Appendix A

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Absolute continuity of the law of the solution to the 3-dimensional stochastic wave equation

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Abstract

We present new results regarding the existence of density of the real-valued solution to a 3-dimensional stochastic wave equation. The noise is white in time and with a spatially homogeneous correlation whose spectral measure μ satisfies that $\int_{\mathbb{R}^3} \mu(d\xi)(1+|\xi|^2)^{-\eta} < \infty$, for some $\eta \in (0, \frac{1}{2})$. Our approach is based on the mild formulation of the equation given by means of Dalang's extended version of Walsh's stochastic integration; we use the tools of Malliavin calculus. Let S_3 be the fundamental solution to the 3-dimensional wave equation. The assumption on the noise yields upper and lower bounds for the integral $\int_0^t ds \int_{\mathbb{R}^3} \mu(d\xi)|\mathscr{F} S_3(s)(\xi)|^2$ and upper bounds for $\int_0^t ds \int_{\mathbb{R}^3} \mu(d\xi)|\xi||\mathscr{F} S_3(s)(\xi)|^2$ in terms of powers of t. These estimates are crucial in the analysis of the Malliavin variance, which can be done by a comparison procedure with respect to smooth approximations of the distribution-valued function $S_3(t)$ obtained by convolution with an approximation of the identity.

MSC: 60H07; 60H15

Keywords: Malliavin calculus; Stochastic partial differential equations; Wave equation

1. Introduction

This paper is devoted to study the probability law of the real-valued solution to the stochastic wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_3\right)u(t,x) = \sigma(u(t,x))\dot{F}(t,x) + b(u(t,x)),$$

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$$u(0,x) = \frac{\partial u}{\partial t}(0,x) = 0, \tag{1}$$

where $(t,x) \in (0,T] \times \mathbb{R}^3$, $T>0$, and Δ_3 denotes the Laplacian operator on $\mathbb{R}^3$.

The aim is to establish sufficient conditions ensuring the law of $u(t,x)$ to be absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$, for any fixed $(t,x) \in (0,T] \times \mathbb{R}^3$.

Eq. (1) is an example of the more general class of stochastic partial differential equations (spde's)

$$Lu(t,x) = \sigma(u(t,x))\dot{F}(t,x) + b(u(t,x)),$$

$$u(0,x) = \frac{\partial u}{\partial t}(0,x) = 0, \tag{2}$$

$t \in (0,T], x \in \mathbb{R}^d$, $T>0$, where L denotes a second-order differential operator and the fundamental solution of $Lu=0$ is, for any $t \in (0,T]$, a non-negative distribution with rapid decrease. We assume that the coefficients σ and b are real Lipschitz functions; the noise F is a mean-zero $L^2(\Omega, \mathscr{F}, P)$-valued Gaussian process indexed by the space of test functions $\mathscr{D}(\mathbb{R}^{d+1})$ with covariance functional given by $J(\varphi,\psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \Gamma(dx)(\varphi(s) * \tilde{\psi}(s))(x)$, where $\tilde{\psi}(s,x) = \psi(s,-x)$ and Γ is a non-negative, non-negative definite tempered measure. According to [22, Chapter VII, Théorème XVII], this implies that Γ is symmetric and there exists a non-negative tempered measure μ on $\mathbb{R}^d$ whose Fourier transform is Γ, that is $J(\varphi,\psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \mu(d\xi)\mathscr{F}\varphi(s)(\xi)\overline{\mathscr{F}\psi(s)(\xi)}$.

We follow the extension of Walsh's approach developed in [5] and give a rigorous meaning to Eq. (2) in the *mild form*, as follows. Let Λ denote the fundamental solution to $Lu=0$. We denote by $M = \{M_t(A), t \in [0,T], A \in \mathscr{B}_b(\mathbb{R}^d)\}$ the martingale measure extension of F (see [6] for the details of this extension) and by $\mathscr{F}_t$ the σ-field generated by the random variables $M_s(A)$, $s \in [0,t]$, $A \in \mathscr{B}_b(\mathbb{R}^d)$, for any $t \in [0,T]$. Then a solution to (2) is a *real-valued* stochastic process $u = \{u(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$, defined on the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, P)$, progressively measurable, satisfying

$$\begin{aligned} u(t,x) = &\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-y)\sigma(u(s,y))M(ds,dy) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} b(u(t-s,x-y))\Lambda(s,dy). \end{aligned} \tag{3}$$

For $L = \frac{\partial^2}{\partial t^2} - \Delta_3$ we will write S_3 instead of Λ; it is well-known that $S_3(t) = \frac{1}{4\pi t}\sigma_t$, where σ_t denotes the uniform measure on the 3-dimensional sphere of radius t. We refer the reader to [19,20] for results related to [5] on the stochastic wave equation.

Malliavin calculus, set up in the seminal paper [10], provides a useful tool for the analysis of densities of Brownian functionals, and more generally for functionals of Gaussian families indexed by a real separable Hilbert space. Applications of this technique to spde's extend to the heat equation [1,11,18] and different examples of hyperbolic spde's [12,16,17,21], including the stochastic wave equation in dimension 1 [4] and dimension 2 [11,13]. In all these works, the fundamental solution of the underlying partial differential equation is a real-valued function, while in our case it is a distribution. This fact involves a new mathematical analysis. In this article, we will consider the Gaussian family described as follows. Let $\mathscr{E}$ be the inner-product space consisting of functions $\varphi \in \mathscr{S}(\mathbb{R}^d)$, the Schwartz space of rapidly decreasing $\mathscr{C}^\infty$ test functions, endowed with the inner-product $\langle \varphi, \psi \rangle_{\mathscr{E}} := \int_{\mathbb{R}^d} \Gamma(dx)(\varphi * \tilde{\psi})(x)$, where $\tilde{\psi}(x) = \psi(-x)$. Notice that

$$\langle \varphi, \psi \rangle_{\mathscr{E}} = \int_{\mathbb{R}^d} \mu(d\xi) \mathscr{F}\varphi(\xi) \overline{\mathscr{F}\psi(\xi)}. \tag{4}$$

Let $\mathscr{H}$ denote the completion of $(\mathscr{E}, \langle \cdot, \cdot \rangle_{\mathscr{E}})$. Set $\mathscr{H}_T = L^2([0,T]; \mathscr{H})$; notice that $\mathscr{H}$ and $\mathscr{H}_T$ may contain not only functions but also distributions. The space $\mathscr{H}_T$ is a real Hilbert separable space. For $h \in \mathscr{H}_T$, set $W(h) = \int_0^t \int_{\mathbb{R}^d} h(s,x) M(ds,dx)$ where the stochastic integral can be interpreted in Dalang's sense. Then $\{W(h), h \in \mathscr{H}_T\}$ is a Gaussian process and we can use the differential Malliavin calculus based on it (see for instance [15]).

In Section 2 of this paper, we extend Dalang's stochastic integral to integrators that are defined by stochastic integration of Hilbert-valued predictable processes with respect to martingale measures. This extension, defined coordinatewise, is needed to give a rigorous meaning to the equation satisfied by the Malliavin derivative of $u(t,x)$.

Section 3 deals with the class of spde's studied in [5], that is, with Eq. (3). We prove in Theorem 2 that the solution of these equations belongs to the stochastic Sobolev space $\mathbb{D}^{1,p}$ of differentiable L^p random variables with L^p derivatives—in the Malliavin sense; we give the equation on $\mathscr{H}_T$ satisfied by the Malliavin derivative. If $\Lambda(t)$ is a real function, for example in the stochastic heat equation in any dimension d and the stochastic wave equation in dimension $d = 1, 2$, it is well-known that the solution of (3) at any fixed point (t,x) belongs to $\mathbb{D}^{N,p}$ for any $N \in \mathbb{N}$ and every $p \in [1, \infty)$ (see for instance [1,4,11,13]); the equation satisfied by the Nth derivative is obtained recursively using the rules of Malliavin calculus, by derivation of each term of the equation satisfied by the $(N-1)$th derivative. If $\Lambda(t)$ is a distribution, for example in the wave equation in dimension $d = 3$, this approach is not possible, the problem being how to differentiate the stochastic integral term. We have been able to solve this problem at the level $N = 1$ by an approximation procedure based on the convolution of $\Lambda(t)$ with an approximation of the identity. We believe that pushing ahead our arguments should give the differentiability of any order of the solution; however, there is yet no hope for obtaining an expression of the Nth derivative for $N \geqslant 2$. The non-negative requirement on $\Lambda(t)$ prevents from estimating L^p moments of differences of stochastic integrals like those appearing in (3), for arbitrary $p \in (2, \infty)$. However, for $p = 2$ one can invoke the isometry property of Dalang's stochastic integral. For these reasons, L^p-convergence of a sequence of approximations of the solution to (3) is proved by checking L^2-convergence and L^p-boundedness for any $p \in (2, \infty)$.

Section 4 is devoted to prove the almost sure non-singularity of the Malliavin matrix. Owing to Bouleau's and Hirsch's criterium (see [3,14]), this property together with the results in Section 3 provide the existence of density for the law of $u(t,x)$ at any fixed $(t,x) \in (0,T] \times \mathbb{R}^3$. We split the Malliavin matrix into a principal and a secondary term. The former one is obtained by differentiating the martingale measure M in the stochastic integral of the right-hand side of Eq. (3); it is an $\mathscr{H}_T$-valued random vector that we formally write here as $Z(t,x) := S_3(t-\cdot, x-*)\sigma(u(\cdot,*))$. Part of the proof of Theorem 2 consists in giving a precise meaning to this random vector. Lower estimates of the $\mathscr{H}_T$-norm of $Z(t,x)$ are achieved using first a suitable approximation of S_3—in order to get rid of the non-linearity $\sigma(u(s,y))$; then we reintroduce S_3, we give a lower bound of its deterministic $\mathscr{H}_T$-norm and we keep control of the error. We prove an upper bound of this error and of the secondary term of the Malliavin matrix. In both, upper and lower bounds related to $\mathscr{H}_T$-norms we use the spectral form derived from (4) and auxiliary results proved in the appendix.

Along the paper we use the notation C for any positive real constant, independently of its value. We refer the reader to [14] for the notions and notations on Malliavin calculus and to [22] to those on distributions invoked along this article.

2. Stochastic integrals with respect to Hilbert-valued martingale measures

In this section, we extend Dalang's results on stochastic integration to a Hilbert-valued setting.

Let $\mathscr{A}$ be a separable real Hilbert space with inner-product and norm denoted by $\langle \cdot, \cdot \rangle_{\mathscr{A}}$ and $\|\cdot\|_{\mathscr{A}}$, respectively. Let $K = \{K(s,z), (s,z) \in [0,T] \times \mathbb{R}^d\}$ be an $\mathscr{A}$-valued predictable process; we assume the following condition:

Hypothesis B. The process K satisfies $\sup_{(s,z)\in[0,T]\times\mathbb{R}^d} E(\|K(s,z)\|^2_{\mathscr{A}}) < \infty$.

Our first purpose is to define a martingale measure with values in $\mathscr{A}$ obtained by integration of K. Let $\{e_j, j \geqslant 0\}$ be a complete orthonormal system of $\mathscr{A}$. Set $K^j(s,z) = \langle K(s,z), e_j \rangle_{\mathscr{A}}$, $(s,z) \in [0,T] \times \mathbb{R}^d$. According to [23], for any $j \geqslant 0$ the process

$$M_t^{K^j}(A) = \int_0^t \int_A K^j(s,z) M(ds,dz), \quad t \in [0,T],\ A \in \mathscr{B}_b(\mathbb{R}^d),$$

defines a martingale measure. Indeed, the process K^j is predictable and

$$\sup_{(s,z)\in[0,T]\times\mathbb{R}^d} E(|K^j(s,z)|^2) \leqslant \sup_{(s,z)\in[0,T]\times\mathbb{R}^d} E(||K(s,z)||^2_{\mathscr{A}}) < \infty.$$

Set, for any $t\in[0,T], A\in\mathscr{B}_b(\mathbb{R}^d)$,

$$M_t^K(A) = \sum_{j\geqslant 0} M_t^{K^j}(A)e_j. \tag{5}$$

The right-hand side of (5) defines an element of $L^2(\Omega;\mathscr{A})$. Indeed, using the isometry property of the stochastic integral, Parseval's identity and Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
\sum_{j\geqslant 0} E(|M_t^{K^j}(A)|^2) &= \sum_{j\geqslant 0} E\left(\left|\int_0^t \int_{\mathbb{R}^d} \mathbf{1}_A(z)K^j(s,z)M(ds,dz)\right|^2\right) \\
&= \sum_{j\geqslant 0} E\left(\int_0^t ds \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy\, \mathbf{1}_A(y)K^j(s,y)\mathbf{1}_A(y-x)K^j(s,y-x)\right) \\
&= \int_0^t ds \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy\, \mathbf{1}_A(y)\, 1_A(y-x)E(\langle K(s,y), K(s,y-x)\rangle_{\mathscr{A}}) \\
&\leqslant \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} E(||K(t,x)||^2_{\mathscr{A}}) \int_0^t ds \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy \mathbf{1}_A(y)\mathbf{1}_A(y-x) \\
&\leqslant C \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} E(||K(t,x)||^2_{\mathscr{A}}).
\end{aligned}$$

This shows that $E(||M_t^K(A)||^2_{\mathscr{A}}) = \sum_{j\geqslant 0} E(|M_t^{K^j}(A)|^2) < \infty$, due to Hypothesis B.

Clearly, the process $\{M_t^K(A), t\in[0,T], A\in\mathscr{B}_b(\mathbb{R}^d)\}$ defines a worthy $\mathscr{A}$-valued martingale measure and by construction we have that $\langle M_t^K(A), e_j\rangle_{\mathscr{A}} = M_t^{K^j}(A)$. By the previous computations

$$E(||M_t^K(A)||^2_{\mathscr{A}}) = \sum_{j\geqslant 0} E\left(\int_0^t ds\, ||\mathbf{1}_A(\cdot)K^j(s,\cdot)||^2_{\mathscr{H}}\right),$$

where we have denoted by a dot the $\mathscr{H}$-variable.

Our next goal is to introduce stochastic integration with respect to M^K, allowing the integrands to take values on some subset of the space of Schwartz distributions. First, we briefly recall Walsh's construction in the Hilbert-valued context.

A stochastic process $\{g(s,z;\omega), (s,z)\in[0,T]\times\mathbb{R}^d\}$ is called *elementary* if

$$g(s,z;\omega) = \mathbf{1}_{(a,b]}(s)\mathbf{1}_A(z)X(\omega), \tag{6}$$

for some $0\leqslant a<b\leqslant T$, $A\in\mathscr{B}_b(\mathbb{R}^d)$ and X a bounded $\mathscr{F}_a$-measurable random variable. For such g, the stochastic integral $g\cdot M^K$ is the $\mathscr{A}$-valued martingale

measure defined by

$$(g \cdot M^K)_t(B)(\omega) = (M^K_{t\wedge b}(A\cap B) - M^K_{t\wedge a}(A\cap B))X(\omega),$$

$t\in[0,T]$, $B\in\mathscr{B}_b(\mathbb{R}^d)$. This definition is extended by linearity to the set $\mathscr{E}_s$ of all linear combinations of elementary processes. For $g\in\mathscr{E}_s$ and $t\geqslant 0$ one easily checks that

$$\begin{aligned} &E(\|(g\cdot M^K)_t(B)\|^2_{\mathscr{A}}) \\ &\quad = \sum_{j\geqslant 0} E\bigg(\int_0^t ds \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy\, 1_B(y) g(s,y) K^j(s,y) 1_B(y-x) \\ &\qquad \times g(s,y-x)K^j(s,y-x)\bigg) \\ &\quad \leqslant \|g\|^2_{+,K}, \end{aligned} \tag{7}$$

where

$$\|g\|^2_{+,K} := \sum_{j\geqslant 0} E\bigg(\int_0^T ds\, \|\,|g(s,\cdot)K^j(s,\cdot)|\,\|^2_{\mathscr{H}}\bigg).$$

Let $\mathscr{P}_{+,K}$ be the set of all predictable processes g such that $\|g\|_{+,K}<\infty$. Then, owing to [23, Exercise 2.5, Proposition 2.3], $\mathscr{P}_{+,K}$ is complete and $\mathscr{E}_s$ is dense in this Banach space. Thus, we use the bound (7) to define the stochastic integral $g\cdot M^K$ for $g\in\mathscr{P}_{+,K}$.

Next, following [5] we aim to extend the above stochastic integral to include a larger class of integrands. Consider the inner-product defined by the formula

$$\langle g_1, g_2\rangle_{0,K} = \sum_{j\geqslant 0} E\bigg(\int_0^T ds\, \langle g_1(s,\cdot)K^j(s,\cdot), g_2(s,\cdot)K^j(s,\cdot)\rangle_{\mathscr{H}}\bigg)$$

and the norm $\|\cdot\|_{0,K}$ derived from it. We notice that this inner-product makes sense for elements in $\mathscr{E}_s$ and we have that $\|\cdot\|^2_{0,K} = \sum_{j\geqslant 0}\|\cdot\|^2_{0,K^j}$, where in the particular case of an absolutely continuous measure Γ, the definition of the norm $\|\cdot\|^2_{0,K^j}$ is given in [5, Equation (22)].

By the first equality in (7) we have that

$$E(\|(g\cdot M^K)_T(\mathbb{R}^d)\|^2_{\mathscr{A}}) = \|g\|^2_{0,K}$$

for any $g\in\mathscr{E}_s$.

Let $\mathscr{P}_{0,K}$ be the completion of the inner-product space $(\mathscr{E}_s, \langle\cdot,\cdot\rangle_{0,K})$. Since $\|\cdot\|_{0,K}\leqslant\|\cdot\|_{+,K}$, the space $\mathscr{P}_{0,K}$ will be in general larger than $\mathscr{P}_{+,K}$. So, we can extend the stochastic integral with respect to M^K to elements of $\mathscr{P}_{0,K}$. Let $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$ be the space of $\mathscr{A}$-valued continuous square integrable martingales endowed with the norm $||X||^2_{\mathcal{M}} = E(||X_T||^2_{\mathcal{A}})$. Then the map $g \mapsto g \cdot M^K$, where $g \cdot M^K$ denotes the martingale $t \mapsto (g \cdot M^K)_t(\mathbb{R}^d)$, is an isometry between the spaces $(\mathcal{P}_{0,K}, ||\cdot||_{0,K})$ and $(\mathcal{M}, ||\cdot||_{\mathcal{M}})$. Here we still have denoted by $||\cdot||_{0,K}$ the norm derived from the inner-product of the completion of $(\mathcal{E}_s, \langle\cdot,\cdot\rangle_{0,K})$. Classical results on Hilbert spaces tell us precisely how this norm is constructed (see for instance [2, Chapter V, Section 2]).

In the sequel, we denote either by $(g \cdot M^K)_t$ or by $\int_0^t \int_{\mathbb{R}^d} g(s,z)K(s,z)M(ds,dz)$ the martingale obtained by stochastic integration of $g \in \mathcal{P}_{0,K}$ with respect to M^K.

Remark 1. The stochastic integral $(g \cdot M^K)_t$, $t \in [0,T]$, introduced before coincides with that given by the formula

$$(g \cdot M^K)_t = \sum_{j \geqslant 0} (g \cdot M^{K^j})_t e_j,$$

where $(g \cdot M^{K^j})_t$ is the stochastic integral of predictable real-valued processes with respect to the martingale measure M^{K^j} defined in [5]. This fact can be easily checked using that both definitions agree on processes belonging to $\mathcal{E}_s$.

Let us consider the particular case where the following stationary assumption is fulfilled.

Hypothesis C. For all $j \geqslant 0$, $s \in [0,T]$, $x, y \in \mathbb{R}^d$,

$$E(K^j(s,x)K^j(s,y)) = E(K^j(s,0)K^j(s,y-x)).$$

Consider the non-negative definite function $G_j^K(s,z) := E(K^j(s,0)K^j(s,z))$. Owing to [22, Theorem XIX, Chapter VII], the measure $\Gamma_{j,s}^K(dz) = G_j^K(s,z)\Gamma(dz)$, is a non-negative definite distribution. Thus, by Bochner's theorem (see for instance [22, Theorem XVIII, Chapter VII]) there exists a non-negative tempered measure $\mu_{j,s}^K$ such that $\Gamma_{j,s}^K(dz) = \mathcal{F}\mu_{j,s}^K$.

Clearly, the measure $\Gamma_s^K(dz) := \sum_{j \geqslant 0} \Gamma_{j,s}^K(dz)$ is a well-defined non-negative definite measure on $\mathbb{R}^d$, because

$$\sum_{j \geqslant 0} G_j^K(s,z) \leqslant \sup_{(s,z) \in [0,T] \in \mathbb{R}^d} E(||K(s,z)||^2_{\mathcal{A}}) < \infty.$$

Consequently, there exists a non-negative tempered measure μ_s^K such that $\mathcal{F}\mu_s^K = \Gamma_s^K$. Furthermore, by the uniqueness and linearity of the Fourier transform, $\mu_s^K = \sum_{j \geqslant 0} \mu_{j,s}^K$.

Thus, if Hypotheses B and C are satisfied then for any *deterministic* function $g(s,z)$ such that $||g||^2_{0,K} < \infty$ and $g(s) \in \mathscr{S}(\mathbb{R}^d)$ we have that

$$\begin{aligned} ||g||^2_{0,K} &= \sum_{j\geqslant 0} \int_0^T ds \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy\, g(s,y) g(s,y-x) G_j^K(s,x) \\ &= \int_0^T ds \int_{\mathbb{R}^d} \Gamma_s^K(dx)(g(s,\cdot) * \tilde{g}(s,\cdot))(x) \\ &= \int_0^T ds \int_{\mathbb{R}^d} \mu_s^K(d\xi) |\mathscr{F} g(s)(\xi)|^2. \end{aligned} \tag{8}$$

We want now to give examples of deterministic distribution-valued functions $t \to S(t)$ belonging to $\mathscr{P}_{0,K}$. A result in this direction is given in the next theorem, which is the Hilbert-valued counterpart of [5, Theorems 2, 5].

Theorem 1. *Let* $\{K(s,z), (s,z) \in [0,T] \times \mathbb{R}^d\}$ *be an* $\mathscr{A}$*-valued process for which Hypotheses B and C are satisfied. Let* $t \mapsto S(t)$ *be a deterministic function with values in the space of non-negative distributions with rapid decrease, such that*

$$\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi) |\mathscr{F} S(t)(\xi)|^2 < \infty.$$

Then S *belongs to* $\mathscr{P}_{0,K}$ *and*

$$E(||(S \cdot M^K)_t||^2_{\mathscr{A}}) = \int_0^t ds \int_{\mathbb{R}^d} \mu_s^K(d\xi) |\mathscr{F} S(s)(\xi)|^2. \tag{9}$$

Moreover, for any $p \in [2, \infty)$,

$$E(||(S \cdot M^K)_t||^p_{\mathscr{A}}) \leqslant C_t \int_0^t ds \sup_{x \in \mathbb{R}^d} E(||K(s,x)||^p_{\mathscr{A}}) \int_{\mathbb{R}^d} \mu(d\xi) |\mathscr{F} S(s)(\xi)|^2, \tag{10}$$

with $C_t = (\int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathscr{F} S(s)(\xi)|^2)^{\frac{p}{2}-1}$, $t \in [0,T]$.

Proof. Let ψ be a non-negative function in $\mathscr{C}^\infty(\mathbb{R}^d)$ with support contained in the unit ball of $\mathbb{R}^d$ and such that $\int_{\mathbb{R}^d} \psi(x)\, dx = 1$. Set $\psi_n(x) = n^d \psi(nx)$, $n \geqslant 1$. Define $S_n(t) = \psi_n * S(t)$. Clearly, $S_n(t) \in \mathscr{S}(\mathbb{R}^d)$ for any $n \geqslant 1$, $t \in [0,T]$ and $S_n(t) \geqslant 0$. According to the arguments in the proof of [5, Theorem 2] we obtain that $S_n \in \mathscr{P}_{+,K} \subset \mathscr{P}_{0,K}$ and

$$\lim_{n\to\infty} ||S_n - S||^2_{0,K^j} = \lim_{n\to\infty} \int_0^T ds \int_{\mathbb{R}^d} \mu_{j,s}^K(d\xi) |\mathscr{F}(S_n - S)(s)(\xi)|^2 = 0,$$

for any $j \geqslant 0$. Consequently, $\lim_{n\to\infty} ||S_n - S||_{0,K} = 0$, and thus $S \in \mathscr{P}_{0,K}$. By the isometry property of the stochastic integral and (8) we see that the equality (9) holds for any S_n; then the construction of the stochastic integral yields

$$\begin{aligned}E(\|(S\cdot M^K)_t\|^2_{\mathscr{A}}) &= \lim_{n\to\infty} E(\|(S_n\cdot M^K)_t\|^2_{\mathscr{A}})\\ &= \lim_{n\to\infty}\int_0^t ds\int_{\mathbb{R}^d}\mu_s^K(d\xi)|\mathscr{F}S_n(s)(\xi)|^2\\ &= \int_0^t ds\int_{\mathbb{R}^d}\mu_s^K(d\xi)|\mathscr{F}S(s)(\xi)|^2,\end{aligned}$$

where the last equality follows from bounded convergence. This proves (9).

For the proof of (10) we refer the reader to that of expression (36) in [5, Theorem 5], with the obvious modifications. Therefore, the theorem is proved. □

Remark 2. From the identity (9) it follows that for any S satisfying the assumptions of Theorem 1 we have

$$\|S\|^2_{0,K} = \int_0^T ds\int_{\mathbb{R}^d}\mu_s^K(d\xi)|\mathscr{F}S(s)(\xi)|^2.$$

3. Malliavin differentiability of the solution of spatially homogeneous spde's

In this section, we consider the spde (2). We assume that the following set of assumptions is satisfied:

Hypothesis D. Let Λ be the fundamental solution of $Lu=0$. Then $\Lambda(t)$ is a non-negative distribution with rapid decrease such that

$$\int_0^T dt\int_{\mathbb{R}^d}\mu(d\xi)|\mathscr{F}\Lambda(t)(\xi)|^2<\infty \tag{11}$$

and

$$\lim_{h\downarrow 0}\int_0^T dt\int_{\mathbb{R}^d}\mu(d\xi)\sup_{t<r<t+h}|\mathscr{F}(\Lambda(r)-\Lambda(t))(\xi)|^2=0.$$

Moreover, Λ is a non-negative measure on $\mathbb{R}_+\times\mathbb{R}^d$ of the form $\Lambda(t,dy)\,dt$ such that $\sup_{0\leqslant t\leqslant T}\Lambda(t,\mathbb{R}^d)\leqslant C_T<\infty$.

Under these hypotheses [5, Theorem 13] establishes the existence of a unique progressively measurable process $u=\{u(t,x),(t,x)\in[0,T]\times\mathbb{R}^d\}$ such that Eq. (3) holds; in addition u satisfies $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}E(|u(t,x)|^p)<+\infty$, for any $p\in[1,\infty)$ and has spatial stationary covariance function. The process u is called a *solution* of (2).

The aim of this section is to prove that for any fixed $(t,x)\in[0,T]\times\mathbb{R}^d$, $u(t,x)$ belongs to the space $\mathbb{D}^{1,p}$ of the Malliavin calculus developed in the framework that has been made precise in the introduction and defined by means of the Malliavin derivative operator D (see [15, Section 1.1]). We recall that for a random variable X in the domain of D, DX defines an $\mathscr{H}_T$-valued random variable. For $h \in \mathscr{H}_T$ set $D_h X = \langle DX, h \rangle_{\mathscr{H}_T}$. Since $\mathscr{H}_T = L^2([0,T]; \mathscr{H})$, for $r \in [0,T]$, $DX(r)$ defines an element in $\mathscr{H}$, which will be denoted by $D_r X$. Then, for any $h \in \mathscr{H}_T$, $D_h X = \int_0^T \langle D_r X, h(r) \rangle_{\mathscr{H}} dr$. We will write $D_{r,\varphi} X = \langle D_r X, \varphi \rangle_{\mathscr{H}}$, $r \in [0,T]$, $\varphi \in \mathscr{H}$ and we shall use the convention $\mathbb{D}^{0,p} = L^p(\Omega)$. Along the section, we will denote by $\cdot$ and $*$ the time and the $\mathscr{H}$ variable, respectively. Here is the main result of this section.

Theorem 2. *Assume that Λ satisfies Hypothesis* D *and the coefficients σ and b are $\mathscr{C}^1$ functions with bounded Lipschitz continuous derivatives. Then,*

(1) *for any $(t,x) \in [0,T] \times \mathbb{R}^d$, $u(t,x)$ belongs to $\mathbb{D}^{1,p}$ for any $p \in [1,\infty)$;*
(2) *there exists an $\mathscr{H}_T$-valued stochastic process $\{Z(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ satisfying $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} ||Z(t,x)||_{L^p(\Omega,\mathscr{H}_T)} < \infty$ such that*

$$\begin{aligned} Du(t,x) = Z(t,x) &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z)\sigma'(u(s,z))Du(s,z)M(ds,dz) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} b'(u(t-s,x-z))Du(t-s,x-z)\Lambda(s,dz). \end{aligned} \tag{12}$$

Moreover, for any $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$\begin{aligned} E(||Z(t,x)||^2_{\mathscr{H}_T}) &= ||\Lambda(t-\cdot, x-*)||^2_{0,\sigma(u)} \\ &= E\left(\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z)\sigma(u(s,z))M(ds,dz)\right)^2. \end{aligned} \tag{13}$$

Since $\{\sigma(u(s,x)), (s,x) \in [0,T] \times \mathbb{R}^d\}$ has a stationary covariance function we have

$$||\Lambda(t-\cdot, x-*)||^2_{0,\sigma(u)} = \int_0^T ds \int_{\mathbb{R}^d} \mu_s^{\sigma(u)}(d\xi)|\mathscr{F}\Lambda(t-s)(\xi)|^2.$$

Let $\mathscr{B}_p^{\mathscr{H}_T}$, $p \in [2,\infty)$, be the class of progressively measurable $\mathscr{H}_T$-valued processes $\{\phi(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ with spatially homogeneous covariance function and satisfying $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(||\phi(t,x)||^p_{\mathscr{H}_T}) < \infty$. Consider the stochastic integral equation for processes in $\mathscr{B}_p^{\mathscr{H}_T}$ for any fixed p,

$$\begin{aligned} U(t,x) = Z(t,x) &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z)\sigma'(u(s,z))U(s,z)M(ds,dz) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)b'(u(t-s,x-z))U(t-s,x-z). \end{aligned} \tag{14}$$

Owing to the results of Section 2 applied to the Hilbert space $\mathscr{A} := \mathscr{H}_T$ and to the $\mathscr{H}_T$-valued stochastic process $K(s,z) := \sigma'(u(s,z))U(s,z)$, all the terms in Eq. (14) are well defined; in particular, the existence of the stochastic integral term is ensured by Theorem 1. Following the same arguments as those in the proof of [5, Theorem 13] based on Picard's approximations, it can be proved that Eq. (14) possesses a unique solution.

