Theories of modied gravity and reconstruction schemes of cosmological models

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PhD thesis

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A mis padres, Loli y Antonio

Resumen

En esta memoria de tesis se expone el trabajo llevado a cabo por el doctorando durante los últimos cuatro años, el cual versa principalmente sobre diversos aspectos de soluciones cosmológicas obtenidas a partir de teorías de gravedad modificada. Para entender el origen y la importancia de las teorías de gravedad modificada es necesario comentar antes algunos hechos acontecidos durante el siglo XX en el marco de la cosmología.

La cosmología como ciencia nació gracias a la Teoría de la Relatividad General de Albert Einstein. Antes de ésta, el espacio no era más que el lugar en el que las estrellas y los planetas residían y el tiempo no era más que algo que iba pasando, siendo espacio y tiempo dos cosas completamente desconectadas y que no se veían afectadas por lo acontecido en el Universo. Sin embargo, la teoría de Einstein derrumbó estas ideas, proponiendo que espacio y tiempo están ligados entre sí y que, además, no son meros espectadores de lo que sucede en el Universo, sino que se ven afectados por su contenido. Fue de esta manera como surgió el concepto de espacio–tiempo, el cual, según Einstein, se curva debido a la presencia de materia y/o energía (ya unificadas en su teoría de la relatividad especial). Las ecuaciones de campo de Einstein son las ecuaciones que permiten a la cosmología ser considerada como una ciencia, y establecen un diálogo entre la forma del Universo y el contenido de materia y energía que en él hay. Las primeras soluciones cosmológicas que se dieron para el Universo eran estáticas, sin embargo éstas se desecharon cuando se verificó que la ley de expansión propuesta por Hubble era cierta. La teoría más aceptada hoy en día para describir el Universo es la Teoría del Big Bang, que predice un universo en expansión que habría empezado tras una gran explosión. Entre los logros de esta teoría se encuentran el estar de acuerdo con la ley de Hubble, haber podido predecir el fondo de radiación cósmica de microondas o el ser capaz de explicar la abundancia relativa de elementos primordiales. Sin embargo, este modelo no se encuentra exento de problemas, ya que hay ciertos aspectos que la teoría no es capaz de explicar, entre ellos se encuentran el problema de la bariogénesis (explicar el proceso que produce la asimetría encontrada entre bariones y antibariones) o los problemas de la planitud y del horizonte. Si bien es cierto que alguno de estos problemas pueden ser subsanados completando la teoría del Big Bang con otras como el modelo inflacionario, se ha demostrado que estos parches también presentan sus propios problemas. Aún así, la teoría del Big Bang está considerada como la mejor descripción que tenemos del Universo.

A pesar de los pequeños o grandes problemas que aún quedaban por resolver, parecía que la cosmología estaba destinada a vivir de manera más o menos plácida. Pero esta aparente calma se vio truncada cuando, a finales del siglo XX, dos grupos liderados por Saul Perlmutter y por Adam Riess y Brian Schmidt, respectivamente, descubrieron, a partir de observaciones de supernovas de tipo Ia, que el Universo se encuentra en una fase de expansión acelerada. Esto contrasta con la visión que aporta la teoría del Big Bang, ya que según este modelo el Universo habría surgido de una gran explosión, fruto de la cual se estaría expandiendo; sin embargo, debido a la atracción gravitatoria de la masa contenida en el Universo, dicha expansión debería ir frenándose. Además, el grupo de Perlmutter determinó que, para poder explicar este hecho en el seno de la teoría del Big Bang, asumiendo un Universo espacialmente plano, la materia ordinaria y la materia oscura aportarían un 28% del total del contenido del Universo, mientras que el 72% restante debería atribuirse a un tipo de energía exótica denominada energía oscura y que ejercería una fuerza repulsiva.

El descubrimiento de la expansión acelerada del Universo fue el origen del surgimiento de un gran número de teorías cuyo objetivo era darle una explicación. La más aceptada actualmente es la teoría

Λ-Cold-Dark-Matter (ΛCDM) la cual propone que la energía oscura no es más que una constante cosmológica que daría cuenta de la energía de vacío del Universo. Otras teorías muy populares entre los cosmólogos para dar una explicación a la energía oscura son las teorías escalar–tensor, en las cuales la aceleración se consigue mediante la introducción de un campo escalar en el lagrangiano de la teoría, de manera similar a como el inflatón consigue la aceleración en el periodo de inflación. Básicamente, las teorías comentadas hasta ahora se basan en la introducción de algún tipo de materia o energía exótica en las ecuaciones de campo de Einstein para conseguir la aceleración deseada en el Universo. Sin embargo, ésta no es la única forma de conseguir el resultado deseado. Otra manera es suponer que las ecuaciones de Einstein son válidas hasta un cierto límite, pero han de ser modificadas más allá de este. De esta forma la aceleración en la expansión no estaría causada por un tipo de materia/energía exótica, sino que sería consecuencia de las nuevas ecuaciones. A este tipo de teorías es a las que se conoce como teorías de gravedad modificada.

Entre los modelos que proponen modificar las ecuaciones de Einstein, para intentar dar una explicación a la actual aceleración en la expansión del Universo, se encuentran las teorías de gravedad modificada $f(R)$. Estas teorías se basan en la sustitución de la curvatura escalar, R , en la acción de Einstein–Hilbert por una función genérica de la misma, $f(R)$. Esta modificación, que a priori puede no parecer especialmente traumática, se traduce finalmente en que las ecuaciones de campo derivadas de la nueva acción sean ecuaciones diferenciales no lineales de cuarto orden, en lugar de ser de segundo orden como es el caso de las ecuaciones de campo de Einstein. Una parte muy importante, si bien no es la única, de los esfuerzos realizados para llevar a cabo este trabajo de tesis se basa en el estudio de diversos aspectos de diferentes teorías de gravedad modificada $f(R)$.

Uno de los bloques fundamentales de la memoria de la tesis es aquél dedicado a la reconstrucción de soluciones cosmológicas a partir de diferentes teorías gravitatorias. El objetivo es determinar si es posible encontrar una acción que sea capaz de reproducir una cosmología, dada por su factor de escala o su función de Hubble, y, en caso afirmativo, determinar la forma de dicha acción. Esta labor se ha llevado a cabo para teorías de gravedad modificada $f(R)$ mediante el uso de dos esquemas de reconstrucción distintos, uno basado en el uso de un campo escalar y otro en el uso de las ecuaciones de campo obtenidas a partir de la acción de la teoría $f(R)$. En el capítulo 2 se presentarán estos esquemas de reconstrucción y se analizarán los resultados obtenidos mediante el uso de ambos para una misma cosmología dada. Posteriormente, en el capítulo 3, se extenderá el uso de estos programas de reconstrucción a modelos cosmológicos acoplados de manera mínima a campos de Yang–Mills, estudiando de nuevo lo que ocurre con las soluciones obtenidas a partir de ambos métodos para una misma cosmología. Además, se llevará a cabo el desarrollo de un programa de reconstrucción para teorías de Yang–Mills acopladas de manera no–mínima a gravedad. Para terminar con el bloque dedicado a la reconstrucción de soluciones cosmológicas, se estudiará el caso de universos cíclicos en el seno de teorías de gravedad de Hořava–Lifshitz modificada. La gravedad de Hořava–Lifshitz es una teoría renormalizable, propuesta por Hořava, basada en la introducción de una anisotropía entre las cordenadas espaciales y la temporal, con la cual se rompe la invariancia bajo difeomorfismos de la Relatividad General. En el capítulo 4, se hará uso de los métodos de reconstrucción estudiados anteriormente para reconstruir un universo cíclico en el seno de teorías de gravedad de Hořava–Lifshitz modificada, dichas teorías se obtienen mediante una generalización del modelo de Hořava–Lifshitz, de manera similar a como se obtienen las teorías $f(R)$ a partir de la acción de Einstein–Hilbert.

El estudio de la historia cósmica, y del crecimiento de las perturbaciones de densidad de materia, para diversos modelos $f(R)$ viables constituye otra de las partes fundamentales de esta memoria de tesis. Debido a la arbitrariedad de la función $f(R)$, existen infinitas teorías de este tipo, tantas como funciones que se puedan proponer; sin embargo, no todas ellas son viables, para ello han de cumplir con una serie de condiciones, como pueden ser pasar los tests de Sistema Solar y tener un acoplo gravitacional efectivo positivo. En el capítulo 5 se hará un estudio de la historia cósmica para dos modelos viables. Se analizarán numéricamente las oscilaciones de alta frecuencia de energía oscura producidas durante la época de dominación de materia, las cuales pueden producir algunas divergencias. Es por ello que se propondrán unos términos correctivos para los modelos que ayudarán a estabilizar estas oscilaciones sin hacer perder la viabilidad a los modelos. Para estas nuevas teorías se hará un estudio de la evolución que tendrían en el futuro y, además, se analizará de manera exhaustiva la historia de crecimiento de las perturbaciones de densidad de materia. Para llevar a cabo esta última tarea se determinará el índice de crecimiento para ambos modelos según tres parametrizaciones distintas. En la segunda parte del capítulo 5 se realizará un análisis de la época inflacionaria para dos modelos exponenciales. Para terminar con este bloque, en el capítulo 6, se estudiará el crecimiento de las perturbaciones de densidad de materia, de manera similar a como se hizo en el capítulo 5, para dos nuevos modelos $f(R)$ viables.

Un tercer bloque, que consta de dos capítulos, se dedica al estudio de otros aspectos importantes para las teorías gravitatorias, como es el caso del problema de la aparición de singularidades y el estudio del límite de campo débil en teorías $f(R,\mathcal{G})$, siendo $\mathcal{G}$ el invariante de curvatura de Gauss–Bonnet. El caso de la existencia de singularidades futuras en el seno de teorías de gravedad modificada y de energía oscura es tratado en el capítulo 7, en el cual también se dará una clasificación de las mismas dependiendo de la magnitud causante de la divergencia. Si bien es cierto que, para tratar de manera rigurosa el tema de las singularidades, es necesaria una teoría cuántica de la gravedad, de la que aún hoy carecemos, también es importante intentar encontrar escenarios naturales a nivel clásico o semiclásico que sean capaces de curar la aparición de estas singularidades. En el capítulo 7 se propondrá una posible cura para este problema, la cual está basada en la adición de un término R^2 en el Lagrangiano de la teoría. Tras este análisis del problema de la aparición de singularidades en el seno de distintas teorías gravitatorias, en el capítulo 8 se afronta el estudio del límite de campo débil para las teorías de gravedad modificada $f(R,\mathcal{G})$. Hasta finales del siglo XX, la Relatividad General de Einstein se había mostrado como la teoría gravitatoria más fiable, debido a la excelente concordancia entre sus predicciones y los datos observacionales que se tenían en ese momento. Sin embargo, el descubrimiento del actual estado de aceleración, en el que se ve inmersa la expansión del Universo, abrió una grieta en la teoría gravitatoria de Einstein, poniendo en duda su validez a grandes escalas y en regímenes de altas energías. Aún así, los excelentes resultados a cortas escalas, como a nivel de sistema solar, de la Relatividad General hacen que el análisis del límite de campo débil de cualquier teoría gravitatoria sea muy relevante, ya que éstas deberían ser capaces de reproducir los resultados obtenidos por la Relatividad General para pequeñas escalas. De esta manera, el estudio del límite de campo débil puede ser usado para desechar o seguir teniendo en consideranción una teoría gravitatoria. En el capítulo 8, se calcularán los límites Newtoniano, post–Newtoniano y post–post–Newtoniano de las teorías $f(R,\mathcal{G})$; además, el límite Newtoniano se resolverá a partir de funciones de Green. Para finalizar con el capítulo se presentarán los límites Newtoniano, post–Newtoniano y post–post–Newtoniano para dos casos especiales, las teorías $f(R)$ y $f(\mathcal{G})$.

La memoria de la tesis finaliza con un bloque dedicado a las conclusiones obtenidas y a las cuestiones que quedan abiertas para un trabajo futuro.

Resum

En aquesta memòria de tesis s'exposa el treball dut a terme pel doctorant durant els últims quatre anys, i que versa principalment sobre diversos aspectes de solucions cosmològiques obtingudes a partir de teories de gravetat modificada. Per entendre l'orígen i la importància de les teories de gravetat modificada és necessari comentar abans alguns fets que ocorregueren durant el segle XX en el marc de la cosmologia.

La cosmologia com a ciència va néixer gràcies a la Teoria de la Relativitat General d'Albert Einstein. Abans d'aquesta, l'espai no era més que el lloc on les estrelles i els planetes residien, i el temps no era més que quelcom que anava passant, essent l'espai i el temps dues coses completament desconectades i que no es veien afectades pel que passava a l'Univers. La teoria d'Einstein va enderrocar aquestes idees, proposant que l'espai i el temps estan conectats i que, a més, no son purs espectadors del que passa a l'Univers, sinó que es veuen afectats pel seu contingut. Va ser d'aquesta manera com va sorgir el concepte d'*espai-temps*, el qual, segons Einstein, es corba degut a la presència de matèria i/o energia (ja unificades en la seva teoria de la relativitat especial). Les equacions de camp d'Einstein són les equacions que permeten a la cosmologia ser considerada com una ciència, i estableixen un diàleg entre la forma de l'Univers i el seu contingut de matèria i energia. Les primeres solucions cosmològiques que es trobaren per a l'Univers eren estàtiques, malgrat que aquestes es rebutjaren quan es verificà que la llei de l'expansió proposada per Hubble era certa.

Malgrat els petits o grans problemes que encara quedaven per resoldre, semblava que la cosmologia estava destinada a viure de manera més o menys plàcida. Però aquesta aparent calma es va veure truncada quan, a finals del segle XX, dos grups liderats per Saul Perlmutter i per Adam Riess i Brian Schmidt, respectivament, descobriren, a partir d'observacions de supernoves de tipus Ia, que l'Univers es troba en una fase d'expansió accelerada. Això constrata amb la visiò que aporta la teoria del Big Bang, ja que segons aquest model l'Univers hauria surgit d'una gran explosió, fruit de la qual s'estaria expandint; no obstant, degut a l'atracció gravitatòria de la massa continguda en l'Univers, l'expansió hauria de frenar-se. A més, el grup de Perlmutter va determinar que, per tal d'explicar aquest fet en el marc de la teoria del Big Bang, assumint un univers espacialment pla, la matèria ordinària i la matèria fosca aportarien un 28% del total del continug de l'Universe, mentres que el 72% restant s'hauria d'atribuïr a un tipus d'energia exòtica denominada *energia fosca* i que exerciria una força repulsiva.

El descobriment de l'expansió accelerada de l'Univers va ser seguit del naixement d'un gran número de teories que pretenien explicar-ne l'origen. La més acceptada actualment és la teoria Λ-Cold-Dark-Matter (ΛCDM), que proposa que l'energia fosca no és més que una constant cosmològica relacionada amb l'energia del buit de l'Univers. Altres teories molt populars entre els cosmòlegs per a donar una explicació a l'energia fosca són les teories *escalar-tensor*, en les que l'acceleració s'aconsegueix mitjançant la introducció d'un camp escalar en el Lagrangià de la teoria, de manera equivalent a com l'inflató aconsegueix l'acceleració en el periode d'inflació. Bàsicament, les teories comentades fins ara es basen en la introducció d'algun tipus de matèria o energia exòtica en les equacions de camp d'Einstein per tal d'aconseguir l'acceleració. Ara bé, aquesta no és l'única forma d'aconseguir el resultat desitjat. Una altra manera es suposar que les equacions d'Einstein només són valides fins a un cert límit, però que han de ser modificades un cop propassat aquest. D'aquesta manera l'acceleració en l'expansió no estaria causada per un tipus de matèria/energia exòtica, sinó que seria conseqüència de les noves equacions. A aquest tipus de teories és a les que es coneixen com a teories de gravetat modificada.

Entre els models que proposen modificar les equacions d'Einstein, per intentar donar una explicació a l'actual acceleració en l'expansió de l'Univers, es troben les teories de gravetat modificada $f(R)$. Aquestes teories es basen en la substitució de la curvatura escalar, R , en l'acció d'Einstein-Hilbert per una funció genèrica de la mateixa, $f(R)$. Aquesta modificació, que a priori pot no semblar especialment traumàtica, es tradueix finalment en que les equacions de camp derivades de la nova acció siguin equacions diferencials no lineals de quart ordre, en lloc de ser de segon order com en el cas de les equacions d'Einstein. Una part molt important, si bé no l'única, dels esforços realitzats per portar a terme aquest treball de tesis es basa en l'estudi de diversos aspectes de diferents teories de gravetat modificada $f(R)$.

Un dels blocs fonamentals de la memòria de la tesis és dedicat a la reconstrucció de solucions cosmològiques a partir de diferents teories gravitatòries. L'objectiu és determinar si és possible trobar una acció que sigui capaç de reproduïr una cosmologia, donada pel seu factor d'escala o la seva funció de Hubble, i, en cas afirmatiu, determinar la forma d'aquesta acció. Aquesta feina s'ha dut a terme per teories de gravetat modificada $f(R)$ mitjançant l'us de dos esquemes de reconstrucció diferents: un basat en l'ús d'un camp escalar i un altre en l'ús d'equacions de camp obtingudes a partir de l'acció de la teoria $f(R)$. En el capítol 2 es presentaran aquests esquemes de reconstrucció i s'analitzaran els resultats obtinguts mitjançant l'ús d'ambdós esquemes per a una cosmologia donada. Posteriorment, en el capítol 3, s'extendrà l'ús d'aquests programes de reconstrucció a models cosmològics acoplats de manera mínima a camps de Yang-Mills, estudiant de nou el que succeeix amb les solucions obtingudes a partir d'ambdós mètodes per a una mateixa cosmologia. També es durà a terme el desenvolupament d'un programa de reconstrucció per a teories de Yang-Mills acoplades de manera no-mínima a la gravetat. Per acabar amb el bloc dedicat a la reconstrucció de solucions cosmològiques, on s'estudiarà el cas d'universos cíclics en el marc de les teories de gravetat d'Hořava-Lifshitz modificada. La gravetat d'Hořava-Lifshitz és una teoria renormalitzable, proposada per Hořava, basada en la introducció d'una anisotropia entre les coordenades espacials i la temporal, amb la que es trenca la invariança sota difeomorfismes de la Relativitat General. En el capítol 4, es farà ús dels mètodes de reconstrucció estudiats anteriorment per tal de reconstruïr un univers cíclic en el marc de les teories de gravetat d'Hořava-Lifshitz modificada, teories obtingudes mitjançant una generalització del model d'Hořava-Lifshitz, de manera similar a com s'obtenen les teories $f(R)$ a partir de l'acció d'Einstein-Hilbert.

L'estudi de la història còsmica, i del creixement de les perturbacions de densitat de matèria, per diversos models $f(R)$ viables constitueix una altra de les parts fonamentals d'aquesta memòria de tesis. Degut a l'arbitrarietat de la funció $f(R)$, existeixen un infinit de teories d'aquest típus, tantes com funcions es puguin proposar; ara bé, no totes elles són viables, ja que per això han de satisfer una sèrie de condicions, com ara els tests de Sistema Solar i tenir un acoblament gravitacional efectiu positiu. En el capítol 5 es farà un estudi de la història còsmica per a dos models viables. S'analitzaran numèricament les oscil·lacions d'alta freqüència d'energia fosca produïdes durant l'època de domini de la matèria, les quals podrien produïr alguna divergència. És per això que es proposaran uns termes correctius per als models que ajudaran a establir aquestes oscil·lacions sense fer perdre la viabilitat dels models. Per a aquestes noves teories es farà un estudi de l'evolució que tindran en el futur i, a més, s'analitzarà de manera exhaustiva la història de creixement de les pertorbacions de densitat de matèria. Per tal de dur a terme aquesta darrera tasca es determinarà l'índex de creixement per ambdós models segons tres parametritzacions diferents. En la segona part del capítol 5 es realitzarà un anàlisis de l'època inflacionària per a dos models exponencials. Per acabar aquest bloc, en el capítol 6, s'estudiarà el creixement de les perturbacions de densitat de matèria, de manera similar a com es fa en el capítol 5, per a dos nous models $f(R)$ viables.

Un tercer bloc, que consta de dos capítols, es dedica a l'estudi d'altres aspectes importants per a les teories gravitatòries, com és el cas del problema de l'aparició de singularitats i l'estudi del límit de camp dèbil en teories $f(R,G)$, essent G l'invariant de curvatura de Gauss-Bonnet. El cas de l'existència de singularitats futures en el marc de teories de gravetat modificada i d'energia fosca és tractat al capítol 7, en el que també es farà una classificació de les mateixes, en funció de la magnitud causant de la divergència. Si bé és cert que, per tal de tractar de manera rigorosa el tema de les singularitats, és necessària una teoria quàntica de la gravetat, teoria que encara ens manca avui en dia, també és important intentar trobar escenaris naturals a nivell clàssic o bé semi–clàssic que siguin capaços de capturar l'aparició d'aquestes singularitats. En el capítol 7 es proposarà una possible cura per a aquest problema, que estarà basada en afegir un terme R^2 en el Lagrangià de la teoria. Després de fer aquesta anàlisi del problema de les singularitats en el context de les diferents teories de gravetat, el capítol 8 es dedicarà a l'estudi del límit de camp feble per les teories de gravetat modificada de tipus f(R,G). Fins a finals del segle XX, la Teoria de la Relativitat General d'Einstein era tinguda com la teoria gravitatòria més fiable, degut a la concordància excel.lent entre les seves prediccions i les dades observacionals que s'havien obtés fins aleshores. Però el descobriment que va tenir lloc a finals de segle sobre l'estat present d'acceleració que afecta l'expansió de l'Univers obrí una escletxa en la teoria gravitatòria d'Einstein, posant en dubte de sobte la seva validea a escales molt grans i en règims de molt alta energia. Així i tot, els seus excel.lents resultats a escales petites, com ara a nivell del nostre sistema solar, no s'han vist en absolut afectats. Això fa que l'anàlisi de camp feble de qualsevol teoria gravitatòria segueixi sent extraordinariament relevant, ja que tota teoria ha de ser capa de reproduïr els molt bons resultats obtinguts per la Relativitat General a escales petites. D'aquesta manera, l'estudi del límit de camp feble pot ser usat per tal de rebutjar o, contràriament, de seguir tenint en consideració una teoria gravitatòria. Dins del capítol 8 es calcularan els límits Newtonià, post–Newtonià i post–post–Newtonià de les teories $f(R,\mathcal{G})$; a més. el límit Newtonià serà resolt mitjanant funcions de Green. Per acabar aquest capítol es presentaran explícitament els límits Newtonià, post–Newtonià i post–post–Newtonià de dos cassos especials importants, les teories $f(R)$ i $f(\mathcal{G})$.

La memòria de la tesis finalitza amb un bloc dedicat a les conclusions obtingudes i a qüestions que queden obertes per a un treball futur.

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- K. Bamba, A. J. Lopez-Revelles, R. Myrzakulov, S. D. Odintsov, and L. Sebastiani. *The universe evolution in exponential $F(R)$ -gravity.* Proceedings of QFTG2013, published in TSPU Bulletin 3 (128), 2012, n 13 p.19-24.
- E. Elizalde and A. J. López-Revelles.*Reconstructing cosmic acceleration from modified and nonminimal gravity: The Yang-Mills case.* Phys. Rev. D, 82:063504 (2010).
- E. Elizalde, A. J. Lopez-Revelles, S. D. Odintsov, and S. Yu. Vernov. *Cosmological models with Yang-Mills fields.* Phys.Atom.Nucl., 76:996 (2013).
- A. J. Lopez-Revelles. *Growth of matter perturbations for realistic $F(R)$ models.* Phys. Rev. D, 87:024021 (2013).
- A. J. Lopez-Revelles. *Reconstructing cosmic acceleration from $f(R)$ modified gravity.* Proceedings of ERE2011, Madrid. ArXiv: 1301.2190.
- A. J. Lopez-Revelles and E. Elizalde. *Universal procedure to cure future singularities of dark energy models.* Gen. Rel. Grav., 44:751 (2012).
- A. J. Lopez-Revelles, R. Myrzakulov and D. Saez-Gomez. *Ekpyrotic universes in $F(R)$ Hořava-Lifshitz gravity.* Phys. Rev. D, 85:103521 (2012).

Contents

Chapter 1

Introducción

The work that will be presented in this Thesis was carried out by Antonio Jesús López Revelles in the period between June 2009^{th} and July 2013^{th}. It is focussed in the study of different aspects related with several theories of modified gravity.

1.1 The origins of Modern Cosmology in the 20th Century

Let us start reviewing the most important events in cosmology throughout the twentieth Century, which gave rise to what is now known as modern cosmology. This epoch may be considered as the most exciting time for this branch of physics, starting with the birth of the theory of General Relativity of Albert Einstein in 1915, and ending with the discovery in 1998 of the current acceleration in the expansion of the Universe. Such state of acceleration gave rise to a large number of theories attempting to give it an explanation. One class of these theories, the so-called modified gravity theories, are based on the modification of the theory of General Relativity.

General Relativity and Cosmology

This historical review begins in the year 1907, when Einstein realized that his theory of Special Relativity was not compatible with the theory of gravitation of Isaac Newton. At this time Einstein began to think about how to introduce gravity in his new theory. This task is non–trivial because Special Relativity only applies to inertial observers, i.e. observers who do not suffer any acceleration, while the observers within a gravitational field are non–inertial.

The theory of General Relativity, i.e. the relativistic theory incorporating gravity, appeared in November 1915, when Einstein submitted the field equations to the Prussian Academy of Sciences.

The theory of General Relativity is based on two fundamental principles [7]:

- **The principle of equivalence:** *Free–falling observers within a gravitational field are locally equivalent to inertial observers. These situations cannot be discriminated using local experiments.*

In order to formulate this principle, Einstein realized that the equality between the gravitational mass (the measure of how the one body interacts gravitationally with other bodies) and the inertial mass (the proportionality constant, which appears in the second Newton law, between the force applied to a body and its acceleration) means that a free–falling body does not feel its weight and, therefore, can consider itself as an inertial observer, at least locally.

The equivalence principle implies that, for each point of the spacetime, there exists a general change of coordinates that makes the spacetime to look, locally, as flat. Mathematically, this principle results in that the spacetime is a four dimensional Lorentzian manifold, in which the free particles move along its geodesics. Then, gravity becomes a manifestation of the spacetime curvature.

- **The principle of general covariance:** *The laws of physics must have the same form in all frames of reference. Therefore, they must change in a covariant way under general changes of coordinates.*

  This principle implies that a physical law is valid if it is in the frame of the Special Relativity and it is written in a covariant way, i.e., as a function of objects that transform properly under general changes of coordinates ($y^\mu = y^\mu(x^\nu)$). Mathematically, this principle means that the equations of the physical laws are tensorial equations.

Summarizing, gravity can be considered as a manifestation of the spacetime curvature. Moreover, as the source of the Newtonian gravitational potential is the mass, and in General Relativity matter and energy are equivalent, this suggests that the source of the spacetime curvature is related directly with the stress–energy tensor, $T_{\mu\nu}$, which is the one that contains all information about the matter and energy of a system. Therefore, the interaction between spacetime and matter should be given by a tensorial equation of the following type

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.1}$$

where $G_{\mu\nu}$ is a purely geometric tensor, $T_{\mu\nu}$ is the stress–energy tensor and κ is a proportionality constant.

There exist several constraints to be satisfied by the tensor $G_{\mu\nu}$, which are

- It must be symmetric in the two indexes.
- It must depend on the metric and its derivatives, only.
- It must cancel for flat spacetime.
- $\nabla_\mu G^{\mu\nu} = 0$.
- It must contain second derivatives of the metric in order to be a dynamical theory and to reproduce the Poisson equation.
- It must be linear in the curvature to provide second–order differential equations.

It can be demonstrated that the only tensor that satisfies every condition is the so–called Einstein tensor, given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{1.2}$$

Finally, the Einstein field equations that govern the interaction between the spacetime geometry and the content of matter and energy in the Universe are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}, \tag{1.3}$$

being G_N the Newtonian gravitational constant.

The Einstein field equations (1.3) can be obtained by varying with respect to the metric the action given by the following expression

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi G_N}R + \mathcal{L}_m\right). \quad (1.4)$$

This action has two different parts. The first one is the Einstein–Hilbert action, given by

$$S_{EH} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}R, \quad (1.5)$$

whose variation with respect to the metric provides the Einstein equations for vacuum spacetime. While the second part of the action (1.4), given by

$$S_m = \int d^4x\sqrt{-g}\mathcal{L}_m, \quad (1.6)$$

accounts for the different energy–matter fields.

From the cosmological point of view, General Relativity is very important, since it allows us to conceive the Universe as a dynamical system, with a closed relationship between its structure and its content of energy and matter. Then, it can be said that cosmology, as a science, was born thanks to General Relativity.

Relativistic cosmology is based on two basic principles:

- **The cosmological principle:** *At any time, the Universe is homogeneous and isotropic on large scales.*

  This principle implies that the Universe is maximally symmetric, i.e., it has the maximum number of symmetries. Mathematically, maximally symmetric manifolds are spaces with constant curvature.

  Observations coming from radiowaves, cosmic X rays and, specially, the cosmic microwave background radiation point out towards the fact that the Universe is very homogeneous.

- **Weyl's postulate:** *On cosmological scale, matter behaves as a perfect fluid, whose components move along temporal geodesics. These geodesics do not intersect, except (possibly) at one point in the past.*

  The peculiar velocities produced by the gravitational interactions are, usually, negligible with respect to the velocities generated by the evolution of the Universe.

  This postulate allows to define a special kind of observers, the so–called comoving observers, whose motion is determined by the evolution of the Universe. It is also possible to define a comoving time, as the one that a comoving observer will measure.

These two principles determine largely the form of the spacetime metric. On the one hand, Weyl's postulate implies that the spacetime can be foliated in spatial hypersurfaces (the ones with a constant cosmological time). On the other hand, the cosmological principle implies that such spatial hypersurfaces are maximally symmetric. Considering these two premises, the metric takes the following form

$$ds^2 = -dt^2 + a(t)^2\left[\frac{1}{1-kr^2}dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)\right]. \quad (1.7)$$

The parameter k can only take three values that represent the three hypersurfaces of constant curvature: the sphere $\mathbb{S}^3$ (positive curvature, $k = 1$), the plane $\mathbb{R}^3$ (null curvature, $k = 0$) and the hyperboloid $\mathbb{H}^3$ (negative curvature, $k = -1$).

The metric given by Eq.(1.7) is the so–called Friedmann-Lemâitre-Robertson-Walker (FLRW) metric. Alexander Friedmann was the first person proposing this ansatz in 1922; Georges Lemâitre developed, independently, the same work done by Friedmann; while Howard Robertson y Arthur Walker showed independently, in 1935 and 1936, that this metric is the most general one describing an homogeneous and isotropic universe.

In 1922, Friedmann introduced the FLRW metric into the Einstein equations and deduced the so–called Friedmann equations, which reproduced a non–statical universe, against the prevailing prejudices in the scientific community at that time, although he did not realize this fact explicitly (he died very young) and his work went unnoticed for some years until Edwin Hubble showed that the Universe was expanding.

Friedmann was not the first scientist proposing a cosmological model based in the use of Einstein's equations. In 1917, Einstein realized that his equations, as he wrote them in 1916, resulted in a non–statical universe when it was supposed a normal content of matter for the Universe. The idea of a non–statical universe seemed senseless to Einstein and irritated him, according to a letter addressed to the astronomer Willem de Sitter when de Sitter deduced the equations of an empty universe which could be expanding. This fact persuaded Einstein to modify his field equations by introducing a new term proportional to a constant Λ, the so–called cosmological constant, which was interpreted as the energy density of the vacuum. The new field equations took the following form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}. \tag{1.8}$$

When Λ is positive the new term generates a repulsive cosmic force, while if it is negative the new force is attractive. Several years after, when Hubble demonstrated that the Universe was expanding, Einstein declared that the introduction of the cosmological constant had been the worst error of his scientific career. Nevertheless, with the discovery at the end of the twentieth century of the accelerated expansion of the Universe, the cosmological constant appeared again in the scientific community.

Expanding Universe and the Big Bang Theory

In the first part of the last Century, the scientific prejudices established that the Universe should be static. Many observational evidences were necessary to overthrow this idea.

The astronomer Vesto Slipher was the first to provide some of these evidences. In 1914, Slipher showed the results obtained from his measures of the radial velocity of twelve galaxies. Only one of these galaxies (Andromeda) was not moving away from us. Moreover, these velocities were higher than the expected ones, a fact that suggested that these objects were outside our galaxy. In order to measure these velocities, Slipher made use of the shift in the spectral lines of the light coming from the different galaxies

In 1929, the astronomers Edwin Hubble y Milton Humason measured the velocities and the distances of forty six galaxies, finding a linear relation between both quantities, the higher the distance to the galaxy the higher the recession velocity measured. This linear relation, experimentally obtained, finally overthrew the idea of a static universe. As, practically, all the galaxies move away from us and from themselves, the conclusion was that the Universe was expanding. This discovery is considered nowadays as one of the most important scientific discoveries of all times.

At the end of the 40's, once discarded the idea of a static universe, two theories seemed to compete for giving an explanation to the current Universe.

The first of them, promoted by the physicist Georges Gamow, could be considered as a continuation of the universe proposed by Lemâitre in 1931. According to him, our universe may be "the ashes and smoke of bright but very rapid fireworks". Specifically, Gamow proposed that the Universe could be the expansion and cooling of a state, the *Ylem*, in which the matter was a mixture of protons, neutrons and electrons with a density and a temperature close to infinity. Moreover, he explained the relative abundances of hydrogen and helium within the Universe in the framework of his theory. Nevertheless, it failed in explaining the relative abundances of heavier elements. Later, the astrophysicists (Fred Hoyle among them) proposed that these heavier elements were produced in nuclear reactions which occurred within the stars from the hydrogen and the helium. Then, the theory proposed by Gamow was not wrong, only incomplete, in this aspect.

The other theory that had large number of followers was the one promoted by Fred Hoyle, Thomas Gold and Hermann Bondi since 1948, the steady–state theory. This theory was based on the assumption of the perfect cosmological principle, which states that the universe is homogeneous and isotropic in the space, but also in the time. As the Universe is expanding, this means that it is necessary a continuous creation of matter in order to keep constant the matter density.

Throughout the 50's and in the first part of the 60's, both theories consolidated their positions with some successes, although the theory of Gamow seemed to take the advantage. The final blow against the steady–state theory was the discovery of the cosmic microwave background radiation in 1965.

In a paper of 1948, Gamow described how the Universe in its first moments would have been dominated by radiation. As the Universe was expanding and cooling, this radiation would have been turning into matter. Ralph Alpher and Robert Herman, Gamow's colleagues, predicted the existence of a remainder of this radiation, whose temperature would be around 5 Kelvin due to the expansion and cooling of the Universe.

In 1964, Arno Penzias and Robert Wilson worked in the construction of an antenna for Bell Labs when they found a source of an isotropic noise in the atmosphere. After several attempts to remove this noise, and discarding the possibility of an interference, they published their results in a paper. Robert Dicke, a physicist of the Princeton University, interpreted this noise as a microwave background radiation, whose temperature would be 3 Kelvin, which would be the result of the expansion and cooling of the Ylem proposed by Gamow. For the physicist Stephen Hawking, the discovery of the cosmic microwave background radiation was 'the final nail in the coffin of the steady-state theory'. From this moment on, the theory of Gamow was considered as the best theory which attempted to explain the Universe.

The theory of Gamow is the so–called Big Bang theory. Its name is due to the astrophysicist Fred Hoyle, one of its most important detractors, who told in an interview for the BBC in 1949, in an attempt to taunt the theory of Gamow, that it was only a big bang.

Summarizing, the Big Bang theory agrees with the expansion law of Hubble, being able to predict the cosmic microwave background radiation and the relative abundances of the primordial elements. Nevertheless, this theory is not without problems. One of the most important is the problem of the baryogenesis, i.e., to explain the process to produce the asymmetry found between baryons and antibaryons in the Universe. Other important problems are:

- The horizon problem: The observational data coming from the cosmic microwave background radiation show that the Universe is quite isotropic, which indicates that in the moment of the recombination the Universe would have been in a thermal equilibrium, and that is not possible, because sky regions separated by a few degrees were causally disconected from the others at that time.

- The flatness problem: The observational data we have nowadays point out that the spatial geometry of the Universe is flat. This fact means that the density of the Universe takes a very specific value, known as critical density. In the framework of the Big Bang theory there seems to be no mechanism to set this value so precisely.

- Magnetic monopoles: Grand Unification Theories predicted the existence of magnetic monopoles, which would be produced in the early Universe, but they are not observed.

Inflation

In 1981, the particle physicist Alan Guth proposed a theory of an inflationary universe, which was able to explain several unresolved problems by the Big Bang theory. According to the model of Guth, an initial phase, that lasted a very small fraction of a second, would have happened before the Big Bang. In this period, the Universe would have suffered an exponential expansion, which would have produced a growth of the scale factor of about thirty orders of magnitude. After this huge expansion, a reheating of the Universe would have resulted in the initial state of the Big Bang.

This exponential expansion may fix the flatness problem. The reason is that, for any initial value, an expansion of this type flats everything. Moreover, the distances in the initial period of an inflationary universe would have been very small, allowing the universe to be in a thermal equilibrium before inflation began, which would give an explanation to the horizon problem. This initial thermal equilibrium may be an inconvenience when trying to explain the existence of structures in the Universe, nevertheless, quantum fluctuations due to the uncertainty principle exist in this equilibrium. These perturbations, which would have been extremely small in their origin, would have grown enormously during the inflationary period, giving as a result the irregularities which would later become the galaxies in the Universe nowadays.

Summarizing, an exponential expansion may fix several problems left by the Big Bang theory. Nevertheless, the first model proposed by Alan Guth (see [141]) was not completely satisfactory. In this model a scalar field, the so–called inflaton, is located in the global minimum of a potential. As the universe cools, this potential evolves in a way that the global minimum (true vacuum) becomes a local one (false vacuum). At the end of the inflationary period, due to the fast expansion, the universe is in a very cold state which makes impossible the creation of radiation and elementary particles. Within the theory of Guth, the reheating needed to begin the Big Bang is achieved by means of collisions between bubbles of true vacuum, which are created in the false vacuum via quantum tunneling. This model is problematic because it does not reheat properly. The reason is that the exponential expansion suffered during inflation pulls apart the bubbles of true vacuum, and the collisions between bubbles becomes very rare. Nowadays the model of Guth is known as old inflation.

This problem was solved by Andrei Linde in 1982 (and, independently, by Andreas Albrecht and Paul Steinhardt) with the so–called Slow–Roll inflation (or new inflation). In this theory, the inflaton rolls down a potential energy hill with a soft slope, in this phase the scalar field rolls very slowly compared with the expansion of the universe and inflation occurs. After this initial phase, the inflaton falls in a minimum of the potential where the field begins to oscillate. By means of these oscillations, heavy particles are created from the vacuum energy. Finally, these particles decay in lighter ones, reheating the universe.

In addition to the use of scalar fields, there exist other mechanisms to produce inflationary models. In this sense, at the same time that Guth was developing his model, the physicist Alexei Starobinsky proposed in 1980 his inflationary theory. Starobinsky suggested that the quantum corrections to General Relativity would be very important in the early universe. These corrections are usually quadratic terms of the curvature which are added to the Einstein–Hilbert action. These new terms in the action may produce an effective cosmological constant which would result in an inflationary phase. In this model, the initial singularity of the Universe would be replaced by the inflationary epoch.

Nowadays, an inflationary period before the Big Bang is widely accepted in the scientific community, but it is very important to point out some of the cons of these theories. One of the most important problems, pointed out by the physicist Roger Penrose, are the very specific initial conditions needed to begin the inflationary period. In this sense, Penrose says the initial condition problem is not solved. Other important criticism against the inflationary universe is that it is not completely clear how reheating works.

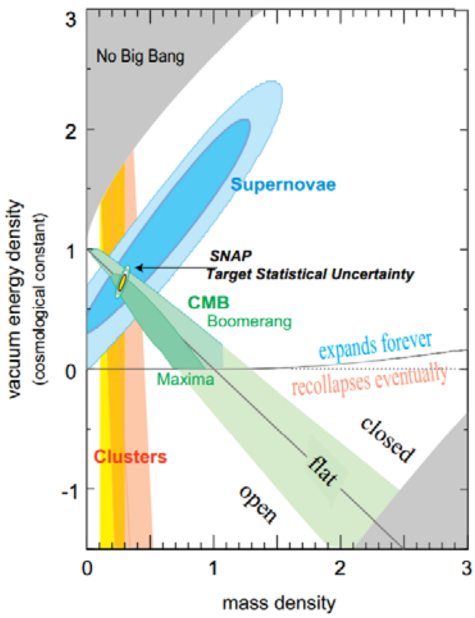
Accelerated Expansion

As it has been shown up to now, the twentieth century was the most exciting period for cosmology because of the very significant advances in this field of the physics. Nevertheless, the last years of the century provided one of the greatest challenge for the cosmology in the new century: the current accelerated expansion of the Universe.

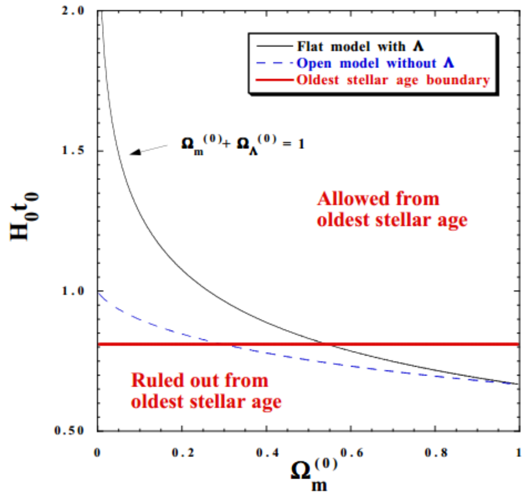
In 1998, two groups leaded by Saul Perlmutter and by Brian Schmidt and Adam Riess, respectively, showed that observational data coming from type Ia supernovae suggests that the Universe is undergoing an accelerated expansion. Type Ia supernovae appears when a white dwarf exceeds the Chandrasekhar mass and it explodes. This kind of supernovae is very important for astrophysicists because they are objects that can be used as standard candles, i.e. as objects with precisely known luminosity. As standard candles, type Ia supernovae can be used to determine distances of very far galaxies. Using the luminosity distance for several type Ia supernovae, the groups of Perlmutter and Riess determined that the luminosity of these objects were less than the one they should have in a decelerating universe; then, they proposed that the expansin of the Universe is accelerating. Specifically, the group of Perlmutter found that, assuming a spatially flat universe, the matter should account for just the 28% of the total content of the Universe, including ordinary matter and dark matter; while the rest of the content of the Universe, the other 72%, should be a exotic kind of energy named as dark energy.

The discovery of the current accelerated expansion resulted in a very large number of new theories, which nowadays still attempt to explain this new behavior of the Universe. The most accepted theory is the so–called Λ-Cold-Dark-Matter (ΛCDM), which proposes that the cosmological constant is the cause of the current accelerated expansion of the Universe.

In addition to the observational data coming from type Ia supernovae, there exist other evidences for the acceleration in the expansion. Among these proofs we find the data coming from the observation of the cosmic microwave background radiation ([145, 169, 170, 272]), large scale structure ([264, 278]), baryonic acoustic oscillations ([102]) and weak lensing ([151]). Finally, other evidence comes from the comparison between the age of the Universe and the age of the oldest stars. In [156] the authors determined the age of a globular cluster in the Milky Way as $\pm 13.5\,\text{Gyr}$, then the age of the universe given by a consistent theory should be bigger than $13.5\,\text{Gyr}$. It has been demonstrated that this is impossible in a universe without dark energy, which is another evidence for the acceleration in the expansion. In Fig. (1.1) two graphics are depicted with the data from observations of type Ia supernovae, cosmic microwave background radiation, galaxy clustering and the age of the universe. It can be easily seen that a universe without dark energy is not possible.



(a) The $\Omega_m^{(0)} - \Omega_\Lambda^{(0)}$ confidence regions constrained from the observations of SN Ia, CMB and galaxy clustering. We also show the expected confidence region from a SNAP satellite for a flat universe with $\Omega_m^{(0)} = 0.28$. From Ref. [10].



(b) The age of the universe (in units of H_0^{-1}) is plotted against $\Omega_m^{(0)}$ for (i) a flat model with $\Omega_m^{(0)} + \Omega_\Lambda^{(0)} = 1$ and a open model. We also show the border $t_0 = 11$ Gyr coming from the bound of the oldest stellar ages. The region above this border is allowed for consistency. From Ref. [90].

Figure 1.1: Evidences for a dark energy universe.

1.2 Trying to explain the accelerated expansion of our Universe

After the great discovery in 1998 of the acceleration state in the expansion of the Universe, a lot of theories appeared (and continue appearing) trying to explain the last observational data (see [26]).

In principle, there seem to be three ways to solve this new problem. The first option may be to consider that the evidences, that lead us to think that the Universe is undergoing an acceleration expansion, are erroneous. This first way seems to be ruled out because of the great number of evidences coming from different kind of independent observations. Other option to face the problem is to accept that even if General Relativity is still the gravitational theory that rules the Universe, it may become necessary to introduce a new kind of fluid (dark energy) which accounts for around the 70% of the content of the Universe. Moreover, in order to reproduce the acceleration required, this dark energy should have an equation of state, $p = \omega\rho$, with $\omega < -1/3$. It is in this framework where we find the ΛCDM theory. Finally, maybe the most traumatic way, we could think that General Relativity is not the gravitational theory that rules the Universe, but simply the limit at low scales of a more general theory. One example of this kind of theories that choose this last option are the so–called modified gravities.

It is very important to point out that every relativistic theory of gravity should fulfill the following requirements, from a phenomenological point of view:

1. In the weak field limit and weak velocities, the theory must reproduce the Newtonian dynamics;

while in the post–Newtonian limit, the theory must pass the Solar system tests.

2. The theory must reproduce the observed galactic dynamics, accounting for the known baryonic matter and radiation, and being able to reproduce the Newtonian potential.

3. The theory must explain the creation of large–scale structure.

4. Finally, the relativistic theory must be able to reproduce the cosmological dynamics, i.e. to predict the Hubble parameter, the deceleration parameter, the density parameters, etc.

In the following subsections, some of the most accepted theories nowadays are review, trying to explain their pros and cons. Naturally, I will focus in modified gravity theories, as they are the main theme of the thesis.

ΛCDM model

The most popular model nowadays is the one known as Λ-Cold-Dark-Matter (ΛCDM), which is based on the introduction of a cosmological constant in the Einstein equations. This cosmological constant may cause the accelerated expansion of the Universe. Then, the equations that rule the Universe according to the ΛCDM theory are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}. \tag{1.9}$$

As it was explained in the previous section, Einstein already introduced a cosmological constant term to reproduce a static cosmological model, nevertheless, after the discovery of the expansion of the Universe, Einstein himself said that the introduction of the cosmological constant had been the worst error he made in his life. After the discovery of the groups of Perlmutter and Riess, the cosmological constant came back to the modern cosmology. It is very interesting to realize that Einstein used the cosmological constant in the first years of the last century to achieve a static universe, while in the last years of the same century, the same cosmological constant term was used to achieve an accelerated expansion for the universe.

Even being the most popular theory, the ΛCDM model is not free of problems. One of the most important is known as the cosmological constant problem [288]. The problem lies in the following fact: a known result in particle physics is that the energy density of the vacuum produces naturally a cosmological constant term; this energy density of the vacuum is given by

$$\rho_v \approx \frac{k_{max}^4}{16\pi^2}, \tag{1.10}$$

being k_{max} the cut–off scale up to where the quantum field theory is expected to be valid. In the case of General Relativity, the limit may be the Planck mass, obtaining $\rho_v \approx 10^{74}\,\mathrm{GeV}^4$. Nevertheless, in order to reproduce the observational data, the cosmological constant should be of the order of the recent Hubble parameter, which should imply that the energy density produced by the cosmological constant should be

$$\rho_\Lambda = \frac{\Lambda m_{PL}^2}{8\pi} \approx 10^{-47}\,\mathrm{GeV}^4. \tag{1.11}$$

Comparing the value of the energy density obtained from the quantum field theory (1.10) with that obtained observationally (1.11), a difference of around 121 orders of magnitude is found. Even if we take the cut–off scale, k_{max}, of quantum chromodynamics, the difference between the theoretical and observational value would be around 44 orders of magnitude. A great number of attempts have been done to solve this problem, a sample of them are the following: anthropic considerations, quantum gravity, string theory, vacuum fluctuations of the energy density, supersymmetry...

Finally, it is important to remark that the equation of state for the cosmological constant is constant, being $\omega = -1$, which seems to agree with the latest observational data.

Scalar-tensor theories

Other popular theories among the cosmologists are the so–called scalar–tensor theories. In this case, the acceleration is achieved by the use of a scalar field, in a similar way as the inflationary phase was obtained with the inflaton. One of the main differences, of this kind of theories with respect to the cosmological constant, is that the equation of state is not constant.

There are a variety of scalar–tensor theories, some of them are based in a minimal coupling between the scalar field and gravity, while for other models the coupling is non–minimal. Among the last kind we have the Brans-Dicke theory, which is given by the following action

$$S = \int d^4x\sqrt{-g}\left(\frac{\phi R - \omega\frac{\partial_\mu\phi\partial^\mu\phi}{\phi}}{16\pi G_N} + \mathcal{L}_m\right), \tag{1.12}$$

where in this case ω is not the equation of state, but the Dicke non–dimensional coupling constant. This theory was born as an attempt to build a gravitational theory which incorporated the Mach principle, because General Relativity only contains some of the Mach's ideas, but it admits explicitly non–Machian solutions, as the Gödel universe.

Within the framework of the scalar–tensor theories minimally coupled to gravity, the most popular is the one known as quintessence, developed by Caldwell et al. in [56]. This theory is based on the use of a scalar field with potentials that give rise to an accelerated expansion. The action for quintessence is given by

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi G_N}R - \frac{1}{2}\left(\nabla\phi\right)^2 - V(\phi)\right), \tag{1.13}$$

being $\left(\nabla\phi\right)^2 = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ and $V(\phi)$ the potential of the scalar field. The equation of state for this theory is given by $\omega = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}$, then it is easy to check that $-1 \leq \omega \leq 1$.

Other important scalar-tensor model is the phantom theory. The reason why it has this name is that the equation of state may be $\omega < -1$ and this region is often named as phantom dark energy. The action that describes the theory is similar to the one of quintessence, it only changes the sign of the kinetic term of the scalar field, i.e.

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi G_N}R + \frac{1}{2}\left(\nabla\phi\right)^2 - V(\phi)\right), \tag{1.14}$$

obtaining that $\omega = \frac{\dot{\phi}^2 + 2V(\phi)}{\dot{\phi}^2 - 2V(\phi)}$, then when $\phi^2 < 2V(\phi)$ it is $\omega < -1$.

There exist many more types of scalar-tensor theories in the literature, as they can be K–essence theories, tachyon field theories, dilatonic dark energy, etc. In Chapter 7 of the thesis I will study the behavior of scalar–tensor models of the following form:

$$S = \int d^4x\,\sqrt{-g}\,\left[\frac{1}{2\kappa^2}R\,-\,\frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi\,-\,V(\phi)\,+\,\mathcal{L}_{matter}\right]. \tag{1.15}$$

Modified gravity theories

In the previous subsections, several theories based on the use of some kind of exotic matter or energy have been discussed. This kind of theories basically modify the right hand side of Einstein equations (1.3); in this sense, these theories propose that the Universe is ruled by General Relativity and it is composed, approximately, of 95% of dark matter and dark energy. Nevertheless, the Einstein–Hilbert action, which is given by (1.5), is not the only possible valid choice according to the basic assumptions considered by Einstein and Hilbert, although it is the only way to build a linear invariant on the second derivatives of the metric.

The idea of a classical gravitational theory ruled by a Lagrangian with a linear dependence with respect to the curvature, as the Einstein–Hilbert theory, may not be as strong as it was since the suspicion that gravity may be considered as the low energy limit of a string theory, or other quantum theory, has grown. In this sense, in semiclassical expansions of quantum Lagrangians and low energy limits for stringy actions, there appear other curvature invariants or non linear functions on the curvature which may be considered as corrections of the theory of General Relativity. Then, by means of inserting non linear term on the curvature in the Einstein–Hilbert action (see, for example, [235]), the geometric part of Einstein's equations can be modified, i.e. the left hand side of (1.3).

This way of thinking has a great advantage, since the great number of different actions, which may be considered, and the freedom in the choice of their parameters allow many of them to reproduce the observational data we have nowadays. But this advantage may be also considered as a great handicap, since the great freedom we have in the choice of the parameters becomes a predictive power loss for these theories. In this sense, it may be of great interest to establish some theoretical requirements which allow us to discriminate between valid and non–valid theories.

As it has been shown, from the theoretical point of view it is possible and it is motivated the development of gravitational theories based on Lagrangians which are non linear on the curvature. These corrections may be very different, some examples of them are the following terms: R^2 , $R_{\mu\nu}R^{\mu\nu}$, $R_{\alpha\beta\rho\tau}R^{\alpha\beta\rho\tau}$, etc. The main theme of this thesis was the development of some aspects of the so–called $f(R)$ modified gravity theories. These theories are based on the replacement of the scalar curvature, R , in the Einstein–Hilbert action by a generic function f of R , i.e. the actions studied are given by

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi G_N}f(R) + \mathcal{L}_m\right). \tag{1.16}$$

[1]The main difference with respect to the Einstein's field equations given by (1.3) is that Einstein ones are second order differential equations, while the new ones, obtained by varying the new action with respect to the metric, are non linear fourth order differential equations; specifically, these new equations, obtained from the action given by (1.16), are the following:

$$f'(R)\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = 8\pi G_N T^{(\mathrm{matter})}_{\mu\nu} + \left[\frac{1}{2}g_{\mu\nu}\left(f(R) - Rf'(R)\right) + \left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)f'(R)\right]. \tag{1.17}$$

It is easy to check that one can recover Einstein's equations from the new ones by taking $f(R) = R$.

In addition to $f(R)$ modified gravity theories, other kind of modified gravities are studied in this thesis. These theories are: non–local gravity theories, inspired by quantum loop corrections, which introduce in the action the inverse of the D'Alembertian operator; or the $f(R,\mathcal{G})$ modified gravity theories based on

[1]From now on, unless otherwise indicated, the function $f(R)$ will be used for an arbitrary function of the scalar curvature, while $F(R)$ will be considered as $F(R) = R + f(R)$.

the change of the scalar curvature in the action for a generic function of the same scalar curvature and the Gauss–Bonnet curvature invariant, which is given by $\mathcal{G} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma}$.

From a purely mathematical point of view, one remark must be done. In the framework of differential geometry, it is known that, for a given metric, there exists a connection which is completely determined by this metric, this is the so–called Levi–Civita connection. There are two possibilities in deriving the equations of motion for a gravitational theory. One way is to establish that the connection of the theory is the Levi–Civita one, then the metric is the only independent magnitude. This is the so–called metric formalism. The other way to derive the equations of motion is considering that the metric and the connection are independent. This is the so–called Palatini formalism and it requires to derive the equations of motion for the metric and the respective ones for the connection. In the case of General Relativity these two approaches are equivalents. The reason of this equivalence lies in the fact that, when using the Palatini approach, the equations of motion derived for the connection only establish that the connection must be the Levi–Civita one, which is the assumption done when using the metric formalism. Nevertheless, this equivalence between the two approaches does not hold anymore for the case of modified gravity theories based on the use of functions of curvature invariants, as $f(R)$ or $f(R,\mathcal{G})$ theories. Hence, even considering the same action, the physical equations obtained using one or the other formalism are not the same. This is the reason why it is very important to explain that throughout this thesis the metric formalism has been used.

It is also important to remark several aspects from the quantum point of view. Up to now, all the attempts done in order to achieve a renormalizable and unitary consistent gravitational theory have failed. It is known that General Relativity is a non–renormalizable theory, but it can be renormalized to one loop level in low energy regimes (with respect to the Planck energy) at large scales. In order to achieve this purpose it is necessary the introduction in the Einstein–Hilbert action of higher order curvature invariants and non–minimal couplings between gravity and matter. In this sense, modified gravity theories may achieve this objective.

A gravitational theory that is renormalizable is the so–called Hořava-Lifshitz gravity. This theory is based on the use of an anisotropy between the spatial and the temporal coordinates, breaking in this way the invariance under diffeomorphisms of General Relativity. As a result of this restriction in the symmetry of the system, it is possible to achieve a renormalizable theory of gravity, but at the cost of introducing a new degree of freedom that generates instabilities in the spectrum of the theory. Some cosmological models have been studied in the framework of this theory and some generalizations of the model have been proposed, as the case of $f(R)$ theories. One of this modifications has been developed in this thesis.

1.3 Discriminating among theories

In the previous section, several aspects were discussed of some of the theories that, nowadays, are trying to give a convincing explanation of the current accelerated expansion of the Universe. In addition to the theories discussed, there exist a large number of models created with the same purpose. Even within the framework of each theory, there are plenty of possibilities which finally result in different models, as is the case of $f(R)$ modified gravity theories, which may be very different depending on the specific choice made for the function f . At this point, because of this large population of theories, it becomes quite clear that it is necessary to have as many observational data as possible in order to discriminate between valid and non–valid theories.

In this sense, the scale factor, $a(t)$, is one of the most important magnitudes in cosmology, because it describes the expansion of the universe throughout time; hence, the measure of the scale factor or some related magnitude is a priority for cosmologists. In this case, the magnitude measure is the so–called Hubble function, as a function of the redshift, $H(z)$; this function is related to the scale function through the following expression: $H(z) \equiv \frac{d}{dt}\ln a$ con $1+z \equiv a(t_{now})/a(t)$. Then, a necessary condition for a theory to be consider as valid is to be able to reproduce the cosmic history obtained from the observational data; this means that the valid theories should begin with an inflation phase, then go through a radiation dominated epoch followed by a matter dominated period and, finally, reproduce the current phase of accelerated expansion of the Universe.

A big deal of gravitational theories may be discarded by comparing the cosmic history derived from the theories with the one obtained from the observational data; nevertheless, other theories are able to reproduce the observed cosmic history. Nowadays, the theory that best reproduces the expansion history is the ΛCDM model. Even so, a large number of models are able to obtain similar results for the cosmic history, fact that makes almost impossible to discriminate, using the expansion history, among a big deal of theories. Hence, physically different theories may reproduce the same global properties of expansion, which finally means that it is impossible to distinguish between these models, only using the cosmic expansion history.

At this point, an other important property of the gravitational theories becomes relevant: the growth history. Fortunately, the global characteristics of the Universe are not the only observable magnitudes. In this sense, it is very important the way in which the original state of the Universe evolved, such that the different structures that, nowadays, we can see in the Universe, as the galaxies, were created. Instead of focusing on the global properties of the expansion, as it is the case of the cosmic history, the most important aspect of the growth history is the study of the tiny perturbations and inhomogeneities which gave rise to the creation of structure in the Universe.

Thereby, by means of the knowledge of the expansion and the growth history it may be easier to discriminate among the different theories that populate the world of cosmology.

In the second part of the thesis, several aspects related to the cosmic and growth history are studied for some $f(R)$ modified gravity models.

1.4 The singularity problem of cosmological models: past and future singularities of different types

Another topic covered in this thesis is an important problem that face a large number of gravitational theories in their way to explain accelerated expansion. This problem is the appearance of finite-time future singularities. Among the theories that suffer this problem are the phantom models, some quintessence models, other models of modified gravity, etc. The importance of the singularity topic lies in the fact that it can produce instabilities in black holes and in stellar physics. Nevertheless, this problem can be understood and/or solved only from the perspective given by a quantum theory of gravity that, up to now, we do not have yet.

The singularities that may appear in these dark energy models can be classified into several types, depending on the divergent magnitude. In [240], the authors propose the following classification for the different types of singularities that may appear in cosmological models:

- Type I ("Big Rip") : For $t \to t_s$, $a \to \infty$, $\rho \to \infty$ and $|p| \to \infty$. This type of singularity is discussed in [20, 21, 23, 37, 53, 57, 84, 85, 88, 93, 94, 96, 108, 112, 119, 134, 136, 137, 140, 143, 181, 188, 195, 196, 210, 214, 216, 258, 265, 266, 284, 291, 297].

- Type II (“sudden”) : For $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$ and $|p| \to \infty$.
- Type III : For $t \to t_s$, $a \to a_s$, $\rho \to \infty$ and $|p| \to \infty$.
- Type IV : For $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$, $|p| \to p_s$ and higher derivatives of H diverge.

Here t_s , a_s , ρ_s y p_s are constants, with $a_s \neq 0$. While a , ρ y p are, respectively, the scale factor, the energy density and the pressure.

An important aspect to remark is that, even if we do not have a quantum theory of gravity that allows us to better understand the singularity problem, it is possible to try to solve the problem using a semiclassical approach. In this sense, terms of the type $\Box R$ have been shown to work against the appearance of future singularities; hence, the addition of a term proportional to R^2 in the action may prevent the existence of these singularities. In Chapter 7, the singularity problem for several dark energy models is discussed and a cure is proposed.

1.5 Organization of the thesis

Several aspects of modified gravities and other gravitational theories were studied during the last four years in order to give rise to this Thesis. As the last and most accurate observations seem to point that the Universe is homogeneous and isotropic, throughout this work I use the FLRW metric; when this is not the case, it will be specified. The Thesis is composed of three main blocks.

The first block, consisting of Chapters 2–4, is devoted to the reconstruction of the cosmic evolution for different modified gravities. As it will be demonstrated throughout the Thesis, one of the most important magnitudes for cosmologists is the scale factor, which relates the proper distance between a pair of objects, moving with the Hubble flow in a FLRW universe, at an arbitrary time, to their distance at some reference time. The knowledge of the scale factor is of great importance because it provides the cosmic history of the Universe. This is the reason why it is mandatory for a realistic gravitational theory to provide a scale factor that matches the observational data and, in this sense, the reconstruction schemes developed for modified gravities play an important role. It will be shown that any given cosmology can be realized through some corresponding modified gravity by means of the use of a convenient reconstruction model. In Chapter 2, two different reconstruction schemes are used in order to reproduce some cosmology in the framework of $f(R)$–modified gravities and some examples are given. It is also shown that these examples support the idea that the Einsteinian and Jordanian frame descriptions actually lead to two physically different theories. In Chapter 3, reconstruction schemes are used in order to reproduce a specific cosmology in the framework of some gravitational theories based on different couplings with the Yang–Mills field. The interesting cases of non–minimal and minimal coupling with the Yang–Mills field are developed and some examples are given. Finally, in Chapter 4, the case of cyclic universes is considered and the reconstruction schemes are used in order to reproduce cyclic universes in the framework of F(R) Hořava–Lifshitz gravity.

The second block, consisting of Chapters 5–6, is devoted to the study of the cosmic history and the growth of matter density perturbation for some $F(R)$–modified gravity models. As it was commented before, mimicking the known cosmic history (i. e. to reproduce the sequence of inflationary epoch, radiation domination epoch, matter domination epoch and late–time acceleration) is mandatory for any realistic gravitational theory. Thus, the study of the cosmic history is of great importance for every gravitational theory, but in the first block of the Thesis we realize that every cosmology can be reproduced through some modified gravity by means of the use of some convenient reconstruction scheme. Then it may be possible that the same cosmic history is reproduced from very different gravitational models, making extremely difficult to discriminate between different physical theories. It is in this framework where the study of the growth of the matter density perturbations can be very helpful. In order to understand the nature of these perturbations it is important to remark that the Universe was in thermal equilibrium before inflation started, as it is accepted nowadays by most cosmologists, but in this equilibrium there were quantum fluctuations, due to the uncertainty principle. These fluctuations, so–called matter density perturbations, would have been the origin of the galaxies in the Universe nowadays. This is why it is also very important to match the growth history given by some gravitational theory to the current observational data. In the first part of Chapter 5, the generic features occurring in the matter dominated era are shown for two well–known viable $F(R)$ gravity models and, by introducing an additive modification to these models, the large frequency oscillations of dark energy are stabilized. For these models, a study of the future evolution and the growth of matter density perturbations is carried out. In the second part of Chapter 5, a study of $F(R)$ modified gravities is done for the unification of the early–time cosmic acceleration (inflation) and the late–time one. Finally, in Chapter 6, a detailed analysis of the growth of the matter density perturbations is performed for two other realistic $F(R)$ modified–gravity models.

The third block is devoted to two very important aspects of the gravitational theories: the singularity problem and the weak field limit. The great importance of the appearance of future singularities at finite time lies in the fact that they may cause various problems of physical nature, as instabilities in current black hole and stellar astrophysics. Even if this problem can be only understood properly with a full theory of quantum gravity (which does not exist up to now), it is very important to try to fix the problem already at the classical and semiclassical levels. In this way, it is known that the addition of a term proportional to R^2 in the action may cure these singularities. In Chapter 7, the singularity problem is studied for different gravitational theories, as $f(R)$ modified gravity, non–local gravity or scalar–tensor gravity. After this analysis of the singularity problem, the weak field limit of gravitational theories is treated. This analysis is of extreme importance, because any theory must reproduce the results obtained by General Relativity at short scales and low energy regimes, for example at the solar system level. In this sense, the analysis of the weak field limit of a gravitational theory can be used in order to retain it or, on the contrary, to ruled it out. In Chapter 8, the weak field limit of $f(R,\mathcal{G})$ is studied, in particular, the Newtonian, post–Newtonian and post–post–Newtonian cases.

The Thesis ends with general conclusions corresponding to the most important results obtained in the present work.

Part I

Reconstruction schemes

Chapter 2

Cosmological reconstruction of realistic modified $f(R)$ gravities

One of the most important problems of modern cosmology (and theoretical physics too) is the explanation of the current universe speed-up, first discovered in [249, 254]. A convenient way to express this situation is to introduce a new form of energy, called Dark Energy (DE). Indeed, it is not easy to generate the necessary amount of repulsive force (not less the very fact that it is repulsive) through quantum vacuum fluctuation contributions of ordinary fields, at cosmological scale [103]. To start with, one needs to find an acceptable solution to the cosmological constant problem. During the last few years several theories, which are extensions of or alternatives to Einstenian Gravity, have been developed in order to formulate and try to explain the dark energy universe. The most accurate observational data we now have indicate that the equation of state (EoS) parameter, ω, for DE is very close to -1 (for a review of observational data from the theoretical point of view, see e.g. [153, 211], and for a description of the observable cosmological parameters, see e.g. [190]).

As gravitational alternatives for DE, modified gravity theories have been formulated, calling for plausible late-time modification of General Relativity (GR). Many modified gravity models have been proposed in the literature (for a review, see [67, 182, 221, 232, 269]), starting from the very simple $1/R$ theory [59, 63, 77, 215] (that soon was declared as problematic) to more elaborated ones which, being still not fundamental, are already quite often inspired by string and M-theory considerations [217, 238].

For any such theory to be valid it is always strictly required that it accurately describes the known sequence of cosmological epochs, specifically it must fit very well an increasing number of more and more precise observational data [14, 15, 47, 51, 60, 61, 62, 71, 168, 177, 197, 198, 253, 268]. The aim of this chapter is to show that it is possible to reconstruct a cosmology given by its scale factor (or Hubble parameter) from a $f(R)$ modified gravity. Specifically, by making one further step in the direction of trying to build a truly realistic theory, a non-trivial variant of the accelerating cosmology reconstruction program is developed for $f(R)$ gravity (see [24] for related work).

The chapter is organized as follows. In Sect. I and Sect. II, two seemingly different reconstruction schemes, (I) in terms of e-foldings (for a general review, see [236]), and (II) by using an auxiliary scalar field (see, e.g. [27, 30, 34, 49, 69, 91, 97, 106, 114, 139, 222, 223, 224, 292]) are reviewed and then explicitly compared, what is done here for the first time. To illustrate the results, the example of a model with a transient phantom behavior without real matter is discussed in both schemes. In Sect. III, a summary of the results obtained and, in particular, of the detailed comparison with pros and cons of the two schemes of reconstruction is presented.

This Chapter is based on the publications: [104, 185].

2.1 General formulation in terms of e-foldings

We will show in this section how one can construct an $f(R)$ model realizing a given cosmology, by using the techniques of [236]. For the benefit of the reader (and self-consistency of the paper), in a simpler situation, and with the help of an explicit example, we shall review the method that will be later applied to the Yang-Mills case. The starting action for $f(R)$ gravity (see e.g. [67, 182, 221, 232, 269], for a general review) is

$$S = \int d^4x\sqrt{-g}\left(\frac{f(R)}{2\kappa^2} + \mathcal{L}_{matter}\right). \tag{2.1}$$

The first FRW equation turns into the following field equation

$$0 = -\frac{f(R)}{2} + 3\left(H^2 + \dot{H}\right) f'(R) - 18\left(4H^2\dot{H} + H\ddot{H}\right) f''(R) + \kappa^2\rho, \tag{2.2}$$

with $R = 6\dot{H} + 12H^2$. Using the e-folding variable, $N = \ln\frac{a}{a_0}$, instead of the cosmological time t, one gets

$$0 = -\frac{f(R)}{2} + 3\left(H^2 + HH'\right) f'(R) - 18\left(4H^3H' + H^2(H')^2 + H^3H''\right) f''(R) + \kappa^2\rho, \tag{2.3}$$

where $H' \equiv \frac{dH}{dN}$. Assuming the matter density ρ is given in terms of a sum of fluid densities with constant EoS parameters, ω_i, we have

$$\rho = \sum_i \rho_{i0} a^{-3(1+\omega_i)} = \sum_i \rho_{i0} a_0^{-3(1+\omega_i)} e^{-3(1+\omega_i)N}. \tag{2.4}$$

Using the Hubble rate $H = g(N) = g(-\ln(1+z))$, with $z = e^{-N} - 1$ the redshift, the scalar curvature takes the form: $R = 6g'(N)g(N) + 12g(N)^2$, which can be solved with respect to N as $N = N(R)$. Defining $G(N) \equiv g(N)^2 = H^2$ and using (2.4), Eq. (2.3) yields

$$0 = -9G(N(R))\left[4G'(N(R)) + G''(N(R))\right]\frac{d^2f(R)}{dR^2} + \left[3G(N(R)) + \frac{3}{2}G'(N(R))\right]\frac{df(R)}{dR} - \frac{f(R)}{2} + \\ + \sum_i \rho_{i0} a_0^{-3(1+\omega_i)} e^{-3(1+\omega_i)N(R)}. \tag{2.5}$$

This is a differential equation for $f(R)$, where the scalar curvature is here $R = 3G'(N) + 12G(N)$.

Example: asymptotically transient phantom behavior

Let us consider an evolution given by the following Hubble parameter:

$$H^2(N) = H_0 \ln\left(\frac{a}{a_0}\right) + H_1 = H_0 N + H_1 = G(N), \tag{2.6}$$

where H_0 and H_1 are positive constants. We can, in this case, achieve a phantom behavior with the possibility to be asymptotically transient, without the presence of real matter. The present is actually a simplified example, but can be of use as a component part of a more elaborated model where, with a modified functionality, the transition can be reached at finite time (this is work in progress, the results of which will be presented in a future publication.) Indeed, from $R = 3G'(N) + 12G(N)$, we find

$$N = \frac{R - 3H_0}{12H_0} - \frac{H_1}{H_0}. \tag{2.7}$$

Eq. (2.5) takes the form (with r being the curvature measured in terms of H_0, $r \equiv R/H_0$)

$$0 = -3(r-3)\frac{d^2 f(r)}{dr^2} + \left(\frac{r+3}{4}\right)\frac{df(r)}{dr} - \frac{f(r)}{2}, \tag{2.8}$$

and changing now the variable from r to x, as $x = \frac{r-3}{12}$, Eq. (2.8) reduces to

$$0 = x\frac{d^2 F(x)}{dx^2} - \left(x + \frac{1}{2}\right)\frac{dF(x)}{dx} + 2F(x), \tag{2.9}$$

which is easily recognized as a degenerate hypergeometric equation, whose solutions are given by the Kummer's series $\Phi(a, b; z)$, the simplest one being

$$f(r) = C\,\Phi\left(-2, -\frac{1}{2}; \frac{r-3}{12}\right) = C_1\left(-\frac{1}{2} + r - \frac{r^2}{18}\right), \tag{2.10}$$

where C is a constant. As a consequence, with *this* $f(R)$ theory, the solution given by Eq. (2.10), we can reproduce the phantom behavior without real matter given by Eq. (2.6).

Taking this into account, for (2.6), we have

$$H(t) = \frac{H_0}{2}(t - t_0), \tag{2.11}$$

and it turns out that, with this model, we have a contribution of an effective cosmological constant and another term which will produce an accelerating phase but, remarkably, without developing a future singularity, in spite of its phantom nature. Hence, the $f(R)$ gravity given by Eq. (2.10) gives rise to a cosmological solution, with an asymptotically transient phantom behavior, which does not evolve into a future singularity. This property relies on the fact that the phantom behavior gets more and more mild with time (asymptotically disappears), at a rate that overcomes the one for the formation of the singularity.

Actually, there is another independent solution of Kummer's equation (2.9), the complete solution being:

$$f(r) = C_1\left(-\frac{1}{2} + r - \frac{r^2}{18}\right) + C_2\left(\frac{r-3}{12}\right)^{3/2} L_{1/2}^{(3/2)}\left(\frac{r-3}{12}\right), \tag{2.12}$$

where the second basic solution is a Laguerre L function, which is well behaved but cannot be represented as a simple rational one. It is interesting to note that this second function asymptotically behaves exactly in the same way as the first, for large negative curvature (e.g., as R^2, when $R \to -\infty$). For large positive one it explodes exponentially, as $R^{-3/2}\, e^{R/12}$ (again, R in units of H_0).

