CELLULAR APPROXIMATIONS OF INFINITE LOOP SPACES AND CLASSIFYING SPACES

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CERTIFICO que la present memòria ha estat realitzada per n'Alberto Gavira Romero, sota la direcció de la Dr. Natàlia Castellana Vila.

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Signat: Dr. Natàlia Castellana Vila .

A mis abuelos.

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Preface

Dennis Sullivan conjectured in [Sul71] that if X is a finite CW-complex, then the pointed mapping space $\mathrm{map}_*(BG, X)$ is contractible for any locally finite group G. This problem was solved by Haynes Miller in [Mil84]. The consequent works of W. Dwyer and A. Zabrodsky ([DZ87]), J. Lannes ([DL99]) and others show the importance of the study of the previous mapping space in understing homotopical properties of X that can be detected by maps from a classifying space of a finite p-group. In this way, E. Dror-Farjoun introduced in [Far96] the notion of A-homotopy theory for an arbitrary connected space A. In this theory A and its suspensions play the role of the spheres in classical homotopy theory. Hence, the A-homotopy groups of a space X are defined to be the homotopy classes of pointed maps $\pi_i(X; A) := [\Sigma^i A, X]_*$. The classical notion of CW-complex is replaced by the one of A-cellular space, that is, spaces that can be constructed from A by means of pointed homotopy colimits. The analogue to (weakly) contractible spaces are those spaces for which all A-homotopy groups are trivial, this means, the pointed mapping space $\mathrm{map}_*(A, X)$ is contractible, these spaces are called A-null spaces.

Thanks to work of A. K. Bousfield ([Bou94]) and E. Dror-Farjoun ([Far96]) there is a functorial way to study X throughh the eyes of A: the nullification functor P_A and the cellularization functor CW_A. Roughly speaking, the A-nullification of a space X is the biggest quotient of X which is A-null and $CW_A(X)$ is the best A-cellular approximation of X, in this sense, $CW_A(X)$ contains all the trascendent information of the mapping space $\mathrm{map}_*(A, X)$, since the latter is equivalent to $\mathrm{map}_*(A, CW_A(X))$.

While many computations of $P_A(X)$ are present in the literature (see for instance, [Bou94] or [Far96]), very few computations of $CW_A(X)$ are available. W. Chachólski describes a strategy to compute the cellularization $CW_A(X)$ in [Cha96]. His method has been successfully applied in some cases to obtain explicit computations or qualitative information: cellularization with respect to Moore spaces ([RS01]), $B\mathbb{Z}/p$-cellularization of classifying spaces of finite groups ([Flo07], [FS07] and [FF11]), of Postnikov pieces ([CCS07]) and of classifying spaces of compact Lie groups ([CF13]).

In this work, we compute the $B\mathbb{Z}/p^m$-cellularization of two families of spaces: $\Sigma B\mathbb{Z}/p$-acyclic spaces up to p-completion (e.g.: infinite loops spaces with some technical conditions) and classifying spaces of p-local compact groups.

Below we summarize briefly the work done in this memory, and we refer the reader to each chapter for further details on a specific subject.

The first chapter introduces the notion of A-nullification in two ways: as it is constructed in [Bou94] by A. K. Bousfield and as a particular case of the localization with respect to a map given in [Far96] by E. Dror-Farjoun. Then, and since homological localizations are also localizations with respect to a map, we start the chapter with a section about localizations with respect to a map, where we list some of its properties and basic results. The last two sections are devoted to original work about relationships between the nullification functor and R-completion functors (in the sense of Bousfield-Kan) and homological localizations. Then, in the third section we present a generalization of some results that appear in [CF13] about how to commute the nullification and completion functors under favorable conditions. The last section of this chapter is centralized in comparing the nullification functor and homological localizations. Basically, in this section we continue the work in the comparison of these functors that presented in [Dwy96] and [CF13]. The main result of this section is

Theorem 1.4.2. *Let R be a subring of $\mathbb{Q}$. Let X be a 1-connected space and let A be a connected $H_*(-;R)$-acyclic space. If $\mathcal{P}$ denotes the set of divisible primes of R. Then there exists a fibration*

$$F \to P_A(X) \to L_R(X),$$

where F is the homotopy fibre of $\prod_{p\in\mathcal{P}}(P_A(X))^\wedge_p \to (\prod_{p\in\mathcal{P}}(P_A(X))^\wedge_p)_{\mathbb{Q}}$.

The second chapter is devoted to A-homotopy and the A-cellular functor, with emphasis when $A = BG$, where G is a discrete group. Another functors used to isolate properties detecting by $B\mathbb{Z}/p$ is the Bousfield-Kan p-completion. It is important to undestand how these two functors commute. That is:

Proposition 2.2.4. *Let X be a connected nilpotent space and G a finite abelian group. Then,*

(i) If X is 1-connected, then the map

$$CW_{BG}(\eta_X)\colon CW_{BG}(X) \to CW_{BG}(X^\wedge_p)$$

is a mod p equivalence for all prime p.

(ii) For all prime $p \mid |G|$, $CW_{BG}(X^\wedge_p) \simeq CW_{BG_p}(X^\wedge_p)$, where G_p is the p-torsion component of G.

One of our goal is to give description as explicit as possible in terms of other localization functor. In this way we have

Theorem 2.3.1. *Let X be a 1-connected space and let p be a prime and $r \geq 0$. Then the $B\mathbb{Z}/p^r$-cellularization of X fits in a fibration sequence*

$$\Omega F \to CW_{B\mathbb{Z}/p^r}(X) \to \overline{L}_{\mathbb{Z}[\frac{1}{p}]}(X),$$

where $\overline{L}_{\mathbb{Z}[\frac{1}{p}]}X$ is the homotopy fibre of the coaugmention map $X \to L_{\mathbb{Z}[\frac{1}{p}]}(X)$ and F is the homotopy fibre of $(P_{\Sigma B\mathbb{Z}/p^r}(C))^\wedge_p \to ((P_{\Sigma B\mathbb{Z}/p^r}(C))^\wedge_p)_{\mathbb{Q}}$, where C is the Chachólski's cofibre.

Each of the following chapters introduces original work on the study of cellularization of some spaces. Where, for instance, we present some cases such the cellularization is the homotopy fibre of the rationalization, solving partially the question exposed in [CF13]: For which class of classifying spaces of compact Lie groups (or spaces in general) is the $B\mathbb{Z}/p$-cellularization equivalent to the homotopy fibre of the rationalization, up to p-completion?

The third section studies, given a finite abelian group G, the BG-cellularization of $\Sigma B\mathbb{Z}/p$-acyclic up to p-completion spaces, that is, spaces X such $P_{\Sigma B\mathbb{Z}/p}(X)^\wedge_p$ is contractible. Recall that E. Dror-Farjoun proves in [Far96] that if a space X verifies $P_{\Sigma A}(X) \simeq *$, then X is A-cellular. Then, in this chapter, we study a p-local version of the Dror-Farjoun's result. That is, if X is a 1-connected space such that $P_{\Sigma B\mathbb{Z}/p}(X)^\wedge_p \simeq *$, then the augmention map $CW_{B\mathbb{Z}/p}(X) \to X$ is a mod p equivalence. This allows us, using the results developed in Section 2.3, the following theorem:

Theorem 3.1.1. *Let X be a 1-connected space. Let p be a prime number. If $(P_{\Sigma B\mathbb{Z}/p^s}(X))^\wedge_p \simeq *$ for some $s \geq 1$, then $CW_{B\mathbb{Z}/p^r}(X)$ has the homotopy type of the homotopy fibre of $X^\wedge_p \to (X^\wedge_p)_\mathbb{Q}$ for all $r \geq 1$.*

As examples of $\Sigma B\mathbb{Z}/p$-acyclic up to p-completion spaces we have the 1-connected infite loop spaces E such that $\pi_2 E$ is a torsion group, thanks to [McG97, Theorem 2]. Therefore in the second section we get that the $B\mathbb{Z}/p^r$-cellularization of an infinite loop spaces E as above is (weak) equivalent to the homotopy fibre of the rationalization $E^\wedge_p \to (E^\wedge_p)_\mathbb{Q}$ (see Corollary 3.2.4). The third section is devoted to present two consequence of the above result that complete previous results from [CCS07]. Specifically, in [CCS07] the authors, on the one hand, prove that a Postnikov piece is $B\mathbb{Z}/p^m$-cellular if and only if it is p-torsion and, on the another hand, compute the cellularization of certain infinite loop spaces related with K-theories with respect to $K(\mathbb{Z}/p, m)$ for all $m \geq 2$. In this sense, we prove that the $B\mathbb{Z}/p^m$-cellularization of a 1-connected Postnikov piece X which $\pi_2 X$ is a torsion group is equivalent to the homotopy fibre of $X^\wedge_p \to (X^\wedge_p)_\mathbb{Q}$, and we compute the $B\mathbb{Z}/p^m$-cellularization of the mentioned infinite loop spaces related with K-theories.

The fourth chapter deals with the cellularization of classifying spaces of p-local compact groups, with emphasis in the finite case and in the compact connected Lie group, where we get stronger progress. This chapter is organized as follows. First section is devoted to compute specifically the A-cellularization of a classifying space of a finite p-group, where A is such that $\pi_1 A$ is a finitely generated abelian group. In the second section the study is focused in the B_{p^m}-cellularization of classifying spaces of discrete p-toral groups, where $B_{p^m} = B\mathbb{Z}/p^\infty \times B\mathbb{Z}/p^m$. More concretely, we compute the $B\mathbb{Z}/p^m$ and $B\mathbb{Z}/p^\infty$-cellularization of these classifying spaces and we get the following existence result:

Proposition 4.2.5. *Let P be a discrete p-toral group. Then there is a non-negative integer m_0 such that BP is B_{p^m}-cellular for all $m \geq m_0$.*

The previous integer m_0 depends on the order of the generators of the group of components of P lifted to P.

The third section is divided in four subsection. In the fisrt one we introduces the notion of kernel of a map $f\colon |\mathcal{L}|^\wedge_p \to Y^\wedge_p$, where $|\mathcal{L}|^\wedge_p$ is the classifying spaces of a p-local compact group and $Y^\wedge_p$ is a p-complete and $\Sigma B\mathbb{Z}/p$-null space, following ideas of D. Notbohm ([Not94]). Given a p-local compact group $(S, \mathcal{F}, \mathcal{L})$, we define $\ker(f) := \{g \in S \mid f|_{B\langle g\rangle} \simeq *\}$. We prove in this section that $\ker(f)$ is a normal subgroup of S and, in addition, it is strongly $\mathcal{F}$-closed. The main result of this subsection is to prove that $f\colon |\mathcal{L}|^\wedge_p \to Y^\wedge_p$ is null-homotopic if and only if $\ker(f) = S$ (Theorem 4.3.16). Therefore, in the second subsection, we apply the thecnology of the kernel with the map $r^\wedge_p : |\mathcal{L}|^\wedge_p \to P_{\Sigma B_{p^m}}(C)^\wedge_p$. Hence we get that for A a classifying space of the type $B\mathbb{Z}/p^m$ or B_{p^m}, and under good conditions over $|\mathcal{L}|^\wedge_p$, if $\ker(r^\wedge_p) = S$, then $CW_A(|\mathcal{L}|^\wedge_p)$ is equivalent to the homotopy fibre of $|\mathcal{L}|^\wedge_p \to (|\mathcal{L}|^\wedge_p)_{\mathbb{Q}}$ (see Theorem 4.3.19 and Corollaries 4.3.20 and 4.3.21). We improve this result in the fourth subsection in the finite case. More concretely, if Cl_{p^m} denotes the smallest strongly $\mathcal{F}$-closed subgroup of S that contains all it p^m-torsion, then we prove:

Theorem 4.3.29. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Then $|\mathcal{L}|^\wedge_p$ is $B\mathbb{Z}/p^m$-cellular if and only if $S = Cl_{p^m}(S)$.*

This improvement with respect to the general case is given from [CL09, Theorem 1,2], this theorem is only proved in the finite case and the key that allows us to prove the theorem. To close this subsection we finalize with example. In particular, we prove that, for all $i \geq 0$, the $B\mathbb{Z}/p^i$-cellularization of $BG^\wedge_p$, when G is a finite group such that the normalizer of a Sylow p-subgroup controls fusion in G, is equivalent to the homotopy fibre of the map $BG^\wedge_p \simeq BN_G(S)^\wedge_p \to B(N_G(S)/Cl_{p^i}(S))^\wedge_p$.

Finishing this section, the fourth subsection is devoted to the particular case of a p-completion of the classifying space of a compact connected Lie group. We prove the "good conditions over $|\mathcal{L}|^\wedge_p$" are not necessary in this case, using the classifying of compact connected Lie group developed by E. Cartan. Furthermore, by the rational structure of a compact Lie group, we conclude that for any compact connected Lie group G there is a integer $m_0 \geq 1$ such that $BG^\wedge_p$ is $K(\mathbb{Q} \times \mathbb{Z}/p^\infty \times \mathbb{Z}/p^m, 1)$-cellular for all $m \geq m_0$ (Corollary 4.3.48). Moreover, the computation of the $B\mathbb{Z}/p^m$-cellularization of $BG^\wedge_p$ is described for all $m \geq 0$, this comes from Theorem 1.5 in [Not94] (see Theorem 4.3.54). In particular, if G is a compact 1-connected simple Lie group, then we get for every $m \geq 1$ that the $B\mathbb{Z}/p^m$-cellularization of $BG^\wedge_p$ is equivalent to the homotopy fibre of the rationalization $BG^\wedge_p \to (BG^\wedge_p)_{\mathbb{Q}}$ in most cases (Proposition 4.3.55).

Te appendix at the end contain notions, definitions and thecnical results about Bousfield-Kan R-completion and homological localizations needed all along this memory. We have chosen to place it at the end due to the extension of the contents.

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Chapter 1

Localizations

In this chapter we introduce certain localizations in homotopy theory, with emphasis in localizations with respect to a map, and list the properties which will be used in the rest of this work. The notions and results are taken from differents sources of A.K. Bousfield ([Bou94]), G. Mislin ([Mis78]) and E. Dror-Farjoun ([Far96]).

More specifically, the first section is devoted to localization with respect to a map, where we will explain certain properties which we will use repeatedly in later sections and chapters. In the second section we present one of the most important examples of localization with respect to a map, the nullification with respect to a space. The third section is centered to describe the relations of nullifications with the Bousfield-Kan p-completion (described in [BK72]). Finally, the fourth section contains a result about when nullifications are homological localizations (described in [Bou75]). We will use this result in Chapter 3 to describe the cellularization of infinite loop spaces.

In this chapter we do not provide proofs of results already proved in other sources for the sake of simplicity. The reader is then refered to the corresponding source for further details.

1.1 Localizations with respect to a map

Given a cofibrant map f between cofibrant spaces (e.g.: a map between CW-complexes), a f-local space is a topological space such that the induced map in mapping spaces is a weak equivalence. Moreover there exists a functor which turn any space into a f-local one. A special case, when the map f is null-homotopic, has particularly pleasant properties and is called nullification (see Section 1.2).

Definition 1.1.1 ([Far96, Definition 1.A.1])**.** Let $f\colon A \to B$ be a map between cofibrant spaces. We say that a fibrant space X is *f-local* if f induces a weak homotopy equivalence on function complexes,

$$\mathrm{map}(f,X)\colon \mathrm{map}(B,X) \xrightarrow{\simeq} \mathrm{map}(A,X).$$

Remark 1.1.2. Note that from the fibration $\mathrm{map}_*(V,X) \to \mathrm{map}(V,X) \to X$ we get that if $f\colon A \to B$ is a pointed map then $\mathrm{map}(f,X)$ is an equivalence if and only if so is $\mathrm{map}_*(f,X)$ for each choice of the base point.

Now we are interested in the existence of a functor that turn an arbitrary space into a f-local one. For this, we have to introduce certain definitions for a functor $F\colon \mathbf{Top} \to \mathbf{Top}$.

Definition 1.1.3 ([Far96, Definition 1.A.2])**.** Let $F\colon \mathbf{Top} \to \mathbf{Top}$ be a functor. Then,

(a) F is *coaugmented* if it comes with a natural transformation $\eta\colon Id \to F$, this means, for each $X \in \mathbf{Top}$ there is a map $\eta_X\colon X \to F(X)$ (it is called the *coaugmentation map*) and for all morphism $g\colon X \to Y \in \mathbf{Top}$, we obtain the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & F(X) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle F(g)} \\ Y & \xrightarrow{\eta_Y} & F(Y) \end{array}$$

(b) If F is a coaugmented functor we say that F is *idempotent* if both natural maps:

$$F(X) \underset{F(\eta_X)}{\overset{\eta_{F(X)}}{\rightrightarrows}} F(F(X))$$

are weak equivalences and are homotopic to each.

(c) The coaugmentation map η_X is said to be *homotopy universal with respect to f-local spaces* if any map $g\colon X \to Y$ into a f-local space Y factors up to homotopy through $\eta_X\colon X \to F(X)$ and the factorization is unique up to homotopy, i.e., there is a map $\tilde{g}\colon F(X) \to Y$ such that the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad g \quad} & Y \\ & {\scriptstyle \eta_X}\searrow \quad \nearrow{\scriptstyle \tilde{g}} & \\ & F(X) & \end{array}$$

is commutative up to homotopy, and if there is another map $h\colon F(X) \to Y$ such that $h \circ \eta_X \simeq g$ then $h \simeq \tilde{g}$.

In this way, E. Dror-Farjoun proves the following theorem:

Theorem 1.1.4 ([Far96, Theorem 1.A.3])**.** *For any map $f\colon A \to B$ in* $\mathbf{Top}$ *(or* $\mathbf{Top}_*$*) there exists a functor $L_f\colon$* $\mathbf{Top} \to \mathbf{Top}$ *(or from* $\mathbf{Top}_*$ *to* $\mathbf{Top}_*$*), called the f-localization functor, which is coaugmented and idempotent. Moreover $L_f(X)$ is f-local and the coaugmentation map $X \to L_f(X)$ is homotopy universal with respect to f-local spaces.*

Remark 1.1.5. Note that in the definition of localization with respect to a map $f\colon A \to B$, the map and the spaces need be cofibrants. Furthermore, to verify the universal property of localization the space X must be fibrant. In the case that they are not cofibrant, taking a cellular approximation of this map we get a cofibrant map between cofibrants spaces which is homotopy equivalent to the initial map. Similarly, if X is not a fibrant space then take the cellular approximation of X to get a fibrant one with the same homotopy type.

The main example of f-localization is given by localization with respect to maps of the form $A \to *$, or $* \to A$. In this case $L_{A\to *} = L_{*\to A}$ is called the A-nullification or A-periodization functor and it is denoted by P_A, because it is related to the Postnikov section functor, but we will talk about nullification with more details in Section 1.2.

Another important example is given by homological localization, see Section A.2.

Example 1.1.6. Let p be a odd prime number and let $M^n(\mathbb{Z}/p)$ denote the Moore space with a top cell at dimension n. Homotopically we can cosider the n-th mod p homotopy group $\pi_n(X;\mathbb{Z}/p) := [M^n(\mathbb{Z}/p), X]_*$ and the v_1-periodicity operator induced by the Adam's map $v_1\colon M^{n+q}(\mathbb{Z}/p) \to M^n(\mathbb{Z}/p)$ where $q = 2p - 2$ (see [CN86] for more details). A v_1-periodic space, naively speaking, is a space for which v_1 induces an isomorphism on mod p homotopy. The functor L_{v_1} turn every space into a v_1-periodic one.

Now we want to list properties of f-localizations which we will use frequently. First, certain immedate consequences of the definition, universality, idempotency and the above theorem given by E. Dror-Farjoun:

Proposition 1.1.7 ([Far96, 1.A.8])**.** *Let $f\colon A \to B$ be a map. Then:*

(i) If T is f-local, then for all X both map(X, T) and map$_*(X, T)$ are f-local for any choice of base point. In particular, if T is f-local then so is $\Omega^n T$ for all $n \geq 0$.

(ii) The natural map $L_f(X \times Y) \to L_f(X) \times L_f(Y)$ has a homotopy inverse and thus is a homotopy equivalence.

*(iii) A connected space $T \in$ **Top**$_*$ is local with respect to $\Sigma f\colon \Sigma A \to \Sigma B$ if and only if ΩT is f-local.*

(iv) If T is f-local, then it is also $\Sigma^k f$-local for all $k \geq 0$.

(v) $L_f X \simeq *$ if and only if for any f-local space P one has map$_*(X, P) \simeq *$.

Another important property of f-localization is that the coaugmentation map is not only universal respect to f-local spaces, it is universal also in certain class of maps: the f-local equivalences or, sometimes called L_f-equivalence:

Definition 1.1.8. A map $g\colon X \to Y$ is called *f-local equivalence* or *L_f-equivalence* if it satisfies one of the two equivalent conditions:

(a) For all f-local space T the induced map map(g, T) is an equivalence.

(b) The map g induces a homotopy equivalence $L_f(g)\colon L_f(X) \to L_f(Y)$.

And in this sense is proved:

Proposition 1.1.9 ([Far96, Proposition 1.C.6])**.** *For any map $g\colon X \to Y$ which is an L_f-equivalence there is an extension $X \to Y \xrightarrow{l} L_f(X)$ that is unique up to homotopy with $l \circ g \simeq \eta_X\colon X \to L_f(X)$.*

About homotopy colimits, the functor L_f does not commute with them in general. But we get the following result:

Theorem 1.1.10 ([Far96, Theorem 1.D.3]). *Given a small category I and a diagram over it $\tilde{X}\colon I \to$ **Top**$_*$, the natural map obtained by applying f-localization to the coaugmentation map: $L_f(a)\colon L_f(\mathrm{hocolim}_{*I}\tilde{X}) \to L_f(\mathrm{hocolim}_{*I} L_f(\tilde{X}))$ is a weak homotopy equivalence. Moreover, there is a natural map $c\colon \mathrm{hocolim}_{*I} L_f(\tilde{X}) \to L_f(\mathrm{hocolim}_{*I}\tilde{X})$ such that $L_f(c)$ is an inverse to $L_f(a)$ and thus it is a homotopy equivalence. (The same is true for unpointed homotopy colimits).*

Corollary 1.1.11 ([Far96, Example 1.D.5]). *Let $f\colon A \to B$ a map. Then:*

(i) For any pointed spaces X, Y one has $L_f(X \vee Y) \simeq L_f(L_f(X) \vee L_f(Y))$.

(ii) For any pointed spaces X, Y one has $L_f(X \wedge Y) \simeq L_f(L_f(X) \wedge L_f(Y))$.

(iii) In any cofibration sequence $X \rightarrowtail Y \xrightarrow{g} Y/X$, if $L_f(X) \simeq *$ then $L_f(g)\colon L_f(Y) \to L_f(Y/X)$ is a weak equivalence.

About the case of localizations with respect to a map and fibrations, there is a section devoted to this in [Far96] (Section 3.D). Nevertheless, we want to state the following result that will use in later chapter:

Theorem 1.1.12 ([Far96, Theorem 1.H.1]). *If $F \to E \xrightarrow{p} B$ is a fibration and $L_f(F) \simeq *$, then $L_f(p)\colon L_f(E) \to L_f(B)$ is a homotopy equivalence.*

Finally, the commutation rule for the f-localization and the loop functor described in the following theorem is very useful in many situation.

Theorem 1.1.13 ([Far96, Theorem 3.A.1]). *Let $f\colon A \to B$ be any map in* **Top**$_*$ *and $X \in$* **Top**$_*$ *a connected space. There is a natural homotopy equivalence*

$$L_f(\Omega X) \simeq \Omega L_{\Sigma f}(X).$$

1.2 Nullifications

Given a connected space A, the concept of A-nullification was introduced by A.K. Bousfield in [Bou94] with the name of A-periodization and, independently, by E. Dror-Farjoun in the sense of f-localization for maps of the form $A \to *$ (or $* \to A$). Roughly speaking, the A-nullification of a space X is the biggest quotient of X in which all the information from A and its suspensions is killed.

Definition 1.2.1. Let A and X be spaces. X is called A-null if the map $A \to *$ induces a weak equivalence $\mathrm{map}(A, X) \to X$. If A and X are pointed spaces and X is connected this condition is equivalence to $\mathrm{map}_*(A, X) \simeq *$.

Note that the A-null space are the f-local space which $f\colon A \to *$ (or $f\colon * \to A$) described in Section 1.1.

Remark 1.2.2. If X is connected the condition $\mathrm{map}_*(A, X) \simeq *$ is equivalence to $[\Sigma^i A, X]_* \cong 0$ for all $i \geq 0$.

Example 1.2.3. If $A = S^n$ then a pointed space X is S^n-null if and only if $\pi_i(X) \cong 0$ for all $i \geq n$.

Example 1.2.4. One of the most important examples of null spaces is given by the Sullivan conjeture solved by Miller in [Mil84, Theorem A]. This theorem says that if G is a discrete group which is locally finite and X is a connected finite CW-complex, then $\mathrm{map}_*(BG, X) \simeq *$. This is means, any connected finite CW-complex X is BG-null for all discrete locally finite group G.

As in Section 1.1 we are interested in a functor that turns a space into an A-null one. Although this is a direct consequence of Theorem 1.1.4, it is also proved by Bousfield in [Bou94]:

Theorem 1.2.5 ([Far96, Theorem 1.A.3], [Bou94, Theorem 2.10])**.** *For any connected space A there exists a functor $P_A\colon$ **Top**$_* \to$ **Top**$_*$, called the A-nullification functor, which is coaugmented and idempotent. Moreover $P_A(X)$ is an A-null space and the coaugmentation map $\eta_X\colon X \to P_A(X)$ is homotopy universal with respect to A-null spaces, this means, if Y is an A-null space and $f\colon X \to Y$ is a pointed map then there is a map $\tilde{f}\colon P_A(X) \to Y$ such that the following diagram*

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ & {}_{\eta_X}\searrow \quad \nearrow_{\tilde{f}} & \\ & P_A(X) & \end{array}$$

is commutative up to homotopy, and if there is another map $g\colon P_A(X) \to Y$ such that $g \circ \eta_X \simeq f$ then $g \simeq \tilde{f}$.

The notation P_A is derived from the following classical example:

Example 1.2.6. If $A = S^n$, then the functor P_{S^n} is P_{n-1}, the $(n-1)$-th Postnikov section functor.

This last functor allows one to introduce an interesting partial order on spaces, the spaces that are "killing" by A:

Definition 1.2.7. We say that a space Y is *A-acyclic* if $P_A(Y) \simeq *$. Hence we write $A < Y$.

Remark 1.2.8. This relation is indeed a "weak" partial order relation in **Top**, in the sense that not verfies the antisymmetric property:

1. Reflexivity: By definition, for all A-null space P we have $\mathrm{map}_*(A, P) \simeq *$, according to Proposition 1.1.7.(v) we get $P_A(A) \simeq *$.

2. Transitivity: Assume that $P_A(B) \simeq *$ and $P_B(C) \simeq *$. By Theorem 1.2.5, an A-null space P is B-null, because $\mathrm{map}_*(B, P) \simeq \mathrm{map}_*(P_A(B), P) \simeq \mathrm{map}_*(*, P) \simeq *$. Hence, by the same theorem, $\mathrm{map}_*(C, P) \simeq \mathrm{map}_*(P_B(C), P) \simeq \mathrm{map}_*(*, P) \simeq *$, and Proposition 1.1.7.(v) shows that $P_A(C) \simeq *$.

3. Not antisymmetry: $X \vee X < X$ and $X < X \vee X$ but, in general, $X \not\simeq X \vee X$.

In this way we define the *A-nullity class* as the class of all A-null space, and as if $A < B$ then the A-nullity class in contained in the B-nullity class, we say that X and Y *have the same nullity class* if $X < Y$ and $Y < X$. For instant $\bigvee_{i=1}^{n} X$ and X have the same nullity class for all n.

There is an alternative construction of the space $P_A(X)$ given by Bousfield in [Bou94, 2.8]:

Construction of $P_A(X)$. 1.2.9. Let X and A be pointed connected CW-complexes, the idea is construct inductively an increasing sequence of CW-complexes:

$$X = X_0 \subset X_1 \subset \ldots \subset X_\alpha \subset X_{\alpha+1} \subset \ldots,$$

as follows. Given X_α, let $X_{\alpha+1}$ be the mapping cone of the evaluation map

$$ev\colon \bigvee_{\substack{i\geq 0\\ [\Sigma^i A, X_\alpha]_*}} \Sigma^i A \to X_\alpha.$$

Then $P_A(X) \simeq \varinjlim_{\alpha} X_\alpha$.

A consequence of this construction is:

Proposition 1.2.10 ([Bou94, Proposition 2.9])**.** *For X and A pointed connected spaces with A n-connected, the homomorphism $\pi_i(\eta_X)\colon \pi_i X \to \pi_i P_A(X)$ is bijective for $i \leq n$ and onto for $i = n+1$.*

And a direct corollary is:

Corollary 1.2.11. *Let X and A be pointed connected spaces. Then $\pi_1(\eta_X)\colon \pi_1 X \to \pi_1 P_A(X)$ is an epimorphism. In particular, if X is* 1*-connected then so is $P_A(X)$.*

From this construction is also easy to show the analogous property in homology with coefficients $\mathbb{Z}$:

Proposition 1.2.12. *For X and A pointed connected spaces with A n-connected, the homomorphism $H_i(\eta_X;\mathbb{Z})\colon H_i(X;\mathbb{Z}) \to H_i(P_A(X);\mathbb{Z})$ is bijective for $i \leq n$ and onto for $i = n+1$.*

Proof. Let X_α and $X_{\alpha+1}$ be as in the construction 1.2.9. Hence we have the cofibration sequence:

$$\bigvee_{\substack{i\geq 0\\ [\Sigma^i A, X_\alpha]_*}} \Sigma^i A \to X_\alpha \to X_{\alpha+1}$$

By the exactness axiom in homology we get the following long exact sequence of abelian groups:

$$\ldots \to H_i(\bigvee_{\substack{i\geq 0\\ [\Sigma^i A, X_\alpha]_*}} \Sigma^i A;\mathbb{Z}) \to H_i(X_\alpha;\mathbb{Z}) \to H_i(X_{\alpha+1};\mathbb{Z}) \to H_{i-1}(\bigvee_{\substack{i\geq 0\\ [\Sigma^i A, X_\alpha]_*}} \Sigma^i A;\mathbb{Z}) \to \ldots$$

From this sequence, for all $i \leq n$ we obtain the exact sequence,

$$\ldots \to 0 \to H_i(X_\alpha;\mathbb{Z}) \to H_i(X_{\alpha+1};\mathbb{Z}) \to 0 \to \ldots$$

because $H_i(\bigvee_{\substack{i\geq 0 \\ [\Sigma^i A, X_\alpha]_*}} \Sigma^i A;\mathbb{Z}) \cong \pi_i(\bigvee_{\substack{i\geq 0 \\ [\Sigma^i A, X_\alpha]_*}} \Sigma^i A)$ by Hurewicz's theorem and A is n-connected. This means, $H_i(X_\alpha;\mathbb{Z}) \cong H_i(X_{\alpha+1};\mathbb{Z})$ for all α and all $i \leq n$, hence

$$H_i(P_A(X);\mathbb{Z}) = H_i(\varinjlim_\alpha X_\alpha;\mathbb{Z}) \cong \varinjlim_\alpha H_i(X_\alpha;\mathbb{Z}) \cong \varinjlim_\alpha H_i(X_0;\mathbb{Z}) = H_i(X;\mathbb{Z}), \text{ for all } i \leq n.$$

For $i = n+1$, we get for all α:

$$\ldots \to H_{n+1}(X_\alpha;\mathbb{Z}) \twoheadrightarrow H_{n+1}(X_{\alpha+1};\mathbb{Z}) \to 0 \to \ldots$$

and hence $H_{n+1}(X;\mathbb{Z}) = H_{n+1}(X_0;\mathbb{Z}) \twoheadrightarrow H_{n+1}(X_\alpha;\mathbb{Z})$ for all α, and finally we have

$$H_{n+1}(X;\mathbb{Z}) = \varinjlim_\alpha H_{n+1}(X;\mathbb{Z}) \twoheadrightarrow \varinjlim_\alpha H_{n+1}(X_\alpha;\mathbb{Z}) \cong H_{n+1}(P_A(X);\mathbb{Z}).$$

□

Obviously, the A-nullification functor verifies all the properties listed in Section 1.1. Moreover P_A verifies the following assumption on fibration:

Proposition 1.2.13 ([Far96, Corollary 3.D.3(2)]). *Let A be a pointed connected space and let $F \to E \to B$ be a fibration over a connected B. If B is A-null, then P_A preserves the fibration, i.e., $P_A(F) \to P_A(E) \to P_A(B) \simeq B$ is a fibration sequence.*

1.3 Commuting nullification and R-completion

Let R be a ring and let R_∞ denote the R-completion functor of Bousfield-Kan (see Section A.1 for more details). The functors P_A and R_∞ do not commute in general. This means, in general $P_A(R_\infty(X)) \not\simeq R_\infty(P_A(X))$. In this section we will give some properties about nullifications and completions. We start with a condition about when the nullification map is a R-equivalence.

Proposition 1.3.1. *Let R be a ring. Let X and A be pointed connected spaces. If $\tilde{H}_*(A;R) \cong 0$ then $H_*(\eta_X;R)\colon H_*(X;R) \to H_*(P_A(X);R)$ is an isomorphism. Hence the coaugmentation map induces a homotopy equivalence $R_\infty(\eta_X)\colon R_\infty(X) \to R_\infty(P_A(X))$.*

Proof. Let X_α and $X_{\alpha+1}$ be as in 1.2.9. As in Proposition 1.2.12 we obtain the long exact sequence:

$$\ldots \to H_i(\bigvee_{\substack{i\geq 0 \\ [\Sigma^i A, X_\alpha]_*}} \Sigma^i A;R) \to H_i(X_\alpha;R) \to H_i(X_{\alpha+1};R) \to H_{i-1}(\bigvee_{\substack{i\geq 0 \\ [\Sigma^i A, X_\alpha]_*}} \Sigma^i A;R) \to \ldots$$

where

$$\tilde{H}_i(\bigvee_{\substack{i\geq 0 \\ [\Sigma^i A, X_\alpha]_*}} \Sigma^i A;R) \cong \bigvee_{\substack{i\geq 0 \\ [\Sigma^i A, X_\alpha]_*}} \tilde{H}_i(\Sigma^i A;R) \cong 0 \text{ for all } i,$$

and hence

$$H_*(X;R) = H_*(X_0;R) \cong H_*(X_\alpha;R) \text{ for all } \alpha.$$

Consequently,

$$H_*(P_A(X);R) = H_*(\varinjlim_\alpha X_\alpha;R) \cong \varinjlim_\alpha H_*(X_\alpha;R) \cong \varinjlim_\alpha H_*(X;R) = H_*(X;R).$$

Finally, $R_\infty(\eta_X)\colon R_\infty(X) \to R_\infty(P_A(X))$ is a homotopy equivalence by [BK72, Lemma I.5.5]. □

In general, as we mentioned in the above paragraph, it is not true that if a space X is A-null then so is $X_p^\wedge$, hence it is not true that $P_A(X) = P_A(X_p^\wedge)$. For instance, $X = B\mathbb{Z}/p^{infty} = K(\mathbb{Z}/p^\infty, 1)$ is S^2-null and $X_p^\wedge = K(\hat{Z}_p, 2)$ is not S^2-null. But thank to a Miller's theorem the assumption is true over certain strongly conditions. This theorem is:

Theorem 1.3.2 ([Mil84, Theorem 1.5])**.** *Let A be a connected space with $\tilde{H}_*(A;\mathbb{Z}[\frac{1}{p}]) \cong 0$ and let X be a nilpotent space. Then the p-completion map $\eta_{X_p^\wedge}\colon X \to X_p^\wedge$ induces an equivalence in mapping spaces*

$$map_*(A, \eta_{X_p^\wedge})\colon map_*(A, X) \xrightarrow{\simeq} map_*(A, X_p^\wedge).$$

Corollary 1.3.3. *Let A be a connected space with $\tilde{H}_*(A;\mathbb{Z}[\frac{1}{p}]) \cong 0$ and let X be a nilpotent space. Then X is A-null if and only if so is $X_p^\wedge$.*

The relationship between completion and nullification is described by N. Castellana and R. Flores in [CF13]. In this source appears a powerful lemma ([CF13, Lemma 3.9]) to understand this relationship. Moreover, it will be crucial in this work. We will present a generalization to R_∞ of this lemma:

Lemma 1.3.4 ([CF13, Lemma 3.9])**.** *Let A be a connected space, and let X such that $P_A(X)$ and $P_A(R_\infty X)$ are R-good spaces. Assume that $R_\infty(P_A(X))$ and $R_\infty(P_A(R_\infty X))$ are A-null spaces. Then the R-completion map $\eta_{R_\infty X}\colon X \to R_\infty X$ induces a R-equivalence*

$$P_A(\eta_{R_\infty X})\colon P_A(X) \to P_A(R_\infty X).$$

Proof. Since $R_\infty(P_A(X))$ is A-null, there is an unique map up to homotopy $\epsilon\colon P_A(R_\infty X) \to R_\infty(P_A(X))$ such that the following diagram is commutative

$$\begin{array}{ccccc}
X & \xrightarrow{\eta_{R_\infty X}} & R_\infty X & = & R_\infty X \\
\downarrow{\scriptstyle \eta_{P_A(X)}} & & \downarrow{\scriptstyle \eta_{P_A(R_\infty X)}} & & \downarrow{\scriptstyle R_\infty(\eta_{P_A(X)})} \\
P_A(X) & \xrightarrow{P_A(\eta_{R_\infty X})} & P_A(R_\infty X) & \xrightarrow{\epsilon} & R_\infty(P_A(X))
\end{array}$$

The left square commutes by naturality of P_A, so

$$R_\infty(\eta_{P_A(X)}) \circ \eta_{R_\infty X} \simeq \epsilon \circ P_A(\eta_{R_\infty X}) \circ \eta_{P_A(X)}.$$

But also, $R_\infty(\eta_{P_A(X)}) \circ \eta_{R_\infty X} \simeq \eta_{R_\infty(P_A(X))} \circ \eta_{P_A(X)}$ by naturality of the completion. Now, by the universality of P_A we get $\epsilon \circ P_A(\eta_{R_\infty X}) \simeq \eta_{R_\infty(P_A(X))}$. Since $P_A(X)$ is R-good,

$$(\eta_{R_\infty(P_A(X))})^*\colon H^*(R_\infty(P_A(X));R) \to H^*(P_A(X);R)$$

is an isomorphism. In particular, ϵ^* is a monomorphism and $P_A(\eta_{R_\infty X})^*$ is an epimorphism.

Now consider the following commutative diagram:

$$\begin{array}{ccccc} R_\infty X & = & R_\infty X & \xrightarrow{\eta_{R_\infty(R_\infty X)}} & R_\infty X \\ \downarrow{\scriptstyle \eta_{P_A(R_\infty X)}} & & \downarrow{\scriptstyle R_\infty(\eta_{P_A(R_\infty X)})} & & \downarrow{\scriptstyle R_\infty(P_A(R_\infty X))} \\ P_A(R_\infty X) & \xrightarrow{\epsilon} & R_\infty(P_A(X)) & \xrightarrow{R_\infty(P_A(\eta_{R_\infty X}))} & R_\infty(P_A(R_\infty X)) \end{array}$$

That is,

$$R_\infty(P_A(R_\infty X)) \circ \eta_{R_\infty(R_\infty X)} \simeq R_\infty(P_A(\eta_{R_\infty X})) \circ \epsilon \circ \eta_{P_A(R_\infty X)}.$$

But also have $R_\infty(P_A(R_\infty X)) \circ \eta_{R_\infty(R_\infty X)} \simeq \eta_{R_\infty(P_A(R_\infty(X)))} \circ \eta_{P_A(R_\infty X)}$. By hypothesis $R_\infty(P_A(R_\infty X))$ is A-null, then the universal property of P_A gives us that $R_\infty(P_A(\eta_{R_\infty X})) \circ \epsilon \simeq \eta_{R_\infty(P_A(R_\infty(X)))}$. And $(\eta_{R_\infty(P_A(R_\infty(X)))})^*$ is an isomorphism in homology with coefficient R, since $P_A(R_\infty X)$ is R-good. Therefore $(R_\infty(P_A(\eta_{R_\infty X})))^*$ is a monomorphism abd hence $P_A(\eta_{R_\infty X})^*$ is so. □

Moreover, according to the Miller's theorem, the Corollary 1.2.11 and the previous lemma, it is follows that:

Corollary 1.3.5 ([CF13, Corollary 3.11]). *If X is a 1-connected space and A is such that $\tilde{H}_*(A;\mathbb{Z}[\frac{1}{p}]) \cong 0$ then $P_A(\eta_{X^\wedge_p})\colon P_A(X) \to P_A(X^\wedge_p)$ is a mod p equivalence.*

Furthermore, the above authors described a general situation in which the nullification of a mod p equivalence is so.

