimension $I \times 2$ that only possesses $\varepsilon$ number of women in the best position and $\varepsilon$ number of men in the worst position in $S$ for $\varepsilon$ small enough, all the other positions being empty.

We classify societies into two distinct types according to the reaction of the measured inequality to a women improving addition. If no women improving addition is resulting in a decrease in the social inequality, then we classify the society as of Women-type. On the contrary if any women improving addition is decreasing the social inequality, then the society is classified as of Men-type. Moreover, we define completely equal societies as of both Women-type and Men-type. Formally; given a society $(S, L_{\mathcal{I}})$ and a social inequality measure $H$, for any $\varepsilon \in \mathbb{R}_{++}$ in a $\delta$ neighborhood of 0, for $\delta$ small enough, $S$ is said to be of $W$**-type** if $H(S + \varepsilon_W, L_{\mathcal{I}}) \geq H(S, L_{\mathcal{I}})$ or $H(S, L_{\mathcal{I}}) = 0$. $S$ is said to be of $M$**-type** if $H(S + \varepsilon_W, L_{\mathcal{I}}) < H(S, L_{\mathcal{I}})$ or $H(S, L_{\mathcal{I}}) = 0$. Notice that, by definition, not being a $W$-type matrix directly implies being an $M$-type matrix. Obviously which matrices are of which type will depend on the particular behavior of the specific functional form of $H$. But for some unambiguous distributions like the ones of $S$ and $S'$ given in the example above, all reasonably monotonic measures should agree on the effect of a women improving addition. For the society $(S, L_{\mathcal{I}})$, clearly no women improving addition should increase the relative inequality, implying that $S$ is of $M$-type. On the contrary, for $(S', L_{\mathcal{I}})$, any women improving addition should not decrease inequality, implying that $S'$ is of $W$-type. This reasoning is applied in those societies in which we can make use of the first-order stochastic dominance to compare women and men distribution.

For a society $(S, L_{\mathcal{I}})$ that has exactly the same number of women and men, we say that the distribution of women dominates the distribution of men if for any position there is always more women than men in total in the positions that are at least as good as that position. To put formally; for $(S, L_{\mathcal{I}})$ with $S_w = S_m$, $S_W$ dominates $S_M$ if for all $k \in \mathcal{I}; \sum_{i:iLk} S_{iw} \geq \sum_{i:iLk} S_{im}$. Symmetrically, we say that $S_M$ dominates $S_W$ if for all $k \in \mathcal{I}; \sum_{i:iLk} S_{im} \geq \sum_{i:iLk} S_{iw}$.

**Monotonicity (MON):** Given a society $(S, L_{\mathcal{I}})$ with $S_w = S_m$, (i) if $S_W$ dominates $S_M$, then $S$ is a $W$-type society matrix; (ii) if $S_M$ dominates $S_W$, then $S$ is a $M$-type society matrix.

For a society in which the women distribution dominates the distribution of men, there is always more women in better positions. The first part of MON ensures that these type of society matrices are of $W$-type and the second part is the symmetric counterpart. Note that MON also guarantees a zero level of inequality for equally distributed societies. The following lemma states that any monotone social inequality measure assigns a value 0 to an equally distributed society.

**Lemma 1.2.** *For any $H$ that satisfies MON, for a society $(S, L_{\mathcal{I}})$ such that for any $i$, $S_{iw} = S_{im}$, we have $H(S, L_{\mathcal{I}}) = 0$.*

The properties that are introduced up to now, SYM, INV and MON are standard properties and are satisfied by many other functions in addition to $D$. The last two properties, however, will narrow down this class of functions extremely, up to a single family. Both properties are about the decomposability of overall inequality into the inequalities in different parts of the society. The concentration of social inequality in specific parts of a society is not an uncommon phenomenon. For instance in explaining the structure of the gender-based inequality in the labor market the theories of "glass ceiling" or "sticky floor" supplement strong evidence for the unbalanced distribution of women and men in the upper and lower tails of the wage distribution.[9] The following property, Decomposability allows to express overall inequality as an aggregation of inequalities in the upper part and the lower part of the society.

Given a society $(S, L_{\mathcal{I}})$, we define **an ordered division** of $(S, L_{\mathcal{I}})$ as a pair of societies $(S^1, L_{\mathcal{I}^1})$ and $(S^2, L_{\mathcal{I}^2})$ such that: (i) $\mathcal{I}^1$ and $\mathcal{I}^2$ define a partition of $\mathcal{I}$ such that for any $i$ in $\mathcal{I}^1$ and any $j$ in $\mathcal{I}^2$ we have $iLj$, (ii) for $k = 1, 2$, $iL_{\mathcal{I}^k}j$ if and only if $iLj$ for any $i, j$ in $\mathcal{I}^k$ and (iii) for $k = 1, 2$, $S^k$ is a $I^k \times 2$ society matrix such that each position possesses the same number of women and men in $S$ and $S^k$. An ordered division of a society is basically a partition of the society respecting the order relation: there is the

[9]See McDowell, Singell and Ziliak (1999), Blau and Kahn (2000), Baker (2003), de la Rica, Dolado and Llorens (2008)

upper part that is composed of the better positions in the society and a lower part that is composed of the worse ones. $D$ expresses the overall inequality as an aggregation of the inequalities in each of these parts and an interaction term between them that stems from the fact that all of the positions in the upper part are actually better than all the positions in the lower part. What we define as the interaction term is equal to the social inequality in a society with two positions. The first position is occupied by all the individuals of the upper part of the original society and the second position is occupied by all the individuals of the lower part. Formally; given an ordered division of a society $(S, L_{\mathcal{I}})$ as $(S^1, L_{\mathcal{I}^1})$ and $(S^2, L_{\mathcal{I}^2})$, **the interaction society** $(S', L'_{\mathcal{I}'})$ is a society defined as the following: (i) It consists of two social positions: $\mathcal{I}' = 1, 2$ with $1L'2$ (ii) The first position contains all individuals of $S^1$: $S'_{1w} = S^1_w$ and $S'_{1m} = S^1_m$ (iii) The second position contains those of $S^2$: $S'_{2w} = S^2_w$ and $S'_{2m} = S^2_m$.

The decomposability property allows to decompose the total social inequality as the weighted sum of the inequalities in the upper part, lower part and the interaction society for any ordered division of a society, as long as all resulting society matrices are of the same type. The specific form of the weighting structure depends on the specific form of the measure. As $D$ counts the number of dominations in pairwise confrontations, the weighting structure depends on the proportion of these pairwise confrontations in each part. Given an ordered division, we define a **population weight** as the proportion of pairs of women and men in each part to the overall number of pairs. Formally; for an ordered division of $(S, L_{\mathcal{I}})$ as $(S^1, L_{\mathcal{I}^1})$ and $(S^2, L_{\mathcal{I}^2})$, the population weight of $(S^k, L_{\mathcal{I}^1})$ for $k = 1, 2$ is $\lambda^k_S = \frac{(S^k_w)(S^k_m)}{(S_w)(S_m)}$. Notice that the weight of the interaction society $(S', L_{\mathcal{I}'})$ will be equal to 1 by this definition.

**Decomposability (DEC):** For any ordered division of $(S, L_{\mathcal{I}})$ as $(S^1, L_{\mathcal{I}^1})$ and $(S^2, L_{\mathcal{I}^2})$ such that $S^1$, $S^2$ and $S'$ are either all of $W$-type or all of $M$-type the following holds:

$$H(S, L_{\mathcal{I}}) = \lambda_{S^1} H(S^1, L_{\mathcal{I}^1}) + \lambda_{S^2} H(S^2, L_{\mathcal{I}^2}) + H(S', L'_{\mathcal{I}'})$$

As the last property, we introduce Subgroup Additivity, that helps to deepen the analysis one step further by differentiating the effects of subgroups to overall social inequality. We define a subgroup as a subset of a social group. For instance Immigrant Women and Local Women refer to two subgroups of the social group Women. Subgroup Additivity will allow to identify how much of the overall inequality is between Men and Immigrant Women and how much of it is between Men and Local Women.

For a society $(S, L_{\mathcal{I}})$, **a partition into subgroups** is a pair of societies $(S', L_{\mathcal{I}})$, $(S'', L_{\mathcal{I}})$ such that for all $i$, either $S_{iw} = S'_{iw} = S''_{iw}$ and $S_{im} = S'_{im} + S''_{im}$ or $S_{im} = S'_{im} = S''_{im}$ and $S_{iw} = S'_{iw} + S''_{iw}$ holds. Hence, a partition of a society into subgroups results in a pair of societies that possess exactly the same distribution of one of the groups with the original society, and sum up to the original distribution of the other group. For the sake of simplicity we have defined a partition into subgroups only for two subgroups, but clearly repeated application of the partition will result in a partition into many subgroups.

Given a partition into subgroups, the Subgroup Additivity property allows to express the overall social inequality between Women and Men as a weighted aggregation of the inequalities between each subgroup and the other social group as long as all societies are of the same type. Similar to DEC, for a partition of a society $(S, L_{\mathcal{I}})$ into subgroups $(S', L_{\mathcal{I}})$ and $(S'', L_{\mathcal{I}})$, the population weight of the subgroups will be as $\lambda_{S'} = \frac{(S'_w)(S'_m)}{(S_w)(S_m)} = \frac{(S'_w)}{(S_w)}$ and $\lambda_{S''} = \frac{(S''_w)(S''_m)}{(S_w)(S_m)} = \frac{(S''_w)}{(S_w)}$. But notice that this time the population weights of the subgroups add up to 1. Hence SAD expresses the overall inequality as a convex combination of the subsociety inequalities.

**Subgroup Additivity (SAD):** For any partition of a society $(S, L_{\mathcal{I}})$ into subgroups $(S', L_{\mathcal{I}})$ and $(S'', L_{\mathcal{I}})$ such that $S'$ and $S''$ are both of $W$-type or $M$-type, the following holds:

$$H(S, L_{\mathcal{I}}) = \lambda_{S'} H(S', L_{\mathcal{I}}) + \lambda_{S''} H(S'', L_{\mathcal{I}})$$

With SAD, since the weights are nonnegative, the original society matrix $S$ will necessarily be of the same type with $S'$ and $S''$. To see this notice that for $S$, $S'$ and $S''$ with $H(S, L_{\mathcal{I}}) = \lambda_{S'} H(S', L_{\mathcal{I}}) + \lambda_{S''} H(S'', L_{\mathcal{I}})$, we have $H(S + \varepsilon_W, L_{\mathcal{I}}) = \lambda_{S'} H(S' + \frac{\varepsilon_W}{2}, L_{\mathcal{I}}) + \lambda_{S''} H(S'' + \frac{\varepsilon_W}{2}, L_{\mathcal{I}})$ for all $\varepsilon$ in a $\delta$ neighborhood around 0, where $\delta$ is determined by the smaller of the neighborhoods that are induced by $S'$ and $S''$. Then, the change in the overall inequality yielded by the addition of $\varepsilon_W$ to $S$ will be in the same direction with the changes in $S'$ and $S''$ created by the addition of $\frac{\varepsilon_W}{2}$.

SAD is a strong property and as will be highlighted in the proof of the characterization result, it has an important role in determining the functional form of $D$. It actually implies INV property. In other words, any function that satisfies SAD is a relative inequality measure, as stated in the following proposition:

**Proposition 1.3.** *Any $H : C \to \mathbb{R}_+$ that satisfies SAD is Scale Invariant.*

We are now ready to introduce the main result of the paper. These properties listed not only are satisfied by $D$, but also they do characterize it up to a positive scalar transformation. As SAD implies INV, we do not include it as an additional axiom.

**Theorem 1.4.** *A social inequality function $H : C \rightarrow \mathbb{R}_+$ satisfies SYM, MON, DEC and SAD if and only if it is a positive scalar transformation of the Domination Index:*

$$D(S, L_{\mathcal{I}}) = |\sum_i (s_{iw} \sum_{j:iLj} s_{jm} - s_{im} \sum_{j:iLj} s_{jw})|$$

The proof of the characterization can be summarized by the following steps: In Step 1, we consider a very specific type of a society and derive the functional form of $H$ for it. We focus our attention to societies for which inequality is always favoring Women, i.e., for any subset of positions Women have a better distribution than Men. These would be the societies with strictly decreasing $\frac{S_{iw}}{S_{im}}$ ratios from the best to the worst position. We call them as Women-perfect societies. Since Women-perfectness allows for iterative application of DEC, together with INV and MON, we first show that for Women-perfect societies, overall inequality can be decomposed into the inequalities between the individuals of a position and all other individuals in worse positions. Hence, we remain with a collection of simpler hypothetical societies with two positions, where the first position of each hypothetical society possesses the individuals of an original position and the second position includes all individuals that are in worse positions than this one. In Step 2, we focus only to those $2 \times 2$ hypothetical societies. SAD ensures that the inequality of these societies is a function of the difference in number of dominations by groups. Then, aggregation of the inequalities of hypothetical societies yields the functional form as a positive multiple of the average number of dominations by Women net of average number of dominations by Men. Step 3 simply shows that by SYM, we arrive to the functional form of the index for societies for which inequality is always favoring Men, Men-perfect societies. In Step 4, we consider any $W$-type society and associate it with a particular Women-perfect society. We do this by adding sufficient number of women to the original society. The functional form of the index for any $W$-type society appears from the difference of the inequalities of the Women-perfect society and the subsociety that includes the women that are added to the original society. Step 5 mimics Step 4 for any $M$-type society. Since any society is either $W$-type or $M$-type, we arrive to the index.

All of the characterizing properties are independent. SYM guarantees equal treatment to Women and Men. Hence, an asymmetric version of $D$ that values dominations by Women and Men differently can be an example to a social inequality function that satisfies all of the other properties but SYM. For instance, $H(S, L_{\mathcal{I}}) = |\sum_i (2s_{iw} \sum_{j:iLj} s_{jm} - s_{im} \sum_{j:iLj} s_{jw})|$.

MON is responsible from the comparison between groups. The function that counts the total average number of dominations instead of the difference in average number of dominations will be an example to a function that only does not satisfy MON out of the stated properties, i.e., $H(S, L_{\mathcal{I}}) = |\sum_i (s_{iw} \sum_{j:iLj} s_{jm} + s_{im} \sum_{j:iLj} s_{jw})|$.

DEC ensures that individuals of each position are taken into account in relation to the relative order of the position. A function that only considers the dominations by some of the individuals will not satisfy DEC. An example that comply with MON, SYM and SAD will be a function that only counts the dominations by the individuals of the best position: $H(S, L_{\mathcal{I}}) = |s_{xw} \sum_{j:xLj} s_{jm} - s_{xm} \sum_{j:xLj} s_{jw}|$, where $x$ denotes the best position according to $L$ over $\mathcal{I}$.

Finally, SAD accounts for considering only the dominations between groups. A function that takes into account the dominations within groups will not satisfy SAD. For instance: $H(S, L_{\mathcal{I}}) = \frac{1}{S_w S_m}(|\sum_i (S_{iw} \sum_{j:iLj} S_{jm} - S_{im} \sum_{j:iLj} S_{jw})| + |\sum_i (S_{iw} \sum_{j:iLj} S_{jw} - S_{im} \sum_{j:iLj} S_{jm})|)$.

## 1.3 Segregation as Inequality

Segregation, in very general terms, is about how separated different groups of a society are. It is the degree to which social groups are distributed differently in the society. It has started to attract attention with the discrepancy in the distributions of different race groups across residential areas in United States at the first half of the 20th century and since then different types of segregation have been recognized, documented and analyzed both theoretically and empirically. A major part of the questions about segregation concerns the relation of segregation to inequalities. Spatial separation of social groups from each other has been detected to be a cause and a consequence of the unequal levels of wellbeing between social groups. Higher levels of residential segregation by ethno-race groups is found to be responsible for low levels of education and occupation outcomes [30, 37]. Lower racial inequality in terms of educational attainment is in turn shown to increase residential segregation [13]. Segregation of black and non-black students into different schools has been blamed for substantial differences in achievement [37, 45]. An important share of wage inequality between women and men has been explained by gender-based occupational segregation [18, 19, 63]. However not always the trend in wage inequality follows exactly the same pattern with occupational segregation [75]. Hence there exists a close link between segregation and inequalities between groups, though the strength or the direction of this link is never clear.

Segregation is defined as a form of inequality. It is the inequality in the distributions of social groups across neither measurable nor comparable categories. Categories need not to be uncompared by nature, but a comparison of these categories is not relevant to the question of segregation. For instance for residential segregation by race, neighborhood quality may very well define an ordering of neighborhoods. However the essence of the idea of residential segregation, "the degree to which two or more groups live separately from each other in different parts of the urban environment" [68] hinges on spatial difference in residential patterns. A quality ordering of the neighborhoods is not a part of the question of segregation itself, but brings in the notion of inequality. Segregation captures the nominal discrepancy of the distributions regardless of how good or bad the distributions are, whereas inequality involves an evaluation of the distributions. Hence the comparison of the categories do matter for inequality analysis in contrast to segregation. However, despite this conceptual distinction we can observe cases such that social groups are as segregated as unequally distributed. In other words, there exist societies in which segregation is actually the inequality between groups. Let us go back to the imaginary building example with residents from two different groups, given in the introduction. In the completely polarized society scenario, in which one of the groups is occupying the nicer flats with the view at the higher floors of the building, whereas all members of the other group are living downstairs facing the facade of the building across the street, both segregation and inequality between groups are at their maximum, hence equal for standardized measures of segregation and inequality. Moreover, this is not a unique example. One can find many other scenarios such that segregation coincides with the inequality. The Domination Index becomes helpful at this point: It allows to identify the societies such that spatial inequality between groups captures the overall inequality, clarifying the structure of the relation between segregation and inequality.

In this section, we show that for particular societies the inequality between social groups measured by the Domination Index is equal to the segregation measured by a well-known segregation measure, Gini Segregation Index. Gini Segregation Index, $G_S$, is one of the oldest methods to measure segregation, suggested in the first paper on the subject (Jahn et. al., 1947). As the name suggests it shares the same underlying logic with Gini Inequality Index in measuring inequality as a normalized mean absolute difference between all pairs of components.[10] Since the order relation $L$ over $\mathcal{I}$ is not an argument for segregation, a society is simply the society matrix over $I$ positions that we denote as $S_{\mathcal{I}}$. Let $S_{it}$ denote the total number of individuals in position $i$, $q_i$ women share in position $i$ and $S_t$ and $q$ the respective amounts for the whole society, i.e.; $S_{it} = S_{iw} + S_{im}$ and $q_i = S_{iw}/S_{it}$. A segregation index is simply a non-zero continuous function defined

[10]The general formulation of Gini inequality indices can be given as $\frac{1/T^2 \sum_{i=1}^{n} \sum_{j=1}^{n} |T_i - T_j|}{2T/n}$, where there are $n$ components (individual, place, position) with component $i$ possessing a $T_i$ share of the $T$ units (income, people) in total. In the context of income inequality it becomes $\frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j|}{2n \sum_{i=1}^{n} y_i}$, where $y_i$ denotes the income level of individual $i$.

from $B = \cup_{\mathcal{I}} \mathbb{R}_+^{I \times 2}$ to $\mathbb{R}_+$. Then, Gini Segregation Index is:

$$G_S(S_{\mathcal{I}}) = \sum_i \sum_j \frac{S_{it} S_{jt} |q_i - q_j|}{2 S_t^2 q(1-q)}$$

$G_S$ measures segregation in terms of the average difference in women shares of positions. It will be equal to zero only if the women shares of all positions are the same; for all $i$ and $j$, $q_i = q_j$. This happens only if the proportion of women in each position, $s_{iw}$ is equal to the proportion of men, $s_{im}$. Notice that in this case, as shown in Lemma 1.2 there will be zero social inequality as well, i.e., $D(S, L_{\mathcal{I}}) = 0$. Hence, no segregation implies no social inequality. On the other hand, $G_S$ will take its maximum value as 1 under complete polarization, when all men occupy better positions than all women, or vice versa. In this case, $D$ gives a value of 1, as well.

These two extreme cases with identical levels of segregation and inequality may seem as opposites. However they share an important property: In both of the cases the order of the positions are in line with the relative masses of groups occupying the positions. Going down in the hierarchy of positions, the ratio of number of women to number of men occupying a position follows a monotone path. Indeed this is a sufficient property to have equal levels of segregation and inequality. If the ratio of number of women to men is increasing or decreasing from the best to worse positions, then segregation of this society equals to the inequality between groups. Let $r_i$ denote the proportion of number of women to number of men in position $i$, i.e.: $r_i = \frac{S_{iw}}{S_{im}}$. Then, we have the following proposition:

**Proposition 1.5.** *For any $(S, L_{\mathcal{I}})$ in $C$, we have $G_S(S_{\mathcal{I}}) = D(S, L_{\mathcal{I}})$ if and only if (i) $r_i \geq r_j$ for all $i$ and $j$ with $iLj$ or (ii) $r_j \geq r_i$ for all $i$ and $j$ with $iLj$.*

The proof of the proposition follows fast from an alternative expression of Gini Segregation Index as $G_S(S_{\mathcal{I}}) = \frac{1}{2} \sum_i \sum_j |s_{iw} s_{jm} - s_{im} s_{jw}|$. This expression of $G_S$ stresses out the relation between $G_S$ and $D$, since $D$ can alternatively be stated as: $D(S, L_{\mathcal{I}}) = |\sum_i \sum_{j:iLj} s_{iw} s_{jm} - s_{im} s_{jw}|$. Both of them evaluate the average difference in cross products of group shares, the amount $(s_{iw} s_{jm} - s_{im} s_{jw})$, for pairs of positions. However there are two main distinctions: (i) In case of inequality this amount refers to the difference in number of dominations by groups, hence $D$ makes use of the order relation $L$ and aggregates over for pairs of positions $i$ and $j$ with $iLj$. In case of segregation this amount is a measure of how differently distributed two groups over $i$ and $j$, hence $G_S$ aggregates it for any pair of positions without reference to an order relation. (ii) $G_S$ is the summation of absolute values over pairs of positions; what matters is the nominal difference in distributions. For any pair of positions $i, j$, the contribution to the overall segregation is always nonnegative. $D$ is the absolute value of a sum over positions. Inequality does not need to be in the same direction over all pairs of positions.

