Strict-Weak Languages. An Analysis of Strict Implication

Félix Bou Moliner

STRICT-WEAK LANGUAGES.
An analysis of strict implication

Félix Bou Moliner

Memoria presentada para optar al grado de Doctor en Lògica i Fonaments de la Matemàtica por la Universitat de Barcelona

Director: **Dr. Ramon Jansana i Ferrer**

STRICT-WEAK LANGUAGES.
An analysis of strict implication

Félix Bou Moliner

Memoria presentada para optar al grado de Doctor en Lògica i
Fonaments de la Matemàtica por la Universitat de Barcelona

Director: **Dr. Ramon Jansana i Ferrer**

UNIVERSITAT DE BARCELONA

Departament de Lògica, Història i Filosofia de la Ciència de la Universitat de Barcelona
dentro del Programa de Doctorado:
Lògica i Fonaments de la Matemàtica. Bienio 1999-2001

STRICT-WEAK LANGUAGES.
An analysis of strict implication[1]

Félix Bou Moliner

Director: **Dr. Ramon Jansana i Ferrer**

Barcelona, julio de 2004

Memoria presentada para optar al grado de Doctor en Lògica i Fonaments de la Matemàtica por la Universitat de Barcelona

[1] Esta memoria ha sido posible gracias al apoyo del *Departament d'Universitats, Recerca i Societat de la Informació de la Generalitat de Catalunya* y a la concesión de la beca 2000FI00100.

Al papa y a la mama

A la mama y al papa

Imagine ghosts, gods and devils.

Imagine hells and heavens, cities floating in the sky and cities sunken in the sea

Unicorns and centaurs. Witches, warlocks, jinns and banshees.

Angels and harpies. Charms and incantations. Elementals, farmiliars, demons.

Easy to imagine all of those things: mankind has been imagining them for thousands of years.

Imagine spaceships and the future.

Easy to imagine; the future is really coming and there'll be spaceships in it.

Is there then anything that's really *hard* to imagine?

Of course there is.

Imagine a piece of matter and yourself inside it, yourself, aware, thinking and therefore knowing you exist, able to move that piece of matter that you're in, to make it sleep or wake, make love or walk uphill.

Imagine a universe—infinite or not, as you wish to picture it—with a billion, billion, billion suns in it.

Imagine a blob of mud whirling madly around one of those suns.

Imagine yourself standing on that blob of mud, whirling with it, whirling through time and space to an unknown destination.

Imagine!

FREDRIC BROWN

Imagination is more important than knowledge. For knowledge is limited to all we now know and understand, while imagination embraces the entire world, and all there ever will be to know and understand.

ALBERT EINSTEIN

Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.

BERTRAND RUSSELL

The essence of mathematics lies in its freedom.

GEORGE CANTOR

Contents

Acknowledgments

Many people have contributed in one way or another to this dissertation.

First of all, I would like to thank my supervisor Ramon Jansana. He gave me the freedom to explore the paths that interested me, and gave me valuable guidance. Finally, his quick reading of several drafts allowed me to conclude the writing in a relatively short period of time. Thanks Ramon.

I am also indebted to all my teachers in the PhD program *Lògica i Fonaments de la Matemàtica* on the *Departament de Lògica, Historia i Filosofia de la Ciència (UB)*; particularly to those in the Mathematics Faculty, with whom I have been in daily contact during the last four years. To all of them I express my gratefulness. To Ventura for his kindness. To Josep Maria for his initial supervision and for leading me towards the final topic of this dissertation. To Toni for his amusing conversation. To Joan for his advice. To Josep for his vigorous discussions. To Juan Carlos for answering my questions.

I also want to mention my companions Jorge Castro, Ángel García-Cerdanya, Andrés Stisman, Pere Pardo, Saverio Trioni, Joan Pons, Alessandra Palmigiano, Roberto Arpaia, José Gil, Carles Noguera, Neus Castells and Rodrigo Peláez. Among them I would like to point out my logic classmate Ángel, with whom I have enjoyed many hours discussing about logic and other topics.

My honest gratitude to my friends Joan Pons, Quim Puig and Lluís Quer, with whom I have shared many pleasant times inside Mathematics Faculty and outside.

During these last four years I had the opportunity to visit *King's College* (London) and *ILLC* (Amsterdam). I am very grateful to Misha Zakharyaschev and Yde Venema for the invitations. Particularly fruitful for this dissertation was the stay in Amsterdam. Besides Misha and Yde, I thank Nick Bezhanishvili, Ulle Endriss, David Gabelaia, Clemens Kupke, Darko Sarenac and Stefan Schlobach for contributing to make my stays so enjoyable.

My acknowledgment is also for that people who *urged* me to finish the dissertation: mainly Hiroakira Ono, for whose trust I am extremely grateful, but also Francesc Esteva, Josep Maria Font, Ramon Jansana and Yde Venema.

Finally, I would like to list some friends, to whom I owe my gratitude. The members of the *Mogauto football team*; friends like Sergio, Iván, Pedro, Javi and Frank; friends from the Math trip and the summer treks; classmates like Oscar, Xavi, Maite, Manuel and Ana; Josep Maria Mondelo; etc. And of course, my parents Félix and M. Carmen, and my sister Laura.

I would not like to finish without thanking the readers; it is always surprising that someone is interested in what I have written.

Barcelona, July 2004

Preface

The purpose of this dissertation is to analyze strict implication. We recall that strict implication is defined in modal languages as

$$\varphi_0 \rightarrow \varphi_1 := \quad \Box(\varphi_0 \supset \varphi_1) \quad \textit{strict implication},$$

where $\supset$ refers to material implication. Strict implication was already considered by Lewis in the birth of modal logic. However, in the literature there is hardly any analysis of strict implication with the tools used today for modal languages: bisimulations, canonical structures, etc. Our aim is to contribute to clarify these questions.

Since it is not hard to check that the same methods that work well for strict implication also work for its dual

$$\varphi_0 \leftharpoondown \varphi_1 := \quad \Diamond(\varphi_0 \smallsetminus \varphi_1) \quad \textit{weak difference},$$

our results will be presented for languages where we can have at the same time strict implications and weak differences. We have termed these languages *strict-weak languages*. They are the fragments of modal languages given by *falsum* ($\bot$), *true* ($\top$), *conjunction* ($\wedge$), *disjunction* ($\vee$), and some strict implications and weak differences (perhaps associated with the same accessibility relations, perhaps not). We emphasize that *material implication* ($\supset$) and *material negation* ($\sim$) are lacking in these languages.

Besides the interest of strict implication and weak difference, strict-weak fragments are also interesting because they are proper fragments of modal languages that give us information about full modal languages. This is the first time in the literature that properties of this kind have been observed for proper fragments of modal languages. The leitmotif surrounding all our research is that *strict-weak languages are the result of removing some symmetrical aspects that we have in* *modal languages*, which explains why we can use strict-weak fragments to talk about full modal languages.

In order to be able to check our claim in the last paragraph we should recall the results for modal languages. We devote the first chapter of the dissertation to this aim. There we analyze modal languages from three different points of view: model theoretic, proof theoretic (normal modal logics) and computational. We do not consider the algebraic approach since we do not develop it for strict-weak languages (the reader can find an initial study of this topic in [CJ00, Bou01]).

Chapter 2 introduces strict-weak languages. In the last part of the chapter several well known examples of logics are shown which can be formulated in the framework of strict-weak languages, intuitionistic propositional logic being the most famous one.

Chapter 3 of this thesis focusses on a model theoretic approach to strict-weak languages. In the first section we prove the Standard Form Theorem, which implies that every modal formula is a Boolean combination of strict-weak formulas. In an informal sense we can consider most of the results in this dissertation as technical consequences of this theorem. Later, we introduce *quasi bisimilarity* as the natural notion to understand strict-weak languages. Quasi bisimilarity is a quasi order that generates bisimilarity relation, and it allows us to prove van Benthem's style theorems for strict-weak languages. Quasi bisimilarity also allows us to introduce in a very natural way the notion of *strongly Hennessy-Milner class*, which can be used to determine which (Kripke) structures have all states characterized up to bisimilarity by a single modal formula. Up to now we have stated that we can consider quasi bisimilarity as a technical notion to understand strict-weak languages (in particular to understand the semantics of intuitionistic propositional logic); but it is also possible to consider certain strict-weak languages as formal languages suited to talking about inclusion, since quasi bisimilarity coincides with inclusion in the theory ($\mathbf{ZFC}^- + \mathbf{AFA}$) of non-well founded sets.

Chapter 4 deals with strict-weak logics, which are designed to be natural counterparts of normal modal logics. First of all, we show that we cannot recuperate normal modal logics using their strict-weak fragments. Nevertheless, we can recuperate them as far as we replace strict-weak formulas with strict-weak sequents. This is the aim of Section 4.2, where we present strict-weak axiomatizations for most common properties on frames. In the next section we develop the notion of *strict-weak logic*. What really makes this notion interesting is the fact that it gives a uniform framework where normal modal logics and superintuitionistic logics site. Finally, we discuss the disjunction property and uniform interpolation in this setting.

Chapter 5 explores computational aspects of the notions that we have already introduced. In Section 5.1 we analyze 'minimal' representations in standard form of modal formulas. In particular we characterize which modal formulas are equivalent to a conjunction of k material implications of strict-weak formulas (where $k \in \omega$). Later, we show that the complexity problem for most common normal modal logics coincides with the complexity problem of the strict-weak fragment based on a single strict implication. In this section we also prove the fact that the minimal normal modal logic $\boldsymbol{K}$ (and its strict-weak fragment based on a single strict implication) is PSpace-complete even in the case that there are no propositions; and we present an embedding from Grzegorczyks logic $\boldsymbol{Grz}$ into intuitionistic propositional logic $\boldsymbol{IPL}$, which unfortunately is not computable in polynomial time.

Each one of the last three chapters ends with a list of open questions. This list is by no means exhaustive, but it can be taken as a pointer for further research.

Chapter 1

Describing Kripke Structures with Modal Languages

> One of the big misapprehensions about mathematics that we perpetrate in our classrooms is that the teacher always seems to know the answer to any problem that is discussed. This gives students the idea that there is a book somewhere with all the right answers to all of the interesting questions, and that teachers know those answers. And if one could get hold of the book, one would have everything settled. That's so unlike the true nature of mathematics.
>
> LEON HENKIN

This dissertation concerns Kripke structures and strict-weak languages, studying them from different points of view: model theoretic, proof theoretic and computational.

Strict-weak languages are fragments of modal languages. These last languages are very well known formalisms used to describe Kripke structures. Thus, before exploring strict-weak languages we should recall the classical results on modal languages. This chapter is devoted first of all to introducing Kripke structures, and later we remind the reader of the results on modal languages that we will use. Readers familiar with modal languages and modal logic can skip this chapter, or skim through it to become acquainted with our notation. The rest of the dissertation focusses on strict-weak languages. There, we will see how the modal results can be extended to our fragments.

1.1 Kripke structures

Kripke structures are directed graphs where both the nodes and the edges may be labelled by certain syntactic symbols. That is, they are models for a first-order signature containing unary predicate symbols and binary relation symbols.

A *vocabulary*, or a *signature*, is a pair $\tau = \langle \mathsf{Mod}, \mathsf{Prop} \rangle$ where Mod and Prop are disjoint (maybe empty) sets[1]. The elements of Mod are called *modalities*, and the elements of Prop are called *(atomic) propositions*. Sometimes we will identify the sets Mod and Prop with disjoint copies of its cardinal number. It is said that the vocabulary τ is *finite* when both Mod and Prop are finite. Given τ and τ' two vocabularies, its *intersection* $\tau \cap \tau'$ is the vocabulary $\langle \mathsf{Mod} \cap \mathsf{Mod}', \mathsf{Prop} \cap \mathsf{Prop}' \rangle$, and its *union* $\tau \cup \tau'$ is the vocabulary $\langle \mathsf{Mod} \cup \mathsf{Mod}', \mathsf{Prop} \cup \mathsf{Prop}' \rangle$.

A *τ-structure* is a triple $\mathfrak{A} = \langle A, \{R_m : m \in \mathsf{Mod}\}, \{V(p) : p \in \mathsf{Prop}\} \rangle$ where (i) A is a non-empty set, called the *universe*, of *states*, (ii) for each $m \in \mathsf{Mod}$, R_m is a binary relation on A called the *accessibility relation associated with* m, and (iii) for each $p \in \mathsf{Prop}$, $V(p)$ is a subset of A. It is usual to present this family of subsets as a map $V : \mathsf{Prop} \longrightarrow \mathcal{P}(A)$ called the *valuation* of the τ-structure. The *global accessibility relation* associated with this τ-structure, denoted by R, is defined as $\bigcup\{R_m : m \in \mathsf{Mod}\}$. $\mathsf{Str}[\tau]$ denotes the class of τ-structures. A *pointed τ-structure* is a pair $\langle \mathfrak{A}, a \rangle$ where $\mathfrak{A}$ is a τ-structure and a is a distinguished point in A. If there is more than one structure involved in our discussion we will use superscripts, for instance, $R_m^{\mathfrak{A}}$, $R^{\mathfrak{A}}$, $V^{\mathfrak{A}}$, etc.

Given a binary relation R and a set X, we define the *image of X under R*, denoted by $R[X]$, as the set $\{a : \langle x, a \rangle \in R \text{ for some } x \in X\}$. The elements of $R_m[\{a\}]$ are called the *m-successors of the state a*, and the elements of $R[\{a\}]$ are the *successors of a*. The *transitive closure* of a binary relation R is the relation R^* defined as follows:

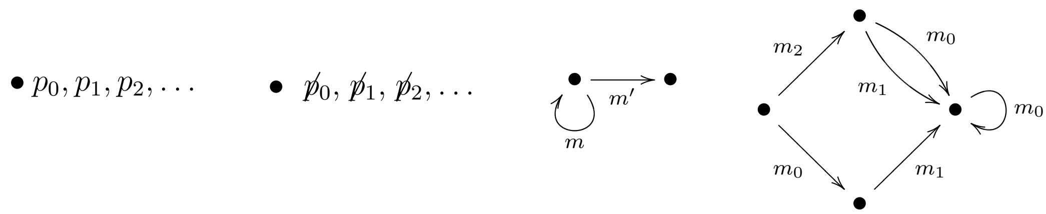
$$\{\langle a, a' \rangle : \exists n \in \omega\ \exists a_0, \ldots, a_{n+1} \in A \text{ such that } a = a_0\ R\ a_1\ R \ldots R\ a_{n+1} = a'\}.$$

A *Kripke structure* is a τ-structure for a certain vocabulary τ. In the same way we can talk of a *pointed Kripke structure*. In the previous notions, and also in those that follow, we will omit the symbol τ when the vocabulary is clear from the context, e.g., we will talk of structures and pointed structures.

Kripke structures are a particular case of structures over first-order signatures [EFT94, Definition III.1.1]. They are the structures obtained when we have in our first-order signature neither functional symbols (including 0-ary symbols) nor n-ary relation symbols with $n \geq 3$. And pointed Kripke structures are obtained when these first-order signatures are enlarged with a constant (0-ary functional symbol). For the sake of simplicity we do not consider arbitrary first-order signatures. In fact, most of our results can be obtained in this more general setting (see section 3.10).

1.1.1. Convention. It is usual to present Kripke structures in the form of diagrams (see Figure 1.1) by depicting states as points and drawing an arrow from a_0

[1] In particular this says that they are not proper classes, i.e., we distinguish between sets and classes. However, we will use the term 'relation' to refer to arbitrary classes of ordered pairs.



(a) $\mathfrak{S}_{IrT}$ (b) $\mathfrak{S}_{IrF}$ (c) (d)

Figure 1.1: Some examples of Kripke structures

to a_1 for each modality relating them. These arrows are labelled with the name of the corresponding modality, unless there is no ambiguity. And the points are labelled with p or $\not p$, for each $p \in \mathsf{Prop}$, depending on if the state is in $V(p)$ or not. We shall often omit this last part under the agreement that all propositions have the same behaviour in all the points (e.g., they are always true). Unless otherwise stated, all arrows are shown explicitly. Nevertheless, to avoid awkwardness we will sometimes consider the transitive closures.

Two Kripke structures will play a role in our work and we introduce them now. The first one is an irreflexive state where all propositions are true, and will be denoted by $\mathfrak{S}_{IrT}$ (see Figure 1.1(a)). And the other one, denoted by $\mathfrak{S}_{IrF}$, is an irreflexive state where all propositions are false (see Figure 1.1(b)). Both structures are available for every vocabulary τ. Sometimes we will consider them as pointed structures where the distinguished point is its unique state. Now we consider two examples of Kripke structures of different entity.

1.1.2. Example. Let us consider a vocabulary with three propositions and a single modality. We take as states of our structure the senior lecturers of the Department of Logic, History and Philosophy of Science at the University of Barcelona (UB). In general, propositions denote properties of states. In this case we may think that our propositions correspond, respectively, to "giving lectures in the Philosophy's Faculty", "giving lectures in the Mathematic's Faculty" and "being a member of the Algebraic Logic Group". And we can consider the modality as saying that "both have published a joint paper". If we want to distinguish a certain state we only need to choose a senior lecturer, e.g., the head of the Department.

1.1.3. Example. [Abstract models of computation] Let us consider a modality for each program. A simple example of program is the one that, given a value for the variable x and a value for the variable y, updates the first variable according to $x := x + 1$. We can consider the states as the possible states of a computer (a state could be an assignment of values to variables). Given a program m then $\langle u, v\rangle \in R_m$ means that there is an execution of the program m that starts in state u and terminates in state v. Observe that our definition of structures does not rule out nondeterministic programs. The propositions denote properties of states, such as $y = x + 1$, or $x = y$. When we want to model the computation of some programs starting from an initial state we can choose it as a distinguished point.

The nature of these examples is very different. In the first case the structure models a static event. On the other hand, the second example gives us a structure to model a dynamic phenomenon, i.e., we can use our structure to follow the dynamic evolution of computations. In so far as we want formalisms to talk about arbitrary structures they will do their work in both (static and dynamic) situations. Although modal languages were originally designed to talk about static situations, in the recent decades the dynamic approach has been more fruitful from the point of view of applications in computer science. Among the first 'pure' modal logicians that have defended this approach to computer science, Goldblatt [Gol92], Parikh [KP81] and van Benthem [vB96] stand out.

Thus, Kripke structures and pointed Kripke structures are also mathematical models proposed in computer science. In this field they are known as *labelled transition systems* (LTSs) and *process graphs* respectively (see [vBvES94, vBB95, PdRV95] for a comparison between the traditions of modal logic and computer science). There the modalities are known as *actions* and the accessibility relations as *transitions*.

Kripke structures can also be seen from a categorical perspective. They are the *coalgebras* associated with the endofunctor $F : \mathsf{Set} \longrightarrow \mathsf{Set}$ defined as $F(X) = \mathcal{P}(\mathsf{Prop}) \times \mathcal{P}(\mathsf{Mod} \times X)$ at the level of objects, and as $F(f) = \mathcal{P}(id_{\mathsf{Prop}}) \times \mathcal{P}(id_{\mathsf{Mod}} \times f)$ at the level of arrows. The reader interested in this categorical approach to processes should consult [JR97, Rut00], and [Mos99] for the relationships with the modal approach.

1.2 Modal languages

Kripke structures are standard mathematical structures, and as such, they may be described using standard logical formalisms.

In particular they can be described using (finitary and non-finitary) first-order (FO) languages. Given a vocabulary τ, let $\mathcal{L}^{FO}(\tau)$ be the (finitary) first-order language with equality ($\approx$) which has unary predicates $\{P : p \in \mathsf{Prop}\}$ and binary relation symbols $\{R_m : m \in \mathsf{Mod}\}$. The logical operators are the usual Boolean connectives and first-order quantifiers with variables $\{v_n : n \in \omega\}$ ranging over states in Kripke structures. The details of this language and of the definition of the corresponding satisfiability relation $\models$ are well known, so we skip them (see for instance [EFT94]). We will use the notation $\mathfrak{A} \models \varphi[\![a_0, \ldots, a_{n-1}]\!]$ to indicate that the formula φ with free variables in $\{v_0, \ldots, v_{n-1}\}$ is satisfied in the τ-structure $\mathfrak{A}$ under the assignment $v_i \longmapsto a_i$. It is well known (see [End00, EFT94]) that these languages have some useful properties such as *recursive enumerability of validities*, *Compactness Theorem* and *Löwenheim-Skolem Theorem*.

Infinitary first-order languages can also describe Kripke structures. We will denote the infinitary version of the previous language by $\mathcal{L}^{FO}_{\infty}(\tau)$. The symbols are the same as above except that now we consider a proper class of variables $\{v_\alpha : \alpha \in \mathbb{O}\mathrm{RD}\}$, and the logical operations now also include arbitrary conjunctions and disjunctions of sets of formulas. That is, if Φ is a set (not a proper class) of $\mathcal{L}^{FO}_{\infty}(\tau)$-formulas then $\bigwedge \Phi$ and $\bigvee \Phi$ are also $\mathcal{L}^{FO}_{\infty}(\tau)$-formulas. We will use the same symbol $\models$ (in practice, this will not cause confusion) to talk about the satisfiability relation in the infinitary case, and $\mathfrak{A} \models \varphi[\![a_0, \ldots, a_{n-1}]\!]$ will mean what is expected. Readers in need of more details about infinitary first-order languages can consult [Bar73][2].

The expressive power of infinitary first-order languages is quite strong; they are well behaved in so far as they capture when two structures are partially isomorphic, that is,

$$\mathfrak{A} \cong_p \mathfrak{B} \quad \text{iff} \quad \mathfrak{A} \text{ and } \mathfrak{B} \text{ satisfy the same } \mathcal{L}^{FO}_{\infty}\text{-sentences.}$$

This statement is known as *Karp's Theorem* [Kar65, Theorem 1]. The last theorem of this paper, sometimes called *Boundedness Theorem* (see [vB99, p. 32]), shows that if a $\mathcal{L}^{FO}_{\infty}$-sentence has models of any cardinality such that the interpretation of a binary relation symbol R is a well-order then this sentence also has models where the interpretation of R is not a well-order. A last interesting theorem involving infinitary first-order languages is *Scott's Theorem* [Sco65, p. 338], which says that countable structures can be characterized up to isomorphism using infinitary first-order sentences in which the conjunction and disjunction is restricted to countable sets.

To be useful for a particular application a language should provide a good balance between expressive power and algorithmic manageability. From a computational point of view the behaviour of (even finitary) first-order languages is not desirable: the set of validities is undecidable [BGG97], it does not have the finite model property, etc. On the other hand, modal languages provide a good balance. It turns out that they are computationally much more tractable than first-order ones, and it is because of this that it is interesting to consider them.

[2] The language $\mathcal{L}^{FO}_{\infty}$ corresponds to what is usually called $\mathcal{L}^{FO}_{\infty\omega}$ [Bar73, p. 9]. The first subindex indicates that we allow arbitrary conjunctions (and disjunctions) over sets and that there is a proper class of variables, and the second subindex indicates that there are quantifiers $\forall_n, \exists_n$ (for each $n \in \omega$) that range over subsets of cardinality $\leq n$.

There is a widespread literature on modal languages (and modal logics). We refer the readers to the neat textbook [BdRV01] and the comprehensive one [CZ97] (but the list of references is huge[3]) if they consider that there is a gap in our presentation of modal languages.

Let $\tau = \langle \mathsf{Mod}, \mathsf{Prop} \rangle$ be a vocabulary. The set $\mathcal{L}^{MOD}(\tau)$ of *(finitary) modal formulas* over τ is given by the rule[4]

$$\varphi ::= \bot \mid \top \mid p \mid \sim\varphi \mid \varphi_0 \wedge \varphi_1 \mid [m]\varphi,$$

where $p \in \mathsf{Prop}$ and $m \in \mathsf{Mod}$. And the set $\mathcal{L}^{MOD}_{\infty}(\tau)$ of *infinitary modal formulas* over τ is defined as follows:

$$\varphi ::= p \mid \sim\varphi \mid \bigwedge \Phi \mid [m]\varphi,$$

where $p \in \mathsf{Prop}$, $m \in \mathsf{Mod}$, and Φ is a set (not a proper class) of $\mathcal{L}^{MOD}_{\infty}(\tau)$-formulas[5]. The connectives here involved are known as *falsum* ($\bot$), *true* ($\top$), *material negation* ($\sim$), *conjunction* ($\wedge$), *boxes* ($[m]$) and *infinitary conjunction* ($\bigwedge$). We also consider the following useful abbreviations:

$\varphi_0 \vee \varphi_1$	$:=$	$\sim(\sim\varphi_0 \wedge \sim\varphi_1)$	*disjunction*
$\varphi_0 \supset \varphi_1$	$:=$	$\sim\varphi_0 \vee \varphi_1$	*material implication*
$\varphi_0 \setminus \varphi_1$	$:=$	$\varphi_0 \wedge \sim\varphi_1$	*material difference*
$\varphi_0 \supset\subset \varphi_1$	$:=$	$(\varphi_0 \supset \varphi_1) \wedge (\varphi_1 \supset \varphi_0)$	*material equivalence*
$\langle m \rangle \varphi$	$:=$	$\sim[m]\sim\varphi$	*diamonds*
$\varphi_0 \rightarrow_m \varphi_1$	$:=$	$[m](\varphi_0 \supset \varphi_1)$	*strict implications*
$\varphi_0 \leftharpoondown_m \varphi_1$	$:=$	$\langle m \rangle(\varphi_0 \setminus \varphi_1)$	*weak differences*
$\neg_m \varphi$	$:=$	$\varphi \rightarrow_m \bot$	*strict negations*
$\ulcorner_m \varphi$	$:=$	$\top \leftharpoondown_m \varphi$	*weak negations*
$\bigvee \Phi$	$:=$	$\sim\bigwedge\{\sim\varphi : \varphi \in \Phi\}$	*infinitary disjunction*

Given $n \in \omega$ we denote by $[m]^n\varphi$ and $[m]^{(n)}\varphi$ the formulas

[3]Other highly recommended references are the books [Seg71, Lem77, Che80, vB82, Pop92, Gol93, HC96, Kra99, GKWZ03] and the surveys [BS84, vB84, Fit93, Gol00, ZWC01, Gol03].

[4]We use the Backus-Naur form (BNF) (also known as Backus normal form) to define the syntax of languages. The notation of BNF is a metasyntax used to express context-free grammars. It was originally named after Backus and later (at the suggestion of Knuth) also after Naur, as part of creating rules for Algol 60 [Bac59, Nau60]. The metasymbols of BNF are ::=, which means "is defined as", and |, which is read "or" (strictly speaking there are also angle brackets ⟨⟩ used to surround category names). At present many books use it to introduce logical languages, e.g., [Gol92, BdRV01].

[5]The transfinite sequence $\langle \xi_\alpha : \alpha \in \mathbb{ORD} \rangle$ defined by $\xi_\alpha = \bigwedge\{\xi_\beta : \beta < \alpha\}$ justifies that the class of $\mathcal{L}^{MOD}_{\infty}(\tau)$-formulas is a proper class (later we will see that this is true even up to equivalence if we have at least one modality). In order to clarify how to formalize this proper class within set theory we refer the reader to [Kei71, p. 6], where this is done for the language $\mathcal{L}^{FO}_{\omega_1\omega}$.

$$\overbrace{[m]\ldots[m]}^{n \text{ times}}\varphi \quad \text{and} \quad \varphi \wedge [m]\varphi \wedge \ldots \wedge [m]^n\varphi,$$

respectively. In the same way we can introduce $\langle m\rangle^n\varphi$ and $\langle m\rangle^{(n)}\varphi$. In the case that the set Mod is a singleton we will use the names $\mathcal{L}^{mod}$ and $\mathcal{L}^{mod}_\infty$, and we will write its box and its diamond as $\Box$ and $\Diamond$ respectively. Even in the case that there is more than one modality, if there is no confusion about which modality we are talking about we will also write $\Box$ and $\Diamond$.

Now it is time to see how modal languages describe Kripke structures. The *modal satisfiability relation* $\Vdash$ is a relation between pointed τ-structures and $\mathcal{L}^{MOD}(\tau)$-formulas defined as follows:

$\mathfrak{A}, a \Vdash \bot$		is never the case
$\mathfrak{A}, a \Vdash \top$		is always the case
$\mathfrak{A}, a \Vdash p$	iff	$a \in V(p)$
$\mathfrak{A}, a \Vdash {\sim}\varphi$	iff	$\mathfrak{A}, a \nVdash \varphi$
$\mathfrak{A}, a \Vdash \varphi_0 \wedge \varphi_1$	iff	$\mathfrak{A}, a \Vdash \varphi_0$ and $\mathfrak{A}, a \Vdash \varphi_1$
$\mathfrak{A}, a \Vdash [m]\varphi$	iff	$\forall a'(\langle a, a'\rangle \in R_m \Rightarrow \mathfrak{A}, a' \Vdash \varphi)$.

In the case of $\mathcal{L}^{MOD}_\infty(\tau)$-formulas the previous definition is expanded with the following clause:[6]

$\mathfrak{A}, a \Vdash \bigwedge\Phi$	iff	$\mathfrak{A}, a \Vdash \varphi$ for every $\varphi \in \Phi$.

It is straightforward to check that the satisfiability conditions for our defined connectives are:

$\mathfrak{A}, a \Vdash \varphi_0 \vee \varphi_1$	iff	$\mathfrak{A}, a \Vdash \varphi_0$ or $\mathfrak{A}, a \Vdash \varphi_1$
$\mathfrak{A}, a \Vdash \varphi_0 \supset \varphi_1$	iff	$\mathfrak{A}, a \nVdash \varphi_0$ or $\mathfrak{A}, a \Vdash \varphi_1$
$\mathfrak{A}, a \Vdash \varphi_0 \setminus \varphi_1$	iff	$\mathfrak{A}, a \Vdash \varphi_0$ and $\mathfrak{A}, a \nVdash \varphi_1$
$\mathfrak{A}, a \Vdash \langle m\rangle\varphi$	iff	$\exists a'(\langle a, a'\rangle \in R_m \,\&\, \mathfrak{A}, a' \Vdash \varphi)$
$\mathfrak{A}, a \Vdash \varphi_0 \rightarrow_m \varphi_1$	iff	$\forall a'(\langle a, a'\rangle \in R_m \,\&\, \mathfrak{A}, a' \Vdash \varphi_0 \Rightarrow \mathfrak{A}, a' \Vdash \varphi_1)$
$\mathfrak{A}, a \Vdash \varphi_0 \leftharpoondown_m \varphi_1$	iff	$\exists a'(\langle a, a'\rangle \in R_m \,\&\, \mathfrak{A}, a' \Vdash \varphi_0 \,\&\, \mathfrak{A}, a' \nVdash \varphi_1)$
$\mathfrak{A}, a \Vdash \neg_m \varphi$	iff	$\forall a'(\langle a, a'\rangle \in R_m \Rightarrow \mathfrak{A}, a' \nVdash \varphi)$
$\mathfrak{A}, a \Vdash \ulcorner_m \varphi$	iff	$\exists a'(\langle a, a'\rangle \in R_m \,\&\, \mathfrak{A}, a' \nVdash \varphi)$
$\mathfrak{A}, a \Vdash \bigvee\Phi$	iff	$\mathfrak{A}, a \Vdash \varphi$ for a certain $\varphi \in \Phi$.

If there is no ambiguity about which structure we are discussing we will write $a \Vdash \varphi$, which is read as 'φ is true at a', 'a satisfies φ' or 'φ holds at a'. As usual,

[6]Strictly speaking we have introduced two satisfiability relations, the finitary $\Vdash$ and the infinitary $\Vdash_\infty$. As the first one is the restriction of the other there is no confusion if we use the same symbol. In fact, we followed the same convention when we introduced the first-order satisfiability relation $\models$. And the same convention will be used to introduce the satisfiability relation for the strict-weak fragments.

given a set Φ of (maybe infinitary) modal formulas we will write $\mathfrak{A}, a \Vdash \Phi$ when all the formulas in this set are satisfiable. Two formulas φ_0, φ_1 are *equivalent*, and we write $\varphi_0 \equiv \varphi_1$, if they are satisfied by exactly the same pointed structures. The *modal theory of a pointed structure* $\langle \mathfrak{A}, a \rangle$, denoted by $\mathrm{Th}_{\mathcal{L}^{MOD}(\tau)}(\mathfrak{A}, a)$, is the set of modal formulas that hold at a. We say that $\langle \mathfrak{A}, a \rangle$ and $\langle \mathfrak{B}, b \rangle$ are *modally equivalent* (notation: $\langle \mathfrak{A}, a \rangle \leftrightsquigarrow_\tau \langle \mathfrak{B}, b \rangle$) if they have the same modal theories. In the same way we introduce $\mathrm{Th}_{\mathcal{L}^{MOD}_\infty(\tau)}(\mathfrak{A}, a)$ (the *infinitary modal theory of this pointed structure*). We will omit the symbol τ in all our definitions when there is no ambiguity.

As $\bot \equiv \bigvee \emptyset$, $\top \equiv \bigwedge \emptyset$ and $\varphi_0 \wedge \varphi_1 \equiv \bigwedge \{\varphi_0, \varphi_1\}$ we know that all (finitary) modal formulas are, up to equivalence, in $\mathcal{L}^{MOD}_\infty$. Later we will see that the expressive power of infinitary modal formulas is much stronger than that of modal formulas. What is true is that modal formulas are closed, up to equivalence, under finitary conjunction and disjunction. Thus[7], for every $\Phi \subseteq_\omega \mathcal{L}^{MOD}$ (including the empty set) we can define modal formulas $\bigwedge \Phi$ and $\bigvee \Phi$ such that:

$$\begin{array}{lll} \mathfrak{A}, a \Vdash \bigwedge \Phi & \text{iff} & \mathfrak{A}, a \Vdash \varphi \text{ for every } \varphi \in \Phi \\ \mathfrak{A}, a \Vdash \bigvee \Phi & \text{iff} & \mathfrak{A}, a \Vdash \varphi \text{ for a certain } \varphi \in \Phi. \end{array}$$

We don't worry about the precise definition of these formulas; our interest lies only in the fact that they satisfy the above properties.

The set of modal formulas can be endowed with an algebraic structure if we look at the connectives as operations. In fact, this algebra[8], denoted by $\boldsymbol{\mathcal{L}^{\mathbf{MOD}}}$, is the absolutely free algebra generated by Prop over the corresponding algebraic signature (which depends on Mod). Then, we define a *modal substitution* as an endomorphism on the algebra $\boldsymbol{\mathcal{L}^{\mathbf{MOD}}}$. In the case that φ is a modal formula with variables in $\{p_0, \ldots, p_{n-1}\}$ we will write $\varphi(\varphi_0, \ldots, \varphi_{n-1})$ as an abbreviation for the result of applying to φ the modal substitution sending $p_i \longmapsto \varphi_i$; in particular $\varphi = \varphi(p_0, \ldots, p_{n-1})$.

There are many measures of the complexity of a modal formula. Here we need only two of the most simple-minded, the modal degree and the length. The *modal degree* of a modal formula is the number of nested boxes (and diamonds). That is, we define $\deg : \mathcal{L}^{MOD} \longrightarrow \omega$ according to the clauses

$$\begin{array}{lll} \deg(\bot) & := & 0 \\ \deg(\top) & := & 0 \\ \deg(p) & := & 0 \\ \deg(\sim\varphi) & := & \deg(\varphi) \\ \deg(\varphi_0 \wedge \varphi_1) & := & \max\{\deg(\varphi_0), \deg(\varphi_1)\} \\ \deg([m]\varphi) & := & \deg(\varphi) + 1. \end{array}$$

[7]The notation $\subseteq_\kappa$ with $\kappa \in \mathbb{C}\mathrm{ARD}$ means that we have a subset of cardinality strictly smaller than κ.

[8]Throughout this dissertation we will use boldface to denote algebras.

Given $n \in \omega$, it is said that $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ are *modally n-equivalent* (notation: $\langle \mathfrak{A}, a\rangle \leftrightsquigarrow_n \langle \mathfrak{B}, b\rangle$) if they satisfy the same modal formulas of modal degree $\leq n$. In the infinitary case the modal degree is given by the map $\deg : \mathcal{L}^{MOD}_{\infty} \longrightarrow \mathbb{O}\mathrm{RD}$ that results from adding the clause

$$\deg(\textstyle\bigwedge \Phi) \quad := \quad \sup\{\deg(\varphi) : \varphi \in \Phi\}.$$

From a computational point of view it is interesting to consider the length, or size, of a modal formula. Of course, this notion only has computational sense when we have a finitary modal formula over a vocabulary with a countable number of modalities and propositions. The *length* of a modal formula is the number of symbol occurrences in the formula, i.e., the natural number defined inductively as follows:

$$\begin{array}{lcl} \mathrm{leng}(\bot) & := & 1 \\ \mathrm{leng}(\top) & := & 1 \\ \mathrm{leng}(p) & := & 1 \\ \mathrm{leng}(\sim\varphi) & := & \mathrm{leng}(\varphi)+1 \\ \mathrm{leng}(\varphi_0 \wedge \varphi_1) & := & \mathrm{leng}(\varphi_0)+\mathrm{leng}(\varphi_1)+1 \\ \mathrm{leng}([m]\varphi) & := & \mathrm{leng}(\varphi)+1. \end{array}$$

A simple induction shows that the number of *subformulas* of φ is bounded by $\mathrm{leng}(\varphi)$. The notion of length that we follow is the same as in [GKWZ03, p. 31]. However, there are other notions of length in the literature. One quite close to ours is to define it as the number of subformulas (see [CZ97, p. 119]). A different approach is to consider the size of memory required to store the symbols in φ, i.e., also taking into account the length of indices: compare for instance the formulas $[0](p_1 \wedge p_2)$ and $[1977](p_{2908} \wedge p_{20032003})$. This last approach has been followed for instance in [Lad77, Sta79, BdRV01], and it seems more accurate for computational purposes. Nevertheless, as long as we restrict ourselves to the standard complexity classes P, NP, PSpace,... the complexity of the decision algorithms is not affected by the choice we make[9]. Thus, we have chosen our definition for the sake of simplicity.

In mathematics, duality is a powerful tool. Perhaps projective geometry is the most famous example of its usefulness. Modal languages also present an interesting duality. The *dual of a $\mathcal{L}^{MOD}(\tau)$-formula* is another $\mathcal{L}^{MOD}(\tau)$-formula defined as follows:

[9]In a subtler analysis it is possible to find differences. For instance, the map $\omega \longrightarrow \mathcal{L}^{mod}$ commonly used to show that the minimal normal modal logic $\boldsymbol{K}$ lacks the polysize model property is in $\mathcal{O}(n^2)$ with our definition (see [CZ97, Theorem 18.10]) while it is not in $\mathcal{O}(n^2)$ with the other definition. For this other definition it is known that the previous map is in $\mathcal{O}(n^2 \log n)$ (see [BdRV01, Theorem 6.42]). The notations for orders of magnitude that we follow is the one proposed in the classical work by Knuth [Knu76] (see [BDG95, Section 2.2] for a more accessible publication), based on several older mathematical notations.

$$\begin{array}{lcl}
\bot^d & := & \top \\
\top^d & := & \bot \\
p^d & := & p \\
(\sim\varphi)^d & := & \sim\varphi^d \\
(\varphi_0 \wedge \varphi_1)^d & := & \varphi_1^d \vee \varphi_0^d \\
([m]\varphi)^d & := & \langle m\rangle\varphi^d.
\end{array}$$

That is, we have just defined a map ${}^d : \mathcal{L}^{MOD}(\tau) \longrightarrow \mathcal{L}^{MOD}(\tau)$. In the case of infinitary modal formulas we obtain a map ${}^d : \mathcal{L}^{MOD}_\infty(\tau) \longrightarrow \mathcal{L}^{MOD}_\infty(\tau)$ when we expand the previous definition with the following clause:

$$(\bigwedge \Phi)^d \quad := \quad \bigvee\{\varphi^d : \varphi \in \Phi\}.$$

A simple checking shows that

$$\begin{array}{lcl}
(\varphi_0 \vee \varphi_1)^d & \equiv & \varphi_1^d \wedge \varphi_0^d \\
(\varphi_0 \supset \varphi_1)^d & \equiv & \varphi_1^d \setminus \varphi_0^d \\
(\varphi_0 \setminus \varphi_1)^d & \equiv & \varphi_1^d \supset \varphi_0^d \\
(\langle m\rangle\varphi)^d & \equiv & [m]\varphi^d \\
(\varphi_0 \rightharpoondown_m \varphi_1)^d & \equiv & \varphi_1^d \leftharpoondown_m \varphi_0^d \\
(\varphi_0 \leftharpoondown_m \varphi_1)^d & \equiv & \varphi_1^d \rightharpoondown_m \varphi_0^d \\
(\neg_m \varphi)^d & \equiv & \ulcorner_m \varphi^d \\
(\ulcorner_m \varphi)^d & \equiv & \neg_m \varphi^d \\
(\bigvee \Phi)^d & \equiv & \bigwedge\{\varphi^d : \varphi \in \Phi\}.
\end{array}$$

We stress that in all binary connectives the order is reversed, and that we have the following pairs of mutual dual connectives: i) material negation and itself, ii) (infinitary) conjunction and (infinitary) disjunction, iii) boxes and diamonds, iv) material implication and material difference, v) strict implications and weak differences, and vi) strict negations and weak negations. Duality can also be defined at the level of τ-structures. The *dual of a τ-structure* $\mathfrak{A} = \langle A, \{R_m : m \in \mathsf{Mod}\}, \{V(p) : p \in \mathsf{Prop}\}\rangle$ is the τ-structure $\mathfrak{A}^d := \langle A, \{R_m : m \in \mathsf{Mod}\}, \{A \setminus V(p) : p \in \mathsf{Prop}\}\rangle$. Of course, $\mathfrak{A} = (\mathfrak{A}^d)^d$ and $\varphi = (\varphi^d)^d$. A straightforward induction shows that

$$\mathfrak{A}, a \Vdash \varphi \quad \text{iff} \quad \mathfrak{A}^d, a \nVdash \varphi^d.$$

This fact is known as the *Duality Principle*. Specifically, we have that

$$\mathfrak{A}, a \Vdash \varphi_0 \supset \varphi_1 \quad \text{iff} \quad \mathfrak{A}^d, a \Vdash \varphi_1^d \supset \varphi_0^d.$$

Here we should note that this is not the standard way to introduce the Duality Principle in the modal case. To my knowledge, duality has always been developed only at the level of formulas (cf. [Che80, Kra99]). The Duality Principle is usually introduced as saying that for every φ with variables in $\{p_0, \ldots, p_{n-1}\}$,

$$\sim\varphi^d \quad \equiv \quad \varphi(\sim p_0, \ldots, \sim p_{n-1}). \tag{1.1}$$

This formulation strengthens the connections between the dual of a formula and the material negation, but it has the disadvantage that it cannot be enunciated without material negation. On the other hand, this problem disappears in our presentation. In fact, the Duality Principle will be very useful when we study the strict-weak languages, where we will not have material negation. To sum up, we are very interested in the Duality Principle but we do not want to formulate it as an equivalence where material negation is involved. We have found a way out by giving a formulation that needs to define the dual of a structure.

Up to now we have introduced the basic notions involved in the presentation of modal languages. Let us make some remarks about their historical origins. The standard semantics for modal languages is based on the possible-world approach, originally proposed by Carnap [Car46]. This semantics was further developed independently by several researchers, including Hintikka, Kanger, Kripke, Meredith, Montague and Prior, reaching its current form with [Kri63], which explains why we call the mathematical structures involved Kripke structures (see [Cop02] for a more detailed analysis of this period, and [Gol03] for the evolution until the present). Modal languages were originally designed to talk about necessity (the box) and possibility (the diamond). Nowadays modal languages are viewed as tools for analyzing the properties of Kripke structures. For instance, the box can mean "according to my knowledge", or "it is provable in Peano arithmetic", or even "after the execution of the program terminates". In recent decades modal languages have been applied to numerous fields of computer science: artificial intelligence [MH69, FHMV95], program verification [Pnu76], database theory [Lip77, CCF82], distributed computing [HM90], spatial reasoning [Aie02], etc.

The modal languages that we have considered are often called propositional (poly)modal languages, to distinguish them from first-order modal languages (see [Gar84, HC96]), in which every state is labelled by a structure over a first-order signature. Nevertheless, modal languages are more accurately viewed as fragments of first-order languages. Modalities are merely quantifiers in a restricted form. We now detail this connection introducing what is known as the standard translation. Let v be a first-order variable, then the *standard translation* ST_v taking $\mathcal{L}^{MOD}(\tau)$-formulas to $\mathcal{L}^{FO}(\tau)$-formulas is defined as follows:

$$\begin{array}{lcl}
\mathrm{ST}_v(\bot) & := & \bot \\
\mathrm{ST}_v(\top) & := & \top \\
\mathrm{ST}_v(p) & := & Pv \\
\mathrm{ST}_v(\sim\varphi) & := & \sim\mathrm{ST}_v(\varphi)
\end{array}$$

$$\begin{array}{lll}\mathrm{ST}_v(\varphi_0 \wedge \varphi_1) & := & \mathrm{ST}_v(\varphi_0) \wedge \mathrm{ST}_v(\varphi_1) \\ \mathrm{ST}_v([m]\varphi) & := & \forall u(R_m vu \supset \mathrm{ST}_u(\varphi))\end{array}$$

where u is a fresh variable (that is, a variable that has not been used so far in the translation). This inductive definition gives, for each first-order variable v, a map ST_v from $\mathcal{L}^{MOD}(\tau)$-formulas into $\mathcal{L}^{FO}(\tau)$-formulas with free variables in $\{v\}$. In the infinitary case what we obtain by adding the clause

$$\mathrm{ST}_v(\bigwedge \Phi) \quad := \quad \bigwedge\{\mathrm{ST}_v(\varphi) : \varphi \in \Phi\}$$

is a map from $\mathcal{L}^{MOD}_\infty(\tau)$-formulas into $\mathcal{L}^{FO}_\infty(\tau)$-formulas with free variables in $\{v\}$. What it is interesting about this translation is that it behaves well from the point of view of satisfiability relations, i.e.,

$$\mathfrak{A}, a \Vdash \varphi \quad \text{iff} \quad \mathfrak{A} \models \mathrm{ST}_v(\varphi)[\![a]\!].$$

Thus, we can identify modal formulas with the first-order fragment obtained as the image under the standard translation. In modal logic, the relationship between modal and first-order languages has been a major research topic [Kam68, Gol75, Sah75, Gab81, vB82, vB84] and is still one of the main themes of study today [GHV03].

As Gabbay observed in [Gab81], a careful examination reveals that in fact propositional modal languages can be viewed as fragments of 2-variable first-order languages (FO^2). Details of how to reuse variables can be found for instance in [BdRV01, Proposition 2.49], where we find the simultaneous definitions of ST_{v_0} and ST_{v_1}. But, in presence of equality, this can be done in an even simpler way (see [Var97]); we only need to define ST_{v_0} using the same clauses as before but replacing the box clause with the following one:

$$\mathrm{ST}_{v_0}([m]\varphi) := \quad \forall v_1\Big(R_m v_0 v_1 \supset \forall v_0\big(v_0 \approx v_1 \supset \mathrm{ST}_{v_0}(\varphi)\big)\Big).$$

Thus, modal languages are fragments of first-order languages that behave well for computational purposes (see [Lew79, DG79, BGG97] for classical references on other fragments of first-order languages). In the early 90's, the decidability of FO^2 (see [Sco62, Hen67, Mor75, GKV97]) was regarded as an explanation for the decidability of many modal logics. More recently, it has been argued that the tree model property [Var97] and the embedding into bounded (or guarded) fragments of first-order logics [ANvB98, Grä01] yield better explanations.

Although modal languages are algorithmically well behaved, their expressive power is less than optimal. Other languages can also be considered over Kripke structures, and have obtained better balances between expressive power and computational tasks. There are powerful extensions of modal logics that are sufficiently expressive for computationally interesting properties and still

Figure 1.2: The clauses (bis2) and (bis3) of bisimulation

algorithmically manageable[10]: CTL (computational tree logic) [Eme90], PDL (propositional dynamic logic) [HKT00], description logics [DLNS96], μ-calculus [Koz83, Wal95, Wal00, SB01], etc. We will not consider these other formalisms in the present research, perhaps on another occasion[11].

1.3 Model theory

The crucial notion to understand the behaviour of modal formulas from the point of view of model theory is bisimilarity. This notion is usually introduced using the notion of bisimulation. Let τ be a vocabulary, and let $\langle \mathfrak{A}, a \rangle$ and $\langle \mathfrak{B}, b \rangle$ be two pointed τ-structures. A relation $Z \subseteq A \times B$ is a *bisimulation* between $\mathfrak{A}$ and $\mathfrak{B}$ in the case that:

(bis1): If $\langle a, b \rangle \in Z$ and $p \in \mathsf{Prop}$, then $\mathfrak{A}, a \Vdash p$ iff $\mathfrak{B}, b \Vdash p$.

(bis2): For every $m \in \mathsf{Mod}$, $R_m^{\mathfrak{B}} \circ Z \subseteq Z \circ R_m^{\mathfrak{A}}$, i.e., if $\langle a, b \rangle \in Z$ and $\langle b, b' \rangle \in R_m^{\mathfrak{B}}$, then there is a' such that $\langle a, a' \rangle \in R_m^{\mathfrak{A}}$ and $\langle a', b' \rangle \in Z$.

(bis3): For every $m \in \mathsf{Mod}$, $R_m^{\mathfrak{B}} \circ Z^{-1} \subseteq Z^{-1} \circ R_m^{\mathfrak{A}}$, i.e., if $\langle a, b \rangle \in Z$ and $\langle a, a' \rangle \in R_m^{\mathfrak{A}}$, then there is b' such that $\langle b, b' \rangle \in R_m^{\mathfrak{B}}$ and $\langle a', b' \rangle \in Z$.

It is said that $\langle \mathfrak{A}, a \rangle$ *is bisimilar to* $\langle \mathfrak{B}, b \rangle$, in symbols $\langle \mathfrak{A}, a \rangle \simeq_\tau \langle \mathfrak{B}, b \rangle$, if there is a bisimulation Z between $\mathfrak{A}$ and $\mathfrak{B}$ such that $\langle a, b \rangle \in Z$. And it is said that $\mathfrak{A}$ *is bisimilar to* $\mathfrak{B}$, denoted by $\mathfrak{A} \simeq_\tau \mathfrak{B}$, if there is a bisimulation Z between $\mathfrak{A}$ and $\mathfrak{B}$ such that its domain is A and its range is B.

Clause (*bis*1) tells us that propositions are invariant. What the other clauses express is clarified on Figure 1.2. Condition (bis2) is also known in the literature under the names (zag) and (back); and (bis3) is known as (zig) or (forth).

[10] We do not consider temporal languages (without the operators *until* and *since*) because they can be seen as a particular case of modal languages (see [vB83, Bur84, GHR94, Ven01]).

[11] An interesting topic for future research would be to extend the ideas considered in this dissertation to the μ-calculus. The interest comes from the fact that this formalism captures the monadic second-order formulas that are invariant under bisimilarity [JW96]. Some ideas around this question are discussed in Section 3.12.

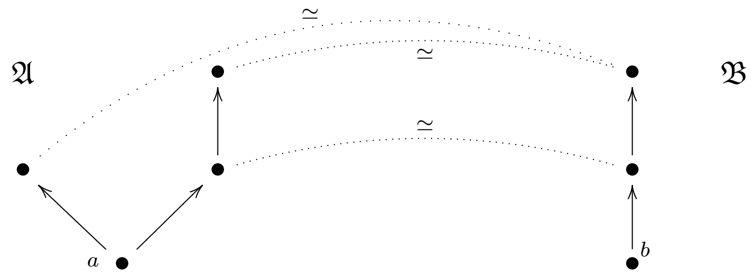


Figure 1.3: An example on bisimilarity

When Z_0 is a bisimulation between $\mathfrak{A}$ and $\mathfrak{B}$, and Z_1 is a bisimulation between $\mathfrak{B}$ and $\mathfrak{C}$, it holds that $Z_1 \circ Z_0$ is a bisimulation between $\mathfrak{A}$ and $\mathfrak{C}$. It is also obvious that if Z is a bisimulation then Z^{-1} is also a bisimulation. Further if $\{Z_i : i \in I\}$ is a family of bisimulations then $\bigcup_{i \in I} Z_i$ is also a bisimulation. Thus, the largest bisimulation between $\mathfrak{A}$ and $\mathfrak{B}$ always exists. Indeed, it is $\{\langle a, b\rangle : \langle \mathfrak{A}, a\rangle \simeq \langle \mathfrak{B}, b\rangle\}$. The relation $\simeq$ is clearly an equivalence relation between pointed structures. And it holds that $\langle \mathfrak{A}, a\rangle \simeq_\tau \langle \mathfrak{B}, b\rangle$ iff $\langle \mathfrak{A}^d, a\rangle \simeq_\tau \langle \mathfrak{B}^d, b\rangle$. Figure 1.3 shows the bisimilarity relation between two concrete structures.

Bisimilarity equivalence was isolated, apparently independently, as an adequate notion of equivalence both in the context of process analysis [Mil80, Par81, HM85] (where the term bisimulation was coined), and in the Ehrenfeucht-Fraïssé style analysis of Kripke structures for propositional modal logics [vB76, vB82, vB84] (under the names p-relation [vB76, p. 10] [vB82, Def. 3.7] and zig-zag connection [vB84, Def. 2.1.4]). In the modal logic world, van Benthem proposed bisimilarity for understanding modal languages (we will develop the details later). In the computer science tradition bisimilarity was initially introduced in [Mil80, Par81] as a natural notion to understand the equivalence between processes[12]. Indeed, processes are usually identified as pointed structures modulo bisimilarity. It was not until the work of Hennessy and Milner [HM85] that a certain modal language was suited to characterize this equivalence notion. So we see that the approaches of the two communities were radically different: whereas bisimilarity was the starting point for computer scientists, it was the goal in the logic tradition. But both communities studied the same concept, and developed the same techniques and ideas. Between the slight differences (forgetting notation) the most notable is perhaps that computer scientists usually have no propositions (see for instance [HM85]), whereas the prototypical modal language has countable propositions and a single modality.

Before recalling the most important results deriving from bisimilarity, we will see some classical constructions inspired by this notion. We start by introducing

[12]Nowadays, in the computer science literature bisimulation is sometimes referred to as *strong bisimulation* to distinguish it from other process-equivalence notions also known as bisimulation.

six classical constructions between structures. All of them, except ultrafilter extensions, can be seen as particular cases of bisimilarity.

- Let $\mathfrak{A}$ and $\mathfrak{B}$ be two τ-structures. $\mathfrak{B}$ is a *substructure* of $\mathfrak{A}$, denoted by $\mathfrak{B} \subseteq \mathfrak{A}$, if B is a subset of A, and the accessibility relations and the valuation are the restrictions to B (that is, if $m \in \mathsf{Mod}$ and $p \in \mathsf{Prop}$ then $R_m^{\mathfrak{B}} = R_m^{\mathfrak{A}} \cap (B \times B)$ and $V^{\mathfrak{B}}(p) = V^{\mathfrak{A}}(p) \cap B$). This notion is commonly used in classical model theory to analyze first-order languages. In the modal case it is more useful to consider generated substructures. A substructure $\mathfrak{B}$ of $\mathfrak{A}$ is a *generated substructure* (notation: $\mathfrak{B} \rightarrowtail \mathfrak{A}$) if B is closed under the global accessibility relation, i.e., $R^{\mathfrak{A}}[B] \subseteq B$. Given a state a of the structure $\mathfrak{A}$, the smallest generated substructure of $\mathfrak{A}$ containing a, which is denoted by $\mathfrak{A}_a$, always exists. It is easy to see that the universe of $\mathfrak{A}_a$ consists of all states that can be reached from a via a finite $R^{\mathfrak{A}}$-path, i.e., the universe is $\{a\} \cup (R^{\mathfrak{A}})^*[\{a\}]$. The connection with bisimilarity is given by the fact that $\langle \mathfrak{A}, a\rangle \simeq \langle \mathfrak{A}_a, a\rangle$ for every state a.

- Let $\{\mathfrak{A}_i : i \in I\}$ be a set of τ-structures[13]. We assume that the universes of these structures are pairwise disjoint — if this is not the case, then we replace our structures containing some isomorphic copies with the property (e.g., replace the universe A_i with $\{\langle a, i\rangle : a \in A_i\}$). The *disjoint union* of this family of τ-structures is the τ-structure $\biguplus_{i\in I} \mathfrak{A}_i$, where (i) the universe is the union of the sets A_i, (ii) the accessibility relation associated with $m \in \mathsf{Mod}$ is the union $\bigcup_{i\in I} R_m^{\mathfrak{A}_i}$, and (iii) the valuation is the map $p \longmapsto \bigcup_{i\in I} V^{\mathfrak{A}_i}(p)$. The connection with bisimilarity is given by the fact that $\langle \mathfrak{A}_i, a\rangle \simeq \langle \biguplus_{i\in I} \mathfrak{A}_i, a\rangle$ for every $i \in I$ and every state a in A_i. In particular this means that $\langle \mathfrak{A}, a\rangle \simeq \langle \mathfrak{B}, b\rangle$ iff $\langle \mathfrak{A} \uplus \mathfrak{B}, a\rangle \simeq \langle \mathfrak{A} \uplus \mathfrak{B}, b\rangle$. Hence, we could have introduced the bisimilarity notion restricting ourselves to the case in which both structures are the same[14].

- Let $\mathfrak{A}$ and $\mathfrak{B}$ be two τ-structures. A mapping $f : A \longrightarrow B$ is a *bounded morphism*, or a *p-morphism*, from $\mathfrak{A}$ into $\mathfrak{B}$ if it satisfies the following conditions:

  (bmor1): For each $a \in A$, $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, f(a)\rangle$ satisfy the same propositions.

  (bmor2): For each $m \in \mathsf{Mod}$, $a \in A$ and $b' \in B$, it holds that if $\langle f(a), b'\rangle \in R_m^{\mathfrak{B}}$, then there is a' such that $\langle a, a'\rangle \in R_m^{\mathfrak{A}}$ and $f(a') = b'$.

  (bmor3): For each $m \in \mathsf{Mod}$ and each $a, a' \in A$, if $\langle a, a'\rangle \in R_m^{\mathfrak{A}}$ then $\langle f(a), f(a')\rangle \in R_m^{\mathfrak{B}}$.

[13] Although we have also used a subindex in the case of generated substructures generated by a state there will be no confusion: the context will clarify whether the subindex refers to an indexed family or to a certain state. That is, we follow the standard convention in the literature.

[14] This restriction is commonly considered in the computer science tradition.

This says that $\{\langle a, f(a)\rangle : a \in A\}$ is a bisimulation between $\mathfrak{A}$ and $\mathfrak{B}$. Thus, $\langle\mathfrak{A}, a\rangle \simeq \langle\mathfrak{B}, f(a)\rangle$ for every state a. To sum up, bounded morphisms are exactly the bisimulations that are functional from $\mathfrak{A}$ into $\mathfrak{B}$.

- Given a set A, an *ultrafilter over* A is any family u of subsets of A such that (i) it is closed under taking intersections: if $X, Y \in u$ then $X \cap Y \in u$, (ii) it is upwards closed: if $X \in u$ and $X \subseteq Y$, then $X \in u$, (iii) it contains, for any subset X of A, either X or its complement $A \setminus X$. For instance, if $a \in A$ then $\pi_a := \{X \subseteq A : a \in X\}$ is an ultrafilter over A. We will refer to it as the *principal ultrafilter associated with* a. Now let $\mathfrak{A}$ be a τ-structure. The *ultrafilter extension* is the τ-structure $\mathfrak{ue}\,\mathfrak{A}$, where (i) the universe is the collection Uf_A of ultrafilters over A, (ii) the accessibility relation associated with $m \in \mathsf{Mod}$ is $\{\langle u_0, u_1\rangle \in Uf_A \times Uf_A$: for all $X \subseteq A$, if $\{a \in A : R_m[\{a\}] \subseteq X\} \in u_0$ then $X \in u_1\}$, and (iii) the valuation is the map $p \longmapsto \{u \in Uf_A : V(p) \in u\}$. This time, we have that $\langle\mathfrak{A}, a\rangle \leftrightsquigarrow \langle\mathfrak{ue}\,\mathfrak{A}, \pi_a\rangle$ for every state a. In general they are not bisimilar.

- Let $\langle\mathfrak{A}, a\rangle$ be a pointed τ-structure, and let $\kappa \in \mathbb{C}\mathrm{ARD}$ such that $\kappa \geq 1$. The *κ-expansion* is the pointed τ-structure $\mathfrak{exp}_\kappa(\mathfrak{A}, a)$, where (i) its universe is the collection of sequences $\rho = \langle a_0, m_1, \alpha_1, a_1, \ldots, m_n, \alpha_n, a_n\rangle$ such that $n \in \omega$, $a_0 = a$ and $\forall i < n$ it holds that $m_{i+1} \in \mathsf{Mod}$, $\alpha_{i+1} < \kappa$, $a_{i+1} \in A$, and $\langle a_i, a_{i+1}\rangle \in R_{m_{i+1}}$, (ii) the accessibility relation associated with $m \in \mathsf{Mod}$ is precisely the relation $\{\langle \rho_0, \rho_1\rangle$: there are m', α', a' such that ρ_1 is the concatenation of ρ_0 and $\langle m', \alpha', a'\rangle\}$, and (iii) the valuation is the map $p \longmapsto \{\rho : \mathrm{end}(\rho) \in V(p)\}$ where $\mathrm{end}(\rho)$ gives us the last component of the sequence ρ, (iv) the distinguished state is $\langle a\rangle$. Indeed, the map end is a surjective bounded morphism of the structure underlying $\mathfrak{exp}_\kappa(\mathfrak{A}, a)$ into $\mathfrak{A}$. Hence, $\mathfrak{exp}_\kappa(\mathfrak{A}, a) \simeq \langle\mathfrak{A}, a\rangle$. The κ-expansions are very useful due to the fact that they replace our initial pointed structure with a bisimilar one that satisfies the following desirable properties:

  1. The global accessibility relation is well-founded (i.e., there is no infinite R-decreasing sequence of states).
  2. Every state can be reached from the distinguished state in a finite number of R-steps. This path is unique, and the number of R-steps is called the *height* of this state.
  3. Every state, except the distinguished one, has a unique R-predecessor, and this predecessor does not coincide with itself.
  4. If m and m' are different modalities then R_m and $R_{m'}$ are disjoint.
  5. If there are m-successors of a state a' then there are at least κ distinct m-successors of a' that are pairwise bisimilar.

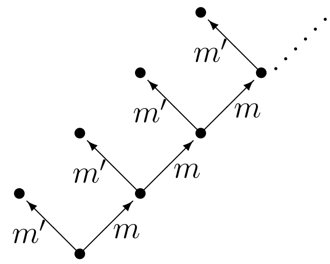


Figure 1.4: The unravelling of Figure 1.1(c)

We define a *tree* (of height $\leq \omega$ and root the distinguished point) as any structure satisfying the first four properties. The κ-expansions give us a connection between bisimilarity and isomorphism, because two pointed structures $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ are bisimilar if there exists a cardinal κ such that $\mathfrak{exp}_\kappa(\mathfrak{A}, a)$ and $\mathfrak{exp}_\kappa(\mathfrak{B}, b)$ are isomorphic (the cardinal κ can be chosen to be $\max\{|A|, |B|\}$). When $\kappa = 1$ this construction will be called *unravelling* (see [DL59, Sah75]); i.e., $\mathfrak{unr}(\mathfrak{A}, a) = \mathfrak{exp}_1(\mathfrak{A}, a)$. In Figure 1.4 we can see an example.

- Let $\mathfrak{A}$ be a τ-structure. The *bisimilarity collapsing* is the τ-structure $\mathfrak{coll}\,\mathfrak{A}$, where (i) its universe is the collection of equivalence classes of A under the equivalence relation $\simeq$ (i.e., $a_0 \simeq a_1$ in the case that $\langle \mathfrak{A}, a_0\rangle \simeq \langle \mathfrak{A}, a_1\rangle$), (ii) the accessibility relation associated with $m \in \mathsf{Mod}$ is precisely the relation $\{\langle [a_0]_\simeq, [a_1]_\simeq\rangle :$ there exists $a_1' \in [a_1]_\simeq$ such that $\langle a_0, a_1'\rangle \in R_m\}$, and (iii) the valuation is the map $p \longmapsto \{[a]_\simeq : a \in V(p)\}$. It is known that $\mathfrak{A} \simeq \mathfrak{coll}\,\mathfrak{A}$, and also that $\mathfrak{A} \simeq \mathfrak{B}$ iff $\mathfrak{coll}\,\mathfrak{A}$ is isomorphic to $\mathfrak{coll}\,\mathfrak{B}$. As an example it is easy to see that the bisimilarity collapsing of Figure 1.4 is precisely Figure 1.1(c).

To finish our definitions concerning structures we will define the statement that two pointed structures are α-bisimilar, where $\alpha \in \mathbb{O}\mathrm{RD}$. These relations are known as *bounded-bisimilarity*. In the case of $\alpha \in \omega$ this notion was introduced in modal logic by Fine [Fin74b, Fin85] (in the computer science tradition the history goes back to [HM85]). But in the arbitrary case this is a recent notion that was first introduced by Gerbrandy [Ger97, Ger99]. The relation of α-bisimilarity is a relation between pointed τ-structures defined by induction on α. Two pointed τ-structures $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ are *α-bisimilar* (notation: $\langle \mathfrak{A}, a\rangle \simeq_\alpha \langle \mathfrak{B}, b\rangle$) iff:

(bbis1): If $p \in \mathsf{Prop}$, then $\mathfrak{A}, a \Vdash p$ iff $\mathfrak{B}, b \Vdash p$.

(bbis2): For every $\beta < \alpha$ and $m \in \mathsf{Mod}$, if $\langle b, b'\rangle \in R_m^{\mathfrak{B}}$, then there is a' such that $\langle a, a'\rangle \in R_m^{\mathfrak{A}}$ and $\langle \mathfrak{A}, a\rangle \simeq_\beta \langle \mathfrak{B}, b\rangle$.

(bbis3): For every $\beta < \alpha$ and $m \in \mathsf{Mod}$, if $\langle a, a'\rangle \in R_m^{\mathfrak{A}}$, then there is b' such that $\langle b, b'\rangle \in R_m^{\mathfrak{B}}$ and $\langle \mathfrak{A}, a\rangle \simeq_\beta \langle \mathfrak{B}, b\rangle$.

Thus two pointed structures $\langle\mathfrak{A}, a\rangle$ and $\langle\mathfrak{B}, b\rangle$ are 0-bisimilar iff they satisfy the same propositions. They are $\alpha+1$-bisimilar iff they satisfy the same propositions, and for each m-successor of b there is an α-bisimilar m-successor of a, and viceversa. In the case that α is a limit ordinal, they are α-bisimilar just in the case they are β-bisimilar for every $\beta < \alpha$.

Now we will recall how the notions that we have introduced relate to modal languages. The first remark is that bisimilar pointed structures satisfy the same modal formulas; in particular we have that for the previous six constructions it holds that

$$\langle\mathfrak{A}, a\rangle \simeq \langle\mathfrak{B}, b\rangle \quad \text{implies} \quad \langle\mathfrak{A}, a\rangle \leftrightsquigarrow \langle\mathfrak{B}, b\rangle. \tag{1.2}$$

In fact, bisimulation can be viewed as a natural adaptation of partial isomorphisms [Bar73] and Ehrenfeucht-Fraïssé games [Ehr61, Doe] to the modal case. The role played by the back and forth clauses of partial isomorphisms is here played, respectively, by (bis3) and (bis2). This connection was strengthened from an heuristic point of view by the work of de Rijke. In his dissertation [dR93, p. 107] he proposed the heuristic "partial isomorphism is to first-order logic what bisimilarity is to modal logic" to generate theorems in modal logic (and to produce new proofs of the classical results). This idea has been very fruitful, and most of the results that we indicate can be proved following this idea. For instance, the first part of the following theorem is the modal version of Karp's Theorem.

1.3.1. Theorem. [Ger99, Sections 2.1–2.2] *Let $\alpha \in \mathbb{O}\mathrm{RD}$, let τ be a vocabulary, and let $\langle\mathfrak{A}, a\rangle$ and $\langle\mathfrak{B}, b\rangle$ be two pointed τ-structures. Then,*

1. *$\langle\mathfrak{A}, a\rangle$ and $\langle\mathfrak{B}, b\rangle$ are bisimilar iff they satisfy the same $\mathcal{L}_\infty^{MOD}(\tau)$-formulas.*
2. *$\langle\mathfrak{A}, a\rangle$ and $\langle\mathfrak{B}, b\rangle$ are α-bisimilar iff they satisfy the same $\mathcal{L}_\infty^{MOD}(\tau)$-formulas of modal degree $\leq \alpha$.*
3. *For every $n \in \omega$, if $\langle\mathfrak{A}, a\rangle \simeq_n \langle\mathfrak{B}, b\rangle$ then $\langle\mathfrak{A}, a\rangle \leftrightsquigarrow_n \langle\mathfrak{B}, b\rangle$. And $\langle\mathfrak{A}, a\rangle \simeq_\omega \langle\mathfrak{B}, b\rangle$ implies $\langle\mathfrak{A}, a\rangle \leftrightsquigarrow \langle\mathfrak{B}, b\rangle$. If τ is finite, then the converse implications also hold.*
4. *If* Mod *is non-empty, then there exists a proper class of pointed structures that are pairwise non bisimilar.*
5. *If* Mod *is non-empty, then there exist pointed structures that are α-bisimilar but not $\alpha+1$-bisimilar.*

1.3.2. Remark. We emphasize that in order to have $\simeq_\omega = \leftrightsquigarrow$ and $\simeq_n = \leftrightsquigarrow_n$ it is necessary to be in a finite vocabulary. In the literature this hypothesis has sometimes been overlooked. To see the necessity of this requirement take a vocabulary such that $\mathsf{Prop} = \{p_n : n \in \omega\}$ and Mod is a singleton. Let $\mathfrak{A}$ be the structure with universe $A := \{a\} \cup \{a_X : X \subseteq \omega,\ \text{and } X \text{ and } \omega \setminus X \text{ are infinite}\}$, accessibility relation $R := \{a\} \times (A \setminus \{a\})$, and valuation $p_n \longmapsto \{a_X \in A : n \in X\}$. Now fix Y an arbitrary subset of ω such that it and its complement are infinite. And let $\mathfrak{B}$ be the substructure of $\mathfrak{A}$ with underlying universe $A \setminus \{a_Y\}$. It is easy to see that $\langle \mathfrak{A}, a \rangle \leftrightsquigarrow \langle \mathfrak{B}, a \rangle$ (because in finitary modal formulas only a finite number of propositions are involved) while $\langle \mathfrak{A}, a \rangle \not\simeq_1 \langle \mathfrak{B}, a \rangle$.

An easy induction on α (based on the fact the Boolean algebra generated by a set is a set) shows that the proper class of infinitary modal formulas of modal degree $\leq \alpha$ is, up to equivalence, a set. We can therefore consider its conjunction, and what we obtain is an infinitary modal formula. Thus, being α-bisimilar can be characterized by a single infinitary modal formula. In fact, the next theorem says that this is also true for bisimilarity, which implies that the class of infinitary modal formulas (when there are modalities) is, up to equivalence, a proper class. The first proof[15] of this result is due to Baltag (see [BM97, Section 11.2]) and is based on a cofinality argument over a transfinite sequence of infinitary modal formulas explicitly defined. It was formulated between non-well-founded sets, but as we will see later these are just pointed structures for a particular vocabulary.

1.3.3. THEOREM. [Ger99, Corollary 2.23] *Let τ be a vocabulary, and let $\langle \mathfrak{A}, a \rangle$ be a pointed τ-structure. Then, there exists a $\mathcal{L}_{\infty}^{MOD}(\tau)$-formula $\phi^{\langle \mathfrak{A}, a \rangle}$ such that for every pointed τ-structure $\langle \mathfrak{B}, b \rangle$,*

$$\langle \mathfrak{A}, a \rangle \simeq \langle \mathfrak{B}, b \rangle \quad \textit{iff} \quad \mathfrak{B}, b \Vdash \phi^{\langle \mathfrak{A}, a \rangle}.$$

It is said that the formula $\phi^{\langle \mathfrak{A}, a \rangle}$ *characterizes* $\langle \mathfrak{A}, a \rangle$ *up to bisimilarity*. This allows us to define the *rank* of a pointed structure, denoted by $\mathrm{rank}(\mathfrak{A}, a)$, as the minimum modal degree of an infinitary modal formula that characterizes $\langle \mathfrak{A}, a \rangle$ up to bisimilarity. This exists because the class of ordinals is well-ordered. It is obvious that $\mathrm{rank}(\mathfrak{A}, a)$ coincides with

$$\min\{\alpha \in \mathbb{O}\mathrm{RD} : \text{for all } \langle \mathfrak{B}, b \rangle,\ \langle \mathfrak{A}, a \rangle \simeq_\alpha \langle \mathfrak{B}, b \rangle \text{ implies } \langle \mathfrak{A}, a \rangle \simeq \langle \mathfrak{B}, b \rangle\}.$$

It is clear that bisimilar pointed structures have the same rank. Gerbrandy saw that for each successor ordinal α, there is a pointed structure with rank α. But it is still an open problem to determine whether this is also true for limit ordinals. Another simple consequence of the definition of rank is that if $\langle a, a' \rangle \in R^{\mathfrak{A}}$ then $\mathrm{rank}(\mathfrak{A}, a') \leq \mathrm{rank}(\mathfrak{A}, a)$. This inequality is strict when $\mathrm{rank}(\mathfrak{A}, a)$ is a successor ordinal. Another interesting remark is the following (see Theorem 1.3.1(3)): if τ is finite, then $\mathrm{rank}(\mathfrak{A}, a) \leq \omega$ means exactly that there is a set of $\mathcal{L}^{MOD}(\tau)$-formulas that characterizes $\langle \mathfrak{A}, a \rangle$ modulo bisimilarity, and $\mathrm{rank}(\mathfrak{A}, a) < \omega$ means that there is a single $\mathcal{L}^{MOD}(\tau)$-formula that characterizes $\langle \mathfrak{A}, a \rangle$ modulo bisimilarity. Once more the structure $\langle \mathfrak{A}, a \rangle$ given in Remark 1.3.2, which has rank 2, shows the necessity of finiteness.

[15]Perhaps we should also consider the study by Fagin [Fag94], which works with certain knowledge structures that are very close to our pointed structures.

1.3.4. Proposition. [D'A98, Proposition 4.1.2] *Let τ be a vocabulary, and let φ be a $\mathcal{L}^{MOD}_{\infty}(\tau)$-formula. The following are equivalent:*

1. *φ characterizes, up to bisimilarity, a certain pointed τ-structure.*

2. *φ is satisfiable by a certain pointed τ-structure, and for each $\varphi' \in \mathcal{L}^{MOD}_{\infty}(\tau)$, either $\top \equiv \varphi \supset \varphi'$ or $\top \equiv \varphi \supset \sim \varphi'$.*

3. *φ is satisfiable by a certain pointed τ-structure, and for each $\varphi' \in \mathcal{L}^{MOD}_{\infty}(\tau)$, if φ' is satisfiable and $\top \equiv \varphi' \supset \varphi$, then $\varphi' \equiv \varphi$.*

Now we enunciate *van Benthem's Theorem* [vB76, Theorem 1.9]. It corresponds to the first three equivalences in the incoming theorem. This is perhaps the most important result correlating bisimilarity and modal languages. Later, van Benthem generalized it to the infinitary case [vB96, vB99]. First of all, let us explain the terms involved in the formulation. It is said that a (maybe infinitary) first-order formula is, up to equivalence, a modal formula in the case that it is equivalent to the standard translation of a certain modal formula. Given a relation R between pointed structures, we say that a first-order formula is *invariant* under this relation if

$$\mathfrak{A} \models \varphi[\![a]\!] \quad \text{iff} \quad \mathfrak{B} \models \varphi[\![b]\!]$$

when $\langle\langle\mathfrak{A}, a\rangle, \langle\mathfrak{B}, b\rangle\rangle \in R$. If we replace the previous condition with

$$\mathfrak{A} \models \varphi[\![a]\!] \quad \text{implies} \quad \mathfrak{B} \models \varphi[\![b]\!]$$

then we say that the first-order formula is *preserved* under the relation. And it is said that it is *reflected* when what we have is

$$\mathfrak{A} \models \varphi[\![a]\!] \quad \text{if} \quad \mathfrak{B} \models \varphi[\![b]\!].$$

As particular cases we obtain the definitions of invariance under bisimilarity, invariance under bounded morphisms (in this case what we consider is the relation that relates $\langle\mathfrak{A}, a\rangle$ and $\langle\mathfrak{B}, b\rangle$ if there is a bounded morphism f from $\mathfrak{A}$ into $\mathfrak{B}$ such that $f(a) = b$), etc. When we restrict the class of (pointed) τ-structures to a certain class K we talk of K-equivalence, K-invariance and K-preservation.

1.3.5. Theorem. [dR93, Theorem 6.5.7] *Let τ be a vocabulary, and let $\varphi(v_0)$ be a $\mathcal{L}^{FO}(\tau)$-formula. The following are equivalent:*

1. *φ is, up to equivalence, in $\mathcal{L}^{MOD}(\tau)$.*

2. *There is $n \in \omega$ such that φ is invariant under n-bisimilarity.*

3. *φ is invariant under bisimilarity.*

4. *φ is invariant under bounded morphisms.*

1.3.6. THEOREM. [vB99, p. 31] *Let τ be a vocabulary, and let $\varphi(v_0)$ be a $\mathcal{L}_{\infty}^{FO}(\tau)$-formula. The following are equivalent:*

1. *φ is, up to equivalence, in $\mathcal{L}_{\infty}^{MOD}(\vartheta)$.*

2. *There is $\alpha \in \mathbb{O}\mathrm{RD}$ such that φ is invariant under α-bisimilarity.*

3. *φ is invariant under bisimilarity.*

4. *φ is invariant under bounded morphisms.*

1.3.7. REMARK. In the references given for the last two theorems the equivalence with the last statement is not proved. The only source where I have found this explicitly enunciated is [Gol95, Theorem 18]. But the formulation given there is for the finitary case when there are no propositions. Nevertheless, it is not difficult, using κ-expansions, to see that formulas invariant under bounded morphisms are also invariant under bisimilarity (the other direction is trivial). Let us examine the details. Assume $\langle \mathfrak{A}, a\rangle \simeq \langle \mathfrak{B}, b\rangle$. Then, we know that (i) there is a cardinal κ such that $\mathfrak{exp}_{\kappa}(\mathfrak{A}, a)$ and $\mathfrak{exp}_{\kappa}(\mathfrak{B}, b)$ are isomorphic, (ii) the map end_A is a bounded morphism from $\mathfrak{exp}_{\kappa}(\mathfrak{A}, a)$ into $\langle \mathfrak{A}, a\rangle$, and (iii) the map end_B is a bounded morphism from $\mathfrak{exp}_{\kappa}(\mathfrak{B}, b)$ into $\langle \mathfrak{B}, b\rangle$. From this we obtain what we need. Indeed, this proof shows that it is enough to require that φ is invariant under surjective bounded morphisms.

In the above theorems we can replace invariance under bisimilarity with preservation under bisimilarity. The reason is that if Z is a bisimulation then Z^{-1} is also a bisimulation. On the other hand, it is false that if f is a bounded morphism (i.e., a functional bisimulation) then f^{-1} is also a bounded morphism. Indeed, it is not possible to replace invariance under bounded morphisms with preservation under bounded morphisms. For instance, formulas like $R_m v_0 v_0$ and $\exists v_1 R_m v_1 v_0$ are preserved under bounded morphisms, but they are not modal formulas. The formulas that are preserved under bounded morphisms were characterized in [vB82, Theorem 15.11] and [Gol89, Theorem 4.2.5].

1.3.8. THEOREM. [Hol98, Theorem 2.4.2] *Let τ be a vocabulary, let $\varphi(v_0)$ be a $\mathcal{L}^{FO}(\tau)$-formula, and let K be a class of τ-structures closed under ultraproducts. The following are equivalent:*

1. *φ is, up to K-equivalence, in $\mathcal{L}^{MOD}(\tau)$.*

2. *There is $n \in \omega$ such that φ is K-invariant under n-bisimilarity.*

3. *φ is* K*-invariant under bisimilarity.*

The class of finite pointed τ-structures is clearly not closed under ultraproducts, but Rosen proved that the theorem remains valid for this class [Ros97]. The proof given there is rather complicated if one is not familiar with finite model theory (a certain acquaintance with Hanf's function [Han65] is recommended), but the argument can be easily reconstructed from [dRS01, Lemma 5.4.5]. A more self-contained presentation of the proof can be found in [Ott02].

1.3.9. THEOREM. [BdRV01, Theorem 2.75] *Let τ be a vocabulary, and let* K *be a class of pointed τ-structures. The following are equivalent:*

1. K *is definable by a set of $\mathcal{L}^{MOD}(\tau)$-formulas.*
2. K *is closed under bisimilarity and under ultraproducts, and its complementary class is closed under ultrapowers.*

1.3.10. THEOREM. [BdRV01, Theorem 2.76] *Let τ be a vocabulary, and let* K *be a class of pointed τ-structures. The following are equivalent:*

1. K *is definable by a single $\mathcal{L}^{MOD}(\tau)$-formula.*
2. *Both* K *and its complement are closed under bisimilarity and ultraproducts.*

1.3.11. THEOREM. [BM98, Proposition 3.5] *Let τ be a vocabulary, and let* K *be a class of pointed τ-structures. The following are equivalent:*

1. K *is definable by a (maybe proper) class of $\mathcal{L}^{MOD}_{\infty}(\tau)$-formulas.*
2. K *is closed under bisimilarity.*

1.3.12. THEOREM. [BM98, Proposition 3.7] *Let τ be a vocabulary, and let* K *be a class of pointed τ-structures. The following are equivalent:*

1. K *is definable by a set of $\mathcal{L}^{MOD}_{\infty}(\tau)$-formulas.*
2. K *is definable by a $\mathcal{L}^{MOD}_{\infty}(\tau)$-formula.*
3. *There exists an ordinal α such that* K *is closed under α-bisimilarity.*

Now we examine the tools usually used to prove van Benthem's Theorem. In the infinitary version it can be proved as an application of the Boundedness Theorem (see the proof given in [vB99]). We will concentrate on the finitary version. What would make the proof very simple is the reciprocal implication of (1.2). Unfortunately, this does not hold in general (see Figure 1.5). Therefore, the trick is to saturate a structure, as in classical model theory [Cha73, CK90, Poi00]. But now, we only need to saturate it from the point of view of modal formulas. Let us consider certain degrees of saturation in a τ-structure $\mathfrak{A}$.

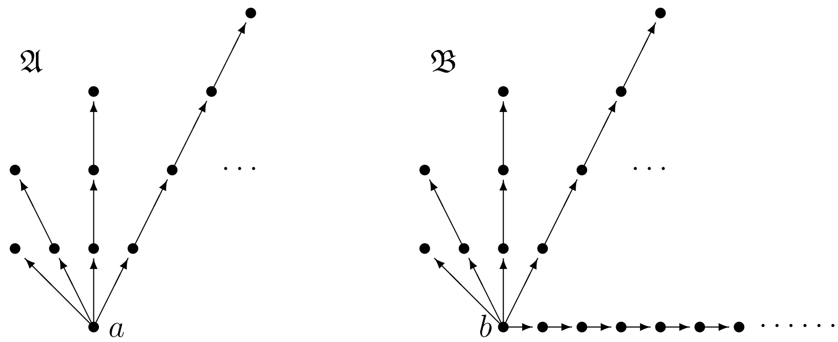


Figure 1.5: $\langle \mathfrak{A}, a\rangle \leftrightsquigarrow \langle \mathfrak{B}, b\rangle$ (indeed, $\langle \mathfrak{A}, a\rangle \simeq_\omega \langle \mathfrak{B}, b\rangle$) but $\langle \mathfrak{A}, a\rangle \not\simeq \langle \mathfrak{B}, b\rangle$

1. $\mathfrak{A}$ is *finite* if the universe A is finite.

2. $\mathfrak{A}$ is *image finite* if for every state a in A the set of successors of a is finite.

3. $\mathfrak{A}$ is *hereditarily finite* if it is image finite and R^{-1} is well-founded (i.e., there is no infinite sequence of states such that $a_0\ R\ a_1\ R\ a_2\ R \ldots$). In hereditarily finite structures, by König's Tree Lemma [Dev91, Theorem 4.4.1], all substructures generated by a certain state are finite. Thus, if $\mathfrak{A}$ is generated by a state then it must be finite.

4. $\mathfrak{A}$ is *modally saturated* if for every modality $m \in \mathsf{Mod}$, every state $a \in A$ and every set Φ of modal formulas, if Φ is finitely satisfiable in the set of m-successors of a (i.e., for every $\Phi' \subseteq_\omega \Phi$ there exists a state a' with $\mathfrak{A}, a' \Vdash \Phi'$ that is an m-successor of a) then Φ is satisfiable in a m-successor of a (i.e., there exists a' a m-successor of a with $\mathfrak{A}, a' \Vdash \Phi$). An hitherto unknown remark is that to obtain a modally saturated structure it is only necessary to impose the previous condition to sets Φ that are maximally satisfiable. Let us prove this remark. Assume $\kappa \in \mathbb{C}\text{ARD}$, $\langle \varphi_\alpha : \alpha < \kappa\rangle$ is an enumeration of all modal formulas, and Φ is finitely satisfiable in the set of m-successors of a. We define $\langle \Sigma_\alpha : \alpha < \kappa\rangle$ by induction on α as the sequence:

   (a) $\Sigma_0 := \Phi$,

   (b) $\Sigma_{\alpha+1} := \begin{cases} \Sigma_\alpha \cup \{\varphi_\alpha\} & \text{if } \Sigma_\alpha \cup \{\varphi_\alpha\} \text{ is finitely satisfiable in } R_m[\{a\}], \\ \Sigma_\alpha \cup \{\sim \varphi_\alpha\} & \text{if not,} \end{cases}$

   (c) $\Sigma_\alpha := \bigcup_{\beta<\alpha} \Sigma_\beta$, if α is a limit ordinal.

   CLAIM: For every $\alpha < \kappa$, Σ_α is finitely satisfiable in the set of m-successors of a.

   Proof of Claim: The proof is by induction on α. The only non-trivial step is when we move from α to its successor. Assume $\Sigma_\alpha \cup \{\varphi_\alpha\}$ is not finitely satisfiable in $R_m[\{a\}]$. Thus, there exists $\Delta \subseteq_\omega \Sigma_\alpha$ such that $\Delta \cup \{\varphi_\alpha\}$ is not satisfiable in any m-successor of a. Let us now see that $\Sigma_\alpha \cup \{\sim \varphi_\alpha\}$ is finitely satisfiable in $R_m[\{a\}]$. We consider $\Sigma' \subseteq_\omega \Sigma_\alpha$. Then, $\Sigma' \cup \Delta \subseteq_\omega \Sigma_\alpha$. Hence, by the inductive hypothesis there exists a' an m-successor of a that satisfies $\Sigma' \cup \Delta$. And $\mathfrak{A}, a' \not\Vdash \varphi_\alpha$ because $\Delta \cup \{\varphi_\alpha\}$ is not satisfiable in $R_m[\{a\}]$. Thus, $\mathfrak{A}, a' \Vdash \Sigma' \cup \{\sim \varphi_\alpha\}$. $\dashv$

It follows from our claim that $\Sigma := \bigcup_{\alpha<\kappa} \Sigma_\alpha$ is finitely satisfiable in the set of m-successors of a. By the Compactness Theorem of first-order logic it is deduced that Σ is satisfiable. And clearly Σ is maximally satisfiable because for every modal formula either it or its negation belongs to Σ. Thus, if we know modal saturation for maximally satisfiable sets, it follows that Σ is satisfiable in an m-successors of a. Hence, Φ is satisfiable in an m-successor of a, and this concludes the proof of the remark.

5. $\mathfrak{A}$ is *$\mathcal{H}$-closed* if for every modality $m \in \mathsf{Mod}$ and all states a and a', if $\mathfrak{A}, a' \Vdash \{\varphi \in \mathcal{L}^{MOD} : \mathfrak{A}, a \Vdash [m]\varphi\}$ then there exists an m-successor a'' of a such that $\langle \mathfrak{A}, a' \rangle \leftrightsquigarrow \langle \mathfrak{A}, a'' \rangle$.

6. $\mathfrak{A}$ is *Hennessy-Milner* if for all states a and a', if $\langle \mathfrak{A}, a \rangle \leftrightsquigarrow \langle \mathfrak{A}, a' \rangle$ then $\langle \mathfrak{A}, a \rangle \simeq \langle \mathfrak{A}, a' \rangle$. That is, the other direction of (1.2) holds whenever we take pointed structures over $\mathfrak{A}$. In other words $\{\langle a, a' \rangle \in A \times A : \langle \mathfrak{A}, a \rangle \leftrightsquigarrow \langle \mathfrak{A}, a' \rangle\}$ is a bisimulation inside $\mathfrak{A}$.

7. $\mathfrak{A}$ is *ω-saturated* if whenever $n \in \omega$, $\Phi(v, v_1, \ldots, v_n)$ is a set of first-order formulas, $a_1, \ldots, a_n$ are states, and $\Phi(v, a_1, \ldots, a_n)$ is finitely satisfiable (i.e., for every $\Phi' \subseteq_\omega \Phi$ there exists a state a with $\mathfrak{A} \models \Phi'[\![a, a_1, \ldots, a_n]\!]$) then $\Phi(v, a_1, \ldots, a_n)$ is satisfiable (i.e., there exists a state a with $\mathfrak{A} \models \Phi[\![a, a_1, \ldots, a_n]\!]$). By [CK90, Proposition 2.3.6] we can replace v with a finite sequence of variables, i.e., if $\mathfrak{A}$ is ω-saturated and $\Phi(\vec{v}, a_1, \ldots, a_n)$ is finitely satisfiable then this set is also satisfiable as a whole.

All these notions are very well known in the literature. The notion of ω-saturation comes from classical model theory. It is well known that each structure can be extended to an ω-saturated elementary extension. Moreover, it is possible to choose this extension as an ultrapower of the original structure[16]. The rest of the concepts can be found in the modal literature. It is known that ultrafilter extensions yield modally saturated structures (see [Hol95, Theorem 2.10]), so we can think of them as the modal counterpart of ultrapowers. A systematic study of these notions, except hereditary finiteness and ω-saturation, can be found in [Hol98, Chapter 5] (a shortened published version is [Hol95]). As an easy consequence of Hollenberg's work (see [Hol98, p. 140]), the relations between

[16]For countable languages, this is shown in [CK90, Theorem 6.1.1]. And for languages of arbitrary cardinality just combine [CK90, Theorem 6.1.4] and [CK90, Theorem 6.1.8].

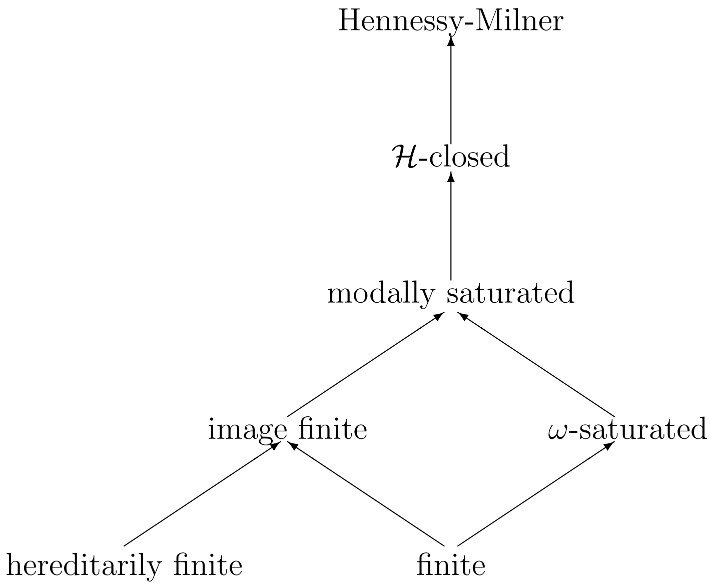


Figure 1.6: The landscape of properties for a structure

these properties can be depicted as in Figure 1.6[17]. The arrows indicate strict inclusions, while the absence of such an arrow refers to non-comparability (modulo transitive closure). Specifically, the figure says that if a structure satisfies any of the previous seven properties then it is Hennessy-Milner. It is also interesting to recall that image finite structures and hereditarily finite structures can be characterized (modulo bisimilarity) through modal languages.

1.3.13. Theorem. *Let τ be a vocabulary, and let $\mathfrak{A}$ be a τ-structure. The following are equivalent:*

1. *There is an image finite $\mathfrak{B}$ such that $\mathfrak{A} \simeq \mathfrak{B}$.*
2. *$\mathfrak{coll}\,\mathfrak{A}$ is image finite.*
3. *For every $a \in A$, $\langle \mathfrak{A}, a \rangle$ is characterized (up to bisimilarity) by a set of $\mathcal{L}^{MOD}(\tau)$-formulas, i.e., for every pointed structure $\langle \mathfrak{B}, b \rangle$ it holds that $\langle \mathfrak{A}, a \rangle \leftrightsquigarrow \langle \mathfrak{B}, b \rangle$ iff $\langle \mathfrak{A}, a \rangle \simeq \langle \mathfrak{B}, b \rangle$.*

1.3.14. Remark. In order to enunciate this theorem for arbitrary (maybe not finite) vocabularies observe that we have not written as another equivalent statement that $\text{rank}(\mathfrak{A}, a) \leq \omega$ for every $a \in A$. Indeed, Remark 1.3.2 gives a counterexample. A proof of the above theorem restricted to finite vocabularies can

[17] The position of ω-saturation is easily established by [BdRV01, Theorem 2.65] together with, for instance, the fact that $\langle \mathbb{Q}, < \rangle$ is an ω-saturated structure (but it is not image finite) while $\langle \{\langle n, k \rangle : n < k \in \omega\}, \{\langle \langle n_0, k \rangle, \langle n_1, k \rangle \rangle : n_0 < n_1 < k \in \omega\} \rangle$ it is not (but it is image finite).

be found in [BM97, Theorem 11.20]. In order to obtain a proof for arbitrary vocabularies simply follow Hollenberg in [Hol95, Proposition 3.12] and [Hol95, Theorem 3.13].

1.3.15. THEOREM. [BM98, Proposition 3.4] *Let τ be a finite vocabulary, and let $\mathfrak{A}$ be a τ-structure. The following are equivalent:*

1. *There is an hereditarily finite $\mathfrak{B}$ such that $\mathfrak{A} \simeq \mathfrak{B}$.*

2. *$\mathfrak{coll}\,\mathfrak{A}$ is hereditarily finite.*

3. *For every $a \in A$, $\langle \mathfrak{A}, a\rangle$ is characterized (up to bisimilarity) by a single $\mathcal{L}^{MOD}(\tau)$-formula, i.e., exists $n \in \omega$ such that for every pointed structure $\langle \mathfrak{B}, b\rangle$ it holds that $\langle \mathfrak{A}, a\rangle \leftrightsquigarrow_n \langle \mathfrak{B}, b\rangle$ iff $\langle \mathfrak{A}, a\rangle \simeq \langle \mathfrak{B}, b\rangle$.*

Until now we have talked about saturation for a certain structure. Obviously we can introduce the same notions for sets of structures whenever the disjoint union satisfies the corresponding notion. But we would like to consider also arbitrary classes of structures. Given K a class of structures, it is said that it is a *Hennessy-Milner class* if for all $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$, all $a \in A$, and all $b \in B$ it holds that:

$$\langle \mathfrak{A}, a\rangle \leftrightsquigarrow \langle \mathfrak{B}, b\rangle \quad \text{implies} \quad \langle \mathfrak{A}, a\rangle \simeq \langle \mathfrak{B}, b\rangle.$$

Whenever K is a set, it is the same to claim K is a Hennessy-Milner class as to claim $\biguplus \mathsf{K}$ is a Hennessy-Milner structure. Thus, there is no ambiguity. Hollenberg showed that the class of modally saturated structures is a (maximal) Hennessy-Milner class, while the class of $\mathcal{H}$-closed structures is not. Indeed, the next two lemmas can be considered as the crucial ideas to prove van Benthem's Theorem. The second is an easy consequence, using Figure 1.6, of the first one.

1.3.16. LEMMA. [Hol95, Theorem 2.5] *Let τ be a vocabulary. Then, the class of structures that are modally saturated is a Hennessy-Milner class.*

1.3.17. LEMMA (DETOUR). [BdRV01, Theorem 2.66] *Let τ be a vocabulary, and let $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ be two τ-pointed structures. Then, the following are equivalent:*

1. *$\langle \mathfrak{A}, a\rangle \leftrightsquigarrow \langle \mathfrak{B}, b\rangle$.*

2. *$\langle \mathfrak{ue}\,\mathfrak{A}, \pi_a\rangle \simeq \langle \mathfrak{ue}\,\mathfrak{B}, \pi_b\rangle$.*

3. *There exist ultrapowers $\Pi_{U_1}\langle \mathfrak{A}, a\rangle$ and $\Pi_{U_2}\langle \mathfrak{B}, b\rangle$ that are bisimilar.*

To sum up, although in general the reciprocal of (1.2) does not hold the Detour Lemma allows us to change our original pointed structures for different ones where it holds. This is the idea underlying the proof of van Benthem's Theorem.

Maximal Hennessy-Milner classes have been studied in depth by Hollenberg in [Hol95] and [Hol98]. There he develops, following a suggestion by Visser, a very elegant characterization of maximal Hennessy-Milner classes based on canonical-like structures[18] (see [Hol95, Theorems 3.5 and 3.7] for the details). In the theorems below we summarize the results obtained by him that we are interested in.

1.3.18. THEOREM. [Hol95] *Let* τ *be a vocabulary.*

1. *Hennessy-Milner classes are closed under the operators* B *and* S*, where* $\mathsf{B}(\mathsf{K}) := \{\mathfrak{B} : \textit{there exists } \mathfrak{A} \in \mathsf{K} \textit{ such that } \mathfrak{A} \simeq \mathfrak{B}\}$ *and* $\mathsf{S}(\mathsf{K}) := \{\mathfrak{B} : \textit{there exists } \mathfrak{A} \in \mathsf{K} \textit{ such that } \mathfrak{B} \rightarrowtail \mathfrak{A}\}$.
2. *Every Hennessy-Milner class can be extended to a maximal one.*
3. *Every maximal Hennessy-Milner class is of the form* $\mathsf{B}\,\mathsf{S}(\{\mathfrak{A}\})$ *for a certain Hennessy-Milner structure* $\mathfrak{A}$ *that shares universe and valuation with the canonical structure for the minimal normal modal logic.*
4. *The class of modally saturated structures is a maximal Hennessy-Milner class. It is precisely* $\mathsf{B}\,\mathsf{S}(\{\mathfrak{H}\})$ *where* $\mathfrak{H}$ *is the canonical structure for the minimal normal modal logic.*
5. *There exist at least* $2^{\aleph_0}$ *different maximal Hennessy-Milner classes.*

1.3.19. THEOREM. [Hol95] *Let* τ *be a vocabulary, and let* $\mathfrak{A}$ *be a* τ*-structure. The following are equivalent:*

1. $\mathfrak{A} \in \bigcap\{\mathsf{K} : \mathsf{K} \textit{ is a maximal Hennessy-Milner class}\}$.
2. *For every state* $a \in A$*, and every pointed structure* $\langle \mathfrak{B}, b\rangle$*, it holds that* $\langle \mathfrak{A}, a\rangle \leftrightsquigarrow \langle \mathfrak{B}, b\rangle$ *implies* $\langle \mathfrak{A}, a\rangle \simeq \langle \mathfrak{B}, b\rangle$.
3. $\mathfrak{c}oll\,\mathfrak{A}$ *is image finite.*

This theorem gives a characterization of image finite structures (modulo bisimilarity) by using maximal Hennessy-Milner classes. We can look at it as a complementary result to Theorem 1.3.13. However, there is no known counterpart for

[18] We will introduce canonical structures on page 38.

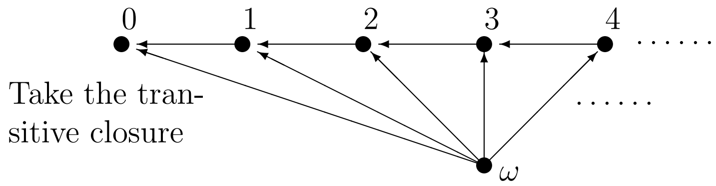


Figure 1.7: The canonical pointed structure associated with the set ω

Theorem 1.3.15 based on Hennessy-Milner classes. We will fill this gap thanks to the strict-weak fragments involved in this dissertation (see Corollary 3.5.18).

From now to the end of the section we will talk about non-well-founded sets. This is a different framework, but closely related to bisimilarity. The presence of the Foundation Axiom

$$(\mathbf{FA}) : \ \forall v_0\Big(\exists v_1(v_1 \in v_0) \supset \exists v_1\big(v_1 \in v_0 \wedge \forall v_2(v_2 \in v_1 \supset v_2 \notin v_0)\big)\Big)$$

in the canonical axiomatization of set theory implies that all sets of the universe are *well-founded*, i.e., there is no infinite sequence $a_0, a_1, a_2, \dots$ of sets such that $a_0 \ni a_1 \ni a_2 \ni \dots$ In order to admit non well-founded sets we must reject this axiom. So first of all, we work in a universe of sets that satisfies the axioms of $\mathbf{ZFC}^-$, i.e., the standard Zermelo-Fraenkel with the Axiom of Choice (see [End77, Dev91]) and without the Foundation Axiom. All that we consider (e.g., structures, etc) is inside this universe of sets.

Every set a can be viewed in a canonical way as a pointed τ-structure, where τ is the vocabulary with a single modality and no propositions. We fix this vocabulary for the rest of the section. The *canonical structure* associated with the set a, denoted by $\mathfrak{G}_a$, is defined as follows. The set of states is given by the transitive closure of $a \cup \{a\}$, i.e., the states are the elements in

$$\{a\} \cup \{b : \exists n \in \omega\ \exists b_0, \dots, b_n \text{ such that } b = b_0 \in b_1 \in \dots \in b_n \in a\}.$$

The accessibility relation is the inverse of the relation $\in$, i.e., it is $\{\langle b, b'\rangle \in G_a \times G_a : b' \in b\}$. Figure 1.7 shows an example.

Given a structure $\mathfrak{A}$, a *decoration of* $\mathfrak{A}$ is a function with domain A such that for every state $a \in A$,

$$d(a) = \{d(b) : \langle a, b\rangle \in R^{\mathfrak{A}}\}. \tag{1.3}$$

By definition of the accessibility relation in canonical structures, (1.3) can be rewritten for these structures as $d(a) = \{d(b) : b \in a\}$. As a consequence of the fact that $a = \{b : b \in a\}$, it is obvious that each canonical structure associated with a set admits a decoration: just take the identity function restricted to the set of states of the canonical structure.

The first results connecting this setting with bisimilarity are the following.

1.3.20. Proposition. [D'A98, p. 34] *(**ZFC**$^-$) Let $\langle\mathfrak{A}, a\rangle$ and $\langle\mathfrak{B}, b\rangle$ be pointed structures. If there are decorations $d_\mathfrak{A}$ and $d_\mathfrak{B}$ of, respectively, $\mathfrak{A}$ and $\mathfrak{B}$ such that $d_\mathfrak{A}(a) = d_\mathfrak{B}(b)$, then $\langle\mathfrak{A}, a\rangle \simeq \langle\mathfrak{B}, b\rangle$.*

1.3.21. Proposition. [D'A98, Proposition 2.6.1] *(**ZFC**$^-$) Let $\mathfrak{A}$ be a structure. If d is a decoration of $\mathfrak{coll}\,\mathfrak{A}$, then the function d^* defined on $\mathfrak{A}$ by $d^*(a) = d([a])$ is a decoration of $\mathfrak{A}$.*

Up to now we have been working in **ZFC**$^-$. In order to prove the existence of non-well-founded sets we need to add an additional axiom. The most famous axiom that allows this is the Antifoundation Axiom

(**AFA**) : For every structure $\mathfrak{A}$, there is one and only one decoration of $\mathfrak{A}$.