Corollary 1.3.6 ([CF13, Corollary 3.13]). *Let A be a connected space which $\tilde{H}_*(A;\mathbb{Z}[\frac{1}{p}]) \cong 0$. If $f\colon X \to Y$ is a mod p equivalence between 1-connected spaces then $P_A(f)\colon P_A(X) \to P_A(Y)$ is a mod p equivalence.*

1.4 Comparing nullification and homological localization

W. G. Dwyer shows in [Dwy96] that if G is an abelian compact Lie group such that $\pi_0 G$ is a p-group, then $P_{B\mathbb{Z}/p}(BG) \simeq L_{\mathbb{Z}[\frac{1}{p}]}(BG)$. Afterwards N. Castellana and R. Flores give in [CF13] a relationship between $L_{\mathbb{Z}[\frac{1}{p}]}$ and $P_{B\mathbb{Z}/p}$ in connected spaces with finite fundamental group. We give general conditions for comparing nullification and homological localization. This will be a fundamental step to study the cellularization of infinite loop spaces in Chapter 3. We start with the next lemma:

Lemma 1.4.1. *If $\widetilde{h}_*(A) = 0$, then for all $X \in$ **Top** there is a map $\mu_X\colon P_A(X) \to L_h(X)$ such that the following diagram*

$$\begin{array}{ccc} & X & \\ {\scriptstyle \eta_{P_A(X)}}\swarrow & & \searrow{\scriptstyle \eta_{L_h(X)}} \\ P_A X & \xrightarrow{\mu_X} & L_h(X). \end{array}$$

is commutative up to homotopy, where $\eta_F(X)$ denotes the coaugmention map $X \to F(X)$ for $F = P_A$ or L_{h_*}.

Proof. Note that the constant map $A \to *$ is an h_*-isomorphism, because A is h-acyclic. Hence, given a space X, by definition of h_*-local space, we get

$$\mathrm{map}(A, L_h(X)) \simeq \mathrm{map}(*, L_h(X)) \simeq L_h(X),$$

i.e., $L_h(X)$ is A-null and, according to Theorem 1.2.5, there is a natural map

$$\mu_X : P_A(X) \to L_h(X)$$

making commutative the desired diagram. □

The previous lemma gives a condition to have a map between nullification and homological localization. Now we want to try to describe the homotopy fibre of this map. In this sense, our main result is the next:

Theorem 1.4.2. *Let R be a subring of $\mathbb{Q}$. Let X be a 1-connected space and let A be a connected $H_*(-;R)$-acyclic space. If $\mathcal{P}$ denotes the set of divisible primes of R. Then there exists a fibration*

$$F \to P_A(X) \to L_R(X),$$

where F is the homotopy fibre of $\prod_{p\in\mathcal{P}}(P_A(X))^\wedge_p \to (\prod_{p\in\mathcal{P}}(P_A(X))^\wedge_p)_\mathbb{Q}$.

Proof. Since X is 1-connected, so are $P_A(X)$, by Corollary 1.2.11, and $L_R(X)$, by Corollary A.2.8, in particular $P_A(X)$ and $L_R(X)$ are nilpotents. Hence we can apply Sullivan's arithmetic square to P_AX and $L_R(X)$ (see Section A.1.2), and we get the following homotopy pull back diagrams

$$\begin{array}{ccc} P_A(X) & \longrightarrow & \prod_{p\ \mathrm{prime}}(P_A(X))^\wedge_p \\ \downarrow & & \downarrow \\ (P_A(X))_\mathbb{Q} & \longrightarrow & (\prod_{p\ \mathrm{prime}}(P_A(X))^\wedge_p)_\mathbb{Q}\,, \end{array} \qquad \begin{array}{ccc} L_R(X) & \longrightarrow & \prod_{p\ \mathrm{prime}}(L_R(X))^\wedge_p \\ \downarrow & & \downarrow \\ (L_R(X))_\mathbb{Q} & \longrightarrow & (\prod_{p\ \mathrm{prime}}(L_R(X))^\wedge_p)_\mathbb{Q} \end{array}$$

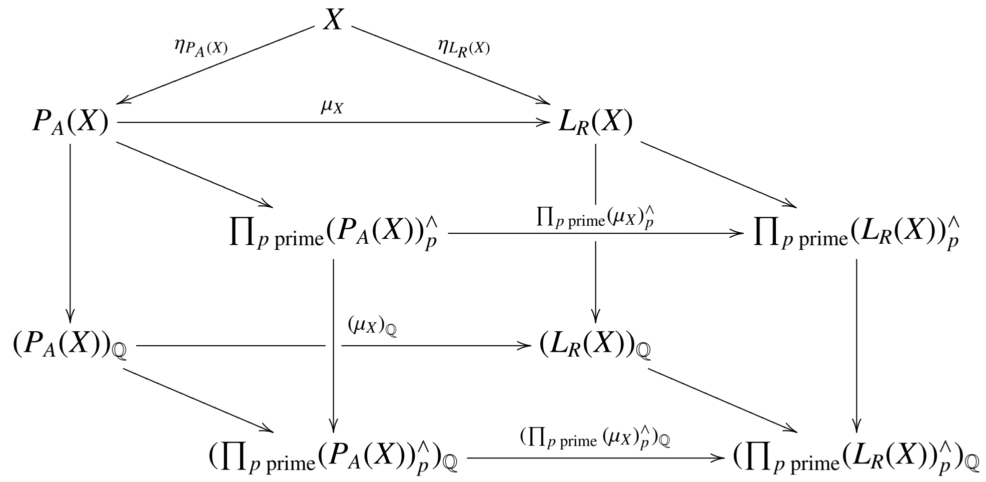
Now, since A is $H_*(-;R)$-acyclic, Lemma 1.4.1 shows that there is a natural map

$$\mu_X : P_A(X) \to L_R(X)$$

making commutative the folowing diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{\eta_{P_A(A)}} & & \searrow^{\eta_{L_R(X)}} & \\ P_A(X) & & \xrightarrow[\mu_X]{} & & L_R(X), \end{array}$$

and therefore we get the following commutative diagram:



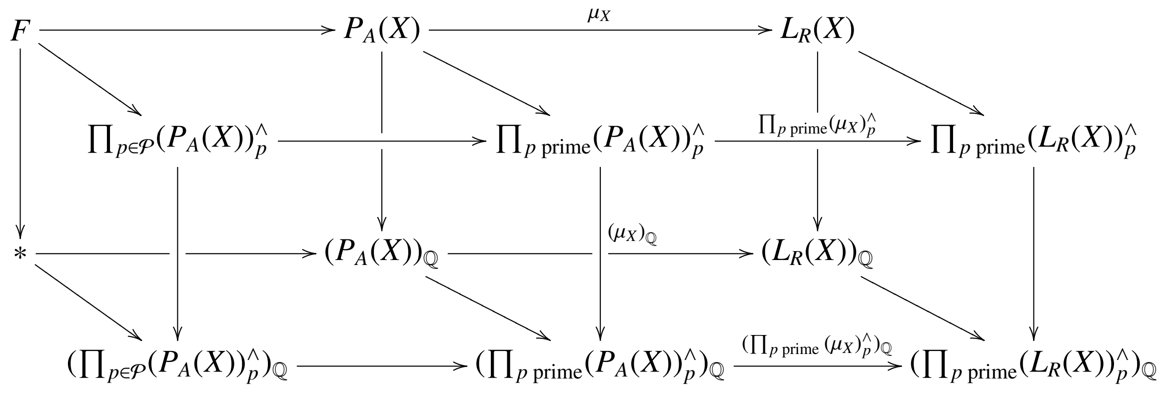
Let $h_* = H_*(-;R)$. Since X is 1-connected, according to Proposition A.3.1, we have $(L_h(X))^\wedge_p \simeq L_{\mathbb{Z}/p}(L_h(X))$ and we conclude from Proposition A.2.12 that $L_{\mathbb{Z}/p}(L_h(X)) \simeq L_{h\mathbb{Z}/p}(X)$ where, by Lemma A.2.13, we get

$$(L_R(X))^\wedge_p \simeq L_{h\mathbb{Z}/p}(X) \simeq \begin{cases} L_{\mathbb{Z}/p}(X) \simeq X^\wedge_p & \text{, if } p \notin \mathcal{P}, \\ * & \text{, if } p \in \mathcal{P}. \end{cases}$$

Let p be a prime not in $\mathcal{P}$. Since $\widetilde{h}_*(A) = 0$, $\widetilde{h}_*(A;\mathbb{Z}/p) = 0$, and hence $\widetilde{H}_*(A;\mathbb{Z}/p) = 0$ since $H_*(-;\mathbb{Z}/p) = h_*(-;\mathbb{Z}/p)$. By Proposition 1.3.1, the coaugmentation map $\eta_{P_A(X)}\colon X \to P_A(X)$ is a mod p equivalence. It follows that $(\mu_X)^\wedge_p\colon (P_A(X))^\wedge_p \to (L_h(X))^\wedge_p$ is an equivalence. Similarly, it is proves that $(\mu_X)_{\mathbb{Q}}\colon (P_A(X))_{\mathbb{Q}} \to (L_R(X))_{\mathbb{Q}}$ is an equivalence. If $p \in \mathcal{P}$ then $(L_R(X))^\wedge_p \simeq *$ and we obtain the following fibration

$$\prod_{p\in\mathcal{P}}(P_A(X))^\wedge_p \longrightarrow \prod_p(P_A(X))^\wedge_p \xrightarrow{\prod_p(\mu_X)^\wedge_p} \prod_p(L_R(X))^\wedge_p.$$

Now let F be the homotopy fibre of $\mu_X\colon P_A(X) \to L_R(X)$. If we consider the homotopy fibres over the horizontal arrows in the above diagram, then we get the next commutative (up to homotopy) diagram, according to [BK72, Example XI.4.3],



where the horizontal arrows are fibrations. Hence we get the following homotopy pull back diagram

$$\begin{array}{ccc} F & \longrightarrow & \prod_{p\in\mathcal{P}}(P_A(X))^\wedge_p \\ \downarrow & & \downarrow \\ * & \longrightarrow & (\prod_{p\in\mathcal{P}}(P_A(X))^\wedge_p)_{\mathbb{Q}} \end{array}$$

this means, F is the homotopy fibre of $\prod_{p\in\mathcal{P}}(P_A(X))^\wedge_p \to (\prod_{p\in\mathcal{P}}(P_A(X))^\wedge_p)_{\mathbb{Q}}$. □

And we finish this section giving a case in which nullification and homological localization are the same.

Corollary 1.4.3. *Let R a subring of $\mathbb{Q}$. Let X be a 1-connected space and let A be a connected $H_*(-;R)$-acyclic space. Let $\mathcal{P}$ denote the set of divisible primes of R. Assume that $(P_A(X))^\wedge_p \simeq *$ for all $p \in \mathcal{P}$. Then $P_A(X) \simeq L_R(X)$.*

Proof. In the above theorem, if $(P_A(X))^\wedge_p \simeq *$ for all $p \in \mathcal{P}$, then $F \simeq *$. Hence $\mu_X \colon P_A(X) \to L_R(X)$ is an equivalence. □

Chapter 2

Cellularization

Given a pointed space A, E. Dror-Farjoun generalizes in [Far96] the concept of homotopy theory, and he introduces the A-homotopy theory, in which the role of the spheres is replaced by A and its suspensions. In this sense, the classical homotopy theory is the S^0-homotopy theory.

We introduce in this chapter the notion of A-homotopy theory. This chapter is then organized as follows. We start by giving the definitions of A-homotopy groups, A-cellular spaces and A-cellular approximation of spaces. Moreover, we finish this chapter by explaining the relationship between A-cellularization and A-nullification. In the second section we generalize some results about $B\mathbb{Z}/p$-cellularization and p-completion that appear in [CF13] to BG-cellularization, where G is a finite abelian group. In the third section we compare the cellularization, of spaces with good properties, with certain homological localizations. Finally the fourth section is dedicated to show the differences between the cellularization of a space with respect to a space B and with respect to a space A which is B-cellular.

As in the previous chapter, we do not provide proofs of results already proved in other sources for the sake of simplicity.

2.1 A-homotopy and A-cellular spaces

It is well known that the n-th homotopy group of a space X is defined by homotopy classes of pointed maps from S^n to X, where $S^n \simeq \Sigma^n S^0$. In this way, E. Dror-Farjoun defines for a pointed cofibrant space A:

Definition 2.1.1 ([Far96, 2.E]). Given a fibrant pointed spaces X, the n-th *A-homotopy group* is defined by $\pi_n(X;A) := [\Sigma^n A, X]_* \cong \pi_0\mathrm{map}_*(\Sigma^n A, X)$.

In this way, the idea of (weak) homotopy equivalence is replaced by A-equivalence, that is:

Definition 2.1.2 ([Far96, Definition 2.A.1]). A pointed map $f\colon X \to Y$ of fibrant spaces is called an *A-equivalence*, if it induces a weak homotopy equivalence on the pointed function complex

$$\mathrm{map}_*(A, f)\colon \mathrm{map}_*(A, X) \to \mathrm{map}_*(A, Y).$$

The concept of CW-complex space is replaced by the concept of A-cellular space:

Definition 2.1.3 ([Far96, Definition 2.D.2.1])**.** A fibrant pointed space is called A-cellular if it can be built from A by means of pointed homotopy colimits, possibly iterated.

The full subcategory of $\mathbf{Top}_*$ which objects are A-cellular spaces is a particular case of closed classes:

Definition 2.1.4 ([Far96, Definition 2.D.1])**.** A full subcategory of pointed spaces $C \subset \mathbf{Top}_*$ is called a *closed class* if it is closed under weak equivalence and arbitrary pointed homotopy colimits.

Remark 2.1.5. In this sense, $C(A)$ denotes the closed class of A-cellular spaces, and it is the smaller closed class that contains A. Moreover, to be A-cellular defines a partial order on spaces, because $A \in C(A)$, if $X \in C(A)$ and $A \in C(B)$ then $X \in C(B)$ and if $A \in C(B)$ and $B \in C(A)$ then $C(A) = C(B)$, according to [Far96, Proposition 2.E.9]. Hence sometimes if X is A-cellular then we write $A << X$.

Example 2.1.6. $C(S^0)$ is the category of CW-complexes and, for $n \geq 1$, $C(S^n)$ is the category of $(n-1)$-connected complexes

The most important properties of closed classes and, in particular, of A-cellular spaces are the following

Proposition 2.1.7 ([Far96, 2.D])**.** *Let C be a closed class. Then:*

(i) C is closed under finite products.

(ii) If $X \in C$ and Y is any (unpointed) space, then $X \rtimes Y = (X \times Y)/ * \times Y$ is in C.

(iii) If $F \to E \to B$ is a fibration sequence with B connected and $F, E \in C$, then $B \in C$.

(iv) If $A \rightarrowtail X \overset{i}{\twoheadrightarrow} X/A$ is a cofibration sequence and $A \in C$, then so is the homotopy fibre of i.

(v) C is closed under retracts.

This result gives us an important consequence for cellular classes:

Corollary 2.1.8. *Let A be a pointed connected space. Then $C(A) = C(A \times A \times \ldots \times A)$.*

Proof. Note that by Proposition 2.1.7.(i), $A \times \ldots \times A$ is A-cellular. Moreover, since A is a retract of $A \times \ldots \times A$, Proposition 2.1.7.(v) shows that A is $(A \times \ldots \times A)$-cellular. Therefore, $C(A) = C(A \times \ldots \times A)$. □

As in the case of nullification we are interested in a functor that turn any space into an A-cellular one. For this, first we have to introduce the dual definition of idempotent coaugmented functor:

Definition 2.1.9 ([Far96, Definition 1.A.2])**.** Let $F\colon \mathbf{Top}_* \to \mathbf{Top}_*$ be a functor. Then,

(a) F is *augmented* if it comes with a natural transformation $a\colon F \to Id$, this means, for each $X \in \mathbf{Top}_*$ there is a map $a_X\colon F(X) \to X$ (it is called the *augmentation map*) and for all morphism $g\colon X \to Y \in \mathbf{Top}_*$, we obtain the following commutative diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{a_X} & X \\ {\scriptstyle F(g)}\downarrow & & \downarrow{\scriptstyle g} \\ F(Y) & \xrightarrow{a_Y} & Y \end{array}$$

(b) If F is an augmented functor we say that F is *idempotent* if both natural maps:

$$F(F(X)) \underset{F(a_X)}{\overset{a_{F(X)}}{\rightrightarrows}} F(X)$$

are weak equivalences and are homotopic to each.

Therefore, E. Dror-Farjoun proves in [Far96, 1.B] the following result:

Theorem 2.1.10. *For any connected space A there exists a functor $CW_A\colon \mathbf{Top}_* \to \mathbf{Top}_*$, called the A-cellularization functor, which is augmented and idempotent. Moreover $CW_A(X)$ is an A-cellular space and the coaugmentation map $a_X\colon CW_A(X) \to X$ is an A-equivalence, this means, a_X induces a weak equivalence $map_*(A, a_X)\colon map_*(A, CW_A(X)) \to map_*(A, X)$.*

Remark 2.1.11. Note that A need be cofibrant and X must be fibrant. As in the case of localization with respect to map, if they are not cofibrant or fibrant then consider the cellular approximation of them.

Example 2.1.12. If $A = S^0$ then the S^0-cellular approximation is the cellular approximation. And for $n \geq 1$, if $A = S^n$, $CW_{S^n}(X) = X\langle n-1\rangle$, the $(n-1)$-connected cover of X.

This functor presents two universal properties:

Proposition 2.1.13. *[Far96, 1.E.8] Let $a_X\colon CW_A(X) \to X$ be the A-cellular approximation of a pointed space X. Then,*

(i) The map a_X is initial among all A-equivalences $f\colon Y \to X$. This means, there exists a map $\tilde{f}\colon CW_A(X) \to Y$ such that the following diagram

$$\begin{array}{ccc} CW_A(X) & \xrightarrow{a_X} & X \\ {\scriptstyle \tilde{f}}\downarrow & \nearrow{\scriptstyle f} & \\ Y & & \end{array}$$

is commutative, up to homotopy, and if $g\colon CW_A(X) \to Y$ is another map such that $f \circ \tilde{f} \simeq a_X$, then $g \simeq \tilde{f}$.

(ii) The map a_X is terminal among all map $\omega\colon W \to X$ from spaces $W \in C(A)$ into X. This means, there exists a map $\tilde{\omega}\colon W \to CW_A(X)$ such that the following diagram

$$\begin{array}{ccc} CW_A(X) & \xrightarrow{a_X} & X \\ {\scriptstyle \tilde{\omega}}\uparrow & \nearrow_{\omega} & \\ W & & \end{array}$$

is commutative, up to homotopy, and if $g\colon W \to CW_A(X)$ is another map such that $a_X \circ \tilde{\omega} \simeq \omega$, then $g \simeq \tilde{\omega}$.

As in the case of localization with respect a map, the cellularization functor commutes with finite product:

Theorem 2.1.14 ([Far96, Theorem 2.E.10])**.** *For any A, X, Y pointed connected spaces there is a homotopy equivalence*

$$CW_A(X \times Y) \to CW_A(X) \times CW_A(Y).$$

And, moreover, there exists a commutation rule for the A-cellularization and the loop space functor:

Theorem 2.1.15 ([Far96, Theorem 3.A.2])**.** *Let A, X be pointed and connected spaces. There is a natural homotopy equivalence*

$$CW_A(\Omega X) \simeq \Omega CW_{\Sigma A}(X).$$

This last theorem proves, in particular, that the cellularization of an infinite loop space is an infinite loop space (see Chapter 3 for more details). Moreover it is also used to prove the following result about cellularization of Eilenberg-MacLane spaces whith respect to Eilenberg-MacLane spaces:

Proposition 2.1.16 ([Far96, Corollary 3.D.10])**.** *Let $A = K(\mathbb{Z}/p^k, n)$ and $X = K(\mathbb{Z}/p^l, n)$, then*

$$CW_A(X) \simeq \begin{cases} A & \text{, if } k \leq l, \\ X & \text{, if } k \geq l. \end{cases}$$

E. Dror-Farjoun studies also the cellularization of Generalized Eilenberg-MacLane spaces and, in particular, of symmetric product of spaces. Let X be a pointed space and $k \geq 0$. Let Σ_k the symmetric group of k-letters. Then the *k-fold symmetric product* of X is defined by $SP^k := X^k/\Sigma_k$, and the *symmetric product* of X by $SP^\infty := \operatorname{hocolim}_{*k} SP^k$.

Proposition 2.1.17 ([Far96, Corollary 4.A.2.1])**.** *Let X be a pointed space. For all $0 \leq k \leq \infty$, $SP^k(X)$ is X-cellular.*

And from this proposition we get:

Corollary 2.1.18. *Let A be a pointed and connected space which $\pi_1(A) \cong \mathbb{Z} \times G$, where G is an abelian group. Then any connected space is A-cellular.*

Proof. Note that since $\pi_1(A)$ is abelian,

$$SP^{\infty}(A) \simeq \prod_{i\geq 1} K(H_i(A;\mathbb{Z}), i) \simeq B\pi_1(A) \times \prod_{i\geq 2} K(H_i(A;\mathbb{Z}), i).$$

Hence $B\pi_1(A)$ is A-cellular, since $SP^{\infty}(A)$ is so and Propositions 2.1.17 and 2.1.7. Moreover, $B\pi_1(A) \simeq B\mathbb{Z} \times BG$ and hence $B\mathbb{Z} = S^1$ is A-cellular by Proposition 2.1.7. Therefore, any connected space is A-cellular. □

2.1.1 Relationship between cellularization and nullifications

In this subsection we present the relations between the functor CW_A and P_A. Intuitively $CW_A(X)$ contains all the "A-information" on X available via $\mathrm{map}_*(A, X)$, while $P_A(X)$ contains what remains of X after all that "A-information" is killed. Thus $CW_A(X)$ should morally be the homotopy fibre of $X \to P_A(X)$. That is "almost" the case but, as E. Dror-Farjoun and W. Chachólski prove it.

Proposition 2.1.19 ([Far96, Proposition 3.B.1]). *For $X, A \in$ **Top**$_*$ one has $P_A(CW_A(X)) \simeq *$ and $CW_A(P_A(X)) \simeq *$.*

The main relation between the nullification and the cellularization with respect to A that E. Dror-Farjoun explains is:

Theorem 2.1.20 ([Far96, Theorem 3.B.2]). *Consider the sequence*

$$CW_A(X) \xrightarrow{a_X} X \xrightarrow{\eta_X} P_{\Sigma A}(X)$$

for arbitrary pointed connected spaces A, X. This sequence is a fibration sequence if and only if the composite $\eta_X \circ a_X \simeq *$ or $[A, X]_* \cong *$.

And the particular case of this theorem:

Proposition 2.1.21 ([Far96, Proposition 3.B.3]). *For any pointed connected space A, X, if $P_{\Sigma A}(X) \simeq *$ then $a_X \colon CW_A(X) \to X$ is a homotopy equivalence.*

In this way, W. Chachólski proves, possibly, the best tool to compute the cellularization of a space:

Theorem 2.1.22 ([Cha96, Theorem 20.3]). *Let A be a pointed and connected space, and let $f \colon X \to Y$ be a map of pointed and connected spaces. Assume that*

(i) Y is the homotopy cofibre of a pointed map $g \colon Z \to X$, where Z is A-cellular,

(ii) the induced map $g_* \colon [A, Z]_* \to [A, X]_*$ is surjective.

Then if F is the homotopy fibre of the composite $X \xrightarrow{f} Y \xrightarrow{\eta_Y} P_{\Sigma A} Y$, then F is A-cellular, and the map $F \to X$ is an A-equivalence.

And the following consequence:

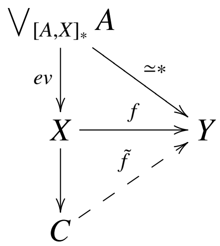
Theorem 2.1.23 ([Cha96, Theorem 20.5])**.** *Let A and X be pointed and connected spaces. Let C be the homotopy cofibre of $ev\colon \bigvee_{[A,X]_*} A \to X$, where the wedge is taken over all the homotopy classes of pointed maps $A \to X$. Then CW_AX has the homotopy type of the homotopy fibre of the composite map $X \to C \xrightarrow{\eta_C} P_{\Sigma A}C$.*

Remark 2.1.24. Sometimes we will call C the Chachólski's cofibre and Chachólski's fibration to the fibration $CW_A(X) \to X \to P_{\Sigma A}(X)$.

From the definition of A-cellular and A-null spaces comes that any map $f\colon X \to Y$ from an A-cellular space X to an A-null spaces Y is null-homotopic. Now we want to find a method for detecing null-homotopic maps if Y is ΣA-null.

Proposition 2.1.25. *Let X and Y pointed connected spaces. Assume that X is A-cellular and Z is ΣA-null. Then a map $f\colon X \to Y$ is null-homotopic if and only if for any map $g\colon A \to X$ the composite $f \circ g$ is null-homotopic.*

Proof. If f is null-homotopic then for any map $g\colon A \to X$ the composite $f \circ g$ is null-homotopic. On the another hand, assume that for all map $g\colon A \to X$ the composite $f \circ g$ is null-homotopic. Let C be the homotopy cofibre over $\bigvee_{[A,X]_*} A \xrightarrow{ev} X$. Since $f \circ ev \simeq *$ by hypothesis, there is a map $\tilde{f}\colon C \to Y$ making commutative the following diagram



and such that $f \simeq *$ if and only if $\tilde{f} \simeq *$. Note that $CW_A(X) \simeq X$ is the homotopy fibre of $X \to P_{\Sigma A}(C)$, hence $P_{\Sigma A}(C) \simeq *$. Therefore Y is C-null, because Y is ΣA-null and hence

$$\mathrm{map}_*(C, Y) \simeq \mathrm{map}_*(P_{\Sigma A}(C), Y) \simeq \mathrm{map}_*(*, Y) \simeq *.$$

Necessarily $\tilde{f} \simeq *$ and finally $f \simeq *$. □

These type of results are quite useful when $A = B\mathbb{Z}/p$, since any classifying space is $\Sigma B\mathbb{Z}/p$-null.

2.2 Commuting cellularization and p-completion

Note that if X is nilpotent, then so is $CW_A(X)$ by [CF13, Lemma 2.5]. Hence we will prove that the cellularization with respect a $\tilde{H}_*(-;\mathbb{Q})$-acyclic space fits in a fibration

$$CW_A(X) \to \prod_{p \text{ prime}} CW_A(X)^\wedge_p \to (\prod_{p \text{ prime}} CW_A(X)^\wedge_p)_\mathbb{Q}$$

In particular, we are interested in the case when $A = BG$ and G is a finite abelian group (in fact, it is necessary that G split in a p-torsion component and in a p'-torsion component, for instance, G a nilpotent group). In this section, if G is a finite abelian group, then G_p denotes the p-torsion component of G.

In Section 1.3 we present when mod p equivalence is preserved by nullification, now we will need a consequence of these results and, for this, we will need the next lemma, that will be used frequently in later chapter of this work, a variation of Dwyer's version of the Zabrodsky's lemma in [Dwy96].

Lemma 2.2.1 ([CCS07, Lemma 2.3],[Dwy96, Proposition 3.4]). *Let $F \to E \xrightarrow{f} B$ be a fibration over a connected base, and let X be a connected space such that ΩX is F-null. Then any map $g\colon E \to X$ such that $g|_F \simeq *$ factors through a map $h\colon B \to X$ up to unpointed homotopy and, moreover, g is pointed null-homotopic if and only if f is so.*

Now, we want to present two thecnical lemmas that we will use later:

Lemma 2.2.2. *Let G be a finite abelian group and p a prime number such that $p \mid |G|$. Let X and Y be 1-connected spaces and let $f\colon X \to Y$ be a mod p equivalence. Then $P_{\Sigma BG}(f)\colon P_{\Sigma BG}(X) \to P_{\Sigma BG}(Y)$ is also a mod p equivalence.*

Proof. We want to apply Lemma 1.3.4 to X and to Y where $A = \Sigma BG$. Note that ΣBG is 1-connected and, in particular, connected. Since X is 1-connected, $X^\wedge_p$ and $P_{\Sigma BG}(X)$ are 1-connected. Moreover, $P_{\Sigma BG}(X^\wedge_p)$ is 1-connected, because $X^\wedge_p$ is so. Consequently, $P_{\Sigma BG}(X)$ and $P_{\Sigma BG}(X^\wedge_p)$ are p-good spaces (Y verifies the same conclusion).

Now, we have to prove that $(P_{\Sigma BG}(X))^\wedge_p$ and $(P_{\Sigma BG}(X^\wedge_p))^\wedge_p$ are ΣBG-null spaces. We will prove that $(P_{\Sigma BG}(X))^\wedge_p$ is a ΣBG-null space (the proofs for $(P_{\Sigma BG}(X^\wedge_p))^\wedge_p$ and Y are analogous). Note first that this is equivalent to prove that $\Omega(P_{\Sigma BG}(X))^\wedge_p$ is a BG-null space, where $\Omega(P_{\Sigma BG}(X))^\wedge_p \simeq (P_{BG}(\Omega X))^\wedge_p$, since $P_{\Sigma BG}(X)$ is 1-connected.

We are interested to apply the Zabrodsky's Lemma to the following fibration

$$BG_p \longrightarrow BG \longrightarrow \prod_{\substack{i=1\\ p_i\neq p}}^{n} BG_{p_i}$$

and the map $BG \to (P_{BG}(\Omega X))^\wedge_p$. Note that $(P_{BG}(\Omega X))^\wedge_p$ is BG_p-null, because $\widetilde{H}_*(BG_p;\mathbb{Z}[\frac{1}{p}]) \cong 0$ and $(P_{BG}(\Omega X))^\wedge_p \simeq \Omega(P_{\Sigma BG}(X))^\wedge_p$ is nilpotent (it is an H-space), hence [Mil84, Theorem 1.5] shows that $\mathrm{map}_*(G_p,(P_{BG}(\Omega X))^\wedge_p) \simeq \mathrm{map}_*(BG_p, P_{BG}(\Omega X))$, and this is contractible because BG_p is BG-cellular (it is a retract) and hence $P_{BG}(BG_p) \simeq *$. Therefore, $\Omega(P_{BG}(\Omega X))^\wedge_p$ is BG_p-null and any map $BG \to (P_{BG}(\Omega X))^\wedge_p$ restricted to BG_p is null-homotopic. Hence the Zabrodsky's lemma shows that

$$\mathrm{map}_*(BG,(P_{BG}(\Omega X))^\wedge_p) \simeq \mathrm{map}_*(\prod_{\substack{i=1\\ p_i\neq p}}^{n} BG_{p_i},(P_{BG}(\Omega X))^\wedge_p).$$

Now, since $(P_{\Sigma BG}(X))^\wedge_p$ is 1-connected, Proposition A.3.1 shows that

$$(P_{\Sigma BG}(X))^\wedge_p = L_{\mathbb{Z}/p}(P_{\Sigma BG}(X)),$$

hence we get

$$\mathrm{map}_*(\prod_{\substack{i=1\\ p_i\neq p}}^{n} BG_{p_i},(P_{BG}(\Omega X))^\wedge_p) \simeq \Omega\mathrm{map}_*(L_{\mathbb{Z}/p}(\prod_{\substack{i=1\\ p_i\neq p}}^{n} BG_{p_i}), L_{\mathbb{Z}/p}(P_{\Sigma BG}(X))),$$

and $L_{\mathbb{Z}/p}(\prod_{\substack{i=1 \\ p_i \neq p}}^{n} BG_{p_i}) \simeq *$. Therefore $(P_{BG}(\Omega X))^\wedge_p$ is BG-null, hence $(P_{\Sigma BG}(X))^\wedge_p$ is ΣBG-null, and then $P_{\Sigma BG}(\eta_X)$ and $P_{\Sigma BG}(\eta_Y)$ are mod p equivalences by Lemma 1.3.4.

Now consider the following commutative diagram

$$\begin{array}{ccc} P_{\Sigma BG}X & \xrightarrow{P_{\Sigma BG}f} & P_{\Sigma BG}Y \\ {\scriptstyle P_{\Sigma BG}(\eta_X)}\downarrow & & \downarrow{\scriptstyle P_{\Sigma BG}(\eta_Y)} \\ P_{\Sigma BG}(X^\wedge_p) & \xrightarrow[P_{\Sigma BG}(f^\wedge_p)]{} & P_{\Sigma BG}(Y^\wedge_p) \end{array}$$

where the vertical arrows are mod p-equivalences. Moreover, the lower row is an equivalence because f is a mod p equivalence and hence $f^\wedge_p$ is an equivalence. Finally $P_{\Sigma BG}(f)$ is a mod p equivalence. □

Lemma 2.2.3. *Let X be a nilpotent space. Let G be a finite abelian group and p a prime number such that $p \mid |G|$. Then $CW_{BG_p}(X)$ is a $(\prod_{\substack{q \| G| \\ q \neq p}} BG_q)$-null space.*

Proof. Let $\{p_1, \dots, p_n\}$ be the set of prime number q such that $q \mid |G|$ and assume that $p = p_1$. We have to prove

$$\mathrm{map}_*(\prod_{i=2}^{n} BG_{p_i}, CW_{BG_p}(X)) \simeq *.$$

Since X is nilpotet, $CW_{BG_p}(X)$ is nilpotent by [CF13, Lemma 2.5]. Note that $CW_{BG_p}(X)$ is BG_{p_i}-null for $i = 2, \dots, n$, because $\widetilde{H}_*(BG_{p_i}; \mathbb{Z}[\frac{1}{p_i}]) = 0$, and [Mil84, Theorem 1.5] shows that $\mathrm{map}_*(BG_{p_i}, CW_{BG_p}(X)) \simeq \mathrm{map}_*(BG_{p_i}, (CW_{BG_p}(X))^\wedge_{p_i})$ and $(CW_{BG_p}(X))^\wedge_{p_i} \simeq *$ according to [CF13, Lemma 2.8].

We now proceed by induction. Assume that, for all $k \leq n-1$,

$$\mathrm{map}_*(\prod_{i=2}^{k} BG_{p_i}, CW_{BG_p}(X)) \simeq *;$$

we will prove that

$$\mathrm{map}_*(\prod_{i=2}^{n} BG_{p_i}, CW_{BG_p}(X)) \simeq *.$$

Consider the fibration

$$BG_{p_n} \to \prod_{i=2}^{n-1} BG_{p_i} \times BG_{p_n} \to \prod_{i=2}^{n-1} BG_{p_i}.$$

Hence $CW_{BG_p}(X)$ is (BG_{p_n})-null by induction. Therefore $\Omega CW_{BG_p}(X)$ is (BG_{p_n})-null and, moreover, any map from $\prod_{i=2}^{n-1} BG_{p_i} \times BG_{p_2} \to CW_{BG_p}(X)$ restricted to BG_{p_n} is homotopic equivalent to a point. Hence, Zabrodsky's lemma proves that

$$\mathrm{map}_*(\prod_{i=2}^{n-1} BG_{p_i} \times BG_{p_n}, CW_{BG_p}(X)) \simeq \mathrm{map}_*(\prod_{i=2}^{n} BG_{p_i}, CW_{BG_p}X),$$

and $\mathrm{map}_*(\prod_{i=2}^{n-1} BG_{p_i}, CW_{BG_p}(X)) \simeq *$ by induction. □

Therefore, our way to compute $CW_{BG}(X)$ is the following:

Proposition 2.2.4. *Let X be a connected nilpotent space and G a finite abelian group. Then,*

(i) If X is 1-connected, then the map

$$CW_{BG}(\eta_X)\colon CW_{BG}(X) \to CW_{BG}(X_p^\wedge)$$

is a mod p equivalence for all prime p.

(ii) For all prime $p \mid |G|$, $CW_{BG}(X_p^\wedge) \simeq CW_{BG_p}(X_p^\wedge)$.

Proof. (i) Let C be the homotopy cofibre of $\bigvee_{[BG,X]_*} BG \xrightarrow{\text{ev}} X$. Let D be the homotopy cofibre of

$$\bigvee_{[BG,X]_*} BG \xrightarrow{\eta_X \circ \text{ev}} X_p^\wedge .$$

and let $\pi\colon X_p^\wedge \to D$ the induced map.

First, We want to prove that $CW_{BG}(X_p^\wedge)$ is the homotopy fibre of $X_p^\wedge \to P_{\Sigma BG}(D)$. On account of Theorem 2.1.22, we have to prove that

- $\bigvee_{[BG,X]_*} BG$ is BG-cellular, but this is true since $\bigvee_{[BG,X]_*} BG$ is a pointed homotopy colimit of BG;
- the induced map $(\eta_X \circ \text{ev})_*\colon [BG, \bigvee_{[BG,X]_*} BG]_* \to [BG, X_p^\wedge]_*$ is surjective. Hence, let $g\colon BG \to X_p^\wedge$ be a pointed map, we have to find a pointed map $f\colon BG \to \bigvee_{[BG,X]_*} BG$ such that $(\eta_X \circ \text{ev})_*[f] = g$.

Note that $(BG)_p^\wedge \simeq \prod_{q \| |G|} (BG_q)_p^\wedge \simeq BG_p$ since

$$(BG_q)_p^\wedge \simeq \begin{cases} BG_p & \text{, if } q = p, \\ * & \text{, if } q \neq p. \end{cases}$$

Hence [BK72, Proposition II.2.8] shows that $(\eta_{BG})^*\colon [BG_p, X_p^\wedge]_* \to [BG, X_p^\wedge]_*$ is a bijection.

Furthermore, X is nilpotent and $\widetilde{H}_*(BG_p; \mathbb{Z}[\frac{1}{p}]) \cong 0$, hence the map

$$(\eta_X)_*\colon \text{map}_*(BG, X) \to \text{map}_*(BG, X_p^\wedge)$$

is an equivalence, by [Mil84, Theorem 1.5]. This proves that the induced map

$$(\eta_X)_*\colon [BG, X]_* \to [BG, X_p^\wedge]_*$$

is a bijection.

Therefore, there is an unique map (up to homotopy) $h\colon BG_p \to X$ such that the following diagram

$$\begin{array}{ccc} BG & \xrightarrow{g} & X_p^\wedge \\ {\scriptstyle \eta_{BG}}\downarrow & & \uparrow{\scriptstyle \eta_X} \\ BG_p & \xrightarrow[h]{} & X \end{array}$$

is commutative (up to homotopy), this means, $((\eta_{BG})^* \circ (\eta_X)_*)[h] = [g]$, i.e., $[\eta_X \circ h \circ \eta_{BG}] = [g]$.

Let $h' = h \circ \eta_{BG} \in [BG, X]_*$ and let f be inclusion in the $([h'])$th-position

$$f = \iota_{([h'])} \colon BG \hookrightarrow \bigvee_{[BG,X]_*} BG.$$

Therefore, $(\eta_X \circ \mathrm{ev})_*[f] = (\eta_X)_*[\mathrm{ev}(f)] = (\eta_X)_*[h'] = [\eta_X \circ h'] = [\eta_X \circ h \circ \eta_{BG}] = [g]$.

Consider now the following commutative diagram:

$$\begin{array}{ccccc} \bigvee_{[BG,X]_*} BG & \xrightarrow{\mathrm{ev}} & X & \longrightarrow & C \\ \| & & \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle f} \\ \bigvee_{[BG,X]_*} BG & \xrightarrow{\eta_X \circ \mathrm{ev}} & X^\wedge_p & \longrightarrow & D. \end{array}$$

Since X is 1-conenected, $X^\wedge_p$ is so. Hence C and D are 1-connected (by Seifert-Van Kampen's theorem). Moreover, f is a mod p equivalence, because id and η_X so are. It follows from Lemma 2.2.2 that $P_{\Sigma BG}(f)$ is a mod p equivalence. Now consider the commutative diagram

$$\begin{array}{ccccc} CW_{BG}(X) & \longrightarrow & X & \longrightarrow & P_{\Sigma BG}(C) \\ {\scriptstyle CW_{BG}(\eta_X)}\downarrow & & \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle P_{\Sigma BG}(f)} \\ CW_{BG}(X^\wedge_p) & \longrightarrow & X^\wedge_p & \longrightarrow & P_{\Sigma BG}(D) \end{array}$$

where the second and third vertical lines are mod p equivalence. Therefore, $CW_{BG}(\eta_X)$ is a mod p equivalence.

(ii) Let $\{p_1, \dots, p_n\}$ be the set of prime number q such that $q \mid |G|$ and assume that $p = p_1$. First, we want to prove that

$$a_{X^\wedge_p} \colon CW_{BG_p}(X^\wedge_p) \to X^\wedge_p$$

is a BG-equivalence. For this, we want to apply the Zabrodsky's Lemma to the fibration

$$\textstyle\prod_{i=2}^n BG_{p_i} \xrightarrow{i} BG \xrightarrow{p} BG_p$$

for any map $BG \to CW_{BG_p}(X^\wedge_p)$. Note that by Lemma 2.2.3, $CW_{BG_p}(X^\wedge_p)$ is $(\prod_{i=2}^n BG_{p_i})$-null. Hence $\Omega CW_{BG_p}(X^\wedge_p)$ is also $(\prod_{i=2}^n BG_{p_i})$-null and any map from $BG \to CW_{BG_p}(X^\wedge_p)$ restricted to $\prod_{i=2}^n BG_{p_i}$ is null homotopic. Therefore

$$\mathrm{map}_*(BG, CW_{BG_p}(X^\wedge_p)) \simeq \mathrm{map}_*(BG_p, CW_{BG_p}(X^\wedge_p)).$$

Now, note that $\mathrm{map}_*(BG_p, CW_{BG_p}(X^\wedge_p)) \simeq \mathrm{map}_*(BG_p, X^\wedge_p) \simeq \mathrm{map}_*(BG, X^\wedge_p)$ (the latter is because $X^\wedge_p = L_{\mathbb{Z}/p}(X)$, by Proposition A.3.1, and use $L_{\mathbb{Z}/p}(BG) \simeq BG_p$), i.e.,

$$\mathrm{map}_*(BG, CW_{BG_p}(X^\wedge_p)) \simeq \mathrm{map}_*(BG, X^\wedge_p),$$

this means, the augmetation $a_{X_p^\wedge}\colon CW_{BG_p}(X_p^\wedge) \to X_p^\wedge$ is a BG-equivalence. Therefore

$$CW_{BG}(CW_{BG_p}(X_p^\wedge)) \simeq CW_{BG}(X_p^\wedge).$$

Finally, BG_p is BG-cellular (it is a retract), and hence $CW_{BG_p}(X_p^\wedge)$ is so. Therefore

$$CW_{BG}(X_p^\wedge) \simeq CW_{BG}(CW_{B\mathbb{Z}/BG_p}(X_p^\wedge)) \simeq CW_{BG_p}(X_p^\wedge).$$

□

Remark 2.2.5. Note that $C(B\mathbb{Z}/p^{r_1} \times \ldots \times B\mathbb{Z}/p^{r_n}) = C(B\mathbb{Z}/p^r)$, where $r = \max\{r_1, \ldots, r_n\}$, because Proposition 2.1.16 gives that $B\mathbb{Z}/p^{r_i}$ is $B\mathbb{Z}/p^r$-cellular for all $i = 1, \ldots n$, and $B\mathbb{Z}/p^r$ is $(B\mathbb{Z}/p^{r_1} \times \ldots \times B\mathbb{Z}/p^{r_n})$-cellular by Proposition 2.1.7 since $B\mathbb{Z}/p^r$ is a retract of $B\mathbb{Z}/p^{r_1} \times \ldots \times B\mathbb{Z}/p^{r_n}$. Hence if G_p is a finite abelian p-group, then $C(G_p) = C(B\mathbb{Z}/p^r)$, where $r = \exp(G_p)$, i.e., the minimal exponent such that $p^r g$ for all $g \in G$.