The proof of Theorem 2 relies on the following lemma quoted from [8].

Lemma 1. *Let $\{F_n\}_{n\geqslant 1}$ be a sequence of random variables belonging to $\mathbb{D}^{1,p}$, for some $p\in[2,\infty)$. Assume that the following two conditions are fulfilled*:

(1) *The sequence $\{F_n\}_{n\geqslant 1}$ converges in $L^p(\Omega)$ to a random variable F*;
(2) $\sup_{n\geqslant 1} E(\|DF_n\|^p_{\mathscr{H}_T}) < +\infty$.

Then F belongs to $\mathbb{D}^{1,p}$ and there is a subsequence of $\{DF_n\}_{n\geqslant 1}$ converging to DF in the weak topology of $L^p(\Omega;\mathscr{H}_T)$.

Let $\Lambda_n(t) = \psi_n * \Lambda(t)$, where $\{\psi_n\}_{n\geqslant 1}$ is an approximation of the identity as has been defined in the proof of Theorem 1. Consider the process $\{u_n(t,x), (t,x)\in[0,T]\times\mathbb{R}^d\}$ solution to the integral stochastic equation

$$\begin{aligned} u_n(t,x) = &\int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s,x-z)\sigma(u_n(s,z))M(ds,dz) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} b(u_n(t-s,x-z))\Lambda(s,dz). \end{aligned} \tag{15}$$

The existence and uniqueness of such a process can be easily deduced from the arguments used in the proof of [5, Theorem 13].

We shall apply the previous Lemma 1 to the sequence $F_n := u_n(t,x),\ n\geqslant 1$. First, we notice that $u_n(t,x)$ belongs to $\mathbb{D}^{1,p}$, for all $n\geqslant 1$. Indeed, this follows from an easy adaptation of the proof of [11, Proposition 2.4], taking into account that $|\mathscr{F}\Lambda_n(t)(\xi)| \leqslant |\mathscr{F}\Lambda(t)(\xi)|$, for each $t\geqslant 0, \xi\in\mathbb{R}^d$. Moreover, the derivative of u_n satisfies the equation in $\mathscr{H}_T$:

$$\begin{aligned} Du_n(t,x) = &\,\Lambda_n(t-\cdot,x-*)\sigma(u_n(\cdot,*)) \\ &+ \int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s,x-z)\sigma'(u_n(s,z))Du_n(s,z)M(ds,dz) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)b'(u_n(t-s,x-z))Du_n(t-s,x-z). \end{aligned} \tag{16}$$

The following propositions provide the necessary arguments for the proof of Theorem 2.

Proposition 1. *Assume that the coefficients σ and b are Lipschitz continuous and that Hypothesis* D *is satisfied. Then, for any $p \in [1, \infty)$*

$$\lim_{n\to\infty}\left(\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|u_n(t,x)-u(t,x)|^p\right) = 0. \tag{17}$$

Proof. We first prove that for any $p \in [1, \infty)$,

$$\sup_{n\geqslant 1}\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|u_n(t,x)|^p) < \infty. \tag{18}$$

Taking into account (15), we have that $E(|u_n(t,x)|^p) \leqslant C(A_{1,n}(t,x) + A_{2,n}(t,x))$, where

$$A_{1,n}(t,x) = E\left(\left|\int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s, x-z)\sigma(u_n(s,z))M(ds,dz)\right|^p\right),$$

$$A_{2,n}(t,x) = E\left(\left|\int_0^t ds \int_{\mathbb{R}^d} b(u_n(t-s, x-z))\Lambda(s,dz)\right|^p\right).$$

Owing to [5, Theorem 5], the properties of σ and the definition of Λ_n, we obtain

$$\begin{aligned} A_{1,n}(t,x) &\leqslant Cv(t)^{\frac{p}{2}-1}\int_0^t ds \sup_{z\in\mathbb{R}^d} E(|\sigma(u_n(s,z))|^p)\int_{\mathbb{R}^d}\mu(d\xi)|\mathscr{F}\Lambda_n(t-s)(\xi)|^2 \\ &\leqslant C\int_0^t ds\left(1+\sup_{z\in\mathbb{R}^d} E(|u_n(s,z)|^p)\right)\int_{\mathbb{R}^d}\mu(d\xi)|\mathscr{F}\Lambda_n(t-s)(\xi)|^2, \end{aligned}$$

with $v(t) = \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)|\mathscr{F}\Lambda(s)(\xi)|^2$. Consequently, since $|\mathscr{F}\Lambda_n(t)(\xi)| \leqslant |\mathscr{F}\Lambda(t)(\xi)|$ we have that

$$A_{1,n}(t,x) \leqslant C\int_0^t ds\left[1 + \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E(|u_n(\tau,y)|^p)\right]J(t-s), \tag{19}$$

where

$$J(t) = \int_{\mathbb{R}^d} \mu(d\xi)|\mathscr{F}\Lambda(t)(\xi)|^2. \tag{20}$$

Hölder's inequality with respect to the finite measure $\Lambda(s,dz)\,ds$, the properties of b and Hypothesis D yield

$$\begin{aligned} A_{2,n}(t,x) &\leqslant C\int_0^t ds\int_{\mathbb{R}^d} E(|b(u_n(t-s,x-z))|^p)\Lambda(s,dz) \\ &\leqslant C\int_0^t ds\left[1+\sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E(|u_n(\tau,y)|^p)\right]\int_{\mathbb{R}^d}\Lambda(t-s,dz) \\ &\leqslant C\int_0^t ds\left[1+\sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E(|u_n(\tau,y)|^p)\right]. \end{aligned} \tag{21}$$

Putting together (19) and (21) we obtain

$$\sup_{(s,x)\in[0,t]\times\mathbb{R}^d} E(|u_n(s,x)|^p)\leqslant C\int_0^t ds\left[1+\sup_{(\tau,x)\in[0,s]\times\mathbb{R}^d} E(|u_n(\tau,x)|^p)\right](J(t-s)+1).$$

We then apply a suitable version of Gronwall's Lemma (see for instance [5, Lemma 15]) to finish the proof of (18).

Next we prove that

$$\lim_{n\to\infty}\left(\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|u_n(t,x)-u(t,x)|^2\right)=0. \tag{22}$$

Indeed, according to the integral equations (3) and (15), we have

$$E(|u_n(t,x)-u(t,x)|^2)\leqslant C(I_{1,n}(t,x)+I_{2,n}(t,x)),$$

where

$$\begin{aligned} I_{1,n}(t,x) &= E\Bigg(\bigg|\int_0^t\int_{\mathbb{R}^d}[\Lambda_n(t-s,x-z)\sigma(u_n(s,z)) \\ &\qquad -\Lambda(t-s,x-z)\sigma(u(s,z))]M(ds,dz)\bigg|^2\Bigg), \\ I_{2,n}(t,x) &= E\left(\left|\int_0^t ds\int_{\mathbb{R}^d}[b(u_n(t-s,x-z))-b(u(t-s,x-z))]\Lambda(s,dz)\right|^2\right). \end{aligned}$$

We have $I_{1,n}(t,x)\leqslant C(I_{1,n}^1(t,x)+I_{1,n}^2(t,x))$ with

$$\begin{aligned} I_{1,n}^1(t,x) &= E\left(\left|\int_0^t\int_{\mathbb{R}^d}\Lambda_n(t-s,x-z)[\sigma(u_n(s,z))-\sigma(u(s,z))]M(ds,dz)\right|^2\right), \\ I_{1,n}^2(t,x) &= E\left(\left|\int_0^t\int_{\mathbb{R}^d}[\Lambda_n(t-s,x-z)-\Lambda(t-s,x-z)]\sigma(u(s,z))M(ds,dz)\right|^2\right). \end{aligned}$$

Owing to [5, Theorem 2], the assumptions on σ and the definition of Λ_n, we obtain

$$I^1_{1,n}(t,x) \leqslant C \int_0^t ds \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E(|u_n(\tau,y)-u(\tau,y)|^2)J(t-s),$$

where J is defined in (20).

Although $\Lambda_n(t-s)-\Lambda(t-s)$ may not be a non-negative distribution, it does belong to the space $\mathscr{P}_{0,\sigma(u)}$ of deterministic processes integrable with respect to the martingale measure $M^{\sigma(u)}$. Hence, by the isometry property of the stochastic integral specified in [5, Section 2, p. 9],

$$I^2_{1,n}(t,x) = ||\Lambda_n(t-\cdot, x-*) - \Lambda(t-\cdot, x-*)||^2_{0,\sigma(u)}.$$

Then, the definition of the norm in the right-hand side of the above equality yields

$$\begin{aligned} I^2_{1,n}(t,x) &= \int_0^t ds \int_{\mathbb{R}^d} \mu_s^{\sigma(u)}(d\xi)|\mathscr{F}(\Lambda_n(t-s)-\Lambda(t-s))(\xi)|^2 \\ &= \int_0^t ds \int_{\mathbb{R}^d} \mu_s^{\sigma(u)}(d\xi)|\mathscr{F}\psi_n(\xi)-1|^2|\mathscr{F}\Lambda(t-s)(\xi)|^2. \end{aligned}$$

Hence, by bounded convergence we conclude that $C_n := \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} I^2_{1,n}(t,x)$ tends to zero as n goes to infinity.

Now we study the term $I_{2,n}(t,x)$. Applying the same techniques as for the term $A_{2,n}(t,x)$ before we obtain

$$I_{2,n}(t,x) \leqslant C \int_0^t ds \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E(|u_n(\tau,y)-u(\tau,y)|^2).$$

Consequently,

$$\begin{aligned} &\sup_{(s,x)\in[0,t]\times\mathbb{R}^d} E(|u_n(s,x)-u(s,x)|^2) \\ &\quad \leqslant C_n + C\int_0^t ds \sup_{(\tau,x)\in[0,s]\times\mathbb{R}^d} E(|u_n(\tau,x)-u(\tau,x)|^2)(J(t-s)+1), \end{aligned}$$

where $\lim_{n\to\infty} C_n = 0$. The proof of (22) concludes with an application of the above-mentioned version of Gronwall's Lemma. The convergence (17) is now a consequence of (18) and (22). $\square$

Proposition 2. *Assume that Hypothesis* D *holds and that the coefficients σ and b are $\mathscr{C}^1$ functions with bounded derivatives. Then, for all $p\in[1,\infty)$*

$$\sup_{n\geqslant 1} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(||Du_n(t,x)||^p_{\mathscr{H}_T}) < \infty.$$

Proof. Fix $p \in [2, \infty)$; set $Z_n(t,x) = \Lambda_n(t-\cdot, x-*)\sigma(u_n(\cdot,*))$. We first prove that

$$\sup_{n \geqslant 1} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(||Z_n(t,x)||^p_{\mathscr{H}_T}) < \infty. \tag{23}$$

Indeed, Hölder's inequality with respect to the finite measure $\Lambda_n(t-s, x-z)\Lambda_n(t-s, x-y+z)\, ds\, \Gamma(dz)\, dy$, Cauchy–Schwarz inequality and the properties of σ imply

$$\begin{aligned}
&E(||\Lambda_n(t-\cdot, x-*)\sigma(u_n(\cdot,*))||^p_{\mathscr{H}_T}) \\
&\quad = E\bigg(\int_0^t ds \int_{\mathbb{R}^d} \Gamma(dz) \int_{\mathbb{R}^d} dy \Lambda_n(t-s, x-y)\Lambda_n(t-s, x-y+z) \\
&\qquad \times \sigma(u_n(s,y))\sigma(u_n(s,y-z))\bigg)^{p/2} \\
&\quad \leqslant CE \int_0^t ds \int_{\mathbb{R}^d} \Gamma(dz) \int_{\mathbb{R}^d} dy\, \Lambda_n(t-s, x-y)\Lambda_n(t-s, x-y+z) \\
&\qquad \times E(|\sigma(u_n(s,y))\sigma(u_n(s,y-z))|^{p/2}) \\
&\quad \leqslant C\bigg(1 + \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|u_n(t,x)|^p)\bigg) \int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi)|\mathscr{F}\Lambda_n(s)(\xi)|^2,
\end{aligned}$$

which is uniformly bounded with respect to n, by Proposition 1 and the definition of Λ_n.

Consider now the second term of the right-hand side of (16), which we denote by $B_{1,n}(t,x)$. By Theorem 1 and the properties of σ it holds that

$$\begin{aligned}
E(||B_{1,n}(t,x)||^p_{\mathscr{H}_T}) &\leqslant C \int_0^t ds \sup_{z\in\mathbb{R}^d} E(||\sigma'(u_n(s,z))Du_n(s,z)||^p_{\mathscr{H}_T}) J(t-s) \\
&\leqslant C \int_0^t ds \sup_{(\tau,z)\in[0,s]\times\mathbb{R}^d} E(||Du_n(\tau,z)||^p_{\mathscr{H}_T}) J(t-s),
\end{aligned}$$

where J is defined as in (20).

Finally, for the third term on the right-hand side of (16), denoted in the sequel by $B_{2,n}(t,x)$, we use Hölder's inequality with respect to the finite measure $\Lambda(s, dz)\, ds$. Then, the assumptions on b and Λ yield

$$E(||B_{2,n}(t,x)||^p_{\mathscr{H}_T}) \leqslant C \int_0^t ds \sup_{(\tau,z)\in[0,s]\times\mathbb{R}^d} E(||Du_n(\tau,z)||^p_{\mathscr{H}_T}).$$

Therefore,

$$\sup_{(s,z)\in[0,t]\times\mathbb{R}^d} E(||Du_n(s,z)||^p_{\mathscr{H}_T}) \leqslant C\left(1+\int_0^t ds \sup_{(\tau,z)\in[0,s]\times\mathbb{R}^d} E(||Du_n(\tau,z)||^p_{\mathscr{H}_T})(J(t-s)+1)\right).$$

Then, by Gronwall's Lemma we finish the proof. □

Proposition 3. *We assume that σ is Lipschitz and that Hypothesis* D *holds. Then, for any $(t,x)\in[0,T]\times\mathbb{R}^d$ the sequence $\{Z_n(t,x)=\Lambda_n(t-\cdot,x-*)\sigma(u_n(\cdot,*)),n\geqslant 1\}$ converges in the topology of $L^p(\Omega;\mathscr{H}_T)$, for any $p\in[1,\infty)$, to a random variable denoted by $Z(t,x)$. Moreover, the process $\{Z(t,x),(t,x)\in[0,T]\times\mathbb{R}^d\}$ satisfies*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} ||Z(t,x)||_{L^p(\Omega;\mathscr{H}_T)}<\infty \tag{24}$$

and

$$E(||Z(t,x)||^2_{\mathscr{H}_T}) = ||\Lambda(t-\cdot,x-*)||^2_{0,\sigma(u)} = E\left(\int_0^t\int_{\mathbb{R}^d}\Lambda(t-s,x-z)\sigma(u(s,z))M(ds,dz)\right)^2. \tag{25}$$

Proof. We first prove that $\{Z_n(t,x),n\geqslant 1\}$ is a Cauchy sequence in $L^2(\Omega;\mathscr{H}_T)$. Indeed, for any $n,m\geqslant 1$ we consider the following decomposition:

$$E(||\Lambda_n(t-\cdot,x-*)\sigma(u_n(\cdot,*))-\Lambda_m(t-\cdot,x-*)\sigma(u_m(\cdot,*))||^2_{\mathscr{H}_T}) \leqslant C(T_{1,n}(t,x)+T_{2,n,m}(t,x)+T_{3,m}(t,x)),$$

where

$$T_{1,n}(t,x)=E(||\Lambda_n(t-\cdot,x-*)[\sigma(u_n(\cdot,*))-\sigma(u(\cdot,*))]||^2_{\mathscr{H}_T}),$$

$$T_{2,n.m}(t,x)=E(||[\Lambda_n(t-\cdot,x-*)-\Lambda_m(t-\cdot,x-*)]\sigma(u(\cdot,*))||^2_{\mathscr{H}_T}),$$

$$T_{3,m}(t,x)=E(||\Lambda_m(t-\cdot,x-*)[\sigma(u(\cdot,*))-\sigma(u_m(\cdot,*))]||^2_{\mathscr{H}_T}).$$

Since Λ_n is a positive test function, the Lipschitz property of σ and the definition of Λ_n yield

$$T_{1,n}(t,x)\leqslant C\sup_{(t,y)\in[0,T]\times\mathbb{R}^d}E(|u_n(t,y)-u(t,y)|^2)\int_0^t ds\int_{\mathbb{R}^d}\mu(d\xi)|\mathscr{F}\Lambda_n(s)(\xi)|^2 \leqslant C\sup_{(t,y)\in[0,T]\times\mathbb{R}^d}E(|u_n(t,y)-u(t,y)|^2).$$

Then, by Proposition 1 we conclude that $\lim_{n\to\infty}\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} T_{1,n}(t,x)=0$. Similarly, $\lim_{m\to\infty}\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} T_{3,m}(t,x)=0$. Owing to the isometry property of the stochastic integral we have

$$\begin{aligned} T_{2,n,m}(t,x) &= E(\|\Lambda_n(t-\cdot,x-*)-\Lambda_m(t-\cdot,x-*)\|^2_{0,\sigma(u)}) \\ &= \int_0^T ds \int_{\mathbb{R}^d} \mu_s^{\sigma(u)}(d\xi)|\mathscr{F}(\psi_n-\psi_m)(\xi)|^2|\mathscr{F}\Lambda(t-s)(\xi)|^2. \end{aligned}$$

Then, by bounded convergence we conclude that

$$\lim_{n,m\to\infty}\ \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} T_{2,n,m}(t,x)=0.$$

Therefore,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(\|\Lambda_n(t-\cdot,x-*)\sigma(u_n(\cdot,*))-\Lambda_m(t-\cdot,x-*)\sigma(u_m(\cdot,*))\|^2_{\mathscr{H}_T}) \xrightarrow[n,m\to\infty]{} 0$$

and consequently the sequence $\{Z_n(t,x), n\geqslant 1\}$ converges in $L^2(\Omega;\mathscr{H}_T)$ to a random variable denoted by $Z(t,x)$. Actually, the convergence holds in $L^p(\Omega;\mathscr{H}_T)$ for any $p\in[1,\infty)$—due to (23)—and the process $\{Z(t,x),(t,x)\in[0,T]\times\mathbb{R}^d\}$ satisfies (24).

We now prove (25). Since $\lim_{n\to\infty}(\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} T_{1,n}(t,x))=0$, we have

$$Z(t,x)=L^2(\Omega;\mathscr{H}_T)-\lim_{n\to\infty}(\Lambda_n(t-\cdot,x-*)\sigma(u(\cdot,*))).$$

Thus, by bounded convergence and the isometry property of the stochastic integral,

$$\begin{aligned} E(\|Z(t,x)\|^2_{\mathscr{H}_T}) &= \lim_{n\to\infty}\int_0^t ds\int_{\mathbb{R}^d}\mu_s^{\sigma(u)}|\mathscr{F}\Lambda_n(t-s)(\xi)|^2 \\ &= \int_0^t ds\int_{\mathbb{R}^d}\mu_s^{\sigma(u)}|\mathscr{F}\Lambda(t-s)(\xi)|^2 = \|\Lambda(t-\cdot,x-*)\|^2_{0,\sigma(u)} \\ &= E\left(\int_0^t\int_{\mathbb{R}^d}\Lambda(t-s,x-z)\sigma(u(s,z))M(ds,dz)\right). \end{aligned}$$

Hence, the proof of the Proposition is complete. □

We can now proceed to the proof of Theorem 2.

Proof of Theorem 2. Owing to Propositions 1 and 2, the assumptions of Lemma 1 are satisfied by the sequence $F_n := u_n(t,x)$, $n\geqslant 1$, for any $p\in[2,\infty)$ and any $(t,x)\in[0,T]\times\mathbb{R}^d$. Therefore, the assertion (1) of the theorem holds and there exists a subsequence of $\{Du_n(t,x), n\geqslant 1\}$ converging in the weak topology of $L^p(\Omega;\mathscr{H}_T)$ to $Du(t,x)$. The next step consists in identifying this limit as the solution to the evolution Eq. (12).

We prove that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(||Du_n(t,x) - U(t,x)||^p_{\mathscr{H}_T}) \to 0, \tag{26}$$

as n tends to infinity, where $\{U(t,x), (t,x)\in[0,T]\times\mathbb{R}^d\}$ is the solution of (14). This implies that the process $\{Du(t,x), (t,x)\in[0,T]\times\mathbb{R}^d\}$ satisfies Eq. (12). Notice that it suffices to check (26) for $p=2$ due to Proposition 2.

Set

$$I^n_Z(t,x) = Z_n(t,x) - Z(t,x),$$

$$\begin{aligned} I^n_\sigma(t,x) = {} & \int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s,x-z)\sigma'(u_n(s,z))Du_n(s,z)M(ds,dz) \\ & - \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z)\sigma'(u(s,z))U(s,z)M(ds,dz), \end{aligned}$$

$$\begin{aligned} I^n_b(t,x) = {} & \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)(b'(u_n(t-s,x-z))Du_n(t-s,x-z) \\ & - b'(u(t-s,x-z))U(t-s,x-z)). \end{aligned}$$

By Proposition 3, $\lim_{n\to\infty}\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E\left(||I^n_Z(t,x)||^2_{\mathscr{H}_T}\right) = 0$. Consider the decomposition

$$E(||I^n_\sigma(t,x)||^2_{\mathscr{H}_T}) \leqslant C(D_{1,n}(t,x) + D_{2,n}(t,x) + D_{3,n}(t,x)),$$

where

$$\begin{aligned} D_{1,n}(t,x) = E\Bigg(\Bigg|\Bigg| \int_0^t \int_{\mathbb{R}^d} & \Lambda_n(t-s,x-z)[\sigma'(u_n(s,z)) \\ & - \sigma'(u(s,z))]Du_n(s,z)M(ds,dz)\Bigg|\Bigg|^2_{\mathscr{H}_T}\Bigg), \end{aligned}$$

$$\begin{aligned} D_{2,n}(t,x) = E\Bigg(\Bigg|\Bigg| \int_0^t \int_{\mathbb{R}^d} & \Lambda_n(t-s,x-z)\sigma'(u(s,z))[Du_n(s,z) \\ & - U(s,z)]M(ds,dz)\Bigg|\Bigg|^2_{\mathscr{H}_T}\Bigg), \end{aligned}$$

$$D_{3,n}(t,x) = E\left(\left\|\int_0^t \int_{\mathbb{R}^d} [\Lambda_n(t-s,x-z) - \Lambda(t-s,x-z)]\sigma'(u(s,z))U(s,z)M(ds,dz)\right\|_{\mathscr{H}_T}^2\right).$$

The inequality (10), Cauchy–Schwarz's inequality and the properties of σ and Λ_n yield

$$D_{1,n}(t,x) \leqslant C \sup_{(t,y)\in[0,T]\times\mathbb{R}^d} (E(|u_n(t,y)-u(t,y)|^4)E(\|Du_n(t,y)\|_{\mathscr{H}_T}^4))^{\frac{1}{2}} \times \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)|\mathscr{F}\Lambda(s)(\xi)|^2.$$

Owing to Propositions 1 and 2 we conclude that $\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} D_{1,n}(t,x) = 0$. Similarly,

$$D_{2,n}(t,x) \leqslant C \int_0^t ds \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E(\|Du_n(\tau,y)-U(\tau,y)\|_{\mathscr{H}_T}^2)J(t-s), \tag{27}$$

where J is defined in (20).

Denote by $\overline{U}$ the $\mathscr{H}_T$-valued process $\{\sigma'(u(s,z))U(s,z), (s,z)\in[0,T]\times\mathbb{R}^d\}$. Then, the isometry property yields

$$\begin{aligned} D_{3,n}(t,x) &= \|\Lambda_n(t-\cdot,x-*) - \Lambda(t-\cdot,x-*)\|_{0,\overline{U}}^2 \\ &= \int_0^T ds \int_{\mathbb{R}^d} \mu_s^{\overline{U}}(d\xi)|\mathscr{F}\psi_n(\xi)-1|^2|\mathscr{F}\Lambda(t-s)(\xi)|^2. \end{aligned}$$

Thus, by bounded convergence $\lim_{n\to\infty}\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} D_{3,n}(t,x) = 0$.

For the deterministic integral term, we have

$$E(\|I_b^n(t,x)\|_{\mathscr{H}_T}^2) \leqslant C(b_{1,n}(t,x) + b_{2,n}(t,x)),$$

with

$$b_{1,n}(t,x) = E\left(\left\|\int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)[b'(u_n(t-s,x-z)) - b'(u(t-s,x-z))] \times Du_n(t-s,x-z)\right\|_{\mathscr{H}_T}^2\right),$$

$$b_{2,n}(t,x) = E\left(\left|\left|\int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz) b'(u(t-s,x-z)) \right.\right.\right.$$
$$\left.\left.\left. \times [Du_n(t-s,x-z) - U(t-s,x-z)]\right|\right|^2_{\mathscr{H}_T}\right).$$

By the properties of the deterministic integral of Hilbert-valued processes, the assumptions on b and Cauchy–Schwarz's inequality we obtain

$$b_{1,n}(t,x) \leqslant \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz) E(|b'(u_n(t-s,x-z)) - b'(u(t-s,x-z))|^2$$
$$\times ||Du_n(t-s,x-z)||^2_{\mathscr{H}_T})$$
$$\leqslant \sup_{(t,y)\in[0,T]\times\mathbb{R}^d} (E|u_n(t,y)-u(t,y)|^4\, E||Du_n(t,y)||^4_{\mathscr{H}_T})^{1/2} \int_0^t ds\, \Lambda(s,dz).$$

Thus, $\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} b_{1,n}(t,x) = 0$.

Similar arguments yield

$$b_{2,n}(t,x) \leqslant C \int_0^t ds \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} E(||Du_n(\tau,y) - U(\tau,y)||^2_{\mathscr{H}_T}.$$

Therefore, we have obtained that

$$\sup_{(s,x)\in[0,t]\times\mathbb{R}^d} E(||Du_n(s,x) - U(s,x)||^2_{\mathscr{H}_T})$$
$$\leqslant C_n + C\int_0^t ds \sup_{(\tau,x)\in[0,s]\times\mathbb{R}^d} E(||Du_n(\tau,x) - U(\tau,x)||^2_{\mathscr{H}_T})(J(t-s)+1),$$

with $\lim_{n\to\infty} C_n = 0$. Thus, applying Gronwall's Lemma we complete the proof of (26) and consequently that of assertion (2) of the theorem. □

Remark 3. The $\mathscr{H}_T$-valued random vector

$$Z(t,x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s,x-z)\sigma'(u(s,z))Du(s,z)M(ds,dz)$$

in Eq. (12) is the Malliavin derivative of the stochastic integral $\int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z)\sigma(u(s,z))M(ds,dz)$. Indeed, this can be proved using Lemma 1 applied to the sequence defined by $\int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s,x-z)\sigma(u(s,z))M(ds,dz),\ n\geqslant 1$.

Remark 4. Assume that $\int_{\mathbb{R}^d} \mu(d\xi)(1+|\xi|^2)^{-1} < \infty$; then the fundamental solution of the wave equation in dimension $d\in\{1,2,3\}$ satisfies Hypothesis D, as has been proved in [5].

4. The stochastic wave equation in dimension 3. Existence of density

In this section, we consider the stochastic wave equation (1). We consider the solution $u(t,x)$ at $(t,x)\in(0,T]\times\mathbb{R}^3$ in the sense given by Eq. (3). The purpose is to prove the following result.

Theorem 3. *Assume that*

(1) *the coefficients σ and b are $\mathscr{C}^1$ functions with bounded Lipschitz continuous derivatives*;
(2) *there exists $\sigma_0>0$ such that* $\inf\{|\sigma(z)|; z\in\mathbb{R}\}\geqslant\sigma_0$;
(3) *there exists $\eta\in(0,\frac{1}{2})$ such that*

$$\sup_{y\in\mathbb{R}^3}\int_{\mathbb{R}^3}\Gamma(dx)\mathscr{F}^{-1}\left(\frac{1}{(1+|\xi|^2)^\eta}\right)(x-y)<\infty.$$

Then, the random variable $u(t,x)$, $(t,x)\in(0,T]\times\mathbb{R}^3$, has a density.

By Bouleau's and Hirsch's criterium this theorem is a consequence of Theorem 2 in Section 3 and the next one.

