The stochastic wave equation: study of the law and approximations

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Certifico que aquesta memòria ha estat
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i dirigida per mi.

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Marta Sanz i Solé

al pare i a la mare

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Introduction

This dissertation is devoted to present some research challenges in the study of stochastic partial differential equations (SPDEs). It is widely recognised that SPDEs provide suitable models for a great variety of applied problems. For instance, we mention the study of growth population, the models of some climate phenomenons or in oceanography, or some applications to mathematical finance (see, for instance, [DS80], [Imk01], [AMR96], [Bjö01], respectively). However, in this thesis we focus more on theoretical mathematical aspects of the SPDEs theory rather than on applied ones.

More precisely, we shall mainly deal with a SPDE of hyperbolic type, namely the stochastic wave equation, which can be formally represented by

$$\frac{\partial^2 u}{\partial t^2}(t,x) - \Delta_d u(t,x) = b(u(t,x)) + \sigma(u(t,x))\dot{F}(t,x), \tag{1}$$

where Δ_d stands for the Laplacian operator on $\mathbb{R}^d$, σ and b are some real-valued functions and $\dot{F}(t,x)$ is some random perturbation. The time and space variables belong to $\mathbb{R}_+$ and $\mathbb{R}^d$, $d \leq 3$, respectively, and $\{u(t,x), (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ is a real-valued stochastic process.

All the results that shall be presented in this manuscript will be developed in the framework set up in the course given by Walsh in Saint-Flour [Wal86] and in further generalizations (see [Dal99] and [DM03]). In Walsh's course, a rigorous formulation for several classes of SPDEs is given, including the stochastic wave and heat equations. Indeed, he constructs a stochastic integral for predictable processes with respect to martingale measures; in Section 1.2.1 we review the main ideas of this theory. With this stochastic integral, Walsh defines a *mild* solution of a SPDE by means of an evolution formulation of the equation and obtains a solution taking values on the real line.

Concerning the random noise, it is known that if we are interested in SPDEs in one space dimension, we can consider random perturbations given by the so called *space-time white noise*. However, for spatial dimension strictly greater than one, if one still wants to have real-valued solutions of the equation, some spatial correlation on the noise shall be required. In the recent literature one can find several contributions dealing with solutions to SPDEs in higher dimensions; see, for instance, [AHR96], [DF98], [Mue97], [OR98], [PZ00], [MSS99], for the wave equation in $\mathbb{R}^d$, with $d = 1, 2$, and

[AHR01], [Nob97], [RO99], [PZ97] for the heat equation in any spatial dimension $d \geq 1$.

The case of the stochastic wave equation with dimension greater than or equal to three has an added difficulty. Namely, the fundamental solution associated to the wave operator is not a function but a Schwartz distribution. Thus, Walsh formulation cannot be used. Indeed, the stochastic integral constructed in [Wal86] only allows function-valued integrands. With this problem in mind, Dalang [Dal99] extended Walsh stochastic integral with respect to martingale measures in order to be able to integrate some deterministic distribution-valued functions. Then, he found conditions relating the fundamental solution and the spatial correlation of the noise which ensure existence and uniqueness of a real-valued solution to Equation (1). For $d = 3$, properties of the sample paths of the solution to this equation are given in the very recent work [DSS]; see also [DSS04] for path properties of a more general class of SPDEs.

The case of space dimension strictly greater than three cannot be analysed with Dalang's approach. In fact, the fundamental solution is no more a positive distribution, and this was one of the requirements of the extension of the stochastic integral. However, positive answer to this problem has been given by Peszat [Pes02] using a more abstract approach via stochastic equations in infinite dimensions ([DPZ92]); see [PZ00] and [KZ01] for related references using also this setting. We refer the reader to [DM03] for stochastic integration of non necessarily positive distribution-valued processes. Let us also mention that the stochastic wave equation (1) when $d = 1, 2$ was studied in the papers [CN88] and [MSS99], respectively.

The content of this dissertation may be split up in two parts. In the first one, we mainly deal with the stochastic wave equation in three space dimension, that is, Equation (1) with $(t, x) \in [0, T] \times \mathbb{R}^3$, for some positive T, and initial conditions given by

$$u(0, x) = \frac{\partial u}{\partial t}(0, x) = 0,$$

$x \in \mathbb{R}^3$. More precisely, we want to find sufficient conditions ensuring existence and regularity of the density of the probability law of the solution u at any point (t, x). In the second part we consider Equation (1) on $(t, x) \in [0, T] \times [0, 1]$, with homogeneous boundary conditions of Dirichlet type, some non vanishing initial conditions and a random perturbation given by the space-time white noise. We study the convergence of a sequence of approximations to the solution obtained by a finite-difference spatial discretisation of the equation.

Concerning the first part, we notice that Equation (1) is an example of the more general class of SPDEs

$$Lu(t, x) = b(u(t, x)) + \sigma(u(t, x))\dot{F}(t, x), \tag{2}$$

with the same vanishing initial conditions as above and where L denotes a second-order differential operator such that the fundamental solution of $Lu = 0$ is a non-negative distribution with rapid decrease. We assume that the coefficients b and σ are globally Lipschitz functions and $\dot{F}(t,x)$ is a Gaussian noise, white in time and with an homogeneous spatial correlation given by a positive definite tempered measure Γ; the rigorous description of the noise is given in the next Section 1.2.1.

We follow the extension of Walsh's approach developed in [Dal99] and give a rigorous meaning to Equation (2) in the *mild form*, as follows. Let Λ denote the fundamental solution to $Lu = 0$. Assume that Λ is a non-negative measure of the form $\Lambda(t,dx)dt$. We fix a probability space $(\Omega, \mathcal{F}, P)$, denote by $M = \{M_t(A), t \in [0,T], A \in \mathcal{B}_b(\mathbb{R}^d)\}$ the martingale measure associated with F (see Section 1.2.1) and by $\mathcal{F}_t$ the σ-field generated by the random variables $M_s(A)$, $s \in [0,t]$, $A \in \mathcal{B}_b(\mathbb{R}^d)$, for any $t \in [0,T]$. Then a solution to (2) is a *real-valued* stochastic process $u = \{u(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$, defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, progressively measurable, satisfying

$$
\begin{aligned}
u(t,x) = &\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-y)\sigma(u(s,y))M(ds,dy) \\
&+ \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dy) b(u(t-s,x-y)).
\end{aligned} \tag{3}
$$

Dalang [Dal99] proved the existence of a unique solution to Equation (3) in the case where the measure Γ is absolutely continuous with respect to Lebesgue measure. However, the proof in the general case follows using the same kind of arguments. In Section 1.2.1 we give an extension of this result by proving existence and uniqueness of solution for a SPDE in a general Hilbert setting; the result is a quotation of Theorem 7.2 in [SSar]. This framework shall be needed in order to give a rigorous meaning to the stochastic evolution equations satisfied by the Malliavin derivatives of the solution to Equation (3). Hence, an extension of Dalang's stochastic integral to Hilbert-valued integrands shall also be needed. More precisely, we introduce an extension of Dalang's stochastic integral to integrators that are defined by stochastic integration of Hilbert-valued predictable processes with respect to martingale measures (see either Section 1.2.1 or Section 2 in [QSSS04a]).

We are interested in the existence and regularity of the probability law of the random variable $u(t,x)$, for any fixed $(t,x) \in [0,T] \times \mathbb{R}^d$; the main results in this direction are provided by Theorem 3 in [QSSS04a] and [QSSS04c], which use the techniques provided by the Malliavin differential calculus. For this, we prove the differentiability in the Malliavin sense of the random variable $u(t,x)$ and we study its associated Malliavin matrix. This is done in two steps; first, in [QSSS04a], we prove that the probability law of $u(t,x)$ has a density and secondly, in [QSSS04c], we check that the random variable $u(t,x)$ has, indeed, an infinitely differentiable density.

Malliavin calculus theory, initially created by Malliavin in [Mal78] and further developed by Bismut, Stroock, Bell, Shigekawa, etc. among others, provides useful tools in order to study probability laws of functionals of families of Gaussian random variables. We refer the reader to [Str83], [Nua95], [Mal97] or [Nua98] for complete manuscripts on this topic. In Section 1.3 we give some preliminaries in a general form in order to present the main tools of the Malliavin calculus needed along the dissertation. At the end of Section 1.3 we specify the setting in which these techniques shall be applied.

If the fundamental solution $\Lambda(t)$ is a real-valued function, for example in the stochastic heat equation in any dimension $d \geq 1$ or the stochastic wave equation in dimension $d = 1, 2$, it is well-known that the solution of (3) at any fixed point (t, x) belongs to $\mathbb{D}^{N,p}$ for any $N \in \mathbb{N}$ and every $p \in [1, \infty)$ (see for instance [BP98], [CN88], [MCMS01], [MSS99]); the equation satisfied by the N-th derivative is obtained recursively using the rules of Malliavin calculus, by derivation of each term of the equation satisfied by the $(N-1)$-th derivative. In this case, the formal derivative of Equation (3) would give the following Hilbert-valued evolution equation:

$$\begin{aligned} Du(t,x) =& \Lambda(t-\cdot, x-*)\sigma(u(\cdot,*)) \\ &+ \int_0^t \int_{\mathbb{R}^3} \Lambda(t-s, x-z)\sigma'(u(s,z))Du(s,z)M(ds,dz) \\ &+ \int_0^t ds \int_{\mathbb{R}^3} \Lambda(s,dz)b'(u(t-s,x-z))Du(t-s,x-z), \end{aligned}$$

where $(\cdot, *)$ stand for the variables of the underlying Hilbert space H in which the Malliavin derivatives take their values. The first two terms in the right hand-side of the above equation come from the differentiation of the stochastic integral in (3). If $\Lambda(t)$ is a distribution, for example for the wave equation with dimension $d = 3$, this approach is not possible, the problem being that the product of $\Lambda(t-\cdot)$ times the function σ is not defined in general.

In spite of this difficulty, we succeed in proving the regularity in the sense of Malliavin, as follows. In the article [QSSS04a] we apply Lemma 1.3.1 to obtain that the solution $u(t, x)$ belongs to the space $\mathbb{D}^{1,p}$, for all $p \in [1, \infty)$ while in [QSSS04c] we make use of Lemma 1.3.2 and we get that $u(t, x) \in \mathbb{D}^\infty$. To apply these lemmas, we need to show that a sequence of regularised processes $u_n(t, x)$, $n \geq 1$, obtained by convolution of the fundamental solution Λ with an approximation of the identity, converges to $u(t, x)$ in $L^p(\Omega)$. To show this convergence, we notice that the difference of two positive distributions is not necessarily positive; however, positivity is one of the requirements in the construction of Dalang's integral and in particular for obtaining $L^p(\Omega)-$bounds, a useful tool to prove $L^p(\Omega)-$convergences. We circumvent this problem by showing that the sequence of processes $u_n(t, x)$ is uniformly bounded in $L^p(\Omega)$, for any $p \in (1, \infty)$, and thus the sequence $(|u_n(t,x)|^p)_{n\geq 1}$ is uniformly integrable. Then we prove convergence in $L^2(\Omega)$ which can be checked with techniques related to the isometry property of the stochastic integral (see Proposition 1 in [QSSS04a]). This tool also plays a crucial role along the proof of the main theorem in [QSSS04a], Theorem 2. Concerning the work [QSSS04c], since the iterated Malliavin derivative operator D^N is closed, it suffices to prove that the sequence $D^N u_n(t,x)$ converges in the topology of $L^p(\Omega; H^{\otimes N})$, for any $N \geq 1$, $p \in [1, \infty)$. This can be also achieved proving first that the sequence is bounded in any $L^p(\Omega; H^{\otimes N})$ and then proving the convergence of order two, which also follows from the isometry property of the stochastic integral.

We next consider the stochastic wave equation (1) with $d = 3$ and we show that, for any $(t,x) \in [0,T] \times \mathbb{R}^3$, the probability law of the random variable $u(t,x)$ has a density which is a $\mathcal{C}^\infty$ function. We remark that in this case the fundamental solution reads

$$\Lambda(t) = \frac{1}{4\pi t}\sigma_t,$$

where σ_t denotes the uniform measure on the 3-dimensional sphere of radius t.

The proof of the existence of density for the random variable $u(t,x)$ is carried out in [QSSS04a]. For this we apply Bouleau and Hirsch criterion (see Theorem 1.3.7 in Section 1.3); hence, due to the differentiability results that we have just explained, we only need to check that the random variable $\|Du(t,x)\|_H$ is non-degenerate, almost surely. More precisely, we check that

$$E(\|Du(t,x)\|_H^{-p}) < \infty,$$

for some $p \in [0, \infty)$.

On the other hand, to prove the regularity of the density we apply one of the criteria provided by the Malliavin calculus (see Proposition 1.3.8); it is sufficient to check the existence of moments of any order of the inverse of the Malliavin variance. For this, we study the integrability in a neighbourhood of zero of the function

$$\varepsilon \longrightarrow \varepsilon^{-(1+p)} P\{||Du(t,x)||^2_{\mathcal{H}_T} < \varepsilon\},$$

for any $p \in [0, \infty)$. Hence, the main issue is to obtain the *size* in ε of the factor $P\{||Du(t,x)||^2_{\mathcal{H}_T} < \varepsilon\}$. The difficulties come from the fact that the fundamental solution of the wave equation is a Schwartz distribution. The natural idea is to smooth this distribution, as we did to study the differentiability. This time we introduce a regularisation kernel which depends on ε in a suitable way so that the error in this approximation is a function of ε as well. This technique is complemented with upper and lower bounds of integrals involving the Fourier transform of the fundamental solution of the wave equation; they are collected in the Appendix of [QSSS04a]. Notice that, though the fundamental solution of the wave equation becomes more irregular as dimension increases, the Fourier transform has a unified expression for any dimension $d \geq 1$:

$$\mathcal{F}\Lambda(t)(\xi) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}, \quad \xi \in \mathbb{R}^d.$$

We mention that if the reader is interested in a non-detailed exposition of the main results and techniques of the reference [QSSS04a], we refer to the preliminary communication [QSSS03].

In addition to the published works [QSSS04a], [QSSS04c], and the prepublication [QSSS04b], in Section 2.3 of Chapter 2 we deal with some SPDEs of parabolic type. More precisely, we consider first Equation (2) on $[0,T] \times [0,1]$, with homogeneous Dirichlet boundary conditions and L a quite general operator of parabolic type. Under non-global Lipschitz conditions on the drift and diffusion coefficients, we give sufficient conditions ensuring existence of density for the probability law of the solution at any fixed point. This extends the work by Pardoux and Zhang [PZ93], where $L = \frac{\partial}{\partial t} - \Delta$. Secondly, we consider Equation (2) on $\mathbb{R}^d$, $d \geq 1$, and also with a general parabolic operator. Now we extend the results given by Márquez-Carreras *et al.* in [MCMS01] concerning existence and smoothness of the density of the process solution under globally Lipschitz assumptions on σ and b.

The motivation of these problems comes from some discussions with Peter Imkeller held during one of the visits of the author of this dissertation at Humboltd Universität-Berlin. Under locally Lipschitz and some dissipativity assumptions on the drift b, we wanted to prove existence of a $\mathcal{C}^\infty$ density for the solution to parabolic SPDE. As far as we know, a positive answer to this question does not exist. However, the author considers that it is worth completing the known results of existence and regularity of densities for the solution to stochastic heat equations to more general parabolic SPDEs.

The proofs of the main results of Section 2.3 do not require new techniques but only putting together known results and some straightforward extensions. Thus, we only lay particular stress on the main ingredients needed. Namely, for the extension of the results in [PZ93], we shall make use of an existence and uniqueness of solutions result and a comparison of solutions theorem (see Theorems 2.3.2 and 2.3.1, respectively). It is worth remarking that comparison results constitute one of the main tools for the proof of existence of solutions of SPDEs with non-global Lipschitz coefficients; see, for instance, [BGP94], [GP93a], [GP93b], [MZ99], [DMP93]. On the other hand, for the extension of the results given in [MCMS01], it suffices to take into account the isometry property of the stochastic integral and the Gaussian upper and lower bounds for the fundamental solution associated to the deterministic problem (see Theorem 2.3.4); for similar results concerning a different type of parabolic SPDEs we refer to [CWM04].

We jump to the second part of the dissertation, which corresponds to the contents in Appendix C ([QSSS04b]). We consider the equation

$$\frac{\partial^2 u}{\partial t^2}(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x) = f(t,x,u(t,x)) + \sigma(t,x,u(t,x))\dot{F}(t,x), \tag{4}$$

on the product space $[0,T] \times [0,1]$, with some initial conditions and homogeneous Dirichlet boundary conditions. The random perturbation is given by a space-time white noise and the initial conditions belong to some subspaces of the so called fractional Sobolev spaces (see Section C.2). Notice that we are now considering more general coefficients than in (1). Our aim is to discretise the above formal equation with respect to space using a finite difference method and define a sequence of processes $(u^n(t,x))_{n\geq 1}$ approximating the solution $u(t,x)$. Then, we study $L^p(\Omega)$ and almost sure convergence of $u^n(t,x)$ and get bounds for the rate of convergence.

The construction of discretisation schemes for SPDEs has been recently become an active topic on stochastic analysis. Different methods may be considered, most of them inspired in the deterministic context. Let us mention for instance, finite differences ([GN97], [GN95], [Gyö98b], [Gyö99], [MM03], [GM04], [GM], [Yoo00]), finite elements ([GP88], [Walar]), splitting up methods ([BGR92], [BGR90], [IR00], [GK03]), Garlekin approximations ([GK96]) and time discretization ([Hau03], [Pri01]). Others are more genuine stochastic, based on the Wiener chaos decomposition ([LMR97], [Lot96]) or on truncations of the Fourier expansion of the noise ([Sha03], [Sha99]). We refer the reader to [Gyö02] for a survey of some of these methods, together with a more extensive list of references.

Roughly speaking, discretisation methods could be split up in two large families: explicit and implicit methods –though there are mixtures of both. For the formers, the value of the discretised solution at some spatial or temporal point is computed directly from the already known values up to this moment. On the other hand, when using an implicit method, the discretised solution at some spatial or temporal point depends non-algebraically on the own solution at instants which are not yet computed. This means that an implicit equation must be solved. However, it is known that these methods provide, among other advantages, better rates of convergence than the explicit one's, while the latters imply less computing time. We refer the reader to [KP92] for a complete manuscript on numerics for stochastic equations.

In our case, since the results presented in [QSSS04b] correspond to the first step in the study of lattice approximations for hyperbolic SPDEs, we consider a spatial discretisation, ending up with a implicit equation for the discretised solution on some spatial grid. As we shall comment later on in this Introduction, the next step would be to discretise the equation both with respect to time and space. Indeed, the reader is addressed to the reference [MPW03] for discretisation schemes of a stochastic wave equation having non-random coefficients.

The techniques used to prove the main result in [QSSS04b] (Theorem C.3.1) are inspired in [Gyö98b]. More precisely, once we have discretised the formal equation (4) and defined the approximating process $u^n(t,*)$ on the spatial grid given by $x_k = \frac{k}{n}$, $k = 1, \dots, n-1$, we extend u^n to any $(t,x) \in [0,T] \times [0,1]$ by linear interpolation . This extension satisfies an evolution equation. This shall allow us to prove $L^p(\Omega)$ and almost sure convergence of u^n to u, as n tends to infinity; for the former, we obtain the rate of convergence.

In comparison with parabolic examples, the rate of convergence differs substantially from the Hölder continuity order of the sample paths of the solution. Indeed, assuming for simplicity that the initial conditions vanish, sample paths are jointly Hölder continuous in (t,x) of order $\alpha < \frac{1}{2}$ (see Proposition C.2.2), while the rate of convergence is of order $\rho < \frac{1}{3}$. In the final appendix from [QSSS04b] (Appendix C.4), we test this result numerically and we conclude that, using this method, we cannot expect better results.

Obviously, the natural step to follow would be to study discretisations of Equation (4) both with respect to time and space. We think that a deep exploration of purely deterministic methods should be useful. In fact, after some preliminary research, it seems that Gyöngy's methods are not the most suitable for hyperbolic equations.

The outline of the dissertation is the following. Chapter 1 is devoted to give some preliminaries. Namely, in Section 1.2 we present the main tools of the theory of SPDEs driven by a Gaussian correlated noise, such as Walsh's and Dalang's stochastic integration, the latter being performed in a Hilbert-valued setting, we give the definition of the so called *mild* solutions and we prove an existence and uniqueness result in a Hilbert-valued context. Section 1.3 is devoted to the framework of the stochastic calculus of variations or Malliavin calculus needed to deal with the study of probability laws of solutions to SPDEs. In Chapter 2, we summarise the contents of the works [QSSS04a], [QSSS04c] and [QSSS04b] in Sections 2.1, 2.2 and 2.4, respectively. In Section 2.3 we present some original material concerning existence and smoothness of densities for some parabolic SPDEs. The above cited references are collected in Appendices A, B and C, respectively. Between Chapter 2 and the appendices we place a summary of conclusions. Finally, in Appendix D we give a summary in Catalan of the whole dissertation.

Chapter 1

Preliminaries

1.1 Introduction

This Chapter is devoted to give the preliminaries needed to develop the results of the works collected in Appendices A, B and C.

First, in Section 1.2, we give a rigorous meaning to SPDEs of the form

$$Lu(t,x) = \sigma(u(t,x))\dot{F}(t,x) + b(u(t,x)), \tag{1.1}$$

with vanishing initial conditions. L is a second order partial differential operator, σ and b are some real-valued functions defined on the real line and $\dot{F}(t,x)$ is the formal notation for a Gaussian random perturbation white in time and with some spatial correlation. We are interested in real-valued solutions to Equation (1.1). We consider solutions of Equation (1.1) in the so called *mild* form by means of either Walsh's formulation ([Wal86]) or, in case the fundamental solution associated to L is not a function but a distribution, by means of Walsh's extended theory given by Dalang in [Dal99].

A brief review of Walsh's theory, the rigorous definition of the noise $\dot{F}$, Dalang's stochastic integration theory in a Hilbert setting and the definition of a *mild* solution to Equation (1.1) are given in Section 1.2.1. In Section 1.2.2, we state a more general SPDE in a Hilbert-valued context, we define the corresponding *mild* solution to this equation and we prove an existence and uniqueness result.

In order to study the existence and smoothness of the density of the law of the solution to (1.1) at any fixed point, we shall make use of the techniques provided by the Malliavin differential calculus. In Section 1.3 we review the main definitions and results from the Malliavin calculus needed along the dissertation in a general context. In the very last part of this section, we define the particular Malliavin setting for the study of the law of the solution to Equation (1.1).

1.2 SPDEs driven by spatially correlated noise

1.2.1 Stochastic integration and *mild* solutions

The purpose of this section is to define the stochastic integration needed to give the mild formulation of Equation (1.1). We briefly review Walsh's stochastic integration theory ([Wal86]). Then we describe the random noise we are going to consider and finally we extend Dalang's stochastic integral (see [Dal99]) to a Hilbert-valued setting. This shall allow us to integrate some deterministic distribution-valued processes and deal with Malliavin derivatives of solutions to SPDEs.