2.2 General formulation using a scalar field

In this section it will be shown how to construct an $f(R)$ model realizing a given cosmology, but using this time a different technique, which involves a scalar field [222]. The final aim will be to apply this procedure to the novel case with a Yang-Mills term, what will be performed in Sect. III. Here we summarize the basic tools necessary in order to understand the procedure and to make the present paper self-contained. We start from the action for modified gravity

$$S = \int d^4x\sqrt{-g}(f(R) + \mathcal{L}_{matter}), \tag{2.13}$$

which is equivalent to

$$S = \int d^4x\sqrt{-g}(P(\phi)R + Q(\phi) + \mathcal{L}_{matter}). \tag{2.14}$$

Here, $\mathcal{L}_{matter}$ is the matter Lagrangian density and P and Q are proper functions of the scalar field, ϕ, which can be regarded as an auxiliary field, because there is no kinetic term depending on ϕ in the Lagrangian. By varying the action with respect to ϕ, $0 = P'(\phi)R + Q'(\phi)$, which can be solved in terms of ϕ, as $\phi = \phi(R)$. Substituting it into (2.14) and comparing with (2.13), one obtains

$$S = \int d^4x\sqrt{-g}(f(R) + \mathcal{L}_{matter}), \qquad f(R) \equiv P(\phi(R))R + Q(\phi(R)), \tag{2.15}$$

and varying the action with respect to the metric $g_{\mu\nu}$,

$$0 = -\frac{1}{2}g_{\mu\nu}(P(\phi)R + Q(\phi)) + R_{\mu\nu}P(\phi) + g_{\mu\nu}\nabla^2 P(\phi) - \nabla_\mu\nabla_\nu P(\phi) - \frac{1}{2}T_{\mu\nu}. \tag{2.16}$$

The equations corresponding to the standard, spatially-flat FRW universe are

$$0 = -Q(\phi) - 6H^2 P(\phi) - 6H\frac{dP(\phi)}{dt} + \rho, \tag{2.17}$$

$$0 = Q(\phi) + \left(4\dot{H} + 6H^2\right) P(\phi) + 4H\frac{dP(\phi)}{dt} + 2\frac{d^2P(\phi)}{dt^2} + p, \tag{2.18}$$

and, by combining them, we find

$$0 = 2\frac{d^2P(\phi(t))}{dt^2} - 2H\frac{dP(\phi(t))}{dt} + 4\dot{H}P(\phi(t)) + p + \rho. \tag{2.19}$$

As we are allowed to redefine the scalar field ϕ properly, we choose the most simple, non-constant, smooth possibility (what is commonly done in this kind of approaches), namely $\phi = t$.

Now, given a cosmology, specified through the scale factor a, given by a proper function $g(t)$ as

$$a = a_0 e^{g(t)}, \tag{2.20}$$

with a constant a_0, and if it is assumed that p and ρ consist of the sum of different matter contributions, each one with constant EoS parameter, ω_i, then Eq. (2.19) reduces to the following second order differential equation

$$0 = 2\frac{d^2P(\phi)}{d\phi^2} - 2g'(\phi)\frac{dP(\phi)}{d\phi} + 4g''(\phi)P(\phi) + \sum_i (1+\omega_i)\rho_{i0}a_0^{-3(1+\omega_i)}e^{-3(1+\omega_i)g(\phi)}, \tag{2.21}$$

from where one can obtain $P(\phi)$ and, using Eq. (2.17),

$$Q(\phi) = -6(g'(\phi))^2 P(\phi) - 6g'(\phi)\frac{dP(\phi)}{d\phi} + \sum_i \rho_{i0}a_0^{-3(1+\omega_i)}e^{-3(1+\omega_i)g(\phi)}. \tag{2.22}$$

As a result, and as anticipated, any given cosmology (2.20) can indeed be realized through some corresponding $f(R)$-gravity. Let us make things even more clear by means of the example considered before.

Example: asymptotically transient phantom behavior

In order to compare the two different methods developed for the reconstruction of $f(R)$ gravities—to reproduce any given cosmology—we consider again the asymptotically transient phantom behavior, without real matter, given by (2.6). The Hubble parameter can be written as

$$H = \sqrt{H_0 g(t) + H_1} = \frac{dg(t)}{dt}, \tag{2.23}$$

and thus

$$g(t) = \frac{H_0}{4}(t-c)^2 - \frac{H_1}{H_0}, \tag{2.24}$$

with c an integration constant. Introducing (2.24) into (2.21),

$$0 = \frac{d^2P(\phi)}{d\phi^2} - \frac{H_0}{2}(\phi - c)\frac{dP(\phi)}{d\phi} + H_0 P(\phi), \tag{2.25}$$

and using a new variable, $x = \phi - c$, we get

$$0 = \frac{d^2P(x)}{dx^2} - \frac{H_0}{2}x\frac{dP(x)}{dx} + H_0 P(x), \tag{2.26}$$

whose solution is

$$P(x) = \frac{1}{2}\left(2 - H_0 x^2\right) C_1 + \frac{1}{2}\left(2 - H_0 x^2\right) C_2 \left(\frac{e^{\frac{H_0}{4}x^2} x}{4(2 - H_0 x^2)} - \frac{i}{4\sqrt{H_0}} \int\limits_0^{i\sqrt{\frac{H_0 x^2}{4}}} e^{-y^2} dy \right). \tag{2.27}$$

Now, using (2.17), we obtain

$$Q(x) = \frac{3}{32} H_0 x \left[8H_0 x \left(2 + H_0 x^2\right) C_1 - \left(\left(8 + 2H_0 x^2\right) e^{\frac{H_0 x^2}{4}} + i\, 2\sqrt{H_0}x \left(2 + H_0 x^2\right) \int\limits_0^{i\sqrt{\frac{H_0 x^2}{4}}} e^{-y^2} dy \right) C_2 \right]. \tag{2.28}$$

Taking into account that $R = 6\dot{H} + 12H^2 = 6g''(x) + 12\left(g'(x)\right)^2$, it follows that

$$x = \sqrt{\frac{R - 3H_0}{3H_0^2}}. \tag{2.29}$$

Introducing at this step (2.27) and (2.28) into (2.15), and considering (2.29), one finally gets the explicit expression

$$f(R) = -\frac{R^2 - 18H_0 R + 9H_0^2}{12H_0} C_1 - \left[e^{\frac{R-3H_0}{12H_0}} \frac{R - 9H_0}{16} - \frac{i}{24H_0}\left(R^2 - 18H_0 R + 9H_0^2\right) \int\limits_0^{i\sqrt{\frac{R-3H_0}{12H_0}}} e^{-y^2} dy \right] C_2. \tag{2.30}$$

We thus have proven that, within this scheme, we are able to obtain the $f(R)$ model (2.30) which reproduces the desired transient phantom behavior without real matter, as given by (2.6).

2.3 Summary and discussion

In this chapter, two different schemes of reconstructing cosmologies for modified gravity are reviewed and compared (for the first time), and also with the help of corresponding examples. The first scheme does not need an auxiliary scalar field, while the second one is thoroughly based on its use. With these reconstruction methods, any explicitly given cosmology can be realized as a corresponding modified gravity. As it is indicated in the chapter, things are far from straightforward, and a very careful analysis of the solutions here obtained (also in relation with the comparison of the two different methods), and of other additional solutions of the differential equations, with potential physical interest, is still necessary. First results indicate that the solutions obtained do pass the solar system tests and the other known physical constraints.

In order to compare both schemes of reconstruction, an example has been explicitly worked out in the two cases. The result obtained in the first scheme (in terms of e-foldings) is

$$f(R) = C\,\Phi\left(-2, -\frac{1}{2}; \frac{R-3H_0}{12H_0}\right) = C\left(-\frac{1}{4} + \frac{1}{2H_0}R - \frac{1}{36H_0^2}R^2\right), \tag{2.31}$$

while for the second scheme (using an auxiliary scalar field), the $f(R)$ obtained has the form

$$f(R) = -\frac{R^2 - 18H_0R + 9H_0^2}{12H_0}\,C_1 - \left[e^{\frac{R-3H_0}{12H_0}}\,\frac{R-9H_0}{16} - \frac{i}{24H_0}\left(R^2 - 18H_0R + 9H_0^2\right)\int_0^{i\sqrt{\frac{R-3H_0}{12H_0}}} e^{-y^2}dy\right] C_2. \tag{2.32}$$

As one can easily see, the results obtained for both methods are in fact different. The reason behind this is the fact that action (2.14) corresponds to a wider class of theories than action (2.13) (for a related and quite detailed discussion, see [239, 252]). Nevertheless, if in Eq. (2.32) we set $C_2 = 0$, then the results coming from both schemes are similar, at least in the sense that, for low curvatures, they behave as constant, while for large curvatures the behavior is in both cases proportional to R^2.

This finding here further supports the point of view that the Einsteinian and the Jordanian frame descriptions actually lead to two physically different theories, making thus clear the physical non-equivalence of the two frames as discussed in [114, 222]. In view of the strong and still on going discussion about this issue in the specialized literature, this additional piece of evidence is very valuable.

Also important is to remark that, sometimes, it is actually more convenient to use one scheme instead of the other, because the final result may be definitely easier to obtain and to interpret in one of the two schemes. To repeat, although these conclusions may not seem really new, since they were already derived in more simplified situations, it will be shown in the next chapter that they continue to be valid in much more realistic situations, from the point of view of physics, as corresponding to the actions there considered, involving Yang-Mills fields.

Chapter 3

Reconstruction of the Yang–Mills theory

Non-Abelian gauge fields are widely used in particle physics and are being actively studied in cosmology [33, 36, 38, 40, 78, 95, 104, 126, 127, 191, 192]. Note that string compactifications may naturally lead to an effective-scalar–Yang–Mills–Einstein theory (plus higher-order corrections). Inflationary cosmology and the late-time accelerated expansion of the Universe in a non-minimal, non-Abelian gauge theory (the Yang–Mills theory), in which a non-Abelian gauge (Yang–Mills) field plays a significant role, has been considered in [33], where the authors show that the appearance of such non-minimal terms in the early Universe can be compatible with current formulations of the Yang–Mills theory coming from a specific choice for the non-minimal function. Also in [33] the cosmological reconstruction of the Yang–Mills theory has been discussed and a corresponding algorithm has been proposed.

A remarkable fact is that the SU(2) Yang–Mills field admits an isotropic and homogeneous parametrization by a single scalar function. This parametrization, which is useful for the reconstruction program [33], has been employed to get an inflationary scenario [191, 192]. As noted in [191, 192], the standard Yang–Mills term, minimally coupling with gravity, does not lead to inflation, and thus one should add new terms in order to get a convenient inflationary scenario. In [191, 192] an inflationary scenario, in which slow-roll inflation is driven by a non-Abelian gauge field minimally coupled to gravity has been proposed. To achieve this, the authors add a fourth-degree term to the Yang–Mills Lagrangian.

In the previous chapter, the importance of the reconstruction schemes for $f(R)$ modified gravities has been revealed and a cosmology given by its scale factor has been reproduced from $f(R)$ gravity by means of two different reconstruction schemes. In this chapter, the reconstruction program is successfully extended to several cosmological models with Yang–Mills fields. It will be demonstrated that the case of a Yang–Mills Lagrangian with a fourth-degree term minimally coupled with gravitation has no de Sitter solution. As is well known, de Sitter solutions play a very important role in cosmological models, because both inflation and the late-time universe acceleration can be described as a de Sitter solution with perturbations. In order to obtain these solutions, a gravitational model with non-minimally coupled Yang–Mills fields will be considered.

The chapter is organized as follows. In Sect. I, reconstruction schemes for the case of a Yang–Mills theory (in a extended, non-minimal version) are developed, one of them by means of an auxiliary scalar field and the other one without it. The specific example of power-law expansion is carefully considered in both schemes. In Sect. II, the reconstruction program is generalized for the case of a Yang–Mills Lagrangian with a fourth-degree term. In Sect. III, a gravitational model with nonminimally coupled Yang–Mills fields is considered. Finally, in Sect. IV the most important results are summarized and discussed.

This Chapter is based on the publications: [104, 105].

3.1 Minimal gravitational coupling with the Yang–Mills field

3.1.1 General formulation

In this section we develop a reconstruction scheme of the Yang–Mills theory without any auxiliary scalar field. Consider the following action

$$S = \int dx^4 \sqrt{-g} \left[\frac{R}{2\kappa^2} + \mathcal{F}\left(F^a_{\mu\nu} F^{a\,\mu\nu} \right) \right], \tag{3.1}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$, and $\mathcal{F}$ may be assumed to be a smooth function (however, strictly speaking it may suffice if it is continuously differentiable). The presence of this function is necessary in order to allow for more freedom in the choice of the theory [27, 30, 34] (since difficulties inherent to the problem may prevent obtaining the standard Yang–Mills case). Anyway, this function will not constitute a problem for the development of our methods, which are thus proven to be even more powerful. For simplicity of the derivation (and in order not to break the line of argument of this paper) we concentrate here on the $SU(2)$ case where $f^{abc} = \epsilon^{abc}$ but, with some more effort, exactly the same procedure can be extended to other gauge groups (more general cases will be treated in a subsequent publication). Taking into account that

$$\frac{\delta \left(F^a_{\mu\nu} F^{a\,\mu\nu} \right)}{\delta A^h_\beta} = -4\epsilon^{hbc} A^b_\gamma F^{c\,\gamma\beta}, \tag{3.2}$$

$$\frac{\delta \left(F^a_{\mu\nu} F^{a\,\mu\nu} \right)}{\delta \left(\partial_\alpha A^h_\beta \right)} = 4F^{h\,\alpha\beta}, \tag{3.3}$$

the equation of motion for the field potential A^a_μ turns into

$$\partial_\nu \left[\frac{\delta S}{\delta \left(\partial_\nu A^a_\mu \right)} \right] - \frac{\delta S}{\delta A^a_\mu} = 0 \tag{3.4}$$

and, from here,

$$\partial_\nu \left[\sqrt{-g}\, \mathcal{F}' \left(F^a_{\alpha\beta} F^{a\,\alpha\beta} \right) F^{a\,\nu\mu} \right] + \sqrt{-g}\, \mathcal{F}' \left(F^a_{\alpha\beta} F^{a\,\alpha\beta} \right) \epsilon^{abc} A^b_\nu F^{c\,\nu\mu} = 0. \tag{3.5}$$

Variation of (3.1) with respect to $g^{\mu\nu}$ yields the following equation of motion

$$\frac{1}{2\kappa^2} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) - \frac{1}{2} g_{\mu\nu} \mathcal{F} \left(F^a_{\alpha\beta} F^{a\,\alpha\beta} \right) + 2\mathcal{F}' \left(F^a_{\alpha\beta} F^{a\,\alpha\beta} \right) F^a_{\mu\rho} F^{a\,\rho}_\nu = 0, \tag{3.6}$$

where we have used $\frac{\delta \left(F^a_{\rho\sigma} F^{a\,\rho\sigma} \right)}{\delta g^{\mu\nu}} = 2\mathcal{F}' \left(F^a_{\alpha\beta} F^{a\,\alpha\beta} \right) F^a_{\mu\gamma} F^{a\,\gamma}_\nu$. Considering now a FRW universe, and the following Ansatz for the gauge field [1],

$$A^a_\mu = \begin{cases} \bar{\alpha} e^{\lambda(t)} \delta^a_\mu, & \mu = i, \\ 0, & \mu = 0, \end{cases} \tag{3.7}$$

[1] Note that our aim here is to demonstrate that the procedure works, and not to find the most general solution which exhausts all possibilities. This issue, which is certainly interesting and more difficult, will be left to further consideration.

the $\mu = 0$ component of (3.4) becomes an identity, the $\mu = i$ component yields

$$\partial_t \left[a(t)\, \mathcal{F}'\left(F^a_{\alpha\beta} F^{a\,\alpha\beta}\right)\, \dot{\lambda}(t)\, e^{\lambda(t)} \right] + \frac{2\bar{\alpha}^2}{a(t)}\, \mathcal{F}'\left(F^a_{\alpha\beta} F^{a\,\alpha\beta}\right)\, e^{3\lambda(t)} = 0, \tag{3.8}$$

while the (t,t) component of (3.6) is

$$\frac{3H^2(t)}{2\kappa^2} + \frac{1}{2}\mathcal{F}\left(F^a_{\alpha\beta} F^{a\,\alpha\beta}\right) + 6\bar{\alpha}^2\, \mathcal{F}'\left(F^a_{\alpha\beta} F^{a\,\alpha\beta}\right)\, \frac{\dot{\lambda}^2(t) e^{2\lambda(t)}}{a^2(t)} = 0, \tag{3.9}$$

and the (i,i) component of (3.6) reduces to

$$-\frac{1}{2\kappa^2}\left[2\dot{H}(t) + 3H^2(t)\right] - \frac{1}{2}\mathcal{F}\left(F^a_{\alpha\beta} F^{a\,\alpha\beta}\right) - 2\bar{\alpha}^2 \frac{e^{2\lambda(t)}}{a^2(t)} \mathcal{F}'\left(F^a_{\alpha\beta} F^{a\,\alpha\beta}\right)\left[\dot{\lambda}^2(t) - 2\bar{\alpha}^2 \frac{e^{2\lambda(t)}}{a^2(t)}\right] = 0. \tag{3.10}$$

Adding (3.9) to (3.10), one arrives at

$$a^2(t)\dot{H}(t) - 4\kappa^2\bar{\alpha}^2\, \mathcal{F}'\left(F^a_{\alpha\beta} F^{a\,\alpha\beta}\right)\, e^{2\lambda(t)} \left[\dot{\lambda}^2(t) + \bar{\alpha}^2 \frac{e^{2\lambda(t)}}{a^2(t)}\right] = 0, \tag{3.11}$$

and then

$$\mathcal{F}'\left(F^a_{\alpha\beta} F^{a\,\alpha\beta}\right) = \frac{a^2(t)\, \dot{H}(t)}{4\kappa^2\bar{\alpha}^2} e^{-2\lambda(t)} \left[\dot{\lambda}^2(t) + \bar{\alpha}^2 \frac{e^{2\lambda(t)}}{a^2(t)}\right]^{-1}. \tag{3.12}$$

Using (3.12), Eq. (3.8) reduces to:

$$\partial_t \left[a^3(t)\, \dot{H}(t)\, \dot{\lambda}(t)\, e^{-\lambda(t)} \left(\dot{\lambda}^2(t) + \bar{\alpha}^2 \frac{e^{2\lambda(t)}}{a^2(t)}\right)^{-1} \right] + 2\bar{\alpha}^2\, a(t)\, \dot{H}(t)\, e^{\lambda(t)} \left(\dot{\lambda}^2(t) + \bar{\alpha}^2 \frac{e^{2\lambda(t)}}{a^2(t)}\right)^{-1} = 0, \tag{3.13}$$

which constitutes a differential equation for $\lambda(t)$. Hence, by using Eq. (3.7), once we have the function $\lambda(t)$, given by (3.13), we can obtain the corresponding Yang–Mills theory which reproduces the selected cosmology. The Ansatz considered above actually leads to a mathematical solution of the problem.

Example: power law expansion

Considering the case of power law expansion: $a(t) = \left(\frac{t}{t_1}\right)^{h_1}$, where t_1 and h_1 are constant, and assuming $\lambda(t) = (h_1 - 1)\ln\left(\frac{t}{t_1}\right) + \lambda_1$, where λ_1 is again a constant, Eq. (3.13) reduces to the following algebraic equation

$$h_1(h_1 - 1) + \bar{\alpha}^2 t_1^2\, e^{2\lambda_1} = 0, \tag{3.14}$$

hence

$$\lambda_1 = \frac{1}{2}\ln\left(\frac{h_1(1-h_1)}{\bar{\alpha}^2 t_1^2}\right) \tag{3.15}$$

and

$$\lambda(t) = (h_1 - 1)\ln\left(\frac{t}{t_1}\right) + \frac{1}{2}\ln\left(\frac{h_1(1-h_1)}{\bar{\alpha}^2 t_1^2}\right). \tag{3.16}$$

With the help of this reconstruction scheme, we have obtained the function $\lambda(t)$ given by (3.16). Then, using (3.7), we are able to reproduce the cosmology given by the power law expansion: $a(t) = \left(\frac{t}{t_1}\right)^{h_1}$.

3.1.2 General formulation using a scalar field

In this section, the reconstruction of a Yang–Mills theory is performed using the technique of [33]. By way of introducing an auxiliary scalar field, ϕ, we can rewrite action (3.1) as

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa^2} + \frac{1}{4}P(\phi)F^a_{\mu\nu}F^{a\,\mu\nu} + \frac{1}{4}Q(\phi)\right). \tag{3.17}$$

Variation of (3.17) with respect to ϕ yields the corresponding equation of motion

$$0 = F^a_{\mu\nu}F^{a\,\mu\nu}\frac{dP(\phi)}{d\phi} + \frac{dQ(\phi)}{d\phi}, \tag{3.18}$$

which can be solved with respect to ϕ as $\phi = \phi(F^a_{\mu\nu}F^{a\,\mu\nu})$. Then, action (3.17) is rewritten as

$$S = \int dx^4\sqrt{-g}\left[\frac{R}{2\kappa^2} + \mathcal{F}\left(F^a_{\mu\nu}F^{a\,\mu\nu}\right)\right], \tag{3.19}$$

where

$$\mathcal{F}\left(F^a_{\mu\nu}F^{a\,\mu\nu}\right) = \frac{1}{4}P(\phi(F^a_{\mu\nu}F^{a\,\mu\nu}))F^a_{\mu\nu}F^{a\,\mu\nu} + \frac{1}{4}Q(\phi(F^a_{\mu\nu}F^{a\,\mu\nu})). \tag{3.20}$$

Taking the variations of this action (3.17) with respect to $g_{\mu\nu}$, we obtain the Einstein equation

$$\frac{1}{2\kappa^2}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = -\frac{1}{2}P(\phi)F^a_{\mu\rho}F^{a\,\rho}_{\nu} + \frac{1}{8}g_{\mu\nu}\left(P(\phi)F^a_{\alpha\beta}F^{a\,\alpha\beta} + Q(\phi)\right). \tag{3.21}$$

Finally, taking the variations of (3.17) with respect to A^a_μ, it follows that

$$0 = \partial_\nu\left(\sqrt{-g}P(\phi)F^{a\,\nu\mu}\right) + \sqrt{-g}P(\phi)f^{abc}A^b_\nu F^{c\,\nu\mu}. \tag{3.22}$$

We restrict our analysis to the case where the gauge algebra is $SU(2)$ and the gauge fields are given by

$$A^a_\mu = \begin{cases} \bar{\alpha}e^{\lambda(t)}\delta^a_\mu, & \mu = i, \\ 0, & \mu = 0. \end{cases} \tag{3.23}$$

With these assumptions, Eq. (3.18) reduces to

$$0 = 6\left(-\bar{\alpha}^2\dot{\lambda}(t)^2e^{2\lambda(t)}a(t)^{-2} + \bar{\alpha}^4e^{4\lambda(t)}a(t)^{-4}\right)\frac{dP(\phi)}{d\phi} + \frac{dQ(\phi)}{d\phi}. \tag{3.24}$$

The (t,t) component of (3.21) is

$$0 = \frac{3H(t)^2}{\kappa^2} + \frac{3}{2}\left(\bar{\alpha}^2\dot{\lambda}(t)^2e^{2\lambda(t)}a(t)^{-2} + \bar{\alpha}^4e^{4\lambda(t)}a(t)^{-4}\right)P(\phi) + \frac{1}{4}Q(\phi), \tag{3.25}$$

the (t,i) component of (3.21) becomes an identity, while the component (i,j) is

$$\left[-\frac{1}{2\kappa^2}\left(2\dot{H}(t) + 3H(t)^2\right)\right]\delta_{ij} = \left[-\frac{1}{4}P(\phi)\left(\bar{\alpha}^2\dot{\lambda}(t)^2e^{2\lambda(t)}a(t)^{-2} + \bar{\alpha}^4e^{4\lambda(t)}a(t)^{-4}\right) + \frac{1}{8}Q(\phi)\right]\delta_{ij}. \tag{3.26}$$

The $\mu = 0$ component of (3.22) becomes an identity, and the $\mu = i$ component yields

$$0 = \partial_t\left(a(t)P(\phi)\dot{\lambda}(t)e^{\lambda(t)}\right) + 2\bar{\alpha}^2a(t)^{-1}P(\phi)e^{3\lambda(t)}. \tag{3.27}$$

Here, we can identify $\phi = t$, because we are always allowed to take the scalar field ϕ properly in order to satisfy this. By differentiating (3.25) with respect to t and eliminating $\dot{Q} = \frac{dQ(\phi)}{d\phi}$, it follows that

$$0 = \frac{2}{\kappa^2} H(t)\dot{H}(t) + \bar{\alpha}^2 \dot{\lambda}(t)^2 e^{2\lambda(t)} a(t)^{-2} \dot{P}(t) +$$

$$+ \left[\bar{\alpha}^2 \left(\dot{\lambda}(t)\ddot{\lambda}(t) + \dot{\lambda}(t)^3 - \dot{\lambda}(t)^2 H(t) \right) e^{2\lambda(t)} a(t)^{-2} + 2\bar{\alpha}^4 \left(\dot{\lambda}(t) - H(t) \right) e^{4\lambda(t)} a(t)^{-4} \right] P(t). \tag{3.28}$$

Using (3.27), we can solve for $\dot{P}$ in (3.28), and obtain

$$P = \frac{a(t)^2 \dot{H}(t)}{\kappa^2 \bar{\alpha}^2 e^{2\lambda(t)} \left[\dot{\lambda}(t)^2 + \bar{\alpha}^2 e^{2\lambda(t)} a(t)^{-2} \right]}. \tag{3.29}$$

Taking into account (3.29), Eq. (3.27) reduces to

$$0 = 2\dot{H}(t) \left(\bar{\alpha}^2 e^{2\lambda(t)} a(t)^{-2} \right)^2 + \bar{\alpha}^2 e^{2\lambda(t)} a(t)^{-2} \left[\dot{\lambda}(t) \left(5H(t)\dot{H}(t) + \ddot{H}(t) \right) - \dot{\lambda}(t)^2 \dot{H}(t) + \ddot{\lambda}(t)\dot{H}(t) \right] +$$

$$+ \dot{\lambda}(t)^2 \left[\dot{\lambda}(t) \left(3H(t)\dot{H}(t) + \ddot{H}(t) \right) - \dot{\lambda}(t)^2 \dot{H}(t) - \ddot{\lambda}(t)\dot{H}(t) \right], \tag{3.30}$$

which constitutes a differential equation for $\lambda(t)$. As was the case for the other reconstruction scheme, developed in Sect. (3.1.1), once we have the function $\lambda(t)$—given here by (3.30)—we can readily obtain the modified Yang–Mills theory that reproduces the desired cosmology, through the use of Eq. (3.23), which was our starting Ansatz. Note that in (3.28) we have positively corrected some missprints of a previous calculation (recognized by the authors). Using Eq. (3.25) and (3.29), it is easy to check that Eq. (3.26) becomes an identity, thus being always fulfilled.

Example: power law expansion

Consider now the case of power law expansion: $a(t) = \left(\frac{t}{t_1}\right)^{h_1}$, where t_1 and h_1 are constants and assume $\lambda(t) = (h_1 - 1)\ln\left(\frac{t}{t_1}\right) + \lambda_1$, where λ_1 is a constant. Eq. (3.30) turns into the algebraic one

$$0 = \bar{\alpha}^4 t_1^4 e^{4\lambda_1} + \bar{\alpha}^2 t_1^2 e^{2\lambda_1} (h_1 - 1)(2h_1 - 1) + h_1 (h_1 - 1)^3, \tag{3.31}$$

hence

$$\lambda_1 = \begin{cases} \frac{1}{2} \ln\left(\frac{(h_1 - 1)(1 - h_1)}{\bar{\alpha}^2 t_1^2} \right), \\ \frac{1}{2} \ln\left(\frac{h_1 (1 - h_1)}{\bar{\alpha}^2 t_1^2} \right), \end{cases} \tag{3.32}$$

and then

$$\lambda(t) = (h_1 - 1)\ln\left(\frac{t}{t_1}\right) + \begin{cases} \frac{1}{2} \ln\left(\frac{(h_1 - 1)(1 - h_1)}{\bar{\alpha}^2 t_1^2} \right), \\ \frac{1}{2} \ln\left(\frac{h_1 (1 - h_1)}{\bar{\alpha}^2 t_1^2} \right). \end{cases} \tag{3.33}$$

With the function $\lambda(t)$ given by (3.33) and using (3.23), we can now reproduce the cosmology given by the power law expansion: $a(t) = \left(\frac{t}{t_1}\right)^{h_1}$.

3.2 Gravitational models with Yang–Mills fields

Let us consider a minimal gravitational coupling of the $SU(2)$ Yang–Mills field in the general theory of relativity, which is described by the following action:

$$S = \int d^4x\,\sqrt{-g}\,\left[\frac{M_P^2}{2}R + \mathcal{F}(Z) + \frac{\tilde{\kappa}}{384}(\epsilon^{\alpha\beta\lambda\sigma}F^a{}_{\alpha\beta}F^a{}_{\lambda\sigma})^2 - \Lambda\right], \tag{3.34}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, R is the scalar curvature, $M_P = M_{Pl}/\sqrt{8\pi}$, the Planck mass $M_{\rm Pl} = 1.2\times 10^{19}$ GeV, and $\tilde{\kappa}$ is a constant. The $SU(2)$ Yang–Mills field A^b_μ has the internal symmetry index a, the field strength tensor being

$$F^a_{\alpha\beta} = \partial_\alpha A^a_\beta - \partial_\beta A^a_\alpha + f^{abc}A^b_\alpha A^c_\beta. \tag{3.35}$$

The function $\mathcal{F}(Z)$ is an arbitrary function of $Z = F^a_{\mu\nu}F^{a\mu\nu}$ (summation in terms of the index a is understood), while the numbers f^{abc} are structure constants and thus completely antisymmetric. For the $SU(2)$ group,

$$f^{abc} = -\,\tilde{g}[abc], \tag{3.36}$$

where $\tilde{g}$ is a constant and $[abc]$ the Levi–Civita antisymmetric symbol (we use this notation instead of ε^{abc} because we reserve the last one for the Levi–Civita antisymmetric tensor). Roman indices, a, b, c, will run over $1, 2, 3$, and the Levi–Civita tensor is given by

$$\epsilon^{\alpha\beta\lambda\sigma} = \sqrt{-g}g^{\rho_1\alpha}g^{\rho_2\beta}g^{\rho_3\lambda}g^{\rho_4\sigma}[\rho_1\rho_2\rho_3\rho_4], \quad [0123] = 1. \tag{3.37}$$

Models of this kind, in the case $\tilde{\kappa} = 0$, have been considered in [104]. The case $\mathcal{F}(Z) = Z$ has been analysed in [191, 192] where an inflationary scenario, in which slow-roll inflation is driven by a non-Abelian gauge field minimally coupled to gravity, has been proposed.

The equation of motion for the field A^a_μ is

$$\begin{aligned}&\partial_\nu\left\{\sqrt{-g}\left[\mathcal{F}'(Z)\,F^{a\,\nu\mu} + \frac{\tilde{\kappa}}{192}J\varepsilon^{\nu\mu\alpha\beta}F^a_{\alpha\beta}\right]\right\} - \\ &-\tilde{g}\sqrt{-g}\,[abc]\,A^b_\nu\left\{\mathcal{F}'(Z)\,F^{c\,\nu\mu} - \frac{\tilde{\kappa}}{192}J\varepsilon^{\mu\nu\alpha\beta}F^c_{\alpha\beta}\right\} = 0,\end{aligned} \tag{3.38}$$

where $J \equiv \epsilon^{\alpha\beta\lambda\sigma}F^b{}_{\alpha\beta}F^b{}_{\lambda\sigma}$. Variation of (3.34) with respect to $g^{\mu\nu}$ yields the field equations:

$$\begin{aligned}&\frac{M_P^2}{2}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) - \frac{1}{2}g_{\mu\nu}\mathcal{F}(Z) + 2\mathcal{F}'(Z)\,F^a_{\mu\rho}F^{a\,\rho}_{\nu} - \\ &-\frac{\tilde{\kappa}}{384}\left\{\frac{3}{2}J^2 g_{\mu\nu} - 8J\sqrt{-g}g^{\rho_2\beta}g^{\rho_3\lambda}g^{\rho_4\sigma}[\mu\rho_2\rho_3\rho_4]\,F^b{}_{\nu\beta}F^b{}_{\lambda\sigma}\right\} + \frac{1}{2}g_{\mu\nu}\Lambda = 0.\end{aligned} \tag{3.39}$$

In the spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) universe

$$ds^2 = -\,dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right), \tag{3.40}$$

the following ansatz for the $SU(2)$ field:

$$A^b_\mu = \left(0, \phi(t)\delta^b_i\right). \tag{3.41}$$

is very useful [33, 104, 191, 192]. The ansatz identifies the combination of the Yang–Mills fields for which the rotation symmetry violation is compensated by the gauge transformations. Thus, we get the rotationally invariant energy–momentum tensor of the Yang–Mills fields, namely

$$F^b_{0j} = -F^b_{j0} = \dot{\phi}\delta^b_j, \quad F^b_{ij} = -\tilde{g}\phi^2[bij], \quad F^{b0j} = -F^{bj0} = -\frac{\dot{\phi}}{a^2}\delta^{bj}, \quad F^{bij} = -\frac{\tilde{g}\phi^2}{a^4}[bij] \tag{3.42}$$

and

$$Z \equiv F^b_{\mu\nu}F^{b\mu\nu} = 6\left(\frac{\tilde{g}^2\phi^4}{a^4} - \frac{\dot{\phi}^2}{a^2}\right), \tag{3.43}$$

where differentiation with respect to time t is denoted by a dot.

Use of the ansatz (3.41) allows to obtain the Yang–Mills energy–momentum tensor, having the same form as an ideal isotropic fluid with the energy density ρ and the pressure P, in other words: $T^{(\mathrm{YM})\mu}_{\ \ \nu} = \mathrm{diag}(-\rho, P, P, P)$.

Taking into account (3.37) and (3.40), the equation of motion (3.38) can be written as follows:

$$\partial_\nu\left\{\sqrt{-g}\mathcal{F}'(Z)\ F^{a\,\nu\mu} - \frac{\tilde{\kappa}}{192}J[\nu\mu\alpha\beta]F^a_{\alpha\beta}\right\} -$$

$$-\tilde{g}\,[abc]\,A^b_\nu\left\{\sqrt{-g}\mathcal{F}'(Z)\ F^{c\,\nu\mu} + \frac{\tilde{\kappa}}{192}J[\mu\nu\alpha\beta]F^c_{\alpha\beta}\right\} = 0. \tag{3.44}$$

It is convenient to write the Friedmann equations in terms of $\psi \equiv \phi/a$. Using

$$\dot{\phi} = a\left(\dot{\psi} + H\psi\right), \qquad \ddot{\phi} = a\left(\ddot{\psi} + 2H\dot{\psi} + \psi\left(\dot{H} + H^2\right)\right), \tag{3.45}$$

where $H = \dot{a}/a$ is the Hubble parameter, we get the equations which follow.

The $(0,0)$ component of (3.39) reduces to:

$$\frac{3M_P^2}{2}H^2 + \frac{1}{2}\mathcal{F}(Z) + 6\mathcal{F}'(Z)\left(\dot{\psi} + H\psi\right)^2 - \frac{3}{4}\tilde{\kappa}\tilde{g}^2\psi^4(\dot{\psi} + H\psi)^2 - \frac{1}{2}\Lambda = 0, \tag{3.46}$$

where $\psi \equiv \phi/a$. Note that

$$Z = 6\left(\tilde{g}^2\psi^4 - (\dot{\psi} + H\psi)^2\right), \quad \dot{Z} = 12\left(2\tilde{g}^2\psi^3\dot{\psi} - (\dot{\psi} + H\psi)(\ddot{\psi} + H\dot{\phi} + \dot{H}\psi)\right),$$

$$J = 24\tilde{g}\psi^2\left(\dot{\psi} + H\psi\right). \tag{3.47}$$

The (i,i) components of (3.39) yield:

$$\frac{1}{2}M_P^2\left[2\dot{H} + 3H^2\right] + \frac{1}{2}\mathcal{F}(Z) + 2\mathcal{F}'(Z)\left[\left(\dot{\psi} + H\psi\right)^2 - 2\tilde{g}^2\psi^4\right] - \frac{3}{4}\tilde{\kappa}\tilde{g}^2\psi^4(\dot{\psi} + H\psi)^2 - \frac{1}{2}\Lambda = 0. \tag{3.48}$$

By subtracting Eq. (3.46) from Eq. (3.48), we get

$$\frac{M_P^2}{2}\dot{H} = 2\mathcal{F}'(Z)\left(\left(\dot{\psi} + H\psi\right)^2 + \tilde{g}^2\psi^4\right). \tag{3.49}$$

From this equation, it follows that the model considered does not have nontrivial de Sitter solutions (H is a constant). Such solutions can exist only if either $\mathcal{F}(Z)$ is a constant, or the function $\psi(t) = 0$. In the next section we show that nontrivial de Sitter solutions do exist in a model which has a non-minimal coupling.

Rewriting the last equation in the following form

$$\mathcal{F}'(Z) = \frac{M_P^2}{4}\dot{H}\left(\left(\dot{\psi}+H\psi\right)^2+\tilde{g}^2\psi^4\right)^{-1}, \tag{3.50}$$

introducing (3.50) into Eq. (3.46) and differentiating, we get:

$$\begin{aligned} &H\dot{H}+\frac{\dot{H}}{2\left(\left(\dot{\psi}+H\psi\right)^2+\tilde{g}^2\psi^4\right)}\left(2\tilde{g}^2\psi^3\dot{\psi}-\left(\dot{\psi}+H\psi\right)\left(\ddot{\psi}+H\dot{\psi}+\dot{H}\psi\right)\right)+\\ &+\frac{1}{2}\left(\dot{\psi}+H\psi\right)^2\frac{d}{dt}\left[\frac{\dot{H}}{\left(\dot{\psi}+H\psi\right)^2+\tilde{g}^2\psi^4}\right]+\frac{\dot{H}\left(\dot{\psi}+H\psi\right)\left(\ddot{\psi}+H\dot{\psi}+\dot{H}\psi\right)}{\left(\dot{\psi}+H\psi\right)^2+\tilde{g}^2\psi^4}-\\ &-\frac{\tilde{\kappa}}{M_P^2}\tilde{g}^2\left(\psi^3\left(\dot{\psi}+H\psi\right)^2\dot{\psi}+\frac{1}{2}\psi^4\left(\dot{\psi}+H\psi\right)\left(\ddot{\psi}+H\dot{\psi}+\dot{H}\psi\right)\right)=0. \end{aligned} \tag{3.51}$$

If we assume, or rather know, the specific form of the Hubble function $H(t)$, then Eq. (3.51) constitutes a differential equation for $\psi(t)$ and, once we determine this function, the corresponding Yang–Mills theory can be found (i.e., the function $\mathcal{F}(Z)$) which reproduces the cosmology given by $H(t)$ in the frame of the spatially flat FLRW universe. From (3.50) we can find $\mathcal{F}(Z)$ up to an integration constant. This constant can be determined from (3.48) and corresponds to the cosmological constant.

3.3 Non-minimal gravitational coupling with the Yang–Mills field

3.3.1 Action and equations

In this section we will consider a non-minimal gravitational coupling of the $SU(2)$ Yang–Mills field in general relativity, which is described by the action:

$$S_{\rm GR} = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R+\mathcal{L}_{\rm YM}-\Lambda\right], \tag{3.52}$$

$$\mathcal{L}_{\rm YM} = -\frac{1}{4}(1+f(R))Z, \tag{3.53}$$

where $f(R)$ is an arbitrary, thrice differentiable function of R.

The field equations can be derived by taking variations of the action in Eq. (3.52) with respect to the metric $g_{\mu\nu}$ and the $SU(2)$ Yang–Mills field A^a_μ, as follows:

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\frac{1}{M_P^2}\left(T^{(\rm YM)}_{\mu\nu}-\Lambda g_{\mu\nu}\right), \tag{3.54}$$

with

$$T_{\mu\nu}^{(\mathrm{YM})} = (1+f(R))\left(g^{\alpha\beta}F_{\mu\beta}^{b}F_{\nu\alpha}^{b}-\frac{1}{4}g_{\mu\nu}\mathcal{F}\right)+ \\ + \frac{1}{2}\left\{f'(R)\mathcal{F}R_{\mu\nu}+g_{\mu\nu}\Box\left[f'(R)\mathcal{F}\right]-\nabla_\mu\partial_\nu\left[f'(R)\mathcal{F}\right]\right\}, \tag{3.55}$$

where the prime denotes derivative with respect to R, ∇_μ is the covariant derivative operator associated with $g_{\mu\nu}$, and $\Box \equiv g^{\mu\nu}\nabla_\mu\partial_\nu$ is the covariant d'Alembertian for the scalar field.

It is convenient to write down the trace equation

$$R = -\frac{1}{M_P^2}g^{\mu\nu}\left(T_{\mu\nu}^{(\mathrm{YM})}-g_{\mu\nu}\Lambda\right) = -\frac{1}{2M_P^2}\left\{f'(R)\mathcal{F}R+3\Box\left[f'(R)\mathcal{F}\right]-8\Lambda\right\}. \tag{3.56}$$

We will show that the trace equation is useful in order to find the de Sitter solutions.

3.3.2 Friedmann–Lemaître–Robertson–Walker metric and equations of motion

Using the ansatz (3.41), we get the following equations in the FLRW metric (see Appendix A, for details):

$$3H^2 = \frac{1}{M_P^2}(\Lambda+\rho) = \frac{\Lambda}{M_P^2}+\frac{3}{2M_P^2}\Big[(1+f(R))\left(\tilde{g}^2\psi^4+(\dot{\psi}+H\psi)^2\right)- \\ -6\left(\dot{H}+H^2\right)f'(R)\left(\tilde{g}^2\psi^4-(\dot{\psi}+H\psi)^2\right)+6H\partial_0\left[f'(R)\left(\tilde{g}^2\psi^4-\left(\dot{\psi}+H\psi\right)^2\right)\right]\Big], \tag{3.57}$$

$$2\dot{H}+3H^2 = \frac{\Lambda-P}{M_P^2} = \frac{\Lambda}{M_P^2}-\frac{1}{2M_P^2}\Big[(1+f(R))\left(\tilde{g}^2\psi^4+\left(\dot{\psi}+H\psi\right)^2\right)+ \\ +6\left(\dot{H}+3H^2\right)f'(R)\left(\tilde{g}^2\psi^4-\left(\dot{\psi}+H\psi\right)^2\right)- \\ -6[\partial_0\partial_0+2H\partial_0]f'(R)\left(\tilde{g}^2\psi^4-\left(\dot{\psi}+H\psi\right)^2\right)\Big], \tag{3.58}$$

It is suitable to get, from system (3.57)–(3.58), the following equivalent one:

$$R = \frac{1}{M_P^2}\left(4\Lambda-3R\vartheta+9\ddot{\vartheta}+27H\dot{\vartheta}\right), \tag{3.59}$$

$$\dot{H} = -\frac{1}{2M_P^2}\left[2(1+f(R))\left(\tilde{g}^2\psi^4+(\dot{\psi}+H\psi)^2\right)-6\dot{H}\vartheta-3\ddot{\vartheta}+3H\dot{\vartheta}\right], \tag{3.60}$$

where

$$\vartheta \equiv f'(R)\left(\tilde{g}^2\psi^4-(\dot{\psi}+H\psi)^2\right). \tag{3.61}$$

We can see that the term $(1+f(R))\left(\tilde{g}^2\psi^4+(\dot{\psi}+H\psi)^2\right)$ corresponds to radiation since, if we neglect other terms, we get $\rho = 3P$. This result is a trivial generalization of the corresponding one in the model with minimal coupling ($f(R)=0$), considered in [191, 192]. In the $f(R)$ modified model, $T_{\mu\nu}^{(\mathrm{YM})}$ has also terms proportional to $f'(R)$. In this paper, we will show that these terms can actually play the role of the cosmological constant.

3.3.3 Solutions with constant Hubble parameter

We now investigate the de Sitter solutions for the model (3.52). Our goal is to see how the Yang–Mills field, which is described by $\mathcal{L}_{\rm YM}$, can change the value of the cosmological constant. In particular, we will demonstrate in this section, that there do exist de Sitter solutions in the case when $\Lambda = 0$.

We seek solutions with $H = H_0 = \text{const}$, in other words, de Sitter and Minkowski solutions. If $H = H_0$, then $R = R_0 = 12H_0^2$, and (3.59) is a linear differential equation in ϑ:

$$\ddot{\vartheta} + 3H_0\dot{\vartheta} - 4H_0^2\vartheta = B, \tag{3.62}$$

where the constant $B = (M_P^2 R_0 - 4\Lambda)/3$. Eq. (3.62) has the following general solution

$$\vartheta = C_1 e^{Ht} + C_2 e^{-4Ht} - \frac{B}{4H^2} \tag{3.63}$$

and, from (3.60), we get

$$2(1 + f(R_0))\left(\tilde{g}^2\psi^4 + (\dot{\psi} + H\psi)^2\right) - 3\ddot{\vartheta} + 3H\dot{\vartheta} = 0. \tag{3.64}$$

If $f(12H_0^2) = -1$, then we have the equation

$$\ddot{\vartheta} - H_0\dot{\vartheta} = 0, \tag{3.65}$$

which has the general solution:

$$\vartheta = C_3 + C_4 e^{H_0 t}. \tag{3.66}$$

Thus, from (3.63) and (3.66), we get that the de Sitter solution corresponds to

$$\vartheta_{dS} = C_1 e^{H_0 t} - \frac{B}{4H_0^2}, \tag{3.67}$$

where C_1 is an arbitrary constant. At $\Lambda = 0$,

$$\vartheta_{dS_0} = C_1 e^{H_0 t} - M_P^2. \tag{3.68}$$

It is easy to see that the Minkowski solutions (at $f(12H_0^2) = -1$) correspond to

$$\vartheta_M = C(t - t_0), \tag{3.69}$$

where C and t_0 are arbitrary constants.

At $\vartheta = 0$, Eqs. (3.57) and (3.58) have the following nontrivial (ψ is not a constant) de Sitter and Minkowski solutions:

- $H = 0$, $\Lambda = 0$, $\psi(t) = \frac{1}{\tilde{g}(t-t_0)}$, $f(0) = -1$, $f_1'(0)$ is an arbitrary number.
- $H = H_0 \neq 0$, $\Lambda = 3M_P^2 H_0^2$, $\psi(t) = \frac{H_0}{\pm\tilde{g} + H_0\exp(H_0(t-t_0))}$, $f(0) = -1$, $f'(0)$ is an arbitrary number.

These solutions do not change the value of the cosmological constant. Solutions, which corresponds to $\vartheta(t) \not\equiv 0$ are more interesting. The following equation for ψ arises

$$f'(R_0)\left(\dot{\psi}^2 - \tilde{g}^2\psi^4 + 2H\dot{\psi}\psi + H^2\psi^2\right) = \frac{B}{4H^2} - C_1 e^{Ht}, \tag{3.70}$$

which, for $C_1 = 0$, yields the following first order differential equation for ψ:

$$\dot{\psi}^2 + 2H_0\dot{\psi}\psi = \tilde{g}^2\psi^4 - H_0^2\psi^2 - \frac{B}{4H_0^2 f'(12H_0^2)}. \tag{3.71}$$

The trivial cases $c = 0$ and $H_0 = 0$ have been considered above. In the general case, Eq. (3.71) does not satisfy the Fuchs conditions and, therefore, its solutions are multivalued functions (see, for example [135]). For $C_1 = 0$ the solution $\psi(t)$ can been found by quadratures, namely

$$-\int_0^{\psi} \frac{1}{H_0\tilde{\psi} \pm \sqrt{\tilde{g}^2\tilde{\psi}^4 - c}} d\tilde{\psi} = t - t_0, \qquad c \equiv \frac{B}{4H_0^2 f'(12H_0^2)}. \tag{3.72}$$

For nonzero values of C_1 the solution $\psi(t)$ can be found numerically.

3.4 Summary and discussion

In the first part of this chapter, after carefully reviewing and comparing (for the first time), and also with the help of corresponding examples, the reconstruction schemes developed in the previous chapter for $f(R)$ modified gravities have been successfully extended to the case of Yang–Mills theories. The first scheme does not need an auxiliary scalar field, while the second one is thoroughly based on its use. With these reconstruction methods, any explicitly given cosmology can be realized as a corresponding Yang–Mills theory. Although this fact had been already anticipated in the specialized literature—for modified gravities and concerning some basic models—it is comforting to see here how it can be also explicitly extended to more realistic physical theories, as the modified Yang–Mills one, with reasonable effort. As we have indicated in Sect. I, things are far from straightforward, and a very careful analysis of the solutions here obtained (also in relation with the comparison of the two different methods), and of other additional solutions of the differential equations, with potential physical interest, is still necessary. First results indicate that the solutions obtained do pass the solar system tests and the other known physical constraints.

For the novel case of the reconstruction of a Yang–Mills theory, the same example for both reconstruction schemes has been considered. The first one yields the result

$$\lambda_1 = \frac{1}{2}\ln\left(\frac{h_1(1-h_1)}{\bar{\alpha}^2 t_1^2}\right), \tag{3.73}$$

while the second scheme gave the following results

$$\lambda_1 = \begin{cases} \frac{1}{2}\ln\left(\frac{(h_1-1)(1-h_1)}{\bar{\alpha}^2 t_1^2}\right), \\ \\ \frac{1}{2}\ln\left(\frac{h_1(1-h_1)}{\bar{\alpha}^2 t_1^2}\right). \end{cases} \tag{3.74}$$

As in the case of modified gravities, in the new situation considered in this chapter of reconstructing a Yang–Mills theory, it also happens that action (3.17) expresses a more extensive class of theories than the action given by (3.1), and it is again for this reason that more solutions are obtained for the scheme based on an auxiliary scalar field. Moreover, with the help of this example we could see explicitly that there is, in fact, a very interesting coincidence between the result obtained in (3.73) and one of the results of (3.74). This finding here further supports, once more, the point of view that the Einsteinian and the Jordanian frame descriptions actually lead to two physically different theories, making thus clear the physical non-equivalence of the two frames as discussed in [114, 222]. This additional piece of evidence is very valuable. Even much more than the one in the previous chapter because it comes from a theory that it is way closer to physics than the ones considered previously.

As in the case of $f(R)$ modified gravity, here is also important to remark that, sometimes, it is actually more convenient to use one scheme instead of the other, because the final result may be definitely easier to obtain and to interpret in one of the two schemes. Even if these conclusions were already derived in more simplified situations, it is actually comforting (and rather non-trivial) to see that they continue to be valid in much more realistic situations, from the point of view of physics, as corresponding to the actions here considered, involving Yang–Mills fields.

In Sect. II, we have studied a model with a minimally coupled $SU(2)$ Yang–Mills field, described by the action (3.34), which includes second- and fourth-order terms of the Yang–Mills field strength tensor. The second-order term can play the role of radiation, whereas the fourth-order one plays the role of the cosmological constant. It has been shown that the function $\mathcal{F}(Z)$ can be reconstructed provided the Hubble parameter is given. In particular, it has been demonstrated that de Sitter solutions exist only in the trivial case, namely when $\mathcal{F}(Z)$ is a constant.

In order to obtain genuine de Sitter solutions, in Sect. III a model in which the Yang–Mills field has a nonminimal coupling with gravity was considered. We have explicitly shown that this model, described by the action in (3.52), has de Sitter solutions even in the absence of a cosmological constant term. The de Sitter solutions correspond to the Yang–Mills fields which satisfy Eq. (3.71). This equation includes an arbitrary parameter. Depending on the value of this parameter, it has been shown that it can be easily solved in quadratures or, in the most general case, numerically.

Chapter 4

Reconstructing cyclic universes: Ekpyrotic universes in f(R) Hořava–Lifshitz gravity

As an alternative for the inflationary universe, the so-called ekpyrotic scenario may avoid the need to provide initial conditions (inherent in every inflationary model), since the universe evolution acquires a periodic behavior, such that in every cycle a new universe is born (see Ref. [159, 160, 275]). In addition, it is argued that the problem of flatness does not appear in this model because the universe initially was in a nearly BPS (Bogolmonyi-Prasad-Sommerfield) state, which is homogeneous (see Ref. [159, 160, 275]). In the last years, very promising models capable to unify the entire cosmic evolution under the same mechanism have been proposed, where the inflationary epoch and the late-time acceleration era are unified under the same mechanism (or alternatively the ekpyrotic scenario), providing a simpler picture of the universe evolution. Most such models are described by scalar fields due to its simple form (see Ref. [110] and references therein), or other kind of fields (see Ref. [104, 296]), but also a large effort has been done in the reconstruction of modified gravity theories (for a general report, see Ref. [86]) available to reproduce the cosmic evolution (for a review, see Ref. [64, 67, 207, 221, 234, 269], and Refs. [222, 256]), which may seem more natural as they are expressed in terms purely of the metric tensor without additional fields.

On the other hand, a new theory of gravity that is power–counting renormalizable has been proposed recently in Ref. [146]. Such theory, already known as Hořava–Lifshitz gravity, breaks the invariance under full diffeomorphisms of General Relativity by introducing an anisotropy between the spatial and time coordinates through a critical exponent z. This restriction of the symmetries allows the theory to be power-counting renormalizable, but an additional scalar degree of freedom is found, which introduces instabilities in the spectrum of the theory (see Refs. [42, 80]). However, some extensions of the theory seem to address the problem of the scalar mode [43, 147], as well as to generalize the action to more complex ones (see Ref. [164]). Moreover, cosmological models have been widely studied in the context of Hořava–Lifshitz gravity (see Ref. [11, 25, 44, 45, 48, 54, 55, 76, 131, 152, 163, 199, 205, 208, 229, 246, 247, 260, 267, 270, 276, 286, 287]), and also generalizations of the original action (similarly to standard $f(R)$ gravity) have been proposed, where the entire cosmological history can be well reproduced, and it has also a good UV behavior (see Refs.[75, 79, 109, 165, 166, 255]).

The aim of this chapter is to study the ekpyrotic scenario in the frame of some extensions of Hořava–Lifshitz gravity, where a universe described in terms purely of gravity is able to pass along the different stages of an ekpyrotic model. This class of cosmological solutions can be realized in standard $f(R)$ gravity as shown in Ref. [237]. Here, some periodic solutions for the Hubble parameter, which may be able to describe the entire evolution of the universe, are reconstructed. In addition, we also analyze the shape of the action for each phase of the ekpyrotic scenario, where the possibility of the occurrence of a *Little Rip* is explored. The so-called *Little Rip* is a postulated phase of the universe evolution, when a very strong accelerating expansion would lead to break some bounded systems, as the Solar System or even the molecules and atoms (see Ref. [121, 122, 123]). Such breaking is shown to be fully compatible with the ekpyrotic scenario in comparison with future singularities as the Big Rip that are not, unless some cure for the future singularity is considered [186]. Moreover, the presence of a Big Bang/Crunch singularity, usual in ekpyrotic cosmologies, is still an open issue for this kind of cyclic scenario, where quantum effects may resolve it (see [89, 92, 175, 213, 282]) or an effective theory that generates a non singular bounce (see [52]). Nevertheless, here we are interested to explore the classical effects of the theory, where some non singular solutions are proposed, while the study of possible UV effects in the presence of the singularity is beyond the purpose of this paper.

The chapter is organized as follows. In Sect. I, $f(R)$ Hořava–Lifshitz gravity is briefly reviewed. In Sect. II, the actions for some cyclic solutions are reconstructed. Finally, Sect. IV is devoted to the analysis of ekpyrotic scenario, where each phase of the cycle is analyzed.

This Chapter is based on the publications: [187].

4.1 Modified $f(R)$ Hořava–Lifshitz gravity

In this section, modified Hořava–Lifshitz $f(R)$ gravity is briefly reviewed [75, 79, 109, 164, 165, 166, 255]. We start by writing a general metric in the so-called Arnowitt-Deser-Misner (ADM) decomposition in a $3+1$ spacetime (for more details see [22, 130]),

$$ds^2 = -N^2 dt^2 + g_{ij}^{(3)}(dx^i + N^i dt)(dx^j + N^j dt)\,, \tag{4.1}$$

where $i, j = 1, 2, 3$, N is the so-called lapse variable, and N^i is the shift 3-vector. In standard general relativity (GR), the Ricci scalar can be written in terms of this metric, and yields

$$R = K_{ij}K^{ij} - K^2 + R^{(3)} + 2\nabla_\mu(n^\mu \nabla_\nu n^\nu - n^\nu \nabla_\nu n^\mu)\,, \tag{4.2}$$

here $K = g^{ij}K_{ij}$, K_{ij} is the extrinsic curvature, $R^{(3)}$ is the spatial scalar curvature, and n^μ a unit vector perpendicular to a hypersurface of constant time. The extrinsic curvature K_{ij} is defined as

$$K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij}^{(3)} - \nabla_i^{(3)} N_j - \nabla_j^{(3)} N_i\right)\,. \tag{4.3}$$

In the original model [146], the lapse variable N is taken to be just time-dependent, so that the projectability condition holds and by using the foliation-preserving diffeomorphisms (4.6), it can be fixed to be $N = 1$. As pointed out in [43], imposing the projectability condition may cause problems with Newton's law in the Hořava gravity. For the non-projectable case, the Newton law could be restored (while keeping stability) by the "healthy" extension of the original Hořava gravity of Ref. [43].

The action for standard $f(R)$ gravity can be written as

$$S = \int d^4x \sqrt{g^{(3)}} N f(R)\,. \tag{4.4}$$

Gravity of Ref. [146] is assumed to have different scaling properties of the space and time coordinates

$$x^i = bx^i \,, \quad t = b^z t \,, \tag{4.5}$$

where z is a dynamical critical exponent that renders the theory renormalizable for $z = 3$ in $3+1$ spacetime dimensions [146]. GR is recovered when $z = 1$. The scaling properties (4.5) render the theory invariant only under the so-called foliation-preserving diffeomorphisms:

$$\delta x^i = \zeta(x^i, t) \,, \quad \delta t = f(t) \,. \tag{4.6}$$

It has been pointed that, in the IR limit, the additional scalar degree of freedom can be removed by means of an additional $U(1)$ symmetry [147]. Here, we are interested on actions as follow,

$$S = \frac{1}{2\kappa^2}\int dtd^3x\sqrt{g^{(3)}}Nf(\tilde{R}) \,, \quad \tilde{R} = K_{ij}K^{ij} - \lambda K^2 + R^{(3)} + 2\mu\nabla_\mu(n^\mu\nabla_\nu n^\nu - n^\nu\nabla_\nu n^\mu) - L^{(3)}(g_{ij}^{(3)}) \,, \tag{4.7}$$

where κ is the dimensionless gravitational coupling, and where, two new constants λ and μ appear, which account for the violation of the full diffeomorphism transformations. Note that in the original Hořava gravity theory [146], the fourth term in the expression for $\tilde{R}$ can be omitted, as it becomes a total derivative. This generalization of the Hořava–Lifshitz action, similar to standard $f(R)$ gravity, may provide the way to describe the entire cosmological evolution with no need to introduce any additional field but where an additional scalar mode is assumed. The possibility of violations of Newtonian law, due to the extra scalar mode coming from $f(\tilde{R})$, can be avoided by the appropriate expression for the action, as it was pointed out in Ref. [109]. In addition, standard $f(R)$ gravity (4.4) can be recovered by setting $\lambda = \mu = 1$. The term $L^{(3)}(g_{ij}^{(3)})$ in the action (4.7) is chosen to be [146]

$$L^{(3)}(g_{ij}^{(3)}) = E^{ij}G_{ijkl}E^{kl} \,, \tag{4.8}$$

where the generalized De Witt metric is given by,

$$G^{ijkl} = \frac{1}{2}\left(g^{(3)ik}g^{(3)jl} + g^{(3)il}g^{(3)jk}\right) - \lambda g^{(3)ij}g^{(3)kl} \,. \tag{4.9}$$

In Ref. [146], the expression for E_{ij} is constructed to satisfy the "detailed balance principle" in order to restrict the number of free parameters of the theory, and it is defined through the variation of an action

$$\sqrt{g^{(3)}}E^{ij} = \frac{\delta W[g_{kl}]}{\delta g_{ij}} \,, \tag{4.10}$$

The action $W[g_{kl}]$ is assumed to be defined by the metric and the covariant derivatives on the three-dimensional hypersurface $\sum_t$. In [146], $W[g_{kl}^{(3)}]$ is explicitly given for the case $z = 2$,

$$W = \frac{1}{\kappa_W^2}\int d^3x\,\sqrt{g^{(3)}}(R - 2\Lambda_W) \,, \tag{4.11}$$

and for the case $z = 3$,

$$W = \frac{1}{w^2}\int_{\Sigma_t}\omega_3(\Gamma) \,. \tag{4.12}$$

Here κ_W in (4.11) is a coupling constant of dimension $-1/2$ and w^2 in (4.12) is the dimensionless coupling constant. $\omega_3(\Gamma)$ in (4.12) is given by

$$\omega_3(\Gamma) = \mathrm{Tr}\left(\Gamma\wedge d\Gamma + \frac{2}{3}\Gamma\wedge\Gamma\wedge\Gamma\right) \equiv \varepsilon^{ijk}\left(\Gamma_{il}^m\partial_j\Gamma_{km}^l + \frac{2}{3}\Gamma_{il}^n\Gamma_{jm}^l\Gamma_{kn}^m\right)d^3x \,. \tag{4.13}$$

Here we are interested in the study of cosmological solutions for the theory described by action (4.7). Spatially-flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric is assumed

$$ds^2 = -N^2 dt^2 + a^2(t) \sum_{i=1}^{3} \left(dx^i\right)^2 , \tag{4.14}$$

where N is taken to be just time-dependent (projectability condition) and, by using the foliation-preserving diffeomorphisms (4.6), it can be set to unity, $N = 1$. Then, just as an assumption of the solution, N is taken to be unity.

For a flat FLRW metric (4.14), and a vanishing cosmological constant, the scalar $\tilde{R}$ is given by

$$\tilde{R} = \frac{3(1-3\lambda+6\mu)H^2}{N^2} + \frac{6\mu}{N}\frac{d}{dt}\left(\frac{H}{N}\right) . \tag{4.15}$$

For the action (4.7), and assuming the FLRW metric (4.15), the second FLRW equation can be obtained by varying the action with respect to the spatial metric $g_{ij}^{(3)}$, what yields

$$0 = f(\tilde{R}) - 2(1-3\lambda+3\mu)\left(\dot{H}+3H^2\right)f'(\tilde{R}) - 2(1-3\lambda)H\dot{\tilde{R}}f''(\tilde{R}) + 2\mu\left(\dot{\tilde{R}}^2 f'''(\tilde{R}) + \ddot{\tilde{R}}f''(\tilde{R})\right) + \kappa^2 p_m , \tag{4.16}$$

here $\kappa^2 = 16\pi G$, p_m is the pressure of a perfect fluid that fills the universe, and $N = 1$. Note that this equation turns out the usual second FLRW equation for standard $f(R)$ gravity (4.4) when $\lambda = \mu = 1$. If we assume the projectability condition, variation over N of the action (4.7) yields the following global constraint

$$0 = \int d^3x \left[f(\tilde{R}) - 6f'(\tilde{R})\left\{(1-3\lambda+3\mu)H^2 + \mu\dot{H}\right\} + 6\mu H\dot{\tilde{R}}f''(\tilde{R}) - \kappa^2\rho_m\right] . \tag{4.17}$$

Now, by using the ordinary conservation equation for the matter fluid $\dot{\rho}_m + 3H(\rho_m + p_m) = 0$, and integrating Eq. (4.16), it yields

$$0 = f(\tilde{R}) - 6\left[(1-3\lambda+3\mu)H^2 + \mu\dot{H}\right]f'(\tilde{R}) + 6\mu H\dot{\tilde{R}}f''(\tilde{R}) - \kappa^2\rho_m - \frac{C}{a^3} , \tag{4.18}$$

where C is an integration constant, taken to be zero, according to the constraint equation (4.17). In [204], however, it has been claimed that C needs not always vanish in a local region, since (4.17) needs to be satisfied in the whole universe. In the region $C > 0$, the Ca^{-3} term in (4.18) may be regarded as dark matter.

If we do not assume the projectability condition, we can directly obtain (4.18), which corresponds to the first FLRW equation, by varying the action (4.7) over N . Hence, starting from a given $f(\tilde{R})$ function, and solving Eqs. (4.16) and (4.17), a cosmological solution can be obtained.

4.2 Reconstructing cyclic universes

The aim of this section is to show that any cosmology may be realized in $f(\tilde{R})$ Hořava–Lifshitz gravity. For this purpose, we present two different methods of reconstruction, the first one is based on the use of the number of e-foldings, while, the second one uses an auxiliary scalar field.

4.2.1 Reconstructing a cyclic universe using e-folding

We will assume the flat FLRW metric defined in (4.14) with $N = 1$, in such a case the first FLRW equation is given by (4.18) with $C = 0$, which can be rewritten as a function of the number of e-foldings $\eta = \ln \frac{a}{a_0}$ instead of the time t. This technique has been developed in [236] for classical $f(R)$ gravity, and for Hořava–Lifshitz f(R)-gravity [109]. Since $\frac{d}{dt} = H\frac{d}{d\eta}$ and $\frac{d^2}{d\eta^2} = H^2\frac{d^2}{d\eta^2} + H\frac{dH}{d\eta}\frac{d}{d\eta}$, the first FLRW equation (22) is rewritten as

$$0 = f(\tilde{R}) - 6\left[\frac{A}{3}H^2 + \mu HH'\right]\frac{df(\tilde{R})}{d\tilde{R}} + 6\mu H^2\left[2AHH' + 6\mu H'^2 + 6\mu H''H'\right]\frac{d^2 f(\tilde{R})}{d^2\tilde{R}} - \rho\ , \tag{4.19}$$

where $A = 3 - 9\lambda + 18\mu$ and the primes denote derivatives respect η. By using the energy conservation equation $\dot{\rho} + 3H(1+w)\rho = 0$, the energy density yields,

$$\rho = \rho_0 a^{-3(1+w)} = \rho_0 a_0^{-3(1+w)} e^{-3(1+w)\eta}\ . \tag{4.20}$$

As the Hubble parameter can be written as a function of the number of e-foldings, $H = H(\eta)$, the scalar curvature in (4.15) takes the form

$$\tilde{R} = AH^2 + 6\mu HH'\ , \tag{4.21}$$

which can be solved respect to η as $\eta = \eta(\tilde{R})$. Then, the equation (4.19) for $f(\tilde{R})$ with the variable $\tilde{R}$ is obtained. This can be a little simplified by writing $G(\eta) = H^2$ instead of the Hubble parameter. In such a case, the differential equation (4.19) gives

$$0 = f(\tilde{R}) - 6\left[\frac{A}{3}G + \frac{\mu}{2}G'\right]\frac{df(\tilde{R})}{d\tilde{R}} + 6\mu\left[AGG' + 3\mu GG''\right]\frac{d^2 f(\tilde{R})}{d^2\tilde{R}} - \rho_0 a_0^{-3(1+w)} e^{-3(1+w)\eta}\ , \tag{4.22}$$

and the scalar curvature is now written as $\tilde{R} = AG + 3\mu G'$. Hence, for a given cosmological solution $H^2 = G(\eta)$, one can solve the equation (4.22), and the corresponding $f(\tilde{R})$ is obtained.

In order to illustrate that cyclic solutions can be reproduced by this kind of theories, let us consider the following example:

$$H(t) = -\frac{2\pi}{T}H_1 \sin\left(\frac{2\pi}{T}t\right) \tag{4.23}$$

where H_1 and T are constants. The number of e-foldings is:

$$H(t) = \frac{1}{a}\frac{da}{dt} = -\frac{2\pi}{T}H_1\sin\left(\frac{2\pi}{T}t\right) \Longrightarrow \frac{da}{a} = -\frac{2\pi}{T}H_1\sin\left(\frac{2\pi}{T}t\right)dt \Longrightarrow$$

$$\Longrightarrow \eta(t) = \ln\left(\frac{a(t)}{a_0}\right) = H_1\left[\cos\left(\frac{2\pi}{T}t\right) - 1\right] \tag{4.24}$$

Using (4.24), the function $G(\eta)$ and its derivatives are given by:

$$G(\eta) = H^2 = -\left(\frac{2\pi}{T}\right)^2(2H_1 + \eta)\,\eta, \quad G'(\eta) = -2\left(\frac{2\pi}{T}\right)^2(H_1 + \eta)\,, \quad G''(\eta) = -2\left(\frac{2\pi}{T}\right)^2. \tag{4.25}$$

Then, we have:

$$\tilde{R} = -3\left(\frac{2\pi}{T}\right)^2\left[2\mu H_1 + 2\eta\left(\mu + (1 - 3\lambda + 6\mu)\,H_1\right) + (1 - 3\lambda + 6\mu)\,\eta^2\right] \Rightarrow$$

$$\Rightarrow \eta = -\left(\frac{\mu}{1-3\lambda+6\mu} + H_1\right) \pm \sqrt{\frac{\mu^2}{(1-3\lambda+6\mu)^2} + H_1^2 - \frac{\tilde{R}}{3(1-3\lambda+6\mu)\left(\frac{2\pi}{T}\right)^2}}. \qquad (4.26)$$

Now, if we call $x = \pm\sqrt{\frac{\mu^2}{(1-3\lambda+6\mu)^2} + H_1^2 - \frac{\tilde{R}}{3(1-3\lambda+6\mu)\left(\frac{2\pi}{T}\right)^2}}$, we can write:

$$\eta = -\left(\frac{\mu}{1-3\lambda+6\mu} + H_1\right) + x. \qquad (4.27)$$

We also have that:

$$\frac{df(\tilde{R})}{d\tilde{R}} = -\frac{1}{6(1-3\lambda+6\mu)\left(\frac{2\pi}{T}\right)^2 x}\frac{dF_1(x)}{dx},$$

$$\frac{d^2 f(\tilde{R})}{d\tilde{R}^2} = \frac{1}{\left[6(1-3\lambda+6\mu)\left(\frac{2\pi}{T}\right)^2 x\right]^2}\left(\frac{d^2F_1(x)}{dx^2} - \frac{1}{x}\frac{dF_1(x)}{dx}\right), \qquad (4.28)$$

where $F_1(x) = F(\tilde{R}(x))$.

We can now rewrite (4.25) in terms of the new variable x by using (4.27), leading to:

$$G(\eta(x)) = -\left(\frac{2\pi}{T}\right)^2\left(\frac{\mu^2}{(1-3\lambda+6\mu)^2} - H_1^2 - \frac{2\mu}{1-3\lambda+6\mu}x + x^2\right),$$

$$G'(\eta(x)) = -2\left(\frac{2\pi}{T}\right)^2\left(-\frac{\mu}{1-3\lambda+6\mu} + x\right),$$

$$G''(\eta(x)) = -2\left(\frac{2\pi}{T}\right)^2. \qquad (4.29)$$

Finally, by introducing (4.28-4.29) into the equation (4.22) and considering the case of vacuum, we arrive to the following differential equation for $F_1(x)$:

$$0 = x^2F_1(x) + \left[\frac{\mu}{1-3\lambda+6\mu}\left(H_1^2 - \frac{\mu^2}{(1-3\lambda+6\mu)^2}\right) + \left(\frac{2\mu^2}{(1-3\lambda+6\mu)^2} + H_1\right)x - x^3\right]\frac{dF_1(x)}{dx} +$$

$$+\frac{\mu}{1-3\lambda+6\mu}x\left[-\left(H_1^2 - \frac{\mu^2}{(1-3\lambda+6\mu)^2}\right) - \frac{2\mu}{1-3\lambda+6\mu}x + x^2\right]\frac{d^2F_1(x)}{dx^2}. \qquad (4.30)$$

Here, we have obtained an equation for the gravitational action, that in principle can not provide an exact expression, but which can be integrated numerically. Hence, this solution reproduces a periodic behavior for the Hubble parameter leading to a cyclic universe.

4.2.2 Reconstructing a cyclic universe using a scalar field

In this subsection it will be shown how to construct an $f(\tilde{R})$ Hořava–Lifshitz gravity model realizing any given cosmology, this time using instead the technique of [222]. We start from the action for $f(\tilde{R})$ Hořava–Lifshitz gravity

$$S = \int dt d^3x\sqrt{g^{(3)}}N(f(\tilde{R}) + \mathcal{L}_{matter}), \qquad (4.31)$$

which is equivalent to

$$S = \int dt d^3x \sqrt{g^{(3)}} N(P(\phi)\tilde{R} + Q(\phi) + \mathcal{L}_{matter}). \tag{4.32}$$

Here, $\mathcal{L}_{matter}$ is the matter Lagrangian density and P and Q are proper functions of the scalar field, ϕ, which can be regarded as an auxiliary field, because there is no kinetic term depending on ϕ in the Lagrangian. By varying the action with respect to ϕ, it follows that

$$0 = P'(\phi)\tilde{R} + Q'(\phi), \tag{4.33}$$

which can be solved in terms of ϕ, as

$$\phi = \phi(\tilde{R}). \tag{4.34}$$

By substituting (4.34) into (4.32) and comparing with (4.31), one obtains

$$S = \int dt d^3x \sqrt{g^{(3)}} N(F(\tilde{R}) + \mathcal{L}_{matter}),$$

$$f(\tilde{R}) \equiv P(\phi(\tilde{R}))\tilde{R} + Q(\phi(\tilde{R})). \tag{4.35}$$

We proceed now in the same way that we did in Section II, assuming the FLRWL metric, the second FLRWL equation can be obtained by varying the action (4.32) with respect to the spatial metric $g_{ij}^{(3)}$. This equation can be written as:

$$P(\phi)\left\{\tilde{R} - 2\left(1-3\lambda+3\mu\right)\left(3H^2+\dot{H}\right)\right\} - 2\left(1-3\lambda\right)H\frac{dP(\phi)}{dt} + 2\mu\frac{d^2P(\phi)}{dt^2} + Q(\phi) + p = 0 \tag{4.36}$$

If we assume now the projectability condition, we can obtain a global constraint doing the variation of the action (4.7) over N, it yields:

$$P(\phi)\left\{\tilde{R} - 6\left[\left(1-3\lambda+3\mu\right)H^2 + \mu\dot{H}\right]\right\} + 6\mu H\frac{dP(\phi)}{dt} + Q(\phi) - \rho = 0 \tag{4.37}$$

We can combine (4.36) and (4.37) in order to eliminate the function $Q(\phi)$, we finally obtain:

$$2\mu\frac{d^2P(\phi(t))}{dt^2} - 2\left(1-3\lambda+3\mu\right)H\frac{dP(\phi(t))}{dt} - 2\left(1-3\lambda\right)\dot{H}P(\phi(t)) + p + \rho = 0 \tag{4.38}$$

As we may redefine the scalar field ϕ properly, we can choose

$$\phi = t. \tag{4.39}$$

Provided the scale factor a is given by a proper function $g(t)$ as

$$a = a_0 e^{g(t)}, \tag{4.40}$$

with a constant a_0, and if it is moreover assumed that p and ρ are the sum of the different matter contributions, with constant equation of state (EoS) parameters ω_i, Eq. (4.38) then reduces to the following second order differential equation

$$2\mu\frac{d^2P(\phi)}{d\phi^2} - 2\left(1-3\lambda+3\mu\right)g'(\phi)\frac{dP(\phi)}{d\phi} - 2\left(1-3\lambda\right)g''(\phi)P(\phi) + \sum_i (1+\omega_i)\rho_{i0}a_0^{-3(1+\omega_i)}e^{-3(1+\omega_i)g(\phi)} = 0 \tag{4.41}$$

From this equation we can obtain $P(\phi)$ and using Eq. (4.37) we find that

$$Q(\phi) = -P(\phi)\left\{\tilde{R} - 6\left[\left(1-3\lambda+3\mu\right)H^2 + \mu\dot{H}\right]\right\} - 6\mu H\frac{dP(\phi)}{dt} + \sum_i \rho_{i0}a_0^{-3(1+\omega_i)}e^{-3(1+\omega_i)g(\phi)} \tag{4.42}$$

As a result, any given cosmology, expressed as (4.40), can indeed be realized (as anticipated) by some specific $f(R)$-gravity. Note that Eq.(4.41) is a second order differential equation on $P(\phi)$ when $g'(\phi)$ is known, but it can also be considered as a first order differential equation on $g'(\phi)$ (i.e. on $H(\phi)$) in the case that the function $P(\phi)$ is given. In the following we will use this last point of view to find out a function $f(\tilde{R})$ that reproduces a cyclic universe.