Corollary 2.2.6. *Let X be a 1-connected space. Let G be a finite abelian group, p a prime number and let $r = \exp(G_p)$. Then $(CW_{BG}(X))_p^\wedge \simeq (CW_{B\mathbb{Z}/p^r}(X))_p^\wedge$.*

Proof. First note that $(CW_{BG}(X))_p^\wedge \simeq (CW_{BG}(X_p^\wedge))_p^\wedge$ and $(CW_{B\mathbb{Z}/p^r}(X))_p^\wedge \simeq (CW_{B\mathbb{Z}/p^r}(X_p^\wedge))_p^\wedge$, by Proposition 2.2.4.(i). Finally, if $r = \exp(G_p)$, then $(CW_{BG}(X_p^\wedge))_p^\wedge \simeq (CW_{B\mathbb{Z}/p^r}(X_p^\wedge))_p^\wedge$ by Proposition 2.2.4.(ii) and Remark 2.2.5. □

These results allow to compute the BG-cellularization in function of the $B\mathbb{Z}/p^r$-cellularization, for all p dividing $|G|$.

Corollary 2.2.7. *Let X be a nilpotent space. Let G be a finite abelian group. Then $CW_{BG}(X)$ is the homotopy fibre of*

$$\prod_{p \| G|} (CW_{B\mathbb{Z}/p^{r_p}}(X))_p^\wedge \to (\prod_{p \| G|} (CW_{B\mathbb{Z}/p^{r_p}}(X))_p^\wedge)_{\mathbb{Q}}.$$

where $r_p = \exp(G_p)$.

Proof. Since X is nilpotent, $CW_{BG}(X)$ is so by [CF13, Lemma 2.5]. Therefore we can to apply Sullivan's arithmetic square to $CW_{BG}(X)$ and we obtain the following pullback diagram

$$\begin{array}{ccc} CW_{BG}(X) & \longrightarrow & \prod_{p \text{ prime}} (CW_{BG}(X))_p^\wedge \\ \downarrow & & \downarrow \\ (CW_{BG}(X))_{\mathbb{Q}} & \longrightarrow & (\prod_{p \text{ prime}} (CW_{BG}(X))_p^\wedge)_{\mathbb{Q}}. \end{array}$$

By [CF13, Lemma 2.8], we have $R_\infty(CW_{BG}(X)) \simeq *$ for $R = \mathbb{Q}$ or $\mathbb{Z}/p$, $p \nmid |G|$, because in this case $\widetilde{H}^*(BG; R) \cong 0$. Therefore, the above diagram becomes

$$\begin{array}{ccc} CW_{BG}(X) & \longrightarrow & \prod_{p \| G|} (CW_{BG}(X))_p^\wedge \\ \downarrow & & \downarrow \\ * & \longrightarrow & (\prod_{p \| G|} (CW_{BG}(X))_p^\wedge)_{\mathbb{Q}} \end{array}$$

and finally we get the fibration $CW_{BG}(X) \to \prod_{p \| G|} (CW_{B\mathbb{Z}/p^{r_p}}(X))_p^\wedge \to (\prod_{p \| G|} (CW_{B\mathbb{Z}/p^{r_p}}(X))_p^\wedge)_{\mathbb{Q}}$, where $r_p = \exp(G_p)$ from Corollary 2.2.6. □

2.3 Comparing cellularization and homological localization

In this section we introduce a general relation between cellularization and homological localization, using the relation of nullification and homological localization describing in Section 1.4. This last result will be use in Chapter 3 to compute the cellularization of infinite loop spaces.

Combaning the Chachólski's theorem (Theorem 2.1.23) with Theorem 1.4.2 it is easy to prove the following theorem.

Theorem 2.3.1. *Let X be a 1-connected space and let p be a prime and $r \geq 0$. Then the $B\mathbb{Z}/p^r$-cellularization of X fits in a fibration sequence*

$$\Omega F \to CW_{B\mathbb{Z}/p^r}(X) \to \overline{L}_{\mathbb{Z}[\frac{1}{p}]}(X),$$

where $\overline{L}_{\mathbb{Z}[\frac{1}{p}]}X$ is the homotopy fibre of the coaugmention map $X \to L_{\mathbb{Z}[\frac{1}{p}]}(X)$ and F is the homotopy fibre of $(P_{\Sigma B\mathbb{Z}/p^r}(C))^{\wedge}_p \to ((P_{\Sigma B\mathbb{Z}/p^r}(C))^{\wedge}_p)_{\mathbb{Q}}$, where C is the Chachólski's cofibre.

This theorem implies the following direct corollary:

Corollary 2.3.2. *Let X be a 1-connected space and let p be a prime. and $r \geq 0$. Let C be the homotopy cofibre of*

$$ev\colon \bigvee_{[B\mathbb{Z}/p^r,X]_*} B\mathbb{Z}/p^r \to X.$$

If $(P_{\Sigma B\mathbb{Z}/p^r}(C))^{\wedge}_p \simeq *$, then $CW_{B\mathbb{Z}/p^r}(X) \simeq \overline{L}_{\mathbb{Z}[\frac{1}{p}]}(X)$.

Before we need a technical lemma about the relation of the homological localization of a space a it Chachólski's cofibre.

Lemma 2.3.3. *Let X be a pointed space and let C be the homotopy cofibre of*

$$ev\colon \bigvee_{[B\mathbb{Z}/p^r,X]_*} B\mathbb{Z}/p^r \to X.$$

Then $L_{\mathbb{Z}[\frac{1}{p}]}(X) \simeq L_{\mathbb{Z}[\frac{1}{p}]}(C)$.

Proof. According to Corollary 1.1.11.(iii) if we prove that $L_{\mathbb{Z}[\frac{1}{p}]}(\bigvee_{[B\mathbb{Z}/p^r,X]_*} B\mathbb{Z}/p^r) \simeq *$, then the induced map $L_{\mathbb{Z}[\frac{1}{p}]}(X) \to L_{\mathbb{Z}[\frac{1}{p}]}(C)$ is a weak equivalence.

First, $* \to B\mathbb{Z}/p^r$ is a $H\mathbb{Z}[\frac{1}{p}]$-isomorphism, because $H_*(B\mathbb{Z}/p;\mathbb{Z}[\frac{1}{p}]) \cong H_*(*;\mathbb{Z}[\frac{1}{p}])$. Hence, by Definition A.2.1, if P is a connected $H\mathbb{Z}[\frac{1}{p}]$-local space, then

$$\mathrm{map}_*(B\mathbb{Z}/p^r, P) \simeq \mathrm{map}_*(*, P) \simeq *,$$

and hence Proposition 1.1.7.(v) shows that $L_{\mathbb{Z}[\frac{1}{p}]}(B\mathbb{Z}/p^r) \simeq *$.

Finally, Theorem 1.1.10 gives us:

$$L_{\mathbb{Z}[\frac{1}{p}]}(\bigvee_{[B\mathbb{Z}/p^r,X]_*} B\mathbb{Z}/p^r) \simeq L_{\mathbb{Z}[\frac{1}{p}]}(\bigvee_{[B\mathbb{Z}/p^r,X]_*} L_{\mathbb{Z}[\frac{1}{p}]}(B\mathbb{Z}/p^r)) \simeq L_{\mathbb{Z}[\frac{1}{p}]}(\bigvee_{[B\mathbb{Z}/p^r,X]_*} *) \simeq L_{\mathbb{Z}[\frac{1}{p}]}(*) \simeq *.$$

□

Hence the proof of the above theorem is the following:

Proof of Theorem 2.3.1. On the one hand, let F be the homotopy fibre of$(P_{\Sigma B\mathbb{Z}/p^r}(C))^{\wedge}_p \to ((P_{\Sigma B\mathbb{Z}/p^r}(C))^{\wedge}_p)_{\mathbb{Q}}$. Theorem 1.4.2 gives the fibration $F \to P_{\Sigma B\mathbb{Z}/p^r}(C) \to L_{\mathbb{Z}[\frac{1}{p}]}(C)$. On the other hand, by Lemma 2.3.3, $L_{\mathbb{Z}[\frac{1}{p}]}(C) \simeq L_{\mathbb{Z}[\frac{1}{p}]}(X)$. Therefore, if

$$f\colon P_{\Sigma B\mathbb{Z}/p}(C) \to L_{\mathbb{Z}[\frac{1}{p}]}(X)$$

is the composite of $P_{\Sigma B\mathbb{Z}/p}(C) \to L_{\mathbb{Z}[\frac{1}{p}]}(C)$ and the homotopic inverse map of

$$L_{\mathbb{Z}[\frac{1}{p}]}(X) \xrightarrow{\simeq} L_{\mathbb{Z}[\frac{1}{p}]}(C),$$

then F is equivalent to the homotopy fibre of $f\colon P_{\Sigma B\mathbb{Z}/p}(C) \to L_{\mathbb{Z}[\frac{1}{p}]}(X)$.

Now consider the following commutative (up to homotopy) diagram

$$\begin{array}{ccccc}
 & & CW_{B\mathbb{Z}/p^r}(X) & \longrightarrow & \overline{L}_{\mathbb{Z}[\frac{1}{p}]}(X) \\
 & & \downarrow & & \downarrow \\
* & \longrightarrow & X & = & X \\
\downarrow & & \downarrow & & \downarrow \\
F & \longrightarrow & P_{\Sigma B\mathbb{Z}/p^r}(C) & \longrightarrow & L_{\mathbb{Z}[\frac{1}{p}]}(X)
\end{array}$$

where the middle vertical line is Chachólski's fibration (Theorem 2.1.23). Completing this diagram yields the desired result. □

2.4 Comparing cellularizations

Let A, B, X be pointed and connected spaces. Assume that A is B-cellular, hence $CW_A(X)$ is B-cellular according to Remark 2.1.5. Therefore, by the universal property given in Proposition 2.1.13.(ii), there exists a map $\omega\colon CW_A(X) \to CW_B(X)$ such that the following diagram

$$\begin{array}{ccc}
CW_A(X) & \xrightarrow{\omega} & CW_B(X) \\
 & {\scriptstyle a_A}\searrow \quad \swarrow{\scriptstyle a_B} & \\
 & X &
\end{array}$$

is commutative up to homotopy, where $a_Y\colon CW_Y(X) \to X$ denotes the augmentation map for X and $Y = A, B$. Moreover ω is unique (up to homotopy) verifying this property.

The goal of this section is to describe the homotopy fibre of this map ω.

Theorem 2.4.1. *Let A, B be pointed and connected spaces. Assume that A is B-cellular. Then for any pointed and connected space X there exists a fibre sequence*

$$P_A(\Omega C) \to CW_A(X) \to CW_B(X)$$

where C is the homotopy cofibre of the map $ev\colon \bigvee_{[A,CW_B(X)]_*} A \to CW_B(X)$.

First to prove this theorem we have to introduce the following result:

Proposition 2.4.2. *Let A, B be pointed and connected spaces. Assume that A is B-cellular. Then $CW_A(CW_B(X)) \simeq CW_A(X)$ for all pointed and connected space X.*

Proof. Let

$$f = CW_A(a_B)\colon CW_A(CW_B(X)) \to CW_A(X)$$

and

$$g = CW_A(\omega)\colon CW_A(CW_A(X)) \to CW_A(CW_B(X)),$$

where $CW_A(CW_A(X)) \simeq CW_A(X)$. Therefore we will prove that $f \circ g \simeq Id_{CW_A(X)}$ and $g \circ f \simeq Id_{CW_A(CW_B(X))}$.

First, $f \circ g = CW_A(a_B) \circ CW_A(\omega) = CW_A(a_B \circ \omega)$. Finally, since $a_B \circ \omega \simeq a_A$, we get $CW_A(a_B \circ \omega) \simeq CW_A(a_A) \simeq Id_{CW_A(X)}$.

Second, to prove that $g \circ f \simeq Id_{CW_A(CW_B(X))}$, let $a_{CW_A(X)}\colon CW_A(CW_B(X)) \to CW_B(X)$ be the augmentation amp and consider the following commutative (up to homotopy) diagram:

$$\begin{array}{ccc} CW_A(CW_B(X)) & \xrightarrow{a_{CW_A(X)}} & CW_A(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle a_B} \\ CW_A(X) & \xrightarrow{a_A} & X \\ {\scriptstyle \omega}\downarrow & \nearrow{\scriptstyle a_B} & \\ CW_B(X) & & \end{array}$$

Hence, since $CW_A(CW_B(X))$ si B-cellular (beacuse so is A) and Proposition 2.1.13.(ii), we get $\omega \circ f \simeq a_{CW_A(X)}$. Now note that $CW_A(f) = CW_A(CW_A(a_B)) \simeq CW_A(a_B) = f$, hence

$$g \circ f \simeq CW_A(\omega) \circ CW_A(f) = CW_A(\omega \circ f) \simeq CW_A(a_{CW_A(X)}) \simeq Id_{CW_A(CW_B(X))}.$$

□

Remark 2.4.3. Note that in Theorem 2.4.1, C is the Chachólski's cofibre. Under these hypothesis, to compute this cofibre we can change the index of the wegde by $[A, X]_*$, because $[A, CW_B(X)]_* \cong [A, CW_A(CW_B(X))]_*$ by Theorem 2.1.10, but $CW_A(CW_B(X)) \simeq CW_A(X)$ by the previous proposition and, moreover, $[A, CW_A(X)]_* \cong [A, X]_*$, one more time, by Theorem 2.1.10. Hence $[A, CW_B(X)]_* \cong [A, X]_*$.

Now, we are able to prove the Theorem 2.4.1.

Proof of Thereom 2.4.1. Let C be the homotopy cofibre of $ev\colon \bigvee_{[A,CW_B(X)]_*} A \to CW_B(X)$. According to Theorem 2.1.23 we get the following fibration

$$CW_A(CW_B(X)) \to CW_B(X) \to P_{\Sigma A}(C),$$

where $CW_A(CW_B(X)) \simeq CW_A(X)$ by Proposition 2.4.2. Finally from Theorem 1.1.13 we obtain the fibration

$$P_A(\Omega C) \to CW_A(X) \to CW_B(X),$$

□

Chapter 3

Cellular approximations of infinite loop spaces

In this chapter we discuss the cellularization of $\Sigma B\mathbb{Z}/p$-acyclic spaces up to p-completion, we mean, X verifies $P_{\Sigma B\mathbb{Z}/p}(X)^{\wedge}_p \simeq *$ We will prove that, under this hypothesis, X is cellular up to p-completion, i.e., $CW_{B\mathbb{Z}/p^r}(X)^{\wedge}_p \simeq X^{\wedge}_p$. An important example of spaces verifying this property is given by C. A. McGibbon in [McG97]. He proves that if E is a 1-connected infinite loop space with $\pi_2(E)$ is a torsion group , then $P_{\Sigma B\mathbb{Z}/p}(E)^{\wedge}_p \simeq *$.

We say that a space E is an *infinite loop space* if there is an infinite numerable set of pointed spaces

$$E = E_0, E_1, \dots, E_n, \dots$$

such that for all n there is a weak equivalences $E_n \xrightarrow{\simeq} \Omega E_{n+1}$, this means,

$$E = E_0 \simeq \Omega E_1 \simeq \Omega^2 E_2 \simeq \dots \simeq \Omega^n E_n \simeq \dots$$

Note that an infinite loop space defines an Ω-Spectra and hence any infinite loop space defines a generalized homology theory (see, for instance, [Ada95] for more details). Given two infinite loop spaces E, E', a pointed map $f\colon E \to E'$ is called an *infinite loop map* if for all n there exists a map $f_n\colon E_n \to E'_n$ such that $f_n \simeq \Omega f_{n+1}$ and $f_0 = f$.

Finally, we will present some examples of BG-cellularization of infinite loop spaces, specifically BO, BU, BSp and some connected covers of BO, where G is a finite abelian group.

3.1 Cellular properties of $\Sigma B\mathbb{Z}/p$-acyclic spaces up to p-completion

Recall that E. Dror-Farjoun proves that if X and A are connected spaces such that $P_{\Sigma A}(X) \simeq *$, then X is A-cellular ([Far96, Proposition 3.B.3]). In this section we want to prove a mod p version of this result:

Theorem 3.1.1. *Let X be a 1-connected space. Let p be a prime number. If $(P_{\Sigma B\mathbb{Z}/p^s}(X))^{\wedge}_p \simeq *$ for some $s \geq 1$, then $CW_{B\mathbb{Z}/p^r}(X)$ has the homotopy type of the homotopy fibre of $X^{\wedge}_p \to (X^{\wedge}_p)_{\mathbb{Q}}$ for all $r \geq 1$.*

The hypothesis $(P_{\Sigma B\mathbb{Z}/p^s}(X))^\wedge_p \simeq *$ for some $s \geq 1$ gives us the $B\mathbb{Z}/p^r$-cellularization for all $r \geq 1$, since if X is 1-connected, then $(P_{\Sigma B\mathbb{Z}/p}(X))^\wedge_p \simeq *$ if and only if $(P_{\Sigma B\mathbb{Z}/p^r}(X))^\wedge_p \simeq *$ for all $r \geq 1$. To prove this fact we need the following generalization of [CF13, Lemma 4.6.1]:

Lemma 3.1.2 ([CF13, Lemma 4.6.1])**.** *Let A and X be connected spaces. Assume that X is 1-connected and $\tilde{H}_*(A;\mathbb{Z}[\frac{1}{p}]) \cong 0$. Then $(P_A(X))^\wedge_p \simeq *$ if and only if $map_*(X,Z) \simeq *$ for any 1-connected A-null p-complete space Z.*

Proof. Note that $(P_A(X))^\wedge_p \simeq L_{\mathbb{Z}/p}(P_A(X))$ and $Z^\wedge_p \simeq L_{\mathbb{Z}/p}(Z)$ by Proposition A.3.1, since X and Z are 1-connected. Hence if Z is p-complete, then

$$\mathrm{map}_*((P_A(X))^\wedge_p, Z) \simeq \mathrm{map}_*(L_{\mathbb{Z}/p}(P_A(X)), Z) \simeq \mathrm{map}_*(P_A(X), Z)$$

by Theorem A.2.3 and since Z is A-null,

$$\mathrm{map}_*((P_A(X))^\wedge_p, Z) \simeq \mathrm{map}_*(X, Z).$$

Hence if $(P_A(X))^\wedge_p \simeq *$, then $\mathrm{map}_*(X,Z) \simeq *$ for any 1-connected A-null p-complete space Z. On the other hand, assume that $\mathrm{map}_*(X,Z) \simeq *$ for any 1-connected A-null p-complete space Z. Note that $(P_A(X))^\wedge_p$ is A-null by Theorem 1.3.2. Since $(P_A(X))^\wedge_p$ is a 1-connected A-null p-complete space, $\mathrm{map}_*((P_A(X))^\wedge_p, (P_A(X))^\wedge_p) \simeq *$, therefore $(P_A(X))^\wedge_p \simeq *$. □

Now we are ready to proof the previous fact:

Proposition 3.1.3. *Let X be a 1-connected space. Then $(P_{\Sigma B\mathbb{Z}/p}(X))^\wedge_p \simeq *$ if and only if $(P_{\Sigma B\mathbb{Z}/p^r}(X))^\wedge_p \simeq *$ for all $r \geq 1$.*

Proof. It is clear that if $(P_{\Sigma B\mathbb{Z}/p^r}(X))^\wedge_p \simeq *$ for all $r \geq 1$ then, in particular, $(P_{\Sigma B\mathbb{Z}/p}(X))^\wedge_p \simeq *$. Hence, assume that $(P_{\Sigma B\mathbb{Z}/p}(X))^\wedge_p \simeq *$ and let $r \geq 1$. By Lemma 3.1.2, $(P_{\Sigma B\mathbb{Z}/p^r}(X))^\wedge_p \simeq *$ if and only $\mathrm{map}_*(X,Z) \simeq *$ for all 1-connected $\Sigma B\mathbb{Z}/p^r$-null and p-complete space Z. Now note that if Z is $\Sigma B\mathbb{Z}/p^r$-null, then it is $\Sigma B\mathbb{Z}/p$-null, because

$$\mathrm{map}_*(\Sigma B\mathbb{Z}/p, Z) \simeq \mathrm{map}_*(P_{\Sigma B\mathbb{Z}/p^r}(\Sigma B\mathbb{Z}/p), Z),$$

and since $\Sigma B\mathbb{Z}/p$ is $\Sigma B\mathbb{Z}/p^r$-cellular (by Proposition 2.1.16), $P_{\Sigma B\mathbb{Z}/p^r}(\Sigma B\mathbb{Z}/p) \simeq *$. Therefore, if Z is a 1-connected $\Sigma B\mathbb{Z}/p^r$-null and p-complete space, then Z is a 1-connected $\Sigma B\mathbb{Z}/p$-null and p-complete space and, by hypothesis, $\mathrm{map}_*(X,Z) \simeq *$. □

In Section 2.3 we saw the relationship between cellularization and homological localization. Recall that according to Corollary 2.3.2 if X is 1-connected and C, the homotopy cofibre of $ev\colon \bigvee_{[B\mathbb{Z}/p^r,X]_*} B\mathbb{Z}/p^r \to X$, verifies that $(P_{\Sigma B\mathbb{Z}/p^r}(C))^\wedge_p \simeq *$, then $CW_{B\mathbb{Z}/p^r}(X) \simeq \overline{L}_{\mathbb{Z}[\frac{1}{p}]}(X)$, but, if $(P_{\Sigma B\mathbb{Z}/p^r}(X))^\wedge_p \simeq *$ then $(P_{\Sigma B\mathbb{Z}/p^r}(C))^\wedge_p \simeq *$?

Lemma 3.1.4. *Let X be a 1-connected space. Let C be homotopy cofibre of*

$$ev\colon \bigvee_{[B\mathbb{Z}/p^r,X]_*} B\mathbb{Z}/p^r \to X.$$

If $(P_{\Sigma B\mathbb{Z}/p^r}(X))^\wedge_p \simeq *$, then $(P_{\Sigma B\mathbb{Z}/p^r}(C))^\wedge_p \simeq *$.

Proof. Let $I = [B\mathbb{Z}/p^r, X]_*$. By Theorem 1.1.10 we get

$$(P_{\Sigma B\mathbb{Z}/p^r}(C))^\wedge_p \simeq \Big(P_{\Sigma B\mathbb{Z}/p^r}\Big(\operatorname{Cofib}(P_{\Sigma B\mathbb{Z}/p^r}(\bigvee_I B\mathbb{Z}/p^r) \to P_{\Sigma B\mathbb{Z}/p^r}(X))\Big)\Big)^\wedge_p.$$

Moreover, since X is 1-connected, $P_{\Sigma B\mathbb{Z}/p^r}(X)$ is so, and hence

$$\operatorname{Cofib}(P_{\Sigma B\mathbb{Z}/p^r}(\bigvee_I B\mathbb{Z}/p^r) \to P_{\Sigma B\mathbb{Z}/p^r}(X))$$

is 1-connected by Seifert-Van Kampen's theorem. Therefore Corollary 1.3.5 shows that

$$\Big(P_{\Sigma B\mathbb{Z}/p^r}\Big(\operatorname{Cofib}(P_{\Sigma B\mathbb{Z}/p^r}(\bigvee_I B\mathbb{Z}/p^r) \to P_{\Sigma B\mathbb{Z}/p^r}(X))\Big)\Big)^\wedge_p \simeq$$
$$\Big(P_{\Sigma B\mathbb{Z}/p^r}\Big((\operatorname{Cofib}(P_{\Sigma B\mathbb{Z}/p^r}(\bigvee_I B\mathbb{Z}/p^r) \to P_{\Sigma B\mathbb{Z}/p^r}(X)))^\wedge_p\Big)\Big)^\wedge_p.$$

Furthermore,

$$\Big(\operatorname{Cofib}\Big(P_{\Sigma B\mathbb{Z}/p^r}(\bigvee_I B\mathbb{Z}/p^r) \to P_{\Sigma B\mathbb{Z}/p^r}(X)\Big)\Big)^\wedge_p \simeq$$
$$\Big(\operatorname{Cofib}\Big((P_{\Sigma B\mathbb{Z}/p^r}(\bigvee_I B\mathbb{Z}/p^r))^\wedge_p \to (P_{\Sigma B\mathbb{Z}/p^r}(X))^\wedge_p\Big)\Big)^\wedge_p,$$

where $(P_{\Sigma B\mathbb{Z}/p^r}(X))^\wedge_p \simeq *$ by hypothesis, and hence

$$\Big(\operatorname{Cofib}\Big((P_{\Sigma B\mathbb{Z}/p^r}(\bigvee_I B\mathbb{Z}/p^r))^\wedge_p \to *\Big)\Big)^\wedge_p \simeq \Big((\Sigma P_{\Sigma B\mathbb{Z}/p^r}(\bigvee_I B\mathbb{Z}/p))^\wedge_p\Big)^\wedge_p.$$

We have proved that

$$\Big(P_{\Sigma B\mathbb{Z}/p^r}\Big((\operatorname{Cofib}(P_{\Sigma B\mathbb{Z}/p^r}(\bigvee_I B\mathbb{Z}/p^r) \to P_{\Sigma B\mathbb{Z}/p^r}(X)))^\wedge_p\Big)\Big)^\wedge_p \simeq$$
$$\Big(P_{\Sigma B\mathbb{Z}/p}\Big(((\Sigma P_{\Sigma B\mathbb{Z}/p^r}(\bigvee_I B\mathbb{Z}/p^r))^\wedge_p)^\wedge_p\Big)\Big)^\wedge_p,$$

and hence if we apply Corollary 1.3.5 two times, then we obtain

$$\Big(P_{\Sigma B\mathbb{Z}/p^r}\Big(((\Sigma P_{\Sigma B\mathbb{Z}/p^r}(\bigvee_I B\mathbb{Z}/p^r))^\wedge_p)^\wedge_p\Big)\Big)^\wedge_p \simeq \Big(P_{\Sigma B\mathbb{Z}/p^r}\Big(\Sigma P_{\Sigma B\mathbb{Z}/p^r}(\bigvee_I B\mathbb{Z}/p^r)\Big)\Big)^\wedge_p,$$

and by Theorem 1.1.10, $(P_{\Sigma B\mathbb{Z}/p^r}(\Sigma(\bigvee_I B\mathbb{Z}/p^r)))^\wedge_p \simeq (P_{\Sigma B\mathbb{Z}/p^r}(\bigvee_I(\Sigma B\mathbb{Z}/p^r)))^\wedge_p$, and Corollary 1.1.11.(i) shows that $P_{\Sigma B\mathbb{Z}/p^r}(\bigvee(\Sigma_I B\mathbb{Z}/p^r))$ is contractible since $P_{\Sigma B\mathbb{Z}/p^r}(\Sigma B\mathbb{Z}/p^r)$ is so. □

Now we can to prove Theorem 3.1.1:

Proof of Theorem 3.1.1. By Proposition 3.1.3 $(P_{\Sigma B\mathbb{Z}/p^r}(X))^\wedge_p \simeq *$ for all $r \geq 1$, since it is so for one $r \geq 1$. Fix $r \geq 1$. Since $(P_{\Sigma B\mathbb{Z}/p^r}(X))^\wedge_p \simeq *$, if C is the homotopy cofibre of $ev\colon \bigvee_{[B\mathbb{Z}/p^r,X]_*} B\mathbb{Z}/p^r \to X$, then $(P_{\Sigma B\mathbb{Z}/p^r}(C))^\wedge_p \simeq *$ by Lemma 3.1.4, and hence Corollary 2.3.2 gives the fibration

$$CW_{B\mathbb{Z}/p^r}(X) \to X \to L_{\mathbb{Z}[\frac{1}{p}]}(X),$$

which is a nilpotent fibration (because the base is 1-connected). Therefore we can to p-complete the fibration

$$(CW_{B\mathbb{Z}/p^r}(X))^\wedge_p \to X^\wedge_p \to (L_{\mathbb{Z}[\frac{1}{p}]}(X))^\wedge_p,$$

where $(L_{\mathbb{Z}[\frac{1}{p}]}(X))^\wedge_p \simeq *$, because $(L_{\mathbb{Z}[\frac{1}{p}]}(X))^\wedge_p \simeq L_{\mathbb{Z}/p}(L_{\mathbb{Z}[\frac{1}{p}]}(X)))$ by Proposition A.2.12 and $L_{\mathbb{Z}/p}(L_{\mathbb{Z}[\frac{1}{p}]}(X))) \simeq *$ by Lemma A.2.13. Therefore, $(CW_{B\mathbb{Z}/p^r}(X))^\wedge_p \simeq X^\wedge_p$. □

Furthermore, now it is not difficult to compute the BG-cellularization of these spaces when G is a finite abelian group:

Proposition 3.1.5. *Let X be a 1-connected space. Let G be a finite abelian group. If for all $p \mid |G|$ there is a $r \geq 1$ such that $(P_{\Sigma B\mathbb{Z}/p^r}(X))^\wedge_p \simeq *$, then $CW_{BG}(X)$ has the homotopy type of the homotopy fibre of $\prod_{p||G|} X^\wedge_p \to (\prod_{p||G|} X^\wedge_p)_{\mathbb{Q}}$.*

Proof. Since X is 1-connected, X is nilpotent and hence by Corollary 2.2.7 there is a fibration $CW_{BG}(X) \to \prod_{p||G|}(CW_{BG}(X))^\wedge_p \to (\prod_{p||G|}(CW_{BG}(X))^\wedge_p)_{\mathbb{Q}}$. Hence, we have to prove that $(CW_{BG}(X))^\wedge_p \simeq X^\wedge_p$ for all $p \mid |G|$. If $p \mid |G|$, then $(CW_{BG}(X))^\wedge_p \simeq (CW_{B\mathbb{Z}/p^r}(X))^\wedge_p$, by Corollary 2.2.6, where $r = \max\{s \geq 1 \mid p^s \mid |G|\}$. Finally $(CW_{B\mathbb{Z}/p^r}(X))^\wedge_p \simeq X^\wedge_p$ by Theorem 3.1.1. □

3.2 Cellularization of infinite loop spaces

In this section we want to describe the BG-cellularization of a family of infinite loop spaces that are $\Sigma B\mathbb{Z}/p^r$-acyclic spaces up to p-completion. We want to prove that if E is a 1-connected infinite loop space with $\pi_2 E$ a torsion group, then $(P_{\Sigma B\mathbb{Z}/p^r}(X))^\wedge_p \simeq *$ for all p prime. We will see that this is a consequence of the following theorem:

Theorem 3.2.1 ([McG97, Theorem 2])**.** *Let E be an infinite loop space with $\pi_1 E$ a torsion group. Then $(P_{B\mathbb{Z}/p}(E))^\wedge_p \simeq *$ for all p prime.*

A first question about the nullification and the cellularization of infinite loop spaces is: the nullification and the cellularization of infinite loop spaces are again infinite loop spaces? And in this case, the coaugmentation map and the augmentation are infinite loop maps? The answer to both questions is yes:

Proposition 3.2.2. *Let E be an infinite loop space and let A be a pointed space. Then $P_A(E)$ and $CW_A(E)$ are infinite loop spaces. Moreover $\eta_E\colon E \to P_A(E)$ and $a_E\colon CW_A(E) \to E$ are infinite loop maps.*

Proof. We give only the proof for the case of the cellularization; the case of the nullification is analogous.

First we have to find an infinite numerable set of pointed spaces $\{(CW_A(E))_n\}_{n\geq 0}$ such that

$$(CW_A(E))_0 = CW_A(E) \text{ and } (CW_A(E))_n \simeq \Omega(CW_A(E))_{n+1}.$$

Then take $(CW_A(E))_n := CW_{\Sigma^n A}(E_n)$. By Theorem 2.1.15,

$$\Omega(CW_A(E))_{n+1} = \Omega CW_{\Sigma^{n+1}A}(E_{n+1}) \simeq CW_{\Sigma^n A}(\Omega E_{n+1}) \simeq CW_{\Sigma^n A}(E_n) = (CW_A(E))_n.$$

Now we have to find infinite numerable set of pointed maps $\{(a_E)_n\}_{n\geq 0}$ such that

$$(a_E)_0 = a_E \text{ and } (a_E)_n \simeq \Omega(a_E)_{n+1}.$$

Take $(a_E)_n = a_{E_n} \colon CW_{\Sigma^n A}(E_n) \to E_n$. From the following commutative (up to homotopy) diagram

$$\begin{array}{ccc} CW_{\Sigma^n A}(E_n) & \xrightarrow{a_{E_n}} & E_n \\ \downarrow \simeq & & \downarrow \simeq \\ \Omega CW_{\Sigma^{n+1}A}(E_{n+1}) & \xrightarrow{\Omega a_{E_{n+1}}} & E_n \end{array}$$

we conclude that $\Omega a_{E_{n+1}}$ is an A-equivalence. Moreover, by the vertical equivalences, we get the A-equivalence $\Omega a_{E_{n+1}} \colon CW_{\Sigma^n A}(E_n) \to E_n$, hence $\Omega a_{E_{n+1}} \simeq a_{E_n}$ by Proposition 2.1.13.(i). □

Let G be a finite abelian group. We want to find 1-connected $\Sigma B\mathbb{Z}/p^r$-acyclic spaces up to p-completion infinite loop spaces for all $p \mid |G|$.

Proposition 3.2.3. *Let E be a 1-connected infinite loop space. If $\pi_2 E$ is a torsion group, then $(P_{\Sigma B\mathbb{Z}/p^r}(E))^\wedge_p \simeq *$ for all prime p and all $r \geq 1$.*

Proof. Let p be a prime number. According to Proposition 3.1.3 it is sufficient to show that $P_{\Sigma B\mathbb{Z}/p}(E)^\wedge_p \simeq *$. Note that ΩE is an infinite loop space whith $\pi_1(\Omega E) \cong \pi_2 E$ a torsion group. Therefore $(P_{B\mathbb{Z}/p}(\Omega E))^\wedge_p \simeq *$, by Theorem 3.2.1. Now, since E is a 1-connected spaces, $P_{\Sigma B\mathbb{Z}/p}(E)$ is so by Proposition 1.2.10, and hence $\Omega(P_{\Sigma B\mathbb{Z}/p}(E))^\wedge_p \simeq (\Omega P_{\Sigma B\mathbb{Z}/p}(E))^\wedge_p$, where $\Omega P_{\Sigma B\mathbb{Z}/p}(E) \simeq P_{B\mathbb{Z}/p}(\Omega E)$ by Theorem 1.1.13. Therefore

$$\Omega(P_{\Sigma B\mathbb{Z}/p}(E))^\wedge_p \simeq (P_{B\mathbb{Z}/p}(\Omega E))^\wedge_p \simeq *$$

and hence $(P_{\Sigma B\mathbb{Z}/p}(E))^\wedge_p \simeq *$, since $(P_{\Sigma B\mathbb{Z}/p}(E))^\wedge_p$ is 1-connected. □

In this situation, using Proposition 3.1.5 we get the cellularization with respect to the classifying space of a finite abelian group:

Corollary 3.2.4. *Let E be a 1-connected infinite loop space such that $\pi_2 E$ is a torsion group. Let G be a finite abelian group. Then $CW_{BG}(E)$ has the homotopy type of the homotopy fibre of $\prod_{p||G|} X^\wedge_p \to (\prod_{p||G|} X^\wedge_p)_{\mathbb{Q}}$.*

Remark 3.2.5. If $G = \mathbb{Z}/p^r$, then under the above hypothesis we get the fibration

$$CW_{B\mathbb{Z}/p^r}(E) \to E^\wedge_p \to (E^\wedge_p)_{\mathbb{Q}}.$$

3.3 Examples

3.3.1 Postnikov pieces

In [CCS07], the authors prove that a simply connected Postnikov piece is $B\mathbb{Z}/p^m$-cellular if and only if it is p-torsion, that is, its homotopy groups are p-torsion groups. Hence in this subsection we want to generalize this result and understand the $B\mathbb{Z}/p^m$-cellularization when not all the homotopy groups are p-torsion groups, as a consequence of Theorem 3.1.1 and Corollary 3.2.4.

Before, we need the following lemma about fibrations and $\Sigma B\mathbb{Z}/p^m$-acyclic spaces up to p-completion. This is also proved in [McG97] using a different strategy.

Lemma 3.3.1. *Let $F \to E \to B$ be a fibration of 1-connected spaces. Let p be a prime number. If $P_{\Sigma B\mathbb{Z}/p^m}(F)^\wedge_p \simeq *$, then $P_{\Sigma B\mathbb{Z}/p^m}(E)^\wedge_p \simeq P_{\Sigma B\mathbb{Z}/p^m}(B)^\wedge_p$.*

Proof. According to the fibrewise localization ([Far96, Theorem 1.F.1]), there exist a space $\bar{E}$ and a commutative diagram of fibrations

$$\begin{array}{ccc} F & \xrightarrow{\eta_F} & P_{\Sigma B\mathbb{Z}/p^m}(F) \\ \downarrow & & \downarrow \\ E & \xrightarrow{f} & \bar{E} \\ \downarrow & & \downarrow \\ B & = & B \end{array}$$

such that f induces a (weak) equivalence $P_{\Sigma B\mathbb{Z}/p^m}(f)\colon P_{\Sigma B\mathbb{Z}/p^m}(E) \to P_{\Sigma B\mathbb{Z}/p^m}(\bar{E})$. Applying the p-completion functor to the right fibration, we get the fibration

$$P_{\Sigma B\mathbb{Z}/p^m}(F)^\wedge_p \to \bar{E}^\wedge_p \to B^\wedge_p,$$

since B is 1-connected and [BK72, Lemma II.5.1]. But $P_{\Sigma B\mathbb{Z}/p^m}(F)^\wedge_p \simeq *$, hence $\bar{E}^\wedge_p \simeq B^\wedge_p$. Moreover, $\bar{E}$ is 1-connected since so are $P_{\Sigma B\mathbb{Z}/p^m}(F)$ and B. Since $P_{\Sigma B\mathbb{Z}/p^m}(\bar{E})^\wedge_p \simeq P_{\Sigma B\mathbb{Z}/p^m}(\bar{E}^\wedge_p)^\wedge_p$ and $P_{\Sigma B\mathbb{Z}/p^m}(B)^\wedge_p \simeq P_{\Sigma B\mathbb{Z}/p^m}(B^\wedge_p)^\wedge_p$ by [CF13, Corollary 3.11], $P_{\Sigma B\mathbb{Z}/p^m}(\bar{E})^\wedge_p \simeq P_{\Sigma B\mathbb{Z}/p^m}(B)^\wedge_p$ and, moreover, $P_{\Sigma B\mathbb{Z}/p^m}(E)^\wedge_p \simeq P_{\Sigma B\mathbb{Z}/p^m}(\bar{E})^\wedge_p$. □

Corollary 3.3.2. *Let p be a prime number. Let X be a 1-connected Postnikov piece. If $\pi_2(X)$ is a torsion group, then $CW_{B\mathbb{Z}/p^m}(X)$ fits in a fibration*

$$CW_{B\mathbb{Z}/p^m}(X) \to X^\wedge_p \to (X^\wedge_p)_\mathbb{Q}.$$

Proof. Let X be a 1-connected Postnikov piece. For some integer n, the n-connected cover $X\langle n\rangle$ is an Eilenberg-Mac Lane space. Consider the principal fibration

$$K(\pi_n X, n-1) \to X\langle n\rangle \to X\langle n-1\rangle$$

If $n \geq 3$, then $K(\pi_n X, n-1)$ and $X\langle n\rangle$ are 1-connected inifite loop spaces whose second homotopy group is a torsion group, hence $P_{\Sigma B\mathbb{Z}/p^m}(K(\pi_n X, n-1))^\wedge_p \simeq P_{\Sigma B\mathbb{Z}/p^m}(X\langle n\rangle)^\wedge_p \simeq *$ by

Proposition 3.2.3. Therefore $P_{\Sigma B\mathbb{Z}/p^m}(X\langle n-1\rangle)^{\wedge}_p \simeq *$ by Lemma 3.3.1. An iteration of the same argument shows that $P_{\Sigma B\mathbb{Z}/p^m}(X\langle 2\rangle)^{\wedge}_p \simeq *$. Thus, look at the fibration

$$X\langle 2\rangle \to X \to K(\pi_2 X, 2)$$

where $P_{\Sigma B\mathbb{Z}/p^m}(X\langle 2\rangle)^{\wedge}_p \simeq P_{\Sigma B\mathbb{Z}/p^m}(K(\pi_2 X, 2))^{\wedge}_p \simeq *$ (the last one happens because $K(\pi_2 X, 2)$ is a 1-connected inifite loop spaces with second homotopy group is a torsion group and Proposition 3.2.3). Hence Lemma 3.3.1 shows that $P_{\Sigma B\mathbb{Z}/p^m}(X)^{\wedge}_p \simeq *$ and finally we get the fibration

$$CW_{B\mathbb{Z}/p^m}(X) \to X^{\wedge}_p \to (X^{\wedge}_p)_{\mathbb{Q}}.$$

on account to Theorem 3.1.1. □

3.3.2 Infinite loop spaces related with K-theories.

The cellularization of the inifite loop spaces BU and BO and some connected covers of BO with respect to $K(\mathbb{Z}/p, m)$ appears in [CCS07] for all $m \geq 2$. Specifically, for all $m \geq 2$, $CW_{K(\mathbb{Z}/p,m)}(BU) \simeq CW_{K(\mathbb{Z}/p,m)}(BSU) \simeq *$ ([CCS07, Example 5.5]), $CW_{K(\mathbb{Z}/p,m)}(BO) \simeq CW_{K(\mathbb{Z}/p,m)}(BSO) \simeq CW_{K(\mathbb{Z}/p,m)}(BSpin) \simeq *$ and $CW_{K(\mathbb{Z}/p,m)}(BString) \simeq *$ for all $m > 2$ and $CW_{K(\mathbb{Z}/p,2)}(BString) \simeq K(\mathbb{Z}/p, 2)$ ([CCS07, Proposition 6.5]). Moreover [CCS07, Proposition 1.6] establishes that the $B\mathbb{Z}/p$-cellularization of these spaces must have infinitely many non-vanishing homotopy groups.