Thus, combining (i) and (ii), if the structure of the society is such that inequality is always favoring the same group, then $D$ would be equal to $G_S$. This is possible only if the positions are ordered according to the relative masses of the groups occupying them. If from the best to the worst position $r_i$ is always decreasing, then Women always have an advantageous distribution, i.e.; number of dominations by Women is larger then Men for any pair of positions. If, on the other hand, $r_i$ is increasing from the best to the worst position, then Men always have an advantageous distribution.

Hence for a particular organization of the society, segregation is the inequality between groups. If the order of importance of the positions is reflected by the relative distribution of the groups, then segregation is actually responsible from the inequality.

This simple result not only helps to understand the theoretical link between segregation and inequality, but also provides a characterization of the Gini Segregation Index. We exploit the relation between $D$ and $G_S$ to adapt the characterizing properties of $D$: SYM and INV properties remain the same. MON property becomes redundant as there is no direction in segregation as opposed to inequality. However we need an additional property to fix the level of no segregation to zero. A normalization property (NORM) requires that if the distribution of Women is exactly equal to the distribution of Men, then there is zero segregation. Notice that, this property was implied by MON in case of inequality. The DEC and SAD properties are the ones that require to be adapted with reference to the $r_i$ ordering instead of the exogenous order of positions. We define r-DEC as the decomposability of overall segregation into two different segments of the society and an interaction term between them where the upper segment consists of the positions with higher $r_i$ ratios with respect to the positions of the lower segment. Similarly, r-SAD ensures that overall segregation could be expressed as a weighted sum of the levels of segregation of the subsocieties if the $r_i$ ordering of the positions is preserved for subsocieties with respect to the original society. As before, r-SAD implies INV. Formal definitions of the properties are introduced in the Appendix as well as the proof of the following theorem:

**Theorem 1.6.** *A segregation index $H : B \to \mathbb{R}_+$ satisfies SYM, NORM, r-DEC, r-SAD if and only if $H$ is a positive scalar transformation of the Gini Segregation Index:* $G_S(S_{\mathcal{I}}) = \frac{1}{2}\sum_i \sum_j |s_{iw}s_{jm} - s_{im}s_{jw}|$

## 1.4 Extension: Inequality with Incomplete Information

The previous section has established that if the order of the positions of a society does not follow a particular pattern, then the level of inequality will be different than the level of segregation. The proof of this claim demonstrates that for any other organization of the society, inequality will actually be less severe than segregation. For the sake of completeness, let us state this observation formally:

Let $\mathcal{L}_{\mathcal{I}}$ denote the set of linear orderings of $\mathcal{I}$.

**Corollary to Proposition 1.5** For any $(S_{\mathcal{I}})$ in $B$, the level of segregation measured by $G_S$ is equal to the maximum level of inequality measured by $D$ over all possible linear orderings of $\mathcal{I}$, i.e;

$$G_S(S_{\mathcal{I}}) = \max_{L_{\mathcal{I}} \in \mathcal{L}_{\mathcal{I}}} D(S, L_{\mathcal{I}})$$

When there is no information about the ordering of positions the only inequality between groups is due to segregation and is equal to the maximum group inequality over all possible linear orderings of $\mathcal{I}$. When the information is not null but not complete either, we could actually follow the same argumentation. For instance, in order to evaluate the inequality of the distributions of gender groups in a firm hierarchy, one could encounter problems in ordering the positions completely. Firm hierarchies do not necessarily show a linear pattern, but they mostly follow a tree structure. This would mean that for some positions the order relation is clear, but not necessarily all positions are compared to each other. In other words, the order relation is incomplete. How could we measure the inequality of distributions if we have incomplete information about the ordering of the positions?

We propose to follow what is suggested by the previous observation and complete the missing information by considering all possible linear orderings of the positions and determining the maximum possible level of group inequality. When there is complete information about the ordering of the positions, the Domination Index makes use of all of the existing information. Under no information regarding the ordering of the positions, it would be safe to consider the maximum level of group inequality over all possible ways of completing the existing information since we have shown that this coincides with segregation. Then under incomplete information about the ordering of the positions a natural extension would be to consider all possible ways of completing it. The maximum level of group inequality over all possible completions would be qualified as the group inequality in that society.

Let $P_{\mathcal{I}}$ be a strict partial order over the set of positions $\mathcal{I}$. A society will be a pair of elements $(S, P_{\mathcal{I}})$, where $S$ is the usual society matrix. Let $\mathcal{L}^{P_{\mathcal{I}}}$ denote the set of linear extensions of $P$ over $\mathcal{I}$, i.e.; the set of complete, transitive and asymmetric binary relations over $\mathcal{I}$ with for all $L_{\mathcal{I}}$ in $\mathcal{L}^{P_{\mathcal{I}}}$, $iLj$ if $iPj$. Then, Maximum Group Inequality Index, $M$, will be a continuous function defined from the set of all possible societies to nonnegative real numbers in the following way:

$$M(S, P_{\mathcal{I}}) = \max_{L_{\mathcal{I}} \in \mathcal{L}^{P_{\mathcal{I}}}} D(S, L_{\mathcal{I}})$$

We know that $M$ is a relative group inequality measure that takes values in $[0, 1]$ as well. If there is no missing information about the ordering of the positions, $M$ is equal to $D$. If there is no ordering information available, then the only inequality between groups is due to segregation and that is completely captured by $M$, since it is equal to $G_S$ for this case. In case of some missing information, $M$ gives the maximum possible level of group inequality, which refers to the worst-case scenario of the society. If two positions remain uncompared by the original ordering, this will be because of the fact that there is no unique universal way of ranking these positions; their ordering may change from time to time, society to society. Considering the worst-case scenario is consistent with a Rawlsian framework of welfare, apart from being a natural outcome of the structural relations between inequality and segregation.

### 1.4.1 An Empirical Exercise: Gender-based Occupational Inequality in Europe

In this section we provide a quick application of Maximum Inequality Index to assess the discrepancy in the distribution of genders across occupational groups in Europe. According to International Standard Classification of Occupations (ISCO), jobs are classified into occupational groups with respect to the skill level and skill specialization required to competently perform the tasks and duties of the occupations.[11] Figure 1 summarizes the gender distribution across 9 major occupational groups in 9 European countries by 2010. [12] [13]

In order to evaluate the inequality between gender groups across these occupational categories we need to take into account welfare attributes of occupations such as income, working conditions or other socio-economic status indicators offered by the occupations. The hinge is that not necessarily all attributes are perfectly correlated. An occupation

[11] We make use of the latest version of the classification system, ISCO08, which is published by International Labor Organization in 2008. (http://www.ilo.org/public/english/bureau/stat/isco/isco08/index.htm)

[12] According to ISCO08, there are 10 major occupational groups. We leave out category 0, "Armed forces occupation" due to data restrictions.

[13] The distribution data used in this exercise is taken from United Nations Economic Comission for Europe (UNECE) Statistical Division Database (http://w3.unece.org/pxweb/).

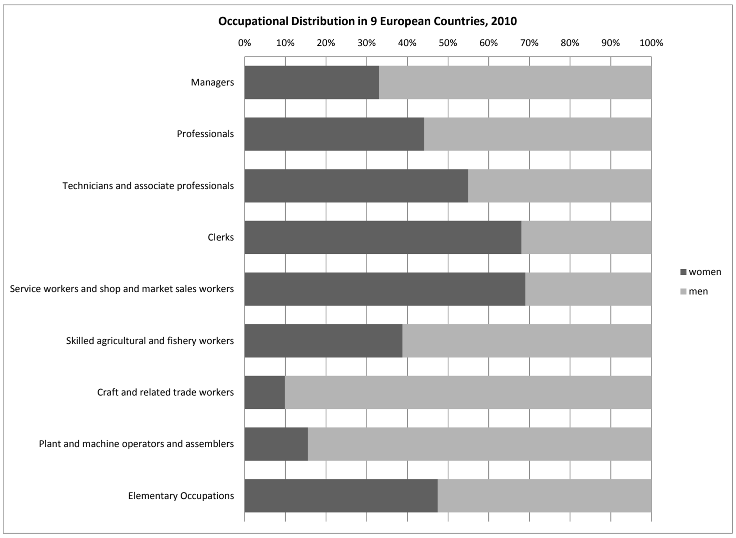



FIGURE 1.1: Occupational distribution of gender groups in Austria, Denmark, Greece, Germany, Iceland, Italy, Luxembourg, Portugal, United Kingdom in 2010

may have quite challenging working conditions, even resulting in health troubles, although offering a very high level of wage. Hence taking multi attributes into account, a linear ordering of occupations will not be possible. However, we can arrive to a partial ordering of occupations that will not contradict with any of the orderings suggested by each welfare attribute. Given this partial ordering, the Maximum Inequality Index will tell us the worst case scenario as the occupational inequality between gender groups.

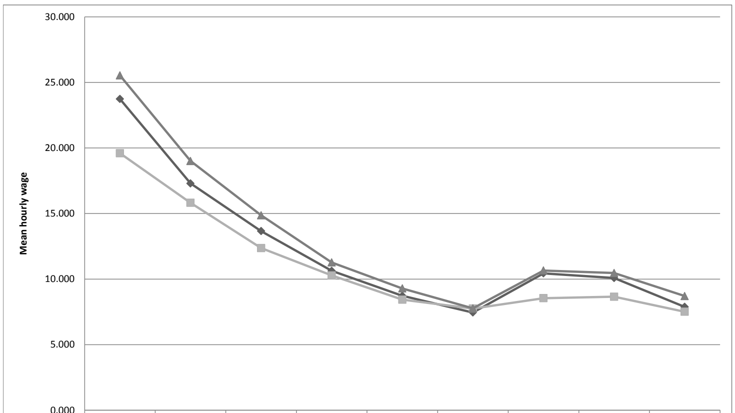
Figure 2 shows the mean hourly wage pattern of occupational groups computed as 2010 European mean of total population and for women and men separately.[14]

Although wage levels for women and men differ, the order of occupational categories suggested by mean hourly wage of women, men and total population do coincide. The wage ordering of the occupations would be, in decreasing order, as the following: Managers, Professionals, Technicians and Associate Professionals, Clerical Support Workers, Craft and Related Trade Workers, Plant and Machine Operators and Assemblers, Service and Sales Workers, Elementary Occupations, Skilled Agricutural, Forestry and Fishery Workers.

The International Socio-Economic Index of occupational status (ISEI-08) is a scale designed for occupations using the required level of education and the earnings offered. It basically assigns an optimal score to each occupation that aims to minimize the direct effect of education on earnings and maximizing the indirect effect of education on earnings via occupation.[15] The ISEI ordering of occupations computed according to ISCO08,

[14]The wage data is taken from the statistical database of Eurostat, Structure of earnings survey, 2010. (http://epp.eurostat.ec.europa.eu/portal/page/portal/statistics/)

[15]For further reference: http://www.harryganzeboom.nl/isco08/qa-isei-08.htm




| | Managers | Professionals | Technicians and associate professionals | Clerical support workers | Service and sales workers | Skilled agricultural, forestry and fishery workers | Craft and related trades workers | Plant and machine operators, and assemblers | Elementary occupations |
|---|---|---|---|---|---|---|---|---|---|
| Total | 23.734 | 17.293 | 13.657 | 10.626 | 8.723 | 7.453 | 10.436 | 10.083 | 7.882 |
| Women | 19.597 | 15.818 | 12.364 | 10.279 | 8.434 | 7.753 | 8.545 | 8.655 | 7.512 |
| Men | 25.534 | 19.002 | 14.863 | 11.269 | 9.287 | 7.767 | 10.651 | 10.461 | 8.704 |

FIGURE 1.2: Mean hourly wage of occupational groups in Europe in 2010

with the corresponding ISEI08 secores in parantheses, gives us: Professionals (65), Managers (62), Technicians and Associate Professionals (51), Clerical Support Workers (41), Craft and Related Trade Workers (35), Plant and Machine Operators and Assemblers (32), Service and Sales Workers (31), Elementary Occupations (20), Skilled Agricutural, Forestry and Fishery Workers (18).

Finally, working conditions offered by the occupation is a significant welfare determinant, especially when health related outcomes are considered. In order to assess the working conditions of occupations we make use of five different variables related to work context: Cramped Work Space-Awkward Positions (How often does this job require working in cramped work spaces that requires getting into awkward positions?), Exposed to Hazardous Conditions (How often does this job require exposure to hazardous conditions?), Spend Time Making Repetitive Motions (How much does this job require making repetitive motions?), Deal With Unpleasant or Angry People (How frequently does the worker have to deal with unpleasant, angry, or discourteous individuals as part of the job requirements?), Lack of Decision Power (How much decision making freedom, without supervision, does the job offer?).[16] Averaging the scores attached to each of these variables for each occupational category, we arrive to the work conditions ordering as the following: Managers (25.08), Professionals (26.61), Technicians and Associate Professionals (35.03), Clerical Support Workers (35.54), Service and Sales

[16]Data related to these variables is taken from Occupational Information Network (ONET) database (http://www.onetonline.org/). ONET database contains information on a variety of standardized and occupation-specific descriptors and provides importance and levels of these descriptors for each occupation. It is based on the Standard Occupational Classification. In order to translate it to ISCO08 we make use of the crosswalk suggested by US Department of Labor, Bureau of Labor Statistics: http://www.bls.gov/soc/soccrosswalks.htm.

Workers (37.35), Skilled Agricutural, Forestry and Fishery Workers (38.73), Craft and Related Trade Workers (43.68), Elementary Occupations (43.83), Plant and Machine Operators and Assemblers (47.77).

The following figure corresponds to the partial ordering of occupations once we consider all three orderings together:

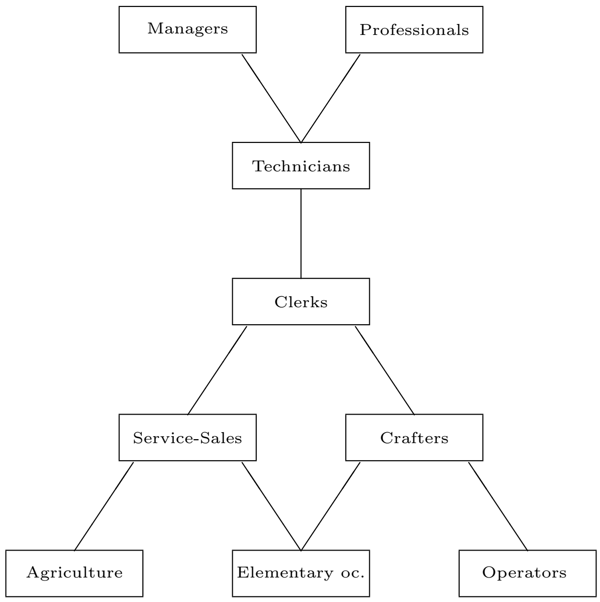



FIGURE 1.3: Order of Occupations according to Wage, Socio-Economic Status and Working Conditions

Below we present the gender-based occupational inequality in 9 European countries in 2000 and 2010, computed by the Maximum Inequality Index by making use of this partial ordering of occupations.

**Table 2. Gender-based Occupational Inequality by Maximum Inequality Index**

| Country | 2000 | 2010 |
|---|---|---|
| Austria | 0.1715 | 0.1928 |
| Denmark | 0.1958 | 0.1442 |
| Germany | 0.1499 | 0.1479 |
| Greece | 0.1993 | 0.2001 |
| Iceland | 0.2436 | 0.2704 |
| Italy | 0.2116 | 0.1577 |
| Luxembourg | 0.1676 | 0.1527 |
| Portugal | 0.1778 | 0.1821 |
| United Kingdom | 0.1867 | 0.1622 |

In 3 out of 9 countries, Austria, Greece and Iceland, occupational inequality has tended to increase in 10 years span. Let us note that, out of these 9 countries, only in Iceland, inequality is actually favoring women. In all countries but Iceland, men have a more advantageous distribution with respect to women.

## 1.5 Concluding Remarks

Unequal distibution of social groups across different levels of welfare is quite commonly observed. When we go beyond income inequality and consider non-cardinal welfare determining variables such as education, health, occupation or subjective well-being, we run short of well-developed inequality measurement techniques. This paper aimed to propose an intuitive and well-founded methodology to evaluate non-income inequalities between social groups without appealing to additional cardinalisation assumptions.

The Domination Index evaluates the discrepancy in group distributions as a function of the number of times a group beats the other group. We showed in a first result that a set of properties, a classical Symmetry property, a Monotonicity property and two decomposability properties characterize the Domination Index up to a positive scalar transformation. The Domination Index is instrumental in clarifying the intimate link between social inequalities and segregation. In a second result, we showed that segregation is actually the inequality for a very specific distribution of the society, where the organization coincides with the socially worst outcome. Furthermore, we exploited this theoretical link between segregation and inequality to propose a technique to evaluate inequalities where the information regarding the ordering of the categories is not necessarily complete and provided a simple empirical exercise to evaluate gender-based occupational inequality across nine European countries.

We provided the index for evaluation of inequalities between two social groups. However there are many real life cases that require a multi-group analysis. A natural way to extend the Domination Index to multi-group case is to consider an aggregation of the differences in pairwise dominations for any pair of groups. When there are more than two social groups, we first focus on pairs of groups and calculate for each pair the average difference in number of dominations, i.e., the Domination Index for two groups. Then, the average of these average differences would be a multi-group version of the Domination Index. Let us state this idea formally: Let $\mathcal{G}$ be a set of social groups with cardinality $G$. Then a society with $G$ groups and $I$ positions will be a pair $(S, L_{\mathcal{I}})$ where $S$ is a society matrix of dimension $I \times G$ and the multi-group Domination Index would be equal to $\frac{1}{2G}\sum_{M\in\mathcal{G}}\sum_{N\in\mathcal{G}} D(S_M, S_N)$, where $S_M$ denotes the vector of group $M$ in

$S$ as usual. Notice that this is again a relative inequality measure that takes values between 0 and 1.

The foundational analysis of the multi-group version of the Domination Index is a question of ongoing research as well as its relation to multi-group segregation indices. In addition, the algorithmic structure and behavior of the Maximum Group Inequality Index remain to be explored.

## 1.6 Appendix

**Extra Notation for the Proofs:**
To denote a society $(S, L_{\mathcal{I}})$ with $1L2L\ldots LI$, we use $S = (S_{1w}, S_{1m}; S_{2w}, S_{2m}; \ldots; S_{Iw}, S_{Im})$ or in a more compact form $S = (S_1; S_2; \ldots; S_I)$ where each $S_i$ is a row vector of dimension $2 \times 1$ such that $S_i = (S_{iw}, S_{im})$. Given $(S, L_{\mathcal{I}})$ with with $1L2L\ldots LI$, $S_j^k$ is used to denote a row vector of dimension $2 \times 1$ where the first entry is the sum of all women in $(S, L_{\mathcal{I}})$ from position $j$ to $k$ and the second entry is the sum of all men in the same positions, i.e.; $S_j^k = \sum_{i=j}^k S_i = (\sum_{i=j}^k S_{iw}, \sum_{i=j}^k S_{im})$. We use $\sum_i S_{iw}$ to denote $\sum_{i\in\mathcal{I}} S_{iw}$ and $\sum_k^l S_{iw}$ to denote $\sum_{i=k}^l S_{iw}$.