When matter can be neglected Eq.(4.41) can be rewritten as:

$$\frac{d}{d\phi}\left(g'(\phi)\,P(\phi)^{\frac{1-3\lambda+3\mu}{1-3\lambda}}\right) = \frac{\mu}{1-3\lambda}\,P(\phi)^{\frac{3\mu}{1-3\lambda}}\,\frac{d^2P(\phi)}{d\phi^2} \tag{4.43}$$

which can be solved as [256]:

$$g'(\phi) = \frac{\mu}{1-3\lambda}\,P(\phi)^{-\frac{1-3\lambda+3\mu}{1-3\lambda}}\int d\phi\,P(\phi)^{\frac{3\mu}{1-3\lambda}}\,\frac{d^2P(\phi)}{d\phi^2} =$$

$$= \frac{\mu}{1-3\lambda}\,\frac{1}{P(\phi)}\,\frac{dP(\phi)}{d\phi} - \frac{3\mu^2}{(1-3\lambda)^2}\,P(\phi)^{-\frac{1-3\lambda+3\mu}{1-3\lambda}}\int d\phi\,P(\phi)^{\frac{3\mu}{1-3\lambda}-1}\left(\frac{dP(\phi)}{d\phi}\right)^2 \tag{4.44}$$

In the second equality, we have used the partial integration. Furthermore by writing $P(\phi)$ as:

$$P(\phi) = U(\phi)^{\frac{2(1-3\lambda)}{1-3\lambda+3\mu}} \tag{4.45}$$

(4.44) is rewritten as follows:

$$g'(\phi) = \frac{2\mu}{1-3\lambda+3\mu}\,\frac{1}{U(\phi)}\,\frac{dU(\phi)}{d\phi} - \frac{12\mu^2}{(1-3\lambda+3\mu)^2}\,\frac{1}{U(\phi)^2}\int d\phi\left(\frac{dU(\phi)}{d\phi}\right)^2. \tag{4.46}$$

We now consider the case given by:

$$P(\phi) = U(\phi)^{\frac{2(1-3\lambda)}{1-3\lambda+3\mu}} = P_0\,[\cos(\omega\phi)]^{-\frac{2(1-3\lambda)}{1-3\lambda+3\mu}} \tag{4.47}$$

where P_0 and ω are constants. Then, using Eq.(4.43), the solution is given by:

$$g'(\phi) = g_0\,[\cos(\omega\phi)]^2 + \frac{2\omega\mu}{1-3\lambda+3\mu}\tan(\omega\phi)\left(1 - \frac{2\mu}{1-3\lambda+3\mu}\,[\sin(\omega\phi)]^2\right) \tag{4.48}$$

where g_0 is an integration constant. Note that the tangent term in (4.48) makes the solutions to contain some divergences that correspond to points where the scale factor becomes null, i.e. $a(t_0) = 0$. These divergences can be identified with a Big Bang/Crunch singularity and they are very common in cyclic universes, where the ekpyrotic scenario is reproduced. In order to have a smooth transition through the Big Bang/Crunch singularity, one expects that the quantum effects of the theory will avoid the occurrence of the singularity. However, this is a large task, even more in a background solution as (4.48), and should be explored separately in the future. In addition, other mechanisms for a smooth transition have been suggested as the introduction of an additional term in the action or a different coupling with the matter lagrangian (see Ref. [158]).

4.3 Ekpyrotic scenario in Hořava–Lifshitz gravity

We have shown above that periodic solutions can be easily reconstructed in the frame of extended Hořava–Lifshitz gravity. Here we are more interested to analyze ekpyrotic models in such kind of theories. The so-called Ekpyrotic/cyclic universe is an alternative explanation to the inflationary paradigm proposed one decade ago in Ref. [159, 160, 275], that can provide a realistic picture of the universe evolution (for a confrontation between both models, see [178]). In the same way as the inflationary scenario, ekpyrotic cosmological models can also predict the origin of primordial inhomogeineties that leads to the formation of large structures and the anisotropies observed in the CMB. In addition, this model does not require initial conditions in comparison with the standard inflationary scenario due to its cyclic nature. In general, the cosmological evolution presented by an ekpyrotic universe consist of infinite cycles, where each cycle contains four stages: a first initial hot state similar to the standard Big Bang model, then a phase of accelerated expansion, after which the universe starts to contract and finally the cycle ends in a Big Bang/Crunch transition, when the cycle starts again. The cosmological problems enumerated above are solved during the contracting phase. In the usual ekpyrotic models, brane scenarios or scalar fields are considered (see [159, 160, 275]). However, it is clear that modified gravity, and precisely $f(\tilde{R})$ gravity, can perfectly reproduce the ekpyrotic scenario [237]. Here we are interested to see how the cosmological problems can be solved during the contracting phase in the context of Horřava-Lifshitz gravity, and to reconstruct the corresponding behavior of the action during each phase of an ekpyrotic universe. The first FLRW equation is given by,

$$\frac{3}{\kappa^2}H^2 = \frac{1}{(1-3\lambda+3\mu)f'(\tilde{R})}\left(\frac{\rho_{m0}}{a^3}+\frac{\rho_{r0}}{a^4}+\frac{\rho_{\sigma 0}}{a^6}-\frac{k}{a^2}\right)+\rho_{f(\tilde{R})} \ , \tag{4.49}$$

where the subscripts refers to matter (m), radiation (r), anisotropies (σ), and k is the spatial curvature, while $\rho_{f(R)}$ is defined as,

$$\rho_{f(\tilde{R})} = \frac{1}{\kappa^2(1-3\lambda+3\mu)f'(\tilde{R})}\left(\frac{1}{2}f(\tilde{R})-3\mu\dot{H}f'(\tilde{R})+3\mu H\dot{\tilde{R}}f''(\tilde{R})\right) \ . \tag{4.50}$$

In order to solve the initial cosmological problems, the last term in (4.49) should dominate over the rest when the scale factor tends to zero, i.e. when the universe approaches the Big Bang (Crunch) singularity. Hence, the effective energy density defined in (4.50) should behave as $\rho_{f(\tilde{R})} \propto 1/a^m$ with $m>6$ when the scale factor tends to zero, such that close to the initial singularity, the FLRW equation (4.49) can be approximated as,

$$\frac{3}{\kappa^2}H^2 \sim \rho_{f(\tilde{R})} \sim \frac{C}{a^m} \ , \tag{4.51}$$

where C is a constant. Then, we can reconstruct the form of the action $f(R)$ close to the Big Bang (Crunch) singularity by solving the FLRW equation. Hence, for the Hubble parameter (4.51), the scalar curvature is given by,

$$\tilde{R} = \left[(1-3\lambda+6\mu)-\mu m\right]\kappa^2\frac{C}{a_m} \ . \tag{4.52}$$

And the FLRW equation (4.51) yields an expression where $f(R)$ is the unknown quantity,

$$\tilde{R}^2 f''(\tilde{R}) + \frac{2\kappa^2(1-3\lambda+3\mu)-\mu m}{2\mu m}\tilde{R}f'(\tilde{R}) - \frac{(1-3\lambda+6\mu)-\mu m}{2\mu m}f(\tilde{R}) = 0 \ . \tag{4.53}$$

This is an Euler equation that can be easily solved, and gives the function for $f(R)$,

$$f(\tilde{R}) = \kappa_1\tilde{R}^{\beta_+} + \kappa_2\tilde{R}^{\beta_-} \ . \tag{4.54}$$

where,

$$\beta_\pm = \frac{3m\mu - 2\kappa^2(1-3\lambda+3\mu) \pm \sqrt{4\kappa^2(1-3\lambda+3\mu)(\kappa^2(1-3\lambda+3\mu)-3m\mu)+m\mu(8-24\lambda+(48+m)\mu)}}{4m\mu} \ . \tag{4.55}$$

Note that the scalar curvature tends to infinity when $a \rightarrow 0$, and in such strong gravity regime, the parameters λ and μ should be different than one, the limit of General Relativity, as the breaking of Lorentz invariance will be present in such kind of regimes, while it is recovered for the weak field systems. Moreover, in order to get a smooth transition along the singularity, the first derivative of $f(\tilde{R})$ should tend to infinity to ensure that the matter energy densities remain finite in (4.49), which can be easily achieved when $(\beta_{\pm} - 1) < 0$ in (4.53).
After this contracting phase, the ekpyrotic model suggests that a hot initial state, similar to the Big Bang model, is created (in the original ekpyrotic model by the collision between branes), and which may be created by the decaying of the extra scalar modes coming from $f(R)$ in this class of theories. Nevertheless, this is beyond the purpose of this paper, where our aim is to show the approximated form that the action should look like for each phase of the cycle. Then, during the matter/radiation dominated epochs, the action may seem as the standard Hilbert-Einstein action with $f(\tilde{R}) \sim \tilde{R}$ and $\tilde{R} = R$, i. e. the parameters responsible of the breaking of full diffeomorphisms should recover the values of GR, $\lambda = \mu \sim 1$. The last phase for each cycle refers to an accelerating era, which may be described by the usual Λ CDM model, whose Hubble parameter can be written in terms of the number of e-foldings as,

$$H^2 = H_0^2 + \frac{\kappa^2}{3}\rho_0 a^{-3} = H_0^2 + \frac{\kappa^2}{3}\rho_0 a_0^{-3} \mathrm{e}^{-3\eta} \ . \tag{4.56}$$

where H_0 and ρ_0 are constants. In the frame of General Relativity, the terms in the r.h.s of equation (4.56) correspond to an effective cosmological constant $\Lambda = 3H_0^2$ and to a pressureless fluid. The corresponding $f(\tilde{R})$ can be reconstructed by following the steps described above. For this case the function $G(\eta)$ is given by

$$G(\eta) = H_0^2 + \frac{\kappa^2}{3}\rho_0 a_0^{-3} \mathrm{e}^{-3\eta} \ . \tag{4.57}$$

And by using the expression for the scalar curvature $\tilde{R} = AG + 3\mu G'$, the relation between $\tilde{R}$ and η is obtained,

$$\mathrm{e}^{-3\eta} = \frac{\tilde{R} - AH_0^2}{k(3 + 9(\mu - \lambda))} \ , \tag{4.58}$$

where $k = \frac{\kappa^2}{3}\rho_0 a_0^{-3}$. Then, by substituting (4.57) and (4.58) in the equation (4.22), one gets the following differential expression,

$$\frac{1 + 3(\mu - \lambda)}{6\mu(1 - 3\lambda)} f(\tilde{R}) - \left[\frac{1 + 3(\mu - \lambda)}{3\mu(1 - 3\lambda)}\tilde{R} - \frac{3H_0^2\mu(1 - 3\lambda + 6\mu)}{2\mu(1 - 3\lambda)}\right] f'(\tilde{R}) - (\tilde{R} - 9\mu H_0^2)(\tilde{R} - 3H_0^2(1 - 3\lambda + 6\mu)) f''(\tilde{R}) = 0 \ , \tag{4.59}$$

here we have neglected the contribution of matter for simplicity. By performing a change of variable $x = \frac{R - 9\mu H_0^2}{3H_0^2(1 + 3(\mu - \lambda))}$ and denoting $F(x) = f(\tilde{R}(x))$, the equation (4.59) can be easily identified as an hypergeometric differential equation,

$$0 = x(1 - x)\frac{d^2F}{dx^2} + (\gamma - (\alpha + \beta + 1)\, x)\,\frac{dF}{dx} - \alpha\beta F \ , \tag{4.60}$$

with the set of parameters (α, β, γ) given by

$$\gamma = -\frac{1}{2(1 + 3(\mu - \lambda))} \ , \quad \alpha + \beta = \frac{1 + \lambda(9\mu - 1)}{3\mu(1 - 3\lambda)} \ , \quad \alpha\beta = -\frac{1 + 3(\mu - \lambda)}{6\mu(1 - 3\lambda)} \ . \tag{4.61}$$

The solution of the equation (4.60) is a Gauss' hypergeometric function [109],

$$F(x) = C_1 F(\alpha, \beta, \gamma; x) + C_2 x^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; x) \ . \tag{4.62}$$

where C_1 and C_2 are constants. Then, this action reproduces the Λ CDM model described by the Hubble parameter (4.56) without including a cosmological constants. Note that this is the same result obtained in [236] for classical $f(R)$ gravity, although in this case the solution depends on the parameters of the theory (μ, λ) whose values differ from the classical theory.
Other kind of accelerating expansions can be also reconstructed in the context of this class of theories as showed in Ref. [109]. However, due to the periodic behavior of ekpyrotic universes, models containing future singularities (usually phantom models) are not allowed in this kind of models unless a mechanism for avoiding the singularity is introduced. Nevertheless, a new class of phantom models that do not contain Big Rip singularities but only affects to bound systems without reaching a singular point, the so-called Little Rip, has been proposed in Ref. [121, 123, 122], and extended to modified gravities in Ref. [237]. Basically, these cosmological models consist on a phantom-like evolution, free of future singularities but whose strong expansion breaks the bond of some coupling systems (as galaxies, solar systems, or even atoms, nuclei...), what has been called as a Little Rip. A simple example of this kind of evolution can be described by the Hubble parameter,

$$H(t) \sim H_0 t \ , \tag{4.63}$$

where H_0 is a constant. In this case, we can also reconstruct the corresponding $f(\tilde{R})$ action by solving the FLRW equation (4.18). The scalar curvature is given by,

$$\tilde{R} = 3(1-3\lambda+6\mu)H_0^2 t^2 + 6\mu H_0 \ . \tag{4.64}$$

Then, the FLRW (4.18) yields,

$$\frac{1}{2}f(\tilde{R}) - \left(3H_0\mu + \frac{(1-3\lambda+3\mu)(R-6H_0\mu)}{1-3\lambda+6\mu}\right) f'(\tilde{R}) + 6H_0\mu(R-6H_0\mu)f''(\tilde{R}) = 0 \ . \tag{4.65}$$

This is also an hypergeometric equation, whose solution is given by,

$$f(\tilde{R}) = \left[C_1 U(\gamma,\beta;x(\tilde{R})) + C_2 L_\gamma^{(\alpha)}(x(\tilde{R}))\right](\tilde{R}-6H_0\mu)^{3/2} \ , \tag{4.66}$$

where $U(\gamma,\beta;x)$ is the confluent hypergeometric function and $L_\gamma^{(\alpha)}(x)$ is the Laguerre polynomial. The variable $x(\tilde{R})$ and the set of parameters (γ,β,α) are defined as,

$$x(\tilde{R}) = \frac{(1-3\lambda+3\mu)(R-6H_0\mu)}{6H_0\mu(1-3\lambda+6\mu)} \ , \quad \gamma = -\frac{(2-6\lambda+3\mu)}{2(1-3\lambda+3\mu)} \ , \quad \beta = \frac{5}{2} \ , \quad \alpha = \frac{3}{2} \ . \tag{4.67}$$

Hence, the $f(\tilde{R})$ action (4.66) corresponds to a series of powers in $\tilde{R}$ that are capable to reproduce a kind of behavior given by the Hubble parameter (4.63). In such case, we have that the effective energy density can be approximated as,

$$\rho_{f(\tilde{R})} \propto t^2 \ . \tag{4.68}$$

Note that for a cyclic universe, as the ones studied in section above, the phase when the universe expansion is accelerated can be approximated by (4.63), such that a Little Rip may occur in the ekpyrotic scenario. In order to show in a qualitative way how this Little Rip occurs, i.e. how some bounded systems are broken, let us compare the effective energy density (4.68) with the energy density of some known systems as the Solar-Earth system, and calculate the time remaining before the Little Rip occurs. By assuming that $\rho_{f(R)}(t_0) = \frac{3}{\kappa^2}H_0^2 \sim 10^{-47}\,\mathrm{GeV}^4$, where the age of the universe is taken to be $t_0 \sim 13.73\,\mathrm{Gyrs}$, according to Ref. [271], and a mean density of the Sun-Earth system given by $\rho_{\odot-\oplus} = 0.594\times10^{-3}\,\mathrm{kg/m}^3 \sim 10^{-21}\,\mathrm{GeV}^4$, according to the evolution (4.68), the time for the little rip is,

$$t_{LR} \sim 10^{13} Gyrs \ , \tag{4.69}$$

which is a large period compared with the current age of the universe. For other kind of expansions, as the an exponential Hubble parameter (studied in [237]), this time can be much shorter ($\sim 300Gyrs$). However, in an ekpyrotic scenario the occurrence of a Little Rip will depend on the duration of the accelerating phase before this ends, and a new contracting phase starts again. Note also that close to the dissolution of the bound structure, gravity will be very strong, and the breaking of Lorentz invariance will be present, such that the values of (λ, μ) will determine the expansion rate, and for instance the occurrence of the Little Rip.

Let us now consider a model that may reproduce a entire cycle of an ekpyrotic universe,

$$H = H_0 - H_1 e^{-\beta t}. \tag{4.70}$$

For $H_1 > H_0$, the Hubble parameter (4.70) represents a universe that crosses through out a contracting phase, and then ends in an accelerating expansion for large times. Obviously, one would need to provide the way to start a cycle again, however for a qualitative description, we assume here that the cycle starts again after the accelerating phase somehow. For the solution (4.70), we have

$$\tilde{R} = AH^2 + 6\mu\dot{H} = A(H_0^2 - 2H_0H_1e^{-\beta t} + H_1^2e^{-2\beta t}) - 6\mu\beta H_1 e^{-\beta t} \tag{4.71}$$

where we recall that $A = 3(1-3\lambda+6\mu)$. From (4.71) we get

$$e^{-\beta t} = \frac{(AH_0 + 3\mu\beta) \pm \sqrt{(AH_0+3\mu\beta)^2 - (AH_0^2 - \tilde{R})}}{H_1} \tag{4.72}$$

For simplicity we consider the case when $AH_0 + 3\mu\beta = 0$. Then Eq. (4.72) gives

$$e^{-\beta t} = \pm \frac{\sqrt{\tilde{R} - AH_0^2}}{H_1} \ . \tag{4.73}$$

And the Hubble parameter (4.70) can be rewritten in terms of the scalar curvature $\tilde{R}$,

$$H = H_0 - H_1 e^{-\beta t} = H_0 \mp \sqrt{\tilde{R} - AH_0^2}. \tag{4.74}$$

In this case the first Friedmann equation (4.18) yields,

$$12\mu\beta(AH_0^2 - \tilde{R})\left(H_0 \mp \sqrt{\tilde{R} - AH_0^2}\right) f''(\tilde{R}) - Bf'(\tilde{R}) + f(\tilde{R}) - \kappa^2\rho_m = 0 \ , \tag{4.75}$$

where $B = 6[(1-3\lambda+3\mu)H^2 + \mu\dot{H}]$. Then, by setting $\beta = \frac{2H_0(1-3\lambda+3\mu)}{\mu}$, we obtain,

$$B = 6(1-3\lambda+3\mu)(1-A)H_0^2 + 6(1-3\lambda+3\mu)\tilde{R} \ . \tag{4.76}$$

Eq. (4.75) is still a very difficult expression, so that the search of exact solutions for $f(\tilde{R})$ is a difficult task. Nevertheless, we can reconstruct some particular exact actions by considering special matter fluids. Let us consider the matter energy density,

$$\rho_m = \kappa^{-2}\left[12\mu\beta(AH_0^2 - \tilde{R})\left(H_0 \mp \sqrt{\tilde{R} - AH_0^2}\right) f''(\tilde{R}) - Ca^{-3}\right] \ . \tag{4.77}$$

Then the FLRW equation (4.75) admits the following particular solution

$$f(\tilde{R}) = C_1[6(1-3\lambda+3\mu)\tilde{R} + 6(1-3\lambda+3\mu)(1-A)H_0^2]^{\frac{1}{6(1-3\lambda+3\mu)}} . \tag{4.78}$$

In a similar way, other particular solutions of the Friedmann equations can be reconstructed. Hence, we have shown here that ekpyrotic universes can be well described in the frame of Hořava–Lifshitz gravity.

4.4 Discussions

In the present chapter, we have analyzed some particular cosmological solutions in the context of Hořava–Lifshitz gravity, where basically some generalizations of the original action [146], similar to standard $f(R)$ gravity, have been studied. It is well known that for a particular Hubble parameter, the corresponding action can be reconstructed in the framework of $f(\tilde{R})$ Hořava–Lifshitz gravity (see Ref [109]), where the presence of the set of parameters $\{\lambda, \mu\}$, consequence of the restriction of the symmetries of the theory, can vary along the cosmological evolution, since their value depends on the energy scale of a particular system. Hence, the presence of this set of parameters will fluctuate along the universe evolution, affecting the corresponding cosmic solution. By assuming that General Relativity should be recovered when $\tilde{R} \sim H_0^2 << m_{pl}^4 \sim 10^{74} GeV^4$, the parameters $\lambda = \mu \sim 1$ during the radiation/matter dominated epoch and the current accelerating era, while it becomes large when $\tilde{R} \propto m_{pl}^4$, where the quantum effects should become important. In this sense, the effects of Hořava–Lifshitz gravity, and specifically the extra scalar mode, may become important when the universe reaches stages as the Little Rip, or other phases from a typical ekpyrotic universe.

Hence, in the particular solutions studied here, the ekpyrotic scenario becomes an important focus for analyzing Hořava–Lifshitz gravity, as the universe owns a periodic behavior, crossing different stages, where the quantum nature of the theory may be relevant. Moreover, we have shown that particular actions which lead to a cyclic nature of the Hubble parameter can be reconstructed. Several techniques have been used for the reconstruction procedure. By using an auxiliary scalar field, coming from the $f(\tilde{R})$ sector, we have shown that cosmological solutions can be easily obtained. In addition, we have studied the shape of the action along each phase of a typical ekpyrotic universe, where the corresponding actions have been obtained. It is straightforward to show that such actions lead to standard $f(R)$ gravity when $\lambda = \mu = 1$, and can be identified with some particular viable theories [236]. Then, we can conclude that this class of actions can perfectly describe the entire universe evolution by means of an ekpyrotic model. Moreover, we have suggested the compatibility between an ekpyrotic universe and the presence of a Little Rip, a non singular point that may lead to the break of some bounded systems, where the effects of Hořava–Lifshitz gravity turn out important, and $\lambda \neq 1, \mu \neq 1$. Future singularities can not be compatible with a cyclic universe unless a cure for the singularity is considered [186]. A next step should be to probe the possibility to reproduce cyclic cosmologies within the frame of so-called viable $f(\tilde{R})$ gravities (see for instance, Ref. [148]). While the violation of Newtonian law can be avoided in $f(\tilde{R})$ Hořava-Lifshitz gravity (see [109]), the presence of instabilities and other features should be studied in more detail.

On the other hand, in order to have a complete picture of the universe evolution, one should specify how reheating occurs. Nevertheless, this is beyond of the scope of this work, but an interesting proposal for a reheating mechanism in the frame of UV complete theory is pointed out in [132].

Therefore, in an ekpyrotic universe, the main implications of $f(\tilde{R})$ Hořava–Lifshitz gravity would come during those phases when the full diffeomorphisms are broken, basically during the early and ending phases, that may affect other *classical* eras, specially by the perturbations, which should be an important point to be studied in the future, where the effects may be distinguishable from other models.

Part II

Perturbation growth and the whole cosmic history

Chapter 5

Cosmic history of viable exponential gravity

In this chapter, we study a generic feature of viable $F(R)$ gravity models, in particular, exponential gravity and a power form model. The conditions for the viability are summarized as follows: (i) Positive definiteness of the effective gravitational coupling. (ii) Matter stability condition [101, 118, 215, 268]. (iii) In the large curvature regime, the model is close to the Λ–Cold–Dark–Matter (ΛCDM) model asymptotically. (iv) Stability of the late–time de Sitter point [16, 117, 206]. (v) The equivalence principle. (vi) Solar–system tests [81, 83, 215, 243]. We find that the behavior of higher derivatives of the Hubble parameter may be influenced by large frequency oscillations of effective dark energy, which makes solutions singular and unphysical at a high redshift. Therefore, in order to stabilize such oscillations, we examine an additional correction term to the model and remove such an instability with keeping the viability properties. We also demonstrate the cosmological evolutions of the universe and growth index of the matter density perturbations in detail. Furthermore, by applying two viable models of exponential gravity to inflationary cosmology and executing the numerical analysis of the inflation process, we illustrate that the exit from inflation can be realized. Concretely, we demonstrate that different numbers of e–folds during inflation can be obtained by taking different model parameters in the presence of ultrarelativistic matter, the existence of which makes inflation end and leads to the exit from inflation. Indeed, we observe that at the end of the inflation, the effective energy density as well as the curvature of the universe decrease. Accordingly, a unified description between inflation and the late time cosmic acceleration is presented. We use units of $k_{\mathrm{B}} = c = \hbar = 1$ and denote the gravitational constant $8\pi G$ by $\kappa^2 \equiv 8\pi/{M_{\mathrm{Pl}}}^2$ with the Planck mass of $M_{\mathrm{Pl}} = G^{-1/2} = 1.2 \times 10^{19}$ GeV.

The chapter is organized as follows. In Sec. I, the formulations of $F(R)$ gravity is briefly review. We use the fluid representation of $F(R)$ gravity [68, 69]. Here, in the Friedmann–Lemaître–Robertson–Walker (FLRW) background, the equations of motion with the addition of an effective gravitational fluid are presented. In Sec. II, we explain two well–known viable $F(R)$ gravity models and show those generic features occurring in the matter dominated era, when large frequency oscillation of dark energy appears and influences on the behavior of higher derivatives of the Hubble parameter in terms of time with the risk to produce some divergence and to render the solution unphysical. Thus, we suggest a way to stabilize such oscillations by introducing an additive modification to the models. We also perform a numerical analysis of the matter dominated era. In Sec. III, we demonstrate that the term added to stabilize the dark energy oscillations in the matter dominated epoch does not cause any problem on the viability of the models, which satisfy the cosmological and local gravity constraints. We investigate their future evolution and show that the effective crossing of the phantom divide, which characterizes the de Sitter epoch, takes place in the very far future. We also analyze the growth index using three different ansatz choices. The second part of the chapter is devoted to the study of $F(R)$ models for the unification of the early–time cosmic acceleration, i.e., inflation, and the late–time one. In Sec. IV, we explore two applications of exponential gravity for inflation. In particular, we show how it is possible to obtain different numbers of e –folds during inflation by making different choices of model parameters in the presence of ultrarelativistic matter in the early universe. In Sec. V, we execute the numerical analysis of inflation and illustrate that at the end of it the effective energy density and the curvature decrease and eventually the cosmology in the ΛCDM model can follow. Finally, the summary and outlook for this chapter are given in Sec. VI. For reference, we also explain the procedure of conformal transformation in Appendix B and asymptotically phantom or quintessence modified gravity in Appendix C.

This Chapter is based on the publications: [31].

5.1 $F(R)$ gravity and its dynamics in the FLRW universe: General overview

In this section, we briefly review formulations in $F(R)$ gravity and derive the gravitational field equations in the FLRW space–time. The action describing $F(R)$ gravity is given by

$$I = \int_{\mathcal{M}} d^4x\sqrt{-g}\left[\frac{F(R)}{2\kappa^2} + \mathcal{L}^{(\mathrm{matter})}\right], \tag{5.1}$$

where $F(R)$ is a generic function of the Ricci scalar R only, g is the determinant of the metric tensor $g_{\mu\nu}$, $\mathcal{L}^{(\mathrm{matter})}$ is the matter Lagrangian and $\mathcal{M}$ denotes the space–time manifold. In a large class of modified gravity models reproducing the standard cosmology in General Relativity (GR), i.e., $F(R) = R$, with a suitable correction to realize current acceleration and/or inflation, one represents

$$F(R) = R + f(R) . \tag{5.2}$$

Thus, the modification of gravity is encoded in the function $f(R)$, which is added to the classical term R of the Einstein–Hilbert action in GR. In what follows, we discuss modified gravity in this form by explicitly separating the contribution of its modification from GR. The field equation simply reads

$$F'(R)\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = \kappa^2 T^{(\mathrm{matter})}_{\mu\nu} + \left[\frac{1}{2}g_{\mu\nu}\left(F(R) - RF'(R)\right) + \left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)F'(R)\right] . \tag{5.3}$$

Here, ∇_μ is the covariant derivative operator associated with $g_{\mu\nu}$, $\Box\phi \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu\phi$ is the covariant d'Alembertian for a scalar field ϕ, and $T^{\mu(\mathrm{matter})}_{\nu} = \mathrm{diag}\left(-\rho_{\mathrm{m}}, P_{\mathrm{m}}, P_{\mathrm{m}}, P_{\mathrm{m}}\right)$ is the contribution to the stress energy–momentum tensor from all ordinary matters, with ρ_{m} and P_{m} being the energy density and pressure of matter, respectively. Moreover, the prime denotes the derivative with respect to the curvature R.

The flat FLRW space–time is described by the metric $ds^2 = -dt^2 + a(t)^2 d\mathbf{x}^2$, where $a(t)$ is the scale factor of the universe. The Ricci scalar reads

$$R = 12H^2 + 6\dot{H} , \tag{5.4}$$

where $H = \dot{a}(t)/a(t)$ is the Hubble parameter and the dot denotes the time derivative of $\partial_t (\equiv \partial/\partial t)$. In the flat FLRW background, from the $(\mu,\nu) = (0,0)$ component and the trace part of $(\mu,\nu) = (i,j)$

(with $i, j = 1, \cdots, 3$) components in Eq. (5.3), we obtain the gravitational field equations [221, 234]

$$\rho_{\mathrm{eff}} = \frac{3}{\kappa^2} H^2 \,, \tag{5.5}$$

$$P_{\mathrm{eff}} = -\frac{1}{\kappa^2} \left(2\dot{H} + 3H^2 \right) . \tag{5.6}$$

Here, ρ_{eff} and P_{eff} are the effective energy density and pressure of the universe, respectively, defined as

$$\rho_{\mathrm{eff}} \equiv \rho_{\mathrm{m}} + \frac{1}{2\kappa^2} \left[(F'R - F) - 6H^2(F' - 1) - 6H\dot{F}' \right] , \tag{5.7}$$

$$P_{\mathrm{eff}} \equiv P_{\mathrm{m}} + \frac{1}{2\kappa^2} \left[-(F'R - F) + (4\dot{H} + 6H^2)(F' - 1) + 4H\dot{F}' + 2\ddot{F}' \right] . \tag{5.8}$$

In this way, we have a fluid representation of the so–called geometrical dark energy in $F(R)$ gravity with the energy density $\rho_{\mathrm{DE}} = \rho_{\mathrm{eff}} - \rho$ and pressure $P_{\mathrm{DE}} = P_{\mathrm{eff}} - P$. However, it is important for us to remember that gravitational terms enter in both left and right sides of Eqs. (5.5) and (5.6). For general relativity in which $F(R) = R$, $\rho_{\mathrm{eff}} = \rho_{\mathrm{m}}$ and $P_{\mathrm{eff}} = P_{\mathrm{m}}$ and therefore Eqs. (5.5) and (5.6) lead to Friedman equations.

We also explain basic equations that we use to carry out our analysis. In order to study the dynamics of $F(R)$ gravity models in the flat FLRW universe, we may introduce the variable [28, 148]

$$y_H(z) \equiv \frac{\rho_{\mathrm{DE}}}{\rho_{\mathrm{m}(0)}} = \frac{H^2}{\tilde{m}^2} - (z+1)^3 - \chi(z+1)^4 \,. \tag{5.9}$$

Here, $\rho_{\mathrm{m}(0)}$ is the energy density of matter at the present time, $\tilde{m}^2$ is the mass scale, given by

$$\tilde{m}^2 \equiv \frac{\kappa^2 \rho_{\mathrm{m}(0)}}{3} \simeq 1.5 \times 10^{-67} \mathrm{eV}^2 \,,$$

and χ is defined as [169]

$$\chi \equiv \frac{\rho_{\mathrm{r}(0)}}{\rho_{\mathrm{m}(0)}} \simeq 3.1 \times 10^{-4} \,,$$

where $\rho_{\mathrm{r}(0)}$ is the current energy density of radiation and $z = 1/a(t) - 1$ is the redshift. Here, we have taken the current value of the scale factor as unity. By using Eqs. (5.5) and (5.9), we find

$$\frac{d^2 y_H(z)}{dz^2} + J_1 \frac{dy_H(z)}{dz} + J_2 \left(y_H(z) \right) + J_3 = 0 \,, \tag{5.10}$$

where

$$J_1 = \frac{1}{(z+1)} \left[-3 - \frac{1}{y_H + (z+1)^3 + \chi(z+1)^4} \frac{1 - F'(R)}{6\tilde{m}^2 F''(R)} \right] , \tag{5.11}$$

$$J_2 = \frac{1}{(z+1)^2} \left[\frac{1}{y_H + (z+1)^3 + \chi(z+1)^4} \frac{2 - F'(R)}{3\tilde{m}^2 F''(R)} \right] , \tag{5.12}$$

$$J_3 = -3(z+1) - \frac{(1 - F'(R))((z+1)^3 + 2\chi(z+1)^4) + (R - F(R))/(3\tilde{m}^2)}{(z+1)^2 (y_H + (z+1)^3 + \chi(z+1)^4)} \frac{1}{6\tilde{m}^2 F''(R)} . \tag{5.13}$$

Furthermore, the Ricci scalar is expressed as

$$R = 3\tilde{m}^2 \left[4y_H(z) - (z+1) \frac{dy_H(z)}{dz} + (z+1)^3 \right] . \tag{5.14}$$

In deriving this equation, we have used the fact that $-(z+1)H(z)d/dz = H(t)d/d(\ln a(t)) = d/dt$, where H could be an explicit function of the red shift as $H = H(z)$, or an explicit function of the time as $H = H(t)$. In general, Eq. (5.10) can be solved in a numerical way, once we write the explicit form of an $F(R)$ gravity model.

5.2 Generic feature of realistic $F(R)$ gravity models in the matter dominated era

In this section, we consider viable $F(R)$ gravity models representing a realistic scenario to account for dark energy, in particular, two well–known ones proposed in Refs. [18, 87, 148, 180, 274, 281] (for more examples and detailed explanations on viable models, see, e.g., [28, 29] and references therein). Here, we mention that in Ref. [294], the gravitational waves in viable $F(R)$ models have been studied, and that the observational constraints on exponential gravity have also been examined in Ref. [295]. We show that for these models, large frequency oscillation of dark energy in the matter dominated era appears, and that it may influence on the behavior of higher derivatives of the Hubble parameter with respect to time. Such a oscillation has the risk to produce some divergence, and therefore we suggest a way to stabilize the frequency oscillation by performing the subsequent numerical analysis. In these models, a correction term to the Hilbert–Einstein action is added as $F(R) = R + f(R)$ in (5.2), so that the current acceleration of the universe can be reproduced in a simple way. Namely, a vanishing (or fast decreasing) cosmological constant in the flat limit of $R \to 0$ is incorporated, and a suitable, constant asymptotic behavior for large values of R is exhibited.

5.2.1 Realistic $F(R)$ gravity models

First, we explore the Hu–Sawicki model [148] (for the related study of such a model, see Ref. [12, 19, 99, 114, 116, 129, 144, 150, 173, 227, 241, 285]),

$$F(R) = R - \frac{\tilde{m}^2 c_1 (R/\tilde{m}^2)^n}{c_2 (R/\tilde{m}^2)^n + 1} = R - \frac{\tilde{m}^2 c_1}{c_2} + \frac{\tilde{m}^2 c_1/c_2}{c_2 (R/\tilde{m}^2)^n + 1}\,, \tag{5.15}$$

where $\tilde{m}^2$ is the mass scale, c_1 and c_2 are positive parameters, and n is a natural positive number. The model is very carefully constructed such that in the high curvature regime, $\tilde{m}^2 c_1/c_2 = 2\Lambda$ can play a role of the cosmological constant Λ and thus the ΛCDM model can be reproduced.

Moreover, in Refs. [87, 180] another simple model which may easily be generalized to reproduce also inflation has been constructed

$$F(R) = R - 2\Lambda \left[1 - \mathrm{e}^{-R/(b\,\Lambda)}\right]\,, \tag{5.16}$$

where $b > 0$ is a free parameter. Also in this model, in the flat space the solution of the Minkowski space–time is recovered, while at large curvatures the ΛCDM model is realized. This kind of models can satisfy the cosmological and local gravity constraints. Both of these models asymptotically approach the ΛCDM model in the high curvature regime. Indeed, however, the mechanisms work in two different manners, i.e., via a power function of R (the first one) and via an exponential function of it (the second one). For our treatment, we reparameterize the model (5.15) by describing $c_1\tilde{m}^2/c_2 = 2\Lambda$ and $(c_2)^{1/n}\,\tilde{m}^2 = b\,\Lambda$ with $b > 0$, so that we can obtain

$$F(R) = R - 2\Lambda \left\{1 - \frac{1}{\left[R/\,(b\,\Lambda)\right]^n + 1}\right\}\,, \quad n = 4\,. \tag{5.17}$$

Through this procedure, in both of these models the term $b\,\Lambda$ corresponds to the curvature for which the cosmological constant is "switched on". This means $b \ll 4$, so that $b\,\Lambda \ll 4\Lambda$ and hence $R = 4\Lambda$ can be the curvature of de Sitter universe describing the current cosmic acceleration. In the model in Eq. (5.17), since n has to be sufficiently large in order to reproduce the ΛCDM model, we have assumed $n = 4$ and we keep only the parameter b free.

5.2.2 Dark energy oscillations in the matter dominated era

Despite the fact that the models in Eqs. (5.16) and (5.17) precisely resemble the ΛCDM model, there is a problem that in the matter dominated era the higher derivatives of the Hubble parameter diverge and thus this can make the solutions unphysical. This problem originates from the stability conditions to be satisfied by these models [113] and from dark energy oscillations during the matter phase [274] in Ref. [16]. Since in matter dominated era $R = 3\tilde{m}^2(z+1)^3$ and $y_H(z) \ll (1+z)^3$ and $\chi(1+z)^4 \ll (z+1)^3$ in order for dark energy and radiation to vanish during this phase, one may locally solve Eq. (5.10) around $z = z_0 + (z - z_0)$, where $|z - z_0| \ll z$. The solution reads to the first order in terms of $(z - z_0)$,

$$y''_H(z) + \frac{\alpha}{(z-z_0)} y'_H(z) + \frac{\beta}{(z-z_0)^2} y_H(z) = \zeta_0 + \zeta_1 (z - z_0)\,, \tag{5.18}$$

where

$$\begin{aligned} \alpha &= -\frac{7}{2} - \frac{(1-F'(R_0))F'''(R_0)}{2F''(R_0)^2}\,, \\ \beta &= 2 + \frac{1}{R_0 F''(R_0)} + \frac{2(1-F'(R_0))F'''(R_0)}{F''(R_0)^2}\,, \end{aligned} \tag{5.19}$$

with ζ_0 and ζ_1 being constants and $R_0 = 3\tilde{m}^2(z_0+1)^3$. Thus, the solution of Eq. (5.18) is derived as

$$y_H(z) = a + b\cdot(z-z_0) + C_0 \cdot \exp \frac{1}{2(z_0+1)} \left(-\alpha \pm \sqrt{\alpha^2 - 4\beta}\right)(z-z_0)\,, \tag{5.20}$$

where a, b and C_0 are constants. Now, for the two models in Eqs. (5.16) and (5.17), when $R \gg b\,\Lambda$, we find

$$\begin{aligned} F'(R) &\simeq 1\,, \\ F''(R) &\simeq 0^+\,. \end{aligned} \tag{5.21}$$

These behaviors guarantee the occurrence of the realistic matter dominated era. Furthermore, since in the expanding universe $(z - z_0) < 0$, it turns out that the dark energy perturbations in Eq. (5.20) remain small around R_0, and that we acquire

$$\frac{(1-F'(R_0))\,F'''(R_0)}{2F''(R_0)^2} > -\frac{7}{2}\,, \quad \frac{1}{R_0 F''(R_0)} > 12\,, \tag{5.22}$$

for both these models. Owing to the fact that $F''(R)$ is very close to 0^+, the discriminant in the square root of Eq. (5.20) is negative and dark energy oscillates as

$$y_H(z) = \frac{\Lambda}{3\tilde{m}^2} + \mathrm{e}^{-\frac{\alpha_{1,2}(z-z_0)}{2(z_0+1)}} \left[A \sin\left(\frac{\sqrt{\beta_{1,2}}}{(z_0+1)}(z-z_0)\right) + B\cos\left(\frac{\sqrt{\beta_{1,2}}}{(z_0+1)}(z-z_0)\right)\right]\,. \tag{5.23}$$

Here, A and B are constants and $\alpha_{1,2}$ and $\beta_{1,2}$ are given by Eq. (5.19), so they correspond to two models under investigation. In particular, $\alpha_1 = -3$ for the model in Eq. (5.16) and $\alpha_2 \simeq -29/10$ for the model in Eq. (5.17), while $\beta_{1,2} \simeq 1/(R_0 F''(R_0))$, i.e.,

$$\beta_1 \simeq \left(\frac{b^2 \Lambda \mathrm{e}^{\frac{R_0}{R}}}{2R_0}\right)\,, \tag{5.24}$$

in case of exponential model in Eq. (5.16) and

$$\beta_2 \simeq \frac{R_0 \left[1+\left(\frac{R_0}{b\Lambda}\right)^n\right]^3 \left(\frac{b\Lambda}{R_0}\right)^n}{2\Lambda\, n \left\{1+n\left[\left(\frac{R_0}{b\Lambda}\right)^n - 1\right] + \left(\frac{R_0}{b\Lambda}\right)^n\right\}} \simeq \frac{R_0}{2\Lambda\, n(n+1)} \left(\frac{R_0}{b\Lambda}\right)^n , \tag{5.25}$$

in case of model in Eq. (5.17). This means that the frequency of dark energy oscillations increases as the curvature (and redshift) becomes large. Moreover, the effects of such oscillations are amplified in the derivatives of the dark energy density, namely,

$$\left|\frac{d^n}{dt^n} y_H(t_0)\right| \propto \left(\mathcal{F}(z_0)\right)^n , \tag{5.26}$$

where $\mathcal{F}(z) \simeq \left(R * F''(R)\right)^{-1/2}/(z{+}1)$ is the oscillation frequency and t_0 is the cosmic time corresponding to the redshift z_0. This is for example the case of the EoS parameter for dark energy defined as[1]

$$\omega_{\rm DE}(z) \equiv \frac{P_{\rm DE}}{\rho_{\rm DE}} = -1 + \frac{1}{3}(z+1)\frac{1}{y_H(z)}\frac{dy_H(z)}{d(z)} . \tag{5.27}$$

For large values of the redshift, the dark energy density oscillates with a high frequency and also its derivatives become large, showing a different feature of the dark energy EoS parameter in the models in Eqs. (5.16) and (5.17) compared with the case of the cosmological constant in GR. During the matter dominated era, the Hubble parameter behaves as

$$H(z) \simeq \sqrt{\tilde{m}^2}\left[(z+1)^{3/2} + \frac{y_H(z)}{2(z+1)^{3/2}}\right] . \tag{5.28}$$

If the frequency $\mathcal{F}(z_0)$ in Eq. (5.26) is extremely large, the derivatives of dark energy density could become dominant in some higher derivatives of the Hubble parameter which may approach an effective singularity and therefore make the solution unphysical. We see it for specific cases. In Refs. [28, 111, 174], the cosmological evolutions in exponential gravity and the Hu–Sawicki model have carefully been explored. It has explicitly been demonstrated that the late–time cosmic acceleration which follows the matter dominated era can occur, according with astrophysical data. A reasonable choice is to take $b = 1$ for both these models. We also put $\Lambda = 7.93\tilde{m}^2$ [169]. We can solve Eq. (5.10) numerically[2] by taking the initial conditions at $z = z_i$, where $z_i \gg 0$ is the redshift at the initial time to execute the numerical calculation, as follows:

$$\begin{aligned} \left.\frac{dy_H(z)}{d(z)}\right|_{z_i} &= 0 , \\ \left. y_H(z)\right|_{z_i} &= \frac{\Lambda}{3\tilde{m}^2} . \end{aligned}$$

Here, we have used the fact that at a high redshift the universe should be very close to the ΛCDM model. We have set $z_i = 2.80$ for the model in Eq. (5.16) and $z_i = 4.5$ for the model in Eq. (5.17), such that $RF''(R) \sim 10^{-8}$ at $R = 3\tilde{m}^2(z_i+1)^3$. We note that it is hard to extrapolate the numerical results to the higher redshifts because of the large frequency of dark energy oscillations.

Using Eq. (5.27) with y_H, we derive $\omega_{\rm DE}$. In addition, by using Eq. (5.14) we obtain R as a function of the redshift. We can also execute the extrapolation in terms of the behavior of $\Omega_{\rm DE}$, given by

$$\Omega_{\rm DE}(z) \equiv \frac{\rho_{\rm DE}}{\rho_{\rm eff}} = \frac{y_H}{y_H + (z+1)^3 + \chi\,(z+1)^4} . \tag{5.29}$$

[1] Throughout this paper, we describe the EoS parameter by "ω" and not "w".
[2] We have used Mathematica 7 ©.

The numerical extrapolation to the present universe leads to the following results: For the model (5.16), $y_H(0) = 2.736$, $\omega_{\rm DE}(0) = -0.950$, $\Omega_{\rm DE}(0) = 0.732$ and $R(z = 0) = 4.365$, whereas for the model (5.17), $y_H(0) = 2.652$, $\omega_{\rm DE}(0) = -0.989$, $\Omega_{\rm DE}(0) = 0.726$ and $R(z = 0) = 4.358$. These resultant data are in accordance with the last and very accurate observations of our current universe [169], which are

$$\begin{aligned} \omega_{\rm DE} &= -0.972^{+0.061}_{-0.060}\,, \\ \Omega_{\rm DE} &= 0.721 \pm 0.015\,. \end{aligned} \tag{5.30}$$

Next, we introduce the deceleration q , jerk j and snap s parameters [82, 257]

$$\begin{aligned} q(t) &\equiv -\frac{1}{a(t)}\frac{d^2a(t)}{dt^2}\frac{1}{H(t)^2} = -\frac{\dot{H}}{H^2} - H^2 \\ j(t) &\equiv \frac{1}{a(t)}\frac{d^3a(t)}{dt^3}\frac{1}{H(t)^3} = \frac{\ddot{H}}{H^3} - 3q - 2 \\ s(t) &\equiv \frac{1}{a(t)}\frac{d^4a(t)}{dt^4}\frac{1}{H(t)^4} = \frac{\dddot{H}}{H^4} + 4j + 3q(q+4) + 6\,. \end{aligned} \tag{5.31}$$

In what follows, we show the values of these cosmological parameters at the present time ($z = 0$) as the result of numerical extrapolation in our two models, which we called Model I in Eq. (5.16) and Model II in Eq. (5.17), and the calculation in the ΛCDM model:

$$\begin{aligned} q(z=0) &= -0.650\,(\Lambda\mathrm{CDM})\,,\ -0.544\,(\mathrm{Model\,I})\,,\ -0.577\,(\mathrm{Model\,II}) \\ j(z=0) &= 1.000\,(\Lambda\mathrm{CDM})\,,\ 0.792(\mathrm{Model\,I})\,,\ 0.972\,(\mathrm{Model\,II}) \\ s(z=0) &= -0.050\,(\Lambda\mathrm{CDM})\,,\ -0.171(\mathrm{Model\,I})\,, -0.152\ (\mathrm{Model\,II})\,. \end{aligned}$$

The deviations of the parameters in Models I and II from those in the ΛCDM model are small at the present. However, since these parameters depend on the time derivatives of the Hubble parameter, it is interesting to analyze those behaviors at high curvature. Therefore, in Fig. 5.1 we plot the cosmological evolutions of q , j and s as functions of the redshift z . From this figure, we see that there exist overlapped regions for Models I and II with those in the ΛCDM model.

The deceleration parameter in Models I and II remains very close to the value in the ΛCDM model, because in the first time derivative of the Hubble parameter the contribution of dark energy is still negligible. Hence, it guarantees the correct cosmological evolution of these models. However, it is clearly seen that in the jerk and snap parameters the derivatives of the dark energy density become relevant and the parameters grow up with an oscillatory behavior. Since the frequency of such oscillations strongly increases in the redshift, it is reasonable to expect that some divergence occurs in the past. We also remark that if from one side at high redshifts the exponential Model I is more similar to the ΛCDM model because of the faster decreasing of exponential function in comparison with the power function of Model II, from the other side it involves stronger oscillations in the matter dominated era.

It may be stated that the closer the model is to the ΛCDM model (i.e., as much $F''(R)$ is close to zero), the bigger the oscillation frequency of dark energy becomes. As a consequence, despite the fact that the dynamics of the universe depends on the matter and the dark energy density remains very small, some divergences in the derivatives of the Hubble parameter can occur. In the models in Eqs. (5.16) and (5.17), although the approaching manners to a model with the cosmological constant are different from each other, it may be interpreted that these models show a generic feature of realistic $F(R)$ gravity models, in which the cosmological evolutions are similar to those in a model with the cosmological constant. The corrections to the Einstein's equations in the small curvature regime lead to undesired effects in the high curvature regime. Thus, we need to investigate additional modifications.

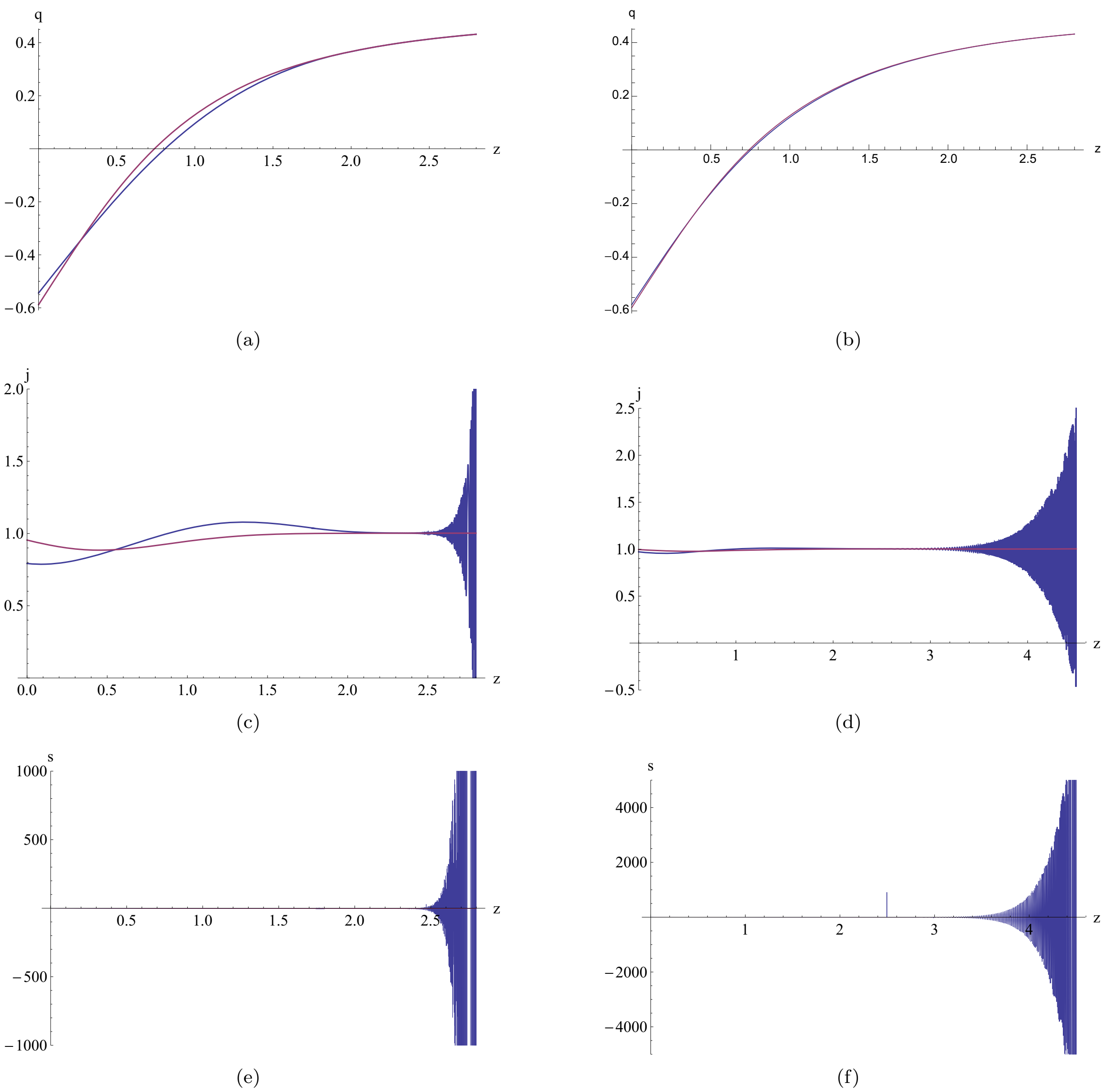


Figure 5.1: Cosmological evolutions of $q(z)$ [(a) and (b)], $j(z)$ [(c) and (d)] and $s(z)$ [(e) and (f)] parameters as functions of the redshift z for Model I [(a),(c),(e)] and Model II [(b),(d),(f)] in the region of $z > 0$.

5.2.3 Proposal of a correction term

In order to remove the divergences in the derivatives of the Hubble parameter, we introduce a function $g(R)$ for which the oscillation frequency of the dark energy density in Eq. (5.23) acquires a constant value $1/\sqrt{\delta}$, where $\delta > 0$, for a generic curvature $R \gg b\Lambda$, and we stabilize the oscillations of dark energy during the matter dominated era with the use of a correction term. Since in the matter dominated era, i.e., $z+1=\left[R/(3\tilde{m}^2)\right]^{1/3}$, we have to require

$$\begin{aligned}\frac{(3\tilde{m}^2)^{2/3}}{R^{5/3}\,g''(R)} &= \frac{1}{\delta}\\ g(R) &= -\tilde{\gamma}\,\Lambda\left(\frac{R}{3\tilde{m}^2}\right)^{1/3}, \quad \tilde{\gamma}>0\,,\end{aligned} \tag{5.32}$$

where $\tilde{\gamma}\equiv(9/2)\delta(3\tilde{m}^2/\Lambda)=1.702\,\delta$. We explore the models in Eqs. (5.16) and (5.17) with adding these correction as

$$F_1(R) = R-2\Lambda(1-\mathrm{e}^{-\frac{R}{b\,\Lambda}})-\tilde{\gamma}\,\Lambda\left(\frac{R}{3\tilde{m}^2}\right)^{1/3}, \tag{5.33}$$

$$F_2(R) = R-2\Lambda\left[1-\frac{1}{(R/b\,\Lambda)^4+1}\right]-\tilde{\gamma}\,\Lambda\left(\frac{R}{3\tilde{m}^2}\right)^{1/3}. \tag{5.34}$$

We note that in both cases $F_{1,2}(0)=0$ and therefore we still have the solution of the flat space in the Minkowski space–time. The effects of the last term vanish in the de Sitter epoch, when $R=4\Lambda$ and these models resemble to a model with an effective cosmological constant, provided that $\tilde{\gamma}\ll(\tilde{m}^2/\Lambda)^{1/3}$. We may also evaluate the dark energy density at high redshifts by deriving $\rho_{\mathrm{DE}}=\rho_{\mathrm{eff}}-\rho_{\mathrm{m}}$ from Eq. (5.7) and by putting $R=3\tilde{m}^2(z+1)^3$ such that

$$y_H(z)\simeq\frac{\Lambda}{3\tilde{m}^2}\left[1+\tilde{\gamma}(1+z)\right]. \tag{5.35}$$

According to the observational data of our universe, the current value of dark energy amount is estimated as $y_H\equiv\Lambda/(3\tilde{m})=2.643$. With the reasonable choice $\tilde{\gamma}\sim1/1000$, the effects of modification of gravity on the dark energy density begin to appear at a very high redshift (for example, at $z=9$, $y_H(9)=1.01\times y_H(0)$), and hence the universe seems to be very close to the ΛCDM model. However, while the pure models in Eqs. (5.16) and (5.17) mimic an effective cosmological constant, the models in Eqs. (5.33) and (5.34) mimic (for the matter solution) a quintessence fluid. Equation (5.27) leads to

$$\omega_{\mathrm{DE}}(z)\simeq-1+\frac{(1+z)\tilde{\gamma}}{3(1+(1+z)\tilde{\gamma})}, \tag{5.36}$$

so that when $z\gg\tilde{\gamma}^{-1}$, $\omega_{\mathrm{DE}}(z)\simeq-2/3$.

Thus, it is simple to verify that all the cosmological constraints [66] are still satisfied. Since $|F'_{1,2}(R\gg b\Lambda)-1|\ll1$, the effective gravitational coupling $G_{\mathrm{eff}}=G/F'_{1,2}(R)$ is positive, and hence the models are protected against the anti–gravity during the cosmological evolution until the de Sitter solution ($R_{\mathrm{dS}}=4\Lambda$) of the current universe is realized. Thus, thanks to the fact that $|F''_{1,2}(R\gg b\Lambda)>0|$, we do not have any problem in terms of the existence of a stable matter. In Sec. 5.3, we also analyze the local constraints in detail, and we see that our modifications do not destroy the feasibility of the models in the solar system. It should be stressed that the energy density preserves its oscillation behavior in the matter dominated era, but that owing to the correction term reconstructed here, such oscillations keep a constant frequency $\mathcal{F}=\sqrt{1.702/\tilde{\gamma}}$ and do not diverge. Despite the small value of $\tilde{\gamma}$, in this way the high redshift divergences and possible effective singularities are removed.

From the point of view of the end of inflation, there is another resolution of this problem. It is well known that the scalar begins to oscillate once the mass m becomes larger than the Hubble parameter, $H<m$. Indeed, for a canonical scalar, the energy density sloshes between the potential energy ($w=-1$, where w is the equation of state of the canonical scalar) and the kinetic energy ($w=+1$). What is done usually is that the oscillations enough rapidly (i.e., those with $m\gg H$) can be averaged over giving an

effective energy–momentum tensor with $w = 0$, i.e., dust. The same procedure should be performed here, once the oscillations are rapid enough. In this interpretation, there would be no problem with any strange rapidly oscillating contributions to the energy momentum tensor. A solution is to choose the potential effectively so that the mass can not increase as the matter energy density increases.

Furthermore, it is significant to remark that in a number of models of $F(R)$ gravity for dark energy, there exists a well–known problem that positions in the field space are a finite distance away from the minimum of the effective potential, so that a curvature singularity in the Jordan frame could appear. This means that large excursions of the scalar could result in a singularity forming in a solution. It is known that the solution for this problem is also adding the higher powers of R so that the behavior at large curvatures can be soften. The oscillations are extremely large at small curvatures too, and the higher power of R or R itself do not change in this range of detection. We also note that this argument is applicable to the so–called type I, II and III finite–time future singularities (where R diverges), which has been classified in Ref. [240], while for a kind of singularities in our work, R does not become singular, and hence the argument would become different from the above.

5.2.4 Analysis of exponential and power–form models with correction terms in the matter dominated era

In this subsection, we carry out the numerical analysis of the models in Eqs. (5.33) and (5.34). In both cases, we assume $b = 1$ and $\tilde{\gamma} = 1/1000$ and solve Eq. (5.10) in a numerical way, by taking accurate initial conditions at $z = z_i$ so that $z_i \gg 2$. By using Eq. (5.35), we acquire

$$\begin{aligned} \frac{dy_H(z)}{d(z)}\Big|_{z_i} &= \frac{\Lambda}{3\tilde{m}^2}\gamma\,, \\ y_H(z)\Big|_{z_i} &= \frac{\Lambda}{3\tilde{m}^2}\left(1+\gamma\left(z_i+1\right)\right)\,, \end{aligned}$$

where we have set $z_i = 9$. The feature of the models in Eqs. (5.33) and (5.34) at the present time is very similar to those of the models in Eqs. (5.16) and (5.17). With the numerical extrapolation to the current universe, for the model in Eq. (5.33) we have $y_H(0) = 2.739$, $\omega_{\mathrm{DE}}(0) = -0.950$, $\Omega_{\mathrm{DE}}(0) = 0.732$ and $R(z = 0) = 4.369$, while for the model in Eq. (5.34), we find $y_H(0) = 2.654$, $\omega_{\mathrm{DE}}(0) = -0.989$, $\Omega_{\mathrm{DE}}(0) = 0.726$ and $R(z = 0) = 4.361$. We analyze those behaviors in the matter dominated era. It follows from the initial conditions $y_H(9) = 2.670$ and $\omega_{\mathrm{DE}}(9) = -0.997$ that the universe is extremely close to the ΛCDM model also at high redshifts. We see how the dynamical correction of the Einstein's equation, which corresponds to, roughly speaking, the fact of having "a dynamical cosmological constant", introduces an oscillatory behavior of dark energy density. Thanks to the contribution of the correction term, we obtain a constant frequency of such oscillations without changing the cosmological evolution described by the theory. In Fig. 5.2, we show the cosmological evolutions of the deceleration, jerk and snap parameters as functions of the redshift z in these models. There is overlapped region of the evolutions with those in the ΛCDM model. We may compare the graphics in Fig. 5.2 with the corresponding ones in Fig. 5.1 of the models in Eqs. (5.16) and (5.17) without the correction term analyzed in Sec. 5.2.2. At high redshifts, the deceleration parameter is not influenced by dark energy and hence the behavior in both these models in Eqs. (5.33) and (5.34) are the same as that in the ΛCDM model. On the other hand, in terms of the jerk and snap parameters, the derivatives of dark energy density become relevant and accordingly these parameters oscillate with the same frequency as that of dark energy, showing a different behavior in comparison with the case of GR with the cosmological constant. However, here such oscillations have a constant frequency and do not diverge. The predicted value of the oscillation frequency is $\mathcal{F} \equiv \sqrt{1.702/\tilde{\gamma}} = 41.255$. The oscillation period is $T = 2\pi/\mathcal{F} \simeq 0.152$. Thus, the numerical data are in good accordance with the predicted ones. (We can also appreciate the result by taking into account the fact that the number of crests per units of the redshift has to be $1/T \simeq 7$).

Consequently, we have shown in both analytical and numerical ways that increasing oscillations of dark energy in the past approach to effective singularities. It is not "a rapid oscillating system" but a system which becomes singular. The effects of such oscillations are evident especially in the higher derivative of the Hubble parameter. It is not a case that if all the numerical simulations presented in the literature start from small redshifts, at higher redshifts this singular problem appears. Eventually, the oscillations may influence also on the behavior of the Ricci scalar (which depends on the first derivative of the Hubble parameter, see Eq. (5.28) and $|d^n H(t)|_{t_0}/dt^n| \propto (\mathcal{F}(z_0))^n$ with $n \gg 1$, following from Eq. (5.26)). Of course, the average value of the dark energy density remains negligible, but the oscillations around this value become huge. Thus, the Ricci scalar may have an oscillatory behavior. We have also evaluated the frequency of the oscillations, so that the result can match with the numerical simulations, and therefore all the analyses in this work are consistent. This behavior of realistic $F(R)$ gravity models has recently been studied also in Ref. [173].

We remark that if the mass of the additional scalar degree of freedom, the so–called scalaron, is too large, the predictability could be lost [283]. Clearly, the mass of the scalaron in the two models in Eqs. (5.33) and (5.34) is not bounded, and thus it would diverge in very dense environment. We have confirmed that in the large curvature regime compared with the current curvature the correction term $g(R)$ in (5.32) in these two models do not strongly affect the scalaron potential in the Einstein frame, namely, the correction term would not be the leading term in the form of the scalaron potential, and thus the scalaron mass is not changed very much. The model parameter of the correction term $g(R)$ mainly related to the scalaron mass as well as its potential is $\tilde{\gamma}$. In the limit that the energy density of the environment becomes infinity, since the contribution of the correction term to the scalaron mass, it would be impossible to constrain the values of $\tilde{\gamma}$, for which the divergence of the scalaron mass can be avoided.

5.3 Cosmological constraints and future evolution

In this section, first we show that the models in Eqs. (5.33) and (5.34) satisfy the cosmological and local gravity constraints [83, 243], and that the term added to stabilize the dark energy oscillations in the matter dominated epoch does not cause any problem to these proprieties. The confrontation of $F(R)$ models with SNIa, BAO, CMB radiation and gravitational lensing has been executed in the past several works [41, 46, 58, 98, 120, 124, 133, 171, 183, 193, 209, 242, 244, 245, 250, 261, 262, 263, 277, 279, 293, 298, 299]. We have just seen that the models with the choice of $b = 1$ can be consistent with the observational data of the universe. Here, we examine the range of b in which the models are compatible with the observations and analyze the behavior of the models near to local (matter) sources in order to check possible Newton law corrections or matter instabilities. Then, we concentrate on the future evolution of the universe in the models and demonstrate that the effective crossing of the phantom divide which characterizes the de Sitter epoch takes place in the very far future.

In the way of trying to explain the several aspects that characterize our universe, there exists the problem of distinguishing different theories. It has been revealed that sometimes the study of the expansion history of the universe is not enough because different theories can achieve the same expansion history. Fortunately, theories with the same expansion history can have a different cosmic growth history. This fact makes the growth of the large scale structure in the universe an important tool in order to discriminate among the different theories proposed. Thus, the characterization of growth of the matter density perturbations become very significant. In order to execute it, the so–called growth index γ [179] is useful. Therefore, in the second part of this section we study the evolution of the matter density perturbation for

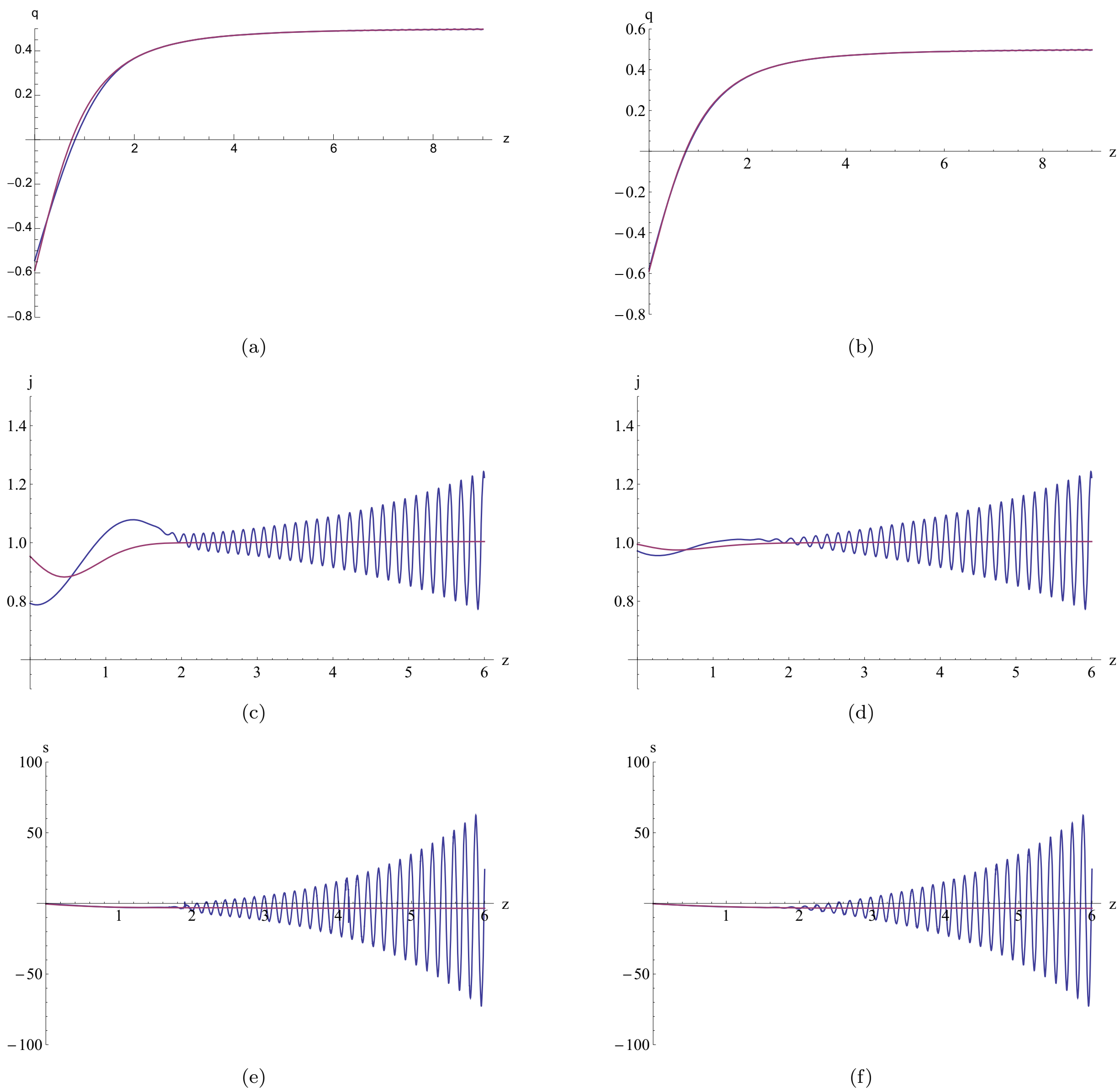


Figure 5.2: Cosmological evolutions of $q(z)$ [(a) and (b)], $j(z)$ [(c) and (d)] and $s(z)$ [(e) and (f)] parameters as functions of the redshift z for the model $F_1(R)$ [(a),(c),(e)] and the model $F_2(R)$ [(b),(d),(f)] in the region of $z > 0$.

our $F(R)$ gravity model.

Again, we clearly state the main purpose of this section. Since the original models, i.e., the Hu–Sawicki model [148] in Eq. (5.17) and exponential gravity [87, 180] in Eq. (5.16), have been studied well, we concentrate on the question whether the corrected models in Eqs. (5.34) and (5.33) lead to any difference in the observables. These modified models have been constructed in order not to alter the background evolution significantly except the oscillatory effect. In Refs. [148, 180] and many follow–up studies of these pioneering works, the cosmological background evolutions and the growth of structures in the two unmodified models in Eqs. (5.17) and (5.16) have been investigated. In order make this work self consistent study of modified gravity, we explicitly demonstrate the cosmological background evolutions and the growth of the matter density perturbations in the modified models in Eqs. (5.34) and (5.33). It is meaningful to investigate these behaviors in the modified models even though the modifications on the observable quantities are small.

5.3.1 Cosmological and local constraints

We take $\tilde{\gamma} = 1/1000$ in the models in Eqs. (5.33) and (5.34), keeping the parameter b free. Now, the dark energy density is a function of z and b, i.e., $y_H(z,b)$. We can again solve Eq. (5.10) numerically, taking the initial conditions at $z_i = 9$ as

$$\left.\frac{dy_H(z,b)}{d(z)}\right|_{z_i} = \frac{\Lambda}{3\tilde{m}^2}\tilde{\gamma}\,,$$
$$\left. y_H(z,b)\right|_{z_i} = \frac{\Lambda}{3\tilde{m}^2}\left(1+\tilde{\gamma}\left(z_i+1\right)\right)\,,$$

as we did in the previous section. We take $0.1 < b < 2$. In Figs. 5.3 and 5.4, we display the resultant values of dark energy EoS parameter $\omega_{\rm DE}(z=0,b)$ and $\Omega_{\rm DE}(z=0,b)$ at the present time as functions of b for the two models. We also show the bounds of cosmological data in Eq. (5.30), namely, the lines in rose denote the upper bounds, while the lines in yellow do the lower ones. By matching the comparison between the two graphics of every model, we find that in order to correctly reproduce the universe where we live with exponential gravity in Eq. (5.33), $0.1 < b < 1.174$, with power–law model in Eq. (5.34), $0.1 < b < 1.699$. The results are consistent with the choices in Sec. 5.2.4.

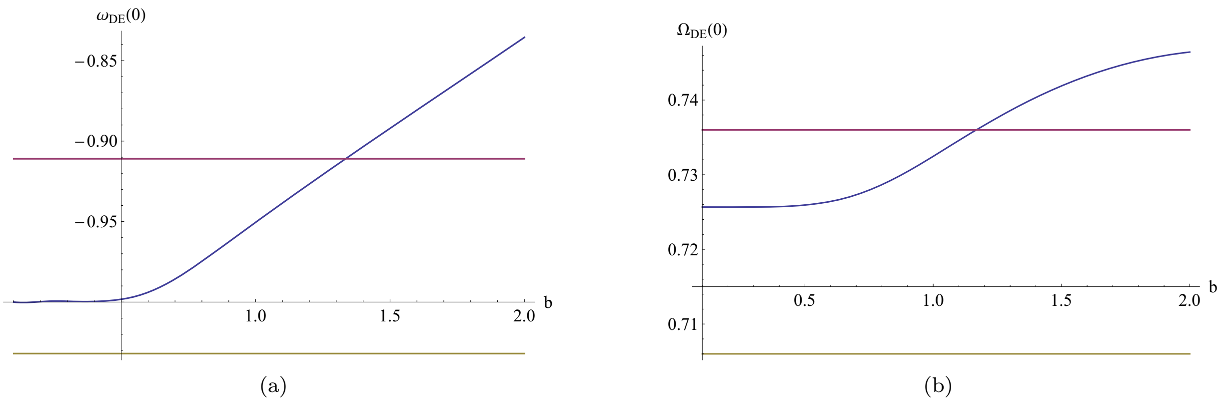
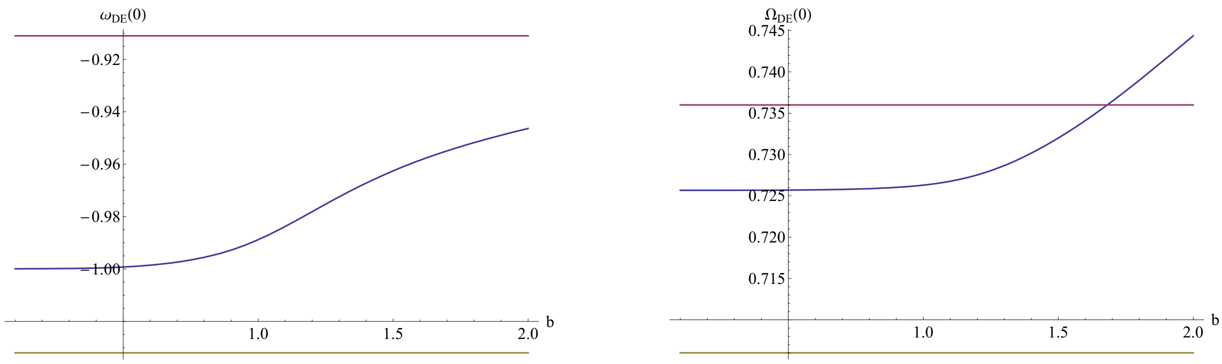


Figure 5.3: Behaviors of $\omega_{\rm DE}(z=0,b)$ and of $\Omega_{\rm DE}(z=0,b)$ as functions of b for exponential model. The observational data bounds (horizontal lines) are also shown.