In this section we want to compute the homotopy groups of these spaces and BSp with respect to $B\mathbb{Z}/p$ and, more generally, with respect to BG, where G is a finite abelian group. Hence, if $\{p_1, \ldots, p_n\}$ is the set of prime dividing $|G|$, then we get the fibration sequence

$$CW_{BG}(E) \to \prod_{i=1}^{n} E^{\wedge}_{p_i} \to (\prod_{i=1}^{n} E^{\wedge}_{p_i})_{\mathbb{Q}},$$

under good hypothesis, according to Corollary 3.2.4. Hence we can compute easily the homotopy group of $CW_{BG}(E)$ in terms of the homotopy groups of E.

Example 3.3.3. Let $U(m)$ be the unitary group of degree m and let $BU = \bigcup_{m=1}^{\infty} BU(m)$. Hence BU is the classifying space of complex vector bundles. Note that BU is an infinite loop space which homotopy groups are

$$\pi_i(BU) \cong \begin{cases} \mathbb{Z} & \text{, if } i \text{ is even,} \\ 0 & \text{, if } i \text{ is odd.} \end{cases}$$

Hence $\pi_2 BU$ is not a torsion group. But from the equivalence $BU \simeq BSU \times BS^1$ and by Theorem 2.1.14, $CW_{BG}(BU) \simeq CW_{BG}(BSU) \times CW_{BG}(BS^1)$, where $BSU = BU\langle 2\rangle$ is a 1-connected infinite loop space which $\pi_2(BSU)$ a torsion group (in fact, it is 2-connected).

We can apply Corollary 3.2.4 to BSU and we get the following fibration

$$CW_{BG}(BSU) \to \prod_{i=1}^{n} BSU^{\wedge}_{p_i} \to (\prod_{p=1}^{n} BSU^{\wedge}_{p_i})_{\mathbb{Q}}.$$

Fix $p \in \{p_1, \dots, p_n\}$. Since BSU is 1-connected, by [BK72, Example 5.2], $\pi_*(BSU^\wedge_p) \cong \hat{\mathbb{Z}}_p \otimes \pi_* BSU$, i.e.,

$$\pi_i(BSU^\wedge_p) \cong \begin{cases} \hat{\mathbb{Z}}_p & \text{, if } i \text{ is even and } i \geq 4, \\ 0 & \text{, otherwise.} \end{cases}$$

and

$$\pi_i((BSU^\wedge_p)_{\mathbb{Q}}) \cong \begin{cases} \hat{\mathbb{Z}}_p \otimes \mathbb{Q} & \text{, if } i \text{ is even and } i \geq 4, \\ 0 & \text{, otherwise.} \end{cases}$$

Therefore, the long exact sequence of homotopy groups associated to the previous fibration is:

- If $j \geq 2$, then

$$\cdots \to 0 \to \pi_{2j} CW_{BG}(BSU) \to \prod_{i=1}^n \hat{\mathbb{Z}}_{p_i} \to \prod_{i=1}^n (\hat{\mathbb{Z}}_{p_i} \otimes \mathbb{Q}) \to \pi_{2j-1} CW_{BG}(BSU) \to 0 \to \cdots$$

- If $j < 2$, then

$$\cdots \to 0 \to \pi_{2j} CW_{BG}(BSU) \to 0 \to 0 \to \pi_{2j-1} CW_{BG}(BSU) \to 0 \to \cdots$$

Therefore,

$$\pi_j CW_{BG}(BSU) \cong \begin{cases} \prod_{i=1}^n \mathbb{Z}/p_i^\infty & \text{, if } j \text{ is odd and } j \neq 1, \\ 0 & \text{, otherwise.} \end{cases}$$

Note now that $CW_{BG}(BS^1) \simeq \prod_{i=1}^n B\mathbb{Z}/p_i^{r_i}$, where $r_i = \max\{r_1, \dots, r_{s_i}\}$. Because BS^1 is nilpotent (it is 1 -connected) and hence $CW_{BG}(BS^1)$ is the homotopy fibre of

$$\prod_{i=1}^n CW_{BG}(BS^1)^\wedge_{p_i} \to (\prod_{p=1}^n CW_{BG}(BS^1)^\wedge_{p_i})_{\mathbb{Q}},$$

by Corollary 2.2.7. Furthermore $CW_{BG}(BS^1)^\wedge_{p_i} \simeq CW_{BG_p}(BS^1)^\wedge_{p_i}$ by Proposition 2.2.4.(ii) and Proposition 2.2.4.(i), and $CW_{BG_p}(BS^1)^\wedge_{p_i} \simeq CW_{B\mathbb{Z}/p_i^{r_i}}(BS^1)^\wedge_{p_i}$ by Remark 2.2.5 and, moreover, $CW_{B\mathbb{Z}/p_i^{r_i}}(BS^1) \simeq B\mathbb{Z}/p_i^{r_i}$ since $\mathrm{map}_*(B\mathbb{Z}/p_i^{r_i}, BS^1)$ is homotopically discrete with components $\mathrm{Hom}(\mathbb{Z}/p_i^{r_i}, S^1) \cong \mathrm{Hom}(\mathbb{Z}/p_i^{r_i}, \mathbb{Z}/p_i^{r_i})$.

Finally, we get

$$\pi_j CW_{BG}(BU) \cong \begin{cases} \prod_{i=1}^n \mathbb{Z}/p_i^{r_i} & \text{, ij } j = 1, \\ \prod_{i=1}^n \mathbb{Z}/p_i^\infty & \text{, if } j \text{ is odd and } j \neq 1, \\ 0 & \text{, otherwise.} \end{cases}$$

Note that we get $CW_{BG}(BSU) \simeq CW_{BG}(BU)\langle 2 \rangle$.

Example 3.3.4. Let $O(m)$ be the ortogonal group of degree m and let $BO = \bigcup_{m=1}^\infty BO(m)$. Hence BO is the classifying space of real vector bundles. Note that BO is a infinite loop space which homotopy groups are

$$\pi_i(BO) \cong \begin{cases} \mathbb{Z}/2 & \text{, if } i \equiv 1,2 \mod 8, \\ \mathbb{Z} & \text{, if } i \equiv 0,4 \mod 8, \\ 0 & \text{, if } i \equiv 3,5,6,7 \mod 8. \end{cases}$$

Note that BO is not 1-connected. Consider $BSO = BO\langle 1\rangle$, we have $BO \simeq BSO \times B\mathbb{Z}/2$, where BSO is a 1-connected inifite loop space which $\pi_2 BSO = \mathbb{Z}/2$, a torsion group. Therefore, by Corollary 3.2.4 we obtain the following fibration

$$CW_{BG}(BSO) \to \prod_{i=1}^{n} BSO^{\wedge}_{p_i} \to (\prod_{p=1}^{n} BSO^{\wedge}_{p_i})_{\mathbb{Q}}.$$

where for all $p \in \{p_1, \ldots, p_n\}$,

$$\pi_i(BSO^{\wedge}_p) \cong \begin{cases} \mathbb{Z}/2 \otimes \hat{\mathbb{Z}}_p & \text{, if } i \equiv 1,2 \mod 8 \text{ and } i \neq 1, \\ \hat{\mathbb{Z}}_p & \text{, if } i \equiv 0,4 \mod 8, \\ 0 & \text{, if } i \equiv 3,5,6,7 \mod 8 \text{ or } i = 1. \end{cases}$$

by [BK72, Example 5.2]; and

$$\pi_i((BSO^{\wedge}_p)_{\mathbb{Q}}) \cong \begin{cases} \mathbb{Z}/2 \otimes \hat{\mathbb{Z}}_p \otimes \mathbb{Q} & \text{, if } i \equiv 1,2 \mod 8 \text{ and } i \neq 1, \\ \hat{\mathbb{Z}}_p \otimes \mathbb{Q} & \text{, if } i \equiv 0,4 \mod 8, \\ 0 & \text{, if } i \equiv 3,5,6,7 \mod 8\text{or } i = 1\ . \end{cases}$$

Therefore, the long exact sequence of homotopy groups associated to the above fibration becomes

- for $j \equiv 1, 2, \mod 8$ and $j \neq 1$,

$$\cdots \longrightarrow 0 \longrightarrow \pi_j CW_{BG}(BSO) \longrightarrow \prod_{j=1}^{n} \mathbb{Z}/2 \otimes \hat{\mathbb{Z}}_{p_j} \longrightarrow 0 \longrightarrow \cdots$$

- for $j \equiv 3, 7 \mod 8$,

$$\ldots \to 0 \to \pi_{j+1} CW_{BG}(BSO) \to \prod_{j=1}^{n} \hat{\mathbb{Z}}_{p_j} \to \prod_{j=1}^{n} (\hat{\mathbb{Z}}_{p_j} \otimes \mathbb{Q}) \to \pi_j CW_{BG}(BSO) \to 0 \to \ldots$$

- for $j \equiv 5, 6 \mod 8$ or $j = 1$,

$$\ldots \longrightarrow 0 \longrightarrow \pi_j(CW_{BG} BO) \longrightarrow 0 \longrightarrow \ldots$$

and hence,

$$\pi_j CW_{BG}(BSO) \cong \begin{cases} \prod_{j=1}^{n} \mathbb{Z}/2 \otimes \hat{\mathbb{Z}}_{p_j} & \text{, if } j \equiv 1,2 \mod 8 \text{ and } j \neq 1, \\ \prod_{i=1}^{n} \mathbb{Z}/p_i^{\infty} & \text{, if } j \equiv 3,7 \mod 8, \\ 0 & \text{, if } j \equiv 0,4,5,6 \mod 8 \text{ or } j = 1. \end{cases}$$

where

$$\prod_{j=1}^{n} \mathbb{Z}/2 \otimes \hat{\mathbb{Z}}_{p_j} \cong \begin{cases} \mathbb{Z}/2 & \text{, if } 2 \in \{p_1, \ldots, p_n\}, \\ 0 & \text{, if } 2 \notin \{p_1, \ldots, p_n\}. \end{cases}$$

Moreover, by Theorem 4.1.7,

$$CW_{BG}(B\mathbb{Z}/2) \simeq \begin{cases} \mathbb{Z}/2 & \text{, if } 2 \in \{p_1, \ldots, p_n\}, \\ 0 & \text{, if } 2 \notin \{p_1, \ldots, p_n\}. \end{cases}$$

And finally:

- If $2 \in \{p_1, \dots, p_n\}$, then

$$\pi_j(CW_{BG}BO) = \begin{cases} \mathbb{Z}/2 & \text{, if } j \equiv 1,2 \mod 8, \\ \prod_{i=1}^n \mathbb{Z}/p_i^\infty & \text{, if } j \equiv 3,7 \mod 8, \\ 0 & \text{, if } j \equiv 0,4,5,6 \mod 8. \end{cases}$$

- If $2 \notin \{p_1, \dots, p_n\}$, then

$$\pi_j(CW_{BG}BO) = \begin{cases} \prod_{i=1}^n \mathbb{Z}/p_i^\infty & \text{, if } j \equiv 3,7 \mod 8, \\ 0 & \text{, if } j \equiv 0,1,2,4,5,6 \mod 8. \end{cases}$$

Ana to the case BO and BSO, we get $CW_{BG}(BSO) = CW_{BG}(BO)\langle 1\rangle$.

Example 3.3.5. Let now consider the spaces $BSpin = BO\langle 4\rangle$ and $BString = BO\langle 8\rangle$. These spaces are 1-connected infinite loop spaces which second homotopy group is a torsion group. Therefore Corollary 3.2.4 shows that

$$CW_{BG}(BSpin) \to \prod_{i=1}^n (BSpin)^\wedge_{p_i} \to (\prod_{i=1}^n (BSpin)^\wedge_{p_i})_{\mathbb{Q}}, \text{ and}$$

$$CW_{BG}(BString) \to \prod_{i=1}^n (BString)^\wedge_{p_i} \to (\prod_{i=1}^n (BString)^\wedge_{p_i})_{\mathbb{Q}}.$$

Hence the long exact sequence of homotopy groups induced by the first fibration becomes

- for $j \equiv 1,2,\mod 8$ and $j \neq 1,2$,

$$\cdots \longrightarrow 0 \longrightarrow \pi_j CW_{BG}(BSpin) \longrightarrow \prod_{j=1}^n \mathbb{Z}/2 \otimes \hat{\mathbb{Z}}_{p_j} \longrightarrow 0 \longrightarrow \cdots$$

- for $j \equiv 3,7 \mod 8$ and $j \neq 3$,

$$\dots \to 0 \to \pi_{j+1} CW_{BG}(BSpin) \to \prod_{j=1}^n \hat{\mathbb{Z}}_{p_j} \to \prod_{j=1}^n (\hat{\mathbb{Z}}_{p_j} \otimes \mathbb{Q}) \to \pi_j CW_{BG}(BSpin) \to 0 \to \dots$$

- for $j \equiv 5,6 \mod 8$ or $j = 1,2,3$,

$$\dots \longrightarrow 0 \longrightarrow \pi_j(CW_{BG}BSpin) \longrightarrow 0 \longrightarrow \dots$$

and hence,

$$\pi_j CW_{BG}(BSpin) \cong \begin{cases} \prod_{j=1}^n \mathbb{Z}/2 \otimes \hat{\mathbb{Z}}_{p_j} & \text{, if } j \equiv 1,2 \mod 8 \text{ and } j \neq 1,2, \\ \prod_{i=1}^n \mathbb{Z}/p_i^\infty & \text{, if } j \equiv 3,7 \mod 8 \text{ and } j \neq 3, \\ 0 & \text{, if } j \equiv 0,4,5,6 \mod 8 \text{ or } j = 1,2,3. \end{cases}$$

Similarly, we obtain

$$\pi_j CW_{BG}(BString) \cong \begin{cases} \prod_{j=1}^n \mathbb{Z}/2 \otimes \hat{\mathbb{Z}}_{p_j} & \text{, if } j \equiv 1,2 \mod 8 \text{ and } j \neq 1,2, \\ \prod_{i=1}^n \mathbb{Z}/p_i^\infty & \text{, if } j \equiv 3,7 \mod 8 \text{ and } j \neq 3,7, \\ 0 & \text{, if } j \equiv 0,4,5,6 \mod 8 \text{ or } j = 1,2,3,7. \end{cases}$$

One more time, we have that $CW_{BG}(BSpin) = CW_{BG}(BO)\langle 4\rangle$ and $CW_{BG}(BString) = CW_{BG}(BO)\langle 8\rangle$.

Example 3.3.6. Let now $Sp(m)$ the compact symplectic group of degree m and let $BSp = \bigcup_{m=1}^{\infty} BSp(m)$. BSp is a infinite loop space which homotopy groups are

$$\pi_i BSp \cong \begin{cases} \mathbb{Z}/2 & , \text{if } i \equiv 5,7 \mod 8, \\ \mathbb{Z} & , \text{if } i \equiv 0,4 \mod 8, \\ 0 & , \text{if } i \equiv 1,2,3,7 \mod 8. \end{cases}$$

Note that BSp is 1-connected which $\pi_2 BSp = 0$, a torsion group. Therefore, by Corollary 3.2.4 we obtain the following fibration

$$CW_{BG}(BSp) \to \prod_{i=1}^{n} (BSp)^{\wedge}_{p_i} \to (\prod_{i=1}^{n} (BSp)^{\wedge}_{p_i})_{\mathbb{Q}}.$$

Therefore there is a long exact sequence of homotopy groups,

$$\begin{array}{c} \cdots \to \pi_{j+2}(\prod_{i=1}^{n}(BSp)^{\wedge}_{p_i}) \to \pi_{j+2}((\prod_{i=1}^{n}(BSp)^{\wedge}_{p_i})_{\mathbb{Q}}) \to \pi_{j+1}CW_{BG}(BSp) \longrightarrow \pi_{j+1}(\prod_{i=1}^{n}(BSp)^{\wedge}_{p_i}) \\ \downarrow \\ \cdots \leftarrow \pi_{j}((\prod_{i=1}^{n}(BSp)^{\wedge}_{p_i})_{\mathbb{Q}}) \longleftarrow \pi_{j}(\prod_{i=1}^{n}(BSp)^{\wedge}_{p_i}) \longleftarrow \pi_{j}CW_{BG}(BSp) \leftarrow \pi_{j+1}((\prod_{i=1}^{n}(BSp)^{\wedge}_{p_i})_{\mathbb{Q}}) \end{array}$$

where

- If $j \equiv 1,2 \mod 8$, then,

$$\ldots \longrightarrow 0 \longrightarrow \pi_j CW_{BG}(BSp) \longrightarrow 0 \longrightarrow \ldots$$

- If $j \equiv 3,7 \mod 8$, then

$$\ldots \to 0 \to \pi_{j+1}CW_{BG}(BSp) \to \prod_{i=1}^{n} \hat{\mathbb{Z}}_{p_i} \to \prod_{i=1}^{n}(\hat{\mathbb{Z}}_{p_i} \otimes \mathbb{Q}) \to \pi_j CW_{BG}(BSp) \to 0 \to \ldots$$

- If $j \equiv 5,6 \mod 8$, then

$$(2 \notin \{p_1, \ldots, p_n\}) \qquad \ldots \longrightarrow 0 \longrightarrow \pi_j CW_{BG}(BSp) \longrightarrow 0 \longrightarrow \ldots$$

$$(2 \in \{p_1, \ldots, p_n\}) \qquad \ldots \longrightarrow 0 \longrightarrow \pi_j CW_{BG}(BSp) \longrightarrow \mathbb{Z}/2 \longrightarrow \ldots$$

Therefore, if $2 \notin \{p_1, \ldots, p_n\}$ then

$$\pi_j CW_{BG}(BSp) = \begin{cases} \prod_{i=1}^{n} \mathbb{Z}/p_i^{\infty} & , \text{if } j \equiv 3,7 \mod 8, \\ 0 & , \text{if } j \equiv 0,1,2,4,5,6 \mod 8. \end{cases}$$

and if $2 \in \{p_1, \ldots, p_n\}$ then

$$\pi_j CW_{BG}(BSp) = \begin{cases} \mathbb{Z}/2 & , \text{if } j \equiv 5,6 \mod 8, \\ \prod_{i=1}^{n} \mathbb{Z}/p_i^{\infty} & , \text{if } j \equiv 3,7 \mod 8, \\ 0 & , \text{if } j \equiv 0,1,2,4 \mod 8. \end{cases}$$

Note that if $2 \notin \{p_1, \ldots, p_n\}$, then $CW_{BG}(BSO), CW_{BG}(BO)$ and $CW_{BG}(BSp)$ are weak equivalent.

Chapter 4

Cellular approximations of classifying spaces

In this chapter we will discuss the cellularization of p-complete classifying spaces of Lie groups, and more generally of classifying spaces of p-local compact groups, with respect to classifying spaces of certain p-torsion groups. The definition and main properties of p-local compact groups are given in subsection 4.3.1.

The first section of this chapter contains a result on the cellular covers of the classifying space of a finite p-group and then, it applies to the cellularization of Sylow p-subgroups of finite group. The second section is devoted to describe the cellularization of discrete p-toral groups, and hence of Sylow p-subgroups of p-local compact groups. In Section 3 we present a strategy to compute the cellularization of classifying spaces of p-local compact groups. Stronger results can bo obtained in the particular cases of p-local finite groups and p-completed classifying spaces of compact connected Lie groups. The kernel of a map from a classifying space will play an important role.

4.1 Classifying spaces of finite p-groups

If the cellularization of the classifying space of a p-group is, again, a classifying space of a p-group was an open question solved recently by W. Chachólski, E. Dror-Farjoun, R. Flores and J. Scherer, with a positive answer. We computed, independently, the cellularization of classifying space of finite p-groups with respect to a space with finitely generated abelian fundamental group. Moreover, this result is an imput in our study of the cellularization of classifying space of p-local finite group. In this section, P denotes a finite p-group and $\Omega_{p^r}P$ the (normal) subgroup of P generated by elements of order p^i with $i \leq r$.

An initial computation of cellularization of classifying space of finite p-groups is given from:

Corollary 4.1.1 ([CCS07, Corollary 2.5])**.** *Let $r \geq 1$ and let G be a nilpotent group generated by elements of order p^i with $i \leq r$. Then BG is $B\mathbb{Z}/p^r$-cellular.*

This corollary allows to compute the $B\mathbb{Z}/p^r$-cellularization of these classifying spaces (see also [Flo07, Propositions 4.8 and 4.14]):

Proposition 4.1.2. *Let P be a finite p-group. Let $r \geq 0$. Then $CW_{B\mathbb{Z}/p^r}(BP) \simeq B\Omega_{p^r}P$.*

Proof. The map $B\Omega_{p^r}P \to BP$ induced by the inclusion $\Omega_{p^r}P \hookrightarrow P$ is a $B\mathbb{Z}/p^r$-equivalence because

$$\mathrm{map}_*(B\mathbb{Z}/p^r, B\Omega_{p^r}P) \simeq \mathrm{Hom}(\mathbb{Z}/p^r, \Omega_{p^r}P) \simeq \mathrm{Hom}(\mathbb{Z}/p^r, P) \simeq \mathrm{map}_*(B\mathbb{Z}/p^r, BP).$$

Then $CW_{B\mathbb{Z}/p^r}(BP) \simeq CW_{B\mathbb{Z}/p^r}(B\Omega_{p^r}P)$ and $B\Omega_{p^r}P$ is $B\mathbb{Z}/p^r$-cellular by [CCS07, Corollary 2.5], since a finite p-group is nilpotent. □

Remark 4.1.3. This proposition gives a condition about when BP is $B\mathbb{Z}/p^r$-cellular: BP is $B\mathbb{Z}/p^r$-cellular if and only if P is generated by elements of order p^i with $i \leq r$. Hence, for any finite p-group there exists a r_0 such that BP is $B\mathbb{Z}/p^r$-cellular for all $r \geq r_0$.

Now, using Remark 2.2.5 and the previous proposition it is easy to prove the following corollary:

Corollary 4.1.4. *Let P be a finite p-group. Let BG be the classifying space of a finite abelian p-group and let $p^r = \exp(G)$. Then $CW_{BG}(BP) \simeq B\Omega_{p^r}P$.*

Proof. By Remark 2.2.5 we get $CW_{\prod_{i=1}^n B\mathbb{Z}/p^{r_i}}(BP) \simeq CW_{B\mathbb{Z}/p^r}(BP)$ and the latter is equivalent to $B\Omega_{p^r}P$, by Proposition 4.1.2. □

Lemma 4.1.5. *Let P be a finite p-group. Let G be a finite abelian group, and G_p be the p-torsion subgroup of G. Let $p^r = \exp(G_p)$. Then $CW_{BG}(BP) \simeq B\Omega_{p^r}P$.*

Proof. By Corollary 4.1.4 we have that $CW_{BG_p}(BP) \simeq B\Omega_{p^r}P$. Note that BG_p is a retract of BG and then BG_p is BG-cellular by Proposition 2.1.7.(v). Hence $B\Omega_{p^r}P \simeq CW_{BG_p}(BP)$ is BG-cellular. We will prove that the augmentation $a_{BP}\colon B\Omega_{p^r}P \simeq CW_{BG_p}(BP) \to BP$ is a BG-equivalence. Let $pr\colon G \to G_p$ the projection over G_p. Consider the following commutative diagram

$$\begin{array}{ccc}
\mathrm{map}_*(BG, B\Omega_{p^r}P) & \xrightarrow{\mathrm{map}_*(BG, a_{BP})} & \mathrm{map}_*(BG, BP) \\
{\scriptstyle Bpr^*}\uparrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\uparrow{\scriptstyle Bpr^*} \\
\mathrm{map}_*(BG_p, B\Omega_{p^r}P) & \xrightarrow[\mathrm{map}_*(BG_p, a_{BP})]{\simeq} & \mathrm{map}_*(BG_p, BP)
\end{array}$$

where the vertical arrows are equivalences, since P and $\Omega_{p^r}P$ are p-groups, and the down arrow is an equivalence, since $B\Omega_{p^r}P$ is the BG_p-cellularization of BP. Therefore $\mathrm{map}_*(BG, a_{BP})$ is an equivalence, and hence the map $CW_{BG}(\eta)\colon CW_{BG}(B\Omega_{p^r}P) \to CW_{BG}(BP)$ is an equivalence, where $CW_{BG}(B\Omega_{p^r}P) \simeq B\Omega_{p^r}P$. □

Now we know the cellularization of BP with respect to the classifying space of a finite abelian group. The following proposition gives us a method to extect the previous results to cellularization with respect to any space with finitely generated abelian fundamental group.

Proposition 4.1.6. *Let A be a space with $\pi_1 A$ an abelian group. Let G and H be discrete groups. Assume that $CW_{B\pi_1 A}(BG) \simeq BH$. Then $CW_A(BG) \simeq BH$.*

Proof. Let $a_{BG}\colon BH \simeq CW_{B\pi_1 A}(BG) \to BG$ be the augmentation map. Note that $B\pi_1 A \simeq K(H_1(A;\mathbb{Z}), 1)$ a retract of $SP^\infty(A)$, an A-cellular space by Proposition 2.1.17. Hence BH is A-cellular because $B\pi_1 A$ is so. Consider now the following commutative diagram

$$\begin{array}{ccc} \mathrm{map}_*(A, BH) & \xrightarrow{\mathrm{map}_*(A, a_{BG})} & \mathrm{map}_*(A, BG) \\ \simeq \uparrow & & \uparrow \simeq \\ \mathrm{map}_*(B\pi_1 A, BH) & \underset{\mathrm{map}_*(B\pi_1 A, a_{BG})}{\xrightarrow{\simeq}} & \mathrm{map}_*(B\pi_1 A, BG) \end{array}$$

where the horinzotal arrows are equivalences because G and H are discrete groups, and the down arrow is an equivalence because BH is the $B\pi_1 A$-cellularization of BG. It follows that $\mathrm{map}_*(A, a_{BG})$ is an equivalence, this implies $CW_A(BG) \simeq CW_A(BH) \simeq BH$, because BH is A-cellular. □

We can summarizethe previous results in the following theorem:

Theorem 4.1.7. *Let P be a finite p-group. Let A be a space with $G = \pi_1 A$ a finitely generated abelian group. Then*

(i) If $\pi_1 A$ is infinite, then any CW-complex is A-cellular. In particular, BP is A-cellular.

(ii) If $\pi_1 A$ is finite and a Sylow p-subgroup of G, G_p, is not trivial, then $CW_A(BP) \simeq B\Omega_{p^r} P$, where $p^r = \exp(G_p)$.

(iii) If $\pi_1 A$ is finite and $(p, |G|) = 1$, then BP is A-null and hence $CW_A(BP)$ is contactible.

Proof. (i) The result follows from Corollary 2.1.18.

(ii) From Lemma 4.1.5 we get $CW_{B\pi_1 A}(BP) \simeq B\Omega_{p^r} P$ and hence Proposition 4.1.6 proves that $CW_A(BP) \simeq BP'$.

(iii) In this case $\mathrm{Hom}(\pi_1 A, P) \cong \{e\}$ because $\pi_1 A$ does not have elements of order p, and $\mathrm{map}_*(A, BP) \simeq \mathrm{Hom}(\pi_1 A, P)$.

□

4.2 Classifying spaces of discrete p-toral groups

In this section we want to present some results on the cellularization of classifying spaces of discrete p-toral groups. As in the previous section we want to understand when this spaces are cellular with respect to the classifying space of a p-torsion group. In this section, P denotes a discrete p-toral group, $P_0 \cong (\mathbb{Z}/p^\infty)^r$ denotes the maximal divisible subgroup, called the maximal torus in P, $\pi = P/P_0$ the group of components of P. Let $B_{p^m} = B\mathbb{Z}/p^\infty \times B\mathbb{Z}/p^m$ for m a non-negative integer. Obviously, not all classifying spaces of discrete p-toral group will be cellular with respect to $B\mathbb{Z}/p^m$ for some m, for instance, $CW_{B\mathbb{Z}/p^m}(B\mathbb{Z}/p^\infty) \simeq B\mathbb{Z}/p^m$. In fact, it is not difficult to compute the cellularization with respect to $B\mathbb{Z}/p^\infty$ or $B\mathbb{Z}/p^m$ of a discrete p-toral group:

Proposition 4.2.1. *Let P be a discrete p-toral group and $P_0 \cong (\mathbb{Z}/p^\infty)^r$ a maximal torus of P. Then the inclusion $BP_0 \hookrightarrow BP$ is a $B\mathbb{Z}/p^\infty$-equivalence and hence $CW_{B\mathbb{Z}/p^\infty}(BP) \simeq BP_0$.*

Proof. Let $\pi = P/P_0$ be the group of components of P. Consider the fibration

$$BP_0 \xrightarrow{f} BP \xrightarrow{g} B\pi$$

and if we apply it the functor $\mathrm{map}_*(B\mathbb{Z}/p^\infty, -)$, then we get the fibration

$$\mathrm{map}_*(B\mathbb{Z}/p^\infty, BP_0) \xrightarrow{f_*} \mathrm{map}_*(B\mathbb{Z}/p^\infty, BP)_{\{c\}} \to \mathrm{map}_*(B\mathbb{Z}/p^\infty, B\pi)_c,$$

where $\mathrm{map}_*(B\mathbb{Z}/p^\infty, B\pi)_c$ is the connected component of the constant map $c\colon B\mathbb{Z}/p^\infty \to B\pi$ and $\mathrm{map}_*(B\mathbb{Z}/p^\infty, BP)_{\{c\}}$ are the connected components of $\mathrm{map}_*(B\mathbb{Z}/p^\infty, BP)$ that map into $\mathrm{map}_*(B\mathbb{Z}/p^\infty, B\pi)_c$ via g_*. Since $\mathrm{map}_*(B\mathbb{Z}/p^\infty, B\pi) \cong \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi) \cong \{e\}$, $\mathrm{map}_*(B\mathbb{Z}/p^\infty, B\pi)$ is connected and hence $\mathrm{map}_*(B\mathbb{Z}/p^\infty, B\pi)_c \simeq \mathrm{map}_*(B\mathbb{Z}/p^\infty, B\pi) \simeq *$. Therefore, f_* is a $B\mathbb{Z}/p^\infty$-equivalence and $CW_{B\mathbb{Z}/p^\infty}(BP) \simeq CW_{B\mathbb{Z}/p^\infty}(BP_0) \simeq BP_0$, since $BP_0 \cong (B\mathbb{Z}/p^\infty)^r$ is $B\mathbb{Z}/p^\infty$-cellular. □

The study of the $B\mathbb{Z}/p$-cellularization of discrete p-toral group is described in [CF13, Exmaple 6.16] and it is not difficult to extend this result to $B\mathbb{Z}/p^m$-cellularization:

Proposition 4.2.2. *Let P be a discrete p-toral group. Then the $B\mathbb{Z}/p^m$-cellularization of BP is equivalent to $B\Omega_{p^m}(P)$.*

Proof. Note that

$$\mathrm{map}_*(B\mathbb{Z}/p^m, BP) \simeq \mathrm{Hom}(\mathbb{Z}/p^m, P) \simeq \mathrm{Hom}(\mathbb{Z}/p^m, \Omega_{p^m}P) \simeq \mathrm{map}_*(B\mathbb{Z}/p^m, B\Omega_{p^m}(P)),$$

that is, the map $B\Omega_{p^m}(P) \to BP$ is a $B\mathbb{Z}/p^m$-equivalence, hence

$$CW_{B\mathbb{Z}/p^m}(BP) \simeq CW_{B\mathbb{Z}/p^m}(B\Omega_{p^m}P)$$

and $B\Omega_{p^m}P$ is $B\mathbb{Z}/p^m$-cellular according to Theorem 4.1.7. □

Nevertheless, $B\mathbb{Z}/p^\infty$ is $B\mathbb{Z}/p^\infty$-cellular and, in particular, $(B\mathbb{Z}/p^\infty \times B\mathbb{Z}/p^m)$-cellular for all $m \geq 0$. The main goal of this section is to prove that for any classifying space of a discrete p-toral group there is a $m_0 \geq 0$ such that it is cellular with respect to $B\mathbb{Z}/p^\infty \times B\mathbb{Z}/p^m$ for all $m \geq m_0$.

If π is a finite group and $x \in \pi$ is an element of order $|x| \leq p^m$, then there is a homomorphism $\alpha_x\colon \mathbb{Z}/p^m \to \pi$ defined by $\alpha_x(1) = x$. We denotes by $f_x\colon B\mathbb{Z}/p^m \to B\pi$ the map induced by the homomorphism α_x in classifying spaces.

The next proposition is a result about lifting of maps from $B\mathbb{Z}/p^m$ to the group of components of a p-toral group.

Proposition 4.2.3. *Let P be a discrete p-toral group, $P_0 \cong (\mathbb{Z}/p^\infty)^r$ a maximal torus of P and π the group of components of P. For any generator x of π there is a $m_x \geq 0$ such that the map*

$f_x\colon B\mathbb{Z}/p^{m_x} \to B\pi$ lifts to BP, that is, there is a map $\tilde{f}_x\colon B\mathbb{Z}/p^{m_x} \to BP$ such that the following diagram

$$\begin{array}{ccc} & & BP \\ & \overset{\tilde{f}_x}{\nearrow} & \downarrow{\scriptstyle Bpr} \\ B\mathbb{Z}/p^{m_x} & \xrightarrow[f_x]{} & B\pi, \end{array}$$

is commutative, i.e., such that $Bpr \circ \tilde{f}_x = f_x$.

Proof. Let x be a generator of π and let $g \in pr^{-1}(x)$. Let Q the cyclic subgroup of P generated by g, hence $Q \cong \mathbb{Z}/p^m$ for certain m since P is locally finite p-group (see [BLO07, Proposition 1.2]). Therefore if we define $\tilde{f}_x\colon B\mathbb{Z}/p^m \to BP$ as the map induced in classifying spaces of the homomorphism $\beta_x\colon \mathbb{Z}/p^m \to P$ given by $\beta_x(1) = g$, then $Bpr \circ \tilde{f}_{x_i} = f_{x_i}$ if and only if $pr \circ \beta_x = \alpha_x$, and $pr(\beta_x(1)) = pr(g) = x = \alpha_x(1)$. □

Remark 4.2.4. Note that given x a generator of π the integer m_x does not depend on the choice of the pre-image $g \in pr^{-1}(x)$, becuase if $g, h \in pr^{-1}(x)$, then $\langle g\rangle \cong \mathbb{Z}/p^{m_g}$, $\langle h\rangle \cong \mathbb{Z}/p^{m_h}$ and there exists a $t \in P$ such that $t \cdot \mathbb{Z}/p^{m_g} \cdot t^{-1} = \mathbb{Z}/p^{m_h}$, that is, $c_t : \mathbb{Z}/p^{m_g} \to \mathbb{Z}/p^{m_h}$ is an isomorphism, and hence $m_g = m_h$.

Then, we are ready to prove the main result of this section (we follow ideas in [CCS07]):

Proposition 4.2.5. *Let P be a discrete p-toral group. Then there is a non-negative integer m_0 such that BP is B_{p^m}-cellular for all $m \geq m_0$.*

Proof. Consider the Chachólski's fibration $CW_{B_{p^m}}(BP) \xrightarrow{c} BP \xrightarrow{g} P_{\Sigma B_{p^m}}(C)$. If we prove that $g \simeq *$ then BP is B_{p^m}-cellular, because if $g \simeq *$, then $CW_{B_{p^m}}(BP) \simeq BP \times \Omega P_{\Sigma B_{p^m}}(C)$, hence

$$* \simeq P_{B_{p^m}}(CW_{B_{p^m}}(BP)) \simeq P_{B_{p^m}}(BP) \times P_{B_{p^m}}(\Omega P_{\Sigma B_{p^m}}(C))$$

and $\Omega P_{\Sigma B_{p^m}}(C) \simeq P_{B_{p^m}}(\Omega P_{\Sigma B_{p^m}}(C)) \simeq *$. Finally $P_{\Sigma B_{p^m}}(C) \simeq *$ because it is connected.

Now consider the fibration

$$BP_0 \xrightarrow{\iota} BP \xrightarrow{Bpr} B\pi.$$

Note that, on the one hand,

$$\mathrm{map}_*(BP_0, \Omega P_{\Sigma B_{p^m}}(C)) \simeq \mathrm{map}_*(BP_0, P_{B_{p^m}}(\Omega C)) \simeq \mathrm{map}_*(P_{B_{p^m}}(BP_0), P_{B_{p^m}}(\Omega C)) \simeq *,$$

because BP_0 is B_{p^m}-cellular and hence $P_{B_{p^m}}(BP_0) \simeq *$. That means, $\Omega P_{\Sigma B_{p^m}}(C)$ is BP_0-null.

On the other hand, since BP_0 is B_{p^m}-cellular, there is a map

$$\tilde{\iota}\colon BP_0 \to CW_{B_{p^m}}(BP)$$

such that the following diagram

$$\begin{array}{ccccc} & & BP_0 & & \\ & \overset{\tilde{\iota}}{\swarrow} & \downarrow{\scriptstyle \iota} & \overset{g\circ\iota}{\searrow} & \\ CW_{B_{p^m}}(BP) & \xrightarrow{c} & BP & \xrightarrow{g} & P_{\Sigma B_{p^m}}(C) \\ & & \downarrow{\scriptstyle Bpr} & & \\ & & B\pi & & \end{array}$$

is commutative. Therefore $g \circ \iota \simeq g \circ c \circ \tilde{\iota} \simeq *$. Hence, by Zabrodsky's Lemma, there is a map $\overline{g}\colon B\pi \to P_{\Sigma B_{p^m}}(C)$ such that the following diagram

$$\begin{array}{ccccc}
 & & BP_0 & & \\
 & {\scriptstyle \tilde{\iota}}\swarrow & \downarrow{\scriptstyle \iota} & \searrow{\scriptstyle g\circ\iota} & \\
CW_{B_{p^m}}(BP) & \xrightarrow{c} & BP & \xrightarrow{g} & P_{\Sigma B_{p^m}}(C) \\
 & & \downarrow{\scriptstyle Bpr} & \nearrow{\scriptstyle \overline{g}} & \\
 & & B\pi & &
\end{array}$$

is commutative and $g \simeq *$ if and only if $\overline{g} \simeq *$.

We will prove that BP is B_{p^m}-cellular or equivalenty that $\overline{g} \simeq *$ by induction over the order of π. Assume first that $\pi \cong \mathbb{Z}/p^r$. By Proposition 4.2.3, there is a $m \geq 0$ and a map $\tilde{f}\colon B\mathbb{Z}/p^m \to BP$ such that $Bpr \circ \tilde{f} = f = B\alpha$, where $\alpha(1)$ is the generator of $\pi \cong \mathbb{Z}/p^r$. Moreover, since $B\mathbb{Z}/p^m$ is B_{p^m}-cellular there is a map $\overline{f}\colon B\mathbb{Z}/p^r \to CW_{B_{p^m}}(BP)$ such that the following diagram

$$\begin{array}{ccccc}
CW_{B_{p^m}}(BP) & \xrightarrow{c} & BP & \xrightarrow{g} & P_{\Sigma B_{p^m}}(C) \\
\uparrow{\scriptstyle \overline{f}} & \nearrow{\scriptstyle \tilde{f}} & \downarrow{\scriptstyle Bpr} & \nearrow{\scriptstyle \overline{g}} & \\
B\mathbb{Z}/p^m & \xrightarrow{f} & B\mathbb{Z}/p^r & &
\end{array}$$

is commutative, i.e., $g \circ f \simeq g \circ c \circ \overline{f} \simeq *$, hence $\overline{g} \circ f \simeq *$. Therefore $\overline{g} \simeq *$, because for $i > 1$, $\pi_i(B\mathbb{Z}/p^r) = 0$ and hence $\pi_i(\overline{g}) = 0$, and since $\alpha(1)$ is the generator of $\mathbb{Z}/p^r$, $\pi_1(\overline{g})(\alpha(1)) = \pi_1(\overline{g} \circ f)(1) = 0$.