Martingale measures and Walsh's stochastic integration

We fix a probability space $(\Omega, \mathcal{F}, P)$ and a right-continuous filtration $\{\mathcal{F}_t\}_t$ on it. We denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel $\sigma-$algebra of $\mathbb{R}^d$. First we define the object that is going to play the role of integrator.

Definition 1.2.1. *A stochastic process* $\{M_t(A), \mathcal{F}_t, t \geq 0, A \in \mathcal{B}(\mathbb{R}^d)\}$ *is a* martingale measure *if*

(a) $M_0(A) = 0$ *a.s. for each* $A \in \mathcal{B}(\mathbb{R}^d)$,

(b) if $t > 0$, M_t *is a* $\sigma-$*finite* $L^2(\Omega)-$*valued measure; that is,* M_t *defines a function on* $\Omega \times \mathcal{B}(\mathbb{R}^d)$ *such that* $E(|M_t(A)|^2) < \infty$, *for* $A \in \mathcal{B}(\mathbb{R}^d)$; *there exists an increasing sequence* $(E_n)_n$ *in* $\mathcal{B}(\mathbb{R}^d)$ *whose union is* $\mathbb{R}^d$, *verifying that for all* $n \geq 1$,

$$\sup\{E(|M_t(A)|^2), A \in \mathcal{E}_n\} < \infty,$$

where $\mathcal{E}_n = \mathcal{B}(\mathbb{R}^d)_{|E_n}$, *and* M_t *is countably additive,*

(c) $\{M_t(A), \mathcal{F}_t\}_t$ *is a martingale, for all* $A \in \mathcal{B}(\mathbb{R}^d)$.

If $M = \{M_t(A), \mathcal{F}_t, t \geq 0, A \in \mathcal{B}(\mathbb{R}^d)\}$ is a martingale measure, we define its *covariance measure* by

$$Q((s,t] \times A \times B) := \langle M(A), M(B)\rangle_t - \langle M(A), M(B)\rangle_s,$$

for $0 \leq s < t$ and $A, B \in \mathcal{B}(\mathbb{R}^d)$. A set of the form $(s,t] \times A \times B$ will be called a *rectangle*.

A signed measure $K(ds, dx, dy)$ on $\mathcal{B}([0,\infty)) \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d)$ is *positive definite* if for each bounded measurable function f for which the integral makes sense, we have

$$\int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x,s) f(y,s) K(ds, dx, dy) \geq 0.$$

We are interested in a particular case of martingale measures. The definition is as follows.

Definition 1.2.2. *A martingale measure M is* worthy *if there exists a random $\sigma-$finite measure K in $\mathcal{B}([0,\infty)) \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d)$, such that*

(a) K is positive definite and symmetric in x and y,

(b) for fixed A and B in $\mathcal{B}(\mathbb{R}^d)$, $\{K((0,t] \times A \times B), t \geq 0\}$ is predictable,

(c) for all n, $E(K([0,T] \times E_n \times E_n)) < \infty$,

(d) for any rectangle R, $|Q(R)| \leq K(R)$.

We call K the dominating measure *of M.*

Using this notion, Walsh ([Wal86]) constructed the stochastic integral of a predictable process $g = \{g(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ with respect to a worthy martingale measure M. Here g belongs to the space $\mathcal{P}_+$, which stands for the completion of the class of finite linear combinations of elementary functions (see (1.4)) with respect to the norm

$$\|g\|_+^2 = E\left(\int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |g(s,x)||g(s,y)| K(ds,dx,dy)\right).$$

The integral of g with respect to M is denoted either by $g \cdot M$ or, for $t \in [0,T]$ and $A \in \mathcal{B}(\mathbb{R}^d)$,

$$(g \cdot M)_t(A) = \int_0^t \int_A g(s,x) M(ds,dx).$$

In the final part of this Section we shall recall in more detail Walsh's construction of this stochastic integral, but in a Hilbert-valued setting.

For the sake of completeness we state Walsh's characterisation of the stochastic integral $g \cdot M$ (see Theorem 2.5 in [Wal86]), as follows.

Theorem 1.2.3. *If $g \in \mathcal{P}_+$, then $g \cdot M$ is a worthy martingale measure. Its covariance and dominating measures are given by*

$$Q_{g \cdot M}(ds,dx,dy) = g(s,x)g(s,y)Q(ds,dx,dy),$$

$$K_{g \cdot M}(ds,dx,dy) = |g(s,x)g(s,y)|K(ds,dx,dy),$$

respectively. Moreover, if $f \in \mathcal{P}_+$ and $A, B \in \mathcal{B}(\mathbb{R}^d)$, then

$$\langle (f \cdot M)(A), (g \cdot M)(B) \rangle_t = \int_0^t \int_A \int_B f(s,x) g(s,y) Q(ds,dx,dy),$$

and

$$E\left(|(g \cdot M)_t(A)|^2\right) \leq \|g\|_+^2.$$

Description of the noise and extension to a martingale measure

We are now going to describe which type of random perturbation we consider in Equation (1.1). We will need to integrate with respect to this noise, therefore the first step shall be to extend it to a martingale measure.

We are interested in a Gaussian noise, white in time and with some homogeneous spatial correlation. Let us consider a mean-zero $L^2(\Omega)-$valued Gaussian process $F = \{F(\varphi), \varphi \in \mathcal{D}(\mathbb{R}^{d+1})\}$, where $\mathcal{D}(\mathbb{R}^{d+1})$ denotes the space of infinitely differentiable functions with compact support, with covariance functional given by

$$J(\varphi,\psi) := E(F(\varphi)F(\psi)) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \Gamma(dx) \left(\varphi(s,\cdot) * \tilde{\psi}(s,\cdot)\right)(x), \qquad (1.2)$$

for $\varphi,\psi \in \mathcal{D}(\mathbb{R}^{d+1})$, where $\tilde{\psi}(s,x) = \psi(s,-x)$ and Γ is a non-negative and non-negative definite tempered measure. The integral with respect to the Lebesgue measure means that the noise is white in time; the spatial correlation is given by Γ. According to [Sch66] (Chap. VII, Theorem XVII), this implies that Γ is symmetric and there exists a non-negative tempered measure μ on $\mathbb{R}^d$ whose Fourier transform is Γ. Hence

$$J(\varphi,\psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \mu(d\xi)\mathcal{F}\varphi(s,\cdot)(\xi)\overline{\mathcal{F}\psi(s,\cdot)(\xi)},$$

for $\varphi,\psi \in \mathcal{D}(\mathbb{R}^{d+1})$ Here, the Fourier transform is defined by

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} \varphi(x)e^{-2\pi i\langle x,\xi\rangle}dx,$$

for $\xi \in \mathbb{R}^d$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, where $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz space of rapidly decreasing $\mathcal{C}^\infty$ test functions and $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product in $\mathbb{R}^d$. We will refer to μ as the *spectral measure* of Γ.

Example 1.2.4. *Assume that the measure Γ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, with density given by a function $f : \mathbb{R}^d \to \mathbb{R}_+$. In this case, the covariance functional of the noise reads*

$$J(\varphi,\psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\varphi(s,x)f(x-y)\psi(s,y).$$

An important example is given by the so called Riesz kernels. In this case, the function f is of the form

$$f(x) = \frac{1}{|x|^\alpha},$$

for some $\alpha \in (0, 2\wedge d)$. This type of correlation has been considered by several authors (see, for instance, [DF98], [KZ00], [MSS99], [PZ00], [DSS]).

As in [DF98], we extend F to a worthy martingale measure M, as follows. Fix a rectangle R in $\mathbb{R}^{d+1}$ and let $(\varphi_n, n \geq 1) \subset \mathcal{D}(\mathbb{R}^{d+1})$ be such that

$$\lim_{n\to\infty} \varphi_n(z) = 1_R(z),$$

for all $z \in \mathbb{R}^{d+1}$. Then, by bounded convergence it follows that

$$\lim_{n,m\to\infty} E(F(\varphi_n) - F(\varphi_m))^2 = \lim_{n,m\to\infty} E(F(\varphi_n - \varphi_m)^2) = 0.$$

Set

$$F(R) = L^2(\Omega) - \lim_{n\to\infty} F(\varphi_n).$$

It is straightforward to check that this limit does not depend on the particular approximating sequence. This extension of F trivially holds for finite unions of rectangles. Moreover, if R_1, R_2 are two such elements, using again bounded convergence one proves that

$$E(F(R_1)F(R_2)) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \Gamma(dx)(1_{R_1}(s) * \tilde{1}_{R_2}(s))(x).$$

In addition, if $(R_n)_{n\geq 0}$ is a sequence of finite unions of rectangles decreasing to $\emptyset$, then the same kind of arguments yield $\lim_{n\to\infty} E(F(R_n)^2) = 0$. Hence the mapping $R \to F(R)$ can be extended to an $L^2(\Omega)-$valued measure defined on $\mathcal{B}_b(\mathbb{R}^{d+1})$, the bounded Borel sets of $\mathbb{R}^{d+1}$.

For any $t \geq 0$ and $A \in \mathcal{B}_b(\mathbb{R}^{d+1})$, set $M_t(A) = F([0,t] \times A)$. Let $\mathcal{F}_t$ be the completion of the $\sigma-$field generated by the random variables $M_s(A)$, with $0 \leq s \leq t$, $A \in \mathcal{B}_b(\mathbb{R}^{d+1})$.

The properties of F ensure that the process $M = \{M_t(A), t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^{d+1})\}$ is a worthy martingale measure with respect to the filtration $(\mathcal{F}_t, t \geq 0)$. The covariance measure is determined by

$$\langle M(A), M(B)\rangle_t = t \int_{\mathbb{R}^d} \Gamma(dx)(1_A * \tilde{1}_B)(x).$$

The dominating measure coincides with the covariance measure.

Extension of the stochastic integral

Owing to Theorem 1.2.3, Walsh theory allows to integrate real-valued stochastic processes $\{X(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ satisfying the integrability condition

$$E\left(\int_0^T ds \int_{\mathbb{R}^d} \Gamma(dx)(|X|(s) * |\tilde{X}|(s))(x)\right) < \infty.$$

In the context of SPDEs this integral is not always appropriate. Consider for instance the stochastic wave equation in dimension $d = 3$, which has a distribution-valued fundamental solution. Therefore in evolution formulations of this equation we shall meet integrands which include deterministic distribution-valued functions. With this problem as motivation, Dalang has extended in [Dal99] Walsh's stochastic integral.

In the remaining of this section, we shall review his ideas in the more general context of Hilbert-valued integrands. This setting is needed in order to state the stochastic evolution equations satisfied by Malliavin derivatives of solutions to SPDEs. The extension, together with the main theorem identifying some distribution-valued processes that can be integrated (Theorem 1.2.5), can be found in [QSSS04a], Section2.

Let $\mathcal{A}$ be a separable real Hilbert space with inner-product and norm denoted by $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{A}}$, respectively. Let $K = \{K(s,z), (s,z) \in [0,T] \times \mathbb{R}^d\}$ be an $\mathcal{A}-$valued predictable process; we assume the following condition:

Hypothesis B The process K satisfies

$$\sup_{(s,z)\in[0,T]\times\mathbb{R}^d} E\left(||K(s,z)||_{\mathcal{A}}^2\right) < \infty.$$

Our first purpose is to define a martingale measure with values in $\mathcal{A}$ obtained by integration of K. Let $(e_j, j \geq 0)$ be a complete orthonormal system of $\mathcal{A}$. Set $K^j(s,z) = \langle K(s,z), e_j \rangle_{\mathcal{A}}$, $(s,z) \in [0,T] \times \mathbb{R}^d$. According to Theorem 1.2.3, for any $j \geq 0$ the process

$$M_t^{K^j}(A) = \int_0^t \int_A K^j(s,z) M(ds,dz),\ t \in [0,T], A \in \mathcal{B}_b(\mathbb{R}^d),$$

defines a martingale measure. We define, for any $t \in [0,T], A \in \mathcal{B}_b(\mathbb{R}^d)$,

$$M_t^K(A) = \sum_{j\geq 0} M_t^{K^j}(A) e_j. \tag{1.3}$$

The right hand-side of (1.3) defines an element of $L^2(\Omega; \mathcal{A})$ and the process $\{M_t^K(A),$ $t \in [0,T], A \in \mathcal{B}_b(\mathbb{R}^d)\}$ defines an $\mathcal{A}-$valued martingale measure; by construction, we have that $\langle M_t^K(A), e_j \rangle_{\mathcal{A}} = M_t^{K^j}(A)$ (the details are developed in [QSSS04a], p. 5).

Our next aim is to introduce stochastic integration with respect to M^K, allowing the integrand to take values on some subset of the space of Schwartz distributions. First we briefly recall Walsh's construction in the Hilbert-valued context.

A stochastic process $\{g(s,z;\omega), (s,z) \in [0,T] \times \mathbb{R}^d\}$ is called *elementary* if

$$g(s,z;\omega) = 1_{(a,b]}(s) 1_A(z) X(\omega), \tag{1.4}$$

for some $0 \leq a < b \leq T,\ A \in \mathcal{B}_b(\mathbb{R}^d)$ and X a bounded $\mathcal{F}_a-$measurable random variable. For such g the stochastic integral $g \cdot M^K$ is the $\mathcal{A}-$valued martingale measure defined by

$$(g \cdot M^K)_t(B)(\omega) = \left(M^K_{t\wedge b}(A \cap B) - M^K_{t\wedge a}(A \cap B)\right) X(\omega),$$

$t \in [0, T],\ B \in \mathcal{B}_b(\mathbb{R}^d)$. This definition is extended by linearity to the set $\mathcal{E}_s$ of all linear combinations of elementary processes. For $g \in \mathcal{E}_s$ and $t \geq 0$, $B \in \mathcal{B}_b(\mathbb{R}^d)$, one easily checks that

$$\begin{aligned}
&E\left(||(g \cdot M^K)_t(B)||^2_{\mathcal{A}}\right)\\
&= \sum_{j\geq 0} E\Big(\int_0^t ds \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy 1_B(y) g(s,y) K^j(s,y) 1_B(y-x)\\
&\quad \times g(s, y-x) K^j(s, y-x)\Big)\\
&\leq ||g||^2_{+,K},
\end{aligned} \tag{1.5}$$

where

$$||g||^2_{+,K} := \sum_{j\geq 0} E\left(\int_0^T ds \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy |g(s,y) K^j(s,y) g(s,y-x) K^j(s,y-x)|\right).$$

Let $\mathcal{P}_{+,K}$ be the set of all predictable processes g such that $||g||_{+,K} < \infty$. Then, owning to [Wal86], Exercise 2.5 and Proposition 2.3, $\mathcal{P}_{+,K}$ is complete and $\mathcal{E}_s$ is dense in this Banach space. Thus, we use the bound (1.5) to define the stochastic integral $g \cdot M^K$ for $g \in \mathcal{P}_{+,K}$.

Next, following [Dal99] we aim to extend the above stochastic integral to include a larger class of integrands. Consider the inner-product defined by the formula

$$\begin{aligned}
\langle g_1, g_2\rangle_{0,K} = &\sum_{j\geq 0} E\left(\int_0^T ds \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy g_1(s,y) K^j(s,y)\right.\\
&\left. \times g_2(s, y-x) K^j(s, y-x)\right)
\end{aligned} \tag{1.6}$$

and the norm $||\cdot||_{0,K}$ derived from it. We notice that this inner-product makes sense for elements in $\mathcal{E}_s$ and we have that $||\cdot||^2_{0,K} = \sum_{j\geq 0} ||\cdot||^2_{0,K^j}$, where in the particular case of an absolutely continuous measure Γ with density f, the definition of the norm $||\cdot||^2_{0,K^j}$ is given by

$$||g||^2_{0,K^j} = E\left(\int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy g(s,x) K^j(s,x) f(x-y) g(s,y) K^j(s,y)\right)$$

(see [Dal99], Equation (22)).

It is worth mentioning that in [QSSS04a] the norm $\| \cdot \|_{+,K}$ and the inner product $\langle \cdot, \cdot \rangle_{0,K}$ are defined by means of the norm and scalar product associated to the Hilbert space $\mathcal{H}$, which in this dissertation is defined in Section 1.3; indeed, the space $\mathcal{H}_T = L^2([0,T]; \mathcal{H})$ shall be the underlying Hilbert space on which we develop the techniques of the Malliavin calculus.

By the first equality in (1.5) we have that

$$E\left(||(g \cdot M^K)_T(\mathbb{R}^d)||_{\mathcal{A}}^2\right) = ||g||_{0,K}^2 \tag{1.7}$$

for any $g \in \mathcal{E}_s$.

Let $\mathcal{P}_{0,K}$ be the completion of the inner-product space $(\mathcal{E}_s, \langle \cdot, \cdot \rangle_{0,K})$. Since we have $|| \cdot ||_{0,K} \leq || \cdot ||_{+,K}$, the space $\mathcal{P}_{0,K}$ will be in general larger than $\mathcal{P}_{+,K}$. So, we can extend the stochastic integral with respect to M^K to elements of $\mathcal{P}_{0,K}$. The extension is done through the isometry provided by (1.7) between $(\mathcal{P}_{0,K}, \| \cdot \|_{0,K})$ and the space $(\mathcal{M}, || \cdot ||_{\mathcal{M}})$ of $\mathcal{A}-$valued continuous square integrable martingales endowed with the norm $||X||_{\mathcal{M}}^2 = E(||X_T||_{\mathcal{A}}^2)$ (see Section 2 in [QSSS04a]).

The stochastic integral of a process $g \in \mathcal{P}_{0,K}$ with respect to M^K is denoted either by $(g \cdot M^K)_t$ or

$$\int_0^t \int_{\mathbb{R}^d} g(s,z) K(s,z) M(ds,dz).$$

Let us consider the particular case where the following stationary assumption is fulfilled.

Hypothesis C For all $j \geq 0,\ s \in [0,T],\ x, y \in \mathbb{R}^d$,

$$E(K^j(s,x)K^j(s,y)) = E(K^j(s,0)K^j(s,y-x)).$$

If Hypotheses B and C are satisfied, then for any *deterministic* function $g(s,z)$ such that $||g||_{0,K}^2 < \infty$ and $g(s) \in \mathcal{S}(\mathbb{R}^d)$ we have that

$$||g||_{0,K}^2 = \int_0^T ds \int_{\mathbb{R}^d} \mu_s^K(d\xi) |\mathcal{F}g(s)(\xi)|^2,$$

where the measure μ_s^K on $\mathbb{R}^d$ is defined as follows.

We consider the non-negative definite function $G_j^K(s,z) = E\left(K^j(s,0)K^j(s,z)\right)$. Owing to [Sch66], Theorem XIX, Chapter VII, the measure $\Gamma_{j,s}^K(dz) = G_j^K(s,z) \times \Gamma(dz)$, is a non-negative definite distribution. Thus, by Bochner's theorem (see for instance [Sch66], Theorem XVIII, Chapter VII) there exists a non-negative tempered measure $\mu_{j,s}^K$ such that $\Gamma_{j,s}^K(dz) = \mathcal{F}\mu_{j,s}^K$.

The measure $\Gamma_s^K(dz) := \sum_{j \geq 0} \Gamma_{j,s}^K(dz)$ is a well defined non-negative definite measure on $\mathbb{R}^d$. Consequently, there exists a non-negative tempered measure μ_s^K such that $\mathcal{F}\mu_s^K = \Gamma_s^K$. Furthermore, by the uniqueness and linearity of the Fourier transform, $\mu_s^K = \sum_{j\geq 0} \mu_{j,s}^K$.

Now we state the main result of the section. It is a quotation of Theorem 1 in [QSSS04a]. For the sake of completeness, we reproduce the proof in a more detailed form as it is performed in [QSSS04a]. This result is, indeed, the Hilbert-valued counterpart of Theorems 2 and 5 in [Dal99].

Theorem 1.2.5. *Let $\{K(s,z),(s,z)\in[0,T]\times\mathbb{R}^d\}$ be an $\mathcal{A}-$valued process for which Hypothesis B and C are satisfied. Let $t\mapsto S(t)$ be a deterministic function with values in the space of non-negative distributions with rapid decrease, such that*

$$\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}S(t)(\xi)|^2 < \infty.$$

Then S belongs to $\mathcal{P}_{0,K}$ and

$$E\big(||(S\cdot M^K)_t||_{\mathcal{A}}^2\big) = \int_0^t ds \int_{\mathbb{R}^d} \mu_s^K(d\xi)|\mathcal{F}S(s)(\xi)|^2. \tag{1.8}$$

Moreover, for any $p\in[2,\infty)$,

$$E\big(||(S\cdot M^K)_t||_{\mathcal{A}}^p\big) \leq C_t \int_0^t ds \sup_{x\in\mathbb{R}^d} E(||K(s,x)||_{\mathcal{A}}^p) \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}S(s)(\xi)|^2, \tag{1.9}$$

with $C_t = (\int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}S(s)(\xi)|^2)^{\frac{p}{2}-1}$, $t\in[0,T]$.

Proof. Let ψ be a non-negative function in $\mathcal{C}^\infty(\mathbb{R}^d)$ with support contained in the unit ball of $\mathbb{R}^d$ and such that $\int_{\mathbb{R}^d}\psi(x)dx = 1$. Set $\psi_n(x) = n^d\psi(nx)$, $n\geq 1$. Define $S_n(t) = \psi_n * S(t)$. We have that $S_n(t)\in\mathcal{S}(\mathbb{R}^d)$ for any $n\geq 1$, $t\in[0,T]$ and $S_n(t)\geq 0$.

The first step is to prove that $S_n\in\mathcal{P}_{+,K}\subset\mathcal{P}_{0,K}$. The definition of the norm $\|\cdot\|_{+,K}$, Cauchy-Schwarz's inequality and the relation between the measures μ and Γ yield

$$\begin{aligned}
\|S_n\|_{+,K}^2 =& \sum_{j\geq 0} E\left(\int_0^T ds \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy S_n(s,y) K^j(s,y)\right.\\
&\left.\times\, S_n(s,y-x)K^j(s,y-x)\right)\\
\leq& E\left(\int_0^T ds \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy S_n(s,y) S_n(s,y-x)\right.\\
&\left.\times\, \|K(s,y)\|_{\mathcal{A}} \|K(s,y-x)\|_{\mathcal{A}}\right)\\
\leq& \int_0^T ds \sup_{z\in\mathbb{R}^d} E(\|K(s,z)\|_{\mathcal{A}}^2) \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}S_n(s)(\xi)|^2.
\end{aligned} \tag{1.10}$$

Notice that $\sup_{n\geq 1} |\mathcal{F}\psi_n(\xi)| \leq 1$, which implies that

$$\sup_{n\geq 1} |\mathcal{F}S_n(s)(\xi)| \leq |\mathcal{F}S(s)(\xi)|. \tag{1.11}$$

Hence, owing to (1.10) we obtain that

$$\sup_{n\geq 1} \|S_n\|_{+,K} < +\infty. \tag{1.12}$$

Let us now show that

$$\lim_{n\to\infty} \|S_n - S\|_{0,K} = 0. \tag{1.13}$$

We have

$$\begin{aligned}\|S_n - S\|^2_{0,K} &= \int_0^T dt \int_{\mathbb{R}^d} \mu_t^K(d\xi) |\mathcal{F}(S_n(t) - S(t))(\xi)|^2 \\ &= \int_0^T dt \int_{\mathbb{R}^d} \mu_t^K(d\xi) |\mathcal{F}\psi_n(\xi) - 1|^2 |\mathcal{F}S(t)(\xi)|^2.\end{aligned}$$

The integrand in the last term of the above equality converges pointwise to zero as n tends to infinity. Then, since $|\mathcal{F}\psi_n(\xi) - 1| \leq 2$, to apply bounded convergence, it suffices to check that

$$\int_0^T dt \int_{\mathbb{R}^d} \mu_t^K(d\xi) |\mathcal{F}S(t)(\xi)|^2 < \infty.$$

We know that $|\mathcal{F}S_n(t)(\xi)|$ converges pointwise to $|\mathcal{F}S(t)(\xi)|$ and

$$\|S_n\|^2_{0,K} = \int_0^T dt \int_{\mathbb{R}^d} \mu_t^K(d\xi) |\mathcal{F}S_n(t)(\xi)|^2.$$

Then, Fatou's lemma imply

$$\|S\|^2_{0,K} \leq \liminf_{n\to\infty} \|S_n\|^2_{0,K} \leq \|S_n\|^2_{+,K} < \infty,$$

by (1.12). This finish the proof of (1.13) and therefore $S \in \mathcal{P}_{0,K}$.