Theorem 4. *Assume that the coefficients σ and b are $\mathscr{C}^1$ functions with bounded derivatives of order one and that the hypotheses* (2) *and* (3) *of the previous theorem are satisfied. Then, $\|Du(t,x)\|_{\mathscr{H}_T}>0$, a.s.*

Let $G_{d,\eta}(x)=\mathscr{F}^{-1}\left(\frac{1}{(1+|\xi|^2)^\eta}\right)(x)$, $d\geqslant1$, $\eta\in(0,1)$. It is well-known (see for instance [7]) that

$$G_{d,\eta}(x)=C_{d,\eta}|x|^{\eta-\frac{d}{2}}K_{\frac{d}{2}-\eta}(|x|),$$

where $C_{d,\eta}$ is some strictly positive constant and K_ρ is the modified Bessel function of second kind of order ρ. Set $F_{d,\eta}(y)=\int_{\mathbb{R}^d}\Gamma(dx)G_{d,\eta}(x-y)$, $y\in\mathbb{R}^d$ and

($\overline{\mathrm{H}}_\eta$) $\sup_{y\in\mathbb{R}^d}F_{d,\eta}(y)<\infty$.

Hypothesis ($\overline{\mathrm{H}}_\eta$) is almost equivalent to the next one:

(H_η)

$$\int_{\mathbb{R}^d}\frac{\mu(d\xi)}{(1+|\xi|^2)^\eta}<\infty. \tag{28}$$

Indeed, as has been enlighted in [9, Proposition 4.4.1] the condition (H_η) implies

$$F_{d,\eta}(0) = \int_{\mathbb{R}^d} \Gamma(dx) G_{d,\eta}(x) \leqslant \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta} < \infty,$$

and on the other hand $(\overline{\mathrm{H}}_\eta)$ implies (H_η).

Notice that the assumption (3) of Theorem 3 is equivalent to $(\overline{\mathrm{H}}_\eta)$ for some $\eta \in (0, \frac{1}{2})$.

Proof of Theorem 4. We will check that $E(\|Du(t,x)\|_{\mathscr{H}_T}^{-p}) < \infty$ for some $p > 0$. This clearly yields the conclusion of the theorem.

Set $Y = \|Du(t,x)\|_{\mathscr{H}_T}^{-p}$. Owing to the classical expression $E(Y) = \int_0^\infty P\{Y > z\} dz$, we easily obtain that

$$E(Y) \leqslant \varepsilon_0 + \frac{p}{2} \int_0^{\varepsilon_0^{-2/p}} \varepsilon^{-\frac{p}{2}-1} P\{\|Du(t,x)\|_{\mathscr{H}_T}^2 < \varepsilon\}\, d\varepsilon,$$

for any $\varepsilon_0 > 0$. Hence, our purpose is to prove that for $\eta_0 > 0$ small enough,

$$\int_0^{\eta_0} \varepsilon^{-\frac{p}{2}-1} P\{\|Du(t,x)\|_{\mathscr{H}_T}^2 < \varepsilon\}\, d\varepsilon < \infty.$$

Let $\varepsilon_1, \delta > 0$ be such that for any $\varepsilon \in (0, \varepsilon_1]$, $t - \varepsilon^\delta > 0$. Then we obviously have

$$P\{\|Du(t,x)\|_{\mathscr{H}_T}^2 < \varepsilon\} \leqslant P\left\{ \int_{t-\varepsilon^\delta}^t dr \|D_{r,*}u(t,x)\|_{\mathscr{H}}^2 < \varepsilon \right\}. \tag{29}$$

Owing to the expression of $Du(t,x)$ given in (12) we consider, as in [13], the decomposition

$$\|D_{r,*}u(t,x)\|_{\mathscr{H}}^2 = \|Z_{r,*}(t,x)\|_{\mathscr{H}}^2 + U(t,r,x). \tag{30}$$

where $Z(t,x)$ is the $\mathscr{H}_T$-valued random vector obtained in Proposition 3 as the limit in $L^2(\Omega; \mathscr{H}_T)$ of the sequence $S_{3,n}(t-\cdot, x-*)\sigma(u(\cdot,*))$, with $S_{3,n}(t) = \psi_n * S_3(t)$, $n \geqslant 1$. Then, by (29) and (30) we get

$$P\{\|Du(t,x)\|_{\mathscr{H}_T}^2 < \varepsilon\} \leqslant P^1(\varepsilon,\delta) + P^2(\varepsilon,\delta),$$

with

$$P^1(\varepsilon,\delta) = P\left\{ \left| \int_{t-\varepsilon^\delta}^t dr\, U(t,r,x) \right| \geqslant \varepsilon \right\},$$

$$P^2(\varepsilon,\delta) = P\left\{ \int_{t-\varepsilon^\delta}^t dr\, \|Z_{r,*}(t,x)\|_{\mathscr{H}}^2 < 2\varepsilon \right\}.$$

The analysis of the term $P^1(\varepsilon,\delta)$ is done following the same lines as in [13, Theorem 3.1]. On the one hand, there are some obvious simplifications implied by the fact that in our case $u(t,x)$ is a random variable instead of a random vector; on the other hand the integrals in [13] defined by

$$\mu_{\varepsilon^\delta} = \int_0^{\varepsilon^\delta} ds \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy\, f(|x-y|) S_2(s,x) S_2(s,y),$$

$$\nu_{\varepsilon^\delta} = \int_0^{\varepsilon^\delta} ds \int_{\mathbb{R}^2} dx\, S_2(s,x),$$

where S_2 is the fundamental solution of the 2-dimensional wave equation, must be replaced by

$$I_1(\varepsilon,\delta) = \int_0^{\varepsilon^\delta} ds \int_{\mathbb{R}^3} \mu(d\xi) |\mathscr{F} S_3(s)(\xi)|^2,$$

$$I_2(\varepsilon,\delta) = \int_0^{\varepsilon^\delta} ds \int_{\mathbb{R}^3} S_3(s,dx),$$

respectively. Indeed, in [13], the correlation measure is assumed to be absolutely continuous with respect to Lebesgue measure with a density denoted by $f(|x|)$. Hence

$$\mu_{\varepsilon^\delta} = \int_0^{\varepsilon^\delta} ds \int_{\mathbb{R}^2} \Gamma(dx)(S_2(s) * \tilde{S}_2(s))(x) = \int_0^{\varepsilon^\delta} ds \int_{\mathbb{R}^2} \mu(d\xi) |\mathscr{F} S_2(s)(\xi)|^2.$$

Notice that the inequalities (A.4) and (A.5) imply

$$I_2(\varepsilon,\delta) \leqslant C\varepsilon^{2\delta}, \quad I_1(\varepsilon,\delta) \leqslant C\varepsilon^{\delta(3-2\eta)}, \tag{31}$$

respectively.

More explicitly, Chebychev's inequality yields

$$P^1(\varepsilon,\delta) \leqslant \varepsilon^{-1} E\left| \int_{t-\varepsilon^\delta}^t dr\, U(t,r,x) \right| \leqslant \varepsilon^{-1} \sum_{k=1}^5 T_k, \tag{32}$$

with

$$T_1 = E\bigg(\bigg| \int_{t-\varepsilon^\delta}^t dr \left\langle Z_{r,*}(t,x), \int_{t-\varepsilon^\delta}^t \int_{\mathbb{R}^3} S_3(t-s,x-y) D_{r,*}u(s,y) \right.$$
$$\left. \times\, \sigma'(u(s,y)) M(ds,dy) \right\rangle_{\mathscr{H}} \bigg|\bigg),$$

$$T_2 = E\bigg(\bigg|\int_{t-\varepsilon^\delta}^t dr \,\langle Z_{r,*}(t,x), \int_{t-\varepsilon^\delta}^t ds \int_{\mathbb{R}^3} D_{r,*}u(t-s,x-y) \times b'(u(t-s,x-y))S_3(s,dy)\rangle_{\mathscr{H}}\bigg|\bigg),$$

$$T_3 = E\bigg(\int_{t-\varepsilon^\delta}^t dr \bigg\|\int_{t-\varepsilon^\delta}^t \int_{\mathbb{R}^3} S_3(t-s,x-y)D_{r,*}u(s,y)\sigma'(u(s,y))M(ds,dy)\bigg\|_{\mathscr{H}}^2\bigg),$$

$$T_4 = E\bigg(\bigg|\int_{t-\varepsilon^\delta}^t dr \bigg\langle \int_{t-\varepsilon^\delta}^t \int_{\mathbb{R}^3} S_3(t-s,x-y)D_{r,*}u(s,y)\sigma'(u(s,y))M(ds,dy), \times \int_{t-\varepsilon^\delta}^t ds \int_{\mathbb{R}^3} D_{r,*}u(t-s,x-y)b'(u(t-s,x-y))S_3(s,dy)\bigg\rangle_{\mathscr{H}}\bigg|\bigg),$$

$$T_5 = E\bigg(\int_{t-\varepsilon^\delta}^t dr \bigg\|\int_{t-\varepsilon^\delta}^t ds \int_{\mathbb{R}^3} D_{r,*}u(t-s,x-y)b'(u(t-s,x-y))S_3(s,dy)\bigg\|_{\mathscr{H}}^2\bigg).$$

The arguments given in the proof of Proposition 3 show that

$$T_{11} := E\bigg(\int_{t-\varepsilon^\delta}^t dr\, \|Z_{r,*}(t,x)\|_{\mathscr{H}}^2\bigg) = E\bigg(\int_0^{\varepsilon^\delta} dr\, \|Z_{t-r,*}(t,x)\|_{\mathscr{H}}^2\bigg) = \int_0^{\varepsilon^\delta} dr \int_{\mathbb{R}^3} \mu_r^{\bar\sigma}(d\xi)|\mathscr{F}S_3(r)(\xi)|^2,$$

where $\bar\sigma$ denotes the stochastic process $\{\sigma(u(t-r,x)), (r,x)\in[0,t]\times\mathbb{R}^3\}$. By the stationarity property of the solution to (3) proved in [5], we know that $\bar\sigma$ satisfies $E(\bar\sigma(s,x)\bar\sigma(s,y)) = E(\bar\sigma(s,0)\bar\sigma(s,x-y))$, for all $x,y\in\mathbb{R}^3$. Therefore, the inequality (28) of [5] together with the uniformly boundedness of the process u in $L^p(\Omega)$ (see [5, Theorem 13]) yield

$$T_{11} \leqslant C\int_0^{\varepsilon^\delta} dr \int_{\mathbb{R}^3} \mu(d\xi)|\mathscr{F}S_3(r)(\xi)|^2 \leqslant C\varepsilon^{\delta(3-2\eta)}.$$

Proceeding as in [13, Theorem 3.1] and taking into account the preceding remarks we obtain from (32),(31) that

$$\begin{aligned} P^1(\varepsilon,\delta) &\leqslant C\varepsilon^{-1}[I_1(\varepsilon,\delta)^{3/2} + I_1(\varepsilon,\delta)I_2(\varepsilon,\delta)] \\ &\leqslant C\varepsilon^{-1}\Big(\varepsilon^{\frac{3}{2}\delta(3-2\eta)} + \varepsilon^{\delta(5-2\eta)}\Big). \end{aligned} \tag{33}$$

Thus, $\int_0^{\eta_0} \varepsilon^{-\frac{p}{2}-1}P^1(\varepsilon,\delta)\,d\varepsilon < \infty$ if and only if

$$\frac{2}{p+2}\bigg(\frac{3}{2}(3-2\eta)\wedge(5-2\eta)\bigg) > \frac{1}{\delta}. \tag{34}$$

Let us now consider $P^2(\varepsilon,\delta)$. The triangle inequality implies $P^2(\varepsilon,\delta)\leqslant P^{21}(\varepsilon,\delta)+P^{22}(\varepsilon,\delta)$, with

$$P^{21}(\varepsilon,\delta)=P\{||\Lambda_{\varepsilon^{-1}}(\cdot,x-*)\sigma(u(t-\cdot,*))||^2_{\mathscr{H}_{\varepsilon^\delta}}<6\varepsilon\},$$

$$P^{22}(\varepsilon,\delta)=P\{||Z_{t-\cdot,*}(t,x)-\Lambda_{\varepsilon^{-1}}(\cdot,x-*)\sigma(u(t-\cdot,*))||^2_{\mathscr{H}_{\varepsilon^\delta}}\geqslant\varepsilon\},$$

where $\Lambda_{\varepsilon^{-1}}(t)=\psi_{\varepsilon^{-1}}*S_3(t)$.

Notice that by assumption (2),

$$\begin{aligned}||\Lambda_{\varepsilon^{-1}}(r,x-*)\sigma(u(t-r,*))||^2_{\mathscr{H}}&\geqslant\sigma_0^2\int_{\mathbb{R}^3}\mu(d\xi)|\mathscr{F}\Lambda_{\varepsilon^{-1}}(r)(\xi)|^2\\&\geqslant\sigma_0^2\left(\frac{1}{2}\int_{\mathbb{R}^3}\mu(d\xi)|\mathscr{F}S_3(r)(\xi)|^2\right.\\&\quad\left.-\int_{\mathbb{R}^3}\mu(d\xi)|\mathscr{F}S_3(r)(\xi)|^2|\mathscr{F}\psi_{\varepsilon^{-1}}(\xi)-1|^2\right).\end{aligned}\tag{35}$$

We have

$$I:=\int_{\mathbb{R}^3}\mu(d\xi)|\mathscr{F}S_3(r)(\xi)|^2|\mathscr{F}\psi_{\varepsilon^{-1}}(\xi)-1|^2\leqslant4\pi\varepsilon\int_{\mathbb{R}^3}\mu(d\xi)|\xi||\mathscr{F}S_3(r)(\xi)|^2.\tag{36}$$

Indeed,

$$\begin{aligned}|\mathscr{F}\psi_{\varepsilon^{-1}}(\xi)-1|^2&=\left|\int_{\mathbb{R}^d}\psi(y)(\exp(-2\pi i\varepsilon(y\cdot\xi))-1)\,dy\right|^2\\&\leqslant\sup_{|y|\leqslant1}|\exp(-2\pi i\varepsilon(y\cdot\xi))-1|^2\\&=2\sup_{|y|\leqslant1}(1-\cos2\pi\varepsilon(y\cdot\xi))\leqslant4\pi|\xi|\varepsilon,\end{aligned}\tag{37}$$

where $(y\cdot\xi)$ denotes the Euclidean inner-product of the vectors $y,\xi\in\mathbb{R}^3$. Owing to (35), the lower bound of (A.3), (36) and (A.8), we obtain

$$\begin{aligned}||\Lambda_{\varepsilon^{-1}}(\cdot,x-*)\sigma(u(t-\cdot,*))||^2_{\mathscr{H}_{\varepsilon^\delta}}&=\int_0^{\varepsilon^\delta}dr\,||\Lambda_{\varepsilon^{-1}}(r,x-\cdot)\sigma(u(t-r,x))||^2_{\mathscr{H}}\\&\geqslant\sigma_0^2\left(\frac{C_1}{2}\varepsilon^{3\delta}-C_5\varepsilon^{\delta(2-2\eta)+1}\right),\end{aligned}\tag{38}$$

with $C_5=4\pi\overline{C}_\eta$ and $\overline{C}_\eta$ given in Lemma 4. Assume that

$$1+2\eta<\frac{1}{\delta}.\tag{39}$$

Then, for $\varepsilon \leqslant (\frac{C_1}{4C_5})^{1/(1-\delta(1+2\eta))}$ the very last expression of (38) is bounded below by $\sigma_0^2 \frac{C_1}{4} \varepsilon^{3\delta}$. Consequently, if

$$3\delta < 1, \tag{40}$$

then, for $\varepsilon \leqslant \varepsilon_1 \wedge (\frac{C_1}{4C_5})^{1/(1-\delta(1+2\eta))} \wedge (\frac{C_1\sigma_0^2}{24})^{1/(1-3\delta)}$ the set

$$\{||\Lambda_{\varepsilon^{-1}}(\cdot, x-*)\sigma(u(t-\cdot,*))||^2_{\mathscr{H}_{\varepsilon^\delta}} < 6\varepsilon\}$$

is empty and therefore $P^{21}(\varepsilon,\delta) = 0$.

We now study the contribution of $P^{22}(\varepsilon,\delta)$. Chebychev's inequality and the identity (27) in [5] yield

$$\begin{aligned} P^{22}(\varepsilon,\delta) &\leqslant \varepsilon^{-1} E(||Z_{t-\cdot,*}(t,x) - \Lambda_{\varepsilon^{-1}}(\cdot, x-*)\sigma(u(t-\cdot,*))||^2_{\mathscr{H}_{\varepsilon^\delta}}) \\ &= \varepsilon^{-1}\int_0^{\varepsilon^\delta} ds \int_{\mathbb{R}^3} \mu_s^{\bar{\sigma}}(d\xi)|\mathscr{F}(S_3(s) - \Lambda_{\varepsilon^{-1}}(s))(\xi)|^2. \end{aligned}$$

Then, using (37) and Lemma 6, we obtain

$$P^{22}(\varepsilon,\delta) \leqslant 4\pi \int_0^{\varepsilon^\delta} ds \int_{\mathbb{R}^3} \mu_s^{\bar{\sigma}}(d\xi)|\xi||\mathscr{F}S_3(s)(\xi)|^2 \leqslant C\varepsilon^{\delta(2-2\eta)}.$$

Therefore, $\int_0^{\eta_0} \varepsilon^{-\frac{p}{2}-1} P^{22}(\varepsilon,\delta)\, d\varepsilon < \infty$ if and only if

$$-\frac{p}{2} + \delta(2-2\eta) > 0. \tag{41}$$

Let us summarize the restrictions encountered so far, that means, (34), (39)–(41); we realize that they are satisfied if one can choose $\delta, p > 0$ such that for any $\eta \in (0, \frac{1}{2})$, $3 < \frac{1}{\delta} < \frac{2}{2+p}(\frac{3}{2}(3-2\eta) \wedge (5-2\eta)) \wedge \frac{2}{p}(2-2\eta)$. This is always possible taking for instance $p < 1-2\eta$, because in this case we have $3 < \frac{2}{2+p}(\frac{3}{2}(3-2\eta) \wedge (5-2\eta)) \wedge \frac{2}{p}(2-2\eta)$. Hence, the theorem is completely proved. $\square$

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Appendix A

This section is devoted to prove all the auxiliary results on bounds for the Fourier transform of the stochastic wave equation that have been applied in the proof of Theorem 4.

Denote by S_d the fundamental solution of the wave equation in dimension $d \geqslant 1$. It is well-known that its Fourier transform is given by

$$\mathscr{F} S_d(t)(\xi) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}.$$

The following result is a consequence of [9, Lemmas 5.4.1, 5.4.3].

Lemma 2. *For any* $t \geqslant 0$ *it holds that*

$$c_1(t \wedge t^3)\frac{1}{1+|\xi|^2} \leqslant \int_0^t ds\, |\mathscr{F} S_d(s)(\xi)|^2 \leqslant c_2(t+t^3)\,\frac{1}{1+|\xi|^2}, \tag{A.1}$$

with $c_1 = (2^4\pi^2)^{-1}$, $c_2 = \frac{2}{3}$.

The hypothesis (11) relating the noise and the differential operator is in this example equivalent to

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty. \tag{A.2}$$

Hence, (A.1) and (A.2) yield

$$C_1(t \wedge t^3) \leqslant \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)|\mathscr{F} S_d(s)(\xi)|^2 \leqslant C_2(t+t^3),$$

with $C_1 = (2^4\pi^2)^{-1} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2}$, $C_2 = \frac{2}{3}\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2}$. In particular, for $t \in [0,1)$ we have

$$C_1 t^3 \leqslant \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)|\mathscr{F} S_d(s)(\xi)|^2 \leqslant 2C_2 t. \tag{A.3}$$

Let $d \in \{1,2,3\}$; a direct computation based on the expression of S_d shows that

$$\int_0^t ds \int_{\mathbb{R}^d} S_d(s, dy) \leqslant C_3 t^2, \tag{A.4}$$

for any $t \in [0,T]$, where C_3 depends on d.

The upper bound provided by Lemma 2 is not sharp enough to fulfil the requirements of the proof of Theorem 4. We are going to show that the stronger condition (28) furnishes an improved version.

Lemma 3. *Assume* (H_η) *for some* $\eta \in (0,1)$. *Then for any* $t \in [0,T]$,

$$\int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)|\mathscr{F}S_d(s)(\xi)|^2 \leqslant C_\eta t^{3-2\eta}, \tag{A.5}$$

with $C_\eta = \frac{\mu\{|\xi|\leqslant 1\}}{3} T^{2\eta} + \frac{1}{(2\pi^2)^\eta(3-2\eta)} \int_{\{|\xi|>1\}} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta}$.

Proof. It is included in the proof of [11, Lemma 3.4], where d is supposed to be either 1 or 2. However, the arguments are valid for any dimension d, because the expression of $\mathscr{F}S_d(s)$ does not depend on the dimension. For the sake of completeness we give the main lines of the proof.

Set

$$I_1 = \int_0^t ds \int_{\{|\xi|\leqslant 1\}} \mu(d\xi) \frac{\sin^2(2\pi s|\xi|)}{(2\pi|\xi|)^2},$$

$$I_2 = \int_0^t ds \int_{\{|\xi|>1\}} \mu(d\xi) \frac{\sin^2(2\pi s|\xi|)}{(2\pi|\xi|)^2}.$$

Clearly,

$$I_1 \leqslant \mu\{|\xi|\leqslant 1\}\frac{t^3}{3}. \tag{A.6}$$

As for I_2 we have,

$$\begin{aligned} I_2 &\leqslant \int_0^t ds \int_{\{|\xi|>1\}} \mu(d\xi) \frac{(\sin(2\pi s|\xi|))^{2(1-\eta)}}{4\pi^2|\xi|^2} \\ &\leqslant \int_{\{|\xi|>1\}} \mu(d\xi) \frac{1}{4\pi^2|\xi|^2} \int_0^t ds(2\pi s|\xi|)^{2(1-\eta)} \\ &\leqslant \frac{1}{(2\pi^2)^\eta(3-2\eta)} \left(\int_{\{|\xi|>1\}} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta}\right) t^{3-2\eta}. \end{aligned} \tag{A.7}$$

The inequalities (A.6) and (A.7) give (A.5) with the announced value of the constant C_η. □

Lemma 4. *Assume that* (H_η) *holds for some* $\eta \in (0,\frac{1}{2})$. *Then for any* $t \in [0,T]$,

$$\int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)|\xi||\mathscr{F}S_d(s)(\xi)|^2 \leqslant \overline{C}_\eta t^{2-2\eta}, \tag{A.8}$$

with $\overline{C}_\eta = \frac{\mu\{|\xi|\leqslant 1\}}{3} T^{1+2\eta} + \frac{1}{(2-2\eta)2^{1+\eta}\pi^{1+2\eta}} \int_{\{|\xi|>1\}} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta}$.

Proof. As in the proof of the preceding lemma we decompose the left-hand side of (A.8) into the sum $J_1 + J_2$, with

$$J_1 = \int_0^t ds \int_{\{|\xi|\leqslant 1\}} \mu(d\xi)|\xi||\mathscr{F} S_d(s)(\xi)|^2,$$

$$J_2 = \int_0^t ds \int_{\{|\xi|>1\}} \mu(d\xi)|\xi||\mathscr{F} S_d(s)(\xi)|^2.$$

Obviously

$$J_1 \leqslant \mu\{|\xi|\leqslant 1\}\frac{t^3}{3}. \tag{A.9}$$

Let $0<\gamma<1$. Then,

$$\begin{aligned} J_2 &\leqslant \int_0^t ds \int_{\{|\xi|>1\}} \mu(d\xi)|\xi| \frac{(\sin 2\pi s|\xi|)^\gamma}{4\pi^2|\xi|^2} \leqslant (2\pi)^{\gamma-2} \int_0^t ds \int_{\{|\xi|>1\}} \mu(d\xi)|\xi|^{\gamma-1}s^\gamma \\ &= (2\pi)^{\gamma-2}\frac{t^{\gamma+1}}{\gamma+1}\int_{\{|\xi|>1\}} \mu(d\xi)\frac{1}{(|\xi|^2)^{\frac{1-\gamma}{2}}} \leqslant \frac{2^{\frac{\gamma-3}{2}}\pi^{\gamma-2}}{\gamma+1} t^{\gamma+1}\int_{\{|\xi|>1\}} \frac{\mu(d\xi)}{(1+|\xi|^2)^{\frac{1-\gamma}{2}}}. \end{aligned}$$

Let $\eta := \frac{1-\gamma}{2}$. Then we obtain

$$J_2 \leqslant Ct^{2-2\eta}, \tag{A.10}$$

with $C = \frac{1}{(2-2\eta)2^{1+\eta}\pi^{1+2\eta}} \int_{\{|\xi|>1\}} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta}$. Consequently, (A.9) and (A.10) yield (A.8) with the value of $\overline{C}_\eta$ given in the statement. $\square$

Let $\{Z(t,x), (t,x)\in[0,T]\times\mathbb{R}^d\}$ be a predictable L^2 process with stationary covariance function and such that $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|Z(t,x)|^2)<\infty$. Set $\Gamma_s^Z(dx) = g(s,x)\Gamma(dx)$ with $g(s,x) = E(Z(s,y)Z(s,x+y))$. The measure Γ_s^Z is non-negative, non-negative definite and tempered. We denote by μ_s^Z the measure $\mathscr{F}^{-1}(\Gamma_s^Z)$.

Lemma 5. *Assume* $(\overline{\mathrm{H}}_\eta)$ *for some* $\eta\in(0,1)$. *Then*

$$\sup_{0\leqslant s\leqslant T} \int_{\mathbb{R}^d} \mu_s^Z(d\xi)\frac{1}{(1+|\xi|^2)^\eta} \leqslant C\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta},$$

for some positive constant C.

Proof. Set

$$F^Z_{d,\eta}(s,y) := \int_{\mathbb{R}^d} \Gamma^Z_s(dx) G_{d,\eta}(x-y),$$

$s \in [0,T]$, $y \in \mathbb{R}^d$. Hypothesis $(\overline{\mathrm{H}}_\eta)$ implies that

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}^d} F^Z_{d,\eta}(s,y) < \infty.$$

Indeed, this follows from the definition of the measure Γ^Z_s and the properties of the process Z. Then, applying [9, Proposition 4.4.1], it follows that for any $s \in [0,T]$

$$\int_{\mathbb{R}^d} \mu^Z_s(d\xi) \frac{1}{(1+|\xi|^2)^\eta} < \infty.$$

Set $p_t := \mathscr{F}^{-1}(\exp^{-t|\xi|^2})$; by bounded convergence we have that

$$\begin{aligned}\int_{\mathbb{R}^d} \mu^Z_s(d\xi) \frac{1}{(1+|\xi|^2)^\eta} &= \lim_{t \searrow 0} \int_{\mathbb{R}^d} \mu^Z_s(d\xi) \frac{\exp^{-t|\xi|^2}}{(1+|\xi|^2)^\eta} \\ &= \lim_{t\searrow 0} \int_{\mathbb{R}^d} \Gamma^Z_s(dx)(G_{d,\eta} * p_t)(x).\end{aligned}$$

Fubini's Theorem yields that

$$\int_{\mathbb{R}^d} \Gamma^Z_s(dx)(G_{d,\eta} * p_t)(x) = \int_{\mathbb{R}^d} dy\, p_t(y) F^Z_{d,\eta}(s,y).$$

But, the definition of Γ^Z_s implies

$$\begin{aligned}\int_{\mathbb{R}^d} dy\, p_t(y) F^Z_{d,\eta}(s,y) &= \int_{\mathbb{R}^d} dy\, p_t(y) \int_{\mathbb{R}^d} \Gamma^Z_s(dx) G_{d,\eta}(x-y) \\ &= \int_{\mathbb{R}^d} dy\, p_t(y) \int_{\mathbb{R}^d} \Gamma(dx) g(s,x) G_{d,\eta}(x-y) \\ &\leqslant \sup_{(s,x)\in[0,T]\times\mathbb{R}^d} E(|Z(s,x)|^2) \int_{\mathbb{R}^d} dy\, p_t(y) \int_{\mathbb{R}^d} \Gamma(dx) G_{d,\eta}(x-y) \\ &= C \int_{\mathbb{R}^d} \Gamma(dx)(G_{d,\eta} * p_t)(x) = C \int_{\mathbb{R}^d} \mu(d\xi) \frac{\exp^{-t|\xi|^2}}{(1+|\xi|^2)^\eta}.\end{aligned}$$

Owing to (H_η) and using again bounded convergence, it follows that

$$\int_{\mathbb{R}^d} \mu_s^Z(d\xi)\,\frac{1}{(1+|\xi|^2)^\eta} \leqslant C \lim_{t \searrow 0} \int_{\mathbb{R}^d} \mu(d\xi)\,\frac{\exp^{-t|\xi|^2}}{(1+|\xi|^2)^\eta} = C \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta}. \qquad \square$$

Lemma 6. *Assume that* $(\overline{\mathrm{H}}_\eta)$ *holds with* η *restricted to the interval* $(0, \frac{1}{2})$. *Then for any* $t \in [0, T]$ *there exists a positive constant* C *such that*

$$\int_0^t ds \int_{\mathbb{R}^d} \mu_s^Z(d\xi)|\xi||\mathscr{F} S_d(s)(\xi)|^2 \leqslant C t^{2-2\eta}. \tag{A.11}$$

Proof. Clearly, by inequality (28) in [5] and Lemma 3,

$$T_1 := \int_0^t ds \int_{\{|\xi| \leqslant 1\}} \mu_s^Z(d\xi)|\xi||\mathscr{F} S_d(s)(\xi)|^2 \leqslant C t^{3-2\eta}. \tag{A.12}$$

Using the same arguments as those in the proof of Lemma 4 to study the term J_2, we obtain that

$$\begin{aligned} T_2 &:= \int_0^t ds \int_{\{|\xi|>1\}} \mu_s^Z(d\xi)|\xi|\,\frac{\sin^2(2\pi s|\xi|)}{4\pi^2|\xi|^2} \\ &\leqslant \int_0^t ds \int_{\{|\xi|>1\}} \mu_s^Z(d\xi)|\xi|\,\frac{(\sin(2\pi s|\xi|))^{1-2\eta}}{4\pi^2|\xi|^2} \\ &\leqslant C \int_0^t ds\, s^{1-2\eta} \int_{\mathbb{R}^d} \mu_s^Z(d\xi)\,\frac{1}{(1+|\xi|^2)^\eta}. \end{aligned}$$

Due to Lemma 5, this last term is bounded by $Ct^{2-2\eta}$, which together with (A.12) imply (A.11). $\square$

References

[1] V. Bally, E. Pardoux, Malliavin calculus for white noise driven parabolic spde's, Potential Anal. 9 (1998) 27–64.