It is clear that this axiom allows the proof of the existence of non-well-founded sets, e.g., the set $d_\mathfrak{A}(a)$ where $d_\mathfrak{A}$ is the decoration of a structure $\mathfrak{A}$ that has a reflexive state a. This axiom was introduced by Forti and Honsell in [FH83], but it was not until the monograph [Acz88] by Aczel that researchers found interesting applications of this theory. Since then, this theory has enjoyed fruitful applications to different fields, e.g., communicating systems [Acz88], situation theory [BE87], modal logic [BM97, Bal98], and epistemic logic [Ger99]. In addition to these references, the author highly recommends [BM91] and [Dev91, Chapter 7] as introductory references to the topic (consistency, equivalent axioms, co-induction, etc.). Now we summarize the results that are relevant for bisimilarity.

1.3.22. Proposition. [Ger99, Proposition 1.6] *(**ZFC**$^-$ + **AFA**) Let $\langle\mathfrak{A}, a\rangle$ and $\langle\mathfrak{B}, b\rangle$ be two pointed structures. The following are equivalent:*

1. *There are decorations $d_\mathfrak{A}$ and $d_\mathfrak{B}$ of, respectively, $\mathfrak{A}$ and $\mathfrak{B}$ such that $d_\mathfrak{A}(a) = d_\mathfrak{B}(b)$.*
2. *$\langle\mathfrak{A}, a\rangle \simeq \langle\mathfrak{B}, b\rangle$*

1.3.23. Proposition. [D'A98, Corollary 2.6.2] *(**ZFC**$^-$ + **AFA**) Let a and b be two sets. Then,*

$$a = b \quad \textit{iff} \quad \mathfrak{G}_a \simeq \mathfrak{G}_b \quad \textit{iff} \quad \langle\mathfrak{G}_a, a\rangle \simeq \langle\mathfrak{G}_b, b\rangle.$$

1.3.24. Proposition. [D'A98, Proposition 2.6.3] *(**ZFC**$^-$ + **AFA**) Let $\mathfrak{A}$ be a structure that is generated by a certain state. The following are equivalent:*

1. *$\mathfrak{A} = \mathfrak{coll}\,\mathfrak{A}$.*

2. $\mathfrak{A}$ *is isomorphic to the canonical structure associated with a certain set.*

Hence, what we have in (**ZFC**$^-$ + **AFA**) is that equality between sets is just bisimilarity (between the canonical structures associated with these sets). Having this in mind it is very simple to obtain some of the results that were stated previously. For instance, as a consequence of the fact that there is a proper class of sets (by Russell's Paradox) Theorem 1.3.1(4) is trivial. Indeed, some of these results were previously proved in this framework, e.g., Theorems 1.3.3, 1.3.13 and 1.3.15[19]. The first of these theorems can be reformulated as saying that sets are univocally characterized by $\mathcal{L}^{mod}_{\infty}$-formulas.

1.3.25. THEOREM. [BM97, Theorem 11.12] *(***ZFC**$^-$ + **AFA***) For every set a, there exists a* $\mathcal{L}^{mod}_{\infty}$*-formula* ϕ^a *such that for every set b,*

$$\mathfrak{G}_b, b \Vdash \phi^a \quad \textit{iff} \quad a = b.$$

As modal languages do not distinguish between bisimilar states, we obtain the same expressive power if we restrict ourselves to pointed structures modulo bisimilarity. Thus, by the above results we could have defined the satisfiability relation between (non-well-founded) sets and formulas. This simple idea was fully developed by Barwise and Moss in [BM97].

1.3.26. REMARK. (What happens when there are propositions?) If there are propositions in the vocabulary τ, all previous considerations also hold if one first enlarges the axiomatic theory (**ZFC**$^-$ + **AFA**) with urelements, one for each proposition (see [BM97] for a development of this consideration). It is only for the sake of simplicity that we have not adopted this general situation.

In the literature there are other axioms, besides (**AFA**), that allow us to prove the existence of non-well-founded sets. Although, as we have seen, (**AFA**) is the axiom that behaves well with respect to the bisimulation notion commonly considered in modal languages, in her dissertation [D'A98] D'Agostino studied some of these other axioms from a bisimulation perspective: there she introduces different bisimulation notions to capture these axioms.

---

[19]These results were first proved by Baltag (see [BM97, Section 11.2–11.3]) using a cofinality argument over a transfinite sequence of infinitary modal formulas explicitly defined. Hence, in some sense, it is a constructive proof. For the case of Theorem 1.3.3 perhaps we should also consider the work of Fagin [Fag94]: there he obtains a version of the theorem for certain knowledge structures that are very close to our pointed structures.

1.4 Normal modal logics

Fix a vocabulary τ for the rest of the section. In the literature it is usually assumed that Prop is countable, but most of what we are going to say does not depend on this fact. The set of modalities Mod is commonly identified with its cardinality. In other words, we fix a vocabulary $\tau = \langle \kappa, \{p_n : n \in \omega\}\rangle$ with $\kappa \in \mathbb{C}\mathrm{ARD} \setminus \{0\}$ for the rest of the section.

A *normal modal logic* $\mathbf{\Lambda}$ is a set of $\mathcal{L}^{MOD}(\tau)$-formulas such that

- it contains all propositional tautologies,
- it is closed under *Modus Ponens*, i.e., if $\varphi_0 \in \mathbf{\Lambda}$ and $\varphi_0 \supset \varphi_1 \in \mathbf{\Lambda}$ then $\varphi_1 \in \mathbf{\Lambda}$,
- it is closed under *modal substitution*, i.e., if φ belongs to $\mathbf{\Lambda}$ then so do all of its modal substitution instances,
- it contains the formulas in $\{[m](p_0 \supset p_1) \supset ([m]p_0 \supset [m]p_1) : m \in \mathsf{Mod}\}$,
- it is closed under *generalization*[20], i.e., if $\varphi \in \mathbf{\Lambda}$ and $m \in \mathsf{Mod}$ then it holds that $[m]\varphi \in \mathbf{\Lambda}$.

It is clear that this definition could have been introduced through a Hilbert-style calculus, as is done sometimes in the literature. By (1.1) it is clear that for all modal formulas φ,

$$\varphi \in \mathbf{\Lambda} \quad \text{iff} \quad \sim\varphi^d \in \mathbf{\Lambda}. \tag{1.4}$$

Hence, for all modal formulas φ_0, φ_1, it holds that $\varphi_0 \supset \varphi_1 \in \mathbf{\Lambda}$ iff $\varphi_1^d \supset \varphi_0^d \in \mathbf{\Lambda}$. If $\mathbf{\Lambda}$ and $\mathbf{\Lambda'}$ are normal modal logics such $\mathbf{\Lambda} \subseteq \mathbf{\Lambda'}$, it is said that $\mathbf{\Lambda'}$ is an *extension* of $\mathbf{\Lambda}$. The set of all normal modal logics that are extensions of $\mathbf{\Lambda}$ is denoted by $\mathrm{Ext}\,\mathbf{\Lambda}$. By definition it is clear that the set of all $\mathcal{L}^{MOD}(\tau)$-formulas is a normal modal logic (called the *inconsistent logic*), and that normal modal logics are closed under intersection. Therefore, there exists a minimum normal modal logic (just consider the intersection of all normal modal logics), which is called $\boldsymbol{K_\kappa}$ in honour of Kripke. Hence the set of all normal modal logics is precisely $\mathrm{Ext}\,\boldsymbol{K_\kappa}$. Given a family $\{\mathbf{\Lambda_i} : i \in I\}$ of normal modal logics its *sum* $\bigoplus_{i\in I} \mathbf{\Lambda_i}$ is the intersection of all normal modal logics containing $\bigcup_{i\in I} \mathbf{\Lambda_i}$. It is a normal modal logic, and it is the smallest one that is an extension of all $\mathbf{\Lambda_i}$. It results that $\langle \mathrm{Ext}\,\boldsymbol{K_\kappa}, \bigcap, \bigoplus\rangle$ is a complete lattice. It is known that it is distributive (indeed, the intersection distributes over the infinite sum[21]). Given a normal modal logic

[20]We notice that this rule has also been called *necessity* in the literature. Here we follow the terminology used in [BdRV01].

[21]However, in general the sum does not distribute over the infinite intersection (cf. [CZ97, Exercise 6.16])

$\mathbf{\Lambda}$ and a set of Φ of modal formulas,

$$\mathbf{\Lambda} \oplus \Phi$$

denotes the smallest normal modal logic containing $\mathbf{\Lambda} \cup \Phi$. It is clear (just consider Φ as $\mathbf{\Lambda}$) that for every normal modal logic $\mathbf{\Lambda}$, there is a set Φ such that $\mathbf{\Lambda} = \boldsymbol{K_\kappa} \oplus \Phi$. In this case it is said that Φ is an *axiomatization* of $\mathbf{\Lambda}$. If Φ can be chosen finite[22], then we call $\mathbf{\Lambda}$ *finitely axiomatizable*. We write $\mathbf{\Lambda} \oplus \varphi$ when $\Phi = \{\varphi\}$. By (1.4) it is obvious that $\mathbf{\Lambda} \oplus \varphi = \mathbf{\Lambda} \oplus \sim\varphi^d$. Hence if φ axiomatizes a normal modal logic then $\sim\varphi^d$ also axiomatizes the same normal modal logic. Specifically this says that if $\varphi_0 \supset \varphi_1$ axiomatizes a normal modal logic then $\varphi_1^d \supset \varphi_0^d$ also axiomatizes it.

1.4.1. Example. Let us consider the case $\mathcal{L}^{mod}$, i.e., $\kappa = 1$. In this case the minimal normal modal logic $\boldsymbol{K_1}$ is simply denoted by $\boldsymbol{K}$. There are many other examples of normal modal logics, of which we highlight the following. The notation of the names is somewhat involved, but it has been the standard one since the monograph [Lem77]. The formulas here considered refer to the ones introduced in Table 1.2.

$$\begin{array}{rlrl}
\boldsymbol{T} := & \boldsymbol{K} \oplus \mathsf{T} & \boldsymbol{K4} := & \boldsymbol{K} \oplus \mathsf{4} \\
\boldsymbol{S4} := & \boldsymbol{T} \oplus \mathsf{4} & \boldsymbol{S5} := & \boldsymbol{S4} \oplus \mathsf{5} \\
\boldsymbol{D} := & \boldsymbol{K} \oplus \mathsf{D} & \boldsymbol{Alt} := & \boldsymbol{K} \oplus \mathsf{Alt} \\
\boldsymbol{DAlt} := & \boldsymbol{D} \oplus \mathsf{Alt} & \boldsymbol{KD45} := & \boldsymbol{D} \oplus \mathsf{4} \oplus \mathsf{5} \\
\boldsymbol{K4.3} := & \boldsymbol{K4} \oplus \mathsf{.3} & \boldsymbol{S4.3} := & \boldsymbol{S4} \oplus \mathsf{.3} \\
\boldsymbol{GL} := & \boldsymbol{K} \oplus \mathsf{GL} & \boldsymbol{GL.3} := & \boldsymbol{GL} \oplus \mathsf{.3} \\
\boldsymbol{Grz} := & \boldsymbol{K} \oplus \mathsf{Grz} & \boldsymbol{Grz.3} := & \boldsymbol{Grz} \oplus \mathsf{.3} \\
\boldsymbol{Verum} := & \boldsymbol{K} \oplus \mathsf{V} & \boldsymbol{Triv} := & \boldsymbol{K} \oplus \mathsf{Tr}
\end{array}$$

In Figure 1.8 (see [GKWZ03, p. 14]) the reader can find all the relationships between these logics. The arrows indicate strict inclusions, while the absence of arrows indicates to non-comparability (modulo transitive closure).

Normal modal logics give us two finitary structural consequence relations: local and global. Recall first that a *consequence relation* over a propositional language $\mathcal{L}$ is a relation $\vdash$ between sets of formulas and formulas satisfying that

1. If $\varphi \in \Phi$, then $\Phi \vdash \varphi$.

2. If $\Phi \vdash \varphi$ and for every $\phi \in \Phi$, $\Psi \vdash \phi$, then $\Psi \vdash \varphi$.

From (1) and (2) it follows that:

[22]Indeed, it is possible to assume that it is a singleton: just take its conjunction.

Table 1.2: Some canonical and 'complete' formulas in the modal case.

Canonical Formulas		
T	$[m]p_0 \supset p_0$	R_m is *reflexive*, i.e., $Id \subseteq R_m$
4	$[m]p_0 \supset [m][m]p_0$	R_m is *transitive*, i.e., $R_m \circ R_m \subseteq R_m$
5	$\langle m\rangle[m]p_0 \supset [m]p_0$	R_m is *Euclidean*, i.e., $R_m \circ R_m^{-1} \subseteq R_m$
B	$\langle m\rangle[m]p_0 \supset p_0$	R_m is *symmetric*, i.e., $R_m \subseteq R_m^{-1}$
	$p_0 \supset [m]p_0$	$R_m \subseteq Id$
V	$[m]\bot$	$R_m = \emptyset$
D	$\langle m\rangle\top$	R_m is *serial*, i.e., $A = \text{dom } R_m$
Alt	$\langle m\rangle p_0 \supset [m]p_0$	R_m is *functional*, i.e., $R_m \circ R_m^{-1} \subseteq Id$
Tr	$p_0 \supset\subset [m]p_0$	$R_m = Id$
	$[m][m]p_0 \supset [m]p_0$	R_m is *dense*, i.e., $R_m \subseteq R_m \circ R_m$
	$[m']p_0 \supset [m]p_0$	$R_m \subseteq R_{m'}$
	$\langle m\rangle[m']p_0 \supset p_0$	$R_m \subseteq R_{m'}^{-1}$
	$[m]p_0 \supset [m][m']p_0$	$R_{m'} \circ R_m \subseteq R_m$
	$[m']p_0 \supset [m][m']p_0$	$R_{m'} \circ R_m \subseteq R_{m'}$
	$[m'][m]p_0 \supset [m][m']p_0$	$R_{m'} \circ R_m \subseteq R_m \circ R_{m'}$
	$[m][m']p_0 \supset p_0$	$Id \subseteq R_{m'} \circ R_m$
	$\langle m\rangle[m']p_0 \supset [m']p_0$	$R_{m'} \circ R_m^{-1} \subseteq R_{m'}$
con	$([m]p_0 \to_m p_1) \lor ([m]p_1 \to_m p_0)$	R_m is *connected*, i.e., $R_m \circ R_m^{-1} \subseteq R_m \cup R_m^{-1}$
.3	$([m]^{(1)}p_0 \to_m p_1) \lor ([m]^{(1)}p_1 \to_m p_0)$	R_m is *weakly connected*, i.e., $R_m \circ R_m^{-1} \subseteq Id \cup R_m \cup R_m^{-1}$
	$\langle m\rangle[m]p_0 \supset [m]\langle m\rangle p_0$	R_m is *directed*, i.e., $R_m \circ R_m^{-1} \subseteq R_m^{-1} \circ R_m$
.2	$\langle m\rangle([m]p_0 \land p_1) \supset [m](\langle m\rangle p_0 \lor p_1)$	R_m is *weakly directed*, i.e., $R_m \circ R_m^{-1} \subseteq Id \cup (R_m^{-1} \circ R_m)$

Formulas axiomatizing complete logics		
GL	$([m]p_0 \to_m p_0) \supset [m]p_0$	R_m is a *Noetherian* (i.e., R_m^{-1} is well-founded) strict order
Grz	$((p_0 \to_m [m]p_0) \to_m p_0) \supset p_0$	R_m is a Noetherian partial order

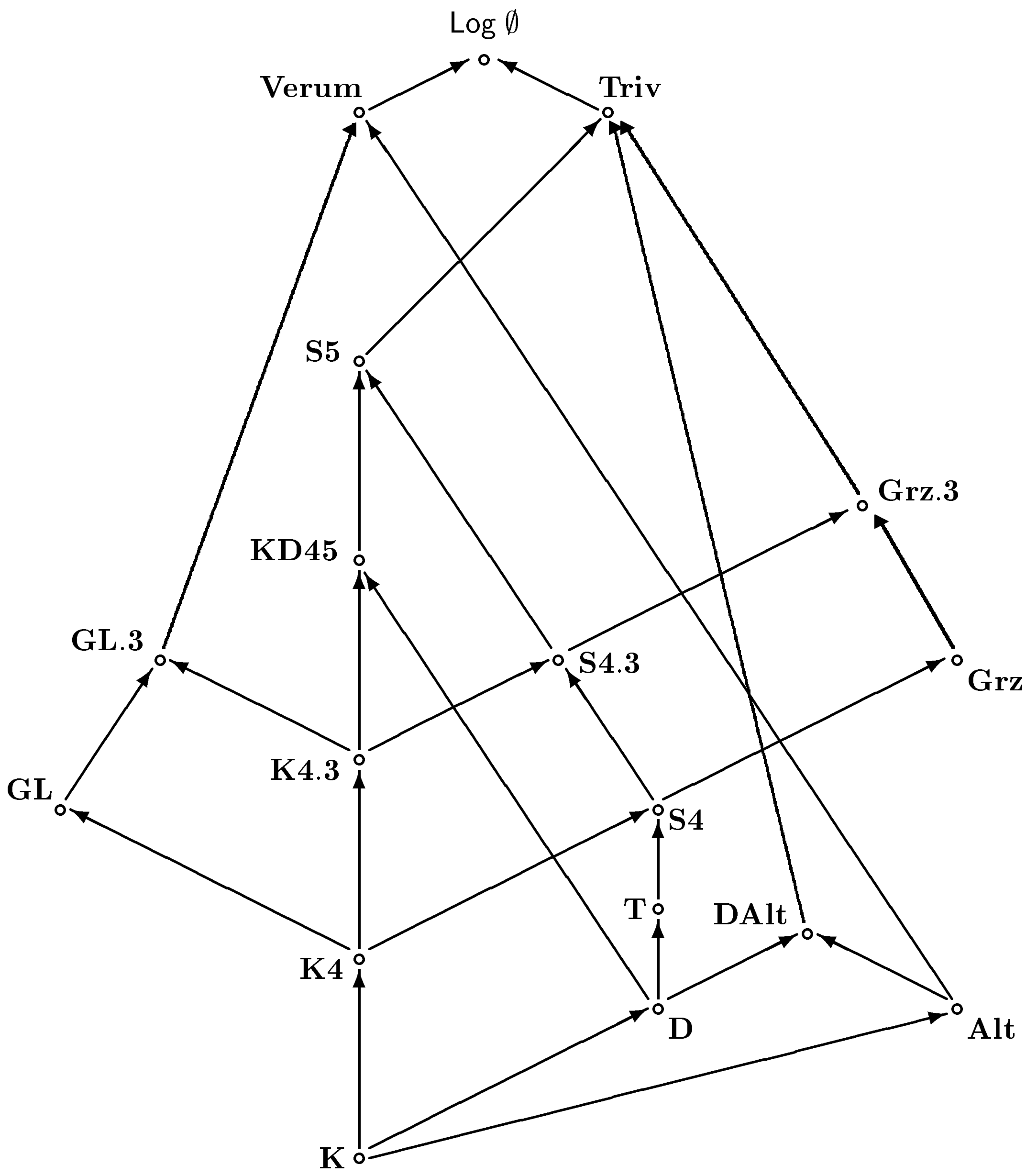


Figure 1.8: The comparison of 'standard' normal modal logics

3. If $\Phi \vdash \varphi$ and $\Phi \subseteq \Phi'$, then $\Phi' \vdash \varphi$.

It is *structural* if, moreover, it satisfies the following condition:

4. If $\Phi \vdash \varphi$, then for any substitution e, $e[\Phi] \vdash e(\varphi)$, where a *substitution* is a homomorphism from the formula algebra $\mathbf{Fm}$ into itself. This property is called *substitution invariance*.

It is said that it is *finitary* if it holds that

5. If $\Phi \vdash \varphi$ then there is a finite subset Φ' of Φ with $\Phi' \vdash \varphi$.

A set Σ of formulas is a *theory of* $\vdash$ if it is closed under $\vdash$, i.e., if $\Sigma \vdash \varphi$ then $\varphi \in \Sigma$. It is a *consistent* theory if it is not the set of all formulas. A *maximal consistent theory* is a consistent theory such that it cannot be properly extended to a consistent theory. The *local consequence* associated with a normal modal logic $\mathbf{\Lambda}$ is the finitary structural consequence relation $\vdash_{l\mathbf{\Lambda}}$ in the language $\mathcal{L}^{MOD}(\tau)$[23] defined as follows:

$$\Phi \vdash_{l\mathbf{\Lambda}} \varphi \quad \text{iff} \quad \text{there is a finite subset } \Phi' \text{ of } \Phi \text{ such that } \bigwedge \Phi' \supset \varphi \in \mathbf{\Lambda}.$$

And the *global consequence* associated with $\mathbf{\Lambda}$ is the finitary structural consequence relation $\vdash_{g\mathbf{\Lambda}}$ in the language $\mathcal{L}^{MOD}(\tau)$ defined in the following way:

$$\Phi \vdash_{g\mathbf{\Lambda}} \varphi \quad \text{iff} \quad \{[m]^n\phi : n \in \omega, m \in \mathsf{Mod}, \phi \in \Phi\} \vdash_{l\mathbf{\Lambda}} \varphi.$$

Hence, $\vdash_{g\mathbf{\Lambda}}$ is precisely the closure of the local consequence under the generalization rule. The generalization rule is derivable in $\vdash_{g\mathbf{\Lambda}}$, while in $\vdash_{l\mathbf{\Lambda}}$ it is only admissible. As normal modal logics are closed under this rule it follows that

$$\vdash_{l\mathbf{\Lambda}} \varphi \quad \text{iff} \quad \varphi \in \mathbf{\Lambda} \quad \text{iff} \quad \vdash_{g\mathbf{\Lambda}} \varphi.$$

It is also known that local consequences satisfy the *deduction-detachment theorem*, i.e., for every set Φ of modal formulas, and every pair of modal formulas ϕ and φ, it holds that

$$\Phi \cup \{\phi\} \vdash_{l\mathbf{\Lambda}} \varphi \quad \text{iff} \quad \Phi \vdash_{l\mathbf{\Lambda}} \phi \supset \varphi.$$

It is not hard to see that maximal consistent theories of $\vdash_{l\mathbf{\Lambda}}$ are precisely the consistent theories such that for every modal formula, it or its material negation is in this theory.

One of the simplest ways of introducing normal modal logics is based on frames. A *frame* is a pair $\mathfrak{F} = \langle F, \{R_m : m \in \mathsf{Mod}\}\rangle$ where (i) F is a non-empty

[23] In this language it is clear that the abstract notion of substitution considered in the substitution invariance coincides precisely with what we have called modal substitutions.

set, called the *universe*, of *states*, and (ii) for each $m \in \mathsf{Mod}$, R_m is a binary relation on F called the *accessibility relation associated with* m. Hence, the only difference with a structure is that there is no valuation. Given a frame $\mathfrak{F}$, a valuation $V : \mathsf{Prop} \longrightarrow \mathcal{P}(F)$ and a state $a \in F$ we can consider the pointed structure $\langle \mathfrak{F}, V, a \rangle$. A modal formula φ is said to be *valid in* $\mathfrak{F}$ (notation: $\mathfrak{F} \Vdash \varphi$) if for every valuation $V : \mathsf{Prop} \longrightarrow \mathcal{P}(F)$ and every state $a \in F$ it holds that $\mathfrak{F}, V, a \Vdash \varphi$. And φ *is satisfiable in* $\mathfrak{F}$ if there is a valuation $V : \mathsf{Prop} \longrightarrow \mathcal{P}(F)$ and a state $a \in F$ where $\mathfrak{F}, V, a \Vdash \varphi$. Analogously to the case of structures, at the level of frames we introduce the notions of *subframe*, *generated subframe*, *disjoint union* of frames and *bounded morphism* between frames. It is known that the validity of a modal formula is preserved under generated subframes, disjoint unions and surjective bounded morphisms. A class C of frames is *definable by a set of modal formulas* if it is exactly the class of frames where these formulas are valid. Obviously definable classes are closed under the previous operations. The problem of characterizing the classes of frames definable by modal formulas was solved by Goldblatt and Thomason in [GT74].

There are many examples of properties on accessibility relations that can be characterized as the validity of certain modal formulas in the frame. In Table 1.2 we find the definition of some of the most famous: reflexivity, transitivity, etc. Below we extend the list by introducing new properties, and we also repeat some of the more involved definitions of Table 1.2 using a clearer presentation. Given a set A, a binary relation R on the set A is:

- a *quasi order* if it is reflexive and transitive.
- an *equivalence relation* if it is a symmetric quasi order.
- *antisymmetric* if $\mathfrak{A} \models \forall xy((Rxy \wedge Ryx) \supset x \approx y)$.
- a *partial order* if it is antisymmetric and also a quasi-order.
- *irreflexive* if $\mathfrak{A} \models \forall x \sim Rxx$.
- a *strict order* if it is irreflexive and transitive.
- *Euclidean* if $\mathfrak{A} \models \forall v_0 v_1 v_2((Rv_0v_1 \wedge Rv_0v_2) \supset Rv_1v_2)$.
- *connected* if $\mathfrak{A} \models \forall v_0 v_1 v_2((Rv_0v_1 \wedge Rv_0v_2) \supset (Rv_1v_2 \vee Rv_2v_1))$.
- *weakly connected* if $\mathfrak{A} \models \forall v_0 v_1 v_2((Rv_0v_1 \wedge Rv_0v_2) \supset (v_1 \approx v_2 \vee Rv_1v_2 \vee Rv_2v_1))$.
- *directed* if $\mathfrak{A} \models \forall v_0 v_1 v_2((Rv_0v_1 \wedge Rv_0v_2) \supset \exists v_3(Rv_1v_3 \wedge Rv_2v_3))$.
- *weakly directed* if $\mathfrak{A} \models \forall v_0 v_1 v_2((Rv_0v_1 \wedge Rv_0v_2) \supset (v_1 \approx v_2 \vee \exists v_3(Rv_1v_3 \wedge Rv_2v_3)))$.

- *Noetherian* if there is no infinite sequence $\langle a_n : n \in \omega\rangle$ of elements (not necessarily distinct) in A such that $\langle a_n, a_{n+1}\rangle \in R$ for every $n \in \omega$.

We note that all these properties except the last one can be expressed by first-order formulas. Hence, the structures satisfying these properties are closed under ultraproducts.

1.4.2. PROPOSITION. [CZ97, Section 3.5] *For each one of the formulas in Table 1.2, a frame validates the formula iff the frame satisfies the condition on the right.*

Given an arbitrary class C of frames it is easy to check that

$$\operatorname{Log}\mathsf{C} := \{\varphi \in \mathcal{L}^{MOD}(\tau) : \forall\mathfrak{F} \in \mathsf{C}\ \mathfrak{F} \Vdash \varphi\}$$

is a normal modal logic. It is called the *normal modal logic of* C. The normal modal logics that are obtained by this method are called *frame complete*. The inconsistent logic is an example of a frame complete one; in fact, it is precisely $\operatorname{Log}\emptyset$. Since the seventies it has been known that there are modal logics that are not obtained through this method, i.e., there are frame incomplete normal modal logics[24]. The class $\operatorname{Fr}\mathbf{\Lambda}$ is the class of frames where all formulas in $\mathbf{\Lambda}$ are valid. By Proposition 1.4.2 we know which classes of frames for the normal modal logics are involved in Figure 1.8, e.g.,

$$\begin{aligned}
\operatorname{Fr}\boldsymbol{S4} &= \{\mathfrak{F} : \mathfrak{F} \text{ is a quasi order}\}\\
\operatorname{Fr}\boldsymbol{GL} &= \{\mathfrak{F} : \mathfrak{F} \text{ is a Noetherian strict order}\}\\
\operatorname{Fr}\boldsymbol{GL.3} &= \{\mathfrak{F} : \mathfrak{F} \text{ is a Noetherian weakly connected strict order}\}\\
\operatorname{Fr}\boldsymbol{Grz} &= \{\mathfrak{F} : \mathfrak{F} \text{ is a Noetherian partial order}\}\\
\operatorname{Fr}\boldsymbol{Grz.3} &= \{\mathfrak{F} : \mathfrak{F} \text{ is a Noetherian weakly connected partial order}\}.
\end{aligned}$$

In general $\mathsf{C} \subseteq \operatorname{Fr}\operatorname{Log}\mathsf{C}$ and $\mathbf{\Lambda} \subseteq \operatorname{Log}\operatorname{Fr}\mathbf{\Lambda}$. It is obvious that a normal modal logic $\mathbf{\Lambda}$ is frame complete iff $\mathbf{\Lambda} = \operatorname{Log}\operatorname{Fr}\mathbf{\Lambda}$; and a class C of frames is definable by modal formulas iff $\mathsf{C} = \operatorname{Fr}\operatorname{Log}\mathsf{C}$. Frames can also be used to introduce structural consequence relations. Before explaining how this is achieved we introduce a new definition. It is said that a modal formula φ is *valid in a structure* $\mathfrak{A}$ (notation: $\mathfrak{A} \Vdash \varphi$) when $\mathfrak{A}, a \Vdash \varphi$ for every state $a \in A$; it is written $\mathfrak{A} \Vdash \Phi$ if all the formulas in the set Φ are satisfiable. Given an arbitrary class C of frames we define

$$\Phi \models_{l\mathsf{C}} \varphi \quad \text{iff} \quad \forall\mathfrak{F} \in \mathsf{C}\ \forall V \in \mathcal{P}(F)^{\mathsf{Prop}}\ \forall a \in F,\ \text{if } \mathfrak{F}, V, a \Vdash \Phi \text{ then } \mathfrak{F}, V, a \Vdash \varphi,$$

and

[24] There is a huge literature on this topic. The most important papers are [Fin74a, Tho74, Blo78] which were the first to solve the problem.

$$\Phi \models_{g\mathsf{C}} \varphi \quad \text{iff} \quad \forall \mathfrak{F} \in \mathsf{C}\ \forall V \in \mathcal{P}(F)^{\mathsf{Prop}},\ \text{if } \mathfrak{F}, V \Vdash \Phi \text{ then } \mathfrak{F}, V \Vdash \varphi.$$

We write $\models_{l\mathfrak{F}}$ and $\models_{g\mathfrak{F}}$ when $\mathsf{C} = \{\mathfrak{F}\}$. For every class C of frames, $\models_{l\mathsf{C}}$ and $\models_{g\mathsf{C}}$ are structural consequence relations on the modal language; but in general they are not finitary. Hence in general they are neither the local nor the global consequence of a normal modal logic. But the problem disappears when we restrict ourselves to finite sets, i.e., by the deduction-detachment theorem it is easy to see that for every finite set Φ of modal formulas and every modal formula φ, it holds that

$$\Phi \models_{l\mathsf{C}} \varphi \quad \text{iff} \quad \Phi \vdash_{l\,\mathrm{Log}\,\mathsf{C}} \varphi.$$[25]

A normal modal logic $\mathbf{\Lambda}$ is *strongly frame complete* if its local consequence $\vdash_{l\mathbf{\Lambda}}$ is obtained by a class of frames through the previous method. That is, $\mathbf{\Lambda}$ is strongly frame complete iff $\vdash_{l\mathbf{\Lambda}} = \models_{l\,\mathrm{Fr}\,\mathbf{\Lambda}}$. Not all frame complete normal modal logics are strongly frame complete. It is said that $\mathbf{\Lambda}$ is *compact* if $\models_{l\,\mathrm{Fr}\,\mathbf{\Lambda}}$ is finitary. It is not hard to see that a normal modal logic $\mathbf{\Lambda}$ is strongly frame complete iff it is frame complete and compact. Hence, $\mathrm{Log}\,\mathsf{C}$ is strongly frame complete iff $\models_{l\mathsf{C}}$ is finitary. By the modal version of [CJ01, Proposition 3] it is known that if $\models_{l\mathsf{C}} = \vdash_{l\mathbf{\Lambda}}$ and C is closed under generated frames, then $\models_{g\mathsf{C}} = \vdash_{g\mathbf{\Lambda}}$. Therefore, if $\mathbf{\Lambda}$ is strongly frame complete, then $\vdash_{g\mathbf{\Lambda}} = \models_{g\,\mathrm{Fr}\,\mathbf{\Lambda}}$.

We have already said that not all normal modal logics are obtained by a class of frames. On the other hand, all normal modal logics can be obtained through a class of structures. It is well known that all normal modal logics are of the form

$$\{\varphi \in \mathcal{L}^{MOD}(\tau) : \forall \mathfrak{A} \in \mathsf{K}\ \mathfrak{A} \Vdash \varphi\} \tag{1.5}$$

for a certain class K of structures. Indeed, it is enough to consider a single structure as it is shown by the canonical structure construction. Assume $\mathbf{\Lambda}$ is a consistent normal modal logic. The *canonical structure* associated with $\mathbf{\Lambda}$ is the structure $\mathfrak{H}^{\mathbf{\Lambda}}$ (in honour of Henkin [Hen49, Hen96]) defined as follows:

- The universe $H^{\mathbf{\Lambda}}$ is the set of all maximal consistent theories of $\vdash_{l\mathbf{\Lambda}}$.

[25] If we replace this finitary condition with $\models_{g\mathsf{C}}$ and $\vdash_{g\,\mathrm{Log}\,\mathsf{C}}$ we do not know if this holds in general. What we know is that this is true if the number of modalities κ is finite, C is closed under generated subframes and there is $n \in \omega$ such that for every modality m, it holds that $[m]^n p_0 \supset [m]^{n+1} p_0 \in \mathrm{Log}\,\mathsf{C}$. This is easily shown by the following chain of equivalences:

$$\begin{array}{rl} \Phi \models_{g\mathsf{C}} \varphi & \text{iff} \\ \bigwedge\{[m]^{(n)} \bigwedge \Phi : m \in \mathsf{Mod}\} \models_{l\mathsf{C}} \varphi & \text{iff} \\ \bigwedge\{[m]^{(n)} \bigwedge \Phi : m \in \mathsf{Mod}\} \vdash_{l\,\mathrm{Log}\,\mathsf{C}} \varphi & \text{iff} \\ \bigwedge\{[m]^{k} \bigwedge \Phi : k \in \omega, m \in \mathsf{Mod}\} \vdash_{l\,\mathrm{Log}\,\mathsf{C}} \varphi & \text{iff} \\ \Phi \vdash_{g\,\mathrm{Log}\,\mathsf{C}} \varphi. & \end{array}$$

The first of the equivalences is obtained reasoning as in [CJ01, Proposition 3].

- For every $m \in \mathsf{Mod}$, the accessibility relation $R_m^{\mathfrak{H}^{\mathbf{\Lambda}}}$ is

$$\{\langle \Sigma_0, \Sigma_1 \rangle \in H^{\mathbf{\Lambda}} : \forall \varphi \in \mathcal{L}^{MOD}(\tau),\ \text{if } [m]\varphi \in \Sigma_0 \text{ then } \varphi \in \Sigma_1\}.$$

- The valuation is the map $p \longmapsto \{\Sigma \in H^{\mathbf{\Lambda}} : p \in \Sigma\}$.

The utility of the canonical structure is due to the following results.

1.4.3. Lemma (Lindenbaum). [BdRV01, Lemma 4.17] *Every consistent theory of* $\vdash_{l\mathbf{\Lambda}}$ *can be extended to a maximal consistent theory.*

1.4.4. Lemma (Truth). [BdRV01, Lemma 4.21] *For every* $\Sigma \in H^{\mathbf{\Lambda}}$ *and every* $\mathcal{L}^{MOD}(\tau)$*-formula* φ*,*

$$\mathfrak{H}^{\mathbf{\Lambda}}, \Sigma \Vdash \varphi \quad \textit{iff} \quad \varphi \in \Sigma.$$

1.4.5. Theorem (Canonical Structure). [BdRV01, Theorem 4.22] *For every (maybe infinite) set* Φ *of* $\mathcal{L}^{MOD}(\tau)$*-formulas and every* $\mathcal{L}^{MOD}(\tau)$*-formula* φ*, then*

$$\Phi \vdash_{l\mathbf{\Lambda}} \varphi \quad \textit{iff} \quad \forall \Sigma \in H^{\mathbf{\Lambda}},\ \textit{if } \mathfrak{H}^{\mathbf{\Lambda}}, \Sigma \Vdash \Phi \textit{ then } \mathfrak{H}^{\mathbf{\Lambda}}, \Sigma \Vdash \varphi.$$

In particular the last theorem says that

$$\mathbf{\Lambda} = \{\varphi \in \mathcal{L}^{MOD}(\tau) : \mathfrak{H}^{\mathbf{\Lambda}} \Vdash \varphi\}.$$

Therefore every normal modal logic can be obtained through a class of structures. At first glance this is wonderful, but we must be careful because not all sets obtained by (1.5) are normal modal logics. In general these sets are not closed under modal substitutions[26]. The same problem appears when we try to define consequence relations from a class of structures. Given a class K of structures we introduce the consequence relations

$$\Phi \models_{l\mathsf{K}} \varphi \quad \text{iff} \quad \forall \mathfrak{A} \in \mathsf{K}\ \forall a \in A,\ \text{if } \mathfrak{A}, a \Vdash \Phi \text{ then } \mathfrak{A}, a \Vdash \varphi,$$

and

$$\Phi \models_{g\mathsf{K}} \varphi \quad \text{iff} \quad \forall \mathfrak{A} \in \mathsf{K},\ \text{if } \mathfrak{A} \Vdash \Phi \text{ then } \mathfrak{A} \Vdash \varphi.$$

[26] We have seen that neither frames nor structures give us a good correspondence with normal modal logics. In the literature this is solved introducing the notion of general frame, but we are not going to consider them in this dissertation. This notion allows a nice correspondence with normal modal logics.

We write $\models_{l\mathfrak{A}}$ and $\models_{g\mathfrak{A}}$ when $\mathsf{C} = \{\mathfrak{A}\}$. For every class K of structures, both $\models_{l\mathsf{K}}$ and $\models_{g\mathsf{K}}$ are consequence relations; but in general they are neither structural nor finitary. In the case that K is the singleton formed by the canonical structure associated with a certain normal modal logic then the Canonical Structure Theorem claims that $\models_{l\mathsf{K}}$ is structural and finitary. In fact, the Canonical Structure Theorem exactly says that $\vdash_{l\mathbf{\Lambda}} = \models_{l\mathfrak{H}^{\mathbf{\Lambda}}}$. Let us define some properties over structures. An structure $\mathfrak{A}$ is *modally compact* if the consequence relation $\models_{l\mathfrak{A}}$ is finitary (perhaps it is not structural). $\mathfrak{A}$ is *differentiated* if there are no different states satisfying the same modal formulas. And it is *tight* if for every modality $m \in \mathsf{Mod}$ and all states a and a', if $\mathfrak{A}, a' \Vdash \{\varphi \in \mathcal{L}^{MOD} : \mathfrak{A}, a \Vdash [m]\varphi\}$ then $\langle a, a' \rangle \in R^{\mathfrak{A}}_m$. All tight structures are $\mathcal{H}$-closed. Over differentiated structures the converse also holds.

1.4.6. PROPOSITION. [CZ97, Propositions 5.6, 5.7 and 5.11] *The canonical structures are differentiated, tight, $\mathcal{H}$-closed, modally saturated*[27] *and modally compact.*

The canonical frame for $\mathbf{\Lambda}$, denoted by $\mathfrak{F}^{\mathbf{\Lambda}}$, is obtained by deleting the valuation on the canonical structure for $\mathbf{\Lambda}$. A normal modal logic $\mathbf{\Lambda}$ is *canonical* if its canonical frame $\mathfrak{F}^{\mathbf{\Lambda}}$ belongs to $\mathrm{Fr}\,\mathbf{\Lambda}$. If $\mathbf{\Lambda}$ is canonical then $\vdash_{l\mathbf{\Lambda}} = \models_{l\mathfrak{F}^{\mathbf{\Lambda}}} = \models_{l\,\mathrm{Fr}\,\mathbf{\Lambda}}$. Specifically, $\vdash_{l\mathbf{\Lambda}}$ is obtained through a class of frames. Hence if $\mathbf{\Lambda}$ is canonical then it is also strongly frame complete, frame complete and compact. A modal formula φ is *canonical* if, for any normal modal logic $\mathbf{\Lambda}$, $\varphi \in \mathbf{\Lambda}$ implies that φ is valid on the canonical frame for $\mathbf{\Lambda}$. It is clear that normal modal logics axiomatized by canonical formulas are strongly frame complete with respect to their class of frames.

1.4.7. PROPOSITION. [CZ97, Theorem 5.16] *All modal formulas in the Table 1.2 except* GL *and* Grz *are canonical.*

Therefore all normal modal logics axiomatized by the formulas in Table 1.2 (except GL and Grz) are frame complete with respect to their frames. For instance, ***S*4** is characterized by the quasi orders. This shows that canonicity is a powerful tool to show completeness, but there are cases in which it does not work. For instance, GL and Grz are non-canonical formulas, but the following proposition holds.

1.4.8. PROPOSITION. [GKWZ03, Theorem 1.4] *The logics **GL**, **GL.3**, **Grz** and **Grz.3** are frame complete.*

[27] We do not know of any reference in which it is proved that the canonical structures are modally saturated, but it is not hard to see this. Indeed, it follows from our Propositions 4.2.25 and 4.3.7.

This proposition tells us that these normal modal logics are characterized by their frames, which were described on page 37. Indeed, it is also known that $\boldsymbol{GL}$ is characterized by the class of finite strict orders, and that $\boldsymbol{Grz}$ is characterized by the class of finite partial orders. These logics are not strongly frame complete.

1.4.9. Remark. The list of modal canonical formulas such that their validity characterizes a certain property on frames is much larger than what we find in Table 1.2, e.g., $\mathsf{G}^{k,l,m,n}$ [Che80, p. 88], Sahlqvist formulas [Sah75, dRV95], etc. We do not consider them because we do not know how to obtain finite axiomatizations of the strict-weak logics given by their classes of frames. Nor do we know finite axiomatizations for the strict-weak logics given by the classes of directed and weakly directed frames.

To finish the section we recall the definition of two interesting properties: the modal disjunction property and (uniform) interpolation.

For the modal disjunction property we restrict our definition to the case of $\mathcal{L}^{mod}$. It is said that a normal modal logic $\mathbf{\Lambda}$ in $\mathcal{L}^{mod}$ has the *modal disjunction property* iff for all modal formulas $\varphi_0, \ldots, \varphi_{n-1}$, it holds that

$$\Box\varphi_0 \vee \ldots \vee \Box\varphi_{n-1} \in \mathbf{\Lambda} \quad \text{iff} \quad \text{there is } i < n \text{ such that } \Box\varphi_i \in \mathbf{\Lambda}.$$

Some examples of normal modal logics having this property are $\boldsymbol{K}$, $\boldsymbol{T}$, $\boldsymbol{K4}$ and $\boldsymbol{S4}$.

Now let us introduce the notion of interpolation. The definition is introduced in general for any consequence relation $\vdash$ over an arbitrary propositional language $\mathcal{L}$. Given a pair $\langle\varphi_0, \varphi_1\rangle$ of $\mathcal{L}$-formulas, it is said that a $\mathcal{L}$-formula φ is an *interpolant for the pair* $\langle\varphi_0, \varphi_1\rangle$ iff it satisfies the following three properties:

- $\mathsf{Prop}(\varphi) \subseteq \mathsf{Prop}(\varphi_0) \cap \mathsf{Prop}(\varphi_1)$, i.e., the propositions appearing in φ must appear in both formulas of the pair,
- $\varphi_0 \vdash \varphi$,
- $\varphi \vdash \varphi_1$.

It is obvious that a necessary condition for a pair of formulas to have an interpolant is that $\varphi_0 \vdash \varphi_1$. It is said that $\vdash$ has *interpolation* (or Craig interpolation) if each pair of formulas has an interpolant[28]. Among the known examples of consequence relations having interpolation we find classical propositional logic, $\vdash_{l\boldsymbol{K}}$,

[28] The definition is usually introduced without the restriction to proposition languages; but then the first condition in the definition of interpolant considers all the symbols of the language appearing in φ and not only the propositions. Indeed, interpolation was first proved for first-order logic by Craig [Cra57].

$\vdash_{l\boldsymbol{T}}$, $\vdash_{l\boldsymbol{K4}}$, $\vdash_{l\boldsymbol{S4}}$, $\vdash_{l\boldsymbol{S5}}$, $\vdash_{l\boldsymbol{GL}}$, and $\vdash_{l\boldsymbol{Grz}}$. In order to prove interpolation the main tools that we have are algebraic. For instance, it is known that if $\boldsymbol{\Lambda}$ is a normal modal logic then: (i) $\vdash_{l\boldsymbol{\Lambda}}$ has interpolation iff the variety of algebras associated with $\boldsymbol{\Lambda}$ has the superamalgamation property, and (ii) $\vdash_{g\boldsymbol{\Lambda}}$ has interpolation iff the variety of algebras associated with $\boldsymbol{\Lambda}$ has the amalgamation property. We refer readers interested in these algebraic characterizations to [CZ97, Chapter 14], for further explanations. An easy consequence of the previous algebraic characterization is that for every normal modal logic, if $\vdash_{g\boldsymbol{\Lambda}}$ has interpolation then $\vdash_{l\boldsymbol{\Lambda}}$ also has interpolation. Recently, Marx managed to prove the interpolation of some modal consequences using the notion of bisimulation (see [Mar98]).

Let us now consider a stronger form of interpolation that is called uniform interpolation. The idea behind it is that the interpolant of a pair only depends on one of its components and on the propositions common to both components. Given a proposition p and a $\mathcal{L}$-formula φ_0, it is said that a $\mathcal{L}$-formula φ is a *p-existential uniform interpolant for φ_0* iff it satisfies the following three properties:

- $\mathsf{Prop}(\varphi) \subseteq \mathsf{Prop}(\varphi_0) \setminus \{p\}$,
- $\varphi_0 \vdash \varphi$,
- For every $\mathcal{L}$-formula φ_1, if $\varphi_0 \vdash \varphi_1$ and $p \notin \mathsf{Prop}(\varphi_1)$, then $\varphi \vdash \varphi_1$.

We will write $\exists p\varphi_0$ for a formula φ satisfying these conditions. The formula $\exists p\varphi_0$ is unique up to equivalence, i.e., if φ and φ' satisfy the previous three properties then $\varphi \dashv\vdash \varphi'$. It is obvious that if $p \notin \mathsf{Prop}(\varphi_0)$, then $\varphi_0 \dashv\vdash \exists p\varphi_0$. It is said that $\vdash$ has *existential uniform interpolation* if for every proposition p and every formula φ_0 there is a p-existential uniform interpolant $\exists p\varphi_0$ for φ_0. It is not hard to check that if $\vdash$ has existential uniform interpolation then $\vdash$ has interpolation. Indeed, if $\varphi_0 \vdash \varphi_1$ and $\{q_0, \ldots, q_{n-1}\} = \mathsf{Prop}(\varphi_0) \setminus \mathsf{Prop}(\varphi_1)$[29] then $\exists q_0 \ldots \exists q_{n-1}\varphi_0$ is an interpolant for $\langle \varphi_0, \varphi_1 \rangle$. In fact, $\exists q_0 \ldots \exists q_{n-1}\varphi_0$ is the minimum interpolant for $\langle \varphi_0, \varphi_1 \rangle$ in the sense that for every interpolant φ' for $\langle \varphi_0, \varphi_1 \rangle$ it holds that $\exists q_0 \ldots \exists q_{n-1}\varphi_0 \vdash \varphi'$.

Analogously we can define universal uniform interpolation. Given a proposition p and a $\mathcal{L}$-formula φ_1, it is said that a $\mathcal{L}$-formula φ is a *p-universal uniform interpolant for φ_1* iff it satisfies the following conditions:

- $\mathsf{Prop}(\varphi) \subseteq \mathsf{Prop}(\varphi_1) \setminus \{p\}$,
- $\varphi \vdash \varphi_1$,
- For every $\mathcal{L}$-formula φ_0, if $\varphi_0 \vdash \varphi_1$ and $p \notin \mathsf{Prop}(\varphi_0)$, then $\varphi_0 \vdash \varphi$.

[29] It is clear that $\mathsf{Prop}(\varphi_0) \setminus \mathsf{Prop}(\varphi_1) = \mathsf{Prop}(\varphi_0) \setminus (\mathsf{Prop}(\varphi_0) \cap \mathsf{Prop}(\varphi_1))$. It justifies the intuitive idea expressed above that the interpolant only depends on one of its components (in this case φ_0) and on the propositions common to both.

We will write $\forall p\varphi_1$ for a formula φ satisfying these conditions. The formula $\forall p\varphi_1$ is unique up to equivalence. If $p \notin \mathsf{Prop}(\varphi_1)$, then $\varphi_1 \dashv\vdash \forall p\varphi_1$. It is said that $\vdash$ has *universal uniform interpolation* if for every proposition p and every formula φ_1 there is a p-universal uniform interpolant $\forall p\varphi_1$ for φ_1. If $\vdash$ has universal uniform interpolation then $\vdash$ has interpolation. Indeed, if $\varphi_0 \vdash \varphi_1$ and $\{q_0, \ldots, q_{n-1}\} = \mathsf{Prop}(\varphi_1) \setminus \mathsf{Prop}(\varphi_0)$ then $\forall q_0 \ldots \forall q_{n-1}\varphi_1$ is an interpolant for $\langle\varphi_0, \varphi_1\rangle$. In fact, $\forall q_0 \ldots \forall q_{n-1}\varphi_1$ is the maximum interpolant for $\langle\varphi_0, \varphi_1\rangle$, i.e., for every interpolant φ' for $\langle\varphi_0, \varphi_1\rangle$ it holds that $\varphi' \vdash \forall q_0 \ldots \forall q_{n-1}\varphi$.

We will say that $\vdash$ has *uniform interpolation* when it has existential uniform interpolation and universal uniform interpolation.

1.4.10. Remark. It is easy to prove that if our propositional language has a connective $\sim$ satisfying that for every φ_0 and φ_1,

$$\varphi_0 \dashv\vdash {\sim}{\sim}\varphi_0 \qquad \text{and} \qquad \varphi_0 \vdash \varphi_1 \text{ iff } {\sim}\varphi_1 \vdash {\sim}\varphi_0,$$

then $\vdash$ has existential uniform interpolation iff it has universal uniform interpolation. The explanation is that in this case existential uniform interpolants are interdefinable with universal uniform interpolants: indeed, $\exists p\varphi \dashv\vdash {\sim}\forall p{\sim}\varphi$ and $\forall p\varphi \dashv\vdash {\sim}\exists p{\sim}\varphi$. It is clear that classical propositional logic and $\vdash_{l\mathbf{\Lambda}}$ for a normal modal logic $\mathbf{\Lambda}$ are examples of consequence relations where the requirement is clearly true (material negation is the connective). We notice that indeed we do not need to have a connective in the language. We refer to the fact that the same argument works under the assumption that there is a function f that transforms $\mathcal{L}$-formulas into $\mathcal{L}$-formulas such that for every φ_0 and φ_1,

$$\varphi_0 \dashv\vdash ff(\varphi_0) \qquad \text{and} \qquad \varphi_0 \vdash \varphi_1 \text{ iff } f(\varphi_1) \vdash f(\varphi_0).$$

Under this assumption we also have interdefinability of existential uniform interpolants and universal uniform interpolants because $\exists p\varphi \dashv\vdash f(\forall pf(\varphi))$ and $\forall p\varphi \dashv\vdash f(\exists pf(\varphi))$. What is interesting here is that the duality map ^{d} is an example of a map satisfying this condition for the local consequence given by a normal modal logic. That is, for every normal modal logic $\mathbf{\Lambda}$ and every modal formulas φ_0 and φ_1, it holds that

$$\varphi_0 \dashv\vdash_{l\mathbf{\Lambda}} (\varphi_0^d)^d \qquad \text{and} \qquad \varphi_0 \vdash_{l\mathbf{\Lambda}} \varphi_1 \text{ iff } \varphi_1^d \vdash_{l\mathbf{\Lambda}} \varphi_0^d.$$

Therefore, if $\mathbf{\Lambda}$ is a normal modal logic and $\vdash_{l\mathbf{\Lambda}}$ has uniform interpolation then for every modal formula φ,

$${\sim}\forall p{\sim}\varphi \dashv\vdash_{l\mathbf{\Lambda}} \exists p\varphi \dashv\vdash_{l\mathbf{\Lambda}} (\forall p\varphi^d)^d \qquad \text{and} \qquad {\sim}\exists p{\sim}\varphi \dashv\vdash_{l\mathbf{\Lambda}} \forall p\varphi \dashv\vdash_{l\mathbf{\Lambda}} (\exists p\varphi^d)^d.$$

It implies that $(\exists p\varphi)^d \dashv\vdash_{l\mathbf{\Lambda}} \forall p\varphi^d$ and that $(\forall p\varphi)^d \dashv\vdash_{l\mathbf{\Lambda}} \exists p\varphi^d$.

It is easy to show that classical propositional logic has uniform interpolation. Indeed, given a formula $\varphi(p, q_0, \ldots, q_{n-1})$ we can check that the formula

$$\varphi(\top, q_0, \ldots, q_{n-1}) \vee \varphi(\bot, q_0, \ldots, q_{n-1})$$

satisfies the properties characterizing $\exists p\varphi$, and the formula

$$\varphi(\top, q_0, \ldots, q_{n-1}) \wedge \varphi(\bot, q_0, \ldots, q_{n-1})$$

satisfies the requirement for being $\forall p\varphi$. For the local consequences of normal modal logics in general it is much harder to prove uniform interpolation[30]. Ghilardi and Zawadowski [GZ95] and Visser [Vis96a, Vis96b] prove that $\vdash_{l\boldsymbol{K}}$, $\vdash_{l\boldsymbol{GL}}$ and $\vdash_{l\boldsymbol{Grz}}$ have uniform interpolation but that $\vdash_{l\boldsymbol{S4}}$ lacks it[31]. It is also known that for every minimal normal modal logic $\boldsymbol{K_\kappa}$ its local consequence has uniform interpolation [D'A98, Section 3.1]. A semantical characterization of the uniform interpolants for $\vdash_{l\boldsymbol{K_\kappa}}$ is known. For every pointed τ-structure $\langle \mathfrak{A}, a\rangle$ it holds that

$$\mathfrak{A}, a \Vdash \exists p\varphi \quad \text{iff} \quad \left\{ \begin{array}{l} \text{there is a pointed } \tau\text{-structure } \langle \mathfrak{B}, b\rangle \text{ such} \\ \text{that } \langle \mathfrak{A}, a\rangle \simeq_{\tau_p} \langle \mathfrak{B}, b\rangle \text{ and } \mathfrak{B}, b \Vdash \varphi, \end{array} \right. \tag{1.6}$$

where τ_p is the same vocabulary τ except for the fact that we have removed p from its propositions, and it also holds that

$$\mathfrak{A}, a \Vdash \forall p\varphi \quad \text{iff} \quad \left\{ \begin{array}{l} \text{for every pointed } \tau\text{-structure } \langle \mathfrak{B}, b\rangle, \text{ if} \\ \langle \mathfrak{A}, a\rangle \simeq_{\tau_p} \langle \mathfrak{B}, b\rangle \text{ then } \mathfrak{B}, b \Vdash \varphi. \end{array} \right. \tag{1.7}$$

A formula $\exists p\varphi$ having the semantic characterization expressed in (1.6) and with propositions in $\mathsf{Prop}(\varphi) \setminus \{p\}$ is called a *p-existential bisimilarity quantifier for* φ. Analogously, a formula $\forall p\varphi$ having the semantic characterization expressed in (1.7) and with propositions in $\mathsf{Prop}(\varphi) \setminus \{p\}$ is called a *p-universal bisimilarity quantifier for* φ. It is easy to check that (i) if a formula is a p-existential bisimilarity quantifier for φ, then it is also a p-existential uniform interpolant for φ, and that (ii) if a formula is a p-universal bisimilarity quantifier for φ, then it is also a p-universal uniform interpolant for φ. The strategy considered in the literature to prove the fact that $\vdash_{l\boldsymbol{K_\kappa}}$ has uniform interpolant is to check for every proposition p and every modal formula φ that there is a p-existential bisimilarity quantifier for φ and also a p-universal bisimilarity quantifier for φ. This method also succeeds in showing uniform interpolation for $\vdash_{l\boldsymbol{GL}}$ and $\vdash_{l\boldsymbol{Grz}}$: but this time we have to restrict the structures used in (1.6) and (1.7) to, respectively, Noetherian strict orders and Noetherian partial orders. Using the semantic characterizations (1.6) and (1.7) it is easy to check that uniform interpolants for $\vdash_{l\boldsymbol{K_\kappa}}$ satisfy that:[32]

[30] As far as the author knows there is only one situation where it is immediate that $\vdash_{l\boldsymbol{\Lambda}}$ has uniform interpolation. It was noticed by Wolter in [Wol98, p. 376], and it is when $\vdash_{l\boldsymbol{\Lambda}}$ has interpolation and the variety of algebras associated with $\boldsymbol{\Lambda}$ is locally finite (i.e., each finitely generated algebra is finite). In particular it follows that $\vdash_{l\boldsymbol{S5}}$ has uniform interpolation.

[31] It is also known that first-order logic does not have uniform interpolation (this was first proved in [Cra63]) and that the μ-calculus has it [DH98, D'A98].

[32] We notice that for every modal formulas φ_0 and φ_1 it holds that $\varphi_0 \equiv \varphi_1$ iff $\varphi_0 \dashv\vdash_{l\boldsymbol{K_\kappa}} \varphi_1$.

$$\exists p \sim \varphi \equiv \sim \forall p\varphi \qquad \forall p \sim \varphi \equiv \sim \exists p\varphi$$
$$\exists p(\varphi_0 \vee \varphi_1) \equiv \exists p\varphi_0 \vee \exists p\varphi_1 \qquad \forall p(\varphi_0 \wedge \varphi_1) \equiv \forall p\varphi_0 \wedge \forall p\varphi_1$$
$$\exists p\langle m\rangle\varphi \equiv \langle m\rangle\exists p\varphi \qquad \forall p[m]\varphi \equiv [m]\forall p\varphi$$
$$\exists p[m]\varphi \equiv [m]\exists p\varphi \qquad \forall p\langle m\rangle\varphi \equiv \langle m\rangle\forall p\varphi.$$

1.4.11. Remark. In modal languages "besides propositions" we have modalities. We could also have required in the definition of interpolant for a pair $\langle\varphi_0, \varphi_1\rangle$ that the modalities appearing in the interpolant are shared between φ_0 and φ_1. In [vB97, p. 277] van Benthem has called the existence of this kind of interpolants *strong interpolation*, a notion that has been occasionally considered in the literature. Analogously, it is also natural to introduce the notion of *strong uniform interpolation* (see [DH98, D'A98]) when in addition to uniform interpolation we require the existence of m-existential and m-universal uniform interpolants for arbitrary modal formulas ($m \in \mathsf{Mod}$). A *m-existential uniform interpolant for φ_0* is by definition a modal formula φ such that:

- $\mathsf{Mod}(\varphi) \subseteq \mathsf{Mod}(\varphi_0) \setminus \{m\}$,
- $\varphi_0 \vdash \varphi$,
- For every $\mathcal{L}$-formula φ_1, if $\varphi_0 \vdash \varphi_1$ and $m \notin \mathsf{Mod}(\varphi_1)$, then $\varphi \vdash \varphi_1$.

Obviously, a *m-universal uniform interpolant for φ_1* is defined as a modal formula φ such that:

- $\mathsf{Mod}(\varphi) \subseteq \mathsf{Mod}(\varphi_1) \setminus \{m\}$,
- $\varphi \vdash \varphi_1$,
- For every $\mathcal{L}$-formula φ_0, if $\varphi_0 \vdash \varphi_1$ and $m \notin \mathsf{Mod}(\varphi_0)$, then $\varphi_0 \vdash \varphi$.

The method of bisimilarity quantifiers can also be used to show that all $\vdash_{\imath\boldsymbol{K}_{\boldsymbol{\kappa}}}$ indeed have strong uniform interpolation (see [D'A98, Section 3.1]).

1.5 Computational aspects

A subset X of ω is said to be *recursive* if there is an algorithm (or a program) that decides, given an arbitrary $n \in \omega$, whether $n \in X$; otherwise, it is said that X is *non-recursive*. It is beyond the scope of this presentation to give a formal treatment of the concepts from *computability theory* such as *algorithm*, *recursive* (or *computable*) *function* and *set*, *recursive enumerability*, etc. The reader can find all these in [Men97, End00, Bar93] and other textbooks on mathematical logic and recursion theory.

In the definitions below we restrict our exposition to the modal language $\mathcal{L}^{mod}$ (i.e., we assume that there is a single modality), but the same can be done for every propositional language over a finite signature. It is clear that finitary modal formulas can be codified as natural numbers (see [BdRV01, Section 6.1] for a more detailed discussion, where also finite structures are codified). Hence, we can develop computability notions for subsets of modal formulas. A set L of modal formulas is called *decidable* if there is an algorithm that decides, given an arbitrary $\varphi \in \mathcal{L}^{mod}$, whether $\varphi \in L$; otherwise, it is said that L is *undecidable*. Sometimes we will denote the problem L by "$\varphi \in L$?". Our interest is mainly when L is a normal modal logic $\mathbf{\Lambda}$. The existence of a decision algorithm for L does not guarantee that it can be used in practice: the amount of computational resources it requires may be enormous. This explains why we are interested in knowing the optimal computational complexity of the decision problem for L.

There are two standard ways of measuring the difficulty of decidable problems like "$\varphi \in L$?". One is the amount of *time* (i.e., number of steps), depending on the length of φ, required by a decision algorithm to solve the problem; and the other one is the amount of *space* (memory), also depending on $\mathrm{leng}(\varphi)$, required by a successful algorithm. Now let us introduce the complexity classes that we use in this dissertation. An algorithm is called *deterministic* if each of its steps is uniquely determined. Hence, the computations of a deterministic algorithm are completely determined by the input. On the other hand, a *non-deterministic* algorithm may guess, at each step, which step to take next out of a finite number of possibilities. It is said that a problem "$\varphi \in L$?" belongs to the *complexity class*

- P if it is solvable by a deterministic algorithm in polynomial time of $\mathrm{leng}(\varphi)$, i.e., there is a deterministic algorithm solving the decision problem for L and a polynomial $p(x)$ such that for every input $\varphi \in \mathcal{L}^{mod}$ the algorithm gives an answer in $\leq p(\mathrm{leng}(\varphi))$ steps.

- NP if it is solvable by a non-deterministic algorithm in polynomial time of $\mathrm{leng}(\varphi)$, i.e., there is a non-deterministic algorithm solving the decision problem for L and a polynomial $p(x)$ such that for every input $\varphi \in \mathcal{L}^{mod}$ all computations of the non-deterministic algorithm give an answer in $\leq p(\mathrm{leng}(\varphi))$ steps.

- PSpace if it is solvable by a deterministic algorithm in polynomial space of $\mathrm{leng}(\varphi)$, i.e., there is an algorithm solving the decision problem for L and a polynomial $p(x)$ such that for every input $\varphi \in \mathcal{L}^{mod}$ the algorithm gives an answer using an amount of space[33] bounded by $p(\mathrm{leng}(\varphi))$.

[33]In the context of Turing machines this corresponds to the number of tape squares scanned in the computation.

According to a result obtained by Savitch [Sav70], non-determinism does not increase the level of space complexity. Hence, the complexity class of non-deterministic polynomial space coincides with PSpace. Here we limit ourselves to recalling some well known facts on complexity theory; the reader interested in a detailed exposition of these complexity classes and others should look at [GJ79, HMU01, BDG95, BDG90, Pap94, Imm99]. By definition it is obvious that

$$\mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSpace}.$$

To determine whether these inclusions are strict is the most famous open problem in complexity theory.

Given a problem "$\varphi \in L$?", its *complement* is the problem "$\varphi \in L^c$?", i.e., to decide given an arbitrary $\varphi \in \mathcal{L}^{mod}$, whether $\varphi \notin L$. It is obvious that

$$L \text{ is decidable} \quad \text{iff} \quad L^c \text{ is decidable.}$$

For any complexity class $\mathcal{C}$, the class $\mathsf{co}\,\mathcal{C}$ consists of all problems whose complement is in $\mathcal{C}$. This introduces the definitions of the complexity classes $\mathsf{co}\,\mathsf{P}$, $\mathsf{co}\,\mathsf{NP}$ and $\mathsf{co}\,\mathsf{PSpace}$. We notice that for deterministic classes $\mathcal{C} = \mathsf{co}\,\mathcal{C}$. In particular $\mathsf{co}\,\mathsf{P} = \mathsf{P}$, $\mathsf{co}\,\mathsf{PSpace} = \mathsf{PSpace}$. It is unknown whether $\mathsf{co}\,\mathsf{NP} = \mathsf{NP}$ holds.

Let $\mathcal{C}$ be a complexity class. A problem L is $\mathcal{C}$*-hard* if for every problem $L' \in \mathcal{C}$ there is a polynomial time *reduction* from L' to L, i.e., a recursive (computable) function f which for every input $\varphi \in \mathcal{L}^{mod}$ in deterministic polynomial time returns a formula $f(\varphi) \in \mathcal{L}^{mod}$ such that

$$\varphi \in L' \quad \text{iff} \quad f(\varphi) \in L.$$

It is clear that if L is $\mathcal{C}$-hard, then L^c is $\mathsf{co}\,\mathcal{C}$-hard. A problem is $\mathcal{C}$*-complete* when it is $\mathcal{C}$-hard and belongs to $\mathcal{C}$. Therefore, the standard method for showing that a problem L is $\mathcal{C}$-complete is based on proving the following two facts:

- L is in $\mathcal{C}$.
- there is a $\mathcal{C}$-complete problem L' and a polynomial time reduction from L' to L.

We notice that the method of reductions is also interesting when one tries to prove the undecidability of a certain problem L: it is enough to show a certain recursive (computable) reduction from a known undecidable problem to L.

A famous result obtained by Cook says that classical propositional logic $\boldsymbol{CPL}$ is $\mathsf{co}\,\mathsf{NP}$-complete [Coo71]. Using that normal modal logics are conservative expansions of classical propositional logic it follows that every normal modal logic $\mathbf{\Lambda}$ is $\mathsf{co}\,\mathsf{NP}$-hard. Now we explain the two theorems that we will be interested in. They talk about families of normal modal logics in $\mathcal{L}^{mod}$. The first result is due to Hemaspaandra (*née* Spaan) [Spa93, Hem96], and gives an upper bound for the complexity class of all normal modal logics extending $\boldsymbol{S4.3}$.

1.5.1. Theorem (Hemaspaandra). *(*Prop *is countable) Let* $\mathbf{\Lambda}$ *be a normal modal logic extending* $\boldsymbol{S4.3}$*. Then,* $\mathbf{\Lambda}$ *is* co NP*-complete.*

The other result is due to Ladner [Lad77] and gives a lower bound for the complexity of normal modal logics that are subsystems of $\boldsymbol{GL}$ and $\boldsymbol{Grz}$.

1.5.2. Theorem (Ladner). *(*Prop *is countable) Let* $\mathbf{\Lambda}$ *be a normal modal logic such that either* $\boldsymbol{K} \subseteq \mathbf{\Lambda} \subseteq \boldsymbol{GL}$ *or* $\boldsymbol{K} \subseteq \mathbf{\Lambda} \subseteq \boldsymbol{Grz}$*. Then,* $\mathbf{\Lambda}$ *is* PSpace*-hard.*

Since it is known that $\boldsymbol{K}$, $\boldsymbol{T}$, $\boldsymbol{K4}$, $\boldsymbol{S4}$, $\boldsymbol{D}$, $\boldsymbol{GL}$ and $\boldsymbol{Grz}$ are in PSpace (see [CZ97, BdRV01, GKWZ03]) it follows that all these normal modal logics are PSpace-complete.

We notice that our version of Ladner's Theorem is not the same one as in Ladner's paper: only subsystems of $\boldsymbol{S4}$ were considered (cf. [Lad77, Theorem 3.1]). But the usual proofs for Ladner's statement (see [Lad77, HM92, BdRV01]) also allow us to prove our statement. In Section 5.3 we will need a certain way to prove Ladner's Theorem. So, we now give an outline of this proof: it is quite close to the one exhibited in [HM92][34].

Let $\mathbf{\Lambda}$ be a normal modal logic such that either $\boldsymbol{K} \subseteq \mathbf{\Lambda} \subseteq \boldsymbol{GL}$ or $\boldsymbol{K} \subseteq \mathbf{\Lambda} \subseteq \boldsymbol{Grz}$. The method of the proof is to show a polynomial time reduction from a known PSpace-complete problem to $\mathbf{\Lambda}^c$. The PSpace-complete problem considered is the logic $\boldsymbol{QBF}$ of quantified Boolean formulas. It was proved by Stockmeyer and Meyer [SM73] that this logic is PSpace-complete.

Let us describe what $\boldsymbol{QBF}$ is. We assume that there is a countable set of propositions. The set of *quantified Boolean formulas*[35] consists of expressions of the form

$$Q_0 p_0 \ldots Q_{n-1} p_{n-1} \varphi(p_0, \ldots, p_{n-1})$$

where each $Q_i \in \{\forall, \exists\}$, and $\varphi(p_0, \ldots, p_{n-1})$ is a Boolean formula (called the *matrix*) with propositions among $p_0, \ldots, p_{n-1}$. The *length* $\mathrm{leng}\,(\beta)$ of a quantified Boolean formula β in the previous form is defined as $n + \mathrm{leng}\,(\varphi)$. The quantifiers range over the truth values 1 (true) and 0 (false), and a quantified Boolean formula without free variables is *true* if and only if it evaluates to 1, i.e., the subformulas $\forall p\, \varphi(p)$ and $\exists p\, \varphi(p)$ are regarded to be true iff $\varphi(\top) \wedge \varphi(\bot)$ and

[34] We refer to the fact that the definition that we will introduce later for the modal formula $g(\beta)$ associated with a quantified Boolean formula β is motivated by the modal formula considered in [HM92]. We notice that the modal formulas considered in [Lad77, BdRV01] are simpler, but they are not satisfiable in structures with a persistent valuation (cf. Lemma 1.5.3).

[35] Sometimes these formulas have been called prenex quantified Boolean formulas. Then the set of quantified Boolean formulas is defined by

$$\varphi ::= \bot \mid \top \mid p \mid \sim\varphi \mid \varphi_0 \wedge \varphi_1 \mid \forall p \varphi \mid \exists p \varphi$$

where p ranges over elements of *Prop*.

(i) q_0

(ii) $\sim(q_1 \vee q_2 \vee \ldots \vee q_n \vee q_{n+1} \vee p_0 \vee p_1 \vee \ldots \vee p_{n-1})$

(iii) $\Diamond(q_1 \wedge \sim q_2)$

(iv) $\Box^1\gamma_{1n} \wedge \Box^2\gamma_{2n} \wedge \ldots \wedge \Box^n\gamma_{nn}$

(v) $\Box^1\theta_1 \wedge \Box^2\theta_2 \wedge \ldots \wedge \Box^{n-1}\theta_{n-1}$

(vi) $\bigwedge_{i\in\{j<n:Q_j=\forall\}} \Box^i\delta_i$

(vii) $\Box^1\psi_1 \wedge \Box^2(\psi_1\wedge\psi_2) \wedge \Box^3(\psi_1\wedge\psi_2\wedge\psi_3) \wedge \ldots \wedge \Box^{n-1}(\psi_1\wedge\psi_2\wedge\psi_3\wedge\ldots\wedge\psi_{n-1})$

(viii) $\Box^n(q_n \supset \varphi)$

Figure 1.9: The modal formula $g(\beta)$

$\varphi(\top) \vee \varphi(\bot)$ are true, respectively[36]. For instance, $\exists p_0 \forall p_1(p_0 \vee p_1)$ is true, while $\forall p_0 \exists p_1 \forall p_2(p_0 \wedge p_1 \wedge p_2)$ is not true. The logic $\boldsymbol{QBF}$ is the set of quantified Boolean formulas that are true. Hence, the $\boldsymbol{QBF}$ problem is the problem of deciding, given an arbitrary quantified Boolean formula β, whether $\beta \in \boldsymbol{QBF}$.

As far as we are interested in a PSpace-complete problem we can restrict[37] our definition of quantified Boolean formulas to the case that $Q_0 = \exists$ and $n \geq 2$. This is obvious by the following trivial polynomial reduction:

$$\begin{array}{c} Q_0p_0 \ldots Q_{n-1}p_{n-1}\varphi(p_0,\ldots,p_{n-1}) \text{ is true} \quad \text{iff} \\ \exists q_0 \exists q_1 Q_0p_0 \ldots Q_{n-1}p_{n-1}\big(q_0 \wedge q_1 \wedge \varphi(p_0,\ldots,p_{n-1})\big) \text{ is true,} \end{array}$$

where q_0 and q_1 are two new propositions. From now on we will assume that all quantified Boolean formulas satisfy the requirements $Q_0 = \exists$ and $n \geq 2$.

Now we present the promised polynomial time reduction from a PSpace-complete problem to $\mathbf{\Lambda}$. Let β be a quantified Boolean formula

$$Q_0p_0 \ldots Q_{n-1}p_{n-1}\varphi(p_0,\ldots,p_{n-1})$$

with $Q_0 = \exists$ and $n \geq 2$. We consider new propositions $q_0,\ldots,q_{n+1}$, and we define $g(\beta)$ as the conjunction of formulas displayed in Figure 1.9, where

$$\gamma_{in} := (q_i \wedge \sim q_{i+1}) \supset \big((q_0\wedge\ldots\wedge q_{i-1})\wedge(\sim q_{i+2}\wedge\ldots\wedge\sim q_{n+1})\wedge(\sim p_i\wedge\ldots\wedge\sim p_{n-1})\big),$$

[36]We notice that the semantics coincides precisely with the semantics that we have in the uniform interpolants for classical propositional logic (see p. 44).

[37]Indeed, there is no reason to adopt this restriction in the modal case. However, in Section 5.3 things are easier if we adopt this restriction.

$$\theta_i := (q_i \wedge \sim q_{i+1}) \supset \Diamond(q_{i+1} \wedge \sim q_{i+2}),$$

$$\delta_i := (q_i \wedge \sim q_{i+1}) \supset \big(\Diamond(q_{i+1} \wedge \sim q_{i+2} \wedge p_i) \wedge \Diamond(q_{i+1} \wedge \sim q_{i+2} \wedge \sim p_i)\big),$$

and

$$\psi_i := \big((q_i \wedge p_{i-1}) \supset \Box p_{i-1}\big) \wedge \big((q_i \wedge \sim p_{i-1}) \supset \Box \sim p_{i-1}\big).$$

It is clear that $\deg(g(\beta)) = n$. It is not hard[38] to give a constant $K \in \omega$ such that for every quantified Boolean formula β,

$$\text{leng}\,(g(\beta)) \leq K\,\text{leng}\,(\beta)^2.$$

Hence, the growth of leng $(g(\beta))$ is bounded by a polynomial of degree 2 in the length of β. Therefore, it is clear that for every quantified Boolean formula β we can compute in deterministic polynomial time the modal formula $g(\beta)$. Hence, to obtain a polynomial reduction from $\boldsymbol{QBF}$ to $\boldsymbol{\Lambda}^c$ it is enough to prove that

$$\beta \in \boldsymbol{QBF} \quad \text{iff} \quad \sim g(\beta) \notin \boldsymbol{\Lambda},$$

for every quantified Boolean formula β. This equivalence is a trivial consequence of the following lemma.

1.5.3. Lemma. *Let β be a quantified Boolean formula*

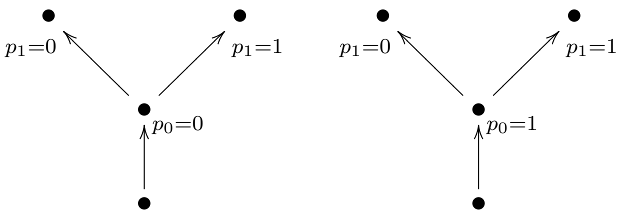
$$Q_0 p_0 \ldots Q_{n-1} p_{n-1} \varphi(p_0, \ldots, p_{n-1})$$

with $Q_0 = \exists$ and $n \geq 2$. The following statements are equivalent:

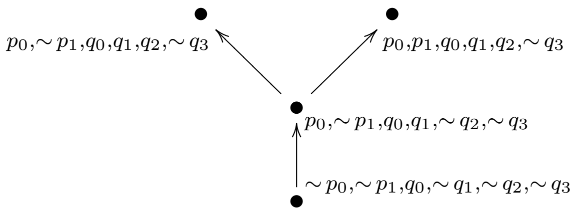
1. *β is true.*
2. *$g(\beta)$ is satisfiable in a structure.*
3. *$g(\beta)$ is satisfiable in a finite strict order with a persistent valuation.*
4. *$g(\beta)$ is satisfiable in a finite partial order with a persistent valuation.*

Sketch of the Proof: Ladner's idea is that when we are evaluating β we are essentially generating a finite number of binary 'trees' $\mathfrak{T}_0, \ldots, \mathfrak{T}_{k-1}$ of height n such that each one of their branches gives us a Boolean valuation in $\{p_0, \ldots, p_{n-1}\}$. These 'trees' consists of the root node, and then — working inwards along the quantifier string — each existential quantifier extends it by adding a single branch, and each universal quantifier extends it by adding two branches (In Figure 1.10(a) the reader can find an example). For every $j < k$, we associate a structure $\mathfrak{A}_j$ over the propositions $\{p_0, \ldots, p_{n-1}, q_0, \ldots, q_{n+1}\}$ with the binary 'tree' $\mathfrak{T}_j$. Let us explain how to define $\mathfrak{A}_j$: the universe and the accessibility relation are given by $\mathfrak{T}_j$, and the behaviour of its valuation at a node of height i ($\leq n$) is the following one:

[38] Hint: For every $n \in \omega$, consider $l_n := \max\{\text{leng}\,(g(\beta)) : \text{leng}\,(\beta) \leq n\}$. It is easy to obtain a constant $k \in \omega$ such that for every $n \geq 1$ it holds that $l_{n+1} \leq l_n + kn$. From here it easily follows that there is a constant $K \in \omega$ such that for every $n \geq 1$ it holds that $l_n \leq Kn^2$.



(a) The binary 'trees' generated by β



(b) A Kripke structure witnessing that β is true

Figure 1.10: Let β be the quantified Boolean formula $\exists p_0 \forall p_1 (p_0 \vee p_1)$

- propositions in $\{q_0, \ldots, q_i\}$ are true.
- propositions in $\{p_i, \ldots, p_n, q_{i+1}, \ldots, q_{n+1}\}$ are false.
- propositions in $\{p_0, \ldots, p_{i-1}\}$ behaves according to the Boolean valuation given by a branch containing the node.

It is clear that $\mathfrak{A}_j$ is really a tree, and that it has a persistent valuation. (In our previous example the tree structure associated with the second 'tree' in Figure 1.10(a) is the one depicted in Figure 1.10(b)).

The connection of the previous ideas with the problem that we are interested in is that β is true iff there is $j < k$ such that the matrix φ is evaluated to 1 in all Boolean valuations recorded by the binary 'tree' $\mathfrak{T}_j$. (In our previous example the witnessing 'tree' is the second one in Figure 1.10(a)). Now let us analyze the different implications that we have in the present lemma.

$(1 \Rightarrow 3)$: Assume that β is true. Then, there is $j < k$ such that the matrix φ is evaluated to 1 in all Boolean valuations recorded by the binary 'tree' $\mathfrak{T}_j$. Now we define $\mathfrak{B}_j$ as the structure $\mathfrak{A}_j$ except for the fact that we take the $R^{\mathfrak{B}_j}$ as the transitive closure of $R^{\mathfrak{A}_j}$. It is clear that $\mathfrak{B}_j$ is a finite strict order with a persistent valuation; and from the fact that φ is evaluated to 1 in all Boolean valuations recorded by the binary 'tree' $\mathfrak{T}_j$ it is simple to check that $\mathfrak{B}_j$ satisfies $g(\beta)$ in the state that is the root of $\mathfrak{A}_j$.

$(1 \Rightarrow 4)$: It is proved as in the previous implication, except for the fact that now we take the reflexive-transitive closure.

$(3 \Rightarrow 2)$: Trivial.

$(4 \Rightarrow 2)$: Trivial.

$(2 \Rightarrow 1)$: We suppose that $g(\beta)$ is satisfiable in a certain pointed structure $\langle \mathfrak{A}, a \rangle$. Replacing it with its unravelling we can assume that $\mathfrak{A}$ is a tree with root a. Using the fact that $\deg(g(\beta)) = n$ we can assume that $\mathfrak{A}$ is a tree of length n: simply remove all states with length $> n$. Using all clauses except (viii) of the definition of $g(\beta)$, it is easily verified that $\mathfrak{A}$ is isomorphic to a (perhaps not generated) substructure of $\mathfrak{A}_j$ for some $j < k$. By clause (viii) of the definition of $g(\beta)$, it follows that φ is evaluated to 1 in all Boolean valuations recorded by the binary 'tree' $\mathfrak{T}_j$. Therefore, β is true. □

We have just concluded the outline of the proof of Ladner's Theorem. We stress that Halpern has been able to extend Ladner's Theorem to the case that there is a single modality [Hal95].

1.5.4. THEOREM (HALPERN). *(*Prop *has cardinal* 1*) Let* $\mathbf{\Lambda}$ *be a normal modal logic such that* $\boldsymbol{K} \subseteq \mathbf{\Lambda} \subseteq \boldsymbol{S4}$*. Then,* $\mathbf{\Lambda}$ *is* PSpace*-hard.*

Halpern's intuitive idea is to imitate the behaviour of a countable set of propositions using formulas with only one proposition. He obtains this by introducing the notion of an infinite family $\{\varphi_n : n \in \omega\}$ of modal formulas being *proposition-like*[39] *for a normal modal logic* $\mathbf{\Lambda}$. For instance, in the case of $\boldsymbol{K}$ Halpern proves that the family $\{\Diamond(\sim p \land \Diamond^{n+1} p) : n \in \omega\}$ is proposition-like. In [Hal95] it is claimed that if we have an infinite pp-like family $\{\varphi_n : n \in \omega\}$, then the modal substitution that replaces, for each $n \in \omega$, proposition p_n with φ_n, is invariant under satisfiability; but this is false[40]. So, the proof given by Halpern in [Hal95] does not work. Fortunately, Halpern knows how to solve this gap using a much more involved method[41]. In Section 5.4 we will use this involved method together with new ideas[42] to prove that indeed the normal modal logic $\boldsymbol{K}$ without any proposition is also PSpace-hard.

To finish the section, we say something about a different kind of problems: problems related to local consequences. Up to now all problems that we have considered are of the form "$\varphi \in \mathbf{\Lambda}$?". What happens if we consider algorithms for problems of the form "$\Phi \vdash_{l\mathbf{\Lambda}} \varphi$?"? First of all, let us specify what we understand by a problem of the form "$\Phi \vdash_{l\mathbf{\Lambda}} \varphi$?" because there are at least two ways to understand this problem.

The first possibility is the problem of deciding, given a $\varphi \in \mathcal{L}^{mod}$ and a finite set $\Phi \subseteq \mathcal{L}^{mod}$, whether $\Phi \vdash_{l\mathbf{\Lambda}} \varphi$. Using the fact that

[39]The paper [Hal95] uses the terminology pp-like where pp refers to primitive propositions.

[40]See Footnote 19 on page 221.

[41]The author thanks Joe Halpern for explaining to him this new (and unpublished) method in several extensive e-mails.

[42]The new idea is that $\{\Diamond(\sim\Diamond\Box\bot \land \Diamond^{n+1}\Diamond\Box\bot) : n \in \omega\}$ is a proposition-like family for $\boldsymbol{K}$.

$$\{\phi_0, \ldots, \phi_{n-1}\} \vdash_{l\mathbf{\Lambda}} \varphi \quad \text{iff} \quad (\phi_0 \wedge \ldots \wedge \phi_{n-1}) \supset \varphi \in \mathbf{\Lambda},$$

it is clear that there is a polynomial time reduction from this problem to the problem "$\varphi \in \mathbf{\Lambda}$?". Therefore, this possibility does not give anything new.

The other possibility is the problem of deciding, given a $\varphi \in \mathcal{L}^{mod}$ and a decidable set $\Phi \subseteq \mathcal{L}^{mod}$, whether $\Phi \vdash_{l\mathbf{\Lambda}} \varphi$. The author has never seen problems of this kind in the literature. Indeed, there is a reason for this fact: these problems are undecidable. Let us prove[43] their undecidability showing a reduction from the Halting Problem, which is known to be undecidable [EFT94, Theorem X.3.2]. Before giving the argument we stress that it works even for the case in which the only formulas that we have are finite conjunctions of $\bot$ and $\top$. Let us assume that A is a deterministic algorithm. We define the map

$$\begin{array}{rrcl} f_A : & \omega & \longrightarrow & \{0,1\} \\ & n & \longmapsto & f_A(n) := \begin{cases} 0 & \text{if } A \text{ halts after } n \text{ steps,} \\ 1 & \text{if not.} \end{cases} \end{array}$$

It is clear that f_A is a computable map. Now we define the set

$$\Phi_A := \{\overbrace{\phi_{f(n)} \wedge \ldots \wedge \phi_{f(n)}}^{n \text{ times}} : n \in \omega\}$$

where $\phi_0 := \bot$ and $\phi_1 := \top$. It is clear that Φ_A is a decidable set, and it is also obvious that

$$A \text{ halts} \quad \text{iff} \quad \Phi_A \vdash_{l\mathbf{\Lambda}} \bot.$$

Thus, we have obtained a reduction from the Halting Problem. This concludes the proof of the undecidability for problems of this kind.

[43] The proof that we present was obtained after several discussions with Ramon Jansana, Rafel Farré and Juan Carlos Martínez. The author thanks them.

Chapter 2

Strict-Weak Languages

This dissertation analyzes certain fragments of modal languages from different points of view. These fragments, which we will term strict-weak fragments, have not been considered before in full generality. As we will see, one of their interests lies in the fact that they are very close to modal languages. These fragments allow us to say things about full modal languages. Indeed, it is the first time in the literature that non-trivial fragments of modal languages have properties of this kind. The leitmotif surrounding all our research is that *strict-weak languages are the result of removing some symmetrical aspects that we have in modal languages.* Thus, as the result of forcing these symmetrical conditions on strict-weak languages we will be able to recuperate full modal languages. The lack of symmetry comes from being material negation-free. Despite their wide applicability material negation-free languages have not been studied as extensively as languages with full Boolean expressivity. This dissertation can be seen as a contribution to this topic.

The other main interest in analyzing strict-weak fragments is that their semantics are natural generalizations (where a certain duality holds) of the semantics that we find in very famous logics. Among these logics intuitionistic propositional logic stands out, but there are many other cases.

The rest of the dissertation is structured in the following way. The first section of this chapter is devoted to the introduction of strict-weak fragments with their semantics. In the second section we survey some logics considered in the literature that can be displayed in the framework of strict-weak languages. And each of the chapters focusses on a certain aspect of the analysis of these new languages.

2.1 Strict-weak languages and its semantics

The main difference vis-à-vis the modal case is that we consider two types of modalities. One is used inside strict implications, and the other is used inside weak differences. This is the reason why we refer to them by the term strict-weak (SW).

2.1.1. Definition. (SW-vocabularies)
A *SW-vocabulary*, or a *SW-signature*, is a triple $\vartheta = \langle \mathsf{SMod}, \mathsf{WMod}, \mathsf{Prop} \rangle$ where SMod, WMod and Prop are sets such that $\mathsf{SMod} \cup \mathsf{WMod}$ and Prop are disjoint. The elements of SMod and WMod are called, respectively, *strict modalities* and *weak modalities*, and the elements of Prop are called *(atomic) propositions*. The *vocabulary associated with* ϑ is the pair $\tau_\vartheta = \langle \mathsf{Mod}, \mathsf{Prop} \rangle$ where $\mathsf{Mod} := \mathsf{SMod} \cup \mathsf{WMod}$. It is said that ϑ is *finite* when SMod, WMod and Prop are finite. Given ϑ and ϑ' two SW-vocabularies, its *intersection* $\vartheta \cap \vartheta'$ is the SW-vocabulary $\langle \mathsf{SMod} \cap \mathsf{SMod}', \mathsf{WMod} \cap \mathsf{WMod}', \mathsf{Prop} \cap \mathsf{Prop}' \rangle$, and its *union* $\vartheta \cup \vartheta'$ is the SW-vocabulary $\langle \mathsf{SMod} \cup \mathsf{SMod}', \mathsf{WMod} \cup \mathsf{WMod}', \mathsf{Prop} \cup \mathsf{Prop}' \rangle$.

Let τ be a vocabulary with $\kappa \in \mathbb{C}\mathrm{ARD}$ modalities. An interesting question is how many SW-vocabularies ϑ there are such that $\tau_\vartheta = \tau$. The answer is 3^κ different SW-vocabularies. The reason is that for each modality there are three different possibilities: (i) it is only a strict modality (then it is said that it is a *pure strict modality*), (ii) it is only a weak modality (then it is said that it is a *pure weak modality*), and (iii) it is at the same time a strict modality and a weak modality . The same argument removing the last possibility shows that there are 2^κ SW-vocabularies ϑ such that $\mathsf{SMod} \cap \mathsf{WMod} = \emptyset$ and $\tau_\vartheta = \tau$. We recall that if κ is an infinite cardinal then $2^\kappa = 3^\kappa$.

2.1.2. Definition. ($\mathcal{L}^{SW}(\vartheta)$-formulas and $\mathcal{L}^{SW}_\infty(\vartheta)$-formulas)
Let $\vartheta = \langle \mathsf{SMod}, \mathsf{WMod}, \mathsf{Prop} \rangle$ be a SW-vocabulary. The set $\mathcal{L}^{SW}(\vartheta)$ of *(finitary) strict-weak formulas* over ϑ is given by the rule

$$\varphi ::= \bot \mid \top \mid p \mid \varphi_0 \wedge \varphi_1 \mid \varphi_0 \vee \varphi_1 \mid \varphi_0 \rightarrow_s \varphi_1 \mid \varphi_0 \leftharpoondown_w \varphi_1,$$

where $p \in \mathsf{Prop}$, $s \in \mathsf{SMod}$ and $w \in \mathsf{WMod}$. And the set $\mathcal{L}^{SW}_\infty(\vartheta)$ of *infinitary strict-weak formulas* over ϑ is defined as follows:

$$\varphi ::= p \mid \bigwedge \Phi \mid \bigvee \Phi \mid \varphi_0 \rightarrow_s \varphi_1 \mid \varphi_0 \leftharpoondown_w \varphi_1,$$

where $p \in \mathsf{Prop}$, $s \in \mathsf{SMod}$, $w \in \mathsf{WMod}$, and Φ is a set (not a proper class) of $\mathcal{L}^{SW}_\infty(\vartheta)$-formulas. We consider the following useful abbreviations: $[s]\varphi := \top \rightarrow_s \varphi$, $\neg_s \varphi := \varphi \rightarrow_s \bot$, $\langle w \rangle \varphi := \varphi \leftharpoondown_w \bot$, and $\urcorner_w \varphi := \top \leftharpoondown_w \varphi$. Given $n \in \omega$ we define $[s]^n\varphi$, $[s]^{(n)}\varphi$, $\langle w \rangle^n \varphi$ and $\langle w \rangle^{(n)} \varphi$ as we did on page 6. If there is no confusion about which modality we refer to we will simply write $\Box$, $\neg$, $\Diamond$ and $\llcorner$. In the case that the set WMod is empty (i.e., all modalities are pure strict modalities) we will call these languages $\mathcal{L}^S$ and $\mathcal{L}^S_\infty$, and if additionally SMod is a singleton then we will write $\mathcal{L}^s$ and $\mathcal{L}^s_\infty$. Dually, in the case that SMod is empty (i.e., all modalities are pure weak modalities) we will call these languages $\mathcal{L}^W$ and $\mathcal{L}^W_\infty$, and if additionally WMod is a singleton then we will write $\mathcal{L}^w$ and $\mathcal{L}^w_\infty$.

2.1.3. Definition. (Semantics for $\mathcal{L}^{SW}(\vartheta)$ and $\mathcal{L}^{SW}_\infty(\vartheta)$)
The *strict-weak satisfiability relation* $\Vdash$ is a relation between pointed τ_ϑ-structures and $\mathcal{L}^{SW}(\vartheta)$-formulas defined as follows:

$\mathfrak{A}, a \Vdash \bot$		is never the case
$\mathfrak{A}, a \Vdash \top$		is always the case
$\mathfrak{A}, a \Vdash p$	iff	$a \in V(p)$
$\mathfrak{A}, a \Vdash \varphi_0 \wedge \varphi_1$	iff	$\mathfrak{A}, a \Vdash \varphi_0$ and $\mathfrak{A}, a \Vdash \varphi_1$
$\mathfrak{A}, a \Vdash \varphi_0 \vee \varphi_1$	iff	$\mathfrak{A}, a \Vdash \varphi_0$ or $\mathfrak{A}, a \Vdash \varphi_1$
$\mathfrak{A}, a \Vdash \varphi_0 \rightarrow_s \varphi_1$	iff	$\forall a'(\langle a, a'\rangle \in R_s \,\&\, \mathfrak{A}, a' \Vdash \varphi_0 \;\Rightarrow\; \mathfrak{A}, a' \Vdash \varphi_1)$
$\mathfrak{A}, a \Vdash \varphi_0 \leftharpoondown_w \varphi_1$	iff	$\exists a'(\langle a, a'\rangle \in R_w \,\&\, \mathfrak{A}, a' \Vdash \varphi_0 \,\&\, \mathfrak{A}, a' \nVdash \varphi_1)$.

In the case of $\mathcal{L}^{SW}_\infty(\vartheta)$-formulas the previous definition is expanded with the following clauses:

$\mathfrak{A}, a \Vdash \bigwedge \Phi$	iff	$\mathfrak{A}, a \Vdash \varphi$ for every $\varphi \in \Phi$
$\mathfrak{A}, a \Vdash \bigvee \Phi$	iff	$\mathfrak{A}, a \Vdash \varphi$ for a certain $\varphi \in \Phi$.

If there is no ambiguity about which structure we are referring to we will write $a \Vdash \varphi$, which is read as 'φ is true at a', 'a satisfies φ' or 'φ holds at a'. The *strict-weak theory of a pointed structure* $\langle \mathfrak{A}, a\rangle$, denoted by $\mathrm{Th}_{\mathcal{L}^{SW}(\vartheta)}(\mathfrak{A}, a)$, is the set of strict-weak formulas that holds at a. We will write $\langle \mathfrak{A}, a\rangle \rightsquigarrow_\vartheta \langle \mathfrak{B}, b\rangle$ if the strict-weak theory of the first is included in the strict-weak theory of the second, i.e., if the satisfiability of strict-weak formulas is preserved. In the same way we introduce $\mathrm{Th}_{\mathcal{L}^{SW}_\infty(\vartheta)}(\mathfrak{A}, a)$ (the *infinitary strict theory of this pointed structure*). We will omit the symbol ϑ when there is no ambiguity, and we will write $\rightsquigarrow_S$, $\rightsquigarrow_s$, $\rightsquigarrow_W$ and $\rightsquigarrow_w$ following the same conventions as in Definition 2.1.2.

The satisfiability conditions for the defined connectives introduced in Definition 2.1.2 are the expected ones, i.e., the same as in the modal case (see Section 1.2). If we compare the satisfiability relation given here and the one for the modal case it is obvious that the strict-weak language given by ϑ is a fragment of the modal language given by the vocabulary associated with ϑ. This means that the map $\sigma : \mathcal{L}^{SW}(\vartheta) \longrightarrow \mathcal{L}^{MOD}(\tau_\vartheta)$ defined by

$$\begin{array}{lcl}
\sigma(\bot) & := & \bot \\
\sigma(\top) & := & \top \\
\sigma(p) & := & p \\
\sigma(\varphi_0 \wedge \varphi_1) & := & \sigma(\varphi_0) \wedge \sigma(\varphi_1) \\
\sigma(\varphi_0 \vee \varphi_1) & := & \sigma(\varphi_0) \vee \sigma(\varphi_1) \\
\sigma(\varphi_0 \rightarrow_s \varphi_1) & := & \sigma(\varphi_0) \rightarrow_s \sigma(\varphi_1) = [s](\sigma(\varphi_0) \supset \sigma(\varphi_1)) \\
\sigma(\varphi_0 \leftharpoondown_w \varphi_1) & := & \sigma(\varphi_0) \leftharpoondown_w \sigma(\varphi_1) = \langle w \rangle(\sigma(\varphi_0) \setminus \sigma(\varphi_1))
\end{array}$$

respects the satisfiability relation, i.e.,

$$\mathfrak{A}, a \Vdash \varphi \quad \text{iff} \quad \mathfrak{A}, a \Vdash \sigma(\varphi) \tag{2.1}$$

where the first $\Vdash$ refers to the strict-weak satisfiability relation and the second $\Vdash$ denotes the modal satisfiability relation. It is clear that the previous map can be extended to the infinitary case adding

$$\begin{array}{lcl}
\sigma(\bigwedge \Phi) & := & \bigwedge\{\sigma(\varphi) : \varphi \in \Phi\} \\
\sigma(\bigvee \Phi) & := & \bigvee\{\sigma(\varphi) : \varphi \in \Phi\}
\end{array}$$

in such a way that it also respects the satisfiability relation. All this explains why there is no problem if we use the same symbol for both satisfiability relations. The map σ has sometimes been considered in the literature, but always in the context of the language $\mathcal{L}^s$. The first appearance under this name is in [Doš93], and since then it has also been considered by other authors, e.g., Celani and Jansana in [CJ01]. As strict implications and weak differences are defined connectives in modal languages we have that the map σ is not the identity. However, (2.1) justifies that there is no problem if we identify φ with $\sigma(\varphi)$. Thus, for simplicity reasons we will be sloppy about this distinction; for instance, given strict-weak formulas $\varphi, \varphi_0, \varphi_1$ we will consider the modal formulas $\sim\varphi$ and $\varphi_0 \supset \varphi_1$ as abbreviations for $\sim\sigma(\varphi)$ and $\sigma(\varphi_0) \supset \sigma(\varphi_1)$. This identification also allows us to talk about the *modal degree* of a $\mathcal{L}^{SW}$-formula or a $\mathcal{L}^{SW}_\infty$-formula. Although we could also use this strategy to introduce the length of these formulas, we will later adopt a different (and more natural) definition. But for the modal degree we in fact adopt this identification. Given $n \in \omega$, $\langle \mathfrak{A}, a \rangle \rightsquigarrow_n \langle \mathfrak{B}, b \rangle$ is written if the satisfiability of finitary strict-weak formulas with modal degree $\leq n$ is preserved.

2.1.4. Remark. If $s \in \mathsf{SMod}$ then $\top \equiv \bot \rightarrow_s \bot$. Thus, in the case $\mathsf{SMod} \neq \emptyset$, we obtain the same expressive power removing $\top$ from our formulas. Analogously, in the case $\mathsf{WMod} \neq \emptyset$, we could have removed $\bot$ from our formulas (because $\bot \equiv \top \leftharpoondown_w \top$). In the case that the three sets $\mathsf{SMod}, \mathsf{WMod}, \mathsf{Prop}$ are non-empty we could simultaneously remove $\bot$ and $\top$ due to the fact that $\bot \equiv p \leftharpoondown_w p$ and $\top \equiv p \rightarrow_s p$.

As $\bot \equiv \bigvee \emptyset$, $\top \equiv \bigwedge \emptyset$ and $\varphi_0 \wedge \varphi_1 \equiv \bigwedge\{\varphi_0, \varphi_1\}$ we know that all (finitary) strict-weak formulas are, up to equivalence, in $\mathcal{L}^{SW}_\infty$. Analogously to what happens in the modal case strict-weak formulas are closed, up to equivalence, under finitary conjunction and disjunction. Thus, for every $\Phi \subseteq_\omega \mathcal{L}^{SW}$ (including the empty set) we can define strict-weak formulas $\bigwedge \Phi$ and $\bigvee \Phi$ such that:

$$\begin{array}{lll} \mathfrak{A}, a \Vdash \bigwedge \Phi & \text{iff} & \mathfrak{A}, a \Vdash \varphi \text{ for every } \varphi \in \Phi \\ \mathfrak{A}, a \Vdash \bigvee \Phi & \text{iff} & \mathfrak{A}, a \Vdash \varphi \text{ for a certain } \varphi \in \Phi. \end{array}$$

As in the modal case, we will not worry about the precise definition of these formulas.

2.1.5. Remark. (A conjunctive, and disjunctive, form). The distributivity law is clearly valid under our semantics and it is also obvious that our language is the closure under conjunction and disjunction of all propositions, strict implications and weak differences. These two facts allow us to see easily that all strict-weak formulas can be written, up to equivalence, as a conjunction of disjunctions of propositions or strict implications or weak differences (*conjunctive form*). In the finitary case the conjunction and the disjunctions are finite, while in the infinitary case they are arbitrary. Analogously all strict-weak formulas can be written as a disjunction of conjunctions of propositions or strict implications or weak differences (*disjunctive form*).

Let us now see what a strict-weak substitution is, or simply a substitution. First of all we introduce the algebra $\mathcal{L}^{\mathbf{SW}}$ as the set of strict-weak formulas endowed with the algebraic structure given by considering its connectives as operations. This algebra is the absolutely free algebra generated by Prop over the corresponding algebraic signature (which depends on its modalities). Then, we define a *substitution* as an endomorphism on the algebra $\mathcal{L}^{\mathbf{SW}}$. We will write $\varphi(\varphi_0, \ldots, \varphi_{n-1})$, adopting the same convention as in the modal case.

Now we define the length of a strict-weak formula. As we have discussed above one possibility is to use the identification with fragments of modal languages. We do not adopt it because then the length of a formula $p \rightarrow_s p$ should be 8 (remember that $\rightarrow_s$ is a defined connective, i.e., $p \rightarrow_s p = [s] \sim(\sim\sim p \wedge \sim p)$). It seems more natural to consider the length of this formula as 3. Thus, the *length of a strict-weak formula* will be the number of symbol occurrences in the formula, i.e., is the natural number defined inductively as follows:

$$\begin{array}{lll} \text{leng}\,(\bot) & := & 1 \\ \text{leng}\,(\top) & := & 1 \\ \text{leng}\,(p) & := & 1 \\ \text{leng}\,(\varphi_0 \wedge \varphi_1) & := & \text{leng}\,(\varphi_0) + \text{leng}\,(\varphi_1) + 1 \\ \text{leng}\,(\varphi_0 \vee \varphi_1) & := & \text{leng}\,(\varphi_0) + \text{leng}\,(\varphi_1) + 1 \\ \text{leng}\,(\varphi_0 \rightarrow_s \varphi_1) & := & \text{leng}\,(\varphi_0) + \text{leng}\,(\varphi_1) + 1 \\ \text{leng}\,(\varphi_0 \leftharpoondown_w \varphi_1) & := & \text{leng}\,(\varphi_0) + \text{leng}\,(\varphi_1) + 1. \end{array}$$

We stress that a simple induction shows that the number of *subformulas* of φ is bounded by leng(φ). Notice that in general leng$(\varphi) \neq$ leng$(\sigma(\varphi))$. However, there is no essential difference. Indeed, it is clear that there is a linear function $p(x)$ such that for every strict-weak formula φ, $\text{leng}^{SW}(\varphi) \leq p(\text{leng}^{MOD}(\varphi))$. As a consequence, both definitions should bring the same computational results.

Finally we introduce the duality for strict-weak languages, which will be a very powerful tool in our future work. The main difference vis-à-vis the modal case is that while there the dual of a modal language is the same language, the dual of a strict-weak language is a different strict-weak language. What it is remarkable here is that it is still a strict-weak language. Therefore, once we know that a certain theorem is valid for all strict-weak languages this theorem will remain valid considering the dual of strict-weak languages.