By the isometry property of the stochastic integral we see that equality (1.8) holds for any S_n; then the construction of the stochastic integral yields

$$\begin{aligned}E(||(S\cdot M^K)_t||^2_{\mathcal{A}}) &= \lim_{n\to\infty} E(||(S_n \cdot M^K)_t||^2_{\mathcal{A}}) \\ &= \lim_{n\to\infty} \int_0^t ds \int_{\mathbb{R}^d} \mu_s^K(d\xi) |\mathcal{F}S_n(s)(\xi)|^2. \end{aligned} \tag{1.14}$$

Here we apply bounded convergence and we get

$$E(||(S\cdot M^K)_t||^2_{\mathcal{A}})=\int_0^t ds\int_{\mathbb{R}^d}\mu_s^K(d\xi)|\mathcal{F}S(s)(\xi)|^2.$$

This proves (1.8).

We now prove (1.9). From the first equality in (1.14) we obtain that there exists a partial sequence $(n_k)_{k\geq 1}$ such that

$$\lim_{k\to\infty}\|(S_{n_k}\cdot M^K)_t\|_{\mathcal{A}}=\|(S\cdot M^K)_t\|_{\mathcal{A}},$$

almost sure. By Fatou's lemma

$$E(\|(S\cdot M^K)_t\|^p_{\mathcal{A}})\leq\liminf_{k\to\infty}E(\|(S_{n_k}\cdot M^K)_t\|^p_{\mathcal{A}}).$$

For the sake of simplicity, in the sequel we shall write S_n instead of S_{n_k}.

Taking into account that each S_n is smooth, the stochastic integral $S_n\cdot M^K$ is defined following Walsh's theory (see the first part at the beginning of this section). The stochastic process $\{(S_n\cdot M^K)_t, t\geq 0\}$ is a $\mathcal{A}-$valued martingale. Then, Burkholder's inequality for Hilbert-valued martingales (see [Mét82]) and Schwarz's inequality ensure

$$\begin{aligned}&E(\|(S_n\cdot M^K)_t\|^p_{\mathcal{A}})\\&\leq CE\left(\sum_{j\geq 0}\int_0^t ds\int_{\mathbb{R}^d}dy\Gamma(dx)\int_{\mathbb{R}^d}S_n(s,y)S_n(s,y-x)K^j(s,y)K^j(s,y-x)\right)^{\frac{p}{2}}\\&\leq CE\left(\int_0^t ds\int_{\mathbb{R}^d}\Gamma(dx)\int_{\mathbb{R}^d}dyS_n(s,y)S_n(s,y-x)\|K(s,y)\|_{\mathcal{A}}\|K(s,y-x)\|_{\mathcal{A}}\right)^{\frac{p}{2}}\end{aligned}\tag{1.15}$$

Notice that, for each $n\geq 1$, $t\in[0,T]$, the measure on $[0,t]\times\mathbb{R}^d\times\mathbb{R}^d$ given by $S_n(s,y)S_n(s,y-x)ds\Gamma(dx)dy$ is finite. Indeed,

$$\begin{aligned}&\sup_{n\geq 1}\sup_{t\in[0,T]}\int_0^t ds\int_{\mathbb{R}^d}\Gamma(dx)\int_{\mathbb{R}^d}dyS_n(s,y)S_n(s,y-x)\\&\leq\sup_{n\geq 1}\sup_{t\in[0,T]}\int_0^t ds\int_{\mathbb{R}^d}\mu(d\xi)|\mathcal{F}S_n(s)(\xi)|^2\\&\leq\int_0^T ds\int_{\mathbb{R}^d}\mu(d\xi)|\mathcal{F}S(s)(\xi)|^2,\end{aligned}$$

which is finite by hypothesis. Thus, Hölder's inequality applied to this measure yields that the last term in (1.15) is bounded by

$$C\left(\int_0^T ds\int_{\mathbb{R}^d}\mu(d\xi)|\mathcal{F}S(s)(\xi)|^2\right)^{\frac{p}{2}-1}$$
$$\times\int_0^t ds\int_{\mathbb{R}^d}\Gamma(dx)\int_{\mathbb{R}^d}dyS_n(s,y)S_n(s,y-x)E(\|K(s,y)\|_{\mathcal{A}}^{\frac{p}{2}}\|K(s,y-x)\|_{\mathcal{A}}^{\frac{p}{2}}).$$

Finally, using Hypothesis B and (1.11) one gets.

$$E(\|(S\cdot M^K)_t\|_{\mathcal{A}}^p)\leq C\left(\int_0^T ds\int_{\mathbb{R}^d}\mu(d\xi)|\mathcal{F}S(s)(\xi)|^2\right)^{\frac{p}{2}-1}$$
$$\times\int_0^t ds\sup_{x\in\mathbb{R}^d}E(\|K(s,x)\|_{\mathcal{A}}^p)\int_{\mathbb{R}^d}\mu(d\xi)|\mathcal{F}S(s)(\xi)|^2.$$

Therefore (1.9) is proved. □

Remark 1.2.6. *From the identity (1.8) it follows that for any S satisfying the assumptions of Theorem 1.2.5 we have*

$$||S||_{0,K}^2=\int_0^T ds\int_{\mathbb{R}^d}\mu_s^K(d\xi)|\mathcal{F}S(s)(\xi)|^2.$$

Definition of mild solutions

In this last part of Section 1.2.1, we define what we understand by a solution of Equation (1.1); this is done by means of the so called *mild* formulation, as follows.

Definition 1.2.7. *A solution to the SPDE (1.1), with vanishing initial conditions, is a predictable real-valued stochastic process $\{u(t,x),(t,x)\in[0,T]\times\mathbb{R}^d\}$ such that*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}E(|u(t,x)|^2)<+\infty,$$

$$E(u(t,x)u(t,y))=E(u(t,0)u(t,x-y))$$

and the following stochastic evolution equation is fulfilled:

$$u(t,x)=\int_0^t\int_{\mathbb{R}^d}\Lambda(t-s,x-y)\sigma(u(s,y))M(ds,dy)$$
$$+\int_0^t ds\int_{\mathbb{R}^d}\Lambda(s,dy)b(u(t-s,x-y)). \tag{1.16}$$

Recall that Λ denotes the fundamental solution associated to the differential operator L on $\mathbb{R}^d$.

The stochastic integral in (1.16) is of the type defined in Theorem 1.2.5. More precisely, here the Hilbert space $\mathcal{A}$ is $\mathbb{R}$ and $K(s,z) := \sigma(u(s,z))$. Notice that, since σ is Lipschitz, the requirements on the process u ensure the validity of Hypothesis B. Concerning Hypothesis C, its validity is a consequence of the proof of Dalang's existence and uniqueness result (see Theorem 13 and Lemma 18 in [Dal99]); these arguments shall be made clearer in the next Section 1.2.2, where a more general result shall be proved (Theorem 1.2.12).

As in Dalang's Theorem 13 ([Dal99]), in Theorem 1.2.12 we will consider the following assumptions relating the differential operator and the spatial correlation of the noise:

Hypothesis D The fundamental solution Λ of $Lu = 0$ is a deterministic function in t taking values in the space of non-negative distributions with rapid decrease such that

$$\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}\Lambda(t)(\xi)|^2 < \infty.$$

Moreover, Λ is a non-negative measure on $\mathbb{R}_+ \times \mathbb{R}^d$ of the form $\Lambda(t,dy)dt$ such that $\sup_{0\leq t\leq T} \Lambda(t,\mathbb{R}^d) < +\infty$.

Under these hypotheses and assuming that the coefficients σ and b are Lipschitz functions, Dalang proved that Equation (1.1) admits a unique solution in the sense of definition 1.2.7 (see Theorem 13 in [Dal99]).

In the next Section 1.2.2 we present an existence and uniqueness result in a more general setting, namely for stochastic equations taking values in some Hilbert space; it is a quotation of Theorem 7.2 in [SSar].

Example 1.2.8. *In [Dal99] Equation (1.1) with* $L = \frac{\partial^2}{\partial t^2} - \Delta_d$ *and* $L = \frac{\partial}{\partial t} - \Delta_d$ *are studied. That is, the stochastic wave and heat equations, respectively.*

For the stochastic wave equation with $d \in \{1,2,3\}$ *and the stochastic heat equation with* $d \geq 1$*, a sufficient condition ensuring existence and uniqueness of solution to the corresponding equation is*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < +\infty.$$

We end this part by quoting a technical result –a version of Gronwall's Lemma proved in [Dal99]– that shall be applied throughout the proof of the main theorem in the next section and in the results collected in Appendices A and B.

Lemma 1.2.9 ([Dal99], Lemma 15). *Let $g : [0,T] \to \mathbb{R}_+$ be a non-negative function such that $\int_0^T g(s)ds < +\infty$. Then there is a sequence $(a_n, n \in \mathbb{N})$ of non-negative real numbers such that for all $p \geq 1$, $\sum_{n=1}^{\infty} a_n^{\frac{1}{p}} < +\infty$, and with the following property: Let $(f_n, n \in \mathbb{N})$ be a sequence of non-negative functions on $[0,T]$ and k_1, k_2 be non-negative numbers such that for $0 \leq t \leq T$,*

$$f_n(t) \leq k_1 + \int_0^t (k_2 + f_{n-1}(s))g(t-s)ds.$$

If $\sup_{0\leq s\leq T} f_0(s) = M$, then for $n \geq 1$,

$$f_n(t) \leq k_1 + (k_1 + k_2)\sum_{i=1}^{n-1} a_i (k_2 + M)a_n.$$

In particular, $\sup_{n\geq 0}\sup_{0\leq t\leq T} f_n(t) < +\infty$, and if $k_1 = k_2 = 0$, then $\sum_{n\geq 0} f_n(t)^{\frac{1}{p}}$ converges uniformly on $[0,T]$.

1.2.2 A result on existence and uniqueness of solution

This section is devoted to present a SPDE in a general Hilbert-valued context. We define a *mild* solution of this equation and we prove an existence and uniqueness result.

The motivation of the above mentioned setting comes from the fact that, when dealing with the study of the probability law of the solution to (1.1) via Malliavin calculus, we need to consider Malliavin derivatives of any order of the solution and show that they satisfy stochastic integral equations obtained by differentiation of (1.16). The equations obtained by this procedure take values in some Hilbert space.

For instance, as it was mentioned in the Introduction, if the coefficients b and σ are differentiable, a formal differentiation of Equation (1.16) reads

$$\begin{aligned} Du(t,x) = Z(t,x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z)\sigma'(u(s,z))Du(s,z)M(ds,dz) \\ + \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)b'(u(t-s,x-z))Du(t-s,x-z), \end{aligned}$$

where $Z(t,x)$ is some Hilbert-valued stochastic process that is made explicit in Section 2.1 (see also Theorem 2 in [QSSS04a]). Thus a Hilbert-valued framework is needed.

Let $\mathcal{K}$ and $\mathcal{A}$ be two separable Hilbert spaces. We consider two operators

$$\sigma, b : \mathcal{K} \times \mathcal{A} \to \mathcal{A}$$

satisfying the following two conditions:

(H1) it holds that

$$\sup_{x\in\mathcal{K}} \left(\|\sigma(x,y)-\sigma(x,z)\|_{\mathcal{A}} + \|b(x,y)-b(x,z)\|_{\mathcal{A}}\right) \le C\|y-z\|_{\mathcal{A}},$$

for all $y,z\in\mathcal{A}$ and $x\in\mathcal{K}$,

(H2) there exists $q\in[1,\infty)$ such that

$$\|\sigma(x,0)\|_{\mathcal{A}} + \|b(x,0)\|_{\mathcal{A}} \le C(1+\|x\|_{\mathcal{K}}^q),$$

for all $x\in\mathcal{K}$,

for some positive constant C. Notice that (H1) and (H2) clearly imply

(H3) $\|\sigma(x,y)\|_{\mathcal{A}} + \|b(x,y)\|_{\mathcal{A}} \le C(1+\|x\|_{\mathcal{K}}^q + \|y\|_{\mathcal{A}})$.

We assume that Hypothesis D from the previous Section 1.2.1 is fulfilled.

Let $\{U_0(t,x),(t,x)\in[0,T]\times\mathbb{R}^d\}$ and $V=\{V(t,x),(t,x)\in[0,T]\times\mathbb{R}^d\}$ be predictable $\mathcal{A}-$valued and $\mathcal{K}-$valued processes, respectively, such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\left(\|V(t,x)\|_{\mathcal{K}}^p + \|U_0(t,x)\|_{\mathcal{A}}^p\right) < +\infty,$$

for any $p\in[1,\infty)$.

We consider the following definition, which corresponds to Definition 5.1 in [Dal99] for our standing setting.

Definition 1.2.10. *For $z\in\mathbb{R}^d$, let $z+B=\{z+y,y\in B\}$, $B\in\mathcal{B}_b(\mathbb{R}^{d+1})$, and define the martingale measure $\{M_s^{(z)}(B),s\in\mathbb{R}_+,B\in\mathcal{B}_b(\mathbb{R}^{d+1})\}$ by $M_s^{(z)}(B)=M_s(z+B)$. Given a Hilbert-valued process $\{X(s,x),(s,x)\in\mathbb{R}_+\times\mathbb{R}^d\}$, we also set $X^{(z)}(s,x)=X(s,z+x)$. We say that the process X has* property (S) *if for all $z\in\mathbb{R}^d$, the joint distribution of the processes $\{V^{(z)}(t,x),(t,x)\in[0,T]\times\mathbb{R}^d\}$, $\{X^{(z)}(s,x),(s,x)\in\mathbb{R}_+\times\mathbb{R}^d\}$ and $\{M_s^{(z)}(B),s\in\mathbb{R}_+,B\in\mathcal{B}_b(\mathbb{R}^{d+1})\}$ does not depend on z.*

We consider the stochastic integral equation on $\mathcal{A}$

$$\begin{aligned} U(t,x) =& U_0(t,x) + \int_0^t\int_{\mathbb{R}^d} \Lambda(t-s,x-y)\sigma(V(s,y),U(s,y))M(ds,dy) \\ &+ \int_0^t ds\int_{\mathbb{R}^d} \Lambda(s,dy)b(V(t-s,x-y),U(t-s,x-y)) \end{aligned} \tag{1.17}$$

The following definition is the analogue of Definition 1.2.7 in the Hilbert-valued context of Equation (1.17).

Definition 1.2.11. *A solution to Equation (1.17) is an $\mathcal{A}-$valued predictable stochastic process $U = \{U(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ such that*

(1) $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(\|U(t,x)\|^2_{\mathcal{A}}) < +\infty$,

(2) $E(\langle U(t,x), U(t,y)\rangle_{\mathcal{A}}) = E(\langle U(t,0), U(t,x-y)\rangle_{\mathcal{A}})$,

and satisfies Equation (1.17).

In (1.17), we consider the stochastic integral of the kind given in Theorem 1.2.5. The next discussion provides a meaning to the pathwise integral.

Let $\{Y(s,y), (s,y) \in [0,T] \times \mathbb{R}^d\}$ be an $\mathcal{A}-$valued stochastic process such that

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}^d} E(\|Y(s,y)\|^2_{\mathcal{A}}) < +\infty.$$

Then, for any orthonormal system $(e_j)_{j\geq 1}$ in $\mathcal{A}$, Cauchy-Schwarz inequality, the boundedness of the measure given by $\Lambda(s,dy)dy$, on $[0,T]\times\mathbb{R}^d$, and Parseval's identity yield

$$\begin{aligned}\sum_{j=1}^{\infty} E\left(\left|\int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dy)\langle Y(s,y), e_j\rangle_{\mathcal{A}}\right|^2\right) &\leq C \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dy) E(\|Y(s,y)\|^2_{\mathcal{A}}) \\ &\leq C \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} E(\|Y(s,y)\|^2_{\mathcal{A}}),\end{aligned}$$

which is finite, by hypothesis. Thus,

$$\left(\int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dy)\langle Y(s,y), e_j\rangle_{\mathcal{A}}, j \geq 1\right)$$

determines a well-defined element of $L^2(\Omega;\mathcal{A})$, which is denoted by

$$\int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dy) Y(s,y).$$

Notice that, by definition,

$$\left\langle \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dy) Y(s,y), e_j \right\rangle_{\mathcal{A}} = \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dy)\langle Y(s,y), e_j\rangle_{\mathcal{A}},$$

for any $j \geq 1$.

Our next purpose is to state and prove a result on existence and uniqueness of solution for Equation (1.17). In particular, we shall obtain the version proved in [Dal99] for the particular case of Equation (1.16). The result is a quotation of Theorem 7.2 in [SSar]. The techniques used in the proof are somehow classical, but they incorporate some arguments that are going to be useful in the proofs of the main results of the papers collected in the Appendices. For this reason we think it is worthy to reproduce the details.

Theorem 1.2.12. *Assume that the coefficients σ and b satisfy conditions (H1) and (H2) above and that Hypothesis D is satisfied. Assume, moreover, that the process $\{U_0(t,x),(t,x) \in [0,T] \times \mathbb{R}^d\}$ satisfies property (S). Then, Equation (1.17) has a unique solution in the sense given in Definition 1.2.11. In addition, the solution satisfies that*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(\|U(t,x)\|_{\mathcal{A}}^p) < +\infty, \tag{1.18}$$

for any $p \in [1,\infty)$.

Proof of Theorem 1.2.12. We define the standard Picard iteration scheme

$$\begin{aligned} U^0(t,x) =& U_0(t,x), \\ U^n(t,x) =& U_0(t,x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-y)\sigma(V(s,y),U^{n-1}(s,y))M(ds,dy) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dy) b(V(t-s,x-y),U^{n-1}(t-s,x-y)), \end{aligned} \tag{1.19}$$

for $n \geq 1$. For any $p \in [1,\infty)$, we prove the following facts:

(i) The sequence of processes $U^n = \{U^n(t,x),(t,x) \in [0,T] \times \mathbb{R}^d\}$, $n \geq 1$, are well defined predictable processes and satisfy property (S).

(ii) It holds that
$$\sup_{n\geq 0} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(\|U^n(t,x)\|_{\mathcal{A}}^p) < +\infty.$$

(iii) For $n \geq 0$, set
$$M_n(t) = \sup_{(s,x)\in[0,t]\times\mathbb{R}^d} E(\|U^{n+1}(t,x) - U^n(t,x)\|_{\mathcal{A}}^p).$$

Then
$$M_n(t) \leq C \int_0^t ds M_{n-1}(s)(J(t-s)+1), \tag{1.20}$$
where the function J is defined by
$$J(t) = \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Lambda(t)(\xi)|^2. \tag{1.21}$$

Proof of (i). We prove by induction on n that U^n is predictable, satisfies property (S) and

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(\|U^n(t,x)\|_{\mathcal{A}}^2) < +\infty. \tag{1.22}$$

This is sufficient to give a rigorous meaning to the integrals appearing in (1.19). Indeed, by assumption this is true for $n = 0$. Assume that the property is true for any $k = 0, 1, \dots, n-1$, $n \geq 2$. Consider the stochastic process given by

$$K(t,x) = \sigma(V(t,x), U^{n-1}(t,x)). \tag{1.23}$$

The induction hypothesis and the assumptions on V and σ ensure the validity of Hypothesis B and C. Thus, $K(t,x)$ satisfies the assumptions of Theorem 1.2.5. In particular (1.9) for $p = 2$ yields

$$\begin{aligned}
&\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\left(\left\|\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-y)\sigma(V(s,y), U^{n-1}(s,y))M(ds,dy)\right\|_{\mathcal{A}}^2\right) \\
&\quad \leq C \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(1 + \|U^{n-1}(t,x)\|_{\mathcal{A}}^2 + \|V(t,x)\|_{\mathcal{K}}^{2q}) \\
&\quad \times \int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Lambda(t)(\xi)|^2.
\end{aligned}$$

This last expression is finite, by assumption.

Let us deal with the pathwise integral in (1.19). By the assumptions on b, we have that

$$\begin{aligned}
&E\left(\left\|\int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dy) b(V(t-s,x-y), U^{n-1}(t-s,x-y))\right\|_{\mathcal{A}}^2\right) \\
&\leq C \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dy) E\left(\|b(V(t-s,x-y), U^{n-1}(t-s,x-y))\|_{\mathcal{A}}^2\right) \\
&\leq C \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(1 + \|U^{n-1}(t,x)\|_{\mathcal{A}}^2 + \|V(t,x)\|_{\mathcal{K}}^{2q}) \int_0^T ds \Lambda(s, \mathbb{R}^d),
\end{aligned}$$

which is finite. Thus we have proved (1.22). By Lemma 1.2.14 below, U^n satisfies property (S).