[2] S.K. Berberian, Introduction to Hilbert Space, 2nd Edition, Chelsea Publ. Co., New York, 1976.

[3] N. Bouleau, F. Hirsch, Dirichlet forms and analysis on Wiener space, in: de Gruyter Studies in Mathematics, Vol. 14, Walter de Gruyter, Berlin, New York, 1991.

[4] R. Carmona, D. Nualart, Random nonlinear wave equations: smoothness of the solutions, Probab. Theory Related Fields 79 (1988) 469–508.

[5] R.C. Dalang, Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e's, Electron. J. Probab. 4 (1999) 1–29.

[6] R.C. Dalang, N.E. Frangos, The stochastic wave equation in two spatial dimensions, Ann. Probab. 26 (1) (1998) 187–212.
[7] W.F. Donoghue, Distributions and Fourier Transforms II, Academic Press, New York, 1969.
[8] D. Lépingle, D. Nualart, M. Sanz-Solé, Dérivation stochastique de diffusions réfléchies, Probabilités et Statistiques 25 (1989) 283–305.
[9] O. Lévêque, Hyperbolic stochastic partial differential equations driven by boundary noises, Thèse, Vol. 2452, EPFL, Lausanne, 2001.
[10] P. Malliavin, Stochastic calculus of variations and hypoelliptic operators, in: Proceedings of the International Symposium on Stochastic Differential Equations, Kyoto, 1976, Wiley, New York, 1978, pp. 195–263.
[11] D. Márquez-Carreras, M. Mellouk, M. Sarrà, On stochastic partial differential equations with spatially correlated noise: smoothness of the law, Stochastics Process Appl. 93 (2001) 269–284.
[12] D. Márquez-Carreras, M. Sanz-Solé, Small perturbations in a hyperbolic stochastic partial differential equation, Stochastic Process. Appl. 68 (1997) 133–154.
[13] A. Millet, M. Sanz-Solé, A stochastic wave equation in two space dimension: smoothness of the law, Ann. Probab. 27 (2) (1999) 803–844.
[14] D. Nualart, Malliavin Calculus and Related Topics, Springer, New York, 1995.
[15] D. Nualart, Analysis on the Wiener space and anticipating calculus, in: P. Bernard (Ed.), École d'été de Probabilités de Saint Flour XXV, Lecture Notes in Mathematics, Vol. 1690, Springer, Berlin, 1998, pp. 863–901.
[16] D. Nualart, M. Sanz-Solé, Malliavin calculus for two-parameter Wiener functionals, Z. für Wahrscheinlichkeitstheorie verw. Gebiete 70 (1985) 573–590.
[17] D. Nualart, M. Sanz-Solé, Stochastic differential equations on the plane: smoothness of the solution, J. Multivar. Anal. 31 (1989) 1–29.
[18] E. Pardoux, Z. Tusheng, Absolute continuity of the law of the solution of a parabolic spde, J. Func. Anal. 112 (1993) 447–458.
[19] S. Peszat, J. Zabczyk, Nonlinear stochastic heat and wave equations, Probab. Theory Related Fields 97 (2000) 421–443.
[20] S. Peszat, The Cauchy problem for a nonlinear stochastic wave equation in any dimension, J. Evol. Equations 3 (2002) 383–394.
[21] C. Rovira, M. Sanz-Solé, The law of a solution to a nonlinear hyperbolic spde, J. Theoret. Probab. 9 (1996) 863–901.
[22] L. Schwartz, Théorie des distributions, Hermann, Paris, 1966.
[23] J.B. Walsh, An introduction to stochastic partial differential equations, in: P.L. Hennequin (Ed.), École d'été de Probabilités de Saint Flour XIV, Lecture Notes in Mathematics, Vol. 1180, Springer, Berlin, 1986, pp. 265–438.

Appendix B

A stochastic wave equation in dimension 3: smoothness of the law

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A stochastic wave equation in dimension 3: smoothness of the law

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We prove the existence and regularity of the density of the real-valued solution to a three-dimensional stochastic wave equation. The noise is white in time and has a spatially homogeneous correlation whose spectral measure μ satisfies $\int_{\mathbb{R}^3}\mu(d\xi)(1+|\xi|^2)^{-\eta}<\infty$, for some $\eta\in(0,\frac{1}{2})$. Our approach uses the mild formulation of the equation given by means of Dalang's extended version of Walsh's stochastic integration. We apply the tools of Malliavin calculus on the appropriate Gaussian space related to the noise. An extension of Dalang's stochastic integral to the Hilbert-valued setting is needed. Let S_3 be the fundamental solution to the three-dimensional wave equation. The assumption on the noise yields upper and lower bounds for the integral $\int_0^t ds\int_{\mathbb{R}^3}\mu(d\xi)|\mathcal{F}S_3(s)(\xi)|^2$ and upper bounds for $\int_0^t ds\int_{\mathbb{R}^3}\mu(d\xi)|\xi||\mathcal{F}S_3(s)(\xi)|^2$ in terms of powers of t. These estimates, together with a suitable mollifying procedure for S_3, are crucial in the analysis of the inverse of the Malliavin variance.

Keywords: Malliavin calculus; stochastic partial differential equations; wave equation

1. Introduction

In this paper we study the probability law of the real-valued solution to the stochastic wave equation

$$\left(\frac{\partial^2}{\partial t^2}-\Delta_3\right)u(t,x)=\sigma(u(t,x))\dot{F}(t,x)+b(u(t,x)),\qquad u(0,x)=\frac{\partial u}{\partial t}(0,x)=0,\tag{1}$$

where $(t,x)\in(0,T]\times\mathbb{R}^3$, $T>0$; Δ_3 denotes the Laplacian operator on $\mathbb{R}^3$ and $\dot{F}$ is a Gaussian noise white in time and correlated in space. Clearly, (1) is a particular case of a class of stochastic partial differential equations (SPDEs) of the form

$$Lu(t,x)=\sigma(u(t,x))\dot{F}(t,x)+b(u(t,x)),\qquad u(0,x)=\frac{\partial u}{\partial t}(0,x)=0,\tag{2}$$

$(t,x)\in(0,T]\times\mathbb{R}^d$, $T>0$, where L is a second-order partial differential operator and the fundamental solution of $Lu=0$ is a non-negative distribution with rapid decrease Λ.

Assume that the coefficients σ and b are Lipschitz continuous real-valued functions and F is a mean-zero $L^2(\Omega,\mathcal{F},P)$-valued Gaussian process indexed by the space of test functions $\mathcal{D}(\mathbb{R}^{d+1})$ with covariance functional $J(\varphi,\psi)=\int_{\mathbb{R}_+}ds\int_{\mathbb{R}^d}\Gamma(dx)(\varphi(s)*\tilde{\psi}(s))(x)$,

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where Γ is a non-negative, non-negative definite tempered measure and $\tilde{\psi}(s, x) = \psi(s, -x)$. Let $\mu = \mathcal{F}^{-1}\Gamma$, where $\mathcal{F}$ is the Fourier transform operator. Then

$$J(\varphi, \psi) = \int_{\mathbb{R}_+} \mathrm{d}s \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi)\mathcal{F}\varphi(s)(\xi)\overline{\mathcal{F}\psi(s)(\xi)}.$$

In Dalang (1999) a suitable extension of Walsh's stochastic integral with respect to martingale measures is developed; with this tool a rigorous meaning is given to equation (2) in a mild form and a theorem on existence and uniqueness of solution is proved. More precisely, there exists a real-valued stochastic process $u = \{u(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ which satisfies the equation

$$u(t, x) = \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-y)\sigma(u(s, y))M(\mathrm{d}s, \mathrm{d}y) + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} b(u(t-s, x-y))\Lambda(s, \mathrm{d}y), \tag{3}$$

where M denotes the martingale measure extension of the process F (see Dalang and Frangos 1998).

Fix $(t, x) \in (0, T] \times \mathbb{R}^3$. Our purpose is to find sufficient conditions ensuring that the law of $u(t, x)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and that the density is a C^∞ function. The existence of the density has been studied in the companion paper by Quer-Sardanyons and Sanz-Solé (2004).

Malliavin calculus provides a suitable tool for the analysis of these problems. The Gaussian family to be considered here is described as follows. Let $\mathcal{E}$ be the inner-product space consisting of functions $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz space of rapidly decreasing C^∞ test functions, endowed with the inner-product $\langle \varphi, \psi \rangle_{\mathcal{E}} := \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi)\mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}$. Let $\mathcal{H}$ denote the completion of $(\mathcal{E}, \langle \cdot, \cdot \rangle_{\mathcal{E}})$ and set $\mathcal{H}_T = L^2([0, T]; \mathcal{H})$. Notice that $\mathcal{H}$ and $\mathcal{H}_T$ may contain distributions. The space $\mathcal{H}_T$ is a real Hilbert separable space. For $h \in \mathcal{H}_T$ we set $W(h) = \int_0^t \int_{\mathbb{R}^d} h(s, x)M(\mathrm{d}s, \mathrm{d}x)$, where the stochastic integral is interpreted in Dalang's sense. Then $\{W(h), h \in \mathcal{H}_T\}$ is a Gaussian process and we can apply the Malliavin calculus based on it (see, for instance, Nualart 1998).

In Theorem 1 of Quer-Sardanyons and Sanz-Solé (2004) we introduce an extension of Dalang's stochastic integral to integrators that are defined by stochastic integration of Hilbert-valued predictable processes with respect to martingale measures. Owing to this extension we have proved that the solution of (3) at any point (t, x) is once differentiable in the Malliavin sense and that the derivative belongs to any L^p and satisfies an SPDE.

We prove in Section 3 below that $u(t, x) \in \mathbb{D}^\infty$ and give the equation satisfied by $D^N u(t, x)$. The standard approach to this problem (see, for instance, Millet and Sanz-Solé 1999; Márquez-Carreras *et al.* 2001) cannot be used here. In fact, the difference of two positive distributions is not necessarily positive; but positivity is one of the requirements in the construction of Dalang's integral and, especially for obtaining L^p bounds, a useful tool for proving L^p convergences. We circumvent this difficulty as follows. We consider a sequence of regularized processes $u_n(t, x)$, $n \geqslant 1$, obtained by convolution of the fundamental solution Λ with an approximation of the identity. The L^p-limit of $u_n(t, x)$ as n tends to infinity is $u(t, x)$, as is proved in Proposition 1 of Quer-Sardanyons and Sanz-Solé (2003); in addition, $u_n(t, x) \in \mathbb{D}^\infty$. Then, since the iterated Malliavin derivative operator D^N is closed, it suffices to prove that the sequence $D^N u_n(t, x)$ converges in the topology of $L^p(\Omega; \mathcal{H}_T^{\otimes N})$, for any $N \geqslant 1$, $p \in [1, \infty)$. This can be achieved by first proving that the sequence is bounded in any L^p and then proving the convergence of order 2, which can be checked with techniques related to the isometry property of the stochastic integral.

The results of Section 4 concern the particular case of equation (1), the stochastic wave equation in spatial dimension 3. We prove that the inverse of the Malliavin variance of $u(t, x)$ belongs to any $L^p(\Omega)$ for all fixed $(t, x) \in (0, T] \times \mathbb{R}^3$. Then, by the results of Section 3, we conclude that the law of $u(t, x)$ has a smooth density.

The existence of moments of any order of the inverse of the Malliavin variance is assured by the integrability in a neighbourhood of zero of the function

$$\varepsilon \rightarrow \varepsilon^{-(1+p)} P\{\|Du(t, x)\|_{\mathcal{H}_T}^2 < \varepsilon\},$$

for any $p \in [0, \infty)$. Hence, the main issue is to obtain the *size* in ε of the factor $P\{\|Du(t, x)\|_{\mathcal{H}_T}^2 < \varepsilon\}$. The difficulties come from the fact that the fundamental solution of the wave equation is a Schwartz distribution. The natural idea is to smooth that distribution, as we did to study the differentiability. This time we introduce a regularization kernel which depends on ε in a suitable way so that the error in this approximation is a function of ε as well. This technique is complemented with upper and lower bounds of integrals involving the Fourier transform of the fundamental solution of the wave equation, which have also played a crucial role in the arguments of Quer-Sardanyons and Sanz-Solé (2004); these are presented in the Appendix.

All positive real constants are denoted by C, regardless of their values. In the following section we give some basic notation for Malliavin calculus used throughout the paper. We refer the reader to Nualart (1995) for a complete account of notions related to this topic.

2. Preliminaries

Consider the stochastic equation (3) as described in the Introduction. Assume that the following set of hypotheses is satisfied:

Hypothesis D. *Let Λ be the fundamental solution of $Lu = 0$. Then $\Lambda(t)$ is a non-negative distribution with rapid decrease such that*

$$\int_0^T \mathrm{d}t \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi) |\mathcal{F}\Lambda(t)(\xi)|^2 < \infty \tag{4}$$

and

$$\lim_{h \downarrow 0} \int_0^T \mathrm{d}t \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi) \sup_{t<r<t+h} |\mathcal{F}(\Lambda(r) - \Lambda(t))(\xi)|^2 + 0.$$

Moreover, Λ is a non-negative measure on $\mathbb{R}_+ \times \mathbb{R}^d$ of the form $\Lambda(t, \mathrm{d}y)\mathrm{d}t$ such that $\sup_{0 \leqslant t \leqslant T} \Lambda(t, \mathbb{R}^d) \leqslant C_T < \infty$.

Then Theorem 5 in Dalang (1999) establishes the existence of a unique progressively measurable process $\{u(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ such that (3) holds; in addition, $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathrm{E}(|u(t, x)|^p) < \infty$, for any $p \in [1, \infty)$, and this has a spatial stationary covariance function. This process will be called the *solution* of (3).

We denote by D the Malliavin derivative operator defined in the framework of the Gaussian space described in the Introduction. Fix any positive integer N; then D^N denotes the Nth iteration of D. For any random variable X, the Nth derivative, if it exists, defines a random vector with values in $\mathcal{H}_T^{\otimes N}$. For any $p \in [1, \infty)$ we denote by $\mathbb{D}^{N,p}$ the Sobolev–Watanabe space of random variables X such that

$$\|X\|_{N,p}^p := \mathrm{E}(|X|^p) + \sum_{j=1}^{N} \mathrm{E}\left(\|D^j X\|_{\mathcal{H}_T^{\otimes j}}^p\right) < +\infty.$$

Let $\mathcal{A}$ be a separable real Hilbert space and $K = \{K(s, z), (s, z) \in [0, T] \times \mathbb{R}^d\}$ be an $\mathcal{A}$-valued predictable process. Set $K^j(s, z) = \langle K(s, z), e_j\rangle_{\mathcal{A}}$, where $\{e_j, j \geqslant 0\}$ is a complete orthonormal system of $\mathcal{A}$. Assume that:

1. $\sup_{(s,z)\in[0,T]\times\mathbb{R}^d} \mathrm{E}(\|K(s, z)\|_{\mathcal{A}}^2) < \infty$;
2. for all $j \geqslant 0$, $s \in [0, T]$, $x, y \in \mathbb{R}^d$,

$$\mathrm{E}(K^j(s, x)K^j(s, y)) = \mathrm{E}(K^j(s, 0)K^j(s, y-x)).$$

For any $j \geqslant 0$, set

$$M_t^{K^j}(A) = \int_0^t \int_A K^j(s, z) M(\mathrm{d}s, \mathrm{d}z), \qquad t \in [0, T], A \in \mathcal{B}_b(\mathbb{R}^d).$$

The process $M_t^K(A) = \sum_{j\geqslant 0} M_t^{K^j}(A) e_j$ defines an $\mathcal{A}$-valued martingale measure.

Set $G_j^K(s, z) = \mathrm{E}(K^j(s, 0)K^j(s, z))$. The measure

$$\Gamma_s^K(\mathrm{d}z) = \sum_{j\geqslant 0} G_j^K(s, z)\Gamma(\mathrm{d}z)$$

is non-negative and tempered. Let μ_s^K be the non-negative tempered measure such that $\mathcal{F}^{-1}\Gamma_s^K = \mu_s^K$.

The next result reproduces Theorem 1 in Quer-Sardanyons and Sanz-Solé (2004). It is an extension to the Hilbert setting of Theorems 2 and 5 in Dalang (1999).

Proposition 1. *Let $t \rightarrow S(t)$ be a deterministic function with values in the space of non-negative distributions with rapid decrease satisfying* (4). *Then the indefinite stochastic integral of S with respect to the martingale measure M^K, $(S \cdot M^K)_t$, $t \in [0, T]$, exists as an $\mathcal{A}$-valued process and satisfies*

$$\mathrm{E}(\|(S \cdot M^K)_t\|_{\mathcal{A}}^2) = \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mu_s^K(\mathrm{d}\xi)|\mathcal{F}S(s)(\xi)|^2.$$

Moreover, for any $p \in [2, \infty)$, $t \in [0, T]$,

$$\mathrm{E}(\|(S\cdot M^K)_t\|_{\mathcal{A}}^p) \leqslant C_t\int_0^t \mathrm{d}s \sup_{x\in\mathbb{R}^d}\mathrm{E}(\|K(s,x)\|_{\mathcal{A}}^p)\int_{\mathbb{R}^d}\mu(\mathrm{d}\xi)|\mathcal{F}S(s)(\xi)|^2, \tag{5}$$

with $C_t = (\int_0^t \mathrm{d}s \int_{\mathbb{R}^d}\mu(\mathrm{d}\xi)|\mathcal{F}S(s)(\xi)|^2)^{p/2-1}$.

We shall use the notation

$$\|S\|_{0,K}^2 = \int_0^T \mathrm{d}s\int_{\mathbb{R}^d}\mu_s^K(\mathrm{d}\xi)|\mathcal{F}S(s)(\xi)|^2.$$

In this paper we will apply this result to $\mathcal{A} := \mathcal{H}_T^{\otimes j}$ and to $\mathcal{H}_T^{\otimes j}$-valued stochastic processes involving Malliavin derivatives up to order $j \geqslant 1$.

3. Malliavin differentiability of spatially homogeneous SPDEs

Suppose that the coefficients of equation (3) are C^1 functions with bounded Lipschitz continuous derivatives. We have proved in Quer-Sardanyons and Sanz-Solé (2004) that, for any fixed $t \geqslant 0$ and $x \in \mathbb{R}^d$, $u(t,x)$ belongs to the space $\mathbb{D}^{1,p}$, for all $p \in [1,\infty)$. The purpose of this section is to extend this result to any differentiability order. That is to say, we wish to prove that $u(t,x) \in \mathbb{D}^\infty = \cap_{N\in\mathbb{N}}\cap_{p\in[1,\infty)}\mathbb{D}^{N,p}$. It is clear that a strengthening of the regularity of the coefficients is needed.

We shall use the notation

$$D^N_{((r_1,\varphi_1),\ldots,(r_N,\varphi_N))}X = \langle D^N_{(r_1,\ldots,r_N)}X, \varphi_1\otimes\ldots\otimes\varphi_N\rangle_{\mathcal{H}^{\otimes N}},$$

for $r_i \in [0,T]$, $\varphi_i \in \mathcal{H}$, $i = 1,\ldots,N$. Thus, we have that

$$\|D^N X\|^2_{\mathcal{H}_T^{\otimes N}} = \int_{[0,T]^N}\mathrm{d}r_1\ldots\mathrm{d}r_N\sum_{j_1,\ldots,j_N}|D_{((r_1,e_{j_1}),\ldots,(r_N,e_{j_N}))}X|^2, \tag{6}$$

where $\{e_j\}_{j\geqslant 0}$ is a complete orthonormal system of $\mathcal{H}$.

Let $N\in\mathbb{N}$, fix a set $A_N = \{\alpha_i = (r_i,\varphi_i)\in\mathbb{R}_+\times\mathcal{H},\ i=1,\ldots,N\}$ and set $\bigvee_i r_i = \max(r_1,\ldots,r_N)$, $\alpha = (\alpha_1,\ldots,\alpha_N)$, $\hat{\alpha}_i = (\alpha_1,\ldots,\alpha_{i-1},\alpha_{i+1},\ldots,\alpha_N)$. Denote by $\mathcal{P}_m$ the set of partitions of A_N consisting of m disjoint subsets $p_1,\ldots,p_m$, $m = 1,\ldots,N$, and by $|p_i|$ the cardinal of p_i. Let X be a random variable belonging to $\mathbb{D}^{N,2}$, $N \geqslant 1$, and g be a real C^N-function with bounded derivatives up to order N. Leibniz's rule for Malliavin's derivatives yields

$$D^N_\alpha(g(X)) = \sum_{m=1}^N\sum_{\mathcal{P}_m} c_m g^{(m)}(X)\prod_{i=1}^m D^{|p_i|}_{p_i}X, \tag{7}$$

with positive coefficients c_m, $m = 1,\ldots,N$, $c_1 = 1$. Let

$$\Delta^N_\alpha(g,X) := D^N_\alpha g(X) - g'(X)D^N_\alpha X.$$

Notice that $\Delta^N_\alpha(g,X) = 0$ if $N = 1$ and it only depends on the Malliavin derivatives up to the order $N-1$ if $N > 1$.

We now state the main result of this section.

Theorem 1. *Assume Hypothesis D and that the coefficients σ and b are C^∞ functions with bounded derivatives of any order greater than or equal to one. Then, for every $(t, x) \in [0, T] \times \mathbb{R}^d$, the random variable $u(t, x)$ belongs to the space $\mathbb{D}^\infty$. Moreover, for any $p \geqslant 1$ and $N \geqslant 1$, there exists an $L^p(\Omega; \mathcal{H}_T^{\otimes N})$-valued random process $\{Z^N(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ such that*

$$\begin{aligned} D^N u(t, x) &= Z^N(t, x) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z)[\Delta^N(\sigma, u(s, z)) + D^N u(s, z)\sigma'(u(s, z))]M(\mathrm{d}s, \mathrm{d}z) \\ &\quad + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \Lambda(s, \mathrm{d}z)[\Delta^N(b, u(t-s, x-z)) \\ &\quad + D^N u(t-s, x-z)b'(u(t-s, x-z))] \end{aligned} \tag{8}$$

and

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}(\|D^N u(s, y)\|^p_{\mathcal{H}_T^{\otimes N}}) < +\infty.$$

We prove this theorem by applying the next lemma, which follows from the fact that D^N is a closed operator defined on $L^p(\Omega)$ with values in $L^p(\Omega; \mathcal{H}_T^{\otimes N})$.

Lemma 1. *Let $\{F_n\}_{n\geqslant 1}$ be a sequence of random variables belonging to $\mathbb{D}^{N,p}$. Assume that:*

(a) *there exists a random variable F such that F_n converges to F in $L^p(\Omega)$ as n tends to ∞,*

(b) *the sequence $\{D^N F_n\}_{n\geqslant 1}$ converges in $L^p(\Omega; \mathcal{H}_T^{\otimes N})$.*

Then F belongs to $\mathbb{D}^{N,p}$ and $D^N F = L^p(\Omega; \mathcal{H}_T^{\otimes N}) - \lim_{n\to\infty} D^N F_n$.

As in Quer-Sardanyons and Sanz-Solé (2004), we consider the sequence of processes $\{u_n(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ solving the equation

$$\begin{aligned} u_n(t, x) = &\int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s, x-z)\sigma(u_n(s, z))M(\mathrm{d}s, \mathrm{d}z) \\ &+ \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} b(u_n(t-s, x-z))\Lambda(s, \mathrm{d}z), \end{aligned}$$

where $\Lambda_n(t) = \psi_n * \Lambda(t)$, with $\psi_n(x) = n^d \psi(nx)$, $n \geqslant 1$, ψ being a non-negative function in $\mathcal{C}^\infty(\mathbb{R}^d)$ with support contained in the unit ball of $\mathbb{R}^d$ and such that $\int_{\mathbb{R}^d} \psi(x)\mathrm{d}x = 1$.

Since Λ_n is smooth, a standard proof (see, for instance, Millet and Sanz-Solé 1999; Márquez-Carreras *et al.* 2001) yields that $u_n(t, x) \in \mathbb{D}^\infty$, for all $n \geqslant 1$. Moreover, the derivative $D^N u_n(t, x)$ satisfies the equation

$$
\begin{aligned}
D_\alpha^N u_n(t,x) = {} & \sum_{i=1}^N \langle \Lambda_n(t-r_i, x-*) D_{\hat{\alpha}_i}^{N-1} \sigma(u_n(r_i,*)), \varphi_i\rangle_{\mathcal{H}} \\
& + \int_{\bigvee_i r_i}^t \int_{\mathbb{R}^d} \Lambda_n(t-s, x-z)[\Delta_\alpha^N(\sigma, u_n(s,z)) \\
& + D_\alpha^N u_n(s,z)\sigma'(u_n(s,z))] M(\mathrm{d}s, \mathrm{d}z) \\
& + \int_{\bigvee_i r_i}^t \mathrm{d}s \int_{\mathbb{R}^d} \Lambda(s, \mathrm{d}z)[\Delta_\alpha^N(b, u_n(t-s, x-z)) \\
& + D_\alpha^N u_n(t-s, x-z) b'(u_n(t-s, x-z))], \qquad (9)
\end{aligned}
$$

where $\alpha = ((r_1, \varphi_1), \ldots, (r_N, \varphi_N))$, with $r_1, \ldots, r_N \geqslant 0$ and $\varphi_1, \ldots, \varphi_N \in \mathcal{H}$.