Newton law corrections and stability on a planet surface

In Ref. [225], it has been shown that some realistic models of $F(R)$ gravity may lead to significant Newton law corrections at large cosmological scales. We briefly review this result. From the trace of the field equation (5.3), we consider the constant background of $R = R_0$, such that $2F(R_0) - R_0F'(R_0) = 0$, by performing a variation with respect to $R = R^{(0)} + \delta R$ and supposing the presence of a matter point



(a) (b)

Figure 5.4: Behaviors of $\omega_{\rm DE}(z=0,b)$ and of $\Omega_{\rm DE}(z=0,b)$ as functions of b for power–law model. Legend is the same as Fig. 5.3.

source (like a planet), that is, $T^{(\rm matter)} = T_0\,\delta(x)$, where $\delta(x)$ is the Dirac's distribution, we find, to first order in δR,

$$\left(\Box - m^2\right)\delta R = \frac{\kappa^2}{3F''(R_0)}T_0\delta(x)\,, \tag{5.37}$$

with

$$m^2 = \frac{1}{3}\left(\frac{F'(R_0)}{F''(R_0)} - R_0\right)\,. \tag{5.38}$$

The solution is given by

$$\delta R = \frac{\kappa^2}{3F''(R_0)}T_0\,G(m^2,|x|)\,, \tag{5.39}$$

where $G(m^2,|x|)$ is the correlation function which satisfies

$$\left(\Box - m^2\right)G(m^2,|x|) = \delta(x)\,. \tag{5.40}$$

Hence, if $m^2 < 0$, there appears a tachyon and thus there could be some instability. Even if $m^2 > 0$, when m^2 is small compared with R_0, $\delta R \neq 0$ at long ranges, which generates the large correction to the Newton law. For the pure exponential model in Eq. (5.16) without correction terms, when $R_0 \gg b\Lambda$, m^2 reads

$$m^2 \simeq \frac{(b^2\Lambda)}{6}\mathrm{e}^{\frac{R_0}{b\Lambda}}\,. \tag{5.41}$$

Therefore, in general m^2/R_0 is very large effectively. The same thing happens in the model in Eq. (5.17). Next, for the models in Eqs. (5.33) and (5.34) with correction terms, we have

$$m^2 \simeq \frac{3^{4/3}\tilde{m}^2 R}{2\Lambda\tilde{\gamma}}\left(\frac{R}{\tilde{m}^2}\right)^{2/3}\,. \tag{5.42}$$

Despite the fact that in this case m^2 is smaller than in Eq. (5.41), it still remains sufficiently large and the correction to the Newton law is very small. For example, the typical value of the curvature in the solar system is $R_0 \simeq 10^{-61}\mathrm{eV}^2$ (it corresponds to one hydrogen atom per cubic centimeter). In this case, from Eq. (5.42) we obtain $m^2/R_0 \simeq 2\times 10^6$.

Concerning the matter instability [101, 118], this might also occur when the curvature is rather large, as on a planet ($R \simeq 10^{-38}\mathrm{eV}^2$), as compared with the average curvature of the universe today ($R \simeq$

$10^{-66}\mathrm{eV}^2$). In order to arrive at a stability condition, we can perturb again the trace of Eq. (5.3) around $R = R_b$, where R_b is the curvature of the planet surface and the perturbation δR is given by the curvature difference between the internal and the external solution. The curvature $R_b = -\kappa^2 T^{(\mathrm{matter})}$ depends on the radial coordinate r . By assuming δR depending on time only, we acquire

$$-\partial_t^2(\delta R) \sim U(R_b)\delta R\,, \tag{5.43}$$

where

$$\begin{aligned} U(R_b) &= \left[\left(\frac{F'''(R_b)}{F''(R_b)}\right)^2 - \frac{F'''(R_b)}{F''(R_b)}\right] g^{rr}\nabla_r R_b \nabla_r R_b - \frac{R_b}{3} + \frac{F'(R_b)}{3F''(R_b)} \\ &\quad \frac{F'''(R_b)}{3(F''(R_b))^2}(2F(R_b) - R_b F'(R_b) - R_b)\,. \end{aligned} \tag{5.44}$$

Here, $g_{\mu\nu}$ is the diagonal metric describing the planet. If $U(R_b)$ is negative, then the perturbation δR becomes exponentially large and the whole system becomes unstable. Thus, the planet stability condition is

$$U(R_b) > 0\,. \tag{5.45}$$

For our models in Eqs. (5.33) and (5.34), $U(R_b) \simeq m^2$, where m^2 is given by Eq. (5.42) again. Also in this case, we do not have any particular problem. For example, by putting $R_b \simeq 10^{-38}\mathrm{eV}^2$, we find $U(R_b)/R_b \simeq 4 \times 10^{21}$. Thus, the models under consideration easily pass these local tests.

We mention that in the past, the non–linear effects on the scalar are much more important, owing to the mechanism of the chameleon effect [161, 162, 200], and that only at late times the linear evolution is a good approximation. For example, if a high–curvature solution is achieved, the Solar–System test is the examination whether the solution is stable against the Dolgov–Kawasaki instability [101]. This is not the same as whether the high–curvature solution can at all be achieved, which is a much more subtle issue and discussed at length by Hu and Sawicki in Ref. [148].

5.3.2 Future universe evolution

In de Sitter universe, we have $R = R_{\mathrm{dS}}$, where R_{dS} is the constant curvature given by the constant dark energy density $y_H = y_0$, such that $y_0 = R_{\mathrm{dS}}/12\tilde{m}^2$. Starting from Eq. (5.10), we are able to study perturbations around the de Sitter solution in the models (5.33) and (5.34) which provide this solution for $R_{\mathrm{dS}} = 4\Lambda$ and well satisfied the de Sitter condition $2F(R_{\mathrm{dS}}) = R_{\mathrm{d}S}F'(R_{\mathrm{dS}})$ as a consequence of the trace of the field equation in vacuum. Performing the variation with respect to $y_H(z) = y_0 + y_1(z)$ with $|y_1(z)| \ll 1$ and assuming the contributions of radiation and matter to be much smaller than y_0 , at the first order in $y_1(z)$ Eq. (5.10) reads

$$\frac{d^2 y_1(z)}{dz^2} + \frac{\alpha}{(z+1)}\frac{dy_1(z)}{dz} + \frac{\beta}{(z+1)^2}y_1(z) = 4\zeta(z+1)\,, \tag{5.46}$$

where

$$\alpha = -2\,, \quad \beta = -4 + \frac{4F'(R_{\mathrm{dS}})}{RF''(R_{\mathrm{dS}})}\,, \quad \zeta = 1 + \frac{1 - F'(R_{\mathrm{dS}})}{R_{\mathrm{dS}}F''(R_{\mathrm{dS}})}\,. \tag{5.47}$$

The solution of Eq. (5.46) is given by

$$y_H(z) = y_0 + y_1(z)\,, \tag{5.48}$$

$$y_1(z) = C_0(z+1)^{\frac{1}{2}\left(1-\alpha\pm\sqrt{(1-\alpha)^2-4\beta}\right)} + \frac{4\zeta}{\beta}(z+1)^3\,, \tag{5.49}$$

where C_0 is a constant. The well–known stability condition for the de Sitter space–time, $F'(R_{\rm dS})/((R_{\rm dS})F''(R_{\rm dS})) > 1$, is also valid. It has also been demonstrated that since in realistic $F(R)$ gravity models for the de Sitter universe $F''(R) \to 0^+$, $F'(R_{\rm dS})/(R_{\rm dS}F''(R_{\rm dS})) > 25/16$ [202] giving negative the discriminant of Eq. (5.49) and an oscillatory behavior to the dark energy density during this phase. Thus, in this case the dark energy EoS parameter $\omega_{\rm DE}$ (5.27) becomes

$$\omega_{\rm DE}(R=R_{\rm dS}) \simeq -1 + 4\tilde{m}^2\frac{(z+1)^{\frac{3}{2}}}{R_{\rm dS}} \times \tag{5.50}$$

$$\left[A_0\cos\left(\sqrt{\left(\frac{4}{R_{\rm dS}F''(R_{\rm dS})}\right)}\log(z+1)\right) + B_0\sin\left(\sqrt{\left(\frac{4}{R_{\rm dS}F''(R_{\rm dS})}\right)}\log(z+1)\right)\right],$$

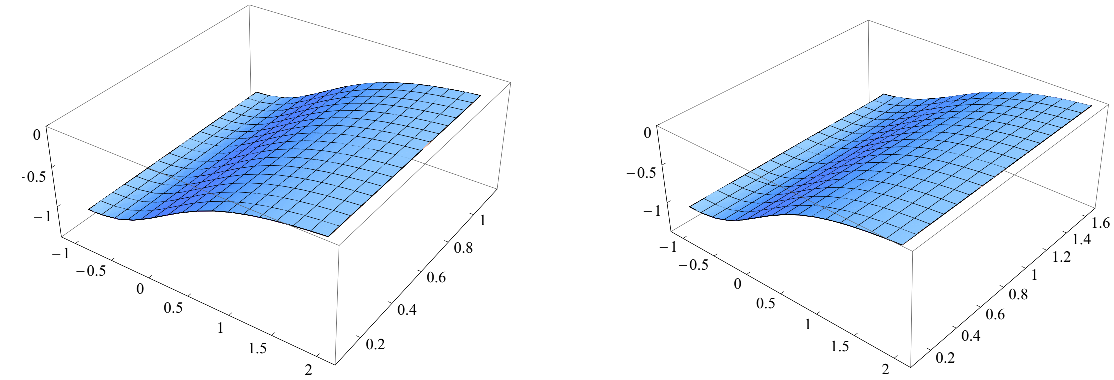
and oscillates infinitely often around the line of the phantom divide $\omega_{\rm DE} = -1$ [202]. According to various recent observational data, the crossing of the phantom divide occurred in the near past [9, 154, 212, 290]. These models possess one crossing in the recent past [28], after the end of the matter dominated era, and infinite crossings in the future (for detailed investigations on the future crossing of the phantom divide, see [29]), but the amplitude of such crossings decreases as $(z+1)^{3/2}$ and it does not cause any serious problem to the accuracy of the cosmological evolution during the de Sitter epoch which is in general the final attractor of the system [28, 111]. However, the existence of a phantom phase can give some undesirable effects such as the possibility to have the Big Rip [57] as an alternative scenario of the universe (in such a case, the model may suddenly exit from ΛCDM description) or the disintegration of bound structures which does not necessarily require to having the final (Big Rip) singularity [121, 122]. In this subsection, we show that in the models in Eqs. (5.33) and (5.34) the effective EoS parameter of the universe (for an alternative study, see [149]) defined as

$$\omega_{\rm eff} \equiv \frac{\rho_{\rm eff}}{P_{\rm eff}} = -1 + \frac{2(z+1)}{3H(z)}\frac{dH(z)}{dz} \tag{5.51}$$

never crosses the phantom divide line in the past, and that only when z is very close to -1 (this means in the very far future), it coincides with $\omega_{\rm DE}$ and the crossings occur. We remark that $\rho_{\rm eff}$ and $P_{\rm eff}$ correspond to the total energy density and pressure of the universe, and hence that if dark energy strongly dominates over ordinary matter, we can consider $\omega_{\rm eff} \approx \omega_{\rm DE}$. In both of the models under investigation, we take again $\tilde{\gamma} = 1/1000$ and keep the parameter b free, such that $0.1 < b < 1.174$ (model in Eq. (5.33)) and $0.1 < b < 1.699$ (model in Eq. (5.34)), according to the realistic representation of current universe. The numerical evaluation of Eq. (5.10) leads to $H(z)$, given by

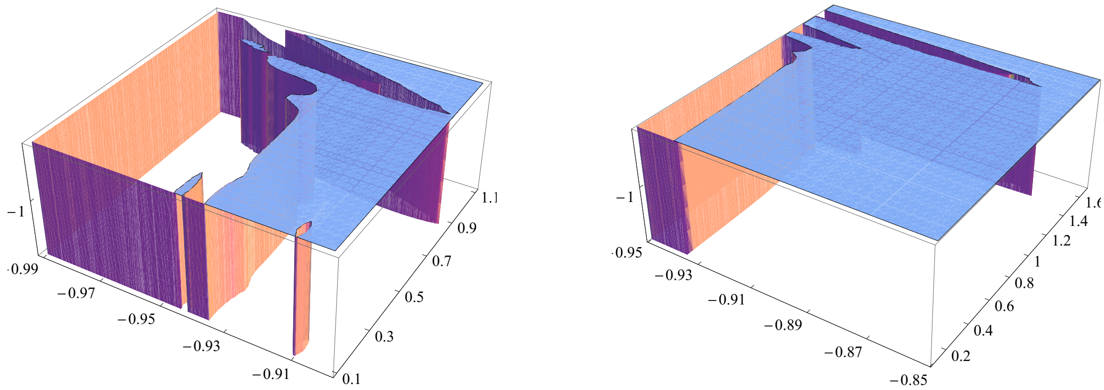
$$H(z) = \sqrt{\tilde{m}^2\left[y_H(z) + (z+1)^3 + \chi(z+1)^4\right]}, \tag{5.52}$$

and therefore $\omega_{\rm eff}(z)$. We depict the cosmological evolution of $\omega_{\rm eff}$ as a function of the red shift z and the b parameter in Fig. 5.5 for the model in Eq. (5.33) and in Fig. 5.6 for the model in Eq. (5.34). On the left panels, we plot the effective EoS parameter for $-1 < z < 2$. We can see that for both of the models, independently on the choice of b, $\omega_{\rm DE}$ starts from zero in the matter dominated era and asymptotically approaches -1 without any appreciable deviation. Only when z is very close to -1 and the matter contribution to $\omega_{\rm eff}$ is effectively zero, we have the crossing of the phantom divide due to the oscillation behavior of dark energy. On the right panels, we display the behavior of the effective EoS parameter around $z = -1$. Here, we focused on the phantom divide line and we excluded the graphic area out of the range $-1.0001 < \omega_{\rm eff} < -0.9999$. The blue region indicates that $\omega_{\rm eff}$ is still in the quintessence phase. We note that especially in the model in Eq. (5.34), the first crossing of phantom divide is very far in the future. For example, with the scale factor $a(t) = \exp(H_0 t)$, where $H_0 \simeq 6.3 \times 10^{-34}{\rm eV}^{-1}$ is the Hubble parameter of the de Sitter universe, $z = -0.90$ (when the crossing of the phantom divide may begin to appear in the exponential models) corresponds to 10^{26} years.



(a) (b)

Figure 5.5: Cosmological evolution of $\omega_{\rm eff}$ as a function of the red shift z and the b parameter for the model in Eq. (5.33). The left panel plots it for $-1 < z < 2$ and the right one displays around $z = -1$.



(a) (b)

Figure 5.6: Cosmological evolution of $\omega_{\rm eff}$ as a function of the red shift z and the b parameter for the model in Eq. (5.34). Legend is the same as Fig. 5.5.

Avoidance of the phantom crossing with (inhomogeneous) fluid

It may be of some interest to check if it is possible to avoid the crossing of the phantom divide by adding a suitable (compensating) fluid in the future cosmological scenario described by the models (5.33) and (5.34). Here, we indicate a possible realization of it. We examine an inhomogeneous fluid with its energy density ρ, pressure P and the Eos parameter ω as a function of ρ, i.e., $\omega = \omega(\rho)$. The EoS is expressed as

$$\frac{d}{dz}\log\rho = \frac{3}{(z+1)}(\omega(\rho)+1)\,. \tag{5.53}$$

We explore the simple case

$$\omega(\rho) = A_0\sigma(z)\rho^{\alpha-1} - 1\,, \tag{5.54}$$

where α is a constant and A_0 is a positive parameter. Moreover, $\sigma(z) = -1$ when $z \geq 0$ and $\sigma(z) = 1$ when $z < 0$, such that the fluid is in the phantom region for $z \geq 0$ and in the quintessence region for

$z < 0$. The fluid energy density reads

$$\rho = \rho_0 \left(B_0 - \sigma(z)\log(z+1)\right)^{\frac{1}{(1-\alpha)}} , \tag{5.55}$$

where $\rho_0 = [3(\alpha-1)A_0]^{1/(1-\alpha)}$ and B_0 are positive parameters depending on the initial conditions.

We note that one can choose $B_0 = 1$ without the loss of generality and in this way the energy density is defined as a positive quantity. If we take $\alpha > 1$, when $z \to +\infty$ or $z \to -1^+$, the energy density asymptotically tends to zero. For $z = 0$, we have a maximum, $\rho(z=0) = \rho_0$, so that we should require $\rho_0 \ll \Lambda/\kappa^2$, namely, the fluid energy density is always small with respect to the dark energy density given by our models for the cosmological constant. A fluid in the form of Eq. (5.55) may asymptotically produce a (Big Rip) singularity $H(t) \sim (t_0 - t)^\beta$, where $t < t_0$ and $\beta > 1$ (for general study of singularities in modified gravity, see [34, 35, 226, 231]), only for $\beta = 1/(2\alpha - 1)$ [138], but in our case $\alpha > 1$, so that this kind of divergence can never appear. If we add this fluid in the scenario described by $F(R)$ gravity models in Eqs. (5.33) and (5.34), when $z \to -1$ we find

$$\omega_{\text{eff}} = \frac{P_{\text{DE}} + P}{\rho_{\text{DE}} + \rho} \simeq -1 + \frac{A_0 \rho^\alpha}{\Lambda/\kappa^2} . \tag{5.56}$$

This means that owing to the presence of fluid, the oscillations of the effective EoS parameter realize not around the phantom divide but around ω_{eff} given by the last equation, namely in the quintessence region. With an accurate fitting of the parameters, in this way we may avoid the crossing of the phantom divide.

We can also add a fluid to the cosmological scenario in order to have an asymptotical phantom phase without the Big Rip singularity. To this purpose, we investigate the EoS parameter of the fluid as in Eq. (5.54) with $A_0 > 0$ and $\sigma(z) = -1$, which describes a phantom fluid. The fluid energy density is given by Eq. (5.55). We put $B_0 = 0$ and $\alpha < 1$ such that $1/(1-\alpha)$ can be an even number and one can have the energy density defined as a positive quantity. In this way, the fluid energy density decreases until $z = 0$ and then it starts to grow up. We can take ρ_0 sufficiently small so that the fluid contribution can become dominant only in the asymptotical limit, when z is close to -1 , avoiding the quintessence region in the final cosmological evolution of our $F(R)$ gravity models. From the equation of motion $3H^2/\kappa^2 = \rho$, we obtain

$$t = -\int_0^{z(t)} \sqrt{\frac{3}{\kappa^2 \rho(z')}} \frac{dz'}{(z'+1)} . \tag{5.57}$$

In our case, it is easy to verify that $t \sim |\log(z+1)^{(2\alpha-1)/(2\alpha-2)}|$ and if $\alpha \leq 1/2$, when $z(t) \to -1$ the integral diverges and $t \to +\infty$, avoiding the Big Rip at a finite time. In this kind of models, the fluid energy density increases with time, but $\omega \to -1$ asymptotically, so that there can be no future singularity. However, in Ref. [122] a careful investigation on the conditions necessary to produce this evolution has been done, and it has been demonstrated that this fluid can rapidly expand in the future, leading to the disintegration of all bound structures (this is the so–called "Little Rip"). For example, a planet in an orbit of radius $\bar{R}$ around a star of mass M will become unbound when $-(4\pi/3)(\rho + 3P)\bar{R}^3 \simeq M$. In our case, $-(\rho + 3P) = A_0 \rho^\alpha$ and in the future every gravitationally bound system will be disintegrated [57].

5.3.3 Growth of the matter density perturbations: growth index

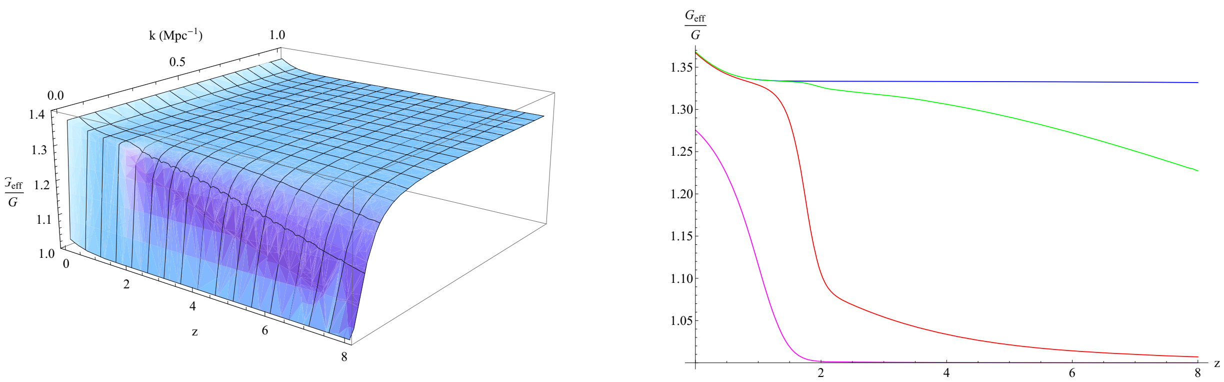
In this subsection, we study the matter density perturbations. The equation that governs the evolution of the matter density perturbations for $F(R)$ gravity has been derived in the literature (see, for example, [280] and references therein). Under the subhorizon approximation (for the case without such an approximation, see [32, 194]), the matter density perturbation $\delta = \frac{\delta\rho_{\text{m}}}{\rho_{\text{m}}}$ satisfies the following equation:

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G_{\text{eff}}(a,k)\rho_{\text{m}}\delta = 0 \tag{5.58}$$

with k being the comoving wavenumber and $G_{\rm eff}(a,k)$ being the effective gravitational "constant" given by

$$G_{\rm eff}(a,k) = \frac{G}{F'(R)}\left[1+\frac{\left(k^2/a^2\right)\left(F''(R)/F'(R)\right)}{1+3\left(k^2/a^2\right)\left(F''(R)/F'(R)\right)}\right]. \tag{5.59}$$

It is worth noting that the appearance of the comoving wavenumber k in the effective gravitational constant makes the evolution of the matter density perturbations dependent on the comoving wavenumber k. It can be checked easily, by taking $F(R) = R$ in Eq. (5.59), that the evolution of the matter density perturbation does not have this kind of dependence in the case of GR. In Fig. 5.7, we show the cosmological evolution as a function of the redshift z and the scale dependence on the comoving wavenumber k of this effective gravitational constant for the case of model $F_1(R)$ in Eq. (5.33), while in Fig. 5.8 we depict those for the case of model $F_2(R)$ in Eq. (5.34). In both these cases, we have fixed $b = 1$ and used $\gamma = 1/1000$.



(a) (b)

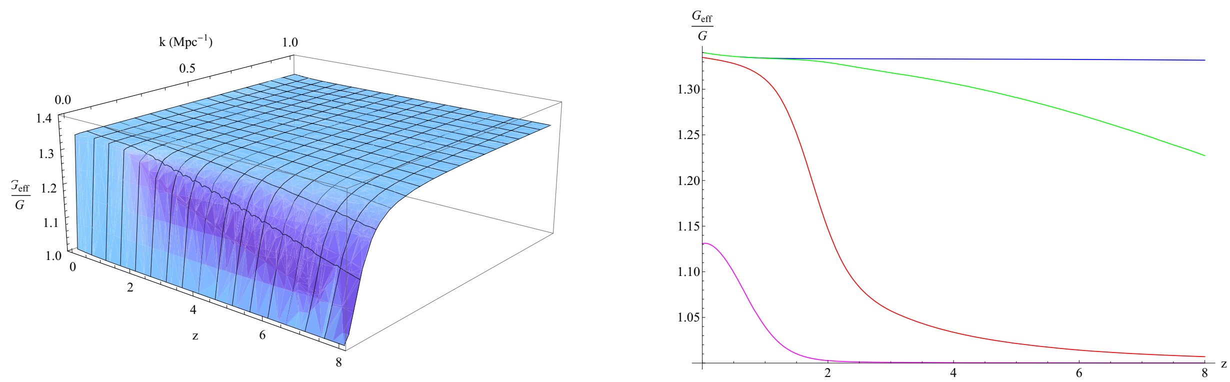
Figure 5.7: (a) Cosmological evolution as a function of z and the scale dependence on k of the effective gravitational constant $G_{\rm eff}$ for the model $F_1(R)$ with $b = 1$ and $\tilde{\gamma} = 1/1000$. (b) Cosmological evolution of $G_{\rm eff}$ as a function of z in the model $F_1(R)$ with $b = 1$ and $\tilde{\gamma} = 1/1000$ for $k = 1{\rm Mpc}^{-1}$ (blue), $k = 0.1{\rm Mpc}^{-1}$ (green), $k = 0.01{\rm Mpc}^{-1}$ (red) and $k = 0.001{\rm Mpc}^{-1}$ (fuchsia).

Another important remark is to state that in deriving Eq. (5.58), we have assumed the subhorizon approximation (see [115]). Namely, comoving wavelengths $\lambda \equiv a/k$ are considered to be much shorter than the Hubble radius H^{-1} as

$$\frac{k^2}{a^2} \gg H^2 . \tag{5.60}$$

This means that we examine the scales of $\log k \geq -3$. On the other hand, as it was pointed out in Ref. [74], for large k we have to take into account deviations from the linear regime. Hence, we do not consider the scales of $\log k > -1$ and take the results obtained for $\log k$ close to -1.

From Figs. 7 and 8, we see that $G_{\rm eff}$ measured today can significantly be different from the Newton's constant in the past. The Newton's constant should be normalized to the current one as $(G_{\rm eff}/G)$. This implies that the Newton's constant at the decoupling epoch must be much lower than what is implicitly assumed in CMB codes such as CAMB [1, 176]. This could significantly change the CMB power spectrum because it changes, for example, the relation between the gravitational interaction and the Thomson scattering rate. Since we use the CMB data when we examine whether the theoretical results are consistent with the observational ones analyzed in the framework of GR, it should be important for us to take into account this point. Therefore, strictly speaking, if we compare our results with the observations, we has



(a) (b)

Figure 5.8: (a) Cosmological evolution as a function of z and the scale dependence on k of the effective gravitational constant $G_{\rm eff}$ for the model $F_2(R)$ with $b = 1$ and $\tilde{\gamma} = 1/1000$. (b) Cosmological evolution of $G_{\rm eff}$ as a function of z for the model $F_2(R)$ with $b = 1$ and $\tilde{\gamma} = 1/1000$. Legend is the same as Fig. 5.7.

to use the observational results obtained by analyzing the CMB data with using the present value of $G_{\rm eff}$ in our $F(R)$ gravity models instead of the Newton's constant G in GR.

Instead of solving Eq. (5.58) for the matter density perturbation δ, we now introduce the growth rate $f_{\rm g} \equiv d\ln\delta/d\ln a$ and solve the equivalent equation to Eq. (5.58) for the growth rate in terms of the redshift z, given by

$$\frac{df_{\rm g}(z)}{dz} + \left(\frac{1+z}{H(z)}\frac{dH(z)}{dz} - 2 - f_{\rm g}(z)\right)\frac{f_{\rm g}(z)}{1+z} + \frac{3}{2}\frac{\tilde{m}^2(1+z)^2}{H^2(z)}\frac{G_{\rm eff}(a(z),k)}{G} = 0\,. \tag{5.61}$$

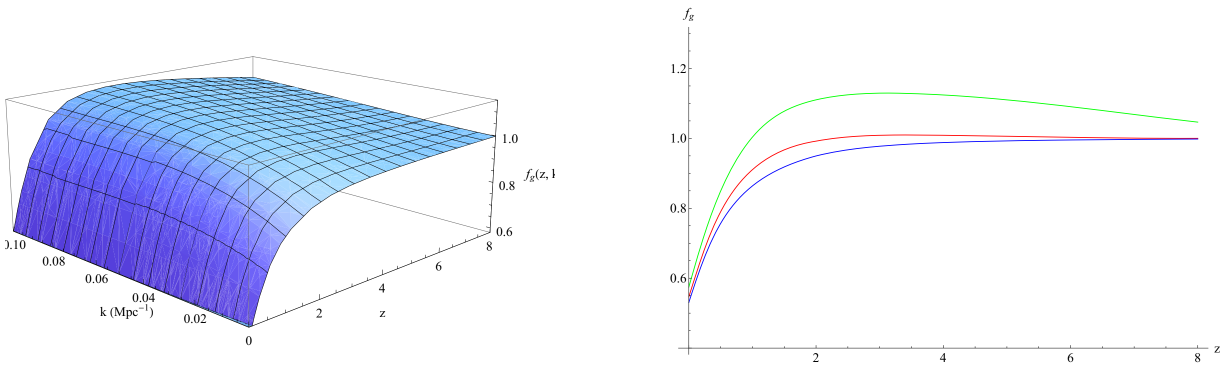
Unfortunately, Eq. (5.61) cannot be solved analytically for the models $F_1(R)$ and $F_2(R)$, but it can be solved numerically by imposing the initial conditions. Therefore, we execute the numerical calculations for both the model $F_1(R)$ and the model $F_2(R)$ with the condition that at a very high redshift the growth rate becomes that in the ΛCDM model. In Fig. 5.9, we illustrate the cosmological evolution as a function of the redshift z and the scale dependence on the comoving wavenumber k of the growth rate for the model $F_1(R)$, while we depict those of the growth rate for the model $F_2(R)$ in Fig. 5.10.

One way of characterizing the growth of the matter density perturbations could be to use the so–called growth index γ, which is defined as the quantity satisfying the following equation:

$$f_{\rm g}(z) = \Omega_{\rm m}(z)^{\gamma(z)}\,, \tag{5.62}$$

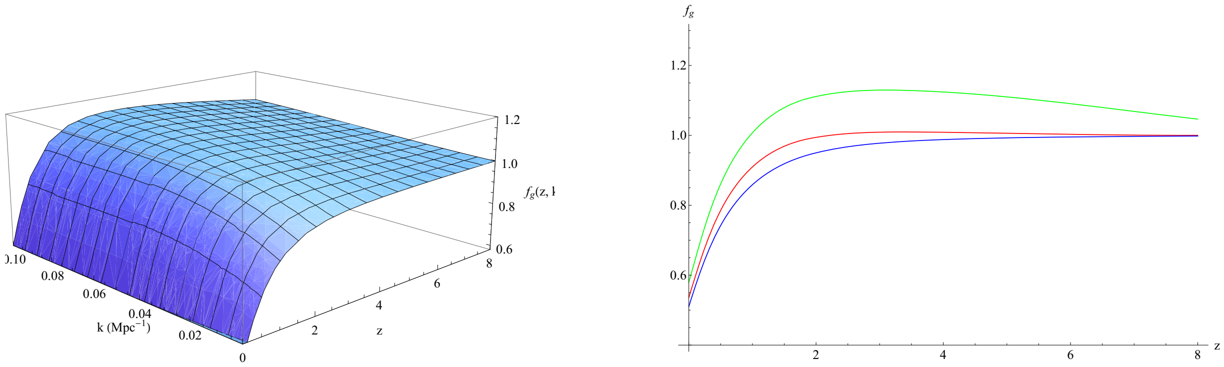
with $\Omega_{\rm m}(z) = \frac{8\pi G\rho_m}{3H^2}$ being the matter density parameter.

It is known that the growth index γ in Eq. (5.62) cannot be observed directly, but it can be determined from the observational data of both the growth factor $f_{\rm g}(z)$ and the matter density parameter $\Omega_{\rm m}(z)$ at the same redshift z. Even if the growth index is not directly observable quantity, it could have a fundamental importance in discriminating among the different cosmological models. One of the reasons is that in general, the growth factor $f_{\rm g}(z)$, which can be estimated from redshift space distortions in the galaxy power spectra at different z [142, 157], may not be expressed in terms of elementary functions and this fact makes the comparison among the different models difficult. If Eq. (5.62) is satisfied with any ansatz for the growth index γ, then its determination could provide an easy and fast way to distinguish between cosmological models.



(a) (b)

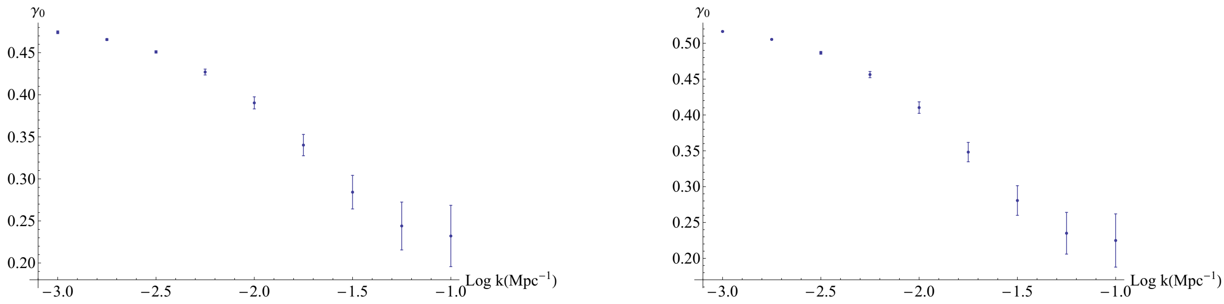
Figure 5.9: (a) Cosmological evolution as a function of the redshift z and the scale dependence on the comoving wavenumber k of the growth rate $f_{\rm g}$ for the model $F_1(R)$. (b) Cosmological evolution of the growth rate $f_{\rm g}$ as a function of z in the model $F_1(R)$ for $k = 0.1{\rm Mpc}^{-1}$ (green), $k = 0.01{\rm Mpc}^{-1}$ (red) and $k = 0.001{\rm Mpc}^{-1}$ (blue).



(a) (b)

Figure 5.10: (a) Cosmological evolution as a function of the redshift z and the scale dependence on the comoving wavenumber k of the growth rate $f_{\rm g}$ for the model $F_2(R)$. (b) Cosmological evolution of the growth rate $f_{\rm g}$ as a function of z for the model $F_2(R)$. Legend is the same as Fig. 5.9.

Various parameterizations for the growth index γ have been proposed in the literature. In the first stage works on this topic, γ was taken constant (see [172, 248]). In the case of dark fluids with the constant EoS ω_0 in GR, it is $\gamma = 3\,(\omega_0 - 1)\,/\,(6\omega_0 - 5)$ (for the ΛCDM model, the growth index is $\gamma \approx 0.545$). Although taking γ constant is very appropriated for a wide class of dark energy models in the framework of GR (for which $|\gamma'(0)| < 0.02$), for modified gravity theories γ is not constant in general (the cases of some viable $F(R)$ gravity models have been investigated in Refs. [74, 128]) and the measurement of $|\gamma'(0)|$ could be very important in order to discriminate between different theories. For this reason, another parameterizations has been proposed. The case of a linear dependence $\gamma(z) = \gamma_0 + \gamma_0' z$ was treated in Ref. [251]. Recently, an ansatz of the type $\gamma(z) = \gamma_0 + \gamma_1 z/(1+z)$ with γ_0 and γ_1 being constants was explored in Ref. [39] and a generalization given by $\gamma(z) = \gamma_0 + \gamma_1 z/(1+z)^{\alpha}$ with α being a constant in Ref. [74]. In the following, we study some of these parameterizations of the growth index for the case of the models $F_1(R)$ and $F_2(R)$.



(a) (b)

Figure 5.11: Constant growth index as a function of $\log k$ for the model $F_1(R)$ (a) and for the model $F_2(R)$ (b). The bars express the 68% CL.

$\gamma = \gamma_0$

We consider the ansatz for the growth index given by

$$\gamma = \gamma_0 \,,$$

where γ_0 is a constant.

In Fig. 5.11, we display the results obtained by fitting Eq. (5.62) to the solution of Eq. (5.61) for different values of the comoving wavenumber k for the two models $F_1(R)$ and $F_2(R)$. We note that in these and following plots, the bars express the 68% confidence level (CL) and the point denotes the median value. The first important result for both models is that the value of the growth index has a strong dependence with $\log k$. This scale dependence seems to be quite similar in both models.

In order to check the goodness of our fits, in Fig. 5.12 we show cosmological evolutions of the growth rate $f_{\rm g}(z)$ and $\Omega_{\rm m}(z)^{\gamma_0}$ as functions of the redshift z together for several values of the comoving wavenumber k for the models $F_1(R)$ and $F_2(R)$. To clarify these results, in Fig. 5.13 we also illustrate the cosmological evolution of the relative difference between $f_{\rm g}(z)$ and $\Omega_{\rm m}(z)^{\gamma_0}$ as a function of z for the same values of k in these models. The first remarkable thing is that for both models the function $\Omega_{\rm m}(z)^{\gamma_0}$ fits the growth rate for large scales (i.e., lower k) very well, but this is not anymore the case for larger values of k. In fact, if we do not consider lower values for z (i.e., $z < 0.2$), for $\log k = -2$ the relative difference is smaller than 3% for both models, while for $\log k = -1$ can arrive up to almost 13%. For $\log k = -3$, we see that the relative difference is always smaller than 1.5% for the model $F_1(R)$ and smaller than 1% for the model $F_2(R)$.

$\gamma = \gamma_0 + \gamma_1 z$

With the same procedure used in the previous subsection, we explore a linear dependence for the growth index

$$\gamma = \gamma_0 + \gamma_1 z \,, \tag{5.63}$$

where γ_1 is a constant.

In Fig. 5.14, we depict the parameters γ_0 and γ_1 for several values of $\log k$ in both the models. As is the same as the case $\gamma = \gamma_0$, it can easily be seen that the scale dependence of the parameters γ_0 and

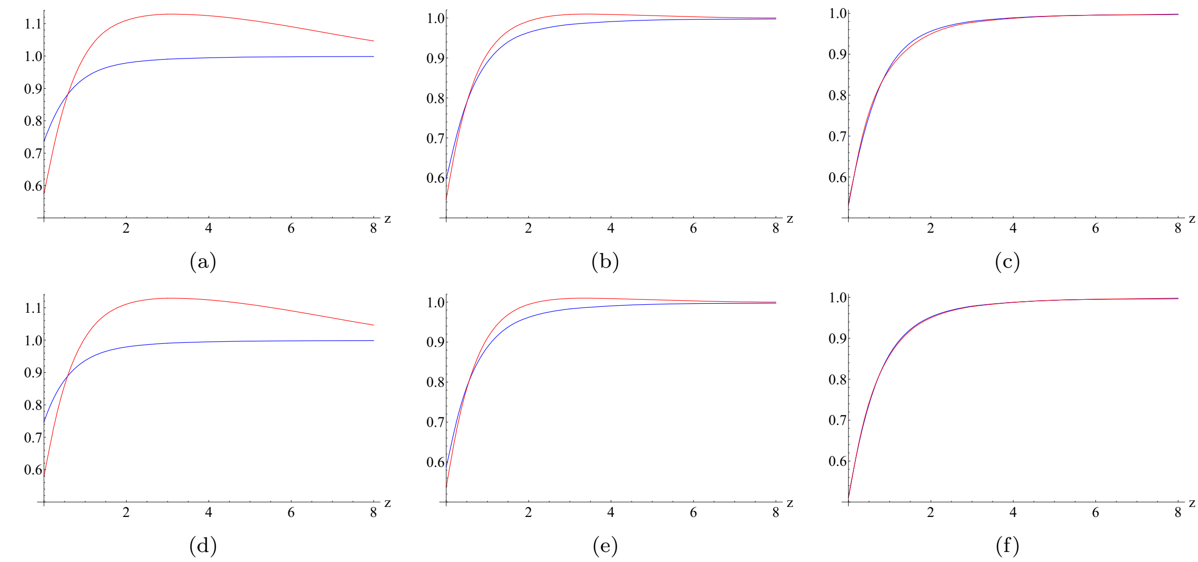


Figure 5.12: Cosmological evolutions of the growth rate f_{g} (red) and $\Omega_{\mathrm{m}}^{\gamma}$ (blue) with $\gamma = \gamma_0$ as functions of the redshift z in the model $F_1(R)$ for $k = 0.1\mathrm{Mpc}^{-1}$ (a), $k = 0.01\mathrm{Mpc}^{-1}$ (b) and $k = 0.001\mathrm{Mpc}^{-1}$ (c), and those in the model $F_2(R)$ for $k = 0.1\mathrm{Mpc}^{-1}$ (d), $k = 0.01\mathrm{Mpc}^{-1}$ (e) and $k = 0.001\mathrm{Mpc}^{-1}$ (f).

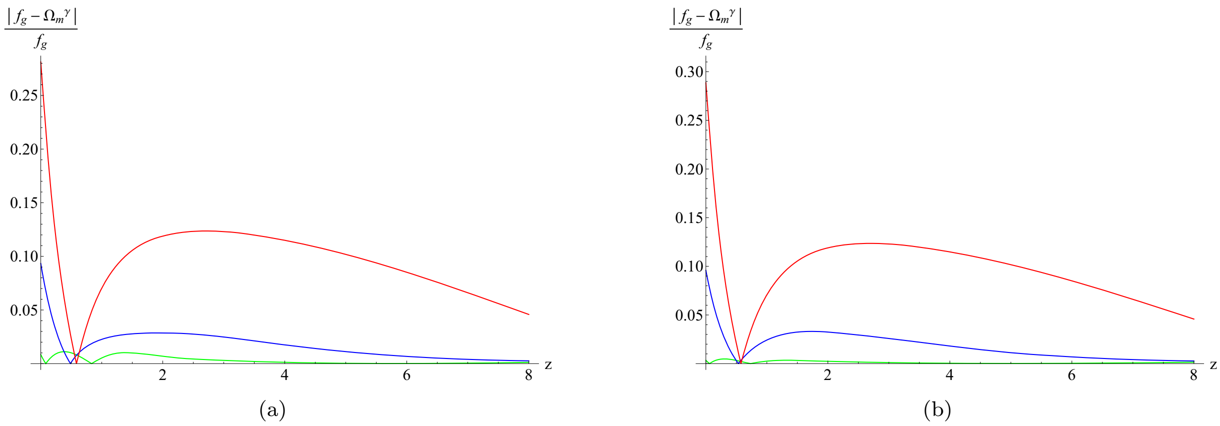


Figure 5.13: Cosmological evolution of the relative difference $\frac{|f_{\mathrm{g}} - \Omega_{\mathrm{m}}^{\gamma}|}{f_{\mathrm{g}}}$ with $\gamma = \gamma_0$ for $k = 0.1\mathrm{Mpc}^{-1}$ (red), $k = 0.01\mathrm{Mpc}^{-1}$ (blue) and $k = 0.001\mathrm{Mpc}^{-1}$ (green) in the model $F_1(R)$ (a) and the model $F_2(R)$ (b).

γ_1 is similar in these models. We can also find that $\gamma_0 \sim 0.46$ for the model $F_1(R)$ when $\log k \leq -2$, whereas $\gamma_0 \sim 0.51$ for the model $F_2(R)$ when $\log k \leq -2.5$. For both these models, the value of γ_1 has a strong dependence on k in the range of $\log k > -2.25$, but in the range of $\log k < -2.25$ this dependence becomes weaker.

In Fig. 5.15, we illustrate cosmological evolutions of the growth rate $f_{\mathrm{g}}(z)$ and $\Omega_{\mathrm{m}}(z)^{\gamma(z)}$ as functions of the redshift z together for the models $F_1(R)$ and $F_2(R)$. We can see that the fits for $\log k = 0.1$ have been improved in comparison with the same fits as the case with a constant growth index. Also, for $\log k < 0.1$ the fits continue to be quite good. In order to demonstrate these facts quantitatively, in

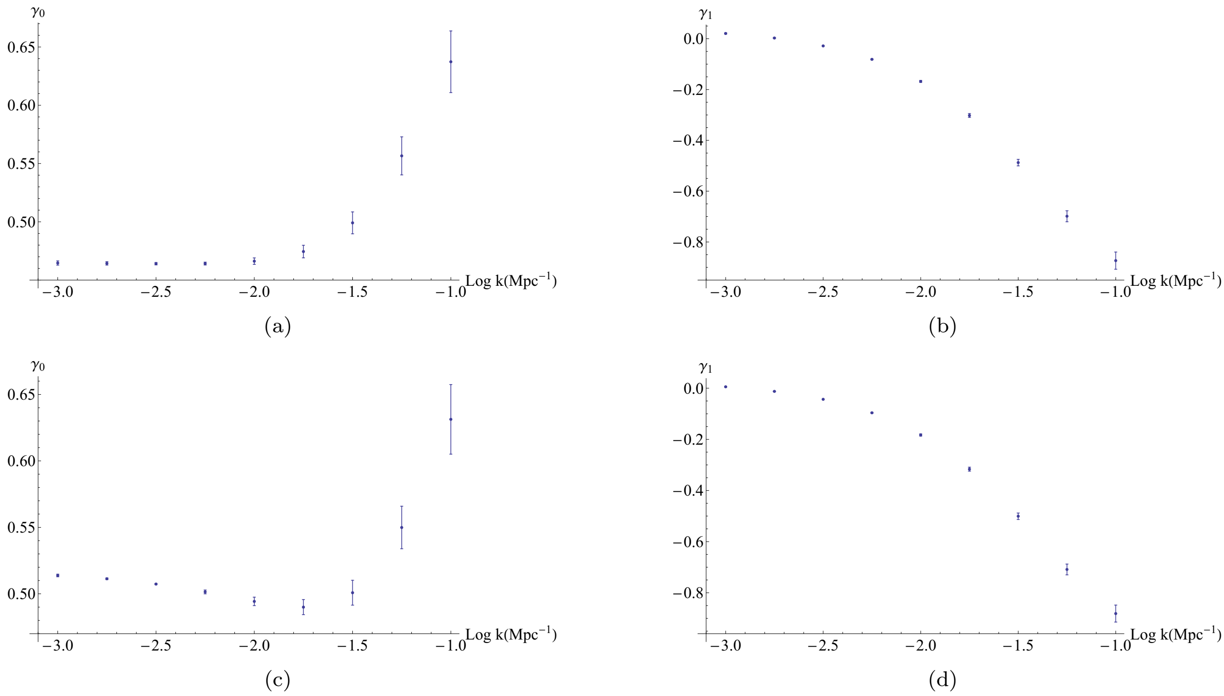


Figure 5.14: Growth index fitting parameters in the case $\gamma = \gamma_0 + \gamma_1 z$ as a function of $\log k$ for the model $F_1(R)$ [(a) and (b)] and the model $F_2(R)$ [(c) and (d)]. Legend is the same as Fig. 5.11.

Fig. 5.16 we plot the cosmological evolution of the relative difference between $f_{\rm g}(z)$ and $\Omega_{\rm m}(z)^{\gamma(z)}$ as a function of z for several values of k in the models $F_1(R)$ and $F_2(R)$. In this case, for $\log k = -1$ the relative difference is smaller than 7.5% in both the models if we do not consider lower values for z (i.e., $z < 0.2$). We also see that the linear growth index improves the fits in both the models for $\log k = -2$ in comparison with those for a constant growth index. In this case, the relative difference for the model $F_1(R)$ is always smaller than 1%, whereas that for model $F_2(R)$ is smaller than 2%. Finally, for $\log k = -3$ the results obtained for a constant growth index are quite similar to those for a linear dependence on z.

$\gamma = \gamma_0 + \gamma_1 \frac{z}{1+z}$

Next, we examine the following ansatz for the growth index:

$$\gamma = \gamma_0 + \gamma_1 \frac{z}{1+z} \,. \tag{5.64}$$

In Fig. 5.17, we depict the parameters γ_0 and γ_1 for several values of $\log k$ for both the models. The scale dependence of these parameters on k is shown. The behavior of the parameter γ_1 seems to be quite similar to that for the previous case $\gamma = \gamma_0 + \gamma_1 z$, but it is worth cautioning that the scale of the figures are different from each other, and that for the present ansatz the scale dependence of γ_1 is stronger than that for the previous case. It can also be seen that $\gamma_0 \sim 0.465$ for the model $F_1(R)$ and $\gamma_0 \sim 0.513$ for the model $F_2(R)$ in the scale $\log k < -2.5$.

In Fig. 5.18, we plot cosmological evolutions of the growth rate $f_{\rm g}(z)$ and $\Omega_{\rm m}(z)^{\gamma(z)}$ in the models

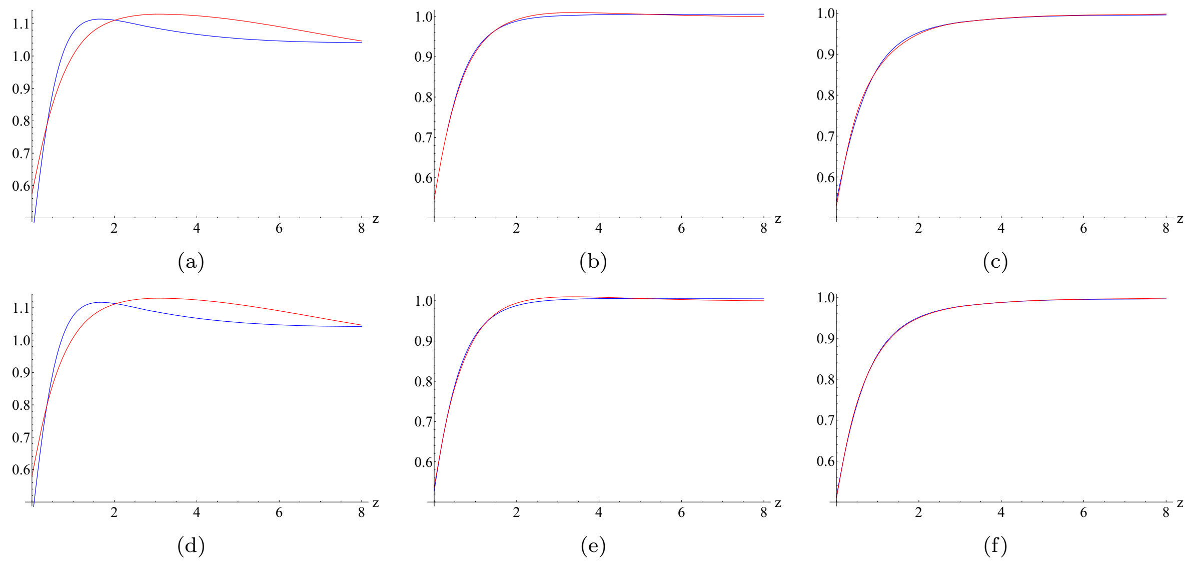


Figure 5.15: Cosmological evolutions of the growth rate f_g (red) and Ω_m^γ (blue) with $\gamma = \gamma_0 + \gamma_1 z$ as functions of the redshift z in the model $F_1(R)$ for $k = 0.1\text{Mpc}^{-1}$ (a), $k = 0.01\text{Mpc}^{-1}$ (b) and $k = 0.001\text{Mpc}^{-1}$ (c), and those in the model $F_2(R)$ for $k = 0.1\text{Mpc}^{-1}$ (d), $k = 0.01\text{Mpc}^{-1}$ (e) and $k = 0.001\text{Mpc}^{-1}$ (f).

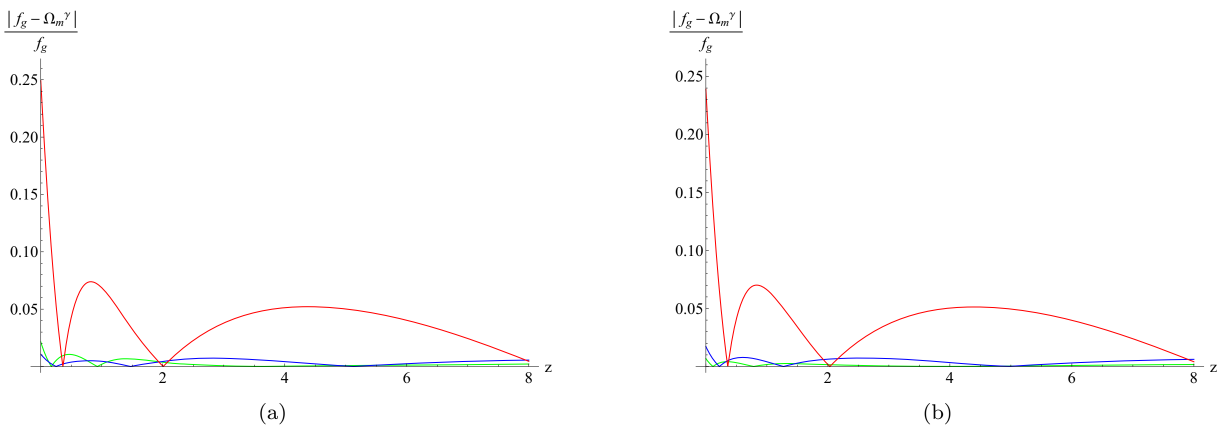


Figure 5.16: Cosmological evolution of the relative difference $\frac{|f_g - \Omega_m^\gamma|}{f_g}$ with $\gamma = \gamma_0 + \gamma_1 z$ for $k = 0.1\text{Mpc}^{-1}$ (red), $k = 0.01\text{Mpc}^{-1}$ (blue) and $k = 0.001\text{Mpc}^{-1}$ (green) in the model $F_1(R)$ (a) and the model $F_2(R)$ (b).

$F_1(R)$ and $F_2(R)$ for several values of k, as demonstrated in the previous subsections. We can see the fits for $\log k \leq -2$ are quite good, as those in the previous ansatz for the growth index. In the case of higher values of $\log k$, it seems that the fits are similar to those for a constant growth rate and these fits do not reach the goodness of those for the case of $\gamma = \gamma_0 + \gamma_1 z$.

In order to analyze the fits quantitatively, in Fig. 5.19 we display the cosmological evolution of the relative difference between $f_g(z)$ and $\Omega_m(z)^{\gamma(z)}$ for several values of k in the models $F_1(R)$ and $F_2(R)$. We see that the relative difference for $\log k = -1$ is smaller than 12% (if we do not consider $z < 0.2$)

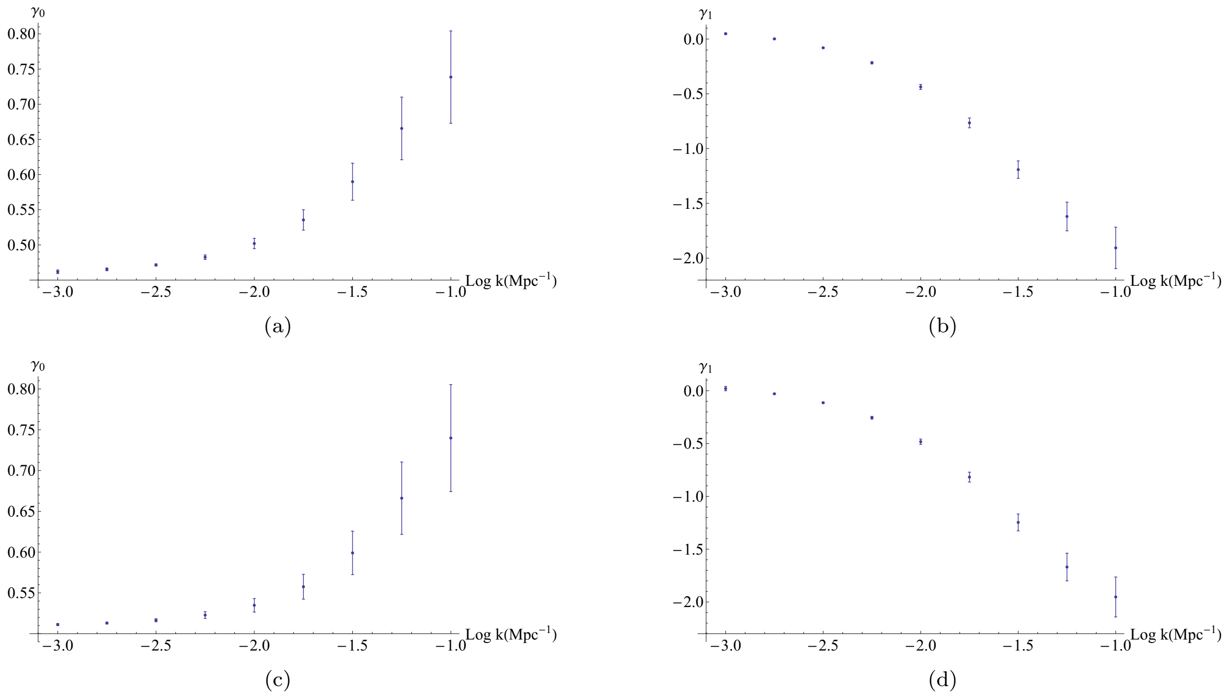


Figure 5.17: Growth index fitting parameters in the case $\gamma = \gamma_0 + \gamma_1 \frac{z}{1+z}$ as a function of $\log k$ for the model $F_1(R)$ [(a) and (b)] and the model $F_2(R)$ [(c) and (d)]. Legend is the same as Fig. 5.11.

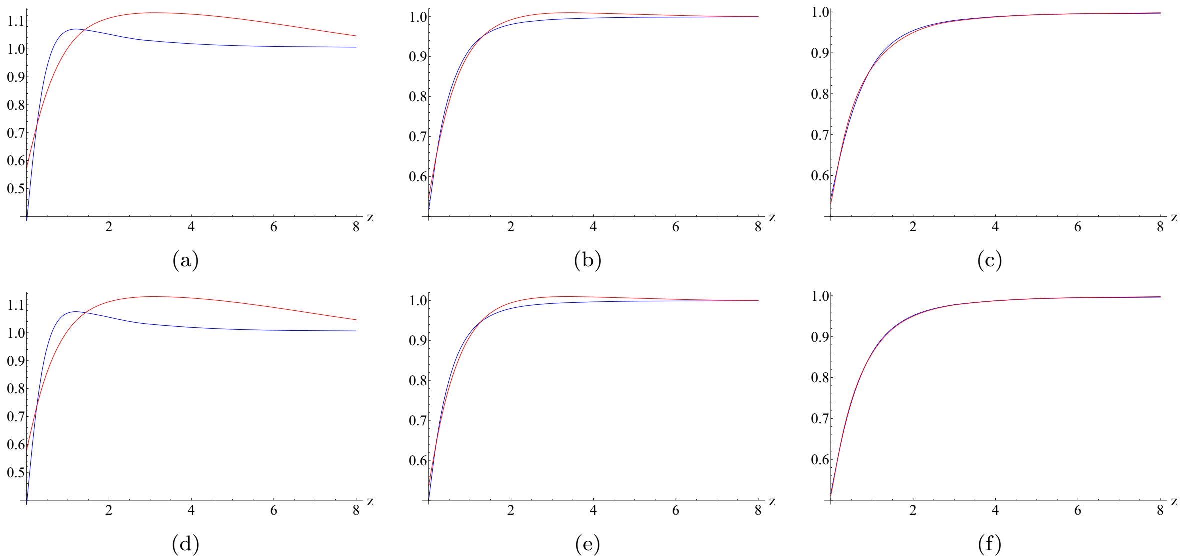


Figure 5.18: Cosmological evolutions of the growth rate f_{g} (red) and $\Omega_{\mathrm{m}}^{\gamma}$ (blue) with $\gamma = \gamma_0 + \gamma_1 \frac{z}{1+z}$ as functions of the redshift z in the model $F_1(R)$ for $k = 0.1\mathrm{Mpc}^{-1}$ (a), $k = 0.01\mathrm{Mpc}^{-1}$ (b) and $k = 0.001\mathrm{Mpc}^{-1}$ (c), and those in the model $F_2(R)$ for $k = 0.1\mathrm{Mpc}^{-1}$ (d), $k = 0.01\mathrm{Mpc}^{-1}$ (e) and $k = 0.001\mathrm{Mpc}^{-1}$ (f).

for both the models. Thus, it is confirmed that these fits are better than those for the constant growth rate, but these are worse than those for $\gamma = \gamma_0 + \gamma_1 z$. For lower values of $\log k$, the relative difference is smaller than 2% in $z > 0.2$.

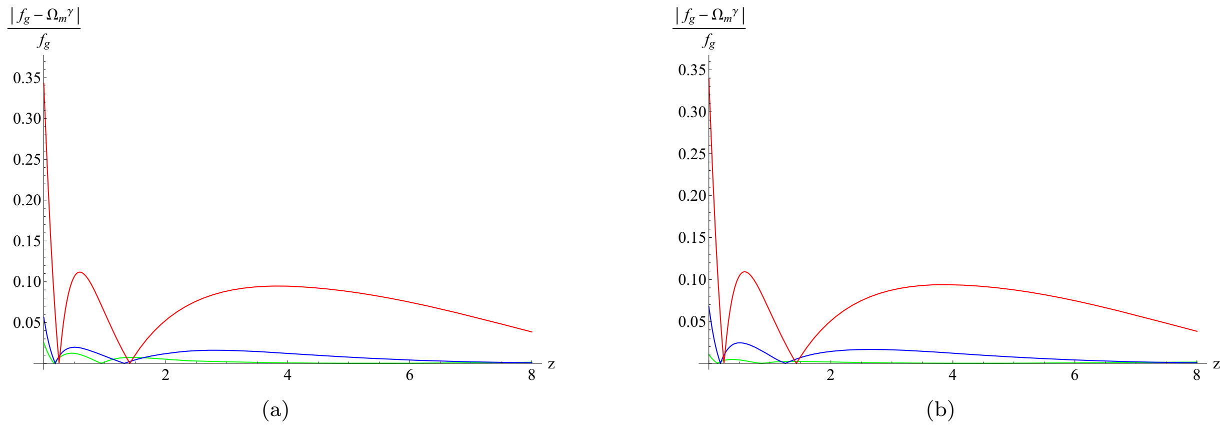


Figure 5.19: Cosmological evolution of the relative difference $\frac{|f_{\rm g}-\Omega_{\rm m}^{\gamma}|}{f_{\rm g}}$ with $\gamma = \gamma_0 + \gamma_1 \frac{z}{1+z}$ for $k = 0.1\mathrm{Mpc}^{-1}$ (red), $k = 0.01\mathrm{Mpc}^{-1}$ (blue) and $k = 0.001\mathrm{Mpc}^{-1}$ (green) in the model $F_1(R)$ (a) and the model $F_2(R)$ (b).

As a consequence, through the investigations of these different ansatz for the growth index, it is concluded that $\gamma = \gamma_0 + \gamma_1 z$ is the parameterization that can fit Eq. (5.62) to the solution of Eq. (5.61) better in a wide range of values for k . Even though the behavior of the parameters γ_0 and γ_1 in the models $F_1(R)$ and $F_2(R)$ is quite similar to each other, in order to distinguish between these models in Fig. 5.14 we can see that the more differences between these models come from the values of γ_0 for $\log k \leq -2$. In fact, as remarked before, for $\log k \leq -2.5$ we have $\gamma_0 \sim 0.46$ for the model $F_1(R)$ and $\gamma_0 \sim 0.51$ for the model $F_2(R)$.

5.4 Unified models for early– and late–time cosmic acceleration

The reason why we also study inflation in $F(R)$ gravity is that one of the most important goals on the study of modified gravity theories is to describe the consistent evolution history of the universe from inflation in the early universe to the dark energy dominated stage at the present time. Namely, the universe starts with an inflationary epoch, followed by the radiation dominated era and the matter dominated universe, and finally the late cosmic acceleration epoch is actually achieved without invoking the presence of dark components in the universe [221, 234] (for a first $F(R)$ theory unifying inflation with dark energy, see Ref. [215]). In Secs. 5.2 and 5.3, it has been demonstrated that the exponential gravity with the correction terms can be a realistic $F(R)$ gravity model. Therefore, in this section we investigate the possibility that in such exponential gravity with additional correction terms, inflation as well as the late–time cosmic acceleration can be realized. Since we examine exponential gravity among various models of $F(R)$ gravity, we consider the unification model between inflation and the late–time cosmic acceleration in this work.

Models of the type (5.16) may be combined in a natural way to obtain the phenomenological description of the inflationary epoch. For example, a 'two–steps' model may be the smooth version, given by

$$F(R) = R - 2\Lambda \left[1 - \mathrm{e}^{-R/(b\,R)}\right] - \Lambda_{\rm i}\,\theta(R - R_{\rm i})\,. \tag{5.65}$$

Here, $\theta(R-R_0)$ is the Heaviside's step distribution, $R_{\rm i}$ is the transition scalar curvature at inflationary scale and $\Lambda_{\rm i}$ is a suitable cosmological constant producing inflation, when $R \gg R_{\rm i}$. The main problem associated with this sharp model is the appearance of a possible antigravity regime in a region around the transition point between inflation and the universe described by the ΛCDM model. The antigravity in a past epoch is not phenomenologically acceptable. Furthermore, adding some terms would be necessary in order for inflation to end.

In this section, we study two applications of exponential gravity to achieve an unified description of the early–time inflation and the late–time cosmic acceleration. In particular, we show how it is possible to obtain inflationary universes with different numbers of e–folds by choosing different models parameters in the presence of ultrarelativistic matter in the early universe.

Following the first proposal of Ref. [111], we start with the form of $F(R)$ with a natural possibility of a unified description of our universe

$$F(R) = R - 2\Lambda\left(1 - {\rm e}^{-\frac{R}{bR}}\right) - \Lambda_{\rm i}\left[1 - {\rm e}^{-\left(\frac{R}{R_{\rm i}}\right)^n}\right] + \bar{\gamma}\left(\frac{1}{\tilde{R}_{\rm i}^{\alpha-1}}\right)R^{\alpha}\,, \tag{5.66}$$

where $R_{\rm i}$ and $\Lambda_{\rm i}$ are the typical values of transition curvature and expected cosmological constant during inflation, respectively, and n is a natural number larger than unity (here, we do not write the correction term for the stability of oscillations in the matter dominated era). In Eq. (5.66), the last term $\bar{\gamma}(1/\tilde{R}_{\rm i}^{\alpha-1})R^{\alpha}$, where $\bar{\gamma}$ is a positive dimensional constant and α is a real number, works at the inflation scale $\tilde{R}_{\rm i}$ and is actually necessary in order to realize an exit from inflation.

We also propose another nice inflation model based on the good behavior of exponential function described as

$$F(R) = R - 2\Lambda\left(1 - {\rm e}^{-\frac{R}{bR}}\right) - \Lambda_{\rm i}\frac{\sin\left(\pi\,{\rm e}^{-\left(\frac{R}{R_{\rm i}}\right)^n}\right)}{\pi\,{\rm e}^{-\left(\frac{R}{R_{\rm i}}\right)^n}} + \bar{\gamma}\left(\frac{1}{\tilde{R}_{\rm i}^{\alpha-1}}\right)R^{\alpha}\,. \tag{5.67}$$

Here, the parameters have the same roles of the corresponding ones in the model in Eq. (5.66). We note that the second term of the model vanishes when $R \ll R_i$ and tends to $\Lambda_{\rm i}$ when $R \gg R_{\rm i}$. We analyze these models, i.e., Model I in Eq. (5.66) and Model II in Eq. (5.67), and explore the possibilities to reproduce the phenomenologically acceptable inflation.

5.4.1 Inflation in an exponential model (Model I)

First, we investigate the model in Eq. (5.66). For simplicity, we describe a part of it as

$$f_{\rm i}(R) \equiv -\Lambda_{\rm i}\left(1 - {\rm e}^{-\left(\frac{R}{R_{\rm i}}\right)^n}\right) + \bar{\gamma}\left(\frac{1}{\tilde{R}_{\rm i}^{\alpha-1}}\right)R^{\alpha}\,. \tag{5.68}$$

We note that if $n > 1$ and $\alpha > 1$, when $R \ll R_{\rm i}(\sim \tilde{R}_{\rm i})$, we obtain

$$R \gg |f_{\rm i}(R)| \simeq \left| -\frac{R^n}{R_{\rm i}^{n-1}} + \bar{\gamma}\frac{R^{\alpha}}{\tilde{R}_{\rm i}^{\alpha-1}} \right| \,, \tag{5.69}$$

and the absence of the effects of inflation during the matter dominated era. We also find

$$f_{\rm i}'(R) = -\frac{\Lambda_{\rm i} n R^{n-1}}{R_{\rm i}^n} {\rm e}^{-\left(\frac{R}{R_{\rm i}}\right)^n} + \bar{\gamma}\alpha \left(\frac{R}{\tilde{R}_{\rm i}}\right)^{\alpha-1}, \tag{5.70}$$

$$f_{\rm i}''(R) = -\frac{\Lambda_{\rm i} n(n-1) R^{n-2}}{R_{\rm i}^n} {\rm e}^{-\left(\frac{R}{R_{\rm i}}\right)^n} + \Lambda_i \left(\frac{n R^{n-1}}{R_{\rm i}^n}\right)^2 {\rm e}^{-\left(\frac{R}{R_{\rm i}}\right)^n} + \bar{\gamma}\alpha(\alpha-1)\frac{R^{\alpha-2}}{\tilde{R}_{\rm i}^{\alpha-1}}. \tag{5.71}$$

Since when $R = R_{\rm i}\left[(n-1)/n\right]^{1/n}$ the negative term of $f_{\rm i}'(R)$ has a minimum, in order to avoid the anti–gravity effects (this means, $|f_{\rm i}'(R)| < 1$), it is sufficient to require

$$R_{\rm i} > \Lambda_{\rm i} n \left(\frac{n-1}{n}\right)^{\frac{n-1}{n}} {\rm e}^{-\frac{n-1}{n}}. \tag{5.72}$$

It is necessary for the modification of gravity describing inflation not to have any influence on the stability of the matter dominated era in the small curvature limit. When $R \ll R_{\rm i}$, the second derivative of $f_{\rm i}''(R)$, given by

$$f_{\rm i}''(R) \simeq \frac{1}{R}\left[-n(n-1)\left(\frac{R}{R_{\rm i}}\right)^{n-1} + \bar{\gamma}\alpha(\alpha-1)\left(\frac{R}{\tilde{R}_{\rm i}}\right)^{\alpha-1}\right], \tag{5.73}$$

must be positive, that is,

$$n > \alpha\,. \tag{5.74}$$

We require the existence of the de Sitter critical point $R_{\rm dS}$ which describes inflation in the high–curvature regime of $f_{\rm i}(R)$, so that $f_{\rm i}(R_{\rm dS} \gg R_{\rm i}) \simeq -\Lambda_{\rm i} + \bar{\gamma}(1/R_{\rm i}^{\alpha-1})\,R^{\alpha}$. In this case, if we put $\tilde{R}_{\rm i} = R_{\rm dS}$, we may solve the trace of the field equation (5.3) in vacuum for a constant curvature, namely $2F(R) - RF'(0) = 0$, and therefore we obtain (in vacuum, namely, if the effective modified gravity energy density is dominant over matter),

$$R_{\rm dS} = \frac{2\Lambda_{\rm i}}{\bar{\gamma}(2-\alpha)+1}, \quad \left(\frac{R_{\rm dS}}{R_{\rm i}}\right)^n \gg 1\,. \tag{5.75}$$

The last two conditions have to be satisfied simultaneously. By using Eq. (5.72), we also acquire

$$\frac{2}{\bar{\gamma}(2-\alpha)+1} > n\left(\frac{n-1}{n}\right)^{\frac{n-1}{n}} {\rm e}^{-\frac{n-1}{n}}. \tag{5.76}$$

Instability and number of e–folds during inflation

The well–known condition to have an instable de Sitter solution (see Sec. 5.3.2) is given by

$$\frac{F'(R_{\rm dS})}{R_{\rm dS}\,F''(R_{\rm dS})} < 1\,, \tag{5.77}$$

which leads to

$$\alpha\,\bar{\gamma}(\alpha-2) > 1\,, \tag{5.78}$$

for our model. Here, we have considered $f_i(R_{\rm dS}) \simeq -\Lambda_{\rm i} + \bar{\gamma}(1/R_{\rm i}^{\alpha-1})\,R^{\alpha}$. From Eqs. (5.76)–(5.78), we have to require

$$2 + 1/\bar{\gamma} > \alpha > 2\,. \tag{5.79}$$

Thus, we may evaluate the characteristic number of e–folds during inflation

$$N = \log\frac{z_{\rm i}+1}{z_{\rm e}+1}, \tag{5.80}$$

where z_i and z_e are the redshifts at the beginning and at the end of early time cosmic acceleration. Given a small cosmological perturbation $y_1(z_\mathrm{i})$ at the redshift z_i, we have from Eq. (5.49) avoiding the matter contribution

$$y_1(z_\mathrm{i}) = C_0(z_\mathrm{i}+1)^x\,, \tag{5.81}$$

with

$$x = \frac{1}{2}\left(3-\sqrt{25-\frac{16F'(R_\mathrm{dS})}{R_\mathrm{dS}F''(\mathrm{R_{dS}})}}\right)\,, \tag{5.82}$$

where $x<0$ if the de Sitter point is unstable. Thus, the perturbation $y_1(z)$ in Eq. (5.49) grows up in expanding universe as

$$y_1(z) = y_1(z_\mathrm{i})\left[\frac{(z+1)}{(z_\mathrm{i}+1)}\right]^x\,. \tag{5.83}$$

Here, we have considered $C_0 = y_1(z_\mathrm{i})/(z_\mathrm{i}+1)^x$. When $y_1(z)$ is on the same order of the effective modified gravity energy density y_0 of the de Sitter solution describing inflation (we remind, $y_0 = R_\mathrm{dS}/(12\tilde{m}^2)$), the model exits from inflation. We can estimate the number of e–folds during inflation as

$$N \simeq \frac{1}{x}\log\left(\frac{y_1(z_\mathrm{i})}{y_0}\right)\,. \tag{5.84}$$

A value demanded in most inflationary scenarios is at least $N = 50-60$.

A classical perturbation on the (vacuum) de Sitter solution may be given by the presence of ultrarelativistic matter in the early universe. The system gives rise to the de Sitter solution where the universe expands in an accelerating way but, suddenly, it exits from inflation and tends towards the minimal attractor at $R=0$ (the trivial de Sitter point). In this way, the small curvature regime arises and the physics of the ΛCDM model is reproduced.