2.1.6. Definition. (SW-duality)
Let $\vartheta = \langle \mathsf{SMod}, \mathsf{WMod}, \mathsf{Prop} \rangle$ be a SW-vocabulary. The *dual of* ϑ is the SW-vocabulary $\vartheta^d := \langle \mathsf{WMod}, \mathsf{SMod}, \mathsf{Prop} \rangle$. And the *dual of a* $\mathcal{L}^{SW}(\vartheta)$*-formula* is the $\mathcal{L}^{SW}(\vartheta^d)$-formula defined as follows:

$$
\begin{array}{lcl}
\bot^d & := & \top \\
\top^d & := & \bot \\
p^d & := & p \\
(\varphi_0 \wedge \varphi_1)^d & := & \varphi_1^d \vee \varphi_0^d \\
(\varphi_0 \vee \varphi_1)^d & := & \varphi_1^d \wedge \varphi_0^d \\
(\varphi_0 \rightharpoondown_s \varphi_1)^d & := & \varphi_1^d \leftharpoondown_s \varphi_0^d \\
(\varphi_0 \leftharpoondown_w \varphi_1)^d & := & \varphi_1^d \rightharpoondown_w \varphi_0^d.
\end{array}
$$

That is, we have just defined a map ${}^d : \mathcal{L}^{SW}(\vartheta) \longrightarrow \mathcal{L}^{SW}(\vartheta^d)$. In the case of infinitary strict-weak formulas we obtain a map ${}^d : \mathcal{L}^{SW}_\infty(\vartheta) \longrightarrow \mathcal{L}^{SW}_\infty(\vartheta^d)$ when we expand the previous definition with the following clauses:

$$
\begin{array}{lcl}
(\bigwedge \Phi)^d & := & \bigvee\{\varphi^d : \varphi \in \Phi\} \\
(\bigvee \Phi)^d & := & \bigwedge\{\varphi^d : \varphi \in \Phi\}.
\end{array}
$$

From the definition it is obvious that the vocabulary associated with ϑ is the same one than the associated with its dual ϑ^d, i.e., $\tau_\vartheta = \tau_{\vartheta^d}$. And it also holds that $\varphi = (\varphi^d)^d$. An immediate consequence of the modal Duality Principle is the following theorem.

2.1.7. Theorem (SW-Duality Principle). *Let ϑ be a SW-vocabulary, let φ be a $\mathcal{L}^{SW}(\vartheta)$-formula or a $\mathcal{L}^{SW}_\infty(\vartheta)$-formula, and let $\langle \mathfrak{A}, a \rangle$ be a pointed τ_ϑ-structure. Then,*

$$\mathfrak{A}, a \Vdash \varphi \quad \textit{iff} \quad \mathfrak{A}^d, a \nVdash \varphi^d.$$

Some easy consequences of this principle that we are keen to stress are the following equivalences:

$$\begin{array}{rcl}\mathfrak{A}, a \Vdash \varphi_0 \supset \varphi_1 & \text{iff} & \mathfrak{A}^d, a \Vdash \varphi_1^d \supset \varphi_0^d \\ \mathfrak{A}, a \Vdash \varphi_0 \smallsetminus \varphi_1 & \text{iff} & \mathfrak{A}^d, a \Vdash \varphi_1^d \smallsetminus \varphi_0^d \\ \langle \mathfrak{A}, a \rangle \rightsquigarrow_\vartheta \langle \mathfrak{B}, b \rangle & \text{iff} & \langle \mathfrak{B}^d, b \rangle \rightsquigarrow_{\vartheta^d} \langle \mathfrak{A}^d, a \rangle \\ \mathrm{Th}_{\mathcal{L}_\infty^{SW}(\vartheta)}(\mathfrak{A}, a) \subseteq \mathrm{Th}_{\mathcal{L}_\infty^{SW}(\vartheta)}(\mathfrak{B}, b) & \text{iff} & \mathrm{Th}_{\mathcal{L}_\infty^{SW}(\vartheta^d)}(\mathfrak{B}^d, b) \subseteq \mathrm{Th}_{\mathcal{L}_\infty^{SW}(\vartheta^d)}(\mathfrak{A}^d, a).\end{array}$$

Since strict-weak formulas are particular cases of modal formulas we know what it means that a strict-weak formula φ is *valid in a frame* $\mathfrak{F}$ (notation: $\mathfrak{F} \Vdash \varphi$) and what it means that a strict-weak formula φ is *valid in a structure* $\mathfrak{A}$ (notation: $\mathfrak{A} \Vdash \varphi$). Analogously to the modal case, given an arbitrary class C of frames we define the consequence relations $\models_{l\mathsf{C}}$ and $\models_{g\mathsf{C}}$ between strict-weak formulas giving by the conditions

$$\Phi \models_{l\mathsf{C}} \varphi \quad \text{iff} \quad \forall \mathfrak{F} \in \mathsf{C}\ \forall V \in \mathcal{P}(F)^{\mathsf{Prop}}\ \forall a \in F,\ \text{if } \mathfrak{F}, V, a \Vdash \Phi \text{ then } \mathfrak{F}, V, a \Vdash \varphi,$$

and

$$\Phi \models_{g\mathsf{C}} \varphi \quad \text{iff} \quad \forall \mathfrak{F} \in \mathsf{C}\ \forall V \in \mathcal{P}(F)^{\mathsf{Prop}},\ \text{if } \mathfrak{F}, V \Vdash \Phi \text{ then } \mathfrak{F}, V \Vdash \varphi.$$

Then, we have two different consequence relations denoted by the same name: a modal consequence $\models_{l\mathsf{C}}$ and a strict-weak consequence $\models_{l\mathsf{C}}$. To distinguish them we will write $\models_{l\mathsf{C}}^{MOD}$ and $\models_{l\mathsf{C}}^{SW}$, respectively. This dissertation mainly concerns strict-weak languages; so, in the strict-weak case we will sometimes remove the superscript (if we think that there is no ambiguity). Of course, the same will apply for the global consequence. It is clear that it is also possible to associate consequence relations between strict-weak formulas with classes of structures. Given a class K of structures we introduce the consequence relations

$$\Phi \models_{l\mathsf{K}} \varphi \quad \text{iff} \quad \forall \mathfrak{A} \in \mathsf{K}\ \forall a \in A,\ \text{if } \mathfrak{A}, a \Vdash \Phi \text{ then } \mathfrak{A}, a \Vdash \varphi,$$

and

$$\Phi \models_{g\mathsf{K}} \varphi \quad \text{iff} \quad \forall \mathfrak{A} \in \mathsf{K},\ \text{if } \mathfrak{A} \Vdash \Phi \text{ then } \mathfrak{A} \Vdash \varphi.$$

Analogously to our remark above we will write $\models_{l\mathsf{K}}^{MOD}$ and $\models_{l\mathsf{K}}^{SW}$ (and also $\models_{g\mathsf{K}}^{MOD}$ and $\models_{g\mathsf{K}}^{SW}$) to distinguish the modal case from the strict-weak case, deleting sometimes the superscript in the strict-weak case. We will say that an structure $\mathfrak{A}$ is *SW-compact* if the consequence relation $\models_{l\mathfrak{A}}^{SW}$ is finitary (perhaps it is not structural). It is obvious that all modally compact structures are SW-compact. However, we will see in Example 4.1.4 that the converse is false. On the other hand, we do not introduce the notion of SW-differentiated structure because it does not offer anything new: later we will prove (see Corollary 3.1.10) that two states satisfy the same modal formulas iff they satisfy the same strict-weak formulas.

2.2 Examples in the literature

We have explained at the beginning of the chapter that there are many famous examples of logics that site in strict-weak fragments. Before introducing the expected examples we define some new properties on valuations that we will need. Let $\mathfrak{A}$ be a structure with valuation V. It is said that the valuation V is R_m*-persistent* if it holds that for every $a, a' \in A$ and $p \in \mathsf{Prop}$,

$$\text{if } a \in V(p) \text{ and } \langle a, a' \rangle \in R_m, \text{ then } a' \in V(p).$$

We say that the valuation is *persistent* when the previous condition holds replacing R_m with the global accessibility relation. The valuation V is said to be R_m*-antipersistent* if it holds that for every $a, a' \in A$ and $p \in \mathsf{Prop}$,

$$\text{if } a \in V(p) \text{ and } \langle a', a \rangle \in R_m, \text{ then } a' \in V(p).$$

We will say that the valuation is *antipersistent* when the last condition holds replacing R_m with the global accessibility relation.

Intuitionistic propositional logic *IPL*. [Dum00, CZ97] We consider the strict-weak language $\mathcal{L}^s$. A structure is an ***IPL****-structure* if it is a quasi order with a persistent valuation. Intuitionistic propositional logic ***IPL*** can be defined as the set of $\mathcal{L}^s$-formulas that are valid in all ***IPL***-structures. It is known that ***IPL*** is closed under substitutions. There is a well known translation from ***IPL*** into the normal modal logic ***S*4** (and also into ***Grz***) that is usually called Gödel's Translation [Göd33]. This translation is the map $\mathsf{t}_0 : \mathcal{L}^s \longrightarrow \mathcal{L}^{mod}$ defined according to the following clauses:

$$\begin{array}{lcl}
\mathsf{t}_0(\bot) & := & \bot \\
\mathsf{t}_0(\top) & := & \top \\
\mathsf{t}_0(p) & := & \Box p \\
\mathsf{t}_0(\varphi_0 \wedge \varphi_1) & := & \mathsf{t}_0(\varphi_0) \wedge \mathsf{t}_0(\varphi_1) \\
\mathsf{t}_0(\varphi_0 \vee \varphi_1) & := & \mathsf{t}_0(\varphi_0) \vee \mathsf{t}_0(\varphi_1) \\
\mathsf{t}_0(\varphi_0 \rightarrow \varphi_1) & := & \mathsf{t}_0(\varphi_0) \rightarrow \mathsf{t}_0(\varphi_1) = \Box(\mathsf{t}_0(\varphi_0) \supset \mathsf{t}_0(\varphi_1)).
\end{array}$$

The fact that t_0 is a *translation* means that for every $\mathcal{L}^s$-formula φ,

$$\varphi \in \boldsymbol{IPL} \quad \text{iff} \quad \mathsf{t}_0(\varphi) \in \boldsymbol{S4}.$$

In this dissertation we will also use the word *embedding* as a synonym of translation. Now we present a construction to prove that t_0 is a translation from ***IPL*** into ***S*4**. Given a structure $\mathfrak{A}$, we define the structure $\mathfrak{A}^{per}$ as the structure with the same universe and the same accessibility relation than $\mathfrak{A}$, but with a valuation satisfying that for every proposition p and every $a \in A$,

$$\mathfrak{A}^{per}, a \Vdash p \quad \text{iff} \quad \mathfrak{A}, a \Vdash \Box p.$$

It can be proved using a straightforward induction that for every $\mathcal{L}^s$-formula φ, every structure $\mathfrak{A}$ (perhaps it is not a quasi order) and every state $a \in A$, it holds that

$$\mathfrak{A}^{per}, a \Vdash \varphi \quad \text{iff} \quad \mathfrak{A}, a \Vdash \mathsf{t}_0(\varphi). \tag{2.2}$$

Using (2.2) together with the transitivity and the persistence of $\boldsymbol{IPL}$-structures it is not hard to check that t_0 is a translation from $\boldsymbol{IPL}$ into $\boldsymbol{S4}$.

Classical propositional logic $\boldsymbol{CPL}$. [End00] It is clear that $\boldsymbol{CPL}$ coincides with the set of $\mathcal{L}^s$-formulas that are valid in the reflexive frame that has a single point. It also holds that $\boldsymbol{CPL}$ is the set consisting of all $\mathcal{L}^s$-formulas that are valid in all reflexive, transitive and symmetric frames under persistent valuations. Using (2.2) it is easily verified that t_0 is a translation from $\boldsymbol{CPL}$ into $\boldsymbol{S5}$.

Superintuitionistic propositional logics. [CZ97] A *superintuitionistic (propositional) logic* $\boldsymbol{\Lambda}$ is a set of $\mathcal{L}^s$-formulas such that:

- $\boldsymbol{IPL} \subseteq \boldsymbol{\Lambda}$,
- $\boldsymbol{\Lambda}$ is closed under Modus Ponens,
- $\boldsymbol{\Lambda}$ is closed under substitutions.

In the literature they are also known as *intermediate logics*. It is well known that for every superintuitionistic logic $\boldsymbol{\Lambda}$, if $\boldsymbol{\Lambda} \neq \mathcal{L}^s$, then $\boldsymbol{IPL} \subseteq \boldsymbol{\Lambda} \subseteq \boldsymbol{CPL}$. Another interesting property is that for every superintuitionistic logic $\boldsymbol{\Lambda}$, there is a class of $\boldsymbol{IPL}$-structures such that $\boldsymbol{\Lambda}$ is the set of $\mathcal{L}^s$-formulas valid in all structures of this class. This is usually proved using a canonical structure construction.

Heyting-Brouwer propositional logic. [Rau74b, Rau74a, Rau77, Rau80] We consider the strict-weak language $\mathcal{L}^{SW}$ that has a pure strict modality and a pure weak modality. Thus, in our language we have a strict implication $\rightarrow_0$ and a weak difference $\leftharpoondown_1$; and they are associated with different accessibility relations. We define a $\boldsymbol{HBPL}$-structure as a structure $\langle A, R_0, R_1, V\rangle$ such that $\langle A, R_0, V\rangle$ is a $\boldsymbol{IPL}$-structure and $R_1 = R_0^{-1}$. Rauszer defined Heyting-Brouwer propositional logic $\boldsymbol{HBPL}$ as the set of $\mathcal{L}^{SW}$-formulas that are valid in all $\boldsymbol{HBPL}$-structures.

Formal propositional logic $\boldsymbol{FPL}$. [Vis81] Let us consider the strict-weak language $\mathcal{L}^s$. A structure is an $\boldsymbol{FPL}$*-structure* if it is a Noetherian strict order with a persistent valuation. Formal propositional logic $\boldsymbol{FPL}$ was introduced by Visser in [Vis81] as the set of $\mathcal{L}^s$-formulas that are valid in all $\boldsymbol{FPL}$-structures. Visser's interest in this logic was based on the fact that t_0 is a translation from $\boldsymbol{FPL}$ into $\boldsymbol{GL}$. In other words, $\boldsymbol{FPL}$ can be considered as a provability logic.

Basic propositional logic *BPL*. [Vis81, Rui91, Rui93, SWZ98] Let us take the strict-weak language $\mathcal{L}^s$. A structure is an ***BPL**-structure* if it is transitive and has a persistent valuation. Basic propositional logic ***BPL*** is the set of $\mathcal{L}^s$-formulas that are valid in all ***BPL***-structures. The map t_0 is a translation from ***BPL*** into ***K*4**. The history of ***BPL*** is quite surprising. Visser introduced this logic [Vis81] for technical reasons: as we have claimed above his interest was in provability logic. However, over the last decade basic propositional logic has acquired an interest of its own. Ruitenburg has argued its philosophical interest as a constructive logic (see [Rui91, Rui93]): he claims that it appears if we replace the Brouwer-Heyting-Kolmogorov interpretation of implication with the weaker interpretation

- a proof of $\varphi_0 \to \varphi_1$ is a construction that uses the assumption φ_0 to produce a proof of φ_1.

Since Ruitenburg's work, many publications on this logic have appeared: it is even possible to cite more than twenty works (including several dissertations around the world) on the topic. They have mainly concerned proof theoretical aspects (using sequent calculus) of this logic.

Subintuitionistic logics. We understand a *subintuitionistic logic* as a set of $\mathcal{L}^s$-formulas that is closed under substitution and that coincides with the set of $\mathcal{L}^s$-formulas that are valid in a certain class of structures extending the class of ***IPL***-structures. In particular, ***BPL*** is a subintuitionistic logic. In the literature logics of this form have been considered several times: we single out [Hac63, EH76, Cor87, Doš93, Res94, Wan97, CJ01].

2.2.1. Remark. (Consequence relations). Up to now we have considered logics as sets of formulas, but in the literature consequence relations are also considered. Given one of the logics $\mathbf{\Lambda}$ previously introduced in this section, we take K as the class of $\mathbf{\Lambda}$-structures, and then we define

$$\Phi \vdash_{\mathbf{\Lambda}} \varphi \quad \text{iff} \quad \text{there is } \Phi' \subseteq_\omega \Phi \text{ such that } \Phi' \models_{l\mathsf{K}} \varphi.$$

This gives us the definition of $\vdash_{\boldsymbol{IPL}}$, $\vdash_{\boldsymbol{CPL}}$, $\vdash_{\boldsymbol{HBPL}}$, etc.

Chapter 3

Model Theory

In this chapter we will discuss a model theoretic approach to strict-weak languages. To this end we will introduce the notion of quasi bisimilarity, which plays for strict-weak languages the role played by bisimilarity for modal languages. The main results of the present chapter are the Standard Form Theorem (Theorem 3.1.3), van Benthem's style theorems (see Section 3.4 and Theorem 3.5.11) and Corollary 3.5.18.

3.1 The Standard Form Theorem

At the end of the present section we obtain the first result witnessing the leitmotif introduced on page 55. To perform this task first of all we will obtain the Standard Form Theorem. This theorem is the key to understanding why strict-weak languages give information on full modal languages. In an informal sense we can consider most of the results in this dissertation as technical consequences of this theorem. The first proof of this theorem was given by Celani and Jansana [CJ02] in the context of $\mathcal{L}^s$. That proof was based on some profound results obtained by Hollenberg on saturation, e.g., the fact that the class of modally saturated Kripke structures is a maximal Hennessy-Milner class. Here we exhibit a simple and constructive proof that works even in the infinitary case.

The Standard Form Theorem will be a natural generalization of the conjunctive normal form that is taught in elementary courses on classical propositional logic. We give a formulation using the material implication.

3.1.1. Theorem (Conjunctive Normal Form). *Let $\tau = \langle \mathsf{Mod}, \mathsf{Prop} \rangle$ be a vocabulary such that* Mod *is empty and* Prop *is a finite set of cardinality $k \in \omega$. Then, for every $\varphi \in \mathcal{L}^{MOD}(\tau)$ there exists a family $\{X_n : n < 2^k\}$ of subsets of*

Prop *such that*

$$\varphi \quad \equiv \quad \bigwedge\{\nu_n \supset \pi_n : n < 2^k\},$$

where $\nu_n := \bigwedge_{p\in\mathsf{Prop}\setminus X_n} p$ *and* $\pi_n := \bigvee_{p\in X_n} p$*. Moreover, this family of subsets is uniquely determined for each* φ*.*[1]

Sometimes the Conjunctive Normal Form Theorem is presented in its dual version called disjunctive normal form. An examination of the proof of these results reveals that they are still valid in the infinitary case[2]. Specifically, we obtain

$$\sim\bigwedge\{q_\alpha \supset p_\alpha : \alpha \in \kappa\} \quad \equiv \quad \bigwedge\{\bigwedge_{i\notin I} p_i \supset \bigvee_{i\in I} q_i : I \in \mathcal{P}(\kappa)\}. \qquad (3.1)$$

This equivalence can be proved directly using as an intermediate step the formula $\bigvee\{q_\alpha \wedge \sim p_\alpha : \alpha \in \kappa\}$. The equivalence between this formula and the formula on the right of (3.1) is due to the well known infinitary distributive law, and the equivalence with the formula on the left is trivial.

In this context, the word "normal" refers to the unicity that is obtained in this representation. If there are some modalities in our language it is not so easy to obtain a normal representation of modal formulas. The reader interested in this subject should consult [Kra99, Section 2.7]. A categorical approach to this problem can be found in [Ghi95].

Another aspect of the Conjunctive Normal Form Theorem is that all formulas are characterized by families of subsets of Prop where the size of the family is $\leq 2^{|\mathsf{Prop}|}$. The bound is not stated in the Conjunctive Normal Form Theorem because all families of subsets of Prop satisfy this requisite. Hence, formulas are characterized by families of subsets of Prop. This means that there are, up to equivalence, exactly $2^{2^{|\mathsf{Prop}|}}$ formulas in classical propositional logic. The theorem also says that all formulas can be written using a conjunction of less or equal than $2^{|\mathsf{Prop}|}$ material implications of $\mathcal{L}^{SW}(\tau_\vartheta)$-formulas (because $\nu_n, \pi_n \in \mathcal{L}^{SW}(\tau_\vartheta)$).

Let us see how the Conjunctive Normal Form Theorem is generalized when modalities are introduced. This is possible as long as we demand neither uniqueness nor a bound in the number of material implications needed in the conjunction. Let us say that a formula is in *standard form* if it is the conjunction of material implications of strict-weak formulas. The antecedents, denoted by ν's, of these material implications are their negative part[3]; and the consequents, denoted by π's, are their positive part. The Conjunctive Normal Form Theorem says, specifically, that all formulas without modalities can be written, up to equivalence, in standard form. The Standard Form Theorem will say that it is true

[1] Note that we are not assuming that if $n \neq n'$ then $X_n \neq X_{n'}$; hence, we know that $|\{X_n : n < 2^k\}| \leq 2^k$ but the equality could fail.

[2] In the case $\mathcal{L}^{MOD}_\infty(\tau)$ we do not need to restrict ourselves to finite sets of propositions, we can consider sets of propositions of any cardinality $\kappa \in \mathbb{C}\mathrm{ARD}$.

[3] It does not mean that formulas ν's are themselves negative.

when there are modalities. As a constructive proof is desirable we start by giving a method for converting modal formulas into equivalent standard form formulas.

3.1.2. DEFINITION. (Transformations tr)
Let ϑ be a SW-vocabulary. Now we define a transformation tr of $\mathcal{L}^{MOD}_{\infty}(\tau_\vartheta)$-formulas into $\mathcal{L}^{MOD}_{\infty}(\tau_\vartheta)$-formulas that are (arbitrary) conjunctions of material implications of $\mathcal{L}^{SW}_{\infty}(\vartheta)$-formulas. The transformation tr is any map satisfying the following conditions (where we assume $\mathrm{tr}(\varphi) = \bigwedge\{\nu_\alpha \supset \pi_\alpha : \alpha < \kappa\}$):

- $\mathrm{tr}(p) = \top \supset p$ if $p \in \mathsf{Prop}$
- $\mathrm{tr}(\sim \varphi) = \bigwedge\{\bigwedge_{i \notin I} \pi_i \supset \bigvee_{i \in I} \nu_i : I \in \mathcal{P}(\kappa)\}$
- $\mathrm{tr}(\bigwedge \Phi) = \bigwedge\{\mathrm{tr}(\varphi) : \varphi \in \Phi\}$
- $\mathrm{tr}([s]\varphi) = \top \supset \bigwedge\{\nu_\alpha \rightarrow_s \pi_\alpha : \alpha < \kappa\}$ if $s \in \mathsf{SMod} \setminus \mathsf{WMod}$
- $\mathrm{tr}([w]\varphi) = \bigvee\{\nu_\alpha \leftharpoondown_w \pi_\alpha : \alpha < \kappa\} \supset \bot$ if $w \in \mathsf{WMod} \setminus \mathsf{SMod}$
- If $m \in \mathsf{SMod} \cap \mathsf{WMod}$ then $\mathrm{tr}([m]\varphi)$ is one of the formulas considered in the two previous conditions replacing s and w with m.

All clauses except the last one can be considered as definitions, and the last clause gives us a choice between two possibilities. Thus, it is obvious that there are maps satisfying the requirements of this definition. An induction shows that if $\mathrm{tr}(\varphi) = \bigwedge\{\nu_\alpha \supset \pi_\alpha : \alpha < \kappa\}$ then the modal degree of all formulas in $\{\nu_\alpha : \alpha < \kappa\} \cup \{\pi_\alpha : \alpha < \kappa\}$ is bounded by $\deg(\varphi)$. It is also clear that tr respects finitary modal formulas, i.e., if φ is a $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula then it results[4] that $\mathrm{tr}(\varphi)$ is a finite conjunction of material implications of $\mathcal{L}^{SW}(\vartheta)$-formulas. Any transformation tr gives us a constructive proof of the Standard Form Theorem.

3.1.3. THEOREM (STANDARD FORM). *Let ϑ be a SW-vocabulary.*

1. *For every $\varphi \in \mathcal{L}^{MOD}(\tau_\vartheta)$ there exists $k \in \omega$ and sets $\{\nu_n : n < k\}$, $\{\pi_n : n < k\}$ of $\mathcal{L}^{SW}(\vartheta)$-formulas with modal degree $\leq \deg(\varphi)$ such that*

$$\varphi \quad \equiv \quad \bigwedge\{\nu_n \supset \pi_n : n < k\}.$$

2. *For every $\varphi \in \mathcal{L}^{MOD}_{\infty}(\tau_\vartheta)$ there exists $\kappa \in \mathbb{C}\mathrm{ARD}$ and sets $\{\nu_\alpha : \alpha < \kappa\}$, $\{\pi_\alpha : \alpha < \kappa\}$ of $\mathcal{L}^{SW}_{\infty}(\vartheta)$-formulas with modal degree $\leq \deg(\varphi)$ such that*

$$\varphi \quad \equiv \quad \bigwedge\{\nu_\alpha \supset \pi_\alpha : \alpha < \kappa\}.$$

[4] This can be proved with a straightforward induction. The crucial step is that if k is a finite cardinal then $\mathcal{P}(k)$ is also finite, i.e., if $k < \omega$ then $2^k < \omega$.

Proof: It is enough to show that $\varphi \equiv \mathrm{tr}(\varphi)$. A straightforward induction does this job. The only non-trivial case is to settle $\sim \varphi \equiv \mathrm{tr}(\sim \varphi)$ from the inductive hypothesis $\varphi \equiv \mathrm{tr}(\varphi)$. This problem was solved in (3.1). □

3.1.4. Remark. Indeed, it is even possible to add some additional restrictions. For example, we can assume that each ν is an arbitrary conjunction of propositions or strict implications or weak differences, and that each π is an arbitrary disjunction of propositions or strict implications or weak differences. This is an easy consequence of the equivalences

$$\bigvee\nolimits_{i\in I}\varphi_i \supset \varphi \equiv \bigwedge_{i\in I}(\varphi_i \supset \varphi) \quad \text{and} \quad \varphi \supset \bigwedge\nolimits_{i\in I}\varphi_i \equiv \bigwedge_{i\in I}(\varphi \supset \varphi_i),$$

and Remark 2.1.5 (just write the ν's in disjunctive form and the π's in conjunctive form).

3.1.5. Remark. Let $\lambda \in \mathbb{C}\mathrm{ARD}$ with $\lambda \geq 2$, and let $\mathrm{tr}(\varphi_\beta) = \bigwedge\{\nu^\beta_\alpha \supset \pi^\beta_\alpha : \alpha < \kappa\}$ for each $\beta \in \lambda$. And let $\mathrm{tr}(\varphi) = \bigwedge\{\nu_\alpha \supset \pi_\alpha : \alpha < \kappa\}$, $s \in \mathsf{SMod}$ and $w \in \mathsf{WMod}$. Then, the reader can check:

$$\begin{aligned}
\mathrm{tr}(\bigvee_{\beta\in\lambda}\varphi_\beta) &\equiv \bigwedge\{\bigwedge_{\beta\in\lambda}\nu^\beta_{f(\beta)} \supset \bigvee_{\beta\in\lambda}\pi^\beta_{f(\beta)} : f \in {}^\lambda\kappa\} \\
\mathrm{tr}(\varphi_0 \supset \varphi_1) &\equiv \bigwedge\{(\nu^1_\alpha \wedge \bigwedge_{i\notin I}\pi^0_i) \supset (\pi^1_\alpha \vee \bigvee_{i\in I}\nu^0_i) : I \in \mathcal{P}(\kappa), \alpha < \kappa\} \\
\mathrm{tr}(\langle s\rangle\varphi) &\equiv \bigwedge\{\bigwedge_{i\notin I}\pi_i \rightarrow_s \bigvee_{i\in I}\nu_i : I \in \mathcal{P}(\kappa)\} \supset \bot \\
\mathrm{tr}(\langle w\rangle\varphi) &\equiv \top \supset \bigvee\{\bigwedge_{i\notin I}\pi_i \leftharpoondown_w \bigvee_{i\in I}\nu_i : I \in \mathcal{P}(\kappa)\}.
\end{aligned}$$

3.1.6. Corollary. *Let ϑ be a SW-vocabulary.*

1. *For every $\varphi \in \mathcal{L}^{MOD}(\tau_\vartheta)$ there exists $k \in \omega$ and sets $\{\nu_n : n < k\}$, $\{\pi_n : n < k\}$ of $\mathcal{L}^{SW}(\vartheta)$-formulas with modal degree $\leq \deg(\varphi)$ such that*

$$\varphi \quad \equiv \quad \bigvee\{\pi_n \searrow \nu_n : n < k\}.$$

2. *For every $\varphi \in \mathcal{L}^{MOD}_\infty(\tau_\vartheta)$ there exists $\kappa \in \mathbb{C}\mathrm{ARD}$ and sets $\{\nu_\alpha : \alpha < \kappa\}$, $\{\pi_\alpha : \alpha < \kappa\}$ of $\mathcal{L}^{SW}_\infty(\vartheta)$-formulas with modal degree $\leq \deg(\varphi)$ such that*

$$\varphi \quad \equiv \quad \bigvee\{\pi_\alpha \searrow \nu_\alpha : \alpha < \kappa\}.$$

Proof: This is an easy application of duality; just apply the Standard Form Theorem to the SW-vocabulary ϑ^d and the formula φ^d. □

3.1.7. Corollary. *Let ϑ be a SW-vocabulary, $s \in \mathsf{SMod}$ and $w \in \mathsf{WMod}$. Then,*

1. *For every $\varphi \in \mathcal{L}^{MOD}(\tau_\vartheta)$ there exists $\varphi' \in \mathcal{L}^{SW}(\vartheta)$ such that $\varphi' \equiv [s]\varphi$ and $\deg(\varphi') \leq \deg(\varphi) + 1$. And for every $\varphi \in \mathcal{L}^{MOD}_\infty(\tau_\vartheta)$ there exists $\varphi' \in \mathcal{L}^{SW}_\infty(\vartheta)$ such that $\varphi' \equiv [s]\varphi$ and $\deg(\varphi') \leq \deg(\varphi) + 1$.*

2. *For every $\varphi \in \mathcal{L}^{MOD}(\tau_\vartheta)$ there exists $\varphi' \in \mathcal{L}^{SW}(\vartheta)$ such that $\varphi' \equiv \langle w\rangle\varphi$ and $\deg(\varphi') \leq \deg(\varphi) + 1$. And for every $\varphi \in \mathcal{L}^{MOD}_\infty(\tau_\vartheta)$ there exists $\varphi' \in \mathcal{L}^{SW}_\infty(\vartheta)$ such that $\varphi' \equiv \langle w\rangle\varphi$ and $\deg(\varphi') \leq \deg(\varphi) + 1$.*

Proof: A moment of reflection shows that we can choose our transformation in such a way that for every $s \in \mathsf{SMod}$, $\mathrm{tr}([s]\varphi)$ is of the form $\top \supset \varphi'$ for a certain strict-weak formula φ' whose modal degree is $\leq \deg([s]\varphi) = \deg(\varphi) + 1$. This gives us the first part. And the second part is the dual of the first part, i.e., apply the first part to ϑ^d and φ^d. $\square$

3.1.8. Corollary. *Let ϑ be a SW-vocabulary.*

1. *All formulas in the closure under conjunction $\wedge$ and disjunction $\vee$ of the set $\mathsf{Prop} \cup \{\bot, \top\} \cup \{[s]\varphi : s \in \mathsf{SMod}, \varphi \in \mathcal{L}^{MOD}(\tau_\vartheta)\} \cup \{\langle w\rangle\varphi : w \in \mathsf{WMod}, \varphi \in \mathcal{L}^{MOD}(\tau_\vartheta)\}$ are, up to equivalence, in $\mathcal{L}^{SW}(\vartheta)$. The inclusion in the other direction also holds.*

2. *All formulas in the closure under arbitrary conjunction $\bigwedge$ and arbitrary disjunction $\bigvee$ of the set $\mathsf{Prop} \cup \{[s]\varphi : s \in \mathsf{SMod}, \varphi \in \mathcal{L}^{MOD}_\infty(\tau_\vartheta)\} \cup \{\langle w\rangle\varphi : w \in \mathsf{WMod}, \varphi \in \mathcal{L}^{MOD}_\infty(\tau_\vartheta)\}$ are, up to equivalence, in $\mathcal{L}^{SW}_\infty(\vartheta)$. The inclusion in the other direction also holds.*

Proof: Trivial. $\square$

Therefore, all modal formulas that are boxes of strict modalities are, up to equivalence, in $\mathcal{L}^{SW}$. And all modal formulas that are diamonds of weak modalities are in $\mathcal{L}^{SW}$. It does not matter how many times we have used material negation or other connectives inside the modal formula. For instance, we know that $\Box\Diamond p_0$ must be in $\mathcal{L}^s$. This can be illustrated by the chain $\Box\Diamond p_0 \equiv \Box \sim \Box \sim p_0 \equiv \neg\neg p_0$. Thus, the Brouwerian Axiom $p_0 \supset \Box\Diamond p_0$[5] used in normal modal logic (see for instance [HC96, p. 62]) can be written, up to equivalence, as $p_0 \supset \neg\neg p_0$. This gives us the intuitionistically acceptable direction of the double negation law, which makes explicit the connection with Brouwer.

[5] This axiom is usually denoted by B and characterizes symmetric frames. However, remember that in this dissertation we have reserved the name B for a different axiom $\Diamond\Box p_0 \supset p_0$ which also characterizes the symmetry.

3.1.9. Example. Let us now see some more examples of formulas written in standard form. Here we consider eight modal formulas and write them as $\mathcal{L}^s$-formulas or as a material implication of $\mathcal{L}^s$-formulas. The interest of these formulas will become clear in Section 4.2 (see Remark 4.2.4).

$$\begin{aligned}
\Box(p_0 \supset p_1) \supset (p_0 \supset p_1) &\equiv \big(p_0 \wedge (p_0 \rightarrow p_1)\big) \supset p_1,\\
\Box(p_0 \supset p_1) \supset \Box\Box(p_0 \supset p_1) &\equiv (p_0 \rightarrow p_1) \supset \Box(p_0 \rightarrow p_1),\\
\Diamond\Box(p_0 \supset p_1) \supset \Box(p_0 \supset p_1) &\equiv (p_0 \rightarrow p_1) \vee \neg(p_0 \rightarrow p_1),\\
\Diamond\Box(p_0 \supset p_1) \supset (p_0 \supset p_1) &\equiv p_0 \supset \big(p_1 \vee \neg(p_0 \rightarrow p_1)\big),
\end{aligned}$$

$$\begin{aligned}
&(\Box(p_0 \supset p_1) \rightarrow (p_2 \supset p_3)) \ \vee \ (\Box(p_2 \supset p_3) \rightarrow (p_0 \supset p_1)) \equiv\\
&\qquad \equiv \Big(((p_0 \rightarrow p_1) \wedge p_2) \rightarrow p_3\Big) \ \vee \ \Big(((p_2 \rightarrow p_3) \wedge p_0) \rightarrow p_1\Big),\\
&(\Box^{(1)}(p_0 \supset p_1) \rightarrow (p_2 \supset p_3)) \ \vee \ (\Box^{(1)}(p_2 \supset p_3) \rightarrow (p_0 \supset p_1)) \equiv\\
&\quad \equiv \Bigg(\Big(((p_0 \rightarrow p_1) \wedge p_2) \rightarrow (p_0 \vee p_3)\Big) \wedge \Big(((p_0 \rightarrow p_1) \wedge p_1 \wedge p_2) \rightarrow p_3\Big)\Bigg) \vee\\
&\quad \vee \Bigg(\Big(((p_2 \rightarrow p_3) \wedge p_0) \rightarrow (p_2 \vee p_1)\Big) \wedge \Big(((p_2 \rightarrow p_3) \wedge p_3 \wedge p_0) \rightarrow p_1\Big)\Bigg),\\
&(\Box(p_0 \supset p_1) \rightarrow (p_0 \supset p_1)) \supset \Box(p_0 \supset p_1) \ \equiv \ \Big((p_0 \ \wedge \ (p_0 \rightarrow p_1)) \rightarrow p_1\Big) \supset \Big(p_0 \rightarrow p_1\Big),\\
&\Big(((p_0 \supset p_1) \rightarrow \Box(p_0 \supset p_1)) \rightarrow (p_0 \supset p_1)\Big) \supset \Big(p_0 \supset p_1\Big) \equiv\\
&\quad \equiv \Bigg(p_0 \wedge \bigg(\Big(p_0 \ \wedge \ \Box\big(p_0 \vee (p_0 \rightarrow p_1)\big) \ \wedge \ \big(p_1 \rightarrow (p_0 \rightarrow p_1)\big)\Big) \rightarrow p_1\bigg)\Bigg) \supset p_1.
\end{aligned}$$

Up to now, we have been able to write all examples of (finitary) modal formulas using only one material implication of strict-weak formulas. Is it possible to bound the number of material implications needed in the Standard Form Theorem? That is, is there any $k \in \omega$ such that all modal formulas are equivalent to a conjunction of k material implications of strict-weak formulas? In the case that there are no modalities the Conjunctive Normal Form Theorem says that $2^{|\mathsf{Prop}|}$ is a bound when Prop is finite[6]. However, if there is a modality then there is no bound (even in the case that Prop is the empty set). For instance, we will see in Lemma 5.1.9(2) that the modal formula $\bigwedge_{n<k}(\Box^{2n+1}\Diamond\top \supset \Box^{2n+2}\Diamond\top)$, where $k \in \omega$, cannot be written as a conjunction of less than k material implications of $\mathcal{L}^s$-formulas. In fact, we will give characterizations, for each $k \in \omega$, of which modal formulas can be written as a conjunction of k material implications of strict-weak formulas (see Theorem 5.1.6).

Let us now see the promised result about full modal languages which uses strict-weak fragments. It is again a trivial consequence of the Standard Form Theorem.

[6] We notice that it is not a minimal bound. In Theorem 5.1.11(3) we will prove that the minimal bound is $\left[\frac{\mathsf{Prop}}{2}\right] + 1$; as far as we know this is the first time that it has been calculated.

3.1.10. Corollary. *Let $\alpha \in \mathbb{O}\mathrm{RD}$, let $n \in \omega$, let ϑ be a SW-vocabulary, and let $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ be two pointed τ_ϑ-structures. Then,*

1. *$\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ satisfy the same $\mathcal{L}^{MOD}(\tau_\vartheta)$-formulas iff they satisfy the same $\mathcal{L}^{SW}(\vartheta)$-formulas.*
2. *$\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ satisfy the same $\mathcal{L}^{MOD}(\tau_\vartheta)$-formulas of modal degree $\leq n$ iff they satisfy the same $\mathcal{L}^{SW}(\vartheta)$-formulas of modal degree $\leq n$.*
3. *$\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ satisfy the same $\mathcal{L}^{MOD}_\infty(\tau_\vartheta)$-formulas iff they satisfy the same $\mathcal{L}^{SW}_\infty(\vartheta)$-formulas.*
4. *$\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ satisfy the same $\mathcal{L}^{MOD}_\infty(\tau_\vartheta)$-formulas of modal degree $\leq \alpha$ iff they satisfy the same $\mathcal{L}^{SW}_\infty(\vartheta)$-formulas of modal degree $\leq \alpha$.*

Proof: Straightforward. □

The first parts of the four items talk about equivalence relations between pointed structures that are familiar to us. They are, respectively, the relations $\leftrightsquigarrow$, $\leftrightsquigarrow_n$, $\simeq$ and $\simeq_\alpha$. In the second halves only in the first two items have we already introduced a relation to talk about this. In the right part of the first item it is said that $\langle \mathfrak{A}, a\rangle \rightsquigarrow \langle \mathfrak{B}, b\rangle$ and $\langle \mathfrak{B}, b\rangle \rightsquigarrow \langle \mathfrak{A}, a\rangle$. And in the second item it is said that $\langle \mathfrak{A}, a\rangle \rightsquigarrow_n \langle \mathfrak{B}, b\rangle$ and $\langle \mathfrak{B}, b\rangle \rightsquigarrow_n \langle \mathfrak{A}, a\rangle$. Hence, these right parts can be formulated easily using the equivalence relations generated by $\rightsquigarrow$ and $\rightsquigarrow_n$, respectively. We recall that the equivalence relation generated by a quasi order is its *symmetric closure*, i.e., if R is a quasi order then it generates the equivalence relation $\{\langle a, b\rangle : \langle a, b\rangle \in R \text{ and } \langle b, a\rangle \in R\}$.

3.1.11. Corollary. *Let ϑ be a SW-vocabulary. Then, $\rightsquigarrow_\vartheta$ is a quasi order that generates the equivalence relation $\leftrightsquigarrow_{\tau_\vartheta}$. And for every $n \in \omega$, $\rightsquigarrow_n$ is a quasi order that generates the equivalence relation $\leftrightsquigarrow_n$.*

Proof: Straightforward. □

Of course $\rightsquigarrow_\vartheta$ is not the only quasi order that generates $\leftrightsquigarrow_{\tau_\vartheta}$. For instance, $\rightsquigarrow_{\vartheta^d}$ is also another quasi order that generates the equivalence relation $\leftrightsquigarrow_{\tau_\vartheta}$. In fact, if ϑ and ϑ' have the same associated vocabulary, then $\rightsquigarrow_\vartheta$ and $\rightsquigarrow_{\vartheta'}$ generate the same equivalence relation.

These quasi orders are proper quasi orders, i.e., in general they are not symmetric[7]. To show this we can consider for instance the pointed structures given in Figure 1.3. It is easy to see that $\langle \mathfrak{A}, a\rangle \rightsquigarrow_s \langle \mathfrak{B}, b\rangle$ while $\langle \mathfrak{B}, b\rangle \not\rightsquigarrow_s \langle \mathfrak{A}, a\rangle$[8].

[7] In fact, the only case where they are symmetric is the case in which $\mathsf{Prop} = \emptyset$ and $\mathsf{SMod} = \mathsf{WMod}$.

[8] These facts will be simple applications of the quasi bisimilarity notion developed in the next section.

What we have seen is that if we can differentiate two pointed structures using a modal formula then we can also distinguish them using a strict-weak formula. But there is a remarkable difference between the way these languages differentiate pointed structures. In the case that $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ are differentiated using a modal formula we know that one of the following situations holds:

- There is a modal formula φ such that $\mathfrak{A}, a \Vdash \varphi$ and $\mathfrak{B}, b \nVdash \varphi$.
- There is a modal formula φ such that $\mathfrak{B}, b \Vdash \varphi$ and $\mathfrak{A}, a \nVdash \varphi$.

If φ satisfies one of them then $\sim\varphi$ satisfies the other. Thus, the two conditions are equivalent. This claims $\mathrm{Th}_{\mathcal{L}^{MOD}(\tau)}(\mathfrak{A}, a) = \mathrm{Th}_{\mathcal{L}^{MOD}(\tau)}(\mathfrak{B}, b)$ iff $\mathrm{Th}_{\mathcal{L}^{MOD}(\tau)}(\mathfrak{A}, a) \subseteq \mathrm{Th}_{\mathcal{L}^{MOD}(\tau)}(\mathfrak{B}, b)$, and also iff $\mathrm{Th}_{\mathcal{L}^{MOD}(\tau)}(\mathfrak{A}, a) \supseteq \mathrm{Th}_{\mathcal{L}^{MOD}(\tau)}(\mathfrak{B}, b)$. That is, the modal theories of pointed structures are maximal. On the other hand, when $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ are differentiated using a strict-weak formula we only really know that one of the following situations holds:

- There is a strict-weak formula φ such that $\mathfrak{A}, a \Vdash \varphi$ and $\mathfrak{B}, b \nVdash \varphi$.
- There is a strict-weak formula φ such that $\mathfrak{B}, b \Vdash \varphi$ and $\mathfrak{A}, a \nVdash \varphi$.

This time these conditions are not equivalent, e.g., take once more the language $\mathcal{L}^s$ and the pointed structures given in Figure 1.3. Therefore, in general the strict-weak theories of pointed structures are not maximal, i.e., it is possible to have $\mathrm{Th}_{\mathcal{L}^{SW}(\vartheta)}(\mathfrak{A}, a) \subseteq \mathrm{Th}_{\mathcal{L}^{SW}(\vartheta)}(\mathfrak{B}, b)$ while $\mathrm{Th}_{\mathcal{L}^{SW}(\vartheta)}(\mathfrak{A}, a) \neq \mathrm{Th}_{\mathcal{L}^{SW}(\vartheta)}(\mathfrak{B}, b)$. To sum up, if we can distinguish two pointed structures using a modal formula then we can choose in which pointed structure our witnessing formula holds, while in the strict-weak case we cannot.

3.2 Quasi (bounded-)bisimilarity

In corollary 3.1.10 we saw how we can obtain the equivalence relations $\leftrightsquigarrow$, $\leftrightsquigarrow_n$, $\simeq$ and $\simeq_\alpha$ in the strict-weak context. This corollary suggests a way to remove the symmetry from these four equivalence relations. For the cases of $\leftrightsquigarrow$ and $\leftrightsquigarrow_n$ we did this at the end of the previous section, and we saw that what we obtain is $\rightsquigarrow$ and $\rightsquigarrow_n$, respectively. Now, we will do the same with the other two relations. Let us start first with the bisimilarity relation.

3.2.1. Definition. (Quasi bisimilarity)
Let ϑ be a SW-vocabulary and let $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ be two pointed τ_ϑ-structures. We will say that $\langle \mathfrak{A}, a\rangle$ *is quasi bisimilar into* $\langle \mathfrak{B}, b\rangle$, denoted by $\langle \mathfrak{A}, a\rangle \preceq_\vartheta \langle \mathfrak{B}, b\rangle$, if the following three conditions hold:

Figure 3.1: The clauses (qbis2) and (qbis3) of quasi bisimilarity

(qbis1): If $p \in \mathsf{Prop}$ and $\mathfrak{A}, a \Vdash p$, then $\mathfrak{B}, b \Vdash p$.

(qbis2): For every $s \in \mathsf{SMod}$ and every b' such that $\langle b, b' \rangle \in R_s^{\mathfrak{B}}$, there is a' such that $\langle a, a' \rangle \in R_s^{\mathfrak{A}}$ and $\langle \mathfrak{A}, a' \rangle \simeq_{\tau_\vartheta} \langle \mathfrak{B}, b' \rangle$.

(qbis3): For every $w \in \mathsf{WMod}$ and every a' such that $\langle a, a' \rangle \in R_w^{\mathfrak{A}}$, there is b' such that $\langle b, b' \rangle \in R_w^{\mathfrak{B}}$ and $\langle \mathfrak{A}, a' \rangle \simeq_{\tau_\vartheta} \langle \mathfrak{B}, b' \rangle$.

We will omit the symbol ϑ when there is no ambiguity, and we will write $\preceq_S$, $\preceq_s$, $\preceq_W$ and $\preceq_w$ following the same conventions than in Definition 2.1.2.

The reasons why we call this relation "quasi bisimilarity" will be clear after Corollary 3.2.4. Let us now examine the three requirements above. The first one is the typical proposition preservation clause. The second one reminds us of a back condition restricted to strict modalities, and the last one does the same with a forth condition restricted to weak modalities. But we must point out that this back condition (and the same for the forth condition) yields a bisimilar pointed structure and not a quasi bisimilar one (see Figure 3.1). To sum up, the last two conditions impose (i) a back condition for pure strict modalities, (ii) a forth condition for pure weak modalities, (iii) a back and a forth condition for modalities that are strict and weak at the same time.

If $\langle \mathfrak{A}, a \rangle \simeq_{\tau_\vartheta} \langle \mathfrak{B}, b \rangle$ then $\langle \mathfrak{A}, a \rangle \preceq_\vartheta \langle \mathfrak{B}, b \rangle$ (and also $\langle \mathfrak{B}, b \rangle \preceq_\vartheta \langle \mathfrak{A}, a \rangle$), but the other direction does not hold in general. As a counterexample we can consider Figure 1.3. There, $\langle \mathfrak{A}, a \rangle \preceq_s \langle \mathfrak{B}, b \rangle$ while $\langle \mathfrak{B}, b \rangle \not\preceq_s \langle \mathfrak{A}, a \rangle$. Thus, quasi bisimilarity is weaker than bisimilarity.

Another trivial consequence of this definition is that for every pointed structure $\langle \mathfrak{A}, a \rangle$,

$$\langle \mathfrak{A}, a \rangle \preceq_S \mathfrak{S}_{IrT} \quad \text{and} \quad \mathfrak{S}_{IrF} \preceq_W \langle \mathfrak{A}, a \rangle.$$

We will say that $\mathfrak{S}_{IrT}$ is *final* in $\preceq_S$, while $\mathfrak{S}_{IrF}$ is *initial* in $\preceq_W$. Using duality we could have seen that these facts are equivalent. In general, the role played by duality in the quasi bisimilarity relation is obtained by the following trivial observation:

$$\langle \mathfrak{A}, a \rangle \preceq_\vartheta \langle \mathfrak{B}, b \rangle \quad \text{iff} \quad \langle \mathfrak{B}^d, b \rangle \preceq_{\vartheta^d} \langle \mathfrak{A}^d, a \rangle. \tag{3.2}$$

Let us now see that quasi bisimilarity captures exactly when infinitary strict-weak formulas are preserved between two pointed structures.

3.2.2. Proposition. *Let ϑ be a SW-vocabulary, and let $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ be two pointed τ_ϑ-structures. Then,*

$$\langle \mathfrak{A}, a\rangle \preceq_\vartheta \langle \mathfrak{B}, b\rangle \quad \textit{iff} \quad \mathrm{Th}_{\mathcal{L}^{SW}_\infty(\vartheta)}(\mathfrak{A}, a) \subseteq \mathrm{Th}_{\mathcal{L}^{SW}_\infty(\vartheta)}(\mathfrak{B}, b).$$

Proof: $(\Rightarrow)$: Use induction on formulas in $\mathcal{L}^{SW}_\infty(\vartheta)$. The only non-trivial cases are strict implications and weak differences.

Here is a proof for strict implications. Assume $s \in \mathsf{SMod}$, $\langle \mathfrak{A}, a\rangle \preceq_\vartheta \langle \mathfrak{B}, b\rangle$ and $\mathfrak{A}, a \Vdash \varphi_0 \rightarrow_s \varphi_1$. We must check that $\mathfrak{B}, b \Vdash \varphi_0 \rightarrow_s \varphi_1$. Let b' be such that $\langle b, b'\rangle \in R^{\mathfrak{B}}_s$ and $\mathfrak{B}, b' \Vdash \varphi_0$. By (qbis2) of Definition 3.2.1 there exists a' such that $\langle a, a'\rangle \in R^{\mathfrak{A}}_s$ and $\langle \mathfrak{A}, a'\rangle \simeq_{\tau_\vartheta} \langle \mathfrak{B}, b'\rangle$. Since bisimilarity is invariant under infinitary modal formulas, it is also invariant under its strict-weak fragment; in particular, $\mathfrak{A}, a' \Vdash \varphi_0$. Hence, $\mathfrak{A}, a' \Vdash \varphi_1$. And using the bisimilarity again we conclude $\mathfrak{B}, b' \Vdash \varphi_1$, as required.

A similar argument works for the weak differences, but we can do without it because duality sets up this part. It is immediate that formulas preserved under quasi bisimilarity are closed under weak differences by (3.2) and the fact that weak differences in ϑ corresponds to strict implications in ϑ^d.

$(\Leftarrow)$: Condition (qbis1) is trivial. And by duality it is enough to prove one of the other two conditions. Let us now see that (qbis2) holds. Suppose $s \in \mathsf{SMod}$ and b' is such that $\langle b, b'\rangle \in R^{\mathfrak{B}}_s$. We know that bisimilar pointed structures can be characterized using only one infinitary modal formula. So, let $\phi^{\langle \mathfrak{B}, b'\rangle}$ be the formula that characterizes the pointed τ_ϑ-structures bisimilar to $\langle \mathfrak{B}, b'\rangle$. Then, $\mathfrak{B}, b \not\Vdash [s]\sim\phi^{\langle \mathfrak{B}, b'\rangle}$. Corollary 3.1.7 says that the formula $[s]\sim\phi^{\langle \mathfrak{B}, b'\rangle}$ is, up to equivalence, in $\mathcal{L}^{SW}_\infty(\vartheta)$. Hence, as our hypothesis claims that infinitary strict-weak formulas are preserved we deduce that $\mathfrak{A}, a \not\Vdash [s]\sim\phi^{\langle \mathfrak{B}, b'\rangle}$. This means that there exists a' such that $\langle a, a'\rangle \in R^{\mathfrak{A}}_s$ and $\mathfrak{A}, a' \Vdash \phi^{\langle \mathfrak{B}, b'\rangle}$, i.e., $\langle \mathfrak{A}, a'\rangle \simeq_{\tau_\vartheta} \langle \mathfrak{B}, b'\rangle$. □

This proposition gives us a method to see that certain modal formulas are not (modulo equivalence) in the infinitary strict-weak fragments. It is enough to see that these formulas are not preserved under quasi bisimilarity. Thus, it is easy to see that formulas like $\sim p$, $\Diamond p$ and $\Box p \supset p$ are not in $\mathcal{L}^s_\infty$. Let us state some more trivial consequences.

3.2.3. Corollary. *Let ϑ be a SW-vocabulary, and let $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ be two pointed τ_ϑ-structures. Then,*

$$\langle \mathfrak{A}, a\rangle \preceq_\vartheta \langle \mathfrak{B}, b\rangle \quad \textit{implies} \quad \langle \mathfrak{A}, a\rangle \rightsquigarrow_\vartheta \langle \mathfrak{B}, b\rangle.$$

Proof: Trivial. □

3.2.4. Corollary. *Let ϑ be a SW-vocabulary. Then, $\preceq_\vartheta$ is a quasi order that generates the equivalence relation $\simeq_{\tau_\vartheta}$.*

Proof: Just remember what we saw in the third item in Corollary 3.1.10. □

This corollary justifies the term "quasi bisimilarity" chosen for this new concept. First of all, this word suggests that this notion is close to bisimilarity. And second, but more important, it reminds us that this relation is a quasi order for bisimilarity.

We notice that in the literature there are some notions obtained by removing symmetry from the clauses considered in the definition of bisimulation, e.g., the notion of simulation in [BdRV01]. On the other hand, what quasi bisimilarity removes is symmetry from bisimilarity (as Corollary 3.2.4 justifies), and not from bisimulation. This is the main difference between quasi bisimilarity and these other notions.

Now it is time to remove the symmetry from $\simeq_\alpha$. From the next definition it is trivial that $\langle \mathfrak{A}, a\rangle \simeq_\alpha \langle \mathfrak{B}, b\rangle$ implies $\langle \mathfrak{A}, a\rangle \preceq_\alpha \langle \mathfrak{B}, b\rangle$ (and also $\langle \mathfrak{B}, b\rangle \preceq_\alpha \langle \mathfrak{A}, a\rangle$).

3.2.5. Definition. (Quasi bounded-bisimilarity)
Let $\alpha \in \mathbb{O}\mathrm{RD}$, let ϑ be a SW-vocabulary and let $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ be two pointed τ_ϑ-structures. We will say that $\langle \mathfrak{A}, a\rangle$ *is quasi α-bisimilar into* $\langle \mathfrak{B}, b\rangle$, denoted by $\langle \mathfrak{A}, a\rangle \preceq_\alpha \langle \mathfrak{B}, b\rangle$, iff:

(qbbis1): If $p \in \mathsf{Prop}$ and $\mathfrak{A}, a \Vdash p$, then $\mathfrak{B}, b \Vdash p$.

(qbbis2): For every $\beta < \alpha$, $s \in \mathsf{SMod}$, and b' such that $\langle b, b'\rangle \in R_s^{\mathfrak{B}}$, there is a' such that $\langle a, a'\rangle \in R_s^{\mathfrak{A}}$ and $\langle \mathfrak{A}, a'\rangle \simeq_\beta \langle \mathfrak{B}, b'\rangle$.

(qbbis3): For every $\beta < \alpha$, $w \in \mathsf{WMod}$ and a' such that $\langle a, a'\rangle \in R_w^{\mathfrak{A}}$, there is b' such that $\langle b, b'\rangle \in R_w^{\mathfrak{B}}$ and $\langle \mathfrak{A}, a'\rangle \simeq_\beta \langle \mathfrak{B}, b'\rangle$.

3.2.6. Proposition. *Let $\alpha \in \mathbb{O}\mathrm{RD}$, let ϑ be a SW-vocabulary, and let $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ be two pointed τ_ϑ-structures. The following are equivalent:*

1. $\langle \mathfrak{A}, a\rangle \preceq_\alpha \langle \mathfrak{B}, b\rangle$.
2. *All $\mathcal{L}_\infty^{SW}(\vartheta)$-formulas of modal degree $\leq \alpha$ satisfied in $\langle \mathfrak{A}, a\rangle$ are also satisfied in $\langle \mathfrak{B}, b\rangle$.*

Proof: The proof is an easy adaptation of the proof of Proposition 3.2.2. The implication $(1 \Rightarrow 2)$ is easily verified by induction on the modal degree. For the converse, use that for every ordinal β and every pointed structure there exists an infinitary modal formula of modal degree $\leq \beta$ that characterizes this pointed structure modulo β-bisimilarity (recall that the class of infinitary modal formulas of modal degree $\leq \beta$ is, up to equivalence, a set). □

3.2.7. COROLLARY. *Let ϑ be a SW-vocabulary. Then, $\preceq_\alpha$ is a quasi order that generates the equivalence relation $\simeq_\alpha$.*

Proof: Just remember what we saw in the fourth item in Corollary 3.1.10. □

3.2.8. PROPOSITION. *Let α be a successor ordinal, and let ϑ be a SW-vocabulary. Let $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ be two pointed τ_ϑ-structures such that $\langle \mathfrak{A}, a\rangle \preceq_\alpha \langle \mathfrak{B}, b\rangle$. Then, exists $\langle \mathfrak{A}', a'\rangle$ and $\langle \mathfrak{B}', b'\rangle$ such that $\langle \mathfrak{A}, a\rangle \simeq_\alpha \langle \mathfrak{A}', a'\rangle$, $\langle \mathfrak{B}, b\rangle \simeq_\alpha \langle \mathfrak{B}', b'\rangle$ and $\langle \mathfrak{A}', a'\rangle \preceq \langle \mathfrak{B}', b'\rangle$.*

Proof: Assume α is the successor of β. Replacing the starting pointed structures with their unravellings we can assume that they are trees of height ω with root the distinguished state.

Let Y be the set of states in $\mathfrak{B}$ of height 1 that are accessible from the root by a pure strict modality, i.e., $Y := \{y \in B : \exists s \in \mathsf{SMod} \setminus \mathsf{WMod},\ \langle b, y\rangle \in R_s^{\mathfrak{B}}\}$. And let $f : Y \longrightarrow \mathsf{SMod} \setminus \mathsf{WMod}$ be the map that assigns to every state in Y the only modality that allows us to reach it. We know that for every $y \in Y$, there exists a generated subtree $\mathfrak{T}_y$ of $\mathfrak{A}$ with root t_y a $f(y)$-successor of a such that $\langle \mathfrak{B}, y\rangle \simeq_\beta \langle \mathfrak{T}_y, t_y\rangle$. Let $\mathfrak{B}'$ be the tree that results when replacing in $\mathfrak{B}$ each subtree of root $y \in Y$ with a copy of $\langle \mathfrak{T}_y, t_y\rangle$. And we impose in $\mathfrak{B}'$ that each t_y is a $f(y)$-successor of b. A moment's reflection on the construction, separating modalities according to whether they are in $\mathsf{SMod} \setminus \mathsf{WMod}$ or in WMod, shows that $\langle \mathfrak{B}, b\rangle \simeq_\alpha \langle \mathfrak{B}', b\rangle$.

Analogously, but using $\mathfrak{A}$ and $\mathsf{WMod} \setminus \mathsf{SMod}$, we define $\mathfrak{A}'$. Let us do the details. Let $X := \{x \in A : \exists w \in \mathsf{WMod} \setminus \mathsf{SMod},\ \langle a, x\rangle \in R_w^{\mathfrak{A}}\}$, and let $g : Y \longrightarrow \mathsf{WMod} \setminus \mathsf{SMod}$ be the map that gives us the modality that allows us to reach the state. We know that for every $x \in X$, there exists a generated subtree $\mathfrak{T}^x$ of $\mathfrak{B}$ with root t^x a $g(x)$-successor of b such that $\langle \mathfrak{A}, x\rangle \simeq_\beta \langle \mathfrak{T}^x, t^x\rangle$. Let $\mathfrak{A}'$ be the tree that results when replacing in $\mathfrak{A}$ each subtree of root $x \in X$ with a copy of $\langle \mathfrak{T}^x, t^x\rangle$. And we impose in $\mathfrak{A}'$ that each t^x is a $g(x)$-successor of a. It is clear by construction, separating modalities according to whether they are in $\mathsf{WMod} \setminus \mathsf{SMod}$ or in SMod, that $\langle \mathfrak{A}, a\rangle \simeq_\alpha \langle \mathfrak{A}', a\rangle$.

And now, separating modalities depending if they are in $\mathsf{SMod} \setminus \mathsf{WMod}$ or in $\mathsf{WMod} \setminus \mathsf{SMod}$ or in $\mathsf{SMod} \cap \mathsf{WMod}$, it is easy to see that $\langle \mathfrak{A}', a\rangle \preceq \langle \mathfrak{B}', b\rangle$ by construction. □

3.2.9. REMARK. If $\alpha \in \omega$ and the starting pointed structures are finite, then introducing some slight changes in the previous argument we can choose $\langle \mathfrak{A}', a'\rangle$ and $\langle \mathfrak{B}', b'\rangle$ in such a way that they are finite. The only difference is that when we consider their unravellings we cut them at height α.

In the modal case it is known that bisimilarity to a certain pointed structure $\langle \mathfrak{A}, a\rangle$ is characterized by a single infinitary modal formula $\phi^{\langle \mathfrak{A}, a\rangle}$ (see Theorem 1.3.3). In the strict-weak case what we can do is stated in the following theorem.

3.2.10. Remark. Let α be a ordinal, and let $\langle \mathfrak{A}, a\rangle$ be a pointed structure. Then, it is clear that there exist of infinitary strict-weak formulas $\pi_\alpha^{\langle \mathfrak{A},a\rangle}$ and $\nu_\alpha^{\langle \mathfrak{A},a\rangle}$ with modal degree $\leq \alpha$ satisfying that for every pointed structure $\langle \mathfrak{B}, b\rangle$,

$$\langle \mathfrak{A}, a\rangle \preceq_\alpha \langle \mathfrak{B}, b\rangle \quad \text{iff} \quad \mathfrak{B}, b \Vdash \pi_\alpha^{\langle \mathfrak{A},a\rangle},$$

and

$$\langle \mathfrak{B}, b\rangle \preceq_\alpha \langle \mathfrak{A}, a\rangle \quad \text{iff} \quad \mathfrak{B}, b \nVdash \nu_\alpha^{\langle \mathfrak{A},a\rangle}.$$

To prove this, consider $\pi_\alpha^{\langle \mathfrak{A},a\rangle} := \bigwedge\{\varphi \in \mathcal{L}_\infty^{SW} : \deg(\varphi) \leq \alpha \text{ and } \langle \mathfrak{A}, a\rangle \Vdash \varphi\}$ and $\nu_\alpha^{\langle \mathfrak{A},a\rangle} := \bigvee\{\varphi \in \mathcal{L}_\infty^{SW} : \deg(\varphi) \leq \alpha \text{ and } \langle \mathfrak{A}, a\rangle \nVdash \varphi\}$. We note that these formulas exist because these classes are modulo equivalence sets (remember that the class of modal formulas modulo equivalence of a certain modal degree is set-sized).

3.2.11. Theorem. *Let ϑ be a SW-vocabulary, and let $\langle \mathfrak{A}, a\rangle$ be a pointed τ_ϑ-structure.*

1. *There is a $\mathcal{L}_\infty^{SW}(\vartheta)$-formula $\pi^{\langle \mathfrak{A},a\rangle}$ such that for every pointed τ-structure $\langle \mathfrak{B}, b\rangle$,*

$$\langle \mathfrak{A}, a\rangle \preceq_\vartheta \langle \mathfrak{B}, b\rangle \quad \textit{iff} \quad \mathfrak{B}, b \Vdash \pi^{\langle \mathfrak{A},a\rangle}.$$

2. *There is a $\mathcal{L}_\infty^{SW}(\vartheta)$-formula $\nu^{\langle \mathfrak{A},a\rangle}$ such that for every pointed τ-structure $\langle \mathfrak{B}, b\rangle$,*

$$\langle \mathfrak{B}, b\rangle \preceq_\vartheta \langle \mathfrak{A}, a\rangle \quad \textit{iff} \quad \mathfrak{B}, b \nVdash \nu^{\langle \mathfrak{A},a\rangle}.$$

Proof: 1) Let $\pi^{\langle \mathfrak{A},a\rangle}$ be the formula

$$\bigwedge\{p \in \mathsf{Prop} : \langle \mathfrak{A}, a\rangle \Vdash p\} \wedge \\ \wedge \bigwedge\left\{[s]\bigvee\{\phi^{\langle \mathfrak{A},a'\rangle} : \langle a, a'\rangle \in R_s\} : s \in \mathsf{SMod}\right\} \wedge \\ \wedge \bigwedge\left\{\langle w\rangle\phi^{\langle \mathfrak{A},a'\rangle} : w \in \mathsf{WMod}, \langle a, a'\rangle \in R_w\right\}.$$

It is clear that this formula does what we want, and it is (modulo equivalence) in $\mathcal{L}_\infty^{SW}(\vartheta)$.

2) By duality it is clear that the formula $(\pi^{\langle \mathfrak{A}^d, a\rangle})^d$ satisfies the requirement. This says that we can consider $\nu^{\langle \mathfrak{A}, a\rangle}$ as the formula

$$\bigvee\{p \in \mathsf{Prop} : \langle \mathfrak{A}, a\rangle \not\Vdash p\} \vee \\ \vee \bigvee\left\{[s] \sim \phi^{\langle \mathfrak{A}, a'\rangle} : s \in \mathsf{SMod}, \langle a, a'\rangle \in R_s\right\} \vee \\ \wedge \bigvee\left\{\langle w\rangle \bigwedge\{\sim \phi^{\langle \mathfrak{A}, a'\rangle} : \langle a, a'\rangle \in R_w\} : w \in \mathsf{WMod}\right\}.$$

□

3.2.12. Definition. (Positive and negative characterizations)
It will be said that an infinitary strict-weak formula φ is a *positive characterization of* $\langle \mathfrak{A}, a\rangle$ *up to quasi bisimilarity* in the case that $\varphi \equiv \pi^{\langle \mathfrak{A}, a\rangle}$, i.e., for every pointed structure $\langle \mathfrak{B}, b\rangle$,

$$\langle \mathfrak{A}, a\rangle \preceq \langle \mathfrak{B}, b\rangle \quad \text{iff} \quad \mathfrak{B}, b \Vdash \varphi.$$

And it is a *negative characterization of* $\langle \mathfrak{A}, a\rangle$ *up to quasi bisimilarity* in the case that $\varphi \equiv \nu^{\langle \mathfrak{A}, a\rangle}$, i.e., for every pointed structure $\langle \mathfrak{B}, b\rangle$,

$$\langle \mathfrak{B}, b\rangle \preceq \langle \mathfrak{A}, a\rangle \quad \text{iff} \quad \mathfrak{B}, b \not\Vdash \varphi.$$

Thus, φ is a positive characterization of $\langle \mathfrak{A}, a\rangle$ iff φ^d is a negative characterization of $\langle \mathfrak{A}^d, a\rangle$. As a consequence of $\phi^{\langle \mathfrak{A}, a\rangle} \equiv \pi^{\langle \mathfrak{A}, a\rangle} \wedge \sim \nu^{\langle \mathfrak{A}, a\rangle}$, it easily follows that $\mathrm{rank}(\mathfrak{A}, a)$ is smaller than the minimum ordinal α such that there are positive and negative characterizations of this pointed structure with modal degree $\leq \alpha$. We call this ordinal the *quasi rank of* $\langle \mathfrak{A}, a\rangle$ (notation: $\mathrm{qrank}_\vartheta(\mathfrak{A}, a)$, or simply $\mathrm{qrank}(\mathfrak{A}, a)$). This notion is preserved under bisimilarity, and it is obvious that $\mathrm{qrank}(\mathfrak{A}, a)$ coincides with the minimum ordinal α such that for all $\langle \mathfrak{B}, b\rangle$,

$$\langle \mathfrak{A}, a\rangle \preceq_\alpha \langle \mathfrak{B}, b\rangle \quad \text{implies} \quad \langle \mathfrak{A}, a\rangle \preceq \langle \mathfrak{B}, b\rangle,$$

and

$$\langle \mathfrak{B}, b\rangle \preceq_\alpha \langle \mathfrak{A}, a\rangle \quad \text{implies} \quad \langle \mathfrak{B}, b\rangle \preceq \langle \mathfrak{A}, a\rangle.$$

Therefore, $\mathrm{qrank}_\vartheta(\mathfrak{A}, a) = \mathrm{qrank}_{\vartheta^d}(\mathfrak{A}^d, a)$.

3.2.13. Example. In the following examples we assume that there is a single modality. The rank and the quasi rank of $\mathfrak{S}_{IrT}$ are 1. And the same for $\mathfrak{S}_{IrF}$. Let $\langle \mathfrak{A}_0, a_0\rangle$ be the pointed structure based on a reflexive state[9] where all propositions hold. Then, $\mathrm{rank}(\mathfrak{A}_0, a_0) = \omega$, while the quasi rank depends on what kind of

[9]Doing the unravelling it should be clear that this corresponds to a tree that only has one branch, which is infinite.

modality we have. Indeed[10], $\text{rank}_s(\mathfrak{A}_0, a_0) = \omega + 1$, $\text{rank}_w(\mathfrak{A}_0, a_0) = \omega + 1$, and when the modality is at the same time a strict modality and a weak modality its quasi rank is ω. Let $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ now be the pointed structures given in Figure 1.5 where all propositions are satisfied in all states. Rank and quasi rank coincide for these structures; all of them are $\omega + 1$. To conclude, let us see that in the definition of quasi rank it is not enough to consider positive characterizations (of course, by duality it is neither enough to consider negative characterizations). Assume that the modality is a pure strict modality. Then, the formula $\bigwedge\{[m]^n p : n \in \omega, p \in \mathsf{Prop}\}$ is in $\mathcal{L}^s$ and gives a positive characterization of $\langle \mathfrak{A}_0, a_0\rangle$. It has modal degree ω. On the other hand, there is no negative characterization with modal degree $\leq \omega$ because $\langle \mathfrak{A}, a\rangle \preceq_\omega \langle \mathfrak{A}_0, a_0\rangle$ while $\langle \mathfrak{A}, a\rangle \not\preceq \langle \mathfrak{A}_0, a_0\rangle$.

3.2.14. Proposition. *Let ϑ be a SW-vocabulary, and let $\langle \mathfrak{A}, a\rangle$ be a pointed τ_ϑ-structure. Then,*

1. $\text{rank}(\mathfrak{A}, a) \leq \text{qrank}(\mathfrak{A}, a) \leq \text{rank}(\mathfrak{A}, a) + 1$.
2. *If* $\text{rank}(\mathfrak{A}, a)$ *is a successor ordinal, then* $\text{rank}(\mathfrak{A}, a) = \text{qrank}(\mathfrak{A}, a)$.
3. *If* $\mathsf{SMod} \cap \mathsf{WMod} = \emptyset$ *and* $\text{rank}(\mathfrak{A}, a) = \omega$, *then* $\text{qrank}(\mathfrak{A}, a) = \omega + 1$.

Proof: 1) The first part is trivial. For the second part we recall that if $\langle a, a'\rangle \in R^{\mathfrak{A}}$ then $\text{rank}(\mathfrak{A}, a') \leq \text{rank}(\mathfrak{A}, a)$. Bearing this in mind a careful examination of the proof of Theorem 3.2.11 shows that $\deg(\pi^{\langle \mathfrak{A}, a\rangle}) \leq \text{rank}(\mathfrak{A}, a)+1$ and $\deg(\nu^{\langle \mathfrak{A}, a\rangle}) \leq \text{rank}(\mathfrak{A}, a) + 1$. Hence, $\text{qrank}(\mathfrak{A}, a) \leq \text{rank}(\mathfrak{A}, a) + 1$.

2) Let $\text{rank}(\mathfrak{A}, a)$ be a successor ordinal. In this case it is known that $\langle a, a'\rangle \in R^{\mathfrak{A}}$ implies $\text{rank}(\mathfrak{A}, a') < \text{rank}(\mathfrak{A}, a)$. Another checking of the definitions given in Theorem 3.2.11 shows that $\deg(\pi^{\langle \mathfrak{A}, a\rangle}) \leq \text{rank}(\mathfrak{A}, a)$ and $\deg(\nu^{\langle \mathfrak{A}, a\rangle}) \leq \text{rank}(\mathfrak{A}, a)$. Hence, $\text{qrank}(\mathfrak{A}, a) \leq \text{rank}(\mathfrak{A}, a)$.

3) Replacing the pointed structure with a bisimilar one we can assume that it is a tree of height ω and root a. As $\text{rank}(\mathfrak{A}, a) = \omega$ we know that there is an infinite branch $a = a_0\ R_{m_0}\ a_1\ R_{m_1}\ a_2 \ldots$ Deleting the duplicated (modulo bisimilarity) states we assume that a_1 is the unique m_0-successor of a_0 that is bisimilar to a_1. For every $n \in \omega$, let $\langle \mathfrak{T}^n, a_1^n\rangle$ be the result of cutting at height n the generated subtree of $\mathfrak{A}$ with root a_1. Hence, for every $n \in \omega$, $\langle \mathfrak{A}, a_1\rangle \simeq_n \langle \mathfrak{T}^n, a_1^n\rangle$ and $\langle \mathfrak{A}, a_1\rangle \not\simeq \langle \mathfrak{T}^n, a_1^n\rangle$. Let $\mathfrak{B}$ be the tree that results when replacing in $\mathfrak{A}$ the subtree of root a_1 with a copy of $\langle \mathfrak{T}^n, a_1^n\rangle$ for each $n \in \omega$. And impose that in $\mathfrak{B}$ each a_1^n is a m_0-successor of a. Now we distinguish two cases.

Case $m_0 \in \mathsf{SMod} \setminus \mathsf{WMod}$: Then, $\langle \mathfrak{B}, a\rangle \preceq_\omega \langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, a\rangle \not\preceq \langle \mathfrak{A}, a\rangle$. Thus, $\text{qrank}(\mathfrak{A}, a) \neq \omega = \text{rank}(\mathfrak{A}, a)$. By the first item we conclude $\text{qrank}(\mathfrak{A}, a) = \omega + 1$.

[10] The subscripts follow the conventions introduced in Definition 2.1.2.

Case $m_0 \in \mathsf{WMod} \setminus \mathsf{SMod}$: Then, $\langle \mathfrak{A}, a\rangle \preceq_\omega \langle \mathfrak{B}, a\rangle$ and $\langle \mathfrak{A}, a\rangle \not\preceq \langle \mathfrak{B}, a\rangle$. Thus, $\mathrm{qrank}(\mathfrak{A}, a) \neq \omega = \mathrm{rank}(\mathfrak{A}, a)$. By the first item we deduce $\mathrm{qrank}(\mathfrak{A}, a) = \omega + 1$.

This completes the proof. □

From now until the end of the section we analyze which infinitary strict-weak formulas are positive characterizations of pointed structures. Analogously to Proposition 1.3.4 we have the following result. We notice that completely prime satisfiable classes of formulas now play the role of maximal satisfiable classes of formulas.

3.2.15. Proposition. *Let ϑ be a SW-vocabulary, and φ be a $\mathcal{L}^{SW}_\infty(\tau_\vartheta)$-formula. The following are equivalent:*

1. *φ characterizes positively, up to quasi bisimilarity, a certain pointed structure.*
2. *φ is satisfiable in a certain pointed structure. And for every class C of infinitary strict-weak formulas, if $\varphi \cup \{\sim \varphi' : \varphi' \in \mathsf{C}\}$ is unsatisfiable, then $\top \equiv \varphi \supset \varphi'$ for a certain $\varphi' \in \mathsf{C}$.*[11]

Proof: $(1 \Rightarrow 2)$: Assume φ characterizes positively $\langle \mathfrak{A}, a\rangle$, and $\varphi \cup \{\sim \varphi' : \varphi' \in \mathsf{C}\}$ is unsatisfiable. As $\mathfrak{A}, a \Vdash \varphi$, it follows that $\mathfrak{A}, a \not\Vdash \{\sim \varphi' : \varphi' \in \mathsf{C}\}$. That is, there exists $\varphi' \in \mathsf{C}$ such that $\mathfrak{A}, a \Vdash \varphi'$. Now we check $\top \equiv \varphi \supset \varphi'$. Assume $\mathfrak{B}, b \Vdash \varphi$. Then, $\langle \mathfrak{A}, a\rangle \preceq \langle \mathfrak{B}, b\rangle$ by the positive characterization. By Proposition 3.2.2 we conclude $\mathfrak{B}, b \Vdash \varphi'$.

$(2 \Rightarrow 1)$: Let C be $\{\pi^{\langle \mathfrak{B}, b\rangle} : \mathfrak{B}, b \Vdash \varphi\}$. Now we see that $\varphi \cup \{\sim \varphi' : \varphi' \in \mathsf{C}\}$ is unsatisfiable. Assume $\langle \mathfrak{A}, a\rangle \Vdash \varphi$. Then, $\pi^{\langle \mathfrak{A}, a\rangle} \in \mathsf{C}$ and $\langle \mathfrak{A}, a\rangle \Vdash \pi^{\langle \mathfrak{A}, a\rangle}$. Hence, $\mathfrak{A}, a \not\Vdash \{\sim \varphi' : \varphi' \in \mathsf{C}\}$. By our hypothesis this unsatisfiability yields a pointed structure $\langle \mathfrak{B}, b\rangle$ such that $\mathfrak{B}, b \Vdash \varphi$ and $\top \equiv \varphi \supset \pi^{\langle \mathfrak{B}, b\rangle}$. To finish the proof it is enough to see that $\varphi \equiv \pi^{\langle \mathfrak{B}, b\rangle}$; it remains to be seen that $\top \equiv \pi^{\langle \mathfrak{B}, b\rangle} \supset \varphi$. Suppose $\langle \mathfrak{A}, a\rangle \Vdash \pi^{\langle \mathfrak{B}, b\rangle}$. Then, $\langle \mathfrak{B}, b\rangle \preceq \langle \mathfrak{A}, a\rangle$. By Proposition 3.2.2 we can deduce $\mathfrak{A}, a \Vdash \varphi$. □

Proposition 1.3.4 can be formulated in **ZFC**, but this is not possible for the previous proposition. The problem comes from the fact that we have a quantification over arbitrary classes. It would be interesting to find equivalent formulations that could be stated in **ZFC**. We will now give a partial solution to this problem. We introduce two different properties very close to positive characterization which can be formulated in **ZFC**. Although they do not coincide in general, under certain requirements they will coincide with positive characterizations.

[11]Informally, the last sentence says that $\top \equiv \varphi \supset \bigvee \mathsf{C}$ implies $\top \equiv \varphi \supset \varphi'$ for a certain $\varphi' \in \mathsf{C}$. The problem with this formulation is that $\bigvee \mathsf{C}$ only exists when C is a set. In the case that C is the empty set this condition says that φ is satisfiable, i.e., the first part of this item is redundant.

3.2.16. Definition. (Completely prime and bounded completely prime formulas)
Let ϑ be a SW-vocabulary. A $\mathcal{L}_{\infty}^{SW}(\tau_\vartheta)$-formula φ is *completely prime* if for every set Φ of infinitary strict-weak formulas, $\top \equiv \varphi \supset \bigvee \Phi$ implies $\top \equiv \varphi \supset \varphi'$ for a certain $\varphi' \in \Phi$. And φ is *bounded completely prime* when in addition there is an ordinal α such that for all pointed structures $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ where φ holds,

$$\langle \mathfrak{A}, a\rangle \preceq_\alpha \langle \mathfrak{B}, b\rangle \quad \text{implies} \quad \langle \mathfrak{A}, a\rangle \preceq \langle \mathfrak{B}, b\rangle.$$

Taking Φ as the empty set it holds that all completely prime formulas are satisfiable.

3.2.17. Proposition. *Let ϑ be a SW-vocabulary, and φ be a $\mathcal{L}_{\infty}^{SW}(\tau_\vartheta)$-formula. Then, the following properties are in decreasing order, i.e., $3 \Rightarrow 2 \Rightarrow 1$.*

1. *φ is completely prime.*
2. *φ characterizes positively, up to quasi bisimilarity, a certain pointed structure.*
3. *φ is bounded completely prime.*

Proof: $(2 \Rightarrow 1)$: This is a weakening of what we saw in Proposition 3.2.15. What we need here is only the case where C is a set.

$(3 \Rightarrow 1)$: Assume α is the ordinal given by the boundedness condition. And let Φ be $\{\varphi' \in \mathcal{L}_{\infty}^{SW}(\tau_\vartheta) : \deg(\varphi') \leq \alpha,\ \top \not\equiv \varphi \supset \varphi'\}$. Although it is a proper class we know that it is, up to equivalence, a set. Clearly, for every $\varphi' \in \Phi$ it holds that $\top \not\equiv \varphi \supset \varphi'$. By being completely prime it follows that $\top \not\equiv \varphi \supset \bigvee \Phi$. Thus, there exists $\langle \mathfrak{A}, a\rangle$ such that $\mathfrak{A}, a \Vdash \varphi$ and $\mathfrak{A}, a \nVdash \bigvee \Phi$. Now we check that φ characterizes positively $\langle \mathfrak{A}, a\rangle$, i.e., for every $\langle \mathfrak{B}, b\rangle$ it holds

$$\langle \mathfrak{A}, a\rangle \preceq \langle \mathfrak{B}, b\rangle \quad \text{iff} \quad \mathfrak{B}, b \Vdash \varphi.$$

The rightward direction is obvious by Proposition 3.2.2 and the fact that $\mathfrak{A}, a \Vdash \varphi$. For the other direction, assume $\mathfrak{B}, b \Vdash \varphi$. By the boundedness condition it is enough to see that $\langle \mathfrak{A}, a\rangle \preceq_\alpha \langle \mathfrak{B}, b\rangle$. To verify this property we use Proposition 3.2.6. Let $\varphi' \in \mathcal{L}_{\infty}^{SW}(\tau_\vartheta)$ be such that $\mathfrak{B}, b \nVdash \varphi'$ and $\deg(\varphi') \leq \alpha$. Then, $\mathfrak{B}, b \nVdash \varphi \supset \varphi'$. In particular, $\top \not\equiv \varphi \supset \varphi'$ and $\varphi' \in \Phi$. As $\mathfrak{A}, a \nVdash \bigvee \Phi$ we conclude that $\mathfrak{A}, a \nVdash \varphi'$. □

3.2.18. Proposition. *Let ϑ be a SW-vocabulary, and φ be a $\mathcal{L}_{\infty}^{SW}(\tau_\vartheta)$-formula.*

1. *If* $\mathsf{SMod} \subseteq \mathsf{WMod}$*, then φ is completely prime iff φ characterizes positively, up to quasi bisimilarity, a certain pointed structure.*

2. *If* $\mathsf{WMod} \subseteq \mathsf{SMod}$*, then* φ *is bounded completely prime iff* φ *characterizes positively, up to quasi bisimilarity, a certain pointed structure.*

Proof: 1) Suppose φ is completely prime. Then, φ is satisfiable, i.e., there exists $\langle\mathfrak{A}, a\rangle$ such that $\mathfrak{A}, a \Vdash \varphi$. The hypothesis $\mathsf{SMod} \subseteq \mathsf{WMod}$ guarantees that the class $\{\langle\mathfrak{B}, b\rangle : \langle\mathfrak{B}, b\rangle \preceq \langle\mathfrak{A}, a\rangle\}$ is modulo bisimilarity a set, which allows us to consider the infinitary modal formula of the following claim.

Claim: $\top \equiv \varphi \supset (\nu^{\langle\mathfrak{A},a\rangle} \vee \bigvee\{\pi^{\langle\mathfrak{B},b\rangle} : \langle\mathfrak{B}, b\rangle \preceq \langle\mathfrak{A}, a\rangle,\ \mathfrak{B}, b \Vdash \varphi\})$.

Proof of Claim: Assume $\mathfrak{C}, c \Vdash \varphi$ and $\mathfrak{C}, c \nVdash \nu^{\langle\mathfrak{A},a\rangle}$. Then, $\langle\mathfrak{C}, c\rangle \preceq \langle\mathfrak{A}, a\rangle$. Using the fact that $\mathfrak{C}, c \Vdash \pi^{\langle\mathfrak{C},c\rangle}$ we conclude that $\mathfrak{C}, c \Vdash \bigvee\{\pi^{\langle\mathfrak{B},b\rangle} : \langle\mathfrak{B}, b\rangle \preceq \langle\mathfrak{A}, a\rangle,\ \mathfrak{B}, b \Vdash \varphi\}$. ⊣

Since φ is completely prime, together with the fact that $\mathfrak{A}, a \nVdash \varphi \supset \nu^{\langle\mathfrak{A},a\rangle}$ it is deduced that there exists $\langle\mathfrak{B}, b\rangle$ such that $\langle\mathfrak{B}, b\rangle \preceq \langle\mathfrak{A}, a\rangle$, $\mathfrak{B}, b \Vdash \varphi$ and $\top \equiv \varphi \supset \pi^{\langle\mathfrak{B},b\rangle}$. To conclude the proof it is enough to see that $\varphi \equiv \pi^{\langle\mathfrak{B},b\rangle}$. For this purpose it only remains to be seen that $\top \equiv \pi^{\langle\mathfrak{B},b\rangle} \supset \varphi$. Suppose $\langle\mathfrak{C}, c\rangle \Vdash \pi^{\langle\mathfrak{B},b\rangle}$. Then, $\langle\mathfrak{B}, b\rangle \preceq \langle\mathfrak{C}, c\rangle$. By Proposition 3.2.2 we can deduce $\mathfrak{C}, c \Vdash \varphi$.

2) Suppose φ gives a positive characterization of $\langle\mathfrak{A}, a\rangle$. By Proposition 3.2.17 it only remains to prove the boundedness condition. Let α be the rank of $\langle\mathfrak{A}, a\rangle$. Now we prove that for all pointed structures $\langle\mathfrak{B}, b\rangle$ and $\langle\mathfrak{C}, c\rangle$ where φ holds,

$$\langle\mathfrak{B}, b\rangle \preceq_{\alpha+1} \langle\mathfrak{C}, c\rangle \quad \text{implies} \quad \langle\mathfrak{B}, b\rangle \preceq \langle\mathfrak{C}, c\rangle.$$

The fact that φ holds in these structures tells us that $\langle\mathfrak{A}, a\rangle \preceq \langle\mathfrak{B}, b\rangle$ and $\langle\mathfrak{A}, a\rangle \preceq \langle\mathfrak{C}, c\rangle$. Assume $\langle\mathfrak{B}, b\rangle \preceq_{\alpha+1} \langle\mathfrak{C}, c\rangle$, and let us check the properties involved in bisimilarity. To check property (qbis1) is trivial, so we concentrate on the other two properties.

(qbis2): Let $s \in \mathsf{SMod}$, and let c' be such that $\langle c, c'\rangle \in R_s^{\mathfrak{C}}$. Then, there exists b' such that $\langle b, b'\rangle \in R_s^{\mathfrak{B}}$ and $\langle\mathfrak{B}, b'\rangle \simeq_\alpha \langle\mathfrak{C}, c'\rangle$. By $\langle\mathfrak{A}, a\rangle \preceq \langle\mathfrak{B}, b\rangle$ there exists a' such that $\langle a, a'\rangle \in R_s^{\mathfrak{A}}$ and $\langle\mathfrak{A}, a'\rangle \simeq \langle\mathfrak{B}, b'\rangle$. Hence, $\langle\mathfrak{A}, a'\rangle \simeq_\alpha \langle\mathfrak{C}, c'\rangle$. By the choice of α it holds that $\langle\mathfrak{A}, a'\rangle \simeq \langle\mathfrak{C}, c'\rangle$. Therefore, $\langle\mathfrak{B}, b'\rangle \simeq \langle\mathfrak{C}, c'\rangle$.

(qbis3): Let $w \in \mathsf{WMod} \subseteq \mathsf{SMod}$, and let b' be such that $\langle b, b'\rangle \in R_w^{\mathfrak{B}}$. By $\langle\mathfrak{A}, a\rangle \preceq \langle\mathfrak{B}, b\rangle$ there exists a' such that $\langle a, a'\rangle \in R_w^{\mathfrak{A}}$ and $\langle\mathfrak{A}, a'\rangle \simeq \langle\mathfrak{B}, b'\rangle$. And since $\langle\mathfrak{A}, a\rangle \preceq \langle\mathfrak{C}, c\rangle$, there exists c' such that $\langle c, c'\rangle \in R_w^{\mathfrak{C}}$ and $\langle\mathfrak{A}, a'\rangle \simeq \langle\mathfrak{C}, c'\rangle$.

□

3.2.19. Proposition. *Let* ϑ *be a SW-vocabulary.*

1. *If there is any pure strict modality (i.e.,* $\mathsf{SMod} \not\subseteq \mathsf{WMod}$*) and* $\mathsf{SMod} \cap \mathsf{WMod} = \emptyset$*, then* $\top$ *is completely prime and it does not characterize any pointed structure positively.*

2. *If there is any pure weak modality (i.e.,* $\mathsf{WMod} \not\subseteq \mathsf{SMod}$*), then the formula that characterizes* $\mathfrak{S}_{IrF}$ *positively is not bounded completely prime*[12]*.*

Proof: 1) By Theorem 1.3.1(4) and the fact that SMod is non-empty there is no pointed structure $\langle\mathfrak{A}, a\rangle$ such that for every pointed structure $\langle\mathfrak{B}, b\rangle$, $\langle\mathfrak{A}, a\rangle \preceq \langle\mathfrak{B}, b\rangle$. Thus, $\top$ does not characterize any pointed structure positively. In order to see that $\top$ is completely prime we introduce a general method to obtain completely prime formulas under the assumptions of the item. Using this method we obtain what we need due to the fact that $\top \equiv [s]\top$.

Claim: If s is a pure strict modality and φ is an infinitary modal formula, then $[s]\varphi$ is completely prime.

Proof of Claim: Assume Φ is a set of infinitary strict-weak formulas such that for every $\varphi' \in \Phi$, $\top \not\equiv [s]\varphi \supset \varphi'$. Hence, for every $\varphi' \in \Phi$, exists $\langle\mathfrak{A}_{\varphi'}, a_{\varphi'}\rangle$ such that $\langle\mathfrak{A}_{\varphi'}, a_{\varphi'}\rangle \Vdash [s]\varphi \wedge \sim\varphi'$. Replacing them with bisimilar copies we suppose that all of them are disjoint trees of height ω. In a node x different from the root we define its associated modality (notation $\mathrm{mod}(x)$) as the first modality used in the branch that allows us to reach x from the root. Given $\varphi' \in \Phi$, let $X_{\varphi'}$ be the set $\{x \in A_{\varphi'} : x \text{ is not the root, } \mathrm{mod}(x) \in \mathsf{SMod}\}$, and let $Y_{\varphi'} := \{x \in X_{\varphi'} : x \text{ is a successor of the root}\}$. We take the pointed structure $\langle\mathfrak{A}, a\rangle$ as the tree where (i) the universe is the disjoint union $\{a\} \cup \bigcup_{\varphi' \in \Phi} X_{\varphi'}$, (iia) the accessibility relation associated with $m \in \mathsf{SMod}$ is $\{\langle a, x\rangle : x \in \bigcup_{\varphi' \in \Phi} Y_{\varphi'}\} \cup \bigcup\{R_m^{\mathfrak{A}_{\varphi'}} \cap (X_{\varphi'} \times X_{\varphi'}) : \varphi' \in \Phi\}$, (iib) the accessibility relation associated with $m \notin \mathsf{SMod}$ (i.e., $m \in \mathsf{WMod}$) is $\bigcup\{R_m^{\mathfrak{A}_{\varphi'}} \cap (X_{\varphi'} \times X_{\varphi'}) : \varphi' \in \Phi\}$, (iii) the valuation is the map $p \longmapsto \bigcup\{V^{\mathfrak{A}_{\varphi'}} \cap X_{\varphi'} : \varphi' \in \Phi\}$. By construction and the fact that $\mathsf{SMod} \cap \mathsf{WMod} = \emptyset$ it is obvious that $\langle\mathfrak{A}, a\rangle \preceq \langle\mathfrak{A}_{\varphi'}, a_{\varphi'}\rangle$ for every $\varphi' \in \Phi$. Hence, $\mathfrak{A}, a \nVdash \bigvee\Phi$ by Proposition 3.2.2. And by construction it is also clear that $\mathfrak{A}, a \Vdash [s]\varphi$. Thus, $\top \not\equiv [s]\varphi \supset \bigvee\Phi$. ⊣

2) Let w be a pure strict-weak modality. We define the vocabulary $\vartheta' = \langle\emptyset, \{w\}, \mathsf{Prop}\rangle$. By Theorem 1.3.1(5) there is a proper class $\{\langle\mathfrak{A}_\alpha, a_\alpha\rangle : \alpha \in \mathbb{ORD}\}$ of pointed $\tau_{\vartheta'}$-structures such that for every α,

$$\langle\mathfrak{A}_\alpha, a_\alpha\rangle \preceq_{\alpha+1} \langle\mathfrak{A}_{\alpha+1}, a_{\alpha+1}\rangle \quad \text{and} \quad \langle\mathfrak{A}_\alpha, a_\alpha\rangle \npreceq_{\alpha+2} \langle\mathfrak{A}_{\alpha+1}, a_{\alpha+1}\rangle.$$

It is obvious that all pointed $\tau_{\vartheta'}$-structures can be considered as pointed τ_ϑ-structures; just consider as the empty relation the accessibility relation associated with a modality different from w. Under this consideration it is clear that $\mathfrak{S}_{IrF} \preceq_\vartheta \langle\mathfrak{A}_\alpha, a_\alpha\rangle$ for every α. Thus, all $\langle\mathfrak{A}_\alpha, a_\alpha\rangle$ satisfy the formula that characterizes positively $\mathfrak{S}_{IrF}$. And by construction it is clear that there is no ordinal satisfying the boundedness condition. □

The last proposition shows that the hypothesis of Proposition 3.2.18 cannot be eliminated. The problem remains of how to find a characterization inside

[12] In the case $\mathsf{SMod} = \emptyset$ this formula is $\top$, but in general it is not $\top$.

ZFC of the formulas that characterizes positively. The main difficulty is to find a candidate. Indeed, the natural candidates are the ones used before, and we have seen that they do not behave well in general.

Now, we point out something that we have seen in a hidden way in the previous proofs. Given a set Φ of infinitary strict-weak formulas, we say that it *generates a completely prime theory* in the case that $\bigwedge \Phi$ is a completely prime formula. And we say that Φ is *exactly satisfied by* $\langle \mathfrak{A}, a\rangle$ when for every infinitary strict-weak formula φ,

$$\mathfrak{A}, a \Vdash \varphi \quad \text{iff} \quad \top \equiv \bigwedge \Phi \supset \varphi.$$

Given a pointed structure $\langle \mathfrak{A}, a\rangle$ and a set Φ of infinitary strict-weak formulas, it is very simple to see the equivalence between (i) $\bigwedge \Phi$ characterizes $\langle \mathfrak{A}, a\rangle$ positively, and (ii) Φ is exactly satisfied by $\langle \mathfrak{A}, a\rangle$. Adopting this new terminology, Proposition 3.2.18(1) says that if $\mathsf{SMod} \subseteq \mathsf{WMod}$, then all sets that generate a completely prime theory are exactly satisfied by a certain pointed structure. However, Proposition 3.2.19(1) shows that this is not true in general. This last situation does not happen in the case of finitary formulas. By the canonical structure construction that we will introduce in the finitary case (see Section 4.2) it will be clear that all sets of strict-weak formulas that generate a prime[13] theory are exactly satisfied, regardless of the interrelations between strict and weak modalities.

3.2.20. Remark. (Negative Characterizations). Up to now we have focussed on formulas that provide positive characterizations. We have ignored negative characterizations because they can be analyzed using duality. It is enough to exploit the fact that φ is a negative characterization of $\langle \mathfrak{A}, a\rangle$ iff φ^d is a positive characterization of $\langle \mathfrak{A}^d, a\rangle$. Here we do not develop all the results about negative characterizations that can obtained from our previous work, we just restrict ourselves to displaying the dual conditions of the ones involved in Proposition 3.2.17.

1. φ is *dually completely prime*, i.e., for every set Φ of infinitary strict-weak formulas, $\bot \equiv \bigwedge \Phi \smallsetminus \varphi$ implies $\bot \equiv \varphi' \smallsetminus \varphi$ for a certain $\varphi' \in \Phi$. In other words, for every set Φ of infinitary strict-weak formulas, $\top \equiv \bigwedge \Phi \supset \varphi$ implies $\top \equiv \varphi' \supset \varphi$ for a certain $\varphi' \in \Phi$.

2. φ characterizes negatively, up to quasi bisimilarity, a certain pointed structure.

3. φ is *dually bounded completely prime*, i.e., it is dually completely prime and in addition exists an ordinal α such that for all pointed structures $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ where φ fails,

$$\langle \mathfrak{A}, a\rangle \preceq_\alpha \langle \mathfrak{B}, b\rangle \quad \text{implies} \quad \langle \mathfrak{A}, a\rangle \preceq \langle \mathfrak{B}, b\rangle.$$

[13] Note that we do not say "completely prime".

3.3 Quasi bounded morphisms

3.3.1. Definition. (Preserving quasi bounded morphisms)
Let ϑ be a SW-vocabulary, and let $\mathfrak{A}$ and $\mathfrak{B}$ be two τ_ϑ-structures. A mapping $f : A \longrightarrow B$ is a *preserving quasi bounded morphism* from $\mathfrak{A}$ into $\mathfrak{B}$ if it satisfies the following conditions:

(pqbmor1.a): For each $a \in A$ and $p \in \mathsf{Prop}$, if $\mathfrak{A}, a \Vdash p$ then $\mathfrak{B}, f(a) \Vdash p$.

(pqbmor1.b): For each $a \in A$ that is the successor of a state and each $p \in \mathsf{Prop}$, if $\mathfrak{B}, f(a) \Vdash p$ then $\mathfrak{A}, a \Vdash p$.

(pqbmor2.a): For each $s \in \mathsf{SMod}$, $a \in A$ and $b' \in B$, it holds that if $\langle f(a), b'\rangle \in R^{\mathfrak{B}}_s$, then there is a' such that $\langle a, a'\rangle \in R^{\mathfrak{A}}_s$ and $f(a') = b'$.

(pqbmor2.b): For each $w \in \mathsf{WMod} \setminus \mathsf{SMod}$, each $a \in A$ that is the successor of a state and each $b' \in B$, it holds that if $\langle f(a), b'\rangle \in R^{\mathfrak{B}}_w$, then there is a' such that $\langle a, a'\rangle \in R^{\mathfrak{A}}_w$ and $f(a') = b'$.

(pqbmor3): For each $m \in \mathsf{SMod} \cup \mathsf{WMod}$ and each $a, a' \in A$, if $\langle a, a'\rangle \in R^{\mathfrak{A}}_m$ then $\langle f(a), f(a')\rangle \in R^{\mathfrak{B}}_m$.

3.3.2. Definition. (Reflecting quasi bounded morphisms)
Let ϑ be a SW-vocabulary, and let $\mathfrak{A}$ and $\mathfrak{B}$ be two τ_ϑ-structures. A mapping $f : A \longrightarrow B$ is a *reflecting quasi bounded morphism!for structures* from $\mathfrak{A}$ into $\mathfrak{B}$ if it satisfies the following conditions:

(rqbmor1.a): For each $a \in A$ that is the successor of a state and each $p \in \mathsf{Prop}$, if $\mathfrak{A}, a \Vdash p$ then $\mathfrak{B}, f(a) \Vdash p$.

(rqbmor1.b): For each $a \in A$ and $p \in \mathsf{Prop}$, if $\mathfrak{B}, f(a) \Vdash p$ then $\mathfrak{A}, a \Vdash p$.

(rqbmor2.a): For each $s \in \mathsf{SMod} \setminus \mathsf{WMod}$, each $a \in A$ that is the successor of a state and each $b' \in B$, it holds that if $\langle f(a), b'\rangle \in R^{\mathfrak{B}}_s$, then there is a' such that $\langle a, a'\rangle \in R^{\mathfrak{A}}_s$ and $f(a') = b'$.

(rqbmor2.b): For each $w \in \mathsf{WMod}$, $a \in A$ and $b' \in B$, it holds that if $\langle f(a), b'\rangle \in R^{\mathfrak{B}}_w$, then there is a' such that $\langle a, a'\rangle \in R^{\mathfrak{A}}_w$ and $f(a') = b'$.

(rqbmor3): For each $m \in \mathsf{SMod} \cup \mathsf{WMod}$ and each $a, a' \in A$, if $\langle a, a'\rangle \in R^{\mathfrak{A}}_m$ then $\langle f(a), f(a')\rangle \in R^{\mathfrak{B}}_m$.

It is clear that if f is a bounded morphism then it is both a preserving quasi bounded morphism and a reflecting quasi bounded morphism. And if all the states in $\mathfrak{A}$ are the successor of someone (e.g., when $R^{\mathfrak{A}}$ is reflexive), then these new notions collapse to the notion of bounded morphism. Preserving quasi bounded morphisms are closed under composition, and also the reflecting ones.

3.3.3. Proposition. *Let ϑ be a SW-vocabulary, and let $\mathfrak{A}$ and $\mathfrak{B}$ be two τ_ϑ-structures.*

1. *If $a \in A$ and f is a preserving quasi bounded morphism from $\mathfrak{A}$ into $\mathfrak{B}$, then $\langle \mathfrak{A}, a\rangle \preceq \langle \mathfrak{B}, f(a)\rangle$.*
2. *If $a \in A$ and f is a reflecting quasi bounded morphism from $\mathfrak{A}$ into $\mathfrak{B}$, then $\langle \mathfrak{B}, f(a)\rangle \preceq \langle \mathfrak{A}, a\rangle$.*

Proof: When f is a preserving quasi bounded morphism it is straightforward to see that $\{\langle a, f(a)\rangle : a \in A,\ a \text{ is the successor of a state}\}$ is a functional bisimulation between $\mathfrak{A}$ and $\mathfrak{B}$. And using this it is easy to prove $\langle \mathfrak{A}, a\rangle \preceq \langle \mathfrak{B}, f(a)\rangle$ for every state a.

On the other hand, if f is a reflecting quasi bounded morphism then we also have that $\{\langle a, f(a)\rangle : a \in A,\ a \text{ is the successor of a state}\}$ is a functional bisimulation between $\mathfrak{A}$ and $\mathfrak{B}$. With this it is simple to check that $\langle \mathfrak{B}, f(a)\rangle \preceq \langle \mathfrak{A}, a\rangle$ for every state a. □

The last proposition, in combination with Proposition 3.2.2, implies that preserving quasi bounded morphisms preserve strict-weak formulas, and also that reflecting quasi bounded morphisms reflect strict-weak formulas.

We could have proved the second item using duality. It would be enough to check that the following two statements are equivalent:

1. f is a preserving quasi bounded morphism from $\mathfrak{A}$ into $\mathfrak{B}$.
2. f is a reflecting quasi bounded morphism from $\mathfrak{A}^d$ into $\mathfrak{B}^d$.

This equivalence is trivial.