Lemma 2. *Assume the same hypothesis as in Theorem 1. Then, for all $p \in [1, \infty)$ and every $N \geqslant 1$,*

$$
\sup_{n\geqslant 1} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathrm{E}\Big(\|D^N u_n(t,x)\|_{\mathcal{H}_T^{\otimes N}}^p\Big) < +\infty. \qquad (10)
$$

Proof. We will use an induction argument with respect to N with $p \geqslant 2$ fixed. For $N = 1$, the property (10) is proved in Quer-Sardanyons and Sanz-Solé (2003, Proposition 2). Assume that

$$
\sup_{n\geqslant 1} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathrm{E}\Big(\|D^k u_n(t,x)\|_{\mathcal{H}_T^{\otimes k}}^p\Big) < +\infty,
$$

for any $k = 1, \ldots, N-1$. Let $\alpha = ((r_1, e_{j_1}), \ldots, (r_N, e_{j_N}))$, $r = (r_1, \ldots, r_N)$, $\mathrm{d}r = \mathrm{d}r_1 \ldots \mathrm{d}r_N$. Then, by (6), we have that

$$
\begin{aligned}
\mathrm{E}\Big(\|D^N u_n(t,x)\|_{\mathcal{H}_T^{\otimes N}}^p\Big) &= \mathrm{E}\left(\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\ldots,j_N} |D_\alpha^N u_n(t,x)|^2\right)^{p/2} \\
&\leqslant C \sum_{i=1}^5 N_i,
\end{aligned}
$$

where

$$N_1 = \mathrm{E}\left(\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\dots,j_N} \left|\sum_{i=1}^{N} \langle \Lambda_n(t-r_i, x-*) \times D_{\hat{\alpha}_i}^{N-1}\sigma(u_n(r_i,*)), e_{j_i}\rangle_{\mathcal{H}}\right|^2\right)^{p/2},$$

$$N_2 = \mathrm{E}\left(\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\dots,j_N} \left|\int_{\bigvee_i^{r_i}}^{t}\int_{\mathbb{R}^d} \Lambda_n(t-s, x-z) \times \Delta_\alpha^N(\sigma, u_n(s,z))M(\mathrm{d}s,\mathrm{d}z)\right|^2\right)^{p/2},$$

$$N_3 = \mathrm{E}\left(\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\dots,j_N} \left|\int_{\bigvee_i^{r_i}}^{t} \mathrm{d}s\int_{\mathbb{R}^d} \Lambda(s,\mathrm{d}z) \times \Delta_\alpha^N(b, u_n(t-s, x-z))\right|^2\right)^{p/2},$$

$$N_4 = \mathrm{E}\left(\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\dots,j_N} \left|\int_{\bigvee_i^{r_i}}^{t}\int_{\mathbb{R}^d} \Lambda_n(t-s, x-z) D_\alpha^N u_n(s,z) \times \sigma'(u_n(s,z))M(\mathrm{d}s,\mathrm{d}z)\right|^2\right)^{p/2},$$

$$N_5 = \mathrm{E}\left(\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\dots,j_N} \left|\int_{\bigvee_i^{r_i}}^{t} \mathrm{d}s\int_{\mathbb{R}^d} \Lambda(s,\mathrm{d}z) D_\alpha^N u_n(t-s, x-z) \times b'(u_n(t-s, x-z))\right|^2\right)^{p/2}.$$

By Parseval's identity and the definition of the $\mathcal{H}$-norm, it follows that

$$\begin{aligned}
N_1 &\leqslant C\sum_{i=1}^{N} \mathrm{E}\left(\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\dots,j_N} \left|\langle \Lambda_n(t-r_i, x-*) \times D_{\hat{\alpha}_i}^{N-1}\sigma(u_n(r_i,*)), e_{j_i}\rangle_{\mathcal{H}}\right|^2\right)^{p/2} \\
&= C\sum_{i=1}^{n} \mathrm{E}\left(\int_{[0,T]^N} \mathrm{d}r \sum_{\hat{j}_i} \|\Lambda_n(t-r_i, x-*) D_{\hat{\alpha}_i}^{N-1}\sigma(u_n(r_i,*))\|_{\mathcal{H}}^2\right)^{p/2} \\
&= C\sum_{i=1}^{n} \mathrm{E}\Bigg(\int_{[0,T]^N} \mathrm{d}r \int_{\mathbb{R}^d} \Gamma(\mathrm{d}z)\int_{\mathbb{R}^d} \mathrm{d}y \Lambda_n(t-r_i, x-y) \\
&\quad \times \Lambda_n(t-r_i, x-y+z)\left[\sum_{\hat{j}_i} D_{\hat{\alpha}_i}^{N-1}\sigma(u_n(r_i,y)) D_{\hat{\alpha}_i}^{N-1}\sigma(u_n(r_i, y-z))\right]\Bigg)^{p/2},
\end{aligned}$$

where $\hat{j}_i = j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_N$. Then, by the Cauchy–Schwarz inequality and Hölder's inequality the preceding expression is bounded by

$$\sum_{i=1}^{n} \mathrm{E}\bigg(\int_0^T \mathrm{d}r_i \int_{\mathbb{R}^d} \Gamma(\mathrm{d}z) \int_{\mathbb{R}^d} \mathrm{d}y \Lambda_n(t-r_i, x-y)\Lambda_n(t-r_i, x-y+z)$$

$$\times \int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \|D_{\hat{r}_i}^{N-1}\sigma(u_n(r_i, y))\|_{\mathcal{H}^{\otimes(N-1)}} \times \|D_{\hat{r}_i}^{N-1}\sigma(u_n(r_i, y-z))\|_{\mathcal{H}^{\otimes(N-1)}}\bigg)^{p/2}$$

$$\leqslant C\sum_{i=1}^{n} \int_0^T \mathrm{d}r_i \int_{\mathbb{R}^d} \Gamma(\mathrm{d}z) \int_{\mathbb{R}^d} \mathrm{d}y \Lambda_n(t-r_i, x-y)\Lambda_n(t-r_i, x-y+z)$$

$$\times \mathrm{E}\bigg(\int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \|D_{\hat{r}_i}^{N-1}\sigma(u_n(r_i, y))\|_{\mathcal{H}^{\otimes(N-1)}} \times \|D_{\hat{r}_i}^{N-1}\sigma(u_n(r_i, y-z))\|_{\mathcal{H}^{\otimes(N-1)}}\bigg)^{p/2}$$

$$\leqslant C\sum_{i=1}^{n} \int_0^T \mathrm{d}r_i \int_{\mathbb{R}^d} \Gamma(\mathrm{d}z) \int_{\mathbb{R}^d} \mathrm{d}y \Lambda_n(t-r_i, x-y)\Lambda_n(t-r_i, x-y+z)$$

$$\times \sup_{v\in\mathbb{R}^d} \mathrm{E}\bigg(\int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \|D_{\hat{r}_i}^{N-1}\sigma(u_n(r_i, v))\|^2_{\mathcal{H}^{\otimes(N-1)}}\bigg)^{p/2}$$

$$\leqslant C \sup_{(s,z)\in[0,T]\times\mathbb{R}^d} \mathrm{E}\Big(\|D^{N-1}\sigma(u_n(s, z))\|^p_{\mathcal{H}_T^{\otimes(N-1)}}\Big),$$

with $\mathrm{d}\hat{r}_i = \mathrm{d}r_1 \ldots \mathrm{d}r_{i-1}\mathrm{d}r_{i+1} \ldots \mathrm{d}r_N$. By (7), the assumptions on σ and the induction hypothesis, it follows that N_1 is uniformly bounded with respect to n, t and x.

In the remaining terms we can replace $\bigvee_i r_i$ by 0, because the Malliavin derivatives involved vanish for $t < \bigvee_i r_i$.

By Proposition 1 (see (5)),

$$N_2 = \mathrm{E}\left(\left|\int_0^t\int_{\mathbb{R}^d} \Lambda_n(t-s, x-z)\Delta^N(\sigma, u_n(s, z))M(\mathrm{d}s, \mathrm{d}z)\right|^p_{\mathcal{H}_T^{\otimes N}}\right)$$

$$\leqslant C\int_0^t \mathrm{d}s \sup_{y\in\mathbb{R}^d} \mathrm{E}\Big(\|\Delta^N(\sigma, u_n(s, y))\|^p_{\mathcal{H}_T^{\otimes N}}\Big) \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi)|\mathcal{F}\Lambda(t-s)(\xi)|^2$$

$$\leqslant C\int_0^t \mathrm{d}s \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} \mathrm{E}\Big(\|\Delta^N(\sigma, u_n(\tau, y))\|^p_{\mathcal{H}_T^{\otimes N}}\Big) J(t-s),$$

with $J(t) = \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi)|\mathcal{F}\Lambda(t)(\xi)|^2$. According to the induction hypothesis, this last term is uniformly bounded with respect to n, t and x.

Using similar arguments – this time for deterministic integration of Hilbert-valued processes – Hölder's inequality and the assumptions on Λ, we obtain

$$N_3 \leqslant C\int_0^t ds\int_{\mathbb{R}^d}\Lambda(s, dz)\mathrm{E}\|\Delta^N(b, u_n(t-s, x-z))\|^p_{\mathcal{H}_T^{\otimes N}}$$

$$\leqslant C \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}\Big(\|\Delta^N(b, u_n(s, y))\|^p_{\mathcal{H}_T^{\otimes N}}\Big),$$

which again, by the induction hypothesis, is uniformly bounded in n, t and x.

For N_4 we proceed as for N_2; this yields

$$N_4 \leqslant C\int_0^t ds \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} \mathrm{E}\Big(\|D^N u_n(\tau, y))\|^p_{\mathcal{H}_T^{\otimes N}}\Big)J(t-s).$$

Finally, as for N_3,

$$N_5 \leqslant C\int_0^t ds \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} \mathrm{E}\Big(\|D^N u_n(\tau, y))\|^p_{\mathcal{H}_T^{\otimes N}}\Big).$$

Summarizing the estimates obtained so far, we obtain

$$\sup_{(s,y)\in[0,t]\times\mathbb{R}^d} \mathrm{E}\Big(\|D^N u_n(s, y))\|^p_{\mathcal{H}_T^{\otimes N}}\Big)$$

$$\leqslant C\left[1+\int_0^t ds \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} \mathrm{E}\Big(\|D^N u_n(\tau, y))\|^p_{\mathcal{H}_T^{\otimes N}}\Big)(J(t-s)+1)\right].$$

An application of a version of Gronwall's lemma (Dalang 1999, Lemma 15) concludes the proof. □

For $N \geqslant 1$, $n \geqslant 1$, $r = (r_1, \ldots, r_N)$, $\alpha = ((r_1, e_{j_1}), \ldots, (r_N, e_{j_N}))$ and $(t, x) \in [0, t] \times \mathbb{R}^d$, we define the $\mathcal{H}^{\otimes N}$-valued random variable $Z_r^{N,n}(t, x)$ as follows:

$$\langle Z_r^{N,n}(t, x), e_{j_1} \otimes \ldots \otimes e_{j_N}\rangle_{\mathcal{H}^{\otimes N}} = \sum_{i=1}^N \langle \Lambda_n(t-r_i, x-*)D^{N-1}_{\hat{\alpha}_i}\sigma(u_n(r_i, *)), e_{j_i}\rangle_{\mathcal{H}}.$$

Applying Lemma 2, it can easily be seen that $Z^{N,n}(t, x) \in L^p(\Omega; \mathcal{H}_T^{\otimes N})$ and

$$\sup_{n\geqslant 1}\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathrm{E}\Big(\|Z^{N,n}(t, x)\|^p_{\mathcal{H}_T^{\otimes N}}\Big) < +\infty, \tag{11}$$

for every $p \in [1, \infty)$. Notice that $Z^{N,n}(t, x)$ coincides with the first term of the right-hand side of (9) for $\alpha = ((r_1, e_{j_1}), \ldots, (r_N, e_{j_N}))$.

On the other hand, for $N \geqslant 1$, we introduce the assumption that the sequence $\{D^j u_n(t, x), n \geqslant 1\}$ converges in $L^p(\Omega; \mathcal{H}_T^{\otimes j})$, $j = 1, \ldots, N-1$, with the convention that $L^p(\Omega; \mathcal{H}_T^{\otimes 0}) = L^p(\Omega)$ We denote this assumption by (H_{N-1}).

Proposition 1 in Quer-Sardanyons and Sanz-Solé (2004) yields the validity of (H_0). Moreover, for $N > 1$, (H_{N-1}) implies that $u(t, x) \in \mathbb{D}^{j,p}$ and the sequences $\{D^j u_n(t, x), n \geqslant 1\}$ converge in $L^p(\Omega; \mathcal{H}_T^{\otimes j})$ to $D^j u(t, x)$. In addition, by Lemma 2,

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}(\|D^j u(s,y)\|^p_{\mathcal{H}_T^{\otimes j}}) < \infty, \tag{12}$$

$j = 1, \ldots, N-1$.

Lemma 3. *Fix $N \geqslant 1$. Assume the same hypothesis as in Theorem 1 and that (H_{N-1}) holds. Then the sequence $\{Z^{N,n}(t,x)\}_{n\geqslant 1}$ converges in $L^p(\Omega; \mathcal{H}_T^{\otimes N})$ to a random variable $Z^N(t,x)$.*

Proof. For $N = 1$ the result is proved in Quer-Sardanyons and Sanz-Solé (2004, Proposition 3). Assume $N > 1$. In view of (11), it suffices to show that $\{Z^{N,n}(t,x)\}_{n\geqslant 1}$ is a Cauchy sequence in $L^2(\Omega; \mathcal{H}_T^{\otimes N})$.

For $n, m \geqslant 1$, set

$$\begin{aligned} Z^{n,m} := \mathrm{E}\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\ldots,j_N} \Bigg| &\sum_{i=1}^N \langle \Lambda_n(t-r_i, x-*) D^{N-1}_{\hat{\alpha}_i} \sigma(u_n(r_i,*)), e_{j_i}\rangle_{\mathcal{H}} \\ &- \sum_{i=1}^N \langle \Lambda_m(t-r_i, x-*) D^{N-1}_{\hat{\alpha}_i} \sigma(u_m(r_i,*)), e_{j_i}\rangle_{\mathcal{H}} \Bigg|^2. \end{aligned}$$

Then

$$Z^{n,m} \leqslant C(Z_1^n + Z_2^{n,m} + Z_3^m),$$

where

$$Z_1^n = \sum_{i=1}^N \mathrm{E}\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\ldots,j_N} |\langle \Lambda_n(t-r_i, x-*) \times [D^{N-1}_{\hat{\alpha}_i} \sigma(u_n(r_i,*)) - D^{N-1}_{\hat{\alpha}_i} \sigma(u(r_i,*))], e_{j_i}\rangle_{\mathcal{H}}|^2,$$

$$Z_2^{n,m} = \sum_{i=1}^N \mathrm{E}\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\ldots,j_N} |\langle D^{N-1}_{\hat{\alpha}_i} \sigma(u(r_i,*)) \times [\Lambda_n(t-r_i, x-*) - \Lambda_m(t-r_i, x-*)], e_{j_i}\rangle_{\mathcal{H}}|^2,$$

$$Z_3^m = \sum_{i=1}^N \mathrm{E}\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\ldots,j_N} |\langle \Lambda_m(t-r_i, x-*) \times [D^{N-1}_{\hat{\alpha}_i} \sigma(u(r_i,*)) - D^{N-1}_{\hat{\alpha}_i} \sigma(u_m(r_i,*))], e_{j_i}\rangle_{\mathcal{H}}|^2.$$

Parseval's identity and the Cauchy–Schwarz inequality ensure that

$$
\begin{aligned}
Z_1^n &= \sum_{i=1}^N \mathrm{E}\int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \sum_{\hat{j}_i} \|\Lambda_n(t-\cdot, x-*)[D_{\hat{\alpha}_i}^{N-1}\sigma(u_n(\cdot,*)) - D_{\hat{\alpha}_i}^{N-1}\sigma(u(\cdot,*))]\|_{\mathcal{H}_T}^2 \\
&\leqslant \sum_{i=1}^N \mathrm{E}\int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \Gamma(\mathrm{d}z)\int_{\mathbb{R}^d} \mathrm{d}y \Lambda_n(t-s, x-y) \\
&\quad \times \Lambda_n(t-s, x-y+z)\|D_{\hat{r}_i}^{N-1}(\sigma(u_n(s,y)) - \sigma(u(s,y)))\|_{\mathcal{H}^{\otimes(N-1)}} \\
&\quad \times \|D_{\hat{r}_i}^{N-1}(\sigma(u_n(s,y-z)) - \sigma(u(s,y-z)))\|_{\mathcal{H}^{\otimes(N-1)}} \\
&\leqslant \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}\Big(|D^{N-1}(\sigma(u_n(s,y)) - \sigma(u(s,y)))|_{\mathcal{H}_T^{\otimes(N-1)}}^2\Big) \times \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi)|\mathcal{F}\Lambda(t-s)(\xi)|^2 \\
&\leqslant C \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}\Big(\|D^{N-1}(\sigma(u_n(s,y)) - \sigma(u(s,y)))\|_{\mathcal{H}_T^{\otimes(N-1)}}^2\Big).
\end{aligned}
$$

Equation (7), Lemma 3 and assumption (H_{N-1}) yield that the last term tends to zero as n goes to infinity. Analogously, Z_3^m tends to zero as m tends to infinity.

Using similar arguments, we obtain

$$
\begin{aligned}
Z_2^{n,m} &= \sum_{i=1}^N E\int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \sum_{\hat{j}_i} \|D_{\hat{\alpha}_i}^{N-1}\sigma(u(\cdot,*))[\Lambda_n(t-\cdot, x-*) - \Lambda_m(t-\cdot, x-*)]\|_{\mathcal{H}_T}^2 \\
&= \sum_{i=1}^N \mathrm{E}\int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \sum_{\hat{j}_i} \int_0^T \mathrm{d}s \int_{\mathbb{R}^d} \Gamma(\mathrm{d}z) \int_{\mathbb{R}^d} \mathrm{d}y D_{\hat{\alpha}_i}^{N-1}\sigma(u(s,y)) \\
&\quad \times D_{\hat{\alpha}_i}^{N-1}\sigma(u(s,y-z))[\Lambda_n(t-s, x-y) - \Lambda_m(t-s, x-y)] \\
&\quad \times [\Lambda_n(t-s, x-y+z) - \Lambda_m(t-s, x-y+z)] \\
&= \sum_{i=1}^N \mathrm{E}\int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \sum_{\hat{j}_i} \int_0^T \mathrm{d}s \int_{\mathbb{R}^d} \mu_s^{D_{\hat{\alpha}_i}^{N-1}\sigma(u)}(\mathrm{d}\xi)|\mathcal{F}(\Lambda_n(t-s) - \Lambda_m(t-s))(\xi)|^2.
\end{aligned}
$$

This term tends to zero as m and n go to infinity. Indeed, arguing as in the proof of Theorem 2 from Dalang (1999), we have that

$$
\|\Lambda(t-\cdot)\|_{0,D_{\hat{\alpha}_i}^{N-1}\sigma(u)}^2 \leqslant \liminf_{k\to\infty} \|\Lambda_k(t-\cdot)\|_{0,D_{\hat{\alpha}_i}^{N-1}\sigma(u)}^2.
$$

Then, by Fatou's lemma,

$$\mathrm{E}\int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \sum_{\hat{j}_i} \int_0^T \mathrm{d}s \int_{\mathbb{R}^d} \mu_s^{D_{\hat{\alpha}_i}^{N-1}\sigma(u)}(\mathrm{d}\xi)|\mathcal{F}\Lambda(t-s)(\xi)|^2$$

$$= \int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \sum_{\hat{j}_i} \|\Lambda(t-\cdot)\|^2_{0,D_{\hat{\alpha}_i}^{N-1}\sigma(u)}$$

$$\leqslant \liminf_{k\to\infty} \int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \sum_{\hat{j}_i} \|\Lambda_k(t-\cdot)\|^2_{0,D_{\hat{\alpha}_i}^{N-1}\sigma(u)}.$$

This last term is bounded by a finite constant not depending on k, as can easily be seen using (12). Then we conclude by bounded convergence. □

Proof of Theorem 1. Fix $(t, x) \in (0, T] \times \mathbb{R}^d$, $p \in [2, \infty)$. We apply Lemma 1 to $F_n := u_n(t, x)$ and $F := u(t, x)$. We know that assumption (a) of the lemma is satisfied.

Let us check that the sequence $\{D^N u_n(t, x)\}_{n\geqslant 1}$ converges in the space $L^p(\Omega; \mathcal{H}_T^{\otimes N})$, for every $N \geqslant 1$ and $p \geqslant 2$, which implies that the random variable $D^N u(t, x)$ exists, belongs to $L^p(\Omega; \mathcal{H}_T^{\otimes N})$ and, by Lemma 2, satisfies

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}(\|D^N u(s, y)\|^p_{\mathcal{H}_T^{\otimes N}}) < +\infty.$$

Owing to Lemma 2, it suffices to check the assertion for $p = 2$. We will use an induction argument on N. For $N = 1$ the proof is given in Theorem 2 of Quer-Sardanyons and Sanz-Solé (2004).

Assume the induction hypothesis (H_{N-1}). Let $\mathcal{B}_{p,N}$ be the class of progressively measurable $\mathcal{H}_T^{\otimes N}$-valued processes $\{\Psi(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ with spatially homogeneous covariance function and satisfying

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}(\|\Psi(s, y)\|_{\mathcal{H}_T^{\otimes N}}) < +\infty.$$

We consider the stochastic integral equation in $\mathcal{B}_{p,N}$,

$$U(t, x) = Z^N(t, x) + \int_0^t\int_{\mathbb{R}^d} \Lambda(t-s, x-z)[\Delta(\sigma, u(s, z)) + U(s, z)\sigma'(u(s, z))]M(\mathrm{d}s, \mathrm{d}z)$$

$$+ \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \Lambda(s, \mathrm{d}z)[\Delta(b, u(t-s, x-z)) + U(t-s, x-z)b'(u(t-s, x-z))],$$

with $Z^N(t, x)$ given in Lemma 3. There exists a unique solution to this equation. Moreover, following arguments similar to those in the proof of Theorem 2 in Quer-Sardanyons and Sanz-Solé (2004), owing to Lemma 3 and (H_{N-1}) it is easy to prove that

$$U(t, x) = L^2(\Omega; \mathcal{H}_T^{\otimes N}) - \lim_{n\to\infty} D^N u_n(t, x),$$

the limit being uniform in (t, x). Then by uniqueness of the solution $U \equiv D^N u$, and the process $D^N u(t, x)$ satisfies equation (8). □

4. Study of the inverse of the Malliavin matrix

In this section we consider the stochastic wave equation (1). Let S_3 be the fundamental solution of $Lu = 0$ where $L = \partial^2/\partial t^2 - \Delta_3$. In this case condition (4) is equivalent to

$$\int_{\mathbb{R}^3} \frac{\mu(\mathrm{d}\xi)}{1+|\xi|^2} < \infty, \tag{13}$$

and this implies Hypothesis D (for details, see Dalang 1999).

Let $\{u(t, x), (t, x) \in [0, T] \times \mathbb{R}^3\}$ be the real-valued process solving (1). The purpose of this section is to study the L^p-integrability of the inverse of the Malliavin variance of $u(t, x)$ for any fixed $(t, x) \in (0, T] \times \mathbb{R}^3$. More precisely, we prove the following result.

Theorem 2. *Assume that the coefficients σ and b are C^1 functions with bounded Lipschitz continuous derivatives and, in addition, that:*

(a) *there exists $\sigma_0 > 0$ such that* $\inf\{|\sigma(z)|, z \in \mathbb{R}\} \geqslant \sigma_0$;
(b) *there exists $\eta \in (0, \frac{1}{2})$ such that*

$$\sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \Gamma(\mathrm{d}x) \mathcal{F}^{-1}\left(\frac{1}{(1+|\xi|^2)^\eta}\right)(x-y) < \infty.$$

Then, for any $p > 0$,

$$\mathrm{E}\left(\|Du(t, x)\|_{\mathcal{H}_T}^{-p}\right) < \infty.$$

This result, together with Theorem 1 applied to equation (1), yields the main result of the paper, as follows.

Theorem 3. *Assume that the coefficients σ and b are C^∞ functions with bounded derivatives of any order greater than or equal to one, and that hypotheses* (a) *and* (b) *of Theorem* 2 *are satisfied. Then the random variable $u(t, x)$, $(t, x) \in (0, T] \times \mathbb{R}^3$, has a density which is a C^∞ function.*

We notice that assumption (b) in Theorem 2 implies (13) (Lévêque 2001, Proposition 4.4.1).

Recall that the Malliavin derivative $Du(t, x)$ of the solution to (1) satisfies the equation

$$\begin{aligned} Du(t, x) &= Z(t, x) + \int_0^t \int_{\mathbb{R}^3} S_3(t-s, x-z)\sigma'(u(s, z))Du(s, z)M(\mathrm{d}s, \mathrm{d}z) \\ &\quad + \int_0^t \mathrm{d}s \int_{\mathbb{R}^3} S_3(t-s, \mathrm{d}z)b'(u(s, x-z))Du(s, x-z), \end{aligned} \tag{14}$$

where $\{Z(t, x), (t, x) \in [0, T] \times \mathbb{R}^3\}$ is the $\mathcal{H}_T$-valued random process given by

$$Z(t, x) = L^p(\Omega; \mathcal{H}_T) - \lim_{n\to\infty} Z^n(t, x),$$

$p \geqslant 1$, where $Z^n(t, x) := S_{3,n}(t - \cdot, x - *)\sigma(u(\cdot, *))$ with $S_{3,n} = S_3 * \psi_n$; see either Theorem 1 or Quer-Sardanyons and Sanz-Solé (2004, Theorem 2).

Lemma 4. *Assume that σ is Lipschitz continuous and that condition* (13) *is satisfied. Then, for any $(t, x) \in (0, T] \times \mathbb{R}^3$, $v \in (0, t]$ and $q \geqslant 1$,*

$$\mathrm{E}\Big(\|Z_{t-\cdot,*}(t, x)\|^{2q}_{\mathcal{H}_v}\Big) \leqslant C\Big(\int_0^v \mathrm{d}s \int_{\mathbb{R}^3} \mu(\mathrm{d}\xi)|\mathcal{F}S_3(s)(\xi)|^2\Big)^q.$$

Proof. Hölder's inequality with respect to the non-negative finite measure $S_{3,n}(s, x - y)S_{3,n}(s, x - y + z)\mathrm{d}s\Gamma(\mathrm{d}z)\mathrm{d}y$ yields

$$\begin{aligned}
\mathrm{E}(\|Z_{t-\cdot,*}(t, x)\|^{2q}_{\mathcal{H}_v}) &= \lim_{n\to\infty} \mathrm{E}(\|Z^n_{t-\cdot,*}(t, x)\|^{2q}_{\mathcal{H}_v}) \\
&= \lim_{n\to\infty} \mathrm{E}(\|S_{3,n}(\cdot, x - *)\sigma(u(t - \cdot, *))\|^{2q}_{\mathcal{H}_v}) \\
&= \lim_{n\to\infty} \mathrm{E}\Bigg(\bigg|\int_0^v \mathrm{d}s \int_{\mathbb{R}^3} \Gamma(\mathrm{d}z) \int_{\mathbb{R}^3} \mathrm{d}y S_{3,n}(s, x - y)\sigma(u(t - s, y)) \\
&\qquad \times S_{3,n}(s, x - y + z)\sigma(u(t - s, y - z))\bigg|^q\Bigg) \\
&\leqslant \lim_{n\to\infty} \Big(\int_0^v \mathrm{d}s \int_{\mathbb{R}^3} \Gamma(\mathrm{d}z) \int_{\mathbb{R}^3} \mathrm{d}y S_{3,n}(s, x - y)S_{3,n}(s, x - y + z)\Big)^{q-1} \\
&\qquad \times \int_0^v \mathrm{d}s \int_{\mathbb{R}^3} \Gamma(\mathrm{d}z) \int_{\mathbb{R}^3} \mathrm{d}y S_{3,n}(s, x - y)S_{3,n}(s, x - y + z) \\
&\qquad \times \mathrm{E}(|\sigma(u(t - s, y))\sigma(u(t - s, y - z))|^q) \\
&\leqslant C\Big(1 + \sup_{(s,z)\in[0,T]\times\mathbb{R}^3} \mathrm{E}(|u(s, z)|^{2q})\Big) \\
&\qquad \times \lim_{n\to\infty} \Big(\int_0^v \mathrm{d}s \int_{\mathbb{R}^3} \Gamma(\mathrm{d}z)(S_{3,n}(s) * \tilde{S}_{3,n}(s))(z)\Big)^q \\
&\leqslant C(\int_0^v \mathrm{d}s \int_{\mathbb{R}^3} \mu(\mathrm{d}\xi)|\mathcal{F}S_3(s)(\xi)|^2)^q,
\end{aligned}$$

where in the last inequality we have used the L^q uniform boundedness of $u(t, x)$. □

Owing to Lemma 4 and Proposition 1, we obtain the following technical result.

Lemma 5. *Under the same hypothesis as in Lemma 4, we have that*

$$\sup_{t-v\leqslant s\leqslant t}\sup_{y\in\mathbb{R}^3}\mathrm{E}(\|D_{t-\cdot,*}u(s,y)\|_{\mathcal{H}_v}^{2q})\leqslant C\left(\int_0^v \mathrm{d}s\int_{\mathbb{R}^3}\mu(\mathrm{d}\xi)|\mathcal{F}S_3(s)(\xi)|^2\right)^q,$$

for all $t\in(0,T]$ *and* $q\geqslant 1$.

We remark that both of the preceding lemmas also hold in the more general setting of Section 3.

Proof of Theorem 2. Fix $p>0$; it suffices to check that, for some $\epsilon_0>0$,

$$\int_0^{\epsilon_0}\epsilon^{-(1+p)}P\{\|Du(t,x)\|_{\mathcal{H}_T}^2<\epsilon\}\mathrm{d}\epsilon<\infty.$$

Let $\epsilon_1,\delta>0$ be such that, for any $\epsilon\in(0,\epsilon_1]$, $t-\epsilon^\delta>0$. Owing to (14), we consider the decomposition

$$P\{\|Du(t,x)\|_{\mathcal{H}_T}^2<\epsilon\}\leqslant P^1(\epsilon,\delta)+P^{2,1}(\epsilon,\delta,\nu)+P^{2,2}(\epsilon,\delta,\nu),$$

where

$$P^1(\epsilon,\delta)=P\left\{\left|\int_{t-\epsilon^\delta}^t \mathrm{d}rU(t,r,x)\right|\geqslant\epsilon\right\},$$

$$P^{2,1}(\epsilon,\delta,\nu)=P\{\|S_{\epsilon^{-\nu}}(\cdot,x-*)\sigma(u(t-\cdot,*))\|_{\mathcal{H}_{\epsilon^\delta}}^2<6\epsilon\},$$

$$P^{2,2}(\epsilon,\delta,\nu)=P\{\|Z_{t-\cdot,*}(t,x)-S_{\epsilon^{-\nu}}(\cdot,x-*)\sigma(u(t-\cdot,*))\|_{\mathcal{H}_{\epsilon^\delta}}^2\geqslant\epsilon\},$$

with $U(t,r,x)=\|D_{r,*}u(t,x)\|_{\mathcal{H}}^2-\|Z_{r,*}(t,x)\|_{\mathcal{H}}^2$ and $S_{\epsilon^{-\nu}}=\psi_{\epsilon^{-\nu}}*S_3$, $\psi_{\epsilon^{-\nu}}(x)=\epsilon^{-3\nu}\psi(\epsilon^{-\nu}x)$, $\nu>0$ and ψ a non-negative function in $C^\infty(\mathbb{R}^3)$ with support contained in the unit ball of $\mathbb{R}^3$ and such that $\int_{\mathbb{R}^3}\psi(x)\mathrm{d}x=1$.