**Proof of Lemma 1.1:** Immediate from the expression of $D$ as:

$D(S, L_{\mathcal{I}}) = |\sum_i (s_{iw} \sum_{j:iLj} s_{jm}) - \sum_i (s_{im} \sum_{j:iLj} s_{jw})|$. □

**Proof of Lemma 1.2**: Consider any $(S, L_{\mathcal{I}})$ such that for any $i$, $S_{iw} = S_{im}$. Notice that since $S_W$ dominates $S_M$, MON guarantees that $S$ is of $W$-type. Similarly since $S_M$ dominates $S_W$ as well, $S$ is of $M$-type. It is immediate to show that by definition, for any $H$, $S$ is of both $W$-type and $M$-type if and only if $H(S, L_{\mathcal{I}}) = 0$. □

**Proof of Proposition 1.3:** Consider any $H$ that satisfies SAD and any society $(S, L_{\mathcal{I}}) = (S_W, S_M)$. (i) Let $\alpha \in \mathbb{N}_{++}$. By using induction, we will show that

$H(\alpha S_W, S_M) = H(S_W, S_M)$. For $\alpha = 2$, SAD implies: $\frac{1}{2}H(S_W, S_M) + \frac{1}{2}H(S_W, S_M) = H(2S_W, S_M) = H(S_W, S_M)$. Now assume that the statement holds for $\alpha - 1$, i.e.: $H((\alpha-1)S_W, S_M) = H(S_W, S_M)$. Since, $H((\alpha-1)S_W, S_M)$ is of the same type with $H(S_W, S_M)$, by SAD: $\frac{\alpha-1}{\alpha}H((\alpha-1)S_W, S_M) + \frac{1}{\alpha}H(S_W, S_M) = H(\alpha S_W, S_M)$, which implies by the inductive argument: $\frac{\alpha-1}{\alpha}H(S_W, S_M) + \frac{1}{\alpha}H(S_W, S_M) = H(S_W, S_M) = H(\alpha S_W, S_M)$ as claimed. (ii) Now consider $\alpha \in \mathbb{Q}_{++}$. Let $\alpha = \frac{p}{q}$ for some $p, q \in \mathbb{N}_{++}$. Then, repeated application of SAD ensures the following: $q\frac{\frac{p}{q}}{p}H(\frac{p}{q}S_W, S_M) = H(pS_W, S_M)$. Since for $p \in \mathbb{N}_{++}$ we have proved that $H(pS_W, S_M) = H(S_W, S_M)$, we arrive; $H(\frac{p}{q}S_W, S_M) = H(S_W, S_M)$ as claimed. (iii) Finally let $\alpha \in \mathbb{R}_{++}$. Since every irrational number can be expressed as the limit value of a sequence of rational numbers, let $\alpha = \lim q_i$ for some $q_i \in \mathbb{Q}_{++}\ \forall i$. Then, $H(\alpha S_W, S_M) = H(\lim q_i S_W, S_M) = \lim H(q_i S_W, S_M)$ by continuity of the function $H$. Since we have already showed that for any rational $\alpha$ the statement holds, we arrive: $H(\alpha S_W, S_M) = H(S_W, S_M)$, establishing that for any $\alpha \in \mathbb{R}_{++}$, $H(\alpha S_W, S_M) = H(S_W, S_M)$. Since the same argumentation could be made for the men distribution, we have proved that SAD implies INV. □

**Proof of Theorem 1:** We omit the proof of necessary part. To prove the sufficiency part, first we introduce a lemma with three parts. We show that INV together with MON imply; (i) any society that has members from only one of the groups has zero inequality; (ii) any society that has members from both of the groups and only one position occupied by a strictly positive number of individuals has zero inequality; (iii) any society that has members from both of the groups and has women only in the best position or men only in the worst position is of $W$-type and any society that has men only in the best position and women only in the worst position is of $M$-type.

**Lemma 1.7.** *Let $H : C \to \mathbb{R}_+$ satisfy INV and MON. For any $(S, L_\mathcal{I})$, (i) if $S_w = 0$ or $S_m = 0$, then $H(S, L_\mathcal{I}) = 0$; (ii) if there exists $i \in \mathcal{I}$ with $S_{iw} > 0$ and $S_{im} > 0$ and for all $j \neq i$, $S_{jw} = S_{jm} = 0$, then $H(S, L_\mathcal{I}) = 0$; (iii) if $S_w \neq 0 \neq S_m$ and for $i \in \mathcal{I}$ such that there does not exist any $j \in \mathcal{I}$ with $j\ L\ i$, $S_{iw} = S_w$ or for $k \in \mathcal{I}$ such that there does not exist any $j \in \mathcal{I}$ with $k\ L\ j$, $S_{km} = S_m$, then $S$ is of $W$-type. Moreover if $S_{im} = S_m$ or $S_{kw} = S_w$, then $S$ is of $M$-type.*

**Proof** (i) Let $(S, L_\mathcal{I})$ be such that $S_w = 0$. Let $(S', L_\mathcal{I})$ be such that $S'_M = S_M$ and for some $\epsilon > 0$, $S'_W = \epsilon S_M$. By INV and Lemma 1.2, $H(S', L_\mathcal{I}) = 0$. By continuity of $H$, $\lim_{\epsilon \to 0} H(S', L_\mathcal{I}) = H(S, L_\mathcal{I}) = 0$. The same argument holds for any $(S, L_\mathcal{I})$ with $S_m = 0$. (ii) Let $(S, L_\mathcal{I})$ be as stated. The result is immediate from INV and Lemma 1.2. (iii) Let $(S, L_\mathcal{I})$ be such that for $i \in \mathcal{I}$ such that there does not exist $j \in \mathcal{I}$ with $j\ L\ i$, $S_{iw} = S_w > 0$ and $S_m > 0$. By INV $H(S_W, S_M) = H(\frac{S_W}{S_w}, \frac{S_M}{S_m}) = H(S'_W, S'_M)$. Since $S'_{iw} = 1$ and for any $j \in \mathcal{I} \setminus \{i\}$, $S'_{jw}$=0, $S'_W$ dominates $S'_M$. Thus, by MON, $S'$ and $S$ are of $W$-type. Similar arguments establish the result for the other society matrices defined in the statement of Lemma. □

Now we start with the proof of Theorem 1. Let $(S, L_\mathcal{I}) \in C$. If $I = 1$, then by Lemma 1.7(ii), $H(S, L_\mathcal{I}) = 0$. Let $I \geq 2$. For notational simplicity, let us name the positions such that $L$ over $\mathcal{I}$ is as $1\ L\ 2\ L\ \ldots\ L\ I$. Later we show that this holds wihout loss of generality.

In Step 1, we consider a very specific type of society and derive the functional form of $H$ for it. We define a **$W$-perfect society matrix** as one with strictly positive number of women and men in each position and a strictly decreasing $r_i$ ordering from the best to the worst position, i.e.; $\infty > r_1 > r_2 > \cdots > r_I > 0$.

**Step 1:** Let $S$ be a $W$-perfect society matrix. Then

$$H(S_1; \dots; S_I) = \sum_i \frac{\sum_i^I S_{jw} \sum_i^I S_{jm}}{S_w S_m} H(S_i; S_{i+1}^I).^{17}$$

**Step 1.1:** $S$ is of $W$-type.
By Proposition 1, $H$ satisfies INV. Then, for $(S', L_{\mathcal{I}}) = (s_{1w}, s_{1m}; \dots; s_{Iw}, s_{Im})$, we have $H(S, L_{\mathcal{I}}) = H(S', L_{\mathcal{I}})$. Notice that $S'$ is $W$-perfect and $S'_w = S'_m = 1$. We now show that $S'_W$ dominates $S'_M$. Then, by MON, $S'$ is of $W$-type, implying that $S$ is of $W$-type. By $W$-perfection, for any $k = 1, \dots, I$, and for $j = k+1, \dots, I$

$$S'_{kw} S'_{jm} > S'_{jw} S'_{km}.$$

Thus, for each $j$, summing up these equations

$$S'_{kw} \sum_{k+1}^{I} S'_{jm} > S'_{km} \sum_{k+1}^{I} S'_{jw}. \tag{1.1}$$

Since (1.1) holds for each $k = 1, \dots, I-1$, summing over all $k$

$$\sum_1^k (S'_{iw} \sum_{k+1}^I S'_{jm}) > \sum_1^k (S'_{im} \sum_{k+1}^I S'_{jw})$$

$$\sum_1^k S'_{iw} (1 - \sum_1^k S'_{jm}) > \sum_1^k S'_{im} (1 - \sum_1^k S'_{jw})$$

$$\sum_1^k S'_{iw} - \sum_1^k S'_{iw} \sum_1^k S'_{jm} > \sum_1^k S'_{im} - \sum_1^k S'_{im} \sum_1^k S'_{jw}$$

$$\sum_1^k S'_{iw} > \sum_1^k S'_{im}.$$

Since $\sum_1^I S'_{iw} = \sum_1^I S'_{im} = 1$, it follows from MON that $S'$ is of $W$-type. Then, $S$ is of $W$-type as well. Hence any $W$-perfect $S$ is of $W$-type.

As a direct implication of Step 1.1, for a $W$-perfect $S$, since for each $i = 1, \dots, I-1$, $(S_i; \dots; S_I)$ is $W$-perfect, it is of $W$-type as well.

[17] Notice that for $i = I$, $H(S_I; S_{I+1}^I)$ does not exist. For the sake of simplicity we keep the notation this way.

**Step 1.2:** For any $i = 1, \ldots, I-1$, $(S_i; S_{i+1}^I)$ is of $W$-type.

Since for any $i \in \mathcal{I}$, $\infty > r_i > r_{i+1} > \cdots > r_I > 0$, then

$$S_{iw} \sum_{i+1}^{I} S_{jm} > S_{im} \sum_{i+1}^{I} S_{jw}$$
$$\frac{S_{iw}}{S_{im}} > \frac{\sum_{i+1}^{I} S_{jw}}{\sum_{i+1}^{I} S_{jm}}.$$

Then, $(S_i; S_{i+1}^I)$ is a $W$-perfect society matrix with 2 positions. By Step 1.1, for each $i = 1, \ldots, I-1$, $(S_i; S_{i+1}^I)$ is of $W$-type.

**Step 1.3:** $H(S_1; \ldots; S_I) = \sum_i \frac{\sum_i^I S_{jw} \sum_i^I S_{jm}}{S_w S_m} H(S_i; S_{i+1}^I)$.

First consider the division of society $(S, L_{\mathcal{I}}) = (S_1; \ldots; S_I)$ as $S_1$ and $(S_2; \ldots; S_I)$. Since $S$ is $W$-perfect, by Step 1.2, these societies and the interaction society are of $W$-type. Thus, by DEC

$$H(S_1; \ldots; S_I) = \frac{S_{1w} S_{1m}}{S_w S_m} H(S_1) + \frac{\sum_2^I S_{iw} \sum_2^I S_{im}}{S_w S_m} H(S_2; \ldots; S_I) + H(S_1; S_2^I). \quad (1.2)$$

By Lemma 1.7(ii), $H(S_1) = 0$. Now, consider the division of the society $(S_2; \ldots; S_I)$ as $S_2$ and $(S_3; \ldots; S_I)$. Again by $W$-perfection, DEC yields

$$H(S_2; \ldots; S_I) = \frac{S_{2w} S_{2m}}{\sum_2^I S_{iw} \sum_2^I S_{im}} H(S_2) + \frac{\sum_3^I S_{iw} \sum_3^I S_{im}}{\sum_2^I S_{iw} \sum_2^I S_{im}} H(S_3; \ldots; S_I) + H(S_2; S_3^I).$$

Since by Lemma 1.7(ii), $H(S_2) = 0$, substitution into (1.2) yields:

$$H(S_1; \ldots; S_I) = \frac{\sum_3^I S_{iw} \sum_3^I S_{im}}{S_w S_m} H(S_3; \ldots; S_I) + \frac{\sum_2^I S_{iw} \sum_2^I S_{im}}{S_w S_m} H(S_2; S_3^I)) + H(S_1; S_2^I).$$

Iterative application of DEC and Lemma 1.7(ii) results in

$$H(S_1; \ldots; S_I) = \sum_i \frac{\sum_i^I S_{jw} \sum_i^I S_{jm}}{S_w S_m} H(S_i; S_{i+1}^I), \quad (1.3)$$

concluding Step 1.[18]

**Step 2:** For each society $(S, L_{\mathcal{I}}) \in C$ with a $W$-perfect society matrix $S$

$$H(S_1; \ldots; S_I) = \sum_i (s_{iw} \sum_{i+1}^{I} s_{jm} - s_{im} \sum_{i+1}^{I} s_{jw}) H(1, 0; 0, 1).$$

[18]Notice that DEC allows to express overall inequality as a weighted sum of inequalities in $2 \times 2$ society matrices naming the best position as 1 and the other position as 2. This shows that neutrality of positions is implied by DEC, naming the positions as $1 \; L \; 2 \; L \ldots L \; I$ is without loss of generality.

**Step 2.1:** For any $i = 1, \dots, I-1$

$$H(S_i; S_{i+1}^I) = \frac{S_{iw}\sum_{i+1}^I S_{jm} - S_{im}\sum_{i+1}^I S_{jw}}{\sum_i^I S_{jw}\sum_i^I S_{jm}} H(1,0;0,1).$$

First notice that by INV, $H(S_{iw}, S_{im}; \sum_{i+1}^I S_{jw}, \sum_{i+1}^I S_{jm}) = H(s_{iw}, s_{im}; \sum_{i+1}^I s_{jw}, \sum_{i+1}^I s_{jm})$. For simplicity let us use the notation $H(a,b;c,d)$ instead of $H(s_{iw}, s_{im}; \sum_{i+1}^I s_{jw}, \sum_{i+1}^I s_{jm})$. By Step 1.2, $(a,b;c,d,)$ is of $W$-type. Since $a+c = b+d = 1$, then $a > b$ and $d > c$. Since $\frac{b}{b} = 1 > \frac{c}{d}$, by Step 1.1, $(b,b;c,d)$ is of $W$-type and by Lemma 1.7(iii), $(a-b,b;0,d)$ is of $W$-type. Then, by SAD

$$H(a,b;c,d) = \frac{b+c}{a+c}H(b,b;c,d) + \frac{a-b}{a+c}H(a-b,b;0,d). \tag{1.4}$$

Moreover, since by Lemma 1.2, $H(b,b;c,c) = 0$, by definition $(b,b;c,c)$ is of $W$-type and by Lemma 1.7(iii), $(b,0;c,d-c)$ is of $W$-type. Then, by SAD

$$H(b,b;c,d) = \frac{b+c}{b+d}H(b,b;c,c) + \frac{d-c}{b+d}H(b,0;c,d-c). \tag{1.5}$$

Combining (1.4) and (1.5)

$$H(a,b,c,d) = \frac{(b+c)(d-c)}{(a+c)(b+d)}H(b,0;c,d-c) + \frac{a-b}{a+c}H(a-b,b;0,d).$$

Similarly, by Lemma 1.7(ii), $H(0,0;c,d-c) = H(a-b,b;0,0) = 0$. Then, by definition $(0,0;c,d-c)$ and $(a-b,b;0,0)$ are of $W$-type. By Lemma 1.7(iii), $(b,0;0,d-c)$ and $(a-b,0;0,d)$ are of $W$-type. Then, by SAD

$$\begin{aligned} H(b,0;c,d-c) &= \frac{c}{b+c}H(0,0;c,d-c) + \frac{b}{b+c}H(b,0;0,d-c), \\ H(a-b,b;0,d) &= \frac{d}{b+d}H(a-b,0;0,d) + \frac{b}{b+d}H(a-b,b;0,0), \end{aligned}$$

resulting in

$$H(a,b,c,d) = \frac{b(d-c)}{(a+c)(b+d)}H(b,0;0,d-c) + \frac{(a-b)(d)}{(a+c)(b+d)}H(a-b,0;0,d).$$

Finally, by INV

$$\begin{aligned} H(a,b;c,d) &= \frac{b(d-c)}{(a+c)(b+d)}H(1,0;0,1) + \frac{(a-b)d}{(a+c)(b+d)}H(1,0;0,1) \\ &= \frac{ad-bc}{(a+c)(b+d)}H(1,0;0,1). \end{aligned}$$

Going back to the original notation

$$H(S_i; S_{i+1}^I) = \frac{S_{iw}\sum_{j=i+1}^I S_{jm} - S_{im}\sum_{j=i+1}^I S_{jw}}{\sum_{j=i}^I S_{jw}\sum_{j=i}^I S_{jm}} H(1,0;0,1). \tag{1.6}$$

**Step 2.2:** $H(S_1; \ldots; S_I) = \sum_i (s_{iw}\sum_{i+1}^I s_{jm} - s_{im}\sum_{i+1}^I s_{jw}) H(1,0;0,1)$.
Combining equation (1.3) and (1.6),

$$\begin{aligned}
H(S_1; \ldots; S_I) &= \sum_i \frac{\sum_i^I S_{jw}\sum_i^I S_{jm}}{S_w S_m} \frac{S_{iw}\sum_{i+1}^I S_{jm} - S_{im}\sum_{i+1}^I S_{jw}}{\sum_i^I S_{jw}\sum_i^I S_{jm}} H(1,0;0,1) \\
&= \sum_i \frac{S_{iw}\sum_{i+1}^I S_{jm} - S_{im}\sum_{i+1}^I S_{jw}}{S_w S_m} H(1,0;0,1) \\
&= \sum_i (s_{iw}\sum_{i+1}^I s_{jm} - s_{im}\sum_{i+1}^I s_{jw}) H(1,0;0,1), \qquad (1.7)
\end{aligned}$$

concluding Step 2.

Now let us define an $M$**-perfect society matrix** as one with strictly positive number of women and men in each position and a strictly increasing $r_i$ ordering, i.e.; $0 < r_1 < r_2 < \ldots < r_I < \infty$.

**Step 3:** For each society $(S, L_{\mathcal{I}}) \in C$ with an $M$-perfect society matrix

$$H(S, L_{\mathcal{I}}) = -\sum_i (s_{iw}\sum_{i+1}^I s_{jm} - s_{im}\sum_{i+1}^I s_{jw}) H(1,0;0,1).$$

Let $(S, L_{\mathcal{I}}) \in C$ be such that $S$ is $M$-perfect. Let $(S', L_{\mathcal{I}}) \in C$ be such that $S'_W = S_M$ and $S'_M = S_W$. Thus $S'$ is $W$-perfect. By SYM, $H(S', L_{\mathcal{I}}) = H(S, L_{\mathcal{I}})$. Then, by Step 2

$$\begin{aligned}
H(S, L_{\mathcal{I}}) &= H(S', L_{\mathcal{I}}) \\
&= \sum_i (s'_{iw}\sum_{i+1}^I s'_{jm} - s'_{im}\sum_{i+1}^I s'_{jw}) H(1,0;0,1) \\
&= \sum_i (s_{im}\sum_{i+1}^I s_{jw} - s_{iw}\sum_{i+1}^I s_{jm}) H(1,0;0,1) \\
&= -\sum_i (s_{iw}\sum_{i+1}^I s_{jm} - s_{im}\sum_{i+1}^I s_{jw}) H(1,0;0,1).
\end{aligned}$$

**Step 4:** For each society $(S, L_{\mathcal{I}}) \in C$ with a $W$-type society matrix $S$

$$H(S, L_{\mathcal{I}}) = \sum_{i}(s_{iw}\sum_{i+1}^{I} s_{jm} - s_{im}\sum_{i+1}^{I} s_{jw})H(1,0;0,1).$$

**Step 4.1:** Let $S$ be of $W$-type such that for all $i \in \mathcal{I}$, $S_{im} \neq 0$. There exist a $W$-perfect society matrix $X$ with $X_M = S_M$ such that $S'$ with $S'_W = S_W + X_W$ and $S'_M = S_M$ is $W$-perfect as well.

Let us denote the total number of women and men in $S$ with $S_t$, i.e.; $S_t = S_w + S_m$. Let $(X, L_{\mathcal{I}}) \in C$ be such that $X_M = S_M$ and for any $i \in \mathcal{I}$, $X_{iw} = \sum_{k=2}^{I-i+1}(S_t)^k$. Then for any $i = 2, \ldots, I$

$$\frac{X_{(i-1)w}}{X_{(i-1)m}} > \frac{X_{iw}}{X_{im}}$$
$$\frac{\sum_{k=2}^{I-i+2}(S_t)^k}{S_{(i-1)m}} > \frac{\sum_{k=2}^{I-i+1}(S_t)^k}{S_{im}}$$
$$\frac{S_t\sum_{k=2}^{I-i+1}(S_t)^k}{S_{(i-1)m}} > \frac{\sum_{k=2}^{I-i+1}(S_t)^k}{S_{im}}$$
$$\frac{S_t}{S_{(i-1)m}} > \frac{1}{S_{im}}.$$

Thus, $\infty > r_{i-1} > r_i > 0$, establishing that $X$ is $W$-perfect.

Now let $(S', L_{\mathcal{I}}) \in C$ be such that $S'_W = S_W + X_W$ and $S'_M = S_M$. For any $i = 2, \ldots, I$, $S'_{im} \neq 0$ and

$$\frac{S'_{(i-1)w}}{S'_{(i-1)m}} > \frac{S'_{iw}}{S'_{im}}$$
$$\frac{S_{(i-1)w} + X_{(i-1)w}}{S_{(i-1)m}} > \frac{S_{iw} + X_{iw}}{S_{im}}$$
$$\frac{S_{(i-1)w} + \sum_{k=2}^{I-i+2}(S_t)^k}{S_{(i-1)m}} > \frac{S_{iw} + \sum_{k=2}^{I-i+1}(S_t)^k}{S_{im}}$$
$$S_{(i-1)w}S_{im} + S_{(i-1)w}S_{im} + \sum_{k=2}^{I-i+2}(S_t)^k S_{im} > S_{iw}S_{(i-1)m} + \sum_{k=2}^{I-i+1}(S_t)^k S_{(i-1)m}.$$

Since $\sum_{k=3}^{I-i+2}(S_t)^k S_{im} > \sum_{k=2}^{I-i+1}(S_t)^k S_{(i-1)m}$ and $(S_t)^2 S_{im} > S_{iw}S_{(i-1)m}$,

$$S_{(i-1)w}S_{im} + S_{(i-1)w}S_{im} + \sum_{k=3}^{I-i+2}(S_t)^k S_{im} + (S_t)^2 S_{im} > S_{iw}S_{(i-1)m} + \sum_{k=2}^{I-i+1}(S_t)^k S_{(i-1)m}.$$

Thus, $\infty > r'_{i-1} > r'_i > 0$, establishing that $S'$ is $W$-perfect.