Let $\{x_1, \ldots, x_n\}$ be a minimal set of generators of π and let $H \trianglelefteq \pi$ the normal subgroup generated by $x_1, \ldots, x_{n-1}$ and its conjugates by powers of x_n, since π is nilpotent (it is a finite p-group) there is a short exact sequence

$$0 \longrightarrow H \xrightarrow{j} \pi \xrightarrow{k} \mathbb{Z}/p^a \longrightarrow 0,$$

where $\mathbb{Z}/p^a$ is generated by the image of x_n. Consider the pull-back of Bpr over Bj, we obtain the diagram

$$\begin{array}{ccccc}
BP_0 & = & BP_0 & & \\
\downarrow & & \downarrow & & \\
BP' & \xrightarrow{h} & BP & \xrightarrow{g} & P_{\Sigma B_{p^m}}(C) \\
\downarrow{\scriptstyle pr'} & & \downarrow{\scriptstyle Bpr} & \nearrow{\scriptstyle \overline{g}} & \\
BH & \xrightarrow{Bj} & B\pi & &
\end{array}$$

where BP' is also the classifying space of a discrete p-toral group. By induction BP' is B_{p^m}-cellular, hence $g \circ h \simeq *$, because $\mathrm{map}_*(BP', P_{\Sigma B_{p^m}}(C)) \simeq \mathrm{map}_*(BP', CW_{B_{p^m}}(P_{\Sigma B_{p^m}}(C))) \simeq *$. Therefore $\overline{g} \circ Bj \simeq *$ by Zabrodsky's Lemma.

Consider now the diagram

$$\begin{array}{ccccc} B(\langle x_n\rangle \cap H) & \longrightarrow & BH & & \\ \downarrow & & \downarrow{\scriptstyle Bj} & \searrow^{\overline{g}\circ Bj\simeq *} & \\ B(\langle x_n\rangle) & \longrightarrow & B\pi & \xrightarrow{\overline{g}} & P_{\Sigma B_{p^m}}(C) \\ \downarrow & & \downarrow & \nearrow_{g'} & \\ B\mathbb{Z}/p^a & = & B\mathbb{Z}/p^a & & \end{array}$$

By induction BH is B_{p^m}-cellular, hence $P_{\Sigma B_{p^m}}(BH) \simeq *$ and therefore

$$\mathrm{map}_*(BH, P_{\Sigma B_{p^m}}(C)) \simeq \mathrm{map}_*(P_{\Sigma B_{p^m}}(BH), P_{\Sigma B_{p^m}}(C)) \simeq *.$$

By Zabrodsky's lemma there is a map $g'\colon B\mathbb{Z}/p^a \to P_{\Sigma B_{p^m}}(C)$ making the previous diagram commutative and such that $\overline{g} \simeq *$ if and only if $g' \simeq *$. Now again Zabrodsky's lemma applied to the left fibration gives $\overline{g}|_{B(\langle x_n\rangle)} \simeq *$ if and only if $g' \simeq *$. Since $B(\langle x_n\rangle)$ is B_{p^m}-cellular by induction, $\mathrm{map}_*(B(\langle x_n\rangle), P_{\Sigma B_{p^m}}(C)) \simeq *$ and we get $\overline{g}|_{B(\langle x_n\rangle)} \simeq *$, hence $g \simeq *$, and finally BP is B_{p^m}-cellular. □

4.3 Classifying spaces of p-local compact groups

In this section we give the strategy to compute the cellularization with respect to A of the classifying space of $(S, \mathcal{F}, \mathcal{L})$, a p-local compact group, where A is a classifying space of type $B\mathbb{Z}/p^m$ or $B\mathbb{Z}/p^\infty \times B\mathbb{Z}/p^m$ for $m \geq 1$. This section is divided in two subsection. In the first subsection we present the basic definitions and result about p-local compact group (we follow the source [BLO07]). The second one is devoted to the concept of the kernel of a map. In the third one we will describe the kernel of the map $r_p^\wedge\colon |\mathcal{L}|_p^\wedge \to P_{\Sigma A}(C)_p^\wedge$, used to compute the A-cellularization of $|\mathcal{L}|_p^\wedge$.

4.3.1 p-local compact groups

In this section we want to introduce the concept of "p-local homotopy theory" of classifying spaces of finite groups, or more generally of compact Lie groups, this means, the homotopy theory of its p-completion. If G is a finite group, then it turns out that there is a close connection between the p-local homotopy theory of BG and the "p-local structure" of G, this means, the conjugacy relations in a Sylow p-subgroup of G. This connection then suggested the construction of certain spaces which have many of the same properties as have p-completed classifying spaces of finite and compact Lie group: the classifying spaces of "p-local finite groups" and "p-local compact groups".

A p-local compact group is an algebraic object which consists of a system of fusion data in a discrete p-local group S, as formalized by Ll. Puig in the finite case and generalized by C. Broto, R. Levi and B. Oliver. Such objects have classifying spaces which satisfy many of the homotopy theoretic properties of p-completed classifying spaces of finite groups.

We need to describe discrete p-toral group, which play the role of Sylow p-group in p-local compact groups:

Definition 4.3.1. A *p-toral group* is a compact Lie group P which contains a normal subgroup $P_0 \trianglelefteq P$, isomorphic to a torus, i.e, $P_0 \cong (S^1)^r$, and such that the quotient P/P_0 is a finite p-group.

For such a group, we say that P_0 is the *connected component* or *maximal torus* of P and P/P_0 is the *group of components* of P.

The *rank* of a p-toral group P is the rank of the maximal torus P_0. That is, if $P_0 \cong (S^1)^r$, then $rk(P) = r$.

We have that a p-toral group is a central extension of a torus and a finite p-group, in this way a discrete p-toral group is a central extension of a discrete p-torus, that is, a finite produt of copies of $\mathbb{Z}/p^\infty$, and a finite p-group.

Definition 4.3.2. A *discrete p-toral group* is a group P which contains a normal subgroup $P_0 \trianglelefteq P$, isomorphic to $(\mathbb{Z}/p^\infty)^r$, and such that the quotient P/P_0 is a finite p-group. In this case, the *rank* of P is $rk(P) = r$, the *connected component* or *maximal torus* of P is P_0, and P/P_0 is called the *group of components* of P. If $P = P_0$, then P is called *connected*.

A particular case of discrete p-toral group are the finite p-groups. In fact, a finite p-group is a discrete p-toral group of rank 0.

Therefore a fusion system over a discrete p-toral group is defined as follows:

Definition 4.3.3. A *fusion system* over a discrete p-toral group S is a category $\mathcal{F}$, where $\mathrm{Ob}(\mathcal{F})$ is the set of all subgroups of S, and which satisfies the following two properties for all $P, Q \leq S$:

(a) $\mathrm{Hom}_S(P, Q) \subset \mathrm{Hom}_{\mathcal{F}}(P, Q) \subset \mathrm{Inj}(P, Q)$; and

(b) each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ is the conposite of an isomorphism in $\mathcal{F}$ followed by an inclusion.

Let G be a finite group and let S be a Sylow p-sugbroup. The *p-fusion system of G over S* is the category $\mathcal{F}_S(G)$ where $\mathrm{Ob}(\mathcal{F}_S(G))$ is the set of all subgroups of S and $\mathrm{Mor}_{\mathcal{F}_S(G)}(P, Q) = \mathrm{Hom}_G(P, Q)$ for all $P, Q \leq S$, where $\mathrm{Hom}_G(P, Q) = \{\varphi \in \mathrm{Hom}(P, Q) \mid \varphi = c_g$ for some $g \in G\}$. $F_S(G)$ is a fusion system over $S \in \mathrm{Syl}_p(G)$.

To define a p-local compact group we need to define a special class of fusion system, the saturated fusion system, and for this we have to introduce the following technical definitions:

Definition 4.3.4. Let $\mathcal{F}$ be a fusion system over a discrete p-toral S. Two subgroups $P, Q \leq S$ are said to be *$\mathcal{F}$-conjugated* if they are isomorphics as objects of $\mathcal{F}$.

Definition 4.3.5. Let $\mathcal{F}$ be a fusion system over a discrete p-toral S:

(i) A subgroup $P \leq S$ is *$\mathcal{F}$-centric* if $C_S(P') = Z(P')$ for all $P' \leq S$ which are $\mathcal{F}$-conjugate to P.

(ii) A subgroup $P \leq S$ is *$\mathcal{F}$-radical* if $\mathrm{Out}_{\mathcal{F}}(P') = Aut_{\mathcal{F}}(P)/\mathrm{Inn}(P)$ is p-reduced, i.e., if $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$.

If $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G, then $P \leq S$ is $\mathcal{F}$-centric if and only if P is p-centric in G (i.e., $Z(P) \in \mathrm{Syl}_p(C_G(P))$), and P is $\mathcal{F}$-radical if and only if $N_G(P)/P{\cdot}C_G(P)$ is p-reduced.

Definition 4.3.6. Let $\mathcal{F}$ be a fusion system over a finite p-group S:

(i) A subgroup $P \leq S$ is *fully centralized* in $\mathcal{F}$ if $|C_S(P)| \geq |C_S(P')|$ for all $P' \leq S$ which is F-conjugate to P.

(ii) A subgroup $P \leq S$ is *fully normalized* in $\mathcal{F}$ if $|N_S(P)| \geq |C_S(P')|$ for all $P' \leq S$ which is F-conjugate to P.

(iii) $\mathcal{F}$ is a *saturated fusion system* if the following two conditions hold:

 (I) For all $P \leq S$ which is fully normalized in $\mathcal{F}$, P is fully centralized in $\mathcal{F}$, $\mathrm{Out}_{\mathcal{F}}(P)$ is finite, and $\mathrm{Out}_S(P) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(P))$.

 (II) If $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ are such that $\varphi(P)$ is fully centralized, and if we set

$$N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi(P))\},$$

 then there is $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\bar{\varphi}|_P = \varphi$.

 (III) If $P_1 \leq P_2 \leq \ldots$ is an increasing sequence of subgroups of S, with $P_\infty = \bigcup_{n=1}^{\infty} P_n$, and $\varphi \in \mathrm{Hom}(P_\infty, S)$ is any homomorphism such that $\varphi|_{P_n} \in \mathrm{Hom}_{\mathcal{F}}(P_n, S)$ for all n, then $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P_\infty, S)$

If G is a finite group and $S \in \mathrm{Syl}_p(G)$, then the category $\mathcal{F}_S(G)$ is a saturated fusion system (see [BLO03b, Proposition 1.3]).

The role of the p-completed classifying space of a finite group G is replaced by the p-completion of the nerve of certain category associated to a saturated fusion system.

Definition 4.3.7. Let $\mathcal{F}$ be a fusion system over a discret p-toral group S. A *centric linking system* associated to $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor $\pi\colon \mathcal{L} \to \mathcal{F}^c$, and "distinguished" monomorphisms $P \xrightarrow{\delta_P} \mathrm{Aut}_{\mathcal{L}}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$, which satisfies the following conditions:

(A) π is the identity on objects and surjective on morphisms. For each pair of object $P, Q \leq \mathcal{L}$, $Z(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leq \mathrm{Aut}_{\mathcal{L}}(P)$), and π induces a bijection

$$\mathrm{Mor}_{\mathcal{L}}(P, Q)/Z(P) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{F}}(P, Q).$$

(B) For each $\mathcal{F}$-centric subgroup $P \leq S$ and each $g \in P$, π sends $(\delta_P(g)) \in \mathrm{Aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.

(C) For each $f \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ and each $g \in P$, $f \circ \delta_P(g) = \delta_Q(\pi f(g)) \circ f$.

Let G be a finite group and $S \in \mathrm{Syl}_p(G)$, then the *associated linking category of G over S* is the category $\mathcal{L}_S(G)$ where $\mathrm{Ob}(\mathcal{L}_S(G))$ is the set of all subgroups of S and $\mathrm{Mor}_{\mathcal{L}_S(G)}(P, Q) = \{x \in G \mid xP^{-1}x \leq Q\}/O^p(C_G(P))$ for all $P, Q \leq S$. The category $\mathcal{L}_S(G)$ is a centric linking system associated to $\mathcal{F}_S(G)$.

Finally, the definition of p-local compact group is the following:

Definition 4.3.8. A *p-local compact group* is defined to be a triple $(S, \mathcal{F}, \mathcal{L})$, where S is a discret p-toral group, $\mathcal{F}$ is a saturated fusion system over S, and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The *classifying space* of the triple $(S, \mathcal{F}, \mathcal{L})$ is the p-completed nerve $|\mathcal{L}|^{\wedge}_p$. If S is a finite p-group, hence $(S, \mathcal{F}, \mathcal{L})$ is called *p-local finite group*.

Given a finite group G with Sylow p-subgroup S, then $(S, \mathcal{F}_S(G); \mathcal{L}_S(G))$ is a p-local finite group with classifying space $|\mathcal{L}_S(G)|^{\wedge}_p \simeq BG^{\wedge}_p$ (see [BLO03a, Proposition 1.1]). Given a compact Lie group G it is possible construct a saturated fusion system $\mathcal{F}_S(G)$ over a certain discret p-toral subgroup S of G and a linking systems $\mathcal{L}_S(G)$ associated to $\mathcal{F}_S(G)$ such that $|\mathcal{L}_S(G)|^{\wedge}_p \simeq BG^{\wedge}_p$ as follows

Let G be a compact Lie group, the *fusion system of a compact Lie group* is defined as follows: For any $S \in \mathrm{Syl}_p(G)$, $\mathcal{F}_S(G)$ is a category which objects are $P \leq S$ and for all $P, Q \leq S$,

$$\mathrm{Mor}_{\mathcal{F}_S(G)}(P, Q) = \mathrm{Hom}_G(P, Q) \cong N_G(P, Q)/C_G(P)$$

is the set of homomorphisms from P to Q induced by conjugation by elements of G.

Theorem 4.3.9 ([BLO07, Theorem 9.10])**.** *Fix a compact Lie group G and a maximal discrete p-toral subgroup $S \in \mathrm{Syl}_p(G)$. Then there exists a centric linking system $\mathcal{L}^c_S(G)$ associated to $F_S(G)$ such that $(S, F_S(G), \mathcal{L}_S(G))$ is a p-local compact group with classifying space $|\mathcal{L}_S(G)|^{\wedge}_p \simeq BG^{\wedge}_p$.*

One of the standard techniques used when stduying maps between p-completed classifying spaces of finite groups is to replace them by the p-completion of a homotopy colimit of simpler spaces.

Definition 4.3.10. Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. The *orbit category* of $\mathcal{F}$ is the category $\mathcal{O}(\mathcal{F})$ whose objects are the subgroups of S, and whose morphisms are defined by

$$\mathrm{Mor}_{\mathcal{O}(\mathcal{F})}(P, Q) = \mathrm{Rep}_{\mathcal{F}}(P, Q) := \mathrm{Inn}(Q)/\mathrm{Hom}_{\mathcal{F}}(P, Q).$$

We let $\mathcal{O}^c(\mathcal{F})$ denote the full subcategory of $\mathcal{O}(\mathcal{F})$ whose objects are the $\mathcal{F}$-centric subgroups of S. If $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$, then $\tilde{\pi}$ denotes the composite functor

$$\tilde{\pi}\colon \mathcal{L} \xrightarrow{\pi} \mathcal{F}^c \twoheadrightarrow \mathcal{O}^c(\mathcal{F})$$

We next look at the homotopy type of the nerve of a centric lonking system.

Proposition 4.3.11. *Fix a saturated fusion system $\mathcal{F}$ over a discrete p-toral group S and an associated centric linking system $\mathcal{L}$, and let $\tilde{\pi} \to \mathcal{O}^c(\mathcal{F})$ be the projection functor. Let*

$$\tilde{B}\colon \mathcal{O}^c(\mathcal{F}) \to \mathrm{Top}$$

be the left homotopy Kan extension over $\tilde{\pi}$ of the constant functor $\mathcal{L} \xrightarrow{*} \mathrm{Top}$. *Then $\tilde{B}$ is a homotopy lifting of the homotopy functor $P \mapsto BP$, and*

$$|\mathcal{L}| \simeq \mathrm{hocolim}_{\mathcal{O}^c(\mathcal{F})}(\tilde{B}).$$

We will use an important class of subgroups of S that are preserved by fusion:

Definition 4.3.12. Let $\mathcal{F}$ be a fusion system over a discrete p-toral S. Then a subgroup $K \subseteq S$ is *strongly $\mathcal{F}$-closed* if for all $P \leq K$ and all morphism $\varphi\colon P \to S$ in $\mathcal{F}$ we have $\varphi(P) \leq K$.

Note that if G is a finite group and $S\, in\, \mathrm{Syl}_p(G)$, $K \subseteq S$ is strongly $\mathcal{F}_S(G)$-closed if and only if K is *strongly closed in* G, i.e., if for all $k \in K$ and $g \in G$ such that $c_g(s) \in S$, then $c_g(s) \in K$.

It is easy to see that an intersection of strongly $\mathcal{F}$-closed subgroups is a strongly $\mathcal{F}$-closed subgroup. Hence note that, given a subgroup $P \leq S$, we can consider the smallest strongly $\mathcal{F}$-closed subgroup of S that contains P.

4.3.2 The kernel of a map from a classifying space

The kernel of map $f\colon BG^\wedge_p \to Y^\wedge_p$, where G is a compact Lie group and $Y^\wedge_p$ is p-complete and $\Sigma B\mathbb{Z}/p$-null space, is defined by D. Notbohm in [Not94]. He defines it, for a fixed Sylow p-subgroup S of G, by $\ker(f) := \{g \in S \mid f|_{B\langle g\rangle} \simeq *\}$. In general, following Notbohm's description,

Definition 4.3.13. Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local compact group and let $Y^\wedge_p$ be a p-complete and $\Sigma B\mathbb{Z}/p$-null space. Let $f\colon |\mathcal{L}|^\wedge_p \to Y^\wedge_p$ be a pointed map. Then

$$\ker(f) := \{g \in S \mid f|_{B\langle g\rangle} \simeq *\}.$$

The first question about $\ker(f) \subseteq S$ is: is $\ker(f)$ a (normal) subgroup of S?

Proposition 4.3.14. *Let $f\colon |\mathcal{L}|^\wedge_p \to Y^\wedge_p$ as in Definition 4.3.13. Then* $\ker(f)$ *is a normal subgroup of S.*

Proof. Let $x \in \ker(f)$, since $\langle x\rangle = \langle x^{-1}\rangle$, $x^{-1} \in \ker(f)$. Hence to prove that $\ker(f)$ is a group we have only prove that if $x, y \in \ker(f)$, then $xy \in \ker(f)$.

Consider the composite $B\langle x, y\rangle \to BS \to X \to Y^\wedge_p$, by [Not94, Proposition 2.4], this map is null-homotopic, because $x, y \in \ker(f)$. Hence $f|_{B\langle xy\rangle} \simeq *$, since $\langle xy\rangle \hookrightarrow \langle x, y\rangle$. Then $xy \in \ker(f)$.

To prove that $\ker(f)$ is normal in S, let $x \in S$ and $y \in \ker(f)$. Hence we have to prove that $xyx^{-1} \in \ker(f)$. Let $\iota_{BP}\colon BP \hookrightarrow BS$ the induced map by the inclusion of a subgroup P of S in S. Since $c_x\colon \langle y\rangle \to \langle xyx^{-1}\rangle$ is an isomorphism, $\iota_{B\langle xyx^{-1}\rangle} \circ Bc_x \simeq \iota_{B\langle y\rangle}$, and hence $f|_{B\langle xyx^{-1}\rangle} \simeq f|_{B\langle y\rangle} \circ Bc_x \simeq *$, because $f|_{B\langle y\rangle} \simeq *$. Then $xyx^{-1} \in \ker(f)$. □

Now, we are interested in proving that $\ker(f)$ is a strongly $\mathcal{F}$-closed subgroup of S.

Proposition 4.3.15. *Let $f\colon |\mathcal{L}|^\wedge_p \to Y^\wedge_p$ as in Definition 4.3.13. Then* $\ker(f)$ *is strongly $\mathcal{F}$-closed.*

Proof. Let $P \leq \ker(f)$ and $\varphi\colon P \to S$. First, let $x \in P$, we want to prove first that $\varphi(x) \in \ker(f)$. as in the previous proposition, $\iota_{B\langle\varphi(x)\rangle} \circ B\varphi \simeq \iota_{B\langle x\rangle}$ and hence $f|_{B\langle\varphi(x)\rangle} \simeq f|_{B\langle x\rangle} \circ B\varphi \simeq *$, because $f|_{B\langle x\rangle} \simeq *$. Then $\varphi(x) \in \ker(f)$. Since $\varphi(P) = \langle\varphi(x) \mid x \in P\rangle$ and $f|_{B\langle\varphi(x)\rangle} \simeq *$ for all $x \in P$, $\varphi(P) \leq \ker(f)$. □

Furthermore, Dwyer proves in [Dwy96, Theorem 5.1] that if we have a compact Lie group G (in particular a finite group), then a map $f\colon BG^\wedge_p \to Y^\wedge_p$, where $Y^\wedge_p$ is p-completed and $\Sigma B\mathbb{Z}/p$-null, is null-homotopic if and only if f is null-homotopic restricted to the classifying space of a Sylow p-subgroup, this means, f is null-homotopic if and only if $\ker(f) = S$. Now we will prove a version of this theorem for p-local compact groups:

Theorem 4.3.16. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local compact group. Let $Y^\wedge_p$ be a p-complete and $\Sigma B\mathbb{Z}/p$-null space. Then a map $f\colon |\mathcal{L}|^\wedge_p \to Y^\wedge_p$ is null-homotopic if and only if* $\ker(f) = S$.

Proof. Obviously if $f \simeq *$ then $f|_{BS} \simeq *$ and hence $\ker(f) = S$. Hence, assume that $f|_{BS} \simeq *$, we have to prove that $f \simeq *$. According to the Proposition 4.3.11, $|\mathcal{L}|^\wedge_p \simeq (\mathrm{hocolim}_{O^c(\mathcal{F})}(BP))^\wedge_p$, where BP denotes the homotopy lifting $\tilde{B}$ for $P \in \mathcal{F}^c$. Note that $f|_{BP} \simeq *$ for all $P \in \mathcal{F}^c$ because the map $BP \to |\mathcal{L}|^\wedge_p$ factorizes by BS. Therefore we have two maps

$$(\mathrm{hocolim}_{O^c(\mathcal{F})}(BP))^\wedge_p \underset{*}{\overset{f}{\rightrightarrows}} Y^\wedge_p\ ,$$

such that $f|_{BP} \simeq *|_{BP} \simeq *$ for all $P \in \mathcal{F}^c$. Moreover, the obstructions for this maps to be homotopic are in $\lim^i_{O^c(\mathcal{F})} \pi_i(\mathrm{map}(BP, Y^\wedge_p)_c)$, for $i \geq 1$. Since a $B\mathbb{Z}/p$-null space is BQ-null for any finite p-group Q (by Lemma 4.3.17), $Y^\wedge_p$ is ΣBP-null and hence $\mathrm{map}_*(BP, Y^\wedge_p)$ is homotopic discret, therefore $\mathrm{map}_*(BP, Y^\wedge_p)_c \simeq *$ and, from the fibration $\mathrm{map}_*(BP, Y^\wedge_p)_c \to \mathrm{map}(BP, Y^\wedge_p)_c \to Y^\wedge_p$, we obtain $\mathrm{map}(BP, Y^\wedge_p)_c \simeq Y^\wedge_p$. Hence the obstructions live in $\lim^i_{O^c(\mathcal{F})} \pi_i(Y^\wedge_p)$ and to finish the proof we will prove that $\lim^i_{O^c(\mathcal{F})} \pi_*(Y^\wedge_p) = 0$. Note first that $\pi_*(Y^\wedge_p)$ is a constant functor in $O^c(\mathcal{F})$ because if we have a homomorphism $\varphi\colon P \to Q$ in $\mathcal{F}^c$, hence we get the following diagram

$$\begin{array}{ccc}
\mathrm{map}(BQ, Y^\wedge_p)_c & \xrightarrow{B\varphi^*} & \mathrm{map}(BP, Y^\wedge_p)_c \\
\downarrow \simeq & & \downarrow \simeq \\
Y^\wedge_p & \xrightarrow[id]{} & Y^\wedge_p
\end{array}$$

Moreover, by hipothesis, $Y^\wedge_p \simeq (Y^\wedge_p)^\wedge_p$, and hence we can consider $\pi_*(Y^\wedge_p)$ as a constant functor $\pi_*(Y^\wedge_p)\colon O^c(\mathcal{F})^{\mathrm{op}} \to \mathbb{Z}_{(p)}$. Let $F := \pi_*(Y^\wedge_p)$. Fix P in $O^c(\mathcal{F})^{\mathrm{op}}$ and consider the functors $O^c(\mathcal{F})^{\mathrm{op}} \to \mathbb{Z}_{(p)}$

$$F_P(Q) := \begin{cases} \pi_* Y^\wedge_p & , \text{if } Q = P, \\ 0 & , \text{if } Q \neq P. \end{cases}$$

and $\tilde{F}_P(Q) := F(Q)/F_P(Q)$. By the exact sequence of $\lim^i$ associated to the exact sequence of functors $0 \to F_P \to F \to \tilde{F}_P \to 0$, if we can prove that $\lim^i_{O^c(\mathcal{F})} F_P = 0$ then we obtain $\lim^i_{O^c(\mathcal{F})} F \cong \lim^i_{O^c(\mathcal{F})} \tilde{F}_P$, where $\tilde{F}_P(Q) = \pi_*(Y^\wedge_p)$ for $Q \neq P$ and $\tilde{F}_P(P) = 0$. Repeating this method, taking as F the functor $\tilde{F}_P$ in each step, a finite number of times we get that if $\lim^i_{O^c(\mathcal{F})} F_P = 0$ for all $P \in O^c(\mathcal{F})$ then $\lim^i_{O^c(\mathcal{F})} \pi_i(Y^\wedge_p) = 0$. Therefore, according to [BLO03b, Proposition 3.2], $\lim^i_{O^c(\mathcal{F})} F_P = \Lambda^i(\mathrm{Out}_{\mathcal{F}}(P); \pi_* Y^\wedge_p)$. But by [JMO92, Proposition 6.1 (i)], if $p \nmid |\mathrm{Out}_{\mathcal{F}}(P)|$ then

$$\Lambda^i(\mathrm{Out}_{\mathcal{F}}(P); \pi_* Y^\wedge_p) = \begin{cases} (\pi_*(Y^\wedge_p))^{\mathrm{Out}_{\mathcal{F}}(P)} & , \text{if } i = 0, \\ 0 & , \text{if } i > 0. \end{cases}$$

and if $p \mid |\mathrm{Out}_{\mathcal{F}}(P)| = |\ker(\mathrm{Out}_{\mathcal{F}}(P) \to \mathrm{Aut}(\pi_*(Y_p^\wedge)) \cong 0)|$ then by [JMO92, Proposition 6.1 (ii)], $\Lambda^i(\mathrm{Out}_{\mathcal{F}}(P); \pi_*(Y_p^\wedge)) = 0$. From this we get that $\lim^i_{O^c(\mathcal{F})} \pi_i(Y_p^\wedge) = 0$ and finally $f \simeq *$. □

In order to complete the previous proof, we need to prove the following result. This statement is a particular case of Theorem 9.8 in Miller's proof of Sullivan conjecture [Mil84].

Lemma 4.3.17. *If X is a $B\mathbb{Z}/p$-null space then X is BP-null for all finite p-group P.*

Remark 4.3.18. The previous lemma can also be proved by using Lemma 6.13 in [Dwy96] which states that $P_{B\mathbb{Z}/p}(BP)$ is contractible. Then $\mathrm{map}_*(BP, X) \simeq \mathrm{map}_*(P_{B\mathbb{Z}/p}(BP), X) \simeq *$, if X is $B\mathbb{Z}/p$-null. A direct proof can be obtained by induction (using the central extension of a p-group) and Zabrodsky's Lemma.

4.3.3 The cellularization of a classifying space

Assume now that there is a map $\varphi\colon \bigvee_I A \to |\mathcal{L}|_p^\wedge$, where I is a finite set, such that the morphism of sets $\varphi_*\colon [A, \bigvee_I A]_* \to [A, |\mathcal{L}|_p^\wedge]_*$ is surjective. Hence if C denotes the homotopy cofibre of φ, Theorem 2.1.22 shows that $CW_A(|\mathcal{L}|_p^\wedge)$ is the homotopy fibre of $r\colon |\mathcal{L}|_p^\wedge \to P_{\Sigma A}(C)$. Moreover, since C is 1-connected, $P_{\Sigma B\mathbb{Z}/p^r}(C)_p^\wedge$ is so, and hence we can consider $\ker(r_p^\wedge)$, because by [Mil84, Theorem 1.5],

$$\begin{aligned}&\mathrm{map}_*(\Sigma B\mathbb{Z}/p, P_{\Sigma B\mathbb{Z}/p^r}(C)_p^\wedge) \simeq \mathrm{map}_*(\Sigma B\mathbb{Z}/p, P_{\Sigma B\mathbb{Z}/p^r}(C)) \simeq \\ &\mathrm{map}_*(P_{\Sigma B\mathbb{Z}/p^r}(\Sigma B\mathbb{Z}/p), P_{\Sigma B\mathbb{Z}/p^r}(C)) \simeq *,\end{aligned}$$

where $P_{\Sigma B\mathbb{Z}/p^r}(\Sigma B\mathbb{Z}/p) \simeq *$ because $\Sigma B\mathbb{Z}/p$ is $\Sigma B\mathbb{Z}/p^r$-cellular, i.e., $P_{\Sigma B\mathbb{Z}/p^r}(C)_p^\wedge$ is $\Sigma B\mathbb{Z}/p$-null. The main goal in this section is to prove the following theorem:

Theorem 4.3.19. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local compact group. Assume that $H_2(|\mathcal{L}|_p^\wedge; \mathbb{Z})$ is a finite group and that there is a map $\varphi\colon \bigvee_I A \to |\mathcal{L}|_p^\wedge$, where I is a finite set, such that $\pi_1(\varphi)$ is an epimorphism and $\varphi_*\colon [A, \bigvee_I A]_* \to [A, |\mathcal{L}|_p^\wedge]_*$ is surjective. If* $\ker(r_p^\wedge) = S$*, then the augmention map $a_{|\mathcal{L}|_p^\wedge}\colon CW_A(|\mathcal{L}|_p^\wedge) \to |\mathcal{L}|_p^\wedge$ is a mod p-equivalence.*

And the following corollaries:

Corollary 4.3.20. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local compact group as in Theorem 4.3.19. If $|\mathcal{L}|_p^\wedge$ is nilpotent and* $\ker(r_p^\wedge) = S$*, then $CW_A(|\mathcal{L}|_p^\wedge)$ fits into a fibration*

$$CW_A(|\mathcal{L}|_p^\wedge) \to |\mathcal{L}|_p^\wedge \to (|\mathcal{L}|_p^\wedge)_{\mathbb{Q}}.$$

Proof. Note that since $|\mathcal{L}|_p^\wedge$ is nilpotent, $CW_A(|\mathcal{L}|_p^\wedge)$ is nilpotent by [CF13, Lemma 2.5] and we can consider the Sullivan arithmetic square:

$$\begin{array}{ccc} CW_A(|\mathcal{L}|_p^\wedge) & \longrightarrow & \prod_{q \text{ prime}} CW_A(|\mathcal{L}|_p^\wedge)_q^\wedge \\ \downarrow & & \downarrow \\ (CW_A(|\mathcal{L}|_p^\wedge))_{\mathbb{Q}} & \longrightarrow & (\prod_{q \text{ prime}} (CW_A(|\mathcal{L}|_p^\wedge)_q^\wedge)_{\mathbb{Q}} \end{array}$$

Now, since $\widetilde{H}_*(A;\mathbb{Q}) \cong 0$ and $\widetilde{H}_*(A;\mathbb{Z}/q) \cong 0$ for $q \neq p$, $(CW_A(|\mathcal{L}|_p^\wedge))_\mathbb{Q} \simeq *$ and $CW_A(|\mathcal{L}|_p^\wedge)_q^\wedge \simeq *$ for $q \neq p$ by [CF13, Lemma 2.8]. Moreover, $CW_A(|\mathcal{L}|_p^\wedge)_p^\wedge \simeq |\mathcal{L}|_p^\wedge$ by Theorem 4.3.19. Therefore the above pullback diagram becomes

$$\begin{array}{ccc} CW_A(|\mathcal{L}|_p^\wedge) & \longrightarrow & |\mathcal{L}|_p^\wedge \\ \downarrow & & \downarrow \\ * & \longrightarrow & (|\mathcal{L}|_p^\wedge)_\mathbb{Q} \end{array}$$

and this finishes the proof. □

Note that if BS is A-cellular, then $P_{\Sigma A}(C) \simeq *$. Therefore $P_{\Sigma A}(C)_p^\wedge \simeq *$ and hence $\ker(r_p^\wedge) = S$. This proves the following corollary:

Corollary 4.3.21. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local compact group as in Theorem 4.3.19. If $|\mathcal{L}|_p^\wedge$ is nilpotent and BS is A-cellular, then $CW_A(|\mathcal{L}|_p^\wedge)$ fits into a fibration*

$$CW_A(|\mathcal{L}|_p^\wedge) \to |\mathcal{L}|_p^\wedge \to (|\mathcal{L}|_p^\wedge)_\mathbb{Q}.$$

Furthermore, there exists a non-negative integer m_0 such that

$$CW_{B_{p^m}}(|\mathcal{L}|_p^\wedge) \to |\mathcal{L}|_p^\wedge \to (|\mathcal{L}|_p^\wedge)_\mathbb{Q}$$

is a fibration for all $m \geq m_0$.

Proof. If BS is A-cellular, then $\ker(r_p^\wedge) = S$ and apply Theorem 4.3.19. Moreover, by Proposition 4.2.5 there is a non-negative integer m_0 such that BS is B_{p^m}-cellular for all $m \geq m_0$. □

Note that if $\pi_1(\varphi)$ is surjective, then C is a 1-connected space, by Seifert-Van Kampen's theorem, and so is $P_{\Sigma A}(C)$. Hence by [BK72, Lemma II.5.1] we get the fibration

$$CW_A(|\mathcal{L}|_p^\wedge)_p^\wedge \xrightarrow{c_p^\wedge} |\mathcal{L}|_p^\wedge \xrightarrow{r_p^\wedge} P_{\Sigma A}(C)_p^\wedge.$$

We will begin proving the following technical lemmas:

Lemma 4.3.22. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local compact group as in Theorem 4.3.19. Then the fundamental group of $P_A(CW_A(|\mathcal{L}|_p^\wedge)_p^\wedge)$ is a finite p-group and hence $P_A(CW_A(|\mathcal{L}|_p^\wedge)_p^\wedge)$ is a p-good space.*

Proof. Let $Y = CW_A(|\mathcal{L}|_p^\wedge)_p^\wedge$. We want to prove that $\pi_1 Y$ is a finite group, because the homomorphism $\pi_1 Y \to \pi_1 P_A(Y)$ is surjective by [Bou94, Proposition 2.9], hence $\pi_1 P_A(Y)$ is a finite group and finally $P_A(Y)$ is a p-good space by [BK72, Proposition VII.5.1].

From the fibration $Y \to |\mathcal{L}|_p^\wedge \to P_{\Sigma A}(C)_p^\wedge$, it is got because C is 1-connected since φ induces an epimorphism in fundamental groups, we get the exacte sequence of groups

$$\ldots \to \pi_2(P_{\Sigma A}(C)_p^\wedge) \to \pi_1 Y \to \pi_1(|\mathcal{L}|_p^\wedge) \to \ldots$$

where $\pi_1(|\mathcal{L}|_p^\wedge)$ is a finite p-group according to [Gon10, Theorem B.1.6] or [Gon13, Theorem B.5]. Therefore we have to prove that $\pi_2(P_{\Sigma A}(C)_p^\wedge)$ is finite.

Hurewicz's theorem shows that $H_2(P_{\Sigma A}(C)^\wedge_p;\mathbb{Z}) \cong \pi_2(P_{\Sigma A}(C)^\wedge_p)$, since $P_{\Sigma A}(C)^\wedge_p$ is simply connected. Note that since ΣA is 1-connected, we obtain an epimorphism $H_2(C;\mathbb{Z}) \twoheadrightarrow H_2(P_{\Sigma A}C;\mathbb{Z})$ by Proposition 1.2.12. From the cofibration

$$\bigvee_I A \to |\mathcal{L}|^\wedge_p \to C$$

we get, by the exactness axiom in homology, a long exact sequence of homology groups

$$\ldots \longrightarrow H_2(|\mathcal{L}|^\wedge_p;\mathbb{Z}) \xrightarrow{f_1} H_2(C;\mathbb{Z}) \xrightarrow{f_2} H_1(\textstyle\bigvee_I A;\mathbb{Z}) \longrightarrow \ldots$$

and from here a short exact sequence

$$0 \to \ker f_1 \to H_2(C;\mathbb{Z}) \to \operatorname{Im} f_2 \to 0,$$

where $\ker f_1 \subset H_2(|\mathcal{L}|^\wedge_p;\mathbb{Z})$, and hence $\ker f_1$ is finite, and $\operatorname{Im} f_2 \subset H_1(\bigvee_I A;\mathbb{Z}) \cong \pi_1(\bigvee_I A)_{ab} \cong \prod_I \pi_1 A$, where $\pi_1 A \cong \mathbb{Z}/p^\infty \times \mathbb{Z}/p^m$ or $\mathbb{Z}/p^m$. If $\pi_1 A \cong \mathbb{Z}/p^m$, then $\operatorname{Im} f_2$ is finite and finally so is $H_2(C;\mathbb{Z})$. Assume now that $\pi_1 A \cong \mathbb{Z}/p^\infty \times \mathbb{Z}/p^m$. Therefore, we get $H_2(C;\mathbb{Z}) \cong (\mathbb{Z}/p^\infty)^{n'} \times H'$ where H' is a finite group and hence $\pi_2 P_{\Sigma A}(C) \cong (\mathbb{Z}/p^\infty)^n \times H$, where H is a finite group. $P_{\Sigma A}(C)$ is 1-connected, and consequently it is connected and nilpotent, hence [BK72, Proposition VI.5.1] shows that there is a splittable short exact sequence

$$0 \to \operatorname{Ext}(\mathbb{Z}/p^\infty, \pi_2 P_{\Sigma A}(C)) \to \pi_2(P_{\Sigma A}(C)^\wedge_p) \to \operatorname{Hom}(\mathbb{Z}/p^\infty, \pi_1 P_{\Sigma A}(C)) \to 0$$

hence, since $P_{\Sigma A}(C)$ is 1-connected,

$$\pi_2(P_{\Sigma A}(C)^\wedge_p) \cong \operatorname{Ext}(\mathbb{Z}/p^\infty, \pi_2(P_{\Sigma A}(C))) \cong \operatorname{Ext}(\mathbb{Z}/p^\infty, (\mathbb{Z}/p^\infty)^n \times H).$$

By [BK72, Example VI.4.4 (i)], $\operatorname{Ext}(\mathbb{Z}/p^\infty, H) \cong \hat{\mathbb{Z}}_p \otimes H$, a finite group, and by [BK72, Example VI.4.2], $\operatorname{Ext}(\mathbb{Z}/p^\infty, \mathbb{Z}/p^\infty) \cong 0$. Therefore $\pi_2(P_{\Sigma A}(C)^\wedge_p)$ is finite. □

One of the known properties of classifying spaces of p-local compact group is that the are $B\mathbb{Z}/p$-acyclic up to p-completion, i.e., $P_{B\mathbb{Z}/p}(|\mathcal{L}|^\wedge_p)^\wedge_p \simeq *$. Now, we want to proof the same property replacing $B\mathbb{Z}/p$ by certain A:

Proposition 4.3.23. *Let W and X be connected spaces such that $B\mathbb{Z}/p$ is W-acyclic, W is $\tilde{H}_*(-;\mathbb{Z}[\frac{1}{p}])$-acyclic and $P_{B\mathbb{Z}/p}(X) \simeq L_{\mathbb{Z}[\frac{1}{p}]}(X)$. Then $P_W(X) \simeq P_{B\mathbb{Z}/p}(X)$ and hence $P_W(X)^\wedge_p$ is contractible.*

Proof. First, $P_W(X)$ is $B\mathbb{Z}/p$-null since $B\mathbb{Z}/p$ is W-acyclic:

$$\operatorname{map}_*(B\mathbb{Z}/p, P_W(X)) \simeq \operatorname{map}_*(P_W(B\mathbb{Z}/p), P_W(X)) \simeq \operatorname{map}_*(*, P_W(X)) \simeq *.$$

Hence, there is a map $f\colon P_{B\mathbb{Z}/p}(X) \to P_W(X)$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_{P_W}} & P_W(X) \\ & {\scriptstyle \eta_{P_{B\mathbb{Z}/p}}} \searrow & \nearrow {\scriptstyle f} \\ & P_{B\mathbb{Z}/p}(X) & \end{array}$$

is commutative.

Second, $P_{B\mathbb{Z}/p}(X)$ is W-null because $P_{B\mathbb{Z}/p}(X) \simeq L_{\mathbb{Z}[\frac{1}{p}]}(X)$, and $L_{\mathbb{Z}[\frac{1}{p}]}(X)$ is W-null since $\mathrm{map}_*(W, L_{\mathbb{Z}[\frac{1}{p}]}(X)) \simeq \mathrm{map}_*(L_{\mathbb{Z}[\frac{1}{p}]}(W), L_{\mathbb{Z}[\frac{1}{p}]}(X)) \simeq \mathrm{map}_*(*, L_{\mathbb{Z}[\frac{1}{p}]}(X)) \simeq *$. Hence there is a map $g\colon P_W(X) \to P_{B\mathbb{Z}/p}(X)$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_{P_{B\mathbb{Z}/p}}} & P_{B\mathbb{Z}/p}(X) \\ & {\scriptstyle \eta_{P_W}}\searrow \quad \nearrow{\scriptstyle g} & \\ & P_W(X) & \end{array}$$

is commutative.