3.3.4. Proposition. *Let ϑ be a SW-vocabulary, and let $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ be two pointed τ_ϑ-structures such that $\langle \mathfrak{A}, a\rangle \preceq \langle \mathfrak{B}, b\rangle$. Then, there are pointed τ_ϑ-structures $\langle \mathfrak{A}', a'\rangle$, $\langle \mathfrak{B}', b'\rangle$ and $\langle \mathfrak{C}, c\rangle$, and maps f, g, i, j such that*

$$\mathfrak{A} \xleftarrow{\ f\ } \mathfrak{A}' \xrightarrow{\ i\ } \mathfrak{C} \xleftarrow{\ j\ } \mathfrak{B}' \xrightarrow{\ g\ } \mathfrak{B}$$

where f and g are surjective bounded morphisms, i is a one-to-one preserving quasi bounded morphism, and j is a one-to-one reflecting quasi bounded morphism. Each of these maps respects the distinguished points, i.e., $f(a') = a$, $i(a') = c$, $j(b') = c$, and $g(b') = b$.

Proof: We take $\langle \mathfrak{A}', a'\rangle$ and $\langle \mathfrak{B}', b'\rangle$ as, respectively, $\mathfrak{e}xp_\kappa(\mathfrak{A}, a)$ and $\mathfrak{e}xp_\kappa(\mathfrak{B}, b)$, where κ is a big enough cardinal. It is clear that the maps end_A and end_B associated with these κ-expansions are surjective bounded morphisms respecting the distinguished points. We know that the elements ρ of a κ-expansion that are different from the root are of the form $\langle a_0, m_1, \alpha_1, a_1, \ldots, m_n, \alpha_n, a_n\rangle$ with $n \geq 1$. In such cases, we define its associated modality (notation: $\mathrm{mod}(\rho)$) as m_1. Now we define

$$\begin{array}{rl} X_0 := & \{\rho \in A' : \rho \neq a',\ \mathrm{mod}(\rho) \in \mathsf{SMod} \setminus \mathsf{WMod}\} \\ X_1 := & \{\rho \in A' : \rho \neq a',\ \mathrm{mod}(\rho) \in \mathsf{WMod} \setminus \mathsf{SMod}\} \\ X_2 := & \{\rho \in A' : \rho \neq a',\ \mathrm{mod}(\rho) \in \mathsf{SMod} \cap \mathsf{WMod}\} \\ Y_0 := & \{\rho \in B' : \rho \neq b',\ \mathrm{mod}(\rho) \in \mathsf{SMod} \setminus \mathsf{WMod}\} \\ Y_1 := & \{\rho \in B' : \rho \neq b',\ \mathrm{mod}(\rho) \in \mathsf{WMod} \setminus \mathsf{SMod}\} \\ Y_2 := & \{\rho \in B' : \rho \neq b',\ \mathrm{mod}(\rho) \in \mathsf{SMod} \cap \mathsf{WMod}\}. \end{array}$$

We take the pointed structure $\langle \mathfrak{C}, c\rangle$ as the tree where (i) the universe is the disjoint union $A' \cup Y_1$, (ii) the accessibility relation associated with m is $R_m^{\mathfrak{A}'} \cup (R_m^{\mathfrak{B}'} \cap Y_1^2) \cup \{\langle a', \rho\rangle : \rho = \langle b_0, m_1, \alpha_1, b_1\rangle \in Y_1\}$, (iii) the valuation is the map $p \longmapsto V^{\mathfrak{A}'}(p) \cup (V^{\mathfrak{B}'}(p) \cap Y_1)$, and (iv) the distinguished point is a'. It is obvious that the inclusion $i : A' \hookrightarrow C$ is a one-to-one preserving quasi bounded morphism from $\mathfrak{A}'$ into $\mathfrak{C}$ such that $i(a') = c$.

We take the pointed structure $\langle \mathfrak{D}, d\rangle$ as the tree where (i) the universe is $B' \cup X_0$, (ii) the accessibility relation associated with m is $R_m^{\mathfrak{B}'} \cup (R_m^{\mathfrak{A}'} \cap X_0^2) \cup \{\langle b', \rho\rangle : \rho = \langle b_0, m_1, \alpha_1, b_1\rangle \in X_0\}$, (iii) the valuation is the map $p \longmapsto (V^{\mathfrak{B}'}(p) \setminus \{b'\}) \cup (V^{\mathfrak{A}'}(p) \cap X_0) \cup \{b' : a' \in V^{\mathfrak{A}'}(p)\}$, and (iv) the distinguished point is b'. It is clear that the inclusion $j : B' \hookrightarrow D$ is a one-to-one reflecting quasi bounded morphism from $\mathfrak{B}'$ into $\mathfrak{D}$ such that $j(b') = d$.

The fact that $\langle \mathfrak{A}, a\rangle \preceq \langle \mathfrak{B}, b\rangle$, in combination with the properties of κ-expansions, tells us that for every $r \leq 2$ the (generated) substructure of $\mathfrak{A}'$ given by the universe X_r and the (generated) substructure of $\mathfrak{B}'$ given by the universe Y_r are isomorphic. Then, it is quite simple to see that $\langle \mathfrak{D}, d\rangle$ and $\langle \mathfrak{C}, c\rangle$ are isomorphic. Thus, we know that

$$\mathfrak{A} \xleftarrow{\mathrm{end}_A} \mathfrak{A}' \xrightarrow{\ i\ } \mathfrak{C} \cong \mathfrak{D} \xleftarrow{\ j\ } \mathfrak{B}' \xrightarrow{\mathrm{end}_B} \mathfrak{B}.$$

Hence, the composition of j with this isomorphism gives what we were looking for. □

3.3.5. Remark. An examination of the proof shows that if $\mathsf{WMod} \subseteq \mathsf{SMod}$ then i is an isomorphism. And in the case $\mathsf{SMod} \subseteq \mathsf{WMod}$ we have that j is an isomorphism.

Up to now we have considered maps between structures. In the rest of the section we will say some things at the level of frames. We do not succeed in obtaining a characterization of the classes of frames that are definable by strict-weak formulas, but at least we will see some necessary conditions that must satisfy these classes.

3.3.6. Definition. (Quasi bounded morphisms between frames)
Let ϑ be a SW-vocabulary, and let $\mathfrak{F}$ and $\mathfrak{F}'$ be two τ_ϑ-frames. A map $f : F \longrightarrow F'$ is a *preserving quasi bounded morphism* from $\mathfrak{F}$ into $\mathfrak{F}'$ if it satisfies the conditions (pqbmor2.a), (pqbmor2.b) and (pqbmor3) of Definition 3.3.1. And it is a *reflecting quasi bounded morphism* from $\mathfrak{F}$ into $\mathfrak{F}'$ if it satisfies the conditions (rqbmor2.a), (rqbmor2.b) and (rqbmor3) of Definition 3.3.2.

3.3.7. Proposition. *Let ϑ be a SW-vocabulary, let φ be a $\mathcal{L}^{SW}_\infty(\vartheta)$-formula, and let $\mathfrak{F}$ and $\mathfrak{F}'$ be two τ_ϑ-frames.*

1. *If $\mathfrak{F} \Vdash \varphi$ and there is a surjective preserving quasi bounded morphism from $\mathfrak{F}$ into $\mathfrak{F}'$, then $\mathfrak{F}' \Vdash \varphi$.*

2. *If $\mathfrak{F}' \Vdash \varphi$ and there is an injective reflecting quasi bounded morphism from $\mathfrak{F}$ into $\mathfrak{F}'$, then $\mathfrak{F} \Vdash \varphi$.*

Proof: 1) Let $f : F \longrightarrow F'$ be the surjective preserving quasi bounded morphism. We assume that V' is a valuation on $\mathfrak{F}'$ and that a' is a state in $\mathfrak{F}'$, and we show that $\mathfrak{F}', V', a' \Vdash \varphi$. Thus, there exists $a \in F$ such that $f(a) = a'$. We define a valuation V on $\mathfrak{F}$ as the map $p \longmapsto \{x \in F : f(x) \in V'(p)\}$. Then, it is easy to see that f is a preserving quasi bounded morphism from $\langle \mathfrak{F}, V \rangle$ into $\langle \mathfrak{F}', V' \rangle$. By $\mathfrak{F}, V, a \Vdash \varphi$ and Proposition 3.3.3(1) it follows that $\mathfrak{F}', V', a' \Vdash \varphi$.

2) Let $f : F \longrightarrow F'$ be the injective reflecting quasi bounded morphism. We assume that V is a valuation on $\mathfrak{F}$ and that a is a state in $\mathfrak{F}$, and we show that $\mathfrak{F}, V, a \Vdash \varphi$. We define a valuation V' on $\mathfrak{F}'$ as the map $p \longmapsto \{f(x) : x \in V(p)\}$. Then, by injectivity it holds that f is a reflecting quasi bounded morphism from $\langle \mathfrak{F}, V \rangle$ into $\langle \mathfrak{F}', V' \rangle$. By $\mathfrak{F}', V', f(a) \Vdash \varphi$ and Proposition 3.3.3(2) it follows that $\mathfrak{F}, V, a \Vdash \varphi$. ☐

This means that classes of frames definable by strict-weak formulas are closed under surjective preserving quasi bounded morphisms. And their complementary classes are closed under injective reflecting quasi bounded morphisms. It remains to be established whether all classes of frames definable by modal formulas that satisfy the previous conditions are necessarily also definable by strict-weak formulas.

As a trivial corollary of our necessary conditions we can show that there are classes of frames definable by a modal formula that are not definable by any set of strict-weak formulas. For instance, we show that the class of reflexive frames[14] is not definable by any $\mathcal{L}^s$-formula. Let $f : F \longrightarrow F'$ be the map given in Figure 3.2.

[14] The same argument presented here also works for the class of transitive frames, and for the class of reflexive-transitive frames. Indeed, these results correspond precisely to what is stated in [Bou01, Lemma 3.15]. The proof given there is completely different; it is based on the disjunction property of the strict-weak logics given by these classes of frames.

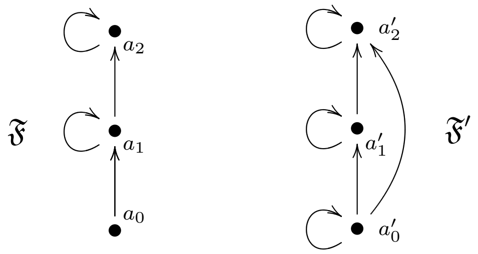


Figure 3.2: The map $a_i \longmapsto a'_i$ is a reflecting quasi bounded morphism (in $\mathcal{L}^s$)

It is easy to see that this map is an injective reflecting bounded morphism between the frames depicted there (we recall that we are working with a single pure strict modality), and $\mathfrak{F}'$ is clearly a reflexive frame. By Proposition 3.3.7(2) it follows that if reflexive frames were definable by $\mathcal{L}^s$-formulas then $\mathfrak{F}$ should be a reflexive frame, which it is not the case. Hence, the class of reflexive frames is not definable by $\mathcal{L}^s$-formulas.

3.4 Van Benthem's style theorems

This section is devoted to giving strict-weak versions of Theorems 1.3.5 and 1.3.6. The proofs are based on direct reductions to the modal theorems. The same methodology succeeds when it is applied to Rosen's Theorem on finite structures (see p. 22). However, in order to generalize Theorem 1.3.8 we will need to introduce the adequate notion of saturation for strict-weak fragments. Hence we leave this until Section 3.5. At the end of this section we characterize the first order formulas that are invariant under quasi bisimilarity.

3.4.1. Remark. It is very simple to see that for every $\varphi(v_0) \in \mathcal{L}^{FO}_\infty(\tau_\vartheta)$ and every $s \in \mathsf{SMod}$, the following are equivalent:

- φ is invariant under bisimilarity.
- $\forall v_1(R_s v_0 v_1 \supset \varphi(v_1))$ is preserved under quasi bisimilarity.

By duality we also have that for every $\varphi(v_0) \in \mathcal{L}^{FO}_\infty(\tau_\vartheta)$ and every $w \in \mathsf{WMod}$, the following are equivalent:

- φ is invariant under bisimilarity.
- $\exists v_1(R_s v_0 v_1 \wedge \varphi(v_1))$ is preserved under quasi bisimilarity.

Indeed, the arguments are not based on the fact that the formulas are first-order. Hence the same works, for instance, for second-order formulas.

3.4.2. THEOREM. *Let ϑ be a SW-vocabulary, and let $\varphi(v_0)$ be a $\mathcal{L}^{FO}_{\infty}(\tau_\vartheta)$-formula. The following are equivalent:*

1. *φ is, up to equivalence, in $\mathcal{L}^{SW}_{\infty}(\vartheta)$.*
2. *There exists $\alpha \in \mathbb{O}\mathrm{RD}$ such that φ is preserved under quasi α-bisimilarity.*
3. *φ is preserved under quasi bisimilarity.*
4. *φ is preserved under preserving quasi bounded morphisms and is reflected under reflecting quasi bounded morphisms.*

Proof: $(1 \Rightarrow 2 \Rightarrow 3)$: They are easy consequences of Propositions 3.2.2 and 3.2.6.

$(3 \Rightarrow 1)$: Then, φ is preserved under bisimilarity. Thus, by Theorem 1.3.6 it is equivalent to an infinitary modal formula. Hence, there is an ordinal β such that φ is preserved under β-bisimilarity. Let us fix $\alpha := \max\{\beta, \deg(\varphi)\}+1$. For every pointed structure $\langle \mathfrak{A}, a\rangle$ let $\pi_\alpha^{\langle \mathfrak{A}, a\rangle}$ be the formula described in Remark 3.2.10. Our proof is based on checking

$$\varphi \quad \equiv \quad \bigvee\{\pi_\alpha^{\langle \mathfrak{A}, a\rangle} : \mathfrak{A}, a \Vdash \varphi\}.$$

The disjunction on the right exists because the class of modal formulas modulo equivalence of a certain modal degree is set-sized. Firstly, we prove the easy direction. Let $\mathfrak{B}, b \Vdash \varphi$. Then, as $\mathfrak{B}, b \Vdash \pi_\alpha^{\langle \mathfrak{B}, b\rangle}$ we conclude that $\mathfrak{B}, b \Vdash \bigvee\{\pi_\alpha^{\langle \mathfrak{A}, a\rangle} : \mathfrak{A}, a \Vdash \varphi\}$. Let us now show the other direction. Assume $\mathfrak{B}, b \Vdash \bigvee\{\pi_\alpha^{\langle \mathfrak{A}, a\rangle} : \mathfrak{A}, a \Vdash \varphi\}$. Then, there exists $\langle \mathfrak{A}, a\rangle$ such that $\mathfrak{A}, a \Vdash \varphi$ and $\mathfrak{B}, b \Vdash \pi_\alpha^{\langle \mathfrak{A}, a\rangle}$. The last part says $\langle \mathfrak{A}, a\rangle \preceq_\alpha \langle \mathfrak{B}, b\rangle$. By Proposition 3.2.8 there exists $\langle \mathfrak{A}', a'\rangle$ and $\langle \mathfrak{B}', b'\rangle$ such that $\langle \mathfrak{A}, a\rangle \simeq_\alpha \langle \mathfrak{A}', a'\rangle$, $\langle \mathfrak{B}, b\rangle \simeq_\alpha \langle \mathfrak{B}', b'\rangle$ and $\langle \mathfrak{A}', a'\rangle \preceq \langle \mathfrak{B}', b'\rangle$. Using the fact that φ is both preserved under quasi bisimilarity and α-bisimilarity we deduce that $\mathfrak{B}, b \Vdash \varphi$.

$(3 \Rightarrow 4)$: By Proposition 3.3.3.

$(4 \Rightarrow 3)$: By Proposition 3.3.4. □

3.4.3. THEOREM. *Let ϑ be a SW-vocabulary, and let $\varphi(v_0)$ be a $\mathcal{L}^{FO}(\tau_\vartheta)$-formula. The following are equivalent:*

1. *φ is, up to equivalence, in $\mathcal{L}^{SW}(\vartheta)$.*
2. *There exists $n \in \omega$ such that φ is preserved under quasi n-bisimilarity.*
3. *φ is preserved under quasi bisimilarity.*
4. *φ is preserved under preserving quasi bounded morphisms and is reflected under reflecting quasi bounded morphisms.*

Proof: Restricting ϑ to the symbols that appear in φ we can assume that ϑ is finite. And now, copy the same proof that was given for Theorem 3.4.2, but this time applying Theorem 1.3.5. □

3.4.4. Remark. As a consequence of Remark 3.3.5 if $\mathsf{WMod} \subseteq \mathsf{SMod}$ then we can replace the fourth condition of Theorems 3.4.2 and 3.4.3 with 'φ is preserved under bounded morphisms and is reflected under reflecting quasi bounded morphisms'. And in the case $\mathsf{SMod} \subseteq \mathsf{WMod}$ we can substitute it with 'φ is preserved under preserving quasi bounded morphisms and is reflected under bounded morphisms'.

3.4.5. Theorem. *Let ϑ be a SW-vocabulary, let $\varphi(v_0)$ be a $\mathcal{L}^{FO}(\tau_\vartheta)$-formula, and let* Fin *be the class of finite τ_ϑ-structures. The following are equivalent:*

1. *φ is, up to* Fin*-equivalence, in $\mathcal{L}^{SW}(\vartheta)$.*
2. *There exists $n \in \omega$ such that φ is* Fin*-preserved under quasi n-bisimilarity.*
3. *φ is* Fin*-preserved under quasi bisimilarity.*

Proof: The same proof works again, this time using Rosen's result discussed on page 22. This time we must be slightly more careful when we apply Proposition 3.2.8, but Remark 3.2.9 takes care of these details. □

For the sake of completeness we now identify which are the formulas invariant under quasi bisimilarity. The following theorem gives a general answer, but the proof can be simplified in certain situations. We consider for instance two cases. In the first one all modalities are pure strict modalities (and similarly in the case that all are pure weak). Then, for all pointed structures $\langle \mathfrak{A}, a\rangle$, it holds that $\langle \mathfrak{A}, a\rangle \preceq \mathfrak{S}_{IrT}$. Hence, it is almost immediate that satisfiable strict-weak formulas invariant under quasi bisimilarity are satisfied in all pointed structures, i.e., they are equivalent to $\top$. In other words, $\bot$ and $\top$ are the only strict-weak formulas that are invariant under quasi bisimilarity. Let us consider now the second case, where $\mathsf{SMod} \cap \mathsf{WMod} = \emptyset$, which includes the case already treated. It is not hard to observe that for every pointed structure $\langle \mathfrak{A}, a\rangle$, there exists a pointed structure $\langle \mathfrak{A}', a'\rangle$ such that $\langle \mathfrak{A}, a\rangle \preceq \langle \mathfrak{A}', a'\rangle \succeq \mathfrak{S}_{IrT}$. If $\langle \mathfrak{A}, a\rangle$ is a tree then we can take $\langle \mathfrak{A}', a'\rangle$ as the result of deleting all nodes different from the root such that the branch that allows us to reach the node starts with a strict modality. By this we have again that satisfiable strict-weak formulas invariant under quasi bisimilarity are satisfied in all pointed structures. Hence, also in this case it results that $\bot$ and $\top$ are the only strict-weak formulas that are invariant under quasi bisimilarity.

3.4.6. Theorem. *Let ϑ be a SW-vocabulary, let $\varphi(v_0)$ be a $\mathcal{L}^{FO}_{\infty}(\tau_\vartheta)$-formula. The following are equivalent:*

1. *φ is invariant under quasi bisimilarity.*

2. *φ is, up to equivalence, in the closure under conjunction $\wedge$ and disjunction $\vee$ of the set $\{\bot, \top\} \cup \{\varphi_0 \rightarrow_m \varphi_1 : m \in \mathsf{SMod} \cap \mathsf{WMod},\ \varphi_0, \varphi_1 \in \mathcal{L}^{SW}_{\infty}(\vartheta)\} \cup \{\varphi_0 \leftharpoondown_m \varphi_1 : m \in \mathsf{SMod} \cap \mathsf{WMod},\ \varphi_0, \varphi_1 \in \mathcal{L}^{SW}_{\infty}(\vartheta)\}$.*

3. *φ is, up to equivalence, in the closure under conjunction $\wedge$ and disjunction $\vee$ of the set $\{\bot, \top\} \cup \{[m]\varphi : m \in \mathsf{SMod} \cap \mathsf{WMod},\ \varphi \in \mathcal{L}^{MOD}_{\infty}(\tau_\vartheta)\} \cup \{\langle m\rangle\varphi : m \in \mathsf{SMod} \cap \mathsf{WMod},\ \varphi \in \mathcal{L}^{MOD}_{\infty}(\tau_\vartheta)\}$.*

4. *φ is, up to equivalence, in the closure under conjunction $\wedge$ and material negation $\sim$ of the set $\{\top\} \cup \{[m]\varphi : m \in \mathsf{SMod} \cap \mathsf{WMod},\ \varphi \in \mathcal{L}^{MOD}_{\infty}(\tau_\vartheta)\}$.*

Proof: $(2 \Rightarrow 3 \Rightarrow 4)$: Trivial.

$(4 \Rightarrow 1)$: Straightforward.

$(1 \Rightarrow 2)$: By Theorem 3.4.2 and Remark 2.1.5 we can assume that φ is a disjunction of conjunctions of propositions or strict implications or weak differences.

Claim I: If $(p \wedge \varphi_0) \vee \varphi_1$ is invariant under quasi bisimilarity, then $(p \wedge \varphi_0) \vee \varphi_1 \equiv \varphi_0 \vee \varphi_1$.

Proof of Claim: It they are not equivalent this means that there exists a pointed structure $\langle \mathfrak{A}, a\rangle$ such that $\mathfrak{A}, a \Vdash \varphi_0 \vee \varphi_1$ and $\mathfrak{A}, a \nVdash (p \wedge \varphi_0) \vee \varphi_1$. Hence, $\mathfrak{A}, a \nVdash \varphi_1$, $\mathfrak{A}, a \Vdash \varphi_0$, and $\mathfrak{A}, a \nVdash p$. We can assume that this structure is a tree. Let $\mathfrak{A}'$ be the same tree except that in the root all propositions hold. Then, $\mathfrak{A}', a \Vdash p$. It is clear that $\langle \mathfrak{A}, a\rangle \preceq \langle \mathfrak{A}', a\rangle$; thus $\mathfrak{A}', a \Vdash \varphi_0$. Hence, $\mathfrak{A}', a \Vdash (p \wedge \varphi_0) \vee \varphi_1$. And this contradicts the fact that $(p \wedge \varphi_0) \vee \varphi_1$ is invariant under quasi bisimilarity. $\dashv$

Claim II: If s is a pure strict modality and $((\varphi_0 \rightarrow_s \varphi_1) \wedge \varphi_2) \vee \varphi_3$ is invariant under quasi bisimilarity, then $((\varphi_0 \rightarrow_s \varphi_1) \wedge \varphi_2) \vee \varphi_3 \equiv \varphi_2 \vee \varphi_3$.

Proof of Claim: It this is not the case, there is a pointed structure $\langle \mathfrak{A}, a\rangle$ such that $\mathfrak{A}, a \Vdash \varphi_2 \vee \varphi_3$ and $\mathfrak{A}, a \nVdash ((\varphi_0 \rightarrow_s \varphi_1) \wedge \varphi_2) \vee \varphi_3$. Therefore, $\mathfrak{A}, a \nVdash \varphi_3$, $\mathfrak{A}, a \Vdash \varphi_2$, and $\mathfrak{A}, a \nVdash \varphi_0 \rightarrow_s \varphi_1$. Let $\langle \mathfrak{B}, b\rangle$ be the unravelling of $\langle \mathfrak{A}, a\rangle$. It is clear that $\mathfrak{B}, b \nVdash ((\varphi_0 \rightarrow_s \varphi_1) \wedge \varphi_2) \vee \varphi_3$ and that $\mathfrak{B}, b \Vdash \varphi_2$. Now we modify slightly the tree $\mathfrak{B}$. Let $\mathfrak{C}$ be the result of deleting from $\mathfrak{B}$ all nodes different from the root such that the branch that allows us to reach the node starts with a pure strict modality. Then, $\mathfrak{C}, b \Vdash \varphi_0 \rightarrow_s \varphi_1$. And it is clear that $\langle \mathfrak{B}, b\rangle \preceq \langle \mathfrak{C}, b\rangle$, which says $\mathfrak{C}, b \Vdash \varphi_2$. Thus, $\mathfrak{C}, b \Vdash ((\varphi_0 \rightarrow_s \varphi_1) \wedge \varphi_2) \vee \varphi_3$. This is in contradiction with the fact that $((\varphi_0 \rightarrow_s \varphi_1) \wedge \varphi_2) \vee \varphi_3$ is invariant under quasi bisimilarity. $\dashv$

The previous two claims allow us to delete the propositions and the strict implications associated with pure strict modalities. Thus, we know that the formula can be written as a disjunction of conjunctions of strict implications associated with modalities that are in $\mathsf{SMod} \cap \mathsf{WMod}$ or weak differences associated with arbitrary modalities. By the distributivity law we can transform it into a conjunction of disjunctions of formulas of these kinds. Finally, the next claim shows that all weak differences associated with pure weak modalities can be deleted.

Claim III: If w is a pure weak modality and $((\varphi_0 \leftharpoondown_w \varphi_1) \vee \varphi_2) \wedge \varphi_3$ is invariant under quasi bisimilarity, then $((\varphi_0 \leftharpoondown_w \varphi_1) \vee \varphi_2) \wedge \varphi_3 \equiv \varphi_2 \wedge \varphi_3$.

Proof of Claim: It they are not equivalent it means that there is a pointed structure $\langle \mathfrak{A}, a\rangle$ such that $\mathfrak{A}, a \nVdash \varphi_2 \wedge \varphi_3$ and $\mathfrak{A}, a \Vdash ((\varphi_0 \leftharpoondown_w \varphi_1) \vee \varphi_2) \wedge \varphi_3$. Hence, $\mathfrak{A}, a \Vdash \varphi_3$, $\mathfrak{A}, a \nVdash \varphi_2$, and $\mathfrak{A}, a \Vdash \varphi_0 \leftharpoondown_w \varphi_1$. Let $\langle \mathfrak{B}, b\rangle$ be the unravelling of $\langle \mathfrak{A}, a\rangle$. It is clear that $\mathfrak{B}, b \Vdash ((\varphi_0 \leftharpoondown_w \varphi_1) \vee \varphi_2) \wedge \varphi_3$ and that $\mathfrak{B}, b \nVdash \varphi_2$. Now we slightly modify the tree $\mathfrak{B}$. Let $\mathfrak{C}$ be the result of deleting from $\mathfrak{B}$ all nodes different from the root such that the branch that allows us to reach the node starts with a pure weak modality. Then, $\mathfrak{C}, b \nVdash \varphi_0 \leftharpoondown_w \varphi_1$. And it is clear that $\langle \mathfrak{C}, b\rangle \preceq \langle \mathfrak{B}, b\rangle$, which says $\mathfrak{C}, b \nVdash \varphi_2$. Thus, $\mathfrak{C}, b \nVdash ((\varphi_0 \leftharpoondown_w \varphi_1) \vee \varphi_2) \wedge \varphi_3$. This contradicts the fact that $((\varphi_0 \leftharpoondown_w \varphi_1) \vee \varphi_2) \wedge \varphi_3$ is invariant under quasi bisimilarity. ⊣

□

3.4.7. Theorem. *Let ϑ be a SW-vocabulary, let $\varphi(v_0)$ be a $\mathcal{L}^{FO}(\tau_\vartheta)$-formula. The following are equivalent:*

1. *φ is invariant under quasi bisimilarity.*
2. *φ is, up to equivalence, in the closure under conjunction $\wedge$ and disjunction $\vee$ of the set $\{\bot, \top\} \cup \{\varphi_0 \rightharpoonup_m \varphi_1 : m \in \mathsf{SMod} \cap \mathsf{WMod},\ \varphi_0, \varphi_1 \in \mathcal{L}^{SW}(\vartheta)\} \cup \{\varphi_0 \leftharpoondown_m \varphi_1 : m \in \mathsf{SMod} \cap \mathsf{WMod},\ \varphi_0, \varphi_1 \in \mathcal{L}^{SW}(\vartheta)\}$.*
3. *φ is, up to equivalence, in the closure under conjunction $\wedge$ and disjunction $\vee$ of the set $\{\bot, \top\} \cup \{[m]\varphi : m \in \mathsf{SMod} \cap \mathsf{WMod},\ \varphi \in \mathcal{L}^{MOD}(\tau_\vartheta)\} \cup \{\langle m\rangle\varphi : m \in \mathsf{SMod} \cap \mathsf{WMod},\ \varphi \in \mathcal{L}^{MOD}(\tau_\vartheta)\}$.*
4. *φ is, up to equivalence, in the closure under conjunction $\wedge$ and material negation $\sim$ of the set $\{\top\} \cup \{[m]\varphi : m \in \mathsf{SMod} \cap \mathsf{WMod},\ \varphi \in \mathcal{L}^{MOD}(\tau_\vartheta)\}$.*

Proof: Copy the same proof that was given for Theorem 3.4.6, but this time applying Theorem 3.4.3. □

3.5 Strongly Hennessy-Milner classes

This section introduces the adequate notion of saturated structure suitable for the study of the strict-weak languages. Surprisingly, we will see that it is the notion of being modally saturated. Once developed, the tools on saturation will allow us to prove Theorem 3.5.11, a strict-weak version of Theorem 1.3.8. In this respect, proofs about quasi bisimilarity generalize the proofs of corresponding results for conventional bisimilarity, and it is often the case that a better understanding of a subject arises from the study of generalizations. At the end of the section we characterize hereditarily finite structures. This solves the gap explained on page 28.

For the sake of completeness we start by introducing strict-weak saturation, which is the natural generalization of a notion of saturation introduced in the context of $\mathcal{L}^s$ by Celani and Jansana in [CJ02]. Then, we generalize one of their results showing the coincidence between strict-weak saturation and modal saturation. The proof here offered is simpler and is based on the Standard Form Theorem.

3.5.1. Definition. (SW-saturation)
Let ϑ be a SW-vocabulary, and let $\mathfrak{A}$ be a τ_ϑ-structure. $\mathfrak{A}$ is *SW-saturated* if for every modality m, for every state $a \in A$ and for every pair of sets Φ_0, Φ_1 of strict-weak formulas, if $\Phi_0 \cup \{\sim\varphi : \varphi \in \Phi_1\}$ is finitely satisfiable in the set of m-successors of a (i.e., for every $\Phi_0' \subseteq_\omega \Phi_0$ and every $\Phi_1' \subseteq_\omega \Phi_1$ there exists a state a' with $\mathfrak{A}, a' \Vdash \Phi_0' \cup \{\sim\varphi : \varphi \in \Phi_1'\}$ that is a m-successor of a) then $\Phi_0 \cup \{\sim\varphi : \varphi \in \Phi_1\}$ is satisfiable in a m-successor of a (i.e., there exists a' a m-successor of a with $\mathfrak{A}, a' \Vdash \Phi_0 \cup \{\sim\varphi : \varphi \in \Phi_1\}$).

3.5.2. Proposition. *Let ϑ be a SW-vocabulary, and let $\mathfrak{A}$ be a τ_ϑ-structure. Then,*

$$\mathfrak{A} \textit{ is SW-saturated} \quad \textit{iff} \quad \mathfrak{A} \textit{ is modally saturated.}$$

Proof: The right-to-left implication is trivial. For the left-to-right implication, we recall that it is enough to check the property that defines modal saturation in the case of maximally satisfiable sets of modal formulas (see what is discussed on page 23). Assume $m \in \mathsf{Mod}$, $a \in A$, and Φ is a set of modal formulas such that is maximally satisfiable and is finitely satisfiable in the set of m-successors of a (i.e., for every $\Phi' \subseteq_\omega \Phi$ there exists a state a' with $\mathfrak{A}, a' \Vdash \Phi'$ that is an m-successor of a). Let Φ_0 be $\{\varphi \in \mathcal{L}^{SW}(\vartheta) : \varphi \in \Phi\}$, and let Φ_1 be $\{\varphi \in \mathcal{L}^{SW}(\vartheta) : \varphi \notin \Phi\}$. Of course, Φ_0 and Φ_1 form a partition of the set of all strict-weak formulas. And Φ_1 coincides with $\{\varphi \in \mathcal{L}^{SW}(\vartheta) : \sim\varphi \in \Phi\}$. Thus, $\Phi_0 \cup \{\sim\varphi : \varphi \in \Phi_1\} \subseteq \Phi$. Hence, it is clear that $\Phi_0 \cup \{\sim\varphi : \varphi \in \Phi_1\}$ is finitely satisfiable in the set of m-successors of a. By the strict-weak saturation we know that $\Phi_0 \cup \{\sim\varphi : \varphi \in \Phi_1\}$ is satisfiable in a m-successor of a. Now the next claim allows us to conclude that Φ is also satisfiable in a m-successor of a.

Claim: For every pointed structure $\langle \mathfrak{B}, b\rangle$,

$$\mathfrak{B}, b \Vdash \Phi \quad \text{iff} \quad \mathfrak{B}, b \Vdash \Phi_0 \cup \{\sim\varphi : \varphi \in \Phi_1\}.$$

Proof of Claim: The rightward implication is trivial because $\Phi_0 \cup \{\sim\varphi : \varphi \in \Phi_1\} \subseteq \Phi$. For the converse, we assume that $\mathfrak{B}, b \Vdash \Phi_0 \cup \{\sim\varphi : \varphi \in \Phi_1\}$ and that φ' is a formula in Φ. By the Standard Form Theorem we can suppose that φ' is of the form $(\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_{k-1} \supset \pi_{k-1})$ where $k \in \omega$, and the ν's and π's are strict-weak formulas. In order to see that $\mathfrak{B}, b \Vdash \varphi'$, consider $n < k$ such that $\mathfrak{B}, b \Vdash \nu_n$, and check that $\mathfrak{B}, b \Vdash \pi_n$. As $\mathfrak{B}, b \Vdash \{\sim\varphi : \varphi \in \Phi_1\}$, it is clear that $\nu_n \notin \Phi_1$. Thus, $\nu_n \in \Phi_0$, i.e., $\nu_n \in \Phi$. Hence, using the fact that $\varphi' \in \Phi$ we deduce that $\pi_n \in \Phi$. Thus, $\pi_n \in \Phi_0$. By the hypothesis $\mathfrak{B}, b \Vdash \Phi_0$ we conclude that $\mathfrak{B}, b \Vdash \pi_n$. ⊣

□

We are interested in structures and classes of structures where the converse of Corollary 3.2.3 holds. Thus, the following definition naturally arises.

3.5.3. Definition. (Strongly Hennessy-Milner)
Let ϑ be a SW-vocabulary. A class K of τ_ϑ-structures is a *ϑ-strongly Hennessy-Milner class* if for all $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$, all $a \in A$, and all $b \in B$ it holds that:

$$\langle \mathfrak{A}, a \rangle \rightsquigarrow_\vartheta \langle \mathfrak{B}, b \rangle \quad \text{implies} \quad \langle \mathfrak{A}, a \rangle \preceq_\vartheta \langle \mathfrak{B}, b \rangle.$$

And a τ_ϑ-structure $\mathfrak{A}$ is *ϑ-strongly Hennessy-Milner* when the class $\{\mathfrak{A}\}$ is a ϑ-strongly Hennessy-Milner class.

It is clear that whenever K is a set, then K is a ϑ-strongly Hennessy-Milner class iff $\biguplus \mathsf{K}$ is a ϑ-strongly Hennessy-Milner structure, and also that a class K is ϑ-strongly Hennessy-Milner iff it is ϑ^d-strongly Hennessy-Milner. We point out that this definition incorporates a symmetrical aspect. When K is a ϑ-strongly Hennessy-Milner class, then it also holds that for all $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$, all $a \in A$, and all $b \in B$,

$$\langle \mathfrak{B}, b \rangle \rightsquigarrow_\vartheta \langle \mathfrak{A}, a \rangle \quad \text{implies} \quad \langle \mathfrak{B}, b \rangle \preceq_\vartheta \langle \mathfrak{A}, a \rangle.$$

Hence, all ϑ-strongly Hennessy-Milner classes are also Hennessy-Milner classes, and all ϑ-strongly Hennessy-Milner structures are also Hennessy-Milner structures. The other direction is false in general, as the following example shows. That is, we have indeed introduced a new property.

3.5.4. Example. (Hennessy-Milner, but not strongly Hennessy-Milner). Let $\mathfrak{A}$ and $\mathfrak{B}$ be the structures given in Figure 3.3. We recall that by Convention 1.1.1 all propositions have the same behaviour in all the states. And let ϑ be the SW-vocabulary that has a single modality, which is pure strict. Then,

- $\mathfrak{A}$ is ϑ-strongly Hennessy-Milner. Why? Assume $\langle \mathfrak{A}, a_0 \rangle \rightsquigarrow \langle \mathfrak{A}, a_1 \rangle$. In order to see $\langle \mathfrak{A}, a_0 \rangle \preceq \langle \mathfrak{A}, a_1 \rangle$ the only condition that non trivially holds is (qbis2). So, let a_2 be such that $\langle a_1, a_2 \rangle \in R$. Then, a_2 is different from the root. This fact allows us to consider $k = \min\{n \in \omega : \mathfrak{A}, a_2 \Vdash \Box^{n+1}\bot\}$. Thus, $\mathfrak{A}, a_2 \Vdash \Box^{k+1}\bot \wedge \sim\Box^k\bot$. Hence, $\mathfrak{A}, a_1 \Vdash \Diamond(\Box^{k+1}\bot \wedge \sim\Box^k\bot)$, i.e.,

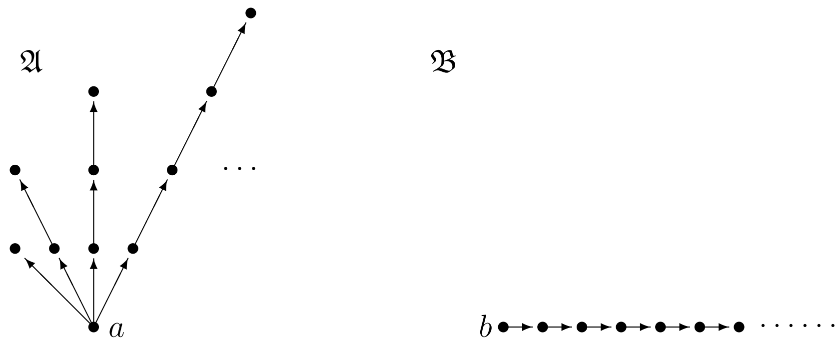


Figure 3.3: Hennessy-Milner, but not strongly Hennessy-Milner.

$\mathfrak{A}, a_1 \not\Vdash \Box \sim(\Box^{k+1}\bot \wedge \sim \Box^k\bot)$. By the fact $\langle \mathfrak{A}, a_0\rangle \rightsquigarrow \langle \mathfrak{A}, a_1\rangle$ it is deduced that $\mathfrak{A}, a_0 \not\Vdash \Box \sim(\Box^{k+1}\bot \wedge \sim \Box^k\bot)$, i.e., $\mathfrak{A}, a_0 \Vdash \Diamond(\Box^{k+1}\bot \wedge \sim \Box^k\bot)$, i.e., exists a_3 such that $\langle a_0, a_3\rangle \in R$ and $\mathfrak{A}, a_3 \Vdash \Box^{k+1}\bot \wedge \sim \Box^k\bot$. We know that both a_2 and a_3 satisfy the formula $\Box^{k+1}\bot \wedge \sim \Box^k\bot$, and by definition of $\mathfrak{A}$ it holds that all points satisfying this formula are bisimilar. Hence, $\langle \mathfrak{A}, a_3\rangle \simeq \langle \mathfrak{A}, a_2\rangle$.

- $\mathfrak{B}$ is ϑ-strongly Hennessy-Milner. This is trivial because all points in $\mathfrak{B}$ are bisimilar.
- $\{\mathfrak{A}, \mathfrak{B}\}$ is a Hennessy-Milner class. Why? By the previous two items we know that $\mathfrak{A}$ and $\mathfrak{B}$ are, separately, Hennessy-Milner structures. And it is clear that there are no states $a' \in A$ and $b' \in B$ such that $\langle \mathfrak{A}, a'\rangle \leftrightsquigarrow \langle \mathfrak{B}, b'\rangle$. Bearing these points in mind it is obvious that we are in a Hennessy-Milner class.
- $\{\mathfrak{A}, \mathfrak{B}\}$ is not a ϑ-strongly Hennessy-Milner class. This is due to the fact that $\langle \mathfrak{A}, a\rangle \rightsquigarrow \langle \mathfrak{B}, b\rangle$ (indeed, $\langle \mathfrak{A}, a\rangle \preceq_\omega \langle \mathfrak{B}, b\rangle$) while $\langle \mathfrak{A}, a\rangle \not\preceq \langle \mathfrak{B}, b\rangle$.

Thus, $\{\mathfrak{A}, \mathfrak{B}\}$ is a Hennessy-Milner class, but not ϑ-strongly Hennessy-Milner. If we replace this $\mathfrak{B}$ with the result of deleting some finite branches from $\mathfrak{B}$ in Figure 1.5 we obtain parallel results.

3.5.5. Remark. In the case that $\mathsf{SMod} = \mathsf{WMod}$ it is straightforward to see than the notion of ϑ-strongly Hennessy-Milner collapses to Hennessy-Milner, i.e., all Hennessy-Milner classes are ϑ-strongly Hennessy-Milner classes. And in the arbitrary case $\mathsf{SMod} \neq \mathsf{WMod}$ slight modifications in the previous example allow us to obtain Hennessy-Milner classes that are not ϑ-strongly Hennessy-Milner.

We now consider the position of this new notion in the landscape of properties displayed in Figure 1.6. We have already seen that it is strictly below Hennessy-Milner. The final answer is shown in Figure 3.4, and it is based on the next proposition and the next example.

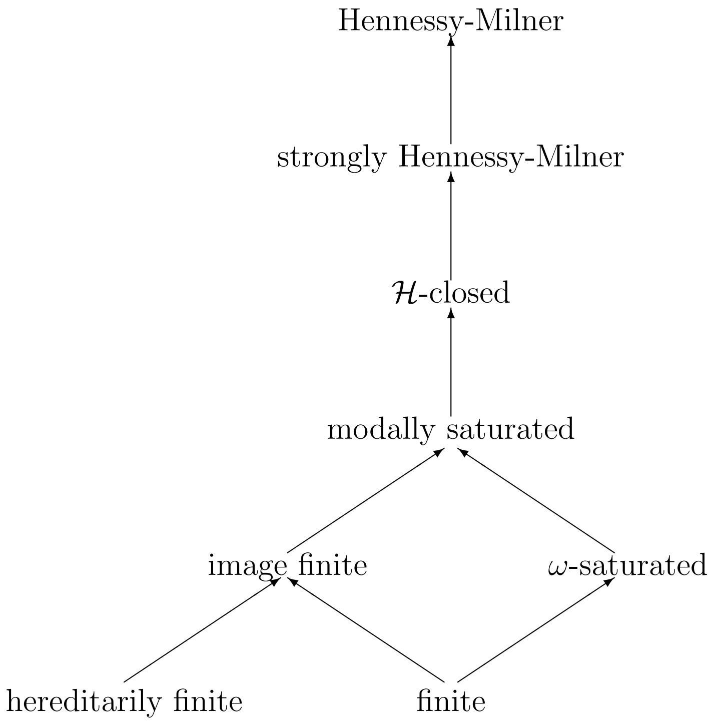


Figure 3.4: The landscape of properties for a structure

3.5.6. PROPOSITION. *Let ϑ be a SW-vocabulary, and let $\mathfrak{A}$ be a τ_ϑ-structure that is $\mathcal{H}$-closed. Then, it is a ϑ-strongly Hennessy-Milner structure.*

Proof: Assume $\langle \mathfrak{A}, a\rangle \rightsquigarrow_\vartheta \langle \mathfrak{A}, a'\rangle$. Let us prove $\langle \mathfrak{A}, a\rangle \preceq_\vartheta \langle \mathfrak{A}, a'\rangle$. Condition (qbis1) obviously holds. And by duality it is enough to prove one of the other two conditions. In order to show (qbis2)[15], let $s \in \mathsf{SMod}$ and let a'' be such that $\langle a', a''\rangle \in R_s^{\mathfrak{A}}$. So, $\mathfrak{A}, a'' \Vdash \{\varphi \in \mathcal{L}^{MOD}(\tau_\vartheta) : \mathfrak{A}, a' \Vdash [s]\varphi\}$. Hence, $\langle \mathfrak{A}, a\rangle \rightsquigarrow_\vartheta \langle \mathfrak{A}, a'\rangle$ and Corollary 3.1.7 shows that $\mathfrak{A}, a'' \Vdash \{\varphi \in \mathcal{L}^{MOD}(\tau_\vartheta) : \mathfrak{A}, a \Vdash [s]\varphi\}$. By $\mathcal{H}$-closure there exists an s-successor a''' of a such that $\langle \mathfrak{A}, a''\rangle \leftrightsquigarrow_{\tau_\vartheta} \langle \mathfrak{A}, a'''\rangle$. And using that $\mathcal{H}$-closed structures are Hennessy-Milner we conclude that $\langle \mathfrak{A}, a''\rangle \simeq_{\tau_\vartheta} \langle \mathfrak{A}, a'''\rangle$. □

3.5.7. EXAMPLE. (Strongly Hennessy-Milner, but not $\mathcal{H}$-closed). Let $\mathfrak{A}$ be the structure displayed on Figure 1.7. This structure was proposed in [Gol95, p. 116] as an example of a Hennessy-Milner structure that is not $\mathcal{H}$-closed (Hint: $\mathfrak{A}, \omega \Vdash \{\varphi \in \mathcal{L}^{MOD} : \mathfrak{A}, \omega \Vdash [m]\varphi\}$). Indeed, we can show that it is ϑ-strongly Hennessy-Milner, where ϑ is the SW-vocabulary that has a single modality, which is pure strict. Assume $\langle \mathfrak{A}, a_0\rangle \rightsquigarrow \langle \mathfrak{A}, a_1\rangle$. In order to see that $\langle \mathfrak{A}, a_0\rangle \preceq \langle \mathfrak{A}, a_1\rangle$ the only

[15] If one tries directly to justify (qbis3) one should begin by observing that if $\langle \mathfrak{A}, a\rangle \rightsquigarrow_\vartheta \langle \mathfrak{A}, a'\rangle$ and $\langle a, a''\rangle \in R_w^{\mathfrak{A}}$, then $\mathfrak{A}, a'' \Vdash \{\varphi \in \mathcal{L}^{MOD}(\tau_\vartheta) : \mathfrak{A}, a' \Vdash [w]\varphi\}$.

condition that non-trivially holds is (qbis2). So, let a_2 be such that $\langle a_1, a_2\rangle \in R$. Then, a_2 is different from ω, i.e., exists $n \in \omega$ such that $n = a_2$. Thus, $\mathfrak{A}, a_2 \Vdash \Diamond^n\Box\bot \wedge \sim\Diamond^{n+1}\Box\bot$. Hence, $\mathfrak{A}, a_1 \Vdash \Diamond(\Diamond^n\Box\bot \wedge \sim\Diamond^{n+1}\Box\bot)$, i.e., $\mathfrak{A}, a_1 \not\Vdash \Box\sim(\Diamond^n\Box\bot \wedge \sim\Diamond^{n+1}\Box\bot)$. By the fact $\langle\mathfrak{A}, a_0\rangle \rightsquigarrow \langle\mathfrak{A}, a_1\rangle$ it follows that $\mathfrak{A}, a_0 \not\Vdash \Box\sim(\Diamond^n\Box\bot \wedge \sim\Diamond^{n+1}\Box\bot)$, i.e., $\mathfrak{A}, a_0 \Vdash \Diamond(\Diamond^n\Box\bot \wedge \sim\Diamond^{n+1}\Box\bot)$, i.e., exists a_3 such that $\langle a_0, a_3\rangle \in R$ and $\mathfrak{A}, a_3 \Vdash \Diamond^n\Box\bot \wedge \sim\Diamond^{n+1}\Box\bot$. Using that n is the only state in $\mathfrak{A}$ that satisfies the formula $\Diamond^n\Box\bot \wedge \sim\Diamond^{n+1}\Box\bot$ we deduce that $a_3 = n = a_2$. Hence, $\langle\mathfrak{A}, a_3\rangle \simeq \langle\mathfrak{A}, a_2\rangle$.

Up to now we have just analyzed Definition 3.5.3 using structures, i.e., using sets of structures. In the next proposition we improve the well known result that says that the proper class of modally saturated structures is Hennessy-Milner.

3.5.8. Proposition. *Let ϑ be a SW-vocabulary, and let $\mathfrak{A}$ and $\mathfrak{B}$ be two τ_ϑ-structures that are modally saturated. Then, for every $a \in A$ and every $b \in B$,*

$$\langle\mathfrak{A}, a\rangle \preceq_\vartheta \langle\mathfrak{B}, b\rangle \quad \textit{iff} \quad \langle\mathfrak{A}, a\rangle \rightsquigarrow_\vartheta \langle\mathfrak{B}, b\rangle.$$

In other words, the class of τ_ϑ-structures that are modally saturated is a ϑ-strongly Hennessy-Milner class.

Proof: The non-trivial direction is from right to left. We must check the three properties of Definition 3.2.1. Condition (qbis1) is obviously implied by our hypothesis. And by duality it is enough to prove one of the other two conditions. In order to see (qbis2), let $s \in \mathsf{SMod}$ and let b' be such that $\langle b, b'\rangle \in R_s^{\mathfrak{B}}$.

Claim: The set $\mathrm{Th}_{\mathcal{L}^{MOD}(\tau_\vartheta)}(\mathfrak{B}, b')$ is finitely satisfiable in an s-successor of a.

Proof of Claim: Let $\Phi \subseteq_\omega \mathrm{Th}_{\mathcal{L}^{MOD}(\tau_\vartheta)}(\mathfrak{B}, b')$. Then, $\mathfrak{B}, b \not\Vdash [s]\sim\bigwedge\Phi$. And Corollary 3.1.7 says that the modal formula $[s]\sim\bigwedge\Phi$ is, up to equivalence, in $\mathcal{L}^{SW}(\vartheta)$. Using the hypothesis $\langle\mathfrak{A}, a\rangle \rightsquigarrow_\vartheta \langle\mathfrak{B}, b\rangle$ it results that $\mathfrak{A}, a \not\Vdash [s]\sim\bigwedge\Phi$. Hence, there exists a' such that $\langle a, a'\rangle \in R_s^{\mathfrak{A}}$ and $\mathfrak{A}, a' \Vdash \bigwedge\Phi$. ⊣

It follows from our claim and the modal saturation that this set is totally satisfiable in an s-successor of a. Let a' be this s-successor of a. Then, $\langle\mathfrak{A}, a'\rangle \leftrightsquigarrow_{\tau_\vartheta} \langle\mathfrak{B}, b'\rangle$. This is the same as $\langle\mathfrak{A}, a'\rangle \simeq_{\tau_\vartheta} \langle\mathfrak{B}, b'\rangle$ because the class of modally saturated structures is Hennessy-Milner. □

3.5.9. Remark. In the modal literature it has sometimes been argued that the interest of modally saturated structures comes from the Hennessy-Milner property. The last proposition suggests that the strongly Hennessy-Milner property offers a better explanation of the interest in modally saturated structures. Indeed, as far as the author is aware all Hennessy-Milner structures and all Hennessy-Milner classes of structures already introduced in the modal literature are strongly

Hennessy-Milner, e.g., $\mathcal{H}$-closed structures, all structures proposed by Hollenberg [Hol98, Section 5.4] to prove the existence of a continuum of maximal Hennessy-Milner classes[16], etc.

We now have the tools needed to prove Theorem 3.5.11. It is time to do so. First of all, we give a simple proof of the Detour Lemma. This lemma solves the crucial step in the proof of Theorem 3.5.11.

3.5.10. LEMMA (DETOUR). *Let ϑ be a SW-vocabulary, and let $\langle\mathfrak{A},a\rangle$ and $\langle\mathfrak{B},b\rangle$ be two τ_ϑ-pointed structures. Then, the following are equivalent:*

1. $\langle\mathfrak{A},a\rangle \rightsquigarrow \langle\mathfrak{B},b\rangle$.
2. $\langle \mathfrak{ue}\,\mathfrak{A},\pi_a\rangle \preceq \langle \mathfrak{ue}\,\mathfrak{B},\pi_b\rangle$.
3. *There exist ultrapowers $\Pi_{U_1}\langle\mathfrak{A},a\rangle$ and $\Pi_{U_2}\langle\mathfrak{B},b\rangle$ such that $\Pi_{U_1}\langle\mathfrak{A},a\rangle \preceq \Pi_{U_2}\langle\mathfrak{B},b\rangle$.*

Proof: $(2 \Rightarrow 1)$: Because $\langle \mathfrak{ue}\,\mathfrak{A},\pi_a\rangle \leftrightsquigarrow \langle\mathfrak{A},a\rangle$ and $\langle \mathfrak{ue}\,\mathfrak{B},\pi_b\rangle \leftrightsquigarrow \langle\mathfrak{B},b\rangle$.

$(3 \Rightarrow 1)$: We know that ultrapowers yield elementary equivalent structures. In particular this means that the ultrapower must satisfy the same modal formulas, i.e., $\Pi_{U_1}\langle\mathfrak{A},a\rangle \leftrightsquigarrow \langle\mathfrak{A},a\rangle$ and $\Pi_{U_2}\langle\mathfrak{B},b\rangle \leftrightsquigarrow \langle\mathfrak{B},b\rangle$.

$(1 \Rightarrow 2)$: As a consequence of Proposition 3.5.8 and the fact that ultrafilter extensions are modally saturated it is enough to see that $\langle \mathfrak{ue}\,\mathfrak{A},\pi_a\rangle \rightsquigarrow \langle \mathfrak{ue}\,\mathfrak{B},\pi_b\rangle$. This is an immediate consequence of the fact that $\langle \mathfrak{ue}\,\mathfrak{A},\pi_a\rangle \leftrightsquigarrow \langle\mathfrak{A},a\rangle$ and $\langle \mathfrak{ue}\,\mathfrak{B},\pi_b\rangle \leftrightsquigarrow \langle\mathfrak{B},b\rangle$.

$(1 \Rightarrow 3)$: It is known that all structures can be extended to an ω-saturated elementary extension, which can be built as an ultrapower of the original one. Let $\Pi_{U_1}\langle\mathfrak{A},a\rangle$ and $\Pi_{U_2}\langle\mathfrak{B},b\rangle$ be ω-saturated elementary extensions of, respectively, $\langle\mathfrak{A},a\rangle$ and $\langle\mathfrak{B},b\rangle$. By Proposition 3.5.8 it is enough to see that $\Pi_{U_1}\langle\mathfrak{A},a\rangle \rightsquigarrow \Pi_{U_2}\langle\mathfrak{B},b\rangle$. And this is obvious from the fact that $\Pi_{U_1}\langle\mathfrak{A},a\rangle \leftrightsquigarrow \langle\mathfrak{A},a\rangle$ and $\Pi_{U_2}\langle\mathfrak{B},b\rangle \leftrightsquigarrow \langle\mathfrak{B},b\rangle$. □

3.5.11. THEOREM. *Let ϑ be a SW-vocabulary, let $\varphi(v_0)$ be a $\mathcal{L}^{FO}(\tau_\vartheta)$-formula, and let K be a class of τ_ϑ-structures closed under ultraproducts. The following are equivalent:*

1. *φ is, up to K-equivalence, in $\mathcal{L}^{SW}(\vartheta)$.*
2. *There exists $n \in \omega$ such that φ is K-preserved under quasi n-bisimilarity.*

[16] A similar argument to the ones exhibited in the previous examples to check the strongly Hennessy-Milner property can be used to prove this.

3. φ is K*-preserved under quasi bisimilarity.*

Proof: $(1 \Rightarrow 2 \Rightarrow 3)$: They are obtained by previous results.

$(3 \Rightarrow 1)$: Assume that $\varphi(v_0)$ is preserved under quasi bisimilarity. Consider the set

$$SWC(\varphi) := \{ST_{v_0}(\phi) : \phi \in \mathcal{L}^{SW}(\vartheta),\ \varphi \models_{\mathsf{K}} ST_{v_0}(\phi)\}.$$

As K is closed under ultraproducts it is known (see [CK90, Corollary 4.1.11]) that $\models_{\mathsf{K}}$ is finitary. By a compactness argument it is enough to show that $SWC(\varphi) \models_{\mathsf{K}} \varphi$; for then a finite subset of $SWC(\varphi)$ will already imply φ, and φ will be K-equivalent to the conjunction of the formulas in this finite subset.

To prove that $SWC(\varphi) \models_{\mathsf{K}} \varphi$, assume that $\mathfrak{A} \in \mathsf{K}$ and $\mathfrak{A} \models SWC(\varphi)[\![a]\!]$; we have to show that $\mathfrak{A} \models \varphi[\![a]\!]$. Let

$$T(v_0) := \{\sim ST_{v_0}(\phi) : \phi \in \mathcal{L}^{SW}(\vartheta),\ \mathfrak{A}, a \nVdash \phi\}.$$

Claim: The set $T \cup \{\varphi\}$ is K-satisfiable.

Proof of Claim: Assume that it is not. Then there are formulas $\phi_0, \ldots, \phi_{n-1} \in \mathcal{L}^{SW}(\vartheta)$ such that $\mathfrak{A}, a \nVdash \phi_0 \vee \ldots \vee \phi_{n-1}$ and $\varphi \models_{\mathsf{K}} ST_{v_0}(\phi_0 \vee \ldots \vee \phi_{n-1})$. By definition, $\phi_0 \vee \ldots \vee \phi_{n-1} \in SWC(\varphi)$. But this implies $\mathfrak{A} \models ST_{v_0}(\phi_0 \vee \ldots \vee \phi_{n-1})[\![a]\!]$, i.e., $\mathfrak{A}, a \Vdash \phi_0 \vee \ldots \vee \phi_{n-1}$, which is a contradiction. ⊣

Let $\langle \mathfrak{B}, b\rangle$ be such that $\mathfrak{B} \in \mathsf{K}$ and $\mathfrak{B} \models T \cup \{\varphi\}[\![b]\!]$. The fact that $\mathfrak{B} \models T[\![b]\!]$ implies that $\langle \mathfrak{B}, b\rangle \rightsquigarrow \langle \mathfrak{A}, a\rangle$. The Detour Lemma yields two ultrapowers $\Pi_{U_1}\langle \mathfrak{B}, b\rangle$ and $\Pi_{U_2}\langle \mathfrak{A}, a\rangle$ such that $\Pi_{U_1}\langle \mathfrak{B}, b\rangle \preceq \Pi_{U_2}\langle \mathfrak{A}, a\rangle$. Then $\langle \Pi_{U_1}\mathfrak{B}, \bar{b}\rangle \preceq \langle \Pi_{U_2}\mathfrak{A}, \bar{a}\rangle$ by the definition of ultrapower. It follows from the closure under ultraproducts that $\Pi_{U_1}\mathfrak{B} \in \mathsf{K}$ and $\Pi_{U_2}\mathfrak{A} \in \mathsf{K}$. By the elementary equivalence of a structure with their ultrapowers it holds that $\Pi_{U_1}\mathfrak{B} \models \varphi[\![\bar{b}]\!]$. So by the K-preservation of φ under quasi bisimilarity we obtain that $\Pi_{U_2}\mathfrak{A} \models \varphi[\![\bar{a}]\!]$. And this allows us to conclude $\mathfrak{A} \models \varphi[\![a]\!]$ by using the elementary equivalence between a structure and their ultrapowers. This completes the proof. □

Now we single out two interesting remarks that can be obtained with the tools used in the proof of the previous theorem. As far as the author knows, they have never been published.

3.5.12. Remark. (Minimal conditions for a preservation theorem)

Let us say that a set F of first-order formulas with at most one free variable *is characterized by a preservation theorem* whenever there exists a certain binary relation E between pointed structures such that for all first-order formulas $\varphi(v_0)$,

φ is equivalent to a formula in F iff φ is preserved under the relation E.

Then, it is not hard to prove that for every set F of first-order formulas with at most one free variable, the following two statements are equivalent:

1. F is characterized by a preservation theorem.

2. F is closed, up to equivalence, under $\bot, \top, \wedge, \vee$.[17]

One direction of the proof is trivial. For the other, reason as in the proof of Theorem 3.5.11 considering this time the binary relation $E := \{\langle\langle\mathfrak{A}, a\rangle, \langle\mathfrak{B}, b\rangle\rangle :$ for all $\varphi \in F$, $\mathfrak{A} \models \varphi[\![a]\!]$ implies $\mathfrak{B} \models \varphi[\![b]\!]\}$. Now the proof is even simpler than before, e.g., it is not necessary to consider the ultrapowers.

Here several interesting and open questions arise naturally[18]. The relation E defined in the proof is not in general decidable (between finite structures); is it always possible to replace this relation with a decidable (or recursively enumerable) one? If not, is there any characterization of the first-order fragments that admits a preservation theorem based on a decidable relation?

3.5.13. Remark. (Minimal conditions for an invariance theorem)
Let us say that a set F of first-order formulas with at most one free variable *is characterized by an invariance theorem* whenever there exists a certain binary relation E between pointed structures such that for all first-order formulas $\varphi(v_0)$,

φ is equivalent to a formula in F iff φ is invariant under the relation E.

Then, for every set F of first-order formulas with at most one free variable, the following two statements are equivalent:

1. F is characterized by an invariance theorem.

2. F is closed, up to equivalence, under $\top, \wedge, \sim$.

This time the proof is based on the relation $E := \{\langle\langle\mathfrak{A}, a\rangle, \langle\mathfrak{B}, b\rangle\rangle :$ for all $\varphi \in F$, $\mathfrak{A} \models \varphi[\![a]\!]$ iff $\mathfrak{B} \models \varphi[\![b]\!]\}$.

From now to the end of the section we will analyze strongly Hennessy-Milner classes. They are of interest mainly because this notion allows us to characterize (modulo bisimilarity) hereditarily finite structures. Hollenberg noticed that Hennessy-Milner classes are closed under B and S (see Theorem 1.3.18). And it is also trivial to see that strongly Hennessy-Milner classes are also closed under these operators. Hence, if K is a strongly Hennessy-Milner class then $\mathsf{B}\,\mathsf{S}\,(\mathsf{K})$ is also strongly Hennessy-Milner.

3.5.14. Proposition. *Let ϑ be a SW-vocabulary. Every ϑ-strongly Hennessy-Milner class of τ_ϑ-structures can be extended to a maximal ϑ-strongly Hennessy-Milner class.*

[17]For $\wedge$ this means that if $\varphi_0, \varphi_1 \in F$ then there exists $\varphi \in F$ such that $\varphi_0 \wedge \varphi_1$ and φ are equivalent.

[18]The author thanks Johan van Benthem for discussions about this topic.

Proof: Let X be the class $\{\mathfrak{A} :$ there is $\mathfrak{B}$ sharing universe and valuation with the canonical structure such that $\mathfrak{A} \rightarrowtail \mathfrak{B}\}$. It is clear that X is a set, i.e., $X = \{\mathfrak{A}_\alpha : \alpha < \kappa\}$ for a certain $\kappa \in \mathbb{C}$ARD. And by Theorem 1.3.18(3) we know that for each Hennessy-Milner structure there is an ordinal $\alpha < \kappa$ such that this structure is in $\mathsf{B}(\{\mathfrak{A}_\alpha\})$. In particular this holds for strongly Hennessy-Milner structures.

We assume K is a strongly Hennessy-Milner class, and we show that there is a maximal strongly Hennessy-Milner class extending it. We define $\langle Y_\alpha : \alpha < \kappa \rangle$ by induction on α as the following sequence of families of structures:

1. $Y_0 := \emptyset$,
2. $Y_{\alpha+1} := \begin{cases} Y_\alpha \cup \{\mathfrak{A}_\alpha\} & \text{if } \mathsf{K} \cup Y_\alpha \cup \{\mathfrak{A}_\alpha\} \text{ is strongly Hennessy Milner,} \\ Y_\alpha & \text{if not,} \end{cases}$
3. $Y_\alpha := \bigcup_{\beta<\alpha} Y_\beta$, if α is a limit ordinal.

By induction it can be easily proved that for every $\alpha < \kappa$, $\mathsf{K} \cup Y_\alpha$ is strongly Hennessy Milner. Thus, $\mathsf{K} \cup \bigcup_{\alpha<\kappa} Y_\alpha$ is strongly Hennessy-Milner. Hence $\mathsf{K}' := \mathsf{B}(\mathsf{K} \cup \bigcup_{\alpha<\kappa} Y_\alpha)$ is a strongly Hennessy-Milner class extending K. Let us show that K' is a maximal strongly Hennessy-Milner class. Assume $\mathfrak{A}$ is a structure such that $\mathsf{K}' \cup \{\mathfrak{A}\}$ is strongly Hennessy-Milner, and let us show that $\mathfrak{A} \in \mathsf{K}'$. As $\mathfrak{A}$ is a strongly Hennessy-Milner structure there is $\alpha < \kappa$ such that $\mathfrak{A} \in \mathsf{B}(\{\mathfrak{A}_\alpha\})$, i.e., $\mathfrak{A} \simeq \mathfrak{A}_\alpha$. Hence $\mathsf{K}' \cup \{\mathfrak{A}_\alpha\}$ is strongly Hennessy-Milner. As $\mathsf{K} \cup Y_\alpha \cup \{\mathfrak{A}_\alpha\} \subseteq \mathsf{K}' \cup \{\mathfrak{A}_\alpha\}$ we know that $\mathsf{K} \cup Y_\alpha \cup \{\mathfrak{A}_\alpha\}$ is strongly Hennessy-Milner. By construction this means that $\mathfrak{A}_\alpha \in Y_{\alpha+1}$. Therefore $\mathfrak{A} \in \mathsf{B}(\{\mathfrak{A}_\alpha\}) \subseteq \mathsf{K}'$. □

3.5.15. Proposition. *Let ϑ be a SW-vocabulary that has pure modalities (i.e., $\mathsf{SMod} \neq \mathsf{WMod}$). Then, (i) there is a maximal Hennessy-Milner class that is not ϑ-strongly Hennessy-Milner, and (ii) there is a maximal ϑ-strongly Hennessy-Milner class that is not a maximal Hennessy-Milner class.*

Proof: We restrict ourselves in the proof to the case that there is a single modality, which is pure strict. Slight changes in this argument give a proof of the general statement.

We saw in Example 3.5.4 that the class $\{\mathfrak{A}, \mathfrak{B}\}$ introduced in Figure 3.3 is Hennessy-Milner but not strongly Hennessy-Milner. By Theorem 1.3.18(2) let K_0 be any maximal Hennessy-Milner class extending $\{\mathfrak{A}, \mathfrak{B}\}$. Then K_0 is a maximal Hennessy-Milner class that is not ϑ-strongly Hennessy-Milner.

We also saw that $\mathfrak{A}$ is strongly Hennessy-Milner. By Proposition 3.5.14 there exists a maximal strongly Hennessy-Milner class K_1 containing $\mathfrak{A}$. By the fact that $\{\mathfrak{A}, \mathfrak{B}\}$ is not strongly Hennessy-Milner we know that $\mathfrak{B} \notin \mathsf{K}_1$. But $\mathfrak{B} \in \bigcap\{\mathsf{K} : \mathsf{K} \text{ is a maximal Hennessy-Milner class}\}$ because $\mathfrak{B} = \mathfrak{c}oll\,\mathfrak{B}$ is image finite (see Theorem 1.3.19). Hence K_1 is not a maximal Hennessy-Milner class. □

By Remark 3.5.5 it is trivial that the requirement of the previous proposition is necessary. While maximal Hennessy-Milner structures were neatly characterized by Hollenberg using canonical-like structures, this proposition tells us that it does not seem plausible to characterize strongly maximal Hennessy-Milner classes using these structures. It remains as an open problem how to obtain a characterization of the maximal strongly Hennessy-Milner classes. As was pointed out in Remark 3.5.9 it is not hard to see that there exist at least $2^{\aleph_0}$ different maximal strongly Hennessy-Milner classes.

3.5.16. THEOREM. *Let ϑ be a finite SW-vocabulary, and let $\mathfrak{A}$ be a τ_ϑ-structure. The following are equivalent:*

1. $\mathfrak{A} \in \bigcap\{\mathsf{K} : \mathsf{K}$ *is a maximal strongly Hennessy-Milner class*$\}$.
2. *For every state $a \in A$, and every pointed structure $\langle \mathfrak{B}, b\rangle$, it holds that*
   - $\langle \mathfrak{A}, a\rangle \rightsquigarrow \langle \mathfrak{B}, b\rangle$ *implies* $\langle \mathfrak{A}, a\rangle \preceq \langle \mathfrak{B}, b\rangle$,
   - $\langle \mathfrak{B}, b\rangle \rightsquigarrow \langle \mathfrak{A}, a\rangle$ *implies* $\langle \mathfrak{B}, b\rangle \preceq \langle \mathfrak{A}, a\rangle$.
3. $\mathfrak{coll}\,\mathfrak{A}$ *satisfies that (i) it is image finite, and (ii) there is no infinite R-increasing sequence of states starting by a modality that is pure (strict or weak).*
4. *There exists $\mathfrak{B} \simeq \mathfrak{A}$ such that $\mathfrak{B}$ satisfies that (i) it is image finite, and (ii) there is no infinite R-increasing sequence of states starting by a modality that is pure (strict or weak).*

Proof: $(1 \Rightarrow 2)$: Suppose $a \in A$, and $\langle \mathfrak{B}, b\rangle$ is a pointed structure. By Proposition 3.5.14 let K be a maximal strongly Hennessy-Milner class containing $\mathfrak{B}$. Then $\mathfrak{A} \in \mathsf{K}$. Thus, $\{\mathfrak{A}, \mathfrak{B}\}$ is a strongly Hennessy-Milner class. And this implies what we wanted to prove.

$(2 \Rightarrow 3)$: In particular we know that for every state $a \in A$, and every pointed structure $\langle \mathfrak{B}, b\rangle$, it holds that $\langle \mathfrak{A}, a\rangle \leftrightsquigarrow \langle \mathfrak{B}, b\rangle$ implies $\langle \mathfrak{A}, a\rangle \simeq \langle \mathfrak{B}, b\rangle$. By Theorem 1.3.19 it is known that $\mathfrak{coll}\,\mathfrak{A}$ is image finite. So it only remains to prove that in $\mathfrak{coll}\,\mathfrak{A}$ there is no infinite R-increasing sequence of states starting by a modality that is pure (strict or weak). If not, there exists in $\mathfrak{coll}\,\mathfrak{A}$ an infinite sequence of states such that $[a_0]\ R_{m_0}\ [a_1]\ R_{m_1}\ [a_2]\ R_{m_2} \ldots$ such that m_0 is a pure modality (strict or weak). Clearly this property is preserved under bisimilarity. Thus in $\langle \mathfrak{C}, c_0\rangle := \mathfrak{unr}(\mathfrak{A}, a_0)$ we know that there exists an infinite sequence of states such that $c_0\ R_{m_0}\ c_1\ R_{m_1}\ c_2\ R_{m_2} \ldots$ Deleting in the tree $\mathfrak{C}$ the duplicated (modulo bisimilarity) states we assume that c_1 is the only m_0-successor of c_0 that is bisimilar to c_1. For every $n \in \omega$, let $\langle \mathfrak{T}^{\mathfrak{n}}, c_1^n\rangle$ be the result of cutting at height n the generated subtree of $\mathfrak{C}$ with root c_1. Hence, for every $n \in \omega$, $\langle \mathfrak{C}, c_1\rangle \simeq_n \langle \mathfrak{T}^{\mathfrak{n}}, c_1^n\rangle$ and $\langle \mathfrak{C}, c_1\rangle \not\simeq \langle \mathfrak{T}^{\mathfrak{n}}, c_1^n\rangle$. Let $\mathfrak{B}$ be the tree that results when replacing in $\mathfrak{C}$ the subtree of root c_1 with a copy of $\langle \mathfrak{T}^n, c_1^n \rangle$ for each $n \in \omega$. We also impose in $\mathfrak{B}$ that each c_1^n is a m_0-successor of c_0. Now we distinguish two cases.

Case m_0 is pure strict: In this case it is clear that $\langle \mathfrak{B}, c_0 \rangle \preceq_\omega \langle \mathfrak{C}, c_0 \rangle$. Hence $\langle \mathfrak{B}, c_0 \rangle \rightsquigarrow \langle \mathfrak{C}, c_0 \rangle$. But, $\langle \mathfrak{B}, c_0 \rangle \not\preceq \langle \mathfrak{C}, c_0 \rangle$ by construction. By definition of $\langle \mathfrak{C}, c_0 \rangle$ what we have just seen is that $\langle \mathfrak{B}, c_0 \rangle \rightsquigarrow \langle \mathfrak{A}, a_0 \rangle$ while $\langle \mathfrak{B}, c_0 \rangle \not\preceq \langle \mathfrak{A}, a_0 \rangle$. And this contradicts the hypothesis of the implication that we are proving.

Case m_0 is pure weak: In this occasion we have that $\langle \mathfrak{C}, c_0 \rangle \preceq_\omega \langle \mathfrak{B}, c_0 \rangle$. Hence $\langle \mathfrak{C}, c_0 \rangle \rightsquigarrow \langle \mathfrak{B}, c_0 \rangle$. But, $\langle \mathfrak{C}, c_0 \rangle \not\preceq \langle \mathfrak{B}, c_0 \rangle$ by construction. By definition of $\langle \mathfrak{C}, c_0 \rangle$ what we have just seen is that $\langle \mathfrak{A}, a_0 \rangle \rightsquigarrow \langle \mathfrak{B}, c_0 \rangle$ while $\langle \mathfrak{A}, a_0 \rangle \not\preceq \langle \mathfrak{B}, c_0 \rangle$, which is a contradiction.

We have thus completed the proof of this implication.

$(3 \Rightarrow 4)$: Trivial.

$(4 \Rightarrow 1)$: We assume K is a maximal strongly Hennessy-Milner class, and we show that $\mathfrak{A} \in \mathsf{K}$. By maximality K is closed under bisimilarity. Thus, it is enough to see that $\mathfrak{B} \in \mathsf{K}$. So by maximality we only need to prove that $\mathsf{K} \cup \{\mathfrak{B}\}$ is strongly Hennessy-Milner. Therefore we must show that for every $\mathfrak{C} \in \mathsf{K}$, $c \in C$ and $b \in B$ it holds that (i) $\langle \mathfrak{B}, b \rangle \rightsquigarrow \langle \mathfrak{C}, c \rangle$ implies $\langle \mathfrak{B}, b \rangle \preceq \langle \mathfrak{C}, c \rangle$, and (ii) $\langle \mathfrak{C}, c \rangle \rightsquigarrow \langle \mathfrak{B}, b \rangle$ implies $\langle \mathfrak{C}, c \rangle \preceq \langle \mathfrak{B}, b \rangle$. We focus on the first of these conditions; the other one can be proved similarly. Let us assume $\mathfrak{C} \in \mathsf{K}$, $c \in C$, $b \in B$ and $\langle \mathfrak{B}, b \rangle \rightsquigarrow \langle \mathfrak{C}, c \rangle$. Condition (qbis1) clearly holds. Let us now see that (qbis2) holds. Suppose $s \in \mathsf{SMod}$ and c' is such that $\langle c, c' \rangle \in R_s^{\mathfrak{C}}$. As $R_s^{\mathfrak{B}}[\{b\}]$ is a finite set we assume it is $\{b_0, \ldots, b_{n-1}\}$. We distinguish two cases.

Case s is pure strict: The generated substructures of $\mathfrak{B}$ generated by the b_i's are clearly hereditarily finite. By Theorem 1.3.15 there are modal formulas $\phi_0, \ldots, \phi_{n-1}$ that characterize up to bisimilarity each one of the s-successors of b. It holds that $\mathfrak{B}, b \Vdash [s](\phi_0 \vee \ldots \vee \phi_{n-1})$. By $\langle \mathfrak{B}, b \rangle \rightsquigarrow \langle \mathfrak{C}, c \rangle$ we obtain that $\mathfrak{C}, c \Vdash [s](\phi_0 \vee \ldots \vee \phi_{n-1})$. Hence $\mathfrak{C}, c' \Vdash \phi_0 \vee \ldots \vee \phi_{n-1}$, i.e., there exists $i < n$ such that $\langle \mathfrak{B}, b_i \rangle \simeq \langle \mathfrak{C}, c' \rangle$.

Case s is not pure strict: Then $s \in \mathsf{SMod} \cap \mathsf{WMod}$. In this case we only know that the generated substructures of $\mathfrak{B}$ generated by the b_i's are image finite. Hence, by Theorem 1.3.13 there are sets of modal formulas $\Phi_0, \ldots, \Phi_{n-1}$ characterizing up to bisimilarity each one of the s-successors of b. For every $i, j < n$ such that $\langle \mathfrak{B}, b_i \rangle \not\leftrightsquigarrow \langle \mathfrak{B}, b_j \rangle$, we choose a modal formula ϕ_i^j such that $\mathfrak{B}, b_i \Vdash \phi_i^j$ and $\mathfrak{B}, b_j \nVdash \phi_i^j$. Define $\phi_i := \bigwedge\{\phi_i^j : j < n, \langle \mathfrak{B}, b_i \rangle \not\leftrightsquigarrow \langle \mathfrak{B}, b_j \rangle\}$ for every $i < n$. Now, for every $i, j < n$, it holds that $\mathfrak{B}, b_j \Vdash \phi_i$ iff $\langle \mathfrak{B}, b_i \rangle \leftrightsquigarrow \langle \mathfrak{B}, b_j \rangle$. And it also holds that $\mathfrak{B}, b \Vdash [s](\phi_0 \vee \ldots \vee \phi_{n-1})$. By $\langle \mathfrak{B}, b \rangle \rightsquigarrow \langle \mathfrak{C}, c \rangle$ we obtain that $\mathfrak{C}, c \Vdash [s](\phi_0 \vee \ldots \vee \phi_{n-1})$. Hence $\mathfrak{C}, c' \Vdash \phi_0 \vee \ldots \vee \phi_{n-1}$, i.e., there exists $i < n$ such that $\mathfrak{C}, c' \Vdash \phi_i$.

CLAIM: $\langle \mathfrak{B}, b_i \rangle \leftrightsquigarrow \langle \mathfrak{C}, c' \rangle$.

Proof of Claim: It is enough to see that the modal formulas holding in $\langle \mathfrak{C}, c' \rangle$ also hold in $\langle \mathfrak{B}, b_i \rangle$. Assume $\mathfrak{C}, c' \Vdash \varphi$. Then $\mathfrak{C}, c \nVdash [s] \sim(\varphi \wedge \phi_i)$. By $\langle \mathfrak{B}, b \rangle \rightsquigarrow \langle \mathfrak{C}, c \rangle$ we deduce that $\mathfrak{B}, b \nVdash [s] \sim(\varphi \wedge \phi_i)$. Hence there exists $j < n$ such that $\mathfrak{B}, b_j \Vdash \varphi \wedge \phi_i$. So $\mathfrak{B}, b_j \Vdash \varphi$ and $\langle \mathfrak{B}, b_i \rangle \leftrightsquigarrow \langle \mathfrak{B}, b_j \rangle$. Thus $\mathfrak{B}, b_i \Vdash \varphi$. $\dashv$

As a consequence of the claim and the fact that bisimilarity with $\langle \mathfrak{B}, b_i \rangle$ is characterized by the set Φ_i we know that $\langle \mathfrak{B}, b_i \rangle \simeq \langle \mathfrak{C}, c' \rangle$.

We have just proved condition (qbis2). Finally, it is time to check (qbis3). The argument is very close to the one exhibited for (qbis2). Suppose $w \in \mathsf{WMod}$ and b' is such that $\langle b, b' \rangle \in R_w^{\mathfrak{B}}$. As $R_w^{\mathfrak{B}}[\{b\}]$ is a finite set we assume it is $\{b_0, \ldots, b_{n-1}\}$ where $b' = b_0$. We distinguish two cases.

Case w is pure weak**:** The generated substructure of $\mathfrak{B}$ generated by the b_0 is clearly hereditarily finite. By Theorem 1.3.15 there is a modal formula ϕ that characterizes up to bisimilarity $\langle \mathfrak{B}, b_0 \rangle$. It holds that $\mathfrak{B}, b \Vdash \langle w \rangle \phi$. By $\langle \mathfrak{B}, b \rangle \rightsquigarrow \langle \mathfrak{C}, c \rangle$ we obtain that $\mathfrak{C}, c \Vdash \langle w \rangle \phi$. Hence there exists c' such that $\langle c, c' \rangle \in R_w^{\mathfrak{C}}$ and $\mathfrak{C}, c' \Vdash \phi$. Therefore $\langle \mathfrak{B}, b_0 \rangle \simeq \langle \mathfrak{C}, c' \rangle$.

Case w is not pure weak**:** Then $w \in \mathsf{SMod} \cap \mathsf{WMod}$. In this case we only know that the substructure of $\mathfrak{B}$ generated by b_0 is image finite. Hence, by Theorem 1.3.13 there exists a set of modal formulas Φ characterizing up to bisimilarity $\langle \mathfrak{B}, b_0 \rangle$. For every $i < n$ such that $\langle \mathfrak{B}, b_0 \rangle \not\leftrightsquigarrow \langle \mathfrak{B}, b_i \rangle$, we choose a modal formula ϕ_i such that $\mathfrak{B}, b_0 \Vdash \phi_i$ and $\mathfrak{B}, b_i \nVdash \phi_i$. Define $\phi := \bigwedge\{\phi_i : i < n, \langle \mathfrak{B}, b_0 \rangle \not\leftrightsquigarrow \langle \mathfrak{B}, b_i \rangle\}$. By construction, for every $i < n$ it holds that $\mathfrak{B}, b_i \Vdash \phi$ iff $\langle \mathfrak{B}, b_0 \rangle \leftrightsquigarrow \langle \mathfrak{B}, b_i \rangle$. And it also holds that $\mathfrak{B}, b \Vdash \langle w \rangle \phi$. By $\langle \mathfrak{B}, b \rangle \rightsquigarrow \langle \mathfrak{C}, c \rangle$ we obtain that $\mathfrak{C}, c \Vdash \langle w \rangle \phi$. Hence there exists c' such that $\langle c, c' \rangle \in R_w^{\mathfrak{C}}$ and $\mathfrak{C}, c' \Vdash \phi$.

CLAIM: $\langle \mathfrak{B}, b_0 \rangle \leftrightsquigarrow \langle \mathfrak{C}, c' \rangle$.

Proof of Claim: It is enough to see that the modal formulas holding in $\langle \mathfrak{C}, c' \rangle$ also holds in $\langle \mathfrak{B}, b_0 \rangle$. Assume $\mathfrak{C}, c' \Vdash \varphi$. Then $\mathfrak{C}, c \nVdash [w] \sim(\varphi \wedge \phi)$. By $\langle \mathfrak{B}, b \rangle \rightsquigarrow \langle \mathfrak{C}, c \rangle$ and the fact that $w \in \mathsf{SMod}$ we deduce that $\mathfrak{B}, b \nVdash [w] \sim(\varphi \wedge \phi)$. Hence there exists $i < n$ such that $\mathfrak{B}, b_i \Vdash \varphi \wedge \phi$. So $\mathfrak{B}, b_i \Vdash \varphi$ and $\langle \mathfrak{B}, b_0 \rangle \leftrightsquigarrow \langle \mathfrak{B}, b_i \rangle$. Thus $\mathfrak{B}, b_0 \Vdash \varphi$. $\dashv$

As a consequence of the claim and the fact that bisimilarity with $\langle \mathfrak{B}, b_0 \rangle$ is characterized by the set Φ we know that $\langle \mathfrak{B}, b_0 \rangle \simeq \langle \mathfrak{C}, c' \rangle$.

□

We emphasize that when $\mathsf{SMod} = \mathsf{WMod}$ the hypothesis of finiteness is not needed in the proof.

3.5.17. Corollary. *Let ϑ be a SW-vocabulary such that* $\mathsf{SMod} = \mathsf{WMod}$, *and let* $\mathfrak{A}$ *be a* τ_ϑ*-structure. The following are equivalent:*

1. $\mathfrak{A} \in \bigcap\{\mathsf{K} : \mathsf{K}$ *is a maximal strongly Hennessy-Milner class*$\}$.

2. $\mathfrak{coll}\,\mathfrak{A}$ *is image finite.*

3. *For every* $a \in A$, $\langle \mathfrak{A}, a\rangle$ *is characterized (up to bisimilarity) by a set of* $\mathcal{L}^{MOD}(\tau_\vartheta)$*-formulas.*

Proof: The first two items are equivalent by the previous theorem. By Theorem 1.3.13 we obtain the equivalence between the last two items.

A different proof is based on Remark 3.5.5. Bearing this in mind it is clear that this theorem is only a reformulation of Theorem 1.3.19. □

3.5.18. Corollary. *Let ϑ be a finite SW-vocabulary such that* $\mathsf{SMod} \cap \mathsf{WMod}$ *is empty, and let* $\mathfrak{A}$ *be a* τ_ϑ*-structure. The following are equivalent:*

1. $\mathfrak{A} \in \bigcap\{\mathsf{K} : \mathsf{K}$ *is a maximal strongly Hennessy-Milner class*$\}$.

2. $\mathfrak{coll}\,\mathfrak{A}$ *is hereditarily finite.*

3. *For every* $a \in A$, $\langle \mathfrak{A}, a\rangle$ *is characterized (up to bisimilarity) by a single* $\mathcal{L}^{MOD}(\tau_\vartheta)$*-formula.*

Proof: The first two items are equivalent by the previous theorem. For the last two items apply Theorem 1.3.15. □

3.5.19. Remark. Let us see that finiteness cannot be eliminated from the hypothesis. Suppose for instance that $\mathsf{Prop} = \{p_n : n \in \omega\}$, SMod is empty and WMod is a singleton. Let $\mathfrak{A}$ be the structure with universe $A := \{a, a'\}$, accessibility relation $R := \{\langle a, a'\rangle\}$, and valuation $p_n \longmapsto \emptyset$. Then $\mathfrak{A}$, which is isomorphic to $\mathfrak{coll}\,\mathfrak{A}$, is image finite. And let $\mathfrak{B}$ be the structure with universe $B := \{a\} \cup \{a'_n : n \in \omega\}$, accessibility relation $R := \{\langle a, a'_n\rangle : n \in \omega\}$, and valuation $p_n \longmapsto \{a'_n\}$. It is easy to see that $\langle \mathfrak{A}, a\rangle \rightsquigarrow \langle \mathfrak{B}, b\rangle$ while $\langle \mathfrak{A}, a\rangle \not\preceq \langle \mathfrak{B}, b\rangle$. Hence, $\mathfrak{A} \notin \bigcap\{\mathsf{K} : \mathsf{K}$ is a maximal strongly Hennessy-Milner class$\}$. It remains as an open question to characterize $\bigcap\{\mathsf{K} : \mathsf{K}$ is a maximal strongly Hennessy-Milner class$\}$ when ϑ is infinite (maybe the non finiteness comes from modalities and not from propositions). The conjecture of the author is that the structures in this intersection are the ones with empty global accessibility relation, but up to now a proof has not been found.

3.6 Definability

The aim of this section is to obtain the definability results corresponding to the strict-weak languages.

3.6.1. THEOREM. *Let ϑ be a SW-vocabulary, and let K be a class of pointed τ_ϑ-structures. The following are equivalent:*

1. *K is definable by a set of $\mathcal{L}^{SW}(\vartheta)$-formulas.*
2. *K is closed under quasi bisimilarity and under ultraproducts, and its complementary class is closed under ultrapowers.*

Proof: The implication ($1 \Rightarrow 2$) is trivial by Proposition 3.2.2. For the converse, assume K satisfies the stated closure conditions. Define T as the set of strict-weak formulas holding in K, i.e.,

$$T = \{\varphi \in \mathcal{L}^{SW}(\vartheta) : \mathfrak{A}, a \Vdash \varphi \text{ for all } \langle \mathfrak{A}, a\rangle \in \mathsf{K}\}.$$

We will show that T defines the class K. First of all, by definition every pointed structure in K is a model satisfying T. Second, to complete the proof we assume that $\mathfrak{A}, a \Vdash T$ and we show that $\langle \mathfrak{A}, a\rangle$ must be in K. Let $\mathfrak{A}, a \Vdash T$, and let Σ be $\{\sim\varphi : \varphi \in \mathcal{L}^{SW}(\vartheta),\ \mathfrak{A}, a \nVdash \varphi\}$.

CLAIM: The set Σ is satisfiable in an ultraproduct of pointed structures in K.

Proof of Claim: By [CK90, Corollary 4.1.11] it is enough to see that Σ is finitely satisfiable in K. Suppose that $\varphi_0, \ldots, \varphi_{n-1}$ are formulas in $\mathcal{L}^{SW}(\vartheta)$ such that $\mathfrak{A}, a \nVdash \varphi_0 \vee \ldots \vee \varphi_{n-1}$ and $\{\sim\varphi_0, \ldots, \sim\varphi_{n-1}\}$ is not satisfiable in K. The last part guarantees that $\varphi_0 \vee \ldots \vee \varphi_{n-1} \in T$. And this yields a contradiction with the fact that $\mathfrak{A}, a \Vdash T$. ⊣

It follows from the claim and the closure of K under taking ultraproducts that Σ is satisfiable in some pointed structure $\langle \mathfrak{B}, b\rangle$ in K. But $\mathfrak{B}, b \Vdash \Sigma$ implies that $\langle \mathfrak{B}, b\rangle \rightsquigarrow \langle \mathfrak{A}, a\rangle$. So by Lemma 3.5.10 there exist ultrapowers $\Pi_{U_1}\langle \mathfrak{B}, b\rangle$ and $\Pi_{U_2}\langle \mathfrak{A}, a\rangle$ such that $\Pi_{U_1}\langle \mathfrak{B}, b\rangle \preceq \Pi_{U_2}\langle \mathfrak{A}, a\rangle$. By closure under ultraproducts $\Pi_{U_1}\langle \mathfrak{B}, b\rangle$ belongs to K. Hence, by closure under quasi bisimilarity, $\Pi_{U_2}\langle \mathfrak{A}, a\rangle$ is in K as well. By the closure condition on the complementary class of K it follows that $\langle \mathfrak{A}, a\rangle$ is in K. And this completes the proof. □

3.6.2. THEOREM. *Let ϑ be a SW-vocabulary, and let K be a class of pointed τ_ϑ-structures. The following are equivalent:*

1. *K is definable by a single $\mathcal{L}^{SW}(\vartheta)$-formula.*
2. *K is closed under quasi bisimilarity and under ultraproducts, and its complementary class is closed under ultraproducts.*

Proof: The implication $(1 \Rightarrow 2)$ is trivial by Proposition 3.2.2. For the converse, by Theorem 1.3.10 we assume that K is definable by a single $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula φ. So as K is closed under quasi bisimilarity we know that φ is preserved under quasi bisimilarity. By Theorem 3.4.3 we conclude that φ is equivalent to a $\mathcal{L}^{SW}(\vartheta)$-formula. □

3.6.3. THEOREM. *Let ϑ be a SW-vocabulary, and let K be a class of pointed τ_ϑ-structures. The following are equivalent:*

1. K *is definable by a (maybe proper) class of $\mathcal{L}^{SW}_\infty(\vartheta)$-formulas.*
2. K *is closed under quasi bisimilarity.*

Proof: $(1 \Rightarrow 2)$ is trivial by Proposition 3.2.2. For the converse, let C be the class $\{\varphi \in \mathcal{L}^{SW}_\infty(\vartheta) : \text{for every } \langle\mathfrak{A}, a\rangle \in \mathsf{K},\ \mathfrak{A}, a \Vdash \varphi\}$. We will show that for every $\langle\mathfrak{B}, b\rangle$,

$$\mathfrak{B}, b \Vdash \mathsf{C} \quad \text{iff} \quad \langle\mathfrak{B}, b\rangle \in \mathsf{K}.$$

The right-to-left implication is trivial by definition of C. For the left-to-right implication, we assume that $\langle\mathfrak{B}, b\rangle \notin \mathsf{K}$. Hence, for every $\langle\mathfrak{A}, a\rangle \in \mathsf{K}$ it holds that $\langle\mathfrak{A}, a\rangle \not\preceq \langle\mathfrak{B}, b\rangle$. By Theorem 3.2.11(2) this means that for every $\langle\mathfrak{A}, a\rangle \in \mathsf{K}$ it holds that $\mathfrak{A}, a \Vdash \nu^{\langle\mathfrak{B},b\rangle}$. Therefore $\nu^{\langle\mathfrak{B},b\rangle} \in \mathsf{C}$. Using that $\mathfrak{B}, b \nVdash \nu^{\langle\mathfrak{B},b\rangle}$ we conclude that $\mathfrak{B}, b \nVdash \mathsf{C}$. □

3.6.4. THEOREM. *Let ϑ be a SW-vocabulary, and let K be a class of pointed τ_ϑ-structures. The following are equivalent:*

1. K *is definable by a set of $\mathcal{L}^{SW}_\infty(\vartheta)$-formulas.*
2. K *is definable by a $\mathcal{L}^{SW}_\infty(\vartheta)$-formula.*
3. *There is an ordinal α such that* K *is closed under quasi α-bisimilarity.*

Proof: $(1 \Rightarrow 2)$: Take the conjunction of the formulas in the set defining K.

$(2 \Rightarrow 3)$: The ordinal α is the modal degree of the infinitary strict-weak formula.

$(3 \Rightarrow 1)$: The proof is an easy adaptation of the proof of Theorem 3.6.3. Let Φ be the class $\{\varphi \in \mathcal{L}^{SW}_\infty(\vartheta) : \deg(\varphi) \leq \alpha, \text{and for every } \langle\mathfrak{A}, a\rangle \in \mathsf{K},\ \mathfrak{A}, a \Vdash \varphi\}$. We know that this class is modulo equivalence a set. We will show that for every $\langle\mathfrak{B}, b\rangle$,

$$\mathfrak{B}, b \Vdash \Phi \quad \text{iff} \quad \langle\mathfrak{B}, b\rangle \in \mathsf{K}.$$

The right-to-left implication is trivial by definition of Φ. For the left-to-right implication, we assume that $\langle\mathfrak{B}, b\rangle \notin \mathsf{K}$. Hence, for every $\langle\mathfrak{A}, a\rangle \in \mathsf{K}$ it holds that $\langle\mathfrak{A}, a\rangle \not\preceq_\alpha \langle\mathfrak{B}, b\rangle$. By Remark 3.2.10 this means that for every $\langle\mathfrak{A}, a\rangle \in \mathsf{K}$ it holds that $\mathfrak{A}, a \Vdash \nu_\alpha^{\langle\mathfrak{B},b\rangle}$. Therefore $\nu_\alpha^{\langle\mathfrak{B},b\rangle} \in \Phi$. Using the fact that $\mathfrak{B}, b \nVdash \nu_\alpha^{\langle\mathfrak{B},b\rangle}$ we conclude that $\mathfrak{B}, b \nVdash \Phi$. □

3.6.5. Remark. In the definability theorems, by Proposition 3.3.4, we could have replaced "K is closed under quasi bisimilarity" with the conjunction of (i) K is closed under preserving quasi bounded morphisms, and (ii) the complementary class of K is closed under reflecting quasi bounded morphisms. And analogously to Remark 3.4.4 it is even possible to simplify these conditions in certain particular cases of ϑ.

3.7 Quasi bisimulations

In the literature the relation of bisimilarity is usually introduced after the notion of bisimulation; this is what we did on page 13. It is clear that the conditions imposed on the definition of bisimulation can be expressed using first-order formulas since the quantifiers range over states (and not over sets of states). When τ is finite a single first-order formula is enough. Hence the relation of being bisimilar inside a structure[19] is explicitly definable by a Σ_1^1-formula[20] of the form $\exists Z\varphi(Z)$ where Z is a second-order variable and φ is a first-order formula.

3.7.1. Remark. (Quasi bisimilarity is not explicitly definable by a first-order formula). It is known that bisimilar states inside a structure are not explicitly definable by a first-order formula, i.e., there is no first-order formula $\varphi(v_0, v_1)$ such that $\forall v_0\forall v_1(v_0 \simeq v_1 \supset\subset \varphi(v_0, v_1))$ holds in all structures (see [vBB95, p. 254][21]). From here it follows that there is no first-order formula $\varphi(v_0, v_1)$ such that $\forall v_0\forall v_1(v_0 \preceq v_1 \supset\subset \varphi(v_0, v_1))$ holds in all structures because if this

[19] This is not an important restriction because bisimilarity between two different structures can be regarded as bisimilarity inside their disjoint union.

[20] We remind the reader that Σ_1^1-*formulas* are, by definition, of the form $\exists Z_0 \ldots \exists Z_{n-1}\varphi(Z_0, \ldots, Z_{n-1})$ where $Z_0, \ldots, Z_{n-1}$ are second-order variables and φ is a first-order formula. Hence they start by a bunch of existential second-order quantifiers.

[21] That paper claims what we have said, but there is no argument. For the sake of completeness we outline here a possible argument. Let τ be $\langle\{m\}, \emptyset\rangle$, let $\varphi(v_0, v_1)$ be a first-order formula that explicitly defines bisimilar states. We define $\mathfrak{A}$ as the τ-structure $\langle\omega, \{\langle n+1, n\rangle : n \in \omega\}\rangle$. It is obvious that $\forall v_0\forall v_1(v_0 \simeq v_1 \supset\subset v_0 \approx v_1)$ holds in $\mathfrak{A}$, i.e., $\forall v_0\forall v_1(\varphi(v_0, v_1) \supset\subset v_0 \approx v_1)$ holds in $\mathfrak{A}$. It is easy to see that the set of first-order formulas $\mathrm{Th}(\mathfrak{A}) \cup \{c_n \not\approx c_{n'} : n \neq n'\} \cup \{R_m c_n c_{n+1} : n \in \omega\}$ is finitely satisfiable. By the compactness theorem it is satisfiable in a structure $\mathfrak{B}$. It results that $\{\langle c_n^{\mathfrak{B}}, c_{n+1}^{\mathfrak{B}}\rangle : n \in \omega\}$ is a τ-bisimulation inside $\mathfrak{B}$. Hence $\langle\mathfrak{B}, c_0^{\mathfrak{B}}\rangle \simeq \langle\mathfrak{B}, c_1^{\mathfrak{B}}\rangle$ and $c_0^{\mathfrak{B}} \neq c_1^{\mathfrak{B}}$. This contradicts the fact that $\forall v_0\forall v_1(\varphi(v_0, v_1) \supset\subset v_0 \approx v_1)$ holds in $\mathfrak{B}$.

formula would exist, then $\forall v_0 \forall v_1 (v_0 \simeq v_1 \supset\subset \varphi(v_0, v_1) \wedge \varphi(v_1, v_0))$ would hold in all structures.

On the other hand, we have introduced the relation of quasi bisimilarity with a definition which is not Σ^1_1. This is due to the fact that when we replace the bisimilarity relation in the definition by its Σ^1_1-definition we have some first-order quantifiers before it. We would like to move the second order quantifiers involved in the definition of the bisimilarity relation to the beginning of the definition. If we manage to do it we would obtain a Σ^1_1-definition of quasi bisimilarity. This section is devoted to this aim. Indeed, we will introduce two different notions (quasi bisimulations and directed quasi bisimulations) that will allow us to express quasi bisimilarity by a Σ^1_1-formula. Quasi bisimulations are closely related to our definition of quasi bisimilarity, and directed quasi bisimulations have already been considered (in a restricted case) by Kurtonina under a different name.

3.7.2. Definition. (Quasi bisimulation)
Let ϑ be a SW-vocabulary, and let $\mathfrak{A}$ and $\mathfrak{B}$ be two τ_ϑ-structures. A *quasi bisimulation* from $\mathfrak{A}$ into $\mathfrak{B}$ is a pair $\langle U, Z\rangle$ such that:

- $Z \subseteq U \subseteq A \times B$.
- Z is a τ_ϑ-bisimulation between $\mathfrak{A}$ and $\mathfrak{B}$.
- If $\langle a, b\rangle \in U$, $p \in \mathsf{Prop}$ and $\mathfrak{A}, a \Vdash p$, then $\mathfrak{B}, b \Vdash p$.
- For every $s \in \mathsf{SMod}$, $R^{\mathfrak{B}}_s \circ U \subseteq Z \circ R^{\mathfrak{A}}_s$, i.e., if $\langle a, b\rangle \in U$ and $\langle b, b'\rangle \in R^{\mathfrak{B}}_s$, then there is a' such that $\langle a, a'\rangle \in R^{\mathfrak{A}}_s$ and $\langle a', b'\rangle \in Z$.
- For every $w \in \mathsf{WMod}$, $R^{\mathfrak{B}}_w \circ U^{-1} \subseteq Z^{-1} \circ R^{\mathfrak{A}}_m$, i.e., if $\langle a, b\rangle \in U$ and $\langle a, a'\rangle \in R^{\mathfrak{A}}_w$, then there is b' such that $\langle b, b'\rangle \in R^{\mathfrak{B}}_w$ and $\langle a', b'\rangle \in Z$.

When $\langle U_0, Z_0\rangle$ is a quasi bisimulation from $\mathfrak{A}$ into $\mathfrak{B}$, and $\langle U_1, Z_1\rangle$ is a quasi bisimulation from $\mathfrak{B}$ into $\mathfrak{C}$, it holds that $\langle U_1 \circ U_0, Z_1 \circ Z_0\rangle$ is a quasi bisimulation from $\mathfrak{A}$ into $\mathfrak{C}$. And in general it is clearly false that $\langle U^{-1}, Z^{-1}\rangle$ is a quasi bisimulation when $\langle U, Z\rangle$ is a quasi bisimulation. What it is true is that if $\langle U, Z\rangle$ is a quasi bisimulation involving ϑ, $\mathfrak{A}$ and $\mathfrak{B}$, then $\langle U^{-1}, Z^{-1}\rangle$ is a quasi bisimulation involving ϑ^d, $\mathfrak{B}^d$ and $\mathfrak{A}^d$. It is also obvious that if $\{\langle U_i, Z_i\rangle : i \in I\}$ is a family of quasi bisimulations then $\langle \bigcup_{i\in I} U_i, \bigcup_{i\in I} Z_i\rangle$ is also a quasi bisimulation. Thus, it always exists the largest quasi bisimulation from $\mathfrak{A}$ into $\mathfrak{B}$. Indeed, the next proposition says that it is precisely $\{\langle a, b\rangle : \langle \mathfrak{A}, a\rangle \preceq \langle \mathfrak{B}, b\rangle\}$.

3.7.3. Proposition. *Let ϑ be a SW-vocabulary, and let $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ be two pointed τ_ϑ-structures. The following statements are equivalent:*

1. *$\langle \mathfrak{A}, a\rangle \preceq \langle \mathfrak{B}, b\rangle$.*

2. *There is a quasi bisimulation $\langle U, Z\rangle$ from $\mathfrak{A}$ into $\mathfrak{B}$ such that $\langle a, b\rangle \in U$.*

Proof: $(1 \Rightarrow 2)$: Let U be $\{\langle a', b'\rangle \in A \times B : \langle \mathfrak{A}, a'\rangle \preceq \langle \mathfrak{B}, b'\rangle\}$, and let Z be $\{\langle a', b'\rangle \in A \times B : \langle \mathfrak{A}, a'\rangle \simeq \langle \mathfrak{B}, b'\rangle\}$. Then, by definition of quasi bisimilarity it is clear that $\langle U, Z\rangle$ is a quasi bisimulation satisfying the conditions that we are interested in.