**Step 4.2:** For any $W$-type $S$ such that for all $i \in \mathcal{I}$, $S_{im} \neq 0$, $H(S, L_{\mathcal{I}}) = \sum_i (s_{iw} \sum_{i+1}^{I} s_{jm} - s_{im} \sum_{i+1}^{I} s_{jw}) H(1,0;0,1)$.
Since both $S$ and $X$ are of $W$-type, by SAD, $H(S', L_{\mathcal{I}}) = \frac{S_w}{S_w+X_w} H(S, L_{\mathcal{I}}) + \frac{X_w}{S_w+X_w} H(X, L_{\mathcal{I}})$.
Since $S'$ and $X$ are W-perfect matrices, using the functional form of $H$ for societies with $W$-perfect society matrices derived in Step 2.2,

$$\begin{aligned} H(S, L_{\mathcal{I}}) &= \frac{S_w + X_w}{S_w} H(S', L_{\mathcal{I}}) - \frac{X_w}{S_w} H(X, L_{\mathcal{I}}) \\ &= \sum_i (s_{iw} \sum_{i+1}^{I} s_{jm} - s_{im} \sum_{i+1}^{I} s_{jw}) H(1,0;0,1). \end{aligned} \tag{1.8}$$

**Step 4.3:** Let $S$ be of $W$-type. $H(S, L_{\mathcal{I}}) = \sum_i (s_{iw} \sum_{i+1}^{I} s_{jm} - s_{im} \sum_{i+1}^{I} s_{jw}) H(1,0;0,1)$. Let $(S', L_{\mathcal{I}}) \in C$ be such that for all $i \in \mathcal{I}$, $S'_{iw} = S_{iw}$, for all $i$ with $S_{im} \neq 0$, $S'_{im} = S_{im}$ and for all $i$ with $S_{im} = 0$, $S'_{im} = \varepsilon$ for some $\varepsilon$ in a small neighborhood of 0. By Step 3 and Step 4.1:

$$H(S', L_{\mathcal{I}}) = \sum_i (s'_{iw} \sum_{i+1}^{I} s'_{jm} - s'_{im} \sum_{i+1}^{I} s'_{jw}) H(1,0;0,1).$$

By continuity of $H$:

$$\begin{aligned} H(S, L_{\mathcal{I}}) &= \lim_{\varepsilon \to 0} H(S', L_{\mathcal{I}}) \\ &= \lim_{\varepsilon \to 0} \sum_i (s'_{iw} \sum_{i+1}^{I} s'_{jm} - s'_{im} \sum_{i+1}^{I} s'_{jw}) H(1,0;0,1) \\ &= \sum_i (s_{iw} \sum_{i+1}^{I} s_{jm} - s_{im} \sum_{i+1}^{I} s_{jw}) H(1,0;0,1) \end{aligned}$$

establishing the functional form for any $W$-type $S$.

**Step 5:** For each society $(S, L_{\mathcal{I}}) \in C$ with an $M$-type society matrix $S$

$$H(S, L_{\mathcal{I}}) = -\sum_i (s_{iw} \sum_{i+1}^{I} s_{jm} - s_{im} \sum_{i+1}^{I} s_{jw}) H(1,0;0,1).$$

Symmetrically, now let $S$ be not $M$-perfect but of $M$-type. Following the same technique in Step 4, one can establish the result.

Hence, for each society $(S, L_{\mathcal{I}}) \in C$ with a $W$-type society matrix $S$,

$$H(S, L_{\mathcal{I}}) = \sum_i (s_{iw} \sum_{i+1}^{I} s_{jm} - s_{im} \sum_{i+1}^{I} s_{jw}) H(1, 0; 0, 1)$$

and with an $M$-type $S$

$$H(S, L_{\mathcal{I}}) = -\sum_i (s_{iw} \sum_{i+1}^{I} s_{jm} - s_{im} \sum_{i+1}^{I} s_{jw}) H(1, 0; 0, 1).$$

Since by construction each $S$ is of $W$-type or of $M$-type, we have derived the functional form for all possible societies. At the beginning, we have assumed that $1 \; L \; 2 \; L \ldots L \; I$. Then in general, for any $(S, L_{\mathcal{I}})$ in $C$

$$H(S, L_{\mathcal{I}}) = |\sum_i (s_{iw} \sum_{j:iLj} s_{jm} - s_{im} \sum_{j:iLj} s_{jw}) H(1, 0; 0, 1)|. \tag{1.9}$$

By definition, $H$ is a nonzero function. Thus, $H(1, 0; 0, 1)$ is a strictly positive real number. □

**Proof of Proposition 1.5:** It is straightforward to show that $G_S(S_{\mathcal{I}}) = \frac{1}{2} \sum_i \sum_j |s_{iw}s_{jm} - s_{im}s_{jw}|$.

First we will show that if (i) or (ii) holds, then $G_S(S_{\mathcal{I}}) = D(S, L_{\mathcal{I}})$. Notice that $(s_{iw}s_{jm} - s_{im}s_{jw}) > 0$ if and only if $r_i > r_j$ and $(s_{iw}s_{jm} - s_{im}s_{jw}) = 0$ if and only if $r_i = r_j$. Hence $G_S$ can equivalently be expressed as:

$$\begin{aligned} G_S(S_{\mathcal{I}}) &= \frac{1}{2} \sum_i \sum_j |s_{iw}s_{jm} - s_{im}s_{jw}| \\ &= \frac{1}{2} (\sum_i \sum_{j:r_i \geq r_j} (s_{iw}s_{jm} - s_{im}s_{jw}) + \sum_i \sum_{j:r_i < r_j} -(s_{iw}s_{jm} - s_{im}s_{jw})). \end{aligned}$$

Notice that $\sum_i \sum_{j:r_i \geq r_j} (s_{iw}s_{jm} - s_{im}s_{jw}) = \sum_i \sum_{j:r_i < r_j} -(s_{iw}s_{jm} - s_{im}s_{jw})$. Since for $i, j$ with $r_i = r_j$, $(s_{iw}s_{jm} - s_{im}s_{jw}) = 0$, then:

$$G_S(S_{\mathcal{I}}) = \sum_i \sum_{j:r_i \geq r_j} (s_{iw}s_{jm} - s_{im}s_{jw}) = \sum_i \sum_{j:r_i \leq r_j} -(s_{iw}s_{jm} - s_{im}s_{jw}).$$

Now let us assume (i) holds. Since for any $i$, $(s_{iw}\sum_{j:iLj}s_{jm}-s_{im}\sum_{j:iLj}s_{jw})\geq 0$, we have:

$$\begin{aligned}D(S,L_{\mathcal{I}})&=\sum_i(s_{iw}\sum_{j:iLj}s_{jm}-s_{im}\sum_{j:iLj}s_{jw})\\&=\sum_i\sum_{j:iLj}(s_{iw}s_{jm}-s_{im}s_{jw})\\&=\sum_i\sum_{j:r_i\geq r_j}(s_{iw}s_{jm}-s_{im}s_{jw})\end{aligned}$$

establishing the claim. Now assume that (ii) holds. Then, we have:

$$\begin{aligned}D(S,L_{\mathcal{I}})&=\sum_i-(s_{iw}\sum_{j:iLj}s_{jm}-s_{im}\sum_{j:iLj}s_{jw})\\&=\sum_i\sum_{j:iLj}-(s_{iw}s_{jm}-s_{im}s_{jw})\\&=\sum_i\sum_{j:r_j\geq r_i}-(s_{iw}s_{jm}-s_{im}s_{jw})\end{aligned}$$

as claimed.
Now we will show that if $G_S(S_{\mathcal{I}})=D(S,L_{\mathcal{I}})$ then (i) or (ii) holds. First notice that $D(S,L_{\mathcal{I}})=|\sum_i\sum_{j:iLj}d_{ij}|a_{ij}||=|\sum_{ij:iLj}d_{ij}|a_{ij}||$ where $a_{ij}=(s_{iw}s_{jm}-s_{im}s_{jw})$ and $d_{ij}=1$ if $r_i\geq r_j$ and $d_{ij}=-1$ if $r_i<r_j$. Hence for any two positions $i$ and $j$, since either $iLj$ or $jLi$, if $a_{ij}$ enters the sum, $a_{ji}$ does not and for sure either $a_{ij}$ or $a_{ji}$ enters. If (i) holds, then $d_{ij}=1$ for all $ij$ with $iLj$. Then, $D(S,L_{\mathcal{I}})=\sum_{ij:iLj}|a_{ij}|=\sum_{ij:iLj}a_{ij}=G_S(S_{\mathcal{I}})$ as shown in the sufficiency part. If (ii) holds, then $d_{ij}=-1$ for all $ij$ with $iLj$. Then, $D(S,L_{\mathcal{I}})=|\sum_{ij:iLj}-|a_{ij}||=\sum_{ij:iLj}|a_{ij}|=\sum_{ij:iLj}-a_{ij}=G_S(S_{\mathcal{I}})$ as shown in the sufficiency part. If neither (i) nor (ii) holds, then for some $ij$ with $iLj$ and $a_{ij}\neq 0$ we have $d_{ij}=1$ and for some other $ij$ with $iLj$ we have $d_{ij}=-1$. Then $D(S,L_{\mathcal{I}})<\sum_{ij:iLj}|a_{ij}|=G_S(S_{\mathcal{I}})$, concluding the proof. □

**Characterizing Properties of Gini Segregation Index**

**(SYM):** For any $(S_{\mathcal{I}})$ and $(S'_{\mathcal{I}})$ with $S_W=S'_M$ and $S_M=S'_W$, $H(S_{\mathcal{I}})=H(S'_{\mathcal{I}})$.
**(INV):** Given $(S_{\mathcal{I}})$ and any $\alpha,\beta\in\mathbb{R}_+$, for $(S'_{\mathcal{I}})$ such that for all $i$, $S'_{iw}=\alpha S_{iw}$ and $S'_{im}=\beta S_{im}$, $H(S_{\mathcal{I}})=H(S'_{\mathcal{I}})$.
**(NORM):** For any $(S_{\mathcal{I}})$ such that for any $i$, $S_{iw}=S_{im}$, we have $H(S_{\mathcal{I}})=0$.
**(r-DEC):** For any r-ordered division of $S_{\mathcal{I}}$ as $S^1_{\mathcal{I}^1}$ and $S^2_{\mathcal{I}^2}$, the following holds: $H(S_{\mathcal{I}})=\lambda_{S^1}H(S^1_{\mathcal{I}^1})+\lambda_{S^2}H(S^2_{\mathcal{I}^2})+H(S'_{\mathcal{I}'})$ , where **an r-ordered division** of $S_{\mathcal{I}}$ is a pair of societies $S^1_{\mathcal{I}^1}$ and $S^2_{\mathcal{I}^2}$ such that: (i) $\mathcal{I}^1$ and $\mathcal{I}^2$ define a partition of $\mathcal{I}$ such that for any $i$ in $\mathcal{I}^1$ and any $j$ in $\mathcal{I}^2$ we have $r_i\geq r_j$. (ii) for $k=1,2$, $S^k$ is a $I^k\times 2$ society matrix such that each position $i$ possess the same number of women and men in $S$ and $S^k$; $S'_{\mathcal{I}'}$

denotes **the interaction society**, which is a society of two positions, $\mathcal{I}' = 1, 2$, with $S'_{1w} = S^1_w$, $S'_{1m} = S^1_m$, $S'_{2w} = S^2_w$ and $S'_{2m} = S^2_m$; $\lambda_{S^k}$ refers to the population weight of society part $S^k$.
**(r-SAD):** For any partition of a society $(S_\mathcal{I})$ into subsocieties $(S'_\mathcal{I})$ and $(S''_\mathcal{I})$ such that for any $i$ and $j$, $r_i \geq r_j$ if and only if $r'_i \geq r'_j$ if and only if $r''_i \geq r''_j$ holds, the following holds: $H(S_\mathcal{I}) = \lambda_{S'} H(S'_\mathcal{I}) + \lambda_{S''} H(S''_\mathcal{I})$.

**Proof of Theorem 1.6:** We omit the necessary part. For sufficiency part we first introduce a couple of lemmas:

**Lemma 1.8.** *Any $H : B \to R_+$ that satisfies r-SAD is Scale Invariant.*

**Proof of Lemma 1.8:** The proof is exactly the same with the proof of Proposition 1.3 with the exception that r-SAD could be applied at any induction step since the $r_i$ ordering of $(\alpha S_W, S_M)$ is the same with the one of $(S_W, S_M)$ for any $\alpha > 0$. □

**Lemma 1.9.** *Given $H$ that satisfies INV and NORM, for any $(S_\mathcal{I})$ (i) if $S_w = 0$ or $S_m = 0$, then $H(S_\mathcal{I}) = 0$; (ii) if $\exists i \in \mathcal{I}$ with $S_{iw} > 0$ and $S_{im} > 0$ and for all $j \neq i$, $S_{jw} = S_{jm} = 0$, then $H(S, L_\mathcal{I}) = 0$.*

**Proof of Lemma 1.9** See part (i) and (ii) of the Proof of Lemma 1.7.□

Now consider any $S_\mathcal{I}$ in $B$. If $I = 1$, then by Lemma 1.9(ii), $H(S_\mathcal{I}) = 0$. Let $I \geq 2$. Let us name the positions such that $r_1 \geq r_2 \geq \ldots \geq r_I$. The proof will closely follow the proof of Theorem 1.

**Step 1:** Consider the division of society $(S_\mathcal{I}) = (S_1, \ldots, S_I)$ as $(S_1)$ and $(S_2, \ldots, S_I)$. Since $r_1 \geq r_j$ for all $j$ in $\{2, 3, \ldots, I\}$, r-DEC is applied:

$$H(S_1, \ldots, S_I) = \frac{S_{1w} S_{1m}}{S_w S_m} H(S_1) + \frac{\sum_2^I S_{iw} \sum_2^I S_{im}}{S_w S_m} H(S_2, \ldots, S_I) + H(S_1, S_2^I).$$

The rest of the iterative decomposition is the same as Step 1.3 of Theorem 1 with the exception that here r-DEC is applied thanks to the decreasing order of $r_i$ ratios. Repeated application of r-DEC, and Lemma 1.9(ii) results in:

$$H(S_1, \ldots, S_I) = \sum_i \frac{\sum_i^I S_{jw} \sum_i^I S_{jm}}{S_w S_m} H(S_i, S_{i+1}^I) \tag{1.10}$$

concluding Step 1.

**Step 2:** In this step, similar to Step 2.1. of Theorem 1, first we focus on one component of the sum over positions derived in Step 1, $H(S_i; S_{i+1}^I)$ and show that for any $i$ the following holds:

$$H(S_i, S_{i+1}^I) = \frac{S_{iw}\sum_{i+1}^I S_{jm} - S_{im}\sum_{i+1}^I S_{jw}}{\sum_i^I S_{jw}\sum_i^I S_{jm}} H(1,0;0,1).$$

First notice that since INV is ensured by Lemma 1.8, we have:

$H(S_{iw}, S_{im}, \sum_{i+1}^I S_{jw}; \sum_{i+1}^I S_{jm}) = H(s_{iw}, s_{im}; \sum_{i+1}^I s_{jw}, \sum_{i+1}^I s_{jm})$. For simplicity let us use the notation $H(a,b;c,d)$ instead of $H(s_{iw}, s_{im}; \sum_{i+1}^I s_{jw}, \sum_{i+1}^I s_{jm})$. Notice that since $r_i \geq r_j$ for all $j \in \{i+1, \dots, I\}$, $r_i = \frac{a}{b} \geq \frac{\sum_{i+1} s_j w}{\sum_{i+1} s_j m} = \frac{c}{d}$. And since $a+c = b+d = 1$, then $a \geq b$ and $c \leq d$. If $a = b$, then $c = d$ yielding $H(a,b;c,d) = 0$. Now let $a > b$ and $c < d$. Let $b \neq 0$. Then for $X = \frac{ad-bc}{d}$, r-SAD results in:

$$H(a,b;c,d) = \frac{a+c-X}{a+c} H(a-X,b;c,d) + \frac{X}{a+c} H(X,b;0,d).$$

Notice that this is admissible since $\frac{a-X}{b} = \frac{c}{d}$ and $\frac{X}{b} > \frac{0}{d}$. As by INV and NORM, $H(a-X,b;c,d) = 0$, we have:

$$H(a,b;c,d) = \frac{X}{a+c} H(X,b;0,d). \tag{1.11}$$

Now notice that for all $\epsilon$ in $(0,d)$, the following holds:

$$H(X,b;0,d) = \frac{d-\varepsilon}{b+d} H(X,0;0,d-\varepsilon) + \frac{b+\varepsilon}{b+d} H(X,b;0,\varepsilon)$$

and hence:

$$\lim_{\varepsilon\to 0} H(X,b;0,d) = \lim_{\varepsilon\to 0}\left(\frac{d-\varepsilon}{b+d} H(X,0;0,d-\varepsilon)\right) + \lim_{\varepsilon\to 0}\left(\frac{b+\varepsilon}{b+d} H(X,b;0,\varepsilon)\right).$$

Then by continuity of $H$:

$$H(X,b;0,d) = \frac{d}{b+d} H(X,0;0,d) + \frac{b}{b+d} H(X,b;0,0).$$

Since, $H(X,b;0,0) = 0$ by NORM, combining with (1.11), we arrive:

$$H(a,b;c,d) = \frac{X}{a+c}\frac{d}{b+d} H(X,0;0,d).$$

By INV,

$$H(a,b;c,d) = \frac{X}{a+c}\frac{d}{b+d} H(1,0;0,1).$$

And finally for values of $X = \frac{ad-bc}{d}$ and $(a, b; c, d) = (s_{iw}, s_{im}; \sum_{i+1}^{I} s_{jw}, \sum_{i+1}^{I} s_{jm})$, we arrive:

$$H(S_i; S_{i+1}^I) = \frac{S_{iw} \sum_{i+1}^{I} S_{jm} - S_{im} \sum_{i+1}^{I} S_{jw}}{\sum_{i}^{I} S_{jw} \sum_{i}^{I} S_{jm}} H(1, 0; 0, 1). \tag{1.12}$$

Notice that for $b = 0$, we would have the same by continuity.

**Step 3:** Combining the results of Step 1 and Step 2, (1.10) and (1.12) we arrive;

$$\begin{aligned} H(S_{\mathcal{I}}) &= \sum_i \frac{\sum_i^I S_{jw} \sum_i^I S_{jm}}{S_w S_m} \frac{S_{iw} \sum_{i+1}^{I} S_{jm} - S_{im} \sum_{i+1}^{I} S_{jw}}{\sum_i^I S_{jw} \sum_i^I S_{jm}} H(1, 0; 0, 1) \\ &= \sum_i (s_{iw} \sum_{i+1}^{I} s_{jm} - s_{im} \sum_{i+1}^{I} s_{jw}) H(1, 0; 0, 1). \end{aligned}$$

We have named the positions as, $r_1 \geq r_2 \geq \ldots \geq r_I$. Then, in general we have:

$$\begin{aligned} H(S_{\mathcal{I}}) &= \sum_i (s_{iw} \sum_{j:r_i \geq r_j} s_{jm} - s_{im} \sum_{j:r_i \geq r_j} s_{jw}) H(1, 0; 0, 1) \\ &= \sum_i \sum_{j:r_i \geq r_j} (s_{iw} s_{jm} - s_{im} s_{jw}) H(1, 0; 0, 1). \end{aligned}$$

Since $G_S(S_{\mathcal{I}}) = \frac{1}{2} \sum_i \sum_j |s_{iw} s_{jm} - s_{im} s_{jw}|$ could equivalently be expressed as;

$\sum_i \sum_{j:r_i \geq r_j} (s_{iw} s_{jm} - s_{im} s_{jw})$ and $H(1, 0; 0, 1)$ is a strictly positive constant, we establish the result. □

# Chapter 2

# Choose what you like or like what you choose? Identifying Influence and Homophily out of Individual Decisions

## 2.1 Introduction

Individuals sharing the same environment, such as members of the same household, colleagues from workplace, friends from school, tend to behave similarly in many occasions. It is no surprise that roommates use the same brand of parfume, colleagues of the same firm happen to crash into each other in the same Italian restaurant or your examplary daughter is caught smoking after beginning to this new school full of adventurous youngsters. As an immediate implication of being part of a society, we do influence each other's behaviors. We take advice from each other, we inspire each other or we convince each other, resulting in behavioral resemblances. For one way or the other, our decisions in life embody and reflect the effect of people that we are in contact with. Understanding the extent of this influence is of particular interest to many economists as well as other social scientists such as psychologists, sociologists or marketing theorists. However it is not straightforward to disentangle the part of our decisions that are due to our social interactions from the part that is not affected by them, since it bears a certain paradoxical situation: We are the ones to decide who we want to be in contact with as well. Indeed most of our social relations are neither given nor random, but built by ourselves consciously. We have a say to choose from whom to take advice or inspiration or with whom to hang out. But then it is highly likely to have built these relations at the first place because we are similar people or we have similar interests, rules or styles. It is very probable to share a flat with someone that we share tastes as well. Thus performance of similar behavior by socially related people does not necessarily imply the influence of one on the other, but performing similarly may have been the reason for which this social relation is built.