Proof of (ii). Fix $p \in [1, \infty)$. The arguments are very similar to those used in the proof of (i). Indeed, we have

$$E(\|U^n(t,x)\|_{\mathcal{A}}^p) \leq C(C_0(t,x) + A_n(t,x) + B_n(t,x)), \tag{1.24}$$

with

$$\begin{aligned}
C_0(t,x) &= E(\|U_0(t,x)\|_{\mathcal{A}}^p) \\
A_n(t,x) &= E\left(\left\|\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-y)\sigma(V(s,y), U^{n-1}(s,y))M(ds,dy)\right\|_{\mathcal{A}}^p\right), \\
B_n(t,x) &= E\left(\left\|\int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dy) b(V(t-s,x-y), U^{n-1}(t-s,x-y))\right\|_{\mathcal{A}}^p\right).
\end{aligned}$$

By assumption

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} C_0(t,x) < +\infty. \tag{1.25}$$

Consider the stochastic process $K(t,x)$ defined in (1.23), which satisfies the assumptions of Theorem 1.2.5. In particular (1.9) and assumption (H3) on σ yield

$$\begin{aligned}\sup_{x\in\mathbb{R}^d} A_n(t,x) \leq & C\int_0^t ds \sup_{x\in\mathbb{R}^d} E(\|\sigma(V(s,x),U^{n-1}(s,x))\|_{\mathcal{A}}^p)\int_{\mathbb{R}^d}\mu(d\xi)|\mathcal{F}\Lambda(t-s)(\xi)|^2\\ \leq & C\int_0^t ds\left(1+\sup_{x\in\mathbb{R}^d} E(\|U^{n-1}(s,x)\|_{\mathcal{A}}^p)\right)\int_{\mathbb{R}^d}\mu(d\xi)|\mathcal{F}\Lambda(t-s)(\xi)|^2.\end{aligned} \tag{1.26}$$

On the other hand, Hölder's inequality with respect to the finite measure on $[0,T]\times\mathbb{R}^d$ given by $\Lambda(s,dz)ds$ and (H3) for b imply

$$\begin{aligned}\sup_{x\in\mathbb{R}^d} B_n(t,x) &\leq CE\left(\int_0^t ds\int_{\mathbb{R}^d}\Lambda(s,dy)\|b(V(t-s,x-y),U^{n-1}(t-s,x-y))\|_{\mathcal{A}}^2\right)^{\frac{p}{2}}\\ &\leq C\int_0^t ds\int_{\mathbb{R}^d}\Lambda(s,dy)E(\|b(V(t-s,x-y),U^{n-1}(t-s,x-y))\|_{\mathcal{A}}^p)\\ &\leq C\int_0^t ds\left(1+\sup_{x\in\mathbb{R}^d}E(\|U^{n-1}(s,x)\|^p)\right)\int_{\mathbb{R}^d}\Lambda(t-s,dy)\\ &\leq C\int_0^t ds\left(1+\sup_{x\in\mathbb{R}^d}E(\|U^{n-1}(s,x)\|^p)\right)\end{aligned} \tag{1.27}$$

Putting together the estimates (1.25) to (1.27) into (1.24) we obtain that

$$\sup_{x\in\mathbb{R}^d} E(\|U^n(t,x)\|_{\mathcal{A}}^p) \leq C + C\int_0^t ds \sup_{x\in\mathbb{R}^d} E(\|U^{n-1}(t,x)\|_{\mathcal{A}}^p)(J(t-s)+1),$$

for any $t\in[0,T]$ and $n\geq 1$.

The conclusion of part (ii) follows applying the version of Gronwall's lemma given in Lemma 1.2.9 to the following situation: $f_n(t)=\sup_{x\in\mathbb{R}^d} E(\|U^n(t,x)\|^p)$, $k_1=C$, $k_2=0$, $g(s)=C(J(s)+1)$, for some positive constant C.

Proof of (iii). We consider the decomposition

$$E(\|U^{n+1}(t,x)-U^n(t,x)\|^p)\leq C(a_n(t,x)+b_n(t,x)),$$

with

$$\begin{aligned} a_n(t,x) =& E\left(\left\|\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-y)\left(\sigma(V(s,y),U^n(s,y))\right.\right.\right. \\ & \left.\left.\left. - \sigma(V(s,y),U^{n-1}(s,y))\right)M(ds,dy)\right\|_{\mathcal{A}}^p\right), \\ b_n(t,x) =& E\left(\left\|\int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dy)\left(b(V(t-s,x-y),U^n(t-s,x-y))\right.\right.\right. \\ & \left.\left.\left. - b(V(t-s,x-y),U^{n-1}(t-s,x-y))\right)\right\|_{\mathcal{A}}^p\right). \end{aligned}$$

Owing to Theorem 1.2.5 and the Lipschitz property of σ, we obtain

$$\begin{aligned} a_n(t,x) \leq& C \int_0^t \sup_{x\in\mathbb{R}^d} E\left(\|\sigma(V(s,x),U^n(s,x)) - \sigma(V(s,x),U^{n-1}(s,x))\|_{\mathcal{A}}^p\right) \\ & \times \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Lambda(t-s)(\xi)|^2 \\ \leq& C \int_0^t ds \left(\sup_{(\tau,x)\in[0,s]\times\mathbb{R}^d} E(\|U^n(\tau,x) - U^{n-1}(\tau,x)\|_{\mathcal{A}}^p\right) J(t-s). \qquad (1.28) \end{aligned}$$

Using similar arguments as those used in the study of the term $B_n(t,x)$, condition (H1) for b and Hölder's inequality with respect to the finite measure on $[0,T]\times\mathbb{R}^d$ given by $\Lambda(s,dy)ds$, we end up with

$$\begin{aligned} b_n(t,x) \leq& E\left(\int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dy)\|U^n(t-s,x-y) - U^{n-1}(t-s,x-y)\|_{\mathcal{A}}^2\right)^{\frac{p}{2}} \\ \leq& C \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dy)E(\|U^n(t-s,x-y) - U^{n-1}(t-s,x-y)\|_{\mathcal{A}}^p) \\ \leq& C \int_0^t ds \sup_{(\tau,x)\in[0,s]\times\mathbb{R}^d} E(\|U^n(\tau,x) - U^{n-1}(\tau,x)\|_{\mathcal{A}}^p). \qquad (1.29) \end{aligned}$$

Then, (1.20) follows from (1.28) and (1.29).

We finish the proof applying Lemma 1.2.9 in the particular case $f_n(t) = M_n(t)$, $k_1 = k_2 = 0$, $g(s) = C(J(s)+1)$, with C given in (1.20). Notice that the results proved in part (ii) show that $M := \sup_{0\leq s\leq T} f_0(s)$ is finite. Then we conclude that $\{U_n(t,x),(t,x)\in[0,T]\times\mathbb{R}^d\}$ converges in $L^p(\Omega)$ to a limit $U = \{U(t,x),(t,x)\in[0,T]\times\mathbb{R}^d\}$, uniformly with respect to $(t,x)\in[0,T]\times\mathbb{R}^d$.

It only remains to show that the process U satisfies Equation (1.17); notice that condition (1.18) follows from (ii).

By Equation (1.19), it suffices to show that the integrals

$$\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-y)\sigma(V(s,y), U^{n-1}(s,y))M(ds, dy)$$

and

$$\int_0^t ds \int_{\mathbb{R}^d} \Lambda(s, dy) b(V(t-s, x-y), U^{n-1}(t-s, x-y))$$

converge in $L^2(\Omega; \mathcal{A})$, as n tends to infinity, to the second and third term in the right hand-side of Equation (1.17), respectively. This can be achieved using the same arguments as for the study of the terms $a_n(t,x)$ and $b_n(t,x)$, respectively, and the convergence

$$\lim_{n\to\infty} \left(\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(\|U^n(t,x) - U(t,x)\|_{\mathcal{A}}^p) \right) = 0.$$

Therefore the theorem is completely proved. □

Example 1.2.13. *Let $\mathcal{A} = \mathcal{K} = \mathbb{R}$ and σ and b depend only on the second variable $y \in \mathbb{R}$. Then condition (H1) states the Lipschitz continuity, (H2) is trivial and (H3) follows from (H1). Equation (1.17) is of the same kind of (1.16), except for the non trivial initial condition. Therefore Theorem 1.2.12 yields the existence of a unique solution in the sense of Definition 1.2.7. Moreover, the process u satisfies*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|u(t,x)|^p) < +\infty.$$

This is a variant of Theorem 13 in [Dal99].

Next, we state a lemma that has been applied along the proof of the above theorem; it is the Hilbert-valued counterpart of Lemma 18 in [Dal99]. For the sake of completeness, we include a detailed proof.

Lemma 1.2.14. *For $n \geq 1$, if $\{U^{n-1}(t,x), (t,x) \in [0,T]\times\mathbb{R}^d\}$ has property (S), then the process $\{U^n(t,x), (t,x) \in [0,T]\times\mathbb{R}^d\}$ defined by Equation (1.19) satisfies property (S) as well.*

Proof. Set $u_0(t) = U_0(t,0)$, $u_0^{(x)}(t) = U_0(t,x)$, for $x \in \mathbb{R}^d$. From (1.19), it is straightforward to check that

$$\begin{aligned} U^n(t,x) =& u_0^{(x)}(t) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,-y)\sigma(V^{(x)}(s,y), (U^{n-1})^{(x)}(s,y))M^{(x)}(ds,dy) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dy) b(V^{(x)}(t-s,-y), (U^{n-1})^{(x)}(t-s,-y)). \end{aligned}$$

Therefore, $U^n(t,x)$ is an abstract function Ψ of $u_0^{(x)}$, $V^{(x)}$, $(U^{n-1})^{(x)}$, and $M^{(x)}$; that is, $U^n(t,x) = \Psi(u_0^{(x)}, V^{(x)}, (U^{n-1})^{(x)}, M^{(x)})$. Similarly,

$$(U^n)^{(z)}(t,x) = \Psi(u_0^{(x+z)}, V^{(x+z)}, (U^{n-1})^{(z+x)}, M^{(z+x)}).$$

Then, for any $z \in \mathbb{R}^d$, $(s_1,\dots,s_k)$, $(t_1,\dots,t_k)$, $(\tau_1,\dots,\tau_k)$ and $(r_1,\dots,r_k)$ in $\mathbb{R}_+^k$, $(y_1,\dots,y_k)$ and $(z_1,\dots,z_k)$ in $(\mathbb{R}^d)^k$ and for all bounded Borel sets $B_1,\dots,B_k$ of $\mathbb{R}^d$, the joint distribution of

$$\big(u_0^{(z)}(s_i), V^{(z)}(t_i,y_i), (U^n)^{(z)}(\tau_i,z_i), M_{r_i}^{(z)}(B_i), i=1,\dots,k\big)$$

is a function of the joint distribution of

$$u_0^{(z)}(\cdot), u_0^{(z+z_i)}(\cdot), V^{(z)}(\cdot,\cdot), V^{(z+z_i)}(\cdot,\cdot), (U^{n-1})^{(z+z_i)}(\cdot,\cdot), M_{\cdot}^{(z+z_i)}(\cdot), M_{r_i}^{(z)}(B_i),$$

$i=1,\dots,k$. By property (S) for U^{n-1}, this joint distribution does not depend on z. Therefore, property (S) holds for U^n. □

1.3 Preliminaries on Malliavin calculus

This section is devoted to present the basic objects and notations of the Malliavin calculus needed along the chapter. We also state some results concerning differentiability (in the sense of Malliavin) and existence and smoothness of densities of random variables. Finally, at the end of the section we give a detailed description of the setting in which we shall apply the Malliavin calculus in the papers collected in the Appendices A and B.

The stochastic calculus of variations or Malliavin calculus, set up in the seminal paper [Mal78], provides a useful tool for the analysis of densities of Brownian functionals, and more generally for functionals of Gaussian families indexed by a real separable Hilbert space (for general manuscripts on this topic we refer to [Nua95], [Mal97], [Nua98] and to [SSar] for a more concrete application to SPDEs). Here we are going to give the main definitions in a general setting. In the next sections we shall specify the Hilbert space and the Gaussian family on which we will base the Malliavin calculus.

Let H be a real separable Hilbert space. Denote by $\|\cdot\|_H$ and $\langle\cdot,\cdot\rangle_H$ the norm and the scalar product of H, respectively. Then, there exists a probability space $(\Omega,\mathcal{F},P)$ and a family $(W(h), h\in H)$ of Gaussian random variables defined on this space, such that $EW(h)=0$ and $E(W(h_1)W(h_2)) = \langle h_1,h_2\rangle_H$, for $h,h_1,h_2\in H$ (see, for instance, [SSar], p. 15).

We begin with the definition of the Malliavin derivative operator. Let $\mathcal{C}_b^\infty(\mathbb{R}^n)$ be the space of infinitely differentiable functions having bounded partial derivatives of any order and let $\mathcal{S}$ denote the class of random variables of the form

$$F = f(W(h_1), \dots, W(h_n)),$$

where f belongs to $\mathcal{C}_b^\infty(\mathbb{R}^n)$, $h_1, \dots, h_n$ are in H, and $n \geq 1$. These random variables are called *smooth*. We define the derivative operator D on $\mathcal{S}$ as the $H-$valued random variable given by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), \dots, W(h_n))h_i.$$

It can be shown that D is closable as an operator from $\mathcal{S} \subset L^p(\Omega)$ into $L^p(\Omega; H)$ (see [Nua98], p. 128). We keep the notation D for the closed extension operator.

We will denote the domain of D in $L^p(\Omega)$ by $\mathbb{D}^{1,p}$, meaning that $\mathbb{D}^{1,p}$ is the closure of the class of smooth random variables $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} = [E(|F|^p) + E(\|DF\|_H^p)]^{\frac{1}{p}}.$$

For $p = 2$, the space $\mathbb{D}^{1,2}$ is a Hilbert space with scalar product

$$\langle F, G\rangle_{1,2} = E(FG) + E(\langle DF, DG\rangle_H).$$

We can define the iteration of the operator D in such a way that for a smooth random variable F, the derivative $D^N F$ is a random variable taking values in $H^{\otimes N}$. Then, for every $p \geq 1$ and any natural number N, we introduce a semi-norm on $\mathcal{S}$ defined by

$$\|F\|_{N,p}^p = E(|F|^p) + \sum_{j=1}^N E(\|D^j F\|_{H^{\otimes j}}^p).$$

As in the case $N = 1$, one can show that the operator D^N is closable from $\mathcal{S} \subset L^p(\Omega)$ into $L^p(\Omega; H^{\otimes N})$, $p \geq 1$, the extension being also denoted by D^N. For any $p \geq 1$ and any natural $N \geq 1$, we will denote by $\mathbb{D}^{N,p}$ the completion of the family of smooth random variables $\mathcal{S}$ with respect to the norm $\|\cdot\|_{N,p}$. Notice that, by definition, $\mathbb{D}^{j,q} \subset \mathbb{D}^{k,p}$ for $j \geq k$ and $q \geq p$. We shall use the notation

$$\mathbb{D}^\infty = \cap_{p\geq 1} \cap_{k \geq 1} \mathbb{D}^{k,p}.$$

The spaces $\mathbb{D}^{k,p}$ can be extended in the following way. Let V be a Hilbert space and $\mathcal{S}_V$ be the set of smooth random vectors taking values in V of the form

$$F = \sum_{j=1}^n F_j v_j,$$

$v_j \in V$, $F_j \in \mathcal{S}$, $j = 1, \ldots, n$, $n \geq 1$. By definition, the N-th derivative of F is given by

$$D^N F = \sum_{j=1}^{n} D^N F_j \otimes v_j.$$

As before, one can prove that D^N is a closable operator from $\mathcal{S}_V \subset L^p(\Omega; V)$ into $L^p(\Omega; H^{\otimes N} \otimes V)$, $p \geq 1$. Then, for any $N \in \mathbb{N}$, $p \in [1, \infty)$, we introduce the seminorm on $\mathcal{S}_V$ given by

$$\|F\|_{N,p,V}^p = E(\|F\|_V^p) + \sum_{j=1}^{N} E(\|D^j F\|_{H^{\otimes j} \otimes V}^p).$$

We define $\mathbb{D}^{N,p}(V)$ as the completion of $\mathcal{S}_V$ with respect to this norm.

The following lemmas give sufficient conditions to ensure regularity of random variables in the Malliavin sense. The first one is proved in [LNS89] and the second one is an immediate consequence of the fact that the operator D^N is a closed operator defined on $L^p(\Omega)$ with values in $L^p(\Omega; H^{\otimes N})$.

Lemma 1.3.1. *Let $(F_n)_{n\geq 1}$ be a sequence of random variables belonging to $\mathbb{D}^{1,p}$, for some $p \in [2, \infty)$. Assume that the following two conditions are fulfilled:*

1. *The sequence $(F_n)_{n\geq 1}$ converges in $L^p(\Omega)$ to a random variable F;*
2. $\sup_{n\geq 1} E(\|DF_n\|_H^p) < +\infty$.

Then F belongs to $\mathbb{D}^{1,p}$ and there is a subsequence of $(DF_n)_{n\geq 1}$ converging to DF in the weak topology of $L^p(\Omega; H)$.

Lemma 1.3.2. *Let $(F_n)_{n\geq 1}$ be a sequence of random variables belonging to $\mathbb{D}^{N,p}$. Assume that*

(a) there exists a random variable F such that F_n converges to F in $L^p(\Omega)$, as n tends to infinity,

(b) the sequence $\left(D^N F_n\right)_{n\geq 1}$ converges in $L^p(\Omega; H^{\otimes N})$.

Then F belongs to $\mathbb{D}^{N,p}$ and $D^N F = L^p(\Omega; H^{\otimes N}) - \lim_{n\to\infty} D^N F_n$.

Now we state the chain rule for the Malliavin derivative, as follows.

Proposition 1.3.3 ([Nua95], Proposition 1.2.2). *Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives. Fix $p \geq 1$ and suppose that*

$F = (F^1, \ldots, F^m)$ *is a random vector whose components belong to the space* $\mathbb{D}^{1,p}$. *Then* $\varphi(F) \in \mathbb{D}^{1,p}$, *and*

$$D(\varphi(F)) = \sum_{i=1}^{m} \frac{\partial \varphi}{\partial x_i}(F) DF^i.$$

Let $N \in \mathbb{N}$. For $h_i \in H$, $i = 1, \ldots, N$, and F a random variable we will write

$$D^N_{(h_1,\ldots,h_N)} F = \langle D^N F, h_1 \otimes \cdots \otimes h_N \rangle_{H^{\otimes N}}.$$

Set $A_N = \{(h_1, \ldots, h_N) \in H^{\otimes N}\}$ and denote by $\mathcal{P}_m$ the set of partitions of A_N consisting of m disjoint subsets $p_1, \ldots, p_m$, $m = 1, \ldots, N$, and by $|p_i|$ the cardinal of p_i. We have the following Leibniz's rule for Malliavin derivatives:

Proposition 1.3.4. *Let* $\varphi : \mathbb{R}^m \to \mathbb{R}$ *be a* $\mathcal{C}^\infty$ *function with bounded partial derivatives of any order. Fix* $p \geq 1$ *and suppose that* F *is a random variable belonging to the space* $\mathbb{D}^{N,p}$. *Then* $\varphi(F) \in \mathbb{D}^{N,p}$, *and*

$$D^N_{(h_1,\ldots,h_N)}(\varphi(F)) = \sum_{m=1}^{N} \sum_{\mathcal{P}_m} \varphi^{(m)}(F) \prod_{i=1}^{m} D^{|p_i|}_{p_i} F.$$

We can localise the domains of the operators D as follows. We will denote by $\mathbb{D}^{1,p}_{loc}$, $p \geq 1$, the set of random variables F such that there exists a sequence $\big((\Omega_n, F_n), n \geq 1\big)$ included in $\mathcal{F} \times \mathbb{D}^{1,p}$ with the following properties:

(i) $\Omega_n \nearrow \Omega$, a.s. as n tends to infinity.

(ii) $F = F_n$, a.s. on Ω_n, for all $n \geq 1$.

We then say that (Ω_n, F_n) is a localising sequence of F in $\mathbb{D}^{1,p}$, and DF is defined without ambiguity by $DF = DF_n$ on Ω_n, $n \geq 1$. The space $\mathbb{D}^{N,p}_{loc}$ can be introduced analogously.

We will denote by δ the adjoint of the operator D as an unbounded operator from $L^2(\Omega)$ into $L^2(\Omega; H)$. That is, the domain of δ, denoted by Dom δ, is the set of $H-$valued square integrable random variables u such that

$$|E(\langle DF, u \rangle_H)| \leq C \|F\|_{L^2(\Omega)},$$

for all $F \in \mathbb{D}^{1,2}$, where C is some constant depending on u. If u belongs to Dom δ, then $\delta(u)$ is the element of $L^2(\Omega)$ characterised by

$$E(F\delta(u)) = E(\langle DF, u \rangle_H)$$

for any $F \in \mathbb{D}^{1,2}$.

We will refer to the operator δ as the *divergence operator* or the *Skorohod integral*. In case H is a Hilbert space of the form $L^2(X, \mathcal{B}, \mu)$, where μ is a $\sigma-$finite measure without atoms on a measurable space $(X, \mathcal{B})$, then the Skorohod integral extends the usual Itô stochastic integral (see [Nua95], Sec. 1.3.2).

We denote by $\mathcal{S}_H$ the class of smooth elementary elements of the form

$$u = \sum_{j=1}^{n} F_j h_j,$$

where the F_j are smooth random variables, and the h_j are elements of H. It can be shown (see [Nua98], p. 130) that an element u of this form belong to the domain of δ and, moreover, it holds that

$$\delta(u) = \sum_{j=1}^{n} F_j W(h_j) - \sum_{j=1}^{n} \langle DF_j, h_j \rangle_H.$$

A useful property of the Skorohod integral is the commutativity relationship between it and the derivative operator. Namely, for smooth elements u we have the following relation:

$$D_h(\delta(u)) = \langle u, h \rangle_H + \delta(D_h u),$$

where we have used the notation $D_h F = \langle DF, h \rangle_H$. Assuming also that the Hilbert space H is of the form $L^2(X, \mathcal{B}, \mu)$, as mentioned above, this property can be extended to random vectors u belonging to $\mathbb{D}^{1,2}(H)$. For this, notice that in this case $L^2(\Omega; H) \cong L^2(\Omega \times X)$; thus DF is a function of two variables, $\omega \in \Omega$ and $t \in X$. We note $DF(t) = D_t F$. The result is as follows (see either [Nua98], Section 1.2, or [SSar], Proposition 4.15):

Proposition 1.3.5. *Let $u \in \mathbb{D}^{1,2}(H)$. Assume that for almost every $t \in X$, the process $\{D_t u(s), s \in X\}$ belongs to $Dom\delta$ and there is a version of the process $\{\delta(D_t u(s)), t \in X\}$ which is in $L^2(\Omega \times X)$. Then $\delta(u)$ belongs to $\mathbb{D}^{1,2}$ and we have*

$$D_t(\delta(u)) = u(t) + \delta(D_t u),$$

$t \in X$.

Along the chapter we shall make use of this property in order to compute Malliavin derivatives of stochastic integrals.

We conclude this enumeration of results by stating two criteria on existence and regularity of densities of random variables; the source can be found in [Mal78]. First we introduce a notion that plays a crucial role.

Definition 1.3.6. *Let $F = (F^1, \dots, F^m)$ be a random vector with components $F^j \in \mathbb{D}^{1,1}_{loc}$, $j = 1, \dots, m$. The* Malliavin matrix *of F is the matrix of size m, denoted by γ_F, whose entries are the random variables $\langle DF^i, DF^j \rangle_H$, $i, j = 1, \dots, m$.*

The next result due to Bouleau and Hirsch (see [BH91]) gives sufficient conditions for the existence of density.

Theorem 1.3.7. *Let $F = (F^1, \dots, F^m)$ be a random vector satisfying the following conditions:*

(a) F^i belongs to the space $\mathbb{D}^{1,p}_{loc}$, $p > 1$, for all $i = 1, \dots, m$.

(b) The Malliavin matrix γ_F is invertible, a.s.

Then the law of F has a density with respect to the Lebesgue measure on $\mathbb{R}^m$.

Finally, we shall need the following result on existence and smoothness of densities of random variables (for a detailed proof see for instance [Nua95], p. 88).

Proposition 1.3.8. *Let $F = (F^1, \dots, F^m)$ be a random vector satisfying the assumptions*

(a) $F^j \in \mathbb{D}^\infty$, for any $j = 1, \dots, m$,

(b) the Malliavin matrix γ_F is invertible, a.s. and

$$(\det \gamma_F)^{-1} \in \cap_{p \geq 1} L^p(\Omega).$$

Then the law of F has an infinitely differentiable density with respect to the Lebesgue measure on $\mathbb{R}^m$.