$(2 \Rightarrow 1)$: We must check the three clauses of the quasi bisimilarity definition. The property (qbis1) is obviously satisfied because $\langle a, b\rangle \in U$. And the other two can be proved in the same way, so we will analyze only (qbis2). Let $s \in \mathsf{SMod}$, and let b' be such that $\langle b, b'\rangle \in R_s^{\mathfrak{B}}$. Then, there exists a' such that $\langle a, a'\rangle \in R_s^{\mathfrak{A}}$ and $\langle a', b'\rangle \in Z$. Hence $\langle \mathfrak{A}, a'\rangle \simeq \langle \mathfrak{B}, b'\rangle$. This completes the proof. □

3.7.4. DEFINITION. (Directed quasi bisimulation)
Let ϑ be a SW-vocabulary, and let $\mathfrak{A}$ and $\mathfrak{B}$ be two τ_ϑ-structures. A *directed quasi bisimulation* from $\mathfrak{A}$ into $\mathfrak{B}$ is a pair $\langle U_0, U_1\rangle$ such that:

- $U_0 \subseteq A \times B$ y $U_1 \subseteq B \times A$.
- If $\langle a, b\rangle \in U_0$, $p \in \mathsf{Prop}$ and $\mathfrak{A}, a \Vdash p$, then $\mathfrak{B}, b \Vdash p$.
- If $\langle b, a\rangle \in U_1$, $p \in \mathsf{Prop}$ and $\mathfrak{B}, b \Vdash p$, then $\mathfrak{A}, a \Vdash p$.
- For every $s \in \mathsf{SMod}$, if $\langle a, b\rangle \in U_0$ and $\langle b, b'\rangle \in R_s^{\mathfrak{B}}$, then there is a' such that $\langle a, a'\rangle \in R_s^{\mathfrak{A}}$, $\langle a', b'\rangle \in U_0$ and $\langle b', a'\rangle \in U_1$.
- For every $s \in \mathsf{SMod}$, if $\langle b, a\rangle \in U_1$ and $\langle a, a'\rangle \in R_s^{\mathfrak{A}}$, then there is b' such that $\langle b, b'\rangle \in R_s^{\mathfrak{B}}$, $\langle a', b'\rangle \in U_0$ and $\langle b', a'\rangle \in U_1$.
- For every $w \in \mathsf{WMod}$, if $\langle a, b\rangle \in U_0$ and $\langle a, a'\rangle \in R_w^{\mathfrak{A}}$, then there is b' such that $\langle b, b'\rangle \in R_w^{\mathfrak{B}}$, $\langle a', b'\rangle \in U_0$ and $\langle b', a'\rangle \in U_1$.
- For every $w \in \mathsf{WMod}$, if $\langle b, a\rangle \in U_1$ and $\langle b, b'\rangle \in R_w^{\mathfrak{B}}$, then there is a' such that $\langle a, a'\rangle \in R_w^{\mathfrak{A}}$, $\langle a', b'\rangle \in U_0$ and $\langle b', a'\rangle \in U_1$.

It is obvious that if $\langle U_0, U_1\rangle$ is a directed quasi bisimulation from $\mathfrak{A}$ into $\mathfrak{B}$, then $\langle U_1, U_0\rangle$ is a directed quasi bisimulation from $\mathfrak{B}$ into $\mathfrak{A}$, and also that if $\{\langle U_0^i, U_1^i\rangle : i \in I\}$ is a family of directed quasi bisimulations then the pair $\langle \bigcup_{i\in I} U_0^i, \bigcup_{i\in I} U_1^i\rangle$ is also a directed quasi bisimulation. Hence the largest directed quasi bisimulation from $\mathfrak{A}$ into $\mathfrak{B}$ always exists. Indeed, we will see that it also coincides with quasi bisimilarity. But first of all we will enunciate some links between quasi bisimulations and directed quasi bisimulations.

3.7.5. PROPOSITION. *Let ϑ be a SW-vocabulary, and let $\mathfrak{A}$ and $\mathfrak{B}$ be two τ_ϑ-structures.*

1. *If $\langle U_0, Z_0\rangle$ is the largest quasi bisimulation from $\mathfrak{A}$ into $\mathfrak{B}$, and $\langle U_1, Z_1\rangle$ is the largest quasi bisimulation from $\mathfrak{B}$ into $\mathfrak{A}$, then $\langle U_0, U_1\rangle$ is a directed quasi bisimulation from $\mathfrak{A}$ into $\mathfrak{B}$.*

2. *If $\langle U_0, U_1\rangle$ is a directed quasi bisimulation from $\mathfrak{A}$ into $\mathfrak{B}$, then $\langle U_0, U_0 \cap U_1^{-1}\rangle$ is a quasi bisimulation from $\mathfrak{A}$ into $\mathfrak{B}$.*

Proof: Both items can be straightforwardly proved. □

3.7.6. Corollary. *Let ϑ be a SW-vocabulary, and let $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ be two pointed τ_ϑ-structures. The following statements are equivalent:*

1. *$\langle \mathfrak{A}, a\rangle \preceq \langle \mathfrak{B}, b\rangle$.*

2. *There is a directed quasi bisimulation $\langle U_0, U_1\rangle$ from $\mathfrak{A}$ into $\mathfrak{B}$ such that $\langle a, b\rangle \in U_0$.*

Proof: By Propositions 3.7.3 and 3.7.5. □

Proposition 3.7.3 and Corollary 3.7.6 give us two different methods to express quasi bisimilarity by a Σ^1_1-formula. By them we know that in the theorems obtained in previous sections we could have replaced 'φ is preserved under quasi bisimilarity' with either of the following conditions:

- φ is preserved under quasi bisimulation.
- φ is preserved under directed quasi bisimulation[22].

We could also have used these notions to introduce the definition of strongly Hennessy-Milner classes. It is easy to see that a class K is strongly Hennessy-Milner iff for every $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$ it holds that the pair $\langle \{\langle a, b\rangle : \langle \mathfrak{A}, a\rangle \rightsquigarrow \langle \mathfrak{B}, b\rangle\}$, $\{\langle a, b\rangle : \langle \mathfrak{A}, a\rangle \leftrightsquigarrow \langle \mathfrak{B}, b\rangle\}\rangle$ is a quasi bisimulation from $\mathfrak{A}$ into $\mathfrak{B}$. And this is also equivalent to the claim that for every $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$ the pair $\langle \{\langle a, b\rangle : \langle \mathfrak{A}, a\rangle \rightsquigarrow \langle \mathfrak{B}, b\rangle\}, \{\langle b, a\rangle : \langle \mathfrak{B}, b\rangle \rightsquigarrow \langle \mathfrak{A}, a\rangle\}\rangle$ is a directed quasi bisimulation from $\mathfrak{A}$ into $\mathfrak{B}$.

3.7.7. Remark. (Relations to Kurtonina's work[23]). Lambek calculus [Lam58, Ono98] is the most outstanding logical system for categorial inference. It is known that the implication involved in Lambek calculus, which we will represent by $\twoheadrightarrow$, can be introduced semantically using structures with a ternary relation S. So

[22] The formulation of Theorem 3.5.11 using directed quasi bisimulations has the advantage of not requiring the Standard Form Theorem in the proof (observe that this theorem was involved in the proof of the auxiliary Proposition 3.5.8). That is, it can be easily proved without using the Standard Form Theorem that in modally saturated structures it holds that $\langle \{\langle a, b\rangle : \langle \mathfrak{A}, a\rangle \rightsquigarrow \langle \mathfrak{B}, b\rangle\}, \{\langle b, a\rangle : \langle \mathfrak{B}, b\rangle \rightsquigarrow \langle \mathfrak{A}, a\rangle\}\rangle$ is a directed quasi bisimulation from $\mathfrak{A}$ into $\mathfrak{B}$. However, in the author's opinion directed quasi bisimulations has the disadvantage that at first glance the connection with bisimulations (and bisimilarity) is not so clear.

[23] The author is grateful to Johan van Benthem for drawing his attention to this connection with Lambek calculus.

$$\mathfrak{A}, a \Vdash \varphi_0 \twoheadrightarrow \varphi_1 \quad \text{iff} \quad \forall a'a''(Sa''a'a \mathbin{\&} \mathfrak{A}, a' \Vdash \varphi_0 \;\Rightarrow\; \mathfrak{A}, a'' \Vdash \varphi_1).$$

Therefore, if we restrict the ternary relations S to those in which the following property holds:

$$\text{if } \langle a'', a', a\rangle \in S \text{ then } a'' = a', \tag{3.3}$$

and define $R = \{\langle a, a'\rangle : \langle a', a', a\rangle \in S\}$ it is clear that the implication $\twoheadrightarrow$ and the strict implication $\to$ associated with R both have the same semantic clauses.

Kurtonina's dissertation [Kur95] analyzed categorial inference, in particular Lambek Calculus, using Kripke semantics. In order to characterize the expressive power of the language of the Lambek Calculus she introduced the notion of directed categorial bisimulation [Kur95, Definition 1.3.5] between structures using a ternary relation. As the previous paragraph suggests it can be checked that directed categorial bisimulations restricted to structures satisfying (3.3) coincide with directed quasi bisimulations (where the SW-vocabulary consists in a pure strict modality).

3.8 Amalgamation

Amalgamation is a classical tool which is often used to prove interpolation. It is commonly formulated in the algebraic framework, but one can find formulations in the literature at the level of structures. They have the advantage of making it possible to prove uniform interpolation (see [D'A98, Vis96a, Vis96b]). This section is devoted to proving two propositions, 3.8.1 and 3.8.3. They will be crucial when we will prove uniform interpolation, and they are natural generalizations, respectively, of [D'A98, Lemma 2.2.10] and [Vis96b, Lemma 6.1].

In the following propositions we will consider the assumption $\mathsf{WMod} \cap \mathsf{SMod}' \subseteq \mathsf{SMod} \cap \mathsf{WMod}' \subseteq \mathsf{SMod}'$. In particular this implies that $\tau_{\vartheta\cap\vartheta'} = \tau_\vartheta \cap \tau_{\vartheta'}$, i.e., $\mathsf{Mod} \cap \mathsf{Mod}' \subseteq (\mathsf{SMod} \cap \mathsf{SMod}') \cup (\mathsf{WMod} \cap \mathsf{WMod}')$; but they are not equivalent in general. We will also consider in Corollary 3.8.5 the dual[24] of this assumption, i.e., $\mathsf{SMod}' \cap \mathsf{WMod} \subseteq \mathsf{WMod}' \cap \mathsf{SMod} \subseteq \mathsf{WMod}$.

We emphasize that if ϑ and ϑ' are only distinguished by their propositions (i.e., $\mathsf{SMod} = \mathsf{SMod}'$ and $\mathsf{WMod} = \mathsf{WMod}'$) then the assumption, and also its dual version, holds. Hence, in these cases we will be able to apply Propositions 3.8.1 and 3.8.3, and Corollary 3.8.5.

3.8.1. Proposition. *Let ϑ and ϑ' be two SW-vocabularies such that* $\mathsf{WMod} \cap \mathsf{SMod}' \subseteq \mathsf{SMod} \cap \mathsf{WMod}' \subseteq \mathsf{SMod}'$. *Let $\langle \mathfrak{A}, a\rangle$ be a pointed τ_ϑ-structure, and let $\langle \mathfrak{B}, b\rangle$ be a pointed $\tau_{\vartheta'}$-structure. The following are equivalent:*

[24] Observe that we interchange strict and weak modalities, but it is also necessary to interchange the roles of ϑ and ϑ'.

1. *$\langle \mathfrak{A}, a\rangle \preceq_{\vartheta\cap\vartheta'} \langle \mathfrak{B}, b\rangle$.*

2. *There is a pointed $\tau_{\vartheta\cup\vartheta'}$-structure $\langle \mathfrak{C}, c\rangle$ such that $\langle \mathfrak{A}, a\rangle \preceq_{\vartheta} \langle \mathfrak{C}, c\rangle \preceq_{\vartheta'} \langle \mathfrak{B}, b\rangle$.*

Proof: The implication ($2 \Rightarrow 1$) is trivial. Let us show ($1 \Rightarrow 2$). Assume $\langle \mathfrak{A}, a\rangle \preceq_{\vartheta\cap\vartheta'} \langle \mathfrak{B}, b\rangle$. We define the set Q as $\{\langle x, y\rangle \in A\times B : \langle \mathfrak{A}, x\rangle \preceq_{\vartheta\cap\vartheta'} \langle \mathfrak{B}, y\rangle\}$. In particular $\langle a, b\rangle \in Q$. The pointed $\tau_{\vartheta\cup\vartheta'}$-structure $\langle \mathfrak{C}, c\rangle$ is defined as follows:

- The universe is the disjoint union of A, B, and Q.
- If $m \in \mathsf{Mod}\cap\mathsf{Mod}'$, then $R^{\mathfrak{C}}_m$ is $\{\langle x, x'\rangle \in A^2 : \langle x, x'\rangle \in R^{\mathfrak{A}}_m\} \cup \{\langle y, y'\rangle \in B^2 : \langle y, y'\rangle \in R^{\mathfrak{B}}_m\} \cup \{\langle\langle x, y\rangle, \langle x', y'\rangle\rangle \in Q^2 : \langle x, x'\rangle \in R^{\mathfrak{A}}_m, \langle y, y'\rangle \in R^{\mathfrak{B}}_m, \langle \mathfrak{A}, x'\rangle \simeq_{\tau_{\vartheta\cap\vartheta'}} \langle \mathfrak{B}, y'\rangle\}$.
- If $m \in \mathsf{Mod}\setminus\mathsf{Mod}'$, then $R^{\mathfrak{C}}_m$ is $\{\langle x, x'\rangle \in A^2 : \langle x, x'\rangle \in R^{\mathfrak{A}}_m\} \cup \{\langle\langle x, y\rangle, x'\rangle \in Q\times A : \langle x, x'\rangle \in R^{\mathfrak{A}}_m\}$.
- If $m \in \mathsf{Mod}'\setminus\mathsf{Mod}$, then $R^{\mathfrak{C}}_m$ is $\{\langle y, y'\rangle \in B^2 : \langle y, y'\rangle \in R^{\mathfrak{B}}_m\} \cup \{\langle\langle x, y\rangle, y'\rangle \in Q\times B : \langle y, y'\rangle \in R^{\mathfrak{B}}_m\}$.
- If $p \in \mathsf{Prop}\cap\mathsf{Prop}'$, then $V^{\mathfrak{C}}(p)$ is $V^{\mathfrak{A}}(p)\cup V^{\mathfrak{B}}(p)\cup\{\langle x, y\rangle \in Q : y \in V^{\mathfrak{B}}(p)\}$.
- If $p \in \mathsf{Prop}\setminus\mathsf{Prop}'$, then $V^{\mathfrak{C}}(p)$ is $V^{\mathfrak{A}}(p)\cup\{\langle x, y\rangle \in Q : x \in V^{\mathfrak{A}}(p)\}$.
- If $p \in \mathsf{Prop}'\setminus\mathsf{Prop}$, then $V^{\mathfrak{C}}(p)$ is $V^{\mathfrak{B}}(p)\cup\{\langle x, y\rangle \in Q : y \in V^{\mathfrak{B}}(p)\}$.
- The distinguished point c is $\langle a, b\rangle$.

CLAIM I: $\{\langle x, x\rangle : x \in A\} \cup \{\langle x, \langle x, y\rangle\rangle \in A\times Q : \langle \mathfrak{A}, x\rangle \simeq_{\tau_{\vartheta\cap\vartheta'}} \langle \mathfrak{B}, y\rangle\}$ is a τ_ϑ-bisimulation between $\mathfrak{A}$ and $\mathfrak{C}$.

Proof of Claim: It is straightforward. (*Hint*: $\mathsf{Mod}\cap\mathsf{Mod}' \subseteq (\mathsf{SMod}\cap\mathsf{SMod}') \cup (\mathsf{WMod}\cap\mathsf{WMod}')$) ⊣

CLAIM II: $\{\langle y, y\rangle : y \in B\} \cup \{\langle\langle x, y\rangle, y\rangle \in Q\times B : \langle \mathfrak{A}, x\rangle \simeq_{\tau_{\vartheta\cap\vartheta'}} \langle \mathfrak{B}, y\rangle\}$ is a $\tau_{\vartheta'}$-bisimulation between $\mathfrak{C}$ and $\mathfrak{B}$.

Proof of Claim: It is also straightforward. ⊣

CLAIM III: $\langle \mathfrak{A}, a\rangle \preceq_{\vartheta} \langle \mathfrak{C}, c\rangle$.

Proof of Claim: We must show the following clauses.

(qbis1): Assume that $\mathfrak{A}, a \Vdash p \in \mathsf{Prop}$, and let us prove that $\mathfrak{C}, c \Vdash p$. If $p \in \mathsf{Prop}\setminus\mathsf{Prop}'$ then it is clear that $\mathfrak{C}, c \Vdash p$. On the other hand, if $p \in \mathsf{Prop}\cap\mathsf{Prop}'$, by $\langle \mathfrak{A}, a\rangle \preceq_{\vartheta\cap\vartheta'} \langle \mathfrak{B}, b\rangle$ it follows that $\mathfrak{B}, b \Vdash p$; and hence $\mathfrak{C}, c \Vdash p$.

(qbis2): Assume that $s \in \mathsf{SMod}$, and let c' be such that $\langle c, c'\rangle \in R^{\mathfrak{C}}_s$. Now we distinguish two cases:

i. Case $s \in \mathsf{SMod} \cap \mathsf{Mod}'$: it holds that $c' = \langle x', y' \rangle$ where $\langle a, x' \rangle \in R_s^{\mathfrak{A}}$, $\langle b, y' \rangle \in R_s^{\mathfrak{B}}$, and $\langle \mathfrak{A}, x' \rangle \simeq_{\tau_{\vartheta \cap \vartheta'}} \langle \mathfrak{B}, y' \rangle$. And by the first claim it follows that $\langle \mathfrak{A}, x' \rangle \simeq_{\tau_\vartheta} \langle \mathfrak{C}, c' \rangle$.

ii. Case $s \in \mathsf{SMod} \setminus \mathsf{Mod}'$: it holds that $c' \in A$ where $\langle a, c' \rangle$. And by the first claim it follows that $\langle \mathfrak{A}, c' \rangle \simeq_{\tau_\vartheta} \langle \mathfrak{C}, c' \rangle$.

(qbis3): Assume that $w \in \mathsf{WMod}$, and let $a' \in A$ be such that $\langle a, a' \rangle \in R_w^{\mathfrak{A}}$. We distinguish two cases:

i. Case $w \in \mathsf{WMod} \cap \mathsf{Mod}'$: then $w \in \mathsf{WMod} \cap \mathsf{WMod}'$ because $\mathsf{WMod} \cap \mathsf{SMod}' \subseteq \mathsf{WMod}'$. By $\langle \mathfrak{A}, a \rangle \preceq_{\vartheta \cap \vartheta'} \langle \mathfrak{B}, b \rangle$ it is deduced that there is b' such that $\langle b, b' \rangle \in R_w^{\mathfrak{B}}$ and $\langle \mathfrak{A}, a' \rangle \simeq_{\tau_{\vartheta \cap \vartheta'}} \langle \mathfrak{B}, b' \rangle$. Hence, $\langle c, \langle a', b' \rangle \rangle \in R_w^{\mathfrak{C}}$. And by Claim I it follows that $\langle \mathfrak{A}, a' \rangle \simeq_{\tau_\vartheta} \langle \mathfrak{C}, \langle a', b' \rangle \rangle$.

ii. Case $w \in \mathsf{WMod} \setminus \mathsf{Mod}'$: it holds that $\langle c, a' \rangle \in R_w^{\mathfrak{C}}$. And by the first claim it follows that $\langle \mathfrak{A}, a' \rangle \simeq_{\tau_\vartheta} \langle \mathfrak{C}, a' \rangle$.

⊣

Claim IV: $\langle \mathfrak{C}, c \rangle \preceq_{\vartheta'} \langle \mathfrak{B}, b \rangle$.

Proof of Claim: The proof is analogous to the one given for the previous claim, but this time using the second claim and the fact that $\mathsf{SMod}' \cap \mathsf{WMod} \subseteq \mathsf{SMod}$. ⊣

The last two claims give us what we want, so the proof is finished. □

3.8.2. Example. Now we show with an example that the hypothesis cannot be eliminated (but it is still an open problem whether there are weaker requirements). Let ϑ be $\langle \{m_0\}, \{m_1\}, \emptyset \rangle$, and ϑ' be $\langle \{m_0, m_1\}, \emptyset, \emptyset \rangle$. Then, $\vartheta \cap \vartheta' = \langle \{m_0\}, \emptyset, \emptyset \rangle$. It holds that $\tau_\vartheta = \tau_{\vartheta'} = \langle \{m_0, m_1\}, \emptyset \rangle$ while $\tau_{\vartheta \cap \vartheta'} = \langle \{m_0\}, \emptyset \rangle$. Let $\mathfrak{A}$ be the τ_ϑ-structure defined as follows: (i) the universe is $A = \{a_0, a_1\}$, (ii) $R_{m_0} = \{\langle a_0, a_1 \rangle\}$ and $R_{m_1} = \emptyset$. And let $\mathfrak{B}$ be the $\tau_{\vartheta'}$-structure defined as follows: (i) the universe is $B = \{b_0, b_1, b_2\}$, (ii) $R_{m_0} = \{\langle b_0, b_1 \rangle\}$ and $R_{m_1} = \{\langle b_1, b_2 \rangle\}$. It is clear that $\langle \mathfrak{A}, a_0 \rangle \preceq_{\vartheta \cap \vartheta'} \langle \mathfrak{B}, b_0 \rangle$. However, let us see that there is no $\langle \mathfrak{C}, c \rangle$ such that $\langle \mathfrak{A}, a_0 \rangle \preceq_\vartheta \langle \mathfrak{C}, c \rangle \preceq_{\vartheta'} \langle \mathfrak{B}, b_0 \rangle$. Assume that there is a pointed structure $\langle \mathfrak{C}, c \rangle$ with this property. By $\langle \mathfrak{C}, c \rangle \preceq_{\vartheta'} \langle \mathfrak{B}, b_0 \rangle$ it follows that there is a m_0-successor c_1 of c such that $\langle \mathfrak{C}, c_1 \rangle \simeq_{\tau_{\vartheta'}} \langle \mathfrak{B}, b_1 \rangle$. Thus there is a m_1-successor c_2 of c_1. Therefore $\langle \mathfrak{A}, a_1 \rangle \not\simeq_{\tau_\vartheta} \langle \mathfrak{C}, c_1 \rangle$. Hence there is no m_0-successor a' of a_0 such that $\langle \mathfrak{A}, a' \rangle \simeq_{\tau_\vartheta} \langle \mathfrak{C}, c_1 \rangle$. And this contradicts the fact that $\langle \mathfrak{A}, a_0 \rangle \preceq_\vartheta \langle \mathfrak{C}, c \rangle$.

3.8.3. Proposition. *Let ϑ and ϑ' be two SW-vocabularies such that* $\mathsf{WMod} \cap \mathsf{SMod}' \subseteq \mathsf{SMod} \cap \mathsf{WMod}' \subseteq \mathsf{SMod}'$, *let $\langle \mathfrak{A}, a \rangle$ be a pointed τ_ϑ-structure, let $\langle \mathfrak{B}, b \rangle$ be a pointed $\tau_{\vartheta'}$-structure, and let $n \in \omega$. The following are equivalent:*

1. $\langle \mathfrak{A}, a \rangle \preceq_{n\, \vartheta \cap \vartheta'} \langle \mathfrak{B}, b \rangle$.

2. *There is a pointed* $\tau_{\vartheta\cup\vartheta'}$*-structure* $\langle\mathfrak{C},c\rangle$ *such that* $\langle\mathfrak{A},a\rangle \preceq_{n\,\vartheta} \langle\mathfrak{C},c\rangle \preceq_{\vartheta'} \langle\mathfrak{B},b\rangle$.

Proof: The upward implication is trivial. Let us show the converse. For $n=0$ this is trivial. So, let us assume $n>0$ and $\langle\mathfrak{A},a\rangle \preceq_{\vartheta\cap\vartheta'} \langle\mathfrak{B},b\rangle$. Let $\bullet$ and $\star$ be two new points. We define the sets $Q_0 := \{\langle x,j,y\rangle \in A\times n\times B : \langle\mathfrak{A},x\rangle \simeq_{j\,\tau_{\vartheta\cap\vartheta'}} \langle\mathfrak{B},y\rangle\}$, $Q_1 := \{\langle x,n,y\rangle \in A\times\{n\}\times B : \langle\mathfrak{A},x\rangle \preceq_{n\,\vartheta\cap\vartheta'} \langle\mathfrak{B},y\rangle\}$, $Q_2 := \{\langle x,-1,\bullet\rangle : x\in A\}$, and $Q_3 := \{\langle\star,-1,y\rangle : y\in B\}$. In particular $\langle a,n,b\rangle \in Q_1$. The pointed $\tau_{\vartheta\cup\vartheta'}$-structure $\langle\mathfrak{C},c\rangle$ is defined as follows:

- The universe is the (disjoint) union of $Q_0\cup Q_1\cup Q_2\cup Q_3$.
- If $m \in \mathsf{Mod}\cap\mathsf{Mod}'$, then $R^{\mathfrak{C}}_m$ is $\{\langle\langle x,-1,\bullet\rangle,\langle\langle x',-1,\bullet\rangle\rangle \in Q_2\times Q_2 : \langle x,x'\rangle \in R^{\mathfrak{A}}_m\} \cup \{\langle\langle\star,-1,y\rangle,\langle\star,-1,y'\rangle\rangle \in Q_3\times Q_3 : \langle y,y'\rangle \in R^{\mathfrak{B}}_m\} \cup \{\langle\langle x,j+1,y\rangle,\langle x',j,y'\rangle\rangle \in (Q_0\cup Q_1)\times Q_0 : \langle x,x'\rangle \in R^{\mathfrak{A}}_m, \langle y,y'\rangle \in R^{\mathfrak{B}}_m\} \cup \{\langle\langle x,0,y\rangle,\langle\star,-1,y'\rangle\rangle \in Q_0\times Q_3 : \langle y,y'\rangle \in R^{\mathfrak{B}}_m\}$.
- If $m \in \mathsf{Mod}\setminus\mathsf{Mod}'$, then $R^{\mathfrak{C}}_m$ is $\{\langle\langle x,j,y\rangle,\langle x',-1,\bullet\rangle\rangle \in (Q_0\cup Q_1)\times Q_2 : \langle x,x'\rangle \in R^{\mathfrak{A}}_m\} \cup \{\langle\langle x,-1,\bullet\rangle,\langle x',-1,\bullet\rangle\rangle \in Q_2\times Q_2 : \langle x,x'\rangle \in R^{\mathfrak{A}}_m\}$.
- If $m \in \mathsf{Mod}'\setminus\mathsf{Mod}$, then $R^{\mathfrak{C}}_m$ is $\{\langle\langle x,j,y\rangle,\langle\star,-1,y'\rangle\rangle \in (Q_0\cup Q_1)\times Q_3 : \langle y,y'\rangle \in R^{\mathfrak{B}}_m\} \cup \{\langle\langle\star,-1,y\rangle\rangle,\langle\star,-1,y'\rangle\rangle \in Q_3\times Q_3 : \langle y,y'\rangle \in R^{\mathfrak{B}}_m\}$.
- If $p \in \mathsf{Prop}\cap\mathsf{Prop}'$, then $V^{\mathfrak{C}}(p)$ is $\{\langle x,j,y\rangle \in Q_0\cup Q_1 : y \in V^{\mathfrak{B}}(p)\} \cup \{\langle x,-1,\bullet\rangle \in Q_2 : x\in V^{\mathfrak{A}}(p)\} \cup \{\langle\star,-1,y\rangle \in Q_3 : y \in V^{\mathfrak{B}}(p)\}$.
- If $p \in \mathsf{Prop}\setminus\mathsf{Prop}'$, then $V^{\mathfrak{C}}(p)$ is $\{\langle x,j,y\rangle \in Q_0\cup Q_1 : x \in V^{\mathfrak{A}}(p)\} \cup \{\langle x,-1,\bullet\rangle \in Q_2 : x \in V^{\mathfrak{A}}(p)\}$.
- If $p \in \mathsf{Prop}'\setminus\mathsf{Prop}$, then $V^{\mathfrak{C}}(p)$ is $\{\langle x,j,y\rangle \in Q_0\cup Q_1 : y \in V^{\mathfrak{B}}(p)\} \cup \{\langle\star,-1,y\rangle \in Q_3 : y \in V^{\mathfrak{B}}(p)\}$.
- The distinguished point is $\langle a,n,b\rangle$.

For every $j<n$, let Z_j be $\{\langle x,\langle x,j,y\rangle\rangle : \langle x,j,y\rangle \in Q_0\} \cup \{\langle x,\langle x,-1,\bullet\rangle\rangle : x \in A\} \subseteq A\times C$.
Claim I: For every $j<n$, the states related by Z_j are j-bisimilar in τ_ϑ.
Proof of Claim: The proof is an easy induction on j. For $j=0$ this is clear. Assume now that the states related by Z_j are j-bisimilar in τ_ϑ, and let us show that the states related by Z_{j+1} are $j+1$-bisimilar in τ_ϑ. We must show the following clauses.

(bbis1): It is trivial.

(bbis2): We distinguish two cases. First of all, assume that $m \in \mathsf{Mod} \setminus \mathsf{Mod}'$, $\langle x, \langle x, z_0, z_1\rangle\rangle \in Z_{j+1}$ and $\langle\langle x, z_0, z_1\rangle, \langle x', z_0', z_1'\rangle\rangle \in R^{\mathfrak{C}}_m$. Then $\langle x', z_0', z_1'\rangle \in Q_2$. Hence, $\langle x', z_0', z_1'\rangle = \langle x', -1, \bullet\rangle$ and $\langle x, x'\rangle \in R^{\mathfrak{A}}_m$. Thus $\langle x', \langle x', z_0', z_1'\rangle\rangle \in Z_j$. By the inductive hypothesis it follows that $\langle \mathfrak{A}, x'\rangle \simeq_{j\,\tau_\vartheta} \langle \mathfrak{C}, \langle x', z_0', z_1'\rangle\rangle$. Now let us see the second case. Assume $m \in \mathsf{Mod} \cap \mathsf{Mod}'$, $\langle x, \langle x, z_0, z_1\rangle\rangle \in Z_{j+1}$ and $\langle\langle x, z_0, z_1\rangle, \langle x', z_0', z_1'\rangle\rangle \in R^{\mathfrak{C}}_m$. If $z_0 = -1$, then it is clear that $\langle x', z_0', z_1'\rangle \in Q_2$, $\langle x', \langle x', z_0', z_1'\rangle\rangle \in Z_j$ and $\langle x, x'\rangle \in R^{\mathfrak{A}}_m$. Hence, we can assume that $z_0 \neq -1$, i.e., $z_0 = j+1$ and $\langle x, z_0, z_1\rangle \in Q_0$. By definition of $R^{\mathfrak{C}}_m$ it holds that $\langle x', z_0', z_1'\rangle \in Q_0$, $z_0' = j$, $\langle x, x'\rangle \in R^{\mathfrak{A}}_m$ and $\langle z_1, z_1'\rangle \in R^{\mathfrak{B}}_m$. In particular, $\langle x, x'\rangle \in R^{\mathfrak{A}}_m$ and $\langle x', \langle x', z_0', z_1'\rangle\rangle \in Z_j$. By the inductive hypothesis it follows that $\langle \mathfrak{A}, x'\rangle \simeq_{j\,\tau_\vartheta} \langle \mathfrak{C}, \langle x', z_0', z_1'\rangle\rangle$.

(bbis3): We also distinguish two cases. First of all, assume that $m \in \mathsf{Mod} \setminus \mathsf{Mod}'$, $\langle x, \langle x, z_0, z_1\rangle\rangle \in Z_{j+1}$ and $\langle x, x'\rangle \in R^{\mathfrak{A}}_m$. Then $\langle x', z_0', z_1'\rangle \in Q_2$. Hence $\langle x', \langle x', z_0', z_1'\rangle\rangle \in Z_j$ and $\langle\langle x, z_0, z_1\rangle, \langle x', z_0', z_1'\rangle\rangle \in R^{\mathfrak{C}}_m$; and by the inductive hypothesis it follows that $\langle \mathfrak{A}, x'\rangle \simeq_{j\,\tau_\vartheta} \langle \mathfrak{C}, \langle x', z_0', z_1'\rangle\rangle$. Now let us show the other case. Assume $m \in \mathsf{Mod} \cap \mathsf{Mod}'$, $\langle x, \langle x, z_0, z_1\rangle\rangle \in Z_{j+1}$ and $\langle x, x'\rangle \in R^{\mathfrak{A}}_m$. Then, the assumption in the SW-vocabularies says that either $m \in \mathsf{SMod} \cap \mathsf{SMod}'$ or $m \in \mathsf{WMod} \cap \mathsf{WMod}'$. If $z_0 = -1$, then it is clear that $\langle x', -1, \bullet\rangle \in Q_2$, $\langle x', \langle x', -1, \bullet\rangle\rangle \in Z_j$ and $\langle\langle x, z_0, z_1\rangle, \langle x', -1, \bullet\rangle\rangle \in R^{\mathfrak{C}}_m$. Hence we can assume that $z_0 \neq -1$, i.e., $z_0 = j+1$ and $\langle x, z_0, z_1\rangle \in Q_0$. Therefore $\langle \mathfrak{A}, x\rangle \simeq_{j+1\,\tau_{\vartheta\cap\vartheta'}} \langle \mathfrak{B}, z_1\rangle$. By this bounded bisimilarity it follows that there is z_1' such that $\langle z_1, z_1'\rangle \in R^{\mathfrak{B}}_m$ and $\langle \mathfrak{A}, x'\rangle \simeq_{j\,\tau_{\vartheta\cap\vartheta'}} \langle \mathfrak{B}, z_1'\rangle$. Thus, $\langle x', j, z_1'\rangle \in Q_0$, $\langle\langle x, z_0, z_1\rangle, \langle x', j, z_1'\rangle\rangle \in R^{\mathfrak{C}}_m$ and $\langle x', \langle x', j, z_1'\rangle\rangle \in Z_j$. By the inductive hypothesis it follows that $\langle \mathfrak{A}, x'\rangle \simeq_{j\,\tau_\vartheta} \langle \mathfrak{C}, \langle x', j, z_1'\rangle\rangle$.

$\dashv$

Claim II: $Z := \{\langle\langle z_0, z_1, y\rangle, y\rangle : \langle z_0, z_1, y\rangle \in Q_0 \cup Q_3\}$ is a $\tau_{\vartheta'}$-bisimulation between $\mathfrak{C}$ and $\mathfrak{B}$.

Proof of Claim: We must show the following clauses.

(bis1): It is straightforward.

(bis2): We distinguish two cases. First of all, assume that $m \in \mathsf{Mod}' \setminus \mathsf{Mod}$, $\langle\langle z_0, z_1, y\rangle, y\rangle \in Z$ and $\langle y, y'\rangle \in R^{\mathfrak{B}}_m$. Then, $\langle\langle z_0, z_1, y\rangle, \langle \star, -1, y'\rangle\rangle \in R^{\mathfrak{C}}_m$ and $\langle\langle \star, -1, y'\rangle, y'\rangle \in Z$. Now let us see the second case. Assume that $m \in \mathsf{Mod} \cap \mathsf{Mod}'$, $\langle\langle z_0, z_1, y\rangle, y\rangle \in Z$ and $\langle y, y'\rangle \in R^{\mathfrak{B}}_m$. Then the assumption in the SW-vocabularies says that either $m \in \mathsf{SMod} \cap \mathsf{SMod}'$ or $m \in \mathsf{WMod} \cap \mathsf{WMod}'$. If $z_1 \in \{-1, 0\}$, then it is clear that $\langle\langle \star, -1, y'\rangle, y'\rangle \in Z$ and $\langle\langle z_0, z_1, y\rangle, \langle \star, -1, y'\rangle\rangle \in R^{\mathfrak{C}}_m$. Hence, we can assume that $z_1 \notin \{-1, 0\}$, i.e., $z_1 \in \{1, \ldots, n\}$. Thus $\langle z_0, z_1, y\rangle \in Q_0$. Hence $\langle \mathfrak{A}, z_0\rangle \simeq_{z_1\,\tau_{\vartheta\cap\vartheta'}} \langle \mathfrak{B}, y\rangle$. Therefore there exists z_0' such that $\langle z_0, z_0'\rangle \in R^{\mathfrak{A}}_m$ and $\langle \mathfrak{A}, z_0'\rangle \simeq_{z_1-1\,\tau_{\vartheta\cap\vartheta'}} \langle \mathfrak{B}, y'\rangle$. It follows that $\langle z_0', z_1 - 1, y'\rangle \in Q_0$, $\langle\langle z_0', z_1 - 1, y'\rangle, y'\rangle \in Z$, and $\langle\langle z_0, z_1, y\rangle, \langle z_0', z_1 - 1, y'\rangle\rangle \in R^{\mathfrak{C}}_m$.

(bis3): Assume $m \in \mathsf{Mod}'$, $\langle\langle z_0, z_1, y\rangle, y\rangle \in Z$ and $\langle\langle z_0, z_1, y\rangle, \langle z_0', z_1', y'\rangle\rangle \in R_m^{\mathfrak{C}}$. A careful reading of our definition shows that $\langle z_0', z_1', y'\rangle \in Q_0 \cup Q_3$ and $\langle y, y'\rangle \in R_m^{\mathfrak{B}}$. Hence it is clear that $\langle\langle z_0', z_1', y'\rangle, y'\rangle \in Z$.

$\dashv$

Claim III: $\langle \mathfrak{A}, a\rangle \preceq_{n\vartheta} \langle \mathfrak{C}, c\rangle$.

Proof of Claim: We must show the following clauses.

(qbbis1): It is straightforward.

(qbbis2): Assume that $s \in \mathsf{SMod}$, and let c' be such that $\langle c, c'\rangle \in R_s^{\mathfrak{C}}$. Now we distinguish two cases:

i. Case $s \in \mathsf{SMod} \cap \mathsf{Mod}'$: it holds that $c' = \langle x', n-1, y'\rangle \in Q_0$ where $\langle a, x'\rangle \in R_s^{\mathfrak{A}}$, $\langle b, y'\rangle \in R_s^{\mathfrak{B}}$, and $\langle \mathfrak{A}, x'\rangle \simeq_{n-1\,\tau_{\vartheta\cap\vartheta'}} \langle \mathfrak{B}, y'\rangle$. And by the first claim it follows that $\langle \mathfrak{A}, x'\rangle \simeq_{n-1\,\tau_\vartheta} \langle \mathfrak{C}, c'\rangle$.

ii. Case $s \in \mathsf{SMod} \setminus \mathsf{Mod}'$: it holds that $c' = \langle x', -1, \bullet\rangle \in Q_2$ and $\langle a, x'\rangle \in R_s^{\mathfrak{A}}$. By the first claim we conclude that $\langle \mathfrak{A}, x'\rangle \simeq_{n-1\,\tau_\vartheta} \langle \mathfrak{C}, c'\rangle$.

(qbbis3): Assume that $w \in \mathsf{WMod}$, and let $a' \in A$ be such that $\langle a, a'\rangle \in R_w^{\mathfrak{A}}$. We distinguish two cases:

i. Case $w \in \mathsf{WMod} \cap \mathsf{Mod}'$: then $w \in \mathsf{WMod} \cap \mathsf{WMod}'$ because $\mathsf{WMod} \cap \mathsf{SMod}' \subseteq \mathsf{WMod}'$. By $\langle \mathfrak{A}, a\rangle \preceq_{n\,\vartheta\cap\vartheta'} \langle \mathfrak{B}, b\rangle$ it is deduced that there is b' such that $\langle b, b'\rangle \in R_w^{\mathfrak{B}}$ and $\langle \mathfrak{A}, a'\rangle \simeq_{n-1\,\tau_{\vartheta\cap\vartheta'}} \langle \mathfrak{B}, b'\rangle$. Hence, $\langle a', n-1, b'\rangle \in Q_0$ and $\langle c, \langle a', n-1, b'\rangle\rangle \in R_w^{\mathfrak{C}}$. And by Claim I it follows that $\langle \mathfrak{A}, a'\rangle \simeq_{n-1\,\tau_\vartheta} \langle \mathfrak{C}, \langle a', n-1, b'\rangle\rangle$.

ii. Case $w \in \mathsf{WMod} \setminus \mathsf{Mod}'$: it holds that $\langle c, \langle a', -1, \bullet\rangle\rangle \in R_w^{\mathfrak{C}}$, and by the first claim it is known that $\langle \mathfrak{A}, a'\rangle \simeq_{n-1\,\tau_\vartheta} \langle \mathfrak{C}, \langle a', -1, \bullet\rangle\rangle$.

$\dashv$

Claim IV: $\langle \mathfrak{C}, c\rangle \preceq_{\vartheta'} \langle \mathfrak{B}, b\rangle$.

Proof of Claim: Let us show the clauses involved in the quasi bisimilarity notion.

(qbis1): It is straightforward.

(qbis2): Assume that $s \in \mathsf{SMod}'$, and let $b' \in B$ be such that $\langle b, b'\rangle \in R_s^{\mathfrak{B}}$. We distinguish two cases:

i. Case $s \in \mathsf{SMod}' \cap \mathsf{Mod}$: then $s \in \mathsf{SMod} \cap \mathsf{SMod}'$ because $\mathsf{SMod}' \cap \mathsf{WMod} \subseteq \mathsf{SMod}$. By $\langle \mathfrak{A}, a\rangle \preceq_{n\,\vartheta\cap\vartheta'} \langle \mathfrak{B}, b\rangle$ it is deduced that there is a' such that $\langle a, a'\rangle \in R_s^{\mathfrak{A}}$ and $\langle \mathfrak{A}, a'\rangle \simeq_{n-1\,\tau_{\vartheta\cap\vartheta'}} \langle \mathfrak{B}, b'\rangle$. Hence, $\langle a', n-1, b'\rangle \in Q_0$ and $\langle c, \langle a', n-1, b'\rangle\rangle \in R_w^{\mathfrak{C}}$. By the second claim we conclude that $\langle \mathfrak{C}, \langle a', n-1, b'\rangle\rangle \simeq_{\tau_{\vartheta'}} \langle \mathfrak{B}, b'\rangle$.

ii. Case $s \in \mathsf{SMod}' \setminus \mathsf{Mod}$: it holds that $\langle c, \langle \star, -1, b' \rangle \rangle \in R_s^{\mathfrak{C}}$, and by the second claim it follows that $\langle \mathfrak{C}, \langle \star, -1, b' \rangle \rangle \simeq_{\tau_{\vartheta'}} \langle \mathfrak{B}, b' \rangle$.

(qbis3): Assume that $w \in \mathsf{WMod}'$, and let c' be such that $\langle c, c' \rangle \in R_w^{\mathfrak{C}}$. Now we distinguish two cases:

i. Case $w \in \mathsf{WMod}' \cap \mathsf{Mod}$: it holds that $c' = \langle x', n-1, y' \rangle \in Q_0$ where $\langle a, x' \rangle \in R_w^{\mathfrak{A}}$, $\langle b, y' \rangle \in R_w^{\mathfrak{B}}$, and $\langle \mathfrak{A}, x' \rangle \simeq_{n-1\, \tau_{\vartheta \cap \vartheta'}} \langle \mathfrak{B}, y' \rangle$. And by the second claim it follows that $\langle \mathfrak{C}, c' \rangle \simeq_{\tau_{\vartheta'}} \langle \mathfrak{B}, y' \rangle$.

ii. Case $w \in \mathsf{WMod}' \setminus \mathsf{Mod}$: it holds that $c' = \langle \star, -1, y' \rangle \in Q_3$ and $\langle b, y' \rangle \in R_s^{\mathfrak{A}}$. By Claim II we conclude that $\langle \mathfrak{C}, c' \rangle \simeq_{\tau_{\vartheta'}} \langle \mathfrak{B}, y' \rangle$.

$\dashv$

The last two claims give us what we want, so the proof is completed. □

3.8.4. Example. Let us show that the previous proposition is false if we consider quasi ω-bisimilarity[25]. Let $\vartheta := \langle \{s\}, \emptyset, \{p_n : n \in \omega\} \rangle$ and $\vartheta' := \langle \{s\}, \emptyset, \emptyset \rangle$. Let $\mathfrak{A}$ be the τ_ϑ-structure depicted in Figure 1.5 on page 23 assuming that different states are distinguished by propositions (this is possible because there is a countable number of propositions). And let $\mathfrak{B}$ be the $\tau_{\vartheta'}$-structure also depicted in the same Figure. It is clear that $\langle \mathfrak{A}, a \rangle \preceq_{\omega\, \vartheta \cap \vartheta'} \langle \mathfrak{B}, b \rangle$ (indeed, they are ω-bisimilar). However, let us see that there is no $\langle \mathfrak{C}, c \rangle$ such that $\langle \mathfrak{A}, a_0 \rangle \preceq_{\omega\, \vartheta} \langle \mathfrak{C}, c \rangle \preceq_{\vartheta'} \langle \mathfrak{B}, b \rangle$. Assume that there is a pointed structure $\langle \mathfrak{C}, c \rangle$ with this property. By $\langle \mathfrak{A}, a \rangle \preceq_{\omega\, \vartheta} \langle \mathfrak{C}, c \rangle$ it follows that there is no infinite path from c (because all states behave differently with respect to propositions). And this contradicts the fact that $\langle \mathfrak{C}, c \rangle \preceq_{\vartheta'} \langle \mathfrak{B}, b \rangle$.

3.8.5. Corollary. *Let ϑ and ϑ' be two SW-vocabularies such that* $\mathsf{SMod}' \cap \mathsf{WMod} \subseteq \mathsf{WMod}' \cap \mathsf{SMod} \subseteq \mathsf{WMod}$, *let $\langle \mathfrak{A}, a \rangle$ be a pointed τ_ϑ-structure, let $\langle \mathfrak{B}, b \rangle$ be a pointed $\tau_{\vartheta'}$-structure, and let $n \in \omega$.*

1. *If $\langle \mathfrak{A}, a \rangle \preceq_{\vartheta \cap \vartheta'} \langle \mathfrak{B}, b \rangle$, then there is a pointed $\tau_{\vartheta \cup \vartheta'}$-structure $\langle \mathfrak{C}, c \rangle$ such that $\langle \mathfrak{A}, a \rangle \preceq_{\vartheta} \langle \mathfrak{C}, c \rangle \preceq_{\vartheta'} \langle \mathfrak{B}, b \rangle$.*
2. *If $\langle \mathfrak{A}, a \rangle \preceq_{n\, \vartheta \cap \vartheta'} \langle \mathfrak{B}, b \rangle$, then there is a pointed $\tau_{\vartheta \cup \vartheta'}$-structure $\langle \mathfrak{C}, c \rangle$ such that $\langle \mathfrak{A}, a \rangle \preceq_{\vartheta} \langle \mathfrak{C}, c \rangle \preceq_{n\, \vartheta'} \langle \mathfrak{B}, b \rangle$.*

Proof: They are the duals of Propositions 3.8.1 and 3.8.3. □

Finally we single out a particular case where the previous constructions yields something stronger. Even in the case that ϑ and ϑ' are only distinguished by its propositions it can happen that the assumptions of the next proposition do not hold.

[25] It is not hard to transform this example to show that [Vis96b, Lemma 6.1] is false for ω-bisimilarity.

3.8.6. Proposition. *Let ϑ and ϑ' be two SW-vocabularies, let $\langle \mathfrak{A}, a\rangle$ be a pointed τ_ϑ-structure, and let $\langle \mathfrak{B}, b\rangle$ be a pointed $\tau_{\vartheta'}$-structure, and let $n \in \omega$.*

1. *If* $\mathsf{WMod} \cap \mathsf{SMod}' \subseteq \mathsf{WMod}'$, $\mathsf{Mod} \cap \mathsf{Mod}' \subseteq \mathsf{SMod} \cap \mathsf{SMod}'$ *and* $\langle \mathfrak{A}, a\rangle \preceq_{\vartheta\cap\vartheta'} \langle \mathfrak{B}, b\rangle$*, then there is a pointed* $\tau_{\vartheta\cup\vartheta'}$*-structure* $\langle \mathfrak{C}, c\rangle$ *such that* $\langle \mathfrak{A}, a\rangle \preceq_{\vartheta} \langle \mathfrak{C}, c\rangle \simeq_{\tau_{\vartheta'}} \langle \mathfrak{B}, b\rangle$.
2. *If* $\mathsf{WMod} \cap \mathsf{SMod}' \subseteq \mathsf{WMod}'$, $\mathsf{Mod} \cap \mathsf{Mod}' \subseteq \mathsf{SMod} \cap \mathsf{SMod}'$ *and* $\langle \mathfrak{A}, a\rangle \preceq_{n\,\vartheta\cap\vartheta'} \langle \mathfrak{B}, b\rangle$*, then there is a pointed* $\tau_{\vartheta\cup\vartheta'}$*-structure* $\langle \mathfrak{C}, c\rangle$ *such that* $\langle \mathfrak{A}, a\rangle \preceq_{n\,\vartheta} \langle \mathfrak{C}, c\rangle \simeq_{\tau_{\vartheta'}} \langle \mathfrak{B}, b\rangle$.
3. *If* $\mathsf{SMod}' \cap \mathsf{WMod} \subseteq \mathsf{SMod}$, $\mathsf{Mod} \cap \mathsf{Mod}' \subseteq \mathsf{WMod} \cap \mathsf{WMod}'$ *and* $\langle \mathfrak{A}, a\rangle \preceq_{\vartheta\cap\vartheta'} \langle \mathfrak{B}, b\rangle$*, then there is a pointed* $\tau_{\vartheta\cup\vartheta'}$*-structure* $\langle \mathfrak{C}, c\rangle$ *such that* $\langle \mathfrak{A}, a\rangle \simeq_{\tau_\vartheta} \langle \mathfrak{C}, c\rangle \preceq_{\vartheta'} \langle \mathfrak{B}, b\rangle$.
4. *If* $\mathsf{SMod}' \cap \mathsf{WMod} \subseteq \mathsf{SMod}$, $\mathsf{Mod} \cap \mathsf{Mod}' \subseteq \mathsf{WMod} \cap \mathsf{WMod}'$ *and* $\langle \mathfrak{A}, a\rangle \preceq_{n\,\vartheta\cap\vartheta'} \langle \mathfrak{B}, b\rangle$*, then there is a pointed* $\tau_{\vartheta\cup\vartheta'}$*-structure* $\langle \mathfrak{C}, c\rangle$ *such that* $\langle \mathfrak{A}, a\rangle \simeq_{\tau_\vartheta} \langle \mathfrak{C}, c\rangle \preceq_{n\,\vartheta'} \langle \mathfrak{B}, b\rangle$.

Proof: Due to duality it is enough to show the first two items. First of all we notice that in these items from the assumptions adopted it follows that $\mathsf{WMod} \cap \mathsf{SMod}' \subseteq \mathsf{SMod} \cap \mathsf{WMod}' \subseteq \mathsf{SMod}'$.

1) Let $\langle \mathfrak{C}, c\rangle$ be the same structure that we built in the proof of Proposition 3.8.1. We saw there that $\langle \mathfrak{A}, a\rangle \preceq_\vartheta \langle \mathfrak{C}, c\rangle$. And now let us see an improvement of the second claim of that proof.

Claim I : $Z := \{\langle y, y\rangle : y \in B\} \cup \{\langle\langle x, y\rangle, y\rangle \in Q \times B : \langle \mathfrak{A}, x\rangle \simeq_{\tau_{\vartheta\cap\vartheta'}} \langle \mathfrak{B}, y\rangle\} \cup \{\langle\langle a, b\rangle, b\rangle\}$ is a $\tau_{\vartheta'}$-bisimulation between $\mathfrak{C}$ and $\mathfrak{B}$.

Proof of Claim: It is straightforward. We only consider the case in which we have a state $y' \in B$ such that $\langle b, y'\rangle \in R^{\mathfrak{B}}_m$ where $m \in \mathsf{Mod} \cap \mathsf{Mod}'$, and let us show that there is $x' \in A$ such that $\langle\langle x', y'\rangle, y'\rangle \in Z$ and $\langle\langle a, b\rangle, \langle x', y'\rangle\rangle \in R^{\mathfrak{C}}_m$. Then $m \in \mathsf{SMod} \cap \mathsf{SMod}'$. By $\langle \mathfrak{A}, a\rangle \preceq_{\vartheta\cap\vartheta'} \langle \mathfrak{B}, b\rangle$ it follows that there exists $x' \in A$ such that $\langle a, x'\rangle \in R^{\mathfrak{A}}_m$ and $\langle \mathfrak{A}, x'\rangle \simeq_{\tau_{\vartheta\cap\vartheta'}} \langle \mathfrak{B}, y'\rangle$. Hence $\langle x', y'\rangle \in Q$ and $\langle\langle a, b\rangle, \langle x', y'\rangle\rangle \in R^{\mathfrak{C}}_m$. Thus $\langle\langle x', y'\rangle, y'\rangle \in Z$. ⊣

By this claim it follows that $\langle \mathfrak{C}, c\rangle \simeq_{\tau_{\vartheta'}} \langle \mathfrak{B}, b\rangle$. This completes the proof.

2) For $n = 0$ this is very easy. Assume now $n > 0$. Let $\langle \mathfrak{C}, c\rangle$ be the same structure that we built in the proof of Proposition 3.8.3. We saw there that $\langle \mathfrak{A}, a\rangle \preceq_{n\,\vartheta} \langle \mathfrak{C}, c\rangle$. Let us show an improvement of the second claim of that proof.

Claim II: $Z := \{\langle\langle z_0, z_1, y\rangle, y\rangle : \langle z_0, z_1, y\rangle \in Q_0 \cup Q_3\} \cup \{\langle\langle a, n, b\rangle, b\rangle\}$ is a $\tau_{\vartheta'}$-bisimulation between $\mathfrak{C}$ and $\mathfrak{B}$.

Proof of Claim: Here we only consider the case in which we have a state $y' \in B$ such that $\langle b, y'\rangle \in R^{\mathfrak{B}}_m$ where $m \in \mathsf{Mod} \cap \mathsf{Mod}'$, and let us show that there is $x' \in A$ such that $\langle\langle x', n-1, y'\rangle, y'\rangle \in Z$ and $\langle\langle a, n, b\rangle, \langle x', n-1, y'\rangle\rangle \in R^{\mathfrak{C}}_m$. Then

$m \in \mathsf{SMod} \cap \mathsf{SMod}'$. By $\langle \mathfrak{A}, a\rangle \preceq_{n\,\vartheta\cap\vartheta'} \langle \mathfrak{B}, b\rangle$ it follows that there is $x' \in A$ such that $\langle a, x'\rangle \in R_m^{\mathfrak{A}}$ and $\langle \mathfrak{A}, x'\rangle \simeq_{n-1\,\tau_{\vartheta\cap\vartheta'}} \langle \mathfrak{B}, y'\rangle$. Hence $\langle x', n-1, y'\rangle \in Q_0$, $\langle\langle a, n, b\rangle, \langle x', n-1, y'\rangle\rangle \in R_m^{\mathfrak{C}}$ and $\langle\langle x', n-1, y'\rangle, y'\rangle \in Z$. ⊣

By this claim it follows that $\langle \mathfrak{C}, c\rangle \simeq_{\tau_{\vartheta'}} \langle \mathfrak{B}, b\rangle$. This completes the proof. □

3.9 Inclusion among non-well-founded sets

In this section we assume that there are no propositions. In the (**ZFC**$^-$ + **AFA**) axiomatization of the theory of non-well founded sets it is known that the equality relation between sets coincides with the bisimilarity relation. We have already seen that $\preceq_w$ is a quasi order that generates the bisimilarity relation. Hence, $\preceq_w$ is a quasi order that generates the equality relation between sets. Among the quasi orders that generate this relation the most famous one is set inclusion. So it is natural to ask whether $\preceq_w$ is the inclusion relation. We will see that the answer is positive. We have also seen that $\preceq_s$ is a quasi order generating the bisimilarity relation. In this case what results is that $\preceq_s$ coincides with the inverse of inclusion.

3.9.1. PROPOSITION. *(**ZFC**$^-$) Let $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ be two pointed structures. If there are decorations $d_{\mathfrak{A}}$ and $d_{\mathfrak{B}}$ of, respectively, $\mathfrak{A}$ and $\mathfrak{B}$ such that $d_{\mathfrak{A}}(a) \subseteq d_{\mathfrak{B}}(b)$, then $\langle \mathfrak{A}, a\rangle \preceq_w \langle \mathfrak{B}, b\rangle$ and $\langle \mathfrak{B}, b\rangle \preceq_s \langle \mathfrak{A}, a\rangle$.*

Proof: By duality[26] it is enough to check that $\langle \mathfrak{A}, a\rangle \preceq_w \langle \mathfrak{B}, b\rangle$. As there are neither propositions nor strict modalities, it only remains to show (qbis3). Let a' be such that $\langle a, a'\rangle \in R_w^{\mathfrak{A}}$. Hence,

$$d_{\mathfrak{A}}(a') \in d_{\mathfrak{A}}(a) \subseteq d_{\mathfrak{B}}(b) = \{d_{\mathfrak{B}}(b') : \langle b, b'\rangle \in R_w^{\mathfrak{B}}\}.$$

Therefore there exists b' such that $\langle b, b'\rangle \in R_w^{\mathfrak{B}}$ and $d_{\mathfrak{A}}(a') = d_{\mathfrak{B}}(b')$. By Proposition 1.3.20 (in fact, this can be easily proved observing that $\{\langle a'', b''\rangle : d_{\mathfrak{A}}(a'') = d_{\mathfrak{B}}(b'')\}$ is a bisimulation) it results that $\langle \mathfrak{A}, a'\rangle \simeq \langle \mathfrak{B}, b'\rangle$. □

3.9.2. PROPOSITION. *(**ZFC**$^-$ + **AFA**) Let $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ be pointed structures. The following are equivalent:*

1. *There are decorations $d_{\mathfrak{A}}$ and $d_{\mathfrak{B}}$ of, respectively, $\mathfrak{A}$ and $\mathfrak{B}$ such that $d_{\mathfrak{A}}(a) \subseteq d_{\mathfrak{B}}(b)$.*
2. *$\langle \mathfrak{A}, a\rangle \preceq_w \langle \mathfrak{B}, b\rangle$.*
3. *$\langle \mathfrak{B}, b\rangle \preceq_s \langle \mathfrak{A}, a\rangle$.*

[26] Here the fact that there are no propositions plays a role. This implies that the dual of a structure coincides with the initial structure.

Proof: By duality it is enough to check the equivalence between the first two statements. We already proved the downward direction in ($\mathbf{ZFC}^-$). For the converse, let $d_{\mathfrak{A}}$ and $d_{\mathfrak{B}}$ be the decorations of, respectively, $\mathfrak{A}$ and $\mathfrak{B}$. Their existence and uniqueness is given by the Antifoundation Axiom. By $\langle \mathfrak{A}, a\rangle \preceq_w \langle \mathfrak{B}, b\rangle$ together with the axiom of choice we consider a map $f : R_w^{\mathfrak{A}}[\{a\}] \longrightarrow R_w^{\mathfrak{B}}[\{b\}]$ such that $\langle \mathfrak{A}, a'\rangle \simeq \langle \mathfrak{B}, f(a')\rangle$ for every $a' \in R_w^{\mathfrak{A}}[\{a\}$. By Proposition 1.3.22 together with the uniqueness of decoration imposed by ($\mathbf{AFA}$) it is clear that $d_{\mathfrak{A}}(a') = d_{\mathfrak{B}}(f(a'))$ for every $a' \in R_w^{\mathfrak{A}}[\{a\}$. Hence,

$$d_{\mathfrak{A}}(a) = \{d_{\mathfrak{A}}(a') : \langle a, a'\rangle \in R_w^{\mathfrak{A}}\} \subseteq \{d_{\mathfrak{B}}(b') : \langle b, b'\rangle \in R_w^{\mathfrak{B}}\} = d_{\mathfrak{B}}(b).$$

This completes the proof. □

3.9.3. Proposition. *(*$\mathbf{ZFC}^- + \mathbf{AFA}$*) Let a and b be two sets. Then,*

$$a \subseteq b \quad \textit{iff} \quad \langle \mathfrak{G}_a, a\rangle \preceq_w \langle \mathfrak{G}_b, b\rangle \quad \textit{iff} \quad \langle \mathfrak{G}_b, b\rangle \preceq_s \langle \mathfrak{G}_a, a\rangle.$$

Proof: Once more, by duality, it is enough to show the first equivalence. In canonical structures, condition (qbis3) can be rewritten as:

- for every a', if $a' \in a$, then there is b' such that $b' \in b$ and $\langle \mathfrak{G}_a, a'\rangle \simeq \langle \mathfrak{G}_b, b'\rangle$.

By Proposition 1.3.23 it is not hard to see that this is equivalent to say that

- for every a', if $a' \in a$, then there is b' such that $b' \in b$ and $b' = a'$.

And this says that $a \subseteq b$. □

3.9.4. Theorem. *(*$\mathbf{ZFC}^- + \mathbf{AFA}$*) Let a be a set.*

1. *There is a $\mathcal{L}_\infty^w$-formula π_w^a such that for every set b,* $a \subseteq b$ *iff* $\mathfrak{G}_b, b \Vdash \pi_w^a$.
2. *There is a $\mathcal{L}_\infty^w$-formula ν_w^a such that for every set b,* $b \subseteq a$ *iff* $\mathfrak{G}_b, b \nVdash \nu_w^a$.
3. *There is a $\mathcal{L}_\infty^s$-formula ν_s^a such that for every set b,* $a \subseteq b$ *iff* $\mathfrak{G}_b, b \nVdash \nu_s^a$.
4. *There is a $\mathcal{L}_\infty^s$-formula π_s^a such that for every set b,* $b \subseteq a$ *iff* $\mathfrak{G}_b, b \Vdash \pi_s^a$.

Proof: This is a trivial consequence of Theorem 3.2.11 together with the last proposition. Nevertheless, it is not hard to give direct arguments, which are more illustrative. Observe that

$$\begin{array}{lllllll}
(1) & a \subseteq b & \text{iff} & \forall x(x \in a \supset x \in b) & \text{iff} & \mathfrak{G}_b, b \Vdash \bigwedge\{\Diamond\phi^x : x \in a\}, \\
(2) & b \subseteq a & \text{iff} & \forall x(x \in b \supset x \in a) & \text{iff} & \mathfrak{G}_b, b \nVdash \Diamond\bigwedge\{\sim\phi^x : x \in a\}, \\
(3) & a \subseteq b & \text{iff} & \forall x(x \in a \supset x \in b) & \text{iff} & \mathfrak{G}_b, b \nVdash \bigvee\{\Box\sim\phi^x : x \in a\}, \\
(4) & b \subseteq a & \text{iff} & \forall x(x \in b \supset x \in a) & \text{iff} & \mathfrak{G}_b, b \Vdash \Box\bigvee\{\phi^x : x \in a\},
\end{array}$$

where the formulas ϕ^x are the ones given by Theorem 1.3.25. □

In previous sections we introduced quasi bisimilarity as a tool suited to analyze the strict-weak fragments. However, in this section we have considered an inverse situation. We have a natural concept in the framework of ($\mathbf{ZFC}^- + \mathbf{AFA}$), the concept of inclusion. We can then design the language $\mathcal{L}^w_\infty$ as a formalism to talk about inclusion, and, analogously, for the inverse of inclusion and the language $\mathcal{L}^s_\infty$.

3.9.5. Remark. (What happens when there are propositions?) We remind the reader that in this general case the universe is expanded with urelements (see Remark 1.3.26). The same proofs work, but now we should be careful in the duality arguments because the dual of a structure is not the same structure. Thus, Proposition 3.9.3 in the general case says that

$$a \subseteq b \quad \text{iff} \quad \langle \mathfrak{G}_a, a\rangle \preceq_w \langle \mathfrak{G}_b, b\rangle \quad \text{iff} \quad \langle \mathfrak{G}^d_b, b\rangle \preceq_s \langle \mathfrak{G}^d_a, a\rangle.$$

Therefore, in the general case inclusion corresponds to $\preceq_w$, but the inverse of inclusion does not coincide with $\preceq_s$. The first two items of Theorem 3.9.4 remain valid in the general case.

3.10 The case of arbitrary first-order structures

It is known that the proof of Theorem 1.3.8 can be generalized a great deal, with first-order definable modalities (see [dR93, Section 6.7] and [Hol98, Section 2.4]). We will see in this section how we can develop the same ideas in the strict-weak setting. But first of all we summarize what is done in the modal case.

Throughout this section we will assume that τ is an arbitrary first-order vocabulary, i.e., we have relation symbols of finite arity (including 0, i.e., constants). Perhaps there are symbols of arity different than 1 or 2. A *first-order normal modality over* τ is a triple $\langle \natural, n, \chi(v_0, \ldots, v_n)\rangle$ such that (i) $\natural$ is a symbol called the *identifier*, (ii) $n \in \omega$ is called the *arity*, and (iii) χ is a first-order formula (called the *table*) in τ with at most free variables in $v_0, \ldots, v_n$. It is said that MOD is a *a family of first-order normal modalities* if it is a set of first-order normal modalities satisfying that their first components are pairwise different. Hence each identifier is univocally attached to a certain first-order modality of the family. Given a family MOD of first-order modalities the set of *MOD-formulas* is defined as follows:

$$\varphi ::= \bot \mid \top \mid \sim\varphi \mid \varphi_0 \wedge \varphi_1 \mid \natural(\varphi_0, \ldots, \varphi_{n-1}),$$

where $\natural$ is an n-ary first-order normal modality in MOD. We introduce the rest of Boolean connectives as is customary. All MOD-formulas can be seen as first-order formulas in τ with at most one free variable. The *standard translation*

$ST_v(\varphi)$ of a MOD-formula φ is defined in such a way that it commutes with the booleans, and if χ is the table of an n-ary $\natural$ then $ST_v(\natural(\varphi_0,\dots,\varphi_{n-1}))$ is the formula

$$\forall v_1 \dots \forall v_n \left(\chi(v_0,\dots,v_n) \supset \bigvee_{i<n} ST_{v_{i+1}}(\varphi_i) \right).$$

3.10.1. Remark. The modal formulas considered on page 6 can be seen as a particular case of this definition; just take MOD as the set $\{\langle p, 0, \sim Pv_0\rangle : p \in \mathsf{Prop}\} \cup \{\langle [m], 1, R_m v_0 v_1\rangle : m \in \mathsf{Mod}\}$. We suggest the reader interested in more examples of the use of first-order (even not normal) modalities to examine [Hol98, p. 20-21].

Let $\mathfrak{A}, \mathfrak{B}$ be a pair of structures in the vocabulary τ. A relation $Z \subseteq A \times B$ is a *MOD-bisimulation* between $\mathfrak{A}$ and $\mathfrak{B}$ if whenever $\natural$ is an n-ary normal modality in MOD with table χ it holds that:

- If $\langle a, b\rangle \in Z$ and $\mathfrak{B} \models \chi[b, b_1, \dots, b_n]$ then there are $a_1, \dots, a_n \in A$ with $\mathfrak{A} \models \chi[a, a_1, \dots, a_n]$ and $\langle a_i, b_i\rangle \in Z$ for $1 \le i \le n$.
- If $\langle a, b\rangle \in Z$ and $\mathfrak{A} \models \chi[a, a_1, \dots, a_n]$ then there are $b_1, \dots, b_n \in B$ with $\mathfrak{B} \models \chi[b, b_1, \dots, b_n]$ and $\langle a_i, b_i\rangle \in Z$ for $1 \le i \le n$.

3.10.2. Theorem. [Hol98, Theorem 2.4.2] *Let τ be a first-order vocabulary, let $\varphi(v_0)$ be a first-order formula in τ, let K be a class of structures in τ closed under ultraproducts, and let MOD be a family of first-order normal modalities. The following are equivalent:*

1. *φ is, up to K-equivalence, a MOD-formula.*
2. *φ is K-invariant under MOD-bisimulations.*

Now it is time to generalize what we have seen for strict-weak fragments to this general framework.

3.10.3. Definition. (SW-formulas and its semantics)
Let τ be a first-order vocabulary. A *family of first-order strict-weak normal modalities* is a pair $\langle SMOD, WMOD\rangle$ of families of first-order normal modalities. Given $SW = \langle SMOD, WMOD\rangle$ a family of first-order strict-weak normal modalities, the set of *SW-formulas* is defined as follows:

$$\varphi ::= \bot \mid \top \mid \varphi_0 \wedge \varphi_1 \mid \varphi_0 \vee \varphi_1 \mid \partial_\sharp(\varphi_0,\dots,\varphi_{2n-1}) \mid \mathfrak{G}_\flat(\varphi_0,\dots,\varphi_{2n-1}),$$

where $\sharp, \flat$ are n-ary first-order normal modalities such that $\sharp \in SMOD$ and $\flat \in WMOD$. The *standard translation* $ST_v(\varphi)$ of a SW-formula φ is defined in such a way that it commutes with the booleans involved in the definition, and

- if χ is the table of an n-ary $\sharp$ then $ST_v(\partial_\sharp(\varphi_0,\dots,\varphi_{2n-1}))$ is the formula

$$\forall v_1 \dots \forall v_n \left(\chi(v_0,\dots,v_n) \supset \bigvee \{ST_{v_{i+1}}(\varphi_i) \supset ST_{v_{i+1}}(\varphi_{n+i}) : i < n\}\right),$$

- if χ is the table of an n-ary $\flat$ then $ST_v(\mathfrak{G}_\flat(\varphi_0,\dots,\varphi_{2n-1}))$ is the formula

$$\exists v_1 \dots \exists v_n \left(\chi(v_0,\dots,v_n) \wedge \bigwedge \{ST_{v_{i+1}}(\varphi_i) \smallsetminus ST_{v_{i+1}}(\varphi_{n+i}) : i < n\}\right).$$

3.10.4. REMARK. Strict-weak formulas introduced in Definition 2.1.2 correspond to the case in which $SMOD = \{\langle p, 0, \sim Pv_0\rangle : p \in \mathsf{Prop}\} \cup \{\langle [s], 1, R_s v_0 v_1\rangle : s \in \mathsf{SMod}\}$ and $WMOD = \{\langle\langle w\rangle, 1, R_s v_0 v_1\rangle : w \in \mathsf{WMod}\}$. Another way to obtain them is to consider $SMOD = \{\langle [s], 1, R_s v_0 v_1\rangle : s \in \mathsf{SMod}\}$ and $WMOD = \{\langle p, 0, Pv_0\rangle : p \in \mathsf{Prop}\} \cup \{\langle\langle w\rangle, 1, R_s v_0 v_1\rangle : w \in \mathsf{WMod}\}$.

It is obvious that $\partial_\sharp(\varphi_0,\dots,\varphi_{2n-1})$ is equivalent, under the standard translation, to

$$\sharp(\varphi_0 \supset \varphi_n, \varphi_1 \supset \varphi_{n+1}, \dots, \varphi_{n-1} \supset \varphi_{2n-1}).$$

And $\mathfrak{G}_\flat(\varphi_0,\dots,\varphi_{2n-1})$ is equivalent to

$$\sim\flat\big(\sim(\varphi_0 \smallsetminus \varphi_n), \sim(\varphi_1 \smallsetminus \varphi_{n+1}), \dots, \sim(\varphi_{n-1} \smallsetminus \varphi_{2n-1})\big).$$

Therefore the SW-formulas correspond to a fragment of the MOD-formulas where $MOD = SMOD \cup WMOD$. The proof that was given for the Standard Form Theorem allows us to show the following generalization.

3.10.5. THEOREM. *Let τ be a first-order vocabulary, and let SW be a family of first-order strict-weak normal modalities. Let us define MOD as $SMOD \cup WMOD$. Then, for every MOD-formula φ there exists $k \in \omega$ and sets $\{\nu_n : n < k\}$, $\{\pi_n : n < k\}$ of SW-formulas such that φ is equivalent, under the standard translation, to*

$$\bigwedge\{\nu_n \supset \pi_n : n < k\}.$$

As a corollary of the proof we obtain the following generalization of Corollary 3.1.7.

3.10.6. COROLLARY. *Let τ be a first-order vocabulary, and let SW be a family of first-order strict-weak normal modalities. Let us define MOD as $SMOD \cup WMOD$. Then,*

1. *whenever $\sharp$ is an n-ary modality in $SMOD$, it holds that for every MOD-formulas $\varphi_0,\dots,\varphi_{n-1}$ there exists a SW-formula φ such that $\sharp(\varphi_0,\dots,\varphi_{n-1})$ and φ are equivalent.*

2. *whenever $\flat$ is an n-ary modality in $WMOD$, it holds that for every MOD-formulas $\varphi_0, \dots, \varphi_{n-1}$ there exists a SW-formula φ such that $\flat(\varphi_0, \dots, \varphi_{n-1})$ and φ are equivalent.*

3.10.7. Definition. (SW-quasi bisimulation)
Let τ be a first-order vocabulary, and let $\mathfrak{A}, \mathfrak{B}$ be a pair of structures in the vocabulary τ. An *SW-quasi bisimulation* from $\mathfrak{A}$ into $\mathfrak{B}$ is a pair $\langle U, Z\rangle$ such that:

- $Z \subseteq U \subseteq A \times B$.
- Z is a MOD-bisimulation between $\mathfrak{A}$ and $\mathfrak{B}$, where MOD is $SMOD \cup WMOD$.
- whenever $\sharp$ is an n-ary normal modality in $SMOD$ with table χ it holds that if $\langle a, b\rangle \in U$ and $\mathfrak{B} \models \chi[b, b_1, \dots, b_n]$ then there are $a_1, \dots, a_n \in A$ with $\mathfrak{A} \models \chi[a, a_1, \dots, a_n]$ and $\langle a_i, b_i\rangle \in Z$ for $1 \leq i \leq n$.
- whenever $\flat$ is an n-ary normal modality in $WMOD$ with table χ it holds that if $\langle a, b\rangle \in U$ and $\mathfrak{A} \models \chi[a, a_1, \dots, a_n]$ then there are $b_1, \dots, b_n \in B$ with $\mathfrak{B} \models \chi[b, b_1, \dots, b_n]$ and $\langle a_i, b_i\rangle \in Z$ for $1 \leq i \leq n$.

The same tools that were used in previous sections can be used to prove the following generalization of Theorem 3.5.11 (this time replacing modally saturated structures with ω-saturated ones).

3.10.8. Theorem. *Let τ be a first-order vocabulary, let $\varphi(v_0)$ be a first-order formula in τ, let K be a class of structures in τ closed under ultraproducts, and let SW be a family of first-order strict-weak normal modalities. The following are equivalent:*

1. *φ is, up to K-equivalence, a SW-formula.*
2. *φ is K-preserved under SW-quasi bisimulations.*

3.11 Historical remarks

Throughout this section we restrict ourselves to the case that ϑ is $\langle\{s\}, \emptyset, \mathsf{Prop}\rangle$, and we will use $\mathcal{L}^s$ to refer to $\mathcal{L}^{SW}(\vartheta)$. The reason for this restriction is that in the context of $\mathcal{L}^s$ there are some examples in the literature of the quasi bisimilarity notion. They refer mainly to the case of intuitionistic structures[27], a case that

[27]Up to now the only publication where $\preceq_s$ is considered over arbitrary structures is the author's abstract [Bou03].

has received the attention of many researchers. We will discuss some of these precedents, but first of all let us mention a result in the literature that is very close to our Theorem 3.5.11 when K is the class of intuitionistic structures. This class is clearly closed under ultraproducts because it is definable by the first-order sentences $\{\forall v Rvv, \forall v_0 \forall v_1 \forall v_2 (Rv_0v_1 \wedge Rv_1v_2 \supset Rv_0v_2)\} \cup \{\forall v_0 \forall v_1 (Rv_0v_1 \wedge Pv_0 \supset Pv_1) : p \in \mathsf{Prop}\}$.

3.11.1. THEOREM. [VvBJL95, Theorem A.1, p. 319] *Let* IPL *be the class of intuitionistic structures*[28]*, and let* $\varphi(v_0)$ *be a* $\mathcal{L}^{FO}(\tau_\vartheta)$*-formula. The following are equivalent:*

1. φ *is, up to* IPL*-equivalence, in* $\mathcal{L}^s$.
2. φ *is* IPL*-preserved under bisimilarity and it is* IPL*-persistent (i.e.,* $\mathsf{IPL} \models \forall v_0 \forall v_1 (Rv_0v_1 \wedge \varphi(v_0) \supset \varphi(v_1))$*).*

In the literature there is also a strengthening of the previous theorem due to Ruitenburg[29].

3.11.2. THEOREM. [Rui99, Theorem 3.4] *Let* K *be any class of transitive and persistent structures, and let* $\varphi(v_0)$ *be a* $\mathcal{L}^{FO}(\tau_\vartheta)$*-formula. The following are equivalent:*

1. *There is* $\varphi' \in \mathcal{L}^s$ *such that* $\mathrm{ST}_{v_0}((\Box\varphi' \to \varphi') \to \Box\varphi')$ *and* φ *are* K*-equivalent.*
2. φ *is* K*-preserved under bisimilarity and it is strongly* K*-persistent (i.e.,* $\mathsf{K} \models \forall v_0 (\varphi(v_0) \supset\subset \forall v_1 (Rv_0v_1 \supset \varphi(v_1)))$*).*

It is a strengthening because (i) φ' and $(\Box\varphi' \to \varphi') \to \Box\varphi'$ are IPL-equivalent for any $\varphi' \in \mathcal{L}^s$, and (ii) strong IPL-persistence coincides with IPL-persistence by the reflexivity.

Our aim is to show that Theorem 3.11.1 is a simple corollary of our Theorem 3.5.11. In order to see this we will analyze what quasi bisimilarity is between intuitionistic structures.

3.11.3. LEMMA. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be two intuitionistic structures, and let* $a \in A$ *and* $b \in B$*. The following are equivalent:*

1. $\langle \mathfrak{A}, a\rangle \preceq_s \langle \mathfrak{B}, b\rangle$.

[28] Strictly speaking in [VvBJL95] the class IPL is defined as the partial orders with a persistent valuation, but it is straightforward to see the equivalence between these different formulations.

[29] Note also [Rui99, Theorem 4.6], which is also a van Benthem style Theorem. We do not state it here because it uses a translation different from the standard one. There $\mathrm{ST}_{v_0}(\varphi_0 \to \varphi_1)$ corresponds to our $\mathrm{ST}_{v_0}((\varphi_0 \to \varphi_1) \wedge (\varphi_0 \supset \varphi_1))$.

2. There is $a' \in A$ such that $\langle a, a' \rangle \in R^{\mathfrak{A}}$ and $\langle \mathfrak{A}, a' \rangle \simeq \langle \mathfrak{B}, b \rangle$.

Proof: $(1 \Rightarrow 2)$: Assume $\langle \mathfrak{A}, a \rangle \preceq_s \langle \mathfrak{B}, b \rangle$. By reflexivity we know that $\langle b, b \rangle \in R^{\mathfrak{B}}$. Hence, there is $a' \in A$ such that $\langle a, a' \rangle \in R^{\mathfrak{A}}$ and $\langle \mathfrak{A}, a' \rangle \simeq \langle \mathfrak{B}, b \rangle$.

$(2 \Rightarrow 1)$: Suppose $a' \in A$ is such that $\langle a, a' \rangle \in R^{\mathfrak{A}}$ and $\langle \mathfrak{A}, a' \rangle \simeq \langle \mathfrak{B}, b \rangle$. Then $\langle \mathfrak{A}, a' \rangle \preceq_s \langle \mathfrak{B}, b \rangle$. By persistence and transitivity it is clear that $\langle \mathfrak{A}, a \rangle \preceq_s \langle \mathfrak{A}, a' \rangle$. As $\preceq_s$ is a quasi order it follows that $\langle \mathfrak{A}, a \rangle \preceq_s \langle \mathfrak{B}, b \rangle$. □

This lemma says that $\preceq_s = \simeq \circ R$ over intuitionistic structures. Therefore, being closed under $\preceq_s$ coincides with being closed under bisimilarity and under the accessibility relation. And this is precisely what is claimed in Theorem 3.11.1(2). Hence, Theorem 3.11.1 is an easy consequence of Theorem 3.5.11 and the previous lemma.

Now it is time to mention several papers where quasi bisimilarity has been considered between intuitionistic structures.

- In [Vis96b, p. 143] Visser considers the relations $\preceq$ (he uses the notation $\preceq_\infty$) and $\preceq_n$. And the idea of our proof of the Standard Form Theorem can be found in the proof of [Vis96b, Lemma 8.2(2)].
- Ghilardi introduces the relations $\preceq_n$ in [Ghi99].
- Our directed quasi bisimulations are precisely the directed intuitionistic bisimulations introduced by Kurtonina and de Rijke in [KdR97]. In [KdR97, Example 5.4] it is proved (but without a constructive proof) that all boxes of modal formulas are IPL-equivalent to a $\mathcal{L}^s$-formula, what can be considered as a precedent of the Standard Form Theorem.

This list does not aim to be comprehensive, merely to show that the quasi bisimilarity concept is not totally new. What we have shown in this chapter is simply that this notion is also useful if we consider arbitrary structures, and not only intuitionistic ones.

3.11.4. Remark. Finally let us say that $\preceq_w$ has also been previously considered in the literature, mainly from the point of view of an inclusion (see Section 3.9). For instance, in [vBB95, p. 276] an inclusion is defined (in the case that there are no propositions) between pointed structures (i.e., processes) which coincides exactly with our $\preceq_w$.

3.12 Open questions

In this chapter we have solved some problems, but (as always) the list of open problems is larger. Here the author summarizes what are, in his opinion, the main open questions.

- For the modal case, Goldblatt and Thomason solved the problem of determining which classes of frames are definable by modal formulas (see [GT74]). How can the classes of frames definable by strict-weak formulas be characterized? At the end of Section 3.3 we gave some necessary conditions; are they enough?

- Given a cardinal κ, let $\mathcal{L}^{MOD}_{\infty\kappa}(\tau_\vartheta)$ and $\mathcal{L}^{SW}_{\infty\kappa}(\vartheta)$ be defined as, respectively, $\mathcal{L}^{MOD}_{\infty}(\tau_\vartheta)$ and $\mathcal{L}^{SW}_{\infty}(\vartheta)$ but restricting arbitrary conjunction and disjunction to conjunction and disjunction of sets with cardinality $< \kappa$. Is it true that for every κ, two pointed structures satisfy the same $\mathcal{L}^{MOD}_{\infty\kappa}(\tau_\vartheta)$-formulas iff they satisfy the same $\mathcal{L}^{SW}_{\infty\kappa}(\vartheta)$-formulas? In order to reproduce the proof of the Standard Form Theorem, we come up against the problem that κ must be strongly inaccessible, i.e., if $\lambda < \kappa$ then $2^\lambda < \kappa$ (see the Footnote 4 on page 67). This is the reason why in our exposition we have restricted ourselves to the case ω and the unbounded case. A related question is: are all boxes of $\mathcal{L}^{MOD}_{\infty\omega_1}(\tau_\vartheta)$-formulas up to equivalence in $\mathcal{L}^{SW}_{\infty\omega_1}(\vartheta)$? For instance, is

$$\Box \bigvee \{p_{2n} \smallsetminus p_{2n+1} : n \in \omega\}$$

equivalent to a $\mathcal{L}^{s}_{\infty\omega_1}$-formula?

- Janin and Walukiewicz proved (using automata theory) in [JW96] that monadic second-order formulas (with at most one free variable, which is first-order) invariant under bisimilarity are exactly the sentences of the μ-calculus. Which are the monadic second-order formulas preserved under quasi bisimilarity? My conjecture is that they are precisely the closure under conjunction $\wedge$ and disjunction $\vee$ of the set $\mathsf{Prop} \cup \{\bot, \top\} \cup \{[s]\varphi : s \in \mathsf{SMod}, \varphi \text{ is a } \mu\text{-sentence}\} \cup \{\langle w\rangle\varphi : w \in \mathsf{WMod}, \varphi \text{ is a } \mu\text{-sentence}\}$. On the other hand, we can define the set of μ^{SW}-formulas as the ones given by:

$$\varphi ::= \bot \mid \top \mid p \mid X \mid \varphi_0 \wedge \varphi_1 \mid \varphi_0 \vee \varphi_1 \mid \mu X.\varphi \mid \nu X.\varphi \mid \varphi_0 \rightarrow_s \varphi_1 \mid \varphi_0 \leftharpoondown_w \varphi_1,$$

where $p \in \mathsf{Prop}$, $s \in \mathsf{SMod}$, $w \in \mathsf{WMod}$, X is a second-order variable, μX is the least fixed point operator (and it is only considered in the case that X occurs only positively in φ) and νX is the greatest fixed point operator (the same restriction applies to this operator). It is easy to see that all μ^{SW}-sentences are preserved under quasi bisimilarity, but are they exactly the second-order formulas preserved under quasi bisimilarity? It is not clear that the expressive power of the μ^{SW}-sentences coincides with the expressive power of what I have conjectured above. Indeed, if we try to obtain a standard form theorem relating μ-formulas and μ^{SW}-formulas we realize that the fixed point operators are a problem, that is, we do not know how to write $\mu X.((\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_{k-1} \supset \pi_{k-1}))$ in standard form.

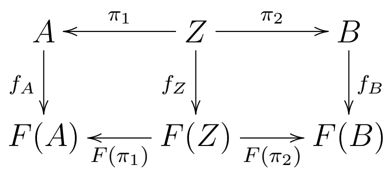
- We have shown that quasi bisimilarity is a quasi order that generates bisimilarity. De Rijke's heuristic idea "partial isomorphism is to first-order logic what bisimilarity is to modal logic" raises a natural question. Is it possible to remove the symmetry from partial isomorphisms? What in the author's opinion is the more natural way to generalize our ideas to this setting collapses with partial isomorphisms, i.e., they do not give anything new. Further ways to remove symmetry from partial isomorphisms must be explored.

- We saw that quasi bisimilarity is explicitly definable using a Σ^1_1-formula where two existential second-order quantifiers are involved. Is it possible to do the same using only one?

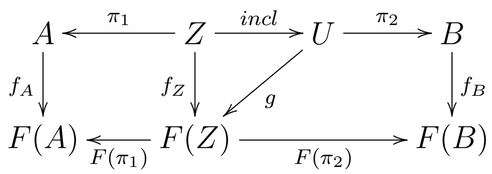
- We have seen how some of the main results in the modal setting can be obtained in the strict-weak setting, but there are other results that we have not considered and that are also interesting. We conjecture that most of these problems can be treated with the same techniques as the modal case. Among them should mention the following.
  - Global definability, i.e., results characterizing classes of (not pointed) structures definable by strict-weak formulas (see [Ven99, dRS01] for the modal case).
  - A Lindström's style Theorem for strict weak languages (see [dR95] for the modal case).
  - A $\mathcal{L}^{FO}(\tau_\vartheta)$-formula $\varphi(v_0, v_1)$ is said to be *safe for strict quasi bisimilarity* if for every ϑ' such that $\vartheta = \vartheta \cap \vartheta'$ and for every $\langle \mathfrak{A}, a\rangle$ and $\langle \mathfrak{B}, b\rangle$ pointed $\tau_{\vartheta'}$-structures, if $\langle \mathfrak{A}, a\rangle \preceq_{\vartheta'} \langle \mathfrak{B}, b\rangle$, then $\langle \mathfrak{A}, X, a\rangle \preceq_{\vartheta''} \langle \mathfrak{B}, Y, b\rangle$ where $\vartheta'' = \langle \mathsf{SMod}' \cup \{s\}, \mathsf{WMod}', \mathsf{Prop}'\rangle$ (s is a new strict modality), $X = \{\langle x, y\rangle \in A^2 : \mathfrak{A} \models \varphi[\![x, y]\!]\}$ and $Y = \{\langle x, y\rangle \in B^2 : \mathfrak{B} \models \varphi[\![x, y]\!]\}$. And analogously we can introduce the notion of *safety for weak quasi bisimilarity* when ϑ'' is obtained adding a new weak modality to ϑ'. These notions can be considered as the natural counterpart of *safety for bisimilarity*, which is defined as before replacing SW-vocabularies with vocabularies and replacing $\preceq$ with $\simeq$ (see [BdRV01, Definition 2.79]). Van Benthem characterized in [vB96, vB98] the formulas that are safe for bisimilarity (see [Hol98, Sections 2.6–2.7] for a different proof and for its generalization to formulas with a finite number of free variables)[30]. Which first-order formulas are safe for strict (and weak) quasi bisimilarity? In the case that in ϑ all modalities are pure strict it is not hard to use the same techniques than Hollenberg to find an answer. But the general situation remains open.

[30]Usually the notion of safety for bisimulations is defined, in which bisimulations substitute bisimilarity in the definition. It is not hard to see the equivalence between the definitions.

– A categorical approach to the quasi bisimilarity notion. We explained that Kripke structures correspond, in a categorical framework, to coalgebras of the endofunctor $F : \mathsf{Set} \longrightarrow \mathsf{Set}$ introduced on page 4. In this framework a *bisimulation* between two coalgebras $\langle A, f_A \rangle$ and $\langle B, f_B \rangle$ is simply a coalgebra $\langle Z, f_Z \rangle$ such that $Z \subseteq A \times B$ and the diagram



commutes (see [JR97, Rut00]). When one tries to do something similar for the quasi bisimilarity notion it seems that there is no natural candidate formulation in terms of commutative diagrams. One of the problems is how to express preservation of propositions in commutative terms (this problems disappears when invariance under propositions is considered). Is it possible to express quasi bisimilarity in a categorical framework? There are some cases where the previous question is simpler. For instance, consider the case $\mathcal{L}^s$ in which there are no propositions. Then, a quasi bisimulation from the coalgebra $\langle A, f_A \rangle$ into the coalgebra $\langle B, f_B \rangle$ could be defined as a triple $\langle U, g, \langle Z, f_Z \rangle \rangle$ where (i) $Z \subseteq U \subseteq A \times B$, (ii) $g \in Hom_{\mathsf{Set}}(U, F(Z))$, and (iii) the following diagram



commutes. It is not hard to see that this definition, once translated to Kripke structures, corresponds to the quasi bisimulations of Definition 3.7.2.

– A matrix approach to quasi bisimulations (see [Fit02] for the modal case).

- In Remark 3.5.12 we showed which first-order fragments enjoy a preservation theorem. When can we guarantee that the relation between pointed structures that allow the preservation theorem is recursively enumerable? When is it decidable?

- In our definition of strict-weak formulas we have included $\bot$ and $\top$. However, there are some cases in the literature where they are not in the language. For instance, the semantics given for Johansson's minimal logic [Joh36] (also called positive logic, e.g., [Bož84, Mak03]) corresponds to $\mathcal{L}^s$ removing $\bot$. Which are the first-order formulas that corresponds to this language? The difficulty comes from the fact that it is not possible to characterize this language using a preservation theorem (see Remark 3.5.12). Therefore, an alternative method must be found.

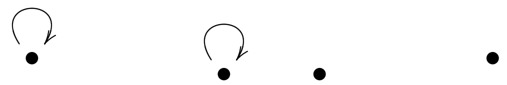
Chapter 4

Strict-Weak Logics

In the previous chapter we have seen that the expressive power of strict-weak fragments is very close to that of modal languages. Now we would like to recuperate normal modal logics from inside strict-weak languages. This is precisely the aim of the present chapter. In the first section we analyze some of the difficulties that appear when we try to do this using strict-weak formulas. The trick suggested by the Standard Form Theorem to solve the problems is to use strict-weak sequents. We develop them in the second section, where we also prove several completeness results. In the third section we introduce strict-weak logics, a framework that will include both normal modal logics and superintuitionistic logics. The next two sections are devoted to the study of two famous properties: disjunction property of several strict-weak logics is proved using proof theoretic methods in the fourth section, and in Section 4.5 we prove that the local consequence given by the class of all structures has uniform interpolation. To finish the chapter we present a list of open problems.

4.1 Strict-weak fragments of normal modal logics

Once we know that the expressive power of strict-weak fragments is very close to that of modal languages we naturally wonder if the strict-weak fragment of a normal modal logic allows us to recuperate the normal modal logic. That is, given a SW-vocabulary ϑ, is the map that assigns to every normal modal logic $\mathbf{\Lambda}$ in τ_ϑ its strict-weak fragment $\mathbf{\Lambda} \cap \mathcal{L}^{SW}(\vartheta)$ an injective map? Unfortunately the answer is negative. Indeed, it is quite simple to find two normal modal logics in $\mathcal{L}^{mod}$ which have the same $\mathcal{L}^s$-fragment. For instance, let us consider the frames in Figure 4.1. It is clear that the modal formula $\Diamond\top$ is evidence of the non-equality



(a) $\mathfrak{F}_0$ (b) $\mathfrak{F}_1$ (c) $\mathfrak{F}$

Figure 4.1: $\operatorname{Log}\mathfrak{F}_0 \neq \operatorname{Log}\mathfrak{F}_1$ while $\operatorname{Log}\mathfrak{F}_0 \cap \mathcal{L}^s = \operatorname{Log}\mathfrak{F}_1 \cap \mathcal{L}^s$

between $\operatorname{Log}\mathfrak{F}_0$ and $\operatorname{Log}\mathfrak{F}_1$. However, it is easy to build an injective reflecting quasi bounded morphism from $\mathfrak{F}$ into $\mathfrak{F}_0$; from which by Proposition 3.3.7(2) it is easily shown that $\operatorname{Log}\mathfrak{F}_0 \cap \mathcal{L}^s = \operatorname{Log}\mathfrak{F}_1 \cap \mathcal{L}^s$.

In the next two propositions we improve what we have shown in the previous paragraph. We prove that there is a continuum of normal modal logics sharing the same strict-weak fragment. Even from the point of view of decidability there is a difference in the behaviour of this continuum of normal logics and the strict-weak fragment. The first proposition talks about the $\mathcal{L}^w$-fragment, while the other one refers to the $\mathcal{L}^s$-fragment.

4.1.1. Proposition. *There is a family $\{\mathbf{\Lambda}_\alpha : \alpha < 2^{\aleph_0}\}$ of normal modal logics (with a single modality) such that for every $\alpha, \beta < 2^{\aleph_0}$ it holds that (i) if $\alpha \neq \beta$ then $\mathbf{\Lambda}_\alpha \cap \mathcal{L}^s \neq \mathbf{\Lambda}_\beta \cap \mathcal{L}^s$, (ii) $\mathbf{\Lambda}_\alpha \cap \mathcal{L}^w = \boldsymbol{K} \cap \mathcal{L}^w$, (iii) $\mathbf{\Lambda}_\alpha \cap \mathcal{L}^w$ is decidable*[1]*, and (iv) $\mathbf{\Lambda}_\alpha \cap \mathcal{L}^s$ is undecidable.*

Proof: Let $\mathfrak{F}$ be the last frame in Figure 4.1. We also consider the frames $\mathfrak{F}_n$ in Figure 4.2. For every $I \subseteq \omega$, let $\mathbf{\Lambda}_I$ be $\operatorname{Log}(\{\mathfrak{F}\} \cup \{\mathfrak{F}_n : n \in I\})$. Let us show that the family $\{\mathbf{\Lambda}_I : I \subseteq \omega\}$ satisfies the first two properties stated. Then, the third condition follows from the fact that $\boldsymbol{K}$ is decidable. By cardinality reasons it follows that there is a subfamily such that all its $\mathcal{L}^s$-fragments are undecidable.

For every $k \in \omega$ we define the modal formula

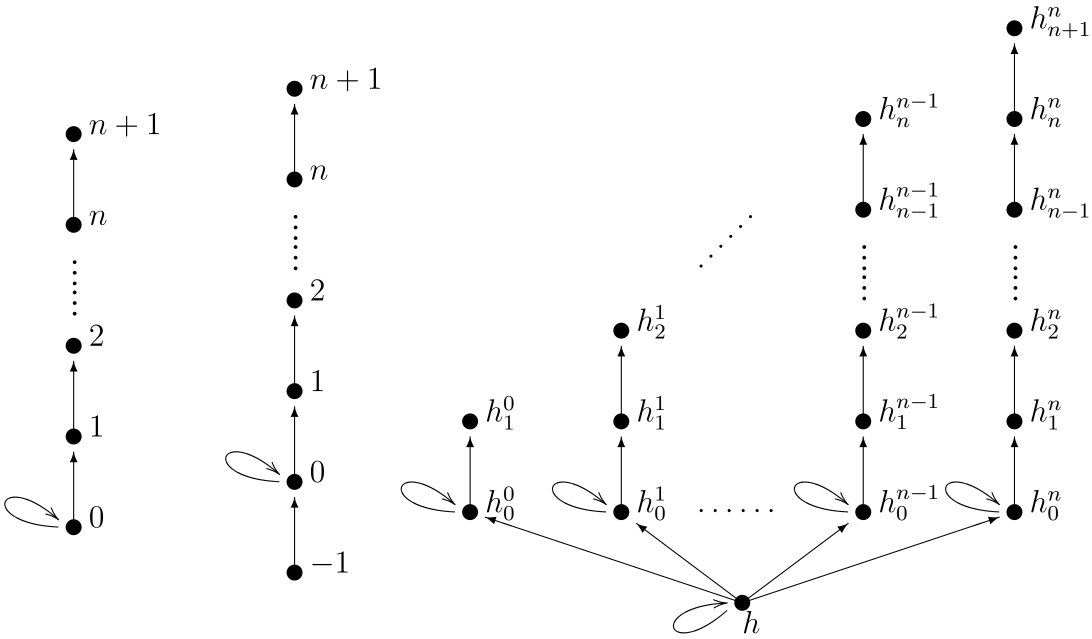
$$\varphi_k := {\sim} p_0 \wedge \Diamond {\sim} p_0 \wedge \Diamond(p_0 \wedge \Diamond^k \Box \bot).$$

By Corollary 3.1.7 it is easy to see that ${\sim}\varphi_k$ is up to equivalence in $\mathcal{L}^s$. It is not hard to see that for every $k \in \omega$, ${\sim}\varphi_k \in \operatorname{Log}(\{\mathfrak{F}\})$. Furthermore, it is easy to see that φ_k is satisfiable in the frame $\mathfrak{F}_n$ iff $k = n$. Thus, $k \in I$ iff ${\sim}\varphi_k \notin \mathbf{\Lambda}_I$[2]. Therefore, if $I_0 \neq I_1$ then $\mathbf{\Lambda}_{I_0} \cap \mathcal{L}^s \neq \mathbf{\Lambda}_{I_1} \cap \mathcal{L}^s$.

Let I be a subset of ω. It is obvious that $\boldsymbol{K} \subseteq \mathbf{\Lambda}_I$. Thus, $\boldsymbol{K} \cap \mathcal{L}^w \subseteq \mathbf{\Lambda}_I \cap \mathcal{L}^w$. Let us see the converse inclusion. Assume that $\varphi \in \mathbf{\Lambda}_I \cap \mathcal{L}^w$. Then, $\mathfrak{F} \Vdash \varphi$. Therefore, $\mathfrak{S}_{IrF} \Vdash \varphi$. Using that $\mathfrak{S}_{IrF} \preceq_w \langle \mathfrak{A}, a \rangle$ for every pointed structure $\langle \mathfrak{A}, a \rangle$ we deduce that $\varphi \in \boldsymbol{K}$. □

[1]Indeed, this decidability is in linear time because we know that $\varphi \in \boldsymbol{K} \cap \mathcal{L}^w$ iff $\mathfrak{S}_{IrF} \Vdash \varphi$. This is a trivial consequence of the fact that $\mathfrak{S}_{IrF}$ is initial in $\preceq_w$ (see p. 73).

[2]In particular this says that if $\omega \setminus I$ is non-recursive then $\mathbf{\Lambda}_I \cap \mathcal{L}^s$ is undecidable.



(a) $\mathfrak{F}_n$ (b) $\mathfrak{G}_n$ (c) $\mathfrak{H}_n$

Figure 4.2: The frames $\mathfrak{F}_n$, $\mathfrak{G}_n$ and $\mathfrak{H}_n$ ($n \in \omega$)

We have shown a continuum of normal modal logics sharing the same $\mathcal{L}^w$-fragment as $\boldsymbol{K}$. It is not hard[3] to see that $\boldsymbol{K}$ is the only normal modal logic with $\boldsymbol{K} \cap \mathcal{L}^s$ as $\mathcal{L}^s$-fragment, but we can prove the following result for $\mathcal{L}^s$.

4.1.2. Proposition. *There is a family* $\{\boldsymbol{\Lambda}_\alpha : \alpha < 2^{\aleph_0}\}$ *of normal modal logics (with a single modality) such that for every* $\alpha, \beta < 2^{\aleph_0}$ *it holds that (i) if* $\alpha \neq \beta$ *then* $\boldsymbol{\Lambda}_\alpha \neq \boldsymbol{\Lambda}_\beta$*, (ii)* $\boldsymbol{\Lambda}_\alpha \cap \mathcal{L}^s = \boldsymbol{\Lambda}_\beta \cap \mathcal{L}^s$*, (iii)* $\boldsymbol{\Lambda}_\alpha \cap \mathcal{L}^s$ *is decidable, and (iv)* $\boldsymbol{\Lambda}_\alpha$ *is undecidable.*

Proof: Let us consider the frames in Figure 4.2. Let $\boldsymbol{\Lambda}$ be $\mathrm{Log}(\{\mathfrak{H}_n : n \in \omega\})$, and for every $I \subseteq \omega$, let $\boldsymbol{\Lambda}_I$ be $\mathrm{Log}(\{\mathfrak{H}_n : n \in \omega\} \cup \{\mathfrak{G}_n : n \in I\})$. Let us show that the family $\{\boldsymbol{\Lambda}_I : I \subseteq \omega\}$ satisfies the first three properties stated. Then, by cardinality it follows that there is a subfamily such that all its normal modal logics are undecidable.

For every $k \in \omega$ we define the modal formula

$$\varphi_k := \Diamond\top \ \wedge\ \Box\left(\sim p_0 \wedge \Diamond \sim p_0 \wedge \Diamond(p_0 \wedge \Diamond^k \Box\bot)\right).$$

It is easy to see that for every $k \in \omega$, $\sim\varphi_k \in \boldsymbol{\Lambda}$. It is not hard[4] to observe that φ_k is satisfiable in the frame $\mathfrak{G}_n$ iff $k = n$. Hence, $k \in I$ iff $\sim\varphi_k \notin \boldsymbol{\Lambda}_I$. Therefore, if $I_0 \neq I_1$ then $\boldsymbol{\Lambda}_{I_0} \neq \boldsymbol{\Lambda}_{I_1}$.

[3] *Hint:* Use Corollary 3.1.7(1) and the fact that if $\Box\varphi \in \boldsymbol{K}$ then $\varphi \in \boldsymbol{K}$.
[4] *Hint:* In all the states different from the root of $\mathfrak{G}_n$ it is clear that φ_k cannot be satisfied.

Let I be a subset of ω. It is obvious that $\mathbf{\Lambda}_I \subseteq \mathbf{\Lambda}$. Thus, $\mathbf{\Lambda}_I \cap \mathcal{L}^s \subseteq \mathbf{\Lambda} \cap \mathcal{L}^s$. Let us prove that the converse inclusion also holds. For every $n \in \omega$ we define the map

$$\begin{array}{rccl} f_n: & G_n & \longrightarrow & H_n \\ & x & \longmapsto & f_n(x) := \begin{cases} h & \text{if } x = -1, \\ h^n_x & \text{if } 0 \leq x \leq n+1. \end{cases} \end{array}$$

It is clear that for every $n \in \omega$, f_n is an injective reflecting quasi bounded morphism (with respect to $\mathcal{L}^s$) from $\mathfrak{G}_n$ into $\mathfrak{H}_n$. By Proposition 3.3.7 it easily follows that $\mathbf{\Lambda} \cap \mathcal{L}^s \subseteq \mathbf{\Lambda}_I \cap \mathcal{L}^s$.

Finally, let us show that $\mathbf{\Lambda} \cap \mathcal{L}^s$ is decidable. Indeed, we will show that $\mathbf{\Lambda}$ is decidable.

CLAIM: For every modal formula φ, $\varphi \in \mathbf{\Lambda}$ iff $\varphi \in \mathrm{Log}(\{\mathfrak{H}_n : n \in \omega, n \leq 2^k\})$ where $k = 1 + \deg(\varphi) \cdot \max\{1, |\mathsf{Prop}(\varphi)|\}$.