This phenomenon is known as the identification problem of homophily and social influence. Homophily refers to the tendency to create social ties with people that are similar to one's self.[1] Peer-to-peer influence, social influence or contagion, on the other hand, is defined as adopting similar behavioral patterns with people that one is socially connected to.[2] Since both homophily and influence result in behavioral resemblances between people that are related to each other, one cannot easily detect the real cause of similar behavior patterns. However, it is important to distinguish these two forces from each other in order to understand the functioning of social structures in general.

In this study, we approach to this identification problem by investigating its microfoundations. We provide a novel framework that focuses on individual decision making in order to identify the social influence and homophily effects. Suggesting to have a closer look to the decision making processes of individuals that interact, we investigate how they affect each other's behaviors. We propose simple and direct measures of homophily and influence by making use of individual preferences of these interacting individuals, hence providing a solution to the identification problem. However since in many occasions, preferences are not easily observed, we extend our analysis to the observables, decision outcomes. In order to infer the underlying preferences of interacting individuals out of their decision outcomes, we follow a foundational approach. We analyze the behavioral characteristics of individual decision making that includes interaction and finally we make use of the tools that are provided by revealed preference theory in order to uncover the underlying preferences of the individuals. Based on revealed preference analysis, we revisit our measurement techniques for homophily and influence.

In the heart of our framework, there lies two important observations. First, we suggest that individuals are susceptible to influence especially in times of indecisiveness. We may refer to each other's opinion, take advice, get inspired or get influenced in some way especially if we are facing a situation that we fail to decide on our own. There may be several explanations for that: We may have informational constraints; we may lack sufficient information to evaluate several alternatives we are facing and we may refer to someone with more expertise or experience on the subject: "These investment

[1] For an overview of research on homophily in general see [70], in couples see [17], on economic networks see [29].

[2] For an overview of research about peer influence in teenage behavior see [41], in crime see [44], in education see [89], in labor market see [67], over social networks see [25] and [23].

decisions are quite difficult for me. Luckily my cousin is a broker. I always ask her when I can't decide what to invest." " Mom! Which detergent should I buy for washing the whites? There are tens of different brands here, I have no idea which one to choose!" Or instead we may have compliance motives; we may especially prefer to comply with another person's preferences although we could be as fine with another option: "I don't mind Caribbeans or Hawai. But Susan from work was in Hawai last year and she did not stop talking about it. Let's go there!" "I was not especially in favor of pizza, but nobody else in the table was having pasta, so I ordered pizza as well." Whatever the underlying motivation is, it is safe to assume that there is room for being influenced by someone else as long as we are not perfectly sure about what to do. In other words, we seek advice, inspiration or an example to imitate especially at times of being indeterminate or indecisive. This observation directly translates into our framework in the form of incomplete preferences. We suggest that individuals are susceptible to influence over those alternatives for which they do not have well-defined preferences.

Second, we observe that interaction is a two-way concept. Quite often those individuals that we ask for advice or get influenced somehow, ask our opinions or copy us in those situations that they are not perfectly sure about what to do. We do influence people that are related to us as well as we get influenced from them. The degree or amount of influence may change, but it, quite often, is a two dimensional issue. Thus, in our approach we do focus on two individuals that are in interaction. The whole of our analysis is based on a pair of individuals that has direct influence on each other's behavior. We suggest that individuals conduct a two stage decision making mechanism where the second stage involves interaction with each other. Facing a decision problem individuals first use their own preferences, maximizing them. If individual preferences are complete enough to single out a best preferred alternative for that specific problem, there would be no room for influence. Whenever this is not the case, in order to be able to choose from those remaining alternatives, they refer to each other's preferences. Hence they directly influence each other's choices. This is what we call as influence via choice completion. Let us follow a simple example in order to demonstrate these two models and how we suggest to measure the influence and homophily with them:

**An Example: Adventures of Ian and Jane.** Ian and Jane are two kids that go to the same school. On their way back from school, they talk about how each of them wants to spend the afternoon. They usually have three different activity options: stealing Apples from Mr. Smith's tree, riding a Bike or chasing Cats. Ian likes riding a Bike more than stealing Apples, but he has never chased Cats in his life, hence he does not have any idea in favor or against chasing Cats. Jane is a fan of Mr. Smith's Apples. She definitely prefers stealing Apples to both riding a Bike and chasing Cats, yet she is never sure how to compare these two. Their preferences can be summarized as: $\succ_i = \{BA\}$ and $\succ_j = \{AB, AC\}$, where $\succ_i$ refers to Ian's preferences, $\succ_j$ to Jane's, and $A, B, C$ to activity options. Hence on a day that the town's cats are away for a cat fight in the neighbor town, leaving the only available activities to the kids as riding a Bike, or stealing Apples, there is no problem. Each of them knows what they would like to do, and could go for their own best. However, the next day, although the cats are back in town, the day is a rainy day, hence it is not possible to ride a Bike. Now, facing the question of 'stealing Apples or chasing Cats?', the decision for Ian is not easy anymore, since he has no idea on whether he likes chasing Cats. Hence he does a very natural move, turns to Jane, and asks her own tastes. Being influenced by Jane's tastes, he goes for stealing Apple over chasing Cats as well. Similarly, on a day that all activity options are available, for Jane maximizing her own preferences would result in her stealing Apples, whereas for Ian maximizing his own preferences leaves him with two possible options, riding a Bike or chasing Cats. Taking Jane's opinion about these two options does not let him to further choose either. The decision outcomes of Ian and Jane under this behavioral model are given in Table 2.1.

| | **Ian** | **Jane** |
|---|---|---|
| **ABC** | BC | A |
| **AB** | B | A |
| **BC** | BC | BC |
| **AC** | A | A |

TABLE 2.1: Behavior of Ian and Jane with Influence via Choice Completion

Let us assume that we observe Ian and Jane behaving exactly as described above. Inevitably, on a rainy day, where riding a Bike is not an option anymore, leaving the kids with activities $A$ and $C$, Mr. Smith catches the two kids stealing his Apples. Immediately he calls Ian's and Jane's mothers to complain about their kids behavior. Hearing about their kid's misbehaviour, both of the moms fall into thoughts: "But it is not my kid, it is that other kid s/he hangs out with!"

In order to figure out if any influence took place, we suggest to compare the decision outcomes of the kids from the binary problems with their own preferences. Any decision from a binary problem that is not supported by individual preferences has to be a result of the social interaction. This would point out Jane's influence on Ian to steal Mr. Smith's Apples instead of chasing Cats as the only influence in this model.

Continuing with the story, Ian's mom becomes happy enough to clear her son's reputation. Afterall, her son has misbehaved under the influence of Jane. But a serious warning comes from Mr. Smith, leaving Ian's mom to deep thoughts once again: "It may be the other kid this time, but kids become friends with kids that are a lot like them." Hence, the question becomes how similar are Ian and Jane?

In order to assess the similarity of kids' preferences we suggest to compare their own preferences, $\succ_i$ and $\succ_j$. Although their preferences do not contain any common binary comparisons, the fact that they fail to compare the same alternatives suggest some level of homophily. In other words Ian is more similar to Jane than another possible kid with binary comparisons $\{AB, AC, BC\}$. We suggest a clear measure of homophily for individuals with incomplete preferences in the following section.

Our approach stands as a novel contribution to the literature on the identification of social influence from homophily since up to our knowledge, this problem is challenged mainly by the application of different econometrical strategies. Many studies document that both effects prevail simultaneously and distinguishing one from the other requires strong parametrical assumptions [4, 59, 65, 74, 81]. A major part of the works concentrates around adolescent behavior such as school achievement, use of drugs and recreational activities among high school children [23, 25, 65] and innovation diffusion [6, 54]. Recently online networks have drawn particular attention since they provide a powerful data source where the network structure is easily observable, hence this structure itself may provide additonal information to solve this identification problem [2, 4, 5, 61].

In the following two sections we present and analyze the model. Section 2 introduces decision making under influence with choice completion, discusses how to evaluate the influence and homophily counterparts under this behavioral model and presents the related foundational analysis. Section 3 deals with the idetification problem related to the revealed preference analysis. We conclude with some further remarks. All the proofs are delayed to an Appendix.

## 2.2 Influence via Choice Completion

### 2.2.1 The Decision Model and the Measurement of Influence and Homophily

Consider two individuals, equipped with transitive but not necessarily complete preferences. The incomplete parts of their preferences constitute the parts that are susceptible to influence. These individuals, simultaneously, refer to each other's preferences over those problems for which their own preferences do not lead them to a single choice.

Let $X$ be a nonempty finite set of alternatives and 1 and 2 denote two individuals with preferences defined over $X$. For $i \in \{1, 2\}$, let $\succ_i$ be the strict preference relation of $i$ over $X$, i.e., an asymmetric and transitive but not necessarily complete binary relation over $X$. Facing a decision problem $S$, individual $i \in \{1, 2\}$ is trying to choose the option she likes better. First she maximizes her own preferences. Given $S$, the set of maximal elemets of $S$ according to $\succ_i$ wil be $Max(S, \succ_i) = \{x \in S : \nexists y \in S \text{ with } yx \in \succ_i\}$. If individual preferences are complete enough to single out a best preferred alternative, i.e., if $Max(S, \succ_i)$ is a singleton, there will be no room for asking other person's opinion, copying her behavior, or any kind of influence process. Therefore individual $i$ would be choosing this maximal element from $S$. However, if there are many alternatives that are deemed to be choosable from $S$, $i$ would be susceptible to influence by $j$'s behavior over these choosable alternatives, hence arriving to $Max(Max(S, \succ_i), \succ_j)$.

To put it formally, given 1 and 2 with initial preferences $\succ_1$ and $\succ_2$ over $X$, respectively, we say that 1 **and** 2 **are influencing each other via choice completion** if for any nonempty decision problem $S \subseteq X$, their respective decision outcomes can be defined as:

$$Max(Max(S, \succ_1), \succ_2)$$
$$Max(Max(S, \succ_2), \succ_1).$$

Notice that unless $j$'s preferences are complete for all those alternatives over which $i$'s preferences are incomplete, the decision outcomes will not be single-valued.

Choice completion suggests a very natural and intutive way of decision-making for interacting individuals with incomplete preferences. It also provides us with direct and simple tools to measure the influence and homophily counterparts of this interaction. Influence is about what happens during interaction that is not justified by the own preferences of the individuals. If two individuals are influencing each other according to this behavioral model, we can actually make use of their own preferences and their decision outcomes to uncover what is happening during interaction. This way we can fully identify who is influencing who and over which decision problems that influence takes place. Homophily, on the other hand, is about why these two individuals have chosen each other to interact at the first place. It suggests the similarity of their own preferences as a possible explanation to the existence of this interaction. Then an assessment of the similarity of individual preferences enables us to identify the level of homophily between these individuals.

Going back to our initial question, if two individuals are behaving according to this influence via choice completion model, how can we assess the influence that they have on each other and the level of homophily of this interaction?

Let 1 and 2 influence each other via choice completion. We can evaluate the influence they create on each other by comparing their preferences with their decision outcomes from the binary problems. Any influence taken from the other individual will show itself in the form of a choice of a single alternative over the other from a binary problem although the preferences do not comply with this choice. We can measure the influence of individual 1 over 2 as $\{xy \in X \times X : x = Max(Max(\{x,y\}, \succ_1), \succ_2)\} \setminus \succ_1$. Similarly, $\{xy \in X \times X : x = Max(Max(\{x,y\}, \succ_2), \succ_1)\} \setminus \succ_2$ summarizes the influence of 2 on 1. Then all social influence that took place during this interaction can be defined by these two sets. A comparison of them will provide an assesment of influence between 1 and 2. For instance, comparing the sizes of these sets, we can point out the more influential individual. Or as a measure of the total influence occured during this interaction we can refer to the sum of the sizes of these two sets relative to the total number of binary comparisons, which gives us a measure in $[0,1]$.

The similarity of two individuals, on the other hand, can be assessed by comparing their preferences. The closer the preferences are to each other, the more similar are individuals' tastes. An immediate idea to quantify how close two incomplete preferences are, is to count the number of binary pairs on which these preferences do not agree, that is, $d(\succ_i, \succ_j) = |\succ_i \setminus \succ_j| + |\succ_j \setminus \succ_i|$.[3] Notice that, this way, the distance between two preferences is not only a function of their complete disagreements, but also depends on the binary pairs that are only compared by one of the preferences. Yet the disagreements count more with respect to the latter. Any pair of alternatives for which $\succ_i$ and $\succ_j$ disagree adds twice as much to the distance as any pair of alternatives that only one of them is able to compare.

Once the distance between the initial preferences is measured, the correlation between them as a function of this distance provides a sound assessment of similarity. Consider:

$$\tau(\succ_1, \succ_2) = 1 - 2\frac{d(\succ_1, \succ_2)}{maxd(\succ_1, \succ_2)} = 1 - 2\frac{d(\succ_1, \succ_2)}{n^2 - n},$$

where $maxd(\succ_1, \succ_2)$ denotes the maximum possible distance between two binary relations defined over $X$ and $n$ is the number of alternatives in $X$. Notice that $\tau$ is equal to 1 when two preferences are identical, and $-1$ when they are completely reverse.[4]

The level of homophily between the individuals is inversely related to the number of the binary pairs over which their preferences do not coincide, which in turn includes those pairs that one is susceptible to influence while the other has clear tastes about. The influence acquired during the interaction has to be a subset of those pairs. Thus, in very rough terms, a higher level of homophily is associated to a lower potential for influence,

[3]This distance function refers to the generalization of Kemeny-Snell distance to incomplete preferences, and first introduced and axiomatized in [22].

[4]As noted in [22], $\tau$ is the only linear function of $d$ that takes a value 1 when two preferences are identical, and $-1$ when they are completely reverse. And also note that $\tau$ reduces to Kendall's correlation coefficient $\tau$ in the case that preferences are complete.

although the exact relation between these two effects certainly depends on the structure of the preferences in question.

The tools that we suggest for the measurement of influence and homophily are quite straightforward tools once the underlying preferences of the individuals are known. However in many decision environments, preferences do not consititute a part of the observable variables. On the contrary we do observe the decision outcomes of the individuals. One possible way to distinguish the influence from homophily in this case is to infer the hidden preferences out of the observables, the decison outcomes. In the following subsection we address this challenge. We first investigate the properties of the decision outcomes that would imply the existence of influence via choice completion between two individuals. Identification of the characterizing properties also allows us to infer the underlying preferences. Following the foundational analysis, we revisit the suggested measurement techniques for influence and homophily and highlight their strengths and weaknesses for the scenarios where the only observables are the decision outcomes.

### 2.2.2 Foundational Analysis

Suppose we observe the individual decision outcomes of two individuals for any problem that they are facing. What kind of properties of the observed behaviors would imply that these two individuals are behaving as if they are influencing each other via choice completion?

Given $X$, let $\Omega_X$ be the set of all nonempty subsets of $X$. For any $i \in \{1, 2\}$, we define the decision outcomes of $i$ on $\Omega_X$ as a choice correspondence $C_i : \Omega_X \rightrightarrows X$ with $\emptyset \neq C_i(S) \subseteq S$ for every $S \in \Omega_X$.

From now on, the object of our analysis will be the choice behaviors of the two individuals: $(C_1, C_2)$. We next introduce the properties over $(C_1, C_2)$ that would indicate that each of the indviduals are completing their choices with influence from each other, that is:

$$C_1(S) = Max(Max(S, \succ_1), \succ_2) \text{ and}$$
$$C_2(S) = Max(Max(S, \succ_2), \succ_1) \text{ for all } S \in \Omega_X.$$

As a two stage maximization process, influence via choice completion is clearly related to other two stage maximization processes studied in the boundedly rational choice literature. The baseline model of two stage maximization would be the Rational Shortlist Method (RSM) proposed by Manzini and Mariotti [66]. RSM essentially refers to a single-valued choice mechanism where a two-stage maximization process yields out the chosen alternative uniquely.[5] In the first stage an acyclic and not necessarily complete binary relation and in the second stage a complete and not necessarily acyclic relation are considered to be maximized. Two rationality axioms, the standard Expansion axiom and a weakening of WARP, Weak WARP are shown to be necessary and sufficient for the choice data to reveal the two binary relations of the RSM. The novelty of RSM lies in explaining cyclical choice as a result of a boundedly rational procedure. A natural and interesting subclass of RSM models, not only for our purposes but also in general, would be RSM with transitive binary relations. Au and Kawai [9] show that an additional axiom that ensures the acyclicity of the revealed preference relation also ensures transitivity. In a recent project, Horan [48] proposes a pair of behavioral axioms that do not impose acyclicity directly but does guarantee the existence of transitive rationales. Our model, differs from the existing two stage maximization procedures in many aspects. First of all we consider two individuals. The second criterion that the individual uses to choose is not simply another criteria in mind, but the preference of another individual that is equipped with a choice structure as well. Hence the behavioral axioms we search for are to reveal the mutual relation between these two individuals, to identify the specific choice problems over which influence is taken. Moreover, we do not impose single-valuedness of the choice structure. Since the basic motivation for getting influenced is the inability to compare all alternatives it would be too restraining to focus only on single-valued choice. It would require individuals to ever interact with each other only if each of them has a unique answer to all of the questions the other makes.

A first axiom, Consistency of Influence links the choice behavior of the two individuals. It allows to detect the binary choice problems over which one of the individuals has influenced the other, and guarantees consistent behavior in larger problems:

**Consistency of Influence (CoI).** Let $x, y \in X$ and $i \in \{1, 2\}$ such that $x = C_i(xy)$ and $C_i(S) \neq C_i(S \setminus y)$ for some $S \in \Omega_X$ with $x \in S$. Then $C_j(S) = C_j(S \setminus y)$ for all $S \in \Omega_X$ with $x \in S$, where $j \in \{1, 2\}$ and $i \neq j$.[6]

The choice of a single alternative from a binary problem may be the result of two alternative scenarios: Either the individual's preferences dictate this choice or being unable to compare these alternatives according to the initial preferences, she gets influenced by the other individual. In the former case, due to the transitivity of the initial preferences we would never observe an inconsistency involving these two alternatives in the choice outcomes: Having the unchosen alternative from the binary problem available in any larger problem that also includes the better alternative would never affect the choice

[5]For further reference on two stage maximization procedures, or sequential decision making in general see [87], [3].

[6]Once again, we abuse the notation and we use $C_i(xy)$ in order to denote $C_i(\{x, y\})$.

from that larger problem. CoI detects the binary problems for which this is indeed not the case. Although $i$ chooses $x$ over $y$ from the binary problem, adding $y$ to a problem that also includes $x$ alters her choice behavior. This is a clear indication of $i$ being influenced by $j$ in her choice of $x$ from the binary problem. But then according to $j$'s preferences, $x$ is clearly a better alternative than $y$. Due to transitivity, $j$ would never choose inconsistently in the larger problems: The availability of $y$ will never change $j$'s choice from any problem that also includes $x$.

CoI ensures that any violation of consistency observed in the choice data has to be the result of influence taken regarding those alternatives. Therefore the individual that is the source of this influence will not show inconsistencies regarding those alternatives in her choice outcomes.

A second property that is required to link the behaviors of the individuals is Full Influence. This property ensures that if an individual is unable to choose one alternative from a binary problem, then the other individual is unable to choose as well:

**Full Influence (FI).** For any $x, y \in X$, if $xy = C_i(xy)$, then $xy = C_j(xy)$, for $i, j \in \{1, 2\}$ with $i \neq j$.

According to FI, whenever an individual is susceptible to influence, she does get influenced as long as the other individual is able to compare the alternatives in question. The last three properties that we introduce will be individual rationality properties. The main axioms of any two stage maximization problem, Expansion and WWARP are satisfied by influence via choice completion as well, but with a slight modification to the case of correspondences.

**Weak WARP (WWARP).** For any $x, y \in X$, if $x = C_i(xy)$ and $x \in C_i(S)$ for some $S \in \Omega_X$ with $y \in S$, then $y \notin C_i(T)$, for any $T \in \Omega_X$ with $\{x, y\} \subseteq T \subseteq S$, for any $i \in \{1, 2\}$.

According to WWARP, if an alternative $x$ is chosen uniquely from a binary problem and a larger problem, then the alternative that is not chosen from the binary problem cannot be chosen from any intermediary problem. Next we introduce the other standard property: Expansion. It simply states that an alternative chosen from two sets, will be chosen from the union of these two sets as well.