Description of the Gaussian context

We set the basic objects needed in the Appendices A and B to apply the techniques of the Malliavin calculus. Namely, using the notation of this section, we precise the Hilbert space H and the Gaussian family $(W(h), h \in H)$ associated to it.

We shall consider $H = \mathcal{H}_T$, the latter being defined as follows. Let $\mathcal{E}$ be the inner-product space consisting of functions $\varphi \in \mathcal{S}(\mathbb{R}^d)$ endowed with the inner-product $\langle \varphi, \psi \rangle_{\mathcal{E}} = \int_{\mathbb{R}^d} \Gamma(dx)(\varphi * \tilde{\psi})(x)$, where $\tilde{\psi}(x) = \psi(-x)$. Recall that the measure Γ is the spatial correlation of the noise $\dot{F}(t, x)$ in Equation (1.1). Notice that

$$\langle \varphi, \psi \rangle_{\mathcal{E}} = \int_{\mathbb{R}^d} \mu(d\xi) \mathcal{F}\varphi(\xi) \overline{\mathcal{F}\psi(\xi)},$$

where μ is the positive and positive definite spectral measure associated to Γ. Let $\mathcal{H}$ denote the completion of $(\mathcal{E}, \langle \cdot, \cdot \rangle_{\mathcal{E}})$. Set $\mathcal{H}_T = L^2([0,T];\mathcal{H})$. The scalar product in $\mathcal{H}_T$ extends that defined in (1.2). Notice that $\mathcal{H}$ and $\mathcal{H}_T$ may contain not only functions but also distributions. The space $\mathcal{H}_T$ is a real separable Hilbert space.

The Gaussian family is given by

$$W(h) = \int_0^T \int_{\mathbb{R}^d} h(s,x) M(ds,dx),$$

for $h \in \mathcal{H}_T$, where the stochastic integral can be interpreted in Dalang's sense as a stochastic integral of a deterministic integrand with respect to the martingale measure M. Then $(W(h), h \in \mathcal{H}_T)$ is a centered Gaussian process such that $E(W(h_1)W(h_2)) = \langle h_1, h_2 \rangle_{\mathcal{H}_T}$, $h_1, h_2 \in \mathcal{H}_T$, and we can use the differential Malliavin calculus based on it.

To conclude this section, we introduce some notation related to the above described setting needed in the results of [QSSS04c] (see Appendix B).

For $r_i \in [0,T]$, $\varphi_i \in \mathcal{H}$, $i = 1, \ldots, N$, and X a random variable we set

$$D^N_{((r_1,\varphi_1),\ldots,(r_N,\varphi_N))} X = \langle D^N_{(r_1,\ldots,r_N)} X, \varphi_1 \otimes \cdots \otimes \varphi_N \rangle_{\mathcal{H}^{\otimes N}}.$$

Thus, we have that

$$\|D^N X\|^2_{\mathcal{H}_T^{\otimes N}} = \int_{[0,T]^N} dr_1 \ldots dr_N \sum_{j_1,\ldots,j_N} |D_{((r_1,e_{j_1}),\ldots,(r_N,e_{j_N}))} X|^2,$$

where $(e_j, j \geq 0)$ is a complete orthonormal system of $\mathcal{H}$.

Let $N \in \mathbb{N}$, fix a set $A_N = \{\alpha_i = (r_i, \varphi_i) \in \mathbb{R}_+ \times \mathcal{H}, i = 1, \ldots, N\}$ and set $\bigvee_i r_i = \max(r_1, \ldots, r_N)$, $\alpha = (\alpha_1, \ldots, \alpha_N)$, $\hat{\alpha}_i = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_N)$. Using the same notation as for the statement of Proposition 1.3.4, we denote by $\mathcal{P}_m$ the set of partitions of A_N consisting of m disjoint subsets $p_1, \ldots, p_m$, $m = 1, \ldots, N$, and by $|p_i|$ the cardinal of p_i. Let X be a random variable belonging to $\mathbb{D}^{N,2}$, $N \geq 1$, and g be a real $\mathcal{C}^N$-function with bounded derivatives up to order N. Leibniz's rule for Malliavin's derivatives (see Proposition 1.3.4) yields

$$D^N_\alpha(g(X)) = \sum_{m=1}^N \sum_{\mathcal{P}_m} c_m g^{(m)}(X) \prod_{i=1}^m D^{|p_i|}_{p_i} X,$$

with positive coefficients c_m, $m = 1, \ldots, N$, $c_1 = 1$. Let

$$\Delta^N_\alpha(g, X) := D^N_\alpha g(X) - g'(X) D^N_\alpha X.$$

Notice that $\Delta^N_\alpha(g,X) = 0$ if $N = 1$ and it only depends on the Malliavin derivatives up to the order $N-1$ if $N > 1$. Finally, $\Delta^N(g,X)$ will denote the $\mathcal{H}_T^{\otimes N}-$valued random variable $D^N g(X) - g'(X) D^N X$.

Chapter 2

Summary of the contents

This chapter is devoted to present the contents of the three articles collected in the Appendices A, B and C. This is carried out, respectively, in Sections 2.1, 2.2 and 2.4. In Section 2.3 we aim to complete some known results on existence and regularity of the solution's density of a stochastic heat equation to more general parabolic SPDEs; first we deal with a stochastic boundary value problem on $(0, 1)$ and secondly with a parabolic SPDE on $\mathbb{R}^d$, $d \geq 1$.

When summarising the works of the above mentioned Appendices, we shall develop in a more detailed way some of the proofs. For the sake of completeness, we shall also add some additional results.

Finally, we mention that the citations of results and sections made along the chapter which do not go accompanied with a particular reference of the Bibliography mean that they refer to the dissertation. However, in order to make the reading easier, sometimes we attach to the citations of the dissertation their exact location in the manuscript.

2.1 Absolute continuity of the law of the solution to the three-dimensional stochastic wave equation

This section is devoted to summarise the contents of the article [QSSS04a] reprinted in Appendix A.

In this paper, we study the probability law of the real-valued solution to the stochastic wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_3\right) u(t,x) = \sigma(u(t,x))\dot{F}(t,x) + b(u(t,x)),$$

$$u(0,x) = \frac{\partial u}{\partial t}(0,x) = 0, \tag{2.1}$$

where $(t,x) \in (0,T] \times \mathbb{R}^3$, $T > 0$, Δ_3 denotes the Laplacian operator on $\mathbb{R}^3$ and the random perturbation $\dot{F}(t,x)$ is the Gaussian noise described in Section 1.2.1.

The aim is to give sufficient conditions ensuring that the law of the random variable $u(t,x)$, for any fixed $(t,x) \in (0,T] \times \mathbb{R}^3$, is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$.

We recall that results on existence of density for the solution of the stochastic wave equation with spatial dimension $d = 1, 2$ can be found in [CN88] and [MSS99], respectively; we also refer the reader to [MCMS01] for an extension of these results that covers the case of the stochastic heat equation in any dimension $d \geq 1$. Indeed, as it shall be mentioned in the next section, the above mentioned works not also deal with existence of density but also with its smoothness. Finally, let us mention the papers [ZN99], [LNP00] and [CW01] for existence of density results for different type of SPDEs.

Notice that Equation (2.1) is an example of the more general class of SPDEs given by Equation (1.1), namely,

$$Lu(t,x) = \sigma(u(t,x))\dot{F}(t,x) + b(u(t,x)), \tag{2.2}$$

with vanishing initial conditions, σ and b are supposed to be real-valued Lipschitz functions and L is a second order partial differential operator such that the fundamental solution associated with $Lu = 0$ takes values in the space of Schwartz distributions.

We give a rigorous meaning to Equation (2.1) by means of the *mild* formulation (see Definition 1.2.7 in Section 1.2.1), as follows. A real-valued predictable stochastic process $\{u(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ is a solution of Equation (2.2) if the following stochastic evolution equation is satisfied:

$$\begin{aligned} u(t,x) = & \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-y)\sigma(u(s,y))M(ds,dy) \\ & + \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dy) b(u(t-s, x-y)), \end{aligned}$$

where Λ denotes the fundamental solution associated to the differential operator L.

We assume that Hypothesis D from Section 1.2.1 is fulfilled. That is,

Hypothesis D The fundamental solution Λ of $Lu = 0$ is a deterministic function in t taking values in the space of non-negative distributions with rapid decrease such that

$$\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}\Lambda(t)(\xi)|^2 < \infty.$$

Moreover, Λ is a non-negative measure on $\mathbb{R}_+ \times \mathbb{R}^d$ of the form $\Lambda(t,dy)dt$ such that $\sup_{0 \leq t \leq T} \Lambda(t, \mathbb{R}^d) < +\infty$.

As it is mentioned in the very last part of Section 1.2.1, under this hypothesis Dalang proved existence and uniqueness of solution to Equation (2.2) (see Theorem 13 from [Dal99]). We denote it by $\{u(t,x),(t,x)\in[0,T]\times\mathbb{R}^3\}$.

We remark that in the definition of Hypothesis D in [QSSS04a], an extra condition is considered:

$$\lim_{h\downarrow 0}\int_0^T dt\int_{\mathbb{R}^d}\mu(d\xi)\sup_{t<r<t+h}|\mathcal{F}(\Lambda(r)-\Lambda(t))(\xi)|^2=0. \tag{2.3}$$

However, this condition is not needed for the proofs in [QSSS04a]; it was used in Theorem 13 [Dal99] to prove the L^2-continuity of the solution (see Lemma 19 in [Dal99] and [Dal01]).

In Section 2 from [QSSS04a], we extend Dalang's results on stochastic integration to a Hilbert-valued setting. This context is needed when dealing with the stochastic evolution equations satisfied by the Malliavin derivatives of the solution. In Section 1.2.1 of this dissertation we have recalled this extension and proved the main result in this direction in a much more detailed way that it appears in [QSSS04a], Theorem 1 (see Theorem 1.3.2 in the dissertation).

In Section 3 in [QSSS04a] we study the Malliavin differentiability of the random variable $u(t,x)$, for any fixed $(t,x)\in[0,T]\times\mathbb{R}^3$. More precisely, we prove that $u(t,x)$ belongs to the space $\mathbb{D}^{1,p}$, for any $p\in[1,\infty)$. Along the paper we make use of the definitions and notations of the Malliavin calculus presented in Section 1.3. In particular, we use the techniques of the Malliavin calculus in the framework defined in the very last part of Section 1.3; the underlying Hilbert space is given by $\mathcal{H}_T=L^2([0,T];\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space defined there.

The main result is the following.

Theorem 2.1.1. *Assume that Λ satisfies Hypothesis D and the coefficients σ and b are $\mathcal{C}^1$ functions with bounded Lipschitz continuous derivatives. Then, for any $(t,x)\in[0,T]\times\mathbb{R}^d$, $u(t,x)$ belongs to $\mathbb{D}^{1,p}$ for any $p\in[1,\infty)$ and there exists an $\mathcal{H}_T$-valued stochastic process $\{Z(t,x),(t,x)\in[0,T]\times\mathbb{R}^d\}$ satisfying*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}||Z(t,x)||_{L^p(\Omega;\mathcal{H}_T)}<+\infty,$$

such that

$$\begin{aligned}Du(t,x)=Z(t,x)&+\int_0^t\int_{\mathbb{R}^d}\Lambda(t-s,x-z)\sigma'(u(s,z))Du(s,z)M(ds,dz)\\&+\int_0^t ds\int_{\mathbb{R}^d}\Lambda(s,dz)b'(u(t-s,x-z))Du(t-s,x-z).\end{aligned}\tag{2.4}$$

Moreover, for any $(t,x) \in [0,T] \times \mathbb{R}^d$*,*

$$E\big(\|Z(t,x)\|^2_{\mathcal{H}_T}\big) = \|\Lambda(t-\cdot, x-*)\|^2_{0,\sigma(u)}$$
$$= E\Big(\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z)\sigma(u(s,z))M(ds,dz)\Big)^2.$$

We recall that the norm $\|\cdot\|_{0,K}$ is defined by means of (1.6).

In order to prove that the Malliavin derivative $Du(t,x)$ satisfies Equation (2.4), we need to consider the following equation (see Equation (14) in [QSSS04a]):

$$U(t,x) = Z(t,x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z)\sigma'(u(s,z))U(s,z)M(ds,dz)$$
$$+ \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)b'(u(t-s,x-z))U(t-s,x-z).$$

We remark that existence and uniqueness of an $\mathcal{H}_T-$valued solution to this equation follows immediately from Theorem 1.2.12 of this dissertation.

The proof of the Theorem 2.1.1 is based on Lemma 1.3.1. More precisely, we regularise the distribution Λ using an approximation of the identity $(\psi_n)_{n\geq 1}$, as the one defined in the proof of Theorem 1.2.5. Then, we obtain a sequence $(\Lambda_n)_{n\geq 1}$ of functions belonging to $\mathcal{S}(\mathbb{R}^d)$ and we take $F_n = u_n(t,x)$ in Lemma 1.3.1, where $\{u_n(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ is the unique real-valued process solution to

$$u_n(t,x) = \int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s,x-z)\sigma(u_n(s,z))M(ds,dz)$$
$$+ \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)b(u_n(t-s,x-z)).$$

Notice that the random variable $u_n(t,x)$ belongs to $\mathbb{D}^{1,p}$, for all $p \geq 1$, and that the Malliavin derivative $Du_n(t,x)$ satisfies the following equation on $\mathcal{H}_T$:

$$Du_n(t,x) = \Lambda_n(t-\cdot,x-*)\sigma(u_n(\cdot,*))$$
$$+ \int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s,x-z)\sigma'(u_n(s,z))Du_n(s,z)M(ds,dz)$$
$$+ \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)b'(u_n(t-s,x-z))Du_n(t-s,x-z).$$

This follows from an easy adaptation of the proof of Proposition 2.4 in [MCMS01].

In order to apply Lemma 1.3.1, first we prove that, assuming Hypothesis D and that the coefficients σ and b are Lipschitz,

$$\lim_{n\to\infty}\left(\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|u_n(t,x)-u(t,x)|^p\right) = 0, \tag{2.5}$$

for all $p \in [1, \infty)$ (see Proposition 1 in [QSSS04a]). For this, as it was pointed out in the Introduction of the dissertation, we prove that

$$\sup_{n \geq 1} \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} E\big(|u_n(t,x)|^p\big) < \infty$$

and then we check (2.5) for $p = 2$.

With this method we avoid dealing with $L^p(\Omega)-$bounds of stochastic integrals of distribution-valued functions which may not be non-negative.

The next step is to show that, under Hypothesis D and assuming that σ and b are $\mathcal{C}^1$ functions with bounded derivatives,

$$\sup_{n \geq 1} \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} E\big(||Du_n(t,x)||^p_{\mathcal{H}_T}\big) < \infty, \tag{2.6}$$

for all $p \in [0, \infty)$ (see Proposition 2 in [QSSS04a]). This yields the validity of condition (2) in Lemma 1.3.1.

To conclude the proof of Theorem 2.1.1, we just need to verify that the Malliavin derivative $Du(t,x)$ satisfies the stochastic evolution equation (2.4). We characterise the $\mathcal{H}_T-$valued element $Z(t,x)$ as the limit on $L^p(\Omega; \mathcal{H}_T)$ of the sequence $Z_n(t,x) = \Lambda_n(t - \cdot, x - *)\sigma(u_n(\cdot, *))$, $n \geq 1$ (see Proposition 3 in [QSSS04a]). Then, we show that the sequence of Malliavin derivatives $Du_n(t,x)$ converges in $L^p(\Omega; \mathcal{H}_T)$ to the unique solution of the equation

$$\begin{aligned} U(t,x) = Z(t,x) &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z)\sigma'(u(s,z))U(s,z)M(ds,dz) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)b'(u(t-s,x-z))U(t-s,x-z). \end{aligned}$$

This is contained in the proof of Theorem 2 in [QSSS04a]. Owing to (2.6), it suffices to show the convergence of order $p = 2$.

We remark that the $\mathcal{H}_T-$valued random vector

$$Z(t,x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z)\sigma'(u(s,z))Du(s,z)M(ds,dz)$$

in Equation (2.4) is the Malliavin derivative of the stochastic integral

$$\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z)\sigma(u(s,z))M(ds,dz)$$

(see Remark 3 in [QSSS04a]).

It is also worthy mentioning that the fundamental solution to the wave equation with dimension $d = 1, 2, 3$ satisfies Hypothesis D if

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^2} < +\infty \tag{2.7}$$

(see [Dal99], Example 6). Notice that, also by [Dal99], Example 8, the fundamental solution of the heat equation in any dimension $d \geq 1$ satisfies Hypothesis D if the above condition (2.7) is fulfilled.

Consider the real-valued solution $u(t, x)$ of Equation (2.1) at a fixed point $(t, x) \in (0, T] \times \mathbb{R}^3$. We prove the following theorem:

Theorem 2.1.2. *Assume that*

(1) the coefficients σ and b are $\mathcal{C}^1$ functions with bounded Lipschitz continuous derivatives;

(2) there exists $\sigma_0 > 0$ such that $\inf\{|\sigma(z)|, z \in \mathbb{R}\} \geq \sigma_0$;

(3) there exists $\eta \in (0, \frac{1}{2})$ such that

$$\sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \Gamma(dx) \mathcal{F}^{-1} \left(\frac{1}{(1 + |\xi|^2)^\eta} \right) (x - y) < \infty.$$

Then, the law of the random variable $u(t, x)$, $(t, x) \in (0, T] \times \mathbb{R}^3$, is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$.

Owing to Bouleau and Hirsch criterion (see Theorem 1.3.7), this theorem is a consequence of Theorem 2.1.1 above and the next one.

Theorem 2.1.3. *Assume that the coefficients σ and b are $\mathcal{C}^1$ functions with bounded derivatives of order one and that hypotheses (2) and (3) of the previous Theorem are satisfied. Then, $\|Du(t, x)\|_{\mathcal{H}_T} > 0$, a.s.*

Along the section of the work that we are summarising, S_3 denotes the fundamental solution associated with the wave equation on $\mathbb{R}^3$.

We define

$$G_{d,\eta}(x) = \mathcal{F}^{-1} \left(\frac{1}{(1 + |\xi|^2)^\eta} \right) (x),$$

for $d \geq 1$, $\eta \in (0, 1)$, and

$$F_{d,\eta}(y) = \int_{\mathbb{R}^d} \Gamma(dx) G_{d,\eta}(x - y),$$

for $y \in \mathbb{R}^d$. We consider the following assumptions:

$(\bar{H}_\eta)$ $\sup_{y\in\mathbb{R}^d} F_{d,\eta}(y) < \infty$,

(H_η) $\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta} < \infty$.

In Proposition 4.4.1 [Lév01] it is proved that condition $(\bar{H}_\eta)$ implies (H_η) and

$$F_{d,\eta}(0) = \int_{\mathbb{R}^d} \Gamma(dx) G_{d,\eta}(x) \leq \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta} < \infty.$$

Notice that assumption (3) of Theorem 2.1.2 is equivalent to $(\bar{H}_\eta)$ for some $\eta \in (0, \frac{1}{2})$.

To prove the above Theorem 2.1.2, we show that $E(\|Du(t,x)\|^{-p}_{\mathcal{H}_T})$ is finite for some $p > 0$. The main ingredients are the following.

We follow the same lines of the proof of Theorem 3.1 in [MSS99] with some modifications; these are due to the fact that the fundamental solution of the three-dimensional wave equation is a distribution. More precisely, to study the contribution of the $\mathcal{H}_T-$valued term $Z(t,x)$, we introduce a regularisation of the distribution $S_3(t)$ given by

$$\Lambda_{\epsilon^{-1}} = \psi_{\epsilon^{-1}} * S_3,$$

where $(\psi_\nu, \nu \in \mathbb{R}_+)$ is an approximation of the identity defined as in the proof of Theorem 1.2.5.

This technique is complemented with upper and lower bounds of integrals of the Fourier transform of the fundamental solution S_3. These bounds are collected in the Appendix of [QSSS04a] and they are actually valid for the fundamental solution of the wave equation in any dimension $d \geq 1$. This is due to the fact that its Fourier transform has a unified expression:

$$\mathcal{F}S_d(t)(\xi) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}, \quad \xi \in \mathbb{R}^d.$$

All the results proved in the above cited Appendix need either condition (H_η) or the stronger one given by $(\bar{H}_\eta)$.

In the very recent reference [SSar], the equivalence between $(\bar{H}_\eta)$ and (H_η) is proved. For the sake of completeness, we quote and prove this result (see Lemma 9.8 in [SSar]).

Lemma 2.1.4. *For any $\eta \in (0,\infty)$, the following conditions are equivalent*

(1) $\sup_{y\in\mathbb{R}^d} F_{d,\eta}(y) < +\infty$,

(2) $\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta} < +\infty$.

In fact, if either (1) or (2) hold, then

$$\sup_{y\in\mathbb{R}^d} F_{d,\eta}(y) = \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta}.$$

Proof. Assume that (1) holds true. For any $t>0$, set

$$p_t = \mathcal{F}^{-1}\left(e^{-2\pi^2 t|\xi|^2}\right).$$

Notice that p_t is the density of a probability measure on $\mathbb{R}^d$. Hence,

$$\sup_{t>0} \int_{\mathbb{R}^d} p_t(y) F_{d,\eta}(y)dy \le \sup_{y\in\mathbb{R}^d} F_{d,\eta}(y).$$

On the other hand, the definition of $F_{d,\eta}$, Fubini's theorem and the fact that $G_{d,\eta} * p_t$ belongs to $\mathcal{S}(\mathbb{R}^d)$ yield

$$\begin{aligned}\int_{\mathbb{R}^d} p_t(y) F_{d,\eta}(y)dy &= \int_{\mathbb{R}^d} \Gamma(dx)(G_{d,\eta} * p_t)(x)\\ &= \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta} e^{-2\pi^2 t|\xi|^2}.\end{aligned}$$

By monotone convergence we get that

$$\lim_{t\searrow 0} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta} e^{-2\pi^2 t|\xi|^2} = \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta}.$$

Thus, we deduce

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta} \le \sup_{y\in\mathbb{R}^d} F_{d,\eta}(y) < +\infty,$$

which proves (2).

Assume now (2). Then, the measure on $\mathbb{R}^d$ given by

$$\frac{\mu(d\xi)}{(1+|\xi|^2)^\eta}$$

is finite. This implies that its Fourier transform is a bounded function. On the other hand, owing to Theorem XV in [Sch66], Chapter VII, we obtain that

$$\mathcal{F}\left(\frac{\mu}{(1+|\cdot|^2)^\eta}\right)(x) = \big(\mathcal{F}\mu) * (\mathcal{F}(1+|\cdot|^2)^{-\eta})\big)(x) = (\Gamma * G_{d,\eta})(x) = F_{d,\eta}(x).$$

Hence the proof of (1) is complete. □

2.2 A stochastic wave equation in dimension 3: smoothness of the law

In this section we summarise the contents of the work [QSSS04c], which can be found in Appendix B.

As in the previous section, we study the probability law of the three-dimensional stochastic wave equation given in (2.1). We want to find conditions under which the law of the solution at any fixed point has an infinitely differentiable density. This means that we are able to go further in comparison to the results in [QSSS04a]. Recall that the analysis of the smoothness of the density for the stochastic wave equation with spatial dimension $d = 1, 2$ is carried out in [CN88] and [MSS99], respectively (see also [MCMS01]).