Proof of Claim: Assume that $\varphi \notin \mathbf{\Lambda}$. Let d be $\deg(\varphi)$, and let $r = |\mathsf{Prop}(\varphi)|$. We can suppose that $d \geq 2$: if not, replace φ with $\varphi \wedge \Box^2\top$. Then, there is $n \in \omega$ such that $\mathfrak{H}_n, V, a \not\Vdash \varphi$ for some valuation V and some state $a \in H_n$. Now we distinguish two cases.

Case a is the root of H_n: We cut the structure at height d, i.e., we consider $\mathfrak{A}$ as the substructure of $\langle \mathfrak{H}_n, V\rangle$ given by the universe $\{h\} \cup \{h^i_j : j < \min\{d, i+1\}\}$. Thus, $\mathfrak{A}, h \not\Vdash \varphi$. It is clear that the frame of $\mathfrak{A}$ can be seen as the result of adding a reflexive state under the structure that is the disjoint union of $\{\mathfrak{F}_i : i < d-2\}$ and a number of copies of $\mathfrak{F}_{d-2}$. As the frame $\mathfrak{F}_{d-2}$ has cardinality d we know that over this frame there are only 2^{dr} different valuations. Deleting all repeated copies it is easy to see that there is $l \leq d - 2 + 2^{dr}$ such that φ is not valid in $\mathfrak{H}_l$. Now it is simple to check that $l \leq 2^k$.

Case a is not the root of H_n: Let l be $\deg(\varphi)$. It is obvious that $l \leq 2^k$. Then it is easy to see that φ is not valid in the frame $\mathfrak{F}_l$. Therefore φ is not valid in the frame $\mathfrak{H}_l$.

⊣

By the claim it is obvious that $\mathbf{\Lambda}$ is decidable. □

Up to now we have observed different behaviours in normal modal logics and their strict-weak fragment. Now we present several examples to illustrate that there is also a different behaviour at the level of consequence relations. In these examples we restrict ourselves to the case that there is a single modality.

4.1.3. EXAMPLE. Let C be the class of all frames, and let Fin be the class of all finite frames. It is well known that $\models^{mod}_{l\mathsf{C}} \neq \models^{mod}_{l\mathsf{Fin}}$. For instance, if $\Phi := \{\Diamond(\Box^{n+1}\bot \wedge \Diamond^n\top) : n \in \omega\}$ then it is easy to see that

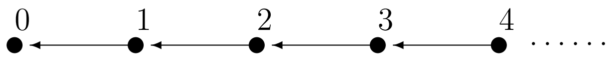


Figure 4.3: Take the transitive closure of the relation depicted

$$\Phi \not\models^{mod}_{l\mathsf{C}} \bot \text{ while } \Phi \models^{mod}_{l\mathsf{Fin}} \bot.$$

This means that Φ is satisfiable in a frame while it is not satisfiable in a finite frame. Then, it is not hard to see that the $\mathcal{L}^s$-fragments $\models^s_{l\mathsf{C}}$ and $\models^s_{l\mathsf{Fin}}$ are also different. This can be proved (we recall that all boxes are up to equivalence in $\mathcal{L}^s$) using that

$$\{\Box\phi : \phi \in \Phi\} \not\models^{mod}_{l\mathsf{C}} \Box\bot \text{ while } \{\Box\phi : \phi \in \Phi\} \models^{mod}_{l\mathsf{Fin}} \Box\bot.$$

But we notice that now the formula on the right is not $\bot$. Is it possible to find a set Φ of $\mathcal{L}^s$-formulas such that $\Phi \not\models^s_{l\mathsf{C}} \bot$ while $\Phi \models^s_{l\mathsf{Fin}} \bot$? That is, is there any set Φ of $\mathcal{L}^s$-formulas such that Φ is satisfiable in a frame while it is not satisfiable in a finite frame? The answer is negative. This follows from the fact that $\mathfrak{S}_{IrT}$ is final in $\preceq_S$ (see p. 73). By this it is clear that if Φ is a satisfiable set of $\mathcal{L}^s$-formulas then it is satisfied in the finite structure $\mathfrak{S}_{IrT}$.

4.1.4. Example. We show a structure $\mathfrak{A}$ such that $\models^s_{l\mathfrak{A}}$ is finitary while $\models^{mod}_{l\mathfrak{A}}$ is non-finitary[5]. Let $\mathfrak{F}$ be the frame depicted in Figure 4.3. Now we pick an arbitrary structure $\mathfrak{A}$ such that its underlying frame is $\mathfrak{F}$ and its valuation is persistent. For every $k \in \omega$ we define the modal formula

$$\varphi_k := \Box^{k+1}\bot \ \wedge \ \Diamond^k\top.$$

It holds that for $n \in \omega$,

$$\mathfrak{A}, n \Vdash \varphi_k \quad \text{ iff } \quad n = k.$$

Using this it is easy to observe that $\{\Diamond\varphi_k : k \in \omega\} \models^{mod}_{l\mathfrak{A}} \bot$ while it is false for every finite subset of the set of hypotheses. Let us now prove that $\models^s_{l\mathfrak{A}}$ is finitary. We assume that $\Phi \models^s_{l\mathfrak{A}} \varphi$, and we seek a finite subset Φ' of Φ such that $\Phi' \models^s_{l\mathfrak{A}} \varphi$. In the case that $\emptyset \models^s_{l\mathfrak{A}} \varphi$ it is clear that this finite subset exists. So, we suppose that $\emptyset \not\models^s_{l\mathfrak{A}} \varphi$. Then $\mathfrak{A} \nVdash \varphi$. Hence we can consider $k := \min\{n \in \omega : \mathfrak{A}, n \nVdash \varphi\}$. Thus, $\mathfrak{A}, k \nVdash \varphi$, and for every $n < k$ it holds that $\mathfrak{A}, n \Vdash \varphi$. As a consequence of the assumption $\Phi \models^s_{l\mathfrak{A}} \varphi$ we know that there is $\phi' \in \Phi$ such that $\mathfrak{A}, k \nVdash \phi'$.

[5]Note that in the example offered $\models^{mod}_{l\mathfrak{A}}$ is not closed under modal substitutions. What it is true is that $\models^s_{l\mathfrak{A}}$ is closed under (strict-weak) substitutions.

In order to finish our proof it is enough to show that $\{\phi'\} \models^s_{l\mathfrak{A}} \varphi$. Assume that $n \in \omega$ and that $\mathfrak{A}, n \Vdash \phi'$. If $n < k$ we already saw that $\mathfrak{A}, n \Vdash \varphi$. So suppose that $n \geq k$. Using that $\mathfrak{A}$ is transitive together with the fact that its valuation is persistent it is clear that $\langle \mathfrak{A}, n\rangle \preceq_s \langle \mathfrak{A}, k\rangle$. This contradicts $\mathfrak{A}, k \nVdash \phi'$ and $\mathfrak{A}, n \Vdash \phi'$.

4.1.5. Example. We show two classes K and K' of structures such that $\models^s_{l\mathsf{K}} = \models^s_{l\mathsf{K}'}$ while $\models^{mod}_{l\mathsf{K}} \neq \models^{mod}_{l\mathsf{K}'}$. This is an immediate consequence of the previous example. There we have found a class K of structures such that $\models^s_{l\mathsf{K}}$ is finitary while $\models^{mod}_{l\mathsf{K}}$ is non-finitary. Let K' be the closure under ultraproducts. It is well known that the first-order consequence $\models_{\mathsf{K}'}$ is finitary (see [CK90, Corollary 4.1.11]). It easily follows that $\models^s_{l\mathsf{K}} = \models^s_{l\mathsf{K}'}$ and that $\models^{mod}_{l\mathsf{K}} \neq \models^{mod}_{l\mathsf{K}'}$.

As noted in a footnote the modal consequences considered in the previous examples are not structural. It is natural to wonder if there are examples like the previous ones where the modal consequences are structural. In particular, are there two classes C and C' of frames such that $\models^s_{l\mathsf{C}} = \models^s_{l\mathsf{C}'}$ while $\models^{mod}_{l\mathsf{C}} \neq \models^{mod}_{l\mathsf{C}'}$? This interesting question remains open. But we note that a negative answer must use the existence of propositions, because if there are no propositions our previous examples are indeed frames.

4.2 Using strict-weak sequents

4.2.1. Definition. ($\mathcal{L}^{SW}(\vartheta)$-sequents and its semantics)
Let ϑ be a SW-vocabulary. An $\mathcal{L}^{SW}(\vartheta)$-*sequent* is an expression of the form $\Gamma \rhd \Delta$ where Γ and Δ are finite (maybe empty) sequences of $\mathcal{L}^{SW}(\vartheta)$-formulas[6]. Finite sequences are usually written without the angles, i.e., we write $\varphi_0, \dots, \varphi_{n-1}$ instead of $\langle \varphi_0, \dots, \varphi_{n-1}\rangle$. The empty sequence is denoted by $\emptyset$. The set of $\mathcal{L}^{SW}(\vartheta)$-sequents is denoted by $Seq\mathcal{L}^{SW}(\vartheta)$. An $\mathcal{L}^{SW}(\vartheta)$-sequent is *simple* if it is of the form $\varphi \rhd \varphi'$, and we will denote the set of simple $\mathcal{L}^{SW}(\vartheta)$-sequents by $SSeq\mathcal{L}^{SW}(\vartheta)$. Given $s \in \mathsf{Smod}$, a finite sequence $\Gamma = \varphi_0, \dots, \varphi_{n-1}$ of $\mathcal{L}^{SW}(\vartheta)$-formulas and φ an $\mathcal{L}^{SW}(\vartheta)$-formula, we define the following finite sequences:

- $\varphi \rightarrow_s \Gamma$ is $\varphi \rightarrow_s \varphi_0, \dots, \varphi \rightarrow_s \varphi_{n-1}$.
- $\Gamma \rightarrow_s \varphi$ is $\varphi_0 \rightarrow_s \varphi, \dots, \varphi_{n-1} \rightarrow_s \varphi$.

Analogously, if $w \in \mathsf{WMod}$ we define

- $\varphi \leftharpoondown_w \Gamma$ is $\varphi \leftharpoondown_w \varphi_0, \dots, \varphi \leftharpoondown_w \varphi_{n-1}$.

[6]We follow the same notation as in [FJP99]. Font, Jansana and Pigozzi decided to use $\rhd$ instead of $\vdash$ or $\Rightarrow$ in order to avoid misunderstandings. However, we notice there are some topics in logic where the symbol $\rhd$ has a concrete meaning, e.g., interpretability logic [Vis98].

- $\Gamma \mathrel{\leftharpoondown_w} \varphi$ is $\varphi_0 \mathrel{\leftharpoondown_w} \varphi, \ldots, \varphi_{n-1} \mathrel{\leftharpoondown_w} \varphi$.

Given a substitution e and a finite sequence $\Gamma = \varphi_0, \ldots, \varphi_{n-1}$, then $e(\Gamma)$ is the finite sequence $e(\varphi_0), \ldots, e(\varphi_{n-1})$. And $e(\Gamma \rhd \Delta)$ is precisely the sequent $e(\Gamma) \rhd e(\Delta)$. If Π is a set of $\mathcal{L}^{SW}(\vartheta)$-sequents, then $\mathrm{sub}(\Pi)$ is the closure under substitutions of Π, i.e.,

$$\mathrm{sub}(\Pi) = \{e(\varsigma) : e \text{ substitution}, \varsigma \in \Pi\}.$$

Given a finite sequence Γ of strict-weak formulas, we will also write Γ to denote its underlying set; and viceversa[7]. For instance, the strict-weak formula $\bigwedge \Gamma$ is the conjunction of the underlying set of Γ, and $\bigvee \Gamma$ is the disjunction of the underlying set of Γ. The satisfiability relation is extended to $\mathcal{L}^{SW}(\vartheta)$-sequents by the following clause:

$$\mathfrak{A}, a \Vdash \Gamma \rhd \Delta \quad \text{iff} \quad \mathfrak{A}, a \Vdash \bigwedge \Gamma \supset \bigvee \Delta.$$

If Π is a set of $\mathcal{L}^{SW}(\vartheta)$-sequents then $\mathfrak{A}, a \Vdash \Pi$ means that all sequents in Π are satisfied in $\langle \mathfrak{A}, a \rangle$. The *dual* Γ^d *of a finite sequence* $\Gamma = \varphi_0, \ldots, \varphi_{n-1}$ of $\mathcal{L}^{SW}(\vartheta)$-formulas is the finite sequence $\varphi^d_{n-1}, \ldots, \varphi^d_0$ of $\mathcal{L}^{SW}(\vartheta^d)$-formulas. The *dual of an* $\mathcal{L}^{SW}(\vartheta)$*-sequent* $\Gamma \rhd \Delta$ is the $\mathcal{L}^{SW}(\vartheta^d)$-sequent $\Delta^d \rhd \Gamma^d$. The *dual* Π^d *of a set* Π *of* $\mathcal{L}^{SW}(\vartheta)$*-sequents* is the set of $\mathcal{L}^{SW}(\vartheta^d)$-sequents formed by the duals of the sequents in Π.

By definition, from a semantic point of view we are identifying a strict-weak sequent $\Gamma \rhd \Delta$ with the modal formula $\bigwedge \Gamma \supset \bigvee \Delta$. In other words, we can extend the map σ considered on page 57 to $\mathcal{L}^{SW}(\vartheta)$-sequents adding the clause:

$$\sigma(\Gamma \rhd \Delta) \;:=\; \sigma(\bigwedge \Gamma) \supset \sigma(\bigvee \Delta).$$

We notice that the simple strict-weak sequent $\bigwedge \Gamma \rhd \bigvee \Delta$ is also identified with the same modal formula. As a consequence of the Duality Principle it is clear that

$$\mathfrak{A}, a \Vdash \Pi \quad \text{iff} \quad \mathfrak{A}^d, a \Vdash \Pi^d.$$

The previous identification allows us to introduce the definition of *equivalence* between modal formulas and sequents. The Standard Form Theorem says that each $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula is equivalent to a finite set of (simple) $\mathcal{L}^{SW}(\vartheta)$-sequents. And every set of $\mathcal{L}^{MOD}(\tau_\vartheta)$-formulas is equivalent to a set of (simple) $\mathcal{L}^{SW}(\vartheta)$-sequents.

[7]This will not cause any ambiguity because in the calculus considered we have all structural rules.

4.2.2. REMARK. Analogously to what is done in Definition 4.2.1 we could have introduced $\mathcal{L}^{MOD}(\tau_\vartheta)$-sequents and its semantics. We do not do this because its expressive power should be the same as we find in $\mathcal{L}^{MOD}(\tau_\vartheta)$-formulas since we can identify a $\mathcal{L}^{MOD}(\tau_\vartheta)$-sequent $\Gamma \rhd \Delta$ with the $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula $\bigwedge \Gamma \supset \bigvee \Delta$. However, we notice that this is not the case when we consider strict-weak sequents. For instance, it is easy to see that the sequent $\Box p \rhd p$ is not equivalent to any $\mathcal{L}^s$-formula because it is not preserved under quasi bisimilarity.

This identification also allows us to define the validity of a sequent in a frame or in a structure. That is, given a frame $\mathfrak{F}$ and a structure $\mathfrak{A}$ we introduce the following definitions:

- $\mathfrak{F} \Vdash \Gamma \rhd \Delta$ iff $\mathfrak{F} \Vdash \bigwedge \Gamma \supset \bigvee \Delta$,
- $\mathfrak{A} \Vdash \Gamma \rhd \Delta$ iff $\mathfrak{A} \Vdash \bigwedge \Gamma \supset \bigvee \Delta$.

And we write $\mathfrak{F} \Vdash \Pi$ and $\mathfrak{A} \Vdash \Pi$ if it holds for all sequents in the set Π. It is clear that (i) $\mathfrak{F} \Vdash \Pi$ iff $\mathfrak{F} \Vdash \Pi^d$, and (ii) $\mathfrak{A} \Vdash \Pi$ iff $\mathfrak{A}^d \Vdash \Pi^d$. As an easy consequence of the Standard Form Theorem we have the following proposition about definability of frames.

4.2.3. PROPOSITION. *Let* C *be a class of* τ_ϑ*-frames such that it is definable by a* $\mathcal{L}^{MOD}(\tau_\vartheta)$*-formula* φ*. Then, there is an algorithm that converts* φ *into a finite set* Π *of* $\mathcal{L}^{SW}(\vartheta)$*-sequents that defines the same class* C*.*

Proof: By the translation tr introduced in the proof of the Standard Form Theorem we have an effective method to convert φ into an equivalent formula $(\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_{k-1} \supset \pi_{k-1})$ where $k \in \omega$, and the ν's and π's are $\mathcal{L}^{SW}(\vartheta)$-formulas. It is clear that $\Pi := \{\nu_i \rhd \pi_i : i < k\}$ satisfies what we want. $\Box$

We can apply the previous algorithm to all classes of frames considered in Table 1.2 (including directedness and weak directedness). As an example we consider the class of frames where R_m is reflexive. If $m \in \mathsf{SMod}$, then the previous algorithm says that

$$\mathfrak{F} \Vdash [m]p_0 \rhd p_0 \quad \text{iff} \quad R_m \text{ is reflexive.}$$

On the other hand, if $m \in \mathsf{WMod}$, then we have that

$$\mathfrak{F} \Vdash p_0 \rhd \langle m \rangle p_0 \quad \text{iff} \quad R_m \text{ is reflexive.}$$

Hence in this example a single sequent is enough. In general this does not seem to be so. For instance, it seems unlikely that we will find a single $\mathcal{L}^s$-sequent such that its validity characterizes the frames that are at the same time transitive and symmetric, but the author does not know any proof of this impossibility. Indeed, it is still an open problem to show the existence of a class of frames definable by a modal formula such that it is not definable by a single strict-weak sequent. The difficulty arises when we want to show that a certain class of frames is not definable by any strict-weak sequent. It seems necessary to develop new tools to solve this problem.

Now we consider other sets of sequents such that their validity also defines the classes of frames considered in Table 1.2 except for directedness and weak directedness. These sets of sequents are displayed in Table 4.1. For some of the sequents considered we have introduced a name in the same table. It may seem unnecessary to introduce them, since they are more complex than the ones obtained in Proposition 4.2.3; the reason why we introduce them is that in general we will need these more complex forms in order to axiomatize the logics given by these classes of frames.

4.2.4. Remark. (An explanation of Table 4.1). In Table 4.1 we have considered all properties in Table 1.2 except directedness and weak directedness. Each one of the properties in Table 1.2 now yields 2^k different properties where k is the number of modalities involved in the modal formula characterizing the property. There are 2^k because each of the modalities involved can be considered as a strict modality or as a weak modality. These blocks of 2^k properties are separated in Table 4.1 by large horizontal lines. Inside these blocks the sequents corresponding to the odd properties are always precisely the duals of the sequents characterizing its predecessor property. And the sequents corresponding to the even properties are obtained in most cases by the following method:

> Assume φ is the $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula characterizing the property in Table 1.2. Let e be the modal substitution such that $e(p_n) = p_{2n} \supset p_{2n+1}$ for every $n \in \omega$. Now take the set $\{\nu_i \rhd \pi_i : i < k\}$ of $\mathcal{L}^{SW}(\vartheta)$-sequents where $\mathrm{tr}(e(\varphi)) = (\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_{k-1} \supset \pi_{k-1})$.

It is easy to check that this method gives most of the sets of sequents[8] that are shown in Table 4.1 for the even properties. For the cases of $\mathsf{T}, \mathsf{4}, \mathsf{5}, \mathsf{B}, \mathsf{con}, .\mathsf{3}, \mathsf{GL}$ and Grz we checked this in Example 3.1.9. The exceptions to the previous method are the sets of sequents corresponding to the following properties:

- R_s is functional: in this case the sequent written is obtained using $\mathrm{tr}(\mathsf{Alt})$, and not using $\mathrm{tr}(e(\mathsf{Alt}))$.
- $R_s = Id$: in this case the set of sequents considered is the union of the sequents considered for the properties R_s is reflexive and R_s is functional.

[8]For most of the properties a single sequent is considered. The only cases where more than one sequent is written are the ones corresponding to the properties $R_s \subseteq Id$, $R_w \subseteq Id$, $R_s = Id$ and $R_w = Id$.

In the case of weak connectedness (and only in this case) besides the sequents obtained by the previous method we have also considered other sequents, which are written on page 145 at the end of the first part of Table 4.1. These alternative sequents are shorter, although they need an additional proposition.

4.2.5. Proposition. *For each of the sets of strict-weak sequents in Table 4.1, a frame validates the set of sequents iff the frame satisfies the condition on the right.*

Proof: By duality it is enough to check the even properties. It is clear that for every modal formula φ,

$$\mathfrak{F} \Vdash \varphi \quad \text{iff} \quad \mathfrak{F} \Vdash e(\varphi),$$

where e is the modal substitution such that $e(p_n) = p_{2n} \supset p_{2n+1}$ for each $n \in \omega$. Therefore, for all the sets of sequents considered in Table 4.1 that are obtained through $\mathrm{tr}(e(\varphi))$ we know that their validity characterizes the property on the right. It is also easy to show that the strict-weak sequent $\mathsf{Alt}^{\mathsf{s}}$ and the set of strict-weak sequents Tr^{s} satisfy the proposition. Hence, it only remains to show the equivalence between

- $\mathfrak{F} \Vdash \top \rhd \Big(\big((p_0 \rightarrow_s p_1) \wedge p_2 \wedge p_4\big) \rightarrow_s p_3\Big) \vee \Big(\big((p_2 \rightarrow_s p_3) \wedge p_0\big) \rightarrow_s \big(p_1 \vee p_4\big)\Big)$,
- $R_s^{\mathfrak{F}}$ is weakly connected.

It is not hard to prove that if $R_s^{\mathfrak{F}}$ is weakly connected then the sequent is valid. For the converse, assume that $R_s^{\mathfrak{F}}$ is not weakly connected. Thus, there are $a_0, a_1, a_2 \in F$ such that $\langle a_0, a_1\rangle \in R_s$, $\langle a_0, a_2\rangle \in R_s$, $a_1 \neq a_2$, $\langle a_1, a_2\rangle \notin R_s$ and $\langle a_2, a_1\rangle \notin R_s$. We consider V as the valuation such that $V(p_0) = \{a_1\}$, $V(p_1) = V(p_2) = V(p_4) = \{a_2\}$ and $V(p_3) = \emptyset$. It is not hard to see that $\mathfrak{F}, V, a_1 \nVdash \big((p_2 \rightarrow_s p_3) \wedge p_0\big) \supset \big(p_1 \vee p_4\big)$ and $\mathfrak{F}, V, a_2 \nVdash \big((p_0 \rightarrow_s p_1) \wedge p_2 \wedge p_4\big) \supset p_3$. Therefore, it follows that $\mathfrak{F}, V, a_0 \nVdash \top \rhd \Big(\big((p_0 \rightarrow_s p_1) \wedge p_2 \wedge p_4\big) \rightarrow_s p_3\Big) \vee \Big(\big((p_2 \rightarrow_s p_3) \wedge p_0\big) \rightarrow_s \big(p_1 \vee p_4\big)\Big)$. □

We have already explained that the Standard Form Theorem says that every set of $\mathcal{L}^{MOD}(\tau_\vartheta)$-formulas is equivalent to a set of (simple) $\mathcal{L}^{SW}(\vartheta)$-sequents. It suggests that $\mathcal{L}^{SW}(\vartheta)$-sequents should now play the role of $\mathcal{L}^{MOD}(\tau_\vartheta)$-formulas. Hence we introduce the notion of finitary structural consequence relation between strict-weak sequents. The definition is the same as in Section 1.4 but replacing modal formulas with strict-weak sequents. To distinguish if the objects of our consequence relation are formulas or sequents we will use the symbol $\mid\!\sim$ whenever the objects are sequents[9]. Hence a *finitary structural consequence relation*

[9]We recall that $\vdash$ is used for formulas. The first use in the literature of the symbol $\mid\!\sim$ for this purpose is in [RV93]. Since then this notation has also been considered in several other places, e.g., [RV95, FJ96].

Table 4.1: Some canonical and 'complete' sequents in the strict-weak case.

Canonical Sequents		
$\mathsf{T^s}$	$p_0 \wedge (p_0 \rightarrow_s p_1) \rhd p_1$	R_s is reflexive
$\mathsf{T^w}$	$p_1 \rhd (p_1 \leftharpoondown_w p_0) \vee p_0$	R_w is reflexive
$\mathsf{4^s}$	$p_0 \rightarrow_s p_1 \rhd [s](p_0 \rightarrow_s p_1)$	R_s is transitive
$\mathsf{4^w}$	$\langle w\rangle(p_1 \leftharpoondown_w p_0) \rhd p_1 \leftharpoondown_w p_0$	R_w is transitive
$\mathsf{5^s}$	$\top \rhd (p_0 \rightarrow_s p_1) \vee \neg_s(p_0 \rightarrow_s p_1)$	R_s is Euclidean
$\mathsf{5^w}$	$\ulcorner_w(p_1 \leftharpoondown_w p_0) \wedge (p_1 \leftharpoondown_w p_0) \rhd \bot$	R_w is Euclidean
$\mathsf{B^s}$	$p_0 \rhd p_1 \vee \neg_s(p_0 \rightarrow_s p_1)$	R_s is symmetric
$\mathsf{B^w}$	$\ulcorner_w(p_1 \leftharpoondown_w p_0) \wedge p_1 \rhd p_0$	R_w is symmetric
	$\top \rhd p_0 \vee (p_0 \rightarrow_s p_1)$ $p_1 \rhd p_0 \rightarrow_s p_1$	$R_s \subseteq Id$
	$(p_1 \leftharpoondown_w p_0) \wedge p_0 \rhd \bot$ $p_1 \leftharpoondown_w p_0 \rhd p_1$	$R_w \subseteq Id$
$\mathsf{V^s}$	$\top \rhd [s]\bot$	$R_s = \emptyset$
$\mathsf{V^w}$	$\langle w\rangle\top \rhd \bot$	$R_w = \emptyset$
$\mathsf{D^s}$	$\neg_s \top \rhd \bot$	R_s is serial
$\mathsf{D^w}$	$\top \rhd \ulcorner_w \bot$	R_w is serial
$\mathsf{Alt^s}$	$\top \rhd [s]p_0 \vee \neg_s p_0$	R_s is functional
$\mathsf{Alt^w}$	$\ulcorner_w p_0 \wedge \langle w\rangle p_0 \rhd \bot$	R_w is functional
$\mathsf{Tr^s}$	$p_0 \wedge (p_0 \rightarrow_s p_1) \rhd p_1$ $\top \rhd [s]p_0 \vee \neg_s p_0$	$R_s = Id$
$\mathsf{Tr^w}$	$p_1 \rhd (p_1 \leftharpoondown_w p_0) \vee p_0$ $\ulcorner_w p_0 \wedge \langle w\rangle p_0 \rhd \bot$	$R_w = Id$
	$[s](p_0 \rightarrow_s p_1) \rhd p_0 \rightarrow_s p_1$	R_s is dense
	$p_1 \leftharpoondown_w p_0 \rhd \langle w\rangle(p_1 \leftharpoondown_w p_0)$	R_w is dense

Canonical Sequents	
$p_0 \rightarrow_{s'} p_1 \rhd p_0 \rightarrow_s p_1$	$R_s \subseteq R_{s'}$
$p_1 \leftharpoondown_w p_0 \rhd p_1 \leftharpoondown_{w'} p_0$	$R_w \subseteq R_{w'}$
$\top \rhd (p_0 \rightarrow_s p_1) \vee (p_0 \leftharpoondown_w p_1)$	$R_s \subseteq R_w$
$(p_1 \rightarrow_s p_0) \wedge (p_1 \leftharpoondown_w p_0) \rhd \bot$	$R_w \subseteq R_s$
$p_0 \rhd p_1 \vee \neg_s(p_0 \rightarrow_{s'} p_1)$	$R_s \subseteq R_{s'}^{-1}$
$\ulcorner_w(p_1 \leftharpoondown_{w'} p_0) \wedge p_1 \rhd p_0$	$R_w \subseteq R_{w'}^{-1}$
$p_0 \rhd p_1 \vee [s](p_0 \leftharpoondown_w p_1)$	$R_s \subseteq R_w^{-1}$
$\langle w\rangle(p_1 \rightarrow_s p_0) \wedge p_1 \rhd p_0$	$R_w \subseteq R_s^{-1}$
$p_0 \rightarrow_s p_1 \rhd [s](p_0 \rightarrow_{s'} p_1)$	$R_{s'} \circ R_s \subseteq R_s$
$\langle w\rangle(p_1 \leftharpoondown_{w'} p_0) \rhd p_1 \leftharpoondown_w p_0$	$R_{w'} \circ R_w \subseteq R_w$
$p_0 \rightarrow_s p_1 \rhd \neg_s(p_0 \leftharpoondown_w p_1)$	$R_w \circ R_s \subseteq R_s$
$\ulcorner_w(p_1 \rightarrow_s p_0) \rhd p_1 \leftharpoondown_w p_0$	$R_s \circ R_w \subseteq R_w$
$p_0 \rightarrow_{s'} p_1 \rhd [s](p_0 \rightarrow_{s'} p_1)$	$R_{s'} \circ R_s \subseteq R_{s'}$
$\langle w\rangle(p_1 \leftharpoondown_{w'} p_0) \rhd p_1 \leftharpoondown_{w'} p_0$	$R_{w'} \circ R_w \subseteq R_{w'}$
$\top \rhd (p_0 \leftharpoondown_w p_1) \vee \neg_s(p_0 \leftharpoondown_w p_1)$	$R_w \circ R_s \subseteq R_w$
$\ulcorner_w(p_1 \rightarrow_s p_0) \wedge (p_1 \rightarrow_s p_0) \rhd \bot$	$R_s \circ R_w \subseteq R_s$
$[s'](p_0 \rightarrow_s p_1) \rhd [s](p_0 \rightarrow_{s'} p_1)$	$R_{s'} \circ R_s \subseteq R_s \circ R_{s'}$
$\langle w\rangle(p_1 \leftharpoondown_{w'} p_0) \rhd \langle w'\rangle(p_1 \leftharpoondown_w p_0)$	$R_{w'} \circ R_w \subseteq R_w \circ R_{w'}$
$\top \rhd \neg_s(p_0 \leftharpoondown_w p_1) \vee \ulcorner_w(p_0 \rightarrow_s p_1)$	$R_w \circ R_s \subseteq R_s \circ R_w$
$\neg_s(p_1 \leftharpoondown_w p_0) \wedge \ulcorner_w(p_1 \rightarrow_s p_0) \rhd \bot$	$R_s \circ R_w \subseteq R_w \circ R_s$
$p_0 \wedge [s](p_0 \rightarrow_{s'} p_1) \rhd p_1$	$Id \subseteq R_{s'} \circ R_s$
$p_1 \rhd \langle w\rangle(p_1 \leftharpoondown_{w'} p_0) \vee p_0$	$Id \subseteq R_{w'} \circ R_w$
$p_0 \wedge \neg_s(p_0 \leftharpoondown_w p_1) \rhd p_1$	$Id \subseteq R_w \circ R_s$
$p_1 \rhd \ulcorner_w(p_1 \rightarrow_s p_0) \vee p_0$	$Id \subseteq R_s \circ R_w$

Canonical Sequents		
	$\top \rhd (p_0 \to_{s'} p_1) \vee \neg_s (p_0 \to_{s'} p_1)$ $⌐_w (p_1 \leftharpoonup_{w'} p_0) \wedge (p_1 \leftharpoonup_{w'} p_0) \rhd \bot$ $p_0 \leftharpoonup_w p_1 \rhd [s](p_0 \leftharpoonup_w p_1)$ $\langle w \rangle (p_1 \to_s p_0) \rhd p_1 \to_s p_0$	$R_{s'} \circ R_s^{-1} \subseteq R_{s'}$ $R_{w'} \circ R_w^{-1} \subseteq R_{w'}$ $R_w \circ R_s^{-1} \subseteq R_w$ $R_s \circ R_w^{-1} \subseteq R_s$
$\mathsf{con}^{\mathsf{s}}$	$\top \rhd \Big(\big((p_0 \to_s p_1) \wedge p_2\big) \to_s p_3\Big) \vee \Big(\big((p_2 \to_s p_3) \wedge p_0\big) \to_s p_1\Big)$	R_s is connected
$\mathsf{con}^{\mathsf{w}}$	$\Big(p_1 \leftharpoonup_w \big(p_0 \vee (p_3 \leftharpoonup_w p_2)\big)\Big) \wedge \Big(p_3 \leftharpoonup_w \big(p_2 \vee (p_1 \leftharpoonup_w p_0)\big)\Big) \rhd \bot$	R_s is connected
$.3^{\mathsf{s}}$	$\top \rhd \Bigg(\Big(\big((p_0 \to_s p_1) \wedge p_2\big) \to_s (p_0 \vee p_3)\Big) \wedge \Big(\big((p_0 \to_s p_1) \wedge p_1 \wedge p_2\big) \to_s p_3\Big)\Bigg) \vee$ $\vee \Bigg(\Big(\big((p_2 \to_s p_3) \wedge p_0\big) \to_s (p_2 \vee p_1)\Big) \wedge \Big(\big((p_2 \to_s p_3) \wedge p_3 \wedge p_0\big) \to_s p_1\Big)\Bigg)$	R_s is weakly connected
$.3^{\mathsf{w}}$	$\Bigg(\Big(p_1 \leftharpoonup_w \big(p_0 \vee p_3 \vee (p_3 \leftharpoonup_w p_2)\big)\Big) \vee \Big((p_1 \wedge p_2) \leftharpoonup_w \big(p_0 \vee (p_2 \leftharpoonup_w p_3)\big)\Big)\Bigg) \wedge$ $\wedge \Bigg(\Big(p_3 \leftharpoonup_w \big(p_2 \vee p_1 \vee (p_1 \leftharpoonup_w p_0)\big)\Big) \vee \Big((p_3 \wedge p_0) \leftharpoonup_w \big(p_2 \vee (p_0 \leftharpoonup_w p_1)\big)\Big)\Bigg) \rhd \bot$	R_w is weakly connected
	$\top \rhd \Big(\big((p_0 \to_s p_1) \wedge p_2 \wedge p_4\big) \to_s p_3\Big) \vee \Big(\big((p_2 \to_s p_3) \wedge p_0\big) \to_s (p_1 \vee p_4)\Big)$	R_s is weakly connected
	$\Big((p_4 \wedge p_1) \leftharpoonup_w \big(p_0 \vee (p_3 \leftharpoonup_w p_2)\big)\Big) \wedge \Big(p_3 \leftharpoonup_w \big(p_4 \vee p_2 \vee (p_1 \leftharpoonup_w p_0)\big)\Big) \rhd \bot$	R_w is weakly connected

Sequents axiomatizing complete logics		
GL^{s}	$\Big(\big(p_0 \wedge (p_0 \to_s p_1)\big) \to_s p_1\Big) \rhd \Big(p_0 \to_s p_1\Big)$	R_s is a Noetherian strict order
GL^{w}	$\Big(p_1 \leftharpoonup_w p_0\Big) \rhd \Big(p_1 \leftharpoonup_w \big((p_1 \leftharpoonup_w p_0) \vee p_0\big)\Big)$	R_w is a Noetherian strict order
$\mathsf{Grz}^{\mathsf{s}}$	$\Bigg(p_0 \wedge \bigg(\Big(p_0 \wedge [s]\big(p_0 \vee (p_0 \to_s p_1)\big) \wedge \big(p_1 \to_s (p_0 \to_s p_1)\big)\Big) \to_s p_1\bigg)\Bigg) \rhd p_1$	R_s is a Noetherian partial order
$\mathsf{Grz}^{\mathsf{w}}$	$p_1 \rhd \Bigg(\bigg(p_1 \leftharpoonup_w \Big(\big((p_1 \leftharpoonup_w p_0) \leftharpoonup_w p_1\big) \vee \langle w \rangle \big((p_1 \leftharpoonup_w p_0) \wedge p_0\big) \vee p_0\Big)\bigg) \vee p_0\Bigg)$	R_w is a Noetherian partial order

between $\mathcal{L}^{SW}(\vartheta)$-sequents is a relation $\mid\!\sim$ between sets of $\mathcal{L}^{SW}(\vartheta)$-sequents and $\mathcal{L}^{SW}(\vartheta)$-sequents satisfying that

1. If $\varsigma \in \Pi$, then $\Pi \mid\!\sim \varsigma$.
2. If $\Pi \mid\!\sim \varsigma$ and for every $\varsigma' \in \Pi$, $\Pi' \mid\!\sim \varsigma'$, then $\Pi' \mid\!\sim \varsigma$.
3. If $\Pi \mid\!\sim \varsigma$ and $\Pi \subseteq \Pi'$, then $\Pi' \mid\!\sim \varsigma$.
4. If $\Pi \mid\!\sim \varsigma$, then $e[\Pi] \mid\!\sim e(\varsigma)$ for any substitution e.
5. If $\Pi \mid\!\sim \varsigma$, then there is a finite subset Π' of Π with $\Pi' \mid\!\sim \varsigma$.

The words structural and finitary refer, respectively, to the fourth and the fifth condition[10]. Given a set Π of $\mathcal{L}^{SW}(\vartheta)$-sequents, a set Σ of $\mathcal{L}^{SW}(\vartheta)$-formulas is a Π-*theory of* $\mid\!\sim$ in the case that if $\Pi \mid\!\sim \Gamma \rhd \varphi$ and $\Gamma \subseteq \Sigma$, then $\varphi \in \Sigma$[11]. It is a *consistent* Π-theory if it is not the set of all $\mathcal{L}^{SW}(\vartheta)$-formulas. A set Σ of $\mathcal{L}^{SW}(\vartheta)$-formulas is a *prime* Π-*theory of* $\mid\!\sim$ if it is consistent and, moreover, if $\Pi \mid\!\sim \Gamma \rhd \Delta$ and $\Gamma \subseteq \Sigma$, then $\Delta \cap \Sigma \neq \emptyset$. Obviously, all prime Π-theories of $\mid\!\sim$ are Π-theories of $\mid\!\sim$.

It is clear that we can use classes of frames and classes of structures to introduce consequence relations between $\mathcal{L}^{SW}(\vartheta)$-sequents. Given an arbitrary class C of frames we define

$$\Pi \mid\!\approx_{\mathsf{C}} \varsigma \quad \text{iff} \quad \forall \mathfrak{F} \in \mathsf{C}\ \forall V \in \mathcal{P}(F)^{\mathsf{Prop}},\ \text{if } \mathfrak{F}, V \Vdash \Pi \text{ then } \mathfrak{F}, V \Vdash \varsigma.$$

And given a class K of structures we define

$$\Pi \mid\!\approx_{\mathsf{K}} \varsigma \quad \text{iff} \quad \forall \mathfrak{A} \in \mathsf{K},\ \text{if } \mathfrak{A} \Vdash \Pi \text{ then } \mathfrak{A} \Vdash \varsigma.$$ [12]

For every class C of frames it holds that $\mid\!\approx_{\mathsf{C}}$ is a structural consequence relation, but in general it is not finitary. If we start with a class K of structures then in general it is neither structural nor finitary.

[10] We emphasize that there is no connection of this use of 'structural' with the rules between sequents called structural in the literature [DSH93].

[11] We notice that in the inclusion $\Gamma \subseteq \Sigma$ the symbol Γ obviously refers to the underlying set.

[12] These definitions have the flavour of a global consequence (although we will see in Remark 4.2.6 that they also tell us something from a local point of view), which is illustrated in Proposition 4.2.7. In order to obtain a version of this proposition replacing global consequences with local consequences it would be interesting to define

$$\Pi \mid\!\approx_{l\mathsf{C}} \varsigma \quad \text{iff} \quad \forall \mathfrak{F} \in \mathsf{C}\ \forall V \in \mathcal{P}(F)^{\mathsf{Prop}}\ \forall a \in F,\ \text{if } \mathfrak{F}, V, a \Vdash \Pi \text{ then } \mathfrak{F}, V, a \Vdash \varsigma,$$

$$\Pi \mid\!\approx_{l\mathsf{K}} \varsigma \quad \text{iff} \quad \forall \mathfrak{A} \in \mathsf{K}\ \forall a \in A,\ \text{if } \mathfrak{A}, a \Vdash \Pi \text{ then } \mathfrak{A}, a \Vdash \varsigma.$$

The author considers that these consequence relations merit study. The reason why we do not consider them in this dissertation is because the author does not know how to axiomatize them using a calculus with strict-weak sequents.

4.2.6. REMARK. It is clear that for every set $\Phi \cup \{\varphi\}$ of $\mathcal{L}^{SW}(\vartheta)$-formulas,

$$\Phi \models_{g\mathsf{F}} \varphi \quad \text{iff} \quad \{\emptyset \rhd \phi : \phi \in \Phi\} \approx_{\mathsf{F}} \emptyset \rhd \varphi,$$

$$\Phi \models_{g\mathsf{K}} \varphi \quad \text{iff} \quad \{\emptyset \rhd \phi : \phi \in \Phi\} \approx_{\mathsf{K}} \emptyset \rhd \varphi.$$

If moreover Φ is finite then we have that

$$\Phi \models_{l\mathsf{F}} \varphi \quad \text{iff} \quad \emptyset \approx_{\mathsf{F}} \Phi \rhd \varphi,$$

$$\Phi \models_{l\mathsf{K}} \varphi \quad \text{iff} \quad \emptyset \approx_{\mathsf{K}} \Phi \rhd \varphi.$$

Hence, it is easy to recuperate the local and global consequences between $\mathcal{L}^{SW}(\vartheta)$-formulas from the consequences between $\mathcal{L}^{SW}(\vartheta)$-sequents.

4.2.7. PROPOSITION. *Let C be a class of τ_ϑ-frames. Then,*

1. *For every class C' of τ_ϑ-frames,*

$$\approx_{\mathsf{C}} = \approx_{\mathsf{C}'} \quad \textit{iff} \quad \models^{MOD}_{g\mathsf{C}} = \models^{MOD}_{g\mathsf{C}'}.$$

2. *$\approx_{\mathsf{C}}$ is finitary iff $\models^{MOD}_{g\mathsf{C}}$ is finitary.*

The same holds for classes K of structures.

Proof: Both items can be shown using the same tools. One direction is obtained observing that

$$\Pi \approx_{\mathsf{C}} \varsigma \quad \text{iff} \quad \sigma[\Pi] \models^{MOD}_{g\mathsf{C}} \sigma(\varsigma).$$

The converse is an easy consequence of the Standard Form Theorem. □

We will now characterize the previous consequence relations using a calculus. The minimal calculus is described in Table 4.2.

4.2.8. DEFINITION. (Consequence relations $\mid\!\sim_L$)
Let ϑ be a SW-vocabulary such that $\mathsf{SMod} \cap \mathsf{WMod} = \emptyset$, and let L be a set of $\mathcal{L}^{SW}(\vartheta)$-sequents. We define $\mid\!\sim_L$ as the finitary structural consequence relation between $\mathcal{L}^{SW}(\vartheta)$-sequents such that $\Pi \mid\!\sim_L \varsigma$ iff there is a derivation of the $\mathcal{L}^{SW}(\vartheta)$-sequent ς with hypotheses in $\Pi \cup \mathrm{sub}(L)$ using the calculus in Table 4.2.

4.2.9. CONVENTION. In the rest of the chapter we will always assume that all SW-vocabularies considered satisfy $\mathsf{SMod} \cap \mathsf{WMod} = \emptyset$.

Table 4.2: The minimal calculus associated with ϑ (when $\mathsf{SMod} \cap \mathsf{WMod} = \emptyset$)

Structural Rules

$$(Ax) \quad \Gamma \rhd \Gamma$$

$$(w\rhd)\ \frac{\Gamma \rhd \Delta}{\Gamma, \varphi \rhd \Delta} \qquad (\rhd w)\ \frac{\Gamma \rhd \Delta}{\Gamma \rhd \varphi, \Delta}$$

$$(c\rhd)\ \frac{\Gamma, \varphi, \varphi \rhd \Delta}{\Gamma, \varphi \rhd \Delta} \qquad (\rhd c)\ \frac{\Gamma \rhd \varphi, \varphi, \Delta}{\Gamma \rhd \varphi, \Delta}$$

$$(e\rhd)\ \frac{\Gamma, \varphi_0, \varphi_1 \rhd \Delta}{\Gamma, \varphi_1, \varphi_0 \rhd \Delta} \qquad (\rhd e)\ \frac{\Gamma \rhd \varphi_0, \varphi_1, \Delta}{\Gamma \rhd \varphi_1, \varphi_0, \Delta}$$

$$(Cut)\ \frac{\Gamma \rhd \varphi, \Delta \qquad \Gamma', \varphi \rhd \Delta'}{\Gamma, \Gamma' \rhd \Delta, \Delta'}$$

Rules for Logical Connectives

$$(\bot\rhd) \quad \Gamma, \bot \rhd \Delta \qquad (\rhd\top) \quad \Gamma \rhd \top, \Delta$$

$$(\wedge\rhd)\ \frac{\Gamma, \varphi_0, \varphi_1 \rhd \Delta}{\Gamma, \varphi_0 \wedge \varphi_1 \rhd \Delta} \qquad (\rhd\wedge)\ \frac{\Gamma \rhd \varphi_0, \Delta \qquad \Gamma \rhd \varphi_1, \Delta}{\Gamma \rhd \varphi_0 \wedge \varphi_1, \Delta}$$

$$(\rhd\vee)\ \frac{\Gamma \rhd \varphi_0, \varphi_1, \Delta}{\Gamma \rhd \varphi_0 \vee \varphi_1, \Delta} \qquad (\vee\rhd)\ \frac{\Gamma, \varphi_0 \rhd \Delta \qquad \Gamma, \varphi_1 \rhd \Delta}{\Gamma, \varphi_0 \vee \varphi_1 \rhd \Delta}$$

$$(\rightarrow_s \rhd \rightarrow_s)\ \frac{\Gamma, \varphi_0 \rhd \varphi_1, \Delta}{\varphi_0 \rightarrow_s \Gamma, \Delta \rightarrow_s \varphi_1 \rhd \varphi_0 \rightarrow_s \varphi_1} \quad s \in \mathsf{SMod}$$

$$(\leftharpoondown_w \rhd \leftharpoondown_w)\ \frac{\Gamma, \varphi_0 \rhd \varphi_1, \Delta}{\varphi_0 \leftharpoondown_w \varphi_1 \rhd \varphi_0 \leftharpoondown_w \Gamma, \Delta \leftharpoondown_w \varphi_1} \quad w \in \mathsf{WMod}$$

4.2.10. Remark. (The case of an arbitrary SW-vocabulary). The restriction to SW-vocabularies such that $\mathsf{SMod} \cap \mathsf{WMod} = \emptyset$ is done for the sake of simplicity. In fact, as a consequence of Proposition 4.2.26 (see the first block on page 144) we also know how to handle arbitrary SW-vocabularies. The only difference is that in the general case it is necessary to add to the calculus in Table 4.2 the axioms

- $\top \rhd (\varphi_0 \rightharpoondown_m \varphi_1) \vee (\varphi_0 \leftharpoondown_m \varphi_1)$,
- $(\varphi_1 \rightharpoondown_m \varphi_0) \wedge (\varphi_1 \leftharpoondown_m \varphi_0) \rhd \bot$,

for every $m \in \mathsf{SMod} \cap \mathsf{WMod}$.

4.2.11. Proposition. *Let L be a set of $\mathcal{L}^{SW}(\vartheta)$-sequents. For every set Π of $\mathcal{L}^{SW}(\vartheta)$-sequents and every $\mathcal{L}^{SW}(\vartheta)$-sequent ς, it holds that*

$$\Pi \mathrel{|\!\sim}^{\vartheta}_{L} \varsigma \quad \textit{iff} \quad \Pi^d \mathrel{|\!\sim}^{\vartheta^d}_{L^d} \varsigma^d.$$

Proof: By idempotency of duality it is enough to show the implication to the right. This is easily proved observing that each rule given in Table 4.2 for ϑ is precisely one of the rules given in Table 4.2 for ϑ^d. We have the following pairs of mutual dual rules: i) (Ax) and itself, ii) $(w\rhd)$ and $(\rhd w)$, iii) $(c\rhd)$ and $(\rhd c)$, iv) $(c\rhd)$ and $(\rhd c)$, v) (Cut) and itself, vi) $(\bot\rhd)$ and $(\rhd\top)$, vii) $(\wedge\rhd)$ and $(\vee\rhd)$, viii) $(\rhd\wedge)$ and $(\rhd\vee)$, and ix) $(\rightharpoondown_m \rhd \rightharpoondown_m)$ and $(\leftharpoondown_m \rhd \leftharpoondown_m)$. □

Now we list several derivable sequents and derivable rules of the calculus given in Table 4.2. In particular, all of them are derivable in the calculus $\mathrel{|\!\sim}_L$ for every set L of $\mathcal{L}^{SW}(\vartheta)$-sequents.

$$(1)\ \varphi_0 \rightharpoondown_s \varphi_1, \varphi_0 \rightharpoondown_s \varphi_2 \rhd \varphi_0 \rightharpoondown_s (\varphi_1 \wedge \varphi_2)$$

$$(2)\ \varphi_0 \rightharpoondown_s \varphi_2, \varphi_1 \rightharpoondown_s \varphi_2 \rhd (\varphi_0 \vee \varphi_1) \rightharpoondown_s \varphi_2$$

$$(3)\ \varphi_0 \rightharpoondown_s \varphi_1, \varphi_1 \rightharpoondown_s \varphi_2 \rhd \varphi_0 \rightharpoondown_s \varphi_2$$

$$(4)\ \bot \rhd \varphi \qquad (5)\ \emptyset \rhd \top$$

$$(1^d)\ (\varphi_2 \vee \varphi_1) \leftharpoondown_w \varphi_0 \rhd \varphi_2 \leftharpoondown_w \varphi_0, \varphi_1 \leftharpoondown_w \varphi_0$$

$$(2^d)\ \varphi_2 \leftharpoondown_w (\varphi_1 \wedge \varphi_0) \rhd \varphi_2 \leftharpoondown_w \varphi_1, \varphi_2 \leftharpoondown_w \varphi_0$$

$$(3^d)\ \varphi_2 \leftharpoondown_w \varphi_0 \rhd \varphi_2 \leftharpoondown_w \varphi_1, \varphi_1 \leftharpoondown_w \varphi_0$$

$$(DT_0)^{13}\ \frac{\varphi_0 \rhd \varphi_1}{\emptyset \rhd \varphi_0 \rightharpoondown \varphi_1} \qquad (DT_0^d)\ \frac{\varphi_0 \rhd \varphi_1}{\varphi_0 \leftharpoondown \varphi_1 \rhd \bot}$$

$$(Pre)\ \frac{\Gamma \rhd \varphi_0}{\varphi_1 \rightharpoondown_s \Gamma \rhd \varphi_1 \rightharpoondown_s \varphi_0} \qquad (Suf)\ \frac{\varphi_0 \rhd \Delta}{\Delta \rightharpoondown_s \varphi_1 \rhd \varphi_0 \rightharpoondown_s \varphi_1}$$

[13] The name (DT_0) comes from the fact that this rule is the deduction theorem when the set of hypotheses has cardinal 0. Similarly the rules (DT_n) can be considered where $n \in \omega$. The reader interested in this family of rules should see [BFGL04].

$$(Pre^d)\ \frac{\varphi_0 \rhd \Delta}{\varphi_0 \leftharpoondown_w \varphi_1 \rhd \Delta \leftharpoondown_w \varphi_1} \qquad (Suf^d)\ \frac{\Gamma \rhd \varphi_0}{\varphi_1 \leftharpoondown_w \varphi_0 \rhd \varphi_1 \leftharpoondown_w \Gamma}$$

$$(6)\ \ (\varphi_0 \wedge \varphi_1) \rightarrow_s \varphi, (\phi_0 \wedge \phi_1) \rightarrow_s \varphi \rhd (\varphi_0 \wedge \phi_0 \wedge (\varphi_1 \vee \phi_1)) \rightarrow_s \varphi$$

To show this, by duality, it is enough to derive the ones where ^{d} does not appear in the name. The reader can find these derivations in Figure 4.4. It is obvious that all Π-theories of $\mid\!\sim_L$ must be closed (in the sense that is considered in the definition of Π-theory of a consequence relation) under the previous axioms and rules.

4.2.12. Remark. (An equivalent calculus using a single formula on the right). In this remark we restrict ourselves to $\mathcal{L}^{SW}(\vartheta)$-sequents where on the right there is a single $\mathcal{L}^{SW}(\vartheta)$-formula. In particular the empty sequence cannot be on the right. Let us consider the calculus given by the restrictions of the rules in Table 4.2 to this kind of sequents, but replacing $(\bot\rhd)$, $(\rhd\top)$, $(\rightarrow_s \rhd \rightarrow_s)$ and $(\leftharpoondown_w \rhd \leftharpoondown_w)$ with the axioms and rules (4), (5), (1), (2), (3), (DT_0), (1^d), (2^d), (3^d) and (DT_0^d). Let us call $\mid\!\sim_L^*$ the consequence relation between this kind of sequents that is introduced with respect to this new calculus as we did in Definition 4.2.8. It is not hard to see that $\mid\!\sim_L$ and $\mid\!\sim_L^*$ are equivalent with respect to this type of sequents in the following strong sense: for every set Π of $\mathcal{L}^{SW}(\vartheta)$-sequents where on the right there is only a single $\mathcal{L}^{SW}(\vartheta)$-formula, it holds that

$$\Pi \mid\!\sim_L \Gamma \rhd \varphi \quad \text{iff} \quad \Pi \mid\!\sim_L^* \Gamma \rhd \varphi.$$

Indeed, this new calculus is the one that was considered by Celani and Jansana in [CJ01] for the particular case of $\mathcal{L}^s$. Hence the calculus that we have previously given is only a reformulation of the one considered by Celani and Jansana. We have changed the presentation for two reasons. First of all, because from the point of view of duality it is better to have the same length on the left as on the right of $\rhd$. And secondly, because our calculus only uses one rule for each strict implication (and for each weak difference).

4.2.13. Proposition (Soundness). *Let L be a set of $\mathcal{L}^{SW}(\vartheta)$-sequents. For every set Π of $\mathcal{L}^{SW}(\vartheta)$-sequents and every $\mathcal{L}^{SW}(\vartheta)$-sequent ς, it holds that*

$$\textit{if } \Pi \mid\!\sim_L \varsigma, \textit{ then } \Pi \approx_{\mathsf{K}} \varsigma,$$

where K *is the class* $\{\mathfrak{A} : \mathfrak{A} \Vdash \mathrm{sub}(L)\}$.

Proof: The proof is by induction on the length of the derivation in $\mid\!\sim_L$. It is easy to see that all rules in Table 4.2 are sound. We illustrate this by checking the case of the rule $(\rightarrow_s \rhd \rightarrow_s)$. Assume that $\mathfrak{A} \Vdash \gamma_0, \ldots, \gamma_{n-1}, \varphi_0 \rhd \varphi_1, \delta_0, \ldots, \delta_{k-1}$ and that $\mathfrak{A}, a \Vdash \{\varphi_0 \rightarrow_s \gamma_0, \ldots, \varphi_0 \rightarrow_s \gamma_{n-1}, \delta_0 \rightarrow_s \varphi_1, \ldots, \delta_{k-1} \rightarrow_s \varphi_1\}$. In order

Figure 4.4: Some examples of derivations in $\mid\!\sim_L$

$$
(\rightarrow_s \rhd \rightarrow_s)\;\dfrac{(w\rhd)\;\dfrac{(\rhd\wedge)\;\dfrac{(w\rhd)\;\dfrac{\varphi_1 \rhd \varphi_1}{\varphi_1, \varphi_2 \rhd \varphi_1} \qquad \dfrac{\dfrac{\varphi_2 \rhd \varphi_2}{\varphi_2, \varphi_1 \rhd \varphi_2}\;(w\rhd)}{\varphi_1, \varphi_2 \rhd \varphi_2}\;(e\rhd)}{\varphi_1, \varphi_2 \rhd \varphi_1 \wedge \varphi_2}}{\varphi_1, \varphi_2, \varphi_0 \rhd \varphi_1 \wedge \varphi_2}}{\varphi_0 \rightarrow_s \varphi_1, \varphi_0 \rightarrow_s \varphi_2 \rhd \varphi_0 \rightarrow_s (\varphi_1 \wedge \varphi_2)}
$$

$$
(\rightarrow_s \rhd \rightarrow_s)\;\dfrac{(\rhd w)\;\dfrac{(\vee\rhd)\;\dfrac{(\rhd e)\;\dfrac{(\rhd w)\;\dfrac{\varphi_0 \rhd \varphi_0}{\varphi_0 \rhd \varphi_1, \varphi_0}}{\varphi_0 \rhd \varphi_0, \varphi_1} \qquad \dfrac{\varphi_1 \rhd \varphi_1}{\varphi_1 \rhd \varphi_0, \varphi_1}\;(\rhd w)}{\varphi_0 \vee \varphi_1 \rhd \varphi_0, \varphi_1}}{\varphi_0 \vee \varphi_1 \rhd \varphi_2, \varphi_0, \varphi_1}}{\varphi_0 \rightarrow_s \varphi_2, \varphi_1 \rightarrow_s \varphi_2 \rhd (\varphi_0 \vee \varphi_1) \rightarrow_s \varphi_2}
$$

$$
(\rightarrow_s \rhd \rightarrow_s)\;\dfrac{(w\rhd)\;\dfrac{\varphi_1 \rhd \varphi_1}{\varphi_1, \varphi_0 \rhd \varphi_2, \varphi_1}}{\varphi_0 \rightarrow_s \varphi_1, \varphi_1 \rightarrow_s \varphi_2 \rhd \varphi_0 \rightarrow_s \varphi_2}
\qquad
(\rightarrow_s \rhd \rightarrow_s)\;\dfrac{\varphi_0 \rhd \varphi_1}{\emptyset \rhd \varphi_0 \rightarrow_s \varphi_1}
$$

$$
(\rightarrow_s \rhd \rightarrow_s)\;\dfrac{(w\rhd)\;\dfrac{\Gamma \rhd \varphi_0}{\Gamma, \varphi_1 \rhd \varphi_0}}{\varphi_1 \rightarrow_s \Gamma \rhd \varphi_1 \rightarrow_s \varphi_0}
\qquad
(\rightarrow_s \rhd \rightarrow_s)\;\dfrac{(\rhd w)\;\dfrac{\varphi_0 \rhd \Delta}{\varphi_0 \rhd \varphi_1, \Delta}}{\Delta \rightarrow_s \varphi_1 \rhd \varphi_0 \rightarrow_s \varphi_1}
$$

$$
(Suf)\;\dfrac{\dfrac{(\vee\rhd)\;\dfrac{\dfrac{(\rhd\wedge)\;\dfrac{\dfrac{\varphi_0 \rhd \varphi_0}{\varphi_0, \varphi_1 \rhd \varphi_0} \qquad \dfrac{\varphi_1 \rhd \varphi_1}{\varphi_0, \varphi_1 \rhd \varphi_1}}{\varphi_0, \varphi_1 \rhd \varphi_0 \wedge \varphi_1}}{\varphi_0, \phi_0, \varphi_1 \rhd \varphi_0 \wedge \varphi_1, \phi_0 \wedge \phi_1} \qquad \dfrac{\dfrac{\dfrac{\phi_0 \rhd \phi_0}{\phi_0, \phi_1 \rhd \phi_0} \qquad \dfrac{\phi_1 \rhd \phi_1}{\phi_0, \phi_1 \rhd \phi_1}}{\phi_0, \phi_1 \rhd \phi_0 \wedge \phi_1}\;(\rhd\wedge)}{\varphi_0, \phi_0, \phi_1 \rhd \varphi_0 \wedge \varphi_1, \phi_0 \wedge \phi_1}}{\varphi_0, \phi_0, \varphi_1 \vee \phi_1 \rhd \varphi_0 \wedge \varphi_1, \phi_0 \wedge \phi_1}}{\varphi_0 \wedge \phi_0 \wedge (\varphi_1 \vee \phi_1) \rhd \varphi_0 \wedge \varphi_1, \phi_0 \wedge \phi_1}}{(\varphi_0 \wedge \varphi_1) \rightarrow_s \varphi, (\phi_0 \wedge \phi_1) \rightarrow_s \varphi \rhd (\varphi_0 \wedge \phi_0 \wedge (\varphi_1 \vee \phi_1)) \rightarrow_s \varphi}
$$

to show that $\mathfrak{A}, a \Vdash \varphi_0 \to \varphi_1$ we suppose that it is not the case. Then there is a' such that $\langle a, a' \rangle \in R_s$, $\mathfrak{A}, a' \Vdash \varphi_0$ and $\mathfrak{A}, a' \not\Vdash \varphi_1$. Using that $\mathfrak{A}, a \Vdash \{\varphi_0 \to_s \gamma_0, \ldots, \varphi_0 \to_s \gamma_{n-1}\}$ it is obvious that $\mathfrak{A}, a' \Vdash \varphi_0 \wedge \gamma_0 \wedge \ldots \wedge \gamma_{n-1}$. Hence, using the fact that $\mathfrak{A}, a' \Vdash \gamma_0, \ldots, \gamma_{n-1}, \varphi_0 \rhd \varphi_1, \delta_0, \ldots, \delta_{k-1}$ and $\mathfrak{A}, a' \not\Vdash \varphi_1$ it follows that there is $i < k$ such that $\mathfrak{A}, a' \Vdash \delta_i$. But this contradicts the fact that $\mathfrak{A}, a \Vdash \delta_i \to_s \varphi_1$. □

Our aim is to show that the converse of the last proposition also holds. Celani and Jansana proved in [CJ01] the converse implication for $\mathcal{L}^s$ in the case that $\Pi = \emptyset$. Their proof is based on the construction of a canonical structure. Here we will generalize this canonical structure construction in order to manage arbitrary Π's. For the case of $\mathcal{L}^s$ this generalization was already developed by the author in [Bou01, Section 3.1]. Before introducing the construction we prove several lemmas.

4.2.14. Lemma. *Let L and Π be sets of $\mathcal{L}^{SW}(\vartheta)$-sequents.*

1. *The set of Π-theories of $\mid\!\sim_L$ is closed under intersections.*
2. *For every set Φ of $\mathcal{L}^{SW}(\vartheta)$-formulas, the smallest Π-theory of $\mid\!\sim_L$ extending Φ is*

   $$\mathrm{Th}^L_\Pi(\Phi) := \{\varphi \in \mathcal{L}^{SW}(\vartheta) : \textit{there is } \Gamma \subseteq_\omega \Phi \textit{ such that } \Pi \mid\!\sim_L \Gamma \rhd \varphi\}.$$

   If Φ is finite then $\mathrm{Th}^L_\Pi(\Phi) = \{\varphi \in \mathcal{L}^{SW}(\vartheta) : \Pi \mid\!\sim_L \Phi \rhd \varphi\}$.
3. *The set of Π-theories of $\mid\!\sim_L$ is an inductive*[14] *closure system.*
4. *A Π-theory Σ of $\mid\!\sim_L$ is prime iff it satisfies that for every $\varphi_0, \varphi_1 \in \mathcal{L}^{SW}(\vartheta)$, if $\varphi_0 \vee \varphi_1 \in \Sigma$ then either $\varphi_0 \in \Sigma$ or $\varphi_1 \in \Sigma$.*

Proof: The first and fourth items are easily proved, and the third one is an easy consequence of the second one. Let us show the second item. By (Ax) we know that $\Phi \subseteq \mathrm{Th}^L_\Pi(\Phi)$. By (Cut) it follows that $\mathrm{Th}^L_\Pi(\Phi)$ is a Π-theory of $\mid\!\sim_L$. It is trivial that all Π-theories of $\mid\!\sim_L$ extending Φ also contain $\mathrm{Th}^L_\Pi(\Phi)$. Hence, $\mathrm{Th}^L_\Pi(\Phi)$ is the smallest Π-theory of $\mid\!\sim_L$ extending Φ. In the case that Φ is finite it is obvious that the previous definition gives us the set $\{\varphi \in \mathcal{L}^{SW}(\vartheta) : \Pi \mid\!\sim_L \Phi \rhd \varphi\}$. □

Now we characterize the consequence relations $\mid\!\sim_L$ using their prime theories. These results can be considered as logical counterparts of the algebraic Prime Filter Theorem [Sto36] that holds in bounded distributive lattices (see for instance [Grä98, Theorem 15]).

[14] This means that it is closed under unions of non-empty upwards directed subfamilies. By Schmidt's Theorem [Sch52] (see [Cze01, Theorem 3] for a more accessible publication) it is enough to require that the closure system is closed under unions of non-empty chains.

4.2.15. LEMMA. *Let L and Π be sets of $\mathcal{L}^{SW}(\vartheta)$-sequents, and let Γ and Δ be two sets of $\mathcal{L}^{SW}(\vartheta)$-formulas. The following are equivalent:*

1. *For every $\Gamma' \subseteq_\omega \Gamma$ and every $\Delta' \subseteq_\omega \Delta$, it holds that $\Pi \not\mid\!\sim_L \Gamma' \rhd \Delta'$.*
2. *There is a prime Π-theory Σ of $\mid\!\sim_L$ such that $\Gamma \subseteq \Sigma$ and $\Delta \cap \Sigma = \emptyset$.*

Proof: $(2 \Rightarrow 1)$: It is a direct consequence of the definition of prime Π-theory.

$(1 \Rightarrow 2)$: Suppose that for every $\Gamma' \subseteq_\omega \Gamma$ and every $\Delta' \subseteq_\omega \Delta$, it holds that $\Pi \not\mid\!\sim_L \Gamma' \rhd \Delta'$. We can assume that Δ is closed under disjunction: if it is not the case, replace Δ with $\{\varphi \in \mathcal{L}^{SW}(\vartheta) : \Pi \mid\!\sim_L \varphi \rhd \Delta' \text{ for some } \Delta' \subseteq_\omega \Delta\}$. Let us consider the family

$$\mathcal{F} := \{\Sigma \subseteq \mathcal{L}^{SW}(\vartheta) : \Sigma \text{ is a } \Pi\text{-theory of } \mid\!\sim_L \text{ such that } \Gamma \subseteq \Sigma \text{ and } \Delta \cap \Sigma = \emptyset\}.$$

This family is non-empty because the set

$$\{\varphi \in \mathcal{L}^{SW}(\vartheta) : \Pi \mid\!\sim_L \Gamma' \rhd \varphi, \Delta' \text{ for some } \Gamma' \subseteq_\omega \Gamma \text{ and } \Delta' \subseteq_\omega \Delta\}$$

is in $\mathcal{F}$. It is clear that every non-empty chain of the partial order $\langle \mathcal{F}, \subseteq \rangle$ admits an upper bound (by Lemma 4.2.14(3) we can take its union). By Zorn's Lemma there is a maximal set Σ in the partial order $\langle \mathcal{F}, \subseteq \rangle$. Let us see that it is a prime Π-theory of $\mid\!\sim_L$. Assume that $\varphi_0 \vee \varphi_1 \in \Sigma$ and $\varphi_0, \varphi_1 \notin \Sigma$. Let $\Sigma_0 := Th^L_\Pi(\Sigma \cup \{\varphi_0\})$ and $\Sigma_1 := Th^L_\Pi(\Sigma \cup \{\varphi_1\})$. It is obvious that $\Gamma \subseteq \Sigma_0$ and $\Gamma \subseteq \Sigma_1$. Since $\varphi_0, \varphi_1 \notin \Sigma$ it follows that $\Sigma \subsetneq \Sigma_0$ and $\Sigma \subsetneq \Sigma_1$. By maximality of Σ in $\langle \mathcal{F}, \subseteq \rangle$, it holds that $\Delta \cap \Sigma_0 \neq \emptyset$ and $\Delta \cap \Sigma_1 \neq \emptyset$. So, there is $\phi_0 \in \Delta$ and $\Gamma_0 \subseteq_\omega \Gamma$ such that $\Pi \mid\!\sim_L \Gamma_0, \varphi_0 \rhd \phi_0$, and there is $\phi_1 \in \Delta$ and $\Gamma_1 \subseteq_\omega \Gamma$ such that $\Pi \mid\!\sim_L \Gamma_1, \varphi_1 \rhd \phi_1$. Therefore, $\Pi \mid\!\sim_L \Gamma_0, \Gamma_1, \varphi_0 \vee \varphi_1 \rhd \phi_0 \vee \phi_1$ where $\Gamma_0 \cup \Gamma_1 \cup \{\varphi_0 \vee \varphi_1\} \subseteq \Sigma$. Since Σ is a Π-theory of $\mid\!\sim_L$ it follows that $\phi_0 \vee \phi_1 \in \Sigma$. Thus, $\phi_0 \vee \phi_1 \notin \Delta$, which is in contradiction with the fact that Δ is closed under disjunction. □

4.2.16. COROLLARY. *Let L and Π be sets of $\mathcal{L}^{SW}(\vartheta)$-sequents, and let $\Gamma \rhd \Delta$ be a $\mathcal{L}^{SW}(\vartheta)$-sequent. The following are equivalent:*

1. $\Pi \mid\!\sim_L \Gamma \rhd \Delta$.
2. *For every prime Π-theory Σ of $\mid\!\sim_L$, if $\Gamma \subseteq \Sigma$, then $\Delta \cap \Sigma \neq \emptyset$.*

Proof: It is obvious from the previous lemma. □

We will say that two sets L and Π of $\mathcal{L}^{SW}(\vartheta)$-sequents are *relatively consistent* in the case that $\Pi \not\mid\!\sim_L \top \rhd \bot$. By the last corollary, it is equivalent to say that there is prime Π-theory of $\mid\!\sim_L$. As a consequence of Proposition 4.2.13 if there is a structure $\mathfrak{A}$ such that $\mathfrak{A} \Vdash \Pi \cup \text{sub}(L)$, then we know that L and Π are relatively consistent.

4.2.17. Definition. (The canonical structures $\mathfrak{H}^L_\Pi$)
Let L and Π be two sets of $\mathcal{L}^{SW}(\vartheta)$-sequents that are relatively consistent. The *canonical structure associated with the sets* L *and* Π is the τ_ϑ-structure $\mathfrak{H}^L_\Pi$ defined as follows:

- The universe H^L_Π is the set of all prime Π-theories of $\mid\!\sim_L$.
- For every strict modality s, the accessibility relation $R_s^{\mathfrak{H}^L_\Pi}$ is

$$\{\langle\Sigma_0,\Sigma_1\rangle \in H^L_\Pi\times H^L_\Pi : \forall\varphi_0,\varphi_1 \in \mathcal{L}^{SW}(\vartheta)(\varphi_0 \to_s \varphi_1 \in \Sigma_0 \& \varphi_0 \in \Sigma_1 \Rightarrow \varphi_1 \in \Sigma_1)\}.$$

- For every weak modality w, the accessibility relation $R_w^{\mathfrak{H}^L_\Pi}$ is

$$\{\langle\Sigma_0,\Sigma_1\rangle \in H^L_\Pi\times H^L_\Pi : \forall\varphi_0,\varphi_1 \in \mathcal{L}^{SW}(\vartheta)(\varphi_0 \in \Sigma_1 \& \varphi_1 \notin \Sigma_1 \Rightarrow \varphi_0 \leftharpoondown_w \varphi_1 \in \Sigma_0)\}.$$

- The valuation is the map $p \longmapsto \{\Sigma \in H^L_\Pi : p \in \Sigma\}$.

The canonical frame for L and Π, denoted $\mathfrak{F}^L_\Pi$, is obtained by deleting the valuation on the canonical structure $\mathfrak{H}^L_\Pi$. When $\Pi = \emptyset$ we also use the notations $\mathfrak{H}^L$ and $\mathfrak{F}^L$.

The previous definition is meaningful because we know by assumption that there is no modality in $\mathsf{SMod} \cap \mathsf{WMod}$, and because $H^L_\Pi \neq \emptyset$ since the sets L and Π are relatively consistent.

4.2.18. Proposition (Duality between canonical structures). *Let* L *and* Π *be two sets of* $\mathcal{L}^{SW}(\vartheta)$*-sequents that are relatively consistent. The map*

$$\begin{array}{rccl} f^L_\Pi: & H^L_\Pi & \longrightarrow & H^{L^d}_{\Pi^d} \\ & \Sigma & \longmapsto & \{\varphi \in \mathcal{L}^{SW}(\vartheta^d) : \varphi^d \notin \Sigma\} \end{array}$$

is an isomorphism between $\mathfrak{H}^L_\Pi$ *and* $\mathfrak{H}^{L^d}_{\Pi^d}$.[15]

Proof: It is clear that $f^L_\Pi(\Sigma)$ also coincides with $\{\varphi^d : \varphi \in \mathcal{L}^{SW}(\vartheta), \varphi \notin \Sigma\}$. By Proposition 4.2.11 it is easy to see that $\Sigma \in H^L_\Pi$ iff $f^L_\Pi(\Sigma) \in H^{L^d}_{\Pi^d}$. The rest of the proof is simple to check. □

Next we seek an explanation of the behaviour of logical connectives in prime Π-theories. The answer is stated in Corollary 4.2.21.

4.2.19. Lemma. *Let* L *and* Π *be sets of* $\mathcal{L}^{SW}(\vartheta)$*-sequents that are relatively consistent, let* Σ *be a* Π*-theory of* $\mid\!\sim_L$*, and let* Γ *and* Δ *be two sets of* $\mathcal{L}^{SW}(\vartheta)$*-formulas. If* $s \in \mathsf{SMod}$*, then the following are equivalent:*

[15] We note that while $\mathfrak{H}^L_\Pi$ corresponds to the vocabulary ϑ, the canonical structure $\mathfrak{H}^{L^d}_{\Pi^d}$ corresponds to ϑ^d.

1. *For every* $\Gamma' \subseteq_\omega \Gamma$ *and every* $\Delta' \subseteq_\omega \Delta$*, it holds that* $\bigwedge \Gamma' \rightarrow_s \bigvee \Delta' \notin \Sigma$.

2. *There is a prime* Π*-theory* Σ' *of* $\mathrel{|\!\sim}_L$ *such that* $\langle \Sigma, \Sigma' \rangle \in R_s^{\mathfrak{H}^L_\Pi}$, $\Gamma \subseteq \Sigma'$ *and* $\Delta \cap \Sigma' = \emptyset$.

Proof: $(2 \Rightarrow 1)$: It is an easy consequence of the definition of $R_s^{\mathfrak{H}^L_\Pi}$.

$(1 \Rightarrow 2)$: Assume that for every $\Gamma' \subseteq_\omega \Gamma$ and every $\Delta' \subseteq_\omega \Delta$, it holds that $\bigwedge \Gamma' \rightarrow_s \bigvee \Delta' \notin \Sigma$. We can assume that Δ is closed under disjunction: if it is not the case, replace Δ with $\{\varphi \in \mathcal{L}^{SW}(\vartheta) : \Pi \mathrel{|\!\sim}_L \varphi \rhd \Delta' \text{ for some } \Delta' \subseteq_\omega \Delta\}$. Let us consider the set

$$\mathcal{F} := \left\{ \Sigma' : \begin{array}{l} \Sigma' \text{ is a } \Pi\text{-theory of } \mathrel{|\!\sim}_L, \Gamma \subseteq \Sigma', \Delta \cap \Sigma' = \emptyset, \text{ and} \\ \forall \varphi_0, \varphi_1 (\varphi_0 \rightarrow_s \varphi_1 \in \Sigma \;\&\; \varphi_0 \in \Sigma' \;\Rightarrow\; \varphi_1 \in \Sigma') \end{array} \right\}.$$

This set is non-empty as

$$\{\varphi \in \mathcal{L}^{SW}(\vartheta) : \left(\bigwedge \Gamma'\right) \rightarrow_s \varphi \in \Sigma \text{ for some } \Gamma' \subseteq_\omega \Gamma\}$$

is a Π-theory of $\mathrel{|\!\sim}_L$ that verifies the conditions for being an element of $\mathcal{F}$. It is clear that every non-empty chain of the partial order $\langle \mathcal{F}, \subseteq \rangle$ admits an upper bound (by Lemma 4.2.14(3) its union works well). Using Zorn's Lemma we obtain a maximal element Σ' in the partial order $\langle \mathcal{F}, \subseteq \rangle$. It only remains to show that Σ' is a prime Π-theory of $\mathrel{|\!\sim}_L$. Assume that $\phi_0 \vee \phi_1 \in \Sigma'$ and $\phi_0, \phi_1 \notin \Sigma'$. Let us define

$$\begin{array}{lcl} \Sigma_0 & := & \{\varphi \in \mathcal{L}^{SW}(\vartheta) : (\phi \wedge \phi_0) \rightarrow \varphi \in \Sigma \text{ for some } \phi \in \Sigma'\} \\ \Sigma_1 & := & \{\varphi \in \mathcal{L}^{SW}(\vartheta) : (\phi \wedge \phi_1) \rightarrow \varphi \in \Sigma \text{ for some } \phi \in \Sigma'\}. \end{array}$$

It is easy to see that both Σ_0 and Σ_1 are Π-theories of $\mathrel{|\!\sim}_L$. These sets also satisfy that

$$\begin{array}{l} \forall \varphi_0, \varphi_1 (\varphi_0 \rightarrow_s \varphi_1 \in \Sigma \;\&\; \varphi_0 \in \Sigma_0 \;\Rightarrow\; \varphi_1 \in \Sigma_0) \\ \forall \varphi_0, \varphi_1 (\varphi_0 \rightarrow_s \varphi_1 \in \Sigma \;\&\; \varphi_0 \in \Sigma_1 \;\Rightarrow\; \varphi_1 \in \Sigma_1). \end{array}$$

Since $\phi_0, \phi_1 \notin \Sigma'$ it follows that $\Sigma' \subsetneq \Sigma_0$ and $\Sigma' \subsetneq \Sigma_1$. By maximality of Σ' in $\langle \mathcal{F}, \subseteq \rangle$, it holds that $\Delta \cap \Sigma_0 \neq \emptyset$ and $\Delta \cap \Sigma_1 \neq \emptyset$. So, there is $\varphi_0 \in \Delta$ and $\phi'_0 \in \Sigma'$ such that $(\phi'_0 \wedge \phi_0) \rightarrow \varphi_0 \in \Sigma$, and there is $\varphi_1 \in \Delta$ and $\phi'_1 \in \Sigma'$ such that $(\phi'_1 \wedge \phi_1) \rightarrow \varphi_1 \in \Sigma$. Thus, $(\phi'_0 \wedge \phi_0) \rightarrow (\varphi_0 \vee \varphi_1) \in \Sigma$ and $(\phi'_1 \wedge \phi_1) \rightarrow (\varphi_0 \vee \varphi_1) \in \Sigma$. Hence, by the axiom (6) on page 150 it follows that $(\phi'_0 \wedge \phi'_1 \wedge (\phi_0 \vee \phi_1)) \rightarrow_s (\varphi_0 \vee \varphi_1) \in \Sigma$ with $\phi'_0 \wedge \phi'_1 \wedge (\phi_0 \vee \phi_1) \in \Sigma'$. Since $\Sigma' \in \mathcal{F}$ we know that $\varphi_0 \vee \varphi_1 \in \Sigma'$. Thus, $\varphi_0 \vee \varphi_1 \notin \Delta$, which is in contradiction with the fact that Δ is closed under disjunction. □

4.2.20. Corollary. *Let* L *and* Π *be sets of* $\mathcal{L}^{SW}(\vartheta)$*-sequents that are relatively consistent, let* Σ *be a* Π*-theory of* $\mathrel{|\!\sim}_L$*, and let* Γ *and* Δ *be two sets of* $\mathcal{L}^{SW}(\vartheta)$*-formulas. If* $w \in \mathsf{WMod}$*, then the following are equivalent:*

1. *For every* $\Gamma' \subseteq_\omega \Gamma$ *and every* $\Delta' \subseteq_\omega \Delta$, *it holds that* $\bigwedge \Gamma' \prec_w \bigvee \Delta' \in \Sigma$.

2. *There is a prime* Π*-theory* Σ' *of* $\mid\!\sim_L$ *such that* $\langle \Sigma, \Sigma' \rangle \in R_w^{\mathfrak{H}_\Pi^L}$, $\Gamma \subseteq \Sigma'$ *and* $\Delta \cap \Sigma' = \emptyset$.

Proof: It is an immediate consequence of Propositions 4.2.19 and 4.2.18. □

4.2.21. Corollary. *Let* L *and* Π *be sets of* $\mathcal{L}^{SW}(\vartheta)$*-sequents that are relatively consistent, let* Σ *be a prime* Π*-theory of* $\mid\!\sim_L$, *and let* φ_0 *and* φ_1 *be two* $\mathcal{L}^{SW}(\vartheta)$*-formulas.*

1. $\bot \notin \Sigma$ *and* $\top \in \Sigma$.

2. $\varphi_0 \wedge \varphi_1 \in \Sigma$ *iff* $\varphi_0 \in \Sigma$ *and* $\varphi_1 \in \Sigma$.

3. $\varphi_0 \vee \varphi_1 \in \Sigma$ *iff* $\varphi_0 \in \Sigma$ *or* $\varphi_1 \in \Sigma$.

4. *If* $s \in \mathsf{SMod}$, *then*
$$\varphi_0 \rightarrow_s \varphi_1 \in \Sigma \quad \textit{iff} \quad \forall \Sigma' \in H_\Pi^L \ \Big(\langle \Sigma, \Sigma' \rangle \in R_s^{\mathfrak{H}_\Pi^L} \ \& \ \varphi_0 \in \Sigma' \ \Rightarrow \ \varphi_1 \in \Sigma' \Big).$$

5. *If* $w \in \mathsf{WMod}$, *then*
$$\varphi_0 \prec_w \varphi_1 \in \Sigma \quad \textit{iff} \quad \exists \Sigma' \in H_\Pi^L \ \Big(\langle \Sigma, \Sigma' \rangle \in R_w^{\mathfrak{H}_\Pi^L} \ \& \ \varphi_0 \in \Sigma' \ \& \ \varphi_1 \notin \Sigma' \Big).$$

Proof: The first three items are very simple. The other two follow by Proposition 4.2.19 and Corollary 4.2.20 when we restrict ourselves to the case that Γ and Δ are singletons. □

4.2.22. Lemma (Truth). *Let* L *and* Π *be sets of* $\mathcal{L}^{SW}(\vartheta)$*-sequents that are relatively consistent. For every* $\Sigma \in H_\Pi^L$ *and every* $\mathcal{L}^{SW}(\vartheta)$*-formula* φ,
$$\mathfrak{H}_\Pi^L, \Sigma \Vdash \varphi \quad \textit{iff} \quad \varphi \in \Sigma.$$

Proof: By induction. For propositions it follows by definition of the valuation in $\mathfrak{H}_\Pi^L$. The inductive steps are solved in Corollary 4.2.21. □

4.2.23. Theorem (Canonical Structure). *Let us take a set* L *of* $\mathcal{L}^{SW}(\vartheta)$*-sequents. For every* $\mathcal{L}^{SW}(\vartheta)$*-sequent* ς *and every set* Π *of* $\mathcal{L}^{SW}(\vartheta)$*-sequents such that* L *and* Π *are relatively consistent, it holds that*
$$\Pi \mid\!\sim_L \varsigma \quad \textit{iff} \quad \mathfrak{H}_\Pi^L \Vdash \varsigma.$$

Proof: It is a consequence of Corollary 4.2.16 together with Lemma 4.2.22. □

4.2.24. Theorem (Completeness). *Let L be a set of $\mathcal{L}^{SW}(\vartheta)$-sequents. For every set Π of $\mathcal{L}^{SW}(\vartheta)$-sequents and every $\mathcal{L}^{SW}(\vartheta)$-sequent ς, it holds that*

$$\Pi \mathrel{|\!\!\sim}_L \varsigma \quad \textit{iff} \quad \Pi \mathrel{|\!\!\approx}_{\mathsf{K}} \varsigma,$$

where K *is the class* $\{\mathfrak{A} : \mathfrak{A} \Vdash \mathrm{sub}(L)\}$.

Proof: The soundness was established in Proposition 4.2.13. For the converse use Theorem 4.2.23. This theorem implies that $\mathfrak{H}^L_\Pi \Vdash \Pi \cup \mathrm{sub}(L)$. □

4.2.25. Proposition. *Let L and Π be sets of $\mathcal{L}^{SW}(\vartheta)$-sequents that are relatively consistent. The canonical structure $\mathfrak{H}^L_\Pi$ is differentiated, tight, $\mathcal{H}$-closed, modally saturated, and modally compact.*

Proof: As a consequence of Corollary 4.2.21 and Lemma 4.2.22 it is not hard to see that $\mathfrak{H}^L_\Pi$ is differentiated, tight and $\mathcal{H}$-closed. In order to show that it is modally saturated we recall that it is enough, by Proposition 3.5.2, to see that it is SW-saturated; and this is an easy consequence of the results previously mentioned together with Lemma 4.2.19 and Corollary 4.2.20. Using Lemma 4.2.15 it is easy to see that $\models_{l\mathfrak{H}^L_\Pi}$ is finitary. However, something stronger holds: it is modally compact. Let us show that $\models^{MOD}_{l\mathfrak{H}^L_\Pi}$ is finitary. We assume that $\Phi \cup \{\varphi\}$ is a set of $\mathcal{L}^{MOD}(\tau_\vartheta)$-formulas such that for every $\Phi' \subseteq_\omega \Phi$ there is $\Sigma_{\Phi'} \in H^L_\Pi$ such that $\mathfrak{H}^L_\Pi, \Sigma_{\Phi'} \Vdash \Phi'$ and $\mathfrak{H}^L_\Pi, \Sigma_{\Phi'} \nVdash \varphi$. Hence $\Phi \cup \{\sim\varphi\} \cup \Pi \cup \mathrm{sub}(L)$ is finitely satisfiable because for every $\Phi' \subseteq_\omega \Phi$ it holds that $\mathfrak{H}^L_\Pi, \Sigma_{\Phi'} \Vdash \Phi' \cup \{\sim\varphi\} \cup \Pi \cup \mathrm{sub}(L)$. By the Compactness Theorem of first-order languages it follows that there is a pointed τ_ϑ-structure $\langle \mathfrak{A}, a\rangle$ such that $\mathfrak{A}, a \Vdash \Phi \cup \{\sim\varphi\} \cup \Pi \cup \mathrm{sub}(L)$. Let Σ be $\mathrm{Th}_{\mathcal{L}^{SW}(\vartheta)}(\mathfrak{A}, a)$. It is not hard to see that $\Sigma \in \mathfrak{H}^L_\Pi$. By the Standard Form Theorem it follows that $\mathfrak{H}^L_\Pi, \Sigma \Vdash \Phi$ and that $\mathfrak{H}^L_\Pi, \Sigma \nVdash \varphi$. □

The next proposition tells us how to axiomatize the consequence relations $\mathrel{|\!\!\approx}_{\mathsf{C}}$ for several famous classes C of frames. We use the sets of sequents considered in Table 4.1. For the cases of T^s and $\mathsf{4}^s$ this was done by Celani and Jansana in [CJ01], but all the other cases are new.

4.2.26. Proposition (Canonicity). *Let L be any set of $\mathcal{L}^{SW}(\vartheta)$-sequents considered in the first part of Table 4.1*[16]*. Then, for every set L' of $\mathcal{L}^{SW}(\vartheta)$-sequents extending L and every set Π of $\mathcal{L}^{SW}(\vartheta)$-sequents such that L' and Π are relatively consistent, it holds that the set L is valid on the canonical frame $\mathfrak{F}^{L'}_\Pi$ (i.e., it satisfies the property on the right).*

[16] This includes all the sets considered there except for the singletons of GL^{s}, GL^{w}, $\mathsf{Grz}^{\mathsf{s}}$ and $\mathsf{Grz}^{\mathsf{w}}$.

Proof: By Proposition 4.2.18 it is clear that if L satisfies a condition then also L^d satisfies it (with respect to ϑ^d). Therefore, it is enough to show only one case for each pair of two consecutive properties in Table 4.1. We recall that by Proposition 4.2.5 we know when these sets of strict-weak sequents are valid. In the proof the relations R_s and R_w refer to, respectively, $R_s^{\mathfrak{F}_\Pi^{L'}}$ and $R_w^{\mathfrak{F}_\Pi^{L'}}$.

(i) Let $p_0 \wedge (p_0 \rightarrow_s p_1) \rhd p_1 \in L'$. Then, $\varphi_0 \wedge (\varphi_0 \rightarrow_s \varphi_1) \rhd \varphi_1 \in \mathrm{sub}(L')$ for every pair φ_0, φ_1 of strict-weak formulas. Hence it is obvious that R_s is reflexive.

(ii) We do not give a proof for the cases of R_s transitive, Euclidean and symmetric because they are particular cases of what we prove in, respectively, (xiii), (xxi) and (xi).

(iii) Suppose now that $\top \rhd p_0 \vee (p_0 \rightarrow_s p_1) \in L'$ and $p_1 \rhd p_0 \rightarrow_s p_1 \in L'$. Let us show that $R_s \subseteq Id$. If not, assume that Σ and Σ' are two different prime Π-theories of $\mid\!\sim_{L'}$ such that $\langle \Sigma, \Sigma' \rangle \in R_s$. We distinguish two cases.

Case there is $\varphi \in \Sigma \setminus \Sigma'$: Since $\varphi \rhd \top \rightarrow_s \varphi \in \mathrm{sub}(L')$ it follows that $\top \rightarrow_s \varphi \in \Sigma$. This is in contradiction with the fact that $\langle \Sigma, \Sigma' \rangle \in R_s$, $\top \in \Sigma'$ and $\varphi \notin \Sigma'$.

Case there is $\varphi \in \Sigma' \setminus \Sigma$: Since $\top \rhd \varphi \vee (\varphi \rightarrow_s \bot) \in \mathrm{sub}(L')$ we deduce that $\varphi \rightarrow_s \bot \in \Sigma$. This is in contradiction with the fact that $\langle \Sigma, \Sigma' \rangle \in R_s$, $\varphi \in \Sigma'$ and $\bot \notin \Sigma'$.

(iv) Assume that $\top \rhd [s]\bot \in L'$, and let us prove that $R_s = \emptyset$. Then it is clear that $[s]\bot \in \Sigma$ for every prime Π-theory Σ of $\mid\!\sim_{L'}$. It easily follows that $R_s = \emptyset$.

(v) Let $\neg_s \top \rhd \bot \in L'$, and let us show that R_s is serial. It is obvious that for every prime Π-theory Σ of $\mid\!\sim_{L'}$ it holds that $\neg_s \top \notin \Sigma$. By Corollary 4.2.21(4) we deduce that R_s is serial.

(vi) Suppose that $\top \rhd [s]p_0 \vee \neg_s p_0 \in L'$, and let us show that R_s is functional. Assume that $\langle \Sigma, \Sigma_0 \rangle \in R_s$ and $\langle \Sigma, \Sigma_1 \rangle \in R_s$. If $\Sigma_0 \neq \Sigma_1$ then there is a strict-weak formula φ distinguishing them. Hence, $[s]\varphi \notin \Sigma$ and $\neg_s \varphi \notin \Sigma$. This is in contradiction with the fact that $[s]\varphi \vee \neg_s \varphi \in \Sigma$ (because $\top \rhd [s]\varphi \vee \neg_s \varphi \in \mathrm{sub}(L')$).

(vii) Assume now that $p_0 \wedge (p_0 \rightarrow_s p_1) \rhd p_1 \in L'$ and $\top \rhd [s]p_0 \vee \neg_s p_0 \in L'$. By (i) and (vi) it follows that R_s is reflexive and functional. Therefore, $R_s = Id$.

(viii) Let $p_1 \leftharpoondown_w p_0 \rhd \langle w \rangle (p_1 \leftharpoondown_w p_0) \in L'$, and show that R_w is dense. So, assume that $\langle \Sigma_0, \Sigma_1 \rangle \in R_w$, and let us prove that there is a prime Π-theory Σ' such that $\langle \Sigma_0, \Sigma' \rangle \in R_w$ and $\langle \Sigma', \Sigma_1 \rangle \in R_w$.

Claim: The set $\{\langle w\rangle\varphi : \varphi \in \mathcal{L}^{MOD}(\tau_\vartheta) \text{ and } \mathfrak{H}^{L'}_{\Pi}, \Sigma_1 \Vdash \varphi\}$ is finitely satisfiable in an w-successor of Σ_0.

Proof of Claim: Let $\varphi_0, \ldots, \varphi_{n-1} \in \mathrm{Th}_{\mathcal{L}^{MOD}(\tau_\vartheta)}(\mathfrak{H}^{L'}_{\Pi}, \Sigma_1)$. By Corollary 3.1.6 we know that $\varphi_0 \wedge \ldots \wedge \varphi_{n-1}$ is equivalent to a formula $(\pi_0 \smallsetminus \nu_0) \vee \ldots \vee (\pi_{k-1} \smallsetminus \nu_{k-1})$ where the π's and the ν's are in $\mathcal{L}^{SW}(\vartheta)$. Since $\mathfrak{H}^{L'}_{\Pi}, \Sigma_0 \Vdash \langle w\rangle(\varphi_0 \wedge \ldots \wedge \varphi_{n-1})$ it holds that $(\pi_0 \leftharpoondown_w \nu_0) \vee \ldots \vee (\pi_{k-1} \leftharpoondown_w \nu_{k-1}) \in \Sigma_0$. Hence, there is $i < k$ such that $\pi_i \leftharpoondown_w \nu_i \in \Sigma_0$. Using the fact that $\pi_i \leftharpoondown_w \nu_i \rhd \langle w\rangle(\pi_i \leftharpoondown_w \nu_i) \in \mathrm{sub}(L')$ it follows that $\langle w\rangle(\pi_i \leftharpoondown_w \nu_i) \in \Sigma_0$. Then, there is a prime Π-theory Σ' such that $\langle \Sigma_0, \Sigma'\rangle \in R_w$ and $\pi_i \leftharpoondown_w \nu_i \in \Sigma'$. Hence $(\pi_0 \leftharpoondown_w \nu_0) \vee \ldots \vee (\pi_{k-1} \leftharpoondown_w \nu_{k-1}) \in \Sigma'$. Therefore $\mathfrak{H}^{L'}_{\Pi}, \Sigma' \Vdash \langle w\rangle(\varphi_0 \wedge \ldots \wedge \varphi_{n-1})$. In particular this implies that $\mathfrak{H}^{L'}_{\Pi}, \Sigma' \Vdash \langle w\rangle\varphi_0 \wedge \ldots \wedge \langle w\rangle\varphi_{n-1}$. $\dashv$

Using the fact that $\mathfrak{H}^{L'}_{\Pi}$ is modally saturated (see Proposition 4.2.25) together with the previous claim we know that there is a prime Π-theory Σ' such that $\langle \Sigma_0, \Sigma'\rangle \in R_w$ and for every $\varphi \in \mathcal{L}^{MOD}(\tau_\vartheta)$, if $\mathfrak{H}^{L'}_{\Pi}, \Sigma_1 \Vdash \varphi$ then $\mathfrak{H}^{L'}_{\Pi}, \Sigma' \Vdash \langle w\rangle\varphi$. Now it is easy to see that $\langle \Sigma', \Sigma_1\rangle \in R_w$.

(ix) Assume now that $p_0 \to_{s'} p_1 \rhd p_0 \to_s p_1 \in L'$, and let us show that $R_s \subseteq R_{s'}$. Suppose that $\langle \Sigma, \Sigma'\rangle \in R_s$, $\varphi_0 \to_{s'} \varphi_1 \in \Sigma$ and $\varphi_0 \in \Sigma'$, and let us see that $\varphi_1 \in \Sigma'$. Since $\varphi_0 \to_{s'} \varphi_1 \rhd \varphi_0 \to_s \varphi_1 \in \mathrm{sub}(L')$ it follows that $\varphi_0 \to_s \varphi_1 \in \Sigma$. Using the fact that $\langle \Sigma, \Sigma'\rangle \in R_s$ we know that $\varphi_1 \in \Sigma'$.

(x) Suppose that $\top \rhd (p_0 \to_s p_1) \vee (p_0 \leftharpoondown_w p_1) \in L'$, and let us show that $R_s \subseteq R_w$. Assume that $\langle \Sigma, \Sigma'\rangle \in R_s$, $\varphi_0 \in \Sigma'$ and $\varphi_1 \notin \Sigma'$. We must prove that $\varphi_0 \leftharpoondown_w \varphi_1 \in \Sigma$. By the assumptions $\varphi_0 \to_s \varphi_1 \notin \Sigma$. As a consequence of $(\varphi_0 \to_s \varphi_1) \vee (\varphi_0 \leftharpoondown_w \varphi_1) \in \Sigma$ we conclude $\varphi_0 \leftharpoondown_w \varphi_1 \in \Sigma$.

(xi) Let $p_0 \rhd p_1 \vee \neg_s(p_0 \to_{s'} p_1) \in L'$, and let us prove that $R_s \subseteq R^{-1}_{s'}$. We assume that $\langle \Sigma, \Sigma'\rangle \in R_s$, $\varphi_0 \to_{s'} \varphi_1 \in \Sigma'$ and $\varphi_0 \in \Sigma$. We must show that $\varphi_1 \in \Sigma$. By the assumptions $\neg_s(\varphi_0 \to_{s'} \varphi_1) \notin \Sigma$. As a consequence of $\varphi_0 \rhd \varphi_1 \vee \neg_s(\varphi_0 \to_{s'} \varphi_1) \in \mathrm{sub}(L')$ it follows that $\varphi_1 \in \Sigma$.

(xii) Suppose that $p_0 \rhd p_1 \vee [s](p_0 \leftharpoondown_w p_1) \in L'$, and let us prove that $R_s \subseteq R^{-1}_w$. Assume that $\langle \Sigma, \Sigma'\rangle \in R_s$, $\varphi_0 \in \Sigma$ and $\varphi_1 \notin \Sigma$. We must show that $\varphi_0 \leftharpoondown_w \varphi_1 \in \Sigma'$. Since $\varphi_0 \rhd \varphi_1 \vee [s](\varphi_0 \leftharpoondown_w \varphi_1) \in \mathrm{sub}(L')$ it follows that $[s](\varphi_0 \leftharpoondown_w \varphi_1) \in \Sigma$. Using the fact that $\langle \Sigma, \Sigma'\rangle \in R_s$ we conclude that $\varphi_0 \leftharpoondown_w \varphi_1 \in \Sigma'$.

(xiii) Let $p_0 \to_s p_1 \rhd [s](p_0 \to_{s'} p_1) \in L'$, and let us prove that $R_{s'} \circ R_s \subseteq R_s$. Suppose that $\langle \Sigma_0, \Sigma_1\rangle \in R_s$, $\langle \Sigma_1, \Sigma_2\rangle \in R_{s'}$, $\varphi_0 \to_s \varphi_1 \in \Sigma_0$ and $\varphi_0 \in \Sigma_2$. We want to show that $\varphi_1 \in \Sigma_2$. As a consequence of $\varphi_0 \to_s \varphi_1 \rhd [s](\varphi_0 \to_{s'} \varphi_1) \in \mathrm{sub}(L')$ it follows that $[s](\varphi_0 \to_{s'} \varphi_1) \in \Sigma_0$. Using the fact that $\langle \Sigma_0, \Sigma_1\rangle \in R_s$ it is deduced that $\varphi_0 \to_{s'} \varphi_1 \in \Sigma_1$. Finally, as $\langle \Sigma_1, \Sigma_2\rangle \in R_{s'}$ we conclude that $\varphi_1 \in \Sigma_2$.

(xiv) Let $p_0 \rightarrow_s p_1 \rhd \neg_s(p_0 \leftharpoondown_w p_1) \in L'$, and let us show that $R_w \circ R_s \subseteq R_s$. Suppose that $\langle \Sigma_0, \Sigma_1 \rangle \in R_s$, $\langle \Sigma_1, \Sigma_2 \rangle \in R_w$, $\varphi_0 \rightarrow_s \varphi_1 \in \Sigma_0$ and $\varphi_0 \in \Sigma_2$. We want to show that $\varphi_1 \in \Sigma_2$. Since $\varphi_0 \rightarrow_s \varphi_1 \rhd \neg_s(\varphi_0 \leftharpoondown_w \varphi_1) \in \mathrm{sub}(L')$ it follows that $\neg_s(\varphi_0 \leftharpoondown_w \varphi_1) \in \Sigma_0$. Hence $\varphi_0 \leftharpoondown_w \varphi_1 \notin \Sigma_1$. Therefore $\varphi_1 \in \Sigma_2$.

(xv) Suppose that $p_0 \rightarrow_{s'} p_1 \rhd [s](p_0 \rightarrow_{s'} p_1) \in L'$, and let us prove that $R_{s'} \circ R_s \subseteq R_{s'}$. Assume that $\langle \Sigma_0, \Sigma_1 \rangle \in R_s$, $\langle \Sigma_1, \Sigma_2 \rangle \in R_{s'}$, $\varphi_0 \rightarrow_{s'} \varphi_1 \in \Sigma_0$ and $\varphi_0 \in \Sigma_2$. We want to show that $\varphi_1 \in \Sigma_2$. As a consequence of $\varphi_0 \rightarrow_{s'} \varphi_1 \rhd [s](\varphi_0 \rightarrow_{s'} \varphi_1) \in \mathrm{sub}(L')$ it follows that $[s](\varphi_0 \rightarrow_{s'} \varphi_1) \in \Sigma_0$. Using the fact that $\langle \Sigma_0, \Sigma_1 \rangle \in R_s$ it is deduced that $\varphi_0 \rightarrow_{s'} \varphi_1 \in \Sigma_1$. Finally, as $\langle \Sigma_1, \Sigma_2 \rangle \in R_{s'}$ we conclude that $\varphi_1 \in \Sigma_2$.

(xvi) Assume now that $\top \rhd (p_0 \leftharpoondown_w p_1) \vee \neg_s(p_0 \leftharpoondown_w p_1) \in L'$, and let us show that $R_w \circ R_s \subseteq R_w$. So, suppose that $\langle \Sigma_0, \Sigma_1 \rangle \in R_s$, $\langle \Sigma_1, \Sigma_2 \rangle \in R_w$, $\varphi_0 \in \Sigma_2$ and $\varphi_1 \notin \Sigma_2$, and let us prove that $\varphi_0 \leftharpoondown_w \varphi_1 \in \Sigma_0$. By the assumptions it follows that $\varphi_0 \leftharpoondown_w \varphi_1 \in \Sigma_1$. Hence $\neg_s(\varphi_0 \leftharpoondown_w \varphi_1) \notin \Sigma_0$. Since $\top \rhd (\varphi_0 \leftharpoondown_w \varphi_1) \vee \neg_s(\varphi_0 \leftharpoondown_w \varphi_1) \in \mathrm{sub}(L')$ we deduce that $\varphi_0 \leftharpoondown_w \varphi_1 \in \Sigma_0$.

(xvii) Let $\langle w \rangle(p_1 \leftharpoondown_{w'} p_0) \rhd \langle w' \rangle(p_1 \leftharpoondown_w p_0) \in L'$, and let us prove that $R_{w'} \circ R_w \subseteq R_w \circ R_{w'}$. Assume that $\langle \Sigma_0, \Sigma_1 \rangle \in R_w$ and $\langle \Sigma_1, \Sigma_2 \rangle \in R_{w'}$. We must show that there is a prime Π-theory Σ' such that $\langle \Sigma_0, \Sigma' \rangle \in R_{w'}$ and $\langle \Sigma', \Sigma_2 \rangle \in R_w$.

Claim: The set $\{\langle w \rangle \varphi : \varphi \in \mathcal{L}^{MOD}(\tau_\vartheta) \text{ and } \mathfrak{H}^{L'}_{\Pi}, \Sigma_2 \Vdash \varphi\}$ is finitely satisfiable in an w'-successor of Σ_0.