**Expansion (EXP).** For any $x \in X$, if $x \in C_i(S)$ and $x \in C_i(T)$ for $S, T \in \Omega_X$, then $x \in C_i(S \cup T)$, for any $i \in \{1, 2\}$.

Expansion forbids not choosing an alternative from a choice problem, if that alternative is chosen somewhere at the presence of each and all of the alternatives of the problem. Finally, the following property, Independence of Inferior Alternatives ensures the choice of an alternative independent of the inferior alternatives. An alternative chosen from a problem continues to be chosen from a larger problem if the additional elements are clearly inferior to the alternatives of the set:

**Independence of Inferior Alternatives (IInA).** Let $x, y, z \in X$ such that (i) there does not exist any $T \in \Omega_X$ with $z \in T$, $y \in C_i(T)$ and (ii) $x \in C_i(T')$ for some $T' \in \Omega_X$ with $y \in T'$. Then for $S \in \Omega_X$ with $x, y, z \in S$, if $x \in C_i(S \setminus y)$, then $x \in C_i(S)$, for $i \in \{1, 2\}$.

Consider $x, y, z \in X$ such that $y$ is never chosen at the presence of $z$, but $x$ is chosen somewhere at the presence of $y$. If $x$ is chosen from a set that includes $z$ but not $y$, then IInA ensures the choice of $x$ when $y$ is added to the set as well.

These five axioms are necessary and sufficient for the representation of influence via choice completion.

**Theorem 2.1.** *Let $C_1$ and $C_2$ be two choice correspondences. Then $C_1$ and $C_2$ satisfy CoI, FI, EXP, WWARP and IInA if and only if, there exist two asymmetric and transitive binary relations $\succ_1$ and $\succ_2$ on $X$ such that $C_i(S) = Max(Max(S, \succ_i), \succ_j)$ for any $S \in \Omega_X$, $i, j \in \{1, 2\}$ with $i \neq j$.*

The representation theorem gives the necessary and sufficient conditions for the existence of influence via choice completion between two individuals, hence identifies a pair of revealed preference relations $(\succ_1, \succ_2)$. How accurate are these relations in capturing the underlying preferences of the individuals is what we investigate in the next section.

## 2.3 Identification of the Underlying Preferences under Influence

Given a particular pair of choice behaviors $(C_1, C_2)$ satisfying the properties listed in the previous section, we can find a pair of revealed preference relations $(\succ_1, \succ_2)$ that would represent $(C_1, C_2)$ according to the influence via choice completion model. We suggest to make use of these revealed preferences in order to assess the influence that this interaction creates and the level of homophily between the individuals. However, the pair of revealed preferences defined in the proof of the representation theorem is not the only pair of binary relations with required properties that would result in this particular choice data according to influence via choice completion. In other words, preferences are not uniquely identified. One can actually find other pairs of revealed preferences that would represent the same choice behavior. Thus conducting the influence and homophily analysis over these pairs of preferences may mislead us. We now investigate how accurate we can be in capturing the underlying preferences of the individuals.

We first point out that all possible pairs of preferences explaining the same pair of choice data must share a common part. Identifying this common part will allow us to determine how much of the underlying preferences are captured accurately by the revealed preference analysis. Notice that the choices of the individuals from the binary problems define the following mutually exclusive sets of binary comparisons: Disagreements, influences on the other one, influences from the other one and agreements, where;

- disagreements of $i$ from $j$: $P_i = \{xy \in X \times X : x = C_i(xy) \neq C_j(xy)\}$
- influence of $i$ over $j$: $Q_i = \{xy \in X \times X : x = C_i(xy) = C_j(xy)$ and $C_j(S) \neq C_j(S \setminus y)$ for some $S \in \Omega_X$ with $x \in S\}$.
- agreements of $i$ and $j$: $R = \{xy \in X \times X : x = C_i(xy) = C_j(xy)$ and $C_i(S) = C_i(S \setminus y)$ and $C_j(S) = C_j(S \setminus y)$ for all $S \in \Omega_X$ with $x \in S\}$.

A first observation is that any pair of preferences explaining the same choice data will recognize the pairs of alternatives that two individuals disagree on, $P_i$. These refer to the pairs of alternatives about which the individuals have reverse tastes. No influence may possibly take place regarding these pairs. Hence, $(P_1, P_2)$ will be common to any preference $(\succ_1, \succ_2)$ for a given $(C_1, C_2)$.

Moreover, we are able to detect the pairs of alternatives such that an influence is taken for sure, $Q_i$ and $Q_j$. Consider a binary problem such that both individuals have chosen $x$ over $y$. If one of the individuals shows inconsistent behavior in a larger problem, then the choice of $x$ over $y$ from the binary problem can only be the result of getting influence and any preference pair resulting in this behavior will recognize that. $Q_i$ identifies the pairs $x, y$ such that individual $j$ has been influenced by $i$ to choose $x$ over $y$. Thus, $(Q_1, Q_2)$ will be common to any preference $(\succ_1, \succ_2)$ for a given $(C_1, C_2)$ as well.

Finally, since preferences are defined to be transitive, the ordered pairs that are not necessarily in $P_i$ or $Q_i$, but implied by transitivity of $\succ_i$ will be common to any $(\succ_1, \succ_2)$ for the given $(C_1, C_2)$: the transitive closure of $P_i \cup Q_i$. Let us denote the transitive closure of $P_i \cup Q_i$ as $tr(P_i \cup Q_i)$.[7] The following theorem states that the intersection of all pairs of preferences $(\succ_1, \succ_2)$, that explain a given choice data is $(tr(P_1 \cup Q_1), tr(P_2 \cup Q_2))$:

**Theorem 2.2.** *Let $C_1$ and $C_2$ be two choice correspondences such that $(C_1, C_2)$ satisfy CoI, FI, EXP, WWARP, and IInA. Then the intersection of all the preference pairs $(\succ_1, \succ_2)$ that represent $(C_1, C_2)$ is $(tr(P_1 \cup Q_1), tr(P_2, Q_2))$.*

[7] The transitive closure of a binary relation is the smallest transitive relation that contains it.

Theorem 2.1 provides conditions for individuals choosing as if they are influencing each other via choice completion. Theorem 2.2 makes sure that we can actually recover a major part of this influence and their underlying preferences.

In the proof of Theorem 2.1, the pair of revealed preference relations of a given $(C_1, C_2)$ was indeed $(P_1 \cup Q_1 \cup R, P_2 \cup Q_2 \cup R)$. Actually these refer to the largest pair of initial preferences that would explain $(C_1, C_2)$. In other words, any other possible pair of initial preferences for the same $(C_1, C_2)$ has to be included in $(P_1 \cup Q_1 \cup R, P_2 \cup Q_2 \cup R)$. And as established by Theorem 2.2, any pair of initial preferences has to include $(tr(P_1 \cup Q_1), tr(P_2 \cup Q_2))$. Thus the part of the underlying preferences that cannot be uniquely identified is included in $R$. They refer to the pairs of alternatives that both of the individuals have agreed on the choice of a unique one out of the binary problem, and none of them has shown any inconsistency in terms of choice outcomes that would indicate the influence taken. Hence the model does not help us to associate these binary pairs uniquely to one of the individuals unless they are a part of the transitive closure of $(P_i \cup Q_i)$. Although this leaves us with an obvious identification problem, we should investigate how severe this problem is and to which extent we can overcome it for the sake of measurement of influence and homophily with the tools we have suggested.

Consider the extreme case where two individuals show exactly the same choice behavior. Then since $\succ_1 = \succ_2 = R$, we cannot conclude if two individuals have actually the same sincere preferences, indicating maximum homophily with $\tau$ equal to 1 and zero influence between the individuals or one of them has null preferences and is getting fully influenced by the other, indicating a level of homophily equal to 0 at the least. On the other extreme, if we observe a pair of choice behaviors with a null $R$, we can completely identify the amount of influence and the exact level of homophily, since we can recover the underlying initial preferences fully. Hence the accurateness of these measures depends on the size of $R$.

Notice that although we cannot determine the exact level of homophily and influence in cases with nonempty $R$, it is always possible to find out the maximum potential homophily level of a pair of individuals as well as the minimum influence. Given $(C_1, C_2)$, the minimum influence between individuals is captured by $Q_i$ and $Q_j$. On the other hand, the minimum possible distance between their underlying preferences will be $d^*(\succ_i, \succ_j) = |P_1| + |P_2| + |Q_1| + |Q_2|$, yielding a maximum potential homophily level equal to $\tau(\succ_1, \succ_2) = 1 - 2\frac{|P_1|+|P_2|+|Q_1|+|Q_2|}{n^2-n}$.

## 2.4 Concluding Remarks

In this study we approached to the identification problem of social influence and homophily from a micro-theoretical perspective. We presented and analyzed a decision model that allows individuals to get influenced from each other and we suggested simple tools to evaluate influence and homophily effects.

With influence with choice completion model, there remains a part of the underlying preferences that is not uniquely identified out of the decision outcomes. The size of this part determines the extent of the accuracy of the suggested measures of influence and homophily. One possible way to overcome this partial identification problem can be found by observing the individual decision outcomes before this particular social interaction. In other words, if we have the chance to observe the individual's decisions both before and after they get related, then we can actually recover the initial preferences completely. This can be valid for several scenarios where the behavior of individuals over a time span, which includes the first time that they get related, can be documented such as recently married individuals, fresh housemates or new colleagues.

In this case, before the interaction individuals will be choosing to maximize their own transitive but not necessarily complete preferences, i.e., $C_i(S) = Max(S, \succ_i)$ for any $S \in \Omega_X$ and for $i \in \{1, 2\}$. Then we can quickly show that, a strong consistency property (SCo) together with EXP characterizes this choice behavior revealing out the underlying sincere preferences $(\succ_1, \succ_2)$ uniquely.

**Strong Consistency (SCo).** For any $i \in \{1, 2\}$ and for any $x, y \in X$, if $x = C_i(xy)$, then $C_i(S) = C_i(S \setminus y)$ for any $S \in \Omega_X$ with $x, y \in S$.

**Theorem 2.3.** *Let $C_i$ be a choice correspondence. Then $C_i$ satisfies EXP and SCo if and only if, there exists a unique asymmetric and transitive binary relation $\succ_i$ on $X$ such that $C_i(S) = Max(S, \succ_i)$ for any $S \in \Omega_X$.*

The underlying preferences can be directly derived from the choices from binary problems: $\succ_i = \{xy \in X \times X : x = C_i(xy)\}$. Once these sincere preferences are derived, identifying the exact level of influence and homophily becomes a simple matter of comparison as described in the previous sections.

All the current analysis is based on interaction of two individuals. Extending this approach to further study more complicated interaction structures, such as social networks, stands as an interesting line of research.

## 2.5 Appendix

**Proof of Theorem 2.1.** Necessity is fairly straightforward, thus omitted. We prove the sufficiency part. The proof consists of two main parts. First we define a pair of auxiliary relations, $(\succ_1^*, \succ_2^*)$, and show that $C_i(S) = Max(Max(S, \succ_i^*), \succ_j^*)$ for any $S \in \Omega_X$, $i, j \in \{1, 2\}$ with $i \neq j$. Then, we show there exists indeed a pair of asymmetric and transitive relations $(\succ_1, \succ_2)$ on $X$ such that $Max(Max(S, \succ_i), \succ_j) = Max(Max(S, \succ_i^*), \succ_j^*)$ for any $S \in \Omega_X$, $i, j \in \{1, 2\}$ with $i \neq j$.

For $i \in \{1, 2\}$, define the binary relation $\succ_i^* \subseteq X \times X$ as follows:

$$xy \in \succ_i^* \text{ iff there does not exist any } S \in \Omega_X \text{ with } y \in C_i(S) \text{ and } x \in S,$$

for any $x, y \in X$. Notice that $\succ_i^*$ is asymmetric and acyclic. Asymmetry is obvious by definition and nonemptiness of the choice correspondence. To see acyclicity, consider $x, z_1, ..., z_k, y \in X$ such that $xz_1, z_1z_2, ..., z_ky \in \succ_i^*$. Assume for a contradiction $yx \in \succ_i^*$. But then, by definition of $\succ_i^*$, $x, z_1, ..., z_k, y \notin C_i(xz_1...z_ky)$, contradicting with the nonemptiness of the choice correspondence.

Fix $i, j \in \{1, 2\}$ with $i \neq j$. Take any $S \in \Omega_X$ and $x \in S$. First we will show that $x \in C_i(S)$ implies $x \in Max(Max(S, \succ_i^*), \succ_j^*)$. By definition of $\succ_i^*$, $x \in Max(S, \succ_i^*)$. Assume for a contradiction, $x \notin Max(Max(S, \succ_i^*), \succ_j^*)$. Then, there exists $z \in Max(S, \succ_i^*)$ with $zx \in \succ_j^*$, which in turn implies, $z = C_j(xz)$. By FI, $xz \neq C_i(xz)$. Let $x = C_i(xz)$. Since $z \in Max(S, \succ_i^*)$, there exists a set $T$ including $x$ such that $z \in C_i(T)$. But then $C_i(T) \neq C_i(T \setminus z)$. By CoI, for any set $T$ including $x$ and $z$, $C_j(T) = C_j(T \setminus z)$, contradicting with $z = C_j(xz)$. Finally, if $z = C_i(xz)$, then since $z \in Max(S, \succ_i^*)$, for any $y \in S$, there exists $S_{zy} \in \Omega_X$ such that $y \in S_{zy}$ and $z \in C_i(S_{zy})$. Then, by EXP, $z \in C_i(\cup_{y \in S} S_{zy})$. But since $S \subseteq \cup_{y \in S} S_{zy}$, and $z = C_i(xz)$, WWARP implies that $x \notin C_i(S)$, giving the desired contradiction.

Now, we will show that $x \in (Max(Max(S, \succ_i^*), \succ_j^*)$ implies $x \in C_i(S)$. First we show that for all $y \in Max(S, \succ_i^*)$, $x \in C_i(xy)$. Assume for a contradiction that this does not hold and consider $y \in Max(S, \succ_i^*)$ with $y = C_i(xy)$. As $x \in Max(S, \succ_i^*)$, there exists a $T \in \Omega_X$ with $x \in C_i(T)$ while $y \in T$, hence $C_i(T) \neq C_i(T \setminus x)$. But then, by CoI, there does not exist any $T \in \Omega_X$ with $x, y \in T$ and $C_j(T) \neq C_j(T \setminus x)$. But this implies that for any $T \in \Omega_X$ with $y \in T$, $x \notin C_j(T)$, which means $yx \in \succ_j^*$, creating a contradiction with $x \in (Max(Max(S, \succ_i^*), \succ_j^*)$. Hence, $x \in C_i(xy)$ for all $y \in Max(S, \succ_i^*)$. Then, by EXP, $x \in C_i(Max(S, \succ_i^*))$. But since for any $y \in S \setminus Max(S, \succ_i^*)$, there exists $z \in S$ with $zy \in \succ_i^*$ and clearly $yx \notin \succ_i^*$, by iterative application of IInA, $x \in C_i(S)$.

Hence we have shown that there exists a pair of acyclic binary relations $(\succ_1^*, \succ_2^*)$ such that $C_i(S) = Max(Max(S, \succ_i^*), \succ_j^*)$ for $i, j \in \{1,2\}$ with $i \neq j$. By making use of these auxiliary relations, we now show that there indeed exists a pair of transitive preferences $(\succ_1, \succ_2)$ such that $Max(Max(S, \succ_i), \succ_j) = Max(Max(S, \succ_i^*), \succ_j^*)$ for any $S \in \Omega_X$, $i, j \in \{1,2\}$ with $i \neq j$.

For $i \in \{1,2\}$, define $\succ_i \subseteq X \times X$ as the following:

$$xy \in \succ_i \text{ if and only if } xy \in \succ_i^* \text{ and } xa \in \succ_i^* \text{ for all } ya \in \succ_i^*.$$

Clearly $\succ_i \subseteq \succ_i^*$. To see transitivity, consider any $xy, yz \in \succ_i$ and notice that $xa \in \succ_i^*$ for any $ya \in \succ_i^*$ implies in particular $xz \in \succ_i^*$. Moreover, since $ya \in \succ_i^*$ for any $za \in \succ_i^*$ and $xa \in \succ_i^*$ for any $ya \in \succ_i^*$, we have $xa \in \succ_i^*$ for any $za \in \succ_i^*$, establishing transitivity of $\succ_i$.

The following lemma clarifies the relation between $\succ_1^*, \succ_2^*, \succ_1$ and $\succ_2$. It shows that not only $\succ_i \subseteq \succ_i^*$, but also $\succ_i \cup \succ_j = \succ_i^* \cup \succ_j^*$, since any ordered pair that belongs to $\succ_i^*$ but not $\succ_i$ is also an element of $\succ_j$:

**Lemma 2.4.** *If $xy \in \succ_i^* \setminus \succ_i$, then $xy \in \succ_j$ for $i \in \{1,2\}$ with $i \neq j$.*

**Proof of Lemma 2.4**. Fix $i, j \in \{1,2\}$ with $i \neq j$. Consider any $xy \in \succ_i^* \setminus \succ_i$. Hence, $x \in S$ implies that $y \notin C_i(S)$ but there exists $z \in X$ with $xz \notin \succ_i^*$ although $yz \in \succ_i^*$. Then, there exists $T \in \Omega_X$ with $x \in T$ and $z \in C_i(T)$. Clearly, $y \notin T$. Since $x = C_i(xy)$, and $z \notin C_i(T \cup \{y\})$ as $yz \in \succ_i^*$ but $z \in C_i(T)$ implies $C_i(T \cup \{y\}) \neq C_i(T)$, by CoI we have $x, y \in S$ implies that $C_j(S) = C_j(S \setminus y)$ for any $S \in \Omega_X$, yielding that $y \notin C_j(S)$ whenever $x \in S$. Hence, $xy \in \succ_j^*$.

Now, for any $t \in X$ with $yt \in \succ_j^*$, if $xt \notin \succ_j^*$, then there exists $T' \in \Omega_X$ with $x \in T'$ and $t \in C_j(T')$. But then, $C_j(T' \cup \{y\}) \neq C_j(T')$, contradicting with CoI. Hence, $xt \in \succ_j^*$ for any $t \in X$ with $yt \in \succ_j^*$, establishing $xy \in \succ_j$. □

Now we are ready to show $(Max(Max(S, \succ_i^*), \succ_j^*) = (Max(Max(S, \succ_i), \succ_j)$ for any $S \in \Omega_X$, $i, j \in \{1,2\}$ with $i \neq j$.

Fix $i, j \in \{1,2\}$ with $i \neq j$. Take any $S \in \Omega_X$ and $x \in (Max(Max(S, \succ_i^*), \succ_j^*)$. Notice that since $\succ_i \subseteq \succ_i^*$, $Max(S, \succ_i^*) \subseteq Max(S, \succ_i)$, in particular $x \in Max(S, \succ_i)$. Now assume for a contradiction, there exists $z \in Max(S, \succ_i)$ with $zx \in \succ_j$. Clearly, $z \notin Max(S, \succ_i^*)$. Then there exists $a_1 \in S$ with $a_1 z \in \succ_i^* \setminus \succ_i$. But then, by Lemma 2.4, $a_1 z \in \succ_j$, which yields $a_1 x \in \succ_j$ by transitivity, and hence we have $a_1 x \in \succ_j^*$. Since $x \in (Max(Max(S, \succ_i^*), \succ_j^*)$, $a_1 \notin Max(S, \succ_i^*)$. But this in turn would imply that there exists $a_2 \in S$ such that $a_2 a_1 \in \succ_1^*$. We have two cases: **(i)** $a_2 a_1 \in \succ_i$, **(ii)** $a_2 a_1 \notin \succ_i$. Below we show that both of the cases yield $a_2 x \in \succ_j$:

**(i)** If $a_2a_1 \in \succ_i$, then by definition $a_2z \in \succ_i^*$. Since $z \in Max(S, \succ_i^*)$, $a_2z \notin \succ_i$. By Lemma 2.4, $a_2z \in \succ_j$. Transitivity of $\succ_j$ establishes $a_2x \in \succ_j$.

**(ii)** If $a_2a_1 \notin \succ_i$, then by Lemma 2.4, $a_2a_1 \in \succ_j$. Transitivity of $\succ_j$ establishes $a_2x \in \succ_j$.

By definition, $a_2x \in \succ_j$ implies $a_2x \in \succ_j^*$. Therefore, since $x \in Max(Max(S, \succ_i^*), \succ_j^*)$, we have $a_2 \notin Max(S, \succ_i^*)$.

But then following a similar argumentation, there exists $a_3 \in S$ such that $a_3a_2 \in \succ_1^*$. We again have two cases: **(i)** $a_3a_2 \in \succ_i$, **(ii)** $a_3a_2 \notin \succ_i$. Using Lemma 2.4 and transitivity of $\succ_j$ iteratively, both cases yield $a_3x \in \succ_2$ as before. If $a_3 \in Max(S, \succ_i^*)$, then $x \notin Max(Max(S, \succ_i^*), \succ_j^*)$, giving the desired contradiction. Hence, $a_3$ cannot be an element of $Max(S, \succ_i^*)$ similar to $a_2$ and $a_1$. But since $S$ finite, and $\succ_i^*$ is acyclic, there has to be a maximal element of this chain. In other words, there exists a finite chain in $S$, such that $a_k \succ_i^* a_{k-1} \succ_i^* ... \succ_i^* a_2 \succ_i^* a_1 \succ_i^* z$ with $a_n \in Max(S, \succ_i^*)$ and $a_nx \in \succ_j$, and hence $a_nx \in \succ_j^*$, for any $n = 1, 2, ..., k$, creating the desired contradiction with $x \in Max(Max(S, \succ_i^*), \succ_j^*)$.