We shall make use of one of the criteria provided by the Malliavin calculus, namely Proposition 1.3.8.

First we consider the more general SPDE given in Equation (1.1) and we denote by $\{u(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ the unique real-valued process solution to this equation. We prove that the random variable $u(t,x)$ has derivatives in the sense of Malliavin of any order, that is, $u(t,x)$ belongs to the space $\mathbb{D}^\infty$.

The setting in which we apply the techniques of the Malliavin calculus is the same as the one considered in the previous section, namely, the one described in the last part of Section 1.3.

In Section 2 from [QSSS04c], we review the main ideas of the extension of Dalang's stochastic integral to a Hilbert-valued setting that is performed in the second section of [QSSS04a] (see also Section 1.2.1 of the present dissertation).

The main result from the article [QSSS04c] concerning differentiability in the sense of Malliavin is the following.

Theorem 2.2.1 ([QSSS04c], Theorem 1). *Assume Hypothesis D and that the coefficients σ and b are $\mathcal{C}^\infty$ functions with bounded derivatives of any order greater or equal than one. Then, for every $(t,x) \in [0,T] \times \mathbb{R}^d$, the random variable $u(t,x)$ belongs to the space $\mathbb{D}^\infty$. Moreover, for any $p \geq 1$ and $N \geq 1$, there exists a $L^p(\Omega; \mathcal{H}_T^{\otimes N})$−valued random process $\{Z^N(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ such that*

$$
\begin{aligned}
&D^N u(t,x) = Z^N(t,x) \\
&+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z)[\Delta^N(\sigma, u(s,z)) + D^N u(s,z)\sigma'(u(s,z))] M(ds,dz) \\
&+ \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)[\Delta^N(b, u(t-s,x-z)) + D^N u(t-s,x-z) b'(u(t-s,x-z))],
\end{aligned}
\tag{2.8}
$$

and

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}^d} E(\|D^N u(s,y)\|^p_{\mathcal{H}_T^{\otimes N}}) < +\infty.$$

Recall that

$$\Delta^N(\sigma, u(s,z)) = D^N\sigma(u(s,z)) - \sigma'(u(s,z))D^N u(s,z).$$

As for the previous section, condition (2.3) in Hypothesis D considered in [QSSS04c] is not needed all along the paper.

To prove the above Theorem 2.2.1, we apply Lemma 1.3.2, which is a consequence of the fact that the iterated Malliavin derivative D^N is a closed operator from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H}_T^{\otimes N})$ (see Section 1.3).

As in the proof of Theorem 2 in [QSSS04a], we consider the sequence of processes $\{u_n(t,x), (t,x) \in [0,T]\times\mathbb{R}^d\}$ solving the equation

$$\begin{aligned} u_n(t,x) = &\int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s, x-z)\sigma(u_n(s,z))M(ds,dz) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz) b(u_n(t-s,x-z)), \end{aligned}$$

where $\Lambda_n(t) = \psi_n * \Lambda$, $n \geq 1$, $(\psi_n)_{n\geq 1}$ being an approximation of the identity.

Using the standard approach given, for instance, in [MSS99] or [MCMS01], we show that that the random variables $u_n(t,x)$ belong to $\mathbb{D}^\infty$, for any $(t,x) \in [0,T]\times\mathbb{R}^d$ and $n \geq 1$. Moreover, the Malliavin derivative $D^N u_n(t,x)$ satisfies the equation

$$\begin{aligned} D^N_\alpha u_n(t,x) &= \sum_{i=1}^N \langle \Lambda_n(t-r_i, x-*) D^{N-1}_{\hat{\alpha}_i}\sigma(u_n(r_i,*)), \varphi_i\rangle_{\mathcal{H}} \\ &+ \int_{\bigvee_i r_i}^t \int_{\mathbb{R}^d} \Lambda_n(t-s,x-z)[\Delta^N_\alpha(\sigma, u_n(s,z)) + D^N_\alpha u_n(s,z)\sigma'(u_n(s,z))]M(ds,dz) \\ &+ \int_{\bigvee_i r_i}^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)[\Delta^N_\alpha(b, u_n(t-s,x-z)) \\ &\quad + D^N_\alpha u_n(t-s,x-z) b'(u_n(t-s,x-z))], \end{aligned} \tag{2.9}$$

where $\alpha = ((r_1,\varphi_1),\dots,(r_N,\varphi_N))$, with $r_1,\dots,r_N \geq 0$ and $\varphi_1,\dots,\varphi_N \in \mathcal{H}$ and we have used the notations presented in the last part of Section 1.3.

Assuming the same hypothesis as in Theorem 2.2.1, we proof the following uniform estimation for the Malliavin derivative $D^N u_n(t,x)$:

$$\sup_{n\geq 1} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\left(||D^N u_n(t,x)||^p_{\mathcal{H}_T^{\otimes N}}\right) < +\infty, \tag{2.10}$$

for any $p \in [1, \infty)$ and $N \geq 1$. A detailed proof of this result can be found in [QSSS04c], Lemma 2.

In order to identify the $\mathcal{H}_T^{\otimes N}-$valued random vector $Z^N(t,x)$ given in Theorem 2.2.1, we give the following definition.

For $N \geq 1$, $n \geq 1$, $r = (r_1, \ldots, r_N)$, $\alpha = ((r_1, e_{j_1}), \cdots, (r_N, e_{j_N}))$ and $(t,x) \in [0,T] \times \mathbb{R}^d$ we define the $\mathcal{H}^{\otimes N}-$valued random variable $Z_r^{N,n}(t,x)$ as follows,

$$\langle Z_r^{N,n}(t,x), e_{j_1} \otimes \cdots \otimes e_{j_N} \rangle_{\mathcal{H}^{\otimes N}} = \sum_{i=1}^N \langle \Lambda_n(t - r_i, x - *) D_{\hat{\alpha}_i}^{N-1} \sigma(u_n(r_i, *)), e_{j_i} \rangle_{\mathcal{H}}.$$

Here $(e_j)_{j \geq 1}$ stands for a complete orthonormal system of $\mathcal{H}$. We prove

$$\sup_{n \geq 1} \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} E(\|Z^{N,n}(t,x)\|_{\mathcal{H}_T^{\otimes N}}^p) < +\infty, \tag{2.11}$$

for every $p \in [1, \infty)$. Notice that $Z^{N,n}(t,x)$ coincides with the first term of the right hand-side of Equation (2.9) for $\alpha = ((r_1, e_{j_1}), \cdots, (r_N, e_{j_N}))$.

On the other hand, for $N \geq 1$ we introduce the assumption

(H_{N-1}) The sequence $(D^j u_n(t,x), n \geq 1)$ converges in $L^p(\Omega; \mathcal{H}_T^{\otimes j})$, for any $j = 0, \cdots, N-1$,

We write $L^p(\Omega; \mathcal{H}_T^{\otimes 0}) = L^p(\Omega)$. Proposition 1 in [QSSS04a] (see also (2.5)) yields the validity of (H_0). Moreover, for $N > 1$, (H_{N-1}) implies that $u(t,x) \in \mathbb{D}^{j,p}$ and the sequence $(D^j u_n(t,x), n \geq 1)$ converges in $L^p(\Omega; \mathcal{H}_T^{\otimes j})$ to $D^j u(t,x)$, for $j = 1, \cdots, N-1$. This is a consequence of the fact that the iterated Malliavin derivative D^j is a closed operator from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H}_T^{\otimes j})$. In addition, by (2.10),

$$\sup_{(s,y) \in [0,T] \times \mathbb{R}^d} E(||D^j u(s,y)||_{\mathcal{H}_T^{\otimes j}}^p) < \infty,$$

for $j = 1, \cdots, N-1$.

In [QSSS04c], Lemma 3, we show that, under the same hypothesis as in Theorem 2.2.1 and assuming (H_{N-1}), for $N \geq 1$, the sequence $(Z^{N,n}(t,x))_{n \geq 1}$ converges in $L^p(\Omega; \mathcal{H}_T^{\otimes N})$ to a random variable $Z^N(t,x)$.

The proof is based on a induction argument with respect to N. The case $N = 1$ is proved in [QSSS04a], Proposition 3. For the rest of the proof, we notice that it suffices to show that $(Z^{N,n}(t,x), n \geq 1)$ is a Cauchy sequence in $L^2(\Omega; \mathcal{H}_T^{\otimes N})$. Indeed, by (2.11), the sequence is uniformly bounded in $L^p(\Omega; \mathcal{H}_T^{\otimes N})$, for any $p \in [1, \infty)$. Consequently, $(|Z^{N,n}(t,x)|^p)_{n \geq 1}$ is uniformly integrable for all $p \in (1, \infty)$.

Let us give some additional details to the proof of the above-mentioned Lemma 3 in [QSSS04c] concerning the convergence of the term

$$\begin{aligned}Z_2^{n,m} :=& \sum_{i=1}^{N} E \int_{[0,T]^N} dr \sum_{j_1,\dots,j_N} |\langle D_{\hat{\alpha}_i}^{N-1}\sigma(u(r_i,*)) \\ &\times [\Lambda_n(t-r_i,x-*) - \Lambda_m(t-r_i,x-*)], e_{j_i}\rangle_{\mathcal{H}}|^2, \\ \leq & \sum_{i=1}^{N} E \int_{[0,T]^{N-1}} d\hat{r}_i \sum_{\hat{j}_i} \int_0^T ds \int_{\mathbb{R}^d} \mu_s^{D_{\hat{\alpha}_i}^{N-1}\sigma(u)}(d\xi) |\mathcal{F}(\Lambda_n(t-s) - \Lambda_m(t-s))(\xi)|^2.\end{aligned} \tag{2.12}$$

We have used the notations

$$\begin{aligned}\hat{\alpha}_i &= (\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_N), \\ d\hat{r}_i &= dr_1 \dots dr_{i-1} dr_{i+1} \dots dr_N, \\ \hat{j}_i &= j_1,\dots,j_{i-1},j_{i+1},\dots,j_N.\end{aligned}$$

We want to show that the last term in (2.12) tends to zero as n and m goes to infinity.

Notice that $|\mathcal{F}(\Lambda_n(t-s) - \Lambda_m(t-s))(\xi)|$ converges to zero pointwise, as n, m tends to infinity. Moreover, it holds that

$$|\mathcal{F}(\Lambda_n(t-s) - \Lambda_m(t-s))(\xi)| \leq 2|\mathcal{F}\Lambda(t-s)(\xi)|.$$

Let us check that

$$I = E\left(\int_{[0,T]^{N-1}} d\hat{r}_i \sum_{j_{\hat{i}}} \int_0^T ds \int_{\mathbb{R}^d} \mu_s^{D_{\hat{\alpha}_i}^{N-1}\sigma(u)}(d\xi) |\mathcal{F}\Lambda(t-s)(\xi)|^2\right)$$

is finite for all $i = 1,\dots,N$. Indeed, from the proof of Theorem 1.2.5 in the particular case of $S = \Lambda(t-\cdot)$, $\mathcal{A} = \mathbb{R}$ and $K(s,y) = D_{\hat{\alpha}_i}^{N-1}\sigma(u(s,y))$, we have

$$\|\Lambda(t-\cdot)\|_{0,D_{\hat{\alpha}_i}^{N-1}\sigma(u)}^2 \leq \liminf_{k\to\infty} \|\Lambda_k(t-\cdot)\|_{0,D_{\hat{\alpha}_i}^{N-1}\sigma(u)}^2.$$

Hence, by Remark 1.2.6 and Fatou's lemma,

$$I = \int_{[0,T]^{N-1}} d\hat{r}_i \sum_{j_{\hat{i}}} \|\Lambda(t-\cdot)\|_{0,D_{\hat{\alpha}_i}^{N-1}\sigma(u)}^2 \leq \liminf_{k\to\infty} \int_{[0,T]^{N-1}} d\hat{r}_i \sum_{\hat{j}_i} \|\Lambda_k(t-\cdot)\|_{0,D_{\hat{\alpha}_i}^{N-1}\sigma(u)}^2.$$

The definition of the norm $\|\cdot\|_{0,D^{N-1}_{\hat{\alpha}_i}\sigma(u)}$ and Parseval's identity yield

$$\begin{aligned}
&\int_{[0,T]^{N-1}} d\hat{r}_i \sum_{\hat{j}_i} \|\Lambda_k(t-\cdot)\|^2_{0,D^{N-1}_{\hat{\alpha}_i}\sigma(u)} \\
&= E\Bigg(\int_{[0,T]^{N-1}} d\hat{r}_i \sum_{\hat{j}_i} \int_0^T ds \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy D^{N-1}_{\hat{\alpha}_i}\sigma(u(s,y)) D^{N-1}_{\hat{\alpha}_i}\sigma(u(s,y-x)) \\
&\quad \times \Lambda_k(t-s,y)\Lambda_k(t-s,y-x)\Bigg) \\
&\le \sup_{(t,z)\in[0,T]\times\mathbb{R}^d} E\left(\sum_{\hat{j}_i} \int_{[0,T]^{N-1}} d\hat{r}_i \left|D^{N-1}_{\hat{\alpha}_i}\sigma(u(t,z))\right|^2\right) \\
&\quad \times \int_0^T ds \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy \Lambda_k(t-s,y)\Lambda_k(t-s,y-x) \\
&= \sup_{(t,z)\in[0,T]\times\mathbb{R}^d} E\left(\|D^{N-1}\sigma(u(t,z))\|^2_{\mathcal{H}_T^{\otimes(N-1)}}\right) \int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}\Lambda_k(t-s)(\xi)|^2.
\end{aligned}$$

By definition of Λ_k, Leibniz rule for Malliavin derivatives (see Proposition 1.3.4) and the hypothesis (H_{N-1}), this last term is finite, uniformly with respect to k. The desired convergence follows by bounded convergence. □

To conclude the proof of Theorem 2.2.1, it remains to show that the sequence $(D^N u_n(t,x))_{n\ge1}$ converges in the space $L^p(\Omega;\mathcal{H}_T^{\otimes N})$, for every $N\ge 1$ and $p\in[2,\infty)$, and that the Malliavin derivative satisfies Equation (2.8).

Owing to (2.10), we only need to check the convergence with $p=2$. More precisely, we show that $(D^N u_n(t,x))_{n\ge1}$ converges in $L^2(\Omega;\mathcal{H}_T^{\otimes N})$ to $U(t,x)$, solution to the stochastic $\mathcal{H}_T^{\otimes N}-$valued evolution equation

$$\begin{aligned}
U(t,x) &= Z^N(t,x) \\
&+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z)[\Delta^N(\sigma,u(s,z)) + U(s,z)\sigma'(u(s,z))] M(ds,dz) \\
&+ \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)[\Delta^N(b,u(t-s,x-z)) + U(t-s,x-z)b'(u(t-s,x-z))],
\end{aligned} \tag{2.13}$$

where

$$Z^N(t,x) = L^p(\Omega;\mathcal{H}_T^{\otimes N}) - \lim_{n\to\infty} Z^{N,n}(t,x).$$

Notice that Theorem 1.2.12 guarantees a unique solution of the above equation.

Since in [QSSS04c] this final part of the proof of Theorem 2.2.1 is only sketched, we develop it here in more detail.

The convergence when $N = 1$ is checked in the proof of Theorem 2 in [QSSS04a].

We assume that $N > 1$. Remark that, owing to Equation (2.9) and the definition of $Z(t, x)$, it suffices to check the convergence to zero of the difference of the stochastic and pathwise integral terms in Equations (2.9) and (2.13), respectively. These differences are denoted by $I_\sigma^{n,N}(t, x)$ and $I_b^{n,N}(t, x)$, respectively.

We have the following decomposition:

$$I_\sigma^{n,N}(t, x) = J_1^n(t, x) + J_2^n(t, x),$$

where

$$\begin{aligned} J_1^n(t, x) = &\int_0^t \int_{\mathbb{R}^d} \Lambda_n(t - s, x - z)\Delta^N(\sigma, u_n(s, z))M(ds, dz) \\ &- \int_0^t \int_{\mathbb{R}^d} \Lambda(t - s, x - z)\Delta^N(\sigma, u(s, z))M(ds, dz), \\ J_2^n(t, x) = &\int_0^t \int_{\mathbb{R}^d} \Lambda_n(t - s, x - z)\sigma'(u_n(s, z))D^N u_n(s, z)M(ds, dz) \\ &- \int_0^t \int_{\mathbb{R}^d} \Lambda(t - s, x - z)\sigma'(u(s, z))U(s, z)M(ds, dz). \end{aligned}$$

For the second term we have that

$$E\left(\|J_2^n(t, x)\|^2_{\mathcal{H}_T^{\otimes N}}\right) \le C(D_{1,n}(t, x) + D_{2,n}(t, x) + D_{3,n}(t, x)),$$

with

$$\begin{aligned} D_{1,n}(t, x) =&E\Bigg(\Bigg\|\int_0^t \int_{\mathbb{R}^d} \Lambda_n(t - s, x - z)[\sigma'(u_n(s, z)) \\ &- \sigma'(u(s, z))]D^N u_n(s, z)M(ds, dz)\Bigg\|^2_{\mathcal{H}_T^{\otimes N}}\Bigg), \\ D_{2,n}(t, x) =&E\Bigg(\Bigg\|\int_0^t \int_{\mathbb{R}^d} \Lambda_n(t - s, x - z)\sigma'(u(s, z))[D^N u_n(s, z) \\ &-U(s, z)]M(ds, dz)\Bigg\|^2_{\mathcal{H}_T^{\otimes N}}\Bigg), \\ D_{3,n}(t, x) =&E\Bigg(\Bigg\|\int_0^t \int_{\mathbb{R}^d} [\Lambda_n(t - s, x - z) \\ &-\Lambda(t - s, x - z)]\sigma'(u(s, z))U(s, z)M(ds, dz)\Bigg\|^2_{\mathcal{H}_T^{\otimes N}}\Bigg). \end{aligned}$$

The study of these three terms follows the same idea ideas developed in the proof of Theorem 2 in [QSSS04a] for real-valued processes and Malliavin derivatives of order 1. We obtain that

$$E\left(\|J_2^n(t,x)\|^2_{\mathcal{H}_T^{\otimes N}}\right) \leq C_n + C\int_0^t ds \sup_{(\tau,x)\in[0,s]\times\mathbb{R}^d} E(||Du_n(\tau,x) - U(\tau,x)||^2_{\mathcal{H}_T^{\otimes N}})J(t-s),$$

where $(C_n)_{n\geq 1}$ is a sequence of positive real numbers decreasing to zero and J is defined by

$$J(t) = \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Lambda(t)(\xi)|^2.$$

Let us deal with the term $J_1^n(t,x)$. We have that

$$E(\|J_1^n(t,x)\|^2_{\mathcal{H}_T^{\otimes N}} \leq C(\tilde{D}_{1,n}(t,x) + \tilde{D}_{2,n}(t,x)),$$

where

$$\begin{aligned}\tilde{D}_{1,n}(t,x) =& E\left(\left\|\int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s,x-z)\big(\Delta^N(\sigma,u_n(s,z))\right.\right.\\ &\left.\left. - \Delta^N(\sigma,u(s,z))\big)M(ds,dz)\right\|^2_{\mathcal{H}_T^{\otimes N}}\right),\\ \tilde{D}_{2,n}(t,x) =& E\left(\left\|\int_0^t \int_{\mathbb{R}^d} (\Lambda_n(t-s,x-z) - \Lambda(t-s,x-z))\right.\right.\\ &\left.\left.\times \Delta^N(\sigma,u(s,z))M(ds,dz)\right\|^2_{\mathcal{H}_T^{\otimes N}}\right).\end{aligned}$$

Theorem 1.2.5, the definition of Λ_n and Hypothesis D yield

$$\begin{aligned}\tilde{D}_{1,n}(t,x) \leq& C\int_0^t ds \sup_{y\in\mathbb{R}^d} E\left(\|\Delta^N(\sigma,u_n(s,z)) - \Delta^N(\sigma,u(s,z))\|^2_{\mathcal{H}_T^{\otimes N}}\right)\\ &\times \int_{\mathbb{R}^d}\mu(d\xi)|\mathcal{F}\Lambda_n(t-s)(\xi)|^2\\ \leq& C \sup_{(r,y)\in[0,T]\times\mathbb{R}^d} E\left(\|\Delta^N(\sigma,u_n(r,z)) - \Delta^N(\sigma,u(r,z))\|^2_{\mathcal{H}_T^{\otimes N}}\right)\\ &\times \int_0^T ds\int_{\mathbb{R}^d}\mu(d\xi)|\mathcal{F}\Lambda(s)(\xi)|^2\\ \leq& C \sup_{(r,y)\in[0,T]\times\mathbb{R}^d} E\left(\|\Delta^N(\sigma,u_n(r,z)) - \Delta^N(\sigma,u(r,z))\|^2_{\mathcal{H}_T^{\otimes N}}\right).\end{aligned}$$

Owing to the definition of Δ^N, Leibniz rule for Malliavin derivatives, (2.10), the induction hypothesis (H_{N-1}) and the assumptions on σ, we have that $\tilde{D}_{1,n}(t,x)$ tends to zero as n goes to infinity, uniformly with respect to $(t,x)\in[0,T]\times\mathbb{R}^d$.

To deal with the term $\tilde{D}_{2,n}(t,x)$, we use the isometry property of the stochastic integral given in Theorem 1.2.5 and we obtain

$$\tilde{D}_{2,n}(t,x) = \|\Lambda_n(t-\cdot, x-*) - \Lambda(t-\cdot, x-*)\|^2_{0,\tilde{U}},$$

where $\tilde{U}$ is the $\mathcal{H}_T^{\otimes N}-$valued process given by $\tilde{U}(s,z) = \Delta^N(\sigma, u(s,z))$, $(s,z) \in [0,T] \times \mathbb{R}^d$. Notice that the process $\tilde{U}$ satisfies the Hypothesis C and D in Section 1.2.1; this is due to the assumptions on σ and (2.10).

Thus,

$$\tilde{D}_{2,n}(t,x) = \int_0^T ds \int_{\mathbb{R}^d} \mu_s^{\tilde{U}}(d\xi) |\mathcal{F}\Lambda_n(t-s)(\xi) - \mathcal{F}\Lambda(t-s)(\xi)|^2.$$

Moreover, since

$$\int_0^T ds \int_{\mathbb{R}^d} \mu_s^{\tilde{U}}(d\xi) |\mathcal{F}\Lambda(t-s)(\xi)|^2 < +\infty,$$

$\tilde{D}_{2,n}(t,x)$ tends to zero as n goes to infinity, by bounded convergence.

The difference of the pathwise integrals terms in Equations (2.9) and (2.13), respectively, can be studied using similar techniques as for the analysis of the term $I_\sigma^{n,N}(t,x)$, but with quite less effort.

Therefore the proof of Theorem 2.2.1 is complete. □

Let us now focus on the three-dimensional stochastic wave equation (2.1). Let $\{u(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ be the unique solution to Equation (2.1).