Proof of Claim: Let $\varphi_0, \ldots, \varphi_{n-1} \in \mathrm{Th}_{\mathcal{L}^{MOD}(\tau_\vartheta)}(\mathfrak{H}^{L'}_{\Pi}, \Sigma_2)$. By Corollary 3.1.6 we know that $\varphi_0 \wedge \ldots \wedge \varphi_{n-1}$ is equivalent to a formula $(\pi_0 \setminus \nu_0) \vee \ldots \vee (\pi_{k-1} \setminus \nu_{k-1})$ where the π's and the ν's are in $\mathcal{L}^{SW}(\vartheta)$. Since $\mathfrak{H}^{L'}_{\Pi}, \Sigma_1 \Vdash \langle w' \rangle(\varphi_0 \wedge \ldots \wedge \varphi_{n-1})$ it holds that $(\pi_0 \leftharpoondown_{w'} \nu_0) \vee \ldots \vee (\pi_{k-1} \leftharpoondown_{w'} \nu_{k-1}) \in \Sigma_1$. Hence, there is $i < k$ such that $\pi_i \leftharpoondown_{w'} \nu_i \in \Sigma_1$. Thus $\langle w \rangle(\pi_i \leftharpoondown_{w'} \nu_i) \in \Sigma_0$. Using the fact that $\langle w \rangle(\pi_i \leftharpoondown_{w'} \nu_i) \rhd \langle w' \rangle(\pi_i \leftharpoondown_w \nu_i) \in \mathrm{sub}(L')$ it follows that $\langle w' \rangle(\pi_i \leftharpoondown_w \nu_i) \in \Sigma_0$. Then, there is a prime Π-theory Σ' such that $\langle \Sigma_0, \Sigma' \rangle \in R_{w'}$ and $\pi_i \leftharpoondown_w \nu_i \in \Sigma'$. Hence $(\pi_0 \leftharpoondown_w \nu_0) \vee \ldots \vee (\pi_{k-1} \leftharpoondown_w \nu_{k-1}) \in \Sigma'$. Therefore $\mathfrak{H}^{L'}_{\Pi}, \Sigma' \Vdash \langle w \rangle(\varphi_0 \wedge \ldots \wedge \varphi_{n-1})$. In particular it implies that $\mathfrak{H}^{L'}_{\Pi}, \Sigma' \Vdash \langle w \rangle \varphi_0 \wedge \ldots \wedge \langle w \rangle \varphi_{n-1}$. ⊣

Using the fact that $\mathfrak{H}^{L'}_{\Pi}$ is modally saturated (see Proposition 4.2.25) together with the claim we know that there is a prime Π-theory Σ' such that $\langle \Sigma_0, \Sigma' \rangle \in R_{w'}$ and for every $\varphi \in \mathcal{L}^{MOD}(\tau_\vartheta)$, if $\mathfrak{H}^{L'}_{\Pi}, \Sigma_2 \Vdash \varphi$ then $\mathfrak{H}^{L'}_{\Pi}, \Sigma' \Vdash \langle w \rangle \varphi$. It is not hard to see that $\langle \Sigma', \Sigma_2 \rangle \in R_w$.

(xviii) Let $\neg_s(p_1 \leftharpoondown_w p_0) \wedge \ulcorner_w(p_1 \rightarrow_s p_0) \rhd \bot \in L'$, and let us show that $R_s \circ R_w \subseteq R_w \circ R_s$. Suppose that $\langle \Sigma_0, \Sigma_1 \rangle \in R_w$ and $\langle \Sigma_1, \Sigma_2 \rangle \in R_s$. We want to prove that there is a prime Π-theory Σ' such that $\langle \Sigma_0, \Sigma' \rangle \in R_s$ and $\langle \Sigma', \Sigma_2 \rangle \in R_w$.

Claim: The set $\{\langle w\rangle\varphi : \varphi \in \mathcal{L}^{MOD}(\tau_\vartheta)$ and $\mathfrak{H}_\Pi^{L'}, \Sigma_2 \Vdash \varphi\}$ is finitely satisfiable in an s-successor of Σ_0.

Proof of Claim: Let $\varphi_0, \ldots, \varphi_{n-1} \in \mathrm{Th}_{\mathcal{L}^{MOD}(\tau_\vartheta)}(\mathfrak{H}_\Pi^{L'}, \Sigma_2)$. By Corollary 3.1.6 $\sim(\varphi_0 \wedge \ldots \wedge \varphi_{n-1})$ is equivalent to a formula $(\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_{k-1} \supset \pi_{k-1})$ where the π's and the ν's are in $\mathcal{L}^{SW}(\vartheta)$. Since $\mathfrak{H}_\Pi^{L'}, \Sigma_1 \not\Vdash [s] \sim(\varphi_0 \wedge \ldots \wedge \varphi_{n-1})$ it holds that $(\nu_0 \rightarrow_s \pi_0) \wedge \ldots \wedge (\nu_{k-1} \rightarrow_s \pi_{k-1}) \notin \Sigma_1$. Hence, there is $i < k$ such that $\nu_i \rightarrow_s \pi_i \notin \Sigma_1$. Thus $\neg_w(\nu_i \rightarrow_s \pi_i) \in \Sigma_0$. Using that $\neg_s(\nu_i \leftharpoondown_w \pi_i) \wedge \neg_w(\nu_i \rightarrow_s \pi_i) \rhd \bot \in \mathrm{sub}(L')$ it follows that $\neg_s(\nu_i \leftharpoondown_w \pi_i) \notin \Sigma_0$. Then, there is a prime Π-theory Σ' such that $\langle \Sigma_0, \Sigma' \rangle \in R_s$ and $\nu_i \leftharpoondown_w \pi_i \in \Sigma'$. Hence $(\nu_0 \leftharpoondown_w \pi_0) \vee \ldots \vee (\nu_{k-1} \leftharpoondown_w \pi_{k-1}) \in \Sigma'$. Therefore $\mathfrak{H}_\Pi^{L'}, \Sigma' \Vdash \langle w\rangle(\varphi_0 \wedge \ldots \wedge \varphi_{n-1})$. In particular this says that $\mathfrak{H}_\Pi^{L'}, \Sigma' \Vdash \langle w\rangle\varphi_0 \wedge \ldots \wedge \langle w\rangle\varphi_{n-1}$. ⊣

Using the fact that $\mathfrak{H}_\Pi^{L'}$ is modally saturated (see Proposition 4.2.25) together with the claim we deduce that there is a prime Π-theory Σ' such that $\langle \Sigma_0, \Sigma' \rangle \in R_s$ and for every $\varphi \in \mathcal{L}^{MOD}(\tau_\vartheta)$, if $\mathfrak{H}_\Pi^{L'}, \Sigma_2 \Vdash \varphi$ then $\mathfrak{H}_\Pi^{L'}, \Sigma' \Vdash \langle w\rangle\varphi$. Now it is not hard to see that $\langle \Sigma', \Sigma_2 \rangle \in R_w$.

(xix) Suppose that $p_1 \rhd \langle w\rangle(p_1 \leftharpoondown_{w'} p_0) \vee p_0 \in L'$, and let us prove that $Id \subseteq R_{w'} \circ R_w$. So, assume that Σ is a prime Π-theory of $\mid\!\sim_{L'}$, and let us see that there is a prime Π-theory Σ' such that $\langle \Sigma, \Sigma' \rangle \in R_w$ and $\langle \Sigma', \Sigma \rangle \in R_{w'}$.

Claim: The set $\{\langle w'\rangle\varphi : \varphi \in \mathcal{L}^{MOD}(\tau_\vartheta)$ and $\mathfrak{H}_\Pi^{L'}, \Sigma \Vdash \varphi\}$ is finitely satisfiable in an w-successor of Σ.

Proof of Claim: Let $\varphi_0, \ldots, \varphi_{n-1} \in \mathrm{Th}_{\mathcal{L}^{MOD}(\tau_\vartheta)}(\mathfrak{H}_\Pi^{L'}, \Sigma)$. By Corollary 3.1.6 we know that $\varphi_0 \wedge \ldots \wedge \varphi_{n-1}$ is equivalent to a formula $(\pi_0 \searrow \nu_0) \vee \ldots \vee (\pi_{k-1} \searrow \nu_{k-1})$ where the π's and the ν's are in $\mathcal{L}^{SW}(\vartheta)$. Since $\mathfrak{H}_\Pi^{L'}, \Sigma \Vdash \varphi_0 \wedge \ldots \wedge \varphi_{n-1}$ it holds that there is $i < k$ such that $\mathfrak{H}_\Pi^{L'}, \Sigma \Vdash \pi_i \searrow \nu_i$. Using the fact that $\pi_i \rhd \langle w\rangle(\pi_i \leftharpoondown_{w'} \nu_i) \vee \nu_i \in \mathrm{sub}(L')$ it follows that $\langle w\rangle(\pi_i \leftharpoondown_{w'} \nu_i) \in \Sigma$. Then, there is a prime Π-theory Σ' such that $\langle \Sigma, \Sigma' \rangle \in R_w$ and $\pi_i \leftharpoondown_{w'} \nu_i \in \Sigma'$. Hence $(\pi_0 \leftharpoondown_{w'} \nu_0) \vee \ldots \vee (\pi_{k-1} \leftharpoondown_{w'} \nu_{k-1}) \in \Sigma'$. Therefore $\mathfrak{H}_\Pi^{L'}, \Sigma' \Vdash \langle w'\rangle(\varphi_0 \wedge \ldots \wedge \varphi_{n-1})$. In particular this says that $\mathfrak{H}_\Pi^{L'}, \Sigma' \Vdash \langle w'\rangle\varphi_0 \wedge \ldots \wedge \langle w'\rangle\varphi_{n-1}$. ⊣

As $\mathfrak{H}_\Pi^{L'}$ is modally saturated (see Proposition 4.2.25) we know that there is a prime Π-theory Σ' such that $\langle \Sigma, \Sigma' \rangle \in R_w$ and for every $\varphi \in \mathcal{L}^{MOD}(\tau_\vartheta)$, if $\mathfrak{H}_\Pi^{L'}, \Sigma \Vdash \varphi$ then $\mathfrak{H}_\Pi^{L'}, \Sigma' \Vdash \langle w'\rangle\varphi$. Now it is easy to see that $\langle \Sigma', \Sigma \rangle \in R_{w'}$.

(xx) Assume now that $p_0 \wedge \neg_s(p_0 \leftharpoondown_w p_1) \rhd p_1 \in L'$, and let us show that $Id \subseteq R_w \circ R_s$. Assume that Σ is a prime Π-theory of $\mid\!\sim_{L'}$, and let us prove that there is a prime Π-theory Σ' such that $\langle \Sigma, \Sigma' \rangle \in R_s$ and $\langle \Sigma', \Sigma \rangle \in R_w$.

Claim: The set $\{\langle w\rangle\varphi : \varphi \in \mathcal{L}^{MOD}(\tau_\vartheta)$ and $\mathfrak{H}_\Pi^{L'}, \Sigma \Vdash \varphi\}$ is finitely satisfiable in an s-successor of Σ.

Proof of Claim: Let $\varphi_0, \ldots, \varphi_{n-1} \in \mathrm{Th}_{\mathcal{L}^{MOD}(\tau_\vartheta)}(\mathfrak{H}^{L'}_\Pi, \Sigma)$. By Corollary 3.1.6 $\sim(\varphi_0 \wedge \ldots \wedge \varphi_{n-1})$ is equivalent to a formula $(\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_{k-1} \supset \pi_{k-1})$ where the π's and the ν's are in $\mathcal{L}^{SW}(\vartheta)$. Since $\mathfrak{H}^{L'}_\Pi, \Sigma \not\Vdash \sim(\varphi_0 \wedge \ldots \wedge \varphi_{n-1})$ it holds that there is $i < k$ such that $\mathfrak{H}^{L'}_\Pi, \Sigma \not\Vdash \nu_i \supset \pi_i$. Using the fact that $\nu_i \wedge \neg_s(\nu_i \leftharpoondown_w \pi_i) \rhd \pi_i \in \mathrm{sub}(L')$ it follows that $\neg_s(\nu_i \leftharpoondown_w \pi_i) \notin \Sigma$. Hence there is a prime Π-theory Σ' such that $\langle \Sigma, \Sigma' \rangle \in R_s$ and $\nu_i \leftharpoondown_w \pi_i \in \Sigma'$. Thus $(\nu_0 \leftharpoondown_w \pi_0) \vee \ldots \vee (\nu_{k-1} \leftharpoondown_w \pi_{k-1}) \in \Sigma'$. Therefore $\mathfrak{H}^{L'}_\Pi, \Sigma' \Vdash \langle w \rangle(\varphi_0 \wedge \ldots \wedge \varphi_{n-1})$. In particular $\mathfrak{H}^{L'}_\Pi, \Sigma' \Vdash \langle w \rangle\varphi_0 \wedge \ldots \wedge \langle w \rangle\varphi_{n-1}$. $\dashv$

From the fact that $\mathfrak{H}^{L'}_\Pi$ is modally saturated (see Proposition 4.2.25) together with the previous claim we deduce that there is a prime Π-theory Σ' such that $\langle \Sigma, \Sigma' \rangle \in R_s$ and for every $\varphi \in \mathcal{L}^{MOD}(\tau_\vartheta)$, if $\mathfrak{H}^{L'}_\Pi, \Sigma \Vdash \varphi$ then $\mathfrak{H}^{L'}_\Pi, \Sigma' \Vdash \langle w \rangle\varphi$. It holds that $\langle \Sigma', \Sigma \rangle \in R_w$.

(xxi) Let $\top \rhd (p_0 \rightarrow_{s'} p_1) \vee \neg_s(p_0 \rightarrow_{s'} p_1) \in L'$, and let us show that $R_{s'} \circ R_s^{-1} \subseteq R_{s'}$. Suppose that $\langle \Sigma_0, \Sigma_1 \rangle \in R_s$, $\langle \Sigma_0, \Sigma_2 \rangle \in R_{s'}$, $\varphi_0 \rightarrow_{s'} \varphi_1 \in \Sigma_1$ and $\varphi_0 \in \Sigma_2$. We must prove that $\varphi_1 \in \Sigma_2$. By the assumptions it follows that $\neg_s(\varphi_0 \rightarrow_{s'} \varphi_1) \notin \Sigma_0$. Using the fact that $\top \rhd (\varphi_0 \rightarrow_{s'} \varphi_1) \vee \neg_s(\varphi_0 \rightarrow_{s'} \varphi_1) \in \mathrm{sub}(L')$ it is deduced that $\varphi_0 \rightarrow_{s'} \varphi_1 \in \Sigma_0$. Finally, as $\langle \Sigma_0, \Sigma_2 \rangle \in R_{s'}$ we conclude that $\varphi_1 \in \Sigma_2$.

(xxii) Suppose that $p_0 \leftharpoondown_w p_1 \rhd [s](p_0 \leftharpoondown_w p_1) \in L'$, and let us prove that $R_w \circ R_s^{-1} \subseteq R_w$. So, assume that $\langle \Sigma_0, \Sigma_1 \rangle \in R_s$, $\langle \Sigma_0, \Sigma_2 \rangle \in R_w$, $\varphi_0 \in \Sigma_2$ and $\varphi_1 \notin \Sigma_2$, and let us see that $\varphi_0 \leftharpoondown_w \varphi_1 \in \Sigma_1$. By the assumptions it follows that $\varphi_0 \leftharpoondown_w \varphi_1 \in \Sigma_0$. Using the fact that $\top \rhd \varphi_0 \leftharpoondown_w \varphi_1 \rhd [s](\varphi_0 \leftharpoondown_w \varphi_1) \in \mathrm{sub}(L')$ we deduce that $[s](\varphi_0 \leftharpoondown_w \varphi_1) \in \Sigma_0$. Therefore, as $\langle \Sigma_0, \Sigma_1 \rangle \in R_s$ we conclude that $\varphi_0 \leftharpoondown_w \varphi_1 \in \Sigma_1$.

(xxiii) Assume now that $\mathsf{con}^{\mathsf{s}} \in L'$, and let us show that R_s is connected. Thus, suppose that $\langle \Sigma_0, \Sigma_1 \rangle \in R_s$, $\langle \Sigma_0, \Sigma_2 \rangle \in R_s$, $\langle \Sigma_1, \Sigma_2 \rangle \notin R_s$, $\varphi_0 \rightarrow_s \varphi_1 \in \Sigma_2$ and $\varphi_0 \in \Sigma_1$. We want to show that $\varphi_1 \in \Sigma_1$. By $\langle \Sigma_1, \Sigma_2 \rangle \notin R_s$ it follows that there are strict-weak formulas φ_2, φ_3 such that $\varphi_2 \rightarrow_s \varphi_3 \in \Sigma_1$, $\varphi_2 \in \Sigma_2$ and $\varphi_3 \notin \Sigma_2$. Thus, $(\varphi_2 \rightarrow_s \varphi_3) \wedge \varphi_0 \in \Sigma_1$ and $(\varphi_0 \rightarrow_s \varphi_1) \wedge \varphi_2 \in \Sigma_2$. From the facts $(\varphi_0 \rightarrow_s \varphi_1) \wedge \varphi_2 \in \Sigma_2$, $\varphi_3 \notin \Sigma_2$ and $\langle \Sigma_0, \Sigma_2 \rangle \in R_s$ we conclude that $((\varphi_0 \rightarrow_s \varphi_1) \wedge \varphi_2) \rightarrow_s \varphi_3 \notin \Sigma_0$. Using the fact that

$$\top \rhd \Big(\big((\varphi_0 \rightarrow_s \varphi_1) \wedge \varphi_2\big) \rightarrow_s \varphi_3\Big) \vee \Big(\big((\varphi_2 \rightarrow_s \varphi_3) \wedge \varphi_0\big) \rightarrow_s \varphi_1\Big)$$

is a substitution instance of $\mathsf{con}^{\mathsf{s}}$ we deduce that $((\varphi_2 \rightarrow_s \varphi_3) \wedge \varphi_0) \rightarrow_s \varphi_1 \in \Sigma_0$. Finally, using the fact that $(\varphi_2 \rightarrow_s \varphi_3) \wedge \varphi_0 \in \Sigma_1$ and $\langle \Sigma_0, \Sigma_1 \rangle \in R_s$ we obtain $\varphi_1 \in \Sigma_1$.

(xxiv) Suppose that $.3^{\mathsf{s}} \in L'$, and let us show that R_s is weakly connected. Hence, assume that $\langle \Sigma_0, \Sigma_1 \rangle \in R_s$, $\langle \Sigma_0, \Sigma_2 \rangle \in R_s$, $\langle \Sigma_1, \Sigma_2 \rangle \notin R_s$ and $\langle \Sigma_2, \Sigma_1 \rangle \notin R_s$, and let us prove that $\Sigma_1 = \Sigma_2$. By $\langle \Sigma_1, \Sigma_2 \rangle \notin R_s$ and $\langle \Sigma_2, \Sigma_1 \rangle \notin R_s$ it follows that there are strict-weak formulas $\varphi_0, \varphi_1, \varphi_2, \varphi_3$ such that $\varphi_0 \to_s \varphi_1 \in \Sigma_1$, $\varphi_0 \in \Sigma_2$, $\varphi_1 \notin \Sigma_2$, $\varphi_2 \to_s \varphi_3 \in \Sigma_2$, $\varphi_2 \in \Sigma_1$, and $\varphi_3 \notin \Sigma_1$. In order to see that $\Sigma_1 = \Sigma_2$ it is enough, by symmetry of the situation, to show that $\Sigma_1 \subseteq \Sigma_2$. If this is not the case, then there is a strict-weak formula φ_4 such that $\varphi_4 \in \Sigma_1$ and $\varphi_4 \notin \Sigma_2$. Let e be the substitution such that $e(p_0) = \varphi_0$, $e(p_1) = \varphi_1 \vee \varphi_4$, $e(p_2) = \varphi_2 \wedge \varphi_4$, and $e(p_3) = \varphi_3$. Now using the fact that $\langle \Sigma_0, \Sigma_1 \rangle \in R_s$ it is easy to see that

$$e\big(((p_0 \to_s p_1) \wedge p_1 \wedge p_2) \to_s p_3\big) \notin \Sigma_0.$$

And using the fact that $\langle \Sigma_0, \Sigma_2 \rangle \in R_s$ it is not hard to show that

$$e\big(((p_2 \to_s p_3) \wedge p_0) \to_s (p_2 \vee p_1)\big) \notin \Sigma_0.$$

This is in contradiction with the fact that $e(.3^{\mathsf{s}}) \in \mathrm{sub}(L')$.

(xxv) Let $\top \rhd \Big(\big((p_0 \to_s p_1) \wedge p_2 \wedge p_4\big) \to_s p_3\Big) \vee \Big(\big((p_2 \to_s p_3) \wedge p_0\big) \to_s (p_1 \vee p_4)\Big)$ be in L', and let us show that R_s is weakly connected. So, assume that $\langle \Sigma_0, \Sigma_1 \rangle \in R_s$, $\langle \Sigma_0, \Sigma_2 \rangle \in R_s$, $\langle \Sigma_1, \Sigma_2 \rangle \notin R_s$ and $\langle \Sigma_2, \Sigma_1 \rangle \notin R_s$, and let us prove that $\Sigma_1 = \Sigma_2$. By $\langle \Sigma_1, \Sigma_2 \rangle \notin R_s$ and $\langle \Sigma_2, \Sigma_1 \rangle \notin R_s$ it follows that there are strict-weak formulas $\varphi_0, \varphi_1, \varphi_2, \varphi_3$ such that $\varphi_0 \to_s \varphi_1 \in \Sigma_1$, $\varphi_0 \in \Sigma_2$, $\varphi_1 \notin \Sigma_2$, $\varphi_2 \to_s \varphi_3 \in \Sigma_2$, $\varphi_2 \in \Sigma_1$, and $\varphi_3 \notin \Sigma_1$. Now we show the two inclusions that we are interested in.[17]

- $\Sigma_1 \subseteq \Sigma_2$: Assume that $\varphi_4 \in \Sigma_1$. Then, $(\varphi_0 \to_s \varphi_1) \wedge \varphi_2 \wedge \varphi_4 \in \Sigma_1$. Therefore, $((\varphi_0 \to_s \varphi_1) \wedge \varphi_2 \wedge \varphi_4) \to_s \varphi_3 \notin \Sigma_0$. Using the fact that

  $$\top \rhd \Big(\big((\varphi_0 \to_s \varphi_1) \wedge \varphi_2 \wedge \varphi_4\big) \to_s \varphi_3\Big) \vee \Big(\big((\varphi_2 \to_s \varphi_3) \wedge \varphi_0\big) \to_s (\varphi_1 \vee \varphi_4)\Big)$$

  is a substitution of a sequent in L' we deduce that $((\varphi_2 \to_s \varphi_3) \wedge \varphi_0) \to_s (\varphi_1 \vee \varphi_4) \in \Sigma_0$. Then, as $(\varphi_2 \to_s \varphi_3) \wedge \varphi_0 \in \Sigma_2$ and $\langle \Sigma_0, \Sigma_2 \rangle \in R_s$ it follows that $\varphi_1 \vee \varphi_4 \in \Sigma_2$. Therefore, since $\varphi_1 \notin \Sigma_2$ we conclude that $\varphi_4 \in \Sigma_2$.
- $\Sigma_2 \subseteq \Sigma_1$: Assume that $\varphi_4 \in \Sigma_2$. Then, $(\varphi_2 \to_s \varphi_3) \wedge \varphi_0 \wedge \varphi_4 \in \Sigma_2$. Therefore, $((\varphi_2 \to_s \varphi_3) \wedge \varphi_0 \wedge \varphi_4) \to_s \varphi_1 \notin \Sigma_0$. Using the fact that

  $$\top \rhd \Big(\big((\varphi_2 \to_s \varphi_3) \wedge \varphi_0 \wedge \varphi_4\big) \to_s \varphi_1\Big) \vee \Big(\big((\varphi_0 \to_s \varphi_1) \wedge \varphi_2\big) \to_s (\varphi_3 \vee \varphi_4)\Big)$$

  is a substitution of a sequent in L' we deduce that $((\varphi_0 \to_s \varphi_1) \wedge \varphi_2) \to_s (\varphi_3 \vee \varphi_4) \in \Sigma_0$. Then, as $(\varphi_0 \to_s \varphi_1) \wedge \varphi_2 \in \Sigma_1$ and $\langle \Sigma_0, \Sigma_1 \rangle \in R_s$ it follows that $\varphi_3 \vee \varphi_4 \in \Sigma_1$. Therefore, since $\varphi_3 \notin \Sigma_1$ we conclude that $\varphi_4 \in \Sigma_1$.

[17] Indeed, for symmetrical reasons it should be enough to show one inclusion.

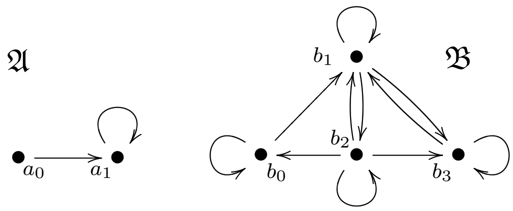


Figure 4.5: Some interesting structures

□

A set L of strict-weak sequents that satisfies the condition on the previous proposition will be called a *canonical generator*, i.e., a set L satisfying the requirement that for every set L' of $\mathcal{L}^{SW}(\vartheta)$-sequents extending L and every set Π of $\mathcal{L}^{SW}(\vartheta)$-sequents such that L' and Π are relatively consistent, it holds that the set L is valid on the canonical frame $\mathfrak{F}^{L'}_{\Pi}$. As was observed at the beginning of the proof, if L is a canonical generator then L^d is one also. As a consequence of Theorem 4.2.24 we know how to axiomatize the consequence $\approx_{\mathsf{K}}$ for all classes K that are sets of properties in Table 4.1. For instance,

- if there is only a single strict modality s, then (i) the consequence associated with reflexive frames corresponds to $\mid\!\sim_L$ where $L = \{\mathsf{T}^s\}$, (ii) the consequence associated with transitive frames corresponds to $\mid\!\sim_L$ where $L = \{\mathsf{4^s}\}$, (iii) the consequence associated with quasi order frames corresponds to $\mid\!\sim_L$ where $L = \{\mathsf{T}^s, \mathsf{4^s}\}$, etc.
- if there is only a single strict modality w, then (i) the consequence associated with reflexive frames corresponds to $\mid\!\sim_L$ where $L = \{\mathsf{T}^w\}$, (ii) the consequence associated with transitive frames corresponds to $\mid\!\sim_L$ where $L = \{\mathsf{4^w}\}$, (iii) the consequence associated with quasi order frames corresponds to $\mid\!\sim_L$ where $L = \{\mathsf{T}^w, \mathsf{4^w}\}$, etc.

We noted above that the sequents used in Table 4.2 are in general more complex than the ones obtained in Proposition 4.2.3. With the help of some examples we will show that this increasing complexity is in general necessary.

4.2.27. Example. In the modal case it is well known that the modal formula $\Box p_0 \supset p_0$ axiomatizes the logic of reflexive frames. In our previous axiomatization we used the more complex sequent $p_0, p_0 \to p_1 \rhd p_1$. Here we show that the sequent $\Box p_0 \rhd p_0$ does not axiomatize the strict-weak consequence between sequents of the class of reflexive frames. Let L be the singleton of this sequent. It is enough to show that $\emptyset \not\mid\!\sim_L p_0, p_0 \to \Box p_0 \rhd \Box p_0$. By Theorem 4.2.24 all we need to show is the existence of a structure $\mathfrak{A}$ such that $\mathfrak{A} \Vdash \mathrm{sub}(L)$ while $\mathfrak{A} \nVdash p_0, p_0 \to p_1 \rhd p_1$. Let us take the structure $\mathfrak{A}$ depicted in Figure 4.5, where the valuation is the map

$\to$	$\emptyset$	$\{a_0\}$	A
$\emptyset$	A	A	A
$\{a_0\}$	A	A	A
A	$\emptyset$	$\emptyset$	A

$\to$	$\emptyset$	$\{b_2\}$	$\{b_0, b_3\}$	$\{b_0, b_2, b_3\}$	B
$\emptyset$	B	B	B	B	B
$\{b_2\}$	$\{b_0, b_3\}$	B	$\{b_0, b_3\}$	B	B
$\{b_0, b_3\}$	$\emptyset$	$\emptyset$	B	B	B
$\{b_0, b_2, b_3\}$	$\emptyset$	$\emptyset$	$\{b_0, b_3\}$	B	B
B	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	B

Table 4.3: Some operations related to the structures in Fig 4.5

$p \longmapsto \{a_0\}$. It is then clear that $\mathfrak{A}, a_0 \nVdash p_0, p_0 \to \Box p_0 \rhd \Box p_0$. Let us show that $\mathfrak{A} \Vdash \text{sub}(L)$, i.e., $\mathfrak{A} \Vdash \varphi \rhd \Box\varphi$ for every strict-weak formula φ. For every strict-weak formula φ we define $V(\varphi) := \{x \in A : \mathfrak{A}, x \Vdash \varphi\}$. Let us consider the family $\mathcal{F} = \{\emptyset, \{a_0\}, A\}$. A simple induction shows that for every strict-weak formula φ, it holds that $V(\varphi) \in \mathcal{F}$[18]. The only non-trivial step in the induction is to check that if $V(\varphi_0) \in \mathcal{F}$ and $V(\varphi_1) \in \mathcal{F}$, then $V(\varphi_0 \to \varphi_1) \in \mathcal{F}$. But this is obvious if we check that the behaviour of the binary operation $\langle V(\varphi_0), V(\varphi_1)\rangle \longmapsto V(\varphi_0 \to \varphi_1)$ is the one described in the first matrix of Table 4.3.

4.2.28. Example. In the modal case it is well known that the modal formula $\Box p_0 \supset \Box\Box p_0$ axiomatizes the logic of transitive frames. Now we show that the sequent $\Box p_0 \rhd \Box\Box p_0$ does not axiomatize the strict-weak consequence between sequents of the class of transitive frames. Let L be the singleton of this sequent. It is enough to show that $\emptyset \not\vdash_L p_0 \to \bot \rhd \Box(p_0 \to \bot)$. To this end we seek a structure $\mathfrak{B}$ such that $\mathfrak{B} \Vdash \text{sub}(L)$ while $\mathfrak{B} \nVdash p_0 \to \bot \rhd \Box(p_0 \to \bot)$. We consider the structure $\mathfrak{B}$ depicted in Figure 4.5, where the valuation is the map $p \longmapsto \{b_2\}$. It is obvious that $\mathfrak{A}, b_0 \nVdash p_0 \to \bot \rhd \Box(p_0 \to \bot)$. In order to show that $\mathfrak{B} \Vdash \text{sub}(L)$ we can use the same idea as in the previous remark, but this time considering the family $\mathcal{F} = \{\emptyset, \{b_2\}, \{b_0, b_3\}, \{b_0, b_2, b_3\}, B\}$ (we have written the operation of the corresponding operation in the second matrix of Table 4.3).

4.2.29. Example. In the modal case we know that the logic of symmetric frames is axiomatized by the modal formula $p_0 \supset \Box\Diamond p_0$[19]. This modal formula is equivalent to the sequent $p_0 \rhd \neg\neg p_0$. Analogously to what we saw above it is also possible to show that this sequent does not axiomatize the consequence relation between sequents of the class of symmetric frames. Indeed, if we consider L as the singleton of it, then $\emptyset \not\vdash_L \top \rhd p_0 \vee \neg\Box p_0$. To show this we can reason as before

[18] This recalls general frames in modal logic. There the condition is required for all modal formulas. We notice that it is clearly false that for all modal formulas φ, $V(\varphi) \in \mathcal{F}$. As a counterexample we can consider the modal formula $\sim p_0$.

[19] Note that it is not the formula given in Table 1.2 for axiomatizing symmetric frames. There the formula $\Diamond\Box p_0 \supset p_0$ appears.

this time considering the structure $\mathfrak{A}$ in Figure 4.5 with the valuation $p \longmapsto \{a_1\}$, and the family $\mathcal{F} = \{\emptyset, \{a_1\}, A\}$.

Up to now we have just considered properties on frames. In the next two propositions we consider properties where the valuation plays a role. They use the following sequents:

$$\begin{aligned} \mathsf{Per^s} &:= p_0 \rhd [s]p_0 \\ \mathsf{APer^w} &:= \langle w \rangle p_0 \rhd p_0. \end{aligned}$$

They are related to persistent and antipersistent valuations. The sequent $\mathsf{Per^s}$ was already considered in [CJ01].

4.2.30. PROPOSITION. *Let L and Π be sets of $\mathcal{L}^{SW}(\vartheta)$-sequents that are relatively consistent, and let s be a strict modality. The following statements are equivalent:*

1. $\Pi \mid\!\sim_L \varphi \rhd [s]\varphi$ *for every $\mathcal{L}^{SW}(\vartheta)$-formula φ.*
2. $R_s^{\mathfrak{H}_\Pi^L} \subseteq \{\langle \Sigma_0, \Sigma_1 \rangle : \Sigma_0 \subseteq \Sigma_1\}$.
3. *It holds that (i) the valuation in $\mathfrak{H}_\Pi^L$ is R_s-persistent, (ii) for every $s' \in \mathsf{SMod}$, $R_{s'} \circ R_s \subseteq R_{s'}$, and (iii) for every $w' \in \mathsf{WMod}$, $R_{w'} \circ R_s^{-1} \subseteq R_{w'}$.*

Proof: $(1 \Leftrightarrow 2)$: It is easy using Theorem 4.2.23.

$(1 \Rightarrow 3)$: Assume that $\Pi \mid\!\sim_L \varphi \rhd [s]\varphi$ for every strict-weak formula φ. From this fact the three things that we want to check easily follow. First of all, it is obvious that the valuation in $\mathfrak{H}_\Pi^L$ is persistent if we restrict ourselves to formulas that are propositions. Secondly, it holds that for every $s' \in \mathsf{SMod}$, $\Pi \mid\!\sim_L \varphi_0 \rightarrow_{s'} \varphi_1 \rhd [s](\varphi_0 \rightarrow_{s'} \varphi_1)$ for all strict-weak formulas φ_0, φ_1; from this we conclude that $R_{s'} \circ R_s \subseteq R_{s'}$ by Proposition 4.2.26. Finally, it holds that for every $w' \in \mathsf{WMod}$, $\Pi \mid\!\sim_L \varphi_0 \leftharpoondown_{w'} \varphi_1 \rhd [s](\varphi_0 \leftharpoondown_{w'} \varphi_1)$ for all strict-weak formulas φ_0, φ_1; from this we conclude that $R_{w'} \circ R_s^{-1} \subseteq R_{w'}$ using again Proposition 4.2.26.

$(3 \Rightarrow 1)$: It is enough to prove by induction on the length of φ that $\mathfrak{H}_\Pi^L \Vdash \varphi \rhd [s]\varphi$ for every $\mathcal{L}^{SW}(\vartheta)$-formula φ. This is straightforwardly shown using our assumptions. □

4.2.31. COROLLARY. *Let L and Π be sets of $\mathcal{L}^{SW}(\vartheta)$-sequents that are relatively consistent, and let w be a weak modality. The following statements are equivalent:*

1. $\Pi \mid\!\sim_L \langle w \rangle\varphi \rhd \varphi$ *for every $\mathcal{L}^{SW}(\vartheta)$-formula φ.*
2. $R_w^{\mathfrak{H}_\Pi^L} \subseteq \{\langle \Sigma_0, \Sigma_1 \rangle : \Sigma_1 \subseteq \Sigma_0\}$.

3. *It holds that (i) the valuation in $\mathfrak{H}^L_\Pi$ is R_w-antipersistent, (ii) for every $s' \in$ SMod, $R_{s'} \circ R_w^{-1} \subseteq R_{s'}$, and (iii) for every $w' \in \mathsf{WMod}$, $R_{w'} \circ R_w \subseteq R_{w'}$.*

Proof: What is claimed here is obtained by duality from Proposition 4.2.30. □

In the case that there is only a single strict modality Proposition 4.2.30 guarantees us that if $p_0 \rhd \Box p_0 \in L$ then the canonical structures for this L (it does not matter what sequents are in Π) are transitive and persistent. Using this it is easy to see that (i) the consequence associated with intuitionistic structures corresponds to $\mid\!\sim_L$ where $L = \{\mathsf{T^s}, \mathsf{Per}^s\}$, and that (ii) the consequence associated with BPL-structures corresponds to $\mid\!\sim_L$ where $L = \{\mathsf{Per^s}\}$. Indeed, for all the properties in Table 4.1 we know how to axiomatize the consequence relation associated with the class of structures such that the underlying frame satisfies the property and the valuation is persistent.

Up to now we have obtained completeness results using the canonical construction. In the next three propositions we obtain some completeness results for sets of sequents that indeed are not canonical generators. We do not give any direct proof of the fact that they are not canonical generators, but this is an easy consequence of Proposition 4.3.7.

4.2.32. PROPOSITION. *Let L be the singleton of $\mathsf{GL^s}$. For every finite set Π of $\mathcal{L}^s$-sequents and every $\mathcal{L}^s$-sequent ς, it holds that*

$$\Pi \mid\!\sim_L \varsigma \quad \textit{iff} \quad \Pi \mid\!\approx_{\mathsf{K}} \varsigma,$$

where K is any class of structures such that $\{\mathfrak{A} : \mathfrak{A}$ is a finite strict order$\} \subseteq \mathsf{K} \subseteq$ $\{\mathfrak{A} : \mathfrak{A}$ is a Noetherian strict order$\}$.

Proof: By Proposition 4.2.5 it suffices to show that if $\Pi \not\mid\!\sim_L \varsigma$ then it is refuted by some finite strict order. As there is a single modality we will write $\rightarrow$ and R without the subscript s. Replacing ς and the sequents in Π with equivalent ones we can assume that all of them are simple $\mathcal{L}^s$-sequents.

CLAIM I: $\emptyset \mid\!\sim_L p_0 \rightarrow p_1 \rhd \Box(p_0 \rightarrow p_1)$.

Proof of Claim: By Theorem 4.2.24 we assume that $\mathfrak{A} \Vdash \mathrm{sub}(L)$ and show that $\mathfrak{A} \Vdash p_0 \rightarrow p_1 \rhd \Box(p_0 \rightarrow p_1)$. Using that $\mathfrak{A} \Vdash \mathsf{GL^s}$ it is easy to see that $\mathfrak{A} \Vdash$ $\big(\Box(p_0 \supset p_1) \rightarrow (p_0 \supset p_1)\big) \supset \Box(p_0 \supset p_1)$ (cf. Example 3.1.9). Now we copy the proof of [Boo93, Theorem 1.18] replacing the formula considered there with $p_0 \supset p_1$. If we do this we will obtain that $\mathfrak{A} \Vdash (p_0 \rightarrow p_1) \supset \Box\Box(p_0 \supset p_1)$, which it is equivalent to $\mathfrak{A} \Vdash p_0 \rightarrow p_1 \rhd \Box(p_0 \rightarrow p_1)$. ⊣

CLAIM II: Suppose that $\varphi_0, \varphi_1 \in \mathcal{L}^s$, $\mathfrak{A} \Vdash \Big(\big(\varphi_0 \wedge (\varphi_0 \rightarrow \varphi_1)\big) \rightarrow \varphi_1\Big) \rhd \Big(\varphi_0 \rightarrow \varphi_1\Big)$ and $\mathfrak{A}, a \nVdash \varphi_0 \rightarrow \varphi_1$. Then, there is a state $a' \in R[\{a\}]$ such that $a \neq a'$, $\langle a', a'\rangle \notin R$, $\mathfrak{A}, a' \Vdash \varphi_0 \wedge (\varphi_0 \rightarrow \varphi_1)$ and $\mathfrak{A}, a' \nVdash \varphi_1$[20]. We will refer by $\rho(a, \varphi_0, \varphi_1)$ to a chosen state satisfying these properties.

[20]The last two conditions imply that $\mathfrak{A}, a' \Vdash \varphi_0 \rightarrow \varphi_1$ and $\mathfrak{A}, a' \nVdash \varphi_0 \rhd \varphi_1$.

Proof of Claim: By the assumptions it follows that $\mathfrak{A}, a \not\Vdash \big(\varphi_0 \wedge (\varphi_0 \to \varphi_1)\big) \to \varphi_1$. Hence, there is $a' \in R[\{a\}]$ such that $\mathfrak{A}, a' \Vdash \varphi_0 \wedge (\varphi_0 \to \varphi_1)$ and $\mathfrak{A}, a' \not\Vdash \varphi_1$. Therefore, $\langle a', a'\rangle \notin R$. $\dashv$

Assume that $\Pi \not\vdash_L \varsigma$. By Theorem 4.2.24 it follows that there is a pointed structure $\langle \mathfrak{A}, a\rangle$ such that $\mathfrak{A} \Vdash \Pi \cup \mathrm{sub}(L)$ and $\mathfrak{A}, a \not\Vdash \varsigma$. As $\mathfrak{H}^L_\Pi$ is transitive (by the first claim together with Proposition 4.2.26) we can assume that $\mathfrak{A}$ is transitive. As $\mathsf{GL^s} \in L$ we know that

$$\mathfrak{A} \Vdash \left\{ \Big(\big(\varphi_0 \wedge (\varphi_0 \to \varphi_1)\big) \to \varphi_1 \Big) \rhd \Big(\varphi_0 \to \varphi_1 \Big) : \varphi_0, \varphi_1 \in \mathcal{L}^s \right\}.$$

Let Φ be the set of $\mathcal{L}^s$-formulas that are $\mathcal{L}^s$-subformulas of a $\mathcal{L}^s$-formula appearing in $\Pi \cup \{\varsigma\}$; and let $\Phi^{\to}$ be $\{\varphi \in \Phi : \varphi = \varphi_0 \to \varphi_1 \text{ for some } \varphi_0, \varphi_1\}$. Both sets are finite. By Claim II we define the map

$$\begin{array}{rccl} f: & A \times \Phi^{\to} & \longrightarrow & A \\ & \langle x, \varphi_0 \to \varphi_1\rangle & \longmapsto & \begin{cases} \rho(x, \varphi_0, \varphi_1)\ (\neq x) & \text{if } \mathfrak{A}, x \not\Vdash \varphi_0 \to \varphi_1 \\ x & \text{if } \mathfrak{A}, x \Vdash \varphi_0 \to \varphi_1. \end{cases} \end{array}$$

Using this we define the map

$$\begin{array}{rccl} g: & A & \longrightarrow & \mathcal{P}_{fin}(A) \\ & x & \longmapsto & \{f(\langle x, \varphi\rangle) : \varphi \in \Phi^{\to}\}. \end{array}$$

By the same claim it is easy to see that if $y \in g(x)$ and $x \neq y$, then $\{\varphi \in \Phi^{\to} : \mathfrak{A}, y \Vdash \varphi\} \not\subseteq \{\varphi \in \Phi^{\to} : \mathfrak{A}, x \Vdash \varphi\}$.

Now we show that we can assume that a is an irreflexive point, i.e., $\langle a, a\rangle \notin R$. Suppose that $\langle a, a\rangle \in R$, and assume that ς is the simple $\mathcal{L}^s$-sequent $\varphi_0 \rhd \varphi_1$. By $\mathfrak{A}, a \not\Vdash \varsigma$ it follows that $\mathfrak{A}, a \not\Vdash \varphi_0 \to \varphi_1$. Hence, $\mathfrak{A}, \rho(a, \varphi_0, \varphi_1) \not\Vdash \varphi_0 \rhd \varphi_1$. Therefore, we can replace a with the irreflexive point $\rho(a, \varphi_0, \varphi_1)$.

We consider the structure $\mathfrak{B}$ as the (perhaps not generated) substructure of $\mathfrak{A}$ given by the universe $\{a\} \cup \{b : \exists n \in \omega\ \exists a_0, \ldots, a_{n+1} \in A \text{ such that } a_0 = a, a_{n+1} = b, a_{n+1} \in g(a_n), a_n \in g(a_{n-1}), \ldots, a_1 \in g(a_0)\}$. Let us prove that it is a finite strict order. The only non-trivial property is finiteness. If it is not finite, then by König's Tree Lemma (see [Dev91, Theorem 4.4.1]) together with the fact that the range of g is included in $\mathcal{P}_{fin}(A)$ it follows that there is an infinite sequence $\langle a_n : n \in \omega\rangle$ of different states such that $a_0 = a$ and for every $n \in \omega$, $a_{n+1} \in g(a_n)$. Hence, $a_0\ R\ a_1\ R\ a_2\ R\ \ldots$ and for every $n \in \omega$ there is a formula $\varphi_n \in \Phi^{\to}$ such that $\mathfrak{A}, a_{n+1} \Vdash \varphi_n$ and $\mathfrak{A}, a_n \not\Vdash \varphi_n$. By transitivity together with the fact that the formulas in $\Phi^{\to}$ are strict implications it is clear that the map $n \longmapsto \varphi_n$ is injective from ω into $\Phi^{\to}$. This is in contradiction with the finiteness of $\Phi^{\to}$.

Claim III: For all $\varphi \in \Phi$, it holds that for every $b \in B$,

$$\mathfrak{A}, b \Vdash \varphi \quad \text{iff} \quad \mathfrak{B}, b \Vdash \varphi.$$

Proof of Claim: The only non-trivial step is to prove $\mathfrak{B}, b \not\Vdash \varphi_0 \to \varphi_1$ under the assumption $\mathfrak{A}, b \not\Vdash \varphi_0 \to \varphi_1$. So, assume that $\mathfrak{A}, b \not\Vdash \varphi_0 \to \varphi_1$. Let b' be $f(\langle b, \varphi_0 \to \varphi_1 \rangle)$. Then $\langle b, b' \rangle \in R^{\mathfrak{A}}$ and $b' \in g(b)$. Hence $b' \in B$ and $\langle b, b' \rangle \in R^{\mathfrak{B}}$. By definition of f we know that $\mathfrak{A}, b' \Vdash \varphi_0 \wedge (\varphi_0 \to \varphi_1)$ and $\mathfrak{A}, b' \not\Vdash \varphi_1$. By the inductive hypothesis it follows that $\mathfrak{B}, b' \Vdash \varphi_0$ and $\mathfrak{B}, b' \not\Vdash \varphi_1$. Using that $\langle b, b' \rangle \in R^{\mathfrak{B}}$ we obtain that $\mathfrak{B}, b \not\Vdash \varphi_0 \to \varphi_1$. ⊣

By the claim it holds that $\mathfrak{B} \Vdash \Pi$ and $\mathfrak{B}, a \not\Vdash \varsigma$. This concludes our proof. □

4.2.33. Proposition. *Let L be the set formed by the sequent $p_0 \rhd \Box p_0$ and the sequent $\top \rhd (\Box p_0 \to p_0) \to \Box p_0$. For every finite set Π of $\mathcal{L}^s$-sequents and every $\mathcal{L}^s$-sequent ς, it holds that*

$$\Pi \mid\!\sim_L \varsigma \quad \textit{iff} \quad \Pi \mid\!\approx_{\mathsf{K}} \varsigma,$$

where K is any class of structures such that $\{\mathfrak{A} : \mathfrak{A}$ is a finite strict order with a persistent valuation$\} \subseteq \mathsf{K} \subseteq \{\mathfrak{A} : \mathfrak{A}$ is a Noetherian strict order with a persistent valuation$\}$.[21]

Proof: Soundness is easily checked. Before looking at the other direction we prove the following claim.

Claim I: For every transitive structure $\mathfrak{A}$, it holds that

$$\mathfrak{A} \Vdash \Box\varphi \to \varphi \rhd \Box\varphi \quad \text{iff} \quad \mathfrak{A} \Vdash \top \rhd (\Box\varphi \to \varphi) \to \Box\varphi.$$

Proof of Claim: One direction is trivial. For the other, assume that $\mathfrak{A} \Vdash \top \rhd (\Box\varphi \to \varphi) \to \Box\varphi$, $\mathfrak{A}, a \Vdash \Box\varphi \to \varphi$ and $\langle a, a' \rangle \in R$. We want to show that $\mathfrak{A}, a' \Vdash \varphi$. By the second assumption together with transitivity it follows that $\mathfrak{A}, a' \Vdash \Box\varphi \to \varphi$. Using the fact that $\mathfrak{A}, a \Vdash (\Box\varphi \to \varphi) \to \Box\varphi$ we deduce that $\mathfrak{A}, a \Vdash \Box\varphi$. Hence $\mathfrak{A}, a' \Vdash \varphi$. ⊣

We assume that $\Pi \not\mid\!\sim_L \varsigma$. By Theorem 4.2.24 there is a pointed structure $\langle \mathfrak{A}, a \rangle$ such that $\mathfrak{A} \Vdash \Pi \cup \mathrm{sub}(L)$ and $\mathfrak{A}, a \not\Vdash \varsigma$. As a consequence of Proposition 4.2.30 we know that $\mathfrak{H}^L_\Pi$ is transitive with a persistent valuation.

Using the filtration method we can assume that $\mathfrak{A}$ is finite. Here we sketch the argument behind it. Let Φ be the set of $\mathcal{L}^s$-formulas that are $\mathcal{L}^s$-subformulas of a $\mathcal{L}^s$-formula appearing in $\Pi \cup \{\varsigma\}$. We define the equivalence relation $\sim_\Phi$ between elements of A in the following way:

$$x \sim_\Phi x' \quad \text{iff} \quad \text{for every } \varphi \in \Phi, \mathfrak{A}, x \Vdash \varphi \text{ iff } \mathfrak{A}, x' \Vdash \varphi.$$

The equivalence class of an element x is denoted by $[x]_\Phi$. Let $\mathfrak{B}$ be the structure such that its universe is $\{[x]_\Phi : x \in A\}$, its accessibility relation is $\{\langle [x]_\Phi, [x']_\Phi \rangle :$

[21]The content of this proposition is inspired by [Vis81, Theorem 2.2] (see also [AR98, Lemma 2.10]).

$(\varphi_0 \to \varphi_1 \in \Phi \; \& \; \mathfrak{A}, x \Vdash \varphi_0 \to \varphi_1) \;\Rightarrow\; (\mathfrak{A}, x' \Vdash \varphi_0 \to \varphi_1 \; \& \; \mathfrak{A}, x' \Vdash \varphi_0 \rhd \varphi_1)\}$, and its valuation is the map $p \longmapsto \{[x]_\Phi : \mathfrak{A}, x \Vdash p\}$. Then, it is easily shown that (i) if $\langle x, x'\rangle \in R^{\mathfrak{A}}$ then $\langle [x]_\Phi, [x']_\Phi\rangle \in R^{\mathfrak{B}}$, and that (ii) if $\langle [x]_\Phi, [x']_\Phi\rangle \in R^{\mathfrak{B}}$, $\varphi_0 \to \varphi_1 \in \Phi$ and $\mathfrak{A}, x \Vdash \varphi_0 \to \varphi_1$ then $\mathfrak{A}, x' \Vdash \varphi_0 \rhd \varphi_1$. Using these two facts it is not hard to show that for all $\varphi \in \Phi$ and all $x \in A$, it holds that

$$\mathfrak{A}, x \Vdash \varphi \quad \text{iff} \quad \mathfrak{B}, [x]_\Phi \Vdash \varphi.$$

Hence $\mathfrak{B} \Vdash \Pi \cup \mathrm{sub}(L)$ and $\mathfrak{B}, [a]_\Phi \not\Vdash \varsigma$. Thus, we can replace $\mathfrak{A}$ with $\mathfrak{B}$, which is a finite transitive structure with a persistent valuation.

We have already shown that we can assume that $\mathfrak{A}$ is finite. Hence, what we have at the moment is a finite transitive pointed structure $\langle \mathfrak{A}, a\rangle$ with a persistent valuation such that $\mathfrak{A} \Vdash \Pi \cup \mathrm{sub}(L)$ and $\mathfrak{A}, a \not\Vdash \varsigma$. Now we consider $\mathfrak{C}$ as the structure that has the same universe and valuation as $\mathfrak{A}$, and has $\{\langle x, x'\rangle \in A \times A : x \neq x'\}$ as accessibility relation.

Claim II: For all $\varphi \in \Phi$, it holds that for every $c \in C$,

$$\mathfrak{A}, c \Vdash \varphi \quad \text{iff} \quad \mathfrak{C}, c \Vdash \varphi.$$

Proof of Claim: The only non-trivial step is to prove $\mathfrak{A}, c \Vdash \varphi_0 \to \varphi_1$ under the assumption $\mathfrak{C}, c \Vdash \varphi_0 \to \varphi_1$. So, assume that $\mathfrak{C}, c \Vdash \varphi_0 \to \varphi_1$, $\langle c, c'\rangle \in R^{\mathfrak{A}}$ and $\mathfrak{A}, c' \Vdash \varphi_0$. We want to show that $\mathfrak{A}, c' \Vdash \varphi_1$. If $c \neq c'$ then this is trivial because $\langle c, c'\rangle \in R^{\mathfrak{C}}$. Hence, we suppose that $c = c'$. Thus, $\mathfrak{A}, c \Vdash \varphi_0$ and $\langle c, c\rangle \in R^{\mathfrak{C}}$.

Let us show that $\mathfrak{A}, c \Vdash \Box\varphi_1 \to \varphi_1$. It is clearly enough to prove that $\mathfrak{A}, c \Vdash (\varphi_0 \to \varphi_1) \to \varphi_1$. We assume that $\langle c, c''\rangle \in R^{\mathfrak{A}}$ and $\mathfrak{A}, c'' \Vdash \varphi_0 \to \varphi_1$. By persistence of arbitrary formulas it follows that $\mathfrak{A}, c'' \Vdash \varphi_0$. As a consequence of the inductive hypothesis it holds that $\mathfrak{C}, c'' \Vdash \varphi_0$. Using that $\mathfrak{C}, c \Vdash \varphi_0 \to \varphi_1$ it holds that $\mathfrak{C}, c'' \Vdash \varphi_1$. Finally, using the inductive hypothesis we obtain that $\mathfrak{A}, c'' \Vdash \varphi_1$.

From the fact that $\mathfrak{A} \Vdash \mathrm{sub}(L)$ we know that $\mathfrak{A} \Vdash (\Box\varphi_1 \to \varphi_1) \to \Box\varphi_1$. By Claim I it holds that $\mathfrak{A} \Vdash \Box\varphi_1 \to \varphi_1 \rhd \Box\varphi_1$. Using what we have seen in the previous paragraph it follows that $\mathfrak{A}, c \Vdash \Box\varphi_1$. Using the fact that $\langle c, c\rangle \in R^{\mathfrak{C}}$ we obtain $\mathfrak{A}, c \Vdash \varphi_1$, as we wanted to prove. ⊣

By the claim it holds that $\mathfrak{C} \Vdash \Pi$ and $\mathfrak{C}, a \not\Vdash \varsigma$, which finishes our proof. □

4.2.34. Proposition. *Let L be the singleton of* $\mathsf{Grz^s}$. *For every finite set Π of $\mathcal{L}^s$-sequents and every $\mathcal{L}^s$-sequent ς, it holds that*

$$\Pi \mid\!\sim_L \varsigma \quad \textit{iff} \quad \Pi \mid\!\approx_{\mathsf{K}} \varsigma,$$

where K is any class of structures such that $\{\mathfrak{A} : \mathfrak{A}$ is a finite partial order$\} \subseteq \mathsf{K} \subseteq \{\mathfrak{A} : \mathfrak{A}$ is a Noetherian partial order$\}$.

Proof: By Proposition 4.2.5 it is enough to prove that if $\Pi \not\mid\!\sim_L \varsigma$ then it is refuted by some finite partial order. As in the previous proof we will write $\rightarrow$ and R without the subscript s, and we will assume that ς and the sequents in Π are simple $\mathcal{L}^s$-sequents.

CLAIM I: $\emptyset \mid\!\sim_L p_0 \rightarrow p_1 \rhd \Box(p_0 \rightarrow p_1)$.

Proof of Claim: The same proof as for the first claim in Proposition 4.2.32 works, this time using [Boo93, p. 157]. $\dashv$

Assume that $\Pi \not\mid\!\sim_L \varsigma$. By Theorem 4.2.24 this means that there is a pointed structure $\langle \mathfrak{A}, a\rangle$ such that $\mathfrak{A} \Vdash \Pi \cup \mathrm{sub}(L)$ and $\mathfrak{A}, a \nVdash \varsigma$. As $\mathfrak{H}^L_\Pi$ is transitive (by the first claim) we can assume that $\mathfrak{A}$ is transitive. Let Φ be the set of $\mathcal{L}^s$-formulas that are $\mathcal{L}^s$-subformulas of a $\mathcal{L}^s$-formula appearing in $\Pi \cup \{\varsigma\}$; and let $\Phi^{\rightarrow}$ be $\{\varphi \in \Phi : \varphi = \varphi_0 \rightarrow \varphi_1$ for some $\varphi_0, \varphi_1\}$. Both sets are finite. We define the equivalence relation $\sim_\Phi$ between elements of A in the following way:

$$x \sim_\Phi x' \quad \text{iff} \quad \text{for every } \varphi \in \Phi,\ \mathfrak{A}, x \Vdash \varphi \text{ iff } \mathfrak{A}, x' \Vdash \varphi.$$

The equivalence class of an element x is denoted by $[x]_\Phi$.

CLAIM II: Assume that $\varphi_0 \rightarrow \varphi_1 \in \Phi$, $x \in A$, $\mathfrak{A}, x \Vdash \varphi_0 \rhd \varphi_1$ and $\mathfrak{A}, x \nVdash \varphi_0 \rightarrow \varphi_1$. Then, there is a state $x' \in R[\{x\}]$ such that $\mathfrak{A}, x' \nVdash \varphi_0 \rhd \varphi_1$ and for every $y \in R[\{x'\}]$, it holds that $x \not\sim_\Phi y$. In particular $x \not\sim_\Phi x'$, $\langle x', x\rangle \notin R$ and $x \neq x'$. We will call $\rho(x, \varphi_0, \varphi_1)$ to a state satisfying these properties.

Proof of Claim: Suppose otherwise. Then, for every $x' \in R[\{x\}]$ such that $\mathfrak{A}, x' \nVdash \varphi_0 \rhd \varphi_1$, there is a state $y \in R[\{x'\}]$ such that $x \sim_\Phi y$. Therefore, for every $x' \in R[\{x\}]$ such that $\mathfrak{A}, x' \nVdash \varphi_0 \rhd \varphi_1$, there is a state $y \in R[\{x'\}]$ such that $\mathfrak{A}, y \Vdash \varphi_0 \rhd \varphi_1$ and $\mathfrak{A}, y \nVdash \varphi_0 \rightarrow \varphi_1$. Now it is easy to see that for every $x' \in R[\{x\}]$ such that $\mathfrak{A}, x' \nVdash \varphi_0 \rhd \varphi_1$, it holds that $\mathfrak{A}, x' \nVdash \Box(\varphi_0 \vee (\varphi_0 \rightarrow \varphi_1)) \wedge (\varphi_1 \rightarrow (\varphi_0 \rightarrow \varphi_1))$ (Hint: Distinguish the cases where φ_0 holds or not in the successor y). Hence, $\mathfrak{A}, x \Vdash (\varphi_0 \wedge \Box(\varphi_0 \vee (\varphi_0 \rightarrow \varphi_1)) \wedge (\varphi_1 \rightarrow (\varphi_0 \rightarrow \varphi_1))) \rightarrow \varphi_1$. As we are in a transitive structure it is clear that for all $x' \in R[\{x\}]$ it holds that $\mathfrak{A}, x' \Vdash (\varphi_0 \wedge \Box(\varphi_0 \vee (\varphi_0 \rightarrow \varphi_1)) \wedge (\varphi_1 \rightarrow (\varphi_0 \rightarrow \varphi_1))) \rightarrow \varphi_1$. Using the fact that $\mathfrak{A} \Vdash \mathrm{sub}(\mathsf{Grz^s})$ it follows that for every $x' \in R[\{x\}]$ it holds that $\mathfrak{A}, x' \Vdash \varphi_0 \rhd \varphi_1$. This is in contradiction with the fact that $\mathfrak{A}, x \nVdash \varphi_0 \rightarrow \varphi_1$. $\dashv$

By the last claim we define the map

$$\begin{array}{llll} f: & A \times \Phi^{\rightarrow} & \longrightarrow & A \\ & \langle x, \varphi_0 \rightarrow \varphi_1\rangle & \longmapsto & \begin{cases} \rho(x, \varphi_0, \varphi_1) & \text{if } \mathfrak{A}, x \Vdash \varphi_0 \rhd \varphi_1,\ \mathfrak{A}, x \nVdash \varphi_0 \rightarrow \varphi_1, \\ x & \text{if not.} \end{cases} \end{array}$$

Using it we define the map

$$\begin{array}{llll} g: & A & \longrightarrow & \mathcal{P}_{fin}(A) \\ & x & \longmapsto & \{f(\langle x, \varphi\rangle) : \varphi \in \Phi^{\rightarrow}\}. \end{array}$$

Using the same claim it is easy to see that if $y \in g(x)$ and $y \neq x$, then $\langle y, x\rangle \notin R$. Now we define an infinite sequence $\langle\langle A_n, R_n, X_n\rangle : n \in \omega\rangle$ such that for every $n \in \omega$ it holds that (i) $\langle A_n, R_n\rangle$ is a finite partial order, (ii) $A_n \subseteq A_{n+1} \subseteq A$, (iii) $R_n = R_{n+1} \cap (A_n \times A_n) \subseteq R$, (iv) $X_n = \{a \in A_n : a$ is maximal in the partial order $\langle A_n, R_n\rangle\}$, and (v) $A_n = X_0 \cup \ldots \cup X_n$. The definition is by induction. For $n = 0$ we take $A_0 := \{a\}$, $R_0 := \{\langle a, a\rangle\}$ and $X_0 := \{a\}$. Assume now that we have already defined $\langle A_n, R_n, X_n\rangle$. Then, $A_{n+1} := A_n \cup \{y \in A :$ there is $x \in X_n$ such that $y \in g(x)\}$ (may not be this union a disjoint one), R_{n+1} is the reflexive and transitive closure of the set $R_n \cup \{\langle x, y\rangle \in A \times A : x \in X_n, y \in g(x)\}$, and $X_{n+1} := \{y \in A :$ there is $x \in X_n$ such that $y \in g(x)\}$. It is not hard to show by induction that this definition satisfies all our requirements. Finiteness can be proved using König's Tree Lemma as in the proof of Proposition 4.2.32. We restrict ourselves to showing that $\langle A_{n+1}, R_{n+1}\rangle$ is a partial order. Otherwise, there is $x \in X_n$ and $y \in g(x)$ such that $x \neq y$ and $\langle y, x\rangle \in R_n$. Then, by condition (iii) of the inductive hypothesis it happens that $\langle y, x\rangle \in R$. But this is in contradiction with the property stated after the definition of g.

Claim III: There is $n \in \omega$ such that $\langle A_n, R_n, X_n\rangle = \langle A_k, R_k, X_k\rangle$ for every $k \geq n$.

Proof of Claim: If it is not the case, then there is an infinite sequence $\langle x_n : n \in \omega\rangle$ with different elements such that $x_0 = a$, $x_n \in A_n$ and $x_{n+1} \in g(x_n)$. By condition (iii) it follows than $\langle x_n, x_{n+1}\rangle \in R$. By transitivity if $n < k$ then $\langle x_{n+1}, x_k\rangle \in R$. By Claim II we know that if $n < k$ then $x_n \not\sim_\Phi x_k$. Thus, the map $n \longmapsto [x_n]_\Phi$ is an injective map from ω into $\{[x]_\Phi : x \in A\}$. This is absurd because $\{[x]_\Phi : x \in A\}$ is finite (Hint: Build an injective map $\{[x]_\Phi : x \in A\} \longrightarrow \mathcal{P}(\Phi)$). $\dashv$

We fix n such that $\langle A_n, R_n, X_n\rangle = \langle A_k, R_k, X_k\rangle$ for every $k \geq n$. Then we define $\mathfrak{B}$ as the structure such that the universe is A_n, the accessibility relation is R_n, and the valuation is the map $p \longmapsto \{x \in A_n : \mathfrak{A}, a \Vdash p\}$.

Claim IV: For all $\varphi \in \Phi$, it holds that for every $b \in B$,

$$\mathfrak{A}, b \Vdash \varphi \quad \text{iff} \quad \mathfrak{B}, b \Vdash \varphi.$$

Proof of Claim: The only non-trivial step is to prove $\mathfrak{B}, b \nVdash \varphi_0 \rightarrow \varphi_1$ under the hypothesis $\mathfrak{A}, b \nVdash \varphi_0 \rightarrow \varphi_1$. So, assume that $\mathfrak{A}, b \nVdash \varphi_0 \rightarrow \varphi_1$. Let us distinguish two cases.

Case $\mathfrak{A}, b \nVdash \varphi_0 \rhd \varphi_1$: By the inductive hypothesis it is clear that $\mathfrak{B}, b \Vdash \varphi_0$ and $\mathfrak{B}, b \nVdash \varphi_1$. As $\langle b, b\rangle \in R^{\mathfrak{B}}$ it holds that $\mathfrak{B}, b \nVdash \varphi_0 \rightarrow \varphi_1$.

Case $\mathfrak{A}, b \Vdash \varphi_0 \rhd \varphi_1$: As $B = A_n = X_0 \cup \ldots \cup X_n$, there is $k \leq n$ such that $b \in X_k$. Let b' be $f(\langle b, \varphi_0 \rightarrow \varphi_1\rangle)$. Hence, $b' \in A_{k+1} \subseteq A_{n+1} = B$. By the fact that $\langle b, b'\rangle \in R_{k+1}$ together with condition (iii) it follows that $\langle b, b'\rangle \in R^{\mathfrak{B}}$. By definition of f we know that $\mathfrak{A}, b' \nVdash \varphi_0 \rhd \varphi_1$. Using the inductive hypothesis it holds that $\mathfrak{B}, b' \Vdash \varphi_0$ and $\mathfrak{B}, b' \nVdash \varphi_1$. Therefore, using that $\langle b, b'\rangle \in R^{\mathfrak{B}}$ we conclude that $\mathfrak{B}, b \nVdash \varphi_0 \rightarrow \varphi_1$.

$\dashv$

By the claim it holds that $\mathfrak{B} \Vdash \Pi$ and $\mathfrak{B}, a \nVdash \varsigma$. We already showed that it is a finite partial order. $\square$

4.3 Strict-weak logics

Now we will see how to talk about normal modal logics using strict-weak sequents. The idea comes from what we have already seen. We define strict-weak logics as the sets of simple strict-weak sequents that are closed under the rules of the calculus in Table 4.2. Let us be more accurate.

4.3.1. Definition. (Strict-weak logics)
A *strict-weak logic* $\mathbf{\Lambda}$ is a set of simple $\mathcal{L}^{SW}(\vartheta)$-sequents such that for every simple $\mathcal{L}^{SW}(\vartheta)$-sequent ς, if $\emptyset \mid\!\sim_{\mathbf{\Lambda}} \varsigma$, then $\varsigma \in \mathbf{\Lambda}$. We denote by $\mathrm{Ext}\,\mathbf{\Lambda}$ the set of all strict-weak logics that are extensions of the strict-weak logic $\mathbf{\Lambda}$.

Strict-weak logics are closed under substitutions. It is obvious that if $\mathbf{\Lambda}$ is a strict-weak logic, then $\mathbf{\Lambda}^d$ is also a strict-weak logic (with respect to ϑ^d). The set of all simple strict-weak sequents is clearly a strict-weak logic, called the *inconsistent* strict-weak logic. It is also trivial that strict-weak logics are closed under intersection, i.e., if $\{\mathbf{\Lambda_i} : i \in I\}$ is a family of strict-weak logics then $\bigcap_{i\in I} \mathbf{\Lambda_i}$ is also a strict-weak logic. Therefore, a minimal strict-weak logic exists. We denote it by $\boldsymbol{K_{\kappa\lambda}}$ where κ is the cardinal of SMod and λ is the cardinal of WMod[22]. From now to the end of the section we consider this κ and this λ fixed. The set of all strict-weak logics is precisely $\mathrm{Ext}\,\boldsymbol{K_{\kappa\lambda}}$. Given a family $\{\mathbf{\Lambda_i} : i \in I\}$ of normal modal logics its *sum* $\bigoplus_{i\in I} \mathbf{\Lambda_i}$ is the intersection of all strict-weak logics containing $\bigcup_{i\in I} \mathbf{\Lambda_i}$. It is clear that $\langle \mathrm{Ext}\,\boldsymbol{K_{\kappa\lambda}}, \bigcap, \bigoplus\rangle$ is a complete lattice. In Proposition 4.3.5 we will show that it is distributive, but before we need to introduce the notion of axiomatization. Given a strict-weak logic $\mathbf{\Lambda}$ and a set of L of simple $\mathcal{L}^{SW}(\vartheta)$-sequents,

$$\mathbf{\Lambda} \oplus L$$

denotes the smallest strict-weak logic containing $\mathbf{\Lambda} \cup L$. It is obvious (just consider L as $\mathbf{\Lambda}$) that for every strict-weak logic $\mathbf{\Lambda}$, there is a set L such that $\mathbf{\Lambda} = \boldsymbol{K_{\kappa\lambda}} \oplus L$. In this case we will say that L is an *axiomatization* of $\mathbf{\Lambda}$. If L can be chosen finite[23], then we call $\mathbf{\Lambda}$ *finitely axiomatizable*. We write $\mathbf{\Lambda} \oplus \varsigma$ when $L = \{\varsigma\}$.

[22] There is no ambiguity because we are assuming in this section that $\mathsf{SMod} \cap \mathsf{WMod} = \emptyset$.

[23] We emphasize that there is no way to transform a finite set of simple strict-weak sequents into an equivalent single strict-weak sequent (see Section 5.1 for a proof of this impossibility).

4.3.2. LEMMA. *Let* $\mathbf{\Lambda}$ *be a strict-weak logic axiomatizable by* L*, and let* L' *be a set of simple* $\mathcal{L}^{SW}(\vartheta)$*-sequents. Then,*

$$\mathbf{\Lambda} \oplus L' = \{\varsigma \in SSeq\mathcal{L}^{SW}(\vartheta) : \emptyset \mid\!\sim_{L''} \varsigma\}$$

where $L'' := L \cup L'$.

Proof: Let $\widehat{L}$ be the set $\{\varsigma \in SSeq\mathcal{L}^{SW}(\vartheta) : \emptyset \mid\!\sim_{L''} \varsigma\}$. We must show that $\widehat{L}$ is the smallest strict-weak logic extending $\mathbf{\Lambda} \cup L'$. It is not hard to see by induction on the length of derivations that if $\emptyset \mid\!\sim_{\widehat{L}} \varsigma$ then $\emptyset \mid\!\sim_{L''} \varsigma$. Hence, $\widehat{L}$ is a strict-weak logic. It is obvious that $\mathbf{\Lambda} \cup L' \subseteq \widehat{L}$, and it is also clear that $\widehat{L}$ contains all strict-weak logics extending $\mathbf{\Lambda} \cup L'$. □

In particular we have that for every set L of simple $\mathcal{L}^{SW}(\vartheta)$-sequents, it holds that

$$\boldsymbol{K_{\kappa\lambda}} \oplus L = \{\varsigma \in SSeq\mathcal{L}^{SW}(\vartheta) : \emptyset \mid\!\sim_{L} \varsigma\}.$$

Now it is also obvious that if L axiomatizes $\mathbf{\Lambda}$, then L^d axiomatizes $\mathbf{\Lambda}^d$.

4.3.3. LEMMA. *For every* $i \in I$*, let* $\mathbf{\Lambda_i}$ *be a strict-weak logic axiomatizable by* L_i*. Then,*

$$\bigoplus_{i \in I} \mathbf{\Lambda_i} = \{\varsigma \in SSeq\mathcal{L}^{SW}(\vartheta) : \emptyset \mid\!\sim_{L} \varsigma\}$$

where $L = \bigcup_{i \in I} L_i$*. That is,* $\bigoplus_{i \in I} \mathbf{\Lambda_i}$ *is axiomatizable by* $\bigcup_{i \in I} L_i$*.*

Proof: It is analogous to the previous proof. □

Thus, to obtain an axiomatization of the sum of a family of strict-weak logics using given axiomatizations of its members we can simply join their axioms. It is more difficult to obtain an axiomatization of the intersection. We need to use again the map tr involved in the proof of the Standard Form Theorem. Given ς and ς' two simple $\mathcal{L}^{SW}(\vartheta)$-sequents, $k, k' \in \omega$, $\overline{m} \in \mathsf{Mod}^k$ and $\overline{m'} \in \mathsf{Mod}^{k'}$, now we associate with them a finite set $\langle \varsigma, k, \overline{m} \rangle \underline{\vee} \langle \varsigma', k', \overline{m'} \rangle$ of simple $\mathcal{L}^{SW}(\vartheta)$-sequents. Let us see how this set is built. Assume that $\varsigma = \varphi_0 \rhd \varphi_1$, that $\varsigma' = \varphi'_0 \rhd \varphi'_1$ and that the propositions of ς and ς' are in $\{p_0, \ldots, p_{n-1}\}$. Let φ be the modal formula

$$[m_0] \ldots [m_{k-1}](\varphi_0(p_0, \ldots, p_{n-1}) \supset \varphi_1(p_0, \ldots, p_{n-1})) \vee$$
$$[m'_0] \ldots [m'_{k'-1}](\varphi'_0(p_n, \ldots, p_{2n-1}) \supset \varphi'_1(p_n, \ldots, p_{2n-1})).$$

We define $\langle \varsigma, k, \overline{m} \rangle \underline{\vee} \langle \varsigma', k', \overline{m'} \rangle$ as the set $\{\nu_i \rhd \pi_i : i < l\}$ where $(\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_{l-1} \supset \pi_{l-1}) = \mathrm{tr}(\varphi)$.

4.3.4. LEMMA. *Let* $\mathbf{\Lambda}$ *and* $\mathbf{\Lambda'}$ *be two strict-weak logics that are axiomatizable, respectively, by* L *and* L'*. Then,* $\mathbf{\Lambda} \cap \mathbf{\Lambda'}$ *is axiomatized by*

$$\bigcup \{\langle \varsigma, k, \overline{m} \rangle \underline{\vee} \langle \varsigma', k', \overline{m'} \rangle : \varsigma \in \mathrm{sub}(L), \varsigma' \in \mathrm{sub}(L'), k, k' \in \omega, \overline{m} \in \mathsf{Mod}^k, \overline{m'} \in \mathsf{Mod}^{k'}\}.$$

Proof: Let $\widehat{L}$ be this set. We recall that $\mathfrak{A}_a$ refers to the smallest generated substructure of $\mathfrak{A}$ containing the state a. It is not hard to show that for every pointed structure $\langle \mathfrak{A}, a \rangle$, it holds that

$$\mathfrak{A}_a \Vdash \mathrm{sub}(\widehat{L}) \quad \text{iff} \quad \mathfrak{A}_a \Vdash \mathrm{sub}(L) \text{ or } \mathfrak{A}_a \Vdash \mathrm{sub}(L').$$

Using this together with Theorem 4.2.24 it easily follows that for every simple $\mathcal{L}^{SW}(\vartheta)$-sequent ς,

$$\emptyset \mid\!\sim_{\widehat{L}} \varsigma \quad \text{iff} \quad \emptyset \mid\!\sim_{L} \varsigma \text{ and } \emptyset \mid\!\sim_{L'} \varsigma.$$

Hence, $\emptyset \mid\!\sim_{\widehat{L}} \varsigma$ iff $\varsigma \in \mathbf{\Lambda} \cap \mathbf{\Lambda'}$. □

4.3.5. Proposition. *The complete lattice $\langle \mathrm{Ext}\, \boldsymbol{K_{\kappa\lambda}}, \bigcap, \bigoplus \rangle$ is a distributive lattice. Moreover, it satisfies the infinitary distributive law*

$$\mathbf{\Lambda} \cap \bigoplus_{i \in I} \mathbf{\Lambda_i} = \bigoplus \{\mathbf{\Lambda} \cap \mathbf{\Lambda_i} : i \in I\}.$$

Proof: It is enough to show that $\mathbf{\Lambda} \cap \bigoplus_{i \in I} \mathbf{\Lambda_i} \subseteq \bigoplus \{\mathbf{\Lambda} \cap \mathbf{\Lambda_i} : i \in I\}$. By Lemma 4.3.4 we know that $\mathbf{\Lambda} \cap \bigoplus_{i \in I} \mathbf{\Lambda_i}$ is axiomatized by

$$\bigcup \{\langle \varsigma, k, \overline{m} \rangle \underline{\vee} \langle \varsigma', k', \overline{m'} \rangle : \varsigma \in \mathrm{sub}(L), \varsigma' \in \mathrm{sub}(L'), k, k' \in \omega, \overline{m} \in \mathsf{Mod}^k, \overline{m'} \in \mathsf{Mod}^{k'}\},$$

where $L = \mathbf{\Lambda}$ and $L' = \bigcup_{i \in I} \mathbf{\Lambda_i}$. Let us call $\widehat{L}$ this union. By the same lemma we also know that for every $i \in I$, $\mathbf{\Lambda} \cap \mathbf{\Lambda_i}$ is axiomatized by

$$\bigcup \{\langle \varsigma, k, \overline{m} \rangle \underline{\vee} \langle \varsigma', k', \overline{m'} \rangle : \varsigma \in \mathrm{sub}(L), \varsigma' \in \mathrm{sub}(L_i), k, k' \in \omega, \overline{m} \in \mathsf{Mod}^k, \overline{m'} \in \mathsf{Mod}^{k'}\},$$

where $L_i = \mathbf{\Lambda_i}$. Let us call $\widehat{L_i}$ (for each $i \in I$) this last union. It is clear that $\widehat{L} \subseteq \bigcup_{i \in I} \widehat{L_i}$. From here it follows that $\mathbf{\Lambda} \cap \bigoplus_{i \in I} \mathbf{\Lambda_i} \subseteq \bigoplus \{\mathbf{\Lambda} \cap \mathbf{\Lambda_i} : i \in I\}$. □

Strict-weak logics are interesting because they give a uniform framework for normal modal logics and superintuitionistic logics. We will show that all of them are particular cases of strict-weak logics. First of all we start by analyzing the case of normal modal logics.

4.3.6. Proposition. *The map*

$$\mathbf{\Lambda} \longmapsto \mathbf{\Lambda}^{SW} := \{\varphi_0 \rhd \varphi_1 : \varphi_0, \varphi_1 \in \mathcal{L}^{SW}(\vartheta), \varphi_0 \supset \varphi_1 \in \mathbf{\Lambda}\}$$

is an embedding of the complete lattice $\langle \mathrm{Ext}\, \boldsymbol{K_{\kappa+\lambda}}, \bigcap, \bigoplus \rangle$ of normal modal logics (in τ_ϑ) into the complete lattice $\langle \mathrm{Ext}\, \boldsymbol{K_{\kappa\lambda}}, \bigcap, \bigoplus \rangle$ of strict-weak logics. This embedding preserves infinitary intersection and infinitary sum.

Proof: Firs of all we check that the map is well defined. If $\mathbf{\Lambda}$ is the inconsistent normal modal logic then it is clear that $\mathbf{\Lambda}^{SW}$ is the inconsistent strict-weak logic. Hence, assume that $\mathbf{\Lambda}$ is a consistent normal modal logic. It is well known that $\mathbf{\Lambda} = \{\varphi \in \mathcal{L}^{MOD}(\tau_\vartheta) : \mathfrak{H}^{\mathbf{\Lambda}} \Vdash \varphi\}$ where $\mathfrak{H}^{\mathbf{\Lambda}}$ is the canonical structure for $\mathbf{\Lambda}$. Therefore,

$$\mathbf{\Lambda}^{SW} = \{\varsigma \in SSeq\mathcal{L}^{SW}(\vartheta) : \mathfrak{H}^{\mathbf{\Lambda}} \Vdash \varsigma\}.$$

Using the fact that $\mathfrak{H}^{\mathbf{\Lambda}} \Vdash \mathbf{\Lambda}^{SW}$ it is obvious that if $\emptyset \mid\!\sim_{\mathbf{\Lambda}^{SW}} \varsigma$, then $\mathfrak{H}^{\mathbf{\Lambda}} \Vdash \varsigma$. Therefore, if $\emptyset \mid\!\sim_{\mathbf{\Lambda}^{SW}} \varsigma$, then $\varsigma \in \mathbf{\Lambda}^{SW}$. This means that $\mathbf{\Lambda}^{SW}$ is a strict-weak logic.

For every normal modal logic it is obvious that $(\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_{k-1} \supset \pi_{k-1})$ belongs to it iff for every $i < k$, $\nu_i \supset \pi_i$ belongs to it. Hence, by the Standard Form Theorem it follows that our map is injective.[24]

Now we show that the map preserves the infinitary operations $\bigcap$ and $\bigoplus$. The case of intersection is trivial. Let us consider the sum $\bigoplus$. Assume that $\{\mathbf{\Lambda_i} : i \in I\}$ is a family of normal modal logics. In the modal literature it is well known that

$$\bigoplus\{\mathbf{\Lambda_i} : i \in I\} = \{\varphi \in \mathcal{L}^{MOD}(\tau_\vartheta) : \forall\mathfrak{A},\ \text{if } \mathfrak{A} \Vdash \bigcup_{i\in I} \mathbf{\Lambda_i} \text{ then } \mathfrak{A} \Vdash \varphi\}.$$

On the other hand, by Theorem 4.2.24 and Lemma 4.3.3 it holds that

$$\bigoplus\{\mathbf{\Lambda_i}^{SW} : i \in I\} = \{\varsigma \in SSeq\mathcal{L}^{SW}(\vartheta) : \forall\mathfrak{A},\ \text{if } \mathfrak{A} \Vdash \bigcup_{i\in I} \mathbf{\Lambda_i}^{SW} \text{ then } \mathfrak{A} \Vdash \varsigma\}.$$

From the Standard Form Theorem it easily follows that for every structure $\mathfrak{A}$, it holds that

$$\mathfrak{A} \Vdash \bigcup\{\mathbf{\Lambda_i} : i \in I\} \quad \text{iff} \quad \mathfrak{A} \Vdash \bigcup\{\mathbf{\Lambda_i}^{SW} : i \in I\}.$$

Considering the three facts that we have stated it is clear that

$$\Big(\bigoplus\{\mathbf{\Lambda_i} : i \in I\}\Big)^{SW} = \bigoplus\{\mathbf{\Lambda_i}^{SW} : i \in I\}.$$

This conclude the proof. □

In the previous proposition we have introduced the definition of $\mathbf{\Lambda}^{SW}$. We will talk of $\mathbf{\Lambda}^{S}$, $\mathbf{\Lambda}^{W}$, $\mathbf{\Lambda}^{s}$ and $\mathbf{\Lambda}^{w}$ following the same convention as in Definition 2.1.2. By the Standard Form Theorem it easily follows that for every normal modal logic $\mathbf{\Lambda}$ and every structure $\mathfrak{A}$, it holds that

[24] This paragraph justifies why it is more interesting to develop the Standard Form Theorem in the version of Theorem 3.1.3 and not in the dual version of Corollary 3.1.6. The problem with the last representation comes from the fact that in general in normal modal logics it is false that $(\pi_0 \smallsetminus \nu_0) \vee \ldots \vee (\pi_{k-1} \smallsetminus \nu_{k-1})$ belongs to it iff there is $i < k$ such that $\pi_i \smallsetminus \nu_i$ belongs to it.

$$\mathfrak{A} \Vdash \mathbf{\Lambda} \quad \text{iff} \quad \mathfrak{A} \Vdash \mathbf{\Lambda}^{SW}.$$

In fact, the previous property is also true when we use pointed structures.

A strict-weak logic that is of the form $\mathbf{\Lambda}^{SW}$ for a certain normal modal logic $\mathbf{\Lambda}$ is said to admit a *modal presentation*. It is clear that the minimal strict-weak logic and the inconsistent strict-weak logic are two trivial examples of strict-weak logics admitting a modal presentation. In Propositions 2.1.2, 4.2.32 and 4.2.34 we have obtained axiomatizations of several strict-weak logics that admit a modal presentation. Among the consequences of these propositions we have the following equalities:

$$\begin{array}{rlrl}
\boldsymbol{T}^s &= \boldsymbol{K}^s \oplus \mathsf{T^s} & \boldsymbol{T}^w &= \boldsymbol{K}^w \oplus \mathsf{T^w}\\
\boldsymbol{K4}^s &= \boldsymbol{K}^s \oplus \mathsf{4^s} & \boldsymbol{K4}^w &= \boldsymbol{K}^w \oplus \mathsf{4^w}\\
\boldsymbol{S4}^s &= \boldsymbol{T}^w \oplus \mathsf{4^s} & \boldsymbol{S4}^w &= \boldsymbol{T}^w \oplus \mathsf{4^w}\\
\boldsymbol{S5}^s &= \boldsymbol{S4}^s \oplus \mathsf{5^s} & \boldsymbol{S5}^w &= \boldsymbol{S4}^w \oplus \mathsf{5^w}\\
\boldsymbol{GL}^s &= \boldsymbol{K}^s \oplus \mathsf{GL^s} & \boldsymbol{GL}^s &= \boldsymbol{K}^s \oplus \mathsf{GL^s}\\
\boldsymbol{Grz}^s &= \boldsymbol{K}^s \oplus \mathsf{Grz^s} & \boldsymbol{Grz}^w &= \boldsymbol{K}^w \oplus \mathsf{Grz^w}\\
\boldsymbol{Verum}^s &= \boldsymbol{K}^s \oplus \mathsf{V^s} & \boldsymbol{Verum}^w &= \boldsymbol{K}^w \oplus \mathsf{V^w}\\
\boldsymbol{Triv}^s &= \boldsymbol{K}^s \oplus \mathsf{Tr^s} & \boldsymbol{Triv}^w &= \boldsymbol{K}^w \oplus \mathsf{Tr^w}.
\end{array}$$

Now we present several results that relate a normal modal logic with its associated strict-weak logic. Among them, Proposition 4.3.12 stands out. All axiomatizations that are obtained as consequence of the three propositions cited above can be seen as simple applications of it.

4.3.7. PROPOSITION. *Let $\mathbf{\Lambda}$ be a consistent normal modal logic. Then, the map*

$$\begin{array}{rccl} f: & \mathfrak{H}^{\mathbf{\Lambda}} & \longrightarrow & \mathfrak{H}^{\mathbf{\Lambda}^{SW}}\\ & \Sigma & \longmapsto & \Sigma \cap \mathcal{L}^{SW}(\vartheta) \end{array}$$

is an isomorphism between the canonical structure for $\mathbf{\Lambda}$ and the canonical structure for the strict-weak logic $\mathbf{\Lambda}^{SW}$.

Proof: First of all we check that the map is well defined. Suppose that Σ is a maximal consistent theory of $\vdash_{l\mathbf{\Lambda}}$. Let us show that $\Sigma \cap \mathcal{L}^{SW}(\vartheta)$ is a prime $\emptyset$-theory of $\mid\!\sim_{\mathbf{\Lambda}^{SW}}$. We suppose that $\emptyset \mid\!\sim_{\mathbf{\Lambda}^{SW}} \Gamma \rhd \Delta$ for a certain strict-weak sequent $\Gamma \rhd \Delta$ such that $\Gamma \subseteq \Sigma$. Then, $\emptyset \mid\!\sim_{\mathbf{\Lambda}^{SW}} \bigwedge \Gamma \rhd \bigvee \Delta$. As $\mathbf{\Lambda}^{SW}$ is a strict-weak theory we deduce that $\bigwedge \Gamma \rhd \bigvee \Delta \in \mathbf{\Lambda}^{SW}$, i.e., $\bigwedge \Gamma \supset \bigvee \Delta \in \mathbf{\Lambda}$. Using the maximality of Σ and the fact that $\Gamma \subseteq \Sigma$ we deduce that there is $\delta \in \Delta$ such that $\delta \in \Sigma$.

Now let us show the injectivity. Suppose that $\Sigma_0 \cap \mathcal{L}^{SW}(\vartheta) = \Sigma_1 \cap \mathcal{L}^{SW}(\vartheta)$ for some $\Sigma_0, \Sigma_1 \in \mathfrak{H}^{\mathbf{\Lambda}}$. In order to show that $\Sigma_0 = \Sigma_1$ it is enough (by maximality) to prove that $\Sigma_0 \subseteq \Sigma_1$. Let $\varphi \notin \Sigma_1$. By the Standard Form Theorem we can assume that φ is of the form $(\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_{k-1} \supset \pi_{k-1})$ where the ν's and the

π's are strict-weak formulas. By maximality it follows that there is $i < k$ such that $\nu_i \in \Sigma_1$ and $\pi_i \notin \Sigma_1$. Using that $\Sigma_0 \cap \mathcal{L}^{SW}(\vartheta) = \Sigma_1 \cap \mathcal{L}^{SW}(\vartheta)$ we know that $\nu_i \in \Sigma_0$ and $\pi_i \notin \Sigma_0$. Therefore, $(\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_{k-1} \supset \pi_{k-1}) \notin \Sigma_0$, i.e., $\varphi \notin \Sigma_0$.

It is time to prove that the map is surjective. Assume that Σ is a prime $\emptyset$-theory of $\mid\!\sim_{\mathbf{\Lambda}^{SW}}$. By Lemma 4.2.22 we know that $\Sigma = \mathrm{Th}_{\mathcal{L}^{SW}(\vartheta)}(\mathfrak{H}^{\mathbf{\Lambda}^{SW}}, \Sigma)$. Using that $\mathfrak{H}^{\mathbf{\Lambda}^{SW}} \Vdash \mathbf{\Lambda}$ it is not hard to see that $\mathrm{Th}_{\mathcal{L}^{MOD}(\tau_\vartheta)}(\mathfrak{H}^{\mathbf{\Lambda}^{SW}}, \Sigma)$ is a theory of $\vdash_{l\mathbf{\Lambda}}$. It easily follows that this set is a maximal consistent theory of $\vdash_{l\mathbf{\Lambda}}$. It is obvious that the image of this set under f is precisely Σ.

The map clearly behaves well with respect to the valuations. Let us now see that it also behaves well with respect the accessibility relations. Here we restrict ourselves to proving the case of strict modalities. The case of weak modalities is analogously proved. Let us fix $s \in \mathsf{SMod}$. We want to prove that for every $\Sigma_0, \Sigma_1 \in \mathfrak{H}^{\mathbf{\Lambda}}$, it holds that

$$\langle \Sigma_0, \Sigma_1 \rangle \in R_s^{\mathfrak{H}^{\mathbf{\Lambda}}} \quad \text{iff} \quad \langle \Sigma_0 \cap \mathcal{L}^{SW}(\vartheta), \Sigma_1 \cap \mathcal{L}^{SW}(\vartheta) \rangle \in R_s^{\mathfrak{H}^{\mathbf{\Lambda}^{SW}}}.$$

The implication to the right is clear. For the converse, assume that $\langle \Sigma_0, \Sigma_1 \rangle \notin R_s^{\mathfrak{H}^{\mathbf{\Lambda}}}$. Then, there is a modal formula φ such that $[s]\varphi \in \Sigma_0$ and $\varphi \notin \Sigma_1$. By the Standard Form Theorem we can assume that φ is of the form $(\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_{k-1} \supset \pi_{k-1})$ where the ν's and the π's are strict-weak formulas. Therefore, $(\nu_0 \rightarrow_s \pi_0) \wedge \ldots \wedge (\nu_{k-1} \rightarrow_s \pi_{k-1}) \in \Sigma_0$, and by maximality there is $i < k$ such that $\nu_i \in \Sigma_1$ and $\pi_i \notin \Sigma_1$. Using the fact that $\nu_i \rightarrow_s \pi_i \in \Sigma_0$ we conclude that $\langle \Sigma_0 \cap \mathcal{L}^{SW}(\vartheta), \Sigma_1 \cap \mathcal{L}^{SW}(\vartheta) \rangle \notin R_s^{\mathfrak{H}^{\mathbf{\Lambda}^{SW}}}$. □

Now we introduce some definitions. Given a modal formula φ such that $\mathrm{tr}(\varphi) = (\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_{k-1} \supset \pi_{k-1})$ we write $\mathrm{seq}(\varphi)$ for the finite set $\{\nu_i \rhd \pi_i : i < k\}$ of simple strict-weak sequents. Given a set Φ of modal formulas we will write $\mathrm{seq}(\Phi)$ for the set $\bigcup\{\mathrm{seq}(\varphi) : \varphi \in \Phi\}$.

4.3.8. Proposition. *Let $\mathbf{\Lambda}$ be a strict-weak logic. The following statements are equivalent:*

1. *$\mathbf{\Lambda}$ admits a modal presentation.*
2. *$\models^{MOD}_{l\mathfrak{H}^{\mathbf{\Lambda}}}$ is closed under modal substitutions.*
3. *$\mathbf{\Lambda}$ is closed under modal substitutions, i.e., if e is a modal substitution and $\varphi_0 \rhd \varphi_1 \in \mathbf{\Lambda}$ then $\mathrm{seq}(e(\varphi_0 \supset \varphi_1)) \subseteq \mathbf{\Lambda}$.*

Proof: $(1 \Rightarrow 2)$: Assume that $\mathbf{\Lambda}$ is the strict-weak logic associated with a normal modal logic $\mathbf{\Lambda}'$. By Proposition 4.3.7 we know that $\mathfrak{H}^{\mathbf{\Lambda}'} \cong \mathfrak{H}^{\mathbf{\Lambda}}$. Therefore, $\models^{MOD}_{l\mathfrak{H}^{\mathbf{\Lambda}'}} = \models^{MOD}_{l\mathfrak{H}^{\mathbf{\Lambda}}}$. Using the fact that $\mathbf{\Lambda}'$ is closed under modal substitutions it is not hard to see that $\models^{MOD}_{l\mathfrak{H}^{\mathbf{\Lambda}'}}$ is closed under modal substitutions. Hence, $\models^{MOD}_{l\mathfrak{H}^{\mathbf{\Lambda}}}$ is also closed under modal substitutions.

$(2 \Rightarrow 3)$: Suppose that $\models^{MOD}_{l\mathfrak{H}^{\mathbf{\Lambda}}}$ is closed under modal substitutions, e is a modal substitution and $\varphi_0 \rhd \varphi_1 \in \mathbf{\Lambda}$. By the last condition we know that $\mathfrak{H}^{\mathbf{\Lambda}} \Vdash \varphi_0 \rhd \varphi_1$. Hence, $\emptyset \models^{MOD}_{l\mathfrak{H}^{\mathbf{\Lambda}}} \varphi_0 \supset \varphi_1$, and by the closure of this local consequence under modal substitutions it follows that $\emptyset \models^{MOD}_{l\mathfrak{H}^{\mathbf{\Lambda}}} e(\varphi_0 \supset \varphi_1)$. Thus, $\mathfrak{H}^{\mathbf{\Lambda}} \Vdash e(\varphi_0 \supset \varphi_1)$. By the Standard Form Theorem it follows that $\mathfrak{H}^{\mathbf{\Lambda}} \Vdash \text{seq}(e(\varphi_0 \supset \varphi_1))$. By Theorem 4.2.23 it holds that $\emptyset \mid\!\sim_{\mathbf{\Lambda}} \varsigma$ for all $\varsigma \in \text{seq}(e(\varphi_0 \supset \varphi_1))$. Finally, as $\mathbf{\Lambda}$ is a strict-weak logic we conclude that $\text{seq}(e(\varphi_0 \supset \varphi_1)) \subseteq \mathbf{\Lambda}$.

$(3 \Rightarrow 1)$: Let us assume that $\mathbf{\Lambda}$ is closed under modal substitutions. We define $\mathbf{\Lambda'}$ as the set

$$\{\varphi \in \mathcal{L}^{MOD}(\tau_\vartheta) : \text{seq}(\varphi) \subseteq \mathbf{\Lambda}\}.$$

It is not hard to see that for all strict-weak formulas φ_0 and φ_1, we have the following chain of equivalences:

$$\begin{array}{ll} \varphi_0 \supset \varphi_1 \in \mathbf{\Lambda'} & \text{iff} \\ \text{seq}(\varphi_0 \supset \varphi_1) \subseteq \mathbf{\Lambda} & \text{iff} \\ \mathfrak{H}^{\mathbf{\Lambda}} \Vdash \text{seq}(\varphi_0 \supset \varphi_1) & \text{iff} \\ \mathfrak{H}^{\mathbf{\Lambda}} \Vdash \varphi_0 \rhd \varphi_1 & \text{iff} \\ \varphi_0 \rhd \varphi_1 \in \mathbf{\Lambda}. & \end{array}$$

Thus, ${\mathbf{\Lambda'}}^{SW} = \mathbf{\Lambda}$. On the other hand, as a trivial consequence of our hypothesis on $\mathbf{\Lambda}$ we know that $\mathbf{\Lambda'}$ is closed under modal substitutions. Then, it is easy to check that $\mathbf{\Lambda'}$ is a normal modal logic. □

4.3.9. COROLLARY. *Let $\mathbf{\Lambda}$ be a consistent strict-weak logic such that its canonical frame $\mathfrak{F}^{\mathbf{\Lambda}}$ validates all sequents in $\mathbf{\Lambda}$. Then, $\mathbf{\Lambda}$ admits a modal presentation.*

Proof: It is an easy consequence of the fact that validity on frames is preserved under modal substitutions. □

In Examples 4.2.27, 4.2.28 and 4.2.29 we showed that

$$\boldsymbol{K}^s \oplus \Box p_0 \rhd p_0 \qquad \boldsymbol{K}^s \oplus \Box p_0 \rhd \Box\Box p_0 \qquad \boldsymbol{K}^s \oplus p_0 \rhd \neg\neg p_0$$

are not closed under modal substitutions. By our last proposition they do not admit a modal presentation. In particular we have that although $\Box p_0 \supset p_0$ axiomatizes the normal modal logic $\boldsymbol{T}$ it holds that $\text{seq}(\Box p_0 \supset p_0)$ does not axiomatize the strict-weak logic $\boldsymbol{T}^s$. Now we analyze how we can transform an axiomatization of a normal modal logic into an axiomatization for its associated strict-weak logic.

4.3.10. PROPOSITION. *Let $\mathbf{\Lambda}$ be a normal modal logic axiomatized by L. Then, the set*

$$\bigcup\{\text{seq}(e(\varphi)) : \varphi \in L,\ e \textit{ is a modal substitution}\}$$

is an axiomatization of the strict-weak logic $\mathbf{\Lambda}^{SW}$.

Proof: Let L' be this set. We must show that for every simple strict-weak sequent ς, it holds that

$$\emptyset \mathrel{|\!\sim}_{\mathbf{\Lambda}^{SW}} \varsigma \quad \text{iff} \quad \emptyset \mathrel{|\!\sim}_{L'} \varsigma.$$

By the canonical structure construction it is enough to prove that for every structure $\mathfrak{A}$, it holds that

$$\mathfrak{A} \Vdash \mathbf{\Lambda}^{SW} \quad \text{iff} \quad \mathfrak{A} \Vdash \mathrm{sub}(L').$$

As we know that for all structures $\mathfrak{A}$, it holds that

$$\mathfrak{A} \Vdash \mathbf{\Lambda}^{SW} \quad \text{iff} \quad \mathfrak{A} \Vdash \mathbf{\Lambda} \quad \text{iff}^{\,25} \quad \mathfrak{A} \Vdash \{e(\varphi) : \varphi \in L,\ e \text{ is a modal substitution}\},$$

then it results that it is enough to show that

$$\mathfrak{A} \Vdash \{e(\varphi) : \varphi \in L,\ e \text{ is a modal substitution}\} \quad \text{iff} \quad \mathfrak{A} \Vdash \mathrm{sub}(L').$$

for all structures $\mathfrak{A}$. This equivalence easily follows from the definition of L'. □

4.3.11. Corollary. *Let $\mathbf{\Lambda}$ be a normal modal logic axiomatized by a recursively enumerable set of modal formulas. Then, $\mathbf{\Lambda}^{SW}$ is axiomatized by a decidable set of strict-weak sequents.*

Proof: Assume that L is an axiomatization of $\mathbf{\Lambda}$. Given a recursive enumeration of L we can obtain a recursive enumeration $\langle \varphi_0^n \rhd \varphi_1^n : n \in \omega\rangle$ of the axiomatization of $\mathbf{\Lambda}^{SW}$ given in the previous proposition. Finally, reasoning as in a famous proof of Craig [Cra53] we can replace this axiomatization of $\mathbf{\Lambda}^{SW}$ with a decidable one: simply replace it with the set

$$\{\varphi_0^n \rhd \overbrace{\varphi_1^n \wedge \ldots \wedge \varphi_1^n}^{n \text{ times}} : n \in \omega\},$$

which is decidable and axiomatizes the same strict-weak logic (i.e., $\mathbf{\Lambda}^{SW}$). □

For the next proposition we need to introduce the meet of two modal substitutions. Given two modal substitutions e_0 and e_1 we define $e_0 \wedge e_1$ as the modal substitution such that $(e_0 \wedge e_1)(p) = e_0(p) \wedge e_1(p)$ for every proposition p.

4.3.12. Proposition. *Let $\mathbf{\Lambda}$ be a normal modal logic axiomatized by L. Assume that the set L of modal formulas satisfies that for every $\varphi \in L$, every structure $\mathfrak{A}$ and every modal substitutions e_0 and e_1, it holds that:*

$$\textit{if } \mathfrak{A} \Vdash e_0(\varphi) \textit{ and } \mathfrak{A} \Vdash e_1(\varphi), \textit{ then } \mathfrak{A} \Vdash (e_0 \wedge e_1)(\varphi).$$

[25] This equivalence is a consequence of the fact that L is an axiomatization of $\mathbf{\Lambda}$.

Let e be the modal substitution such that for every $k \in \omega$, $e(p_k) = p_{2k} \supset p_{2k+1}$. Then, the set

$$\bigcup\{\mathrm{seq}(e(\varphi)) : \varphi \in L\}$$

axiomatizes $\mathbf{\Lambda}^{SW}$.

Proof: Let L' be this set. Reasoning as in the last proposition it is enough to show that

$$\mathfrak{A} \Vdash \{e'(\varphi) : \varphi \in L,\ e' \text{ is a modal substitution}\} \quad \text{iff} \quad \mathfrak{A} \Vdash \mathrm{sub}(L').$$

for all structures $\mathfrak{A}$. The implication to the right is trivial. For the converse, assume that $\mathfrak{A} \nVdash \{e'(\varphi) : \varphi \in L,\ e' \text{ is a modal substitution}\}$. Then, there is a modal substitution e' and a modal formula $\varphi \in L$ such that $\mathfrak{A} \nVdash e'(\varphi)$. Let $n \in \omega$ be such that the propositions in φ are in $\{p_0, \ldots, p_{n-1}\}$. By the Standard Form Theorem we can assume that there is $k \in \omega$ such that for every $j < n$,

$$e'(p_j) = (\nu_0^j \supset \pi_0^j) \wedge \ldots \wedge (\nu_{k-1}^j \supset \pi_{k-1}^j)$$

where the ν's and the π's are strict-weak formulas. For every $i < k$ we define the modal substitution e'_i as the one such that for every $j < n$, $e'_i(p_j) = \nu_i^j \supset \pi_i^j$. It is clear that e' and $e'_0 \wedge \ldots \wedge e'_{k-1}$ coincide in $\{p_0, \ldots, p_{n-1}\}$. Thus, $e'(\varphi) = (e'_0 \wedge \ldots \wedge e'_{k-1})(\varphi)$. Using the fact that $\mathfrak{A} \nVdash e'(\varphi)$ together with the assumption stated in the proposition[26] it follows that there is $i < k$ such that $\mathfrak{A} \nVdash e'_i(\varphi)$. We define e'' as the modal substitution such that for every $j < n$, $e(p_{2j}) = \nu_i^j$ and $e(p_{2j+1}) = \pi_i^j$. We can consider e'' as a strict-weak substitution because the image of each proposition is a strict-weak formula. It is clear that $e'' \circ e$ coincides with e'_i in $\{p_0, \ldots, p_{n-1}\}$. Therefore, as $\mathfrak{A} \nVdash e'_i(\varphi)$ we know that $\mathfrak{A} \nVdash e''(e(\varphi))$. It implies that $\mathfrak{A} \nVdash e''[\mathrm{seq}(e(\varphi))]$. Using the fact that $e''[\mathrm{seq}(e(\varphi))] \subseteq \mathrm{sub}(L')$ we conclude that $\mathfrak{A} \nVdash \mathrm{sub}(L')$. □

It is easy to show that for each of the modal formulas in Table 1.2 except the ones for directedness and weak directedness, the condition stated in the last proposition holds[27]. Hence, the last proposition gives us a general method to axiomatize the classes of frames involving these properties (cf. Remark 4.2.4).