Now take any $x \in Max(Max(S, \succ_i), \succ_j)$. We will show that $x \in Max(Max(S, \succ_i^*), \succ_j^*)$ as well. Assume for a contradiciton $x \notin Max(Max(S, \succ_i^*), \succ_j^*)$. Then we have two cases: **(i)** There exists $y \in S$ with $yx \in \succ_i^*$ or **(ii)** There exists $z \in Max(S, \succ_i^*)$ with $zx \in \succ_j^*$. Let us consider the cases one by one:

**(i)** $yx \in \succ_i^*$, but since $yx \notin \succ_i$, by Lemma 2.4, $yx \in \succ_2$. If $y \in Max(S, \succ_i)$, then $x \notin Max(Max(S, \succ_i), \succ_j)$, the desired contradiction. Thus, $y \notin Max(S, \succ_i)$, implying in turn that there exists $a_1 \in S$ with $a_1y \in \succ_i$ By definition of $\succ_i^*$ and since $yx \in \succ_i^*$, $a_1x \in \succ_i^*$. But since $x \in Max(S, \succ_i)$, $a_1x \notin \succ_i$, hence by Lemma 2.4, $a_1x \in \succ_j$. And if $a_1 \in Max(S, \succ_i)$, then $x \notin Max(Max(S, \succ_i), \succ_j)$, giving the desired contradiction. Thus, $a_1 \notin Max(S, \succ_i)$ similar to $y$. But since $S$ is finite, there exists $a_n \in Max(S, \succ_i)$ with $a_ny \in \succ_i$ And since by definition, $a_nx \in \succ_i^*$, but $a_nx \notin \succ_i$, by Lemma 2.4, $a_nx \in \succ_j$, creating the desired contradiction with $x \in Max(Max(S, \succ_i), \succ_j)$.

**(ii)** $x \in Max(S, \succ_i^*)$ and there exists $z \in Max(S, \succ_i^*)$ with $zx \in \succ_j^*$. As $Max(S, \succ_i^*) \subseteq Max(S, \succ_i)$, $z \in Max(S, \succ_i)$. By maximality of $x$, $zx \notin \succ_j$. But then, by Lemma 2.4 we have $zx \in \succ_i$, contradicting with $x \in Max(S, \succ_i)$. Thus, $Max(Max(S, \succ_i), \succ_j) = Max(Max(S, \succ_i^*), \succ_j^*)$, concluding the sufficiency part of the proof. □

**Proof of Theorem 2.2** Let $(C_1, C_2)$ be as defined. Let $\rhd_i$ denote the intersection of all preference relations $\succ_i$ such that $(\succ_1, \succ_2)$ represent $(C_1, C_2)$ for some $\succ_j$, i.e., define for $i \in \{1, 2\}$, $\rhd_i \subseteq X \times X$ such that: $xy \in \rhd_i$ if and only if $xy \in \succ_i$ for all $(\succ_1, \succ_2)$ that represent $(C_1, C_2)$, for any $x, y \in X$.

First we show that for any $xy \in tr(P_i \cup Q_i)$, we have $xy \in \rhd_i$. Consider any $(\succ_1, \succ_2)$ explaining $(C_1, C_2)$. Fix $i, j \in \{1, 2\}$ with $i \neq j$. Take any $xy \in P_i$. Since $y = C_j(xy)$, we have $xy \notin \succ_j$. But then $x = C_i(xy)$ implies that $xy \in \succ_i$. Now take any $xy \in Q_i$. As there exists $S \in \Omega_X$ with $x \in S$ and $C_j(S) \neq C_j(S \setminus y)$, we have $xy \notin \succ_j$. But then, $x = C_i(xy)$ implies that $xy \in \succ_i$. Hence we have shown that $xy \in (P_i \cup Q_i)$ is included in $\succ_i$ as well. Transitivity of $\succ_i$ proves the claim.

To prove the other direction, we first introduce the following lemma:

**Lemma 2.5.** *If $C_j(S) = C_j(S \setminus y)$ for any $S$ with $x \in S$, then the following holds:*

*(i) $yb \in \succ_j$ implies $xb \in (\succ_i \cup \succ_j)$ for any $b$ in $X$:*

*(ii) $ax, yb \in \succ_j$ implies $ay, ab \in (\succ_i \cup \succ_j)$ for any $a, b \in X$.*

**Proof of Lemma 2.5:** (i). Assume for a contradiction that there exists $b \in X$ with $yb \in \succ_j$ but $xb \notin (\succ_i \cup \succ_j)$. Then, $C_j(bxy) \neq xb = C_j(xb)$, absurd.

(ii). Assume that there exists $a, b \in X$ with $ax, yb \in \succ_j$ but $ab \notin (\succ_i \cup \succ_j)$. But then $C_j(abxy) \neq ab = C_j(abx)$, absurd. Now assume for a contradiction that there exists $a, b \in X$ with $ax, yb \in \succ_j$ but $ay \notin (\succ_i \cup \succ_j)$. But then $C_j(abxy) = ay \neq C_j(abx)$, which is absurd. $\square$.

Now we are ready to prove the other direction. We now show that $(tr(P_i \cup Q_i), \succ_j^*)$ explains $(C_i, C_j)$, for some $\succ_j^*$, for $i, j \in \{1, 2\}$ with $i \neq j$. Consider any $(\succ_1, \succ_2)$ explaining $(C_1, C_2)$. Fix $i, j \in \{1, 2\}$ with $i \neq j$. We will construct $\succ_j^*$, by making use of $(\succ_1, \succ_2)$, such that $(tr(P_i \cup Q_i), \succ_j^*)$ explains $(C_i, C_j)$.

Let us denote $(\succ_i \setminus tr(P_i \cup Q_i))$ as $T_i$. Our claim is that $\succ_j^* = tr(\succ_j \cup T_i)$. We construct $\succ_j^*$ in three steps:

*Step 1:* For any $xy \in T_i$, $(\succ_i, tr(\succ_j \cup \{xy\}))$ explains $(C_i, C_j)$ as well.

*Step 2:* Recursive application of Step 1 implies that $(\succ_i, tr(\succ_j \cup T_i))$ explains $(C_i, C_j)$ as well.

*Step 3:* Finally, $(tr(P_i \cup Q_i), tr(\succ_j \cup T_i))$ explains $(C_i, C_j)$ as well.

*Step 1:* First, take any $xy \in T_i$. Since $x = C_i(xy)$ and $xy \notin P_i$, we have $x = C_j(xy)$. And since $xy \notin Q_i$, $C_j(S) = C_j(S \setminus y)$ for any $S \in \Omega_X$ with $x \in S$. Hence for any $xy \in T_i$, we have $C_j(S) = C_j(S \setminus y)$ for any $S \ni x$.

Now let us denote $tr(\succ_j \cup \{xy\})$ as $\succ_j'$. If $xy \in \succ_j$, the claim of Step 1 holds. Assume $xy \notin \succ_j$. Notice that $\succ_j' = \succ_j \cup \{xb : yb \in \succ_j\} \cup \{ay : ax \in \succ_j\} \cup \{ab : ax, yb \in \succ_j\} \cup \{xy\}$. But then by Lemma 2.5, $(\succ_j' \setminus \succ_j) \subseteq \succ_i$. That is, any binary relation that is added to $\succ_j$ to extend it to $\succ'_j$, is already included in $\succ_i$. It is now easy to check indeed, $Max(Max(S, \succ_i), \succ_j) = Max(Max(S, \succ_i), \succ'_j)$ and $Max(Max(S, \succ_j), \succ_i) = Max(Max(S, \succ'_j), \succ_i)$. Thus, $(\succ_i, \succ'_j)$ explains $(C_1, C_2)$, as claimed.

*Step 2:* Consider $(\succ_i, \succ'_j)$. By Step 1, for any $xy \in T_i$, $(\succ_i, tr(\succ'_j \cup \{xy\}))$ explains $(C_i, C_j)$. Indeed recursive application of Step 1 implies that $(\succ_i, tr(\succ_j \cup T_i))$ explains $(C_i, C_j)$ as well. Let $\succ^*_j$ denote $tr(\succ_j \cup T_i)$.

*Step 3:* Let us denote $tr(P_i \cup Q_i)$ as $\succ^*_i$. Now consider $(\succ^*_i, \succ^*_j)$. Since $T_i \subseteq \succ^*_j$, we have $\succ_i \setminus \succ^*_i \subseteq \succ^*_j$. Hence $Max(Max(S, \succ_i), \succ^*_j) = Max(Max(S, \succ^*_i), \succ^*_j)$ and similarly $Max(Max(S, \succ^*_j), \succ_i) = Max(Max(S, \succ^*_j), \succ^*_i)$. Thus, $(\succ^*_i, \succ^*_j)$ explains $(C_1, C_2)$, concluding the proof. □

**Proof of Theorem 2.3:** Necessity is fairly straightforward, thus omitted. We prove the sufficiency part. Take any $i, j \in \{1, 2\}$ with $i \neq j$. Define $\succ_i$ over $X$ as follows: $xy \in \succ_i$ iff $x = C_i(xy)$ for any $x, y \in X$. Asymmetry is by definition of $\succ_i$. To see transitivity, consider any $xy, yz \in \succ_i$. By SCo, $y \notin C_i(xyz)$ and $z \notin C_i(xyz)$. Hence, $x = C_i(xyz)$. Since $xy \succ_i$, once again by SCo, $C_i(xyz) = C_i(xz) = x$, establishing $xz \in \succ_i$.

Consider any $S \in \Omega_X$. We will now show that $C_i(S) = Max(S, \succ_i)$. Take any $x \in C_i(S)$. Assume for a contradiction that there exists $y \in S$ with $yx \in \succ_i$. But then $y = C_i(xy)$ and by SCo $x \notin C_i(S)$, giving the desired contradiction. Hence, $x \in Max(S, \succ_i)$.

Finally take any $x \in Max(S, \succ_i)$. By definition of $\succ_i$, for all $y \in S \setminus \{x\}$, either $x = C_i(xy)$ or $xy = C_i(xy)$. But then, by EXP, $x \in C_i(S)$, establishing sufficiency. □

# Chapter 3

# Tell Me Who You Are, I Tell You Who Your Friends are: Understanding Social Networks out of Individual Decisions

## 3.1 Introduction

Social networks are known to affect the way individuals behave. From consumption habits, to voting behavior, from school achievement of teenagers to investment interests of business people, a variety of decisions that people make in everyday life have an intimate relation with their social interactions. Extensive research has been conducted in order to understand the "countless ways in which network structures affect behavior" [55]. In this study, we take the reverse approach and suggest to investigate individual behavior in order to understand the underlying social network structure.

Quite often the structure of the social network is not clearly observable. Having individuals in the same environment may give clues about the possible set of social interactions, but not necessarily all people of the same environment are related to each other in the same way. Teenagers going to the same school are probably more likely to be friends than the teenagers registered to different schools, but obviously not all the students in a school have active friendship relations. Or individuals of the same work place probably know each other at some degree, but there exist different types of relations including hierarchical ones such as the one with the boss or nonhierarhical ones such as the one between colleagues working on the same project. Consider on-line network structures such as Facebook. Befriending someone at some point in time is not at all informative about the actual relation that you have with her. It is very possible to 'add' people just for social courtesy reasons, or to even forget that you had met at some point somewhere. Hence many social circles are full of non-active, non-working links and assuming that people of the same circle are all connected to each other in the same way will be quite misleading especially if policy suggestions are made depending on this assumption.

In this study we suggest to uncover the underlying structure of the social network by analyzing individual behavior patterns, individual decision outcomes, to be more specific. Our approach is based on the observation that socially connected individuals, influence each other's decisions through different channels, especially at times of indecisiveness. It is quite common practice for all of us to refer to the opinions of people around, especially if we do not have sufficient information or experience to compare several options that we are facing. We may use different communication tools such as asking for advice, gathering suggestions or we may observe people behaving in a particular way and adapt similar behavioral patterns. With this purpose we first present a decision model that allows individuals to refer to their social contacts when they need it. Then we investigate the properties on their decision outcomes that will reveal out the specific way in which they are connected.

We suggest that individuals prioritize their own preferences over what they observe from their social environment. However they are susceptible to influence over those alternatives that their own preferences are not well-defined. Thus their decision outcomes can be described as the result of a two stage procedure, where the first stage involves maximization of their own preferences. The second stage allows them to refine their choices further, if needed, by making use of the information they gather from their social environment.

In a social network one individual may be connected to a group of individuals in total in four different ways: she may be in touch with only one individual that somehow connects her to the group, she may be influencing all or a subset of the group, she may be getting influenced by all or a subset of the group or finally she may be connecting two individuals that are themselves connected to the rest of the group in some specific way. Our objective is to find out which one of these possible connections is taking place by comparing the decision outcomes of the individuals of the group. With this purpose we present the properties on the decision outcomes of a group of individuals that will reveal out the structure of their connections. Overall we characterize all four possible interaction structures.

Social interactions literature on social networks presents a fruitful line of research since network structure of relations provides a rich environment to overcome the identification problems related to the estimation of the effects of social interactions on individual decisions. Existing research mainly focuses on the influence of peers in contexts such as choice of college education [31], academic achievement [25, 42], consumption decisions [23] or over online networks [2, 4, 5]. A major part of the literature assumes that the structure of the network is known. Noting that this assumption is critical in terms of restricting the emprical work to contexts in which survey data can be used to measure network structure, several directions for the development of empirical strategies when network structure is unknown are suggested in [21]. A second line of research that focuses on the structure of social interactions includes equilibrium analysis on binary choice models [14, 20, 49].[1] Research on these discrete choice models focuses on the analysis of aggregate behavioural outcomes when individual utility exhibits social interaction effects.

The following section introduces the individual decision model and focuses on the analysis with the most simple interaction structure, a dyad. Section 3 extends the analysis to more general network forms, that includes a star, inverse star and a chain. Finally, we conclude. All proofs are delayed to an Appendix.

## 3.2 Decision Making with Influence in a Dyad

Let $X$ be a finite nonempty set of alternatives and consider individual $j$. Let $\succ_j$ denote the preference relation of $j$ over $X$. We assume that preferences are asymmetric, transitive but not necessarily complete. Asymmetry is assumed to keep the analysis simple. Transitivity is seen as an individual rationality requirement. Incompleteness, on the other hand is the main reason for individuals to be susceptible to influence.

Individuals use their social environment as a means of gathering information about the alternatives they fail to compare. Endowed with not necessarily complete preferences, they make use of the behaviors of the individuals they observe around to further refine their decision outcomes. In other words they get influenced through their social interactions over those alternatives that they are initially indecisive. Then individual decision making becomes a function of not only the preferences of the individuals, but also the decision outcomes of the individuals that they get influenced by. We suggest that individuals prioritize their own preferences over what they observe from their society. Then, facing a decision problem, first they try to solve it on their own, maximizing their own preferences. If this maximization leads them to a single outcome, then then there will not be any room for being influenced. However, if they are not able to choose on their

[1] For an overview of theoretical approaches in this literature see [34]. For parallel work with multinomial choice models see [24].

own uniquely, but rather left with a set of choosable alternatives, they refer to their social interactions in order to be able to further refine their choice. We suggest that in this second stage of decision, they maximize a binary relation that summarizes what they have observed from their social environment. In this section we restrict our analysis to the minimal possible group of individuals in which social influence takes place: A dyad. In the following section, based on our analysis on dyads, we will extend our focus to more general types of social interactions.

Now assume that $i$ is the only individual that $j$ is directly in touch with. Facing a a decision problem, say $A \subseteq X$, $j$ first maximizes her own preferences, leaving her with a set of maximal elements: $Max(A, \succ_j) = \{x \in A$: there does not exist $y \in A$ with $yx \in \succ_j\}$.[2] If $Max(A, \succ_j)$ has a unique element, then that is what $j$ chooses from $A$. Otherwise, she refers to what she observes from her social environment, which is only $i$ in this case. We suggest that in this second stage, $j$ maximizes a binary relation, say $\succ_j^i$ that depends on the binary decision outcomes of $i$: $xy \in \succ_j^i$ if and only if $x$ is the unique outcome of $i$ from the decision problem $A = \{x, y\}$.[3] As long as $j$ does not observe cyclic behavior from $i$ for those alternatives that she cannot compare on her own, this two stage maximization process will result in nonempty decision outcomes. We suggest that this is indeed the case, otherwise $j$ would use other social connections that would not create further decision complexities for herself. Notice that the decision outcomes need not to be single-valued though. Thus, the decison outcomes of $j$ can actually be represented by a choice correspondence, $C_j$.

Notice that we have not discussed the way that $i$ decides, since it is not the focus right now. Here we only define the case that $i$ influences $j$, but we simply assume that $i$'s decision outcomes can be represented by a choice correspondence as well.

Formally, let $\Omega_X$ be the set of all nonempty subsets of $X$. For an individual, say $k$, we define the decision outcomes of $k$ on $\Omega_X$ as a choice correspondence $C_k : \Omega_X \rightrightarrows X$ such that $\emptyset \neq C_k(A) \subseteq A$ for every $A \in \Omega_X$.

Given $C_i, C_j$ we say that $i$ **is influencing** $j$, if there exists an asymmetric and transitive $\succ_j$ such that $C_j(A) = Max(Max(A, \succ_j), \succ_j^i)$ for all $A \in \Omega_X$, where $\succ_j^i = \{xy : x = C_i(xy)\}$.

Now we are ready to investigate the main question of this section: Given the decision outcomes of the individuals $C_i, C_j$, what kind of properties on $C_j$ in relation to $C_i$ ensure that $i$ is influencing $j$?

[2] We abuse the notation a bit and denote an ordered pair as '$xy$' instead of '$\{x, y\}$'.

[3] Certainly there are alternative definitions for this second stage relation that summarizes what one observes from her social environment. We discuss several alternatives to $\succ_j^i$ in Concluding Remarks.

The first property we introduce is the classical expansion property. It states that if an alternative is chosen from two different sets, it has to be chosen from the union of them as well:

**(A1).** For any $x$, if $x \in C_k(A)$ and $x \in C_k(B)$, then $x \in C_k(A \cup B)$.

The remaining properties will allow us to identify the directed influence relation between $i$ and $j$. Hence they will combine the choice outcomes of the individuals. Before presenting them let us introduce a few definitions, that will ease the statement of the properties especially for larger networks structures that we will investigate in the following section.

For a given $C_k$, we say that an alternative $x$ is **strongly revealed preferred to** $y$, if the availability of $y$ has never changed the choice behavior of $k$ from a set that includes $x$: $C_k(A) = C_k(A \setminus y)$ for any $A \ni x$. Notice that this also implies that $x$ is uniquely chosen from the binary problem: $x = C_k(xy)$.

We say that an alternative $x$ is **weakly revealed preferred to** $y$ if although $x$ is chosen uniquely from the binary problem, we can find some set that includes $x$ and the inclusion of $y$ changes the choice behavior of $k$ from that set: $x = C_k(xy), C_k(A) \neq C_k(A \setminus y)$ for some $A \ni x$.

Finally, we say that a pair of alternatives $x, y$ **are not revealed preferred to each other** if neither of them is uniquely chosen from the binary problem: $xy = C_k(xy)$.

Given $C_k$, we define three mutually exclusive sets of binary pairs according to such revelations of $k$: The set of strong revelations of $k$, $S_k$; the set of weak revelations of $k$, $W_k$ and finally the set of binary pairs without a particular revelation, $I_k$, where

- $S_k = \{(xy) : x = C_k(xy), C_k(A) = C_k(A \setminus y) \text{ for any } A \ni x\}$.
- $W_k = \{(xy) : x = C_k(xy), C_k(A) \neq C_k(A \setminus y) \text{ for some } A \ni x\}$.
- $I_k = \{(xy) : xy = C_k(xy)\}$.

A second property, that is required to connect the two individuals, (A2), states that a weak revelation of $j$ cannot include alternatives that are not reveal preferred to each other by $i$:

**(A2).** $I_i \cap W_j = \emptyset$.

(A2) is simply a consistency type of property for influence: If $xy$ is a weak revelation of $j$, then although $j$ has chosen $x$ uniquely from the binary problem, she has not chosen consistently in some larger problem: availability of $y$ has changed $j$' s choice from a set that also includes $x$. Then the choice of $x$ from the binary problem is not because $j$

actually likes $x$ better than $y$, but because she has been influenced by $i$'s choice over this problem. Then obviously, $i$ cannot be indecisive for $x$ and $y$.