5.4.2 Inflation in a different model (Model II)

Next, we study the inflation model in Eq. (5.67). By performing a similar analysis to that in the previous subsection, we find that also in this case, if $\alpha > 1$ and $n > 1$, we avoid the effects of inflation at small curvatures and it does not influence the stability of the matter dominated era. The de Sitter point exists if $\tilde{R}_\mathrm{i} = R_\mathrm{dS}$ and it reads as in Eq. (5.75) under the condition $(R/R_\mathrm{i})^n \gg 1$. Thus, the inflation is unstable if the condition in Eq. (5.78) is satisfied. The bigger difference between the two models exists in those behaviors in the transition phase between the small curvature region (where the physics of the ΛCDM model emerges) and the high curvature region. This means that the no antigravity condition is different in the two models and such a condition becomes more critical in the transition region. Therefore, in the following we are able to make the different choices of parameters in the two models. We note that since dark energy sector of the above models only originates from exponential gravity, all qualitative results in terms of the behavior of the dark energy component in exponential gravity found in the previous sections remain to be valid.

5.5 Analysis of inflation

In this section, we perform the numerical analysis of the early time acceleration for the unified models in Eqs. (5.66) and (5.67), by choosing appropriate parameters according with the analysis in Sec. 5.4. For this aim, it is worth rewriting Eq. (5.10) by introducing a suitable scale factor M^2 at the inflation. We can choose $M^2 = \Lambda_{\rm i}$. The effective modified gravity energy density $y_H(z)$ is now defined as

$$y_H(z) \equiv \frac{\rho_{\rm DE}}{M^2/\kappa^2} = \frac{3H^2}{M^2} - \tilde{\chi}(z+1)^4 \,. \tag{5.85}$$

Here, we have neglected the contribution of standard matter and supposed the presence of ultrarelativistic matter/radiation in the hot universe scenario, whose energy density $\rho_{\rm rad}$ at the redshift equal to zero is related with the scale as

$$\tilde{\chi} = \frac{\kappa^2 \rho_{\rm rad}}{M^2} \,. \tag{5.86}$$

Since the results are independent of the redshift scale, we set $z = 0$ at some times around the end of inflation. Equation (5.10) reads

$$\begin{aligned} & y_H''(z) - \frac{y'(z)}{z+1}\left\{3 + \frac{1-F'(R)}{2M^2F''(R)\,[y_H(z)+\tilde{\chi}(z+1)^4]}\right\} \\ & + \frac{y_H(z)}{(z+1)^2}\frac{2-F'(R)}{M^2F''(R)\,[y_H(z)+\tilde{\chi}(z+1)^4]} \\ & + \frac{(F'(R)-1)2\tilde{\chi}(z+1)^4 + (F(R)-R)/M^2}{(z+1)^2\,2M^2F''(R)\,[y_H(z)+\tilde{\chi}(z+1)^4]} = 0 \,. \end{aligned} \tag{5.87}$$

Moreover, the Ricci scalar is expressed as

$$R = M^2\left[4y_H(z) - (z+1)\frac{dy_H(z)}{dz}\right] . \tag{5.88}$$

Thus, it is easy to verify that in the de Sitter universe with $R = R_{\rm dS}$ the perturbation $y_1(z)$ on the solution $y_0 = R_{\rm dS}/(4\,M^2)$ is effectively given by Eq. (5.81), i.e., $y_1 = C_0(z+1)^x$, according with Eq. (5.49) if we neglect the contribution of standard matter. In this derivation, we have assumed the contribute of ultrarelativistic matter to be much smaller than y_0 . However, as stated above, this small energy contribution may originate from the perturbation $y_1(z_{\rm i})$ at the beginning of inflation, which, if $x < 0$, grows up in the expanding universe making inflation unstable.

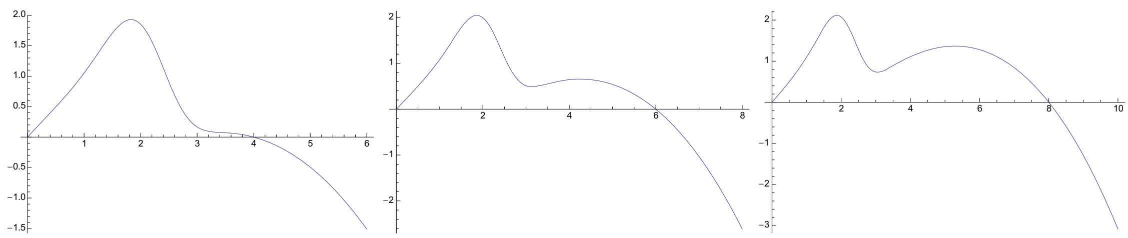
Model I

First, we explore the model in Eq. (5.66). We have to choose the parameters as $\Lambda_{\rm i} \simeq 10^{100-120}\Lambda$. The dynamics of the system is independent of this choice. Here, we summarize the conditions for inflation already stated in Sec. 5.4.1:

$$R_{\rm i} > \Lambda_{\rm i} n \left(\frac{n-1}{n}\right)^{\frac{n-1}{n}} {\rm e}^{-\frac{n-1}{n}} \,, \quad \text{(no antigravity effects)}$$

$$\tilde{R}_{\rm i} = R_{\rm dS} \,, \quad \alpha\bar{\gamma}(\alpha-2) > 1 \,, \quad \left(\frac{R_{\rm dS}}{R_{\rm i}}\right)^n \gg 1 \,, \quad \text{(existence of unstable dS solution)}$$

with $n > 1$, $2 + 1/\bar{\gamma} > \alpha > 2$ and $R_{\rm dS} = 2\Lambda_{\rm i}/\left[\bar{\gamma}(2-\alpha)+1\right]$. Since $\bar{\gamma}$ and α are combined in $\gamma(\alpha-2)$, we can fix $\bar{\gamma} = 1$, so that $R_{\rm dS} = 2\Lambda_{\rm i}/(3-\alpha)$ and $3 > \alpha > 2$. The instability factor x in Eq. (5.82) only depends on $R_{\rm dS}$. Hence, by studying the phenomenology of inflation, we examine the variation of α parameter (and, as a consequence, that of $\tilde{R}_{\rm i}$). We take $n = 4$ and $R_{\rm i} = 2\Lambda_{\rm i}$, which satisfy the condition for no antigravity well. We analyze three different cases of $\alpha = 5/2$, $8/3$, and $11/4$. In these cases, we have $R_{\rm dS} = 4\Lambda_{\rm i}$, $6\Lambda_{\rm i}$, and $8\Lambda_{\rm i}$, respectively.



(a) (b) (c)

Figure 5.20: Cosmological evolution of the quantity $2F(R/\Lambda_\mathrm{i}) - (R/\Lambda_\mathrm{i})F'(R/\Lambda_\mathrm{i})$ as a function of the redshift z for exponential model with $\alpha = 5/2$ (a), $\alpha = 8/3$ (b) and $\alpha = 11/4$ (c). The "zeros" of these graphics indicate the de Sitter solutions of the model.

In Fig. 5.20, we plot the cosmological evolution of the quantity $2F(R/\Lambda_\mathrm{i}) - (R/\Lambda_\mathrm{i})F'(R/\Lambda_\mathrm{i})$ as a function of the redshift z in the three cases. The value of "zero" of this quantity corresponds to the de Sitter points of the model. We can recognize the unstable de Sitter solutions of inflation and the attractor in zero (the de Sitter point of current acceleration is out of scale).

Despite the fact that the three considered values of α are very close each other, the values of R_dS and x significantly change and the reactions of the system to small perturbations are completely different. By starting from Eq. (5.84), we may reconstruct the rate $y_1(z_\mathrm{i})/y_0$ between the abundances of ultrarelativistic matter/radiation and modified gravity energy at the beginning of inflation in order to obtain a determined number of e–folds during inflation in the three different cases, by taking into account that $x = -0.086$, -0.218, and -0.270 for $\alpha = 5/2$, $8/3$, and $11/4$, respectively. For example, in order to have $N = 70$, for $\alpha = 5/2$, a perturbation of $y_1(z_\mathrm{i})/y_0 \sim 10^{-3}$ is necessary; for $\alpha = 8/3$, a perturbation of $y_1(z_\mathrm{i})/y_0 \sim 10^{-7}$ is sufficient; whereas for $\alpha = 11/4$, $y_1(z_\mathrm{i})/y_0 \sim 10^{-9}$. The system becomes more unstable, as $(3 - \alpha)$ is closer to zero.

In studying the behavior of the cosmic evolution in Model I for the three different cases, we set $\tilde{\chi} = 10^{-4}\, y_0/(z_\mathrm{i} + 1)^4$ in Eq. (5.87) for the case $\alpha = 5/2$ and $\tilde{\chi} = 10^{-6}\, y_0/(z_\mathrm{i} + 1)^4$ for the cases $\alpha = 8/3, 11/4$. In these choices, the effective energy density originating from the modification of gravity is 10^4 and 10^6 times larger than that of ultrarelativistic matter/radiation during inflation. By using Eq. (5.84), we can predict the following numbers of e–folds:

$$
\begin{aligned}
N &\simeq 107 \quad (\text{for } \alpha = 5/2)\,, \\
N &\simeq 64 \quad (\text{for } \alpha = 8/3)\,, \\
N &\simeq 51 \quad (\text{for } \alpha = 11/4)\,.
\end{aligned}
\tag{5.89}
$$

In order to solve Eq. (5.87) numerically, we use the initial conditions

$$
\begin{aligned}
\left.\frac{dy_H(z)}{d(z)}\right|_{z_i} &= 0\,, \\
\left. y_H(z)\right|_{z_i} &= \frac{R_\mathrm{dS}}{4\Lambda_\mathrm{i}}\,,
\end{aligned}
\tag{5.90}
$$

at the redshift $z_\mathrm{i} \gg 0$ when inflation starts. We put $z_\mathrm{i} = 10^{46}$, 10^{27}, and 10^{22} for $\alpha = 5/2$, $8/3$, and $11/4$, respectively (just for a more comfortable reading of the graphics). We also remark that the initial conditions are subject to an artificial error that we can estimate to be in the order of $\exp\left[-\left(R_\mathrm{dS}/R_\mathrm{i}\right)^n\right] \sim 10^{-7}$. This is the reason for which we only consider $\tilde{\chi} > 10^{-7}$.

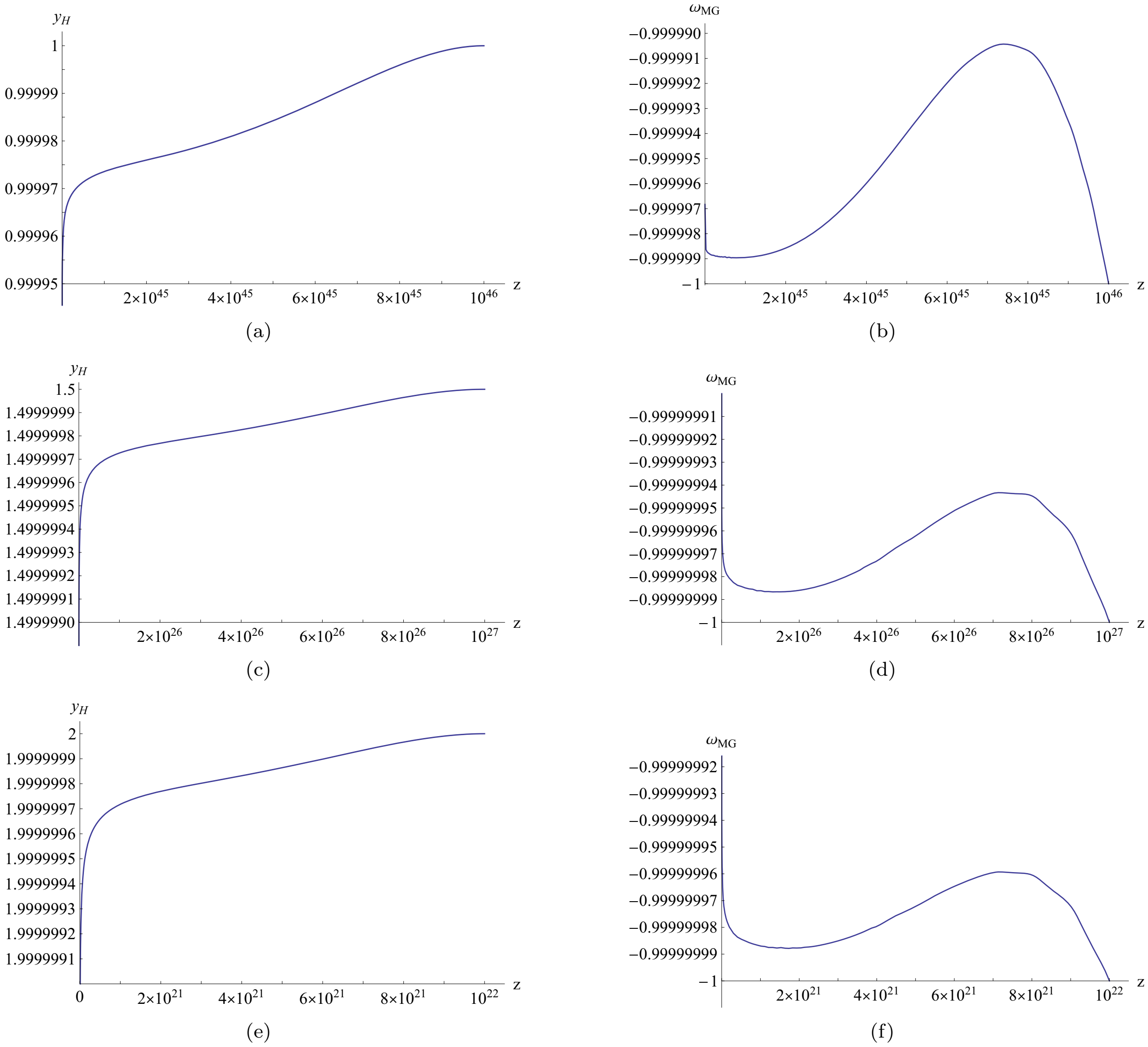


Figure 5.21: Plots of y_H [a,c,e] and $\omega_{\rm MG}$ [b,d,f] as functions of the redshift z for Model I with $\alpha = 5/2$ [a-b], $\alpha = 8/3$ [c-d] and $\alpha = 11/4$ [e-f].

In Fig. 5.21, we illustrate the cosmological evolutions of y_H and the corresponding modified gravity EoS parameter $\omega_{\rm MG}$ (defined as in Eq. (5.27)) as functions of the redshift z in the three cases. We can see, during inflation $\omega_{\rm MG}$ is indistinguishable from the value of -1 and y_H tends to decrease very slowly with respect to $y_H = 1, 3/2, 2$ for $\alpha = 5/2, 8/3, 11/4$, so that the curvature can be the expected de Sitter one, $R_{\rm dS}(= 4y_H) = 4\Lambda_{\rm i}, 6\Lambda_{\rm i}, 8\Lambda_{\rm i}$. The expected values of $z_{\rm e}$ at the end of inflation may be derived from the number of e–folds in (5.89) during inflation and read $z_{\rm e} \simeq -0.47$ for $\alpha = 5/2$; $z_{\rm e} \simeq -0.74$ for

$\alpha = 8/3$; $z_{\rm e} \simeq -0.39$ for $\alpha = 11/4$. The numerical extrapolation yields

$$\begin{aligned} y_H(z_{\rm e}) &= 0.83 y_H(z_{\rm i})\,, \quad R(z_{\rm e}) = 0.825 R_{\rm dS}\,, \quad (\text{for } \alpha = 5/2) \\ y_H(z_{\rm e}) &= 0.88 y_H(z_{\rm i})\,, \quad R(z_{\rm e}) = 0.853 R_{\rm dS}\,, \quad (\text{for } \alpha = 8/3) \\ y_H(z_{\rm e}) &= 0.92 y_H(z_{\rm i})\,, \quad R(z_{\rm e}) = 0.911 R_{\rm dS}\,. \quad (\text{for } \alpha = 11/4) \end{aligned}$$

To confirm the exit from inflation, in Fig. 5.22 we plot the cosmological evolutions of y_H and $R/\Lambda_{\rm i}$ as functions of the redshift z in the region $-1 < z < 1$, where $z_{\rm e}$ is included. The effective modified gravity energy density and the curvature decrease at the end of inflation and the physical processes described by the ΛCDM model can appear.

Model II

Next, we investigate Model II in Eq. (5.67). Here, in order to satisfy the condition for no antigravity we choose $n = 3$ and $R_{\rm i} = 2\Lambda_{\rm i}$, so that $F'(R > 0) > 0$. We take $\bar{\gamma} = 1$ again and we execute the same numerical evaluation for $\alpha = 5/2, 13/5, 21/8$ in this model as that in the previous case for Model I. The corresponding de Sitter curvatures of inflation are $R_{\rm dS} = 4\Lambda_{\rm i}, 5\Lambda_{\rm i}, 16\Lambda_{\rm i}/3$. Now, we obtain the factor in Eq. (5.82) for instability as $x = -0.086$, -0.170 , and -0.188 for $\alpha = 5/2$, $13/5$, and $21/8$, respectively. Hence, we set $\tilde{\chi} = 10^{-3}\, y_0/(z_{\rm i}+1)^4$ for $\alpha = 5/2$, $\tilde{\chi} = 10^{-4}\, y_0/(z_{\rm i}+1)^4$ for $\alpha = 13/5$, and $\tilde{\chi} = 10^{-5}\, y_0/(z_{\rm i}+1)^4$ for $\alpha = 21/8$. As a consequence, the numbers of e–folds during inflation result in $N = 80$, 54 , and 61 . The initial conditions are the same as those in the previous case in (5.90). Furthermore, we put $z_{\rm i} = 10^{34}$, 10^{22} , and 10^{26} for $\alpha = 5/2$, $13/5$, and $21/8$.

Through the numerical extrapolation, we acquire the expected values of $z_{\rm e}$ at the end of inflation as $z_{\rm e} = -0.80$, -0.97 , and -0.71 , and the following values for the effective modified gravity energy density and the Ricci scalar:

$$\begin{aligned} y_H(z_{\rm e}) &= 0.82 y_H(z_{\rm i})\,, \quad R(z_{\rm e}) = 0.813 R_{\rm dS}\,, \quad (\text{for } \alpha = 5/2) \\ y_H(z_{\rm e}) &= 0.84 y_H(z_{\rm i})\,, \quad R(z_{\rm e}) = 0.884 R_{\rm dS}\,, \quad (\text{for } \alpha = 13/5) \\ y_H(z_{\rm e}) &= 0.79 y_H(z_{\rm i})\,, \quad R(z_{\rm e}) = 0.780 R_{\rm dS}\,. \quad (\text{for } \alpha = 21/8) \end{aligned}$$

For this model, in Fig. 5.23 we depict the cosmological evolutions of y_H and $R/\Lambda_{\rm i}$ as functions of the redshift z in the region $-1 < z < 1$ at the end of inflation. Again in this case, the effective modified gravity energy density and curvature decrease, and therefore inflation ends and then the physical processes described by the ΛCDM model can be realized.

Here, we note that at the inflationary stage, radiation is negligible, as in the ordinary inflationary scenario. It causes the perturbations at the origin of instability. This point has been shown in a numerical way by using radiation, whose energy density is six order of magnitude smaller than that of dark energy.

It should be emphasized that in this work, as a first step, we have concentrated on only the possibility of the realization of inflation, and hence those important issues in inflationary cosmology such as the graceful exit problem of inflation, the following reheating process, and the generation of the curvature perturbations, whose power spectrum has to be consistent with the anisotropies of the CMB radiation obtained from the Wilkinson Microwave Anisotropy Probe (WMAP) Observations [169, 170, 271, 272], are the crucial future works of our unified scenario between inflation and the late–time cosmic acceleration.

In the future works, if we analyze the power spectrum of the curvature perturbations in our models, the next question becomes not what the total number of e–folds is, but how many e–folds one could obtain from the point when the power–law index of the primordial power spectrum n_s is close to its

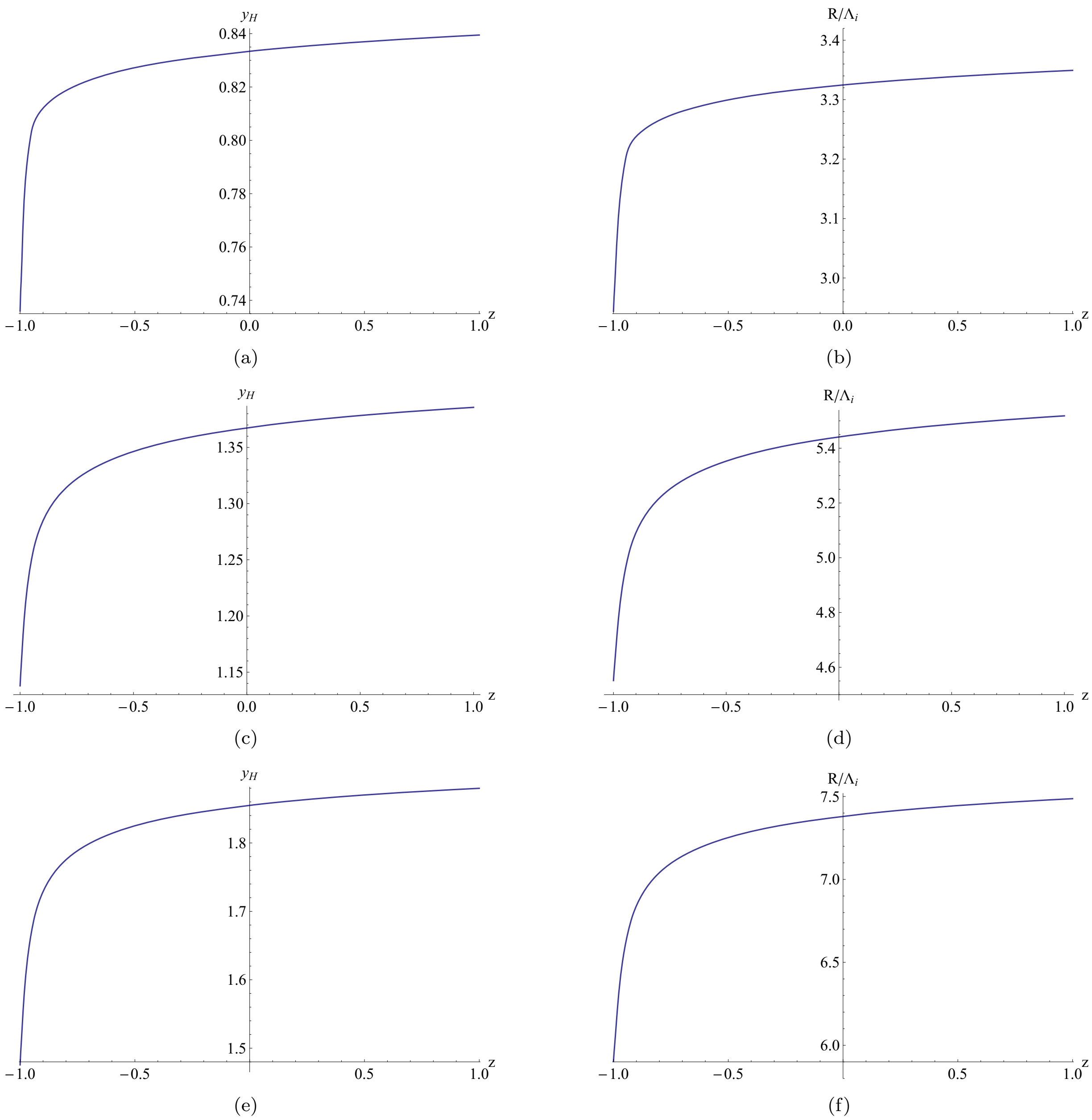


Figure 5.22: Cosmological evolution of y_H [a,c,e] and $R/\Lambda_{\rm i}$ [b,d,f] as functions of the redshift z in the region $-1 < z < 1$ for Model I with $\alpha = 5/2$ [a-b], $\alpha = 8/3$ [c-d] and $\alpha = 11/4$ [e-f].

observed value. It is presumed that since the equation of state w at the inflationary stage is so close to the model, e.g., with $\alpha = 11/4$, the number of e−folds from the point when $n_s \simeq 0.96$ [169, 170] until the end of inflation is much smaller. Accordingly, we should examine whether it is enough for the galaxy power spectrum to be reasonably close to the scale invariance of the power spectrum of the curvature perturbations. Moreover, as a more relevant question which remains is the mechanism for reheating. The problem is how the universe becomes the radiation dominated stage again after the inflationary period.

In order to construct complete models of inflation, we need a discussion of the reheating mechanism and that of exactly how the power spectrum of anisotropies is transferred to the matter. These are significant future subjects in our studies.

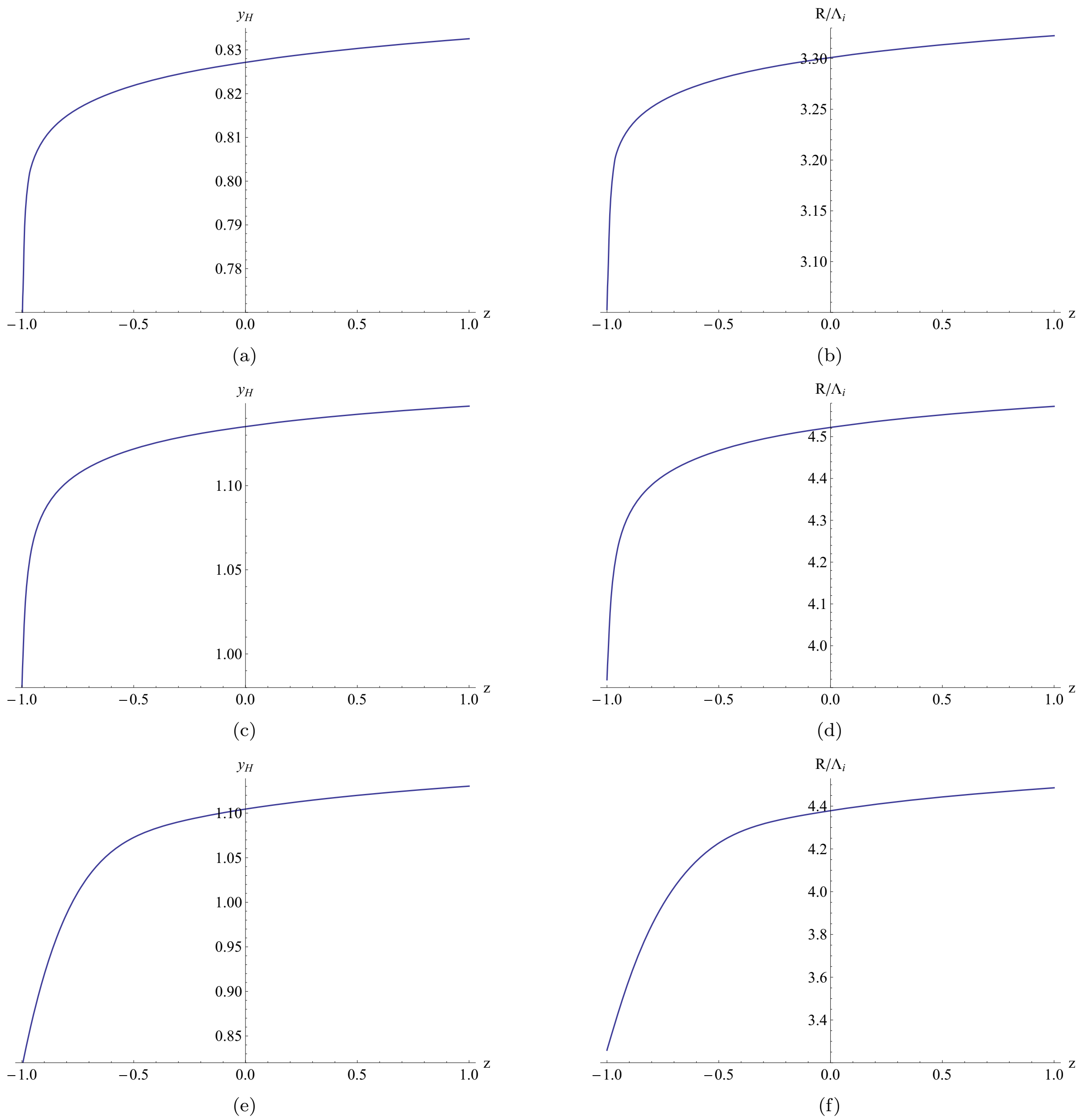


Figure 5.23: Cosmological evolution of y_H [a,c,e] and $R/\Lambda_{\rm i}$ [b,d,f] as functions of the redshift z in the region $-1 < z < 1$ for Model II with $\alpha = 5/2$ [a-b], $\alpha = 13/5$ [c-d] and $\alpha = 21/8$ [e-f].

5.6 Conclusions and general remarks

In this chapter, we have examined a generic feature of viable $F(R)$ gravity models, in particular, exponential gravity as well as a power form model. It has been shown that the behavior of higher derivatives of the Hubble parameter may be affected by large frequency oscillations of effective dark energy, and consequently solutions may become singular and unphysical at a high redshift. The analyzed models approach to a model with the cosmological constant in a manner different from each other, and hence it is reasonable to expect that the found results can be generalized to realistic $F(R)$ gravity models, in which the cosmological evolutions are similar to those in a model with the cosmological constant. To support this claim, in Sect. I it has been explicitly demonstrated how the origin of the problem influences the stability conditions satisfied by these models in order to reproduce the realistic matter dominated era. Since the corrections to the Einstein equations at the small curvature regime may lead to undesired effects at the high curvature regime, we have reconstructed a correcting (compensating) term added to the models in order to stabilize the oscillations of the effective dark energy in the matter dominated era with retaining the viability properties. It is emphasized that all the results we have found in an analytical way via studying the perturbation theory are confirmed by the numerical analysis performed on the models under consideration. Moreover, a detailed investigation on the cosmological evolutions of the universe described by those models has been executed. In particular, it has been demonstrated that the correction term does not cause any problem to the viability of the models, and that the obtained results are consistent with recent very accurate observational data of our current universe and easily pass the local tests of the solar system. Furthermore, it has been shown that the effective crossing of the phantom divide, which characterizes the de Sitter epoch, occurs in the very far future. A way to avoid the crossing of phantom divide by using inhomogeneous fluids has also been explored.

After the discovery of the accelerated expansion of our universe, a lot of theories are proposed in order to explain it. The issue of discriminating among all of these theories has become very important. The first step in order to distinguish between theories can be the study of their expansion history, but it has been revealed that sometimes different models exhibit the same (or very similar) expansion history. For this reason, the investigation of growth of the matter density perturbations by using the so–called growth index can provide a significant tool in order to distinguish among the different gravitational theories. In this context, the growth of the matter density perturbations has been examined for our models. Several ansatz for the growth index have been considered, and consequently it has been concluded that the choice of the growth index as $\gamma = \gamma_0 + \gamma_1 z$ is the most appropriate parameterization for these theories.

In addition, in the second part of this chapter we have discussed the inflationary cosmology in two exponential gravity models. It has explicitly been shown that different numbers of e–folds during inflation can be obtained by taking different model parameters in the presence of ultrarelativistic matter, the existence of which makes inflation to end and realize the exit from it. We have performed the numerical analysis of the inflationary stage in two viable exponential gravity models. It has been found that at the end of the inflation, the effective energy density and therefore the curvature of the universe become small. As a result, we have proved that it is possible to acquire a gravitational alternative scenario for a unified description of inflation in the early universe with the late–time cosmic acceleration due to the ΛCDM–like dark energy domination.

It should be cautioned that in this work, we have constructed a unified description of inflation with the late–time cosmic acceleration in $F(R)$ gravity by examining the cosmological evolutions of inflation in Secs. IV and V and the late–time cosmic acceleration in Sec. II one by one, and therefore that the evolution equation expressing all the processes from inflation to the current cosmic acceleration has not been obtained yet. In order to obtain such a gravitational field equation, the detailed considerations on the reheating process after inflation is also necessary (for a very recent analysis, see, e.g, [201]). Qualitatively, from our results it can presumably be considered that at the inflationary stage the EoS parameter $w_{\rm eff}$ is approximately equal to -1 and after that it becomes close to $1/3$ during the reheating stage because of the appearance of radiation, and after the radiation–dominated stage with $w_{\rm eff} \approx 1/3$ following the matter–dominated stage with $w_{\rm eff} \approx 0$, the dark energy dominated stage with $w_{\rm eff} \approx -1$ can be realized. If we successfully acquire the equation and solve it analytically or numerically, it would be possible to plot the evolution of the Hubble expansion rate H or $w_{\rm eff}$ from the inflationary stage in the early universe to the present time. This is very interesting and significant task in our aim, hence it would be one of the important future works of our study.

We also mention that, as another important future work in terms of our present investigations, at the next step we plan to study cosmological perturbations [167, 203] in such resultant $F(R)$ gravity theories. We calculate the power spectrum of the cosmological perturbations as well as the tensor–to–scalar ratio in these models and compare those with the observational data from such as WMAP satellite [170], future PLANCK satellite [5, 6], QUIET [3, 259], B–Pol [4] and LiteBIRD [2] in terms of the polarization of the CMB radiation. Furthermore, it is meaningful to remark that the growth of the matter density perturbations in modified gravity affects the spectrum of weak lensing (for a concrete way of comparing the theoretical predictions with the observations, see [17]), and therefore more precise future observations of weak lensing effects have a potential to present the chance to find out the signal of the modification of gravity.

It is considered that the consequences obtained in this work can be a clue of explore the features of dark energy as well as inflation. By developing this work further, it is strongly expected that we are able to construct a more sophisticated and realistic inflation model, in which the power spectrum of the curvature perturbations is consistent with the observations, the reheating mechanism is well understood, and the structure formation can be explained more naturally.

Chapter 6

Growth of matter perturbations for realistic $F(R)$ models

As a consequence of the large number of different gravitational theories, which try to give an explanation to the actual acceleration in the expansion of the Universe, a problem of distinction among some of them has appeared. The fact that different models can achieve the same expansion history has revealed that another tool, which may provide a way for discriminating among different gravitational theories, may be required. The study of the growth of matter density perturbations may become the tool that we need, due to the fact that theories with the same expansion history can have a different cosmic growth history. In order to characterize the growth of matter density perturbations, the so-called growth index γ (see [179]) can be very useful.

In this previous chapter, the study of the growth history was done for two different $F(R)$ modified gravity models. In this chapter, the growth history for another two viable $F(R)$ modified gravity models is considered and the growth index has been determined for both models. The chapter is organized as follows. In Sec. I, two different modified $F(R)$ gravity models are considered, and the values of the parameters are adjusted for both models to reach coherence with recent observations of the Universe. In Sec. II, the study of the growth of matter density perturbations is done for these two $F(R)$ gravity models. In Sect. III, several parametrizations for the growth index are studied for both models. Finally, a summary for this work is given in Sec. IV.

This Chapter is based on the publications: [184].

6.1 Realistic $F(R)$ models

In this section, two different kinds of modified $F(R)$ gravity models will be considered. The parameters of these models will be set in order to reproduce recent observations of our current Universe.

In [170], Komatsu *et al.* determined important cosmological parameters by combining the seven-year WMAP data with the latest distance measurements from the (BAO) in the distribution of galaxies, the Hubble constant (H_0) measurement and the last observations coming from the luminosity distances out to high-z type Ia supernovae (SN). The determined values for the dark energy equation of state parameter

ω_{DE} and for the dark energy density parameter Ω_{DE} are given by

$$\omega_{DE} = -0.980 \pm 0.053 \text{ from (WMAP+BAO+SN)} ,$$
$$\Omega_{DE} = 0.725 \pm 0.016 \text{ from (WMAP+BAO+H}_0) . \tag{6.1}$$

From now on, these results will be used as a constraint for the two model parameters.

6.1.1 First $F(R)$ model

In the first place, a model that appeared in [227] which could unify inflation and current acceleration will be considered. This model is given by

$$F(R) = R + \frac{\alpha R^{m+l} - \beta R^n}{1 + \gamma R^l}, \tag{6.2}$$

where α, β and γ are positive constants and m, n and l are positive integers satisfying the condition $m+l > n$. The model given by (6.2) is a generalization of the Hu-Sawicki model (see [148]) which has been proposed by Hu and Sawicki as a model which is in agreement with the constraints imposed by the Solar System tests. Model (6.2) can be reparametrized by choosing $n = l$, $\beta = 2\Lambda/(b\Lambda)^n$ and $\gamma = 1/(b\Lambda)^n$ yielding

$$F(R) = R - 2\Lambda \left(1 - \frac{1}{1 + \left(\frac{R}{b\Lambda}\right)^n}\right) + \frac{\alpha R^{m+n}}{1 + \left(\frac{R}{b\Lambda}\right)^n}, \tag{6.3}$$

with Λ being the current cosmological constant. It is worth noting that the new constant b can be negative when n is a positive even; in any other it must be positive.

The next step should be to solve Eq.(5.10) for model (6.3) and to find out the constraints on the values of the model parameters needed to fulfill the conditions given by (6.1). Unfortunately, because Eq.(5.10) cannot be solved in an analytical way for the $F(R)$ model given by (6.3), this is not possible. Thus, the way to solve this problem is to suggest a set of parameters for (6.3), to solve Eq.(5.10) numerically and to check if the results are in accordance with (6.1).

For the model given by (6.3), the following values for the parameters have been chosen:

$$n = 4, \quad m = 1, \quad b = \frac{3}{8}, \quad \alpha = 10^{-10}\tilde{m}^{-8}, \tag{6.4}$$

with $\Lambda = 7.93\tilde{m}^2$ in accordance with [170]. In order to obtain the initial conditions needed to solve Eq.(5.10) numerically, we may evaluate the dark energy density $\rho_{DE} = \rho_{eff} - \rho_m$ from Eq.(5.7) at the matter dominated era (high redshifts) by putting $R = 3\tilde{m}^2(1+z)^3$. In the case of the first model with the set of parameters given by (6.4), the initial conditions can be written as

$$y_H(z)\big|_{z_i} = \frac{\Lambda}{3\tilde{m}^2} - 81\alpha\tilde{m}^4(1+z)^3,$$
$$\left.\frac{dy_H(z)}{dz}\right|_{z_i} = -243\alpha\tilde{m}^4(1+z)^2. \tag{6.5}$$

In the case of model (6.3) with the set of parameters given by (6.4) and the initial conditions given by (6.5), I set $z_i = 3.40$, obtaining $\omega_{DE}(0) = -1.000$ and $\Omega_{DE} = 0.725$, which are in accordance with the observational data given by (6.1). Note that it is hard to solve Eq.(5.10) for higher values of the redshift due to the large frequency of the dark energy oscillations.

In the following, model (6.3) with the set of parameters given by (6.4) will be called **model I**.

6.1.2 Second $F(R)$ model

The second model considered [233] is given by

$$F(R) = R - \alpha_0 \left[\tanh\left(\frac{b_0(R-R_0)}{2}\right) + \tanh\left(\frac{b_0 R_0}{2}\right)\right] - \alpha_I \left[\tanh\left(\frac{b_I(R-R_I)}{2}\right) + \tanh\left(\frac{b_I R_I}{2}\right)\right]. \quad (6.6)$$

It will be assumed that $R_I \gg R_0$, $\alpha_I \gg \alpha_0$ and $b_I \ll b_0$, with $b_I R_I \gg 1$. By choosing $2\Lambda_0 = \alpha_0 \left[1 + \tanh\left(\frac{b_0 R_0}{2}\right)\right]$ and $2\Lambda_I = \alpha_I \left[1 + \tanh\left(\frac{b_I R_I}{2}\right)\right]$, Eq.(6.6) reduces to

$$F(R) = R - 2\Lambda \left[1 - \frac{1 - \tanh\left(\frac{b_0(R-R_0)}{2}\right)}{1 + \tanh\left(\frac{b_0 R_0}{2}\right)}\right] - 2\Lambda_I \left[1 - \frac{1 - \tanh\left(\frac{b_I(R-R_I)}{2}\right)}{1 + \tanh\left(\frac{b_I R_I}{2}\right)}\right], \quad (6.7)$$

where Λ is the current cosmological constant, while Λ_I accounts for the effective cosmological constant in the early Universe. In the following it will not be considered the last part, the one that accounts for inflation, in (6.7). Thus, the second model we take into consideration will be the one given by

$$F(R) = R - 2\Lambda \left[1 - \frac{1 - \tanh\left(\frac{b_0(R-R_0)}{2}\right)}{1 + \tanh\left(\frac{b_0 R_0}{2}\right)}\right]. \quad (6.8)$$

As already shown in the first model, analytical solutions for Eq.(5.10) cannot be found for (6.8). The procedure followed in order to solve the problem is the same one used in the previous subsection; i.e., a set of parameters will be chosen for model (6.8), then Eq.(5.10) will be solved numerically, and, finally, I will check whether the results are in accordance with (6.1) or not.

For the model (6.8), I set

$$R_0 = 10^{-66} eV^2, \quad b = 1.16 R_0^{-1}, \quad \Lambda = 7.93 \tilde{m}^2. \quad (6.9)$$

Following the same steps as in the previous subsection, it is found that the initial conditions are given by

$$y_H(z)|_{z_i} = \frac{\Lambda}{3\tilde{m}^2} \left(1 - \frac{1 - \tanh\left(b\frac{3\tilde{m}^2(1+z)^3 - R_0}{2}\right)}{1 + \tanh\left(\frac{bR_0}{2}\right)}\right),$$

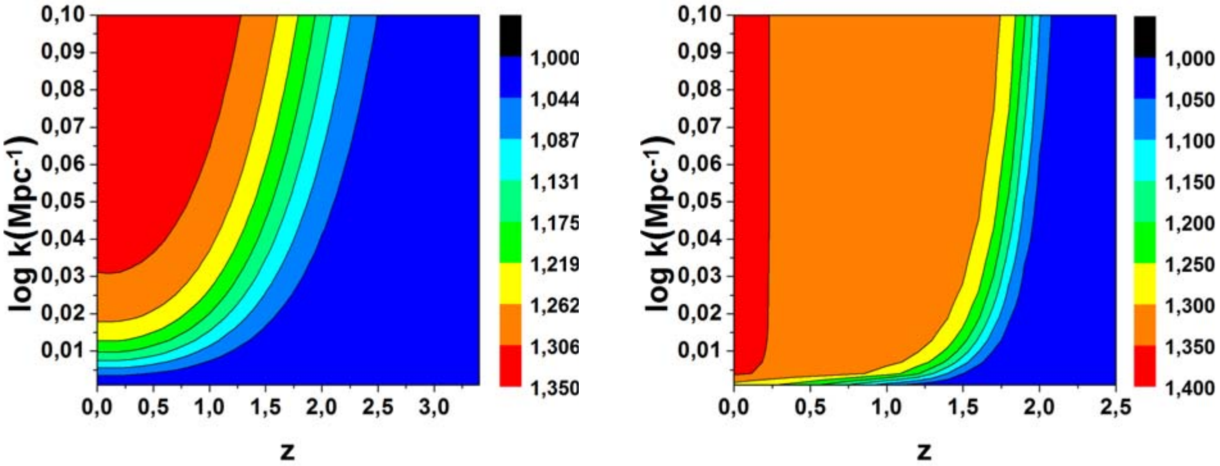
$$\left.\frac{dy_H(z)}{dz}\right|_{z_i} = \frac{3b\Lambda}{2} \frac{\left[\cosh\left(b\frac{3\tilde{m}^2(1+z)^3 - R_0}{2}\right)\right]^{-2}}{1 + \tanh\left(\frac{bR_0}{2}\right)} (1+z)^2. \quad (6.10)$$

And, finally, for model (6.8) with the set of parameters (6.9) and initial conditions given by (6.10), I set $z_i = 2.51$, obtaining $\omega_{DE}(0) = -0.969$ and $\Omega_{DE} = 0.735$, which are also in accordance with the observational data given by (6.1).

From now on, model (6.8) with the set of parameters given by (6.9) will be called **model II**.

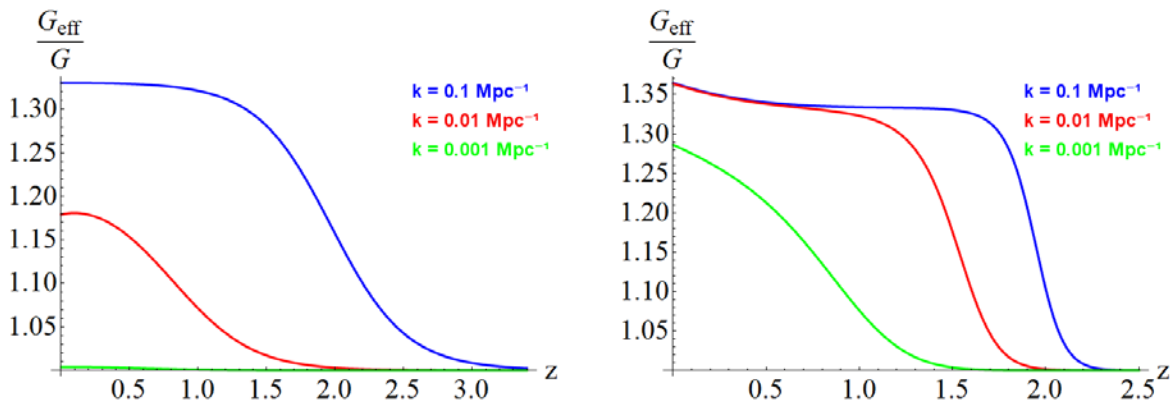
6.2 Growth of matter perturbations: Growth rate

In this section, the growth of matter density perturbations, $\delta = \frac{\delta\rho_m}{\rho_m}$, for model I and model II is studied. Since it is known that many different gravitational theories can mimic the Λ CDM unverse, which is



(a) G_{eff}/G (model I) (b) G_{eff}/G (model II)

Figure 6.1: Contour plot of the effective gravitational constant $G_{\rm eff}/G$ as a function of z and $\log k({\rm Mpc}^{-1})$ for model I (a) and model II (b).



(a) model I (b) model II

Figure 6.2: Cosmological evolution of $G_{\rm eff}/G$ as a function of z for $k = 0.1{\rm Mpc}^{-1}$ (blue line), $k = 0.01{\rm Mpc}^{-1}$ (red line) and $k = 0.001{\rm Mpc}^{-1}$ (green line) for model I (a) and model II (b).

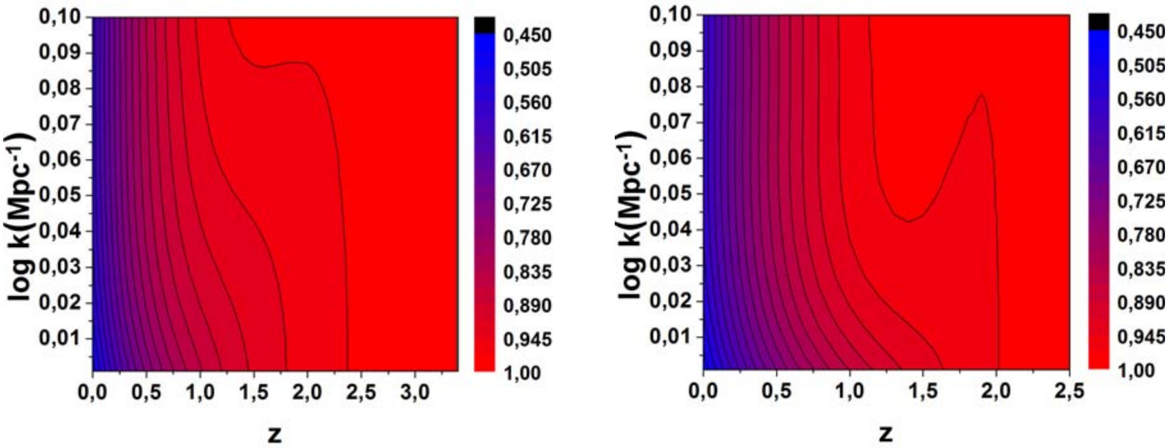
commonly accepted as the Universe in which we live, the study of the growth history of these theories may be considered an essential tool to discriminate among them.

In the previous chapter it was shown that, under the subhorizon approximation, the matter density perturbation $\delta = \frac{\delta\rho_{\rm m}}{\rho_{\rm m}}$ satisfies the following equation [280] (and references therein):

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G_{\rm eff}(a,k)\rho_{\rm m}\delta = 0, \tag{6.11}$$

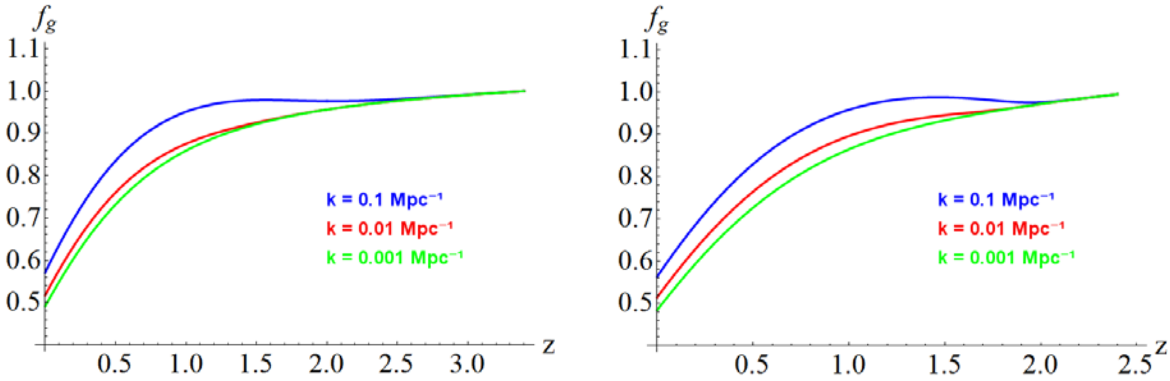
with k being the comoving wave number and $G_{\rm eff}(a,k)$ being the effective gravitational "constant" given by

$$G_{\rm eff}(a,k) = \frac{G}{F'(R)}\left[1 + \frac{(k^2/a^2)\,(F''(R)/F'(R))}{1+3\,(k^2/a^2)\,(F''(R)/F'(R))}\right]. \tag{6.12}$$



(a) $f_{\rm g}$ (model I) (b) $f_{\rm g}$ (model II)

Figure 6.3: Contour plot of the growth rate $f_{\rm g}$ as a function of z and $\log k(\mathrm{Mpc}^{-1})$ for model I (a) and model II (b).



(a) model I (b) model II

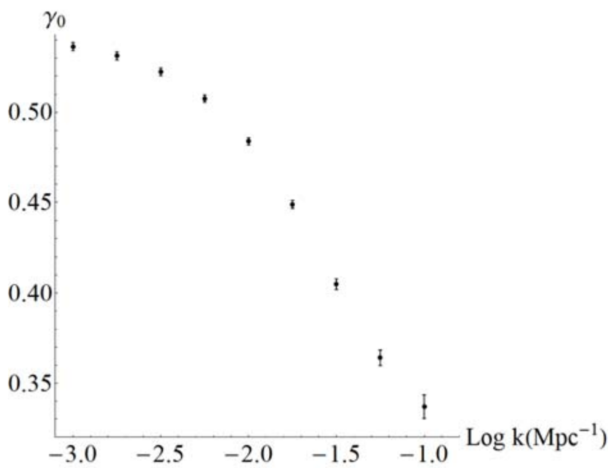
Figure 6.4: Cosmological evolution of the growth rate $f_{\rm g}$ as a function of z for $k = 0.1\mathrm{Mpc}^{-1}$ (blue line), $k = 0.01\mathrm{Mpc}^{-1}$ (red line) and $k = 0.001\mathrm{Mpc}^{-1}$ (green line) for model I (a) and model II (b).

In Figs. 6.1 and 6.2, the cosmological evolution of the ratio G_{eff}/G as a function of redshift z and the comoving wave number k for both model I and model II is depicted.

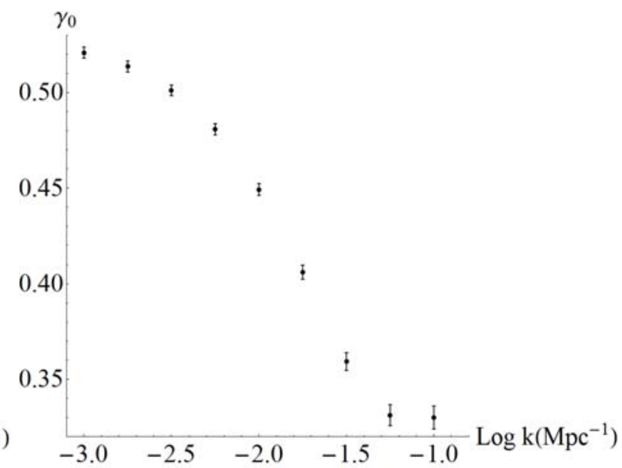
The appearance of the comoving wave number k in the expression of the effective gravitational constant $G_{\rm eff}$ has a huge importance due to the fact that now the evolution of the matter density perturbations also depends on k. This kind of dependence does not appear in the framework of general relativity. This fact can be easily checked by taking $F(R) = R$ in Eq. (6.12).

In deriving Eq. (6.11), I assume the subhorizon approximation (see [115]), for which the comoving wavelengths $\lambda \equiv a/k$ are considered to be much shorter than the Hubble radius H^{-1}. In terms of the comoving wave number, the subhorizon approximation can be written as follows:

$$\frac{k^2}{a^2} \gg H^2 \,. \tag{6.13}$$



(a) model I



(b) model II

Figure 6.5: Constant growth index as a function of $\log k(\mathrm{Mpc}^{-1})$ for model I (a) and for model II (b). The bars express the 68% CL.

For model I, the subhorizon approximation states that $k \gg 0.000116\mathrm{Mpc}^{-1}$. For model II, it states that $k \gg 0.000118\mathrm{Mpc}^{-1}$. In order to fulfill (6.13), the values considered for k in this work will always satisfy $\log k \geq -3$, with k written in Mpc^{-1}, for both model I and model II. From now on, the expression $\log k(\mathrm{Mpc}^{-1})$ will be used to specify taking the logarithm of k, with k written in Mpc^{-1}. On the other hand, deviations from the linear regime have to be taken into account [74] when $\log k(\mathrm{Mpc}^{-1}) > -1$. Thus, the range of values considered for k throughout this work for both model I and model II is

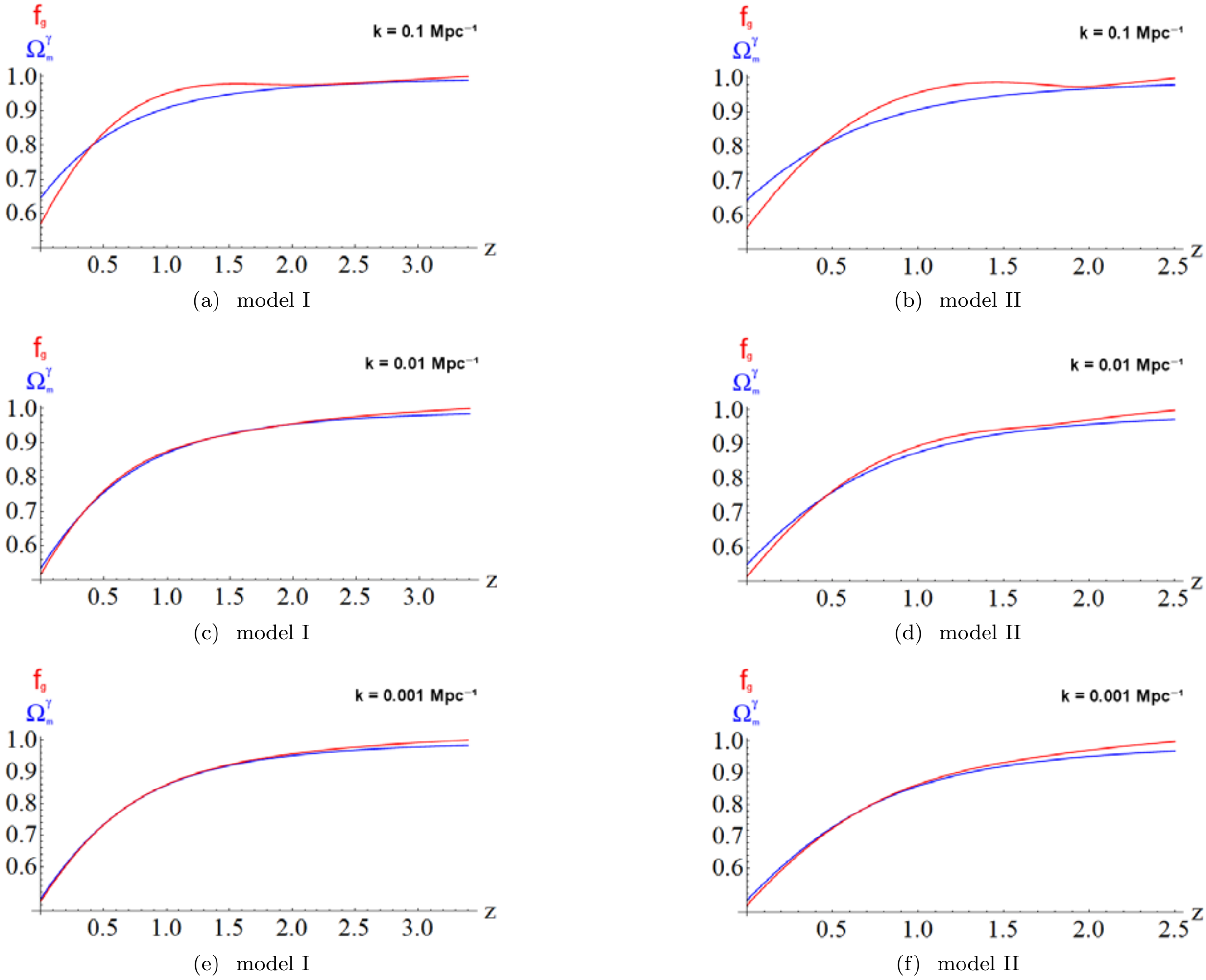
$$-3 \leq \log k(\mathrm{Mpc}^{-1}) \leq -1\,.$$

Equation (6.11) can be written in a different way by using the so-called growth rate f_{g} given by $f_{\mathrm{g}} \equiv d\ln\delta/d\ln a$. In terms of this growth rate, Eq. (6.11) reduces to

$$\frac{df_{\mathrm{g}}(z)}{dz} + \left(\frac{1+z}{H(z)}\frac{dH(z)}{dz} - 2 - f_{\mathrm{g}}(z)\right)\frac{f_{\mathrm{g}}(z)}{1+z} + \frac{3}{2}\frac{\tilde{m}^2(1+z)^2}{H^2(z)}\frac{G_{\mathrm{eff}}(a(z),k)}{G} = 0\,. \tag{6.14}$$

This equation can be solved numerically for model I and model II by imposing the condition that at high redshift the results for the ΛCDM universe are recovered. In Figs. 6.3 and 6.4, the growth rate is shown as a function of the redshift z and the comoving wave number k for model I and model II.

The next step should be to use the growth rate of these two models to compare them, but a new problem comes up when we face this task. Equation (6.14) usually must be solved numerically because of its complexity. This means that, generally, we will not have an analytic expression for the growth rate to deal with. In order to compare and discriminate among different theories it would be helpful to have one or more parameters that characterize their growth history.



(a) model I (b) model II (c) model I (d) model II (e) model I (f) model II

Figure 6.6: Cosmological evolutions of the growth rate $f_{\rm g}$ (red line) and $\Omega_{\rm m}^{\gamma}$ (blue line) with $\gamma = \gamma_0$ as functions of the redshift z in model I for $k = 0.1{\rm Mpc}^{-1}$ (a), $k = 0.01{\rm Mpc}^{-1}$ (c) and $k = 0.001{\rm Mpc}^{-1}$ (e), and those in model II for $k = 0.1{\rm Mpc}^{-1}$ (b), $k = 0.01{\rm Mpc}^{-1}$ (d) and $k = 0.001{\rm Mpc}^{-1}$ (f).

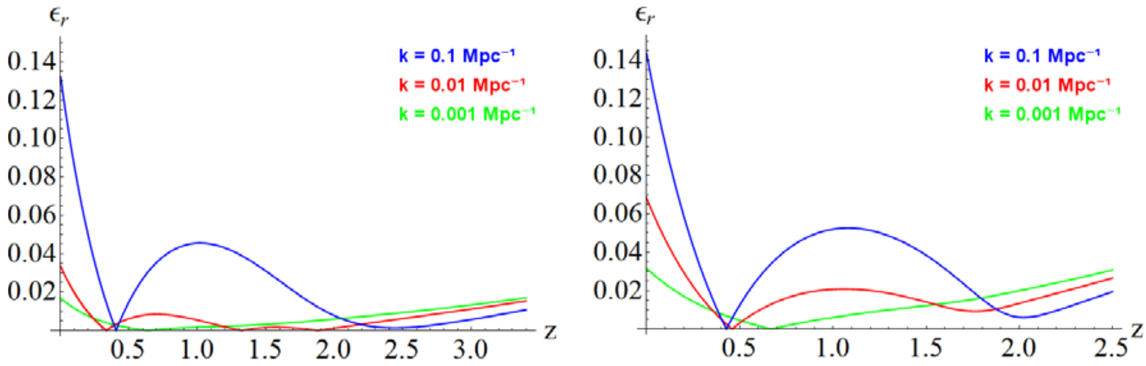
6.3 Characterizing the growth history: Growth index

In this section the concept of the growth index is developed. The growth index γ appears as an important quantity in characterizing the growth of matter density perturbations.

In order to compare the growth of matter density perturbations between different theories, the so-called growth index γ appears. This index is given by

$$f_{\rm g}(z) = \Omega_{\rm m}(z)^{\gamma(z)}, \tag{6.15}$$

where $\Omega_{\rm m}(z) = \frac{8\pi G\rho_m}{3H^2}$ is the matter density parameter. The growth index γ cannot be directly observed, but it could have a huge importance in discriminating among different gravitational theories; it can be inferred from the observational data of both the growth factor $f_{\rm g}(z)$ and the matter density parameter $\Omega_{\rm m}(z)$ at the same redshift z.



(a) model I (b) model II

Figure 6.7: Cosmological evolution of the relative difference $\epsilon_r = \frac{|f_g - \Omega_m^\gamma|}{f_g}$ with $\gamma = \gamma_0$ for $k = 0.1\mathrm{Mpc}^{-1}$ (blue line), $k = 0.01\mathrm{Mpc}^{-1}$ (red line) and $k = 0.001\mathrm{Mpc}^{-1}$ (green line) in model I (a) and model II (b).

As it was done in [31], different parametrizations for the growth index γ will be considered for both model I and model II. In a first stage, a constant γ will be assumed [142, 157]; afterwards, a linear dependence [251] given by $\gamma(z) = \gamma_0 + \gamma_0' \cdot z$ will be considered; and, finally, an ansatz of the type $\gamma(z) = \gamma_0 + \gamma_1 \cdot z/(1+z)$ will be suggested.

In what follows, we will study these different parametrizations of the growth index for model I and model II.

6.3.1 Case $\gamma = \gamma_0$

We consider the ansatz for the growth index given by

$$\gamma = \gamma_0, \tag{6.16}$$

where γ_0 is a constant.

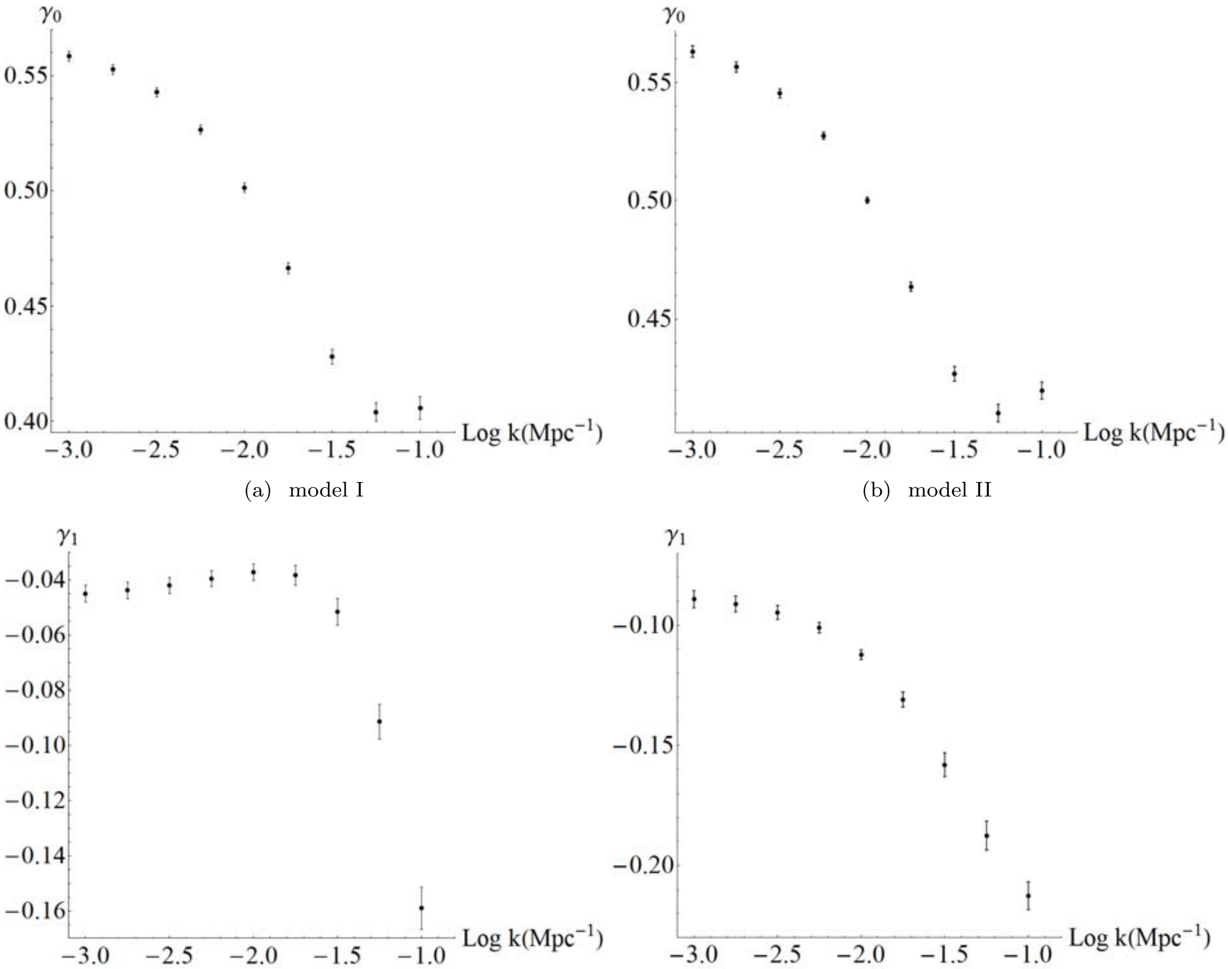
The results obtained by fitting Eq. (6.15) to the solution of Eq. (6.14) for different values of the comoving wave number k for model I and model II are shown in Fig. 6.5, where the points denote the median value while the bars express the 68% confidence level (CL). It can be easily observed that both models exhibit a strong and quite similar dependence on $\log k(\mathrm{Mpc}^{-1})$. Moreover, γ seems to be worse determined for model II.

The cosmological evolutions of the growth rate $f_g(z)$ and the expression $\Omega_m(z)^{\gamma_0}$ as functions of the redshift z for several values of the comoving wave number k for model I and model II are depicted in Fig. 6.6. From what is shown in this figure it is clear that the worst fit is given for the highest value of the comoving wave number k.

In order to clarify the results obtained, the relative difference between $f_g(z)$ and $\Omega_m(z)^{\gamma_0}$ is defined as

$$\epsilon_r(z,k) = \frac{|f_g(z,k) - \Omega_m(z)^\gamma|}{f_g(z,k)}. \tag{6.17}$$

In Fig. 6.7, the cosmological evolution of ϵ_r as a function of z for the same values of k in both models are shown. For model I the relative difference is less than 13% for $\log k(\mathrm{Mpc}^{-1}) = -1$, 3.5% for



(a) model I (b) model II

(c) model I (d) model II

Figure 6.8: Growth index fitting parameters in the case $\gamma = \gamma_0 + \gamma_1 \cdot z$ as a function of $\log k(\mathrm{Mpc}^{-1})$ for model I [(a) and (c)] and model II [(b) and (d)]. The legend is the same as Fig. 6.5.

$\log k(\mathrm{Mpc}^{-1}) = -2$ and 2% for $\log k(\mathrm{Mpc}^{-1}) = -3$; while for model II, the highest value of ϵ_r is 14% for $\log k(\mathrm{Mpc}^{-1}) = -1$, 7% for $\log k(\mathrm{Mpc}^{-1}) = -2$ and 3% for $\log k(\mathrm{Mpc}^{-1}) = -3$. Thus, two points can be made. First of all, the fits for model I are, generally, better than the ones for model II. Second of all, the fits are better for lower values of $\log k(\mathrm{Mpc}^{-1})$ for both models.

6.3.2 Case $\gamma = \gamma_0 + \gamma_1 \cdot z$

In this subsection, the case of a growth index given by

$$\gamma = \gamma_0 + \gamma_1 \cdot z, \tag{6.18}$$

where γ_0 and γ_1 are constants, will be studied following the same steps taken in the case of a constant growth index. The results obtained with this ansatz should improve those for a constant growth index.

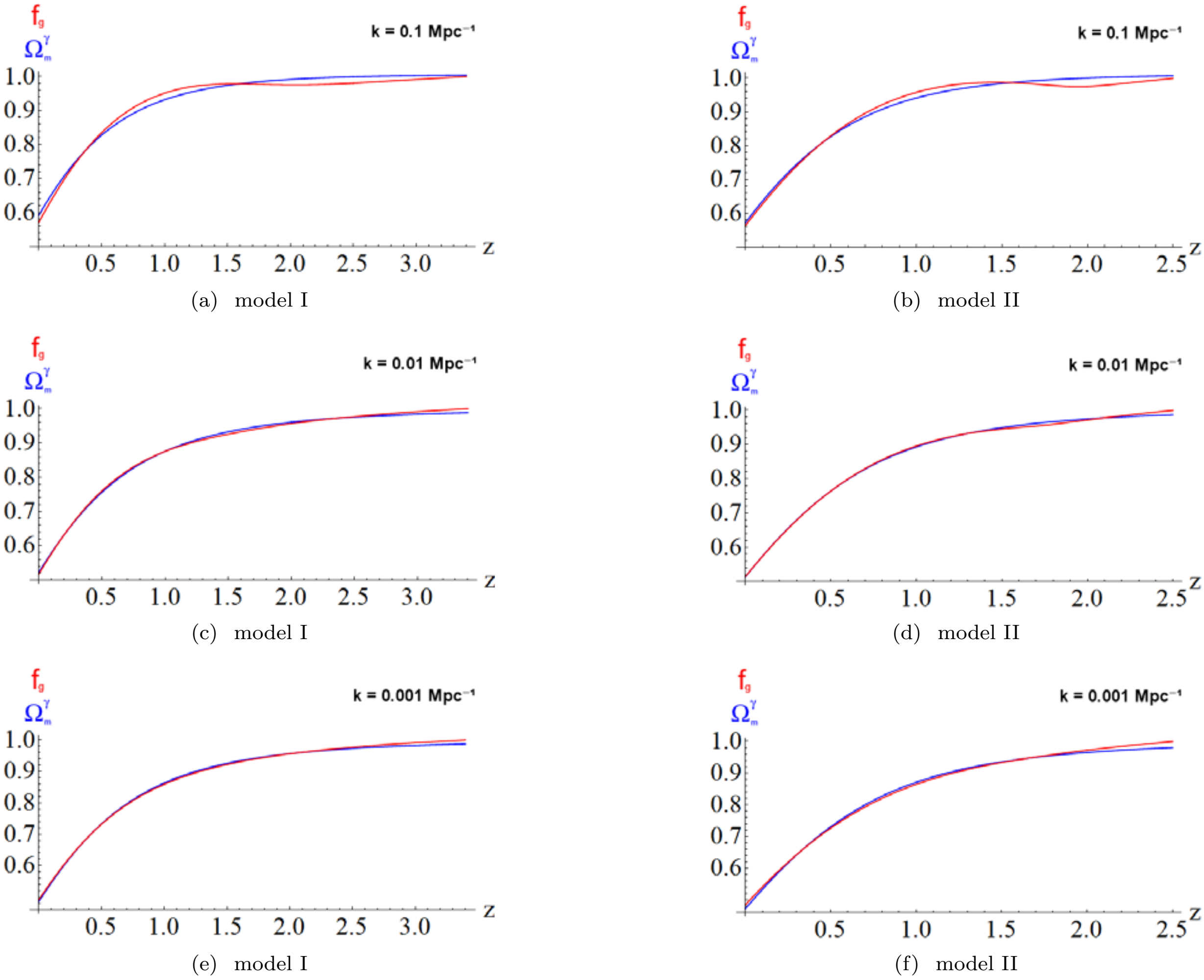
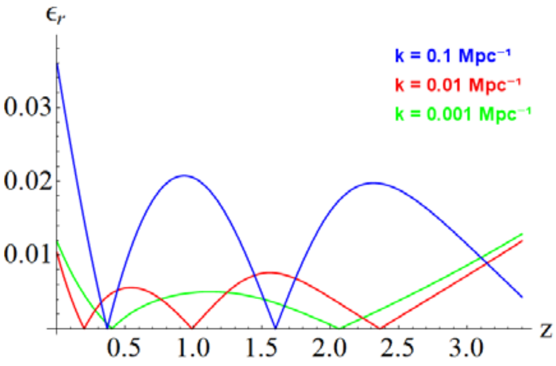


Figure 6.9: Cosmological evolutions of the growth rate f_{g} (red line) and $\Omega_{\mathrm{m}}^{\gamma}$ (blue line) with $\gamma = \gamma_0 + \gamma_1 \cdot z$ as functions of the redshift z in model I for $k = 0.1\mathrm{Mpc}^{-1}$ (a), $k = 0.01\mathrm{Mpc}^{-1}$ (c) and $k = 0.001\mathrm{Mpc}^{-1}$ (e), and those in model II for $k = 0.1\mathrm{Mpc}^{-1}$ (b), $k = 0.01\mathrm{Mpc}^{-1}$ (d) and $k = 0.001\mathrm{Mpc}^{-1}$ (f).

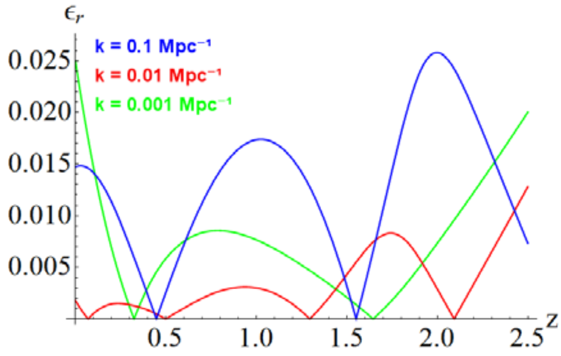
In Fig. 6.8, the parameters γ_0 and γ_1 for several values of $\log k(\mathrm{Mpc}^{-1})$ in both models are shown. For model I, γ_0 exhibits a clear dependence on $\log k(\mathrm{Mpc}^{-1})$, while γ_1 is almost constant for $-3 \leq \log k(\mathrm{Mpc}^{-1}) \leq -1.75$. For model II, the dependence on $\log k(\mathrm{Mpc}^{-1})$ is strong for both γ_0 and γ_1 throughout the range of values considered for $\log k(\mathrm{Mpc}^{-1})$. It may also be noted that the parameter γ_1 gives the main difference between model I and model II.