It follows that $f \circ g \simeq id_{P_W(X)}$ and $g \circ f \simeq id_{P_{B\mathbb{Z}/p}(X)}$, i.e., $P_W(X) \simeq P_{B\mathbb{Z}/p}(X)$. Finally, $\tilde{H}_*(P_W(X);\mathbb{Z}/p) \cong \tilde{H}^*(P_{B\mathbb{Z}/p}(X);\mathbb{Z}/p) \cong \tilde{H}^*(L_{\mathbb{Z}[\frac{1}{p}]}(X);\mathbb{Z}/p) \cong 0$. □

Corollary 4.3.24. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local compact group. Then $P_A(|\mathcal{L}|^\wedge_p)^\wedge_p \simeq *$.*

Proof. Note that $B\mathbb{Z}/p$ is A-acyclic because $B\mathbb{Z}/p$ is $B\mathbb{Z}/p^m$-cellular and hence A-cellular. Moreover $L_{\mathbb{Z}[\frac{1}{p}]}(A) \simeq *$ because $\tilde{H}_*(A;\mathbb{Z}[\frac{1}{p}]) \cong 0$. Finally, from [CF13, Proposition 4.11], $P_{B\mathbb{Z}/p}(|\mathcal{L}|^\wedge_p) \simeq L_{\mathbb{Z}[\frac{1}{p}]}(|\mathcal{L}|^\wedge_p)$. The result follows directly from the Proposition 4.3.23. □

The next thecnical lemma is leading up to the proof or Theorem 4.3.19:

Lemma 4.3.25. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local compact group as in Lemma 4.3.22. If* $\ker(r^\wedge_p) = S$, *then $P_A(CW_A(|\mathcal{L}|^\wedge_p)^\wedge_p)^\wedge_p$ is an A-null space.*

Proof. Let $Y = CW_A(|\mathcal{L}|^\wedge_p)^\wedge_p$. We want to prove that $P_A(Y)^\wedge_p$ is an A-null space, i.e.,

$$\mathrm{map}_*(A, P_A(Y)^\wedge_p) \simeq *.$$

Let F be the homotopy fibre of the p-completion map $P_A(Y) \to P_A(Y)^\wedge_p$ and let $\pi = \pi_1(P_A(Y))$. By Lemma 4.3.22, π is a finite p-group, hence $B\pi$ is a nilpotent p-complete space and according to [BK72, Lemma II.5.1] we can to p-complete the fibration

$$\widetilde{P_A(Y)} \to P_A(Y) \to B\pi,$$

where $\widetilde{P_A(Y)}$ is the 1-connected cover of $P_A(Y)$, because $\pi_1 B\pi = \pi$ acts nilpotently on each $H_i(\widetilde{P_A(Y)};\mathbb{Z}/p)$ (a finite p-group always acts nilpotently on a $\mathbb{Z}/p$-module). We get the commutative digaram of fibrations

$$\begin{array}{ccccc} F & \longrightarrow & \widetilde{P_A(Y)} & \longrightarrow & (\widetilde{P_A(Y)})^\wedge_p \\ \| & & \downarrow & & \downarrow \\ F & \longrightarrow & P_A(Y) & \longrightarrow & P_A(Y)^\wedge_p \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & B\pi & = & B\pi \end{array}$$

Since $\widetilde{P_A(Y)}$ is 1-connected, it is nilpotent and hence F is nilpotent by [BK72, Lemma V.5.2]. Moreover, we can to p-complete the top fibration because both spaces are 1-connected and hence nilpotent and we obtain $F^\wedge_p \simeq *$. It follows that $\pi_i F$ are uniquely p-divisible for all i (see the proof of [BK72, Lemma V.9.4] for more details).

On the other hand, we have the fibration

$$\mathrm{map}_*(A, \Omega(P_A(Y)^\wedge_p)) \to \mathrm{map}_*(A, F)_{\{c\}} \to \mathrm{map}_*(A, P_A(Y))_c$$

where $\mathrm{map}_*(A, P_A(Y)) \simeq *$. Therefore

$$\Omega\mathrm{map}_*(A, P_A(Y)^\wedge_p)_c \simeq \mathrm{map}_*(A, \Omega(P_A(Y)^\wedge_p)) \simeq \mathrm{map}_*(A, F)$$

and $\mathrm{map}_*(A, F) \simeq \mathrm{map}_*(A, F^\wedge_p)$ by [Mil84, Theorem 1.5], but since $F^\wedge_p \simeq *$,

$$\Omega\mathrm{map}_*(A, P_A(Y)^\wedge_p)_c \simeq *.$$

We have $\pi_i\mathrm{map}_*(A, P_A(Y)^\wedge_p)_c \cong 0$ for all $i \geq 1$, Therefore the proof is completed by showing that

$$\pi_0\mathrm{map}_*(A, P_A(Y)^\wedge_p) \cong [A, P_A(Y)^\wedge_p]_* \cong *.$$

For this, we want to apply obstruction theory to the following extension problem

$$\begin{array}{ccc} & & F \\ & & \downarrow \\ & \nearrow & P_A(Y) \\ & & \downarrow{\scriptstyle \eta_P} \\ A & \longrightarrow & P_A(Y)^\wedge_p \end{array}$$

Since $\ker(r^\wedge_p) = S$, Theorem 4.3.16 shows that $r^\wedge_p \simeq *$, hence we get $Y \simeq |\mathcal{L}|^\wedge_p \times \Omega(P_{\Sigma A}(C)^\wedge_p)$. Hence

$$P_A(Y) \simeq P_A(|\mathcal{L}|^\wedge_p) \times P_A(\Omega(P_{\Sigma A}(C)^\wedge_p)) \simeq P_A(|\mathcal{L}|^\wedge_p) \times \Omega P_{\Sigma A}(P_{\Sigma A}(C)^\wedge_p)$$

and

$$P_A(Y)^\wedge_p \simeq P_A(|\mathcal{L}|^\wedge_p)^\wedge_p \times (\Omega P_{\Sigma A}(P_{\Sigma A}(C)^\wedge_p))^\wedge_p$$

where $(\Omega P_{\Sigma A}(P_{\Sigma A}(C)^\wedge_p))^\wedge_p \simeq \Omega(P_{\Sigma A}(P_{\Sigma A}(C)^\wedge_p)^\wedge_p)$, becuase C is 1-connected and hence so is $P_{\Sigma A}(P_{\Sigma A}(C)^\wedge_p)$; and $P_A(|\mathcal{L}|^\wedge_p)^\wedge_p \simeq *$ by Corollary 4.3.24. Hence the extension problem becomes

$$\begin{array}{ccc} & & P_A(|\mathcal{L}|^\wedge_p) \times F' \\ & & \downarrow \\ & & P_A(|\mathcal{L}|^\wedge_p) \times \Omega P_{\Sigma A}(P_{\Sigma A}(C)^\wedge_p) \\ & \nearrow & \downarrow \\ A & \longrightarrow & \Omega(P_{\Sigma A}(P_{\Sigma A}(C)^\wedge_p))^\wedge_p \end{array}$$

where F' is the homotopy fibre over $\Omega P_{\Sigma A}(P_{\Sigma A}(C)^\wedge_p) \to \Omega(P_{\Sigma A}(P_{\Sigma A}(C)^\wedge_p)^\wedge_p)$, a fibrations of H-spaces. Therefore we can apply obstruction theory over the extension problem

$$\begin{array}{ccc} & & F' \\ & & \downarrow \\ & & \Omega P_{\Sigma A}(P_{\Sigma A}(C)^\wedge_p) \\ & \nearrow & \downarrow \\ A & \longrightarrow & \Omega(P_{\Sigma A}(P_{\Sigma A}(C)^\wedge_p)^\wedge_p) \end{array}$$

and, obviously, if we can to resolve this extension problem then we can to resolve the above extension problem. As $\pi_i F$ are uniquely p-divisible for all i, so are $\pi_i F'$ for all i, hence $\widetilde{H}^i(A;\pi_j F') \cong 0$ and, by obstruction theory, there are unique (up to homotopy) lifts over the above extension problems. It follows that $[A, P_A(Y)^\wedge_p]_* \cong [A, P_A(Y)]_*$ and $[A, P_A(Y)]_* \cong *$. □

Now, we are ready to proof Theorem 4.3.19:

Proof of Theorem 4.3.19. Since $\ker(r^\wedge_p) = S$, Theorem 4.3.16 shows that $r^\wedge_p \simeq *$. As in Lemma 4.3.25 we get $CW_A(|\mathcal{L}|^\wedge_p)^\wedge_p \simeq |\mathcal{L}|^\wedge_p \times \Omega(P_{\Sigma A}(C)^\wedge_p)$, and applying P_A and p-completion we get

$$P_A(CW_A(|\mathcal{L}|^\wedge_p)^\wedge_p)^\wedge_p \simeq P_A(|\mathcal{L}|^\wedge_p)^\wedge_p \times \Omega(P_{\Sigma A}(P_{\Sigma A}(C)^\wedge_p)^\wedge_p)$$

On the one hand, since $P_{\Sigma A}(C)$ is 1-connected, according to [CF13, Corollary 3.11],

$$P_{\Sigma A}(P_{\Sigma A}(C)^\wedge_p)^\wedge_p \simeq P_{\Sigma A}(P_{\Sigma A}(C))^\wedge_p \simeq P_{\Sigma A}(C)^\wedge_p.$$

On the other hand, $P_A(CW_A(|\mathcal{L}|^\wedge_p)) \simeq *$ is a p-good space, $P_A(CW_A(|\mathcal{L}|^\wedge_p))^\wedge_p \simeq *$ is A-null, $P_A(CW_A(|\mathcal{L}|^\wedge_p)^\wedge_p)$ is p-good by Lemma 4.3.22 and $P_A(CW_A(|\mathcal{L}|^\wedge_p)^\wedge_p)^\wedge_p$ is A-null by Lemma 4.3.25, hence [CF13, Lemma 3.9] shows that $P_A(CW_A(|\mathcal{L}|^\wedge_p)^\wedge_p)^\wedge_p \simeq P_A(CW_A(|\mathcal{L}|^\wedge_p))^\wedge_p \simeq *$. Therefore $\Omega(P_{\Sigma A}(C)^\wedge_p) \simeq *$, and since $P_{\Sigma A}(C)^\wedge_p$ is connected, $P_{\Sigma A}(C)^\wedge_p \simeq *$. This implies that $CW_A(|\mathcal{L}|^\wedge_p)^\wedge_p \simeq |\mathcal{L}|^\wedge_p$. □

According to the proof of Theorem 4.3.19, it is important that C is 1-connected. If C is not 1-connected, then we can reduce the cellularization to another p-local compact group, which is A-equivalent to the first one, and whose Chachólski's cofibre is simply connected. For this we need the next version of a result of Castellana-Crespo-Scherer:

Proposition 4.3.26 ([CCS07, Proposition 2.1]). *Let $m \geq 1$ and let $F \to E \xrightarrow{\pi} BG$ be a fibration, where G is a discrete group. Let N be the (normal) subgroup generated by all elements $g \in G$ of order p^i for some $i \leq m$ such that the map $f_g \colon B\mathbb{Z}/p^m \to BG$ induced by $\alpha_g \colon \mathbb{Z}/p^m \to G$ given by $\alpha_g(1) = g$ lifts to E. Then the pullback of the fibration along $BN \to BG$*

$$\begin{array}{ccccc} E' & \xrightarrow{f} & E & \xrightarrow{p} & B(G/S) \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \| \\ BN & \longrightarrow & BG & \xrightarrow{p'} & B(G/S) \end{array}$$

induces a $B\mathbb{Z}/p^m$-equivalence $f \colon E' \to E$ on the total space level.

Proof. We want to show that f induces a $B\mathbb{Z}/p^m$-equivalence. The top fibration in the diagram yields a fibration

$$\mathrm{map}_*(B\mathbb{Z}/p^m, E') \xrightarrow{f_*} \mathrm{map}_*(B\mathbb{Z}/p^m, E)_{\{c\}} \xrightarrow{p_*} \mathrm{map}_*(B\mathbb{Z}/p^m, B(G/S))_c,$$

where $\mathrm{map}_*(B\mathbb{Z}/p^m, B(G/S))_c$ is the component of the constant map, and $\mathrm{map}_*(B\mathbb{Z}/p^m, E)_{\{c\}}$ are the components sent to $\mathrm{map}_*(B\mathbb{Z}/p^m, B(G/S))_c$ via p_*. Since the base space is homotopically discrete, we only need to check that all components of $\mathrm{map}_*(B\mathbb{Z}/p^m, E)$ are sent by p_* to $\mathrm{map}_*(B\mathbb{Z}/p^m, B(G/S))_c$. Thus consider a pointed map $h\colon B\mathbb{Z}/p^m \to E$. The composite $p \circ h$ is homotopy equivalent to a map induced by a group homomorphism $\alpha\colon \mathbb{Z}/p^m \to G$ whose image $\alpha(1) = g$ is in S by construction. Therefore $p \circ h \simeq p' \circ \pi \circ h$ is null-homotopic. □

Remark 4.3.27. In the original version the authors consider $\bar{N}$ to be the (normal) subgroup generated by all elements $g \in G$ of order p^i for some $i \le r$ such that the inclusion $B\langle g\rangle \to BG$ lifts to E, but this is not correct. Consider the fibration $B\mathbb{Z}/2 \xrightarrow{\iota} B\mathbb{Z}/4 \xrightarrow{p} B\mathbb{Z}/2$, this fibration has not section and hence $\bar{N} = \{0\}$. Then $E' \simeq B\mathbb{Z}/2$ and $CW_{B\mathbb{Z}/4}(B\mathbb{Z}/2) \simeq B\mathbb{Z}/2 \not\simeq B\mathbb{Z}/4 = CW_{B\mathbb{Z}/4}(B\mathbb{Z}/4)$, contradicting the proposition. However, $N \cong \mathbb{Z}/2 \cong \langle g\rangle$, since $f_g = p$. Hence $E' \simeq B\mathbb{Z}/4$ and $f : E' \to B\mathbb{Z}/4$ is an equivalence, in particular it is a $B\mathbb{Z}/4$-equivalence.

And hence the way to construct this new p-local compact group is the follows:

Proposition 4.3.28. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local compact group. Then there exists a p-local compact group $(S, \mathcal{F}_N, \mathcal{L}_N)$ and a $B\mathbb{Z}/p^m$-equivalence $f\colon |\mathcal{L}_N|^\wedge_p \to |\mathcal{L}|^\wedge_p$ such that the homotopy cofibre of $ev\colon \bigvee_{[B\mathbb{Z}/p^m, |\mathcal{L}_N|^\wedge_p]_*} B\mathbb{Z}/p^m \to |\mathcal{L}_N|^\wedge_p$ is 1-connected.*

Proof. Let N be the (normal) subgroup generated by all elements $g \in \pi_1(|\mathcal{L}|^\wedge_p)$ of order p^i for some $i \le m$ such that the map $f_g\colon B\mathbb{Z}/p^m \to B\pi_1(|\mathcal{L}|^\wedge_p)$ induced by $\alpha_g\colon \mathbb{Z}/p^m \to \pi_1(|\mathcal{L}|^\wedge_p)$, $\alpha_g(1) = g$, lifts to $|\mathcal{L}|^\wedge_p$. Let X is the pullback of the fibration

$$\widetilde{|\mathcal{L}|^\wedge_p} \to |\mathcal{L}|^\wedge_p \to B\pi_1(|\mathcal{L}|^\wedge_p)$$

where $\widetilde{|\mathcal{L}|^\wedge_p}$ is the 1-connected cover of $|\mathcal{L}|^\wedge_p$, along $BN \to B\pi_1(|\mathcal{L}|^\wedge_p)$. Then

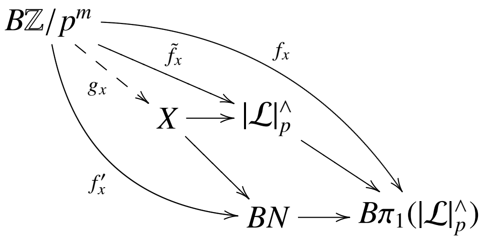
$$\begin{array}{ccc} \widetilde{|\mathcal{L}|^\wedge_p} & = & \widetilde{|\mathcal{L}|^\wedge_p} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & |\mathcal{L}|^\wedge_p \\ \downarrow & & \downarrow \\ BN & \to & B\pi_1(|\mathcal{L}|^\wedge_p) \end{array}$$

According to [Gon10, Theorem B.4.4], there is a p-local compact group $(S, \mathcal{F}_N, \mathcal{L}_N)$ such that $X \simeq |\mathcal{L}_N|^\wedge_p$. Furthermore, $f\colon X \to |\mathcal{L}|^\wedge_p$ is a $B\mathbb{Z}/p^m$-equivalence by Proposition 4.3.26.

Let now C be the homotopy cofibre of $ev\colon \bigvee_{[B\mathbb{Z}/p^m, X]_*} B\mathbb{Z}/p^m \to X$. By Seifert-Van Kampen's theorem, we get the push out of groups

$$\begin{array}{ccc} *_{[B\mathbb{Z}/p^m, X]_*}\mathbb{Z}/p^m & \xrightarrow{\pi_1(ev)} & \pi_1 X \\ \downarrow & & \downarrow \\ \{e\} & \longrightarrow & \pi_1 C \end{array}$$

where $\pi_1 X \cong N$. If we prove that $\pi_1(ev)$ is an epimorphism, then $\pi_1(C) \cong \{e\}$. Hence, let $x \in \pi_1 X \cong N \lhd \pi_1(|\mathcal{L}|_p^\wedge)$ a generator, we have to find a map $g_x\colon B\mathbb{Z}/p^m \to X$ such that $\pi_1(g_x)(1) = x$. Let $f'_x\colon B\mathbb{Z}/p^m \to BN$ such that $\pi_1(f_x)(1) = x$ and let f_x the composite $B\mathbb{Z}/p^m \to BN \to B\pi_1(|\mathcal{L}|_p^\wedge)$. By definition of N, there is a map $\tilde{f}_x\colon B\mathbb{Z}/p^m \to |\mathcal{L}|_p^\wedge$ such that the following diagram



commutes. Since X is the pull back of the diagram, there is an unique map (up to homotopy) $g_x\colon B\mathbb{Z}/p^m \to X$ closing the above diagram. Therefore $\pi_1(g_X)(1) = \pi_1(f'_x)(1) = x$. □

4.3.4 Classifying spaces of p-local finite groups

In this section we want to go into detail about the $B\mathbb{Z}/p^m$-cellularization of classifying spaces of p-local finite groups. As a p-local finite group is a particular case of p-local compact group, we can use Theorem 4.3.19. Given a p-local finite group we will identify, under some hypothesis, $\ker(r_p^\wedge)$ with $Cl_{p^m}(S)$, the smallest strongly $\mathcal{F}$-closed subgroup of S that contains all the p^i-torsion of S, for all $i \leq m$. In fact, the main goal of this section is to prove that $|\mathcal{L}|_p^\wedge$ is $B\mathbb{Z}/p^m$-cellular if and only if $S = Cl_{p^m}(S)$. Hence, in this section, $(S, \mathcal{F}, \mathcal{L})$ denotes a p-local finite group, $\Omega_{p^m}(S)$ denotes the (normal) subgroup of S generated by its elements of order p^i, which $i \leq m$, $Cl_{p^m}(S)$ denotes the smallest strongly $\mathcal{F}$-closed subgroup of S that contains $\Omega_{p^m}(S)$ and C will be the homotopy cofibre of the evaluation map

$$\bigvee_{[B\mathbb{Z}/p^m, |\mathcal{L}|_p^\wedge]_*} B\mathbb{Z}/p^m \xrightarrow{ev} |\mathcal{L}|_p^\wedge.$$

Therefore, we will prove the next theorem:

Theorem 4.3.29. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Then $|\mathcal{L}|_p^\wedge$ is $B\mathbb{Z}/p^m$-cellular if and only if $S = Cl_{p^m}(S)$.*

Note that if $S = \Omega_{p^m}(S)$, then $S = Cl_{p^m}(S)$, because $\Omega_{p^m}(S) \leq Cl_{p^m}(S) \leq S$. Moreover there exists a non-negative integer m_0 such that $S = \Omega_{p^m}(S)$ for all $m \geq m_0$. Then we obtain the following corollary:

Corollary 4.3.30. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Then there exists a $m_0 \geq 0$ such that $|\mathcal{L}|_p^\wedge$ is $B\mathbb{Z}/p^m$-cellular for all $m \geq m_0$.*

According to Theorem 4.3.19 we need that C is 1-connected. Then the next lemma is devoted to prove this when $S = Cl_{p^m}(S)$.

Lemma 4.3.31. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group such that $S = Cl_{p^m}(S)$. Then C is 1-connected.*

Proof. Let C be the homotopy cofibre of $ev\colon \bigvee_{[B\mathbb{Z}/p^m,|\mathcal{L}|^\wedge_p]_*} B\mathbb{Z}/p^m \to |\mathcal{L}|^\wedge_p$. Hence $\pi_1 C \cong \pi_1(|\mathcal{L}|^\wedge_p)/N$, where N is the minimal normal subgroup of $\pi_1(|\mathcal{L}|^\wedge_p)$ that contains $\mathrm{Im}(ev)$. Moreover, by [BCG+07, Theorem B], $\pi_1(|\mathcal{L}|^\wedge_p) \cong S/O^p_{\mathcal{F}}(S)$, where

$$O^p_{\mathcal{F}}(S) := \langle [P, O^p(\mathrm{Aut}_{\mathcal{F}}(P)] \mid P \leq S \rangle.$$

Furthermore, as all map $B\mathbb{Z}/p^m \to |\mathcal{L}|^\wedge_p$ factorizes by BS we get the following commutative diagram

$$\begin{array}{ccc} \bigvee_{[B\mathbb{Z}/p^m,|\mathcal{L}|^\wedge_p]_*} B\mathbb{Z}/p^m & \xrightarrow{ev} & |\mathcal{L}|^\wedge_p \\ & \searrow^{ev_S} & \uparrow \\ & & BS \end{array}$$

where $\mathrm{Im}(ev_S) = \Omega_{p^m}(S)$ and, moreover, $\mathrm{Im}(ev) \leq \mathrm{Im}(ev_S)$. Hence $\pi_1 C \cong S/\Omega_{p^m}(S)\cdot O^p_{\mathcal{F}}(S)$. By [DGPS11, Proposition A.9], $\Omega_{p^m}(S)\cdot O^p_{\mathcal{F}}(S)$ is strongly $\mathcal{F}$-closed and contains $\Omega_{p^m}(S)$. Therefore $S = \Omega_{p^m}(S)\cdot O^p_{\mathcal{F}}(S)$ since $S = Cl_{p^m}(S)$ and hence $\pi_1 C \cong \{e\}$. □

In order to be ready to prove that $\ker(r^\wedge_p) = Cl_{p^m}(S)$, first note that $\ker(r^\wedge_p)$ is a strongly $\mathcal{F}$-closed subgroup by Proposition 4.3.15 and $\Omega_{p^m}(S) \leq \ker(r^\wedge_p)$, hence $Cl_{p^m}(S) \leq \ker(r^\wedge_p)$. To prove the other inclusion we need, for a given strongly $\mathcal{F}$-closed subgroup K, to construct a map $f\colon |\mathcal{L}|^\wedge_p \to Z$, where Z is p-complete and $\Sigma B\mathbb{Z}/p$-null, such that $f|_{BK} \simeq *$.

Proposition 4.3.32. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Let K be a strongly $\mathcal{F}$-closed subgroup. Let ρ be the composite $S \xrightarrow{\pi} S/K \xrightarrow{reg} \Sigma_{|S/K|}$, where π is the quotient homomorphism and reg is the regular representation of S/K. Then there are a non-negative integer $m \geq 0$ and a map $f\colon |\mathcal{L}| \to B(\Sigma_{|S/K|} \wr \Sigma_{p^m})^\wedge_p$ such that the following diagram*

$$\begin{array}{ccc} BS & \xrightarrow{(\Delta B\rho)^\wedge_p} & (B(\Sigma_{|S/K|})^{p^m})^\wedge_p \\ {\scriptstyle \iota_S}\downarrow & & \downarrow{\scriptstyle \Delta^\wedge_p} \\ |\mathcal{L}| & \xrightarrow[f]{} & B(\Sigma_{|S/K|} \wr \Sigma_{p^m})^\wedge_p \end{array}$$

is commutative up to homotopy.

Proof. Let $n = |S/K|$. According to [CL09, Theorem 1.2], if ρ is fusion invariant then there are a non-negative integer $m \geq 0$ and a map $f\colon |\mathcal{L}| \to B(\Sigma_{|S/K|} \wr \Sigma_{p^m})^\wedge_p$ such that $f|_{BS}$ is homotopic to the composite $BS \xrightarrow{(\Delta B\rho)^\wedge_p} (B(\Sigma_n)^{p^m})^\wedge_p \xrightarrow{\Delta^\wedge_p} B(\Sigma_n \wr \Sigma_{p^m})^\wedge_p$. Therefore, it is sufficient to show that ρ is fusion invariant, this means, for all $P \leq S$ and $\varphi\colon P \to S$ in $\mathcal{F}$ there is an $\omega \in \Sigma_n$ such that $\rho|_{\varphi(P)} \circ \varphi = c_\omega \circ \rho|_P$.

Note that the homomorphisms $\rho|_P$ and $c_\omega \rho|_P$ equip to S/K with a structure of P-set, and moreover the induced P-set are Σ_n-isomorphic. Hence, to prove tha above equality, we only need to show that $(S/K, \leq)$ and $(S/K, \leq_\varphi)$ are equivalents as P-sets.

Note that for any $\varphi\colon P \to S \in \mathcal{F}$,

$$(S/K, \leq_\varphi) \cong \mathrm{Iso}^*(\varphi)\,\mathrm{Res}^S_{\varphi(P)}(S/K) \cong \mathrm{Iso}^*(\varphi)\,\mathrm{Res}^S_{\varphi(P)}\,\mathrm{Ind}^S_K(*).$$

Applying the Mackey formula to $\mathrm{Res}^S_{\varphi(P)}\mathrm{Ind}^S_K$, we get

$$\begin{aligned}(S/K, \leq_\varphi) &\cong \coprod_{[x]\in\varphi(P)\backslash S/K} \mathrm{Iso}^*(\varphi)\,\mathrm{Ind}^{\varphi(P)}_{\varphi(P)\cap K^x}\,\mathrm{Iso}^*(c_x)\,\mathrm{Res}^K_{(\varphi(P)\cap K)^x}(*)\\ &= \coprod_{[x]\in\varphi(P)\backslash S/K} \mathrm{Ind}^P_{\varphi^{-1}(\varphi(P)\cap K)}\,\mathrm{Iso}^*(\varphi)\,\mathrm{Iso}^*(c_x)\,\mathrm{Res}^K_{(\varphi(P)\cap K)^x}(*).\end{aligned}$$

where the second equality comes from the commutativity of isogation and induction and, where $K^x = K$ because K is strongly $\mathcal{F}$-closed and $c_x\colon K \to S$ is in $\mathcal{F}$, hence $c_x(K) = K^x \leq K$ and since c_x is an isomorphism, $K^x = K$. Now, note that $\varphi^{-1}(\varphi(P)\cap K) = P\cap K$, because as $\varphi^{-1}|_{\varphi(P)\cap K}\colon \varphi(P)\cap K \to S$ is in $\mathcal{F}$, $\varphi(P)\cap K \leq K$ and K is strongly $\mathcal{F}$-closed, $\varphi^{-1}(\varphi(P)\cap K) \leq K$ but also $\varphi^{-1}(\varphi(P)\cap K) \leq P$, hence $\varphi^{-1}(\varphi(P)\cap K) \leq P\cap K$. If we prove that $|\varphi^{-1}(\varphi(P)\cap K)| = |P\cap K|$, then we show that $\varphi^{-1}(\varphi(P)\cap K) = P\cap K$, where since φ^{-1} is an isomorphism, $|\varphi^{-1}(\varphi(P)\cap K)| = |\varphi(P)\cap K|$. We have $|\varphi(P)\cap K| = |\varphi^{-1}(\varphi(P)\cap K)| \leq |P\cap K|$ and since $\varphi|_{P\cap K}\colon P\cap K \to S$ is in $\mathcal{F}$, $P\cap K \leq K$ and K is strongly $\mathcal{F}$-closed, $\varphi(P\cap K) \leq K$ but also $\varphi(P\cap K) \leq \varphi(P)$, hence $\varphi(P\cap K) \leq \varphi(P)\cap K$ and therefore $|\varphi(P\cap K)| \leq |\varphi(P)\cap K|$, where $|\varphi(P\cap K)| = |P\cap K|$ because φ is an isomorphism. We conclude from this that $|P\cap K| = |\varphi(P)\cap K|$ and $|\varphi^{-1}(\varphi(P)\cap K)| = |\varphi(P)\cap K|$ and finally that $\varphi^{-1}(\varphi(P)\cap K) = P\cap K$. Therefore in the above formula since $\mathrm{Iso}^*(\varphi)\,\mathrm{Iso}^*(c_x)\,\mathrm{Res}^K_{(\varphi(P)\cap K)^x}(*) = *$ as $(P\cap K)$-set and $\varphi^{-1}(\varphi(P)\cap K) = P\cap K$, we get for all $\varphi\colon P\to S \in \mathcal{F}$

$$(S/K, \leq_\varphi) \cong \coprod_{[x]\in\varphi(P)\backslash S/K} \mathrm{Ind}^P_{P\cap K}(*) \cong \coprod_{l_\varphi} P/P\cap K,$$

where $l_\varphi = |\varphi(P)\backslash S/K|$ and, since $K \lhd S$, the number of double cosets in $\varphi(P)\backslash S/K$ is the same as the number of cosets in $S/\varphi(P)\cdot K$, this means, $l_\varphi = |S/\varphi(P)\cdot K| = |S|/|\varphi(P)\cdot K|$. In particular, if $\varphi = id_P$, then

$$(S/K, \leq) \cong \coprod_{[x]\in P\backslash S/K} \mathrm{Ind}^P_{P\cap K}(*) \cong \coprod_{l} P/P\cap K,$$

where $l = |S|/|P\cdot K|$.

Therefore, the proof is completed by showig that $l = l_\varphi$, for this it is enough to show $|\varphi(P)\cap K| = |P\cap K|$ and this is proved in the above paragraph. □

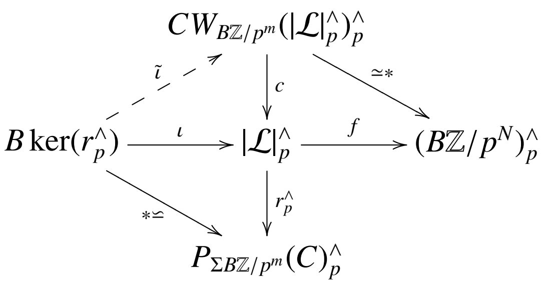
Corollary 4.3.33. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Let K be a strongly $\mathcal{F}$-closed subgroup. Then there exist some $N > 0$ and a map $f\colon |\mathcal{L}|^\wedge_p \to (B\Sigma_N)^\wedge_p$ such that $f|_{BK} \simeq *$.*

Proof. By the previous theorem there exist some $m \geq 0$ and a map $\bar f\colon |\mathcal{L}| \to B(\Sigma_{|S/K|} \wr \Sigma_{p^m})^\wedge_p$ such that $\bar f|_{BS} = \Delta^\wedge_p \circ (\Delta B\rho)^\wedge_p$. Note that $\rho|_K = e$ and hence $\bar f|_{BK} \simeq *$. Now consider the regular representation of $reg\colon \Sigma_{|S/K|}\wr\Sigma_{p^m} \to \Sigma_N$, where $N = |\Sigma_{|S/K|}\wr\Sigma_{p^m}|$. Therefore take f as one of the representation of the homotopy class $(\eta^*)^{-1}([\bar f])$, where $\eta^*\colon [|\mathcal{L}|^\wedge_p, (B\Sigma_N)^\wedge_p]_* \to [|\mathcal{L}|, (B\Sigma_N)^\wedge_p]_*$ is the bijection induced by the p-completion map $\eta\colon |\mathcal{L}| \to |\mathcal{L}|^\wedge_p$. □

Therefore, we can compute expefically $\ker(r^\wedge_p)$:

Proposition 4.3.34. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group such that C is 1-connected. Then* $\ker(r^\wedge_p) = Cl_{p^m}(S)$.

Proof. Let K be a strongly $\mathcal{F}$-closed subgroup that contains $\Omega_{p^m}(S)$. According to Corollary 4.3.33, there exist $N > 0$ and a map $f : |\mathcal{L}|^\wedge_p \to (B\mathbb{Z}/p^N)^\wedge_p$ such that $\ker(f) = K$. Note now that since $\Omega_{p^m}(S) \leq K$, for all $g : B\mathbb{Z}/p^m \to |\mathcal{L}|^\wedge_p$ we get $f \circ g \simeq *$. Furthermore, if $c : CW_{B\mathbb{Z}/p^m}(|\mathcal{L}|^\wedge_p) \to |\mathcal{L}|^\wedge_p$, then $c_* : \mathrm{map}_*(B\mathbb{Z}/p^m, CW_{B\mathbb{Z}/p^m}(|\mathcal{L}|^\wedge_p)) \simeq \mathrm{map}_*(B\mathbb{Z}/p^m, |\mathcal{L}|^\wedge_p)$ and hence $k \circ (f \circ c) \simeq *$ for all $k : B\mathbb{Z}/p^m \to CW_{B\mathbb{Z}/p^m}(|\mathcal{L}|^\wedge_p)$. Therefore $f \circ c \simeq *$ by Proposition 2.1.25 (note that $(B\mathbb{Z}/p^N)^\wedge_p$ is $\Sigma B\mathbb{Z}/p$-null and hence $\Sigma B\mathbb{Z}/p^m$-null by Lemma 4.3.17). Consider now $\iota : B\ker(r^\wedge_p) \to P_{\Sigma B\mathbb{Z}/p^m}(C)^\wedge_p$, since $r^\wedge_p \circ \iota \simeq *$, there is a map $\tilde{\iota} : B\ker(r^\wedge_p) \to CW_{B\mathbb{Z}/p^m}(|\mathcal{L}|^\wedge_p)$ such that the following diagram



is commutative. Moreover, $\tilde{\iota} \circ c \circ f \simeq *$, because $c \circ f \simeq *$, and hence $\iota \circ f \simeq *$. Therefore $\ker(r^\wedge_p) \leq \ker(f) = K$. We proved that if K is a strongly $\mathcal{F}$-closed subgroup that contains $\Omega_{p^m}(S)$, then $\ker(r^\wedge_p) \leq K$ and $\ker(r^\wedge_p)$ is strongly $\mathcal{F}$-closed by Proposition 4.3.15, hence $\ker(r^\wedge_p) = Cl_{p^m}(S)$. □

Then, now we can prove the Theorem 4.3.29:

Proof of Theorem 4.3.29. If $|\mathcal{L}|^\wedge_p$ is $B\mathbb{Z}/p^m$-cellular, then $P_{\Sigma B\mathbb{Z}/p^m}(C)^\wedge_p \simeq *$, and hence $S = \ker(r^\wedge_p)$, and $\ker(r^\wedge_p) = Cl_{p^m}(S)$ by Proposition 4.3.34. If $S = Cl_{p^m}(S)$, then by Lemma 4.3.31 and Proposition 4.3.34, $\ker(r^\wedge_p) = Cl_{p^m}(S) = S$. Moreover, $|\mathcal{L}|^\wedge_p$ is nilpotent since $\pi_i(|\mathcal{L}|^\wedge_p)$ are finite groups for all $i \geq 0$ by [CL09, Lemma 7.6], $|\mathcal{L}|^\wedge_p \simeq (|\mathcal{L}|^\wedge_p)^\wedge_p$ is a nilpotent space by [BK72, Proposition VII.4.3(ii)], $\pi_1(|\mathcal{L}|^\wedge_p)$ is a finite p-group by [BLO03b, Proposition 1.12] and, by [BLO03b, Theorem B], $H^*(|\mathcal{L}|^\wedge_p; \mathbb{Z}) \hookrightarrow H^*(BS; \mathbb{Z})$ and hence, from the exact sequence

$$0 \to \mathrm{Ext}(H_*(|\mathcal{L}|^\wedge_p; \mathbb{Z}), \mathbb{Z}) \to H^*(|\mathcal{L}|^\wedge_p; \mathbb{Z}) \to \mathrm{Hom}(H_*(|\mathcal{L}|^\wedge_p; \mathbb{Z}), \mathbb{Z}) \to 0$$

it follows that $\mathrm{Hom}(H_*(|\mathcal{L}|^\wedge_p; \mathbb{Z}), \mathbb{Z})$ is finite and, necessarily, $H_*(|\mathcal{L}|^\wedge_p; \mathbb{Z})$ is finite. In particular, $H_2(|\mathcal{L}|^\wedge_p; \mathbb{Z})$ is finite. Furthermore $[B\mathbb{Z}/p^m, |\mathcal{L}|^\wedge_p]_*$ is finite because there is an epimorphism of sets $[B\mathbb{Z}/p^m, S]_* \twoheadrightarrow [B\mathbb{Z}/p^m, |\mathcal{L}|^\wedge_p]_*$, where $[B\mathbb{Z}/p^m, BS]_* \cong \mathrm{Hom}(\mathbb{Z}/p^m, S)$ is finite since S is a finite group. Therefore $|\mathcal{L}|^\wedge_p$ is $B\mathbb{Z}/p^m$-cellular by Corollary 4.3.20. □

Question. What is the $B\mathbb{Z}/p^m$-cellularization of the classifying space of p-local finite group if $S \neq Cl_{p^m}(S)$?

Let G be a finite group. The case when G is generated by elements of order p is well studied by R. Flores and R. Foote in [FF11]. Then now we will give some examples of cellularizations of $BG^\wedge_p$, when G is not generated by elements of order p^i.

Example 4.3.35. Let $G = \Sigma_3$, the permutation group of 3 elements. Σ_3 is generated by traspositions, i.e, by elements of order 2, but the Sylow 3-subgroup of Σ_3 is $S = \mathbb{Z}/3$. Therefore $BS = B\mathbb{Z}/3$ is $B\mathbb{Z}/3$-cellular and hence $S = \Omega_{3^r}(S)$ for all $r \geq 1$. Therefore, $S = Cl_{3^r}(S)$ and hence $(B\Sigma_3)^\wedge_3$ is $B\mathbb{Z}/3^r$-cellular for all $r \geq 1$ by Theorem 4.3.29.

Example 4.3.36. Let G be a finite group and S a Sylow p-subgroup. Assume that the normalizer $N_G(S)$ of the Sylow p-subgroup in G *controls fusion* in G, that is, whenever $P < G$ is a p-subgroup such that $gPg^{-1} < N_G(S)$, we have $g = hc$, wuth $h \in N_G(S)$ and $c \in C_G(P)$. Under this condition, the inclusion $N_G(S) \hookrightarrow G$ induces an isomorphism in mod p cohomology (see for instance [MP97, Proposition 2.1]). In other words, $BN_G(S)^\wedge_p \simeq BG^\wedge_p$. Furthermore, we get $\Omega_{p^i}(S) < Cl_{p^i}(S) \lhd S \lhd N_G(S)$, for all $i \geq 0$, and there is an integer $n \geq 0$ such that $Cl_{p^i}(S) = S$ if and only if $i \geq n$. Therefore, $BG^\wedge_p$ is $B\mathbb{Z}/p^i$-cellular if and only if $n \geq i$, according to Therem 4.3.29. Now we will compute the $B\mathbb{Z}/p^i$-cellularization of $BG^\wedge_p$ for $1 \leq i < n$. First, we have the fibration

$$BS \xrightarrow{B\iota} BN_G(S) \longrightarrow B(N_G(S)/S),$$

where $CW_{B\mathbb{Z}/p^i}(BS) \simeq BCl_{p^i}(S)$ and $CW_{B\mathbb{Z}/p^i}(B(N_G(S)/S)) \simeq *$ by Theorem 4.1.7 since p and $|N_G(S)/S|$ are coprime. Therefore $B\iota$ is a $B\mathbb{Z}/p^i$-equivalence and hence $CW_{B\mathbb{Z}/p^i}(BG) \simeq CW_{B\mathbb{Z}/p^i}(BS) \simeq BCl_{p^i}(S)$. Now we want to compute the cellularization of $BG^\wedge_p \simeq BN_G(S)^\wedge_p$. Note that $[B\mathbb{Z}/p^i, BN_G(S)]_* \cong [B\mathbb{Z}/p^i, BN_G(S)^\wedge_p]_*$ by [BK02, Proposition 7.5] and consider the following diagram of horizontal cofibrations,

$$\begin{array}{ccccc} BCl_{p^i}(S) & \longrightarrow & BN_G(S) & \longrightarrow & C \\ \| & & \| & & \downarrow \\ BCl_{p^i}(S) & \longrightarrow & BN_G(S)^\wedge_p & \longrightarrow & D \end{array}$$

hence $C^\wedge_p \simeq D^\wedge_p$ and Theorem 2.1.22 gives the following fibrations

$$CW_{B\mathbb{Z}/p^i}(BN_G(S)) \simeq BCl_{p^i}(S) \longrightarrow BN_G(S) \xrightarrow{r} P_{\Sigma B\mathbb{Z}/p^i}(C),$$

$$CW_{B\mathbb{Z}/p^i}(BN_G(S)^\wedge_p) \longrightarrow BN_G(S)^\wedge_p \longrightarrow P_{\Sigma B\mathbb{Z}/p^i}(D).$$

We need to identify $P_{\Sigma B\mathbb{Z}/p^i}(D)$, specifically, we want now to prove that

$$P_{\Sigma B\mathbb{Z}/p^i}(D)^\wedge_p \simeq P_{\Sigma B\mathbb{Z}/p^i}(C)^\wedge_p \simeq B(N_G(S)/Cl_{p^i}(S))^\wedge_p$$

using Lemma 1.3.4. Hence we need to verify that if $X = C$ or D, then $P_{\Sigma B\mathbb{Z}/p^i}(X)$ and $P_{\Sigma B\mathbb{Z}/p^i}(X^\wedge_p)$ are p-good spaces and $P_{\Sigma B\mathbb{Z}/p^i}(X)^\wedge_p$ and $P_{\Sigma B\mathbb{Z}/p^i}(X^\wedge_p)^\wedge_p$ are $B\mathbb{Z}/p^i$-null spaces. $\pi_1(C^\wedge_p)$ and $\pi_1(D^\wedge_p)$ are finite groups, since $\pi_1 C$ and $\pi_1 D$ are finite groups. Hence $\pi_1(P_{\Sigma B\mathbb{Z}/p^i}(C))$, $\pi_1(P_{\Sigma B\mathbb{Z}/p^i}(D))$, $\pi_1(P_{\Sigma B\mathbb{Z}/p^i}(C^\wedge_p))$ and $\pi_1(P_{\Sigma B\mathbb{Z}/p^i}(D^\wedge_p))$ are finite groups and then $P_{\Sigma B\mathbb{Z}/p^i}(C)$, $P_{\Sigma B\mathbb{Z}/p^i}(D)$, $P_{\Sigma B\mathbb{Z}/p^i}(C^\wedge_p)$ and $P_{\Sigma B\mathbb{Z}/p^i}(D^\wedge_p)$ are p-good spaces.