Under the same hypothesis needed to show the existence of density for the probability law of $u(t,x)$ (see Theorem 2.1.2), we prove now that the inverse of the Malliavin matrix has moments of all orders $p > 0$. Owing to Proposition 1.3.8, this implies that the random variable $u(t,x)$ has a density which is a $\mathcal{C}^\infty$ function. The theorems are the following:

Theorem 2.2.2 ([QSSS04c], Theorem 2). *Assume that the coefficients σ and b are $\mathcal{C}^1$ functions with bounded Lipschitz continuous derivatives and in addition,*

(1) there exists $\sigma_0 > 0$ such that $\inf\{|\sigma(z)|, z \in \mathbb{R}\} \geq \sigma_0$*;*

(2) there exists $\eta \in (0, \frac{1}{2})$ such that

$$\sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \Gamma(dx) \mathcal{F}^{-1} \left(\frac{1}{(1+|\xi|^2)^\eta} \right) (x-y) < \infty.$$

Then, for any $p > 0$,

$$E(\|Du(t,x)\|_{\mathcal{H}_T}^{-p}) < \infty.$$

Theorem 2.2.3 ([QSSS04c], Theorem 3). *Assume that the coefficients σ and b are $\mathcal{C}^\infty$ functions with bounded derivatives of any order greater or equal than one, and that hypothesis (1) and (2) of Theorem 2.2.2 are satisfied. Then, the random variable $u(t,x)$, $(t,x) \in (0,T] \times \mathbb{R}^3$, has a density which is a $\mathcal{C}^\infty$ function.*

We recall that the Malliavin derivative $Du(t,x)$ satisfies the equation

$$\begin{aligned} Du(t,x) = Z(t,x) &+ \int_0^t \int_{\mathbb{R}^3} S_3(t-s,x-z)\sigma'(u(s,z))Du(s,z)M(ds,dz) \\ &+ \int_0^t ds \int_{\mathbb{R}^3} S_3(t-s,dz)b'(u(s,x-z))Du(s,x-z), \end{aligned} \tag{2.14}$$

where $\{Z(t,x), (t,x) \in [0,T] \times \mathbb{R}^3\}$ is the $\mathcal{H}_T-$valued random process given by

$$Z(t,x) = L^p(\Omega;\mathcal{H}_T) - \lim_{n\to\infty} Z^n(t,x),$$

$p \geq 1$, where $Z^n(t,x) := S_{3,n}(t-\cdot, x-*)\sigma(u(\cdot,*))$ and with $S_{3,n} = S_3 * \psi_n$ (see either the proof of Theorem 2.2.1 or Theorem 2 in [QSSS04a]).

For the proof of Theorem 2.2.2 we use the following two technical results (see Lemma 4 and 5 in [QSSS04c], respectively).

Assume that σ is Lipschitz continuous and that condition (2.7) is satisfied. Then, for any $(t,x) \in (0,T] \times \mathbb{R}^3$, $v \in (0,t]$ and $q \geq 1$,

$$E(\|Z_{t-\cdot,*}(t,x)\|_{\mathcal{H}_v}^{2q}) \leq C\left(\int_0^v ds \int_{\mathbb{R}^3} \mu(d\xi)|\mathcal{F}S_3(s)(\xi)|^2\right)^q \tag{2.15}$$

and

$$\sup_{t-v\leq s\leq t}\ \sup_{y\in\mathbb{R}^3} E(\|D_{t-\cdot,*}u(s,y)\|_{\mathcal{H}_v}^{2q}) \leq C\left(\int_0^v ds \int_{\mathbb{R}^3} \mu(d\xi)|\mathcal{F}\Lambda(s)(\xi)|^2\right)^q. \tag{2.16}$$

Since the latter estimation is not explicitly proved in [QSSS04c], to make this dissertation more complete, we add the proof, as follows.

Proof of (2.16). Owing to the equation satisfied by the Malliavin derivative $Du(t,x)$ (see (2.14)) we have that

$$E(\|D_{t-\cdot,*}u(t,x)\|_{\mathcal{H}_v}^{2q}) \leq C(A(v,t,x) + B(v,t,x) + C(v,t,x)),$$

where

$$\begin{aligned} A(v,t,x) =& E(\|Z_{t-\cdot,*}(t,x)\|_{\mathcal{H}_v}^{2q}), \\ B(v,t,x) =& E\left(\left\|\int_0^t \int_{\mathbb{R}^3} \Lambda(t-s,x-z)\sigma'(u(s,z))D_{t-\cdot,*}u(s,z)M(ds,dz)\right\|_{\mathcal{H}_v}^{2q}\right), \\ C(v,t,x) =& E\left(\left\|\int_0^t ds \int_{\mathbb{R}^3} \Lambda(t-s,dz)b'(u(s,x-z))D_{t-\cdot,*}u(s,x-z)\right\|_{\mathcal{H}_v}^{2q}\right). \end{aligned}$$

By the above bound (2.15) we obtain that

$$A(v,t,x) \leq \left(\int_0^v ds \int_{\mathbb{R}^3} \mu(d\xi) |\mathcal{F}\Lambda(s)(\xi)|^2 \right)^q .$$

To study the term $B(v,t,x)$ we apply Theorem 1.2.5 and the fact that the $\mathcal{H}_v-$norm of the Malliavin derivative $D_{t-\cdot,*}u(s,z)$ vanishes for $s < t-v$. Thus,

$$\begin{aligned} B(v,t,x) &\leq C \int_0^t ds \sup_{y\in\mathbb{R}^3} E\left(\|D_{t-\cdot,*}u(s,y)\|_{\mathcal{H}_v}^{2q}\right) \int_{\mathbb{R}^3} \mu(d\xi)|\mathcal{F}\Lambda(t-s)(\xi)|^2 \\ &\leq C \int_0^t ds \sup_{t-v\leq\tau\leq s} \sup_{y\in\mathbb{R}^3} E\left(\|D_{t-\cdot,*}u(\tau,y)\|_{\mathcal{H}_v}^{2q}\right) J(t-s), \end{aligned}$$

where J is defined by (1.21) and C is a positive constant which only depends on q.

Owing to the properties of deterministic Hilbert-valued integrals and Hölder's inequality with respect to the finite measure on $[0,T]\times\mathbb{R}^3$ given by $\Lambda(s,dz)ds$, one gets that

$$C(v,t,x) \leq C \int_0^t ds \sup_{t-v\leq\tau\leq s} \sup_{y\in\mathbb{R}^3} E\left(\|D_{t-\cdot,*}u(\tau,y)\|_{\mathcal{H}_v}^{2q}\right),$$

with C a positive constant depending only on q. Hence we have obtained that

$$\begin{aligned} \sup_{t-v\leq r\leq t} \sup_{y\in\mathbb{R}^3} E\left(\|D_{t-\cdot,*}u(r,y)\|_{\mathcal{H}_v}^{2q}\right) &\leq \left(\int_0^v ds \int_{\mathbb{R}^3} \mu(d\xi)|\mathcal{F}\Lambda(s)(\xi)|^2 \right)^q \\ &+ C\int_0^t ds \sup_{0\leq\tau\leq s} \sup_{y\in\mathbb{R}^3} E\left(\|D_{t-\cdot,*}u(\tau,y)\|_{\mathcal{H}_v}^{2q}\right)(J(t-s)+1). \end{aligned}$$

Applying Gronwall's Lemma 1.2.9 we end the proof. □

Concerning the proof of Theorem 2.2.2, we notice that to prove existence of density (Theorem 2.1.1) it was sufficient to show the existence of the pth moment of the inverse of the Malliavin matrix, for some $p > 0$, while to show existence and regularity existence of moments of any order $p > 0$ are needed.

The regularisation of the fundamental solution $S_3(t)$ needed here is more sophisticated. We consider $S_{\epsilon^{-\nu}} = \psi_{\epsilon^{-\nu}} * S_3$, where $\psi_{\epsilon^{-\nu}}(x) = \epsilon^{-3\nu}\psi(\epsilon^{-\nu}x)$, $\nu > 0$ and ψ is a non-negative function in $\mathcal{C}^\infty(\mathbb{R}^3)$ with support contained in the unit ball of $\mathbb{R}^3$ and such that $\int_{\mathbb{R}^3}\psi(x)dx = 1$.

As for the proof of the existence of density, we also need the same upper and lower bounds of some integrals of the Fourier transform of the fundamental solution S_3. These are collected in [QSSS04c] in an appendix; they are actually quotations of the results from the Appendix in [QSSS04a].

2.3 Some results for parabolic SPDEs

The purpose of this section is to complete the study of probability laws of solutions to SPDEs of parabolic type. Firstly, we shall extend the existence of density result proved by Pardoux and Zhang in [PZ93] for a stochastic heat equation on $(0,1)$. Secondly, we consider a Cauchy problem in $\mathbb{R}^d$, $d \geq 1$, given by a non-linear parabolic SPDE with a general elliptic operator and we obtain necessary conditions to ensure existence and smoothness of the density of the process solution; this extends the results proved by Márquez *et al.* in [MCMS01]. The proofs of the main results of this section do not require essentially new techniques and barely diverge from the arguments used by the above mentioned authors. For this reason, they shall not be developed in detail but only sketched, pointing out the main steps.

2.3.1 Existence of density

In this section we study the existence of density of the law to the real-valued solution of the following stochastic boundary value problem:

$$\frac{\partial u}{\partial t}(t,x) - A(x)u(t,x) = b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x), \tag{2.17}$$

with initial condition

$$u(0,x) = u_0(x),\ x \in (0,1),$$

and Dirichlet boundary conditions

$$u(t,0) = u(t,1) = 0,\ t > 0.$$

Throughout the section $A(x)$ denotes the differential operator

$$A(x) = a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x} + c(x),$$

where the functions $a, b, c : [0,1] \to \mathbb{R}$ are assumed to be Hölder-continuous of order $\alpha \in (0,1)$; moreover, $a(x)$ is uniformly elliptic and bounded, that is,

$$0 < a_0 < a(x) < a_1,$$

for all $x \in [0,1]$, and some positive constants a_0, a_1.

Since we are in space dimension one, it is known that we can consider the random perturbation $\dot{W}$ given by a space-time white noise; indeed, the process $\{W(t,x), (t,x) \in \mathbb{R}_+ \times [0,1]\}$ is the Brownian sheet on $\mathbb{R}_+ \times [0,1]$, that is, W is a Gaussian stochastic process defined on some probability space $(\Omega, \mathcal{F}, P)$ with mean zero and covariance function

$$E(W(t,x)W(s,y)) = (s \wedge t)(x \wedge y).$$

We also denote by $\mathcal{F}_t$ the $\sigma-$field generated by the random variables $W(s,x)$, $(s,x) \in [0,t] \times [0,1]$.

The coefficients σ and b are supposed to be measurable and locally bounded real-valued functions defined on the whole real line. Moreover, we consider the following sets of assumptions for σ and b.

(D) σ and b are differentiable, their derivatives are locally bounded and there exists a constant C such that

$$zb(z) + |\sigma(z)|^2 \leq C(1 + |z|^2), \tag{2.18}$$

for every $z \in \mathbb{R}$.

(L) There exists a positive constant C such that

$$|\sigma(z) - \sigma(y)| + |b(z) - b(y)| \leq C|z - y|,$$

for any $z, y \in \mathbb{R}$.

Notice that the above condition (2.18) on the drift coefficient b is of dissipative type; it is satisfied, for instance, by polynomials of odd degree having a negative dominant coefficient.

We assume that the initial condition u_0 belongs to $\mathcal{C}([0,1])$ and satisfies $u_0(0) = u_0(1) = 0$.

Equation (2.17) is formulated rigorously as follows:

$$\begin{aligned} u(t,x) = & \int_0^1 G(t,x,y)u_0(y)dy \\ & + \int_0^t \int_0^1 G(t-s,x,y)\sigma(u(s,y))W(ds,dy) \\ & + \int_0^t \int_0^1 G(t-s,x,y)b(u(s,y))dyds, \end{aligned} \tag{2.19}$$

where $t \in [0,T]$, for some $T > 0$. The stochastic integral in the right hand-side of the above equality is understood in Walsh's sense (see Section 1.3) as a stochastic integral with respect to the martingale measure associated to the space-time white noise ([Wal86]). The function $G = G(t,x,y)$ denotes the Green function associated with the operator $\frac{\partial}{\partial t} - A(x)$, $(t,x) \in \mathbb{R}_+ \times (0,1)$, with homogeneous Dirichlet boundary conditions. Recall that under the above conditions on the coefficients a, b, c of the operator

$A(x)$, the function $G : [0,T] \times [0,1] \times [0,1] \to \mathbb{R}$ is non-negative, continuous, twice continuously differentiable in x, once continuously differentiable in t and satisfies the following estimate:

$$|\partial_x^i \partial_t^j G(t,x,y)| \leq C_1 t^{-\frac{1+i+2j}{2}} \exp\left(-C_2 \frac{(x-y)^2}{t}\right), \tag{2.20}$$

where i and j are non-negative integer numbers such that $i + 2j \leq 2$ and C_1, C_2 are positive constants (see, for instance, [ÈI70] Theorem 1.1 and [Aro68] p. 669). In particular,

$$0 \leq G(t,x,y) \leq C_1 t^{-\frac{1}{2}} \exp\left(-C_2 \frac{(x-y)^2}{t}\right). \tag{2.21}$$

In order to ensure the existence and uniqueness of a real-valued $\mathcal{F}_t-$measurable stochastic process $\{u(t,x), (t,x) \in [0,T] \times [0,1]\}$ solution to Equation (2.19), we first focus on the globally Lipschitz case. Namely, if the coefficients b and σ satisfy condition (L), then existence and uniqueness of Equation (2.19) follows from a classical Picard-approximation argument using the Gaussian bound given by (2.21) (see also [Wal86], Theorem 3.2). Moreover, owing to (2.20) and following similar techniques as in [SSS02], it can be proved that the process u has Hölder-continuous trajectories of order $\frac{1}{2}$ and $\frac{1}{4}$ with respect to the space and time variable, respectively (see also [Wal86], Corollary 3.4).

Staying at the globally Lipschitz setting, we can also state a comparison-of-solutions result, which is a consequence of Theorem 3.3.1 in [MZ99]. Notice that in this work solutions to SPDEs taking values in some infinite-dimensional spaces are considered and the framework defined in [DPZ92] is used. However, since in the space-time white noise case this formulation is equivalent to the one constructed by Walsh in [Wal86] (see Section 4.3.3 in [DPZ92]), the comparison result shall still work for Equation (2.17). We let u_0^1, u_0^2 and b^1, b^2 be real-valued functions, the formers being defined in $[0,1]$, and u^1 and u^2 the unique solution, in case it exists, of Equation (2.19) when considering the initial condition and the drift coefficient u_0^1, b^1 and u_0^2, b^2, respectively. The comparison theorem is stated as follows.

Theorem 2.3.1. *Assume that b_1, b_2 and σ are globally Lipschitz functions. We suppose that*

$$b^1(z) \leq b^2(z), \ z \in \mathbb{R},$$

and

$$u_0^1(x) \leq u_0^2(x), \ x \in [0,1].$$

Then $u^1(t,x) \leq u^2(t,x)$, a.s. for any $(t,x) \in [0,T] \times [0,1]$.

We now state an existence and uniqueness result in the non-Lipschitz setting.

Theorem 2.3.2. *Assume that the coefficient b is a continuous function, σ is locally Lipschitz and that condition (2.18) is fulfilled. Then, there exists a unique real-valued continuous stochastic process solution to Equation (2.19).*

The proof of this result can be obtained from [GP93c], Theorems 4.2.1 and 4.3.1. In this work, a stochastic heat equation is considered, that is, the authors deal with Equation (2.17) in the particular case of $a \equiv 1$ and $b \equiv c \equiv 0$. It can be checked that the above mentioned results of [GP93c] still hold true for Equation (2.17) by using the existence and uniqueness result in the globally Lipschitz case and the comparison result given in Theorem 2.3.1. It is worth mentioning that in [GP93c] a set of slightly weaker assumptions for the coefficients b and σ than the ones assumed in Theorem 2.3.2 are considered.

Clearly, if b and σ satisfy condition (D), then we can apply Theorem 2.3.2 and we obtain that there exists a unique solution $\{u(t,x),(t,x) \in [0,T] \times [0,1]\}$ of Equation (2.19).

The Malliavin calculus framework to be considered in order to deal with the differentiability of the random variable $u(t,x)$ is defined as follows. The underlying Hilbert space is given by $H = L^2([0,T] \times [0,1])$ and the Gaussian family $(W(h), h \in H)$ is defined by

$$W(h) = \int_0^T \int_0^1 h(s,y) W(ds,dy),$$

for $h \in H$.

Now we are ready to state the main result of the section, which is an extension of Theorem 1.1 in [PZ93].

Theorem 2.3.3. *Assume that b and σ satisfy condition (D). Let $(t,x) \in (0,T] \times (0,1)$. Then, the law of the random variable $u(t,x)$ is absolutely continuous with respect to the Lebesgue measure if, and only if, there exists $s \in [0,t)$ such that $\sigma(u(s,\cdot))$ is not identically zero.*

Let us briefly sketch the proof of Theorem 2.3.3. It is a consequence of Bouleau and Hirsch criterion (see Proposition 1.3.7). Thus, first we need to check that the random variable $u(t,x)$ belongs to $\mathbb{D}^{1,2}_{loc}$. For this, for each $n \geq 1$ we consider differentiable functions $b_n, \sigma_n : \mathbb{R} \to \mathbb{R}$ such that b'_n and σ'_n are bounded and $b_n(z) = b(z)$ and $\sigma_n(z) = \sigma(z)$, for all $z \in [-n,n]$; in particular b_n and σ_n are globally Lipschitz continuous. We denote by $u^n = \{u^n(t,x),(t,x) \in [0,T] \times [0,1]\}$ the unique real-valued continuous solution to the stochastic evolution equation

$$\begin{aligned} u^n(t,x) = &\int_0^1 G(t,x,y)u_0(y)dy \\ &+ \int_0^t \int_0^1 G(t-s,x,y)\sigma_n(u^n(s,y))W(ds,dy) \\ &+ \int_0^t \int_0^1 G(t-s,x,y)b_n(u^n(s,y))dyds. \end{aligned} \tag{2.22}$$

We define the stopping time

$$\tau_n = \inf\{t \in [0,T], \sup_{x\in[0,1]} |u(t,x)| > n\},$$

$n \geq 1$. Since the process u has continuous paths, we have that τ_n converges to infinity almost surely. This implies that the sequence of sets $(\Omega_n, n \geq 1)$ defined by

$$\Omega_n = \{\omega \in \Omega, \tau_n(\omega) > T\}$$

is increasing and its a.s.-limit is Ω. Moreover, notice that the pathwise uniqueness for Equation (2.22) ensures that $u(t,x) = u^n(t,x)$ a.s. on Ω_n.

Thus, to conclude that $u(t,x) \in \mathbb{D}^{1,2}_{loc}$, we only need to check that, for any $n \geq 1$, $u^n(t,x)$ belongs to the space $\mathbb{D}^{1,2}$. This follows from standard arguments because the coefficients b_n and σ_n are of class $\mathcal{C}^1$ and have bounded derivatives (see, for instance, [MSS99], [MCMS01]). The Malliavin derivative $Du^n(t,x)$ satisfies the following equation on $L^2([0,T]\times[0,1])$:

$$\begin{aligned} Du^n(t,x) =&G(t-\cdot,x,*)\sigma_n(u^n(\cdot,*)) \\ &+ \int_0^t \int_0^1 G(t-s,x,y)\sigma_n'(u^n(s,y))Du^n(s,y)W(ds,dy) \\ &+ \int_0^t \int_0^1 G(t-s,x,y)b_n'(u^n(s,y))Du^n(s,y)dyds, \end{aligned}$$

Restricting the above evolution equation to each Ω_n, by definition of the space $\mathbb{D}^{1,2}_{loc}$, we obtain an analogous equation for $Du(t,x)$, as follows.

$$\begin{aligned} Du(t,x) =&G(t-\cdot,x,*)\sigma(u(\cdot,*)) \\ &+ \int_0^t \int_0^1 G(t-s,x,y)\sigma'(u(s,y))Du(s,y)W(ds,dy) \\ &+ \int_0^t \int_0^1 G(t-s,x,y)b'(u(s,y))Du(s,y)dyds. \end{aligned}$$

To deal with the analysis of the Malliavin matrix we proceed as in [PZ93], Section 3. For this, a comparison result, namely Theorem 2.3.1, the Gaussian bound for $G(t,x,y)$ (see (2.21)) and the Hölderianity of the trajectories for the solution to Equation (2.19) when σ and b are globally Lipschitz are needed. □

2.3.2 Smoothness of the density

We consider the following stochastic Cauchy problem:

$$\frac{\partial u}{\partial t}(t,x) = A(t,x)u(t,x) + b(u(t,x)) + \sigma(u(t,x))\dot{F}(t,x), \tag{2.23}$$

$t > 0$, $x \in \mathbb{R}^d$, with initial condition $u(0,x) = 0$, $x \in \mathbb{R}^d$. The random perturbation $\dot{F}$ is the formal derivative of the Gaussian noise described in Section 1.3. The coefficients b and σ are globally Lipschitz functions and $A(t,x)$ is an elliptic operator of the form

$$A(t,x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}[a_{i,j}(t,x)\frac{\partial}{\partial x_i} + a_j(t,x)] + \sum_{i=1}^{d} b_i(t,x)\frac{\partial}{\partial x_i} + c(t,x),$$

where the functions $a_{i,j}, a_j, b_i, c : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ are bounded and continuous. We consider the following assumption for the operator $A(t,x)$.

(E) The functions $a_{i,j}, a_j, b_i, c$ are of class $\mathcal{C}^\infty$ on $[0,T]\times\mathbb{R}^d$. Moreover, the operator $A(t,x)$ is uniformly elliptic, that is, there exists $a_0 > 0$ such that

$$\sum_{i,j=1}^{d} a_{i,j}(t,x)\xi_i\xi_j \geq a_0|\xi|^2,$$

for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\xi = (\xi_1,\ldots,\xi_d) \in \mathbb{R}^d$.

Denote by $f = f(t,s;x,y)$, $0 \leq s < t$ and $x,y \in \mathbb{R}^d$, the fundamental solution associated with the differential operator $\frac{\partial}{\partial t} - A(t,x)$ on $\mathbb{R}^d$. Owing to Theorem 7 in [Aro68], under condition (E) we have the following lower and upper bounds for the function F:

$$C_1(t-s)^{-\frac{d}{2}} \exp\left(-C_2\frac{(x-y)^2}{t-s}\right) \leq f(t,s;x-y) \leq C_3(t-s)^{-\frac{d}{2}} \exp\left(-C_4\frac{(x-y)^2}{t-s}\right), \tag{2.24}$$

where C_k, $k = 1,\ldots,4$, are positive constants. In particular, f is non-negative.

A solution to Equation (2.23) is an $\mathcal{F}_t-$adapted (filtration generated by F) stochastic process $\{u(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ such that

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} f(t,s;x,y)\sigma(u(s,y))M(ds,dy) + \int_0^t \int_{\mathbb{R}^d} f(t,s;x,y)b(u(s,y))dyds, \tag{2.25}$$

where the stochastic integral is of Walsh's type and M is the martingale measure associated with the process F (see Section 1.3). Making use of a standard Picard iteration method and of the upper bound in (2.24), it can be shown that a sufficient condition ensuring existence and uniqueness of process solution to Equation (2.25) is

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < +\infty,$$

where μ is the spectral measure of Γ, the spatial correlation of the noise (for similar results see [Wal86], Theorem 3.2, [MSS99], Theorem 1.2 or [Dal99], Theorem 13). Notice that this condition was also sufficient to deal with the existence and uniqueness of solution of the heat equation in any dimension $d \geq 1$ and the wave equation for $d = 1,2,3$ (see Section 1.4).