In Fig. 6.9, the cosmological evolutions of the growth rate $f_{\mathrm{g}}(z)$ and $\Omega_{\mathrm{m}}(z)^{\gamma(z)}$ as functions of the redshift z together for model I and model II are depicted. Compared with the fits for a constant growth index, it can be easily noticed that the linear ansatz improves the results obtained, especially in the case of $\log k(\mathrm{Mpc}^{-1}) = -1$.

The cosmological evolution of the relative difference ϵ_r as a function of z for several values of k in model I and model II is shown in Fig. 6.10. For model I the relative difference is less than 3.5% for



(a) model I



(b) model II

Figure 6.10: Cosmological evolution of the relative difference $\epsilon_r = \frac{|f_{\rm g} - \Omega_{\rm m}^{\gamma}|}{f_{\rm g}}$ with $\gamma = \gamma_0 + \gamma_1 \cdot z$ for $k = 0.1{\rm Mpc}^{-1}$ (blue line), $k = 0.01{\rm Mpc}^{-1}$ (red line) and $k = 0.001{\rm Mpc}^{-1}$ (green line) in model I (a) and model II (b).

$\log k({\rm Mpc}^{-1}) = -1$ and 1.5% for $\log k({\rm Mpc}^{-1}) \leq -2$; while for model II, the highest value of ϵ_r is 2.5% for $\log k({\rm Mpc}^{-1}) = -1$, 1.5% for $\log k({\rm Mpc}^{-1}) = -2$ and 2.5% for $\log k({\rm Mpc}^{-1}) = -3$. It might be accurate to note that these results improve those obtained for the case given by $\gamma = \gamma_0$, particulary those for the case $\log k({\rm Mpc}^{-1}) = -1$.

6.3.3 Case $\gamma = \gamma_0 + \gamma_1 \cdot \frac{z}{1+z}$

Finally, we assume the following ansatz for the growth index:

$$\gamma = \gamma_0 + \gamma_1 \cdot \frac{z}{1+z}, \tag{6.19}$$

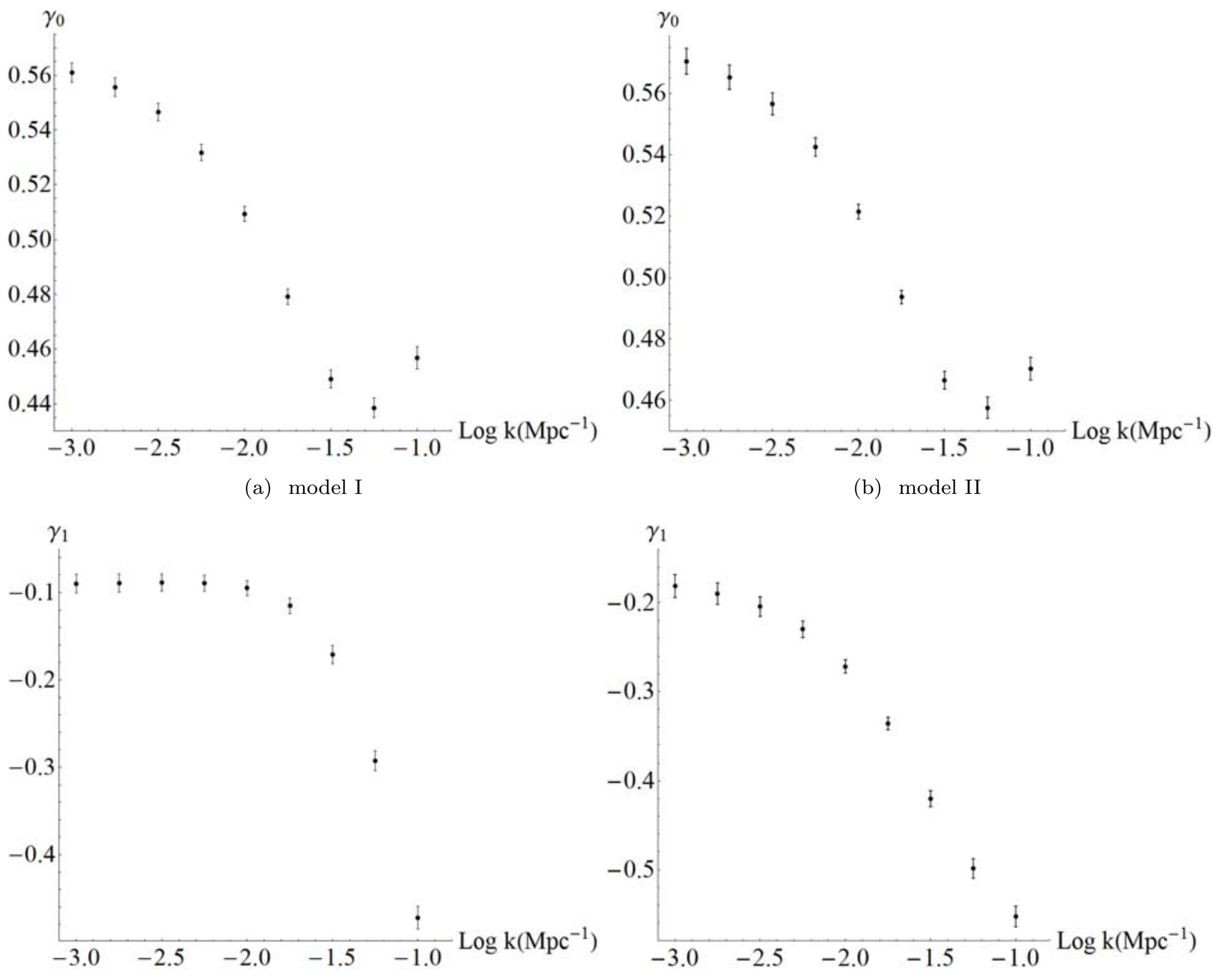
where γ_0 and γ_1 are constants.

The parameters γ_0 and γ_1 for several values of $\log k$ are shown in Fig. 6.11 for both models. For model I, as it happened in the linear case, γ_0 exhibits a strong dependence on $\log k({\rm Mpc}^{-1})$, while γ_1 is almost constant for $-3 \leq \log k({\rm Mpc}^{-1}) \leq -2$. In the case of model II, the dependence on $\log k({\rm Mpc}^{-1})$ is clear for both parameters γ_0 and γ_1.

The cosmological evolutions of the growth rate $f_{\rm g}(z)$ and $\Omega_{\rm m}(z)^{\gamma(z)}$ in model I and model II for several values of k are depicted in Fig. 6.12. The fits in this case improve the results obtained for a constant growth index, but they seem similar to those obtained for the linear case.

As in the previous subsections, the relative difference ϵ_r for several values of k in model I and model II is shown in Fig. 6.13 in order to analyze the fits quantitatively. For model I the relative difference is less than 3% for $\log k({\rm Mpc}^{-1}) = -1$ and 1.5% for $\log k({\rm Mpc}^{-1}) \leq -2$; while for model II, the highest value of ϵ_r is 5% for $\log k({\rm Mpc}^{-1}) = -1$, 3% for $\log k({\rm Mpc}^{-1}) = -2$ and 3.5% for $\log k({\rm Mpc}^{-1}) = -3$. These results improve those obtained for the case given by $\gamma = \gamma_0$. For model I the results are very similar to the corresponding ones of $\gamma = \gamma_0 + \gamma_1 \cdot z$, but for model II they are worse.

To conclude, it is important to point out that three parametrizations for the growth index have been studied for both model I and model II. As it may have been expected, the fits obtained for a constant growth



(a) model I (b) model II

(c) model I (d) model II

Figure 6.11: Growth index fitting parameters in the case $\gamma = \gamma_0 + \gamma_1 \cdot \frac{z}{1+z}$ as a function of $\log k(\mathrm{Mpc}^{-1})$ for model I [(a) and (c)] and model II [(b) and (d)]. The legend is the same as Fig. 6.5.

index give the worst results. The results obtained for the other two ansatz considered, i.e. $\gamma = \gamma_0 + \gamma_1 \cdot z$ and $\gamma = \gamma_0 + \gamma_1 \cdot \frac{z}{1+z}$, are quite similar for model I, but $\gamma = \gamma_0 + \gamma_1 \cdot z$ gives better fits for model II than those corresponding to the ansatz $\gamma = \gamma_0 + \gamma_1 \cdot \frac{z}{1+z}$. In conclusion, the linear ansatz, $\gamma = \gamma_0 + \gamma_1 \cdot z$, is the best parametrization for the growth index for the two models considered.

6.4 Discussion

In this section, the content of the paper is summarized and the results obtained for model I and model II are analyzed.

Two models of $F(R)$ modified gravity given by (6.3) and (6.8) have been considered throughout this

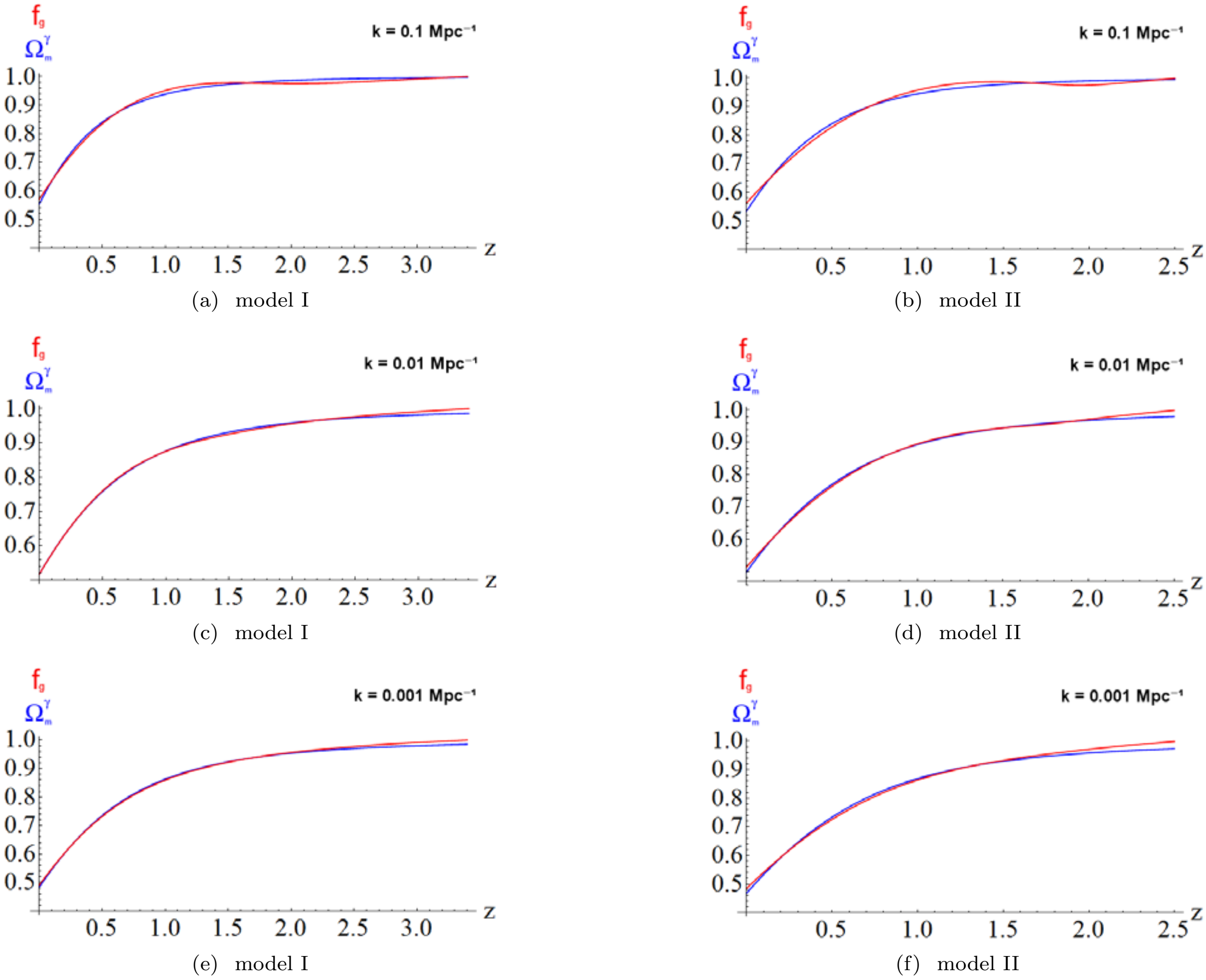
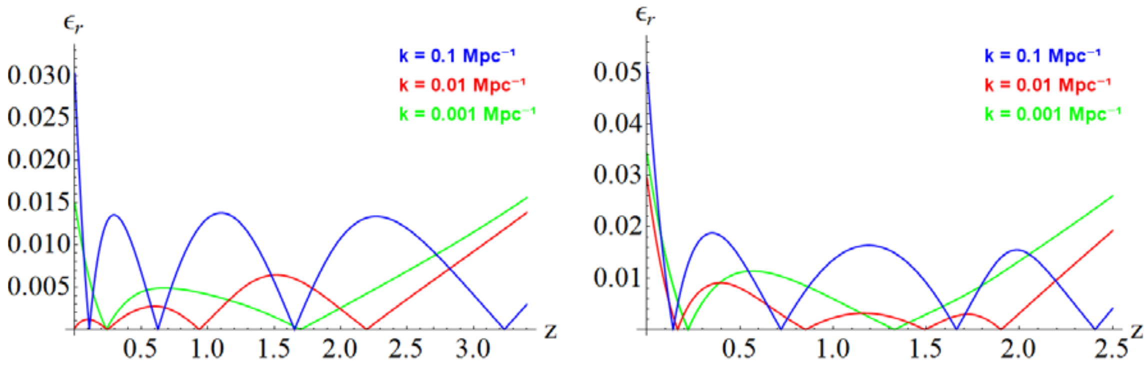


Figure 6.12: Cosmological evolutions of the growth rate f_g (red line) and Ω_m^{γ} (blue line) with $\gamma = \gamma_0 + \gamma_1 \cdot \frac{z}{1+z}$ as functions of the redshift z in model I for $k = 0.1\text{Mpc}^{-1}$ (a), $k = 0.01\text{Mpc}^{-1}$ (c) and $k = 0.001\text{Mpc}^{-1}$ (e), and those in model II for $k = 0.1\text{Mpc}^{-1}$ (b), $k = 0.01\text{Mpc}^{-1}$ (d) and $k = 0.001\text{Mpc}^{-1}$ (f).

work. The parameters of these models have been set in order to agree with current observational data coming from [170]. Model (6.3) with the set of parameters given by (6.4) is the so-called model I, while model (6.8) with the set of parameters given by (6.9) is the so-called model II.

The growth of matter density perturbations has been studied for model I and model II. The so-called growth rate has been obtained numerically for both models and three ansatz for the so-called growth index have been considered. In Figs. 6.5, 6.8 and 6.11, the results of the different parametrizations for the growth index are shown for both models.

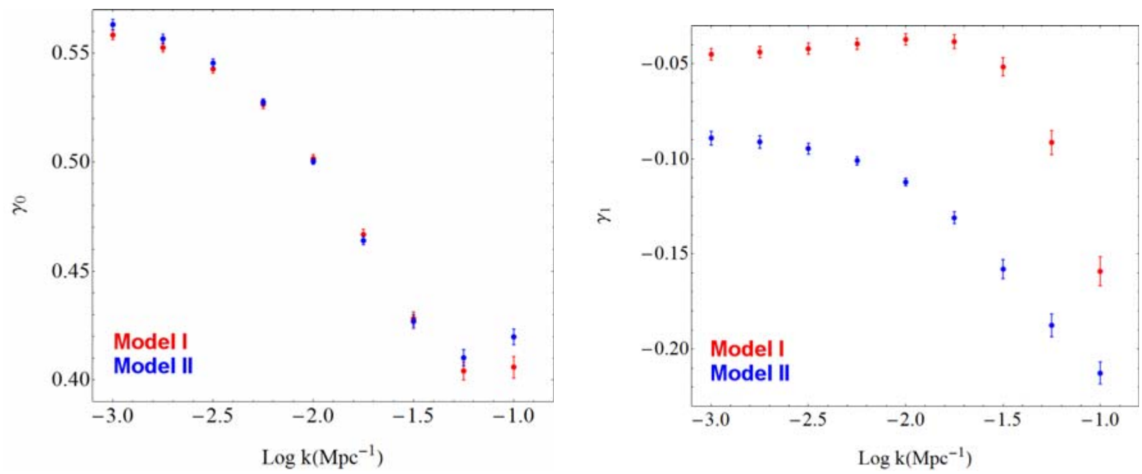
To determine which ansatz of those considered for the growth index fits better Eq. (6.15) to the solution of Eq. (6.14), the results obtained for the three parametrizations have been analyzed in the previous section. The ansatz given by $\gamma = \gamma_0 + \gamma_1 \cdot z$ seems to be the best choice for both models.



(a) model I (b) model II

Figure 6.13: Cosmological evolution of the relative difference $\epsilon_r = \frac{|f_g - \Omega_m^\gamma|}{f_g}$ with $\gamma = \gamma_0 + \gamma_1 \cdot \frac{z}{1+z}$ for $k = 0.1\text{Mpc}^{-1}$ (blue line), $k = 0.01\text{Mpc}^{-1}$ (red line) and $k = 0.001\text{Mpc}^{-1}$ (green line) in model I (a) and model II (b).

Thus, in order to discriminate between model I and model II (or with the models considered in [31]) using the growth history, the values of γ_0 and γ_1 in $\gamma = \gamma_0 + \gamma_1 \cdot z$ could have an essential importance. In Fig. 6.14 these two parameters, γ_0 and γ_1, are depicted for the two models together. We see that the behavior of γ_0 is very similar for both models. However, the values for γ_1 are totally different for model I to model II and they could be used to discriminate between these two models.



(a) (b)

Figure 6.14: Growth index fitting parameters in the case $\gamma = \gamma_0 + \gamma_1 \cdot z$ as a function of $\log k(\text{Mpc}^{-1})$. The legend is the same as Fig. 6.5.

One final note must be made. As it was pointed out in Eq.(6.11), the evolution of matter density perturbations for $F(R)$ modified gravity theories depends on the comoving wave number k, which does not occur in the framework of general relativity. Throughout this work, this fact has been confirmed, in the first place with the results obtained for the growth rate f_g, and finally, with those obtained for each of the three parametrizations considered for the growth index γ. Nevertheless, these parametrizations do not depend on the comoving wave number k, and it may be very interesting in the future to propose some scale-dependent ansatz for these $F(R)$ modified gravity theories.

Part III

The singularity problem and the weak field limit of modified gravity theories

Chapter 7

Future singularities in dark energy models

In order to explain the late-time cosmic acceleration, and to unify it with inflation, a new family of theories has been proposed, based on modifications of Einstein's gravity (see [221, 234] for a recent review). These theories, such as $f(R)$ and $f(G)$ modified gravities (R for the curvature and G for the Gauss-Bonnet invariant), and also non-local gravity, have the power to unify primordial inflation with late-time inflation in a natural way. While a number of the original models had to be rejected by several reasons, some subfamilies of them (see [87, 111]) fulfill all available cosmological constraints and have been checked to pass the solar system tests and cosmological bounds, e.g., the fact that, at these scales, Einstein's gravity is valid to a very high degree of accuracy. But even these, so called viable, modified gravities are not free from other problems, one of the most important being the frequent appearance of finite-time, future singularities.

The singularity problem is indeed of fundamental importance in modern cosmology. In order to address this issue rigorously it would be necessary to develop a fully-fledged theory of quantum gravity, but this has proven to be a very difficult, up to now impossible, challenge. Anyhow, the presence of a finite-time, future singularity may cause various problems of physical nature, as instabilities in current black hole and stellar astrophysics. It turns out that, even without the support of a quantum gravity theory, it is still meaningful at first instance (and of great importance) to try to find natural scenarios, already at the classical and semiclassical levels, that may cure the emergence of these finite-time, future singularities. Usually, one starts with some given theory and then solves the corresponding equations of motion in order to define the associated background dynamics. But, for several models of modified gravity or scalar-tensor gravity, there is in fact another possibility owing to the fact that those models are defined in terms of some arbitrary functions or potentials. The new possibility consists in using the freedom in the choice of such arbitrary functions or potentials, with the aim to reconstruct the—in general complicated—background cosmology which complies with the latest observational data.

In this chapter, a reconstruction program of this sort is applied to a number of theories which do give rise to models that exhibit finite-time, future singularities, with the purpose to investigate its structure in detail and try then to cure this common problem. Specifically, it will be found in all cases that the addition of an R^2 term provides a universal tool capable to cure these finite-time future singularities.

The chapter is organized as follows. In Sect. I the case of a fluid with an equation of state (EoS) depending on a parameter, α, which can give rise to finite-time future singularities, is considered. Different possibilities for the evolution of this universe, depending on the value of α, were studied in [230, 220]. In the subsections it will be explicitly shown that, adding a function $G(H,\dot{H},\ddot{H},...)$ to the EoS of the fluid, the different singularities can indeed be cured. Such function G is actually to be considered as a modification of Einsteinian gravity. Sect. II is devoted to the case of non-minimal coupling of modified gravity to a matter Lagrangian. The reconstruction scheme is developed for this case and the specific example of the cosmology given by the Hubble function $H(t) = h_s/(t_s - t)$ is analyzed. The calculation of the Friedmann equations for this example of modified gravity non-minimally coupled to the matter Lagrangian is explicitly carried out in Appendix D. In Sect. III the case of non-local gravity is discussed. The example of the de Sitter space is reproduced in the framework of non-local gravity. It is pointed out there that theories of this kind can give rise to a finite-time, future singularity. Finally, Sect. IV is devoted to the case of isotropic turbulence in the dark fluid universe. It will be shown there that the contribution of the turbulent part of dark energy can be reproduced through the use of a scalar-tensor theory. Several examples are discussed in detail. The chapter ends with some conclusions and an outlook.

This Chapter is based on the publications: [186].

7.1 Accelerating universe with and without a future singularity

Different accelerating universes, with and without finite-time future singularities, are considered in this section. We work with a particular fluid with EoS given by:

$$p = -\rho + A\rho^{\alpha}, \tag{7.1}$$

The nature of each singularity depends on the value of the parameter α. All possibilities, corresponding to the different values of α, have been studied in [220, 230]. It will be shown below that introducing a specific function, $G(H)$, into Eq. (7.1),

$$p = -\rho + A\rho^{\alpha} + G(H), \tag{7.2}$$

the singularity can be avoided.

The spatially flat Friedmann-Robertson-Walker (FRW) universe in the frame of General Relativity will be considered

$$ds^2 = -dt^2 + a(t)^2\left(dx^2 + dy^2 + dz^2\right), \tag{7.3}$$

$a(t)$ being the scale factor. The Friedmann equations are

$$H^2 = \frac{\kappa^2}{3}\rho, \tag{7.4}$$

$$\dot{H} + H^2 = -\frac{\kappa^2}{6}(\rho + 3p), \tag{7.5}$$

with $\kappa^2 = 8\pi G$. For future use, it is useful to classify the future singularities as in [240], namely,

- Type I ("Big Rip") : For $t \to t_s$, $a \to \infty$, $\rho \to \infty$ and $|p| \to \infty$. This type of singularity is discussed in [20, 21, 23, 37, 53, 57, 84, 85, 88, 93, 94, 96, 108, 112, 119, 134, 136, 137, 140, 143, 181, 188, 195, 196, 210, 214, 216, 258, 265, 266, 284, 291, 297].
- Type II ("sudden") : For $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$ and $|p| \to \infty$.

- Type III : For $t \to t_s$, $a \to a_s$, $\rho \to \infty$ and $|p| \to \infty$.
- Type IV : For $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$, $|p| \to p_s$ and higher derivatives of H diverge.

Here t_s , a_s , ρ_s and p_s are constants, with $a_s \neq 0$.

In the next subsections we will consider four different cases for the dark fluid (7.1), which lead to the four possible different singularities. We will produce a particular function $G(H)$ for each case, which will cure each specific singularity. Finally, in the last subsection, a general function $G(H, \dot{H}, \ddot{H}, ...)$ that cures all possible finite-time future singularities of (7.1) will be constructed.

7.1.1 Case $p = -\rho + A\rho^2$

For this particular EoS a Type III singularity occurs $(H(t) \propto (t_0 - t)^{\frac{1}{1-2\alpha}}$, $\alpha > 1)$. However, if the following EoS is considered

$$p = -\rho + A\rho^2 + G(H), \tag{7.6}$$

with

$$G(H) = -\frac{9A}{\kappa^4} H^4 + C, \tag{7.7}$$

being C real, then the singularity is avoided. In order to explain this fact, one must take into account that, using the Friedmann equation (7.4), Eq. (7.7) reduces to

$$G(H) = -A\rho^2 + C, \tag{7.8}$$

and using now Eq. (7.8) the EoS (7.6) yields

$$p = -\rho + C, \tag{7.9}$$

which does not have future finite-time singularities of any kind. It is interesting to note that the above specific choice of $G(H)$ can be motivated by modified gravity (see [221, 234]).

7.1.2 Case $p = -\rho + A\rho^{\frac{2}{3}}$

In this case, a Type I singularity occurs, namely $H(t) = \frac{-\frac{2}{3A}}{t_0 - t}$, $\alpha = 1$ and $A < 0$ or $H(t) \propto (t_0 - t)^{\frac{1}{1-2\alpha}}$, $1/2 < \alpha < 1$. Considering now Eq. (7.6), with

$$G(H) = BH^2, \tag{7.10}$$

and using the Friedmann equation given by (7.3), the EoS (7.6) reduces to

$$p = -\rho + A\rho^{\frac{2}{3}} + B'\rho, \tag{7.11}$$

where $B' = B\frac{\kappa^2}{3}$. When $\rho \to \infty$ the term $B'\rho$ dominates over the term $A\rho^{\frac{2}{3}}$ which was the one that caused the singularity. If $B > 0$ then this Type I singularity is removed.

7.1.3 Case $p = -\rho + A\rho^{\frac{1}{5}}$

With this EoS there appears a Type IV singularity, namely $H(t) \propto (t_0 - t)^{\frac{1}{1-2\alpha}}$ and $\frac{1}{1-2\alpha}$ is not an integer. Now, if Eq. (7.6) is considered, with

$$G(H) = C, \tag{7.12}$$

where C is real, then the EoS (7.6) turns into

$$p = -\rho + A\rho^{\frac{1}{5}} + C. \tag{7.13}$$

Whenever $\rho \to 0$ the term C dominates over the term $A\rho^{\frac{1}{5}}$ and the existing singularity, of Type IV, is cured.

7.1.4 Case $p = -\rho + A\rho^{-2}$

In this case a Type II singularity shows up, namely $H(t) \propto (t_0 - t)^{\frac{1}{1-2\alpha}}$, $\alpha < 0$. By taking in Eq. (7.6)

$$G(H) = -A\frac{\kappa^4}{9}H^{-4} + C, \tag{7.14}$$

with C real, and using Friedmann's equation (7.3), Eq. (7.14) yields

$$G(H) = -A\rho^{-2} + C. \tag{7.15}$$

Then, the EoS (7.6) reduces to

$$p = -\rho + C, \tag{7.16}$$

which does no more exhibit any kind of finite-time future singularity.

Summing up, we have shown in all previous situations that, for each case, a particular function, $G(H)$, can be found which cures the singularity which can possibly appear in the model given by Eq. (7.1). It is interesting to realize that a function $G(H)$ of this kind can be interpreted as a contribution of modified gravity, as was shown in [34, 221, 226, 234]. Putting everything together, we thus have demonstrated, in a very explicit way, how modified gravity is able to cure all finite-time future singularities that can possibly appear in a fluid with the particular EoS given by Eq. (7.1).

7.1.5 A generic function $G(H, \dot{H}, \ddot{H}, ...)$ which cures all singularities for the fluid with $p = -\rho + A\rho^{\alpha}$

In this subsection we will develop a systematic method for finding a function $G(H, \dot{H}, \ddot{H}, ...)$ which avoids any possible singularity for the model with $p = -\rho + A\rho^{\alpha}$. Let us recall [34, 226] that every $F(R)$-modified gravity can be seen as Einsteinian gravity with a particular EoS which absorbs the effects of $F(R)$.

We consider the case

$$F(R) = R + f(R), \tag{7.17}$$

being $f(R) = aR^2$ (it is known that the term R^2 cures the possible appearance of all future finite-time singularities [8, 65, 221, 234]). For Eq. (7.17), using the results obtained in [34, 226], it follows that

$$\rho_{eff} = \frac{1}{\kappa^2}\left[-\frac{1}{2}f(R) + 3\left(H^2 + \dot{H}\right)f'(R) - 18\left(4H^2\dot{H} + H\ddot{H}\right)f''(R)\right] + \rho_{matter}, \tag{7.18}$$

$$p_{eff} = \frac{1}{\kappa^2}\left[\frac{1}{2}f(R) - \left(3H^2 + \dot{H}\right)f'(R) + 6\left(8H^2\dot{H} + 4\dot{H}^2 + 6H\ddot{H} + \dddot{H}\right)f''(R) + \right.$$
$$\left. +36\left(4H\dot{H} + \ddot{H}\right)^2 f'''(R)\right] + p_{matter}. \tag{7.19}$$

It is also known that the Friedmann equations can be written as

$$\rho_{eff} = \frac{3H^2}{\kappa^2}, \tag{7.20}$$

$$p_{eff} = -\frac{1}{\kappa^2}\left(2\dot{H} + 3H^2\right). \tag{7.21}$$

In the case of $f(R) = aR^2$ and taking into account Eqs. (7.20) and (7.21), Eqs. (7.18) and (7.19) reduce to:

$$\rho_{matter} = \rho_{eff} - \frac{18a}{\kappa^2}\left(\dot{H}^2 - 6H^2\dot{H} - 2H\ddot{H}\right), \tag{7.22}$$

$$p_{matter} = p_{eff} - \frac{6a}{\kappa^2}\left(9\dot{H}^2 + 18H^2\dot{H} + 12H\ddot{H} + 2\dddot{H}\right), \tag{7.23}$$

respectively. If we now consider the EoS

$$p_{matter} = -\rho_{matter} + A\rho_{matter}^{\alpha}, \tag{7.24}$$

introducing into Eq. (7.24), the results obtained in Eqs. (7.22) and (7.23), we get

$$p_{eff} = -\rho_{eff} + A\rho_{eff}^{\alpha} + G(H, \dot{H}, ...), \tag{7.25}$$

where $G(H, \dot{H}, ...)$, in the case of $F(R) = R + aR^2$, is given by:

$$G(H, \dot{H}, ...) = \frac{12a}{\kappa^2}\left(6\dot{H}^2 + 3H\ddot{H} + \dddot{H}\right) + \frac{A}{\kappa^{2\alpha}}\left\{\left[3H^2 + 18a\left(\dot{H}^2 - 6H^2\dot{H} - 2H\ddot{H}\right)\right]^{\alpha} - \left(3H^2\right)^{\alpha}\right\}. \tag{7.26}$$

Thus, using $F(R)$ modified gravity (adding in the action a term proportional to R^2, see [8, 65, 221, 234]), a function $G(H, \dot{H}, ...)$ has been found which cures all the singularities which appeared in the model given by Eq. (7.24), in the frame of Einstein's gravity.

7.2 $f(R)$ modified gravity with possible future singularities for the case: $\mathcal{L} = \frac{1}{\kappa^2}R + f(R)\mathcal{L}_m$

In this section we will investigate $f(R)$ modified gravities non-minimally coupled to matter-like Lagrangians that lead to future finite-time singularities. A general Lagrangian density of this sort is (see [13, 218, 221, 234])

$$\mathcal{L} = \frac{1}{\kappa^2}R + f(R)\mathcal{L}_m. \tag{7.27}$$

By varying with respect to the metric, the following field equations are obtained

$$\frac{1}{\kappa^2}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + R_{\mu\nu}f'(R)\mathcal{L}_m + \left(g_{\mu\nu}\nabla^2 - \nabla_\mu\nabla_\nu\right)\left(f'(R)\mathcal{L}_m\right) - \frac{1}{2}f(R)T_{\mu\nu} = 0. \tag{7.28}$$

Furthermore, considering the particular case given by $\mathcal{L}_m = \partial^\mu\phi\partial_\mu\phi$, and varying with respect to the field, we get

$$\partial_\mu\left(\sqrt{-g}f(R)\partial^\mu\phi\right) = 0, \tag{7.29}$$

and, if it is now assumed that $\phi = \phi(t)$, Eq. (7.29) reduces to

$$\partial_t \left(\sqrt{-g} f(R) \partial^t \phi\right) = 0 \Rightarrow \sqrt{-g} f(R) \partial_t \phi = 0. \tag{7.30}$$

Considering a spatially flat FRW universe, this is $\sqrt{-g} = a(t)^3$, and writing $\partial_t \phi = \dot{\phi}$, Eq. (7.30) yields

$$a(t)^3 f(R) \dot{\phi} = C, \tag{7.31}$$

where C is a constant. Then,

$$\dot{\phi} = \frac{C}{a(t)^3 f(R)}. \tag{7.32}$$

Taking once more into account that $\phi = \phi(t)$, one gets

$$\mathcal{L}_m = -\dot{\phi}^2 = -\frac{C^2}{a(t)^6 f(R)^2}. \tag{7.33}$$

Thus, considering a spatially-flat FRW universe and Eq. (7.33), Friedmann's equations for the Lagrangian density given by (7.27) can be derived. Details of the long calculations leading to these equations are given in Appendix D. One should note that (D.7) and (D.8) constitute a pair of differential equations for $f(R)$ with the variable being the scalar curvature, R . Thus, starting from a given Hubble function $H(t)$ and taking into account that $t = t(R)$, from the relation $R = 6\dot{H}(t) + 12H(t)^2$ and by using (D.7) or (D.8) we obtain a specific function $f(R)$ that reproduces the given Hubble function.

Another important remark is the following. From the stress-energy tensor $T_{\mu\nu}$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta g^{\mu\nu}} \Rightarrow \begin{cases} T_{00} = -\dot{\phi}^2 = -\frac{C^2}{a(t)^6 f(R)^2} \\ T_{ii} = -a(t)^2 \dot{\phi}^2 = -\frac{C^2}{a(t)^4 f(R)^2} \end{cases}, \tag{7.34}$$

by comparison of

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta g^{\mu\nu}} \Rightarrow \begin{cases} T_{00} = -\rho g_{00} = \rho \\ T_{ii} = p g_{ii} = a(t)^2 p \end{cases} \tag{7.35}$$

with (7.34), we obtain the result:

$$p = \rho = -\dot{\phi}^2 = -\frac{C^2}{a(t)^6 f(R)^2}. \tag{7.36}$$

Now, from (7.36) we know that, for this model, a singularity of type II is avoided (for the classification of the future singularities, see [240] or Sect. II above).

In order to find a model with the Lagrangian density (7.27) that ends in a future finite-time singularity, we now consider the case:

$$H(t) = \frac{h_s}{t_s - t} \tag{7.37}$$

where, from $R = 6\dot{H}(t) + 12H(t)^2$, it follows that

$$t = t_s - \frac{h_1}{\sqrt{R}}, \tag{7.38}$$

where $h_1 = \sqrt{6h_s(1 + 2h_s)}$. Then, we can write

$$H = h_2 R^{\frac{1}{2}}, \tag{7.39}$$

$$\dot{H} = h_3 R^{\frac{2}{2}}, \tag{7.40}$$

$$\ddot{H} = h_4 R^{\frac{3}{2}}, \tag{7.41}$$

being $h_2 = \sqrt{\frac{h_s}{6(1+2h_s)}}$, $h_3 = \frac{1}{6(1+2h_s)}$ and $h_4 = \frac{1}{3(1+2h_s)\sqrt{6h_s(1+2h_s)}}$. For (7.37), we have

$$\int_{t_0}^{t(R)} H(t')dt' = \ln\left(\frac{t_s - t_0}{h_1}\sqrt{R}\right)^{h_s}, \tag{7.42}$$

hence, Eq. (D.7) reduces to

$$f(R) + a_1 R^{a_2} f(R)^2 + a_3 R \frac{df(R)}{dR} + 2a_4 \frac{R^2}{f(R)}\left(\frac{df(R)}{dR}\right)^2 - a_4 R^2 \frac{d^2 f(R)}{dR^2} = 0, \tag{7.43}$$

where $a_1 = \frac{a_0^6 h_2^2}{\kappa^2 C^2}\left(\frac{t_s - t_0}{h_1}\right)^{6h_s}$, $a_2 = 3h_s + 1$, $a_3 = h_3 + 7h_2^2$ and $a_4 = 6h_2(h_4 + 4h_2 h_3)$. The solution for Eq. (7.43) gives us the function $f(R)$ that reproduces the Hubble function given by (7.37). Eq. (7.43) can be solved, but the solution obtained is too long and not particularly insightful to be written here.

To conclude, we should mention that quantum gravity effects (which usually contain different powers of the curvature) become very important near the future singularity (see [107, 219]). There, classical considerations are not valid. It is known [8, 65, 221, 234] that the $\Box R$ term works against the singularity. Thus, an R^2 term (which will generate a $\Box R$ term) would cure the possible singularities that could arise in the theory.

7.3 Future finite-time singularities in non-local gravity

The case of non-local gravity will be here considered. This theory gives a natural unification of inflation with the current cosmic acceleration and it is inspired by quantum loop corrections (see [100, 155, 221, 228, 234]). Non-local effects come from the introduction in the action of the inverse of the D'Alembertian, $\Box^{-1}$, and the simplest action of non-local gravity is therefore

$$S = \int d^4x \sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R\left(1 + f(\Box^{-1}R)\right) - 2\Lambda\right] + \mathcal{L}_{matter}(Q; g)\right\}, \tag{7.44}$$

where Q stands for the matter fields and Λ is the cosmological constant. Introducing two scalar fields, η and ξ , action (7.44) can be rewritten as

$$S = \int d^4x \sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R\left(1 + f(\eta)\right) + \xi\left(\Box\eta - R\right) - 2\Lambda\right] + \mathcal{L}_{matter}\right\} \tag{7.45}$$

If we assume a spatially-flat FRW metric, and that $\eta = \eta(t)$ and $\xi = \xi(t)$, the equations of motion for the scalar fields and the Friedmann equations read (see [221, 234])

$$0 = \ddot{\eta} + 3H\dot{\eta} + 6\dot{H} + 12H^2, \tag{7.46}$$

$$0 = \ddot{\xi} + 3H\xi - (6\dot{H} + 12H^2)f'(\eta), \tag{7.47}$$

$$0 = -3H^2\left(1 + f(\eta) - \xi\right) + \frac{1}{2}\dot{\xi}\dot{\eta} - 3H\left(f'(\eta)\dot{\eta} - \dot{\xi}\right) + \Lambda + \kappa^2\rho_{matter}, \tag{7.48}$$

$$0 = \left(2\dot{H} + 3H^2\right)\left(1 + f(\eta) - \xi\right) + \frac{1}{2}\dot{\xi}\dot{\eta} + \left(\frac{d^2}{dt^2} + 2H\frac{d}{dt}\right)\left(f(\eta) - \xi\right) - \Lambda + \kappa^2 p_{matter}. \tag{7.49}$$

Given a Hubble function, $H(t)$, Eq. (7.46) can then be solved to obtain $\eta = \eta(t)$; moreover, the function $\xi = \xi(t)$ can be obtained from Eq. (7.47) if one assumes a form for the function $f(\eta)$. Once we have the functions $\eta(t)$ and $\xi(t)$, assuming an EoS for p_{matter} and ρ_{matter}, Eqs. (7.48) and (7.49) yield the relation between the different parameters that appear in the model (i.e. the constants of integration of the functions $\eta(t)$ and $\xi(t)$, the cosmological constant Λ, etc.).

In [221, 234] the previous scheme is used to show that de Sitter space ($H(t) = H_0$) can be a solution in non-local gravity. This is the case for matter of constant EoS ω, when

$$H = H_0, \tag{7.50}$$

$$\eta(t) = -4H_0 t, \tag{7.51}$$

$$f(\eta) = f_0 e^{\frac{\eta}{\beta}} = f_0 e^{-\frac{4H_0 t}{\beta}}, \tag{7.52}$$

$$\xi(t) = -\frac{3f_0\beta}{3\beta - 4} e^{-\frac{4H_0 t}{\beta}} - \xi_1, \tag{7.53}$$

with H_0, f_0, β and ξ_1 being constants. For de Sitter space and for matter with constant EoS ω, the energy density is

$$\rho_{matter}(t) = \rho_0 e^{-3(1+\omega)H_0 t}. \tag{7.54}$$

In order to fulfill Eqs. (7.46), (7.47), (7.48) and (7.49), it is necessary that

$$\beta = \frac{4}{3(1+\omega)}, \tag{7.55}$$

$$f_0 = \frac{\kappa^2 \rho_0}{3H_0^2(1+3\omega)}, \tag{7.56}$$

$$\xi_1 = -1 + \frac{\Lambda}{3H_0^2}. \tag{7.57}$$

Thus, de Sitter space can indeed be a solution of non-local gravity.

There are also singular solutions; however, they are very involved and will not be considered here. These singular solutions could also be cured by the addition of an R^2 term (inspired by quantum gravity effects near the singularity), a procedure which could again turn into a universal tool in order to suppress all finite-time future singularities, as before (see [8, 65, 221, 234]).

7.4 Reproducing isotropic turbulence in a dark fluid universe with scalar-tensor gravity

We will now emphasize the fact that a scalar-tensor theory can be used [69] in order to reproduce isotropic turbulence in a dark fluid universe [50]. To this end, let us consider the following scalar-tensor theory action

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} R - \frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi) + \mathcal{L}_{matter} \right], \tag{7.58}$$

which leads to the Friedmann equations

$$\frac{3H^2}{\kappa^2} = \frac{1}{2}\omega(\phi)\dot{\phi}^2 + V(\phi) + \rho_{matter}, \tag{7.59}$$

$$-\frac{2\dot{H}+3H^2}{\kappa^2} = \frac{1}{2}\omega(\phi)\dot{\phi}^2 - V(\phi) + p_{matter}. \tag{7.60}$$

If we consider that the scalar part of the action dominates over the matter part, these Friedmann equations reduce to

$$\frac{3H^2}{\kappa^2} = \frac{1}{2}\omega(\phi)\dot{\phi}^2 + V(\phi), \tag{7.61}$$

$$-\frac{2\dot{H}+3H^2}{\kappa^2} = \frac{1}{2}\omega(\phi)\dot{\phi}^2 - V(\phi). \tag{7.62}$$

On the other hand, the Friedmann equations describing isotropic turbulence in a dark fluid universe (see [50]) are

$$\frac{3H^2}{\kappa^2} = \rho_{dark} + \rho_{turb} + \rho_{rad} + \rho_{matter}, \tag{7.63}$$

$$-\frac{2\dot{H}+3H^2}{\kappa^2} = p_{dark} + p_{turb} + p_{rad} + p_{matter}. \tag{7.64}$$

Now, under the proviso that the turbulent part ρ_{turb} dominates, Eqs. (7.63) and (7.64) reduce to, respectively,

$$\frac{3H^2}{\kappa^2} = \rho_{turb}, \tag{7.65}$$

$$-\frac{2\dot{H}+3H^2}{\kappa^2} = p_{turb}. \tag{7.66}$$

Thus, comparing Eqs. (7.61) and (7.62) with Eqs. (7.65) and (7.66),

$$\rho_{turb} = \frac{1}{2}\omega(\phi)\dot{\phi}^2 + V(\phi), \tag{7.67}$$

$$p_{turb} = \frac{1}{2}\omega(\phi)\dot{\phi}^2 - V(\phi), \tag{7.68}$$

and assuming at this point that $\phi = t$, we get the expressions

$$\rho_{turb} = \frac{1}{2}\omega(\phi) + V(\phi), \tag{7.69}$$

$$p_{turb} = \frac{1}{2}\omega(\phi) - V(\phi), \tag{7.70}$$

respectively. Finally, the functions $\omega(\phi)$ and $V(\phi)$ are given by

$$\omega(\phi) = \rho_{turb} + p_{turb}, \tag{7.71}$$

$$V(\phi) = \rho_{turb} - p_{turb}. \tag{7.72}$$

In [50] some examples of isotropic turbulence were given. In the following subsections, by using Eqs. (7.71) and (7.72), scalar-tensor gravities that reproduce these specific examples will be constructed. In what follows below, the EoS $p_{turb} = \omega_{turb}\rho_{turb}$ is assumed for the isotropic turbulence.

7.4.1 De Sitter space

In this subsection we consider the case of de Sitter space, for which the scale factor is given by $a(t) = a_0 e^{H_0 t}$, with constant a_0 and H_0. For this case, the energy density of the turbulent part is

$$\rho_{turb} = e^{-3\gamma_{turb}H_0 t}\left[\frac{C}{3\gamma_{turb}}\left(e^{-\frac{5}{2}\gamma_{turb}H_0 t_{in}} - e^{-\frac{5}{2}\gamma_{turb}H_0 t}\right) + \frac{2C_0 a_0^{\frac{5}{2}\gamma_{turb}H_0}}{5\gamma_{turb}}\right]^{-\frac{6}{5}}, \tag{7.73}$$

with $C = \frac{6}{5t_{in}}\left[\rho_{turb}(t_{in})\right]^{-\frac{5}{6}}$ and where t_{in} is an initial time, from which the universe starts its development onwards. C_0 is an integration constant, which can be determined from the initial condition $t = t_{in}$ and $\gamma_{turb} = 1 + \omega_{turb}$. Then, use of Eqs. (7.71) and (7.72) yields

$$\begin{aligned} \omega(\phi) &= \rho_{turb} + p_{turb} = \gamma_{turb}\rho_{turb} = \\ &= \gamma_{turb}\, e^{-3\gamma_{turb}H_0\phi}\left[\frac{C}{3\gamma_{turb}}\left(e^{-\frac{5}{2}\gamma_{turb}H_0 t_{in}} - e^{-\frac{5}{2}\gamma_{turb}H_0\phi}\right) + \frac{2C_0 a_0^{\frac{5}{2}\gamma_{turb}H_0}}{5\gamma_{turb}}\right]^{-\frac{6}{5}} \end{aligned} \tag{7.74}$$

and

$$\begin{aligned} V(\phi) &= \rho_{turb} - p_{turb} = (1-\omega_{turb})\rho_{turb} = \\ &= (1-\omega_{turb})\, e^{-3\gamma_{turb}H_0\phi}\left[\frac{C}{3\gamma_{turb}}\left(e^{-\frac{5}{2}\gamma_{turb}H_0 t_{in}} - e^{-\frac{5}{2}\gamma_{turb}H_0\phi}\right) + \frac{2C_0 a_0^{\frac{5}{2}\gamma_{turb}H_0}}{5\gamma_{turb}}\right]^{-\frac{6}{5}}. \end{aligned} \tag{7.75}$$

As a consequence, considering the action (7.58), with Eqs. (7.74) and (7.75), together with the assumption of being $\phi = t$, is just equivalent to consider isotropic turbulence in a de Sitter universe.

7.4.2 Effective quintessence-like power-law expansion

In this subsection we discuss the situation when the scale factor is given by $a(t) = a_0 t^{h_0}$, with a_0 and h_0 constant. For this case, the energy density of the turbulent part is

$$\rho_{turb} = t^{-3\gamma_{turb}h_0}\left[\frac{5C}{6\left(1-\frac{5}{2}\gamma_{turb}h_0\right)}\left(t^{1-\frac{5}{2}\gamma_{turb}h_0} - t_{in}^{1-\frac{5}{2}\gamma_{turb}h_0}\right) + C_0 a_0^{\frac{5}{2}\gamma_{turb}h_0}\right]^{-\frac{6}{5}}, \tag{7.76}$$

being $C = \frac{6}{5t_{in}}\left[\rho_{turb}(t_{in})\right]^{-\frac{5}{6}}$ and t_{in}, as before, an initial time from which the universe starts its development onwards. Again, C_0 is a constant of integration, to be determined from the initial condition $t = t_{in}$ and $\gamma_{turb} = 1 + \omega_{turb}$. Repeating the procedure of the previous subsection, we can write

$$\begin{aligned} \omega(\phi) &= \rho_{turb} + p_{turb} = \gamma_{turb}\rho_{turb} = \\ &= \gamma_{turb}\, t^{-3\gamma_{turb}h_0}\left[\frac{5C}{6\left(1-\frac{5}{2}\gamma_{turb}h_0\right)}\left(t^{1-\frac{5}{2}\gamma_{turb}h_0} - t_{in}^{1-\frac{5}{2}\gamma_{turb}h_0}\right) + C_0 a_0^{\frac{5}{2}\gamma_{turb}h_0}\right]^{-\frac{6}{5}} \end{aligned} \tag{7.77}$$

and

$$\begin{aligned} V(\phi) &= \rho_{turb} - p_{turb} = (1-\omega_{turb})\rho_{turb} = \\ &= (1-\omega_{turb})\, t^{-3\gamma_{turb}h_0}\left[\frac{5C}{6\left(1-\frac{5}{2}\gamma_{turb}h_0\right)}\left(t^{1-\frac{5}{2}\gamma_{turb}h_0} - t_{in}^{1-\frac{5}{2}\gamma_{turb}h_0}\right) + C_0 a_0^{\frac{5}{2}\gamma_{turb}h_0}\right]^{-\frac{6}{5}}. \end{aligned} \tag{7.78}$$

Also, as remarked above, the scalar-tensor gravity given by the action (7.58), with Eqs. (7.77) and (7.78), together with the assumption $\phi = t$, is here equivalent to consider isotropic turbulence in the dark energy component, now in an effective quintessence-like power-like expanding universe.

7.4.3 Phantom-like power law expansion

Here we will deal with a phantom-like power law expansion, given by $a(t) = a_0(t_s - t)^{-h_0}$, which is known to give rise to Type I finite-time future singularities, with constant a_0 and h_0. The energy density of the turbulent part is

$$\rho_{turb} = (t_s - t)^{3\gamma_{turb}h_0} \left\{ \frac{5C}{6\left(1+\frac{5}{2}\gamma_{turb}h_0\right)} \left[(t_s-t)^{1+\frac{5}{2}\gamma_{turb}h_0} - (t_s - t_{in})^{1+\frac{5}{2}\gamma_{turb}h_0} \right] + C_0 a_0^{\frac{5}{2}\gamma_{turb}h_0} \right\}^{-\frac{6}{5}}, \tag{7.79}$$

where $C = \frac{6}{5t_{in}}\left[\rho_{turb}(t_{in})\right]^{-\frac{5}{6}}$ and t_{in} and C_0 have exactly the same meaning as before. In this case

$$\omega(\phi) = \rho_{turb} + p_{turb} = \gamma_{turb}\rho_{turb} =$$

$$= \gamma_{turb}\,(t_s - t)^{3\gamma_{turb}h_0} \left\{ \frac{5C}{6\left(1+\frac{5}{2}\gamma_{turb}h_0\right)} \left[(t_s-t)^{1+\frac{5}{2}\gamma_{turb}h_0} - (t_s - t_{in})^{1+\frac{5}{2}\gamma_{turb}h_0} \right] + C_0 a_0^{\frac{5}{2}\gamma_{turb}h_0} \right\}^{-\frac{6}{5}} \tag{7.80}$$

and

$$V(\phi) = \rho_{turb} - p_{turb} = (1-\omega_{turb})\rho_{turb} =$$

$$= (1-\omega_{turb})\,(t_s - t)^{3\gamma_{turb}h_0} \left\{ \frac{5C}{6\left(1+\frac{5}{2}\gamma_{turb}h_0\right)} \left[(t_s-t)^{1+\frac{5}{2}\gamma_{turb}h_0} - (t_s - t_{in})^{1+\frac{5}{2}\gamma_{turb}h_0} \right] + C_0 a_0^{\frac{5}{2}\gamma_{turb}h_0} \right\}^{-\frac{6}{5}}. \tag{7.81}$$

Therefore, a scalar-tensor gravity with functions $\omega(\phi)$ and $V(\phi)$ given by expressions (7.80) and (7.81), respectively, is equivalent to isotropic turbulence in a phantom-like power-like expanding universe.

Thus, we have shown that an equivalence exists between isotropic turbulence and scalar-tensor gravity and that in the isotropic turbulence theory, finite-time future singularities appear too (see Sect. 7.4.3). As in previous sections, addition of a R^2 term to the Langrangian density can avoid the development of these finite-time future singularities [234, 221, 8, 65].

7.5 Conclusions

In this chapter, a reconstruction program has been dealt with which uses the freedom in the choice of arbitrary functions or potentials, for several models of modified gravity or scalar-tensor gravity, with the aim to reconstruct a background cosmology—quite complicated in general—which complies with the latest observational data. Along this line, a systematic search for different viable models of the dark energy universe, all of which give rise to finite-time, future singularities, has been undertaken, having as goal their detailed study to try to find common features, in the search for a general solution to this important problem. Specifically, it has been checked that the addition of an R^2 term provides indeed a universal tool to cure these finite-time future singularities.

More specifically, a universal procedure to cure all future singularities has been defined and carefully tested with the help of explicit examples, corresponding to each of the four different types of possible singularities, as classified in the literature. To start, the case of a fluid with an EoS which depends on a parameter α, which can give rise to finite-time future singularities, has been considered. We have shown explicitly that, adding a specific function $G(H, \dot{H}, \ddot{H}, ...)$ to the EoS of the fluid, the different singularities can be cured, and it has been seen that this function can actually be considered as a modification of Einsteinian gravity.

The case of the non-minimal coupling of modified gravity to a matter Lagrangian has been investigated, too. The reconstruction scheme has been run for this case and the example of the cosmology given by the Hubble function $H(t) = h_s/(t_s - t)$ was analyzed. In Appendix B, the calculation has been done of the Friedmann equations for this non-minimal coupling of modified gravity to a matter Lagrangian. Further, the case of non-local gravity has been discussed. The example of the de Sitter space has been reproduced in the framework of non-local gravity, having been pointed out that such kind of theories can also give rise to finite-time, future singularities. Finally, the case of isotropic turbulence in the dark fluid universe was discussed, with the conclusion that the contribution of the turbulent part of dark energy can indeed be reproduced through the use of a scalar-tensor theory. As already indicated, several examples, corresponding to the different cases, have been studied in the paper in detail.

Concerning future perspectives, it is rather clear that, in order to address the singularity issue in all rigor it will be necessary to develop a fully-fledged theory of quantum gravity, what has proven to be a very difficult, up to now impossible, task. In any case, the presence of a finite-time, future singularity may cause various problems of physical nature, as instabilities in current black hole and stellar astrophysics. And, even without the recourse to a quantum theory of gravity, it is still meaningful to try to find natural scenarios, already at the classical level, that may cure this possible finite-time, future singularities. This has been successfully addressed in the paper, with the explicit construction of a general, universal procedure to cure all future singularities—what has been carefully tested with the help of specific examples corresponding to each one of the four different types of possible singularities, as classified in the literature.

To conclude, we should mention that quantum gravity effects (which usually contain different powers of the curvature) may become very important near future singularities. Even if classical considerations are, in principle, not valid there, it is known that the $\Box R$ term works against the singularity. Thus, an R^2 term (which will on its turn generate a $\Box R$ term) would, in principle, cure the possible singularities that could arise in the quantum theory, too. As a consequence, our method here could presumably be also extended to the quantum case without much trouble.

Chapter 8

Weak field limit of $f(R, \mathcal{G})$ modified gravity

The discovery of the current acceleration in the expansion of the Universe showed up that General Relativity fails at large scales; but it still provides fantastic results at short scales and low energies, as at the solar system level (see [289]). This is the reason why it is mandatory for a new gravitational theory to reproduce the results obtained with General Relativity in the weak field limit. Thus, the study of the weak field limit is of great importance and can be used in order to retain or rule out a gravitational theory.

The purpose of this chapter is to obtain the Newtonian, PN and PPN limits for $f(R, \mathcal{G})$ modified gravity. In order to do that, we will work under the following two hypothesis: (i) asking for low velocities with respect to the speed of light and (ii) asking for weak fields. In this case, the Newtonian limit consists in obtaining the equations of motion of the system to the same power of $\frac{\bar{v}^2}{c^2}$ than the ones given by Newtonian mechanics, with $\bar{v}$ being the typical value of the velocities of the particles in the system and c being the speed of light in vacuum. The PN limit is given by the motion of system to the next higher power of $\frac{\bar{v}^2}{c^2}$, while the PPN limit is given by the motion of system to the following higher power of $\frac{\bar{v}^2}{c^2}$ as for the PN limit. In this sense, these limits can be seen as an expansion in powers of $\frac{\bar{v}^2}{c^2}$ (note that as we work with speeds squared, the order increase is in half-integer powers, what sometimes leads to confusion among non-experts). We will consider, in the following, that $c = 1$, then our expansion will be in powers of $\bar{v}^2$.

The chapter is organized as follows. In Sect. I, the field equations for $f(R, \mathcal{G})$ modified gravity theories are reviewed and the Newtonian, post–Newtonian and post–post–Newtonian are obtained for this kind of theories. In Sect. II, the Newtonian limit for $f(R, \mathcal{G})$ gravities is solved in terms of Green's functions. In Sect. III, the weak field limit of the special cases of $f(R)$ and $f(\mathcal{G})$ modified gravities are, respectively, studied. Finally, in Sect. IV, the results obtained in this chapter are summarized.

8.1 $f(R,\mathcal{G})$ modified gravity: the field equations and the Newtonian limit

This section is devoted to the study of the field equations for $f(R,\mathcal{G})$ modified gravity and their Newtonian, Post–Newtonian (PN) and Post–Post–Newtonian (PPN) limits (for other examples of weak field limit in modified gravity theories, see [70, 72, 73, 273]). The first remarkable characteristic of $f(R,\mathcal{G})$ modified gravity is that, in this case, we obtain fourth–order field equations, instead of the standard second–order ones obtained in the case of General Relativity (GR). This fact is due to the existence of some boundary terms that disappear in GR thanks to the divergence theorem, but they remain in other theories, as in the case of $f(R,\mathcal{G})$ modified gravity.

8.1.1 General form of the field equations

This subsection is devoted to find out the field equations for $f(R,\mathcal{G})$ modified gravity. The starting action for $f(R,\mathcal{G})$ modified gravity is given by:

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}f(R,\mathcal{G}) + \mathcal{L}_{\text{matter}}\right\}, \tag{8.1}$$

where $\kappa^2 = 8\pi G_N$, G_N is the Newton constant, and $\mathcal{G}$ is the Gauss-Bonnet invariant, defined by:

$$\mathcal{G} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma}. \tag{8.2}$$

The variation of (8.1) with respect to the metric tensor $g_{\mu\nu}$ gives rise to the field equations for $f(\mathcal{G})$ modified gravity, which are:

$$\begin{aligned}
&-\frac{1}{2}g_{\mu\nu}f(R,\mathcal{G}) + f_R(R,\mathcal{G})R_{\mu\nu} + g_{\mu\nu}\nabla^2\left(f_R(R,\mathcal{G})\right) - \nabla_\mu\nabla_\nu\left(f_R(R,\mathcal{G})\right) + \\
&+2f_\mathcal{G}(R,\mathcal{G})RR_{\mu\nu} - 4f_\mathcal{G}(R,\mathcal{G})R_{\mu\rho}R_\nu{}^\rho + 2f_\mathcal{G}(R,\mathcal{G})R_{\alpha\beta\rho\mu}R^{\alpha\beta\rho}{}_\nu - 4f_\mathcal{G}(R,\mathcal{G})R_{\mu\rho\nu\sigma}R^{\rho\sigma} + \\
&+2g_{\mu\nu}R\nabla^2 f_\mathcal{G}(R,\mathcal{G}) - 4g_{\mu\nu}R_{\rho\sigma}\nabla^\rho\nabla^\sigma f_\mathcal{G}(R,\mathcal{G}) - 2R\nabla_\mu\nabla_\nu f_\mathcal{G}(R,\mathcal{G}) - 4R_{\mu\nu}\nabla^2 f_\mathcal{G}(R,\mathcal{G}) + \\
&+4R_{\nu\rho}\nabla^\rho\nabla_\mu f_\mathcal{G}(R,\mathcal{G}) + 4R_{\mu\rho}\nabla^\rho\nabla_\nu f_\mathcal{G}(R,\mathcal{G}) + 4R_{\mu\rho\nu\sigma}\nabla^\rho\nabla^\sigma f_\mathcal{G}(R,\mathcal{G}) = 2\kappa^2 T_{\mu\nu},
\end{aligned} \tag{8.3}$$

and the trace equation is given by:

$$-2f(R,\mathcal{G}) + f_R(R,\mathcal{G})R + 3\nabla^2 f_R(R,\mathcal{G}) + 2f_\mathcal{G}(R,\mathcal{G})\mathcal{G} + 2R\nabla^2 f_\mathcal{G}(R,\mathcal{G}) - 4R_{\rho\sigma}\nabla^\rho\nabla^\sigma f_\mathcal{G}(R,\mathcal{G}) = 2\kappa^2 T. \tag{8.4}$$

In Eq.(8.3) and Eq.(8.4), the following notation has been used: $f_R(R,\mathcal{G}) = \frac{df(R,\mathcal{G})}{dR}$ and $f_\mathcal{G}(R,\mathcal{G}) = \frac{df(R,\mathcal{G})}{d\mathcal{G}}$.

The fact that the scalar curvature, R, and the Gauss-Bonnet invariant, $\mathcal{G}$, involves second derivatives of the metric tensor $g_{\mu\nu}$ makes of Eq.(8.3) and Eq.(8.4) fourth-order differential equations for the metric, $g_{\mu\nu}$.

8.1.2 The Newtonian, PN and PPN limits

In this subsection, all the quantities involved in the Newtonian, PN and PPN limits of the $f(R,\mathcal{G})$-gravity given by (8.1) will be expanded in powers of $\bar{v}^2$.

We expect that it should be possible to find a coordinate system in which the metric tensor is nearly equal to the Minkowski tensor $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, the corrections being expandable in powers of $\bar{v}^2$. We will consider the following ansatz for the metric tensor:

$$\begin{cases} g_{00} = g^{(0)}_{00} + g^{(2)}_{00} + g^{(4)}_{00} + O(6) \quad \text{with} \quad \begin{cases} g^{(0)}_{00} = 1 \\ g^{(2)}_{00} = -2U \end{cases} \\ \\ g_{0i} = g^{(3)}_{0i} + O(5) \\ \\ g_{ij} = g^{(0)}_{ij} + g^{(2)}_{ij} + g^{(4)}_{ij} + O(6) \quad \text{with} \quad \begin{cases} g^{(0)}_{ij} = -\delta_{ij} \\ g^{(2)}_{ij} = -\delta_{ij}2V \end{cases} \end{cases} \tag{8.5}$$

The inverse metric of (8.5) can be calculated using the relation $g^{\alpha\rho}g_{\rho\beta} = \delta^\alpha_\beta$, giving the following results:

$$\begin{cases} g^{00} = g^{(0)00} + g^{(2)00} + g^{(4)00} + O(6) \quad \text{with} \quad \begin{cases} g^{(0)00} = 1 \\ g^{(2)00} = 2U \\ g^{(4)00} = -g^{(4)}_{00} + 4U^2 \end{cases} \\ \\ g^{0i} = g^{(3)0i} + O(5) \quad \text{with} \quad g^{(3)0i} = \delta^{ij}g^{(3)}_{0j} \\ \\ g^{ij} = g^{(0)ij} + g^{(2)ij} + g^{(4)ij} + O(6) \quad \text{with} \quad \begin{cases} g^{(0)ij} = -\delta^{ij} \\ g^{(2)ij} = \delta^{ij}2V \\ g^{(4)ij} = -4V^2\delta^{ij} - \delta^{ik}\delta^{jl}g^{(4)}_{kl} \end{cases} \end{cases} \tag{8.6}$$

Given a metric tensor, the connection associated can be calculated with the equation: $\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right)$, which can be expanded in powers of $\bar{v}^2$ introducing, in the last equation, the expressions given by (8.5) and (8.6), namely:

$$\begin{cases} \Gamma^0_{00} = \Gamma^{(3)0}{}_{00} + O(5) \quad \text{with} \quad \Gamma^{(3)0}{}_{00} = -\partial_0 U \\ \\ \Gamma^0_{0i} = \Gamma^{(2)0}{}_{0i} + \Gamma^{(4)0}{}_{0i} + O(6) \quad \text{with} \quad \begin{cases} \Gamma^{(2)0}{}_{0i} = -\partial_i U \\ \\ \Gamma^{(4)0}{}_{0i} = \frac{1}{2}\left(\partial_i g^{(4)}_{00} - 4U\partial_i U\right) \end{cases} \\ \\ \Gamma^0_{ij} = \Gamma^{(3)0}{}_{ij} + O(5) \quad \text{with} \quad \Gamma^{(3)0}{}_{ij} = \frac{1}{2}\left(\partial_j g^{(3)}_{0i} + \partial_i g^{(3)}_{0j} + 2\delta_{ij}\partial_0 V\right) \\ \\ \Gamma^i_{00} = \Gamma^{(2)i}{}_{00} + \Gamma^{(4)i}{}_{00} + O(6) \quad \text{with} \quad \begin{cases} \Gamma^{(2)i}{}_{00} = -\delta^{il}\partial_l U \\ \\ \Gamma^{(4)i}{}_{00} = \frac{1}{2}\delta^{il}\left(\partial_l g^{(4)}_{00} + 4V\partial_l U - 2\partial_0 g^{(3)}_{0l}\right) \end{cases} \\ \\ \Gamma^i_{0j} = \Gamma^{(3)i}{}_{0j} + O(5) \quad \text{with} \quad \Gamma^{(3)i}{}_{0j} = \frac{1}{2}\delta^{il}\left(\partial_l g^{(3)}_{0j} - \partial_j g^{(3)}_{0l} + 2\delta_{lj}\partial_0 V\right) \\ \\ \Gamma^i_{jk} = \Gamma^{(2)i}{}_{jk} + \Gamma^{(4)i}{}_{jk} + O(6) \quad \text{with} \quad \begin{cases} \Gamma^{(2)i}{}_{jk} = \delta^{il}\left(-\delta_{jk}\partial_l V + \delta_{lj}\partial_k V + \delta_{lk}\partial_j V\right) \\ \\ \Gamma^{(4)i}{}_{jk} = -\frac{1}{2}\delta^{il}\left[\partial_j g^{(4)}_{kl} + \partial_k g^{(4)}_{jl} - \partial_l g^{(4)}_{jk}\right] - \\ \\ \qquad -2V\delta^{il}\left[\delta_{lk}\partial_j V + \delta_{lj}\partial_k V - \delta_{jk}\partial_l V\right] \end{cases} \end{cases} \tag{8.7}$$

Given a metric tensor, the following can be immediately calculated: the Riemann tensor, the Ricci tensor and the scalar curvature, using the following expressions

$$R_{\alpha\beta\rho\sigma}=\frac{1}{2}\left(\partial_\sigma\partial_\alpha g_{\beta\rho}-\partial_\sigma\partial_\beta g_{\alpha\rho}-\partial_\rho\partial_\alpha g_{\beta\sigma}+\partial_\rho\partial_\beta g_{\alpha\sigma}\right)+g_{\mu\nu}\left(\Gamma^\mu_{\sigma\alpha}\Gamma^\nu_{\beta\rho}-\Gamma^\mu_{\rho\alpha}\Gamma^\nu_{\beta\sigma}\right)$$

$$R_{\mu\nu}=\partial_\rho\Gamma^\rho_{\mu\nu}-\partial_\mu\Gamma^\rho_{\rho\nu}+\Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma}-\Gamma^\rho_{\sigma\nu}\Gamma^\sigma_{\rho\mu},$$

$$R=g^{\mu\nu}R_{\mu\nu}. \tag{8.8}$$

The components of the Riemann tensor that we will need later can be expanded in powers of $\bar{v}^2$ using (8.5)-(8.8):

$$\begin{cases}
R_{i0j0}=R^{(2)}_{i0j0}+R^{(4)}_{i0j0}+O(6) \quad \text{with} & \begin{cases} R^{(2)}_{i0j0} = \partial_i\partial_j U \\ R^{(4)}_{i0j0} = \frac{1}{2}\left(\partial_0\partial_i g^{(3)}_{0j}+\partial_0\partial_j g^{(3)}_{0i}+2\delta_{ij}\partial_0\partial_0 V-\partial_i\partial_j g^{(4)}_{00}\right)+ \\ \quad +\left(\partial_i U\right)\left(\partial_j U\right)-\left(\partial_i U\right)\left(\partial_j V\right)-\left(\partial_j U\right)\left(\partial_i V\right)+ \\ \quad +\delta_{ij}\delta^{kl}\left(\partial_k U\right)\left(\partial_l V\right)\end{cases} \\
R_{0ij0}=R^{(2)}_{0ij0}+R^{(4)}_{0ij0}+O(6) \quad \text{with} & \begin{cases} R^{(2)}_{0ij0} = -R^{(2)}_{i0j0} \\ R^{(4)}_{0ij0} = -R^{(4)}_{i0j0}\end{cases} \\
R_{ijk0}=R^{(3)}_{ijk0}+O(5) \quad \text{with} & R^{(3)}_{ijk0}=\frac{1}{2}\left[\partial_k\left(\partial_j g^{(3)}_{0i}-\partial_i g^{(3)}_{0j}\right)+2\partial_0\left(\delta_{ik}\partial_j V-\delta_{jk}\partial_i V\right)\right]
\end{cases} \tag{8.9}$$

By assuming the harmonic gauge, given by $g^{\mu\nu}\Gamma^\rho_{\mu\nu}=0$ (in order to simplify the expressions), and using (8.5)-(8.8), we can expand the Ricci tensor in powers of $\bar{v}^2$:

$$\begin{cases}
R_{00}=R^{(2)}_{00}+R^{(4)}_{00}+O(6) \quad \text{with} & \begin{cases} R^{(2)}_{00}=-\triangle U \\ R^{(4)}_{00}=\frac{1}{2}\triangle g^{(4)}_{00}+2V\triangle U+\partial_0\partial_0 U-2\delta^{mn}\left(\partial_m U\right)\left(\partial_n U\right)\end{cases} \\
R_{0i}=R^{(3)}_{0i}+O(5) \quad \text{with} & R^{(3)}_{0i}=\frac{1}{2}\triangle g^{(3)}_{0i} \\
R_{ij}=R^{(2)}_{ij}+R^{(4)}_{ij}+O(6) \quad \text{with} & \begin{cases} R^{(2)}_{ij} = -\delta_{ij}\triangle V \\ R^{(4)}_{ij} = \frac{1}{2}\left[\partial_0\left(\partial_j g^{(3)}_{0i}+\partial_i g^{(3)}_{0j}+2\delta_{ij}\partial_0 V\right)-\right. \\ \quad \left.-\delta^{mn}\left(\partial_m\left(\partial_i g^{(4)}_{nj}+\partial_j g^{(4)}_{ni}\right)-\partial_i\partial_j g^{(4)}_{mn}\right)+\triangle g^{(4)}_{ij}-\partial_i\partial_j g^{(4)}_{00}\right]+ \\ \quad +2V\partial_i\partial_j V+2U\partial_i\partial_j U+\partial_i U\,\partial_j(U-V)+ \\ \quad +\partial_i V\,\partial_j(3V-U)+\delta_{ij}\left(\delta^{kl}\partial_l V\partial_k(U+V)+2V\triangle V\right)\end{cases}
\end{cases} \tag{8.10}$$

and the scalar curvature too:

$$R = R^{(2)} + R^{(4)} + O(6) \;\; \text{with} \;\; \begin{cases} R^{(2)} = 3\triangle V - \triangle U \\ \\ R^{(4)} = \frac{1}{2}\triangle g_{00}^{(4)} + 2V\left(\triangle U - 3\triangle V\right) + \partial_0\partial_0 U - 2\delta^{ij}\left(\partial_i U\right)\left(\partial_j U\right) - 2U\triangle U \end{cases} \tag{8.11}$$

On the matter side, we start with the general definition of the energy-momentum tensor of a perfect fluid:

$$T_{\mu\nu} = \left(\rho + \Pi\rho + p\right) u_\mu u_\nu - p g_{\mu\nu} \tag{8.12}$$

where Π denotes the internal energy density, ρ the energy density and p the pressure. Taking into account that:

$$\begin{aligned} u^0 &= \frac{1}{\sqrt{1-\mathbf{v}^2}} = 1 + \frac{1}{2}\mathbf{v}^2 + \frac{3}{8}\mathbf{v}^4 + O(6) \\ u^i &= \frac{v^i}{\sqrt{1-\mathbf{v}^2}} = v^i\left(1 + \frac{1}{2}\mathbf{v}^2 + O(4)\right) \end{aligned} \tag{8.13}$$

and the expressions (8.5) and (8.6), we can calculate the different components of (8.12):

$$\begin{cases} T_{00} = T_{00}^{(0)} + T_{00}^{(2)} + T_{00}^{(4)} + O(6) \quad \text{with} & \begin{cases} T_{00}^{(0)} = \rho \\ T_{00}^{(2)} = \rho\left(\Pi + \mathbf{v}^2 - 4U\right) \\ T_{00}^{(4)} = \rho\Big[\mathbf{v}^4 - 3U\mathbf{v}^2 + 4U^2 + 2g_{00}^{(4)} + 2g_{0i}^{(3)}v^i + \\ \qquad + \Pi(\mathbf{v}^2 - 4U)\Big] + p\left(\mathbf{v}^2 - 2U\right) \end{cases} \\ T_{0i} = T_{0i}^{(1)} + T_{0i}^{(3)} + O(5) \quad \text{with} & \begin{cases} T_{0i}^{(1)} = -v^i\rho \\ T_{0i}^{(3)} = -v^i\rho\left(\Pi + \frac{1}{2}\mathbf{v}^2 - 2U + \frac{p}{\rho}\right) \end{cases} \\ T_{ij} = T_{ij}^{(2)} + O(4) \quad \text{with} & \begin{cases} T_{ij}^{(2)} = \rho\delta_{ik}\delta_{jl}v^k v^l + p\delta_{ij} \\ T_{ij}^{(4)} = \rho\Big\{-\delta_{ik}v^k g_{0j}^{(3)} - \delta_{jl}v^l g_{0i}^{(3)} + \\ \qquad + \delta_{ik}\delta_{jl}v^k v^l\left(\mathbf{v}^2 + \Pi + 4V\right)\Big\} + \\ \qquad + p\left\{\delta_{ik}\delta_{jl}v^k v^l + 2V\delta_{ij}\right\} \end{cases} \end{cases} \tag{8.14}$$

Finally, the expansion of the Gauss-Bonnet invariant $\mathcal{G}$ in orders of $\bar{v}^2$ can be calculated using (8.5)-(8.11):

$$\begin{aligned} &\mathcal{G} = \mathcal{G}^{(4)} + O(6) \\ &\text{with } \mathcal{G}^{(4)} = -3\left(\triangle U\right)^2 + \left(\triangle V\right)^2 - 6\left(\triangle U\right)\left(\triangle V\right) + 4\delta^{im}\delta^{jn}\left[\partial_i\partial_j\left(U+V\right)\right]\left[\partial_m\partial_n\left(U+V\right)\right] \end{aligned} \tag{8.15}$$

We have now all the ingredients, expanded in orders of $\bar{v}^2$, needed to write the field equations and the trace equation in the Newtonian, PN and PPN limits. In order to do this, we assume that:

$$f^*(R,\mathcal{G}) = f^*(0,0) + f^*_R(0,0)R + f^*_{\mathcal{G}}(0,0)\mathcal{G} + \frac{1}{2}\left(f^*_{RR}(0,0)R^2 + 2f^*_{R\mathcal{G}}(0,0)R\mathcal{G} + f^*_{\mathcal{G}\mathcal{G}}\mathcal{G}^2\right) + ... \tag{8.16}$$

where $f^*(R,\mathcal{G})$ denotes the function $f(R,\mathcal{G})$ or any of its derivatives, i.e. $f_R(R,\mathcal{G})$, $f_{\mathcal{G}}(R,\mathcal{G})$, $f_{RR}(R,\mathcal{G})$ ... Considering (8.11) and (8.15), we can write (8.16) as an expansion in orders of $\bar{v}^2$:

$$f^*(R,\mathcal{G}) = f^*(0,0) + f^*_R(0,0)R^{(2)} + \left(\frac{1}{2}f^*_{RR}(0,0)R^{(2)\,2} + f^*_R(0,0)R^{(4)} + f^*_{\mathcal{G}}(0,0)\mathcal{G}^{(4)}\right) + O(6). \quad (8.17)$$

We now proceed to calculate the Newtonian, PN and PPN limits for the first Friedmann equation and the trace equation.