Moreover, if $X = D, C^\wedge_p$ or $D^\wedge_p$, then $\pi_1 X$ is a finite p-group, and hence so is $\pi_1 P_{\Sigma B\mathbb{Z}/p^i}(X)$. Therefore, as in the proof of Lemma 4.3.25, we get $P_{\Sigma B\mathbb{Z}/p^i}(X)$ is a $\Sigma B\mathbb{Z}/p^i$-null space. Consider the following fibration

$$BCl_{p^i}(S) \xrightarrow{B\iota_i} BN_G(S) \longrightarrow B(N_G(S)/Cl_{p^i}(S)),$$

and note that

$$\mathrm{map}_*(BCl_{p^i}(S), \Omega P_{\Sigma B\mathbb{Z}/p^i}(C)) \simeq \mathrm{map}_*(BCl_{p^i}(S), P_{B\mathbb{Z}/p^i}(\Omega C)) \simeq *,$$

i.e., $\Omega P_{\Sigma B\mathbb{Z}/p^i}(C)$ is $BCl_{p^i}(S)$-null. Moreover $r \circ B\iota_i \simeq *$ because $BCl_{p^i}(S) \simeq CW_{B\mathbb{Z}/p^i}(BN_G(S))$. Hence Zabrodsky Lema proves that there exists a map $\tilde{r}\colon B(G/Cl_{p^i}(S)) \to P_{\Sigma B\mathbb{Z}/p^i}(C)$ such that the following diagram

$$\begin{array}{ccc} BCl_{p^i}(S) & \xrightarrow{\simeq} & CW_{B\mathbb{Z}/p^i}(BN_G(S)) \\ {\scriptstyle B\iota_i}\downarrow & & \downarrow \\ BG & = & BN_G(S) \\ \downarrow & & \downarrow{\scriptstyle r} \\ B(G/Cl_{p^i}(S)) & \xrightarrow[\tilde{r}]{} & P_{\Sigma B\mathbb{Z}/p^i}(C) \end{array}$$

Furthermore, $\tilde{r}$ is a (weak) equivalence, hence $P_{\Sigma B\mathbb{Z}/p^i}(C) \simeq B(G/Cl_{p^i}(S))$ and finally

$$P_{\Sigma B\mathbb{Z}/p^i}(C)^\wedge_p \simeq B(G/Cl_{p^i}(S))^\wedge_p, \text{ a } \Sigma B\mathbb{Z}/p^i\text{-null space.}$$

Then Lemma 1.3.4 gives us

$$B(G/Cl_{p^i}(S))^\wedge_p \simeq P_{\Sigma B\mathbb{Z}/p^i}(C)^\wedge_p \simeq P_{\Sigma B\mathbb{Z}/p^i}(C^\wedge_p)^\wedge_p$$

and

$$P_{\Sigma B\mathbb{Z}/p^i}(D)^\wedge_p \simeq P_{\Sigma B\mathbb{Z}/p^i}(D^\wedge_p)^\wedge_p.$$

Hence $P_{\Sigma B\mathbb{Z}/p^i}(D)^\wedge_p \simeq B(G/Cl_{p^i}(S))^\wedge_p$, since $C^\wedge_p \simeq D^\wedge_p$, and we get the fibration

$$CW_{B\mathbb{Z}/p^i}(BN_G(S)^\wedge_p) \to BN_G(S)^\wedge_p \to B(N_G(S)/Cl_{p^i}(S))^\wedge_p,$$

since $CW_{B\mathbb{Z}/p^i}(BN_G(S)^\wedge_p)$ is p-complete. Finally, recall that $BN_G(S)^\wedge_p \simeq BG^\wedge_p$ and hence $CW_{B\mathbb{Z}/p^i}(G^\wedge_p)$ is equivalent to the homotopy fibre of $BN_G(S)^\wedge_p \to B(N_G(S)/Cl_{p^i}(S))^\wedge_p$.

As example of this we can consider $G = \mathbb{Z}/p^n \wr \mathbb{Z}/q$, with $p \neq q$ and $n \geq 1$. Note that $G = \mathbb{Z}/p^n \wr \mathbb{Z}/q = (\mathbb{Z}/p^n)^q \rtimes \mathbb{Z}/q$, where the action of $\mathbb{Z}/q$ in $(\mathbb{Z}/p^n)^q$ is given by permutation. Hence G is not generated by elements of order p^i, $i \geq 1$, because if $g \in G$, then $g = ((x_1, \ldots, x_q), \sigma)$, where $x_j \in \mathbb{Z}/p^n$ and $\sigma \in \mathbb{Z}/q$, hence

$$g^{p^i} = ((x_1, \ldots, x_q), \sigma) \cdot ((x_1, \ldots, x_q), \sigma) \cdot \ldots \cdot ((x_1, \ldots, x_q), \sigma) = ((x_{\sigma^{p^i}(1)}, \ldots, x_{\sigma^{p^i}(q)}), \sigma^{p^i}),$$

and $\sigma^{p^i} = 1$ if and only if $i = 0$.

Moreover, $N_G(S) = G$ because if $S \in \mathrm{Syl}_p(G)$, then $S = (\mathbb{Z}/p^n)^q \lhd G$. Furthermore, $Cl_{p^i}(S) = \Omega_{p^i}(S) = (\mathbb{Z}/p^i)^q$ (obviously, $\Omega_{p^i}(S) = (\mathbb{Z}/p^i)^q$ and if $\sigma \in \mathbb{Z}/q$ and $g \in \Omega_{p^i}(S)$, then $g\sigma \in \Omega_{p^i}(S)$). Hence $Cl_{p^i}(S) = S$ if and only if $i \geq n$. Therefore, $B(\mathbb{Z}/p^n \wr \mathbb{Z}/q)^\wedge_p$ is $B\mathbb{Z}/p^i$-cellular if and only $i \geq n$ and if $1 \leq i < n$, then $CW_{B\mathbb{Z}/p^i}(B(\mathbb{Z}/p^n \wr \mathbb{Z}/q)^\wedge_p)$ is equivalent to the homotopy fibre of $B(\mathbb{Z}/p^n \wr \mathbb{Z}/q)^\wedge_p \to B((\mathbb{Z}/p^n \wr \mathbb{Z}/q)/(\mathbb{Z}/p^i)^q)^\wedge_p$.

Another example studied in [FS07] is given by the Suzuki group $\mathrm{Sz}(2^n)$, with n an odd integer at least 3. On account of [Gor80, Section 16.4] the Sylow 2-subgroup of $\mathrm{Sz}(2^n)$ is an extension

$$0 \to (\mathbb{Z}/2)^n \to S \to (\mathbb{Z}/2)^n \to 0,$$

where the kernel in the center of the group and contains all its order 2 elements. Hence $Cl_2(S) \cong (\mathbb{Z}/2)^n$ and BS is $B\mathbb{Z}/2^m$-cellular for all $m \geq 2$ and. Moreover, the normalizer

$N_{\mathrm{Sz}(2^n)}(S) = S \rtimes \mathbb{Z}/(2^n-1)$ wich is maximal in $\mathrm{Sz}(2^n)$. In [FS07, Example 5.2] the authors prove that $N_{\mathrm{Sz}(2^n)}(S)$ controls fusion in $\mathrm{Sz}(2^n)$. Therefore, $B\mathrm{Sz}(2^n)^\wedge_2$ is $B\mathbb{Z}/p^m$-cellular for all $m \geq 2$ and $CW_{B\mathbb{Z}/2}(B\mathrm{Sz}(2^n)^\wedge_2)$ is equivalent to the homotoy fibre of $B(S \rtimes \mathbb{Z}/(2^n-1))^\wedge_2 \to B((S \rtimes \mathbb{Z}/(2^n-1))/(\mathbb{Z}/2)^n)^\wedge_2$.

4.3.5 Classifying spaces of compact Lie groups

In this section we want to compute the A-cellularization of the p-completed of the classifying space of compact Lie groups, where $A = B\mathbb{Z}/p^m$ or $A = B\mathbb{Z}/p^m \times B\mathbb{Z}/p^\infty$. We want to conclude that if a Sylow p-subgroup is cellular then we can compute the cellularization as the fibre of the rationalization, using the results developed in Section 4.3. As is usual, in this section, B_{p^m} detones the space $B\mathbb{Z}/p^\infty \times B\mathbb{Z}/p^m$.

Following Theorem 4.3.19, in the case when G is a compact Lie group, we can check some of the hypothesis. More precisely, there is a map $\varphi\colon \bigvee_I A \to |\mathcal{L}|^\wedge_p$, where I is a finite set, such that $\pi_1(\varphi)$ is an epimorphism and $\varphi_*\colon [A, \bigvee_I A]_* \to [A, |\mathcal{L}|^\wedge_p]_*$ is surjective.

By thecnical reasons, we have to modify the "standard" Chachólski's cofibration. Note that if we have a pointed map $f'\colon B\mathbb{Z}/p^\infty \times B\mathbb{Z}/p^m \to BG^\wedge_p$ then we obtain a pointed map $g\colon B\mathbb{Z}/p^\infty \to \mathrm{map}(B\mathbb{Z}/p^m, BG^\wedge_p)_f$, where $f = f'|_{B\mathbb{Z}/p^m}$. Moreover, $\mathrm{map}(B\mathbb{Z}/p^m, BG^\wedge_p)_f \simeq B(C_G(\mathbb{Z}/p^m))^\wedge_p$, by [DZ87]. Hence

$$g \simeq (g_1, \dots, g_{r(f)})\colon B\mathbb{Z}/p^\infty \to ((BS^1)^\wedge_p)^{r(f)} \subset \mathrm{map}(B\mathbb{Z}/p^m, BG^\wedge_p)_f.$$

where $r(f)$ denotes the rank of $C_G(\mathbb{Z}/p^m)$ Let $B = \bigvee_{[f]\in[B\mathbb{Z}/p^m, BG^\wedge_p]_*}((B\mathbb{Z}/p^\infty)^{r(f)} \times B\mathbb{Z}/p^m)$. Note now if we define $\psi\colon B \to BG^\wedge_p$ by $\psi((x_1, \dots, x_{r(f)}, y)_f) = (g_1(x_1), \dots, g_{r(f)}(x_{r(f)}))(y)$, where $g = (g_i)_i$ is the induced by a pointed map in $[f]$, and if given a map $f'\colon B\mathbb{Z}/p^\infty \times B\mathbb{Z}/p^m \to BG^\wedge_p$ we define $F\colon B\mathbb{Z}/p^\infty \times B\mathbb{Z}/p^m \to (B\mathbb{Z}/p^\infty)^{r(f)} \times B\mathbb{Z}/p^m$ by $F(x, y) = (x, \dots, x, y)$ then $\psi(F) = f$. That is, we have proved the following lemma:

Lemma 4.3.37. *The evaluation map $\psi\colon B \to BG^\wedge_p$ induces an epimorphism of sets*

$$\psi_*\colon [B_{p^m}, B]_* \longrightarrow [B_{p^m}, BG^\wedge_p]_*.$$

The cofibre of this map will play the role of the Chachólski's cofibre:

Proposition 4.3.38. *Let C be the homotopy cofibre of $\psi\colon B \to BG^\wedge_p$. Then $CW_{B_{p^m}}(BG^\wedge_p)$ is equivalent to the homotopy fibre of $BG^\wedge_p \to P_{\Sigma B_{p^m}}(C)$.*

Proof. Consider the map $\psi\colon B \to BG^\wedge_p$. On the one hand, note that $B\mathbb{Z}/p^\infty$ and $B\mathbb{Z}/p^m$ are retracts of B_{p^m} and hence they are B_{p^m}-cellular spaces (by Proposition 2.1.7.(v)). Therefore B is a B_{p^m}-cellular space. On the other hand, by Lemma 4.3.37 we obtain that the induced map $\psi_*\colon [B_{p^m}, B]_* \to [B_{p^m}, BG^\wedge_p]_*$ is surjective. The result follows from Theorem 2.1.22. □

We will use the Thereom 4.3.19, hence we have to verify the hypothesis of this theorem. One of this is, in our case, $[B\mathbb{Z}/p^m, BG^\wedge_p]$ is finite.

Lemma 4.3.39. *Let G be a compact Lie group. Then $[B\mathbb{Z}/p^m, BG^\wedge_p]$ is finite for all $m \geq 0$.*

Proof. Note that $[B\mathbb{Z}/p^m, BG] = Rep(\mathbb{Z}/p^m, G)$ is defined by conjugacy classes of elements of order $\leq p^m$. Let n be a integer large enough to have the inclusion $G \hookrightarrow U(n)$, where $U(n)$ is the unitary group of dimension n. Let $T = T_{U(n)}$ the maximal torus in $U(n)$. For all $\mathbb{Z}/p^r \leq G$, $r \leq m$, there is a $g \in U(n)$ such that $g\mathbb{Z}/p^r g^{-1} \leq T$, this means for all $\mathbb{Z}/p^r \leq G$ and $h \in H$ there is a $g \in U(n)$ such that $h\mathbb{Z}/p^r h^{-1} \leq gTg^{-1}$, and [Pal60, Corollary 1.7.29] shows that the number of conjugation classes of this subgroups is finite. Hence $[B\mathbb{Z}/p^m, BG]$ is finite and finally so is $[B\mathbb{Z}/p^m, BG^\wedge_p]$. □

Other condition is the cofibre C must be 1-connected. If G is connected, then $BG^\wedge_p$ is 1-connected and hence so is C. But, if G is not connected (and G is not a finite group) then we can find a integer $m \geq 0$ such that this condition holds. Before we need the following proposition about lifting of maps from $B\mathbb{Z}/p^m$ to the classifying space of the group of components of G:

Proposition 4.3.40. *Let G be a compact Lie group and G_e the connected component of the identity element. Let $\pi = G/G_e$ be the group of components of G. For any element $x \in \pi$ with order a power of p there is a non-negative integer m_x such that $f_x \colon B\mathbb{Z}/p^{m_x} \to B\pi$ lifts to BG, that is, there is a map $\tilde{f}_x \colon B\mathbb{Z}/p^{m_x} \to BG$ such that the following diagram*

$$\begin{array}{ccc} & & BG \\ & \overset{\tilde{f}_x}{\nearrow} & \downarrow{\scriptstyle Bpr} \\ B\mathbb{Z}/p^{m_x} & \xrightarrow[f_x]{} & B\pi, \end{array}$$

is commutative, i.e., such that $Bpr \circ \tilde{f}_x = f_x$.

Proof. Let $x_0 \in \pi$ such that $o(x_0) = p^r$, $r \geq 0$, and consider $\pi_{x_0} = \langle x_0 \rangle \cong \mathbb{Z}/p^r$.

Let $g_0 \in G$ such that $pr(g_0) = x_0$ and let $A = \langle g_0 \rangle = \{g_0^n \mid n \in \mathbb{Z}\} \leq G$. Note that A is abelian and hence $\overline{A}$ is an abelian closed subgroup of G, a compact Lie group. Therefore $\overline{A}$ is an abelian compact Lie group. By [BtD85, Corollary I.3.7] $\overline{A} \cong (S^1)^k \times \pi'$, where π' is an abelian finite group.

Note that $(S^1)^k \times \{0\} \subset G_e$ because $e = (1, 0) \in (S^1)^k \times \{0\}$ and $(S^1)^k \times \{0\}$ is connected.

Let $g_0 = (\omega_0, y_0) \in (S^1)^k \times \pi'$ and take $h_0 = (\omega_0^{-1}, 0) \cdot (\omega_0, \overline{y_0}) = (1, \overline{y_0}) \in \{1\} \times \pi'$, where $\overline{y_0}$ is the projection of y_0 over the p-torsion component of π'. Therefore $pr(h_0) = pr((\omega_0^{-1}, 0) \cdot (\omega_0, \overline{y_0})) = pr(\omega_0^{-1}, 0) + pr(\omega_0, \overline{y_0}) = pr(g_0) = x_0$, because $(\omega_0^{-1}, 0) \in (S^1)^k \times \{0\}$ and hence $pr(\omega_0^{-1}, 0) = 0$.

Since $h_0 = (1, \overline{y_0}) \in \{1\} \times \mathbb{Z}/p^m$ for some $m \geq 0$, $p^m x_0 = 0$ and hence $h_0^{p^m} = (1^{p^m}, p^m y_0) = (1, 0) = e$. Furthermore $m \geq r$ because on the one hand $pr(h_0^{p^m}) = pr(e) = 0$ and on the other hand $pr(h_0^{p^m}) = p^m pr(h_0) = p^m$, hence $p^m = 0 \in \mathbb{Z}/p^r$ and therefore $p^r \mid p^m$, i.e., $m \geq r$.

Consider now $B = \langle h_0 \rangle = \{e, h_0, h_0^2, \ldots, h_0^{p^m - 1}\} \leq \overline{A} \leq G$ and take $\alpha \colon \mathbb{Z}/p^m \to \mathbb{Z}/p^r$ given by $\alpha(1) = 1$. We define $\tilde{\alpha} \colon \mathbb{Z}/p^m \to G$ by

$$\begin{array}{rcccc} \tilde{\alpha} \colon \mathbb{Z}/p^m & \longrightarrow & B & \overset{\iota}{\hookrightarrow} & G \\ 1 & \longmapsto & h_0 & \longmapsto & \iota(h_0), \end{array}$$

and $pr \circ \tilde{\alpha}(1) = pr(h_0) = 1 = \alpha(1)$, i.e., $pr \circ \tilde{\alpha} = \alpha$.

Therefore if we take $f = B\alpha$ and $\tilde{f} = B\tilde{\alpha}$ then we obtain the following commutative diagram

$$\begin{array}{ccc} & & BG \\ & \nearrow^{\tilde{f}} & \downarrow{\scriptstyle Bpr} \\ B\mathbb{Z}/p^m & \xrightarrow[f]{} & B\pi. \end{array}$$

□

Lemma 4.3.41. *Let C be the homotopy cofibre of $\psi\colon B \to BG^\wedge_p$. Then there is a non-negative integer m such that C is 1-connected. Therefore $P_{\Sigma B_{p^m}}(C)$ is also 1-connected.*

Proof. Let m' such that $p^{m'} \geq \max\{o(x) \mid x \in \pi\}$. Note that C is the homotopy push out

$$\begin{array}{ccc} B & \xrightarrow{\psi} & BG^\wedge_p \\ \downarrow & & \downarrow \\ * & \longrightarrow & C, \end{array}$$

hence by Seifert-Van Kampen's theorem we obtain the following push out diagram of groups

$$\begin{array}{ccc} *_{[B\mathbb{Z}/p^{m'},BG^\wedge_p]}\mathbb{Z}/p^\infty \times \mathbb{Z}/p^{m'} & \xrightarrow{\pi_1(\psi)} & \pi_1(BG^\wedge_p) \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \pi_1(C), \end{array}$$

where $[B\mathbb{Z}/p^{m'}, BG^\wedge_p]$ is finite by Proposition 4.3.39. Hence $\pi_1(C) \cong \pi_1(BG^\wedge_p)/ \lhd \operatorname{Im}(\pi_1(\psi))\rhd$. Therefore if $\pi_1(\psi)$ is surjective then $\pi_1(C) \cong 0$.

Note that any homomorphism from $\mathbb{Z}/p^\infty$ to a finite group is trivial, hence $\operatorname{Hom}(\mathbb{Z}/p^\infty \times \mathbb{Z}/p^{m'}, \pi_1(BG^\wedge_p)) \cong \operatorname{Hom}(\mathbb{Z}/p^{m'}, \pi_1(BG^\wedge_p))$. Let $x \in \pi_0 G/O^p(\pi_0 G)$, hence by Proposition 4.3.40 there is a non-negative integer m such that if we consider the homomorphism $\alpha\colon \mathbb{Z}/p^m \to \pi_0 G/O^p(\pi_0 G)$, defined by $\alpha(1) = x$, then there is a map $\tilde{B\alpha}\colon \mathbb{Z}/p^m \to BG^\wedge_p$ such that the following diagram

$$\begin{array}{ccc} & & BG^\wedge_p \\ & \nearrow^{\tilde{B\alpha}} & \downarrow \\ B\mathbb{Z}/p^\infty \times B\mathbb{Z}/p^m & \xrightarrow[B\alpha]{} & B\pi_1(BG^\wedge_p). \end{array}$$

is commutative. Hence $[\tilde{B\alpha}] \in [B\mathbb{Z}/p^\infty \times B\mathbb{Z}/p^m, BG^\wedge_p]$ and $\pi_1(\psi)([\tilde{B\alpha}]) = \alpha(1) = x$, i.e., $\pi_1(\psi)$ is surjective. □

Let $S \in \operatorname{Syl}_p(G)$. By Lemma 4.3.41, there is a $m_0 \geq 0$ such that C is 1-connected for all $m \geq m_0$. Let $r\colon BG^\wedge_p \to P_{\Sigma B_{p^m}}(C)$, then we can consider $\ker(r^\wedge_p) \leq S$.

Proposition 4.3.42. *Let G be a compact Lie group and $S \in \operatorname{Syl}_p(G)$. Assume that $H_2(BG^\wedge_p; \mathbb{Z})$ is finite. Then there is an integer $m_0 \geq 1$ such that for each $m \geq m_0$ such that $\ker(r^\wedge_p) = S$ the augmention map $CW_{B_{p^m}}(BG^\wedge_p) \to BG^\wedge_p$ is a mod p equivalence.*

Proof. Since $[B\mathbb{Z}/p^r, BG^\wedge_p]$ is finite by Lemma 4.3.39 and $\psi\colon B \to BG^\wedge_p$ induces an epimorphism in pointed homotopy classes of maps from B_{p^m} by Lemma 4.3.37, Theorem 4.3.19 gives us that $CW_{B_{p^m}}(BG^\wedge_p)^\wedge_p \simeq BG^\wedge_p$. □

Corollary 4.3.43. *Let G be a compact connected Lie group. Assume that $\pi_1 G$ is finite. Then if* $\ker(r^\wedge_p) = S$*, then the homotopy fibre of $BG^\wedge_p \to (BG^\wedge_p)_\mathbb{Q}$ is homotopic to $CW_{B_{p^m}}(BG^\wedge_p)$.*

Proof. If G is connected, then $BG^\wedge_p$ is 1-connected, and in particular nilpotent. Moreover by Hurewicz Theorem $H_2(BG^\wedge_p;\mathbb{Z}) \cong \pi_2(BG^\wedge_p)$ and it is finite because $\pi_1 G$ is so. Then the result is follows from Corollary 4.3.20 □

Corollary 4.3.44. *Let G be a compact connected Lie group. Assume that $\pi_1 G$ is finite. Then there is an integer $m_0 \geq 1$ such that the homotopy fibre of $BG^\wedge_p \to (BG^\wedge_p)_\mathbb{Q}$ is homotopic to $CW_{B_{p^m}}(BG^\wedge_p)$ for all $m \geq m_0$.*

Proof. By Proposition 4.2.5, there exists a $m_1 \geq 0$ such that BS is B_{p^m}-cellular for all $m \geq m_1$. By Lemma 4.3.41, there is a $m_2 \geq 0$ such that C is 1-connected for all $m \geq m_2$. Take $m_0 = \max\{1, m_1, m_2\}$, then BS is B_{p^m}-cellular and C is 1-connected for all $m \geq m_0$. Since BS is B_{p^m}-cellular, $\ker(r^\wedge_p) = S$ by [Dwy96, Theorem 1.4]. Then, for all $m \geq m_0$, we can apply Corollary 4.3.43. □

Next we will remove the hypothesis $\pi_1 G$ finite using the classification of compact connected Lie group:

Example 4.3.45. $G = S^1$ is a compact connected Lie group such that $\pi_1 S^1 \cong \mathbb{Z}$ is not finite, but the fibration

$$K(\mathbb{Z}/p^\infty, 1) \xrightarrow{\iota} K(\hat{\mathbb{Z}}_p, 2) \longrightarrow K(\hat{\mathbb{Z}}_p, 2)_\mathbb{Q}$$

induces a fibration

$$\mathrm{map}_*(B_{p^m}, K(\mathbb{Z}/p^\infty, 1)) \xrightarrow{\iota_*} \mathrm{map}_*(B_{p^m}, K(\hat{\mathbb{Z}}_p, 2))_{\{c\}} \longrightarrow \mathrm{map}_*(B_{p^m}, K(\hat{\mathbb{Z}}_p, 2)_\mathbb{Q})_c,$$

for all $m \geq 0$, where $K(\hat{\mathbb{Z}}_p, 2)_\mathbb{Q} \simeq L_\mathbb{Q}(K(\hat{\mathbb{Z}}_p, 2))$ because $K(\hat{\mathbb{Z}}_p, 2)$ is 1-connected, and hence

$$\mathrm{map}_*(B_{p^m}, K(\hat{\mathbb{Z}}_p, 2)_\mathbb{Q}) \simeq \mathrm{map}_*(L_\mathbb{Q}(B_{p^m}), K(\hat{\mathbb{Z}}_p, 2)_\mathbb{Q}) \simeq *,$$

because $\tilde{H}_*(B_{p^m};\mathbb{Q}) \cong 0$ and hence $L_\mathbb{Q}(B_{p^m}) \simeq *$. Therefore the above fibration becomes

$$\mathrm{map}_*(B_{p^m}, K(\mathbb{Z}/p^\infty, 1)) \xrightarrow{\iota_*} \mathrm{map}_*(B_{p^m}, K(\hat{\mathbb{Z}}_p, 2)) \longrightarrow *,$$

this means, ι is a B_{p^m}-equivalence. Moreover $K(\mathbb{Z}/p^\infty, 1) \simeq B\mathbb{Z}/p^\infty$ is B_{p^m}-cellular for all $m \geq 0$, hence we obtain $CW_{B_{p^m}}((BS^1)^\wedge_p) \simeq B\mathbb{Z}/p^\infty$ and finally we have the fibration

$$CW_{B_{p^m}}((BS^1)^\wedge_p) \to (BS^1)^\wedge_p \to ((BS^1)^\wedge_p)_\mathbb{Q}.$$

Lemma 4.3.46. *Let G_i be compact Lie groups, $i = 1, \ldots, k$, such that for any i there exists an integer $m_i > 0$ and the fibration $CW_{B_{p^{m_i}}}((BG_i)^\wedge_p) \to (BG_i)^\wedge_p \to ((BG_i)^\wedge_p)_\mathbb{Q}$. If $m = \max\{m_1, \ldots, m_k\}$ then $CW_{B_{p^m}}((BG_1)^\wedge_p \times \ldots \times (BG_k)^\wedge_p)$ fits in a fibration*

$$CW_{B_{p^m}}((BG_1)^\wedge_p \times \ldots \times (BG_k)^\wedge_p) \to (BG_1)^\wedge_p \times \ldots \times (BG_k)^\wedge_p \to ((BG_1)^\wedge_p \times \ldots \times (BG_k)^\wedge_p)_\mathbb{Q}$$

Proof. Note that $CW_{B_{p^{m_i}}}((BG_i)^\wedge_p)$ is B_{p^m}-cellular (since $m \geq m_i$, $B\mathbb{Z}/p^{m_i}$ is $B\mathbb{Z}/p^m$-cellular) and hence for all i we have the fibration

$$CW_{B_{p^m}}((BG_i)^\wedge_p) \to (BG_i)^\wedge_p \to ((BG_i)^\wedge_p)_{\mathbb{Q}},$$

and the homotopy fibre of

$$(G_1)^\wedge_p \times \ldots \times (G_k)^\wedge_p \to ((G_1)^\wedge_p \times \ldots \times (G_k)^\wedge_p)_{\mathbb{Q}}$$

is $CW_{B_{p^m}}((BG_1)^\wedge_p) \times \ldots \times CW_{B_{p^m}}((BG_k)^\wedge_p) \simeq CW_{B_{p^m}}((G_1)^\wedge_p \times \ldots \times (G_k)^\wedge_p)$. □

Theorem 4.3.47. *Let G be a compact connected Lie group. Then there is an integer $m_0 \geq 1$ such that the homotopy fibre of $BG^\wedge_p \to (BG^\wedge_p)_{\mathbb{Q}}$ is homotopic to $CW_{B_{p^m}}(BG^\wedge_p)$ for all $m \geq m_0$.*

Proof. A compact connected Lie group G is homeomorphism to $G \cong H/K$, where $H = G_1 \times \ldots \times G_k \times T^r$, G_i is 1-connected simple Lie groups for all $i \in \{1, \ldots, k\}$ and K is a finite subgroup of the center of H (see [BtD85, Theorem V.8.1]). Therefore we obtain the central fibration, a hence principal fibration, $BK \to BH \to BG$. Since the fibration is principal tha action of $\pi_1 BG$ in $H_j(BK; \mathbb{Z}/p)$ is trivial for all j and, in particular, nilpotent. Hence by [BK72, Lemma II.5.1] we have the fibration $BK^\wedge_p \to BH^\wedge_p \to BG^\wedge_p$.

Moreover, for all $i \in \{1, \ldots, k\}$ we have that G_i is a compact Lie group, because is simple, and $\pi_1 G_i \cong 0$ is a finite group. Therefore by Corollary 4.3.44, there is a $m_i \geq 0$ and a fibration $CW_{B_{p^{m_i}}}((BG_i)^\wedge_p) \to (BG_i)^\wedge_p \to ((BG_i)^\wedge_p)_{\mathbb{Q}}$. Furthermore for any factor $BT^r = (BS^1)^r$ we obtain the fibration $CW_{B_{p^n}}((BS^1)^\wedge_p) \to (BS^1)^\wedge_p \to ((BS^1)^\wedge_p)_{\mathbb{Q}}$ for all $n \geq 0$. It follows from Lemma 4.3.46 that there is a non-negative integer l_1 such that $CW_{B_{p^{l_1}}}(BH^\wedge_p)$ fits in a fibration $CW_{B_{p^{l_1}}}(BH^\wedge_p) \to BH^\wedge_p \to (BH^\wedge_p)_{\mathbb{Q}}$. Moreover $BK^\wedge_p$ is the classifying space of an abelian finite p-group, hence $BK^\wedge_p$ is $B\mathbb{Z}/p^{l_2}$-cellular for certain non-negative integer l_2 and hence it is $B_{p^{l_2}}$-cellular. Take $m = \max\{l_1, l_2\}$, hence $BK^\wedge_p$ is B_{p^m}-cellular and we have the fibration $CW_{B_{p^m}}(BH^\wedge_p) \to BH^\wedge_p \to (BH^\wedge_p)_{\mathbb{Q}}$.

Note that since $BK^\wedge_p$ is the classifying space of an abelian finite p-group, $(BK^\wedge_p)_{\mathbb{Q}} \simeq *$. Furthermore since BG is 1-connected, $BG^\wedge_p$ is it and hence tha action of $\pi_1(BG^\wedge_p)$ in $H_j(BK^\wedge_p; \mathbb{Q})$ is trivial (and nilpotent) and hence from [BK72, Lemma II.5.1] we obtain the fibration $(BK^\wedge_p)_{\mathbb{Q}} \to (BH^\wedge_p)_{\mathbb{Q}} \to (BG^\wedge_p)_{\mathbb{Q}}$, where $(BK^\wedge_p)_{\mathbb{Q}} \simeq *$, hence $(BH^\wedge_p)_{\mathbb{Q}} \simeq (BG^\wedge_p)_{\mathbb{Q}}$. Let F be the homotopy fibre of $BG^\wedge_p \to (BG^\wedge_p)_{\mathbb{Q}}$ and consider the following commutative diagram of fibrations

$$\begin{array}{ccccc}
BK^\wedge_p & \longrightarrow & CW_{B_{p^m}}(BH^\wedge_p) & \longrightarrow & F \\
\| & & \downarrow & & \downarrow f \\
BK^\wedge_p & \longrightarrow & BH^\wedge_p & \longrightarrow & BG^\wedge_p \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & (BH^\wedge_p)_{\mathbb{Q}} & \longrightarrow & (BG^\wedge_p)_{\mathbb{Q}}
\end{array}$$

Now the right vertical fibration induces a fibration in map spaces

$$\mathrm{map}_*(B_{p^m}, F) \to \mathrm{map}_*(B_{p^m}, BG^\wedge_p)_{\{c\}} \to \mathrm{map}_*(B_{p^m}, (BG^\wedge_p)_{\mathbb{Q}})_c$$

but since $BG^\wedge_p$ is 1-connected, $(BG^\wedge_p)_{\mathbb{Q}} \simeq L_{\mathbb{Q}}(BG^\wedge_p)$ and hence

$$\begin{aligned}&\mathrm{map}_*(B_{p^m}, (BG^\wedge_p)_{\mathbb{Q}}) \simeq \mathrm{map}_*(B_{p^m}, L_{\mathbb{Q}}(BG^\wedge_p)) \simeq \\ &\mathrm{map}_*(L_{\mathbb{Q}}(B_{p^m}), L_{\mathbb{Q}}(BG^\wedge_p)) \simeq \mathrm{map}_*(*, L_{\mathbb{Q}}(BG^\wedge_p)) \simeq *.\end{aligned}$$

Therefore $\mathrm{map}_*(B_{p^m}, F) \to \mathrm{map}_*(B_{p^m}, BG^\wedge_p)$ is an equivalence, this means that f is a B_{p^m}-equivalence and hence $CW_{B_{p^m}}F \simeq CW_{B_{p^m}}(BG^\wedge_p)$. Furthermore, since $BK^\wedge_p$ and $CW_{B_{p^m}}(BH^\wedge_p)$ are B_{p^m}-cellular spaces, [Far96, Theorem 2.D.11] gives that F is B_{p^m}-cellular, hence $F \simeq CW_{B_{p^m}}(BG^\wedge_p)$. □

Corollary 4.3.48. *For any compact connected Lie group G there exists an integer $m_0 \geq 1$ such that $BG^\wedge_p$ is $K(\mathbb{Q} \times \mathbb{Z}/p^\infty \times \mathbb{Z}/p^m, 1)$-cellular for all $m \geq m_0$.*

Proof. Let G be a compact connected Lie group G. Then there is an integer $m_0 \geq 1$ such that for all $m \geq m_0$ we have the fibration $CW_{B_{p^m}}(BG^\wedge_p) \to BG^\wedge_p \to (BG^\wedge_p)_{\mathbb{Q}}$ by Theorem 4.3.47. This fibration induces the following fibration

$$(G^\wedge_p)_{\mathbb{Q}} \to CW_{B_{p^m}}(BG^\wedge_p) \to BG^\wedge_p$$

since $\Omega((BG^\wedge_p)_{\mathbb{Q}}) \simeq ((\Omega BG)^\wedge_p)_{\mathbb{Q}} \simeq (G^\wedge_p)_{\mathbb{Q}}$. Moreover, $(G^\wedge_p)_{\mathbb{Q}} \simeq (\prod_{i=1}^n (S^{k_i})^\wedge_p)_{\mathbb{Q}} \simeq K(\hat{\mathbb{Q}}_p, k_i)$, where k_i are odd numbers (see [BT82, Section 19]). Note now that $K(\hat{\mathbb{Q}}_p, k_i)$ is $K(\hat{\mathbb{Q}}_p, 1)$-cellular by [Far96, Proposition 3.C.8]. Furthermore, $\hat{\mathbb{Q}}_p$ is an infinite $\mathbb{Q}$-vector space and hence $K(\hat{\mathbb{Q}}_p, 1)$ is $K(\mathbb{Q}, 1)$-cellular since so is $K(V_r, 1) \simeq K(\mathbb{Q}^r, 1)$. Therefore, $(G^\wedge_p)_{\mathbb{Q}}$ is $K(\mathbb{Q}, 1)$-cellular and hence it is $K(\mathbb{Q} \times \mathbb{Z}/p^\infty \times \mathbb{Z}/p^m, 1)$-cellular for all $m \geq 0$ and hence $BG^\wedge_p$ is $K(\mathbb{Q} \times \mathbb{Z}/p^\infty \times \mathbb{Z}/p^m, 1)$-cellular for all $m \geq m_0$, since so is $CW_{B_{p^m}}(BG^\wedge_p)$ and Proposition 2.1.7.(iii). □

Next, we will give some examples of cellularization of $BG^\wedge_p$:

Example 4.3.49. Let $G = S^1$. In Example 4.3.45 we have seen that for all $m \geq 0$,

$$CW_{B_{p^m}}((BS^1)^\wedge_p) \to (BS^1)^\wedge_p \to ((BS^1)^\wedge_p)_{\mathbb{Q}}.$$

Basically because we have the fibration

$$CW_{B\mathbb{Z}/p^\infty}((BS^1)^\wedge_p) \to (BS^1)^\wedge_p \to ((BS^1)^\wedge_p)_{\mathbb{Q}}.$$

In fact, in this case, $S \cong \mathbb{Z}/p^\infty$ and BS is $(B\mathbb{Z}/p^\infty \times B\mathbb{Z}/p^m)$-cellular for all $m \geq 0$.

Furthermore, $CW_{B\mathbb{Z}/p^m}(BS) \simeq B\mathbb{Z}/p^m$ for all $m \geq 0$ because $B\mathbb{Z}/p^m \hookrightarrow B\mathbb{Z}/p^\infty$ is a $B\mathbb{Z}/p^m$-equivalence. Moreover, as in Example 4.3.45, from the fibration

$$K(\mathbb{Z}/p^\infty, 1) \xrightarrow{\iota} K(\hat{\mathbb{Z}}_p, 2) \longrightarrow K(\hat{\mathbb{Z}}_p, 2)_{\mathbb{Q}}$$

we get the fibration

$$\mathrm{map}_*(B\mathbb{Z}/p^m, K(\mathbb{Z}/p^\infty, 1)) \xrightarrow{\iota_*} \mathrm{map}_*(B\mathbb{Z}/p^m, K(\hat{\mathbb{Z}}_p, 2))_{\{c\}} \longrightarrow \mathrm{map}_*(B\mathbb{Z}/p^m, K(\hat{\mathbb{Z}}_p, 2)_{\mathbb{Q}})_c,$$

for all $m \geq 0$, where $\mathrm{map}_*(B\mathbb{Z}/p^m, K(\hat{\mathbb{Z}}_p, 2)_{\mathbb{Q}}) \simeq *$. This means, ι is a $B\mathbb{Z}/p^m$-equivalence and hence $CW_{B\mathbb{Z}/p^m}((BS^1)^\wedge_p) \simeq CW_{B\mathbb{Z}/p^m}(B\mathbb{Z}/p^\infty) \simeq B\mathbb{Z}/p^m$.

Example 4.3.50. Let $G = S^3$ and $p = 2$. Then $S = P_\infty$ is constructed as follows: $P_\infty = \varinjlim_n P_n$, and $P_n \cong Q_{2^{n+1}}$, where Q_{2^n} denotes the generalized quaternion group given by $Q_{2^n} = (\mathbb{Z}/2^{n-1} \rtimes \mathbb{Z}/4)/\langle(2^{n-2}, 2)\rangle$. Moreover, for $n \geq 2$, $Q_{2^{n+1}} = \langle x, y\rangle$ such that

(i) $x^{2^n} = y^4 = 1$,

(ii) If $g \in Q_{2^{n+1}}$, then $g = x^a$ or $g = x^a y$ for certain $a \in \mathbb{Z}$,

(iii) $x^{2^{n-1}} = y^2$,

(iv) For all $g \in Q_{2^{n+1}}$ such that $g \notin \langle x\rangle$, then $gxg^{-1} = x^{-1}$, in particular, $yxy^{-1} = x^{-1}$.

Note first that $Q_{2^{n+1}} = \langle y, yx^{-1}\rangle$, because $x = yx^{-1} \cdot y^{-1}$, and $o(y) = o(yx^{-1}) = 4$ since $yx^{-1} \cdot yx^{-1} = yyxx^{-1} = y^2$. Hence BP_n is $B\mathbb{Z}/4$-cellular for all $n \geq 2$ by Proposition 4.1.2.

For $n = 1$, $P_1 = \langle x, y\rangle$, where $o(x) = o(y) = 2$, hence BP_1 is $B\mathbb{Z}/2$-cellular and, in particular, $B\mathbb{Z}/4$-cellular.

Therefore, BP_n is $B\mathbb{Z}/4$-cellular for all $n \geq 1$ and hence $BS = BP_\infty$ is $B\mathbb{Z}/4$-cellular because is a pointed homotopy colimit of $B\mathbb{Z}/4$-cellular spaces. Then, Theorem 4.3.47 shows that there is a fibration

$$CW_{B\mathbb{Z}/2^m}((BS^3)^\wedge_2) \to (BS^3)^\wedge_2 \to ((BS^3)^\wedge_2)_{\mathbb{Q}},$$

for all $m \geq 2$.

The case $m = 1$ is result, using a different method, in [CF13, Example 6.10], where they obtain that $CW_{B\mathbb{Z}/2}((BS^3)^\wedge_2) \cong B\mathbb{Z}/2$.