The last property, (A3), also connects the choice behaviors of the individuals:

**(A3).** $xy \in (W_i \cup S_i)$ and $A$ is such that for all $z \in A$, $zx \notin S_j$, then $y \notin C_j(A)$.

$xy \in (W_i \cup S_i)$ simply means that $i$ has chosen $x$ uniquely from the binary problem $x, y$. So if $j$ needs to refer to $i$'s behavior, choice of $x$ over $y$ is what she is going to observe. And $A$ is such a set that no other alternative is clearly better than $x$ for $j$: none of the alternatives are strongly revealed preferred to $x$, including $y$. Then, $y$ cannot be chosen from $A$. Otherwise, if $y$ is chosen from $A$, that would have been definitely due to $j$'s own preferences, since $i$ is behaving differently, but then $y$ would have been strongly revealed preferred to $x$ for $j$, which would be a contradiction.

Now we are ready to state the first result of this text. The previous three properties are necessary and sufficient for $j$ to be influenced by $i$:

**Theorem 3.1.** *Given $C_i$, $C_j$ satisfies A1, A2 and A3 if and only if $i$ is influencing $j$.*

## 3.3 Social Influence on a Network

Our main interest in this study is to uncover the structure of the interactions between individuals by evaluating their individual decision outcomes. The first section tells us that one way to understand if an individual is linked to another is to follow the choice inconsistencies in the choice data. If all the inconsistencies in the choice data of an individual $j$ can be traced back to another individual $i$, then it is safe to assume that $i$ is influencing $j$. In this section we make use of this observation to extend our analysis to larger networks.

Our strategy is to focus on different types of interactions that could occur in a network one by one. Afterall what we observe in a network in general is a possibly complicated structure of individuals influencing and being influenced by each other. There can be individuals that are only being influenced by one person as well as individuals that are being influenced by many others or influencing many others. In this section we analyze each of these cases separately. First we concentrate on the case where one individual influences many others around her. This suggests a star shaped set of interactions where the individual in the center is the one that has a direct influence on the choice outcomes of the others. Second we discuss exactly the opposite case, where an individual is being influenced by many others. We call this set of interactions an inverse star. Finally we focus on a chain, where there is actually an order of individuals such that each of them is influencing the one that comes after. Notice that in this case in addition to the direct influence of one on the individual that comes right after, there is the possibility of indirect influence. An individual may have an effect on the choice outcomes of another individual that is not directly linked to her, but being influenced by the individual that she influences.

Throughout this section we assume that each of the individuals is endowed with an individual choice correspondence $C_k$ over $X$. Our objective is to find out the properties on the choice data of the individuals that would allow us to specify the particular form of interaction that they are a part of: a star, an inverse star or a chain.

### 3.3.1 Star

We say that a group of individuals forms **a star** if there exists an individual $i$ that influences all the other individuals, $j_1, j_2, ..., j_n$.

Notice that we do not put any restriction to the choice behavior of the individual in the center, $i$. She may be getting influenced by someone in the group, or out of the group, or she may be an individual with complete preferences, without any room for being influenced. For a group of individuals to form a star, we only require that the center individual is influencing the others. We suggest that (A1), as a standard property, is satisfied by the choice correspondences of all of the individuals. The remaining properties will allow us to identify the star structure of their relations. Thus they are not about each $C_k$ but they will be combining different choice correspondences to each other, similar to (A2) and (A3) in the previous section. Indeed they will extend these two properties to the case of many individuals since a star is nothing but an individual $i$ that influences many other individuals, $j_1, j_2, ..., j_n$.

**(A2*).** For any pair of individuals $i, j$, we have $I_i \cap W_j = \emptyset$.

(A2*) is a generalization of (A2) to many individuals. Considering any pair of individuals, we have two options: Either one of them is in the center and the other is being influenced by her, or none is. In the former case, if the one in the periphery, is revealing $x$ weakly preferred to $y$, this means that she has been influenced to do so by the one in the center. Hence the center cannot be indecisive over this pair. If the one in the center is revealing $x$ weakly preferred to $y$, observing her, the one in the periphery will do the same, whenever she is susceptible to influence for this pair. And finally, if both of the individuals are in the periphery, and one of them is weakly revealing $x$ preferred to $y$, then the influence coming from the center will exactly be the same for the other periphery individual as long as she is susceptible to influence for this pair of alternatives.

**(A3*).** There exists an individual $i$ such that for any $j$ with $i \neq j$; $xy \in (W_i \cup S_i)$ and $A$ is such that for all $z \in A$, $zx \notin S_j$, then $y \notin C_j(A)$.

(A3*) implies (A3) for two individuals. The center of the star, $i$ has chosen $x$ uniquely from the binary problem $xy$. So if any other $j$ needs to refer to $i$'s behavior, choice of $x$ over $y$ is what she is going to observe. And $A$ is such a set that no other alternative is clearly better than $x$ for $j$. Then, $y$ cannot be chosen from $A$ for $j$.

**Theorem 3.2.** *$C_1, C_2, ..., C_n$ satisfy A1, A4 and A5 if and only if $1, 2, ..., n$ form a star.*

### 3.3.2 Inverse Star

Now we focus on the parts of the network where an individual is being influenced by many others around her. In this case we need to define an aggregation rule for the individual that is being influenced since not necessarily all the others around her always behave in the same way. Let us explain this point with a simple example: Let $j$ and $i_1, i_2, i_3$ be in the same social environment and assume that $j$ is linked to all the others. Consider $x, y$ uncompared according to the preferences of $j$. In order to be able to choose from $xy$, $j$ refers to her social environment and observes that $x = C_{i_1}(xy), y = C_{i_2}(xy)$ and $xy = C_{i_3}(xy)$. How would $j$ compare $x, y$ up on this observation? Naturally, individuals have different thresholds in order to accept support favoring one option convincing enough. In other words, for some individuals observing that a majority of their social environment is strictly deciding in favor of $x$ can be sufficient to go for option $x$, while for some others it will require stronger consensus. At this point, we suggest unanimity rule as a plausible aggregation rule that everbody will agree on: Being connected to many other individuals, if $j$ observes that everybody else is deciding in favor of $x$ from the binary problem, then she will definitely consider $x$ over $y$ in a second stage. Let us define this formally:

Given $C_{i_1}, C_{i_2}, ..., C_{i_n}$, we say that $i_1, i_2, ..., i_n$ **are influencing** $j$, if there exists an asymmetric and transitive $\succ_j$ such that:

$$C_j(A) = Max(Max(A, \succ_j), \succ_j^{i_1 - i_n}) \text{ for all } A \in \Omega_X,$$

where $\succ_j^{i_1 - i_n} = \{xy : x = C_i(xy) \text{ for all } i = i_1, i_2, ..., i_n\}$.

We say that a group of individuals forms **an inverse star** if there exists an individual $j$ that is being influenced by all the other individuals, $i_1, i_2, ..., i_n$.

Now let us present the properties that characterize the inverse star. In addition to (A1), we introduce two properties. The first of them, (A4), is a property that is actually satisfied by any choice correspondence that can be represented by a two stage maximization procedure where the first relation is transitive. And it is indeed implied by the properties (A2) and (A3) for the case of two individuals:

**(A4).** $xy \in (W_j \cup S_j)$ and $A$ is such that for all $z \in A$, $zx \notin S_j$, then $y \notin C_j(A)$.

(A4) basically states that if $x$ is chosen uniquely over $y$ from the binary problem and $A$ is a set that any other alternative in the set is not clearly better than $x$, then $y$ cannot be chosen from $A$. Choosing $x$ over $y$ means that either strongly or weakly $x$ is revealed to be a better alternative compared to $y$. Apparently $x$ is a good alternative in the set $A$ as well, since there is no other alternative that is strongly revealed preferred to $x$. The condition says that in this case, $j$ will not be choosing $y$.

Finally the following property is required to connect the choice data of the individuals to each other:

**(A5).** There exists $j$ such that $W_j \subseteq \bigcap_i (W_i \cup S_i)$ and $(\bigcap_i (W_i \cup S_i)) \cap I_j = \emptyset$.

Obviously $j$ is in the center of the inverse star. The first part of the property ensures that whenever $j$ chooses inconsistently with a choice from a binary problem, then this choice has to be justified by all the other individuals that she is connected to. (A5) is the only property that brings unanimity as the aggregation rule for $j$. If we consider different aggregation rules instead of unanimity, modifying (A5) will be sufficient for the characterization. For instance, if $j$ aggregates the information that she observes according to majority rule, then $W_j \subseteq \{xy : |i : x = C_i(xy)| > |i : y = C_i(yx)|\}$ would be the corresponding property.

The second part of (A5) makes sure that $j$ does not continue to be indecisive if everbody that she is connected to unanimously agree on the choice of a single alternative from a binary problem, ensuring that if $j$ is susceptible to influence and if there is sufficient information, then she must be influenced.

**Theorem 3.3.** *$C_1, C_2, ..., C_n$ satisfy A1, A4 and A5 if and only if $1, 2, ..., n$ form an inverse star.*

### 3.3.3 Chain

Finally, we analyze the parts of the networks that possibly combine different groups to each other. Up to now, all the structures that we have considered only included paths of lenght 1, where one individual directly influences some other. However in social networks, we do observe longer paths, in which individuals are connected to each other via other individuals. We call this kind of interaction structures as chains:

We say that a group of individuals form a **chain** if there exists an order of the individuals such that each individual $i$ influences the individual $j$ that comes after $i$.

The interesting feature of chains is that they allow for indirect influence between the individuals. For instance, consider three individuals $i, j, l$ such that $xy \in \succ_i$, but $xy, yx \notin (\succ_j \cup \succ_l)$, where $i$ is influencing $j$ and $j$ is influencing $l$. Then clearly $x = C_i(xy)$. Since $j$ is being influenced by $i$, $x = C_j(xy)$, allowing $l$ to copy it as well, yielding $x = C_l(xy)$. Hence, the comparison $xy$ coming from the preferences of $i$ may indirectly cause $l$ to choose in favor of $x$.

To present the characterizing property of chains, we first introduce a couple of definitions:

We say that **a pair** $i, j$ is **a team** with $i$ as the team leader if $i$ is influencing $j$.

In a chain, any consecutive pair refers to a team. Thus, we intoduce the following recursive definition of a team that is formed by a group of individuals:

We say that **a group** of $n$ individuals, $N_n$, defines **a team** if:

- There exists a team leader $i \in N_n$ such that $N_n \setminus \{i\}$ defines a team.
- $i$ and the team leader of $N_n \setminus \{i\}$ is a team.

This is indeed sufficient to capture the chain structure. Notice that if $i$ influences $j$ and $j$ influences $l$, then by definition of a team, $W_j \cap W_l$ actually captures the indirect influence of $i$ on $l$. Hence if we consider all the individuals of a team, we have $\bigcap_k W_k \subseteq (W_i \cup S_i)$, where $i$ is the leader of the team. In general, any common inconsistency that is shared by the choice behavior of individuals that are coming after one individual, say $i$, has to be justified by $i$'s behavior as well. Either $i$ makes the same inconsistency or she is the one that has this particular comparison in her initial preferences.

Once we are equipped with this recursive definition of influence, the characterizing property will simply say that the group of individuals we are considering forms a team:

**(A6).** $1, 2, ..., n$ define a team.

**Theorem 3.4.** *$C_1, C_2, ..., C_n$ satisfy A1 and A6 if and only if $1, 2, ..., n$ form a chain.*

## 3.4 Concluding Remarks

In this study we aimed to present a framework that allows us to exploit individual decision outcomes in order to understand the underlying structure of interactions between individuals of a network. This paper can be seen as a first attempt to apprehend the rich environment that this approach is to offer. Much has to be done yet.

First of all, we considered a very particular type of second stage relation to summarize the information that individual gathers from her social environment. An immediate alternative to this would be to directly consider the underlying preferences of the individuals as the second stage relation. In this case, individuals would not be getting influence as a result of observation but probably using more direct forms of communication such as asking for advice. We can easily show that slight modifications of the properties we suggested indeed would characterize the decision procedure in this case, and hence the specific network structures. However, notice that this modelling would not allow for indirect influence. Hence, the only influence one gets is what she obtains with directly asking to the people that she is in contact with. Moreover, if we consider that $i$ is influencing $j$, who is influencing $l$, the influence that $l$ gets from $j$ need not to be equal to how $j$, herself, behaves if she is getting influenced by $i$.

Another interesting alternative would be to allow the second stage relation not to be defined as a function of binary decision outcomes but to depend on the choice from any problem. In other words, once again we consider a two stage mechanism, but not a sequential elimination one. Instead, upon maximization of own preferences individual $j$ exactly copies the choice outcome of $i$ from that set of choosable alternatives; $C_j(A) = C_i(Max(A, \succ_j))$ for any $A$. This model suggests a very interesting decision procedure to investigate, however brings new complexities since it does not satisfy any known properties.

Apart from the discussion of how social influence should be integrated into individual decision making, there is much more to explore. For instance, we focused on special interaction structures that could appear on a social network separately. Bringing all the identified parts together and providing a more integrated approach is of immediate future work. We also need to investigate how accurate our analysis is in terms of identification. Obviously the social interactions revealed out of the choice data are not uniquely identified. Thus we shall explore the boundaries of identification strategies. And finally emprical application of the theory provided will open an interesting line of future research.

## 3.5 Appendix

**Proof of Theorem 3.1:** We only prove the sufficiency part since necessity is fairly easy. Given $C_i$, let $C_j$ satisfy A1, A2 and A3.

Define $\succ_j$ and $\succ'_j$ as follows:

$xy \in \succ_j$ whenever $xy \in S_j$ .

$xy \in \succ'_j$ whenever $xy \in W_j$.

Asymmetry of $\succ_j$ is obvious. To see transitivity of $\succ_j$, consider any $xy, yz \in \succ_j$ and consider any $A \in \Omega_X$ with $x, z \in A$. If $y \in A$, by $yz \in \succ_j$ we have: $C_j(A \setminus z) = C_j(A)$. Let $y \notin A$. But then, once again by $yz \in \succ_j$: $C_j((A \cup y) \setminus z) = C_j(A \cup y)$. Since $xy \in \succ_j$ implies $C_j(A \cup y) = C_j(A)$ and $C_j((A \cup y) \setminus z) = C_j(A \setminus z)$, we have $C_j(A \setminus z) = C_j(A)$, establishing transitivity of $\succ_j$.

We now show that $C_j(A) = Max(Max(A, \succ_j), \succ'_j)$. Then we will show that indeed $Max(Max(A, \succ_j), \succ'_j) = Max(Max(A, \succ_j), \succ^i_j)$ for $\succ^i_j = \{xy : x = C_i(xy)\}$.

Take any $A$ and any $x \in C_j(A)$. By definition, for any $y \in A$, $yx \notin S_j$ since $x \in C_j(A)$ implies $C_j(A) \neq C_(A \setminus x)$. Then, $x \in Max(A, \succ_j)$. Now assume for a contradiction that there exists $y \in Max(A, \succ_j)$ with $yx \in \succ'_j$; $yx \in W_j$. By A2, $yx \notin I_i$. Hence either $y = C_i(xy)$ or $x = C_i(xy)$. If $y = C_i(xy)$, since $x \in C_j(A)$, by A3, there must exist $z \in A$ such that $zy \in S_j$, contradicting with $y \in Max(S, \succ_j)$. Hence $x = C_i(xy)$. But then, again by A3, choosing $y$ from a set that $x$ exists implies that there has to be an element in that set that is strongly revealed preferred to $y$, i.e., $y = Cj(xy)$ implies that $yx \in S_j$, contradicting with $x \in Max(S, \succ_j)$. Hence, $x \in Max(Max(A, \succ_j), \succ'_j)$.

Now take any $x \in Max(Max(A, \succ_j), \succ'_j)$. Notice that $x \in C_j(xy)$ for all $y \in Max(A, \succ_j$ ), since $y = C_j(xy)$ implies $yx \in (W_j \cup S_j)$, which contradicts with maximality of $x$ in $A$ or in $Max(A, \succ_j)$. But then, by A1, $x \in C_j(Max(A, \succ_j)$. Notice that for any $z \in A \setminus (Max(A, \succ_j)$, there exists $t \in A$ such that $tz \in S_j$. But then by transitivity of $\succ_j$ and finiteness of $X$, there exists $w \in Max(A, \succ_j)$ such that $wz \in S_j$, and thus $C_j(S) = C_j(Max(A, \succ_j) \cup \{z\})$. Since this holds for any $z \in A \setminus (Max(A, \succ_j)$, $C_j(S) = C_j(Max(S, \succ_j)$.

Finally we show that $Max(Max(A, \succ_j), \succ'_j) = Max(Max(A, \succ_j), \succ^i_j)$. Since the first stage relation is the same for both, we only need to show that for $x, y \in Max(A, \succ_j)$, $xy \in \succ'_j$ if and only if $xy \in \succ^i_j$. First, notice $x, y \in Max(A, \succ_j)$, means that $xy \notin S_j$ and $yx \notin S_j$. Let $xy \in \succ'_j$; $xy \in W_j$. By A2, $yx \notin I_i$. Hence either $y = C_i(xy)$ or $x = C_i(xy)$. If $y = C_i(xy)$, since $x \in C_j(xy)$, by A3, $xy \in S_j$, contradicting with $y \in Max(S, \succ_j)$. Hence $x = C_i(xy)$, and hence, $xy \in \succ^i_j$.

Now let $xy \in \succ^i_j$, $x = C_i(xy)$. If $y \in C_j(xy)$, by A3, $yx \in S_j$, contradiction. Thus, $x = C_j(xy)$ and since $xy \notin S_j$, we have $xy \in W_j$; $xy \in \succ'_j$, concluding the proof. $\square$.

**Proof of Theorem 3.2:** We only prove the sufficiency. Let $C_1, C_2, ..., C_n$ satisfy A1, A2*and A3*. We only need to show that there exists an $i$ such that for any $j \in \{1, 2, ..., n\} \setminus \{i\}$, $i, j$ satisfy A2 and A3. Then by Theorem 1, we will be done. But this is immediate by A2*, A3* and by the definition of sets $S_k, W_k$ and $I_k$. $\square$.

**Proof of Theorem 3.3:** We only show sufficiency. Let $C_1, C_2, ..., C_n$ satisfy A1, A4 and A5. We will show that there exists $j$ such that $\{1, 2, ..., n\} \setminus \{j\}$ influence $j$. By A5, there exists $j$ such that $W_j \subseteq \bigcap_i (W_i \cup S_i)$.

Define $\succ_j$ and $\succ'_j$ as usual: $xy \in \succ_j$ whenever $xy \in S_j$ and $xy \in \succ'_j$ whenever $xy \in W_j$. Asymmetry and transitivity of $\succ_j$ is as already guaranteed in the Proof of Theorem 3.1.

We now show that $C_j(A) = Max(Max(A, \succ_j), \succ'_j)$. Then we will show that indeed $Max(Max(A, \succ_j), \succ'_j) = Max(Max(A, \succ_j), \succ_j^{i_1 - i_n})$ for $\succ_j^{i_1 - i_n} = \{xy : x = C_i(xy)$ for all $i\}$.

Take any $A$ and any $x \in C_j(A)$. By definition, for any $y \in A$, $yx \notin S_j$ since $x \in C_j(A)$. Then, $x \in Max(A, \succ_j)$. Now assume for a contradiction that there exists $y \in Max(A, \succ_j)$ with $yx \in \succ'_j$; $yx \in W_j$. Hence $y = C_j(xy)$. But since $x \in C_j(A)$, A4 implies that there exists $z \in A$ with $zy \in S_j$, contradicting with $y \in Max(S, \succ_j)$. Hence $x \in Max(Max(A, \succ_j), \succ'_j)$.

Now take any $x \in Max(Max(A, \succ_j), \succ'_j)$. By A1 and transitivity of $\succ_j$, $x \in C_j(S)$ can be shown as exactly the same way by using the arguements of the Proof of Theorem 3.1.

Finally we show that $Max(Max(A, \succ_j), \succ'_j) = Max(Max(A, \succ_j), \succ_j^{i_1 - i_n})$. Since the first stage relation is the same for both, we only need to show that for $x, y \in Max(A, \succ_j)$, $xy \in \succ'_j$ if and only if $xy \in \succ_j^{i_1 - i_n}$. First, notice $x, y \in Max(A, \succ_j)$, means that $xy \notin S_j$ and $yx \notin S_j$. Let $xy \in \succ'_j$; $xy \in W_j$. A5 directly implies $xy \in \succ_j^{i_1 - i_n}$.

Now let $xy \in \succ_j^{i_1 - i_n}$. If $y = C_j(xy)$, since $yx \notin S_j$, $yx \in W_j$. But then by A5 we contradict with $xy \in \succ_j^{i_1 - i_n}$. If $xy = C_j(xy)$, by the second part of A5 we contradict with $xy \in \succ_j^{i_1 - i_n}$. Thus, $x = C_j(xy)$ and since $xy \notin S_j$, we have $xy \in W_j$; $xy \in \succ'_j$, concluding the proof. $\square$.

**Proof of Theorem 3.4:** We only prove the sufficiency. Let $C_1, C_2, ..., C_n$ satisfy A1 and A6. By A6, there is a team leader, say $i$ and $i$ forms a team with the team leader of $\{1, 2, ..., n\} \setminus i$, say $j$. Then, by definition $i$ influences $j$. But then, application of the same argumentation recursively shows that there exists an order of individuals such that each individual influences the one that comes after. $\square$

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