$(0,0)$ -field equation

The $(0,0)$ -field equation is given from (8.3) by:

$$\begin{aligned}
-\frac{1}{2}g_{00}f(R,\mathcal{G}) + f_R(R,\mathcal{G})R_{00} + g_{00}\nabla^2\left(f_R(R,\mathcal{G})\right) - \nabla_0\nabla_0\left(f_R(R,\mathcal{G})\right) + \\
+2f_{\mathcal{G}}(R,\mathcal{G})RR_{00} - 4f_{\mathcal{G}}(R,\mathcal{G})R_{0\rho}R_0{}^{\rho} + 2f_{\mathcal{G}}(R,\mathcal{G})R_{\alpha\beta\rho0}R^{\alpha\beta\rho}{}_0 - 4f_{\mathcal{G}}(R,\mathcal{G})R_{0\rho0\sigma}R^{\rho\sigma} + \\
+2g_{00}R\nabla^2 f_{\mathcal{G}}(R,\mathcal{G}) - 4g_{00}R_{\rho\sigma}\nabla^{\rho}\nabla^{\sigma}f_{\mathcal{G}}(R,\mathcal{G}) - 2R\nabla_0\nabla_0 f_{\mathcal{G}}(R,\mathcal{G}) - 4R_{00}\nabla^2 f_{\mathcal{G}}(R,\mathcal{G}) + \\
+8R_{0\rho}\nabla^{\rho}\nabla_0 f_{\mathcal{G}}(R,\mathcal{G}) + 4R_{0\rho0\sigma}\nabla^{\rho}\nabla^{\sigma}f_{\mathcal{G}}(R,\mathcal{G}) = 2\kappa^2 T_{00},
\end{aligned} \quad (8.18)$$

At the lowest order in the velocity, we obtain: $f(0,0) = 0$.

In the Newtonian limit, i.e. at $O(2)$ order in the velocity, Eq.(8.18) reduces to:

$$-\frac{1}{2}g^{(0)}_{00}f_R(0,0)R^{(2)} + f_R(0,0)R^{(2)}_{00} + \left[g_{00}\nabla^2\left(f_R(R,\mathcal{G})\right) - \nabla_0\nabla_0\left(f_R(R,\mathcal{G})\right)\right]^{(2)} = 2\kappa^2 T^{(0)}_{00}, \quad (8.19)$$

where it has been considered that $f(0,0) = 0$. Introducing (8.5), (8.10), (8.11) and (8.14) into (8.19), we finally obtain the following equation:

$$f_R(0,0)\left(3\triangle V + \triangle U\right) + 2f_{RR}(0,0)\left(3\triangle^2 V - \triangle^2 U\right) = -4\kappa^2\rho. \quad (8.20)$$

Where the notation: $\triangle^2 := \triangle \cdot \triangle$, has been introduced, being $\triangle = \delta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}$.

In the PN limit, i.e. at $O(4)$ order in the velocity, Eq.(8.18) reduces to:

$$\begin{aligned}
& f_R(0,0) && \left\{-\tfrac{1}{2}g^{(2)}_{00}R^{(2)} - \tfrac{1}{2}g^{(0)}_{00}R^{(4)} + R^{(4)}_{00}\right\} + \\
+ \; & f_{\mathcal{G}}(0,0) && \left\{-\tfrac{1}{2}\mathcal{G}^{(4)} + 2R^{(2)}R^{(2)}_{00} - 4R^{(2)\,2}_{00} + 2\left(R_{\alpha\beta\rho0}R^{\alpha\beta\rho}{}_0\right)^{(4)} - 4\left(R_{0\rho0\sigma}R^{\rho\sigma}\right)^{(4)}\right\} + \\
+ \; & f_{RR}(0,0) && \left\{-\tfrac{1}{4}R^{(2)\,2} + R^{(2)}R^{(2)}_{00} + \left(g^{(2)}_{00}g^{(0)\,ij} + g^{(0)}_{00}g^{(2)\,ij}\right)\partial_i\partial_j R^{(2)} + g^{(0)\,ij}\left(\partial_i\partial_j R^{(4)} - \Gamma^{(2)\,k}_{\;ij}\partial_k R^{(2)}\right)\right\} + \\
+ \; & f_{R\mathcal{G}}(0,0) && \left\{g^{(0)\,ij}\partial_i\partial_j\mathcal{G}^{(4)} + 2\partial_i\partial_j R^{(2)}\left[g^{(0)\,ij}\left(R^{(2)} - 2R^{(2)}_{00}\right) + 2g^{(0)\,ik}g^{(0)\,jl}\left(-R^{(2)}_{kl} + R^{(2)}_{0k0l}\right)\right]\right\} + \\
+ \; & f_{RRR}(0,0) && g^{(0)\,ij}\left\{\partial_i R^{(2)}\partial_j R^{(2)} + R^{(2)}\partial_i\partial_j R^{(2)}\right\} = 2\kappa^2 T^{(2)}_{00},
\end{aligned} \quad (8.21)$$

where it has been considered, once more, that $f(0,0)=0$. The calculations that make possible to derive Eq.(8.21) from Eq.(8.18) are written in Appendix E. Introducing in (8.21) the results obtained in the previous section, we finally obtain:

$$\begin{aligned}
& f_R(0,0) && \left\{\tfrac{1}{4}\triangle g^{(4)}_{00}+3(U+V)\triangle V+V\triangle U+\tfrac{1}{2}\partial_0\partial_0 U-\delta^{ij}\partial_i U\partial_j U\right\}+\\
+\quad & f_{RR}(0,0) && \left\{-\tfrac{1}{2}\triangle^2 g^{(4)}_{00}+\tfrac{15}{4}\left(\triangle V\right)^2-\tfrac{7}{2}\triangle U\triangle V+\tfrac{11}{4}\left(\triangle U\right)^2+6(U+2V)\triangle^2 V-4V\triangle^2 U-\right.\\
& && \left.-\partial_0\partial_0\triangle U+\delta^{ij}\left[3\partial_i V\,\partial_j\left(3\triangle V-\triangle U\right)+8\partial_i U\,\partial_j\triangle U+4\delta^{mn}\partial_i\partial_m U\,\partial_j\partial_n U\right]\right\}+\\
+\quad & 2f_{R\mathcal{G}}(0,0) && \left\{4\left(\triangle U+\triangle V\right)\triangle^2 U-4\triangle V\triangle^2 V+\delta^{ij}\left[3\partial_i\triangle U\,\partial_j\left(\triangle U+2\triangle V\right)-\partial_i\triangle V\,\partial_j\triangle V\right]+\right.\\
& && \left.+2\delta^{ij}\delta^{mn}\left[\partial_i\partial_m\left(3\triangle V-\triangle U\right)\partial_j\partial_n U-\triangle\left(\partial_i\partial_m(U+V)\,\partial_j\partial_n(U+V)\right)\right]\right\}-\\
-\quad & f_{RRR}(0,0) && \left\{\delta^{ij}\partial_i\left(3\triangle V-\triangle U\right)\partial_j\left(3\triangle V-\triangle U\right)+\left(3\triangle V-\triangle U\right)\triangle\left(3\triangle V-\triangle U\right)\right\}+\\
+\quad & f_{\mathcal{G}}(0,0) && \left\{-\tfrac{1}{2}\left(\triangle U-\triangle V\right)^2+2\delta^{im}\delta^{jn}\partial_i\partial_j\left(U-V\right)\partial_m\partial_n\left(U-V\right)\right\}=2\kappa^2\rho\left(\Pi+\mathbf{v}^2-4U\right)
\end{aligned}\tag{8.22}$$

In the PPN limit, i.e. at $O(6)$ order in the velocity, using the results obtained in Appendix E, Eq.(8.18) reduces to:

$$\begin{aligned}
& f_R(0,0) && \left\{-\tfrac{1}{2}\left[g^{(4)}_{00}R^{(2)}+g^{(2)}_{00}R^{(4)}+g^{(0)}_{00}R^{(6)}\right]+R^{(6)}_{00}\right\}+\\
+\quad & f_{\mathcal{G}}(0,0) && \left\{-\tfrac{1}{2}\left[g^{(2)}_{00}\mathcal{G}^{(4)}+g^{(0)}_{00}\mathcal{G}^{(6)}\right]+2\left[R^{(2)}R^{(4)}_{00}+R^{(4)}R^{(2)}_{00}\right]-\right.\\
& && \left.-4\left[2g^{(0)\,00}R^{(2)}_{00}R^{(4)}_{00}+g^{(2)\,00}R^{(2)\,2}_{00}+g^{(0)\,ij}R^{(3)}_{0i}R^{(3)}_{0j}\right]+2\left(R_{\alpha\beta\rho0}R^{\alpha\beta\rho}{}_{0}\right)^{(6)}-4\left(R_{0\rho0\sigma}R^{\rho\sigma}\right)^{(6)}\right\}+\\
+\quad & f_{RR}(0,0) && \left\{-\tfrac{1}{2}\left[\tfrac{1}{2}g^{(2)}_{00}R^{(2)\,2}+g^{(0)}_{00}R^{(2)}R^{(4)}\right]+R^{(2)}R^{(4)}_{00}+R^{(4)}R^{(2)}_{00}+\right.\\
& && +g^{(0)}_{00}\left[g^{(2)\,00}\left(\partial_0\partial_0R^{(2)}-\Gamma^{(2)\,i}{}_{00}\partial_iR^{(2)}\right)+\right.\\
& && \quad +g^{(0)\,ij}\left(\partial_i\partial_jR^{(6)}-\Gamma^{(3)\,0}{}_{ij}\partial_0R^{(2)}-\Gamma^{(2)\,k}{}_{ij}\partial_kR^{(4)}-\Gamma^{(4)\,k}{}_{ij}\partial_kR^{(2)}\right)+\\
& && \quad \left.+g^{(2)\,ij}\left(\partial_i\partial_jR^{(4)}-\Gamma^{(2)\,k}{}_{ij}\partial_kR^{(2)}\right)+g^{(4)\,ij}\partial_i\partial_jR^{(2)}+2g^{(3)\,0i}\partial_0\partial_iR^{(2)}\right]+\\
& && +g^{(2)}_{00}\left[g^{(0)\,00}\left(\partial_0\partial_0R^{(2)}-\Gamma^{(2)\,i}{}_{00}\partial_iR^{(2)}\right)+g^{(0)\,ij}\left(\partial_i\partial_jR^{(4)}-\Gamma^{(2)\,k}{}_{ij}\partial_kR^{(2)}\right)+g^{(2)\,ij}\partial_i\partial_jR^{(2)}\right]+\\
& && \left.+g^{(4)}_{00}g^{(0)\,ij}\partial_i\partial_jR^{(2)}\right\}+\\
+\quad & f_{R\mathcal{G}}(0,0) && \left\{-\tfrac{1}{2}g^{(0)}_{00}R^{(2)}\mathcal{G}^{(4)}+R^{(2)}_{00}\mathcal{G}^{(4)}-\partial_0\partial_0\mathcal{G}^{(4)}+\Gamma^{(2)\,i}{}_{00}\partial_i\mathcal{G}^{(4)}+2R^{(2)}\left[R^{(2)}R^{(2)}_{00}-2g^{(0)\,00}R^{(2)\,2}_{00}\right.\right.\\
& && \left.+\left(R_{\alpha\beta\rho0}R^{\alpha\beta\rho}{}_{0}\right)^{(4)}-2\left(R_{0\rho0\sigma}R^{\rho\sigma}\right)^{(4)}-\partial_0\partial_0R^{(2)}+\Gamma^{(2)\,i}{}_{00}\partial_iR^{(2)}\right]+
\end{aligned}$$

$$
\begin{aligned}
&+8\left[R^{(2)}_{00}g^{(0)\,00}\left(\partial_0\partial_0 R^{(2)}-\Gamma^{(2)\,i}{}_{00}\partial_i R^{(2)}\right)+R^{(3)}_{0i}g^{(0)\,ij}\partial_j\partial_0 R^{(2)}\right]+\\
&+g^{(0)}_{00}\left[g^{(0)\,00}\left(\partial_0\partial_0\mathcal{G}^{(4)}-\Gamma^{(2)\,i}{}_{00}\partial_i\mathcal{G}^{(4)}\right)+g^{(0)\,ij}\left(\partial_i\partial_j\mathcal{G}^{(6)}-\Gamma^{(2)\,k}{}_{ij}\partial_k\mathcal{G}^{(4)}\right)+g^{(2)\,ij}\partial_i\partial_j\mathcal{G}^{(4)}\right]+\\
&+g^{(2)}_{00}g^{(0)\,ij}\partial_i\partial_j\mathcal{G}^{(4)}+2\left[g^{(0)}_{00}R^{(2)}-2R^{(2)}_{00}\right]\left[g^{(0)\,00}\left(\partial_0\partial_0 R^{(2)}-\Gamma^{(2)\,i}{}_{00}\partial_i R^{(2)}\right)+\right.\\
&\left.+g^{(0)\,ij}\left(\partial_i\partial_j R^{(4)}-\Gamma^{(2)\,k}{}_{ij}\partial_k R^{(2)}\right)+g^{(2)\,ij}\partial_i\partial_j R^{(2)}\right]+\\
&+2\left[g^{(0)}_{00}R^{(4)}+g^{(2)}_{00}R^{(2)}-2R^{(4)}_{00}\right]g^{(0)\,ij}\partial_i\partial_j R^{(2)}-\\
&-4g^{(0)}_{00}R^{(2)}_{00}\left(g^{(0)\,00}\right)^2\left[\partial_0\partial_0 R^{(2)}-\Gamma^{(2)\,i}{}_{00}\partial_i R^{(2)}\right]+\\
&+4\left[R^{(2)}_{k0l0}-g^{(0)}_{00}R^{(2)}_{kl}\right]\left[g^{(0)\,ki}g^{(0)\,lj}\left(\partial_i\partial_j R^{(4)}-\Gamma^{(2)\,m}{}_{ij}\partial_m R^{(2)}\right)+\right.\\
&\left.+\left(g^{(2)\,ki}g^{(0)\,lj}+g^{(0)\,ki}g^{(2)\,lj}\right)\partial_i\partial_j R^{(2)}\right]-8g^{(0)}_{00}R^{(3)}_{0j}g^{(0)\,00}g^{(0)\,ij}\partial_0\partial_i R^{(2)}+\\
&\left.+4g^{(0)\,ik}g^{(0)\,jl}\left[R^{(4)}_{k0l0}-g^{(0)}_{00}R^{(4)}_{kl}-g^{(2)}_{00}R^{(2)}_{kl}\right]\partial_i\partial_j R^{(2)}\right\}+\\
+\quad f_{\mathcal{G}\mathcal{G}}(0,0)\quad&\left\{2\left[g^{(0)}_{00}R^{(2)}-2R^{(2)}_{00}\right]g^{(0)\,ij}\partial_i\partial_j\mathcal{G}^{(4)}+4g^{(0)\,ik}g^{(0)\,jl}\left[R^{(2)}_{k0l0}-g^{(0)}_{00}R^{(2)}_{kl}\right]\partial_i\partial_j\mathcal{G}^{(4)}\right\}+\\
+\quad f_{RRR}(0,0)\quad&\left\{R^{(2)}\left[R^{(2)}\left(-\tfrac{1}{12}R^{(2)}+\tfrac{1}{2}R^{(2)}_{00}\right)-\partial_0\partial_0 R^{(2)}+\Gamma^{(2)\,i}{}_{00}\partial_i R^{(2)}\right]-\partial_0 R^{(2)}\,\partial_0 R^{(2)}+\right.\\
&+g^{(0)}_{00}\left[g^{(0)\,00}\left(\partial_0 R^{(2)}\,\partial_0 R^{(2)}+R^{(2)}\partial_0\partial_0 R^{(2)}-\Gamma^{(2)\,i}{}_{00}R^{(2)}\partial_i R^{(2)}\right)+\right.\\
&+g^{(0)\,ij}\left(2\partial_i R^{(2)}\partial_j R^{(4)}+R^{(2)}\partial_i\partial_j R^{(4)}+\right.\\
&\left.\left.+R^{(4)}\partial_i\partial_j R^{(2)}-\Gamma^{(2)\,k}{}_{ij}R^{(2)}\partial_k R^{(2)}\right)+g^{(2)\,ij}\left(\partial_i R^{(2)}\partial_j R^{(2)}+R^{(2)}\partial_i\partial_j R^{(2)}\right)\right]+\\
&\left.+g^{(2)}_{00}g^{(0)\,ij}\left[\partial_i R^{(2)}\partial_j R^{(2)}+R^{(2)}\partial_i\partial_j R^{(2)}\right]\right\}+\\
+\quad f_{RR\mathcal{G}}(0,0)\quad&\left\{g^{(0)}_{00}g^{(0)\,ij}\left[2\partial_i R^{(2)}\partial_j\mathcal{G}^{(4)}+\mathcal{G}^{(4)}\partial_i\partial_j R^{(2)}+R^{(2)}\partial_i\partial_j\mathcal{G}^{(4)}\right]+2\left[g^{(0)\,ij}\left(g^{(0)}_{00}R^{(2)}-2R^{(2)}_{00}\right)+\right.\right.\\
&\left.\left.+2g^{(0)\,ik}g^{(0)\,jl}\left(R^{(2)}_{k0l0}-g^{(0)}_{00}R^{(2)}_{kl}\right)\right]\left[\partial_i R^{(2)}\partial_j R^{(2)}+R^{(2)}\partial_i\partial_j R^{(2)}\right]\right\}+\\
+\quad f_{RRRR}(0,0)\quad&g^{(0)}_{00}g^{(0)\,ij}R^{(2)}\left\{\partial_i R^{(2)}\partial_j R^{(2)}+\tfrac{1}{2}R^{(2)}\partial_i\partial_j R^{(2)}\right\}=2\kappa^2 T^{(4)}_{00}.
\end{aligned}
\tag{8.23}
$$

And introducing in (8.23) the results obtained in the previous section, we finally obtain:

$$
\begin{aligned}
& f_R(0,0) && \Big\{\tfrac{1}{2}\left(R^{(6)}_{00}+\delta^{ij}R^{(6)}_{ij}\right)-\delta^{ij}g^{(3)}_{0i}R^{(3)}_{0j}-\tfrac{1}{2}\triangle V\left[3\left(g^{(4)}_{00}+4UV-4V^2\right)+\delta^{ij}g^{(4)}_{ij}\right]-\\
& && -(U+V)\left[\tfrac{1}{2}\left(6\partial_0\partial_0V-\triangle g^{(4)}_{00}+2\delta^{ij}\left(\triangle g^{(4)}_{ij}+\partial_0\partial_i g^{(3)}_{0j}\right)-2\delta^{ij}\delta^{mn}\partial_i\partial_m g^{(4)}_{jn}\right)\right]+\\
& && +8V\triangle V+2U\triangle U+\delta^{ij}\left[\partial_iU\partial_j(U-V)+2\partial_iV\partial_j(3V+U)\right]\Big\}+\\
+\; & f_{\mathcal{G}}(0,0) && \Big\{-\tfrac{1}{2}\mathcal{G}^{(6)}+(8V-7U)\left(\triangle U\right)^2+U\left(\triangle V\right)^2+2(4V-3U)\triangle U\triangle V+\\
& && +2\triangle U\left[\triangle g^{(4)}_{00}+2\partial_0\partial_0(U+2V)-4\delta^{ij}\partial_iU\partial_j(U+V)\right]+\\
& && +\triangle V\left[\triangle g^{(4)}_{00}+6\partial_0\partial_0(U+2V)+4\delta^{ij}\left(\partial_0\partial_i g^{(3)}_{0j}+\partial_iU\partial_j(U-2V)\right)\right]+\\
& && +\delta^{ij}\left[\left(\triangle g^{(3)}_{0i}+4\partial_0\partial_iV\right)\triangle g^{(3)}_{0j}-8\partial_0\partial_iV\partial_0\partial_jV\right]+4\delta^{ij}\delta^{mn}\left[U\partial_i\partial_m(U+V)\partial_j\partial_n(U+V)-\right.\\
& && -\partial_i\partial_m g^{(3)}_{0j}\partial_0\partial_nV+\tfrac{1}{2}\left[\partial_0\left(\partial_n g^{(3)}_{0j}+\partial_j g^{(3)}_{0n}-2\delta_{jn}\partial_0V\right)+\right.\\
& && +\delta^{kl}\left(\partial_k\left(\partial_j g^{(4)}_{ln}+\partial_n g^{(4)}_{lj}\right)-\partial_j\partial_n g^{(4)}_{kl}\right)-\triangle g^{(4)}_{jn}-\partial_j\partial_n g^{(4)}_{00}\Big]-2V\partial_j\partial_nV-\\
& && -\,(U+2V)\partial_j\partial_nU+\partial_jU\,\partial_n(U-V)-\partial_jV\,\partial_n(3V+U)-\delta_{jn}\left(\delta^{kl}\partial_lV\partial_k(U+V)+2V\triangle V\right)\big]-\\
& && -\tfrac{1}{2}\delta^{ij}\delta^{mn}\delta^{kl}\partial_k\left(\partial_m g^{(3)}_{0i}+\partial_i g^{(3)}_{0m}\right)\partial_l\left(\partial_n g^{(3)}_{0j}+\partial_j g^{(3)}_{0n}\right)\Big\}+\\
+\; & f_{RR}(0,0) && \Big\{-\triangle R^{(6)}+(U+V)\triangle^2 g^{(4)}_{00}+\triangle^2U\left(g^{(4)}_{00}-4U^2+4UV+8V^2\right)-\\
& && -3\triangle^2V\left(g^{(4)}_{00}+4UV+8V^2\right)-\left(\triangle U\right)^2\left(\tfrac{5}{2}U+7V\right)-3\left(\triangle V\right)^2\left(\tfrac{5}{2}U+V\right)+\\
& && +2\triangle U\triangle V\left(2U+5V\right)+3\left(\triangle V-\triangle U\right)\left(\tfrac{1}{4}\triangle g^{(4)}_{00}+\tfrac{1}{2}\partial_0\partial_0U-\delta^{ij}\partial_iU\partial_jU\right)+2V\partial_0\partial_0\triangle U+\\
& && +\delta^{ij}\left[\tfrac{1}{2}\left(2\partial_i g^{(3)}_{0j}-\partial_0 g^{(2)}_{ij}\right)\partial_0\left(3\triangle V-\triangle U\right)-\partial_iV\partial_0\partial_0\partial_jU+2\left(3\triangle V-\triangle U\right)\partial_iV\partial_jV+\right.\\
& && +2\triangle U\partial_iU\partial_jV-\tfrac{1}{2}\partial_iV\partial_j\triangle g^{(4)}_{00}+2g^{(3)}_{0i}\partial_0\partial_j\left(3\triangle V-\triangle U\right)+2\partial_i\triangle U\left(U\partial_j\left(V-8U\right)-8V\partial_jU\right)-\\
& && \left.-2\left(V+3U\right)\partial_iV\partial_j\left(3\triangle V-\triangle U\right)\right]+\\
& && +\delta^{ij}\delta^{mn}\left[-\tfrac{1}{2}\left(2\partial_i g^{(4)}_{mj}-\partial_m g^{(4)}_{ij}\right)\partial_n\left(3\triangle V-\triangle U\right)-g^{(4)}_{im}\partial_j\partial_n\left(3\triangle V-\triangle U\right)-\right.\\
& && \left.-4\left(2(U+V)\partial_i\partial_mU-\partial_iV\partial_mU\right)\partial_j\partial_nU\right]\Big\}+\\
+\; & f_{\mathcal{G}\mathcal{G}}(0,0) && \Big\{12\left(\triangle U+\triangle V\right)^2\triangle^2U+4\left(3\triangle V-\triangle U\right)\left(\triangle U+\triangle V\right)\triangle^2V+\\
& && +4\left(\triangle U+\triangle V\right)\delta^{ij}\left[3\partial_i\triangle U\partial_j\left(\triangle U+2\triangle V\right)-\partial_i\triangle V\partial_j\triangle V\right]+
\end{aligned}
$$

$$
\begin{aligned}
&+4\delta^{ij}\delta^{mn}\Big[\triangle\left(\partial_i\partial_m(U+V)\partial_j\partial_n(U+V)\right)+\partial_i\partial_m U\partial_j\partial_n\Big(-3\left(\triangle U\right)^2+\left(\triangle V\right)^2-6\triangle U\triangle V+\\
&+4\delta^{kl}\delta^{rs}\partial_k\partial_r(U+V)\partial_l\partial_s(U+V)\Big)\Big]\Big\}+\\
+\quad f_{R\mathcal{G}}(0,0)\quad &\Big\{-\triangle\mathcal{G}^{(6)}+\tfrac{15}{2}\left(\triangle U\right)^3-7\triangle V\left(\triangle U\right)^2+\tfrac{21}{2}\triangle U\left(\triangle V\right)^2+\tfrac{21}{2}\left(\triangle V\right)^3-2\left(\triangle U+\triangle V\right)\partial_0\partial_0\triangle U+\\
&+\triangle^2 U\Big[-\triangle g^{(4)}_{00}-2\partial_0\partial_0 U-8(U+3V)\triangle U-4(U+2V)\triangle V+4\delta^{ij}\left(\partial_i U\partial_j U+\partial_i V\partial_j V\right)\Big]+\\
&+\triangle^2 V\Big[3\triangle g^{(4)}_{00}+6\partial_0\partial_0 U-4(3U-2V)\triangle U-8(U+7V)\triangle V-12\delta^{ij}\left(\partial_i U\partial_j U+\partial_i V\partial_j V\right)\Big]+\\
&+\triangle U\Big[-\triangle^2 g^{(4)}_{00}+2\delta^{ij}\left\{5\partial_i V\partial_j\left(3\triangle V-\triangle U\right)+8\partial_i U\partial_j\triangle U+\delta^{mn}\left[\partial_i\partial_m U\partial_j\partial_n(U-V)-\right.\right.\\
&-\left.\left.\ \partial_i\partial_m V\partial_j\partial_n(U+V)\right]\right\}\Big]+\\
&+\triangle V\Big[-\triangle^2 g^{(4)}_{00}+2\delta^{ij}\left\{3\partial_i V\partial_j\left(3\triangle V-\triangle U\right)+8\partial_i U\partial_j\triangle U+\delta^{mn}\left[\partial_i\partial_m U\partial_j\partial_n(7U-3V)-\right.\right.\\
&\left.\left.3\partial_i\partial_m V\partial_j\partial_n(U+V)\right]\right\}\Big]-\delta^{ij}\partial_i V\partial_j\Big[-3\left(\triangle U\right)^2+\left(\triangle V\right)^2-6\left(\triangle U\right)\left(\triangle V\right)+\\
&+\ 4\delta^{mn}\delta^{kl}\partial_m\partial_k\left(U+V\right)\partial_n\partial_l\left(U+V\right)\Big]+\\
&+4(U+V)\delta^{ij}\left[\partial_i\triangle V\partial_j\triangle V-3\partial_i\triangle U\partial_j\left(\triangle U+2\triangle V\right)+2\delta^{mn}\triangle\left(\partial_i\partial_m(U+V)\partial_j\partial_n(U+V)\right)\right]+\\
&+2\delta^{ij}\delta^{mn}\Big[\partial_i\partial_m U\Big(\partial_j\partial_n\Big(\triangle g^{(4)}_{00}-4V(3\triangle V-\triangle U)+2\partial_0\partial_0 U-4U\triangle U\Big)-\\
&-4\partial_j V\partial_n\left(3\triangle V-\triangle U\right)-8V\partial_j\partial_n\left(3\triangle V-\triangle U\right)\Big)+\partial_i\partial_m\left(3\triangle V-\triangle U\right)\Big(-\tfrac{1}{2}\triangle g^{(4)}_{jn}-\\
&-2V\partial_j\partial_n V-2U\partial_j\partial_n U-3\partial_j V\partial_n V+\tfrac{1}{2}\delta^{kl}\Big[\partial_k\Big(\partial_j g^{(4)}_{ln}+\partial_n g^{(4)}_{lj}\Big)-\partial_j\partial_n g^{(4)}_{kl}\Big]\Big)\Big]-\\
&-8\delta^{ij}\delta^{mn}\delta^{kl}\partial_i\partial_m U\partial_j\partial_n\left(\partial_k U\partial_l U\right)\Big\}+\\
+\quad f_{RRR}(0,0)\quad &\Big\{\triangle^2 U\Big[\tfrac{1}{2}\triangle g^{(4)}_{00}+\partial_0\partial_0 U+2(3V-U)\triangle U-18V\triangle V-2\delta^{ij}\partial_i U\partial_j U\Big]+\\
&+\triangle^2 V\Big[-\tfrac{1}{2}\triangle g^{(4)}_{00}-\partial_0\partial_0 U-2(7V+2U)\triangle U+6(7V+3U)\triangle V+2\delta^{ij}\partial_i U\partial_j U\Big]-\\
&-\Big(\tfrac{1}{2}\triangle g^{(4)}_{00}+\partial_0\partial_0\triangle U\Big)\left(3\triangle V-\triangle U\right)-\tfrac{29}{12}\left(\triangle U\right)^3+\tfrac{111}{12}\triangle V\left(\triangle U\right)^2-\tfrac{57}{4}\triangle U\left(\triangle V\right)^2+\\
&+\tfrac{63}{4}\left(\triangle V\right)^3+\delta^{ij}\Big[\partial_i\left(3\triangle V-\triangle U\right)\Big(3\left(3\triangle V-\triangle U\right)\partial_j V+2\left(3\triangle V+\triangle U\right)\partial_j\left(3\triangle V-\triangle U\right)-\\
&2\partial_j\Big(\tfrac{1}{2}\triangle g^{(4)}_{00}+\partial_0\partial_0 U-2U\triangle U\Big)\Big)+8\left(3\triangle V-\triangle U\right)\partial_i U\partial_j\triangle U\Big]+\\
&+4\delta^{ij}\delta^{mn}\left[\partial_i\partial_m\left(3\triangle V-\triangle U\right)\partial_j\partial_n\left(3\triangle V-\triangle U\right)+\left(3\triangle V-\triangle U\right)\partial_i\partial_m U\partial_j\partial_n U\right]\Big\}+
\end{aligned}
$$

$$
\begin{aligned}
+ \quad f_{RR\mathcal{G}}(0,0) \quad & \Big\{\triangle^2 U\Big[-11(\triangle U)^2+25(\triangle V)^2+10\triangle U\triangle V+4\delta^{ij}\delta^{mn}\partial_i\partial_m(U+V)\partial_j\partial_n(U+V)\Big]+\\
& +\triangle^2 V\Big[9(\triangle U)^2-39(\triangle V)^2+26\triangle U\triangle V-12\delta^{ij}\delta^{mn}\partial_i\partial_m(U+V)\partial_j\partial_n(U+V)\Big]+\\
& +2\delta^{ij}\left[(3\triangle V-\triangle U)(3\partial_i\triangle U\partial_j(\triangle U+2\triangle V)-\partial_i\triangle V\partial_j\triangle V)-\right.\\
& -\partial_i(3\triangle V-\triangle U)\partial_j\Big(-3(\triangle U)^2+(\triangle V)^2-6\triangle U\triangle V+\\
& \left.+4\delta^{mn}\delta^{kl}\partial_m\partial_k(U+V)\partial_n\partial_l(U+V)\Big)-(\triangle U+\triangle V)\partial_i(3\triangle V-\triangle U)\partial_j(3\triangle V-\triangle U)\right]+\\
& +4\delta^{ij}\delta^{mn}\left[\partial_i\partial_m U\left[\partial_j(3\triangle V-\triangle U)\partial_n(3\triangle V-\triangle U)+(3\triangle V-\triangle U)\partial_j\partial_n(3\triangle V-\triangle U)\right]-\right.\\
& \left.-(3\triangle V-\triangle U)\triangle(\partial_i\partial_m(U+V)\partial_j\partial_n(U+V))\right]\Big\}+\\
+ \quad f_{RRRR}(0,0) \quad & \Big\{-\tfrac{1}{2}(3\triangle V-\triangle U)^2\big(3\triangle^2V-\triangle^2U\big)-\delta^{ij}(3\triangle V-\triangle U)\partial_i(3\triangle V-\triangle U)\partial_j(3\triangle V-\triangle U)\Big\}=\\
= \quad 2\kappa^2 \quad & \Big\{\rho\Big[\mathbf{v}^4-3U\mathbf{v}^2+4U^2+2g^{(4)}_{00}+2g^{(3)}_{0i}v^i+\Pi(\mathbf{v}^2-4U)\Big]+p\big(\mathbf{v}^2-2U\big)\Big\}
\end{aligned}
\tag{8.24}
$$

Then, Eq.(8.20), Eq.(8.22) and Eq.(8.24) constitute the Newtonian, PN and PPN limits, respectively, for the $(0,0)$-field equation of $f(R,\mathcal{G})$-gravity when the metric tensor (8.5) is assumed.

Trace equation

We proceed to find the Newtonian, PN and PPN limits for the trace equation. From Eq.(8.4) at $O(0)$ order in the velocity we obtain again: $f(0,0)=0$.

In the Newtonian limit, i.e. at $O(2)$ order in the velocity, Eq.(8.4) reduces to:

$$-f_R(0,0)R^{(2)}+3f_{RR}(0,0)\left(\nabla^2R\right)^{(2)}=2\kappa^2T^{(0)}. \tag{8.25}$$

Using (8.6), (8.11) and (8.14), Eq.(8.25) is given by:

$$f_R(0,0)(3\triangle V-\triangle U)+3f_{RR}(0,0)\left(3\triangle^2V-\triangle^2U\right)=-2\kappa^2\rho. \tag{8.26}$$

In the PN limit, i.e. at $O(4)$ order in the velocity, using the calculations given in Appendix E, Eq.(8.4) reduces to:

$$
\begin{aligned}
+ \quad 3f_{RR}(0,0) \quad & \Big\{g^{(0)\,00}\Big(\partial_0\partial_0R^{(2)}-\Gamma^{(2)\,i}{}_{00}\partial_iR^{(2)}\Big)+g^{(0)\,ij}\Big(\partial_i\partial_jR^{(4)}-\Gamma^{(2)\,k}{}_{ij}\partial_kR^{(2)}\Big)+g^{(2)\,ij}\partial_i\partial_jR^{(2)}\Big\}+\\
+ \quad 3f_{RRR}(0,0) \quad & g^{(0)\,ij}\left\{\partial_iR^{(2)}\partial_jR^{(2)}+R^{(2)}\partial_i\partial_jR^{(2)}\right\}+\\
+ \quad f_{R\mathcal{G}}(0,0) \quad & \Big\{3g^{(0)\,ij}\partial_i\partial_j\mathcal{G}^{(4)}+2R^{(2)}g^{(0)\,ij}\partial_i\partial_jR^{(2)}-4g^{(0)\,im}g^{(0)\,jn}R^{(2)}_{ij}\partial_m\partial_nR^{(2)}\Big\}-\\
- \quad f_R(0,0)R^{(4)} \quad & =2\kappa^2\Big\{g^{(0)\,00}T^{(2)}_{00}+g^{(2)\,00}T^{(0)}_{00}+g^{(0)\,ij}T^{(2)}_{ij}\Big\}.
\end{aligned}
\tag{8.27}
$$

Using in (8.27) the results obtained in the previous section, the PN limit for the trace equation is given by:

$$\begin{aligned}
- \quad & f_R(0,0) \quad \left\{\tfrac{1}{2}\triangle g^{(4)}_{00} + 2V\left(\triangle U - 3\triangle V\right) + \partial_0\partial_0 U - 2\delta^{ij}\left(\partial_i U\right)\left(\partial_j U\right) - 2U\triangle U\right\} + \\
+ \quad & 3f_{RR}(0,0) \quad \left\{-\tfrac{1}{2}\triangle^2 g^{(4)}_{00} + 2\left(\triangle U\right)^2 - 2\triangle U\,\triangle V + 6\left(\triangle V\right)^2 + \partial_0\partial_0\left(3\triangle V - 2\triangle U\right) + 2\left(U - 2V\right)\triangle^2 U + \right. \\
& \left. +12V\triangle^2 V + \delta^{ij}\left[\partial_i\left(3V + U\right)\partial_j\left(3\triangle V - \triangle U\right) + 8\partial_i U\,\partial_j\triangle U + 4\delta^{mn}\left(\partial_i\partial_m U\right)\left(\partial_j\partial_n U\right)\right]\right\} - \\
- \quad & 3f_{RRR}(0,0) \quad \left\{\delta^{ij}\partial_i\left(3\triangle V - \triangle U\right)\partial_j\left(3\triangle V - \triangle U\right) + \left(3\triangle V - \triangle U\right)\triangle\left(3\triangle V - \triangle U\right)\right\} + \\
+ \quad & 2f_{R\mathcal{G}}(0,0) \quad \left\{2\left(4\triangle U + 5\triangle V\right)\triangle^2 U + 6\left(2\triangle U - \triangle V\right)\triangle^2 V + 3\delta^{ij}\left[3\partial_i\left(\triangle U\right)\partial_j\left(\triangle U + 2\triangle V\right) - \right.\right. \\
& \left.\left. -\partial_i\left(\triangle V\right)\partial_j\left(\triangle V\right)\right] - 6\delta^{ij}\delta^{mn}\triangle\left[\partial_i\partial_m(U+V)\partial_j\partial_n(U+V)\right]\right\} = 2\kappa^2\left\{\rho\left(\Pi - 2U\right) - 3p\right\}
\end{aligned} \tag{8.28}$$

In the PPN limit, i.e. at $O(6)$ order in the velocity, using the calculations given in Appendix E, Eq.(8.4) reduces to:

$$\begin{aligned}
- \quad & f_R(0,0)R^{(6)} \quad + \\
+ \quad & 3f_{RR}(0,0) \quad \left\{g^{(0)\,00}\left[\partial_0\partial_0 R^{(4)} - \Gamma^{(3)\,0}{}_{00}\partial_0 R^{(2)} - \Gamma^{(2)\,i}{}_{00}\partial_i R^{(4)} - \Gamma^{(4)\,i}{}_{00}\partial_i R^{(2)}\right] + \right. \\
& +g^{(2)\,00}\left[\partial_0\partial_0 R^{(2)} - \Gamma^{(2)\,i}{}_{00}\partial_i R^{(2)}\right] + g^{(0)\,ij}\left[\partial_i\partial_j R^{(6)} - \Gamma^{(3)\,0}{}_{ij}\partial_0 R^{(2)} - \Gamma^{(2)\,k}{}_{ij}\partial_k R^{(4)} - \right. \\
& \left.\left. -\Gamma^{(4)\,k}{}_{ij}\partial_k R^{(2)}\right] + 2g^{(3)\,0i}\partial_0\partial_i R^{(2)} + g^{(2)\,ij}\left[\partial_i\partial_j R^{(4)} - \Gamma^{(2)\,k}{}_{ij}\partial_k R^{(2)}\right] + g^{(4)\,ij}\partial_i\partial_j R^{(2)}\right\} + \\
+ \quad & f_{R\mathcal{G}}(0,0) \quad \left\{R^{(2)}\mathcal{G}^{(4)} + 2R^{(4)}g^{(0)\,ij}\partial_i\partial_j R^{(2)} - 4R^{(2)}_{00}\left(g^{(0)\,00}\right)^2\left[\partial_0\partial_0 R^{(2)} - \Gamma^{(2)\,i}{}_{00}\partial_i R^{(2)}\right] + \right. \\
& +3\left[g^{(0)\,00}\left(\partial_0\partial_0\mathcal{G}^{(4)} - \Gamma^{(2)\,i}{}_{00}\partial_i\mathcal{G}^{(4)}\right) + g^{(0)\,ij}\left(\partial_i\partial_j\mathcal{G}^{(6)} - \Gamma^{(2)\,k}{}_{ij}\partial_k\mathcal{G}^{(4)}\right) + g^{(2)\,ij}\partial_i\partial_j\mathcal{G}^{(4)}\right] + \\
& +2R^{(2)}\left[g^{(0)\,00}\left(\partial_0\partial_0 R^{(2)} - \Gamma^{(2)\,i}{}_{00}\partial_i R^{(2)}\right) + g^{(0)\,ij}\left(\partial_i\partial_j R^{(4)} - \Gamma^{(2)\,k}{}_{ij}\partial_k R^{(2)}\right) + \right. \\
& \left. +g^{(2)\,ij}\partial_i\partial_j R^{(2)}\right] - 4R^{(2)}_{jn}g^{(0)\,ij}g^{(0)\,mn}\left[\partial_i\partial_m R^{(4)} - \Gamma^{(2)\,k}{}_{im}\partial_k R^{(2)}\right] - \\
& -4R^{(2)}_{jn}\left[g^{(2)\,ij}g^{(0)\,mn} + g^{(0)\,ij}g^{(2)\,mn}\right]\partial_i\partial_m R^{(2)} - \\
& \left. -8R^{(3)}_{0i}g^{(0)\,00}g^{(0)\,ij}\partial_0\partial_j R^{(2)} - 4R^{(4)}_{jn}g^{(0)\,ij}g^{(0)\,mn}\partial_i\partial_m R^{(2)}\right\} + \\
+ \quad & 2f_{\mathcal{G}\mathcal{G}}(0,0) \quad g^{(0)\,ij}\left\{R^{(2)}\partial_i\partial_j\mathcal{G}^{(4)} - 2R^{(2)}_{jn}g^{(0)\,mn}\partial_i\partial_m\mathcal{G}^{(4)}\right\} + \\
+ \quad & f_{RRR}(0,0) \quad \left\{\tfrac{1}{6}\left(R^{(2)}\right)^3 + 3\left[g^{(0)\,00}\left(\partial_0 R^{(2)}\partial_0 R^{(2)} + R^{(2)}\partial_0\partial_0 R^{(2)} - \Gamma^{(2)\,i}{}_{00}R^{(2)}\partial_i R^{(2)}\right) + \right.\right.
\end{aligned}$$

$$
\begin{aligned}
& & & +g^{(0)\,ij}\left(2\partial_i R^{(2)}\partial_j R^{(4)} + R^{(2)}\partial_i\partial_j R^{(4)} + R^{(4)}\partial_i\partial_j R^{(2)} - \Gamma^{(2)\,k}_{\ \ ij} R^{(2)}\partial_k R^{(2)}\right) + \\
& & & +g^{(2)\,ij}\left(\partial_i R^{(2)}\partial_j R^{(2)} + R^{(2)}\partial_i\partial_j R^{(2)}\right)\Big]\Big\} + \\
+ & & f_{RR\mathcal{G}}(0,0) & \Big\{3\left[g^{(0)\,ij}\left(2\partial_i R^{(2)}\partial_j \mathcal{G}^{(4)} + \mathcal{G}^{(4)}\partial_i\partial_j R^{(2)} + R^{(2)}\partial_i\partial_j \mathcal{G}^{(4)}\right)\right] + \\
& & & +2R^{(2)} g^{(0)\,ij}\left[\partial_i R^{(2)}\partial_j R^{(2)} + R^{(2)}\partial_i\partial_j R^{(2)}\right] - \\
& & & -4R^{(2)}_{jn} g^{(0)\,ij} g^{(0)\,mn}\left[\partial_i R^{(2)}\partial_m R^{(2)} + R^{(2)}\partial_i\partial_m R^{(2)}\right]\Big\} + \\
+ & & 3f_{RRRR}(0,0) & \ g^{(0)\,ij} R^{(2)}\left\{\partial_i R^{(2)}\partial_j R^{(2)} + \tfrac{1}{2} R^{(2)}\partial_i\partial_j R^{(2)}\right\} = \\
= & & 2\kappa^2 & \left\{g^{(0)\,00} T^{(4)}_{00} + g^{(2)\,00} T^{(2)}_{00} + g^{(4)\,00} T^{(0)}_{00} + 2g^{(3)\,0i} T^{(1)}_{0i} + g^{(0)\,ij} T^{(4)}_{ij} + g^{(2)\,ij} T^{(2)}_{ij}\right\}
\end{aligned}
\tag{8.29}
$$

Introducing in (8.29) the results obtained in the previous section, we finally obtain:

$$
\begin{aligned}
- & \ f_R(0,0) & & R^{(6)} + \\
+ & \ f_{RR}(0,0) & & 3\Big\{\partial_0\partial_0\left[\tfrac{1}{2}\triangle g^{(4)}_{00} - 2V\left(3\triangle V - \triangle U\right) + \partial_0\partial_0 U - 2\delta^{ij}\left(\partial_i U\right)\left(\partial_j U\right) - 2U\triangle U\right] + \\
& & & +2U\partial_0\partial_0\left(3\triangle V - \triangle U\right) + \partial_0(3V+U)\,\partial_0\left(3\triangle V - \triangle U\right) - \triangle R^{(6)} + V\triangle^2 g^{(4)}_{00} + \\
& & & +4V(2V-U)\triangle^2 U - 24V\triangle^2 V - 4V\triangle U\triangle\left(U - V\right) - \\
& & & -12V\left(\triangle V\right)^2 + 2g^{(3)\,0i}\partial_0\partial_i\left(3\triangle V - \triangle U\right) + \delta^{ij}\Big[-\left(\tfrac{1}{2}\partial_i g^{(4)}_{00} - \partial_0 g^{(3)}_{0i}\right)\partial_j\left(3\triangle V - \triangle U\right) + \\
& & & +\partial_i(U-V)\left(\tfrac{1}{2}\partial_j\triangle g^{(4)}_{00} - 2\partial_j V\,\left(3\triangle V - \triangle U\right) + \partial_j\partial_0\partial_0 U - 2\partial_j U\,\triangle U - 2U\partial_j\triangle U\right) - \\
& & & -2\left((2V-U)\partial_i U + V\partial_i V\right)\,\partial_j\left(3\triangle V - \triangle U\right) - 16V\partial_i U\partial_j\triangle U + \partial_i g^{(3)}_{0j}\partial_0\left(3\triangle V - \triangle U\right)\Big] - \\
& & & -\delta^{ij}\delta^{mn}\Big[8V\partial_i\partial_m U\,\partial_j\partial_n U + \tfrac{1}{2}\left(2\partial_i g^{(4)}_{jn} - \partial_n g^{(4)}_{ij}\right)\partial_m\left(3\triangle V - \triangle U\right) + \\
& & & +g^{(4)}_{im}\partial_j\partial_n\left(3\triangle V - \triangle U\right) + 4\partial_i(U-V)\partial_m U\partial_j\partial_n U\Big]\Big\} + \\
+ & \ f_{R\mathcal{G}}(0,0) & & \Big\{\triangle^2 U\Big[\triangle g^{(4)}_{00} - 4\left(U+6V\right)\triangle U - 4\left(2U+9V\right)\triangle V + 2\partial_0\partial_0\left(U+2V\right) + \\
& & & +4\delta^{ij}\left(\partial_i V\partial_j\left(U+V\right) - \partial_i U\partial_j U\right)\Big] + \triangle^2 V\Big[-3\triangle g^{(4)}_{00} + 12\left(U-6V\right)\triangle U + 24V\triangle V - \\
& & & -2\partial_0\partial_0\left(3U+2V\right) + 4\delta^{ij}\left(3\partial_i U\partial_j U - \partial_i V\partial_j\left(U+V\right)\right)\Big] - 3\triangle\mathcal{G}^{(6)} + 3\left(\triangle U\right)^3 + \\
& & & +3\partial_0\partial_0\left[-3\left(\triangle U\right)^2 + \left(\triangle V\right)^2 - 6\left(\triangle U\right)\left(\triangle V\right) + 4\delta^{im}\delta^{jn}\partial_i\partial_j\left(U+V\right)\partial_m\partial_n\left(U+V\right)\right] - \\
& & & -11\triangle U\triangle V\triangle\left(U+V\right) + 2\triangle U\left[\partial_0\partial_0\left(3\triangle V - \triangle U\right) - 2\delta^{ij}\delta^{mn}\partial_i\partial_m(U+V)\partial_j\partial_n(U+V)\right] -
\end{aligned}
$$

$$-21\left(\triangle V\right)^3+2\triangle V\left[\triangle^2 g^{(4)}_{00}+\partial_0\partial_0\left(9\triangle V-\triangle U\right)+6\delta^{ij}\delta^{mn}\partial_i\partial_m(U+V)\partial_j\partial_n(U+V)\right]+$$

$$+4\delta^{ij}\left[\partial_i U\left(-\left(5\triangle U+14\triangle V\right)\partial_j\triangle U+3\left(2\triangle V-\triangle U\right)\partial_j\triangle V+\right.\right.$$

$$\left.+\,3\delta^{mn}\delta^{kl}\partial_j\left[\partial_m\partial_k(U+V)\partial_n\partial_l(U+V)\right]\right)-\partial_i V\left(-2\left(2\triangle U+5\triangle V\right)\partial_j\triangle U+\right.$$

$$\left.+6\left(3\triangle V-\triangle U\right)\partial_j\triangle V+3\delta^{mn}\delta^{kl}\partial_j\left[\partial_m\partial_k(U+V)\partial_n\partial_l(U+V)\right]\right)+$$

$$\left.+\,\triangle g^{(3)}_{0i}\partial_0\partial_j\left(3\triangle V-\triangle U\right)-3V\left(3\partial_i\triangle U\partial_j\left(\triangle U+2\triangle V\right)-\partial_i\triangle V\partial_j\triangle V\right)\right]+$$

$$+4\delta^{ij}\delta^{mn}\left[6V\triangle\left(\partial_i\partial_m(U+V)\partial_j\partial_n(U+V)\right)-4\triangle V\partial_i\partial_m U\partial_j\partial_n U-\right.$$

$$-\partial_i\partial_m\left(3\triangle V-\triangle U\right)\left\{\frac{1}{2}\left(\partial_0\partial_j g^{(3)}_{0n}+\partial_0\partial_n g^{(3)}_{0j}+\triangle g^{(4)}_{jn}-\partial_j\partial_n g^{(4)}_{00}-\right.\right.$$

$$\left.\left.-\,\delta^{kl}\left(\partial_k\partial_j g^{(4)}_{ln}+\partial_k\partial_n g^{(4)}_{lj}-\partial_j\partial_n g^{(4)}_{kl}\right)\right)+2U\partial_j\partial_n U+2V\partial_j\partial_n V+\right.$$

$$\left.\left.+\partial_j U\partial_n(U-V)+\partial_j V\partial_n(3V-U)\right\}\right]\Big\}+$$

$$+\quad f_{\mathcal{G}\mathcal{G}}(0,0)\quad 2\left(\triangle U-\triangle V\right)\triangle\left\{-3\left(\triangle U\right)^2+\left(\triangle V\right)^2-6\left(\triangle U\right)\left(\triangle V\right)+4\delta^{im}\delta^{jn}\partial_i\partial_j\left(U+V\right)\,\partial_m\partial_n\left(U+V\right)\right\}+$$

$$+\quad f_{RRR}(0,0)\quad\left\{-\frac{3}{2}\left(3\triangle V-\triangle U\right)\triangle^2 g^{(4)}_{00}-\frac{37}{6}\left(\triangle U\right)^3+\frac{51}{2}\left(\triangle U\right)^2\triangle V-\frac{81}{2}\triangle U\left(\triangle V\right)^2+\frac{117}{2}\left(\triangle V\right)^3+\right.$$

$$+3\triangle^2 U\left[\frac{1}{2}\triangle g^{(4)}_{00}+\partial_0\partial_0 U-2(U+3V)\left(3\triangle V-\triangle U\right)-2\delta^{ij}\partial_i U\partial_j U-2U\triangle U\right]+$$

$$+9\triangle^2 V\left[-\frac{1}{2}\triangle g^{(4)}_{00}-\partial_0\partial_0 U+6V\left(3\triangle V-\triangle U\right)+2\delta^{ij}\partial_i U\partial_j U+2U\triangle U\right]+$$

$$+3\left(3\triangle V-\triangle U\right)\partial_0\partial_0\left(3\triangle V-2\triangle U\right)+3\partial_0\left(3\triangle V-\triangle U\right)\,\partial_0\left(3\triangle V-\triangle U\right)+$$

$$+3\delta^{ij}\left[8\left(3\triangle V-\triangle U\right)\partial_i U\partial_j\triangle U+\left(-\partial_i\triangle g^{(4)}_{00}+\left(3\triangle V-\triangle U\right)\partial_i\left(U+7V\right)+\right.\right.$$

$$\left.\left.+6V\partial_i\left(3\triangle V-\triangle U\right)-2\partial_i\partial_0\partial_0 U+4\triangle U\partial_i U+4U\partial_i\triangle U\right)\partial_j\left(3\triangle V-\triangle U\right)\right]+$$

$$\left.+12\delta^{ij}\delta^{mn}\left[\left(3\triangle V-\triangle U\right)\partial_i\partial_m U\,\partial_j\partial_n U+2\partial_i U\partial_m\left(3\triangle V-\triangle U\right)\partial_j\partial_n U\right]\right\}+$$

$$+\quad f_{RR\mathcal{G}}(0,0)\quad\left\{\triangle^2 U\left[-25\left(\triangle U\right)^2+63\left(\triangle V\right)^2+10\triangle U\triangle V+12\,\delta^{ij}\delta^{mn}\,\partial_i\partial_m(U+V)\,\partial_j\partial_n(U+V)\right]+\right.$$

$$+3\triangle^2 V\left[\left(\triangle U\right)^2-15\left(\triangle V\right)^2+46\triangle U\triangle V-12\,\delta^{ij}\delta^{mn}\,\partial_i\partial_m(U+V)\,\partial_j\partial_n(U+V)\right]+$$

$$+2\delta^{ij}\left[3\left(3\triangle V-\triangle U\right)\left(3\partial_i\triangle U\partial_j\triangle U-\partial_i\triangle V\partial_j\triangle V+6\partial_i\triangle U\partial_j\triangle V\right)-\right.$$

$$-3\partial_i\left(3\triangle V-\triangle U\right)\partial_j\left(-3\left(\triangle U\right)^2+\left(\triangle V\right)^2-6\left(\triangle U\right)\left(\triangle V\right)++\right.$$

$$\left.\left.+4\delta^{im}\delta^{jn}\,\partial_i\partial_j\left(U+V\right)\,\partial_m\partial_n\left(U+V\right)\right)+\left(5\triangle V-\triangle U\right)\partial_i\left(3\triangle V-\triangle U\right)\partial_j\left(3\triangle V-\triangle U\right)\right]-$$

$$\left.-12\delta^{ij}\delta^{mn}\left(3\triangle V-\triangle U\right)\triangle\left[\partial_i\partial_j\left(U+V\right)\,\partial_m\partial_n\left(U+V\right)\right]\right\}-$$

$$- \quad f_{RRRR}(0,0) \quad 3\,(3\triangle V - \triangle U)\left\{\partial_i\,(3\triangle V - \triangle U)\,\partial_j\,(3\triangle V - \triangle U) + \tfrac{1}{2}\,(3\triangle V - \triangle U)\left(3\triangle^2 V - \triangle^2 U\right)\right\} =$$

$$= \quad 2\kappa^2 \quad \left\{\rho\left[-\mathbf{v}^2\,(U+2V) + g^{(4)}_{00} + 2g^{(3)}_{0i}v^i - 2U\Pi\right] - 2Up\right\}. \tag{8.30}$$

Then, Eq.(8.26), Eq.(8.28) and Eq.(8.30) constitute the Newtonian, PN and PPN limit, respectively, for the trace equation of $f(R,\mathcal{G})$ modified gravity when the metric tensor given by (8.5) is assumed.

8.2 Solving the Newtonian limit

In this section our aim is to solve the system of equation for the Newtonian limit of $f(R,\mathcal{G})$ modified gravity, i.e. the system constituted by Eq.(8.20) and Eq.(8.26), in the most general way. In order to do this, we will search for solutions in terms of Green's functions (see [70]).

We start considering the set of equations given by Eq.(8.20) and Eq.(8.26), this is:

$$\begin{aligned} f_R(0,0)\,(3\triangle V + \triangle U) + 2f_{RR}(0,0)\left(3\triangle^2 V - \triangle^2 U\right) &= -4\kappa^2\rho \\ f_R(0,0)\,(3\triangle V - \triangle U) + 3f_{RR}(0,0)\left(3\triangle^2 V - \triangle^2 U\right) &= -2\kappa^2\rho \end{aligned} \tag{8.31}$$

By introducing the new auxiliary functions $A = f_R(0,0)(3V+U)$ and $B = 2f_{RR}(0,0)\,(3V-U)$, we can write (8.31) as:

$$\begin{aligned} -4\kappa^2\rho &= \triangle A + \triangle^2 B \\ -4\kappa^2\rho &= \tfrac{f_R(0,0)}{f_{RR}(0,0)}\triangle B + 3\triangle^2 B \end{aligned} \tag{8.32}$$

Considering now the new function $\Phi = A + \triangle B$, (8.32) reduces to:

$$\begin{aligned} -4\kappa^2\rho &= \triangle\Phi \\ -4\kappa^2\rho &= \tfrac{f_R(0,0)}{f_{RR}(0,0)}\triangle B + 3\triangle^2 B \end{aligned} \tag{8.33}$$

It is important to remark that (8.33) is a set of uncoupled equations. We are interested in the solution of the second equation in (8.33) in terms of the Green's function $\mathbb{G}(\mathbf{x},\mathbf{x'})$ defined by:

$$B = -4\kappa^2 C\int d^3\mathbf{x'}\mathbb{G}(\mathbf{x},\mathbf{x'})\rho(\mathbf{x'}) \tag{8.34}$$

where C is a constant, which is introduced for dimensional reasons. Now the set of equations given by (8.33) is equivalent to:

$$\begin{aligned} -4\kappa^2\rho &= \triangle\Phi \\ \tfrac{1}{C}\delta(\mathbf{x}-\mathbf{x'}) &= \tfrac{f_R(0,0)}{f_{RR}(0,0)}\triangle_{\mathbf{x}}\mathbb{G}(\mathbf{x},\mathbf{x'}) + 3\triangle^2_{\mathbf{x}}\mathbb{G}(\mathbf{x},\mathbf{x'}) \end{aligned} \tag{8.35}$$

where $\delta(\mathbf{x}-\mathbf{x'})$ is the three-dimensional Dirac δ-function. The general solutions of equations (8.31) for $U(\mathbf{x})$ and $V(\mathbf{x})$, in terms of the Green's function $\mathbb{G}(\mathbf{x},\mathbf{x'})$ and the function $\Phi(\mathbf{x})$, are:

$$\begin{aligned} U(\mathbf{x}) &= \frac{1}{2f_R(0,0)}\Phi(\mathbf{x}) + 2\kappa^2 C\left(\frac{\triangle_{\mathbf{x}}}{f_R(0,0)} + \frac{1}{2f_{RR}(0,0)}\right)\int d^3\mathbf{x'}\mathbb{G}(\mathbf{x},\mathbf{x'})\rho(\mathbf{x'}) \\ V(\mathbf{x}) &= \frac{1}{6f_R(0,0)}\Phi(\mathbf{x}) + \tfrac{2}{3}\kappa^2 C\left(\frac{\triangle_{\mathbf{x}}}{f_R(0,0)} - \frac{1}{2f_{RR}(0,0)}\right)\int d^3\mathbf{x'}\mathbb{G}(\mathbf{x},\mathbf{x'})\rho(\mathbf{x'}) \end{aligned} \tag{8.36}$$

In summary, the functions $U(\mathbf{x})$ and $V(\mathbf{x})$, which are related with $g_{00}^{(2)}$ and $g_{ij}^{(2)}$, respectively, by Eq.(8.5) have been found in terms of the Green's function $\mathbb{G}(\mathbf{x},\mathbf{x'})$ and the function $\Phi(\mathbf{x})$, giving in this way a general solution to the Newtonian limit for $f(R,\mathcal{G})$ modified gravity theories.

8.3 Weak field limit in two especial cases: $f(R)$ and $f(\mathcal{G})$ modified gravities

In this section, the results obtained previously will be used for two special cases: $f(R)$ modified gravity and $f(\mathcal{G})$ modified gravity, respectively.

8.3.1 $f(R)$ modified gravity

The starting action is given by:

$$S=\int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}f(R)+\mathcal{L}_m\right). \tag{8.37}$$

In order to obtain the Newtonian, PN and PPN limits for this theory we will use the equations of the previous section considering the change: $f(R,\mathcal{G})\to f(R)$. The field equations for $f(R)$ modified gravity are obtained from Eq.(8.3):

$$-\frac{1}{2}g_{\mu\nu}f(R)+f'(R)R_{\mu\nu}+g_{\mu\nu}\nabla^2 f'(R)-\nabla_\mu\nabla_\nu f'(R)=2\kappa^2T_{\mu\nu}, \tag{8.38}$$

while the trace equation is obtained from Eq.(8.4):

$$-2f(R)+f'(R)R+3\nabla^2 f'(R)=2\kappa^2T. \tag{8.39}$$

Before analyzing the Newtonian, PN and PPN limits for this theory, it is important to remark that at the lowest order in the velocity, i.e. $O(0)$-order, from Eq.(8.38) and Eq.(8.39), we obtain: $f(0)=0$.

Newtonian limit

The Newtonian limit of $f(R)$ modified gravity corresponds to $O(2)$-order for Eq.(8.38) and Eq.(8.39).

The $(0,0)$-field equation for $f(R)$ modified gravity at Newtonian order can be obtained from (8.20) and it is given by:

$$f'(0)\left(3\triangle V+\triangle U\right)+2f''(0)\left(3\triangle^2V-\triangle^2U\right)=-4\kappa^2\rho. \tag{8.40}$$

The trace equation for $f(R)$ modified gravity at Newtonian order can be obtained from Eq.(8.26) and it is given by:

$$f'(0)\left(3\triangle V-\triangle U\right)+3f''(0)\left(3\triangle^2V-\triangle^2U\right)=-2\kappa^2\rho. \tag{8.41}$$

PostNewtonian limit

In this case, the aim is to obtain Eq.(8.38) and Eq.(8.39) at $O(4)$-order in the velocity.

For the $(0,0)$-field equation, we use Eq.(8.22). For $f(R)$ modified gravity it reduces to:

$$\begin{aligned}
& f'(0) \quad \left\{\tfrac{1}{4}\triangle g_{00}^{(4)} + 3(U+V)\triangle V + V\triangle U + \tfrac{1}{2}\partial_0\partial_0 U - \delta^{ij}\partial_i U\partial_j U\right\} + \\
+ \quad & f''(0) \quad \left\{-\tfrac{1}{2}\triangle^2 g_{00}^{(4)} + \tfrac{15}{4}\left(\triangle V\right)^2 - \tfrac{7}{2}\triangle U\triangle V + \tfrac{11}{4}\left(\triangle U\right)^2 + 6(U+2V)\triangle^2 V - 4V\triangle^2 U - \partial_0\partial_0\triangle U + \right. \\
& \left. + \delta^{ij}\left[3\partial_i V\,\partial_j\left(3\triangle V - \triangle U\right) + 8\partial_i U\,\partial_j\triangle U + 4\delta^{mn}\partial_i\partial_m U\,\partial_j\partial_n U\right]\right\} - \\
- \quad & f'''(0) \quad \left\{\delta^{ij}\partial_i\left(3\triangle V - \triangle U\right)\partial_j\left(3\triangle V - \triangle U\right) + \left(3\triangle V - \triangle U\right)\triangle\left(3\triangle V - \triangle U\right)\right\} = 2\kappa^2\rho\left(\Pi + \mathbf{v}^2 - 4U\right).
\end{aligned} \tag{8.42}$$

While for the trace equation, Eq.(8.28) gives:

$$\begin{aligned}
- \quad & f'(0) \quad \left\{\tfrac{1}{2}\triangle g_{00}^{(4)} + 2V\left(\triangle U - 3\triangle V\right) + \partial_0\partial_0 U - 2\delta^{ij}\left(\partial_i U\right)\left(\partial_j U\right) - 2U\triangle U\right\} + \\
+ \quad & 3f''(0) \quad \left\{-\tfrac{1}{2}\triangle^2 g_{00}^{(4)} + 2\left(\triangle U\right)^2 - 2\triangle U\,\triangle V + 6\left(\triangle V\right)^2 + \partial_0\partial_0\left(3\triangle V - 2\triangle U\right) + 2\left(U - 2V\right)\triangle^2 U + \right. \\
& \left. + 12V\triangle^2 V + \delta^{ij}\left[\partial_i\left(3V+U\right)\partial_j\left(3\triangle V - \triangle U\right) + 8\partial_i U\,\partial_j\triangle U + 4\delta^{mn}\left(\partial_i\partial_m U\right)\left(\partial_j\partial_n U\right)\right]\right\} - \\
- \quad & 3f'''(0) \quad \left\{\delta^{ij}\partial_i\left(3\triangle V - \triangle U\right)\partial_j\left(3\triangle V - \triangle U\right) + \left(3\triangle V - \triangle U\right)\triangle\left(3\triangle V - \triangle U\right)\right\} = 2\kappa^2\left\{\rho\left(\Pi - 2U\right) - 3p\right\}
\end{aligned} \tag{8.43}$$

PostPostNewtonian limit

Finally, the PPN limit corresponds to $O(6)$-order for Eq.(8.38) and Eq.(8.39).

For the $(0,0)$-field equation we obtain from Eq.(8.24) the following result:

$$\begin{aligned}
& f'(0) \quad \left\{\tfrac{1}{2}\left(R_{00}^{(6)} + \delta^{ij}R_{ij}^{(6)}\right) - \delta^{ij}g_{0i}^{(3)}R_{0j}^{(3)} - \tfrac{1}{2}\triangle V\left[3\left(g_{00}^{(4)} + 4UV - 4V^2\right) + \delta^{ij}g_{ij}^{(4)}\right] - \right. \\
& -(U+V)\left[\tfrac{1}{2}\left(6\partial_0\partial_0 V - \triangle g_{00}^{(4)} + 2\delta^{ij}\left(\triangle g_{ij}^{(4)} + \partial_0\partial_i g_{0j}^{(3)}\right) - 2\delta^{ij}\delta^{mn}\partial_i\partial_m g_{jn}^{(4)}\right)\right] + \\
& \left. + 8V\triangle V + 2U\triangle U + \delta^{ij}\left[\partial_i U\partial_j(U-V) + 2\partial_i V\partial_j(3V+U)\right]\right\} + \\
+ \quad & f''(0) \quad \left\{-\triangle R^{(6)} + (U+V)\triangle^2 g_{00}^{(4)} + \triangle^2 U\left(g_{00}^{(4)} - 4U^2 + 4UV + 8V^2\right) - 3\triangle^2 V\left(g_{00}^{(4)} + 4UV + 8V^2\right) - \right. \\
& - \left(\triangle U\right)^2\left(\tfrac{5}{2}U + 7V\right) - 3\left(\triangle V\right)^2\left(\tfrac{5}{2}U + V\right) + 2\triangle U\triangle V\left(2U + 5V\right) + \\
& + 3\left(\triangle V - \triangle U\right)\left(\tfrac{1}{4}\triangle g_{00}^{(4)} + \tfrac{1}{2}\partial_0\partial_0 U - \delta^{ij}\partial_i U\partial_j U\right) + 2V\partial_0\partial_0\triangle U + \\
& + \delta^{ij}\left[\tfrac{1}{2}\left(2\partial_i g_{0j}^{(3)} - \partial_0 g_{ij}^{(2)}\right)\partial_0\left(3\triangle V - \triangle U\right) - \partial_i V\partial_0\partial_0\partial_j U + 2\left(3\triangle V - \triangle U\right)\partial_i V\partial_j V + \right. \\
& + 2\triangle U\partial_i U\partial_j V - \tfrac{1}{2}\partial_i V\partial_j\triangle g_{00}^{(4)} + 2g_{0i}^{(3)}\partial_0\partial_j\left(3\triangle V - \triangle U\right) + 2\partial_i\triangle U\left(U\partial_j\left(V - 8U\right) - 8V\partial_j U\right) -
\end{aligned}$$

$$
\begin{aligned}
& && -2\left(V+3U\right)\partial_i V\partial_j\left(3\triangle V-\triangle U\right)\big]+\delta^{ij}\delta^{mn}\Big[-\tfrac{1}{2}\left(2\partial_i g^{(4)}_{mj}-\partial_m g^{(4)}_{ij}\right)\partial_n\left(3\triangle V-\triangle U\right)-\\
& && -g^{(4)}_{im}\partial_j\partial_n\left(3\triangle V-\triangle U\right)-4\left(2(U+V)\partial_i\partial_m U-\partial_i V\partial_m U\right)\partial_j\partial_n U\Big]\Big\}+\\
+ & f'''(0) && \Big\{\triangle^2 U\Big[\tfrac{1}{2}\triangle g^{(4)}_{00}+\partial_0\partial_0 U+2(3V-U)\triangle U-18V\triangle V-2\delta^{ij}\partial_i U\partial_j U\Big]+\\
& && +\triangle^2 V\Big[-\tfrac{1}{2}\triangle g^{(4)}_{00}-\partial_0\partial_0 U-2(7V+2U)\triangle U+6(7V+3U)\triangle V+2\delta^{ij}\partial_i U\partial_j U\Big]-\\
& && -\left(\tfrac{1}{2}\triangle g^{(4)}_{00}+\partial_0\partial_0\triangle U\right)\left(3\triangle V-\triangle U\right)-\tfrac{29}{12}\left(\triangle U\right)^3+\tfrac{63}{4}\left(\triangle V\right)^3+\tfrac{111}{12}\triangle V\left(\triangle U\right)^2-\tfrac{57}{4}\triangle U\left(\triangle V\right)^2+\\
& && +\delta^{ij}\left[\partial_i\left(3\triangle V-\triangle U\right)\left(3\left(3\triangle V-\triangle U\right)\partial_j V+2\left(3\triangle V+\triangle U\right)\partial_j\left(3\triangle V-\triangle U\right)-\right.\right.\\
& && 2\partial_j\left(\tfrac{1}{2}\triangle g^{(4)}_{00}+\partial_0\partial_0 U-2U\triangle U\right)\Big)+8\left(3\triangle V-\triangle U\right)\partial_i U\partial_j\triangle U\Big]+\\
& && +4\delta^{ij}\delta^{mn}\left[\partial_i\partial_m\left(3\triangle V-\triangle U\right)\partial_j\partial_n\left(3\triangle V-\triangle U\right)+\left(3\triangle V-\triangle U\right)\partial_i\partial_m U\partial_j\partial_n U\right]\}+\\
+ & f''''(0) && \Big\{-\tfrac{1}{2}\left(3\triangle V-\triangle U\right)^2\left(3\triangle^2 V-\triangle^2 U\right)-\delta^{ij}\left(3\triangle V-\triangle U\right)\partial_i\left(3\triangle V-\triangle U\right)\partial_j\left(3\triangle V-\triangle U\right)\Big\}=\\
= & 2\kappa^2 && \Big\{\rho\Big[\mathbf{v}^4-3U\mathbf{v}^2+4U^2+2g^{(4)}_{00}+2g^{(3)}_{0i}v^i+\Pi(\mathbf{v}^2-4U)\Big]+p\left(\mathbf{v}^2-2U\right)\Big\}
\end{aligned}
\tag{8.44}
$$

And Eq.(8.30) implies that the trace equation is given by:

$$
\begin{aligned}
- & f'(0)R^{(6)} && +\\
+ & 3f''(0) && \Big\{\partial_0\partial_0\Big[\tfrac{1}{2}\triangle g^{(4)}_{00}-2V\left(3\triangle V-\triangle U\right)+\partial_0\partial_0 U-2\delta^{ij}\left(\partial_i U\right)\left(\partial_j U\right)-2U\triangle U\Big]+\\
& && +2U\partial_0\partial_0\left(3\triangle V-\triangle U\right)+\ \partial_0(3V+U)\,\partial_0\left(3\triangle V-\triangle U\right)-\triangle R^{(6)}+V\triangle^2 g^{(4)}_{00}+\\
& && +4V(2V-U)\triangle^2 U-24V\triangle^2 V-4V\triangle U\triangle\left(U-V\right)-12V\left(\triangle V\right)^2+\\
& && +2g^{(3)\,0i}\partial_0\partial_i\left(3\triangle V-\triangle U\right)+\delta^{ij}\Big[-\left(\tfrac{1}{2}\partial_i g^{(4)}_{00}-\partial_0 g^{(3)}_{0i}\right)\partial_j\left(3\triangle V-\triangle U\right)+\\
& && +\partial_i(U-V)\left(\tfrac{1}{2}\partial_j\triangle g^{(4)}_{00}-2\partial_j V\ \left(3\triangle V-\triangle U\right)+\partial_j\partial_0\partial_0 U-2\partial_j U\,\triangle U-2U\partial_j\triangle U\right)-\\
& && -2\left((2V-U)\partial_i U+V\partial_i V\right)\ \partial_j\left(3\triangle V-\triangle U\right)-16V\partial_i U\partial_j\triangle U+\partial_i g^{(3)}_{0j}\partial_0\left(3\triangle V-\triangle U\right)\Big]-\\
& && -\delta^{ij}\delta^{mn}\Big[8V\partial_i\partial_m U\,\partial_j\partial_n U+\tfrac{1}{2}\left(2\partial_i g^{(4)}_{jn}-\partial_n g^{(4)}_{ij}\right)\partial_m\left(3\triangle V-\triangle U\right)+\\
& && +g^{(4)}_{im}\partial_j\partial_n\left(3\triangle V-\triangle U\right)+4\partial_i(U-V)\partial_m U\partial_j\partial_n U\Big]\Big\}+\\
+ & f'''(0) && \Big\{-\tfrac{3}{2}\left(3\triangle V-\triangle U\right)\triangle^2 g^{(4)}_{00}+3\triangle^2 U\Big[\tfrac{1}{2}\triangle g^{(4)}_{00}+\partial_0\partial_0 U-2(U+3V)\left(3\triangle V-\triangle U\right)-
\end{aligned}
$$

$$
\begin{aligned}
& -2\delta^{ij}\partial_i U\partial_j U - 2U\triangle U\Big] - \tfrac{37}{6}\left(\triangle U\right)^3 + \tfrac{51}{2}\left(\triangle U\right)^2\triangle V - \tfrac{81}{2}\triangle U\left(\triangle V\right)^2 + \tfrac{117}{2}\left(\triangle V\right)^3 + \\
& +9\triangle^2 V\left[-\tfrac{1}{2}\triangle g^{(4)}_{00} - \partial_0\partial_0 U + 6V\left(3\triangle V - \triangle U\right) + 2\delta^{ij}\partial_i U\partial_j U + 2U\triangle U\right] - \\
& +3\left(3\triangle V - \triangle U\right)\partial_0\partial_0\left(3\triangle V - 2\triangle U\right) + 3\partial_0\left(3\triangle V - \triangle U\right)\,\partial_0\left(3\triangle V - \triangle U\right) + \\
& +3\delta^{ij}\Big[8\left(3\triangle V - \triangle U\right)\partial_i U\partial_j\triangle U + \Big(-\partial_i\triangle g^{(4)}_{00} + \left(3\triangle V - \triangle U\right)\partial_i\left(U + 7V\right) + \\
& +6V\partial_i\left(3\triangle V - \triangle U\right) - 2\partial_i\partial_0\partial_0 U + 4\triangle U\partial_i U + 4U\partial_i\triangle U\Big)\,\partial_j\left(3\triangle V - \triangle U\right)\Big] + \\
& +12\delta^{ij}\delta^{mn}\left[\left(3\triangle V - \triangle U\right)\partial_i\partial_m U\,\partial_j\partial_n U + 2\partial_i U\partial_m\left(3\triangle V - \triangle U\right)\partial_j\partial_n U\right]\Big\} - \\
- \quad & 3f''''(0)\quad \left(3\triangle V - \triangle U\right)\left\{\partial_i\left(3\triangle V - \triangle U\right)\partial_j\left(3\triangle V - \triangle U\right) + \tfrac{1}{2}\left(3\triangle V - \triangle U\right)\left(3\triangle^2 V - \triangle^2 U\right)\right\} = \\
= \quad & 2\kappa^2 \quad \left\{\rho\left[-\mathbf{v}^2\left(U + 2V\right) + g^{(4)}_{00} + 2g^{(3)}_{0i}v^i - 2U\Pi\right] - 2Up\right\}.
\end{aligned}
\tag{8.45}
$$

Summing up, the Newtonian limit of $f(R)$ modified gravity theories is given by Eq.(8.40) and Eq.(8.41); the post–Newtonian limit is given by Eq.(8.42) and Eq.(8.43); and the post–post–Newtonian limit is given by Eq.(8.44) and Eq.(8.45), respectively.