Let us first consider the term $P^1(\epsilon,\delta)$. By Chebyshev's inequality, for every $q\geqslant 1$ we have that

$$P^1(\epsilon,\delta)\leqslant\epsilon^{-q}\mathrm{E}\left(\left|\int_{t-\epsilon^\delta}^t \mathrm{d}rU(t,r,x)\right|^q\right)\leqslant C\epsilon^{-q}\sum_{k=1}^5 T_k, \tag{15}$$

with

$$T_1 = \mathrm{E}\left(\left|\int_{t-\epsilon^\delta}^t \mathrm{d}r\left\langle Z_{r,*}(t,x), \int_{t-\epsilon^\delta}^t\int_{\mathbb{R}^3} S_3(t-s,x-z)D_{r,*}u(s,z)\times\sigma'(u(s,z))M(\mathrm{d}s,\mathrm{d}z)\right\rangle_{\mathcal{H}}\right|^q\right),$$

$$T_2 = \mathrm{E}\left(\left|\int_{t-\epsilon^\delta}^t \mathrm{d}r\left\langle Z_{r,*}(t,x), \int_{t-\epsilon^\delta}^t \mathrm{d}s\int_{\mathbb{R}^3} S_3(t-s,\mathrm{d}z)D_{r,*}u(s,x-z)\times b'(u(s,x-z))\right\rangle_{\mathcal{H}}\right|^q\right),$$

$$T_3 = \mathrm{E}\left(\left|\int_{t-\epsilon^\delta}^t \mathrm{d}r\left\|\int_{t-\epsilon^\delta}^t\int_{\mathbb{R}^3} S_3(t-s,x-z)D_{r,*}u(s,z)\times\sigma'(u(s,z))M(\mathrm{d}s,\mathrm{d}z)\right\|_{\mathcal{H}}^2\right|^q\right),$$

$$T_4 = \mathrm{E}\left(\left|\int_{t-\epsilon^\delta}^t \mathrm{d}r\left\langle \int_{t-\epsilon^\delta}^t\int_{\mathbb{R}^3} S_3(t-s,x-z)D_{r,*}u(s,z)\sigma'(u(s,z))M(\mathrm{d}s,\mathrm{d}z),\right.\right.\right.$$
$$\left.\left.\left.\times\int_{t-\epsilon^\delta}^t \mathrm{d}s\int_{\mathbb{R}^3} S_3(t-s,\mathrm{d}z)D_{r,*}u(s,x-z)b'(u(s,x-z))\right\rangle_{\mathcal{H}}\right|^q\right),$$

$$T_5 = \mathrm{E}\left(\left|\int_{t-\epsilon^\delta}^t \mathrm{d}r\left\|\int_{t-\epsilon^\delta}^t \mathrm{d}s\int_{\mathbb{R}^3} S_3(t-s,\mathrm{d}z)D_{r,*}u(s,x-z)\times b'(u(s,x-z))\right\|_{\mathcal{H}}^2\right|^q\right).$$

Schwarz's inequality yields

$$T_1 \leqslant T_{11}^{1/2}T_{12}^{1/2},$$

with

$$T_{11} = \mathrm{E}\left(\left|\int_{t-\epsilon^\delta}^t \mathrm{d}r\|Z_{r,*}(t,x)\|_{\mathcal{H}}^2\right|^q\right),$$

$$T_{12} = \mathrm{E}\left(\left|\int_{t-\epsilon^\delta}^t \mathrm{d}r\left\|\int_{t-\epsilon^\delta}^t\int_{\mathbb{R}^3} S_3(t-s,x-z)D_{r,*}u(s,z)\times\sigma'(u(s,z))M(\mathrm{d}s,\mathrm{d}z)\right\|_{\mathcal{H}}^2\right|^q\right).$$

By Lemma 4 and (27),

$$T_{11} = \mathrm{E}\left(\|Z_{t-\cdot,*}(t,x)\|_{\mathcal{H}_{\epsilon^\delta}}^{2q}\right) \leqslant C\epsilon^{q\delta(3-2\eta)}. \tag{16}$$

We have that

$$T_{12} = \mathrm{E}\left(\left\|\int_{t-\epsilon^\delta}^t\int_{\mathbb{R}^3} S_3(t-s,x-z)D_{t-\cdot,*}u(s,z)\sigma'(u(s,z))M(\mathrm{d}s,\mathrm{d}z)\right\|_{\mathcal{H}_{\epsilon^\delta}}^{2q}\right).$$

Here we apply Proposition 1 to $\mathcal{A} := \mathcal{H}_{\epsilon^\delta}$, $K(s,z) := D_{t-\cdot,*}u(s,z)\sigma'(u(s,z))$ and $S := S_3$. Thus, Lemma 5 and (27) ensure that

$$T_{12} \leqslant C\left(\int_0^{\epsilon^\delta}\mathrm{d}s\int_{\mathbb{R}^3}\mu(\mathrm{d}\xi)|\mathcal{F}S_3(s)(\xi)|^2\right)^{2q} \leqslant \epsilon^{2q\delta(3-2\eta)}.$$

Hence,

$$T_1 \leqslant C\epsilon^{3q\delta(3-2\eta)/2}. \tag{17}$$

We now consider the term

$$T_{22} := \mathrm{E}\left(\left\|\int_{t-\epsilon^\delta}^{t} \mathrm{d}s \int_{\mathbb{R}^3} S_3(t-s, \mathrm{d}z) D_{t-\cdot,*} u(s, x-z) b'(u(s, x-z))\right\|_{\mathcal{H}_{\epsilon^\delta}}^{2q}\right).$$

Hölder's inequality with respect to the finite measure $S_3(t-s, \mathrm{d}z)\mathrm{d}s$ on $[t-\epsilon^\delta, t] \times \mathbb{R}^3$ yields

$$T_{22} \leqslant \left(\int_{t-\epsilon^\delta}^{t} \mathrm{d}s \int_{\mathbb{R}^3} S_3(t-s, \mathrm{d}z)\right)^{2q-1}$$
$$\times \mathrm{E}\left(\int_{t-\epsilon^\delta}^{t} \mathrm{d}s \int_{\mathbb{R}^3} S_3(t-s, \mathrm{d}z) \| D_{t-\cdot,*} u(s, x-z) b'(u(s, x-z))\|_{\mathcal{H}_{\epsilon^\delta}}^{2q}\right).$$

Notice that

$$\int_{t-\epsilon^\delta}^{t} \mathrm{d}s \int_{\mathbb{R}^3} S_3(t-s, \mathrm{d}z) = \int_{0}^{\epsilon^\delta} \mathrm{d}s \int_{\mathbb{R}^3} S_3(s, \mathrm{d}z) \leqslant C\epsilon^{2\delta},$$

because $S_3(t) = \sigma_t/4\pi t$, where σ_t denotes the uniform measure on the three-dimensional sphere of radius t. Then, since b' is bounded, Lemma 5 and (27) imply

$$T_{22} \leqslant C\epsilon^{4q\delta + q\delta(3-2\eta)} = C\epsilon^{q\delta(7-2\eta)}. \tag{18}$$

Schwarz's inequality and the estimates (16), (17), (18) yield

$$T_2 \leqslant T_{11}^{1/2} T_{22}^{1/2} \leqslant C\epsilon^{q\delta(5-2\eta)},$$

$$T_3 = T_{12} \leqslant C\epsilon^{2q\delta(3-2\eta)},$$

$$T_4 \leqslant T_{12}^{1/2} T_{22}^{1/2} \leqslant C\epsilon^{q\delta(13/2-3\eta)},$$

$$T_5 = T_{22} \leqslant C\epsilon^{q\delta(7-2\eta)}. \tag{19}$$

Therefore, (15), (17) and (19) imply

$$P^1(\epsilon, \delta) \leqslant C\epsilon^{q(-1+3\delta(3-2\eta)/2)}.$$

Consequently, $\int_0^{\epsilon_0} P^1(\epsilon, \delta)\epsilon^{-(1+p)}\mathrm{d}\epsilon < \infty$ if

$$\frac{1}{\delta} < \frac{\frac{3}{2}q(3-2\eta)}{p+q}. \tag{20}$$

We now study the term $P^{2,1}(\epsilon, \delta, \nu)$. Our purpose is to choose some positive δ and ν such that, for ϵ sufficiently small, $\{\|S_{\epsilon^{-\nu}}(\cdot, x-*)\sigma(u(t-\cdot, *))\|_{\mathcal{H}_{\epsilon^\delta}}^2 < 6\epsilon\}$ is the empty set and therefore $P^{2,1}(\epsilon, \delta, \nu) = 0$. Assumption (a) in Theorem 2 yields

$$\|S_{\epsilon^{-\nu}}(r, x-*)\sigma(u(t-r,*))\|^2_{\mathcal{H}} \geqslant \sigma_0^2 \int_{\mathbb{R}^3} \mu(\mathrm{d}\xi)|\mathcal{F}S_{\epsilon^{-\nu}}(r)(\xi)|^2$$
$$\geqslant \sigma_0^2\left(\frac{1}{2}\int_{\mathbb{R}^3}\mu(\mathrm{d}\xi)|\mathcal{F}S_3(r)(\xi)|^2 - \int_{\mathbb{R}^3}\mu(\mathrm{d}\xi)|\mathcal{F}(S_{\epsilon^{-\nu}} - S_3)(r)(\xi)|^2\right).$$

We have that

$$|\mathcal{F}(S_{\epsilon^{-\nu}} - S_3)(r)(\xi)|^2 = |\mathcal{F}\psi_{\epsilon^{-\nu}}(\xi) - 1|^2|\mathcal{F}S_3(r)(\xi)|^2$$
$$\leqslant 4\pi|\mathcal{F}S_3(r)(\xi)|^2|\xi|\epsilon^\nu.$$

Therefore, the lower bounds (26) and (28) yield

$$\|S_{\epsilon^{-\nu}}(\cdot, x-*)\sigma(u(t-\cdot,*))\|^2_{\mathcal{H}_{\epsilon^\delta}}$$
$$\geqslant \sigma_0^2\left(\frac{1}{2}\int_0^{\epsilon^\delta}\mathrm{d}r\int_{\mathbb{R}^3}\mu(\mathrm{d}\xi)|\mathcal{F}S_3(r)(\xi)|^2 - 4\pi\epsilon^\nu\int_0^{\epsilon^\delta}\mathrm{d}r\int_{\mathbb{R}^3}\mu(\mathrm{d}\xi)|\xi||\mathcal{F}S_3(r)(\xi)|^2\right)$$
$$\geqslant \sigma_0^2\left(\frac{1}{2}C_1\epsilon^{3\delta} - C_2\epsilon^{\nu+\delta(2-2\eta)}\right),$$

for some positive constants C_1, C_2. Let ν, $\delta > 0$ be such that

$$\frac{1+2\eta}{\nu} < \frac{1}{\delta}; \tag{21}$$

then

$$\frac{1}{2}C_1\epsilon^{3\delta} - C_2\epsilon^{\nu+\delta(2-2\eta)} \geqslant \tfrac{1}{4}C_1\epsilon^{3\delta}, \qquad \text{for all } \epsilon \leqslant \epsilon_2 := \left(\frac{C_1}{4C_2}\right)^{1/(\nu-\delta(1+2\eta))}.$$

Thus, for any $\epsilon \leqslant \epsilon_2$,

$$\|S_{\epsilon^{-\nu}}(\cdot, x-*)\sigma(u(t-\cdot,*))\|^2_{\mathcal{H}_{\epsilon^\delta}} \geqslant \sigma_0^2\tfrac{1}{4}C_1\epsilon^{3\delta}.$$

Moreover, the condition

$$3\delta < 1 \tag{22}$$

implies

$$6\epsilon < \sigma_0^2\tfrac{C_1}{4}\epsilon^{3\delta}, \qquad \text{for } \epsilon \leqslant \epsilon_3 := \left(\frac{C_1\sigma_0^2}{24}\right)^{1/(1-3\delta)}.$$

Hence, if ν, $\delta > 0$ satisfy (21) and (22) then $P^{2,1}(\epsilon, \delta, \nu) = 0$, for any $\epsilon \leqslant \epsilon_2 \wedge \epsilon_3$.

Consider now the term $P^{2,2}(\epsilon, \delta, \nu)$. By Chebyshev's inequality and (29), we have that

$$
\begin{aligned}
P^{2,2}(\epsilon, \delta, \nu) &\leqslant \epsilon^{-1}\mathrm{E}(\|Z_{t-\cdot,*}(t,x) - S_{\epsilon^{-\nu}}(\cdot, x-*)\sigma(u(t-\cdot,*))\|^2_{\mathcal{H}_{\epsilon^\delta}}) \\
&= \epsilon^{-1}\int_0^{\epsilon^\delta} \mathrm{d}s \int_{\mathbb{R}^3} \mu_s^{\overline{\sigma}}(\mathrm{d}\xi)|\mathcal{F}(S_3(s) - S_{\epsilon^{-\nu}}(s))(\xi)|^2 \\
&\leqslant 4\pi\epsilon^{-1+\nu}\int_0^{\epsilon^\delta} \mathrm{d}s \int_{\mathbb{R}^3} \mu_s^{\overline{\sigma}}(\mathrm{d}\xi)|\xi||\mathcal{F}S_3(s)(\xi)|^2 \\
&\leqslant C\epsilon^{-1+\nu+\delta(2-2\eta)},
\end{aligned}
$$

for some positive constant C, where $\overline{\sigma}$ denotes the process $\{\sigma(u(t-r,x)), (r,x) \in [0,t] \times \mathbb{R}^3\}$.

Thus, $\int_0^{\epsilon_0} \epsilon^{-(1+p)} P^{2,2}(\epsilon, \delta, \nu)\mathrm{d}\epsilon < \infty$ if and only if

$$
-1 - p + \nu + \delta(2-2\eta) > 0. \tag{23}
$$

We finish the proof by analysing the compatibility of the conditions (20)–(23). We recall that $\eta \in (0, \frac{1}{2})$ and $p \in [0, \infty)$ are fixed. Choose $\nu > 0$ such that

$$
\frac{1+2\eta}{3} < \nu. \tag{24}
$$

Then (20)–(23) are equivalent to (23) and

$$
3 < \frac{1}{\delta} < \frac{\frac{3}{2}q(3-2\eta)}{p+q}. \tag{25}
$$

Let $q_0 \geqslant 1$ be such that $3 < \frac{3}{2}q_0(3-2\eta)/(p+q_0)$, or equivalently $2p/(1-2\eta) < q_0$. Then let $\delta_0 > 0$ satisfy (25) with $q = q_0$. For this δ_0, choose $\nu_0 > 0$ sufficiently large such that (23) and (24) hold. The proof of the theorem is complete. □

Appendix

In this appendix we present some of the technical results that have been used in the proofs of Section 4. These provide bounds for integrals involving the Fourier transform of the fundamental solution of the wave equation in any spatial dimension d, denoted here by S_d. The proofs of these results are given in Quer-Sardanyons and Sanz-Solé (2004, Appendix).

We recall that, for every $d \geqslant 1$,

$$
\mathcal{F}S_d(t)(\xi) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}.
$$

For any $\eta \in (0, 1]$, we introduce the assumption

$$
\int_{\mathbb{R}^d} \frac{\mu(\mathrm{d}\xi)}{(1+|\xi|^2)^\eta} < \infty
$$

which we denote by (H_η). We observe that (H_η) is weaker than assumption (b) of Theorem 2 (Lévêque 2001, Proposition 4.4.1).

Assume that (H_η) holds for $\eta = 1$. Then there exist two positive constants C_i, $i = 1, 2$, such that, for any $t \in (0, 1)$,

$$C_1 t^3 \leqslant \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi)|\mathcal{F}S_d(s)(\xi)|^2 \leqslant C_2 t. \tag{26}$$

Suppose that (H_η) holds for some $\eta \in (0, 1)$. Then, there exists a positive constant C_3, such that for any $t \in [0, T]$,

$$\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi)|\mathcal{F}S_d(s)(\xi)|^2 \leqslant C_3 t^{3-2\eta}. \tag{27}$$

Assume that (H_η) holds for some $\eta \in (0, \frac{1}{2})$. Then there exists a positive constant C_4 such that for any $t \in [0, T]$,

$$\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi)|\xi||\mathcal{F}S_d(s)(\xi)|^2 \leqslant C_4 t^{2-2\eta}. \tag{28}$$

Let $\{Z(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ be a predictable L^2-valued process with stationary covariance function and such that $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathrm{E}(|Z(t, x)|^2) < \infty$. Assume that hypothesis (b) of Theorem 2 holds. Then, there exists a positive constant C_5 such that

$$\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mu_s^Z(\mathrm{d}\xi)|\xi||\mathcal{F}S_d(s)(\xi)|^2 \leqslant C_4 t^{2-2\eta}. \tag{29}$$

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References

Dalang, R.C. (1999) Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e's. *Electron. J. Probab.*, **4**(6).

Dalang, R.C. and Frangos, N. (1998) The stochastic wave equation in two spatial dimensions. *Ann. Probab.*, **26**, 187–212.

Lévêque, O. (2001) *Hyperbolic Stochastic Partial Differential Equations Driven by Boundary Noises.* Doctoral thesis no. 2452, École Polytechnique Fédérale de Switzerland, Lausanne, Switzerland.

Márquez-Carreras, D., Mellouk, M. and Sarrà, M. (2001) On stochastic partial differential equations with spatially correlated noise: smoothness of the law. *Stochastic Process. Appl.*, **93**, 269–284.

Millet, A. and Sanz-Solé , M. (1999) A stochastic wave equation in two space dimensions: smoothness of the law. *Ann. Probab.*, **27**, 803–844.

Nualart, D. (1995) *Malliavin Calculus and Related Topics.* New York: Springer-Verlag.

Nualart, D. (1998) Analysis on Wiener space and anticipating stochastic calculus. In P. Bernard (ed.), *Lecture Notes in Probability Theory and Statistics: École d'Été de Probabilités de Saint Flour XXV–1995*, Lecture Notes in Math. 1690, pp. 863–901. Berlin: Springer-Verlag.

Quer-Sardanyons, L. and Sanz-Solé , M. (2004) Absolute continuity of the law of the solution to the three-dimensional stochastic wave equation. *J. Funct. Anal.*, **206**(1), 1–32.

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Appendix C

Lattice approximation for a stochastic wave equation

Lattice approximation for a stochastic wave equation

by

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Abstract: We study an approximation scheme for a nonlinear stochastic wave equation in one dimensional space, driven by a space-time white noise. The sequence of approximations is obtained by discretisation of the Laplacian operator. We prove L^p-convergence to the solution of the equation and determine the rate of convergence. As a corollary, almost sure convergence, uniformly in time and space, is also obtained. Finally, the speed of convergence is tested numerically.

Key words: Discretisation schemes, stochastic partial differential equations, wave equation.

AMS Subject Classification: 60H35, 60H15.

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C.1 Introduction

Nowadays, stochastic partial differential equations are accepted as being a very suitable framework to understand complex phenomena. An aspect of the development of the theory consists in seeking methods of finding solutions numerically. Some of them are inspired on those used in the deterministic context. Let us mention for instance, finite differences ([GN97], [GN95], [Gyö98b], [Gyö99], [Yoo00]), finite elements ([GP88]), splitting up methods ([BGR92], [BGR90], [IR00], [GK03]), Garlekin approximations ([GK96]) and time discretisation ([Hau03], [Pri01]). Others are more genuine stochastic, based on the Wiener chaos decomposition ([Lot96], [LMR97]) or on truncations of the Fourier expansion of the noise ([Sha03], [Sha99]). We refer the reader to [Gyö98a] for a survey of some of these methods, together with a more extensive list of references.

Lattice approximation schemes for parabolic spde's in one spatial dimension, developed in [Gyö98b], [Gyö99], have been the starting point of several further investigations. In [MM03], lattice schemes for parabolic spde's in any spatial dimension are considered and the influence of the particular covariance density of the noise given by Riesz kernels is studied. A class of parabolic evolution equations on Banach spaces with monotone operators are analized in [GM04]. In [GM], a finite difference approximation scheme for an elliptic spde in dimension $d = 1, 2, 3$ is studied. The results show how much the behaviour of this kind of approximations depends on the differential operator driving the spde and are one of the very few attempts of looking beyond the parabolic case. Let us also mention [DZ02] for some results on numerical approximations for elliptic equations.

In this paper we consider strong approximations for a stochastic wave equation in spatial dimension one by a sequence obtained substituting the derivatives in space by finite differences. This is a first step towards the analysis of lattice approximations for hyperbolic spde's. In fact, to our best knowledge there are very few results on numerical approximation for the stochastic wave equation ([MPW03]).

We consider the non-linear stochastic wave equation

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x,u(t,x)) + \sigma(t,x,u(t,x))\frac{\partial^2 W}{\partial t \partial x}(t,x),$$

$t > 0$, $x \in (0,1)$, with initial conditions

$$u(0,x) = u_0,\ \frac{\partial u}{\partial t}(0,x) = v_0,\ x \in (0,1),$$

and Dirichlet boundary conditions

$$u(t,0) = u(t,1) = 0,\ t > 0.$$

We assume that u_0 and v_0 are functions defined on $[0,1]$, u_0 vanishes at $x = 0$ and $x = 1$, and that W is the Brownian sheet on $\mathbb{R}_+ \times [0,1]$; that is, $\{W(t,x), (t,x) \in \mathbb{R}_+ \times [0,1]\}$ is a Gaussian stochastic process defined on some probability space $(\Omega, \mathcal{F}, P)$ with mean zero and covariance function

$$E(W(t,x)W(s,y)) = (s \wedge t)(x \wedge y).$$

Following a classical approach to spde's, we attach a rigorous meaning to the formal problem described above by means of the evolution formulation, as follows:

$$\begin{aligned} u(t,x) = &\int_0^1 G(t,x,y)v_0(y)dy + \frac{\partial}{\partial t}\left(\int_0^1 G(t,x,y)u_0(y)dy\right) \\ &+ \int_0^t \int_0^1 G(t-s,x,y)\sigma(s,y,u(s,y))W(ds,dy) \\ &+ \int_0^t \int_0^1 G(t-s,x,y)f(s,y,u(s,y))dsdy, \end{aligned} \tag{C.1}$$

$t \geq 0$, $x \in (0,1)$, where G is the Green function of the wave equation with homogeneous Dirichlet boundary conditions.

For any $n \geq 1$, we fix the spatial grid $x_k = \frac{k}{n}$, $k = 1, \cdots, n-1$, and consider the system of stochastic differential equations obtained by substituting the Laplacian by its discretisation and freezing the evolution equation (C.1) at the points of the grid (see (C.7)). This provides an implicit finite dimensional scheme. By linear interpolation, we obtain a sequence of evolution equations which is proved to converge in any $L^p(\Omega)$, uniformly in t, x, to the solution of (C.1) with a given rate of convergence (see Theorem C.3.1).

In comparison with parabolic examples, the rate of convergence differs substantially from the Hölder continuity order of the sample paths of the solution. Indeed, assuming for simplicity that the initial conditions vanishes, sample paths are jointly Hölder continuous in (t,x) of order $\alpha < \frac{1}{2}$, while the rate of convergence is of order $\rho < \frac{1}{3}$. We have checked with a numerical analysis that one cannot expect better results.

The paper is organized as follows. In the second section, we study the Hölder continuity of the sample paths of equation (C.1). Section three is devoted to prove the main result on the approximation scheme. Finally, in an appendix we analyze numerically the optimality of the result proved in section three.

C.2 Some properties of the solution

In this section we prove some properties of the solution of Equation (C.1). In particular, we analyse sufficient conditions on the initial data ensuring joint Hölder continuity, in time and in space, of the sample paths of the solution.

We fix a finite time horizon T and assume that the coefficients f, σ are real-valued functions defined on $[0,T] \times [0,1] \times \mathbb{R}$, satisfying the following conditions:

(L)

$$\sup_{t\in[0,T]} \big(|f(t,x,z)-f(t,y,v)|+|\sigma(t,x,z)-\sigma(t,y,v)|\big) \le C(|x-y|+|z-v|),$$

(LG)

$$\sup_{(t,x)\in[0,T]\times[0,1]} \big(|f(t,x,z)|+|\sigma(t,x,z)|\big) \le C(1+|z|),$$

for every $x, y \in [0,1]$ and $z, v \in \mathbb{R}$.

Along the paper we shall use the expansion of the Green function

$$G(t,x,y) = \sum_{j=1}^{\infty} \frac{\sin(j\pi t)}{j\pi} \varphi_j(x)\varphi_j(y), \tag{C.2}$$

where $\varphi_j(x) = \sqrt{2}\sin(j\pi x)$, $j \ge 1$, is a complete orthonormal system of $L^2([0,1])$ (see for instance [Duf03], pag. 94).

Assume that u_0, v_0 belong to $L^2([0,1])$. By the classical approach to the deterministic wave equation on $[0,1]$ with Dirichlet boundary conditions, we know that

$$\frac{\partial}{\partial t}\left(\int_0^1 G(t,x,y)u_0(y)dy\right) = \sum_{j=1}^{\infty} \langle u_0, \varphi_j\rangle \cos(j\pi t)\varphi_j(x),$$

where $\langle \cdot,\cdot\rangle$ stands for the usual scalar product in $L^2([0,1])$ (see [Joh82], pag. 44).

Let $\mathcal{F}_t$, $t \in [0,T]$, be the σ-field generated by the random variables $W(s,x)$, $s \in [0,t]$, $x \in [0,1]$. Assume that the process $u = \{u(t,x), (t,x) \in [0,T] \times [0,1]\}$ in (C.1) is $\mathcal{F}_t$ -adapted and satisfies $\sup_{(t,x)\in[0,T]\times\mathbb{R}} E\big(|u(t,x)|^2\big) < \infty$, then all terms in the right hand-side of Equation (C.1) are well defined, when choosing as stochastic integral the extension of Itô's integral with respect to martingale measures developed by Walsh in [Wal86].

By the standard technique based on Picard's iterations scheme, it is not difficult to prove the existence and uniqueness of a measurable, $\mathcal{F}_t$-adapted stochastic process $\{u(t,x), (t,x) \in [0,T] \times [0,1]\}$ such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} E\big(|u(t,x)|^2\big) < \infty$$

and satisfying (C.1). Existence only requires the condition (LG), while uniqueness needs (L). We refer the reader to [CN88] (see also [MSS99]) for a similar result on different types of equations that can be easily adapted to Equation (C.1).

For a function $g : [0,1] \to \mathbb{R}$ and $\alpha \in \mathbb{R}$, we define

$$\|g\|_{\alpha,2} := \left(\sum_{j=1}^{\infty} (1+j^2)^{\alpha} |\langle g, \varphi_j \rangle|^2 \right)^{\frac{1}{2}}$$

and denote by $H^{\alpha,2}([0,1])$ the set of functions $g : [0,1] \to \mathbb{R}$ such that $\|g\|_{\alpha,2} < \infty$. Notice that $H^{\alpha,2}([0,1])$ is a subspace of the fractional Sobolev space of fractional differential order α and integrability order $p = 2$ (see [Tri92]).

The next result gives additional information on the existence and uniqueness of solution of Equation (C.1).

Proposition C.2.1. *Assume that $v_0 \in H^{\beta,2}([0,1])$ for some $\beta > -\frac{1}{2}$ and $u_0 \in H^{\alpha,2}([0,1])$ for some $\alpha > \frac{1}{2}$; suppose also that the coefficients σ and f satisfy condition (LG). Then, for every $p \geq 1$,*

$$\sup_{(t,x)\in[0,T]\times[0,1]} E(|u(t,x)|^p) < +\infty.$$

Proof. Consider the decomposition

$$E(|u(t,x)|^p) \leq C \sum_{k=1}^{4} J_k(t,x),$$

with

$$\begin{aligned}
J_1(t,x) &= \left| \int_0^1 G(t,x,y) v_0(y) dy \right|^p, \\
J_2(t,x) &= \left| \sum_{j=1}^{\infty} \langle u_0, \varphi_j \rangle \cos(j\pi t) \varphi_j(x) \right|^p, \\
J_3(t,x) &= E\left(\left| \int_0^t \int_0^1 G(t-s,x,y) \sigma(s,y,u(s,y)) W(ds,dy) \right|^p \right), \\
J_4(t,x) &= E\left(\left| \int_0^t \int_0^1 G(t-s,x,y) f(s,y,u(s,y)) ds dy \right|^p \right).
\end{aligned}$$

Since

$$\int_0^1 G(t,x,y) v_0(y) dy = \sum_{j=1}^{\infty} \frac{\sin(j\pi t)}{j\pi} \langle v_0, \varphi_j \rangle \varphi_j(x), \tag{C.3}$$

Cauchy-Schwarz inequality and the assumptions on v_0 yield

$$\begin{aligned}\sup_{(t,x)\in[0,T]\times[0,1]} J_1(t,x) &\leq \left(\sum_{j=1}^{\infty} j^{2\beta}|\langle v_0,\varphi_j\rangle|^2\right)^{\frac{p}{2}} \\ &\times \sup_{(t,x)\in[0,T]\times[0,1]} \left(\sum_{j=1}^{\infty} \frac{\sin^2(j\pi t)}{j^2\pi^2}|\varphi_j(x)|^2 j^{-2\beta}\right)^{\frac{p}{2}} \\ &\leq C\|v_0\|_{\beta,2}^p \left(\sum_{j=1}^{\infty} j^{-2(1+\beta)}\right)^{\frac{p}{2}} \leq C.\end{aligned}$$

Using similar arguments, we obtain that

$$\sup_{(t,x)\in[0,T]\times[0,1]} J_2(t,x) \leq C\|u_0\|_{\alpha,2}^p \left(\sum_{j=1}^{\infty} j^{-2\alpha}\right)^{\frac{p}{2}} \leq C.$$

Owing to the expansion (C.2) and the fact that $(\varphi_j, j \geq 1)$ is an orthonormal system of $L^2([0,1])$, it follows that

$$\int_0^1 |G(t,x,y)|^2 dy = \sum_{j=1}^{\infty} \frac{\sin^2(j\pi t)}{j^2\pi^2}\varphi_j^2(x) \leq C\sum_{j=1}^{\infty}\frac{1}{j^2} \leq C,$$

which therefore implies that

$$\sup_{(t,x)\in[0,T]\times[0,1]} \int_0^1 |G(t,x,y)|^2 dy < +\infty. \tag{C.4}$$

Hence, the measure on $[0,T]\times[0,1]$ defined by $\mu_{t,x}(ds,dy) = |G(t-s,x,y)|^2 dsdy$ is finite, uniformly with respect to $(t,x) \in [0,T]\times[0,1]$.
Applying Burkholder's inequality and then Hölder's inequality with respect to $\mu_{t,x}(ds,dy)$ yield

$$\begin{aligned}\sup_{x\in[0,1]} J_3(t,x) &\leq C \sup_{x\in[0,1]} E\left(\left|\int_0^t\int_0^1 |G(t-s,x,y)|^2|\sigma(s,y,u(s,y))|^2 dsdy\right|^{\frac{p}{2}}\right) \\ &\leq C \sup_{x\in[0,1]} \Big(\int_0^t\int_0^1 |G(t-s,x,y)|^2 E(|\sigma(s,y,u(s,y))|^p)dsdy\Big) \\ &\leq C\left(1 + \int_0^t \sup_{x\in[0,1]} E(|u(s,x)|^p)ds\right),\end{aligned}$$

where we have used the bound (C.4) and condition (LG) on σ.