The main result of the section is as follows.

Theorem 2.3.4. *Assume that the coefficients b and σ are $\mathcal{C}^\infty$ functions with bounded derivatives of any order greater than or equal to one and that the following condition is fulfilled for some $\eta \in (0, \frac{1}{2})$:*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta} < +\infty.$$

Moreover, suppose that $|\sigma(z)| \geq \sigma_0 > 0$, for some $\sigma_0 > 0$. Then, for all $(t,x) \in [0,T] \times \mathbb{R}^d$, the law of the random variable $u(t,x)$ is absolutely continuous with respect to the Lebesgue measure and its density is a $\mathcal{C}^\infty$ function.

The proof can be deduced from Lemma 3.1 and Theorem 3.2 in [MCMS01]. Indeed, it is sufficient to notice that the isometry property of the stochastic integral gives

$$E\left(\left|\int_0^t \int_{\mathbb{R}^d} f(t,s;x,y)g(s,y)M(ds,dy)\right|^2\right) = E\left(\int_0^t ds \int_{\mathbb{R}^d} \Gamma(dy) \int_{\mathbb{R}^d} dz f(t,s;x,z)f(t,s;x,z-y)g(s,z)g(s,z-y)\right), \tag{2.26}$$

where $g = \{g(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ is any predictable process satisfying

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|g(t,x)|^2) < +\infty$$

(see Theorem 1.2.3). Owing to the estimations given in (2.24), the right hand-side of equality (2.26) taking $g \equiv 1$ has the same lower and upper bounds, up to multiplication by a constant, as the integral

$$\int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Lambda_h(s)(\xi)|^2, \tag{2.27}$$

where Λ_h is the fundamental solution to the heat equation in any dimension $d \geq 1$ (see (i) and (ii) of Lemma 3.1 [MCMS01]). Moreover, again by (2.24) it holds that

$$\sup_{x\in\mathbb{R}^d} \int_0^t ds \int_{\mathbb{R}^d} dy F(t,s;x,y) \leq Ct \tag{2.28}$$

(see (iii) of Lemma 3.1 [MCMS01]).

From (2.27) and (2.28), it is straightforward to check that the same conclusion as in Theorem 3.2 in [MCMS01] can be obtained; therefore we end the proof of the Theorem. □

2.4 Lattice approximation for a stochastic wave equation

This section is devoted to present the contents of third article of this dissertation. It can be found in a prepublication form in Appendix C.

The aim of this paper is to construct an approximation scheme for a one-dimensional stochastic wave equation on $[0,1]$, with some Dirichlet boundary conditions, using a spatial finite-difference method. We study the convergence in $L^p(\Omega)$ and a.s. of the approximations to the solution. For the former we obtain bounds of the rate of convergence and we test them numerically.

Lattice approximation schemes for parabolic SPDEs in one spatial dimension, developed in [Gyö98b], [Gyö99], have been the starting point of several further investigations. In [MM03], lattice schemes for parabolic SPDEs in any spatial dimension are considered and the influence of the particular covariance density of the noise given by Riesz kernels is studied. A class of parabolic evolution equations on Banach spaces with monotone operators are analysed in [GM04]. In [GM], a finite difference approximation scheme for an elliptic SPDEs in dimension $d = 1,2,3$ is studied. The results show how much the behaviour of this kind of approximations depends on the differential operator driving the SPDE and are one of the very few attempts of looking beyond the parabolic case. Let us also mention [DZ02] for some results on numerical approximations for elliptic equations.

To our best knowledge, the results stated in [QSSS04b] (Appendix C) correspond to the first step towards the analysis of lattice approximations for hyperbolic SPDEs. In fact, we are only aware of [MPW03] for some results on numerical approximations for the stochastic wave equation. The authors consider non-random coefficients b and σ -therefore the solution is a Gaussian process- and construct algorithms to simulate the solution numerically on some grid.

Here, we consider the non-linear stochastic wave equation

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x,u(t,x)) + \sigma(t,x,u(t,x))\frac{\partial^2 W}{\partial t \partial x}(t,x), \tag{2.29}$$

$t > 0$, $x \in (0,1)$, with initial conditions

$$u(0,x) = u_0,\ \frac{\partial u}{\partial t}(0,x) = v_0,\ x \in (0,1),$$

and boundary conditions of Dirichlet type given by

$$u(t,0) = u(t,1) = 0,\ t > 0.$$

The functions u_0 and v_0 are defined on $[0,1]$, u_0 vanishes at $x=0$ and $x=1$. W is the Brownian sheet on $\mathbb{R}_+ \times [0,1]$; that is, $\{W(t,x), (t,x) \in \mathbb{R}_+ \times [0,1]\}$ is a Gaussian stochastic process defined on some probability space $(\Omega, \mathcal{F}, P)$ with mean zero and covariance function

$$E(W(t,x)W(s,y)) = (s \wedge t)(x \wedge y).$$

The real-valued solution $\{u(t,x), (t,x) \in [0,T] \times [0,1]\}$, $T > 0$, to Equation (2.29) is interpreted in the *mild* form,

$$\begin{aligned} u(t,x) = & \int_0^1 G(t,x,y)v_0(y)dy + \frac{\partial}{\partial t}\left(\int_0^1 G(t,x,y)u_0(y)dy\right) \\ & + \int_0^t \int_0^1 G(t-s,x,y)\sigma(s,y,u(s,y))W(ds,dy) \\ & + \int_0^t \int_0^1 G(t-s,x,y)f(s,y,u(s,y))dsdy, \end{aligned} \tag{2.30}$$

$t \geq 0$, $x \in (0,1)$, where G is the Green function associated with the wave equation with homogeneous Dirichlet boundary conditions on $(0,1)$. The third term on the right hand-side of the above equation corresponds to a stochastic integral with respect to the martingale measure associated to the space-time white noise; since the integrand is a real-valued function, this is a stochastic integral of Walsh's type (see Section 1.2.1).

We fix a positive constant T and assume that the coefficients f and σ are real-valued functions defined on $[0,T]\times[0,1]\times\mathbb{R}$, satisfying the following conditions:

(L)

$$\sup_{t\in[0,T]}\big(|f(t,x,z)-f(t,y,v)|+|\sigma(t,x,z)-\sigma(t,y,v)|\big)\leq C(|x-y|+|z-v|),$$

(LG)

$$\sup_{(t,x)\in[0,T]\times[0,1]}\big(|f(t,x,z)|+|\sigma(t,x,z)|\big)\leq C(1+|z|),$$

for every $x,y\in[0,1]$ and $z,v\in\mathbb{R}$.

Existence and uniqueness of a real-valued process solution to Equation (2.30) follows from standard techniques based on Picard iterations. For similar results, we refer the reader to [CN88] and [MSS99]. In these papers the authors deal with stochastic wave equations in dimension $d=1,2$, respectively.

Throughout the paper we use the following decomposition of the Green function G:

$$G(t,x,y)=\sum_{j=1}^{\infty}\frac{\sin(j\pi t)}{j\pi}\varphi_j(x)\varphi_j(y), \tag{2.31}$$

where $\varphi_j(x)=\sqrt{2}\sin(j\pi x)$, $j\geq 1$, is a complete orthonormal system of $L^2([0,1])$ (see, for instance, [Duf03], p. 94).

We remark that, by the classical approach on construction of solutions via expansion into eigenfunctions to the deterministic wave equation on $[0,1]$ with Dirichlet boundary conditions , we know that the contribution of the initial condition u_0 in Equation (2.30) is given by

$$\frac{\partial}{\partial t}\left(\int_0^1 G(t,x,y)u_0(y)dy\right)=\sum_{j=1}^{\infty}\langle u_0,\varphi_j\rangle\cos(j\pi t)\varphi_j(x),$$

where $\langle\cdot,\cdot\rangle$ stands for the usual scalar product in $L^2([0,1])$ (see [Joh82], p. 44).

For a function $g:[0,1]\to\mathbb{R}$ and $\alpha\in\mathbb{R}$, we define

$$\|g\|_{\alpha,2}:=\left(\sum_{j=1}^{\infty}(1+j^2)^{\alpha}|\langle g,\varphi_j\rangle|^2\right)^{\frac{1}{2}}$$

and denote by $H^{\alpha,2}([0,1])$ the set of functions $g:[0,1]\to\mathbb{R}$ such that $\|g\|_{\alpha,2}<\infty$. Notice that the above defined norm is finite if and only if the function $(1-\Delta)^{\alpha}g$ belongs to $L^2([0,1])$, where Δ stands for the Laplacian operator in dimension one. This implies that $H^{\alpha,2}([0,1])$ is a subspace of the fractional Sobolev space of fractional differential order α and integrability order $p = 2$ (see [Tri92]).

First we study some properties of the solution $u(t,x)$. More precisely, we have the following result, whose proof can be found in Appendix C, Propositions C.2.1 and C.2.2.

Proposition 2.4.1. *Assume that $v_0 \in H^{\beta,2}([0,1])$, for some $\beta > -\frac{1}{2}$, and $u_0 \in H^{\alpha,2}([0,1])$, for some $\alpha > \frac{1}{2}$; suppose also that the coefficients σ and f satisfy condition (L) and (LG). Then, for every $p \geq 1$, there exists a positive constant C depending on α and β, such that*

$$\sup_{(t,x)\in[0,T]\times[0,1]} E(|u(t,x)|^p) < +\infty$$

and

$$\begin{aligned} E(|u(s,x) - u(t,y)|^{2p}) \leq C(&|t-s|^{p(1+2\beta)} + |x-y|^{p(1+2\beta)} \\ &+ |t-s|^{p(2\alpha-1)} + |x-y|^{p(2\alpha-1)} \\ &+ |t-s|^p + |x-y|^p), \end{aligned}$$

for every $s,t \in [0,T]$ and $x,y \in [0,1]$. Consequently, the process u has a.s. Hölder-continuous sample paths of order δ, for all $\delta \in (0,\delta_0)$, where $\delta_0 = (\frac{1}{2}+\beta)\wedge(\alpha-\frac{1}{2})\wedge\frac{1}{2}$.

In order to construct the approximations to the solution u, we fix any $n \geq 1$ and the spatial grid $x_k = \frac{k}{n}$, $k = 1, \cdots, n-1$. Then, we consider the system of stochastic differential equations obtained by substituting the Laplacian by its finite-difference discretisation (see (2.32) and (2.33) below). This provides an implicit finite dimensional scheme. By linear interpolation, we obtain a sequence of evolution equations which is proved to converge in any $L^p(\Omega)$, uniformly in t,x, to the solution of (2.29) with a given rate of convergence (see Theorem C.3.1).

Let us briefly sketch the construction of the approximations. The detailed procedure can be found in Section C.3.1.

Notice first that Equation (2.29) is equivalent to the following one:

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t}(t,x) = v(t,x) \\ \frac{\partial v}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x,u(t,x)) + \sigma(t,x,u(t,x))\frac{\partial^2 W}{\partial t \partial x}(t,x), \end{array} \right. \tag{2.32}$$

$t > 0$, $x \in (0,1)$, with initial conditions

$$u(0,x) = u_0(x), \;\; v(0,x) = v_0(x), \; x \in (0,1),$$

and Dirichlet boundary conditions

$$u(t,0) = u(t,1) = 0,\ t > 0.$$

For any integer $n \geq 1$, set $x_k = \frac{k}{n}$, $k = 1, \ldots, n-1$. We consider the system of stochastic differential equations

$$\begin{cases} du^n(t,x_k) &= v^n(t,x_k)dt \\ dv^n(t,x_k) &= n^2(u^n(t,x_{k+1}) - 2u^n(t,x_k) + u^n(t,x_{k-1}))dt \\ & \quad +f(t,x_k,u^n(t,x_k))dt \\ & \quad +n\sigma(t,x_k,u^n(t,x_k))d\left(W(t,x_{k+1}) - W(t,x_k)\right), \end{cases} \tag{2.33}$$

with initial conditions

$$u^n(0,x_k) = u_0(x_k),\ v^n(0,x_k) = v_0(x_k),$$

where

$$u_0(x_k) = \sum_{j=1}^{\infty} \langle u_0, \varphi_j \rangle \varphi_j(x_k), \quad v_0(x_k) = \sum_{j=1}^{\infty} \langle v_0, \varphi_j \rangle \varphi_j(x_k),$$

$k = 1, \ldots, n-1$. System (2.33) is the formal spatial discretisation of (2.32) using finite differences.

The aim is to put together the above system in a suitable way so that we obtain a stochastic differential equation on $\mathbb{R}^{2(n-1)}$. Then, we apply Itô's formula and we obtain a new system of stochastic differential equations for $u^n(t,x_k)$, $k = 1, \ldots, n-1$. Finally, we extend the process $u^n(t,\cdot)$ to the whole interval $[0,1]$ by linear interpolation and we show that the resulting process $\{u^n(t,x), (t,x) \in [0,T] \times [0,1]\}$ satisfies the stochastic evolution equation

$$\begin{aligned} u^n(t,x) = &\int_0^1 G^n(t,x,y) v_0(\kappa_n(y)) dy \\ &+ \frac{\partial}{\partial t}\left(\int_0^1 G^n(t,x,y) u_0(\kappa_n(y)) dy\right) \\ &+ \int_0^t \int_0^1 G^n(t-s,x,y) f(s,\kappa_n(y), u^n(s,\kappa_n(y))) ds dy \\ &+ \int_0^t \int_0^1 G^n(t-s,x,y) \sigma(s,\kappa_n(y), u^n(s,\kappa_n(y))) W(ds,dy), \end{aligned}$$

$t \in (0,T]$ and $x \in (0,1)$, where

$$G^n(t,x,y) = \sum_{j=1}^{n-1} \frac{\sin\left(j\pi t\sqrt{c_j^n}\right)}{j\pi\sqrt{c_j^n}} \varphi_j^n(x) \varphi_j(\kappa_n(y)),$$

with $\kappa_n(y) = [ny]/n$, $\varphi_j^n(x) = \varphi_j(x_l)$ for $x = x_l$ and

$$\varphi_j^n(x) = \varphi_j(x_l) + (nx - l)\left(\varphi_j(x_{l+1}) - \varphi_j(x_l)\right)$$

for $x \in (x_l, x_{l+1})$. For any $n \geq 1$ and $j = 1, \ldots, n-1$, c_j^n is given by

$$c_j^n = \frac{\sin^2\left(\frac{j\pi}{2n}\right)}{\left(\frac{j\pi}{2n}\right)^2}.$$

The main result concerning the convergence of the approximations $u^n(t,x)$ to $u(t,x)$ is the following (see Theorem C.3.1 in Appendix C).

Theorem 2.4.2. *Suppose that $u_0 \in H^{\alpha,2}([0,1])$, with $\alpha > \frac{3}{2}$, $v_0 \in H^{\beta,2}([0,1])$, with $\beta > \frac{1}{2}$. We fix $p \geq 1$ and assume that the coefficients σ and f satisfy conditions (LG) and (L). Then, there exists a positive constant C depending on α, β such that, for any $n \geq 1$,*

$$\sup_{(t,x)\in[0,T]\times[0,1]} E(|u^n(t,x) - u(t,x)|^{2p}) \leq \frac{C}{n^{2p\rho}},$$

for all $\rho \in (0, \rho_0)$, with $\rho_0 = \frac{1}{3} \wedge (\alpha - \frac{3}{2}) \wedge (\beta - \frac{1}{2})$. Moreover, $u^n(t,x)$ converges to $u(t,x)$ almost surely, as n tends to infinity, uniformly with respect to $(t,x) \in [0,T] \times [0,1]$.

We remark that, for any $\gamma > \frac{1}{2}$, $H^{\gamma,2}([0,1])$ is embedded in the space of δ-Hölder continuous functions on $(0,1)$, for any $\delta \in (0, \gamma - \frac{1}{2})$ (see, for instance, [Shi92], Theorem E12). Actually, one could state an analogue to Theorem 2.4.2 assuming Hölder continuity of the initial conditions.

The first result needed for the proof of Theorem 2.4.2 is the following uniform estimation of the $L^2([0,1])-$norm of the difference $G(t,x,\cdot) - G^n(t,x,\cdot)$. Namely, for every $\delta \in (0, \frac{2}{3})$, there exists a positive constant C, depending on δ, such that

$$\sup_{(t,x)\in[0,T]\times[0,1]} \int_0^1 |G(t,x,y) - G^n(t,x,y)|^2 dy \leq \frac{C}{n^\delta}, \tag{2.34}$$

for every $n \geq 1$. For a detailed proof we refer to Lemma C.3.3.

The above bound determines the rate of convergence in the statement of Theorem 2.4.2. Indeed, following the proof of Lemma C.3.3, it turns out that the bound $\frac{1}{n^\delta}$ in (2.34) comes from the analysis of the term

$$I_3^n(t,x) = \sum_{j=1}^{n-1} \left(\frac{\sin(j\pi t)}{j\pi} - \frac{\sin\left(j\pi t\sqrt{c_j^n}\right)}{j\pi\sqrt{c_j^n}} \right)^2 \varphi_j^2(x),$$

$(t, x) \in [0, T] \times [0, 1]$.

In the last section of [QSSS04b] (Appendix C), we attach an appendix in which we simulate numerically the term $I_3^n(t, x)$ in order to test the optimality of the bound $\frac{1}{n^\delta}$ in (2.34). The consequence is that this bound is optimal. Considering the particular case of vanishing initial conditions, the variation of the rate of convergence in Theorem 2.4.2 is determined by $\rho \in (0, \frac{1}{3})$. Unlike for the parabolic case, this factor $\frac{1}{3}$ differs substantially from the order of hölderianity with respect to space for the solution u, which is $\frac{1}{2}$. The numerical tests show that this phenomenon is something intrinsic to the method.

For the proof of Theorem 2.4.2 we also need a uniform bound of the $L^p(\Omega)-$norm of the approximations $u^n(t, x)$. Indeed, we assume that $v_0 \in L^2([0, 1])$ and $u_0 \in H^{\alpha,2}([0, 1])$, for some $\alpha > \frac{1}{2}$, and that the coefficients f and σ satisfy condition (LG). Then we prove that for every $p \geq 1$

$$\sup_{n \geq 1} \sup_{(t,x) \in [0,T] \times [0,1]} E\left(|u^n(t, x)|^{2p}\right) < +\infty$$

(see Proposition C.3.4).

In order to check the $L^p(\Omega)-$convergence, we set

$$\begin{aligned}
\nu(t, x) &= \int_0^1 G(t, x, y) v_0(y) dy, \\
\nu^n(t, x) &= \int_0^1 G^n(t, x, y) v_0(\kappa_n(y)) dy, \\
\mu(t, x) &= \sum_{j=1}^{\infty} \langle u_0, \varphi_j \rangle \cos(j\pi t) \varphi_j(x), \\
\mu^n(t, x) &= \sum_{j=1}^{n-1} \langle u_0, \varphi_j \rangle^n \cos\left(j\pi t \sqrt{c_j^n}\right) \varphi_j^n(x), \\
w(t, x) &= u(t, x) - \nu(t, x) - \mu(t, x) \\
w^n(t, x) &= u^n(t, x) - \nu^n(t, x) - \mu^n(t, x).
\end{aligned}$$

Then, assuming that u_0 belongs to $H^{\alpha,2}([0, 1])$, with $\alpha > \frac{3}{2}$, $v_0 \in H^{\beta,2}([0, 1])$, with $\beta > \frac{1}{2}$, and that the coefficients σ and f satisfy conditions (LG) and (L), we prove that, for any $p \geq 1$, there exists a positive constant C, such that

$$\sup_{(t,x) \in [0,T] \times [0,1]} |\nu^n(t, x) - \nu(t, x)| \leq \frac{C}{n^\epsilon},$$

for any $\epsilon \in (0, \epsilon_0)$, with $\epsilon_0 = \frac{1}{3} \wedge (\beta - \frac{1}{2})$;

$$\sup_{(t,x) \in [0,T] \times [0,1]} |\mu^n(t, x) - \mu(t, x)| \leq \frac{C}{n^\tau},$$

for any $\tau \in (0, \tau_0)$, with $\tau_0 = (\alpha - \frac{3}{2}) \wedge 1$;

$$\sup_{(t,x)\in[0,T]\times[0,1]} E(|w^n(t,x) - w(t,x)|^{2p}) \leq \frac{C}{n^{2p\rho}},$$

for each $n \geq 1$ and any $\rho \in (0, \rho_0)$, with $\rho_0 = \frac{1}{3} \wedge (\alpha - \frac{3}{2}) \wedge (\beta - \frac{1}{2})$.

We refer the reader to Propositions C.3.6, C.3.7 and C.3.8, respectively, for a detailed proof of the above estimations. This bounds yield the convergence in $L^p(\Omega)$ of $u^n(t,x)$ to $u(t,x)$, as n tends to infinity, and the bound for the rate of convergence given in the statement of Theorem 2.4.2.

The proofs of the above estimations are mainly based on the expansion of the Green function given in (2.31) and arguments making use of Cauchy-Schwarz, Hölder and Burkholder's inequalities.

The almost sure convergence is a consequence of the first part of Theorem 2.4.2 and the hölderianity of the sample paths of the processes $w(t,x)$ and $w^n(t,x)$, $(t,x) \in [0,T] \times [0,1]$; notice that the former is a consequence of the Hölder property of the trajectories of u when taking vanishing initial conditions. For the latter see Lemma C.3.5. The details of the almost surely convergence are given in the proof of Theorem C.3.1 in Appendix C.

Conclusions

In this dissertation we have mainly studied a stochastic partial differential equation of hyperbolic type: the stochastic wave equation.

In the first part ([QSSS04a] and [QSSS04c]), we have considered a stochastic wave equation with spatial dimension three and we have given sufficient condition ensuring that the law of the solution at any fixed point has an infinitely differentiable density. In comparison to the one and two-dimensional cases ([CN88], [MSS99], respectively), the main mathematical motivation in the three-dimensional case is the fact that the fundamental solution associated to the wave operator is not a function but a distribution. This requires a more sophisticated stochastic integration theory ([Dal99]) and new techniques in order to apply the Malliavin calculus to the equation.

In the same spirit, we have added some contributions to parabolic SPDEs. We have extended some known results for the stochastic heat equation to SPDEs driven by more general operators than the Laplacian. As has been mentioned in the Introduction of this dissertation, the main motivation to deal with these problems comes from the study of the existence and smoothness of the density for a stochastic heat equation with non-global Lipschitz drift coefficient satisfying some dissipative condition. This problem remains still open.

The second part of the dissertation ([QSSS04b]) is devoted to study lattice approximations for a stochastic wave equation in one space dimension, with boundary conditions of Dirichlet type. We have used a spatial finite-difference method and we have studied the convergence of the discretisation processes to the original solution. As it was pointed out in Section 2.4, the bound for the rate of convergence is not as sharp as we expected. Clearly, discretisations both in time and in space is the next topic to be investigated.