Example 4.3.51. Let $G = SO(3)$ and $p = 2$. In this case $S = D_{2^\infty}$, where $D_{2^\infty} = \varinjlim_n D_{2^n}$, the colimit of the diedral groups. BD_{2^∞} is $B\mathbb{Z}/2$-cellular by [Flo07, Example 5.1]. Then, Theorem 4.3.47 gives us the fibration

$$CW_{B\mathbb{Z}/2^m}(BSO(3)^\wedge_2) \to BSO(3)^\wedge_2 \to (BSO(3)^\wedge_2)_{\mathbb{Q}},$$

for all $m \geq 1$. The case $m = 1$ is described in [CF13, Proposition 6.17] using a different method.

Question. What can we say about the $B\mathbb{Z}/p^m$-cellularization of $BG^\wedge_p$ if BS is not $B\mathbb{Z}/p^m$-cellular?

Example 4.3.52. Let G be a compact connected Lie group and let p be a prime number such that $(p, |W_G|) = 1$. Then, $N_G(S)$ controls fusion in G. Hence proceding as in Example 4.3.36, we get the fibration

$$CW_{B\mathbb{Z}/p^m}(BG^\wedge_p) \to BN_G(S)^\wedge_p \to B(N_G(S)/Cl_{p^m}(S))^\wedge_p,$$

for all $m \geq 1$.

In the case of compact connected Lie group we get a stronger result thanks to the following theorem of D. Notbohm. From now on and following his convention, $\ker(f)$ is the clousure of $\{g \in S \mid f|_{B\langle g\rangle} \simeq *\}$.

Theorem 4.3.53 ([Not94, Theorem 1.5]). *Let G be a compact connected Lie group and let $f\colon BG^\wedge_p \to Y^\wedge_p$, where $Y^\wedge_p$ is a p-complet and $\Sigma B\mathbb{Z}/p$-null space. Then, there exists a compact Lie group H and a commutative diagram*

$$\begin{array}{ccc} BG^\wedge_p & \xrightarrow{f} & Y^\wedge_p \\ {\scriptstyle\varphi}\downarrow & & \| \\ BH^\wedge_p & \xrightarrow{\bar{f}} & Y^\wedge_p \end{array}$$

such that $\ker(\bar{f}) = \{e\}$. *Moreover, the homotopy fibre of q is equivalent to $B\Gamma^\wedge_p$, where Γ is a compact Lie group.*

The construction of H and φ is given by the classification of compact connected Lie groups as follows: First, if G is a compact connected Lie group, there is a extension of compact Lie groups $1 \to K \to \tilde{G} \xrightarrow{\alpha} G \to 1$, where $\tilde{G} = G_1 \times \ldots \times G_k \times T$, G_i is 1-connected simple Lie groups for all $i \in \{1, \ldots, k\}$ and K is a finite subgroup of the center of $\tilde{G}$. The idea is, given a simply connected Lie group M, to associate a p-subgroup $H(M, p)$ for every prime p as follows:

$$H(M,p) = \begin{cases} N_M(T) & \text{, if } (p, |W_M|) = 1, \\ SU(2) \rtimes \mathbb{Z}/2 & \text{, if } M = G_2 \text{ and } p = 3, \\ M & \text{, otherwise.} \end{cases}$$

Therefore, D. Notbohm proves that the inclusion $H(G_s, p) \hookrightarrow \tilde{G}$ induces a mod p equivalence in classifying spaces, where $G_s = G_1 \times \ldots \times G_k$ and $H(G_s, p) = H(G_1, p) \times \ldots \times H(G_k, p)$. Moreover, for $\ker(f \circ B\alpha)$, we can split $G_s \cong G' \times G''$ such that $\ker(f \circ B\alpha) \cong S' \times \Gamma$, where $S' \in \mathrm{Syl}_p(G')$ and $\Gamma \subset T_{G''} \times T$ that is normal in $H(G'', p) \times T$ (according to [Not94, Proposition 4.3]). Finally, D. Notbohm proves that the inclusion $\iota\colon (G' \times H(G'', p) \times T)/K \hookrightarrow G$ induces a mod p equivalence in classifying spaces and he defines $H = (H(G'', p) \times T)/\Gamma$ and $q : (G' \times H(G'', p) \times T)/K \to (H(G'', p) \times T)/K \to (H(G'', p) \times T)/\Gamma = H$, since the (classical) kernel of the projection $G' \times H(G'', p) \times T \to H$ is $G' \times \Gamma$, which contains K. Therefore, $\varphi = Bq^\wedge_p \circ (B\iota^\wedge_p)^{-1}$. Note that the homotopy fibre of φ is $B(G' \times \Gamma/K)^\wedge_p$. See [Not94, Section 4] for more details of this construction.

Theorem 4.3.54. *Let $m \geq 0$. Let G be a compact connected Lie group. Then, there exists a compact Lie group H and a map $\varphi\colon BG^\wedge_p \to BH^\wedge_p$ such that $CW_{B\mathbb{Z}/p^m}(BG^\wedge_p)$ is mod p equivalent to the homotopy fibre of φ.*

Proof. By Theorem 4.3.53 there exists a compact Lie group H and a commutative diagram

$$\begin{array}{ccc} BG^\wedge_p & \xrightarrow{r^\wedge_p} & P_{\Sigma B\mathbb{Z}/p^m}(C)^\wedge_p \\ {\scriptstyle\varphi}\downarrow & & \| \\ BH^\wedge_p & \xrightarrow{\eta} & P_{\Sigma B\mathbb{Z}/p^m}(C)^\wedge_p \end{array}$$

where C is the Chachólski cofibre and such that $\ker(\eta) = \{e\}$. Since $CW_{B\mathbb{Z}/p^m}(BG^\wedge_p)^\wedge_p$ is equivalent to the homotopy fibre of $r^\wedge_p$, we have to prove that $\eta\colon BH^\wedge_p \to P_{\Sigma B\mathbb{Z}/p^m}(C)^\wedge_p$ is a homotopy equivalence. We will construc a map $\mu\colon P_{\Sigma B\mathbb{Z}/p^m}(C)^\wedge_p \to BH^\wedge_p$ such that $\eta \circ \mu \simeq id_{BH^\wedge_p}$ and $\mu \circ \eta \simeq id_{P_{\Sigma B\mathbb{Z}/p^m}(C)^\wedge_p}$.

(i) *Definition of* $\mu\colon P_{\Sigma B\mathbb{Z}/p^m}(C)^{\wedge}_p \to BH^{\wedge}_p$*:* Consider the fibration

$$CW_{B\mathbb{Z}/p^m}(BG^{\wedge}_p)^{\wedge}_p \longrightarrow BG^{\wedge}_p \xrightarrow{r^{\wedge}_p} P_{\Sigma B\mathbb{Z}/p^m}(C)^{\wedge}_p$$

and the map $\varphi\colon BG^{\wedge}_p \to BH^{\wedge}_p$. If $\Omega(BH^{\wedge}_p)$ is $CW_{B\mathbb{Z}/p^m}(BG^{\wedge}_p)^{\wedge}_p$-null and $\varphi|_{CW_{B\mathbb{Z}/p^m}(BG^{\wedge}_p)^{\wedge}_p} \simeq *$, then there is a map $\mu\colon P_{\Sigma B\mathbb{Z}/p^m}(C)^{\wedge}_p \to BH^{\wedge}_p$ such that the following diagram

$$\begin{array}{ccc} BG^{\wedge}_p & \xrightarrow{\varphi} & BH^{\wedge}_p \\ {\scriptstyle r^{\wedge}_p}\downarrow & \nearrow_{\mu} & \\ P_{\Sigma B\mathbb{Z}/p^m}(C)^{\wedge}_p & & \end{array}$$

is commutative, i.e., $\mu \circ r^{\wedge}_p \simeq \varphi$, by Zabrodsky's Lemma (Lemma 2.2.1). On the one hand we have to see that $\mathrm{map}_*(CW_{B\mathbb{Z}/p^m}(BG^{\wedge}_p)^{\wedge}_p, \Omega(BH^{\wedge}_p)) \simeq *$. Since $BH^{\wedge}_p$ is a p-good space, $BH^{\wedge}_p \simeq L_{\mathbb{Z}/p}(BH)$ by Proposition A.3.1, and hence

$$\mathrm{map}_*(CW_{B\mathbb{Z}/p^m}(BG^{\wedge}_p)^{\wedge}_p, \Omega(BH^{\wedge}_p)) \simeq \mathrm{map}_*(CW_{B\mathbb{Z}/p^m}(BG^{\wedge}_p), \Omega(BH^{\wedge}_p)),$$

that is contactible since $\Omega(BH^{\wedge}_p)$ is $B\mathbb{Z}/p^m$-null. On the other hand, to show that

$$\varphi|_{CW_{B\mathbb{Z}/p^m}(BG^{\wedge}_p)^{\wedge}_p} \simeq *$$

is equivalent to show that $\varphi|_{CW_{B\mathbb{Z}/p^m}(BG^{\wedge}_p)} \simeq *$, according to [BK72, Proposition II.2.8]. This is equivalent to show that for any map $B\mathbb{Z}/p^m \to CW_{B\mathbb{Z}/p^m}(BG^{\wedge}_p)$ the composite $B\mathbb{Z}/p^m \longrightarrow CW_{B\mathbb{Z}/p^m}(BG^{\wedge}_p) \xrightarrow{\varphi\circ c} BH^{\wedge}_p$ is null-homotopic, by Proposition 2.1.25, and hence it is equivalent to prove that for any map $B\mathbb{Z}/p^m \to BG^{\wedge}_p$ the composite

$$g\colon B\mathbb{Z}/p^m \to BG^{\wedge}_p \xrightarrow{\varphi} BH^{\wedge}_p$$

is null-homotopic. Note that $\eta \circ g \simeq *$ and hence $g \simeq *$ since $\ker(\eta) = \{e\}$.

(ii) $\eta\circ\mu \simeq id_{P_{\Sigma B\mathbb{Z}/p^m}(C)^{\wedge}_p}$: Consider the above fibration and the map $\eta\circ\varphi\colon BG^{\wedge}_p \to P_{\Sigma B\mathbb{Z}/p^m}(C)^{\wedge}_p$. If $\Omega(P_{\Sigma B\mathbb{Z}/p^m}(C)^{\wedge}_p)$ is $CW_{B\mathbb{Z}/p^m}(BG^{\wedge}_p)^{\wedge}_p$-null, then Zabrodsky's Lemma shows that the map $(r^{\wedge}_p)^*\colon \mathrm{map}_*(P_{\Sigma B\mathbb{Z}/p^m}(C)^{\wedge}_p, P_{\Sigma B\mathbb{Z}/p^m}(C)^{\wedge}_p) \to \mathrm{map}_*(BG^{\wedge}_p, P_{\Sigma B\mathbb{Z}/p^m}(C)^{\wedge}_p)_{CW_{B\mathbb{Z}/p^m}(BG^{\wedge}_p)^{\wedge}_p}$ is an equivalence, where $\mathrm{map}_*(BG^{\wedge}_p, P_{\Sigma B\mathbb{Z}/p^m}(C)^{\wedge}_p)_{CW_{B\mathbb{Z}/p^m}(BG^{\wedge}_p)^{\wedge}_p}$ denotes the pointed maps from $BG^{\wedge}_p$ to $P_{\Sigma B\mathbb{Z}/p^m}(C)^{\wedge}_p$ that are null-homotopic restricted to $CW_{B\mathbb{Z}/p^m}(BG^{\wedge}_p)^{\wedge}_p$. Moreover, $(r^{\wedge}_p)^*(id_{P_{\Sigma B\mathbb{Z}/p^m}(C)^{\wedge}_p}) \simeq \eta \circ \varphi$ since $\eta \circ \varphi \simeq r^{\wedge}_p$, and $(r^{\wedge}_p)^*(\eta \circ \mu) \simeq \eta \circ \varphi$ since $\eta \circ \mu \circ r^{\wedge}_p \simeq \eta \circ \varphi$. Then, $id_{P_{\Sigma B\mathbb{Z}/p^m}(C)^{\wedge}_p}$ and $\eta \circ \mu$ are in the same connected component of $\mathrm{map}_*(P_{\Sigma B\mathbb{Z}/p^m}(C)^{\wedge}_p, P_{\Sigma B\mathbb{Z}/p^m}(C)^{\wedge}_p)$, that is, $\eta \circ \mu \simeq id_{P_{\Sigma B\mathbb{Z}/p^m}(C)^{\wedge}_p}$. Therefore, it is sufficient to prove that $\Omega(P_{\Sigma B\mathbb{Z}/p^m}(C)^{\wedge}_p)$ is $CW_{B\mathbb{Z}/p^m}(BG^{\wedge}_p)^{\wedge}_p$-null, that is,

$$\mathrm{map}_*(CW_{B\mathbb{Z}/p^m}(BG^{\wedge}_p)^{\wedge}_p, \Omega(P_{\Sigma B\mathbb{Z}/p^m}(C)^{\wedge}_p)) \simeq *$$

Since C is 1-connected,so is $P_{\Sigma B\mathbb{Z}/p^m}(C)$. Hence, $\Omega(P_{\Sigma B\mathbb{Z}/p^m}(C)^{\wedge}_p) \simeq P_{B\mathbb{Z}/p^m}(\Omega C)^{\wedge}_p$. Moreover, as $P_{B\mathbb{Z}/p^m}(\Omega C)$ is a p-good space, $P_{B\mathbb{Z}/p^m}(\Omega C)^{\wedge}_p \simeq L_{\mathbb{Z}/p}(P_{B\mathbb{Z}/p^m}(\Omega C))$, and hence

$$\mathrm{map}_*(CW_{B\mathbb{Z}/p^m}(BG^{\wedge}_p)^{\wedge}_p, P_{B\mathbb{Z}/p^m}(\Omega C)^{\wedge}_p) \simeq \mathrm{map}_*(CW_{B\mathbb{Z}/p^m}(BG^{\wedge}_p), P_{B\mathbb{Z}/p^m}(\Omega C)^{\wedge}_p).$$

Furthermore, $\tilde{H}_*(CW_{B\mathbb{Z}/p^m}(BG^\wedge_p);\mathbb{Z}[\frac{1}{p}]) \cong 0$ and $P_{B\mathbb{Z}/p^m}(\Omega C)$ is a nilpotent spaces (it is a H-space), then

$$\mathrm{map}_*(CW_{B\mathbb{Z}/p^m}(BG^\wedge_p), P_{B\mathbb{Z}/p^m}(\Omega C)^\wedge_p) \simeq \mathrm{map}_*(CW_{B\mathbb{Z}/p^m}(BG^\wedge_p), P_{B\mathbb{Z}/p^m}(\Omega C)) \simeq *,$$

by Theorem 1.3.2.

(iii) $\mu \circ \eta \simeq id_{BH^\wedge_p}$: Let $\tilde{\Gamma} = G' \times \Gamma/K$ and consider the fibration

$$B\tilde{\Gamma}^\wedge_p \to BG^\wedge_p \to BH^\wedge_p,$$

and the map $\mu \circ r^\wedge_p \colon BG^\wedge_p \to BH^\wedge_p$. By Zabrodsky's Lemma, if $\Omega(BH^\wedge_p)$ is $B\tilde{\Gamma}^\wedge_p$-null, then $\varphi^* \colon \mathrm{map}_*(BH^\wedge_p, BH^\wedge_p) \to \mathrm{map}_*(BG^\wedge_p, BH^\wedge_p)_{B\tilde{\Gamma}^\wedge_p}$ is an equivalence. Furthermore, $\varphi^*(id_{BH^\wedge_p}) \simeq \mu \circ r^\wedge_p$ since $\mu \circ r^\wedge_p \simeq \varphi$, and $\varphi^*(\mu \circ \eta) \simeq \mu \circ r^\wedge_p$ since $\mu \circ \eta \circ \varphi \simeq \mu \circ r^\wedge_p$. Then, $id_{BH^\wedge_p}$ and $\mu \circ \eta$ are in the same connected component of $\mathrm{map}_*(BH^\wedge_p, BH^\wedge_p)$, that is, $\mu \circ \eta \simeq id_{BH^\wedge_p}$. Finally, $\mathrm{map}_*(B\tilde{\Gamma}^\wedge_p, BH^\wedge_p) \simeq \mathrm{Hom}(\Gamma, H = G/\Gamma) \simeq *$, and hence $\mathrm{map}_*(B\tilde{\Gamma}^\wedge_p, \Omega BH^\wedge_p) \simeq \Omega\mathrm{map}_*(B\tilde{\Gamma}^\wedge_p, BH^\wedge_p)_c \simeq *$.

□

We can describe the situation for compact 1-connected simple Lie group G by the description of the strongly closed subgroup in G given in [Not94, Proposition 4.3]. Basically, D. Notbohm proves that if G is a 1-connected compact simple Lie group and $K \leq S \in \mathrm{Syl}_p(G)$ is a strongly closed subgroup in G, then $K = S$ or K is a finite p-group. Moreover, if K is a finite group then

(a) If $(p, |W_G|) = 1$, then K is central in $N_G(S)$.

(b) If $(p, |W_G|) \neq 1$, then:

  (i) If $G \neq G_2$ or $p \neq 3$, then K is central in G.

  (ii) If $G = G_2$ and $p = 3$, then K is central in $SU(3) \leq G_2$.

Note that situation (a) is described in example 4.3.52.

Proposition 4.3.55. *Let G be a compact 1-connected simple Lie group. Let p be a prime such that $p \mid |W_G|$. Then for all $m \geq 1$, the $B\mathbb{Z}/p^m$-cellularization of $BG^\wedge_p$ is equivalent to the homotopy fibre of the rationalization $BG^\wedge_p \to (BG^\wedge_p)_\mathbb{Q}$.*

Proof. Fix $m \geq 1$ and take $K = Cl_{p^m}(S)$. Note that $Cl_{p^m}(S) \leq \ker(r^\wedge_p)$.

If $Cl_{p^m}(S) = S$, then $\ker(r^\wedge_p) = S$ and by Corollary 4.3.43, we get the fibration

$$CW_{B_{p^m}}(BG^\wedge_p) \to BG^\wedge_p \to (BG^\wedge_p)_\mathbb{Q}.$$

Assume that $Cl_{p^m}(S)$ is a finite p-group, then

(i) If $G \neq G_2$ or $p \neq 3$, then $Cl_{p^m}(S)$ is central in G, where $Z(G) \cong \mathbb{Z}/n$ or $\mathbb{Z}/2 \times \mathbb{Z}/2$, for some $n \geq 1$ depending on the group G (according to the classification), but there are not central elements of order p. Therefore $Cl_{p^m}(S)$ cannot be central, that is, $Cl_{p^m}(S)$ is not finite and hence $Cl_{p^m}(S) = S$.

(ii) If $G = G_2$ and $p = 3$, then $Cl_{3^m}(S)$ is central in $SU(3) \leq G_2$, where $Z(SU(3)) = \mathbb{Z}/3$. Therefore, $Cl_{3^m}(S) = \mathbb{Z}/3$, but there exits not central elements of order 3, hence $Cl_{3^m}(S)$ not contains all the elements of order 3. This is not possible, hence $Cl_{3^m}(S) = S$.

□

Remark 4.3.56. The Theorem 6.9 in [CF13] says us that the $B\mathbb{Z}/p$-cellularization of $BG^\wedge_p$, G is a compact connected Lie group is the classifying space of a p-group generated by order p elements, or else it has an infinite of non-trivial homotopy groups. Note that the previous proposition gives us a more specific result in the case of a compact 1-connected simple Lie group, that is, if the $B\mathbb{Z}/p^m$-cellularization of $BG^\wedge_p$ has infinite non-trivial homotopy groups, then it is equivalent to the homotopy fibre of the rationalization $BG^\wedge_p \to (BG^\wedge_p)_{\mathbb{Q}}$.

Question. Is it possible generalize this result to classifying spaces of p-compact groups and p-local compact groups?

Appendix A

R-completion of Bousfield-Kan and Homological localizations

In this chapter we want to introduce two classical functors which isolate homological information on a ring R (usually $\mathbb{Z}/p$ or a subring $R \subset \mathbb{Q}$): the R-completion of Bousfield-Kan introduced in [BK72]; and the homological localization with respect to $H_*(-; R)$, introduced in [Bou75]. This chapter is then organized in three sections as follows. The first section is devoted to R-completion of Bousfield-Kan, with appear the basic definition, properties and examples. In the second section appear an introduction to homological localizations with emphasis in localize with recpect to homological theories with coefficients. Finally section three gives us the similarities and differences between these two functors.

This chapter is to be understood as a summary of classical results, hence we do not provide all the proofs. The reader is then refered to the corresponding source.

A.1 R-completion of Bousfield-Kan

Let R be a ring. The R-completion functor trie to isolate the homological information on a ring R. Hence, a map $f\colon X \to Y$ is an *R-equivalence* if it induces an isomorphism $f_*\colon H_*(X; R) \to H_*(Y; R)$. The *$R$-completion functor* is a coaugmented functor $R_\infty\colon \mathbf{Top}_* \to \mathbf{Top}_*$ with the following fundamental property:

Lemma A.1.1 ([BK72, Lemma I.5.5])**.** *A map $f\colon X \to Y$ is a R-equivalence if and only if $R_\infty(f)\colon R_\infty(X) \to R_\infty(Y)$ is an equivalence.*

Important classes of spaces in R-completon are the following:

Definition A.1.2 ([BK72, Definition I.5.1])**.** A space X is called:

(a) *R-good* if the coaugmention map $\eta_X\colon X \to R_\infty(X)$ is a R-equivalence,

(b) *R-bad* if it is not R-good,

(c) *R-complete* if $\eta_X\colon X \to R_\infty(X)$ is a weak equivalence, i.e., $R_\infty(X) \simeq X$.

And these spaces are related as follows:

Proposition A.1.3 ([BK72, Proposition I.5.2]). *For a space X the following conditions are equivalent:*

(i) X is R-good,

(ii) $R_\infty(X)$ is R-complete,

(iii) $R_\infty(X)$ is R-good.

As the authors mention: this implies that, roughly speaking, "a good space is very good and a bad space is very bad". Moreover this result shows that the coaugmentation functor is not idempotent in general, only over R-good spaces.

Examples of these types of spaces are the following:

Proposition A.1.4 ([BK72, Proposition VII.5.1]). *Let X be a space which $\pi_1(X)$ is a finite group. Then X is p-good for all prime p.*

A finite wedge of circles is p-bad for all prime p, because from [BK72, Proposition 5.3] if $A = \mathbb{Z} * \ldots * \mathbb{Z}$ is a free product of n copies of $\mathbb{Z}$, then $(S^1 \vee \ldots \vee S^1)^\wedge_p \simeq K(A,1)^\wedge_p \simeq K(\hat{A}_p, 1)$ and by [Bou92, Theorem 1.11], $H_m(K(\hat{A}_p, 1); \mathbb{F}_p)$ is uncountable for $m = 2$ or $m = 3$ or both.

Proposition A.1.5 ([AKO11, Proposition III.1.10]). *If P is a finite p-group, then BP is p-complete.*

An important result about homotopy classes of maps and R-completion is given in the following proposition:

Proposition A.1.6 ([BK72, Proposition II.2.8]). *Let $f\colon X \to Y$ an R-equivalence between connected spaces. Then f induces , for every connected spaces W, a bijection of pointed homotopy classes of maps $f^*\colon [Y, R_\infty(W)]_* \to [X, R_\infty(W)]_*$.*

And in particular:

Corollary A.1.7. *Let X and W connected spaces. Then the coaugmentation map $\eta_X\colon X \to R_\infty(X)$ induces a bijection of pointed homotopy classes of maps $(\eta_X)^*\colon [R_\infty(X), R_\infty(W)]_* \to [X, R_\infty(W)]_*$.*

A.1.1 Nilpotent spaces and *R*-completion of fibrations

A nilpotent space is a space which the action of the fundamental group on higher homotopy group is finite in certain filtration quotients. More precisely:

Definition A.1.8 ([BK72, Definition II.4.1]). Let π and G be groups and let $\alpha\colon \pi \to \mathrm{Aut}(G)$ be an action of π on G. The action α is called *nilpotent* if there exists a finite sequence of subgroups of G:

$$\{e\} = G_n \trianglelefteq \ldots \trianglelefteq G_i \trianglelefteq \ldots \trianglelefteq G_1 = G,$$

such that for all $i = 1 \ldots n$:

(a) G_i is closed under α,

(b) G_i/G_{i+1} is abelian, and

(c) the induced action on G_i/G_{i+1} is trivial.

A group is *nilpotent* if the action on itself via conjugation is nilpotent.

Definition A.1.9 ([BK72, Definition II.4.3])**.** A connected space X is called *nilpotent* if the action of $\pi_1(X)$ on each $\pi_i(X)$ is nilpotent.

The nilpotent spaces have good properties under p-completion, as we see in the following results:

Proposition A.1.10. *If X is a nilpotent space, then*

(i) [BK72, Proposition VI.5.1] $X_p^\wedge$ is nilpotent and, for $n \geq 1$, there is a splittable short exact sequence

$$0 \to \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_n(X)) \to \pi_n(X_p^\wedge) \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1}(X)) \to 0.$$

(ii) [BK72, Proposition VI.5.3] X is a p-good space, and hence $X_p^\wedge$ is p-complete.

A fibration $E \to B$ of connected spaces is preserved by R-completion if it is nilpotent (this means, if it fibre F is connected and the action of $\pi_1(E)$ on each $\pi_i(F)$ is nilpotent) according to [BK72, Lemma II.4.8]. Moreover there is a more general lemma about fibration preserved by R-completion:

Lemma A.1.11 ([BK72, Lemma II.5.1] **Mod-R fibre lemma**)**.** *Let $p\colon E \to B$ be a fibration of connected spaces with connected fibre $F = p^{-1}(*)$ and let the action of $\pi_1(B)$ on $H_i(F; R)$ be nilpotent for all $i \geq 0$. Then $R_\infty(p)\colon R_\infty(E) \to R_\infty(B)$ is a fibration and the map $R_\infty(F) = R_\infty(p^{-1}(*)) \to (R_\infty(p))^{-1}(*)$ is a homotopy equivalence.*

Example A.1.12 ([BK72, Example II.5.2 and proof of Proposition VII.5.1])**.** The condition of the mod-R fibre lemma are satisfied if, for instance:

(i) B is 1-connected,

(ii) $E = F \times B$ and p is the projection on the second factor,

(iii) the fibration $p\colon E \to B$ is principal.

(iv) $\pi_1(B)$ and $H_i(F; R)$ $(i \geq 1)$ are all finite p-groups for a prime p (a finite p-group always acts nilpotently on finite p-groups),

(v) $\pi_1(B)$ is a finite p-group and $R = \mathbb{Z}/p$ (a finite p-group always acts nilpotently on $\mathbb{Z}/p$-modules).

Corollary A.1.13. *If X is 1-connected, then $R_\infty(\Omega X) \simeq \Omega(R_\infty(X))$.*

Proof. If X is 1-connected then ΩX is connected. Therefore applying the mod-R fibre lemma to the fibration $\Omega X \to * \to X$, we get the fibration $R_\infty(\Omega X) \to * \to R_\infty(X)$ and hence $R_\infty(\Omega X) \simeq \Omega(R_\infty(X))$. □

A.1.2 Sullivan's arithmetic square

D. Sullivan noted in [Sul71] that the homotopy type of a simply connected finite complex is determined by "primary" information, "rational" information and certain "coherence" data. In this way E. Dror-Farjoun, W. Dwyer and D. Kan generalize this result for virtually nilpotent spaces in [DDK77].

Definition A.1.14 ([DDK77, Definition 2.2])**.** A space X is called *virtually nilpotent* if for every integer $n \geq 1$, $\pi_1(X)$ has a normal subgroup of finite index which acts nilpotently on $\pi_n(X)$.

The main theorem of [DDK77] is the following:

Theorem A.1.15 ([DDK77, Theorem 4.1] **First Arithmetic Square Theorem**)**.** *If X is a virtually nilpotent space and $R \subset \mathbb{Q}$ is a subring, then the arithmetic square for X*

$$\begin{array}{ccc} R_\infty(X) & \longrightarrow & \prod_{p\,prime}(\mathbb{Z}/p \otimes R)_\infty(X) \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} \simeq (R_\infty(X))_{\mathbb{Q}} & \longrightarrow & (\prod_{p\,prime}(\mathbb{Z}/p \otimes R)_\infty(X))_{\mathbb{Q}} \end{array}$$

is a homotopy pull back.

In the case of X be a nilpotent space the authors first prove that $\mathbb{Z}_\infty(X) \simeq X$ (by [DDK77, Proposition 3.3.(i)]) and since $\mathbb{Z}/p \otimes \mathbb{Z} \cong \mathbb{Z}/p$ for all prime p, we get the following corollary:

Corollary A.1.16. *If X is a nilpotent space, then the arithmetic square for X*

$$\begin{array}{ccc} X & \longrightarrow & \prod_{p\,prime} X_p^\wedge \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & (\prod_{p\,prime} X_p^\wedge)_{\mathbb{Q}} \end{array}$$

is a homotopy pull back.

A.2 Homological localizations

Now we introduce the concept of homological localizations defined by A.K. Bousfield in [Bou75]. In this section, in addition to presenting the main properties and definitions of homological locatizations also explain the main properties which we use in this work about homological localization with respect to a homology theory with coefficients relation to the localization with respect to a map.

Definition A.2.1 ([Bou75, 2.1 & 3.1])**.** Let h_* be a generalized homology theory (defined by a spectrum). A space $X \in \mathbf{Top}_*$ is called *h-local* if for every *h-isomorphism* $f\colon U \to V$ (i.e., f induces an isomorphism $f_*\colon h_*(U) \to h_*(V)$) the induced map $f^*\colon \mathrm{map}(V, X) \to \mathrm{map}(U, X)$ is a weak equivalence.

Remark A.2.2. If $h_* = H_*R = H_*(-;R)$, then an HR-isomorphism is a R-equivalence.

As is usual when we define a local space, we are interested in a functor hat turn a space into a local one:

Theorem A.2.3 ([Bou75, Theorem 3.2]). *Let h_* be a generalized homology theory. There is an idempotent coaugmented functor $L_h\colon \mathbf{Top}_* \to \mathbf{Top}_*$ called the h-localization functor. Moreover $L_h(X)$ is an h-local space and the coaugmentation map $\eta_X\colon X \to L_h(X)$ is an h-isomorphism which is homotopy universal with respect to h-local spaces, this means, if Y is an h-local space and $f\colon X \to Y$ is a pointed map then there is a map $\tilde{f}\colon L_h(X) \to Y$ such that the following diagram*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle \eta_X}\searrow \quad \nearrow{\scriptstyle \tilde{f}} & \\ & L_h(X) & \end{array}$$

is commutative up to homotopy, and if there is another map $g\colon L_h(X) \to Y$ such that $g \circ \eta_X \simeq f$, then $g \simeq \tilde{f}$.

Remark A.2.4. In particular if we have an h-isomorphism $X \to Y$ and Y is h-local, then $L_h(X) \simeq Y$.

Remark A.2.5. Given a generalized homology h_* take the map $f : \bigvee f_i : U_i \to V_i$, a wegde over all h-isomorphisms between spaces of cardinality not bigger than the cardinality of $h_*(S^0)$, taking one copy for each homotopy type. Hence $L_h = L_f$. This means that homological localizations are localizations with respect a map, in particular homological localizations verify the properties listed in Section 1.1.

Remark A.2.6. For h_* a generalized homology theory we have $h_*(X) \cong h_*(\Sigma^n X)$ for all X and n. Hence if $f\colon U \to V$ is an h-isomorphism, then so is $\Sigma^n f$. Therefore the condition f induces a weak equivalence $f^*\colon \mathrm{map}(V, X) \to \mathrm{map}(U, X)$ in Definition A.2.1 is equivalent to f induces a bijection $f^*\colon [V, X] \to [U, X]$.

It is not easy to compute the homotopy groups of $L_h(X)$, but over good conditions it is well-known. This case is when $h_* = H_*(-;R)$ for some ring R, and in this case we denote by L_R the functor $L_h(X)$, and X is a nilpotent space:

Proposition A.2.7 ([Bou75, Proposition 4.3]). *Let X be a connected nilpotent space and let $\mathcal{P}$ be a finite set of prime numbers. Then,*

(i) If $R = \mathbb{Z}[\mathcal{P}^{-1}]$, where $\mathbb{Z}[\mathcal{P}^{-1}]$ denotes the localization of $\mathbb{Z}$ with respect the multiplicatively closed set $\mathcal{P}$, then $\pi_* L_R(X) \cong \mathbb{Z}[\mathcal{P}^{-1}] \otimes \pi_* X$, and $\tilde{H}_*(L_R(X);\mathbb{Z}) \cong \mathbb{Z}[\mathcal{P}^{-1}] \otimes \tilde{H}_*(X;\mathbb{Z})$.

(ii) If $R = \mathbb{Z}/p$, the there is a splittable short exact sequence

$$0 \to \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_n X) \to \pi_n L_R(X) \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1} X) \to 0$$

(iii) If $R = \prod_{p\in\mathcal{P}} \mathbb{Z}/p$, then $L_R(X) \simeq \prod_{p\in\mathcal{P}} L_{\mathbb{Z}/p}(X)$.

Corollary A.2.8. *Let X be a 1-connected space and let $\mathcal{P}$ be a finite set of prime numbers. If $R = \mathbb{Z}[\mathcal{P}^{-1}]$ or $R = \prod_{p\in\mathcal{P}} \mathbb{Z}/p$, then $L_R(X)$ is 1-connected.*

A.2.1 Homological localizations with coefficients

Some of the properties of localizations with respect to a homology theory with coefficient was developed by G. Mislin in [Mis78]. We will use homology with coefficients in an abelian grup G. For h_* a homology theory defined by a spectrum E, we define $h_*G = h_*(-;G) := [-, E \wedge M(G)]$, where $M(G)$ is the Moore spectrum of type G, (see [Ada95, Part III.6]) . There exist universal coefficient sequences:

Proposition A.2.9 ([Ada95, Proposition III.6.6])**.** *Let h_* be a homology theory defined by a spectrum E and G be an abelian group. Let X be a spectrum. Then*

(i) There exists an exact sequence (it need not split) for all n:

$$0 \to \pi_n(E) \otimes G \to \pi_n(EG) \to Tor^1_{\mathbb{Z}}(\pi_{n-1}(E), G) \to 0.$$

(ii) More generally, there exists exact sequence for all n:

$$0 \to h_n(X) \otimes G \to h_n(X;G) \to Tor^1_{\mathbb{Z}}(h_{n-1}(X), G) \to 0.$$

and, if X is a finite spectrum of G is finitely generated,

$$0 \to h^n(X) \otimes G \to h^n(X;G) \to Tor^1_{\mathbb{Z}}(h^{n+1}(X), G) \to 0.$$

Remark A.2.10. This implies that an h-isomorphism is also an hG-isomorphism, or, a hG-local space is also h-local.

G. Mislin in ([Mis78]) develops the following results about homological localization with coefficients that we will use later:

Proposition A.2.11 ([Mis78, Corollary 1.5])**.** *If $f\colon X \to Y$ is an HG-isomorphism, then it is also an hG-isomorphism.*

Proposition A.2.12 ([Mis78, Proposition 1.10])**.** *If X is* 1*-connected, then*

$$L_{H\mathbb{Z}/p}(L_h(X)) \simeq L_{h\mathbb{Z}/p}(X).$$

We proceed now to describe two thecnical lemmas about homology with coefficients and homological localizations:

Lemma A.2.13. *Let h_* be a homology theory and let R be a subring of $\mathbb{Q}$. Let $g_* = h_*(-;R)$. Let $\mathcal{P}$ denotes the set of divisible primes of R. Then*

$$g_*(-;\mathbb{Z}/p) = \begin{cases} h_*(-;\mathbb{Z}/p) & , \text{if } p \notin \mathcal{P}, \\ * & , \text{if } p \in \mathcal{P}. \end{cases}$$

and $g_*(-;\mathbb{Q}) = h_*(-;\mathbb{Q})$.

Proof. By definition of homology theory with coefficients (see [Ada95, p.200]), it is sufficient to show that

$$M(R) \wedge M(\mathbb{Z}/p) = \begin{cases} M(\mathbb{Z}/p) & \text{, if } p \notin \mathcal{P}, \\ * & \text{, if } p \in \mathcal{P}. \end{cases}$$

and $M(R) \wedge M(\mathbb{Q}) = M(\mathbb{Q})$.

According to the definition of a Moore spectrum $M(G)$ (see [Ada95, p. 200]) we have

$$\begin{aligned} &\pi_r(M(G)) = 0 \text{ for } r < 0, \\ &\pi_0(M(G)) = H_0(M(G)) = G, \\ &H_r(M(G)) = 0 \text{ for } r > 0. \end{aligned}$$

Consider the short exact sequence in Proposition A.2.9.(i) applied to $E = M(\mathbb{Z}/p)$ or $M(\mathbb{Q})$ and $G = R$, we get the short exact sequence:

$$0 \to \pi_r(E) \otimes R \to \pi_r(E \wedge M(R)) \to Tor_1^{\mathbb{Z}}(\pi_{r-1}(E), R) \to 0.$$

Note that $Tor_1^{\mathbb{Z}}(\pi_{r-1}(E), R) = 0$, because R is free torsion and hence $\pi_r(E \wedge M(R)) \cong \pi_r(E) \otimes R$. Moreover since $\pi_r(E) = 0$ for all $r < 0$, $\pi_r(E \wedge M(R)) \cong 0$ for all $r < 0$. Furthermore, since $\pi_0(M(\mathbb{Z}/p)) = \mathbb{Z}/p$ and $\pi_0(M(\mathbb{Q})) = \mathbb{Q}$, we obtain

$$\pi_0(M(R) \wedge M(\mathbb{Z}/p)) \cong R \otimes \mathbb{Z}/p \cong \begin{cases} \mathbb{Z}/p & \text{, if } p \notin \mathcal{P}, \\ 0 & \text{, if } p \in \mathcal{P}. \end{cases}$$

and $\pi_0(M(R) \wedge M(\mathbb{Q})) \cong R \otimes \mathbb{Q} \cong \mathbb{Q}$.

For a spectrum X it is defined $H_r(X) := \pi_r(X \wedge H\mathbb{Z})$, hence we now consider the short exact sequence in Proposition A.2.9.(i) applied to $E = M \wedge H\mathbb{Z}$ and $G = R$ where $M = M(\mathbb{Z}/p)$ or $M(\mathbb{Q})$, we obtain the following short exact sequence:

$$0 \to \pi_r(M \wedge H\mathbb{Z}) \otimes R \to \pi_r(M \wedge M(R) \wedge H\mathbb{Z}) \to Tor_1^{\mathbb{Z}}(\pi_{r-1}(M \wedge H\mathbb{Z}), R) \to 0.$$

As R is free torsion, $Tor_1^{\mathbb{Z}}(\pi_{r-1}(M \wedge H\mathbb{Z}), R) = 0$ and therefore $\pi_r(M \wedge M(R) \wedge H\mathbb{Z}) \cong \pi_r(M \wedge H\mathbb{Z}) \otimes R$, where $\pi_r(M \wedge M(R) \wedge H\mathbb{Z}) = H_r(M \wedge M(R))$ and $\pi_r(M \wedge H\mathbb{Z}) = H_r(M)$. Finally, if $r > 0$ then $H_r(M) = 0$ and hence $H_r(M \wedge M(R)) = 0$, this finishes the proof. □

Lemma A.2.14. *Let R be a subring of $\mathbb{Q}$. Then $L_{\mathbb{Q}}(L_R(X)) \simeq L_{\mathbb{Q}}(X)$.*

Proof. By Theorem A.2.3 and on account of Remark A.2.4, we only need to show that there is an $H\mathbb{Q}$-isomorphism $L_R(X) \to L_{\mathbb{Q}}(X)$.

Since $X \to L_R(X)$ is an HR-isomorphism, by Proposition A.2.11 is an hR-isomorphism, where $h_* = H_*\mathbb{Q}$. By Lemma A.2.13, $h_*R = H_*\mathbb{Q}$, i. e., $X \to L_R(X)$ is a $H\mathbb{Q}$-isomorphism and by Theorem A.2.3, we get the equivalence

$$\operatorname{map}(L_R(X), L_{\mathbb{Q}}(W)) \simeq \operatorname{map}(X, L_{\mathbb{Q}}(X)),$$

hence there is a natural map $L_R(X) \to L_{\mathbb{Q}}(X)$ such that the following diagram:

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ L_R(X) & \longrightarrow & L_{\mathbb{Q}}(X), \end{array}$$

is commutative (up to homotopy), where $X \to L_R(X)$ and $X \to L_{\mathbb{Q}}(X)$ are $H\mathbb{Q}$-isomorphism, hence $L_R(X) \to L_{\mathbb{Q}}(X)$ is an $H\mathbb{Q}$-isomorphism. □

A.3 Comparing R_∞ and L_R

The R-completion of Bousfield-Kan and the HR-localization functors verify the same universal properties up to the idempotency, but there exists a family of spaces which the R-completion is idempotent, the called R-good spaces (see Section A.1). Hence it is natural to think that the R-completion of R-good spaces is equivalence to the HR-localization.

Proposition A.3.1. *Let R be a ring. If X is a R-good space then $R_\infty(X) \simeq L_R(X)$.*

Proof. On account of [BK72, Definition I.5.1], since X is R-good, the coaugmentation map $X \to R_\infty(X)$ is a R-equivalence. Therefore it is sufficient to prove that $R_\infty(X)$ is an HR-local space by Remark A.2.4. Given a R-equivalence $f\colon U \to V$ the induced map $f^*\colon [V, R_\infty(X)] \to [U, R_\infty(X)]$ is a bijection by [BK72, Proposition II.2.8], hence according to Remark A.2.6, $R_\infty(X)$ is HR-local. Therefore $L_R(X) \simeq R_\infty(X)$. □

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