For the term $J_4(t,x)$, we apply Hölder's inequality with respect to $\mu_{t,x}(ds,dy)$ and the property (LG) on f. We obtain

$$\sup_{x\in[0,1]} J_4(t,x) \le C\Big(1+\int_0^t \sup_{x\in[0,1]} E(|u(s,x)|^p)ds\Big).$$

Bringing together the above estimates, we obtain

$$\sup_{x\in[0,1]} E(|u(t,x)|^p) \le C\Big(1+\int_0^t \sup_{x\in[0,1]} E(|u(s,x)|^p)ds\Big),$$

with a constant C independent of t. We conclude applying Gronwall's lemma. □

We next study the Hölder property of the trajectories of the solution of equation (C.1).

Proposition C.2.2. *We assume that $v_0 \in H^{\beta,2}([0,1])$, for some $\beta > -\frac{1}{2}$, $u_0 \in H^{\alpha,2}([0,1])$, for some $\alpha > \frac{1}{2}$, and that the coefficients σ and f satisfy conditions (LG) and (L). Then, for all $p \ge 1$ there exists a positive constant C, depending on α, β, such that*

$$\begin{aligned} E(|u(s,x)-u(t,y)|^{2p}) \le C(&|t-s|^{p(1+2\beta)} + |x-y|^{p(1+2\beta)} \\ &+ |t-s|^{p(2\alpha-1)} + |x-y|^{p(2\alpha-1)} \\ &+ |t-s|^p + |x-y|^p), \end{aligned} \tag{C.5}$$

for every $s,t \in [0,T]$ and $x,y \in [0,1]$. Consequently, the process u has a.s. Hölder-continuous sample paths of order δ, for all $\delta \in (0,\delta_0)$, where $\delta_0 = (\frac{1}{2}+\beta)\wedge(\alpha-\frac{1}{2})\wedge\frac{1}{2}$.

Proof. Assume that $s \le t$ and $y \le x$. We set

$$\begin{aligned} H(t,x) &= \int_0^t \int_0^1 G(t-s,x,z)\sigma(s,z,u(s,z))W(ds,dz), \\ F(t,x) &= \int_0^t \int_0^1 G(t-s,x,z)f(s,z,u(s,z))dsdz. \end{aligned}$$

Thus we have the decomposition

$$E(|u(s,x)-u(t,y)|^{2p}) \le C\sum_{k=1}^4 J_k(s,t,x,y),$$

where

$$
\begin{aligned}
J_1(s,t,x,y) &= \left| \int_0^1 (G(s,x,z) - G(t,y,z)) v_0(z) dz \right|^{2p}, \\
J_2(s,t,x,y) &= \left| \sum_{j=1}^{\infty} \langle u_0, \varphi_j \rangle (\cos(j\pi s)\varphi_j(x) - \cos(j\pi t)\varphi_j(y)) \right|^{2p}, \\
J_3(s,t,x,y) &= E(|H(s,x) - H(t,y)|^{2p}), \\
J_4(s,t,x,y) &= E(|F(s,x) - F(t,y)|^{2p}).
\end{aligned}
$$

The identity (C.3) and Cauchy-Schwarz inequality yield

$$
\begin{aligned}
J_1(s,t,x,y) &= \left| \sum_{j=1}^{\infty} \left(\frac{\sin(j\pi s)}{j\pi} \varphi_j(x) - \frac{\sin(j\pi t)}{j\pi} \varphi_j(y) \right) \langle v_0, \varphi_j \rangle \right|^{2p} \\
&\leq \|v_0\|_{\beta,2}^p \left(\sum_{j=1}^{\infty} \left(\frac{\sin(j\pi s)}{j\pi} \varphi_j(x) - \frac{\sin(j\pi t)}{j\pi} \varphi_j(y) \right)^2 j^{-2\beta} \right)^p .
\end{aligned}
$$

Hence $J_1(s,t,x,y) \leq C(A_1(s,t,x,y) + A_2(s,t,x,y))$, where

$$
\begin{aligned}
A_1(s,t,x,y) &= \left(\sum_{j=1}^{\infty} \left(\frac{\sin(j\pi s)}{j\pi} - \frac{\sin(j\pi t)}{j\pi} \right)^2 |\varphi_j(x)|^2 j^{-2\beta} \right)^p, \\
A_2(s,t,x,y) &= \left(\sum_{j=1}^{\infty} (\varphi_j(x) - \varphi_j(y))^2 \left| \frac{\sin(j\pi t)}{j\pi} \right|^2 j^{-2\beta} \right)^p .
\end{aligned}
$$

The mean value theorem yields

$$
A_1(s,t,x,y) \leq C \left(\sum_{j=1}^{\infty} j^{-2(1+\beta)} \left(1 \wedge j^2 (t-s)^2\right) \right)^p .
$$

If $\beta > \frac{1}{2}$, we clearly have $A_1(s,t,x,y) \leq C(t-s)^{2p}$, for some positive constant C depending on β.

Assume now that $\beta \in (-\frac{1}{2}, \frac{1}{2}]$. Obviously,

$$
\begin{aligned}
&\left(\sum_{j=1}^{\infty} j^{-2(1+\beta)} \left(1 \wedge j^2 (t-s)^2\right) \right)^p \\
&\leq C \left((t-s)^2 \sum_{j=1}^{N} j^{-2\beta} + \sum_{j=N+1}^{\infty} j^{-2(1+\beta)} \right)^p ,
\end{aligned}
$$

where $N = \left[\frac{1}{t-s}\right]$ and $[\cdot]$ stands for the integer value.

Since

$$\sum_{j=1}^{N} j^{-2\beta} \le C N^{-2\beta+1}, \quad \sum_{j=N+1}^{\infty} j^{-2(1+\beta)} \le C(N+1)^{-1-2\beta},$$

and $1 + 2\beta \le 2$, if $\beta \in (-\frac{1}{2}, \frac{1}{2}]$, we obtain

$$A_1(s,t,x,y) \le C(t-s)^{p(1+2\beta)}.$$

Analogously, $A_2(s,t,x,y) \le C(x-y)^{p(1+2\beta)}$. Thus,

$$J_1(s,t,x,y) \le C\big((t-s)^{p(1+2\beta)} + (x-y)^{p(1+2\beta)}\big).$$

Let us now deal with the term J_2. By Cauchy-Schwarz inequality,

$$J_2(s,t,x,y) \le C\|u_0\|_{\alpha,2}^{p} \left(\sum_{j=1}^{\infty} j^{-2\alpha} (\cos(j\pi s)\varphi_j(x) - \cos(j\pi t)\varphi_j(y))^2 \right)^{p}.$$

Therefore $J_2(s,t,x,y) \le C(B_1(s,t,x,y) + B_2(s,t,x,y))$, with

$$B_1(s,t,x,y) = \left(\sum_{j=1}^{\infty} j^{-2\alpha} (\cos(j\pi s) - \cos(j\pi t))^2 \right)^{p},$$

$$B_2(s,t,x,y) = \left(\sum_{j=1}^{\infty} j^{-2\alpha} (\varphi_j(x) - \varphi_j(y))^2 \right)^{p}.$$

The same arguments used in the analysis of the terms $A_1(s,t,x,y)$ and $A_2(s,t,x,y)$ yield

$$J_2(s,t,x,y) \le C\big((x-y)^{p(2\alpha-1)} + (t-s)^{p(2\alpha-1)}\big).$$

Let us now study the stochastic integral term by considering the decomposition

$$J_3(s,t,x,y) \le C(D_1(s,t,x) + D_2(t,x,y)),$$

with

$$D_1(s,t,x) = E(|H(s,x) - H(t,x)|^{2p}),$$
$$D_2(t,x,y) = E(|H(t,x) - H(t,y)|^{2p}).$$

Set $h(r,z) = \sigma(r,z,u(r,z))$, $r \in [0,T]$, $z \in \mathbb{R}$. Observe that the assumption (LG) and Proposition C.2.1 yield

$$\sup_{(r,z)\in[0,T]\times[0,1]} E(|h(r,z)|^q) < C, \tag{C.6}$$

for any $q \in [2, \infty)$.

Clearly, $D_1(s,t,x) \le C(D_{11}(s,t,x) + D_{12}(s,t,x))$, with

$$D_{11}(s,t,x) = E\left(\left|\int_0^s \int_0^1 [G(t-r,x,z) - G(s-r,x,z)]h(r,z)W(dr,dz)\right|^{2p}\right),$$

$$D_{12}(s,t,x) = E\left(\left|\int_s^t \int_0^1 G(t-r,x,z)h(r,z)W(dr,dz)\right|^{2p}\right).$$

We apply first Burkholder's inequality and then Hölder's inequality with respect to the finite measure on $[0,T] \times [0,1]$ defined by $|G(t-r,x,y) - G(s-r,x,y)|^2 dr dy$. By virtue of (C.6) we obtain

$$\begin{aligned} D_{11}(s,t,x) &\le CE\left(\left|\int_0^s \int_0^1 |G(t-r,x,z) - G(s-r,x,z)|^2 |h(r,z)|^2 dz dr\right|^p\right) \\ &\le C\left(\int_0^s \int_0^1 |G(t-r,x,z) - G(s-r,x,z)|^2 dz dr\right)^{p-1} \\ &\quad \times \left(\int_0^s \int_0^1 |G(t-r,x,z) - G(s-r,x,z)|^2 E(|h(r,z)|^{2p}) dz dr\right)^p \\ &\le C\left(\int_0^s \int_0^1 |G(t-r,x,z) - G(s-r,x,z)|^2 dz dr\right)^p. \end{aligned}$$

Replace the Green function G by its expansion given in (C.2). Since the family $(\varphi_j, j \ge 1)$ is orthonormal in $L^2([0,1])$, we obtain

$$\begin{aligned} D_{11}(s,t,x) &\le C\left(\int_0^s \sum_{j=1}^{\infty} \frac{1}{j^2} |\sin(j\pi(t-r)) - \sin(j\pi(s-r))|^2 dr\right)^p \\ &\le C\left(\sum_{j=1}^{\infty} j^{-2}\left(1 \wedge j^2(t-s)^2\right)\right)^p. \end{aligned}$$

Therefore, $\sup_{x\in[0,1]} D_{11}(s,t,x) \le C(t-s)^p$.

We can obtain an upper bound for $D_{12}(s,t,x)$ by similar arguments, yielding

$$\sup_{x\in[0,1]} D_{12}(s,t,x) \le C\left(\int_s^t \left(\sum_{j=1}^{\infty} \frac{1}{j^2}\sin^2(j\pi(t-r))\right) dr\right)^p \le C(t-s)^p.$$

Thus, $\sup_{x\in[0,1]} D_1(s,t,x) \le C(t-s)^p$. Similarly, one checks that

$$\sup_{t\in[0,T]} D_2(t,x,y) \le C(x-y)^p.$$

Summarising, we have proved

$$J_3(s,t,x,y) \le C\big((t-s)^p + (x-y)^p\big).$$

With the same type of arguments, but less effort one can check that

$$J_4(s,t,x,y) \le C\big((t-s)^p + (x-y)^p\big).$$

We leave the details to the reader. This finishes the proof of the upper bound (C.5).

The last statement of the proposition follows from Kolmogorov's continuity criterion. □

C.3 Strong approximations

This section is devoted to the proof of the main result of the paper. We start with the description of the discretisation of the stochastic wave equation and end up with a result on the rate of the convergence in $L^p(\Omega)$ of the approximations. As a by-product we also obtain almost sure convergence.

C.3.1 Discretisation of the one-dimensional stochastic wave equation

Throughout this section we shall assume that u_0, v_0 belong to $L^2([0,1])$.

One can express the stochastic boundary value problem we are studying in this paper by means of a system of two first order spde's, as follows:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = v(t,x) \\ \frac{\partial v}{\partial t}(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + f(t,x,u(t,x)) + \sigma(t,x,u(t,x))\frac{\partial^2}{\partial t \partial x}W(t,x), \end{cases}$$

$t > 0$, $x \in (0,1)$, with initial conditions

$$u(0,x) = u_0(x), \;\; v(0,x) = v_0(x), \; x \in (0,1),$$

and Dirichlet boundary conditions

$$u(t,0) = u(t,1) = 0, \; t > 0.$$

For any integer $n \ge 1$, set $x_k = \frac{k}{n}$, $k = 1, \ldots, n-1$. Consider the system of stochastic differential equations

$$\begin{cases} du^n(t,x_k) &=& v^n(t,x_k)dt \\ dv^n(t,x_k) &=& n^2(u^n(t,x_{k+1}) - 2u^n(t,x_k) + u^n(t,x_{k-1}))dt \\ && +f(t,x_k,u^n(t,x_k))dt \\ && +n\sigma(t,x_k,u^n(t,x_k))d\left(W(t,x_{k+1}) - W(t,x_k)\right), \end{cases} \tag{C.7}$$

with initial conditions

$$u^n(0, x_k) = u_0(x_k),\ v^n(0, x_k) = v_0(x_k),$$

where

$$u_0(x_k) = \sum_{j=1}^{\infty} \langle u_0, \varphi_j \rangle \varphi_j(x_k), \quad v_0(x_k) = \sum_{j=1}^{\infty} \langle v_0, \varphi_j \rangle \varphi_j(x_k),$$

$k = 1, \ldots, n-1$.

Conditions (LG) and (L) on the coefficients σ and f guarantee the existence and uniqueness of solution to the above system of equations.

We would like to write the stochastic system (C.7) in an evolution-like form. Let us introduce a simplified notation by setting

$$\begin{aligned} u_k^n(t) &= u^n(t, x_k), \\ v_k^n(t) &= v^n(t, x_k), \\ W_k^n(t) &= \sqrt{n}\left(W(t, x_{k+1}) - W(t, x_k)\right), \end{aligned}$$

$k = 1, \ldots, n-1$. Notice that $W^n(t) = \big(W_1^n(t), \ldots, W_{n-1}^n(t)\big)$ is a $(n-1)$-dimensional standard Brownian motion.

Then (C.7) is equivalent to

$$\left\{ \begin{array}{lcl} du_k^n(t) & = & v_k^n(t)dt \\ dv_k^n(t) & = & n^2 \sum_{i=1}^{n-1} d_{ki} u_i^n(t)dt + f(t, x_k, u_k^n(t))dt \\ & & +\sqrt{n}\sigma(t, x_k, u_k^n(t))dW_k^n(t), \end{array} \right. \tag{C.8}$$

with $u_k^n(0) = u_0(x_k)$, $v_k^n(0) = v_0(x_k)$, $k = 1, \ldots, n-1$, where $d_{kk} = -2$, $d_{ki} = 1$ if $|k-i| = 1$ and $d_{ki} = 0$ if $|k-i| > 1$.

In the sequel we denote by D the square $(n-1)$-dimensional matrix whose entries are d_{ik}.

The system (C.8) can be written as the $\mathbb{R}^{2(n-1)}$-valued stochastic differential equation

$$d\underline{w}^n(t) = \big(A^n \underline{w}^n(t) + F(\underline{w}^n(t))\big)dt + \Sigma(\underline{w}^n(t))d\underline{W}^n(t),$$

$\underline{w}^n(0) = (\underline{u}^n(0), \underline{v}^n(0))^*$, with the following notations: $\underline{u}^n(t) = \big(u_k^n(t), k = 1, \ldots, n-1\big)$, $\underline{v}^n(t) = \big(v_k^n(t), k = 1, \ldots, n-1\big)$, $\underline{w}^n(t) = \big(\underline{u}^n(t), \underline{v}^n(t)\big)^*$, where the superscript $*$ means the transpose of the vector, the drift F is given by

$$F(\underline{w}^n(t)) = (0\overset{n-1}{\overbrace{\cdots}}0, f(t, x_1, u_1^n(t)), \ldots, f(t, x_{n-1}, u_{n-1}^n(t)))^*,$$

and $\underline{W}^n = (Z, W^n)^*$, with Z a $(n-1)$-dimensional Brownian motion independent of W^n. Finally

$$A^n = \begin{pmatrix} 0 & I_{n-1} \\ n^2 D & 0 \end{pmatrix} \quad \text{and} \quad \Sigma(\underline{w}^n(t)) = \sqrt{n} \begin{pmatrix} 0 & 0 \\ 0 & B_\sigma \end{pmatrix},$$

where I_{n-1} denotes the identity matrix in $\mathbb{R}^{n-1}$ and B_σ is the diagonal matrix of size $n-1$ with diagonal elements $\sigma(t, x_k, u_k^n(t))$, for $k = 1, \dots, n-1$.

Next we apply Itô's formula (see, for instance, [KS91], Theorem 3.6) to the function $f : \mathbb{R}_+ \times \mathbb{R}^{2(n-1)} \to \mathbb{R}$ given by $f(t,x) = e^{-tA^n}x$, and the semimartingale $\{\underline{w}^n(t), t \in [0,T]\}$. Then,

$$\begin{aligned} e^{-tA^n}\underline{w}^n(t) &= \underline{w}^n(0) + \int_0^t -A^n e^{-sA^n}\underline{w}^n(s)ds + \int_0^t e^{-sA^n} d\underline{w}^n(s) \\ &= \underline{w}^n(0) + \int_0^t e^{-sA^n} F(\underline{w}^n(s))ds + \int_0^t e^{-sA^n} d\underline{W}^n(s). \end{aligned}$$

Therefore we obtain the following stochastic differential equation for the process $\underline{w}$:

$$\underline{w}^n(t) = e^{tA^n}\underline{w}^n(0) + \int_0^t e^{(t-s)A^n} F(\underline{w}^n(s))ds + \int_0^t e^{(t-s)A^n}\Sigma(\underline{w}^n(s))d\underline{W}^n(s). \tag{C.9}$$

The aim is to compute the exponential matrix e^{rA^n}, $r \geq 0$, and then obtain a system of stochastic differential equations for the first $n-1$ components of the vector $\underline{w}(t)$, that is, $u_k^n(t)$, $k = 1, \dots, n-1$. For this we use the fact that, as mentioned in [Gyö98b], p. 4 , the $(n-1)$-dimensional vectors

$$e_j = \left(\sqrt{\frac{2}{n}} \sin\left(j\frac{k}{n}\pi \right), k = 1, \dots, n-1 \right), \tag{C.10}$$

$j = 1, \dots, n-1$ are an orthonormal basis of $\mathbb{R}^{n-1}$. In addition, they are eigenvectors of $n^2 D$ with eigenvalues

$$\lambda_j^n = -4n^2 \sin^2\left(\frac{j}{2n}\pi \right) = -j^2\pi^2 c_j^n,$$

respectively, where

$$c_j^n = \frac{\sin^2\left(\frac{j\pi}{2n}\right)}{\left(\frac{j\pi}{2n}\right)^2}.$$

We consider the function $g : (0,1) \to \mathbb{R}_+$ defined by

$$g(x) = \frac{\sin^2(x\frac{\pi}{2})}{\left(x\frac{\pi}{2}\right)^2}.$$

It turns out that this function is strictly decreasing on the interval $(0,1)$. This implies that we have the following upper and lower bounds for the constant c_j^n:

$$\frac{4}{\pi^2} \leq c_j^n \leq 1,$$

for any $j = 1, \dots, n-1$ and $n \geq 1$.

With the above ingredients, we compute the matrix e^{rA^n}, as follows. First we notice that, by the definition of the matrix A^n, we have that

$$(A^n)^{2k} = \begin{pmatrix} n^{2k}D^k & 0 \\ 0 & n^{2k}D^k \end{pmatrix}, \quad (A^n)^{2k+1} \begin{pmatrix} 0 & n^{2k}D^k \\ n^{2(k+1)}D^{k+1} & 0 \end{pmatrix}, \ k \in \mathbb{N} \cup \{0\}.$$

This implies that

$$e^{tA^n} = \begin{pmatrix} E_1(t,n) & E_2(t,n) \\ E_3(t,n) & E_1(t,n) \end{pmatrix},$$

where

$$\begin{aligned} E_1(t,n) &= \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} n^{2k} D^k, \\ E_2(t,n) &= \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} n^{2k} D^k, \\ E_3(t,n) &= \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} n^{2k} D^{k+1}. \end{aligned}$$

Then, from Equation (C.9) and the definition of F and Σ,

$$\begin{aligned} \underline{u}^n(t) =& \underline{u}^n(0) + \int_0^t E_2(t-s,n) \Big(f(u_1^n(s)), \dots, f(u_{n-1}^n(s)) \Big)^* ds \\ &+ \int_0^t E_2(t-s,n) \Big(dW_1^n(s), \dots, dW_{n-1}^n(s) \Big)^*. \end{aligned}$$

Using the fact that the vectors given in (C.10) are eigenvectors of $n^2 D$ with eigenvalues λ_j^n, $j = 1, \dots, n-1$, we obtain that, for instance, each component of the stochastic integral in the right hand-side of the above equality read

$$\int_0^t \frac{1}{\sqrt{n}} \sum_{j=1}^{n-1} \sum_{l=1}^{n-1} \Lambda_j^n(t-s) \varphi_j(x_k) \varphi_j(x_l) \sigma(s, x_l, u_l^n(s)) dW_l^n(s),$$

where

$$\Lambda_j^n(t) = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} (\lambda_j^n)^k = \frac{\sin\left(t\sqrt{-\lambda_j^n}\right)}{\sqrt{-\lambda_j^n}} = \frac{\sin\left(j\pi t\sqrt{c_j^n}\right)}{j\pi\sqrt{c_j^n}},$$

for any $k = 1, \dots, n-1$. Thus, the processes $\{u_k^n(t), t \in [0,T]\}$, $k = 1, \dots, n-1$, satisfy the following system of stochastic differential equations:

$$\begin{aligned}
u_k^n(t) &= \frac{1}{n} \sum_{l=1}^{n-1} \sum_{j=1}^{n-1} \cos\left(j\pi t\sqrt{c_j^n}\right) \varphi_j(x_k)\varphi_j(x_l)u_0(x_l) \\
&+ \frac{1}{n} \sum_{l=1}^{n-1} \sum_{j=1}^{n-1} \frac{\sin\left(j\pi t\sqrt{c_j^n}\right)}{j\pi\sqrt{c_j^n}} \varphi_j(x_k)\varphi_j(x_l)v_0(x_l) \\
&+ \int_0^t \frac{1}{n} \sum_{l=1}^{n-1} \sum_{j=1}^{n-1} \frac{\sin\left(j\pi(t-s)\sqrt{c_j^n}\right)}{j\pi\sqrt{c_j^n}} \varphi_j(x_k)\varphi_j(x_l) f(s, x_l, u_l^n(s)) ds \\
&+ \int_0^t \frac{1}{n} \sum_{l=1}^{n-1} \sum_{j=1}^{n-1} \frac{\sin\left(j\pi(t-s)\sqrt{c_j^n}\right)}{j\pi\sqrt{c_j^n}} \varphi_j(x_k)\varphi_j(x_l) \sigma(s, x_l, u_l^n(s)) dW_l^n(s),
\end{aligned}$$

where $\varphi_j(x) = \sqrt{2}\sin(j\pi x)$.

Set

$$G^n(t,x,y) = \sum_{j=1}^{n-1} \frac{\sin\left(j\pi t\sqrt{c_j^n}\right)}{j\pi\sqrt{c_j^n}} \varphi_j^n(x)\varphi_j(\kappa_n(y)),$$

where $\kappa_n(y) = [ny]/n$, $\varphi_j^n(x) = \varphi_j(x_l)$ for $x = x_l$ and

$$\varphi_j^n(x) = \varphi_j(x_l) + (nx - l)\left(\varphi_j(x_{l+1}) - \varphi_j(x_l)\right)$$

for $x \in (x_l, x_{l+1})$.

We extend the definition of $u_k^n(t) = u^n(t, x_k)$ to any $x \in [0,1]$ by linear interpolation, by setting

$$u^n(t,x) = u^n(t,x_k) + (nx-k)\left(u^n(t,x_{k+1}) - u^n(t,x_k)\right),$$

if $x \in [x_k, x_{k+1})$.

The sequence of processes $u^n = \{u^n(t,x), (t,x) \in [0,T] \times (0,1)\}$, $n \geq 1$, is the approximation scheme of the solution of the stochastic wave equation we are considering in this paper. Notice that u^n satisfies the evolution equation

$$
\begin{aligned}
u^n(t,x) = &\int_0^1 G^n(t,x,y)v_0(\kappa_n(y))dy \\
&+ \frac{\partial}{\partial t}\left(\int_0^1 G^n(t,x,y)u_0(\kappa_n(y))dy\right) \\
&+ \int_0^t\int_0^1 G^n(t-s,x,y)f(s,\kappa_n(y),u^n(s,\kappa_n(y)))dsdy \\
&+ \int_0^t\int_0^1 G^n(t-s,x,y)\sigma(s,\kappa_n(y),u^n(s,\kappa_n(y)))W(ds,dy), \qquad \text{(C.11)}
\end{aligned}
$$

$t \in (0,T]$ and $x \in (0,1)$.

C.3.2 The rate of convergence in L^p

This section is devoted to the proof of the main result of this paper, as follows.

Theorem C.3.1. *Suppose that $u_0 \in H^{\alpha,2}([0,1])$, with $\alpha > \frac{3}{2}$, $v_0 \in H^{\beta,2}([0,1])$, with $\beta > \frac{1}{2}$. We also assume that the coefficients σ and f satisfy conditions (LG) and (L). There exists a positive constant C depending on α, β such that, for any $n \geq 1$,*

$$
\sup_{(t,x)\in[0,T]\times[0,1]} E(|u^n(t,x)-u(t,x)|^{2p}) \leq \frac{C}{n^{2p\rho}}, \qquad \text{(C.12)}
$$

for all $\rho \in (0,\rho_0)$, with $\rho_0 = \frac{1}{3} \wedge (\alpha - \frac{3}{2}) \wedge (\beta - \frac{1}{2})$.

Moreover, $u^n(t,x)$ converges to $u(t,x)$ almost surely, as n tends to infinity, uniformly with respect to $(t,x) \in [0,T]\times[0,1]$.

Remark C.3.2. *Remember that, for any $\gamma > \frac{1}{2}$, $H^{\gamma,2}([0,1])$ is imbedded in the space of δ-Hölder continuous functions on $(0,1)$, for any $\delta \in (0,\gamma-\frac{1}{2})$. Actually, one could state an analogue to Theorem C.3.1 assuming Hölder continuity of the initial conditions.*

We prepare the proof of this theorem with some preliminary results.

In the next lemma we will use the inequality

$$
\int_0^1 |h(y)-h(\kappa_n(y))|^2 dy \leq \frac{C}{n^2}\int_0^1 \left|\frac{d}{dy}h(y)\right|^2 dy, \qquad \text{(C.13)}
$$

valid for every function h in $\mathcal{C}^1([0,1])$ and any $n \geq 1$. This inequality is proved as follows. We shall make use of Cauchy-Schwarz inequality and the defintion of the function κ_n:

$$\begin{aligned}
\int_0^1 |h(y)-h(\kappa_n(y))|^2 dy &= \int_0^1 \left|\int_{\kappa_n(y)}^y \frac{d}{dy}h(z)dz\right|^2 dy \\
&\leq \frac{1}{n}\int_0^1 \int_{\kappa_n(y)}^y \left|\frac{d}{dy}h(z)\right|^2 dzdy \\
&= \frac{1}{n}\int_0^1 dz\left|\frac{d}{dy}h(z)\right|^2 \int_z^{\kappa_n(z)+1} dy \\
&\leq \frac{C}{n^2}\int_0^1 \left|\frac{d}{dy}h(z)\right|^2 dz.
\end{aligned}$$

Hence, (C.13) is proved.

Lemma C.3.3. *For every $\delta \in (0, \frac{2}{3})$, there exists a positive constant C, depending on δ, such that*

$$\sup_{(t,x)\in[0,T]\times[0,1]} \int_0^1 |G(t,x,y)-G^n(t,x,y)|^2 dy \leq \frac{C}{n^\delta},$$

for every $n \geq 1$.

Proof. Set

$$G_n(t,x,y) = \sum_{j=1}^{n-1} \frac{\sin(j\pi t)}{j\pi}\varphi_j(x)\varphi_j(y).$$

We consider the upper bound

$$\int_0^1 |G(t,x,y)-G^n(t,x,y)|^2 dy \leq C\sum_{k=1}^4 I_k^n(t,x),$$

where

$$\begin{aligned}
I_1^n(t,x) &= \int_0^1 |G(t,x,y)-G_n(t,x,y)|^2 dy, \\
I_2^n(t,x) &= \int_0^1 |G_n(t,x,y)-G_n(t,x,\kappa_n(y))|^2 dy, \\
I_3^n(t,x) &= \int_0^1 \left|\sum_{j=1}^{n-1}\left(\frac{\sin(j\pi t)}{j\pi} - \frac{\sin\left(j\pi t\sqrt{c_j^n}\right)}{j\pi\sqrt{c_j^n}}\right)\varphi_j(x)\varphi_j(\kappa_n(y))\right|^2 dy, \\
I_4^n(t,x) &= \int_0^1 \left|\sum_{j=1}^{n-1}\frac{\sin\left(j\pi t\sqrt{c_j^n}\right)}{j\pi\sqrt{c_j^n}}(\varphi_j(x)-\varphi_j^n(x))\varphi_j(\kappa_n(y))\right|^2 dy.
\end{aligned}$$

Owing to (C.2) and to the orthonormality of the family $(\varphi_j, j \geq 1)$,

$$I_1^n(t,x) \leq C \sum_{j=n}^{\infty} \frac{1}{j^2} \leq \frac{C}{n}.$$

Notice that $(\psi_j, j \geq 1)$ defined by $\psi_j(y) = \cos(j\pi y)$, is an orthogonal system in $L^2([0,1])$. Thus, by (C.13),

$$I_2^n(t,x) \leq \frac{C}{n^2} \int_0^1 \left| \frac{\partial}{\partial y} G_n(t,x,y) \right|^2 dy \leq \frac{C}{n^2} \sum_{j=1}^{n-1} \cos^2(j\pi t) |\varphi_j(x)|^2 \leq \frac{C}{n},$$

for all $n \geq 1$.