8.3.2 $f(\mathcal{G})$ modified gravity

We will consider the following action:

$$
S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R + f(\mathcal{G})\right] + \mathcal{L}_{\text{matter}}\right\}. \tag{8.46}
$$

We proceed in the same way as we did for $f(R)$ modified gravity, but in this case the change is given by $f(R,\mathcal{G}) \to R + f(\mathcal{G})$.

The field equations for this theory can be obtained from (8.3):

$$
\begin{aligned}
& -\frac{1}{2}g_{\mu\nu}\left(R + f(\mathcal{G})\right) + R_{\mu\nu} + 2f'(\mathcal{G})RR_{\mu\nu} - 4f'(\mathcal{G})R_{\mu\rho}R_\nu{}^\rho + 2f'(\mathcal{G})R_{\alpha\beta\rho\mu}R^{\alpha\beta\rho}{}_\nu - 4f'(\mathcal{G})R_{\mu\rho\nu\sigma}R^{\rho\sigma} + \\
& +2g_{\mu\nu}R\nabla^2 f'(\mathcal{G}) - 4g_{\mu\nu}R_{\rho\sigma}\nabla^\rho\nabla^\sigma f'(\mathcal{G}) - 2R\nabla_\mu\nabla_\nu f'(\mathcal{G}) - 4R_{\mu\nu}\nabla^2 f'(\mathcal{G}) + 4R_{\nu\rho}\nabla^\rho\nabla_\mu f'(\mathcal{G}) + 4R_{\mu\rho}\nabla^\rho\nabla_\nu f'(\mathcal{G}) + \\
& +4R_{\mu\rho\nu\sigma}\nabla^\rho\nabla^\sigma f'(\mathcal{G}) = 2\kappa^2 T_{\mu\nu},
\end{aligned}
\tag{8.47}
$$

while the trace equation is obtained from Eq.(8.4):

$$
-R - 2f(\mathcal{G}) + 2f'(\mathcal{G})\mathcal{G} + 2R\nabla^2 f'(\mathcal{G}) - 4R_{\rho\sigma}\nabla^\rho\nabla^\sigma f'(\mathcal{G}) = 2\kappa^2 T. \tag{8.48}
$$

From the lowest order we obtain again: $f(0) = 0$.

Newtonian limit

In this case, the $(0,0)$-field equation obtained from Eq.(8.20) is given by:

$$
\triangle U + 3\triangle V = -4\kappa^2\rho. \tag{8.49}
$$

While the trace equation can be obtained from Eq.(8.26) and it reduces to:

$$\triangle U - 3\triangle V = 2\kappa^2\rho. \tag{8.50}$$

PostNewtonian limit

The PN limit for the $(0,0)$-field equation can be calculated from Eq.(8.22) and it reduces to:

$$\begin{aligned} &\tfrac{1}{4}\triangle g^{(4)}_{00} + 3(U+V)\triangle V + V\triangle U + \tfrac{1}{2}\partial_0\partial_0 U - \delta^{ij}\partial_i U\partial_j U + \\ + \quad & f'(0)\left\{-\tfrac{1}{2}\left(\triangle U - \triangle V\right)^2 + 2\delta^{im}\delta^{jn}\partial_i\partial_j\left(U-V\right)\partial_m\partial_n\left(U-V\right)\right\} = 2\kappa^2\rho\left(\Pi + \mathbf{v}^2 - 4U\right) \end{aligned} \tag{8.51}$$

For the case of the trace equation, Eq.(8.28) reduces to:

$$- \quad \tfrac{1}{2}\triangle g^{(4)}_{00} + 2V\left(\triangle U - 3\triangle V\right) + \partial_0\partial_0 U - 2\delta^{ij}\left(\partial_i U\right)\left(\partial_j U\right) - 2U\triangle U = 2\kappa^2\left\{\rho\left(\Pi - 2U\right) - 3p\right\} \tag{8.52}$$

PostPostNewtonian limit

The PPN limit for the $(0,0)$-field equation is obtained from Eq.(8.22) and can be written as:

$$\begin{aligned} &\tfrac{1}{2}\left(R^{(6)}_{00} + \delta^{ij}R^{(6)}_{ij}\right) - \delta^{ij}g^{(3)}_{0i}R^{(3)}_{0j} - \tfrac{1}{2}\triangle V\left[3\left(g^{(4)}_{00} + 4UV - 4V^2\right) + \delta^{ij}g^{(4)}_{ij}\right] + \\ &+\delta^{ij}\left[\partial_i U\partial_j(U-V) + 2\partial_i V\partial_j(3V+U)\right] + 8V\triangle V + 2U\triangle U - \\ &-(U+V)\left[\tfrac{1}{2}\left(6\partial_0\partial_0 V - \triangle g^{(4)}_{00} + 2\delta^{ij}\left(\triangle g^{(4)}_{ij} + \partial_0\partial_i g^{(3)}_{0j}\right) - 2\delta^{ij}\delta^{mn}\partial_i\partial_m g^{(4)}_{jn}\right)\right] + \\ + \quad f'(0) \quad &\left\{-\tfrac{1}{2}\mathcal{G}^{(6)} + (8V-7U)\left(\triangle U\right)^2 + U\left(\triangle V\right)^2 + 2(4V-3U)\triangle U\triangle V + 2\triangle U\left[\triangle g^{(4)}_{00} + 2\partial_0\partial_0(U+2V) - \right.\right. \\ &\left. -4\delta^{ij}\partial_i U\partial_j(U+V)\right] + \triangle V\left[\triangle g^{(4)}_{00} + 6\partial_0\partial_0(U+2V) + 4\delta^{ij}\left(\partial_0\partial_i g^{(3)}_{0j} + \partial_i U\partial_j(U-2V)\right)\right] + \\ &+\delta^{ij}\left[\left(\triangle g^{(3)}_{0i} + 4\partial_0\partial_i V\right)\triangle g^{(3)}_{0j} - 8\partial_0\partial_i V\partial_0\partial_j V\right] + 4\delta^{ij}\delta^{mn}\left[U\partial_i\partial_m(U+V)\partial_j\partial_n(U+V) - \right. \\ &-\partial_i\partial_m g^{(3)}_{0j}\partial_0\partial_n V + \tfrac{1}{2}\left[\partial_0\left(\partial_n g^{(3)}_{0j} + \partial_j g^{(3)}_{0n} - 2\delta_{jn}\partial_0 V\right) + \delta^{kl}\left(\partial_k\left(\partial_j g^{(4)}_{ln} + \partial_n g^{(4)}_{lj}\right) - \partial_j\partial_n g^{(4)}_{kl}\right) - \right. \\ &\left. - \triangle g^{(4)}_{jn} - \partial_j\partial_n g^{(4)}_{00}\right] - 2V\partial_j\partial_n V - (U+2V)\partial_j\partial_n U + \partial_j U\,\partial_n(U-V) - \partial_j V\,\partial_n(3V+U) - \\ &\left.\left. -\delta_{jn}\left(\delta^{kl}\partial_l V\partial_k(U+V) + 2V\triangle V\right)\right]\tfrac{1}{2}\delta^{ij}\delta^{mn}\delta^{kl}\partial_k\left(\partial_m g^{(3)}_{0i} + \partial_i g^{(3)}_{0m}\right)\partial_l\left(\partial_n g^{(3)}_{0j} + \partial_j g^{(3)}_{0n}\right)\right\} + \end{aligned}$$

$$
\begin{aligned}
+ \quad f''(0) \quad & \Big\{12\left(\triangle U+\triangle V\right)^2\triangle^2 U+4\left(3\triangle V-\triangle U\right)\left(\triangle U+\triangle V\right)\triangle^2 V+ \\
& +4\left(\triangle U+\triangle V\right)\delta^{ij}\left[3\partial_i\triangle U\partial_j\left(\triangle U+2\triangle V\right)-\partial_i\triangle V\partial_j\triangle V\right]+ \\
& +4\delta^{ij}\delta^{mn}\Big[\triangle\left(\partial_i\partial_m(U+V)\partial_j\partial_n(U+V)\right)+\partial_i\partial_m U\partial_j\partial_n\Big(-3\left(\triangle U\right)^2+\left(\triangle V\right)^2-6\triangle U\triangle V+ \\
& +4\delta^{kl}\delta^{rs}\partial_k\partial_r(U+V)\partial_l\partial_s(U+V)\Big)\Big]\Big\}= \\
= \quad 2\kappa^2 \quad & \Big\{\rho\Big[\mathbf{v}^4-3U\mathbf{v}^2+4U^2+2g^{(4)}_{00}+2g^{(3)}_{0i}v^i+\Pi(\mathbf{v}^2-4U)\Big]+p\left(\mathbf{v}^2-2U\right)\Big\}
\end{aligned}
\tag{8.53}
$$

And for the trace equation, Eq.(8.30) reduces to:

$$
\begin{aligned}
-R^{(6)}+2f''(0)\left(\triangle U-\triangle V\right)\triangle\Big\{-3\left(\triangle U\right)^2+\left(\triangle V\right)^2-6\left(\triangle U\right)\left(\triangle V\right)+4\delta^{im}\delta^{jn}\partial_i\partial_j\left(U+V\right)\,\partial_m\partial_n\left(U+V\right)\Big\}= \\
=2\kappa^2\Big\{\rho\Big[-\mathbf{v}^2\left(U+2V\right)+g^{(4)}_{00}+2g^{(3)}_{0i}v^i-2U\Pi\Big]-2Up\Big\}
\end{aligned}
\tag{8.54}
$$

Summarizing, the Newtonian limit of $f(\mathcal{G})$ modified gravity theories is given by Eq.(8.49) and Eq.(8.50); the post–Newtonian limit is given by Eq.(8.51) and Eq.(8.52); and the post–post–Newtonian limit is given by Eq.(8.53) and Eq.(8.54), respectively.

8.4 Conclusions

In this chapter, the weak field limit of $f(R,\mathcal{G})$ modified gravity has been considered in much detail. The importance of this analysis lies in the fact that it is mandatory, for any relativistic theory of gravity, to reproduce the extremely precise experimental results of the theory of General Relativity of Albert Einstein at this level. As a most ubiquitous example, any viable gravitational theory must pass the solar system tests.

The Newtonian, post–Newtonian and post–post–Newtonian limits of $f(R,\mathcal{G})$ modified gravity have been calculated and they have been applied to the special cases of $f(R)$ and $f(\mathcal{G})$ modified gravities respectively. The next step, for future works, could be compare these limits for viable $f(R,\mathcal{G})$ with those of General relativity and with the most accurate observational data.

In the case of the Newtonian limit of $f(R,\mathcal{G})$ modified gravity, a most convenient, general solution in terms of the Green's functions has been calculated.

Part IV

General conclusions and perspectives

This memory is the product of work done during the last four years. In it I will present the results obtained in the study I have carried out of some aspects related to different gravitational theories, focusing overall in the so–called $f(R)$ modified gravity. In this section, the conclusions obtained throughout the memory are summarized.

The first part of the memory was devoted to the reconstruction program of cosmological solutions for different gravitational theories. With the help of these tools, it has been shown that any cosmology, given by its scale factor or its Hubble parameter, can be reproduced in the framework of a theory of gravity. In first place, two well–known reconstruction schemes for $f(R)$ modified gravity were reviewed and, for the first time, the results obtained, by means of their use, are compared for the same example. These two reconstruction schemes are successfully extended to more realistic physical theories, the Yang–Mills theories and, as in the case of $f(R)$ modified gravity, they are compared. One of the reconstruction schemes is based on the use of an auxiliary scalar field while the other scheme does not require it. By developing the same example for both methods, it is clear that they do not give the same results; in the case of $f(R)$ modified gravity, the result obtained by using the reconstruction program with an auxiliary scalar field seems to be more general than the one obtained with the other method, but under some circumstances the results coming from both schemes have a similar behavior at low and large curvatures; in the case of Yang–Mills theory, two results are obtained by applying the reconstruction scheme with the auxiliary scalar field, one of these results coincides with the one obtained by using the other method. The non–equality between the results, given by the two methods, seems to support the point of view that the Einsteinian and the Jordanian frame descriptions are not equivalent, and actually lead to two physically different theories. This non–equivalence between the results obtained is due to the fact that the action with the auxiliary scalar field expresses a more extensive class of theories than the other action. Any cosmology can be realized in the framework of $f(R)$ modified gravity, or Yang–Mills theory, with the help of any of these reconstruction programs; even though, it is worth saying that the choice of one of the schemes over the other may greatly ease the task. In fact, there are examples in which there exist analytical solutions for one of the reconstruction schemes, but not for the other. In a future work, it would be of great interest to perform a numerical analysis, of several cosmological models, following the two different reconstruction schemes in order to further compare them and to provide more evidences that support the non–equivalence between the Einsteinian and Jordanian frame descriptions. After the comparison between the two methods, a reconstruction scheme is developed for a minimal gravitational coupling of the Yang–Mills field, which includes second– and fourth–order terms of the Yang–Mills field strength tensor, in the general theory of relativity. This kind of theories is usually more complicated than the one considered before and analytical results are hard to find, but it is shown that de Sitter solutions exist only in the trivial case, i.e. when the arbitrary function of the Yang–Mills field, that appears in the action of the theory, is a constant. In order to find out a non–trivial de Sitter solution, the case of a non–minimal gravitational coupling of the Yang–Mills field is considered. The reconstruction scheme is developed for this kind of theories and the equation that must be satisfied in order to obtain the desired de Sitter solutions is shown. This equation includes an arbitrary parameter and, depending on its value, it is demonstrated that the equation can be easily solved in quadratures or, in the most general case, numerically. The performance of the numerical analysis for this model could be interesting for a future work. In order to finish the study of the reconstruction schemes for the gravitational theories, the case of cyclic universes in the framework of $f(R)$ Hořava–Lifshitz gravity is also considered. The Hořava–Lifshitz gravity is a power–counting renormalizable gravitational theory introduced by Hořava, in which the invariance under full diffeomorphisms of General Relativity is broken by introducing an anisotropy between the spatial and time coordinates through a critical exponent z. In this work, a modification of this theory, achieved by changing the scalar curvature for a generic function of it, is considered in order to reproduce cyclic universes. The two reconstruction schemes reviewed for the cases of $f(R)$–gravity and Yang–Mills theory are now used to realize cosmologies with a cyclic behavior. In the case of the reconstruction program in terms of e–folding, the final equation that gives us the function $f(\tilde{R})$, for an example of a cyclic universe is shown, but the solution is not analytic. Instead, a solution of a cyclic universe is found by using the reconstruction scheme with an auxiliary scalar field. As in the previous cases of $f(R)$ gravities and Yang–Mills theories, the numerical analysis of the cosmological models is of great importance for future works in order to be able to compare the results obtained for cyclic universes by using both reconstruction methods. In addition, the shape of the action along each phase of a typical ekpyrotic universe is studied.

After the first part, devoted to the study and development of the reconstruction program for some gravitational theories, the memory of the thesis goes on with a second block dedicated to the detailed analysis of the cosmic history and growth of the matter density perturbations for some viable $F(R)$ modified gravities. In a first place, two viable $F(R)$ models, one exponential and one power–form, are studied. A matter domination era is checked to take place in the framework of both models, but the large frequency of the oscillations of dark energy, which appear in this epoch, together with the stability conditions imposed, cause a divergence in the high derivatives of the Hubble parameter. This problem grows worse at high redshift because the frequency of the dark energy oscillations increases with this parameter. To avoid this problem, a corrective term is added to the models in order to stabilize the frequency of the oscillations. It is demonstrated that this new term does not cause any problem to the viability of the models. The new models are checked to be in agreement with the observational data and they easily pass the local test of the solar system. The cosmological future of the two viable $F(R)$ gravities is also studied and the crossing of the phantom divide, which characterizes the de Sitter epoch, is demonstrated to take place in a very far future. It is also proposed that the adding of inhomogeneous fluids to the models may avoid this crossing. It is very important to remark that all the results obtained in an analytical way, using the perturbation theory, are in agreement with the numerical analysis performed on the two models. After the analysis of the expansion history is done, the study of the growth of matter density perturbations is performed for the two viable $F(R)$ modified gravities. The fact that different gravitational theories can exhibit very similar cosmic history can make it very difficult to discriminate among them. In this sense, the study of the growth of matter density perturbations can provide a significant tool in order to distinguish among the different theories. In a first step, the growth rate for the two $F(R)$ models considered is obtained via numerical analysis. Even if the growth rate can always be obtained, at least in a numerical way, for every gravitational model, it is clear that, in order to discriminate among theories, the growth rate is not very useful. In this sense, one way of characterizing the growth of matter density perturbations can be the so–called growth index. A detailed analysis of several ansätze for the growth index is performed for the two viable models, concluding that the choice of the growth index with a linear dependence with the redshift is the most appropriate parameterization for the models considered. To end with the first part of the block dedicated to the cosmic and growth histories, two new viable exponential models, which can unify early– and late–time cosmic acceleration, are proposed and a detailed analysis of inflation is done for them. It is demonstrated that the number of e–folds obtained for the two models depends on the model parameters in the presence of ultrarelativistic matter, whose existence makes inflation end. A numerical analysis of inflation is performed for both exponential models. Thus, an unified description of early– and late–time acceleration is built in the framework of $F(R)$ modified gravity, but these different periods of the cosmic history has been studied one by one; for a future work, it would be of great importance to obtain the evolution equation expressing all the processes from inflation to the current cosmic acceleration. To finish this part of the memory, the analysis of the growth of matter density perturbations is performed for two new viable $F(R)$ modified gravities, which parameters are set in order to be in agreement with the last observational data. The growth rate is obtained for both models in a numerical way and the same parameterizations of the growth index proposed for the previous models are studied for the new theories. As it happened for the previous viable models, the best choice for the parameterization of the growth index is the one with a linear dependence with respect to the redshift. Concerning future works, it would be of great interest to study the different parameterizations of the growth index for more realistic modified gravity models and other dark energy theories in order to determine the best parameterization and to generalize its use for the characterization of the growth of the matter density perturbations in cosmological theories.

The first part of the last block of the memory is dedicated to the appearance of future finite–time singularities in some dark energy models. The singularity problem is of fundamental importance in modern cosmology, but a quantum theory of gravity is needed in order to address this issue rigorously. Even if we do not have it up to now, it is also important, even at classical or semiclassical level, to try to find a way to cure the emergence of this kind of singularities. In this sense, it is known that the addition of an R^2 –term in the action cures the possible appearance of all future finite–time singularities. In a first step, a particular fluid with a given equation of state with one free parameter is considered. It is demonstrated that the four types of future singularities can be realized with this fluid, depending on the value of the parameter in the equation of state. For each special case, a function depending on the Hubble parameter and its derivatives is found in order to avoid the appearance of the singularity. As it is demonstrated in the literature, this function can be interpreted as a contribution of modified gravity. It is also shown that, with the help of modified gravity, a function of the Hubble parameter and its derivatives can be found in order to cure all the singularities of the considered fluid with a given equation of state. The next case considered is an $f(R)$ modified gravity non-minimally coupled to matter–like Lagrangians. It is shown that, for this kind of theories, the appearance of a singularity of type II is avoided. It is also considered an example of this kind of theories that leads to a future finite–time singularity. The differential equation for $f(R)$, which gives rise to this singularity, is given explicitly. Even if this differential equation has an analytical solution, it is not written here because it is too long. The case of non–local gravity is also considered and this part of the memory finishes with a study of isotropic turbulence in the dark fluid universe. It is demonstrated that the contribution of the turbulent part of dark energy can indeed be reproduced through the use of a scalar-tensor theory and several examples are developed in this framework, showing that future finite–time singularities can appear in this theory. An important conclusion is that, even in the absence of a quantum theory of gravity, an R^2 –term would, in principle, cure the singularities. Concerning future perspectives, it is clear that a more fully–fledged theory of quantum gravity is necessary in order to address the singularity problem. In any case, the fact that future finite–time singularities can cause various problems, as instabilities in current black hole and stellar astrophysics, suggests that finding natural scenarios, even at classical and semiclassical level, that cure this kind of singularities, is of fundamental importance. After the analysis of the singularity problem, the second part of the last block is devoted to the study of the weak field limit of $f(R,\mathcal{G})$ modified gravity. This analysis is of great importance because it is mandatory for any gravitational theory to reproduce the results obtained with General Relativity at short scales. The Newtonian, post–Newtonian and post–post–Newtonian limits of $f(R,\mathcal{G})$ modified gravity are calculated and they are applied to the special cases of $f(R)$ and $f(\mathcal{G})$ modified gravities respectively. In the case of the Newtonian limit of $f(R,\mathcal{G})$ modified gravity, a most convenient, general solution in terms of the Green's functions is calculated. For future work, it would be interesting to consider viable $f(R,\mathcal{G})$ theories and compare their limits with the corresponding ones in General Relativity. The differences could be relevant in order to discriminate some of these theories by invoking the accurate cosmological data to be obtained in ongoing and future sky surveys.

Appendix A

Equations in the FLRW metric for non–minimal gravitational coupling with the Yang–Mills field

In the FLRW spatially flat space–time

$$\begin{aligned}\Gamma^0_{ij} = Ha^2\delta_{ij}, \quad \Gamma^i_{j0} = H\delta^i_j, \quad \Gamma^\mu_{00} = 0, \quad \Gamma^0_{\mu 0} = 0,\\ \Box = -\partial_0\partial_0 - 3H\partial_0 + \frac{1}{a(t)^2}\left(\partial_1\partial_1 + \partial_2\partial_2 + \partial_3\partial_3\right),\end{aligned} \tag{A.1}$$

$$R_{00} = -3\left(\dot{H} + H^2\right), \quad R_{0i} = 0\,, \quad R_{ij} = \left(\dot{H} + 3H^2\right)g_{ij}\,, \quad R = 6\left(\dot{H} + 2H^2\right). \tag{A.2}$$

Using the field A^b in the form (3.41), we get two independent equations. The first one reads:

$$R_{00} - \frac{1}{2}Rg_{00} = 3H^2 = \frac{1}{M_P^2}\left(T^{(\mathrm{YM})}_{00} + \Lambda\right). \tag{A.3}$$

For any twice-differentiable functions $f(R)$ and $\mathcal{W}(t)$, we get

$$\Box\left[f'(R)\mathcal{W}\right] = -f'''(R)\dot{R}^2\mathcal{W} - 2f''(R)\ddot{R}\dot{\mathcal{W}} - f'(R)\ddot{\mathcal{W}} - 3H\left[f''(R)\dot{R}\mathcal{W} + f'(R)\dot{\mathcal{W}}\right],$$

$$g_{00}\Box\left[f'(R)\mathcal{W}\right] - \nabla_0\partial_0\left[f'(R)\mathcal{W}\right] = 3H\left[f''(R)\dot{R}\mathcal{W} + f'(R)\dot{\mathcal{W}}\right].$$

Using

$$F^b_{\beta 0}F^b_{\alpha 0}g^{\alpha\beta} = F^b_{i0}F^b_{i0}g^{ii} = 3\frac{\dot{\phi}^2}{a^2}, \quad g^{\alpha\beta}F^b_{0\beta}F^b_{0\alpha} - \frac{1}{4}g_{00}\mathcal{F} = \frac{3}{2}\left(\frac{\dot{\phi}^2}{a^2} + \frac{\tilde{g}^2\phi^4}{a^4}\right),$$

we obtain that Eq. (A.3) is equivalent to

$$\begin{aligned}2M_P^2H^2 = (1 + f(R))\left(\frac{\dot{\phi}^2}{a^2} + \frac{\tilde{g}^2\phi^4}{a^4}\right) - 6\left(\dot{H} + H^2\right)f'(R)\left(\frac{\tilde{g}^2\phi^4}{a^4} - \frac{\dot{\phi}^2}{a^2}\right) + \frac{2\Lambda}{3} +\\ + 6H\left(\dot{R}f''(R)\left(\frac{\tilde{g}^2\phi^4}{a^4} - \frac{\dot{\phi}^2}{a^2}\right) + 2f'(R)\left(\frac{2\tilde{g}^2\phi^3(\dot{\phi} - H\phi)}{a^4} - \frac{\dot{\phi}(\ddot{\phi} - H\dot{\phi})}{a^2}\right)\right).\end{aligned} \tag{A.4}$$

The second equation reads:

$$R_{ii}-\frac{1}{2}Rg_{ii}=\,-\,g_{ii}\left(2\dot{H}+3H^2\right)=\frac{1}{M_P^2}\left(T_{ii}^{(\mathrm{YM})}+T_{ii}^{(4)}\right). \tag{A.5}$$

To calculate $T_{ii}^{(\mathrm{YM})}$ we use the following formulae (no summation over i)

$$F^b_{\beta i}F^b_{\alpha i}g^{\alpha\beta}=F^b_{0i}F^b_{0i}g^{00}+F^b_{ji}F^b_{ji}g^{jj}\,=\,-\,\dot{\phi}^2+2\tilde{g}^2\frac{\phi^4}{a^2}, \tag{A.6}$$

$$g^{\alpha\beta}F^b_{i\beta}F^b_{i\alpha}-\frac{1}{4}g_{ii}\mathcal{F}=\frac{1}{2}\left(\frac{\tilde{g}^2\phi^4}{a^2}+\dot{\phi}^2\right), \tag{A.7}$$

and get (A.5) in the following form

$$\begin{aligned}-\,2\dot{H}-3H^2=\frac{1}{2M_P^2}\Bigg[&(1+f(R))\left(\frac{\tilde{g}^2\phi^4}{a^4}+\frac{\dot{\phi}^2}{a^2}\right)-2\Lambda+\\ &+6(\dot{H}+3H^2)f'(R)\left(\frac{\tilde{g}^2\phi^4}{a^4}-\frac{\dot{\phi}^2}{a^2}\right)-6[\partial_0\partial_0+2H\partial_0]\left(f'(R)\left(\frac{\tilde{g}^2\phi^4}{a^4}-\frac{\dot{\phi}^2}{a^2}\right)\right)\Bigg].\end{aligned} \tag{A.8}$$

Using (3.45), we rewrite Eqs. (A.4) and (A.8) in terms of $\psi(t)$, to get (3.57) and (3.58).

Appendix B

Conformal transformation of the exponential model for inflation

In several cases, a suitable conformal frame to study inflation may be the so-called "Einstein frame". An $F(R)$ gravity theory can be rewritten in the scalar field theory form via the conformal transformation. We can rewrite the action in Eq. (5.1) by introducing a scalar field which couples to the curvature. Of course, this is not exactly physically-equivalent formulation, but the formulation in the Einstein frame may be used to obtain some of intermediate results in simpler form (especially, the case that the matter is not taken into account).

We introduce a scalar field A into the action

$$I_{\rm JF} = \frac{1}{2\kappa^2}\int_{\mathcal{M}} \sqrt{-g}\,[F'(A)\,(R-A)+F(A)]\,d^4x\,. \tag{B.1}$$

Here, the subscript "JF" means "the Jordan frame" and we neglect the contribute of matter. By making the variation of the action with respect to A, we have $A=R$. We define the scalar field σ as

$$\sigma = -\frac{\sqrt{3}}{\sqrt{2\kappa^2}}\ln[F'(A)]\,. \tag{B.2}$$

We make the conformal transformation of the metric

$$\tilde{g}_{\mu\nu} = \mathrm{e}^{-\sigma} g_{\mu\nu}\,, \tag{B.3}$$

for which we acquire the "Einstein frame" (EF) action of the scalar field σ [125, 189]

$$\begin{aligned} I_{\rm EF} &= \int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\left\{\frac{\tilde{R}}{2\kappa^2}-\frac{1}{2}\left(\frac{F''(A)}{F'(A)}\right)^2\tilde{g}^{\mu\nu}\partial_\mu A\partial_\nu A-\frac{1}{2\kappa^2}\left(\frac{A}{F'(A)}+\frac{F(A)}{F'(A)^2}\right)\right\}\\ &= \int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\left(\frac{\tilde{R}}{2\kappa^2}-\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma+V(\sigma)\right)\,, \end{aligned} \tag{B.4}$$

where

$$V(\sigma)\equiv -\frac{1}{2\kappa^2}\left(\frac{A}{F'(A)}-\frac{F(A)}{F'(A)^2}\right) = -\frac{1}{2\kappa^2}\left\{\mathrm{e}^{\sigma}R(\mathrm{e}^{-\sigma})-\mathrm{e}^{2\sigma}F[R(\mathrm{e}^{-\sigma})]\right\}\,. \tag{B.5}$$

Here, $R(\mathrm{e}^{-\sigma})$ is the solution of Eq. (B.2) with $A = R$, becoming R a function of $\mathrm{e}^{-\sigma}$, and $\tilde{R}$ denotes the Ricci scalar evaluated with respect to the conformal metric $\tilde{g}_{\mu\nu}$. Furthermore, $\tilde{g} = \mathrm{e}^{-4\sigma} g$ is the determinant of conformal metric.

As an example, we explore our unified model (5.66) with $\gamma = 1$. Since we are interested in the de Sitter solution, we take $\exp[-(R/R_{\mathrm{i}})^n] \to 0$ and neglect the cosmological constant Λ. In this case the potential $V(\sigma)$ reads

$$V(\sigma) = -\frac{1}{2\kappa^2}\left[\tilde{R}\left(\frac{\mathrm{e}^{-\tilde{\sigma}}-1}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(\mathrm{e}^{\tilde{\sigma}} - 2\mathrm{e}^{2\tilde{\sigma}}\right) + \Lambda_{\mathrm{i}}\,\mathrm{e}^{2\tilde{\sigma}}\right]. \tag{B.6}$$

According with Sec. 5.4.1, we put $\tilde{R}_{\mathrm{i}} = R_{\mathrm{dS}}$. It is clearly seen that for $R = R_{\mathrm{dS}}$, $\sigma_{\mathrm{dS}} = -\sqrt{3/(2\kappa^2)}\log(1+\alpha)$ and $V'(\sigma_{\mathrm{dS}}) = 0$, where the prime denotes the derivative with respect to the inflation field σ. Since $V''(\sigma_{\mathrm{dS}}) > 0$, the scalar potential has a minimum, that is a necessary condition for a slow-roll inflation. For slow-roll parameters, we have to require

$$\begin{aligned} \epsilon(\sigma) &= \frac{1}{2\kappa^2}\left(\frac{V'(\sigma)}{V(\sigma)}\right)^2 \ll 1\,, \\ |\eta(\sigma)| &= \frac{1}{\kappa^2}\left|\frac{V''(\sigma)}{V(\sigma)}\right| \ll 1\,. \end{aligned} \tag{B.7}$$

By defining the energy density and pressure of σ as $\rho_\sigma = \dot{\sigma}^2/2 - V(\sigma)$ and $P_\sigma = \dot{\sigma}^2/2 + V(\sigma)$, these conditions imply that the gravitational field equations in the flat FLRW space-time are given by $3H^2/\kappa^2 = -V(\sigma)$, $3H\dot{\sigma} \simeq -V'(\sigma)$, and that $\ddot{a}(t) > 0$, and hence guarantee a sufficiently long time inflation. In our case, since $V(\sigma_{\mathrm{dS}}) \neq 0$, these two conditions are well satisfied around the de Sitter solution. Thus, since $\dot{\sigma} \simeq 0$, we find $H_{\mathrm{dS}} = R_{\mathrm{dS}}/\,[12(1+\alpha)] = \tilde{R}_{\mathrm{dS}}/12$.

Appendix C

Asymptotically phantom or quintessence modified gravity

In general, realistic models of modified gravity are similar to GR with the cosmological constant, i.e., the dark energy fluid with the EoS parameter $\omega_{\rm DE} = -1$ and the de Sitter universe as the final scenario for the cosmological evolution. Since in principle quintessence/phantom-dark energy phases are not excluded by observations, it may be of some interest to try to reconstruct an $F(R)$ gravity theory where the quintessence or phantom dark energy (with a constant $\omega_{\rm DE}$) emerges. The big difficulty is due to the fact that in the dark energy density $\rho_{\rm DE}$ and pressure $P_{\rm DE}$ of modified gravity, the effective gravitational terms appear. In this appendix, we reconstruct the form of $F(R)$ gravity which resembles to a fluid with $\omega_{\rm DE}$ being very close but not equal to -1 .

If the energy density of a quintessence/phantom fluid is given by

$$\rho = \rho_0 (z+1)^{3(1+\omega)} \,, \tag{C.1}$$

where ω is the EoS parameter, the Hubble parameter reads

$$H(z) = \sqrt{\frac{\kappa^2}{3}\rho} \simeq \sqrt{\frac{\kappa^2\rho_0}{3}} + \frac{1}{2}\sqrt{3\kappa^2\rho_0}(1+\omega)\log[z+1] \,. \tag{C.2}$$

Here, we have taken into account that ω is very close to -1 . We can write R as a function of the redshift as

$$R(z) = \frac{1}{2}\kappa^2\rho_0 \left[2 + 3(1+\omega)\log(z+1)\right]\left[1 - 3\omega + 6(1+\omega)\log(z+1)\right] \,. \tag{C.3}$$

In addition, from Eq. (5.7), in vacuum we find

$$\begin{aligned} \rho_{\rm eff} \;\equiv\; \rho_{\rm DE} = \frac{1}{2\kappa^2}\Bigg\{ & \left[\left(\frac{dR(z)}{dz}\right)^{-1}\frac{dF(z)}{dz}R(z) - F(z)\right] - 6H^2(z)\left[\left(\frac{dR(z)}{dz}\right)^{-1}\frac{dF(z)}{dz} - 1\right] \\ & + 6H^2(z)(z+1)\frac{dR(z)}{dz}\left[\left(\frac{d^2R(z)}{dz^2}\right)^{-1}\frac{dF(z)}{dz} + \left(\frac{dR(z)}{dz}\right)^{-2}\frac{d^2F(z)}{dz^2}\right]\Bigg\} \,. \end{aligned} \tag{C.4}$$

Here, $F(R)$ model is expressed as a function of the redshift $F(z)$. By equating $\rho_{\rm eff}$ to ρ of Eq. (C.1), we can find the $F(R)$ model realizing this cosmology. For $|\omega - 1| \ll 0$, the solution of Eq. (C.4) is given by

$$F(z) \simeq \frac{6\kappa^2\rho_0\left[11+(34-9\omega)\omega\right]}{(5-3\omega)^2} - 6\kappa^2\rho_0(1+\omega)\log(z+1)\,, \tag{C.5}$$

From Eq. (C.3), we have

$$z = -1 + \exp\left\{\frac{\rho_0\kappa^2(5-3\omega) \pm (1+\omega)\sqrt{\rho_0\kappa^2\left[16R + 9\rho_0\kappa^2(1+\omega)^2\right]}}{12\rho_0\kappa^2(1+\omega)}\right\}\,, \tag{C.6}$$

where the plus sign corresponds to the quintessence solution, whereas the minus sign does to the phantom one. We can now write the modified gravity model as a function of the Ricci scalar as

$$F(R) = \frac{\rho_0\kappa^2(257+183\omega+27\omega^2-27\omega^3)}{2(5-3\omega)^2} \pm \frac{\sqrt{\rho_0\kappa^2\left[16R+9\rho_0\kappa^2(1+\omega)^2\right]}}{2}\,, \tag{C.7}$$

where ρ_0 is a free parameter of the theory, and ω is the EoS parameter of dark energy coming from the modification of gravity and equivalent to ω_{DE}. In this way, we have reconstructed the form of $F(R)$ gravity that gives the quintessence or phantom fluid solution in the empty universe. Remind that this reconstruction is valid for ω_{DE} close to -1.

Appendix D

Friedmann equations for modified gravity non-minimally coupled to the matter Lagrangian

In this appendix we carry out, in some detail, the calculations which are necessary to obtain the Friedmann equations for the Lagrangian density of Eq. (7.27). Assuming a spatially-flat FRW universe and taking into account Eqs. (7.33) and (7.34), Eq. (t,t) from Eq. (7.28) reads

$$\frac{3H(t)^2}{\kappa^2} + 3C^2\left(\dot{H}(t) + H(t)^2\right)\frac{f'(R)}{a(t)^6 f(R)^2} - 3C^2 H(t)\partial_t\left(\frac{f'(R)}{a(t)^6 f(R)^2}\right) + \frac{1}{2}\frac{C^2}{a(t)^6 f(R)} = 0, \tag{D.1}$$

and Eq. (i,i) from Eq. (7.28),

$$\frac{2\dot{H}(t) + 3H(t)^2}{\kappa^2} + C^2\left(\dot{H}(t) + 3H(t)^2\right)\frac{f'(R)}{a(t)^6 f(R)^2} - 2C^2 H(t)\partial_t\left(\frac{f'(R)}{a(t)^6 f(R)^2}\right) -$$
$$-C^2\partial_t\partial_t\left(\frac{f'(R)}{a(t)^6 f(R)^2}\right) - \frac{1}{2}\frac{C^2}{a(t)^6 f(R)} = 0. \tag{D.2}$$

Taking derivatives with respect to time, one gets

$$\partial_t\left(\frac{f'(R)}{a(t)^6 f(R)^2}\right) = \frac{1}{a(t)^6 f(R)^2}\left(f''(R)\dot{R} - 6f'(R)H(t) - \frac{2f'(R)^2\dot{R}}{f(R)}\right) \tag{D.3}$$

and

$$\partial_t\partial_t\left(\frac{f'(R)}{a(t)^6 f(R)^2}\right) = \frac{1}{a(t)^6 f(R)^2}\left[6f'(R)\left(6H(t)^2 - \dot{H}(t)\right) + f''(R)\left(\ddot{R} - 12\dot{R}H(t)\right) +\right.$$
$$\left. + f'''(R)\dot{R}^2 + 2f'(R)^2\left(\frac{12\dot{R}H(t) - \ddot{R}}{f(R)}\right) + 6f'(R)^3\frac{\dot{R}^2}{f(R)^2} - 6f'(R)f''(R)\frac{\dot{R}^2}{f(R)}\right]. \tag{D.4}$$

Introducing Eq. (D.3) into Eq. (D.1) and taking into account that

$$a(t) = a_0 \exp\left[\int_{t_0}^{t} H(t')dt'\right]$$

yields

$$\frac{1}{6}f(R) + \frac{a_0^6 H(t)^2 \exp\left[6\int_{t_0}^{t} H(t')dt'\right]}{\kappa^2 C^2} f(R)^2 + \left(\dot{H}(t) + 7H(t)^2\right) f'(R) + + \frac{2H(t)\dot{R}}{f(R)} f'(R)^2 - \dot{R}H(t)f''(R) = 0, \tag{D.5}$$

and putting Eqs. (D.3) and (D.4) into Eq. (D.2), this reduces to

$$-\frac{1}{2}f(R) + \frac{2\dot{H}(t)+3H(t)^2}{\kappa^2 C^2} a_0^6 \exp\left[6\int_{t_0}^{t} H(t')dt'\right] f(R)^2 + 7\left(\dot{H}(t) - 3H(t)^2\right) f'(R) + +2\left(\frac{\ddot{R} - 10\dot{R}H(t)}{f(R)}\right) f'(R)^2 - 6\frac{\dot{R}^2}{f(R)^2} f'(R)^3 + \left(10\dot{R}H(t) - \ddot{R}\right) f''(R) + +6\frac{\dot{R}^2}{f(R)} f'(R)f''(R) - \dot{R}^2 f'''(R) = 0. \tag{D.6}$$

The other possibilities for Eq. (7.28) are identities.

We also know that $R = 6\dot{H}(t) + 12H(t)^2$, which could be solved in terms of t as $t = t(R)$. Taking this into account, Eq. (D.5) can be written as

$$\frac{1}{6}f(R) + \frac{a_0^6 H(t(R))^2 \exp\left[6\int_{t_0}^{t(R)} H(t')dt'\right]}{\kappa^2 C^2} f(R)^2 + \left[\dot{H}(t(R)) + 7H(t(R))^2\right]\frac{df(R)}{dR} + + \frac{12H(t(R))\left[\ddot{H}(t(R)) + 4\dot{H}(t(R))H(t(R))\right]}{f(R)}\left(\frac{df(R)}{dR}\right)^2 - -6\left[\ddot{H}(t(R)) + 4H(t(R))\dot{H}(t(R))\right] H(t(R))\frac{d^2 f(R)}{dR^2} = 0, \tag{D.7}$$

and Eq. (D.6) as

$$-\frac{1}{2}f(R) + \frac{2\dot{H}(t(R)) + 3H(t(R))^2}{\kappa^2 C^2} a_0^6 \exp\left[6\int_{t_0}^{t(R)} H(t')dt'\right] f(R)^2 + 7\left[\dot{H}(t(R)) - 3H(t(R))^2\right]\frac{df(R)}{dR} + +12\frac{-40H(t(R))^2\dot{H}(t(R)) + 4\dot{H}(t(R))^2 - 6H(t(R))\ddot{H}(t(R)) + \dddot{H}(t(R))}{f(R)}\left(\frac{df(R)}{dR}\right)^2 - -216\frac{\left[\ddot{H}(t(R)) + 4H(t(R))\dot{H}(t(R))\right]^2}{f(R)^2}\left(\frac{df(R)}{dR}\right)^3 - -6\left[-40H(t(R))^2\dot{H}(t(R)) + 4\dot{H}(t(R))^2 - 6H(t(R))\ddot{H}(t(R)) + \dddot{H}(t(R))\right]\frac{d^2 f(R)}{dR^2} + +216\frac{\left[\ddot{H}(t(R)) + 4H(t(R))\dot{H}(t(R))\right]^2}{f(R)}\frac{df(R)}{dR}\frac{d^2 f(R)}{dR^2} - 36\left[\ddot{H}(t(R)) + 4H(t(R))\dot{H}(t(R))\right]^2\frac{d^3 f(R)}{dR^3} = 0. \tag{D.8}$$

Eqs. (D.7) and (D.8) are the Friedmann equations for the Lagrangian density given by (7.27) and constitute the two differential equations we were looking for $f(R)$.

Appendix E

Calculations needed for the PostNewtonian and PostPostNewtonian limits

In this Appendix, the different calculations needed to write the PostNewtonian and PostPostNewtonian limits for the $(0,0)$-field equation and the trace equation are presented.

E.1 The PostNewtonian approximation

I present now the calculations needed to obtain the PN limit for the field equations and the trace equation.

First, I write some calculations that will be needed after:

$$\begin{aligned}
\left(\nabla^2 f^*(R,\mathcal{G})\right)^{(2)} &= g^{(0)\,ij} f^*_R(0,0)\partial_i\partial_j R^{(2)} \\
\left(\nabla^2 f^*(R,\mathcal{G})\right)^{(4)} &= g^{(0)\,00}\left\{f^*_R(0,0)\partial_0\partial_0 R^{(2)} - \Gamma^{(2)\,i}{}_{00} f^*_R(0,0)\partial_i R^{(2)}\right\} + \\
&\quad + g^{(0)\,ij}\left\{f^*_{RR}(0,0)\partial_i R^{(2)}\partial_j R^{(2)} + f^*_R(0,0)\partial_i\partial_j R^{(4)} + f^*_{RR}(0,0)R^{(2)}\partial_i\partial_j R^{(2)} + \right. \\
&\quad \left. + f^*_{\mathcal{G}}(0,0)\partial_i\partial_j\mathcal{G}^{(4)} - \Gamma^{(2)\,k}{}_{ij} f^*_R(0,0)\partial_k R^{(2)}\right\} + g^{(2)\,ij} f^*_R(0,0)\partial_i\partial_j R^{(2)} \\
\left(\nabla^2 f^*(R,\mathcal{G})\right)^{(6)} &= g^{(0)\,00}\left\{f^*_{RR}(0,0)\partial_0 R^{(2)}\partial_0 R^{(2)} + f^*_R(0,0)\partial_0\partial_0 R^{(4)} + f^*_{RR}(0,0)R^{(2)}\partial_0\partial_0 R^{(2)} + \right. \\
&\quad + f^*_{\mathcal{G}}(0,0)\partial_0\partial_0\mathcal{G}^{(4)} - \Gamma^{(3)\,0}{}_{00} f^*_R(0,0)\partial_0 R^{(2)} - \Gamma^{(4)\,i}{}_{00} f^*_R(0,0)\partial_i R^{(2)} - \\
&\quad \left. - \Gamma^{(2)\,i}{}_{00}\left[f^*_R(0,0)\partial_i R^{(4)} + f^*_{RR}(0,0)R^{(2)}\partial_i R^{(2)} + f^*_{\mathcal{G}}(0,0)\partial_i\mathcal{G}^{(4)}\right]\right\} +
\end{aligned}$$

$$+g^{(2)\,00}\left\{f^*_R(0,0)\partial_0\partial_0R^{(2)}-\Gamma^{(2)\,i}_{\ \ 00}f^*_R(0,0)\partial_iR^{(2)}\right\}+$$

$$+2g^{(3)\,0i}f^*_R(0,0)\partial_0\partial_iR^{(2)}+$$

$$+g^{(0)\,ij}\left\{2f^*_{RR}(0,0)\partial_iR^{(2)}\partial_jR^{(4)}+f^*_{RRR}(0,0)R^{(2)}\partial_iR^{(2)}\partial_jR^{(2)}+2f^*_{R\mathcal{G}}(0,0)\partial_iR^{(2)}\partial_j\mathcal{G}^{(4)}+\right.$$

$$+f^*_R(0,0)\partial_i\partial_jR^{(6)}+f^*_{RR}(0,0)R^{(2)}\partial_i\partial_jR^{(4)}+$$

$$+\left[\tfrac{1}{2}f^*_{RRR}(0,0)R^{(2)\,2}+f^*_{RR}(0,0)R^{(4)}+f^*_{R\mathcal{G}}(0,0)\mathcal{G}^{(4)}\right]\partial_i\partial_jR^{(2)}+$$

$$+f^*_{\mathcal{G}}(0,0)\partial_i\partial_j\mathcal{G}^{(6)}+f^*_{R\mathcal{G}}(0,0)R^{(2)}\partial_i\partial_j\mathcal{G}^{(4)}-\Gamma^{(3)\,0}_{\ \ ij}f^*_R(0,0)\partial_0R^{(2)}-$$

$$-\Gamma^{(2)\,k}_{\ \ ij}\left[f^*_R(0,0)\partial_kR^{(4)}+f^*_{RR}(0,0)R^{(2)}\partial_kR^{(2)}+f^*_{\mathcal{G}}(0,0)\partial_k\mathcal{G}^{(4)}\right]-$$

$$\left.-\Gamma^{(4)\,k}_{\ \ ij}f^*_R(0,0)\partial_kR^{(2)}\right\}+$$

$$+g^{(2)\,ij}\left\{f^*_{RR}(0,0)\partial_iR^{(2)}\partial_jR^{(2)}+f^*_R(0,0)\partial_i\partial_jR^{(4)}+f^*_{RR}(0,0)R^{(2)}\partial_i\partial_jR^{(2)}+\right.$$

$$\left.+f^*_{\mathcal{G}}(0,0)\partial_i\partial_j\mathcal{G}^{(4)}-\Gamma^{(2)\,k}_{\ \ ij}f^*_R(0,0)\partial_kR^{(2)}\right\}+$$

$$+g^{(4)\,ij}f^*_R(0,0)\partial_i\partial_jR^{(2)}$$

$$\left(R_{\alpha\beta\rho0}R^{\alpha\beta\rho}{}_0\right)^{(4)} = 2\delta^{ij}\delta^{mn}\left(\partial_i\partial_mU\right)\left(\partial_j\partial_nU\right)$$

$$\left(R_{\alpha\beta\rho0}R^{\alpha\beta\rho}{}_0\right)^{(6)} = 4\triangle U\left(\partial_0\partial_0V+\delta^{ij}\partial_iU\,\partial_jV\right)+2\delta^{ij}\delta^{mn}\partial_i\partial_mU\left[\partial_0\partial_jg^{(3)}_{0n}+\partial_0\partial_ng^{(3)}_{0j}-\partial_j\partial_ng^{(4)}_{00}+\right.$$

$$\left.+2\partial_jU\,\partial_n(U-V)-2\partial_nU\,\partial_jV+(U-2V)\partial_j\partial_nU\right]-$$

$$-\tfrac{1}{4}\delta^{ij}\delta^{mn}\delta^{kl}\partial_k\left(\partial_mg^{(3)}_{0i}-\partial_ig^{(3)}_{0m}\right)\partial_l\left(\partial_ng^{(3)}_{0j}-\partial_jg^{(3)}_{0n}\right)-$$

$$-2\delta^{ij}\delta^{mn}\partial_i\partial_mg^{(3)}_{0j}\partial_0\partial_nV+2\delta^{ij}\partial_0\partial_iV\left(\triangle g^{(3)}_{0j}-2\partial_0\partial_jV\right)$$

$$\left(R_{0\rho0\sigma}R^{\rho\sigma}\right)^{(4)} = -\left(\triangle U\right)\left(\triangle V\right)$$

$$\left(R_{0\rho0\sigma}R^{\rho\sigma}\right)^{(6)} = \triangle V\left\{\tfrac{1}{2}\triangle g^{(4)}_{00}-3\,\partial_0\partial_0V+4V\triangle U-\delta^{ij}\left[\tfrac{1}{2}\partial_0\left(\partial_ig^{(3)}_{0j}+\partial_jg^{(3)}_{0i}\right)+2\partial_iU\,\partial_j(2U-V)\right]\right\}+$$

$$+\delta^{ij}\delta^{mn}\partial_i\partial_mU\,\left\{\tfrac{1}{2}\left[\partial_0\left(\partial_ng^{(3)}_{0j}+\partial_jg^{(3)}_{0n}+2\delta_{jn}\partial_0V\right)-\right.\right.$$

$$\left.-\delta^{kl}\left(\partial_k\left(\partial_jg^{(4)}_{ln}+\partial_ng^{(4)}_{lj}\right)-\partial_j\partial_ng^{(4)}_{kl}\right)+\triangle g^{(4)}_{jn}-\partial_j\partial_ng^{(4)}_{00}\right]+2V\partial_j\partial_nV+2U\partial_j\partial_nU+$$

$$\left.+\partial_jU\,\partial_n(U-V)+\partial_jV\,\partial_n(3V-U)+\delta_{jn}\left(\delta^{kl}\partial_lV\partial_k(U+V)+2V\triangle V\right)\right\}$$

$$\left(\nabla^\rho\nabla^\sigma f_{\mathcal{G}}(R,\mathcal{G})\right)^{(2)} = g^{(0)\,\rho i}g^{(0)\,\sigma j}f_{R\mathcal{G}}(0,0)\partial_i\partial_jR^{(2)}$$

$$\left(\nabla^\rho\nabla^\sigma f_{\mathcal{G}}(R,\mathcal{G})\right)^{(3)} = \left\{g^{(0)\,\rho0}g^{(0)\,\sigma i}+g^{(0)\,\rho i}g^{(0)\,\sigma0}\right\}f_{R\mathcal{G}}(0,0)\partial_0\partial_iR^{(2)}$$

$$\begin{aligned}(\nabla^\rho\nabla^\sigma f_{\mathcal{G}}(R,\mathcal{G}))^{(4)} &= g^{(0)\,\rho 0}g^{(0)\,\sigma 0}\left\{f_{R\mathcal{G}}(0,0)\partial_0\partial_0 R^{(2)} - \Gamma^{(2)\,i}_{\ \ \ 00} f_{R\mathcal{G}}(0,0)\partial_i R^{(2)}\right\} + \\ &+ g^{(0)\,\rho i}g^{(0)\,\sigma j}\left\{f_{RR\mathcal{G}}(0,0)\partial_i R^{(2)}\partial_j R^{(2)} + f_{R\mathcal{G}}(0,0)\partial_i\partial_j R^{(4)} + \right. \\ &+ \left. f_{RR\mathcal{G}}(0,0)R^{(2)}\partial_i\partial_j R^{(2)} + f_{\mathcal{G}\mathcal{G}}(0,0)\partial_i\partial_j G^{(4)} - \Gamma^{(2)\,k}_{\ \ \ ij} f_{R\mathcal{G}}(0,0)\partial_k R^{(2)}\right\} + \\ &+ \left\{g^{(2)\,\rho i}g^{(0)\,\sigma j} + g^{(0)\,\rho i}g^{(2)\,\sigma j}\right\} f_{R\mathcal{G}}(0,0)\partial_i\partial_j R^{(2)}\end{aligned}$$

For the $(0,0)$-field equation the following results are necessary:

$$\begin{aligned}\left(-\tfrac{1}{2}g_{00}f(R,\mathcal{G})\right)^{(4)} &= -\tfrac{1}{2}g^{(4)}_{00}f(0,0) - \tfrac{1}{2}g^{(2)}_{00}f_R(0,0)R^{(2)} - \\ &\quad -\tfrac{1}{2}g^{(0)}_{00}\left(\tfrac{1}{2}f_{RR}(0,0)R^{(2)\,2} + f_R(0,0)R^{(4)} + f_{\mathcal{G}}(0,0)\mathcal{G}^{(4)}\right) \\ (f_R(R,\mathcal{G})R_{00})^{(4)} &= f_R(0,0)R^{(4)}_{00} + f_{RR}(0,0)R^{(2)}R^{(2)}_{00} \\ \left(g_{00}\nabla^2 f_R(R,\mathcal{G})\right)^{(4)} &= g^{(2)}_{00}g^{(0)\,ij}f_{RR}(0,0)\partial_i\partial_j R^{(2)} + g^{(0)}_{00}\left[g^{(0)\,00}f_{RR}(0,0)\left(\partial_0\partial_0 R^{(2)} - \Gamma^{(2)\,i}_{\ \ \ 00}\partial_i R^{(2)}\right) + \right. \\ &\quad + g^{(0)\,ij}\left(f_{RRR}(0,0)\left\{\partial_i R^{(2)}\partial_j R^{(2)} + R^{(2)}\partial_i\partial_j R^{(2)}\right\} + f_{R\mathcal{G}}(0,0)\partial_i\partial_j\mathcal{G}^{(4)} + \right. \\ &\quad \left.\left. + f_{RR}(0,0)\left\{\partial_i\partial_j R^{(4)} - \Gamma^{(2)\,k}_{\ \ \ ij}\partial_k R^{(2)}\right\}\right) + g^{(2)\,ij}f_{RR}(0,0)\partial_i\partial_j R^{(2)}\right] \\ (-\nabla_0\nabla_0 f_R(R,\mathcal{G}))^{(4)} &= -f_{RR}(0,0)\left(\partial_0\partial_0 R^{(2)} - \Gamma^{(2)\,i}_{\ \ \ 00}\partial_i R^{(2)}\right) \\ (2f_{\mathcal{G}}(R,\mathcal{G})RR_{00})^{(4)} &= 2f_{\mathcal{G}}(0,0)R^{(2)}R^{(2)}_{00} \\ (-4f_{\mathcal{G}}(R,\mathcal{G})g^{\rho\sigma}R_{0\rho}R_{0\sigma})^{(4)} &= -4f_{\mathcal{G}}(0,0)R^{(2)}_{00}R^{(2)}_{00} \\ \left(2f_{\mathcal{G}}(R,\mathcal{G})R_{\alpha\beta\rho 0}R^{\alpha\beta\rho}_{\ \ \ \ \ 0}\right)^{(4)} &= 2f_{\mathcal{G}}(0,0)\left(R_{\alpha\beta\rho 0}R^{\alpha\beta\rho}_{\ \ \ \ \ 0}\right)^{(4)} \\ (-4f_{\mathcal{G}}(R,\mathcal{G})R_{0\rho 0\sigma}R^{\rho\sigma})^{(4)} &= -4f_{\mathcal{G}}(0,0)\left(R_{0\rho 0\sigma}R^{\rho\sigma}\right)^{(4)} \\ \left(2g_{00}R\nabla^2 f_{\mathcal{G}}(R,\mathcal{G})\right)^{(4)} &= 2g^{(0)}_{00}R^{(2)}g^{(0)\,ij}f_{\mathcal{G}R}(0,0)\partial_i\partial_j R^{(2)} \\ (-4g_{00}R_{\rho\sigma}\nabla^\rho\nabla^\sigma f_{\mathcal{G}}(R,\mathcal{G}))^{(4)} &= -4g^{(0)}_{00}R^{(2)}_{kl}g^{(0)\,ik}g^{(0)\,jl}f_{\mathcal{G}R}(0,0)\partial_i\partial_j R^{(2)} \\ (-2R\nabla_0\nabla_0 f_{\mathcal{G}}(R,\mathcal{G}))^{(4)} &= 0 \\ \left(-4R_{00}\nabla^2 f_{\mathcal{G}}(R,\mathcal{G})\right)^{(4)} &= -4R^{(2)}_{00}g^{(0)\,ij}f_{\mathcal{G}R}(0,0)\partial_i\partial_j R^{(2)} \\ (8R_{0\rho}\nabla^\rho\nabla_0 f_{\mathcal{G}}(R,\mathcal{G}))^{(4)} &= 0 \\ (4R_{0\rho 0\sigma}\nabla^\rho\nabla^\sigma f_{\mathcal{G}}(R,\mathcal{G}))^{(4)} &= 4R^{(2)}_{0k0l}g^{(0)\,ik}g^{(0)\,jl}f_{\mathcal{G}R}(0,0)\partial_i\partial_j R^{(2)} \\ \left(2\kappa^2 T_{00}\right)^{(4)} &= 2\kappa^2 T^{(2)}_{00}\end{aligned}$$

In the case of the trace equation, we need:

$$
\begin{aligned}
(-2f(R,\mathcal{G}))^{(4)} &= -2f_R(0,0)R^{(4)} - f_{RR}(0,0)R^{(2)\,2} - 2f_{\mathcal{G}}(0,0)\mathcal{G}^{(4)} \\
(f_R(R,\mathcal{G})R)^{(4)} &= f_R(0,0)R^{(4)} + f_{RR}(0,0)R^{(2)\,2} \\
\left(3\nabla^2 f_R(R,\mathcal{G})\right)^{(4)} &= 3f_{RR}(0,0)\left\{g^{(0)\,00}\left(\partial_0\partial_0 R^{(2)} - \Gamma^{(2)\,i}{}_{00}\partial_i R^{(2)}\right) + g^{(0)\,ij}\left(\partial_i\partial_j R^{(4)} - \Gamma^{(2)\,k}{}_{ij}\partial_k R^{(2)}\right) + \right. \\
&\quad \left. + g^{(2)\,ij}\partial_i\partial_j R^{(2)}\right\} + 3f_{RRR}(0,0)g^{(0)\,ij}\left\{\partial_i R^{(2)}\partial_j R^{(2)} + R^{(2)}\partial_i\partial_j R^{(2)}\right\} + \\
&\quad + 3f_{R\mathcal{G}}(0,0)g^{(0)\,ij}\partial_i\partial_j\mathcal{G}^{(4)} \\
(2f_{\mathcal{G}}(R,\mathcal{G})\mathcal{G})^{(4)} &= 2f_{\mathcal{G}}(0,0)\mathcal{G}^{(4)} \\
\left(2R\nabla^2 f_{\mathcal{G}}(R,\mathcal{G})\right)^{(4)} &= 2R^{(2)}g^{(0)\,ij}f_{R\mathcal{G}}(0,0)\partial_i\partial_j R^{(2)} \\
(-4R_{\rho\sigma}\nabla^\rho\nabla^\sigma f_{\mathcal{G}}(R,\mathcal{G}))^{(4)} &= -4g^{(0)\,im}g^{(0)\,jn}R^{(2)}_{ij}f_{R\mathcal{G}}(0,0)\partial_m\partial_n R^{(2)} \\
\left(2\kappa^2 T\right)^{(4)} = 2\kappa^2 T^{(2)} &= 2\kappa^2\left\{g^{(0)\,00}T^{(2)}_{00} + g^{(2)\,00}T^{(0)}_{00} + g^{(0)\,ij}T^{(2)}_{ij}\right\}
\end{aligned}
$$

E.2 The PostPostNewtonian approximation

I present now the calculations needed to obtain the PPN limit for the field equations and the trace equation.

In the case of the $(0,0)$-field equation we need to know the following expressions:

$$
\begin{aligned}
\left(-\tfrac{1}{2}g_{00}f(R,\mathcal{G})\right)^{(6)} &= -\tfrac{1}{2}g^{(6)}_{00}f(0,0) - \tfrac{1}{2}f_R(0,0)\left\{g^{(4)}_{00}R^{(2)} + g^{(2)}_{00}R^{(4)} + g^{(0)}_{00}R^{(6)}\right\} - \\
&\quad -\tfrac{1}{2}f_{\mathcal{G}}(0,0)\left\{g^{(2)}_{00}\mathcal{G}^{(4)} + g^{(0)}_{00}\mathcal{G}^{(6)}\right\} - \tfrac{1}{2}f_{RR}(0,0)\left\{\tfrac{1}{2}g^{(2)}_{00}R^{(2)\,2} + g^{(0)}_{00}R^{(2)}R^{(4)}\right\} \\
&\quad -\tfrac{1}{2}g^{(0)}_{00}f_{R\mathcal{G}}(0,0)R^{(2)}\mathcal{G}^{(4)} - \tfrac{1}{12}f_{RRR}(0,0)R^{(2)\,3} \\
(f_R(R,\mathcal{G})R_{00})^{(6)} &= f_R(0,0)R^{(6)}_{00} + f_{RR}(0,0)\left\{R^{(2)}R^{(4)}_{00} + R^{(4)}R^{(2)}_{00}\right\} + f_{R\mathcal{G}}(0,0)\mathcal{G}^{(4)}R^{(2)}_{00} + \\
&\quad + \tfrac{1}{2}f_{RRR}(0,0)R^{(2)\,2}R^{(2)}_{00} \\
\left(g_{00}\nabla^2 f_R(R,\mathcal{G})\right)^{(6)} &= g^{(0)}_{00}\left(\nabla^2 f_R(R,\mathcal{G})\right)^{(6)} + g^{(2)}_{00}\left(\nabla^2 f_R(R,\mathcal{G})\right)^{(4)} + g^{(4)}_{00}\left(\nabla^2 f_R(R,\mathcal{G})\right)^{(2)} \\
(-\nabla_0\nabla_0 f_R(R,\mathcal{G}))^{(6)} &= -f_{RR}(0,0)\left\{\partial_0\partial_0 R^{(4)} - \Gamma^{(3)\,0}{}_{00}\partial_0 R^{(2)} - \Gamma^{(2)\,i}{}_{00}\partial_i R^{(4)} - \Gamma^{(4)\,i}{}_{00}\partial_i R^{(2)}\right\} - \\
&\quad - f_{R\mathcal{G}}(0,0)\left\{\partial_0\partial_0\mathcal{G}^{(4)} - \Gamma^{(2)\,i}{}_{00}\partial_i\mathcal{G}^{(4)}\right\} - \\
&\quad - f_{RRR}(0,0)\left\{\partial_0 R^{(2)}\partial_0 R^{(2)} + R^{(2)}\partial_0\partial_0 R^{(2)} - \Gamma^{(2)\,i}{}_{00}R^{(2)}\partial_i R^{(2)}\right\}
\end{aligned}
$$

$$\left(2f_{\mathcal{G}}(R,\mathcal{G})RR_{00}\right)^{(6)} = 2f_{\mathcal{G}}(0,0)\left\{R^{(2)}R^{(4)}_{00}+R^{(4)}R^{(2)}_{00}\right\}+2f_{R\mathcal{G}}(0,0)R^{(2)\,2}R^{(2)}_{00}$$

$$\left(-4f_{\mathcal{G}}(R,\mathcal{G})g^{\rho\sigma}R_{0\rho}R_{0\sigma}\right)^{(6)} = -4f_{\mathcal{G}}(0,0)\left\{2g^{(0)\,00}R^{(2)}_{00}R^{(4)}_{00}+g^{(2)\,00}R^{(2)\,2}_{00}+g^{(0)\,ij}R^{(3)}_{0i}R^{(3)}_{0j}\right\}- -4f_{R\mathcal{G}}(0,0)g^{(0)\,00}R^{(2)}R^{(2)\,2}_{00}$$

$$\left(2f_{\mathcal{G}}(R,\mathcal{G})R_{\alpha\beta\rho0}R^{\alpha\beta\rho}{}_{0}\right)^{(6)} = 2f_{\mathcal{G}}(0,0)\left(R_{\alpha\beta\rho0}R^{\alpha\beta\rho}{}_{0}\right)^{(6)}+2f_{R\mathcal{G}}(0,0)R^{(2)}\left(R_{\alpha\beta\rho0}R^{\alpha\beta\rho}{}_{0}\right)^{(4)}$$

$$\left(-4f_{\mathcal{G}}(R,\mathcal{G})R_{0\rho0\sigma}R^{\rho\sigma}\right)^{(6)} = -4f_{\mathcal{G}}(0,0)\left(R_{0\rho0\sigma}R^{\rho\sigma}\right)^{(6)}-4f_{R\mathcal{G}}(0,0)R^{(2)}\left(R_{0\rho0\sigma}R^{\rho\sigma}\right)^{(4)}$$

$$\left(2g_{00}R\nabla^2 f_{\mathcal{G}}(R,\mathcal{G})\right)^{(6)} = 2g^{(0)}_{00}R^{(2)}\left(\nabla^2 f_{\mathcal{G}}(R,\mathcal{G})\right)^{(4)}+2g^{(0)}_{00}R^{(4)}\left(\nabla^2 f_{\mathcal{G}}(R,\mathcal{G})\right)^{(2)}+ +2g^{(2)}_{00}R^{(2)}\left(\nabla^2 f_{\mathcal{G}}(R,\mathcal{G})\right)^{(2)}$$

$$\left(-4g_{00}R_{\rho\sigma}\nabla^\rho\nabla^\sigma f_{\mathcal{G}}(R,\mathcal{G})\right)^{(6)} = -4g^{(0)}_{00}R^{(2)}_{\rho\sigma}\left(\nabla^\rho\nabla^\sigma f_{\mathcal{G}}(R,\mathcal{G})\right)^{(4)}-4g^{(0)}_{00}R^{(3)}_{\rho\sigma}\left(\nabla^\rho\nabla^\sigma f_{\mathcal{G}}(R,\mathcal{G})\right)^{(3)}- -4g^{(0)}_{00}R^{(4)}_{\rho\sigma}\left(\nabla^\rho\nabla^\sigma f_{\mathcal{G}}(R,\mathcal{G})\right)^{(2)}-4g^{(2)}_{00}R^{(2)}_{\rho\sigma}\left(\nabla^\rho\nabla^\sigma f_{\mathcal{G}}(R,\mathcal{G})\right)^{(2)}$$

$$\left(-2R\nabla_0\nabla_0 f_{\mathcal{G}}(R,\mathcal{G})\right)^{(6)} = -2f_{R\mathcal{G}}(0,0)R^{(2)}\left\{\partial_0\partial_0R^{(2)}-\Gamma^{(2)\,i}{}_{00}\partial_iR^{(2)}\right\}$$

$$\left(-4R_{00}\nabla^2 f_{\mathcal{G}}(R,\mathcal{G})\right)^{(6)} = -4R^{(2)}_{00}\left(\nabla^2 f_{\mathcal{G}}(R,\mathcal{G})\right)^{(4)}-4R^{(4)}_{00}\left(\nabla^2 f_{\mathcal{G}}(R,\mathcal{G})\right)^{(2)}$$

$$\left(8R_{0\rho}\nabla^\rho\nabla_0 f_{\mathcal{G}}(R,\mathcal{G})\right)^{(6)} = 8f_{R\mathcal{G}}(0,0)\left\{R^{(2)}_{00}g^{(0)\,00}\left(\partial_0\partial_0R^{(2)}-\Gamma^{(2)\,i}{}_{00}\partial_iR^{(2)}\right)+R^{(3)}_{0i}g^{(0)\,ij}\partial_j\partial_0R^{(2)}\right\}$$

$$\left(4R_{0\rho0\sigma}\nabla^\rho\nabla^\sigma f_{\mathcal{G}}(R,\mathcal{G})\right)^{(6)} = 4R^{(2)}_{\rho0\sigma0}\left(\nabla^\rho\nabla^\sigma f_{\mathcal{G}}(R,\mathcal{G})\right)^{(4)}+4R^{(4)}_{\rho0\sigma0}\left(\nabla^\rho\nabla^\sigma f_{\mathcal{G}}(R,\mathcal{G})\right)^{(2)}$$

$$\left(2\kappa^2T_{00}\right)^{(6)} = 2\kappa^2T^{(4)}_{00}$$

Finally, in the case of the trace equation, the expressions needed are:

$$\left(-2f(R,\mathcal{G})\right)^{(6)} = -2\left\{f_R(0,0)R^{(6)}+f_{\mathcal{G}}(0,0)\mathcal{G}^{(6)}+f_{RR}(0,0)R^{(2)}R^{(4)}+f_{R\mathcal{G}}(0,0)R^{(2)}\mathcal{G}^{(4)}+ +\tfrac{1}{6}f_{RRR}(0,0)R^{(2)\,3}\right\}$$

$$\left(f_R(R,\mathcal{G})R\right)^{(6)} = f_R(0,0)R^{(6)}+2f_{RR}(0,0)R^{(2)}R^{(4)}+f_{R\mathcal{G}}(0,0)R^{(2)}\mathcal{G}^{(4)}+\tfrac{1}{2}f_{RRR}(0,0)R^{(2)\,3}$$

$$\left(3\nabla^2 f_R(R,\mathcal{G})\right)^{(6)} = 3\left(\nabla^2 f_R(R,\mathcal{G})\right)^{(6)}$$

$$\left(2f_{\mathcal{G}}(R,\mathcal{G})\mathcal{G}\right)^{(6)} = 2f_{\mathcal{G}}(0,0)\mathcal{G}^{(6)}+2f_{R\mathcal{G}}(0,0)R^{(2)}\mathcal{G}^{(4)}$$

$$\left(2R\nabla^2 f_{\mathcal{G}}(R,\mathcal{G})\right)^{(6)} = 2R^{(2)}\left(\nabla^2 f_{\mathcal{G}}(R,\mathcal{G})\right)^{(4)}+2R^{(4)}\left(\nabla^2 f_{\mathcal{G}}(R,\mathcal{G})\right)^{(2)}$$

$$\left(-4R_{\rho\sigma}\nabla^{\rho}\nabla^{\sigma}f_{\mathcal{G}}(R,\mathcal{G})\right)^{(6)} = -4R^{(2)}_{\rho\sigma}\left(\nabla^{\rho}\nabla^{\sigma}f_{\mathcal{G}}(R,\mathcal{G})\right)^{(4)} - 4R^{(3)}_{\rho\sigma}\left(\nabla^{\rho}\nabla^{\sigma}f_{\mathcal{G}}(R,\mathcal{G})\right)^{(3)} - 4R^{(4)}_{\rho\sigma}\left(\nabla^{\rho}\nabla^{\sigma}f_{\mathcal{G}}(R,\mathcal{G})\right)^{(2)}$$

$$\left(2\kappa^2 T\right)^{(6)} = 2\kappa^2 T^{(4)} = 2\kappa^2\left(g^{(0)\,00}T^{(4)}_{00} + g^{(2)\,00}T^{(2)}_{00} + g^{(4)\,00}T^{(0)}_{00} + g^{(3)\,0i}T^{(1)}_{0i} + g^{(0)\,ij}T^{(4)}_{ij} + g^{(2)\,ij}T^{(2)}_{ij}\right)$$

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