For $h, g \in L^2([0,1])$, set $\langle h, g \rangle^n = \int_0^1 h(\kappa_n(y)) g(\kappa_n(y)) dy$. By its very definition, for any $j, l \geq 1$,

$$\langle \varphi_j, \varphi_l \rangle^n = \langle e_j, e_l \rangle_{n-1} = \delta_{j,l}, \tag{C.14}$$

where $\langle \cdot, \cdot \rangle_{n-1}$ denotes the Euclidean inner product in $\mathbb{R}^{n-1}$, $(e_j, j = 1 \ldots, n-1)$ is the basis of $\mathbb{R}^{n-1}$ defined in (C.10) and $\delta_{j,l}$ is the Kronecker symbol. Consequently,

$$I_3^n(t,x) = \sum_{j=1}^{n-1} \left(\frac{\sin(j\pi t)}{j\pi} - \frac{\sin\left(j\pi t \sqrt{c_j^n}\right)}{j\pi \sqrt{c_j^n}} \right)^2 \varphi_j^2(x)$$

and

$$I_4^n(t,x) = \sum_{j=1}^{n-1} \left(\frac{\sin\left(j\pi t \sqrt{c_j^n}\right)}{j\pi \sqrt{c_j^n}} \right)^2 (\varphi_j(x) - \varphi_j^n(x))^2.$$

Set $I_3^n(t,x) \leq C(A_n + B_n)$, where

$$A_n = \sum_{j=1}^{n-1} \frac{1}{j^2} \left(1 - \frac{1}{\sqrt{c_j^n}} \right)^2 \sin^2(j\pi t),$$

$$B_n = \sum_{j=1}^{n-1} \frac{1}{j^2 c_j^n} \left(\sin(j\pi t) - \sin\left(j\pi t \sqrt{c_j^n}\right) \right)^2.$$

A Taylor expansion of the function $\sin x$ at $x = 0$ yields

$$1 - \sqrt{c_j^n} \leq C \frac{j^2}{n^2}, \tag{C.15}$$

for any $j = 1, \dots, n-1$, and $n \geq 1$. Then, since c_j^n is bounded below by $\frac{4}{\pi^2}$, we obtain

$$A_n \leq C \sum_{j=1}^{n-1} \frac{1}{j^2} \left(1 - \sqrt{c_j^n}\right)^2 \leq \frac{C}{n}. \tag{C.16}$$

Let $\gamma \in (0, \frac{1}{6})$; the mean value theorem and (C.15) yield

$$\begin{aligned} B_n &\leq C \sum_{j=1}^{n-1} \frac{1}{j^2 c_j^n} \left(\sin(j\pi t) - \sin\left(j\pi t \sqrt{c_j^n}\right)\right)^{2\gamma} \\ &\leq C \sum_{j=1}^{n-1} \frac{1}{j^{2(1-\beta)}} \left(1 - \sqrt{c_j^n}\right)^{2\gamma} \\ &\leq C \frac{1}{n^{4\beta}} \sum_{j=1}^{\infty} \frac{1}{j^{2(1-3\gamma)}} \leq \frac{C}{n^{4\gamma}}, \end{aligned}$$

for some positive constant C depending on γ. Thus, for every $\delta \in (0, \frac{2}{3})$

$$B \leq \frac{C(\delta)}{n^\delta}. \tag{C.17}$$

Putting together (C.16) and (C.17) we obtain that

$$I_3^n(t,x) \leq \frac{C}{n^\delta},$$

for all $\delta \in (0, \frac{2}{3})$.

Observe that

$$|\varphi_j(x) - \varphi_j^n(x)|^2 \leq C \frac{j^2}{n^2}. \tag{C.18}$$

Hence

$$I_4^n(t,x) \leq C \sum_{j=1}^{n-1} \frac{1}{j^2} |\varphi_j(x) - \varphi_j^n(x)|^2 \leq \frac{C}{n}.$$

The proof of the lemma is complete. □

Proposition C.3.4. *Assume that $v_0 \in L^2([0,1])$, $u_0 \in H^{\alpha,2}([0,1])$, for some $\alpha > \frac{1}{2}$, and that the coefficients f and σ satisfy condition (LG). Then for every $p \geq 1$*

$$\sup_{n \geq 1} \sup_{(t,x) \in [0,T] \times [0,1]} E\left(|u^n(t,x)|^{2p}\right) < +\infty.$$

Proof. By virtue of Equation (C.11) we have

$$E\left(|u^n(t,x)|^{2p}\right) \leq C\sum_{k=1}^{4} A_k(n,t,x),$$

with

$$\begin{aligned}
A_1(n,t,x) &= \left|\int_0^1 G^n(t,x,y)v_0(\kappa_n(y))dy\right|^{2p},\\
A_2(n,t,x) &= \left|\sum_{j=1}^{n-1}\langle u_0,\varphi_j\rangle^n \cos\left(j\pi t\sqrt{c_j^n}\right)\varphi_j^n(x)\right|^{2p},\\
A_3(n,t,x) &= E\left(\left|\int_0^t\int_0^1 G^n(t-s,x,y)\sigma(s,\kappa_n(y),u^n(s,\kappa_n(y)))W(ds,dy)\right|^{2p}\right),\\
A_4(n,t,x) &= E\left(\left|\int_0^t\int_0^1 G^n(t-s,x,y)f(s,\kappa_n(y),u^n(s,\kappa_n(y)))dsdy\right|^{2p}\right).
\end{aligned}$$

We can easily prove that

$$\sup_{n\geq 1}\sup_{(t,x)\in[0,T]\times[0,1]}\int_0^1 |G^n(t,x,y)|^2 dy < \infty. \tag{C.19}$$

Let $h \in H^{\alpha,2}([0,1])$, with $\alpha > \frac{1}{2}$, then

$$\langle h,\varphi_j\rangle^n = \langle h,\varphi_j\rangle, \tag{C.20}$$

for every $n \geq 1$ and $j = 1,\ldots,n-1$. Indeed, Fubini's theorem and (C.14) yield

$$\begin{aligned}
\langle h,\varphi_j\rangle^n &= \int_0^1 \sum_{l=1}^{\infty}\langle h,\varphi_l\rangle\varphi_l(\kappa_n(y))\varphi_j(\kappa_n(y))dy\\
&= \sum_{l=1}^{\infty}\langle h,\varphi_l\rangle\int_0^1 \varphi_l(\kappa_n(y))\varphi_j(\kappa_n(y))dy\\
&= \langle h,\varphi_j\rangle.
\end{aligned}$$

By Cauchy-Schwarz inequality,

$$A_1(n,t,x) \leq \left(\int_0^1 |G^n(t,x,y)|^2 dy\right)^p \left(\int_0^1 |v_0(\kappa_n(y))|^2 dy\right)^p.$$

Since $\sup_n \int_0^1 |v_0(\kappa_n(y))|^2 dy$ is finite, using (C.19) we obtain

$$\sup_{n\geq 1} \sup_{(t,x)\in[0,T]\times[0,1]} A_1(n,t,x) \leq C.$$

The identity (C.20) and Cauchy-Schwarz inequality yield

$$\begin{aligned} A_2(n,t,x) &\leq C \left(\sum_{j=1}^{\infty} |\langle u_0, \varphi_j \rangle|^2 j^{2\alpha} \right)^p \left(\sum_{j=1}^{\infty} j^{-2\alpha} \right)^p \\ &\leq C \|u_0\|_{\alpha,2}^{2p}. \end{aligned}$$

Applying first Burkholder's inequality and then Hölder's inequality with respect to the finite (uniformly with respect to n, t and x) measure $|G^n(t-s,x,y)|^2 dsdy$ on $[0,T]\times[0,1]$ yield

$$\begin{aligned} A_3 \leq & CE\left(\left| \int_0^t \int_0^1 |G^n(t-s,x,y)|^2 |\sigma(s,\kappa_n(y), u^n(s,\kappa_n(y))|^2 dsdy \right|^p \right) \\ &\leq C\left(1 + \int_0^t \int_0^1 |G^n(t-s,x,y)|^2 E\left(|u^n(s,\kappa_n(y))|^{2p} \right) dsdy \right) \\ &\leq C\left(1 + \int_0^t \sup_{x\in[0,1]} E\left(|u^n(s,x)|^{2p} \right) ds \right). \end{aligned}$$

Similarly,

$$A_4 \leq C\Big(1 + \int_0^t \sup_{x\in[0,1]} E\left(|u^n(s,x)|^{2p} \right) ds \Big).$$

Therefore

$$\sup_{x\in[0,1]} E\left(|u^n(t,x)|^{2p} \right) \leq C + C\int_0^t \sup_{x\in[0,1]} E\left(|u^n(s,x)|^{2p} \right) ds,$$

with a constant C independent of n.

We apply Gronwall's lemma to conclude the proof. □

With analogous arguments as those used in the study of the terms $J_3(s,t,x,y)$ and $J_4(s,t,x,y)$ in the proof of Proposition C.2.2 we obtain the following.

Lemma C.3.5. *Let $\{h(t,x),(t,x)\in[0,T]\times[0,1]\}$ be an $\mathcal{F}_t-$adapted stochastic process such that for any $p\geq 1$*

$$\sup_{(t,x)\in[0,T]\times[0,1]} E\left(|h(t,x)|^{2p} \right) < \infty.$$

The stochastic processes defined by

$$
\begin{aligned}
H^n(t,x) &= \int_0^t \int_0^1 G^n(t-s,x,z)h(s,z)W(ds,dz), \\
F^n(t,x) &= \int_0^t \int_0^1 G^n(t-s,x,z)h(s,z)dzds,
\end{aligned}
$$

$(t,x) \in [0,T] \times [0,1]$*, are well defined. Moreover, there exists a positive constant* C *such that*

$$
\begin{aligned}
E\left(|H^n(s,x) - H^n(t,y)|^{2p}\right) &\leq C\left(|t-s|^p + |x-y|^p\right), \\
E\left(|F^n(s,x) - F^n(t,y)|^{2p}\right) &\leq C\left(|t-s|^p + |x-y|^p\right),
\end{aligned}
$$

for all $s,t \in [0,T]$, $x,y \in [0,1]$, $n \geq 1$.

Therefore, almost all the sample paths of both, H^n *and* F^n*, are jointly Hölder continuous in time and in space, of any order* $\delta \in (0, \frac{1}{2})$.

In order to shorten the notation, we set

$$
\begin{aligned}
\nu(t,x) &= \int_0^1 G(t,x,y)v_0(y)dy, \\
\nu^n(t,x) &= \int_0^1 G^n(t,x,y)v_0(\kappa_n(y))dy, \\
\mu(t,x) &= \sum_{j=1}^{\infty} \langle u_0, \varphi_j \rangle \cos(j\pi t)\varphi_j(x), \\
\mu^n(t,x) &= \sum_{j=1}^{n-1} \langle u_0, \varphi_j \rangle^n \cos\left(j\pi t\sqrt{c_j^n}\right)\varphi_j^n(x), \\
w(t,x) &= u(t,x) - \nu(t,x) - \mu(t,x) && \text{(C.21)} \\
w^n(t,x) &= u^n(t,x) - \nu^n(t,x) - \mu^n(t,x). && \text{(C.22)}
\end{aligned}
$$

From the proof of Proposition C.2.2 and the above Lemma C.3.5, we clearly have

$$
\begin{aligned}
&\sup_n \left(E(|w^n(s,x) - w^n(t,y)|^{2p}) + E(|w(s,x) - w(t,y)|^{2p})\right) \\
&\quad \leq C\left(|t-s|^p + |x-y|^p\right), && \text{(C.23)}
\end{aligned}
$$

for every $p \geq 1$, $s,t \in [0,T]$, $x,y \in [0,1]$.

In particular, these estimates imply that, if the initial conditions v_0, u_0 vanish, the trajectories of the stochastic processes u^n and u are a.s. jointly Hölder continuous in time and in space, of any order $\delta \in (0, \frac{1}{2})$.

Proposition C.3.6. *Assume that v_0 belongs to $H^{\beta,2}([0,1])$, for some $\beta > \frac{1}{2}$. There exists a positive constant C depending on β, such that*

$$\sup_{(t,x)\in[0,T]\times[0,1]} |\nu^n(t,x) - \nu(t,x)| \le \frac{C}{n^{\epsilon}},$$

for each $n \ge 1$ and every $\epsilon \in (0, \epsilon_0)$, with $\epsilon_0 = \frac{1}{3} \wedge (\beta - \frac{1}{2})$.

Proof. Owing to Cauchy-Schwarz inequality and (C.19) we have

$$|\nu^n(t,x) - \nu(t,x)| \le C(N_1(n,t,x) + N_2(n)),$$

where

$$N_1(n,t,x) = \left(\int_0^1 |G^n(t,x,y) - G(t,x,y)|^2 dy\right)^{\frac{1}{2}},$$
$$N_2(n) = \left(\int_0^1 |v_0(\kappa_n(y)) - v_0(y)|^2 dy\right)^{\frac{1}{2}}.$$

From Lemma C.3.3, it follows that $\sup_{(t,x)\in[0,T]\times[0,1]} N_1(n,t,x) \le \frac{C}{n^{\gamma}}$, for every $\gamma \in (0, \frac{1}{3})$.

Moreover,

$$\begin{aligned} N_2(n) &= \left(\int_0^1 |\sum_{j=1}^{\infty} \langle v_0, \varphi_j\rangle(\varphi_j(\kappa_n(y)) - \varphi_j(y))|^2 dy\right)^{\frac{1}{2}} \\ &\le \|v_0\|_{\beta,2} \left(\sum_{j=1}^{\infty} j^{-2\beta}\big(1 \wedge \frac{j^2}{n^2}\big)\right)^{\frac{1}{2}} \end{aligned} \tag{C.24}$$

Hence, for $\beta \in (\frac{1}{2}, \frac{3}{2}]$, (C.24) is bounded by $\frac{C}{n^{\beta-\frac{1}{2}}}$. If $\beta > \frac{3}{2}$, since the series $\sum_{j=1}^{\infty} j^{2(1-\beta)}$ is convergent, we can estimate (C.24) by $\frac{C}{n}$. Consequently,

$$N_2(n) \le \frac{C}{n^{\beta-\frac{1}{2}}}.$$

The proof is complete. □

Proposition C.3.7. *Assume that $u_0 \in H^{\alpha,2}([0,1])$, with $\alpha > \frac{3}{2}$. There exists a positive constant C depending on α such that*

$$\sup_{(t,x)\in[0,T]\times[0,1]} |\mu^n(t,x) - \mu(t,x)| \le \frac{C}{n^{\tau}},$$

for each $n \ge 1$ and every $\tau \in (0, \tau_0)$, with $\tau_0 = (\alpha - \frac{3}{2}) \wedge 1$.

Proof. By (C.20) we have that

$$|\mu^n(t,x) - \mu(t,x)| \leq C(I_1(t,x,n) + I_2(t,x,n) + I_3(t,x,n)),$$

with

$$I_1(t,x,n) = \left| \sum_{j=n}^{\infty} \langle u_0, \varphi_j \rangle \cos(j\pi t)\varphi_j(x) \right|,$$

$$I_2(t,x,n) = \left| \sum_{j=1}^{n-1} \langle u_0, \varphi_j \rangle \left(\cos(j\pi t) - \cos\left(j\pi t \sqrt{c_j^n} \right) \right) \varphi_j(x) \right|,$$

$$I_3(t,x,n) = \left| \sum_{j=1}^{n-1} \langle u_0, \varphi_j \rangle \cos\left(j\pi t \sqrt{c_j^n} \right) (\varphi_j(x) - \varphi_j^n(x)) \right|.$$

Cauchy-Schwarz inequality yields

$$\begin{aligned} I_1(t,x,n) &\leq C \left(\sum_{j=n}^{\infty} |\langle u_0, \varphi_j \rangle|^2 j^{2\alpha} \right)^{\frac{1}{2}} \left(\sum_{j=n}^{\infty} j^{-2\alpha} \right)^{\frac{1}{2}} \\ &\leq \frac{C}{n^{\alpha - \frac{1}{2}}}. \end{aligned}$$

Cauchy-Schwarz inequality and (C.15) yield

$$\begin{aligned} I_2(t,x,n) &\leq C\|u_0\|_{\alpha,2} \left(\sum_{j=1}^{n-1} j^{-2\alpha} \left| \cos(j\pi t) - \cos\left(j\pi t \sqrt{c_j^n} \right) \right|^2 \right)^{\frac{1}{2}} \\ &\leq C \left(\sum_{j=1}^{n-1} j^{-2(\alpha-1)} \left(1 - \sqrt{c_j^n} \right)^2 \right)^{\frac{1}{2}} \\ &\leq C \left(\frac{1}{n^4} \sum_{j=1}^{n-1} j^{2(3-\alpha)} \right)^{\frac{1}{2}}. \end{aligned} \tag{C.25}$$

For $\alpha < \frac{7}{2}$, the last term in (C.25) is bounded by $\frac{C}{n^{\alpha - \frac{3}{2}}}$. For $\alpha > \frac{7}{2}$, the series $\sum_{j=1}^{\infty} j^{2(3-\alpha)}$ converges and therefore the last term in (C.25) is bounded by $\frac{C}{n^2}$. Hence, since $\alpha > \frac{3}{2}$, for any $n \geq 1$,

$$I_2(t,x,n) \leq \frac{C}{n^{\tau}},$$

for every $\tau \in (0, \tilde{\tau}_0)$, with $\tilde{\tau}_0 = (\alpha - \frac{3}{2}) \wedge 2$.

Similarly, by virtue of (C.18),

$$I_3(t,x,n) \leq \frac{C}{n}.$$

Thus the proof of the statement is complete. $\square$

Proposition C.3.8. *Suppose that u_0 belongs to $H^{\alpha,2}([0,1])$, with $\alpha > \frac{3}{2}$, $v_0 \in H^{\beta,2}([0,1])$, with $\beta > \frac{1}{2}$, and that the coefficients σ and f satisfy conditions (LG) and (L). There exists a positive constant C, depending on α, β, such that*

$$\sup_{(t,x)\in[0,T]\times[0,1]} E(|w^n(t,x) - w(t,x)|^{2p}) \leq \frac{C}{n^{2p\rho}},$$

for each $n \geq 1$ and any $\rho \in (0, \rho_0)$, with $\rho_0 = \frac{1}{3} \wedge (\alpha - \frac{3}{2}) \wedge (\beta - \frac{1}{2})$.

Proof. By definition of w^n and w

$$E(|w^n(t,x) - w(t,x)|^{2p}) \leq C(A_1^n(t,x) + A_2^n(t,x)),$$

with

$$\begin{aligned} A_1^n(t,x) = E\Bigg(\Bigg|&\int_0^t \int_0^1 G^n(t-s,x,y)\sigma(s,\kappa_n(y),u^n(s,\kappa_n(y)))W(ds,dy) \\ &- \int_0^t \int_0^1 G(t-s,x,y)\sigma(s,y,u(s,y))W(ds,dy)\Bigg|^{2p}\Bigg), \\ A_2^n(t,x) = E\Bigg(\Bigg|&\int_0^t \int_0^1 G^n(t-s,x,y)f(s,\kappa_n(y),u^n(s,\kappa_n(y)))dsdy \\ &- \int_0^t \int_0^1 G(t-s,x,y)f(s,y,u(s,y))dsdy\Bigg|^{2p}\Bigg). \end{aligned}$$

Burkholder's inequality and Hölder's inequality with respect to the measures on $[0,T]\times[0,1]$ given by $|G^n(t-s,x,y) - G(t-s,x,y)|^2 dsdy$ and $|G(t-s,x,y)|^2 dsdy$, respectively, yield

$$A_1^n(t,x) \leq C(B_1^n(t,x) + B_2^n(t,x)),$$

where

$$
\begin{aligned}
B_1^n(t,x) &= \left(\int_0^t \int_0^1 |G^n(t-s,x,y) - G(t-s,x,y)|^2 dsdy\right)^{p-1} \\
&\quad \times \left(\int_0^t \int_0^1 |G^n(t-s,x,y) - G(t-s,x,y)|^2\right. \\
&\quad \left. \times\ E(|\sigma(s,\kappa_n(y),u^n(s,\kappa_n(y)))|^{2p}) dsdy\right), \\
B_2^n(t,x) &= \int_0^t \int_0^1 |G(t-s,x,y)|^2 \\
&\quad \times E(|\sigma(s,\kappa_n(y),u^n(s,\kappa_n(y))) - \sigma(s,y,u(s,y))|^{2p}) dsdy.
\end{aligned}
$$

The assumption (LG), Proposition C.3.4 and Lemma C.3.3 yield, for any $n \geq 1$,

$$
B_1^n(t,x) \leq \left(\int_0^t \int_0^1 |G^n(t,x,y) - G(t,x,y)|^2 dyds\right)^p \leq \frac{C}{n^{\delta p}},
$$

for all $\delta \in (0, \frac{2}{3})$.

From the Lipschitz condition (L) on σ and (C.4) we have

$$
\begin{aligned}
B_2^n(t,x) \leq C \Bigg(\frac{1}{n^{2p}} + \int_0^t \big(&\sup_{z\in[0,1]} E(|u^n(s,z) - u(s,z)|^{2p}) \\
&+ \sup_{z\in[0,1]} E(|u(s,\kappa_n(z)) - u(s,z)|^{2p})\big) ds\Bigg)).
\end{aligned}
$$

Owing to (C.21), (C.22) and the upper bounds provided by Propositions C.2.2, C.3.6 and C.3.7, we obtain

$$
B_2^n(t,x) \leq C \left(\frac{1}{n^{2p\rho}} + \int_0^t \sup_{z\in[0,1]} E(|w^n(s,z) - w(s,z)|^{2p}) ds\right),
$$

for any $\rho \in (0, \rho_0)$, with $\rho_0 = \frac{1}{3} \wedge (\alpha - \frac{3}{2}) \wedge (\beta - \frac{1}{2})$.

Taking into account the upper bound obtained for the term $B_1^n(t,x)$, it follows that

$$
\sup_{z\in[0,1]} E(|w^n(t,z) - w(t,z)|^{2p}) \leq C \left(\frac{1}{n^{2p\rho}} + \int_0^t \sup_{z\in[0,1]} E(|w^n(s,z) - w(s,z)|^{2p}) ds\right),
$$

for all $t \in [0,T]$, every $n \geq 1$ and the same range of ρ described before.

With Gronwall's lemma we complete the proof. □

We are now ready to end up with the proof of the main theorem.

Proof of Theorem C.3.1. The first statement (see (C.12)) is a consequence of the previous Propositions C.3.6, C.3.7 and C.3.8.

Owing to Propositions C.3.6 and C.3.7, in order to complete the proof, we only need to check the almost sure convergence $w^n(t,x) \to w(t,x)$, uniformly in $(t,x) \in [0,T] \times [0,1]$.

Notice that, for any $p \in [2,\infty)$,

$$\sup_{(t,x)\in[0,T]\times[0,1]} |w^n(t,x) - w(t,x)|^{2p} \le C(J_1 + J_2 + J_3),$$

with

$$J_1 := \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} |w^n(t_k^n, x_l^n) - w(t_k^n, x_l^n)|^{2p},$$

$$J_2 := \sup_k \sup_l \sup_{|t-t_k^n|\le 1/n} \sup_{|x-x_l^n|\le 1/n} |w^n(t_k^n, x_l^n) - w^n(t,x)|^{2p},$$

$$J_3 := \sup_k \sup_l \sup_{|t-t_k^n|\le 1/n} \sup_{|x-x_l^n|\le 1/n} |w(t_k^n, x_l^n) - w(t,x)|^{2p},$$

where $t_k^n := k\frac{T}{n}$, $x_l^n := \frac{l}{n}$, $k,l = 0,1,\ldots,n-1$. By Proposition C.3.8,

$$E(J_1) \le \frac{C}{n^{2(p\rho-1)}},$$

for all $\rho \in (0,\rho_0)$ and $n \ge 1$.
The joint Hölder continuity of the sample paths of the processes w^n and w (see (C.23)) yields

$$E(J_2 + J_3) \le \frac{C}{n^{2p\delta}},$$

for every $\delta \in (0,\frac{1}{2})$, $n \ge 1$.

Consequently,

$$E\left(\sup_{(t,x)\in[0,T]\times[0,1]} |w^n(t,x) - w(t,x)|^{2p}\right) \le C\frac{C}{n^{2(p\rho-1)}},$$

for all $\rho \in (0,\rho_0)$.
Hence

$$P\left(\sup_{(t,x)\in[0,T]\times[0,1]} |w^n(t,x) - w(t,x)|^{2p} \ge \frac{1}{n^2}\right) \le \frac{C}{n^{2p\rho-4}},$$

for all $n \ge 1$.

Let $p > \frac{5}{2\rho}$, Borel-Cantelli's Lemma yields, with probability one,

$$\sup_{(t,x)\in[0,T]\times[0,1]} |w^n(t,x) - w(t,x)| < \left(\frac{1}{n}\right)^{\frac{1}{p}},$$

for n sufficiently large. The proof is now complete. $\square$

C.4 Appendix

This section is devoted to a numerical analysis of the speed of convergence of the sequence of approximations u^n, $n \geq 1$, given in Theorem C.3.1. We study the optimality of the restriction given by the value $\frac{1}{3}$ in the rate of convergence.

According to the proof of Propositions C.3.8 and C.3.6, the above-mentioned restriction comes from the uniform bound of the $L^2([0,1])$ norm of the difference $G^n(t, x, \cdot) - G(t, x, \cdot)$ stated in Lemma C.3.3 and more precisely, from the upper bound proved for the term

$$I_3^n(t,x) = \sum_{j=1}^{n-1} \left(\frac{\sin(j\pi t)}{j\pi} - \frac{\sin\left(j\pi t\sqrt{c_j^n}\right)}{j\pi\sqrt{c_j^n}} \right)^2 \varphi_j^2(x),$$

$(t,x) \in [0,T]\times[0,1]$.

With a computer program (written in C language) we compute $I_3^n(t,x)$ for different values of t, x and n. First, we check that the term $\varphi_j^2(x) = 2\sin^2(j\pi x)$ has no significant influence in the behaviour of $I_3^n(t,x)$, for fixed $t \in [0,T]$ and large n. Thus, since $\varphi_j^2(x)$ can be uniformly bounded by a constant, we focus our attention on

$$I_3^n(t) = \sum_{j=1}^{n-1} \left(\frac{\sin(j\pi t)}{j\pi} - \frac{\sin\left(j\pi t\sqrt{c_j^n}\right)}{j\pi\sqrt{c_j^n}} \right)^2.$$

For a fixed t, we compute $I_3^n(t)$ for many different natural values of n in some range $[n_0, n_1]$. Then, we consider the function f_t defined by $f_t(z) = I_3^z(t)$, $z \in \mathbb{N}\cap[n_0,n_1]$, and use the least square optimization method to fit f_t to a function of the form $g_t(z) = \frac{e^b}{z^a}$, for some $a > 0$ and $b \in \mathbb{R}$. In the following table, the values of a, b and the range of variation on n for some values of t are displayed.

t	range of n	a	b
1	$2 \leq n \leq 10000$	0.662039	-2.54703
1.002	$2 \leq n \leq 50000$	0.665005	-2.10701
$\sqrt{2}$	$2000 \leq n \leq 15000$	0.659738	-2.00993
8.7	$2 \leq n \leq 10000$	0.65826	-1.40663

We observe that the values of a in the above table are slightly less than $\frac{2}{3}$. It is worthy mentioning that for values of t in the neighbourhood of integer numbers we need to compute $I_3^n(t)$ for larger values of n in order to obtain a suitable convergence.

In Figure C.1 we simultaneously plot the functions $I_3^n(t)$ and $g_t(z)$, for $t = 1.002$ and $t = \sqrt{2}$, respectively. The coefficients a, b of the function g_t and the range of variation of n -and therefore of z- are specified in the above table. Notice the almost perfect matching for $t = \sqrt{2}$.

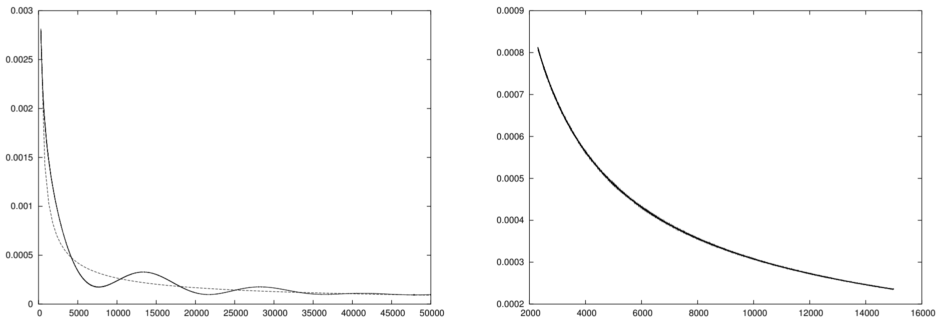

Figure C.1: The doted line corresponds to g_t and the continuous one to $I_3^n(t)$.

We conclude that the bound

$$I_3^n(t,x) \leq \frac{C}{n^\delta},$$

$\delta \in (0, \frac{2}{3})$, is optimal. Therefore, the restriction in Theorem C.3.1 given by the value $\frac{1}{3}$ is intrinsic to the model and it is not due to the method of the proof.

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