We point out that it is still unknown whether finite axiomatizability is preserved when we move from a normal modal logic $\mathbf{\Lambda}$ to the strict-weak logic $\mathbf{\Lambda}^{SW}$. For instance, is the strict-weak logic associated with the modal normal logic of directed frames finitely axiomatizable? All that we know about finite axiomatizability is that it is preserved when the condition in Proposition 4.3.12 holds: just check that the set proposed there is finite in this case.

[26] The written assumption only talks about the case $k = 2$, but an easy induction shows that it also holds for all $k \in \omega$.

[27] This explains why, for the sake of symmetry, in Table 1.2 we have written the modal formula $\Diamond\Box p_0 \supset p_0$ and not the standard one $p_0 \supset \Box\Diamond p_0$. The first one satisfies the condition in Proposition 4.3.12, but the other does not.

Now we state the dual of Proposition 4.3.12. In this case we need to define the joint of two modal substitutions. Given two modal substitutions e_0 and e_1 we define $e_0 \vee e_1$ as the modal substitution such that $(e_0 \vee e_1)(p) = e_0(p) \vee e_1(p)$ for every proposition p.

4.3.13. Corollary. *Let $\mathbf{\Lambda}$ be a normal modal logic axiomatized by L. Assume that the set L of modal formulas satisfies that for every $\varphi \in L$, every structure $\mathfrak{A}$ and every modal substitutions e_0 and e_1, it holds that:*

$$\textit{if } \mathfrak{A} \Vdash e_0(\varphi) \textit{ and } \mathfrak{A} \Vdash e_1(\varphi), \textit{ then } \mathfrak{A} \Vdash (e_0 \vee e_1)(\varphi).$$

Let e be the modal substitution such that for every $k \in \omega$, $e(p_k) = p_{2k} \searrow p_{2k+1}$. Then, the set

$$\bigcup\{\text{seq}(e(\varphi)) : \varphi \in L\}$$

axiomatizes $\mathbf{\Lambda}^{SW}$.

Proof: The same proof works as for Proposition 4.3.12, but this time using the dual version of the Standard Form Theorem (i.e., Corollary 3.1.6). □

Finally, let us see how superintuitionistic logics site inside the framework of strict-weak logics. Obviously, we restrict ourselves to the language $\mathcal{L}^s$. In the $\mathcal{L}^s$ case it holds that for every consistent strict-weak logic, either $\mathbf{\Lambda} \subseteq \boldsymbol{Verum}^s$ or $\mathbf{\Lambda} \subseteq \boldsymbol{Triv}^s$. Let us prove it by distinguishing cases on the shape of canonical structure $\mathfrak{H}^{\mathbf{\Lambda}}$. We consider the two possible frames that have a single state in their universe. We denote by $\mathfrak{F}_0$ the one that has empty accessibility relation, and $\mathfrak{F}_1$ the one in which the accessibility relation is total. It is well known that $\boldsymbol{Verum} = \text{Log}\,\mathfrak{F}_0$ and that $\boldsymbol{Triv} = \text{Log}\,\mathfrak{F}_1$. In the case that there is a state Σ in $\mathfrak{H}^{\mathbf{\Lambda}}$ without any successor then the constant map is a surjective bounded morphism from the smallest generated subframe of $\mathfrak{F}^{\mathbf{\Lambda}}$ containing Σ into $\mathfrak{F}_o$: from this it follows that $\mathbf{\Lambda} \subseteq \boldsymbol{Verum}^s$. On the other hand, if each state in $\mathfrak{H}^{\mathbf{\Lambda}}$ has a successor then the constant map is a surjective bounded morphism from $\mathfrak{F}^{\mathbf{\Lambda}}$ into $\mathfrak{F}_1$: from this it follows that $\mathbf{\Lambda} \subseteq \boldsymbol{Triv}^s$.

Given a superintuitionistic logic $\mathbf{\Lambda}$ we associate with it the strict weak logic

$$\mathbf{\Lambda}^s := \{\varphi_0 \rhd \varphi_1 \in SSeq\mathcal{L}^s : \varphi_0 \to \varphi_1 \in \mathbf{\Lambda}\}.$$

Using the canonical structure for the superintuitionistic logic $\mathbf{\Lambda}$ we can easily prove (analogously to Proposition 4.3.6) that $\mathbf{\Lambda}^s$ is a strict-weak logic. We will say that a strict-weak logic admits a *superintuitionistic presentation* if it is of the form $\mathbf{\Lambda}^s$ for a certain superintuitionistic logic $\mathbf{\Lambda}$. Thanks to Propositions 4.2.26 and 4.2.30 we know that

$$\boldsymbol{IPL}^s = \boldsymbol{K}^s \oplus \mathsf{T}^s \oplus \mathsf{Per}^s \qquad \boldsymbol{CPL}^s = \boldsymbol{IPL}^s \oplus \mathsf{B}^s.$$

4.3.14. PROPOSITION. *Let $\mathbf{\Lambda}$ be a consistent strict-weak logic. The following statements are equivalent:*

1. *$\mathbf{\Lambda}$ admits a superintuitionistic presentation.*
2. *$\mathbf{\Lambda}$ is closed under the deduction-detachment theorem, i.e., for all $\varphi_0, \varphi_1, \varphi_2 \in \mathcal{L}^s$ it holds that*

$$\varphi_0 \rhd \varphi_1 \to \varphi_2 \in \mathbf{\Lambda} \quad \textit{iff} \quad \varphi_0 \wedge \varphi_1 \rhd \varphi_2 \in \mathbf{\Lambda}.$$

3. *$\mathsf{T}^{\mathsf{s}} \in \mathbf{\Lambda}$ and $\mathsf{Per}^{\mathsf{s}} \in \mathbf{\Lambda}$.*

Proof: $(1 \Rightarrow 2)$: It is a consequence of the well known fact that $\varphi_0 \to (\varphi_1 \to \varphi_2)$ is in a superintuitionistic logic iff $(\varphi_0 \wedge \varphi_1) \to \varphi_2$ also is in it.

$(2 \Rightarrow 3)$: As $\mathbf{\Lambda}$ is a strict-weak logic it is clear that $p_0 \to p_1 \rhd p_0 \to p_1 \in \mathbf{\Lambda}$ and that $p_0 \wedge \top \rhd p_0 \in \mathbf{\Lambda}$. Using the closure under the deduction-detachment theorem it easily follows that $p_0 \wedge (p_0 \to p_1) \rhd p_1 \in \mathbf{\Lambda}$ and that $p_0 \rhd \Box p_0 \in \mathbf{\Lambda}$.

$(3 \Rightarrow 1)$: Assume that $\mathsf{T}^{\mathsf{s}} \in \mathbf{\Lambda}$ and $\mathsf{Per}^{\mathsf{s}} \in \mathbf{\Lambda}$. Then, $\boldsymbol{K}^s \oplus \mathsf{T}^{\mathsf{s}} \oplus \mathsf{Per}^{\mathsf{s}} \subseteq \mathbf{\Lambda}$. That is, $\boldsymbol{IPL}^s \subseteq \mathbf{\Lambda}$. Now we define $\mathbf{\Lambda'} := \{\varphi \in \mathcal{L}^s : \top \rhd \varphi \in \mathcal{L}^s\}$. It is very easy to check that $\mathbf{\Lambda'}$ is a superintuitionistic logic and that $\mathbf{\Lambda'}^s = \mathbf{\Lambda}$. Therefore, $\mathbf{\Lambda}$ admits a superintuitionistic presentation. $\Box$

By the last proposition it is clear that the consistent strict-weak logics that admit a superintuitionistic presentation form an interval inside the complete lattice $\langle \mathrm{Ext}\, \boldsymbol{K}^s, \bigcap, \bigoplus \rangle$. It is the interval $[\boldsymbol{IPL}^s, \boldsymbol{CPL}^s]$.

On the other hand, it is easy to see that $\boldsymbol{CPL}^s$ admits a modal presentation because

$$\boldsymbol{CPL}^s = \boldsymbol{Triv}^s.$$

Now we show that it is the only strict-weak logic in the interval $[\boldsymbol{IPL}^s, \boldsymbol{CPL}^s]$ with this property.

4.3.15. PROPOSITION. *Let $\mathbf{\Lambda}$ be a consistent strict-weak logic such that it admits a modal presentation and it admits a superintuitionistic presentation. Then, $\mathbf{\Lambda} = \boldsymbol{CPL}^s$.*

Proof: By Proposition 4.3.14 we know that $\boldsymbol{IPL}^s \subseteq \mathbf{\Lambda} \subseteq \boldsymbol{CPL}^s$. As $p_0 \rhd \Box p_0 \in \boldsymbol{IPL}^s$, it follows that $p_0 \rhd \Box p_0 \in \mathbf{\Lambda}$. Hence, $\mathfrak{H}^{\mathbf{\Lambda}} \Vdash p_0 \rhd \Box p_0$. Using Proposition 4.3.8 and the fact that $\mathbf{\Lambda}$ admits a modal presentation it results that $\mathfrak{H}^{\mathbf{\Lambda}} \Vdash {\sim} p_0 \supset \Box {\sim} p_0$. Therefore, $\mathfrak{H}^{\mathbf{\Lambda}} \Vdash \top \rhd p_0 \vee \neg p_0$. Thus, $\top \rhd p_0 \vee \neg p_0 \in \mathbf{\Lambda}$. Using this and the fact that $\mathbf{\Lambda}$ extends $\boldsymbol{IPL}^s$ we can conclude that $\mathbf{\Lambda} = \boldsymbol{CPL}^s$. $\Box$

4.3.16. PROPOSITION. *The strict-weak logic $\boldsymbol{D}^s$ satisfies that* *if* $\mathbf{\Lambda}$ *is a consistent strict-weak logic extending it, then* $\mathbf{\Lambda} \subseteq \boldsymbol{CPL}^s$.

And all strict-weak logics satisfying the condition are extensions of $\boldsymbol{D}^s$.

Proof: First of all, we check that $\boldsymbol{D}^s$ satisfies the property. Let $\mathbf{\Lambda}$ be a consistent strict-weak logic such that $\boldsymbol{D}^s \subseteq \mathbf{\Lambda}$. We saw on page 182 that either $\mathbf{\Lambda} \subseteq \boldsymbol{Verum}^s$ or $\mathbf{\Lambda} \subseteq \boldsymbol{Triv}^s = \boldsymbol{CPL}^s$. Therefore, it is enough to show that $\mathbf{\Lambda} \not\subseteq \boldsymbol{Verum}^s$. Since $\neg\top \rhd \bot \in \boldsymbol{D}^s \subseteq \mathbf{\Lambda}$ and $\neg\top \rhd \bot \notin \boldsymbol{Verum}^s$ it is clear that $\mathbf{\Lambda} \not\subseteq \boldsymbol{Verum}^s$.

Assume now that $\mathbf{\Lambda}$ is a strict-weak logic such that if $\mathbf{\Lambda}'$ is a consistent strict-weak logic and $\mathbf{\Lambda} \subseteq \mathbf{\Lambda}'$ then $\mathbf{\Lambda}' \subseteq \boldsymbol{CPL}^s$. Let us prove that $\boldsymbol{D}^s \subseteq \mathbf{\Lambda}$. By Proposition 4.2.26 we know that $\boldsymbol{D}^s = \boldsymbol{K}^s \oplus \mathsf{D}^s$. Therefore, it is enough to show that $\mathsf{D}^s \in \mathbf{\Lambda}$, i.e., $\neg\top \rhd \bot \in \mathbf{\Lambda}$. If not, we have that $\neg\top \rhd \bot \notin \mathbf{\Lambda}$. Thus, there is a structure $\mathfrak{A}$ such that $\mathfrak{A} \Vdash \mathbf{\Lambda}$ and $\mathfrak{A} \nVdash \neg\top \rhd \bot$. Then, it easily follows that there is a generated substructure $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{B} \Vdash \mathbf{\Lambda}$ and $\mathfrak{B} \Vdash \top \rhd \Box\bot$. Therefore, $\mathbf{\Lambda} \oplus \top \rhd \Box\bot$ is a consistent strict-weak logic extending $\mathbf{\Lambda}$. Let us call it $\mathbf{\Lambda}'$. Using the property that $\mathbf{\Lambda}$ satisfies we deduce that $\mathbf{\Lambda}' \subseteq \boldsymbol{CPL}^s$. Hence, $\top \rhd \Box\bot \in \boldsymbol{CPL}^s$, which is false. $\Box$

4.3.17. Remark. We could also have defined strict-weak logics that are counterparts of $\boldsymbol{BPL}$ and $\boldsymbol{FPL}$. Let us define

$$\boldsymbol{BPL}^s = \boldsymbol{K}^s \oplus \mathsf{Per^s} \qquad \boldsymbol{FPL}^s = \boldsymbol{BPL}^s \oplus \top \rhd (\Box p_0 \to p_0) \to \Box p_0.$$

In Proposition 4.2.30 we saw that $\boldsymbol{BPL}^s$ is precisely the set of simple strict-weak sequents that are valid over BPL-structures. It is in this sense that we can consider $\boldsymbol{BPL^s}$ as a natural counterpart of $\boldsymbol{BPL}$. Analogously, we can consider $\boldsymbol{FPL^s}$ as a natural counterpart of $\boldsymbol{FPL}$ because in Proposition 4.2.33 we proved that $\boldsymbol{FPL}^s$ is the set of simple strict-weak sequents that are valid over FPL-structures. Using the fact that $\top \rhd p_0 \vee \neg p_0$ belongs neither to $\boldsymbol{BPL}^s$ nor to $\boldsymbol{FPL}^s$ it is easy to show, following the same reasoning as in the proof of Proposition 4.3.15, that neither of them admits a modal presentation.

4.4 Disjunction property

We will say that a strict-weak logic $\mathbf{\Lambda}$ has the *disjunction property* if , for every strict-weak formulas φ_0 and φ_1, if $\top \rhd \varphi_0 \vee \varphi_1 \in \mathbf{\Lambda}$ then either $\top \rhd \varphi_0 \in \mathbf{\Lambda}$ or $\top \rhd \varphi_1 \in \mathbf{\Lambda}$.

4.4.1. Remark. If we allow that $\mathsf{Smod} \cap \mathsf{Wmod} \neq \emptyset$ then the minimal strict-weak logic does not have the disjunction property. This follows using the fact that if $m \in \mathsf{Smod} \cap \mathsf{Wmod}$ then $\top \rhd (p_0 \to_m p_1) \vee (p_0 \leftharpoondown_m p_1)$ is in the minimal strict-weak logic.

It has been well known since Gentzen [Gen35, Section 4.1] that $\boldsymbol{IPL}^s$ has the disjunction property. His famous proof is a syntactic one, and it is a trivial consequence of the Cut Elimination Theorem. Using semantic tools, Corsi, in [Cor87, Section 4], proved that $\boldsymbol{K}^s$, $\boldsymbol{T}^s$, $\boldsymbol{K4}^s$ and $\boldsymbol{S4}^s$ have the disjunction property. For the case of $\boldsymbol{K}^s$ we have already given a semantic argument in the proof of Proposition 3.2.19. Another logic with the disjunction property is $\boldsymbol{BPL}^s$. It was proved by Ardeshir in his dissertation [Ard95].

Now we develop a syntactic argument that will allow us to prove in a uniform way all the previous results. Note that the method can also be used for other strict-weak logics. The aim of our proof is to define a certain Kleene's slash (cf. [BD84, Doš85]).

Let L be a set of $\mathcal{L}^{SW}(\vartheta)$-sequents. We inductively define the map $[\ ]^L : \mathcal{L}^{SW}(\vartheta) \longrightarrow 2$ using the following clauses:

- $[\top]^L = 1$.
- $[\bot]^L = 0$.
- If $p \in \mathsf{Prop}$, then $[p]^L = 1$ iff $\emptyset \mathrel{|\!\!\sim}_L \top \rhd p$.
- $[\varphi_0 \wedge \varphi_1]^L = 1$ iff it holds that $[\varphi_0]^L = 1$ and $[\varphi_1]^L = 1$.
- $[\varphi_0 \vee \varphi_1]^L = 1$ iff it holds that

$$\text{either} \quad [\varphi_0]^L = 1 \text{ and } \emptyset \mathrel{|\!\!\sim}_L \top \rhd \varphi_0 \quad \text{or} \quad [\varphi_1]^L = 1 \text{ and } \emptyset \mathrel{|\!\!\sim}_L \top \rhd \varphi_1$$

- If $s \in \mathsf{SMod}$, then $[\varphi_0 \rightarrow_s \varphi_1]^L = 1$ iff it holds that

$$[\varphi_0]^L = 1 \text{ and } \emptyset \mathrel{|\!\!\sim}_L \top \rhd \varphi_0 \quad \text{implies} \quad [\varphi_1]^L = 1 \text{ y } \emptyset \mathrel{|\!\!\sim}_L \top \rhd \varphi_1.$$

- If $w \in \mathsf{WMod}$, then $[\varphi_0 \leftharpoondown_w \varphi_1]^L = 0$ iff it holds that

$$[\varphi_0]^L = 1 \text{ and } \emptyset \mathrel{|\!\!\sim}_L \top \rhd \varphi_0 \quad \text{implies} \quad [\varphi_1]^L = 1 \text{ and } \emptyset \mathrel{|\!\!\sim}_L \top \rhd \varphi_1.$$

We will refer to $[\varphi]^L$ as the *Kleene slash* of φ. Using the Kleene slash we define the map $[\![\ ]\!]^L : \mathcal{L}^{SW}(\vartheta) \longrightarrow 2$ such that for every $\varphi \in \mathcal{L}^{SW}(\vartheta)$,

$$[\![\varphi]\!]^L = 1 \quad \text{iff} \quad [\varphi]^L = 1 \text{ and } \emptyset \mathrel{|\!\!\sim}_L \top \rhd \varphi.$$

In 2 we can consider the operations $\wedge, \vee, \supset$ and $\smallsetminus$ as the usual Boolean operations corresponding to conjunction, disjunction, material implication and weak difference. A simple checking shows that for all strict-weak formulas φ_0 and φ_1, it holds that

- $[\![\varphi_0 \wedge \varphi_1]\!]^L = [\![\varphi_0]\!]^L \wedge [\![\varphi_1]\!]^L$.

- $[\![\varphi_0 \vee \varphi_1]\!]^L = [\varphi_0 \vee \varphi_1]^L = [\![\varphi_0]\!]^L \vee [\![\varphi_1]\!]^L$.
- If $s \in \mathsf{SMod}$, then $[\varphi_0 \to_s \varphi_1]^L = [\![\varphi_0]\!]^L \supset [\![\varphi_1]\!]^L$.
- If $w \in \mathsf{WMod}$, then $[\varphi_0 \leftharpoondown_w \varphi_1]^L = [\![\varphi_0]\!]^L \smallsetminus [\![\varphi_1]\!]^L$.

We will say that a $\mathcal{L}^{SW}(\vartheta)$-sequent $\gamma_0, \dots, \gamma_{n-1} \rhd \delta_0, \dots, \delta_{k-1}$ is *adequate for the disjunction property with respect to* L if for every substitution e, it holds that

$$\text{if } [\![e(\gamma_0)]\!]^L = \dots = [\![e(\gamma_{n-1})]\!]^L = 1 \text{ then there is } i < k \text{ such that } [\![e(\delta_i)]\!]^L = 1.$$

A set of $\mathcal{L}^{SW}(\vartheta)$-sequents is *adequate for the disjunction property with respect to* L when all the sequents in the set are adequate for the disjunction property with respect to L.

4.4.2. Example. Let L be a set that is the union of some of the following sequents: $\mathsf{T^s}$, $\mathsf{T^w}$, $\mathsf{4^s}$, $\mathsf{4^w}$, $\mathsf{D^s}$, $\mathsf{D^w}$, $\mathsf{Per^s}$ and $\mathsf{APer^w}$. It is easy to see that L is adequate for the disjunction property with respect to L[28]. We restrict ourselves to showing it for the case of the singleton of $\mathsf{T^s}$. Assume that $L = \{\mathsf{T^s}\}$ and that $[\![\varphi_0 \wedge (\varphi_0 \to_s \varphi_1)]\!]^L = 1$, and let us prove that $[\![\varphi_1]\!]^L = 1$. Then, $[\![\varphi_0]\!]^L = 1$ and $[\![\varphi_0 \to_s \varphi_1]\!]^L = 1$. Therefore, $[\varphi_0]^L = 1$, $\emptyset \mathrel{|\!\sim}_L \top \rhd \varphi_0$, $[\varphi_0 \to_s \varphi_1]^L = 1$ and $\emptyset \mathrel{|\!\sim}_L \top \rhd \varphi_0 \to_s \varphi_1$. Using the fact that $[\varphi_0 \to_s \varphi_1]^L = 1$ it follows that $[\![\varphi_0]\!]^L \supset [\![\varphi_1]\!]^L$. As we know that $[\![\varphi_0]\!]^L = 1$ we deduce that $[\![\varphi_1]\!]^L = 1$.

4.4.3. Lemma. *Let L be a set of $\mathcal{L}^{SW}(\vartheta)$-sequents that is adequate for the disjunction property with respect to L. If $\emptyset \mathrel{|\!\sim}_L \varsigma$, then ς is adequate for the disjunction property with respect to L.*

Proof: The proof is by induction on the length of the derivation of ς in $\mathrel{|\!\sim}_L$. The fact that L is adequate for the disjunction property with respect to L takes care of the case that $\varsigma \in \text{sub}(L)$. On the other hand, in the case that ς is obtained using one of the rules in Table 4.2 it is enough to show that all these rules preserve the fact of being adequate for the disjunction property with respect to L. This is easily checked. We show only the case of the rule $(\to_s \rhd \to_s)$.

Assume that $\emptyset \mathrel{|\!\sim}_L \gamma_0, \dots, \gamma_{n-1}, \varphi_0 \rhd \varphi_1, \delta_0, \dots, \delta_{k-1}$, that this sequent is adequate for the disjunction property with respect to L, and that $1 = [\![\varphi_0 \to_s \gamma_0]\!]^L = \dots = [\![\varphi_0 \to_s \gamma_{n-1}]\!]^L = [\![\delta_0 \to_s \varphi_1]\!]^L = [\![\delta_{k-1} \to_s \varphi_1]\!]^L$. In particular

$$1 = [\varphi_0 \to_s \gamma_0]^L = \dots = [\varphi_0 \to_s \gamma_{n-1}]^L = [\delta_0 \to_s \varphi_1]^L = [\delta_{k-1} \to_s \varphi_1]^L.$$

Therefore,

$$1 = [\![\varphi_0]\!]^L \supset [\![\gamma_0]\!]^L = \dots = [\![\varphi_0]\!]^L \supset [\![\gamma_{n-1}]\!]^L \tag{4.1}$$

[28] Of course, the above list does not claim to be comprehensive. For instance, we can also enlarge it with the sequents $[s]p_0 \rhd p_0$ and $[s]p_0 \rhd [s][s]p_0$.

and

$$1 = [\![\delta_0]\!]^L \supset [\![\varphi_1]\!]^L = [\![\delta_{k-1}]\!]^L \supset [\![\varphi_1]\!]^L. \tag{4.2}$$

We want to show that $1 = [\![\varphi_0 \rightarrow_s \varphi_1]\!]^L$, i.e., that $\emptyset \mid\!\sim_L \varphi_0 \rightarrow_s \varphi_1$ and $1 = [\varphi_0 \rightarrow_s \varphi_1]^L$. The fact that $\emptyset \mid\!\sim_L \varphi_0 \rightarrow_s \varphi_1$ is a trivial consequence of the first assumption in the paragraph. So, it only remains to show that $1 = [\varphi_0 \rightarrow_s \varphi_1]^L$. That is, we must show that $1 = [\![\varphi_0]\!]^L \supset [\![\varphi_1]\!]^L$. So, let us suppose that $1 = [\![\varphi_0]\!]^L$, and let us prove that $1 = [\![\varphi_1]\!]^L$. If $1 = [\![\varphi_0]\!]^L$, then by (4.1) we know that

$$1 = [\![\varphi_0]\!]^L = [\![\gamma_0]\!]^L = \ldots = [\![\gamma_{n-1}]\!]^L.$$

Using that $\gamma_0, \ldots, \gamma_{n-1}, \varphi_0 \rhd \varphi_1, \delta_0, \ldots, \delta_{k-1}$ is adequate for the disjunction property with respect to L we deduce that either $1 = [\![\varphi_1]\!]^L$ or there is $i < k$ such that $1 = [\![\delta_i]\!]^L$. Using (4.2) we obtain that in both cases $1 = [\![\varphi_1]\!]^L$, as we wanted to prove. □

4.4.4. Proposition. *Let L be a set of simple $\mathcal{L}^{SW}(\vartheta)$-sequents that is adequate for the disjunction property with respect to L. Then, the strict-weak logic axiomatized by L has the disjunction property.*

Proof: Let $\mathbf{\Lambda}$ be the strict-weak logic axiomatized by L. Suppose that $\top \rhd \varphi_0 \vee \varphi_1 \in \mathbf{\Lambda}$. By Lemma 4.3.2 this means that $\emptyset \mid\!\sim_L \top \rhd \varphi_0 \vee \varphi_1$. Using the previous lemma we deduce that $\top \rhd \varphi_0 \vee \varphi_1$ is adequate for the disjunction property with respect to L. As $[\![\top]\!]^L = 1$ we obtain that $[\![\varphi_0 \vee \varphi_1]\!]^L = 1$. Therefore, $[\![\varphi_0]\!]^L \vee [\![\varphi_1]\!]^L = 1$, i.e., either $[\![\varphi_0]\!]^L = 1$ or $[\![\varphi_1]\!]^L = 1$. In the first case we deduce that $\emptyset \mid\!\sim_L \top \rhd \varphi_0$, and in the second case we obtain that $\emptyset \mid\!\sim_L \top \rhd \varphi_1$. Using Lemma 4.3.2 again we conclude that either $\top \rhd \varphi_0 \in \mathbf{\Lambda}$ or $\top \rhd \varphi_1 \in \mathbf{\Lambda}$. □

Taking L as the empty set we deduce that all minimal strict-weak logics have the disjunction property, and using what we saw in Example 4.4.2 it follows that the disjunction property holds for the strict-weak logics $\boldsymbol{IPL}^s$, $\boldsymbol{K}^s$, $\boldsymbol{T}^s$, $\boldsymbol{K4}^s$, $\boldsymbol{S4}^s$ and $\boldsymbol{BPL}^s$.

4.4.5. Proposition. *Let $\mathbf{\Lambda}$ be a normal modal logic in $\mathcal{L}^{mod}$. We consider its associated strict-weak logic $\mathbf{\Lambda}^s$. Then,*

$\mathbf{\Lambda}$ has the modal disjunction property iff $\mathbf{\Lambda}^s$ has the disjunction property.

Proof: The implication to the left is a trivial consequence of the fact that all boxes of modal formulas are up to equivalence in $\mathcal{L}^s$. For the converse, assume that $\mathbf{\Lambda}$ has the modal disjunction property. First of all, let us show the next claim.

Claim: If $X \subseteq_\omega \mathsf{Prop}$, $\varphi_0, \ldots, \varphi_{n-1} \in \mathcal{L}^{mod}$ and $\bigvee X \vee \Box\varphi_0 \vee \ldots \vee \Box\varphi_{n-1} \in \mathbf{\Lambda}$, then there is $i < n$ such that $\Box\varphi_i \in \mathbf{\Lambda}$.

Proof of Claim: Let φ be the modal formula $\Box\varphi_0 \vee \ldots \vee \Box\varphi_{n-1}$. For every $Y \subseteq X$ we define e_Y as the modal substitution such that (i) $e_Y(q) = q$ if $q \in \mathsf{Prop} \setminus X$, (ii) $e_Y(p) = p$ if $p \in Y$, and (iii) $e_Y(p) = \sim p$ if $p \in X \setminus Y$. By definition it is clear that for every $Y \subseteq X$, $(e_Y \circ e_Y)(\phi) \equiv \phi$ for all modal formulas ϕ. Using that $\mathbf{\Lambda}$ is closed under modal substitution we know that $e_Y(\bigvee X \vee \varphi) \in \mathbf{\Lambda}$ for every $Y \subseteq X$. Therefore $\bigwedge\{e_Y(\bigvee X \vee \varphi) : Y \subseteq X\} \in \mathbf{\Lambda}$. From this together with the fact that $\bigvee\{\sim e_Y(\bigvee X) : Y \subseteq X\}$ is satisfied in all pointed structures, it is not hard to see that $\bigvee\{e_Y(\varphi) : Y \subseteq X\} \in \mathbf{\Lambda}$. It is clear that $\bigvee\{e_Y(\varphi) : Y \subseteq X\}$ is equivalent to the modal formula

$$\bigvee\{\Box e_Y(\varphi_0) \vee \ldots \vee \Box e_Y(\varphi_{n-1}) : Y \subseteq X\}.$$

Using the fact that $\mathbf{\Lambda}$ has the modal disjunction property it follows that there is $Y \subseteq X$ and $i < n$ such that $\Box e_Y(\varphi_i) \in \mathbf{\Lambda}$. By the closure under modal substitution it follows that $e_Y(\Box e_Y(\varphi_i)) \in \mathbf{\Lambda}$. Using the fact that $e_Y(\Box e_Y(\varphi_i)) = e_Y(e_Y(\Box\varphi_i)) \equiv \Box\varphi_i$ we conclude that $\Box\varphi_i \in \mathbf{\Lambda}$. ⊣

Using Remark 2.1.5 and the previous claim it is straightforward to check that $\mathbf{\Lambda}^s$ has the disjunction property. □

4.5 Uniform interpolation

For the rest of the section we fix the class K of all τ_ϑ-structures. In this section we show that $\models_{l\mathsf{K}}$ has uniform interpolation. The method used in the proof is based on showing the existence of certain quasi bisimilarity quantifiers.

4.5.1. PROPOSITION (QUASI BISIMILARITY QUANTIFIERS). *Let p be a proposition, and let φ be a $\mathcal{L}^{SW}(\vartheta)$-formula.*

1. *There is a $\mathcal{L}^{SW}(\vartheta)$-formula $\exists p\varphi$ such that:*
   - $\mathsf{Prop}(\exists p\varphi) \subseteq \mathsf{Prop}(\varphi) \setminus \{p\}$,
   - *For every pointed τ_ϑ-structure $\langle\mathfrak{A}, a\rangle$, it holds that*

     $$\mathfrak{A}, a \Vdash \exists p\varphi \quad \text{iff} \quad \begin{cases} \text{there is a pointed } \tau_\vartheta\text{-structure } \langle\mathfrak{B}, b\rangle \text{ such} \\ \text{that } \langle\mathfrak{B}, b\rangle \preceq_{\vartheta_p} \langle\mathfrak{A}, a\rangle \text{ and } \mathfrak{B}, b \Vdash \varphi, \end{cases}$$

     where ϑ_p is the same SW-vocabulary ϑ except for the fact that we have removed p from its propositions.

2. *There is a $\mathcal{L}^{SW}(\vartheta)$-formula $\forall p\varphi$ such that:*
   - $\mathsf{Prop}(\forall p\varphi) \subseteq \mathsf{Prop}(\varphi) \setminus \{p\}$,
   - *For every pointed τ_ϑ-structure $\langle\mathfrak{A}, a\rangle$, it holds that*

$$\mathfrak{A}, a \Vdash \forall p\varphi \quad \textit{iff} \quad \begin{cases} \textit{for every pointed } \tau_\vartheta\textit{-structure } \langle \mathfrak{B}, b\rangle\textit{, if} \\ \langle \mathfrak{A}, a\rangle \preceq_{\vartheta_p} \langle \mathfrak{B}, b\rangle \textit{ then } \mathfrak{B}, b \Vdash \varphi, \end{cases}$$

where ϑ_p is the same than above.

Proof: It is enough to check the first item because then by duality we can take $\forall p\varphi := (\exists p\varphi^d)^d$. We can assume that ϑ is finite restricting ourselves to the modalities and propositions appearing in φ. Let $n \in \omega$ be the modal degree of φ. Given a pointed τ_ϑ-structure $\langle \mathfrak{B}, b\rangle$ we will use the notation $\langle \mathfrak{B}, b\rangle \restriction \vartheta_p$ to denote the pointed τ_{ϑ_p}-structure that results when we restrict the initial one to ϑ_p. Using the fact that ϑ_p is finite it is clear that for every pointed τ_ϑ-structure $\langle \mathfrak{B}, b\rangle$ the $\mathcal{L}^{SW}_\infty(\vartheta_p)$-formula $\pi_n^{\langle \mathfrak{B}, b\rangle \restriction \vartheta_p}$ characterizing positively quasi n-bisimilarity in ϑ_p (see Remark 3.2.10) is indeed (up to equivalence) a $\mathcal{L}^{SW}(\vartheta_p)$-formula. We take the formula

$$\exists p\varphi := \bigvee \{\pi_n^{\langle \mathfrak{B}, b\rangle \restriction \vartheta_p} : \mathfrak{B}, b \Vdash \varphi\}$$

which is a $\mathcal{L}^{SW}(\vartheta_p)$-formula because there is a finite number of $\mathcal{L}^{SW}(\vartheta_p)$-formulas with modal degree $\leq n$ (since ϑ_p is finite). Hence, $\mathsf{Prop}(\exists p\varphi) \subseteq \mathsf{Prop}(\varphi) \setminus \{p\}$. By definition it is immediate that for every pointed τ_ϑ-structure $\langle \mathfrak{A}, a\rangle$,

$$\mathfrak{A}, a \Vdash \exists p\varphi \quad \text{iff} \quad \begin{cases} \text{there is a pointed } \tau_\vartheta\text{-structure } \langle \mathfrak{B}, b\rangle \text{ such} \\ \text{that } \langle \mathfrak{B}, b\rangle \preceq_{n\,\vartheta_p} \langle \mathfrak{A}, a\rangle \text{ and } \mathfrak{B}, b \Vdash \varphi, \end{cases} \tag{4.3}$$

Therefore to finish our proof we only need to show that if there is a pointed τ_ϑ-structure $\langle \mathfrak{B}, b\rangle$ such that $\langle \mathfrak{B}, b\rangle \preceq_{n\,\vartheta_p} \langle \mathfrak{A}, a\rangle$ and $\mathfrak{B}, b \Vdash \varphi$, then there is also a pointed τ_ϑ-structure $\langle \mathfrak{B}', b'\rangle$ such that $\langle \mathfrak{B}', b'\rangle \preceq_{\vartheta_p} \langle \mathfrak{A}, a\rangle$ and $\mathfrak{B}', b' \Vdash \varphi$. So, assume that $\langle \mathfrak{B}, b\rangle$ is a pointed τ_ϑ-structure such that $\langle \mathfrak{B}, b\rangle \preceq_{n\,\vartheta_p} \langle \mathfrak{A}, a\rangle$ and $\mathfrak{B}, b \Vdash \varphi$. By Proposition 3.8.3 we know that there is a pointed τ_ϑ-structure $\langle \mathfrak{B}', b'\rangle$ such that $\langle \mathfrak{B}, b\rangle \preceq_{n\,\vartheta} \langle \mathfrak{B}', b'\rangle \preceq_{\vartheta_p} \langle \mathfrak{A}, a\rangle$. Using the fact that $\langle \mathfrak{B}, b\rangle \preceq_{n\,\vartheta} \langle \mathfrak{B}', b'\rangle$ and that $\mathfrak{B}, b \Vdash \varphi$ we conclude that $\mathfrak{B}', b' \Vdash \varphi$. □

A formula $\exists p\varphi$ satisfying the properties on the first item in Proposition 4.5.1 is called a *p-existential quasi bisimilarity quantifier for φ*, and a formula $\forall p\varphi$ satisfying the properties on the second item is called a *p-universal quasi bisimilarity quantifier for φ*. Now we show that these formulas play the role of uniform interpolants for $\models_{IK}$.

4.5.2. Proposition. $\models_{IK}$ *has uniform interpolation.*

Proof: By duality it is enough to prove that there is existential uniform interpolation. Suppose there is a $\mathcal{L}^{SW}(\vartheta)$-formula $\exists p\varphi$ that is a p-existential quasi bisimilarity quantifier for φ. We show that it satisfies the three properties in the definition of p-existential uniform interpolant for φ.

- Since $\exists p\varphi$ is a p-existential quasi bisimilarity quantifier for φ we know that $\mathsf{Prop}(\exists p\varphi) \subseteq \mathsf{Prop}(\varphi) \setminus \{p\}$.

- Let us check that $\varphi \models_{l\mathsf{K}} \exists p\varphi$. So, assume that $\mathfrak{A}, a \Vdash \varphi$. It is clear that $\langle \mathfrak{A}, a\rangle \preceq_{\vartheta_p} \langle \mathfrak{A}, a\rangle$ and that $\mathfrak{A}, a \Vdash \varphi$. Therefore, $\mathfrak{A}, a \Vdash \exists p\varphi$.

- Suppose that we have a $\mathcal{L}^{SW}(\vartheta)$-formula φ_1 such that $\varphi \models_{l\mathsf{K}} \varphi_1$ and $p \notin \mathsf{Prop}(\varphi_1)$. Let us prove that $\exists p\varphi \models_{l\mathsf{K}} \varphi_1$. So, we assume that $\mathfrak{A}, a \Vdash \exists p\varphi$, i.e., that there is a pointed τ_ϑ-structure $\langle \mathfrak{B}, b\rangle$ such that $\langle \mathfrak{B}, b\rangle \preceq_{\vartheta_p} \langle \mathfrak{A}, a\rangle$ and $\mathfrak{B}, b \Vdash \varphi$. Let ϑ' be the SW-vocabulary containing only the modalities and propositions that appear in φ. By Proposition 3.8.1 we know that there is a pointed τ_ϑ-structure $\langle \mathfrak{C}, c\rangle$ such that $\langle \mathfrak{B}, b\rangle \preceq_{\vartheta} \langle \mathfrak{C}, c\rangle \preceq_{\vartheta_p} \langle \mathfrak{A}, a\rangle$. Using the fact that $\langle \mathfrak{B}, b\rangle \preceq_{\vartheta} \langle \mathfrak{C}, c\rangle$ and that $\mathfrak{B}, b \Vdash \varphi$ it follows that $\mathfrak{C}, c \Vdash \varphi$. It implies that $\mathfrak{C}, c \Vdash \varphi_1$ since $\varphi \models_{l\mathsf{K}} \varphi_1$. Finally, using the fact that $\langle \mathfrak{C}, c\rangle \preceq_{\vartheta_p} \langle \mathfrak{A}, a\rangle$ and that $p \notin \mathsf{Prop}(\varphi_1)$ we obtain that $\mathfrak{A}, a \Vdash \varphi_1$.

This concludes the proof. □

4.6 Open questions

We list below some interesting open problems which in the author's view, need further investigation.

- Are there two classes C and C' of frames such that $\models^s_{l\mathsf{C}} = \models^s_{l\mathsf{C}'}$ while $\models^{mod}_{l\mathsf{C}} \neq \models^{mod}_{l\mathsf{C}'}$? This question was already discussed on page 138.

- It would be interesting to obtain a kind of Sahlqvist Theorem for sets of strict-weak sequents. That is, is it possible to define in a syntactic way a wide class of sets of sequents such that all these sets of sequents are canonical generators? A starting point to discuss this problem could be [GM97].[29]

- In the modal literature there are places where one can find the canonical structure method for proving strong completeness of infinitary consequence relations $\vdash$ (see for instance [Sun77, Gol82, Gol92, Seg94, RdLKV], [Gol93, Chapter 9] and [Koo03, Chapter 3]). In these cases the main difficulty is the proof of Lindenbaum's Lemma[30]. Is it possible to do something similar for the strict-weak fragments? Example 4.1.4 suggests that perhaps in the infinitary situation the parallelism between what happens for the modal case and what happens for the strict-weak fragments is not so neat as in the finitary case (see Proposition 4.3.7).

- The calculus given in Table 4.2 does not have the proof theoretic property of cut elimination: for instance, the sequent $p_0 \rightarrow_s p_1, p_1 \rightarrow_s p_2 \rhd p_0 \rightarrow_s (p_0 \wedge$

[29]The author thanks Yde Venema for stating this gap in the dissertation.
[30]We notice that since there are infinitary rules it is not so clear that the lattice of theories is closed under directed unions of non-empty chains.

p_2) is not derivable without using the Cut rule. Is it possible to find an equivalent calculus with cut elimination? Perhaps it is better to start by a simple question: is it possible to find an equivalent calculus with analytic cut [Smu68] (see [Ono98])?

- Suppose that $\mathbf{\Lambda}$ is a normal modal logic which is finitely axiomatizable. Is the strict-weak logic $\mathbf{\Lambda}^{SW}$ finitely axiomatizable? As a previous step we suggest analysis of the case in which the modal formula axiomatizing $\mathbf{\Lambda}$ is canonical. In particular, is the strict-weak logic $(\boldsymbol{K} \oplus .2)^{SW}$ finitely axiomatizable? A related question is whether every normal modal logic which is finitely axiomatizable is axiomatizable by a finite set of modal formulas satisfying the condition stated in Proposition 4.3.12.

- Let us consider the languages $\mathcal{L}^s$ and $\mathcal{L}^{mod}$. We fix K as the class of all structures. We know that both $\models^s_{l\mathsf{K}}$ and $\models^{mod}_{l\mathsf{K}}$ have uniform interpolation. It is not hard to see that there is a certain relationship between their uniform interpolants. Assume that φ is a $\mathcal{L}^s$ formula and that p is a proposition. It is easy to prove that the p-existential uniform interpolant for φ in $\models^{mod}_{l\mathsf{K}}$ is indeed a $\mathcal{L}^s$-formula. In order to prove this it is enough to show that the p-existential uniform interpolant for φ in $\models^{mod}_{l\mathsf{K}}$ is preserved under $\preceq_s$, which is an easy consequence of its definition as an existential bisimilarity interpolant (see (1.6) on page 44) together with Proposition 3.8.6(1)(2). On the other hand, it is still unknown whether the p-universal uniform interpolant for φ in $\models^{mod}_{l\mathsf{K}}$ is indeed a $\mathcal{L}^s$-formula. The question is related to knowing whether under the hypotheses of the first (second) item in Proposition 3.8.6 we can also obtain a pointed $\tau_{\vartheta\cup\vartheta'}$-structure $\langle \mathfrak{C}, c\rangle$ such that $\langle \mathfrak{A}, a\rangle \simeq_{\tau_\vartheta} \langle \mathfrak{C}, c\rangle \preceq_{\vartheta'} \langle \mathfrak{B}, b\rangle$ ($\langle \mathfrak{A}, a\rangle \simeq_{\tau_\vartheta} \langle \mathfrak{C}, c\rangle \preceq_{n\,\vartheta'} \langle \mathfrak{B}, b\rangle$).

- The first proof of uniform interpolation for $\vdash_{\boldsymbol{IPL}}$ is due to Pitts [Pit92]. It is a rather involved proof based on a certain contraction-free sequent calculus for intuitionistic propositional logic [Dyc92] and on a certain well-founded relation on sequents. Since then, some semantic (and simple) proofs have been developed. For instance, in [Vis96b] Visser proves uniform interpolation using the method of bisimilarity quantifiers. His proof uses in addition to bisimilarity the canonical structure for $\boldsymbol{IPL}$. We think that the use of quasi bisimilarity could perhaps simplify the proof. It should be analyzed whether using quasi bisimilarity (and not bisimilarity) it is possible to prove uniform interpolation for $\vdash_{\boldsymbol{IPL}}$ without the canonical structure. First of all, we need to establish whether the bisimilarity quantifiers for $\vdash_{\boldsymbol{IPL}}$ considered by Visser (see [Vis96b, Theorem 5.2]) are equivalent to the quasi bisimilarity quantifiers that we have introduced in Proposition 4.5.1 once we restrict ourselves to intuitionistic structures. For existential quantifiers it is easy to check that this is the case, but the answer for universal quantifiers is unknown.

- There are many consequence relations for which it is unknown whether uniform interpolation holds. Among them we single out $\vdash_{l\boldsymbol{K4}}$, $\vdash_{l\boldsymbol{BPL}}$ and $\vdash_{l\boldsymbol{FPL}}$.

- In Section 4.5 we have proved uniform interpolation, but we have not said anything about strong uniform interpolation. This question needs to be explored. Indeed, the first problem is to clarify how to define strong uniform interpolation, since in this context we have two types of modalities: strict and weak.[31]

- In Section 4.3 we saw how normal modal logics site inside the framework of strict-weak logics. It would be interesting to explore the situation when we consider normal intuitionistic modal logics (see [Ono77, FS77, FS84, BD84, Fon84, Doš85, Fon86, WZ97, WZ99a, WZ99b]). In normal intuitionistic modal logics it is usual to have an intuitionistic implication $\to_0$ and a unary necessity operator [1] (sometimes there is also a unary possibility operator $\langle 2\rangle$)[32], so what we expect to obtain in the strict-weak framework is a conservative expansion of any normal intuitionistic modal logic (because we replace $\Box$ with a strict implication associated with the same accessibility relation).

- It would be interesting to give alternative semantics for strict-weak logics. We are thinking in particular of two kinds of semantics. First of all, is it possible to introduce a categorical semantics? (see [Mos99, Pal03] for the modal case). And secondly, a game semantics? In the case of $\mathcal{L}^s$ it is known that the semantics of intuitionistic propositional logic can be formulated in terms of certain games called dialogue games [Fel86]. Is it possible to modify the rules governing these games in such a way that we obtain completeness for $\boldsymbol{K}^s$?

To finish the chapter we simply notice that the author has given in [Bou01, Section 3.2] an effective method to transform an axiomatization (using sequents) of a strict-weak logic $\boldsymbol{\Lambda}$ in $\mathcal{L}^s$ into a Hilbert-style calculus that is strongly complete for the consequence relation:

$$\Phi \vdash \varphi \quad \text{iff} \quad \text{there is } \Phi' \subseteq_\omega \Phi \text{ such that } \Phi' \rhd \varphi \in \boldsymbol{\Lambda}.$$

In [Bou02] the reader can find an exposition of the method for the cases of $\boldsymbol{BPL}^s$ and $\boldsymbol{FPL}^s$. These axiomatizations are really ugly: for instance, the axiomatization for the case of $\boldsymbol{K}^s$ has 11 axioms and 13 rules. So, as we are interested

[31] The counterexample for amalgamation of structures given in Example 3.8.2 is worth considering when strong uniform interpolation is analyzed.

[32] One of the few exceptions to this can be found in [Fon84, p. 37] where a binary operation $\to_1$ is considered instead of [1]. However, the semantics of this $\to_1$ does not correspond to a strict implication of the box [1]. Indeed, the semantics of $\varphi_0 \to_1 \varphi_1$ corresponds to the semantics of our $[1]([1]\varphi_0 \to_0 [1]\varphi_1)$.

in complete axiomatizations it is much better to work with sequents, as we have done in the present chapter. However, if one is interested in these consequence relations from an Abstract Algebraic Logic point of view then a Hilbert-style axiomatization has the advantage of characterizing the filters of the consequence relation (which in general are non-protoalgebraic[33]). Using these Hilbert-style axiomatizations it is possible to prove some interesting and 'pathological' properties for these consequence relations (see [Bou01]).

[33]This is what makes the display of a Hilbert-style axiomatization difficult. To an extent sense it explains the larger number of axioms and rules used in the axiomatizations given by the author.

Chapter 5

Computational Approach

Computers are useless. They can only give answers.
PABLO PICASSO

In this chapter we analyze several computational aspects of the notions already introduced in this dissertation. In the first section we consider 'minimal' standard form representations of modal formulas. The word 'minimal' refers to the number of material implications of strict-weak formulas used in standard form representations. The main results of the first section are Theorems 5.1.6 (a characterization of modal formulas equivalent to a conjunction of k material implications of strict-weak formulas) and 5.1.11. In Sections 5.2, 5.3 and 5.4 we consider reductions from normal modal logics into strict-weak fragments. The reductions considered in Section 5.2 are easily obtained by the proof of the Standard Form Theorem, but they are not computable in polynomial time. The main results of Section 5.2 are Propositions 5.2.2 and 5.2.4. On the other hand, reductions considered in Sections 5.3 and 5.4 are polynomial time and allow us to characterize the complexity problem for the $\mathcal{L}^s$-fragment of most common normal modal logics. The main results of Sections 5.3 and 5.4 are Theorems 5.3.3 (a version of Ladner's Theorem), 5.3.4, 5.4.5 (an improvement of Halpern's Theorem for the case of $\boldsymbol{K}$) and 5.4.9. Finally, we state several open problems in Section 5.5.

5.1 Standard form representations

Let us fix an arbitrary SW-vocabulary ϑ for the rest of the section. We devote the section to studying the complexity of representations in standard form of a modal formula. We can consider at least two measures for this complexity. The first possibility (and the most natural for computational purposes) is to consider the map $\xi : \mathcal{L}^{MOD}(\tau_\vartheta) \longrightarrow \omega$ defined by the following clause:

$$\xi(\varphi) \quad := \quad \min\{\text{leng}\,(\varphi') : \varphi' \text{ in standard form and } \varphi \equiv \varphi'\}.$$

Unfortunately, we do not know the growing behaviour of this map.

5.1.1. Question. Is there any polynomial $p(x)$ such that for every $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula φ, it holds that $\xi(\varphi) \leq p(\text{leng}\,(\varphi))$?

It is trivial that if $\mathsf{Mod} = \emptyset$ and Prop is finite, then the answer is positive: we can take as a bound a constant (a polynomial of degree 0). But in the rest of cases we do not know the answer. These are the cases in which either $\mathsf{Mod} \neq \emptyset$ or Prop is infinite. The author's conjecture is that the answer is negative, but up to now all attempts to prove it have failed. In the case that Prop is infinite our conjecture is supported by the fact that the 'minimal' representation in standard form of

$$\sim\big((q_0 \supset p_0) \wedge \ldots \wedge (q_{k-1} \supset p_{k-1})\big) \tag{5.1}$$

which we know[1] is the one stated in (3.1) on page 66, i.e.,

$$\bigwedge\{\bigwedge_{i \notin I} p_i \supset \bigvee_{i \in I} q_i : I \in \mathcal{P}(k)\}.^2 \tag{5.2}$$

This formula corresponds to the conjunctive normal form of our initial formula. We recall that the conjunctive normal form is unique: this is the main difference with the standard form. If we had defined ξ with respect to conjunctive form (and not standard form) we could use the previous family of formulas to answer Question 5.1.1 in the negative. Let us provide some details on the argument. For every $k \in \omega$, let φ_k be the formula in (5.1). A simple checking (recall that the conjunctive form is unique) shows that $2^k \leq \chi(\varphi_k)$ for every $k \in \omega$; and it is clear that there is a polynomial $p_0(x)$ (indeed it is lineal) such that $\text{leng}\,(\varphi_k) \leq p_0(k)$ for every $k \in \omega$. Using these two facts it is easy to obtain a negative answer to Question 5.1.1.

The other measure is based on the number of material implications appearing in standard form representations. We define the maps $\rho : \mathcal{L}^{MOD}(\tau_\vartheta) \longrightarrow \omega$ and $\rho_\infty : \mathcal{L}^{MOD}_\infty(\tau_\vartheta) \longrightarrow \mathbb{C}\text{ARD}$ by the following clauses:

$$\begin{aligned} \rho(\varphi) \quad &:= \quad \min\{k \in \omega : \varphi \equiv \textstyle\bigwedge_{n<k}(\nu_n \supset \pi_n) \text{ with } \nu_n, \pi_n \in \mathcal{L}^{SW}(\vartheta)\} \\ \rho_\infty(\varphi) \quad &:= \quad \min\{\kappa \in \mathbb{C}\text{ARD} : \varphi \equiv \textstyle\bigwedge_{\alpha<\kappa}(\nu_\alpha \supset \pi_\alpha) \text{ with } \nu_\alpha, \pi_\alpha \in \mathcal{L}^{SW}_\infty(\vartheta)\}. \end{aligned}$$

---

[1] Cf. Remark 5.1.3.

[2] In the case that there is a pure strict modality s (analogously if there is a pure weak modality) we also have a motivation for our conjecture: now consider the 'minimal' known representations for

$$\sim \bigwedge_{n<k} ([s]^{2n+1}\langle s\rangle\top \supset [s]^{2n+2}\langle s\rangle\top),$$

where $k \in \omega$.

It is obvious that $\rho^{\vartheta}(\varphi) = \rho^{\vartheta^d}(\sim\varphi^d)$, and that the equality also holds for ρ_∞. By Corollary 3.1.7 we know that for every $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula φ, it holds that (i) if $s \in \mathsf{SMod}$ then $\rho([s]\varphi) \leq 1$, and (ii) if $w \in \mathsf{WMod}$ then $\rho(\langle w\rangle\varphi) \leq 1$. The same also holds for ρ_∞. Now we define the constants[3] $\varrho \in \omega \cup \{\infty\}$ and $\varrho_\infty \in \mathbb{C}\mathrm{ARD} \cup \{\infty\}$ in the following way:

$$\varrho := \begin{cases} \max \rho[\mathcal{L}^{MOD}(\tau_\vartheta)] & \text{if the maximum exists,} \\ \infty & \text{if not,} \end{cases}$$
$$\varrho_\infty := \begin{cases} \sup \rho_\infty[\mathcal{L}^{MOD}_\infty(\tau_\vartheta)] & \text{if the supremum exists,} \\ \infty & \text{if not.} \end{cases}$$

It is clear that $\varrho^{\vartheta} = \varrho^{\vartheta^d}$ and that $\varrho^{\vartheta}_\infty = \varrho^{\vartheta^d}_\infty$.

Let us analyze the map ρ (occasionally we will consider ρ_∞). First of all, we seek upper bounds for its growth. Our proof of the Standard Form Theorem implies that $\rho(\sim\varphi) \leq 2^{\rho(\varphi)}$. At first glance it seems that ρ grows exponentially. Surprisingly, we are going to show that the growth is in fact linear in the length of the formula. We will constructively prove that $\rho(\sim\varphi) \leq \rho(\varphi) + 1$.

5.1.2. THEOREM.

1. *For every $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula φ, it holds that $\rho(\sim\varphi) \leq \rho(\varphi) + 1$.*
2. *For every $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula φ, it holds that $\rho(\varphi) \leq \mathrm{leng}\,(\varphi)$.*

Proof: The second item is easily proved by induction using the first item. Now let us prove the first item. Suppose that

$$\varphi \equiv (\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_{k-1} \supset \pi_{k-1}),$$

where $k \in \omega$ and $\nu_n, \pi_n \in \mathcal{L}^{SW}(\vartheta)$. For every $n \leq k$, we define

$$\nu'_n := \bigvee\{\bigwedge_{i\in I} \nu_i \wedge \bigwedge_{i\in I} \pi_i : I \subseteq k, |I| = n\}$$

and

$$\pi'_n := \bigwedge\{\bigvee_{j\in J} \nu_j : J \subseteq k, |J| = k - n\}.$$

CLAIM: $\sim\varphi \equiv (\nu'_0 \supset \pi'_0) \wedge \ldots \wedge (\nu'_k \supset \pi'_k)$.

Proof of Claim: First of all, we assume that $\mathfrak{A}, a \nVdash \varphi$, $n \leq k$ and $\mathfrak{A}, a \Vdash \nu'_n$. We must prove that $\mathfrak{A}, a \Vdash \pi'_n$. So, let us assume that $J \subseteq k$ and $|J| = k - n$. Using the fact that $\mathfrak{A}, a \Vdash \nu'_n$ it follows that there is $I \subseteq k$ such that $|I| = n$ and $\mathfrak{A}, a \Vdash \bigwedge_{i\in I} \nu_i \wedge \bigwedge_{i\in I} \pi_i$. Now we distinguish two cases.

[3]They are constants because we have supposed that ϑ is a fixed SW-vocabulary.

Case $I \cap J \neq \emptyset$: Using that $\mathfrak{A}, a \Vdash \bigwedge_{i \in I} \nu_i$ it easily follows that $\mathfrak{A}, a \Vdash \bigvee_{j \in J} \nu_j$.

Case $I \cap J = \emptyset$: Since $k \in \omega$, by finiteness we know that $I = k \setminus J$. Using that $\mathfrak{A}, a \not\Vdash (\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_{k-1} \supset \pi_{k-1})$ and that $\mathfrak{A}, a \Vdash \bigwedge_{i \in I} \pi_i$ it follows that $\mathfrak{A}, a \Vdash \bigwedge_{i \notin I} \nu_i$. Therefore, $\mathfrak{A}, a \Vdash \bigvee_{j \in J} \nu_j$.

Hence, in both cases we have that $\mathfrak{A}, a \Vdash \bigvee_{j \in J} \nu_j$. Thus, we have just proved that $\mathfrak{A}, a \Vdash \pi'_n$.

For the converse let us assume that $\mathfrak{A}, a \Vdash \varphi$. We seek $n \leq k$ such that $\mathfrak{A}, a \not\Vdash \nu'_n \supset \pi'_n$. Let I be $\{i < k : \mathfrak{A}, a \Vdash \nu_i\}$, and let n be the cardinality of I. Using the fact that $\mathfrak{A}, a \Vdash \varphi$ it is clear that $\mathfrak{A}, a \Vdash \bigwedge_{i \in I} \nu_i \wedge \bigwedge_{i \in I} \pi_i$. Hence, $\mathfrak{A}, a \Vdash \nu'_n$. Let J be $k \setminus I$, i.e., $J = \{j < k : \mathfrak{A}, a \not\Vdash \nu_j\}$. It is clear that $|J| = k - n$ and that $\mathfrak{A}, a \not\Vdash \bigvee_{j \in J} \nu_j$. Therefore, $\mathfrak{A}, a \not\Vdash \pi'_n$. Thus, we have already obtained that $\mathfrak{A}, a \not\Vdash \nu'_n \supset \pi'_n$. ⊣

This concludes our proof. □

5.1.3. Remark. In the previous proof we have obtained a new representation in standard form of the formula $\sim\big((q_0 \supset p_0) \wedge \ldots \wedge (q_{k-1} \supset p_{k-1})\big)$. This representation behaves better than (5.2) from the point of view of the number of material implications. However, the reader can check that this new representation is longer than the formula stated in (5.2). Thus, for the purpose of the measure ξ the representation given by the Standard Form Theorem is better.

5.1.4. Remark. We stress that the proof of Theorem 5.1.2 uses finiteness. In fact, in the infinitary case the best bound that we know is that $\rho_\infty(\sim\varphi) \leq 2^{\rho_\infty(\varphi)}$. This bound is given by the proof of the Standard Form Theorem. In order to know if there are better bounds we suggest to start answering whether there is a formula equivalent to

$$\sim \bigwedge \{q_n \supset p_n : n \in \omega\}$$

that is in standard form and only uses a countable number of material implications[4].

Now it is time to find lower bounds for the map ρ. As far as we know this problem has not been previously considered in the literature (even for the case $\mathsf{Mod} = \emptyset$). Our main results will be Theorems 5.1.6 and 5.1.11, but let us start by considering a simple case to show the difficulties.

5.1.5. Example. (Case $\mathsf{Mod} = \emptyset$ and $\mathsf{Prop} = \{p_n : n \in \omega\}$). We are going to prove that $\varrho = \infty$ using several tools from the literature. As $\mathsf{Mod} = \emptyset$ we have that $\mathcal{L}^{MOD}(\tau_\vartheta)$ is the set of Boolean formulas, and that $\mathcal{L}^{SW}(\vartheta)$ is the set

[4]This question is related to the second problem listed in Section 3.12

of *monotone* formulas (i.e., formulas builded using propositions and $\bot, \top, \wedge$ and $\vee$). It is also clear that pointed structures corresponds (up to bisimilarity) to Boolean valuations, and that quasi bisimilarity $\preceq$ between pointed structures corresponds to monotonicity $\leq$ of Boolean valuations. We recall that given two Boolean valuations v and v', $v \leq v'$ means that $v(p) \leq v'(p)$ for every proposition p. We have that monotonicity preserves validity of monotone formulas. In order to prove that $\varrho = \infty$ we need to check that for every $k \in \omega$ there is a Boolean formula φ_k such that φ_k is not (up to equivalence) a conjunction of k material implications of monotone functions. This is not trivial at all. The only proof that we know which is based on facts one can find in the literature uses the known upper bounds for Dedekind's problem.

Let us briefly explain Dedekind's problem. For every $n \in \omega$, let D_n (the *Dedekind number* associated with n) be the number of elements in the free bounded distributive lattice generated by n elements. The problem of determining D_n is what is known as Dedekind's problem. The problem goes back to Dedekind [Ded97], and it is a difficult one (see [Chu40, War46, Gil54, Chu65, Han66, Kle69, Sha70, KM75, Kor77, Kis88, Sap00, Kah02]). To date the known values are

$$D_0 = 2 \qquad D_1 = 3 \qquad D_2 = 6$$
$$D_3 = 20 \qquad D_4 = 168 \qquad D_5 = 7581$$
$$D_6 = 7828354 \qquad D_7 = 2414682040998$$
$$D_8 = 56130437228687557907788.$$

As can be observed, the numbers grow very rapidly. It was only 'recently' that D_8 was computed by Wiedemann (see [Wie91]). The tricks needed to calculate the Dedekind numbers are given from the different formulations of the problem. We refer to the fact that D_n coincides with:

- the number of monotone Boolean functions that it is possible to build using n propositions. We recall that a *monotone Boolean function* using n propositions is a function $f : 2^n \longrightarrow 2$ such that $f(x_0, \ldots, x_{n-1}) \leq f(y_0, \ldots, y_{n-1})$ whenever $x_0 \leq y_0, \ldots, x_{n-1} \leq y_{n-1}$.

- the number of antichains in the partial order $\langle \mathcal{P}(n), \subseteq \rangle$. We recall that an *antichain* is a family $\mathcal{F}$ of subsets of n (i.e., $\mathcal{F} \subseteq \mathcal{P}(n)$) such that if $X, Y \in \mathcal{F}$ and $X \neq Y$, then neither $X \subseteq Y$ nor $Y \subseteq X$.

- the number of ideals in the partial order $\langle \mathcal{P}(n), \subseteq \rangle$[5]. We recall that an *ideal*[6] is a family $\mathcal{F}$ of subsets of n (i.e., $\mathcal{F} \subseteq \mathcal{P}(n)$) such that if $X \in \mathcal{F}$ and $Y \subseteq X$, then $Y \in \mathcal{F}$.

[5] Dually, D_n is also the number of filters in the partial order $\langle \mathcal{P}(n), \subseteq \rangle$.

[6] We emphasize that we are talking about an ideal of a partial order, and not about an ideal of a lattice.

A definite answer to Dedekind's problem was given by Kisielewicz in [Kis88]:

$$D_n = \sum_{k=1}^{2^{2^n}} \prod_{j=1}^{2^n-1} \prod_{i=0}^{j-1} \Big(1 - b_i^k b_j^k \prod_{m=0}^{\log_2 i} (1 - b_m^i + b_m^i b_m^j)\Big),$$

where[7] $b_i^k = [k/2^i] - 2[k/2^{i+1}]$. Some of the first bounds obtained for the Dedekind Problem were the following ones:

$$2^{\binom{n}{[n/2]}} \leq D_n \leq 3^{\binom{n}{[n/2]}}. \tag{5.3}$$

The first inequality was proved in [Gil54][8], and the second one in [Han66]. For our purposes all that we need is the second inequality in (5.3). However, we notice that from an asymptotic point of view this upper bound can be improved: in [Kle69] it is proved that

$$\lim_{n\to\infty} \frac{\log_2 D_n}{\binom{n}{[n/2]}} = 1.$$

Using (5.3) we will be able to prove that $\varrho = \infty$. If not, $\varrho = k \in \omega$. This implies that every Boolean formula is up to equivalence a conjunction of k material implications of monotone formulas (using variables of the initial Boolean formula). Using that 2^{2^n} is the number of Boolean formulas up to equivalence that use n variables (see Theorem 3.1.1) we obtain that for every $n \in \omega$,

$$2^{2^n} \leq D_n^{2k}.$$

In particular, for every $n \in \omega$,

$$2^{2^{2n}} \leq D_{2n}^{2k}.$$

By (5.3) it follows that for every $n \in \omega$, it holds that $2^{2^{2n}} \leq 3^{2k\binom{2n}{n}}$. This means that for every $n \in \omega$,

$$2^{2n} \leq K\binom{2n}{n}, \tag{5.4}$$

where $K = 2k\log_2 3 > 0$. It is not hard to show that the previous claim is false. To this end we use an important formula in applied mathematics[9] as well as in probability, the *Stirling's formula* (see [AS92]). It says that

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\, n^n e^{-n}} = 1.$$

[7] $[x]$ is the greatest integer function (see p. 205).

[8] It trivially follows from the fact that all elements in $\mathcal{P}(\{X \subseteq n : |X| = \left[\frac{n}{2}\right]\})$ are antichains in $\langle \mathcal{P}(n), \subseteq \rangle$.

[9] The author thanks the applied mathematician Quim Puig for reminding him of this formula.

From the Stirling formula it easily follows that

$$\lim_{n\to\infty} \frac{\binom{2n}{n}}{2^{2n}\frac{1}{\sqrt{\pi n}}} = 1.$$

This last limit implies that

$$\lim_{n\to\infty} \frac{\binom{2n}{n}}{2^{2n}} = 0,$$

which is in contradiction with (5.4). This concludes our proof of the fact that $\varrho = \infty$.

The proof just given is not constructive. It also has the disadvantage of needing a very powerful machinery. Later we will remove these objections, thanks to the following theorem.

5.1.6. THEOREM. *Let φ be a $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula, and let $k \in \omega$. The following statements are equivalent:*

1. *φ is equivalent to a conjunction of k material implications of $\mathcal{L}^{SW}(\vartheta)$-formulas, i.e., there are sets $\{\nu_n : n < k\}$ and $\{\pi_n : n < k\}$ of $\mathcal{L}^{SW}(\vartheta)$-formulas such that*

$$\varphi \quad \equiv \quad \bigwedge\{\nu_n \supset \pi_n : n < k\}.$$

2. *There is no sequence $\langle\langle\mathfrak{A}_i, a_i\rangle : i < 2k+1\rangle$ of pointed τ_ϑ-structures such that:*
   - $\langle\mathfrak{A}_0, a_0\rangle \preceq \langle\mathfrak{A}_1, a_1\rangle \preceq \ldots \preceq \langle\mathfrak{A}_{2k-1}, a_{2k-1}\rangle \preceq \langle\mathfrak{A}_{2k}, a_{2k}\rangle$.
   - *If $i < 2k+1$ and i is even, then $\mathfrak{A}_i, a_i \not\Vdash \varphi$.*
   - *If $i < 2k+1$ and i is odd, then $\mathfrak{A}_i, a_i \Vdash \varphi$.*

3. *φ is equivalent to a conjunction of k material implications of $\mathcal{L}^{SW}(\vartheta)$-formulas with modal degree $\leq \deg(\varphi)$.*

Proof: For the case $k = 0$ it is clear that the three conditions are equivalent to saying that $\varphi \equiv \top$. Hence, we can assume that $k \geq 1$.

$(3 \Rightarrow 1)$: Trivial.

$(1 \Rightarrow 2)$: Suppose that $\varphi \equiv \bigwedge\{\nu_n \supset \pi_n : n < k\}$ where the ν's and the π's are $\mathcal{L}^{SW}(\vartheta)$-formulas. Let us assume that there is a sequence of pointed τ_ϑ-structures satisfying the conditions stated in the second item. We know that for every $r \leq k$ it holds that $\mathfrak{A}_{2r}, a_{2r} \not\Vdash \varphi$. Therefore, for every $r \leq k$ there is a $n_r < k$ such that $\mathfrak{A}_{2r}, a_{2r} \not\Vdash \nu_{n_r} \supset \pi_{n_r}$, i.e., $\mathfrak{A}_{2r}, a_{2r} \Vdash \nu_{n_r} \wedge \sim\pi_{n_r}$.

Claim: If $r, r' \leq k$ and $r \neq r'$ then $n_r \neq n_{r'}$.

Proof of Claim: Suppose that $r \neq r'$. By symmetry we can assume that $r < r'$. Then $\langle \mathfrak{A}_{2r}, a_{2r}\rangle \preceq \langle \mathfrak{A}_{2r+1}, a_{2r+1}\rangle \preceq \langle \mathfrak{A}_{2r'}, a_{2r'}\rangle$ because $2r < 2r+1 < 2r'$. Using this together with the fact that $\mathfrak{A}_{2r}, a_{2r} \Vdash \nu_{n_r}$ and $\mathfrak{A}_{2r'}, a_{2r'} \nVdash \pi_{n_{r'}}$ we conclude that $\mathfrak{A}_{2r+1}, a_{2r+1} \Vdash \nu_{n_r} \wedge \sim \pi_{n_{r'}}$, i.e., $\mathfrak{A}_{2r+1}, a_{2r+1} \nVdash \nu_{n_r} \supset \pi_{n_{r'}}$. Using the fact that $\mathfrak{A}_{2r+1}, a_{2r+1} \Vdash \varphi$ (because $2r+1$ is even) it follows that $n_r \neq n_{r'}$. ⊣

By the previous claim we have that the map $r \longmapsto n_r$ is an injective map from $k+1$ into k, which is absurd.

$(2 \Rightarrow 3)$: Let us assume that there is no sequence satisfying the three conditions stated in the second item. Restricting ϑ to modalities and propositions appearing in φ we can assume that ϑ is finite. Let $l \in \omega$ be the modal degree of φ.

Claim I: There is no sequence $\langle\langle \mathfrak{A}_i, a_i\rangle : i < 2k+1\rangle$ of pointed τ_ϑ-structures such that:

- $\langle \mathfrak{A}_0, a_0\rangle \preceq_l \langle \mathfrak{A}_1, a_1\rangle \preceq_l \ldots \preceq_l \langle \mathfrak{A}_{2k-1}, a_{2k-1}\rangle \preceq_l \langle \mathfrak{A}_{2k}, a_{2k}\rangle$.
- If $i < 2k+1$ and i is even, then $\mathfrak{A}_i, a_i \nVdash \varphi$.
- If $i < 2k+1$ and i is odd, then $\mathfrak{A}_i, a_i \Vdash \varphi$.

Proof of Claim: Suppose there is a sequence satisfying the conditions. For every $i < 2k+1$ we define $\langle \mathfrak{A'}_i, a'_i\rangle$ as the result of cutting at height l the tree $\mathfrak{u}nr(\mathfrak{A}_i, a_i)$. It is easy to show that $\langle \mathfrak{A'}_0, a'_0\rangle \preceq \langle \mathfrak{A'}_1, a'_1\rangle \preceq \ldots \preceq \langle \mathfrak{A'}_{2k-1}, a'_{2k-1}\rangle \preceq \langle \mathfrak{A'}_{2k}, a'_{2k}\rangle$, i.e., the sequence satisfies the first condition on the second item. Using that $\langle \mathfrak{A}_i, a_i\rangle \simeq_l \langle \mathfrak{A'}_i, a'_i\rangle$ it also follows that the sequence satisfies the other two conditions stated there, which is absurd. ⊣

We recall (see Remark 3.2.10) that for every pointed τ_ϑ-structure $\langle \mathfrak{A}, a\rangle$ there is a $\mathcal{L}^{SW}_\infty(\vartheta)$-formula $\pi_l^{\langle \mathfrak{A}, a\rangle}$ with modal degree $\leq l$ satisfying that for every pointed structure $\langle \mathfrak{B}, b\rangle$,

$$\langle \mathfrak{A}, a\rangle \preceq_l \langle \mathfrak{B}, b\rangle \quad \text{iff} \quad \mathfrak{B}, b \Vdash \pi_l^{\langle \mathfrak{A}, a\rangle}.$$

Using the fact that ϑ is finite it is clear that we can assume that $\pi_l^{\langle \mathfrak{A}, a\rangle}$ is in $\mathcal{L}^{SW}(\vartheta)$. For every $n < k$, we consider the class K_n of pointed τ_ϑ-structures $\langle \mathfrak{A}, a\rangle$ such that there is a sequence $\langle\langle \mathfrak{A}_i, a_i\rangle : i < 2n+1\rangle$ of pointed τ_ϑ-structures satisfying that (i) $\langle \mathfrak{A}_0, a_0\rangle \preceq \langle \mathfrak{A}_1, a_1\rangle \preceq \ldots \preceq \langle \mathfrak{A}_{2n-1}, a_{2n-1}\rangle \preceq \langle \mathfrak{A}_{2n}, a_{2n}\rangle$, (ii) if $i < 2n+1$ and i is even, then $\mathfrak{A}_i, a_i \nVdash \varphi$, (iii) if $i < 2n+1$ and i is odd, then $\mathfrak{A}_i, a_i \Vdash \varphi$, and (iv) $\langle \mathfrak{A}, a\rangle = \langle \mathfrak{A}_{2n}, a_{2n}\rangle$. Now for every $n < k$ we define the $\mathcal{L}^{SW}(\vartheta)$-formulas

$$\nu_n = \bigvee\{\pi_l^{\langle \mathfrak{A}, a\rangle} : \langle \mathfrak{A}, a\rangle \in \mathsf{K}_n\},$$

and

$$\pi_n = \bigvee\{\pi_l^{\langle \mathfrak{A}, a\rangle} : \mathfrak{A}, a \models \varphi \wedge \nu_n\}.$$

They are $\mathcal{L}^{SW}(\vartheta)$-formulas because there is a finite number of $\mathcal{L}^{SW}(\vartheta)$-formulas with modal degree $\leq l$. It is clear that for every $n < k$, $\deg(\nu_n) \leq l$ and $\deg(\pi_n) \leq l$. It is also clear by definition that for every pointed τ_ϑ-structure $\langle \mathfrak{A}, a\rangle$ and every $n < k$, it holds that

- $\mathfrak{A}, a \Vdash \nu_n$ iff there is a sequence $\langle\langle \mathfrak{A}_i, a_i\rangle : i < 2n{+}1\rangle$ of pointed τ_ϑ-structures satisfying that (i) $\langle \mathfrak{A}_0, a_0\rangle \preceq \langle \mathfrak{A}_1, a_1\rangle \preceq \ldots \preceq \langle \mathfrak{A}_{2n-1}, a_{2n-1}\rangle \preceq \langle \mathfrak{A}_{2n}, a_{2n}\rangle \preceq \langle \mathfrak{A}, a\rangle$, (ii) if $i < 2n+1$ and i is even, then $\mathfrak{A}_i, a_i \nVdash \varphi$, and (iii) if $i < 2n+1$ and i is odd, then $\mathfrak{A}_i, a_i \Vdash \varphi$.

- $\mathfrak{A}, a \Vdash \pi_n$ iff there is a sequence $\langle\langle \mathfrak{A}_i, a_i\rangle : i < 2n+2\rangle$ of pointed τ_ϑ-structures satisfying that (i) $\langle \mathfrak{A}_0, a_0\rangle \preceq \langle \mathfrak{A}_1, a_1\rangle \preceq \ldots \preceq \langle \mathfrak{A}_{2n-1}, a_{2n-1}\rangle \preceq \langle \mathfrak{A}_{2n}, a_{2n}\rangle \preceq \langle \mathfrak{A}_{2n+1}, a_{2n+1}\rangle \preceq \langle \mathfrak{A}, a\rangle$, (ii) if $i < 2n+2$ and i is even, then $\mathfrak{A}_i, a_i \nVdash \varphi$, and (iii) if $i < 2n+2$ and i is odd, then $\mathfrak{A}_i, a_i \Vdash \varphi$.

Claim II: $\varphi \equiv (\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_{k-1} \supset \pi_{k-1})$.

Proof of Claim: First of all, let us prove the easy part. We suppose that $\mathfrak{A}, a \Vdash \varphi$, $n < k$, and $\mathfrak{A}, a \Vdash \nu_n$. Then, it is obvious that $\mathfrak{A}, a \Vdash \pi_l^{\langle \mathfrak{A}, a\rangle} \wedge \varphi \wedge \nu_n$. Therefore, $\mathfrak{A}, a \Vdash \pi_n$.

For the converse we assume that $\mathfrak{A}, a \nVdash \varphi$. We seek $n < k$ such that $\mathfrak{A}, a \Vdash \nu_n \wedge \sim\pi_n$. By Claim I together with the fact that $\mathfrak{A}, a \nVdash \varphi$ we know that $\mathfrak{A}, a \nVdash \pi_{k-1}$. Using the fact that $\langle \mathfrak{A}, a\rangle \preceq \langle \mathfrak{A}, a\rangle$ and that $\mathfrak{A}, a \nVdash \varphi$ it is obvious that $\mathfrak{A}, a \Vdash \nu_0$. So, we can define $n := \max\{r < k : \mathfrak{A}, a \Vdash \nu_r\}$. If $n = k-1$, then we have finished our proof because $\mathfrak{A}, a \Vdash \nu_{k-1} \wedge \sim\pi_{k-1}$. Hence, let us assume that $n < k-1$. Then, $\mathfrak{A}, a \Vdash \nu_n$ while $\mathfrak{A}, a \nVdash \nu_{n+1}$. Using that $\mathfrak{A}, a \nVdash \nu_{n+1}$, that $\mathfrak{A}, a \nVdash \varphi$ and that $\langle \mathfrak{A}, a\rangle \preceq \langle \mathfrak{A}, a\rangle$ it easily follows that $\mathfrak{A}, a \nVdash \pi_n$. Therefore, $\mathfrak{A}, a \Vdash \nu_n \wedge \sim\pi_n$. ⊣

This concludes the proof. □

We emphasize that Theorem 5.1.6 implies that a modal formula φ is equivalent to a conjunction of k $(\in \omega)$ material implications of strict-weak formulas iff there is no $\preceq$-increasing[10] sequence of pointed structures with length $2k+1$ such that φ fails in the first pointed structure of the sequence and the sequence alternates the satisfiability of φ.

5.1.7. Corollary. *Let φ be a $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula. Then, $\rho(\varphi)$ is the minimum $k \in \omega$ such that there is no sequence $\langle\langle \mathfrak{A}_i, a_i\rangle : i < 2k{+}1\rangle$ of pointed τ_ϑ-structures such that:*

- $\langle \mathfrak{A}_0, a_0\rangle \preceq \langle \mathfrak{A}_1, a_1\rangle \preceq \ldots \preceq \langle \mathfrak{A}_{2k-1}, a_{2k-1}\rangle \preceq \langle \mathfrak{A}_{2k}, a_{2k}\rangle$.

- *If $i < 2k+1$ and i is even, then $\mathfrak{A}_i, a_i \nVdash \varphi$.*

[10] As the situation is symmetric we can also write $\preceq$-decreasing. The symmetry comes from the fact that φ fails in both the first and the last pointed structure in the sequence.

- *If $i < 2k+1$ and i is odd, then $\mathfrak{A}_i, a_i \Vdash \varphi$.*

Proof: It follows by Theorem 5.1.6. □

We notice that using the previous corollary what we stated in Theorem 5.1.2 is trivial, but the proof given there has the advantage of being constructive.

5.1.8. Corollary. *Let φ be a $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula. Suppose that there is an infinite sequence $\langle\langle\mathfrak{A}_n, a_n\rangle : n < \omega\rangle$ of pointed τ_ϑ-structures such that (i) if $n < \omega$ then $\langle\mathfrak{A}_n, a_n\rangle \preceq \langle\mathfrak{A}_{n+1}, a_{n+1}\rangle$, and (ii) if $n < \omega$ and n is even, then $\mathfrak{A}_n, a_n \Vdash \varphi$. Then, there is $n < \omega$ such that n is odd and $\mathfrak{A}_n, a_n \Vdash \varphi$.*

Proof: It follows from the fact that $\rho(\varphi) \in \omega$. □

Now we can use Corollary 5.1.7 to calculate the value of ρ for several modal formulas.

5.1.9. Lemma. *Let $k \in \omega$.*

1. $\rho\left(\bigwedge_{n<k}(p_{2n} \supset p_{2n+1})\right) = k$ *and* $\rho\left(\sim \bigwedge_{n<k}(p_{2n} \supset p_{2n+1})\right) = k+1$.
2. *If s is a pure strict modality, then* $\rho\left(\bigwedge_{n<k}([s]^{2n+1}\langle s\rangle\top \supset [s]^{2n+2}\langle s\rangle\top)\right) = k$ *and* $\rho\left(\sim \bigwedge_{n<k}([s]^{2n+1}\langle s\rangle\top \supset [s]^{2n+2}\langle s\rangle\top)\right) = k+1$.
3. *If w is a pure weak modality, then* $\rho\left(\bigwedge_{n<k}(\langle w\rangle^{2n+2}[w]\bot \supset \langle w\rangle^{2n+1}[w]\bot)\right) = k$ *and* $\rho\left(\sim \bigwedge_{n<k}(\langle w\rangle^{2n+2}[w]\bot \supset \langle w\rangle^{2n+1}[w]\bot)\right) = k+1$.

Proof: 1) Let φ be $\bigwedge_{n<k}(p_{2n} \supset p_{2n+1})$. Using the fact that φ is in standard form it is trivial that $\rho(\varphi) \leq k$. By Theorem 5.1.2 it follows that $\rho(\sim\varphi) \leq \rho(\varphi)+1 \leq k+1$. Thus, in order to obtain that $\rho(\varphi) = k$ and $\rho(\sim\varphi) = k+1$ we only need to prove that $k < \rho(\sim\varphi)$. By Corollary 5.1.7 it is enough to find a sequence $\langle\langle\mathfrak{A}_i, a_i\rangle : i < 2k+1\rangle$ of pointed τ_ϑ-structures satisfying that (i) $\langle\mathfrak{A}_0, a_0\rangle \preceq \langle\mathfrak{A}_1, a_1\rangle \preceq \ldots \preceq \langle\mathfrak{A}_{2k}, a_{2k}\rangle$, (ii) if $i < 2k+1$ and i is even, then $\mathfrak{A}_i, a_i \Vdash \varphi$, and (iii) if $i < 2k+1$ and i is odd, then $\mathfrak{A}_i, a_i \nVdash \varphi$. Let us show a sequence satisfying these properties. We take a point $\bullet$. For every $i < 2k+1$ we define $\mathfrak{A}_i$ as the structure such that its universe is $\{\bullet\}$, its accessibility relations are empty, and the propositions that holds in $\bullet$ are exactly $\{p_j : j < i\}$; and for every $i < 2k+1$ we take a_i as $\bullet$. It is simple to check that this sequence satisfies what we want.

2) It is clear that we can assume that s is the only modality in the SW-vocabulary ϑ. Hence, there is no ambiguity if we simply write $\Box$ and $\langle m\rangle$. Let φ be $\bigwedge_{n<k}(\Box^{2n+1}\Diamond\top \supset \Box^{2n+2}\Diamond\top)$. Reasoning as in the previous item it is enough to prove that $k < \rho(\sim\varphi)$. By Corollary 5.1.7 it is enough to exhibit a sequence $\langle\langle\mathfrak{A}^k_i, a^k_i\rangle : i < 2k+1\rangle$ of pointed τ_ϑ-structures such that (i) $\langle\mathfrak{A}^k_0, a^k_0\rangle \preceq_s \langle\mathfrak{A}^k_1, a^k_1\rangle \preceq_s \ldots \preceq_s \langle\mathfrak{A}^k_{2k}, a^k_{2k}\rangle$, (ii) if $i < 2k+1$ and i is even, then $\mathfrak{A}^k_i, a^k_i \Vdash \varphi$, and (iii) if $i < 2k+1$ and i is odd, then $\mathfrak{A}^k_i, a^k_i \nVdash \varphi$. Let us present a sequence satisfying

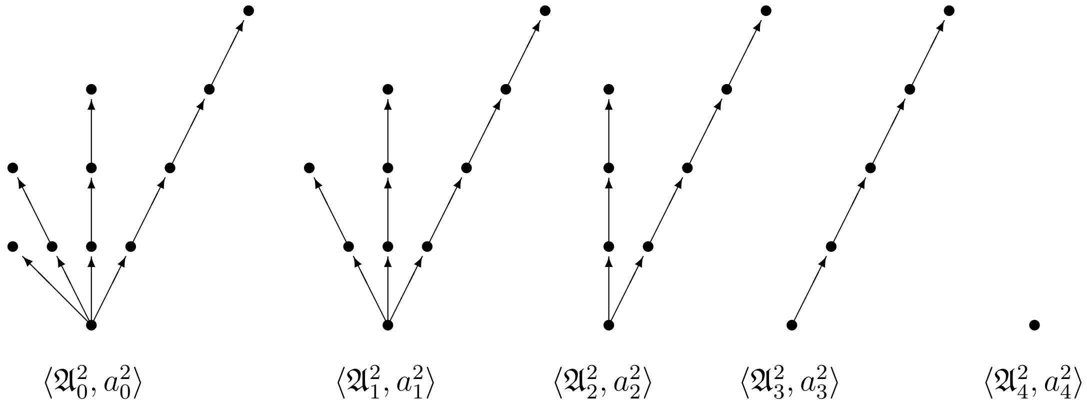


Figure 5.1: $\langle \mathfrak{A}_0^2, a_0^2 \rangle \preceq_s \langle \mathfrak{A}_1^2, a_1^2 \rangle \preceq_s \langle \mathfrak{A}_2^2, a_2^2 \rangle \preceq_s \langle \mathfrak{A}_3^2, a_3^2 \rangle \preceq_s \langle \mathfrak{A}_4^2, a_4^2 \rangle$

these conditions. For every $i < 2k + 1$ we define $\langle \mathfrak{A}_i^k, a_i^k \rangle$ as a tree such that (i) if $j \in \{i, \ldots, 2k-1\}$ then there is a single branch of length $j+1$, and (ii) if $j \notin \{i, \ldots, 2k-1\}$ then there is no branch of length $j+1$ (In Figure 5.1 we have depicted the structures for the case $k = 2$). It is clear that for every $j < 2k - 1$ it holds that

$$\mathfrak{A}_i^k, a_i^k \Vdash \Box^{j+1}\Diamond\top \quad \text{iff} \quad i > j.$$

From this it is easy to check that this sequence satisfies what we want.

3) It is a consequence of the previous item obtained by duality (we recall that $\rho^{\vartheta}(\varphi) = \rho^{\vartheta^d}(\sim\varphi^d)$). □

5.1.10. Remark. Let $k \in \omega$. We notice that there is no way to characterize the modal formulas that are equivalent to a conjunction of k material implications of strict-weak formulas by a preservation theorem (see Remark 3.5.12). For $k = 0$ it follows from the fact that $\bot$ is always preserved. For $k \geq 1$ it is a consequence of the fact that if conjunctions of k material implications of strict-weak formulas are preserved then all modal formulas are preserved (by the Standard Form Theorem).

In the next statement we use the greatest integer function $[\,] : \mathbb{R} \longrightarrow \mathbb{Z}$, which is defined as

$$[x] := \max\{n \in \mathbb{Z} : n \leq x\}.$$

We recall that

$$[x] \leq x < [x] + 1 \quad \text{and} \quad 2\left[\tfrac{x}{2}\right] \leq x \leq 2\left[\tfrac{x}{2}\right] + 1$$

for every $x \in \mathbb{R}$.

5.1.11. Theorem.

1. *If* Prop *is infinite, then* $\varrho = \infty$.
2. *If* SMod $\neq$ WMod*, then* $\varrho = \infty$.
3. *If* Prop *is finite and* SMod $=$ WMod*, then* $\varrho = \left[\frac{l}{2}\right] + 1$ *where* $l := |\mathsf{Prop}|$.

Proof: The first item is a consequence of Lemma 5.1.9(1), and the second item follows by Lemma 5.1.9(2)(3). Now let us prove the third item. Let $k = [\frac{l}{2}]$.
Claim I: Every $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula is equivalent to a formula $(\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_k \supset \pi_k)$ where $\nu_n, \pi_n \in \mathcal{L}^{SW}(\vartheta)$.
Proof of Claim: Using the fact that $l = |\mathsf{Prop}|$ and SMod $=$ WMod it is clear that there is no sequence $\langle\langle \mathfrak{A}_i, a_i\rangle : i < l + 2\rangle$ of pairwise non-bisimilar pointed τ_ϑ-structures such that $\langle \mathfrak{A}_0, a_0\rangle \preceq \langle \mathfrak{A}_1, a_1\rangle \preceq \ldots \preceq \langle \mathfrak{A}_l, a_l\rangle \preceq \langle \mathfrak{A}_{l+1}, a_{l+1}\rangle$. As $l \leq 2k + 1$ it follows that there is no sequence $\langle\langle \mathfrak{A}_i, a_i\rangle : i < 2k + 3\rangle$ of pairwise non-bisimilar pointed τ_ϑ-structures such that $\langle \mathfrak{A}_0, a_0\rangle \preceq \langle \mathfrak{A}_1, a_1\rangle \preceq \ldots \preceq \langle \mathfrak{A}_{2k+2}, a_{2k+2}\rangle \preceq \langle \mathfrak{A}_{2k+3}, a_{2k+3}\rangle$. By Theorem 5.1.6 we conclude what we want. ⊣
Claim II: There is a $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula that is not equivalent to any formula in standard form that uses k material implications.
Proof of Claim: Assume that Prop $= \{p_0, p_1, \ldots, p_{l-1}\}$. We are going to define a propositional formula φ using only Boolean connectives such that it satisfies our requirement. For every $i \leq l$ we define v_i as the Boolean valuation such that (i) if $j < i$ then $v_i(p_j) = 1$, and (ii) if $i \leq j$ then $v_i(p_j) = 0$. It is clear that $v_0 \leq v_1 \leq \ldots \leq v_{l-1} \leq v_l$. We define φ as the Boolean formula with propositions in $\{p_0, p_1, \ldots, p_{l-1}\}$ that is only satisfied by the Boolean valuations in $\{v_i : i \leq l \text{ and } i \text{ is odd}\}$. Using the fact that $2k \leq l$ we know that there is a sequence $\langle v_i : i < 2k + 1\rangle$ of Boolean valuations such that (i) $v_0 \leq v_1 \leq \ldots \leq v_{2k-1} \leq v_{2k}$, (ii) if $i < 2k + 1$ and i is even, then $v_i(\varphi) = 0$, and (iii) if $i < 2k + 1$ and i is odd, then $v_i(\varphi) = 1$. By Theorem 5.1.6 it easily follows that φ is not equivalent to any formula in standard form that uses k material implications. ⊣
The previous two claims imply that $\varrho = k + 1$. □

Finally we will treat the case of the map ρ_∞. In general we have not been able to obtain an equivalence as the one stated in Theorem 5.1.6: it only works in one direction. However, under certain requirements the equivalence holds.

5.1.12. Theorem. *Let φ be a $\mathcal{L}^{MOD}_\infty(\tau_\vartheta)$-formula with* $\deg(\varphi) \in \omega$*, and let $k \in \omega$. The following statements are equivalent:*

1. *φ is equivalent to a conjunction of k material implications of $\mathcal{L}^{SW}_\infty(\vartheta)$-formulas, i.e., there are sets $\{\nu_n : n < k\}$ and $\{\pi_n : n < k\}$ of $\mathcal{L}^{SW}_\infty(\vartheta)$-formulas such that*

$$\varphi \quad \equiv \quad \bigwedge\{\nu_n \supset \pi_n : n < k\}.$$

2. *There is no sequence $\langle\langle\mathfrak{A}_i, a_i\rangle : i < 2k+1\rangle$ of pointed τ_ϑ-structures such that:*

   - $\langle\mathfrak{A}_0, a_0\rangle \preceq \langle\mathfrak{A}_1, a_1\rangle \preceq \ldots \preceq \langle\mathfrak{A}_{2k-1}, a_{2k-1}\rangle \preceq \langle\mathfrak{A}_{2k}, a_{2k}\rangle$.
   - *If $i < 2k+1$ and i is even, then $\mathfrak{A}_i, a_i \not\Vdash \varphi$.*
   - *If $i < 2k+1$ and i is odd, then $\mathfrak{A}_i, a_i \Vdash \varphi$.*

3. *φ is equivalent to a conjunction of k material implications of $\mathcal{L}^{SW}(\vartheta)$-formulas with modal degree $\leq \deg(\varphi)$.*

Proof: The proof is analogous to the one given for Theorem 5.1.6. □

5.1.13. Theorem. *Let φ be a $\mathcal{L}^{MOD}_\infty(\tau_\vartheta)$-formula, and let $\kappa \in \mathbb{C}$ARD. Assume that there are sets $\{\nu_\alpha : \alpha < \kappa\}$ and $\{\pi_\alpha : \alpha < \kappa\}$ of $\mathcal{L}^{SW}_\infty(\vartheta)$-formulas such that*

$$\varphi \quad \equiv \quad \bigwedge\{\nu_\alpha \supset \pi_\alpha : \alpha < \kappa\}.$$

Then, there are no sequences $\langle\langle\mathfrak{A}_\alpha, a_\alpha\rangle : \alpha < \kappa^+\rangle$ and $\langle\langle\mathfrak{B}_\alpha, b_\alpha\rangle : \alpha < \kappa^+\rangle$ of pointed τ_ϑ-structures such that:

- *If $\alpha + 1 < \kappa^+$, then $\langle\mathfrak{A}_\alpha, a_\alpha\rangle \preceq \langle\mathfrak{B}_\alpha, b_\alpha\rangle \preceq \langle\mathfrak{A}_{\alpha+1}, a_{\alpha+1}\rangle$.*
- *If $\alpha < \kappa^+$, then $\mathfrak{A}_\alpha, a_\alpha \not\Vdash \varphi$.*
- *If $\alpha + 1 < \kappa^+$, then $\mathfrak{B}_\alpha, b_\alpha \Vdash \varphi$.*

Proof: The idea is that the existence of such sequences allows us to define an injective map $f : \kappa^+ \longrightarrow \kappa$ with the property that $\mathfrak{A}_\alpha, a_\alpha \Vdash \nu_{f(\alpha)} \wedge \sim\pi_{f(\alpha)}$ for every $\alpha < \kappa$. The proof is again analogous to the one given for Theorem 5.1.6. □

Under the hypotheses of the last theorem the proof given in Theorem 5.1.6 for the converse implication breaks down in at least two steps. First of all, we notice that we need $\deg(\varphi) \in \omega$ in order to prove Claim I[11]. Secondly, we need finiteness of κ to prove Claim II. Finiteness is necessary to guarantee the existence of $\max\{\alpha < \kappa : \mathfrak{A}, a \Vdash \nu_\alpha\}$.

5.1.14. Lemma. *Let $\kappa \in \mathbb{C}\text{ARD} \setminus \omega$, and let $\langle\langle\mathfrak{A}_\alpha, a_\alpha\rangle : \alpha < \kappa\rangle$ be a sequence of non-bisimilar pointed τ_ϑ-structures such that $\langle\mathfrak{A}_\alpha, a_\alpha\rangle \preceq \langle\mathfrak{A}_{\alpha+1}, a_{\alpha+1}\rangle$ for every $\alpha < \kappa$. Then, $\kappa \leq \varrho_\infty$.*

[11]At first glance in the infinitary case it seems that we do not need Claim I because we can work directly with $\preceq$ (replacing $\preceq_l$) using $\nu^{\langle\mathfrak{A},a\rangle}$ and $\pi^{\langle\mathfrak{A},a\rangle}$ (replacing $\nu_l^{\langle\mathfrak{A},a\rangle}$ and $\pi_l^{\langle\mathfrak{A},a\rangle}$). However, if we do this then we cannot be sure that the classes used in the definitions of ν_n and π_n are set-sized.

Proof: We define $f : \kappa \longrightarrow \kappa$ by the following recursion: (i) $f(0) := 0$, (ii) $f(\alpha + 1) := f(\alpha) + 2$, and (iii) $f(\alpha) := \alpha$ if α is a limit ordinal. For every $\alpha < \kappa$, we define $\langle \mathfrak{B}_\alpha, b_\alpha \rangle := \langle \mathfrak{A}_{f(\alpha)}, a_{f(\alpha)} \rangle$ and $\langle \mathfrak{C}_\alpha, c_\alpha \rangle := \langle \mathfrak{A}_{f(\alpha)+1}, a_{f(\alpha)+1} \rangle$. Now we define the $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula

$$\varphi := \bigvee \{ \phi^{\langle \mathfrak{C}_\alpha, c_\alpha \rangle} : \alpha < \kappa \},$$

where $\phi^{\langle \mathfrak{C}_\alpha, c_\alpha \rangle}$ is the $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula characterizing $\langle \mathfrak{C}_\alpha, c_\alpha \rangle$ up to bisimilarity (see Theorem 1.3.3). It is easy to check that (i) if $\alpha + 1 < \kappa$ then $\langle \mathfrak{B}_\alpha, b_\alpha \rangle \preceq \langle \mathfrak{C}_\alpha, c_\alpha \rangle \preceq \langle \mathfrak{B}_{\alpha+1}, b_{\alpha+1} \rangle$, (ii) if $\alpha < \kappa$ then $\mathfrak{B}_\alpha, b_\alpha \nVdash \varphi$, and (iii) if $\alpha < \kappa$ then $\mathfrak{C}_\alpha, c_\alpha \Vdash \varphi$. By Theorem 5.1.13 it follows that $\rho_\infty(\varphi) \nless \kappa$. Therefore, $\kappa \leq \varrho_\infty$. □

5.1.15. Theorem.

1. *If* SMod $\neq$ WMod*, then* $\varrho_\infty = \infty$.
2. *If* Prop *is infinite, then* $|\text{Prop}| \leq \varrho_\infty$.
3. *If* Prop *is finite, then* $\left[\frac{l}{2}\right] + 1 \leq \varrho_\infty$ *where* $l := |\text{Prop}|$.
4. *If* Mod $= \emptyset$*, then* $\varrho_\infty \neq \infty$.
5. *If* Prop *is finite and* Mod $= \emptyset$*, then* $\varrho_\infty = \left[\frac{l}{2}\right] + 1$ *where* $l := |\text{Prop}|$.

Proof: 1) As SMod $\neq$ WMod there is a pure strict modality or a pure weak modality. By duality it is enough to consider the case in which there is a pure weak modality w. It is clear that we can assume that $\vartheta = \langle \emptyset, \{w\}, \emptyset \rangle$. In this situation we saw in Section 3.9 that pointed structures correspond to non-well founded sets and that $\preceq_w$ corresponds to inclusion. For every $\alpha < \kappa$, we define $\langle \mathfrak{A}_\alpha, a_\alpha \rangle$ to be the canonical structure associated with the set α. Using this transfinite sequence together with Lemma 5.1.14 it easily follows that $\kappa \leq \varrho_\infty$ for every cardinal κ. Therefore, $\varrho_\infty = \infty$.

2) Assume that $\{p_\beta : \beta < \kappa\} =$ Prop where $\kappa = |\text{Prop}| \geq \omega$. For every $\alpha < \kappa$, we define the Boolean valuation v_α as the one such that (i) if $\beta < \alpha$ then $v_\alpha(p_\beta) = 1$, and (ii) if $\beta \geq \alpha$ then $v_\alpha(p_\beta) = 0$. By Lemma 5.1.14 it is easy to obtain that $\kappa \leq \varrho_\infty$.

3) The proof is the same one as for Claim II in Theorem 5.1.11 (replacing Theorem 5.1.6 with Theorem 5.1.12).

4) Assume that Mod is empty. Hence, the class of $\mathcal{L}^{MOD}(\tau_\vartheta)$-formulas is up to equivalence a set Φ. Using the fact that the supremum of a set of cardinals is a cardinal we conclude that $\varrho_\infty = \sup \rho_\infty[\Phi] \in \mathbb{C}\text{ARD}$. Hence, $\varrho_\infty \neq \infty$.

5) The proof is the same as for Theorem 5.1.11(3), but replacing Theorem 5.1.6 with Theorem 5.1.12. □

The reader can observe that in the above theorem we have not considered all possibilities. The cases not stated there are still open. Among them we single out the problem of whether SMod $=$ WMod implies that $\varrho_\infty \neq \infty$.

5.2 Embeddings based on standard form

In this section we show how to use the Standard Form Theorem to obtain embeddings from any of the most common normal modal logics into its strict-weak fragments. We will consider two types of embeddings, the box type and the diamond type. From now to the end of the section we assume that we have fixed a SW-vocabulary ϑ.

We start by introducing the box embeddings. For every $s \in \mathsf{SMod}$, we define the map $B_s : \mathcal{L}^{MOD}(\tau_\vartheta) \longrightarrow \mathcal{L}^{SW}(\vartheta)$ as any map satisfying that for every $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula φ, $B_s(\varphi)$ is a $\mathcal{L}^{SW}(\vartheta)$-formula equivalent to $[s]\varphi$. Using the map tr considered in the proof of the Standard Form Theorem, it is clear that we know how to define inductively a map B_s satisfying the previous property: if $\mathrm{tr}(\varphi) = (\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_{k-1} \supset \pi_{k-1})$ then take $B_s(\varphi) = (\nu_0 \rightarrow_s \pi_0) \wedge \ldots \wedge (\nu_{k-1} \rightarrow_s \pi_{k-1})$. We will call B_s the *box map associated with* s. We will say that a normal modal logic $\mathbf{\Lambda}$ is *invariant for* $[s]$ if it satisfies that for every $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula φ, it holds that

$$\varphi \in \mathbf{\Lambda} \quad \text{iff} \quad [s]\varphi \in \mathbf{\Lambda}.$$

By the generalization rule the implication to the right always holds. The other direction in general is not true. For instance, ***Verum***, ***KD*45** and $\boldsymbol{K} \oplus \mathsf{B}$ are examples of normal modal logics where it fails: as witnesses of this failure we can consider the formulas $\bot$, $\Box p_0 \supset p_0$ and $\Diamond\top$ respectively. However, most common examples of normal logics satisfy the above equivalence. In the case of a single modality it is clear that all extensions of $\boldsymbol{T}$ satisfy the equivalence. Now we give another condition that guarantees the equivalence.

We will say that a normal modal logic $\mathbf{\Lambda}$ is *closed under extensions by an* s*-predecessor* if it is characterized by a class C of frames such that

> for every $\mathfrak{F} \in \mathsf{C}$ and every state f in the frame $\mathfrak{F}$ there is a frame $\mathfrak{F}' \in \mathsf{C}$ with a state f' such that (i) f' is not an s-initial point (i.e., $(R_s^{\mathfrak{F}'})^{-1}[\{f'\}] \neq \emptyset$), and (ii) the subframe generated by f and the subframe generated by f' are isomorphic.

It is obvious that if the state f is not an s-initial point in $\mathfrak{F}$, then the previous condition trivially holds: take $\mathfrak{F}'$ as $\mathfrak{F}$, and f' as f. The reason for introducing this definition comes from the following lemma.

5.2.1. Lemma. *Let $\mathbf{\Lambda}$ be a normal modal logic that is closed under extensions by an s-predecessor. Then, $\mathbf{\Lambda}$ is invariant for $[s]$.*

Proof: Let us assume that $[s]\varphi \in \mathbf{\Lambda}$, and let C be the class of frames given by the closure under extensions by an s-predecessor condition. Let us prove that $\varphi \in \mathbf{\Lambda}$.

So, we consider a frame $\mathfrak{F} \in \mathsf{C}$, a valuation V in the frame, and a state $f \in F$; and let us prove that $\mathfrak{F}, V, f \Vdash \varphi$. By the closure under extensions by an s-predecessor we know that there is a frame $\mathfrak{F}' \in \mathsf{C}$ and a state $f' \in F'$ such that (i) f' is not an s-initial point (i.e., $(R_s^{\mathfrak{F}'})^{-1}[\{f'\}] \neq \emptyset$), and (ii) the subframe generated by f and the subframe generated by f' are isomorphic. By (i) let $f_0' \in F'$ be a state such that $\langle f_0', f'\rangle \in R_s$. Using the fact that $\mathfrak{F}' \in \mathsf{C}$ and that $[s]\varphi \in \mathbf{\Lambda}$ we know that $\mathfrak{F}' \Vdash [s]\varphi$. Hence, from (ii) together with the fact that $\langle f_0', f'\rangle \in R_s$ we conclude that $\mathfrak{F}, V, f \Vdash \varphi$. □

Using this lemma it is easy to check that all normal modal logics defined on page 32 except ***Verum*** and ***KD*45** are invariant with respect to the box of its unique modality. For instance, for the case of ***K*4.3** we can use the class of frames

$$\{\mathfrak{F} \in \mathrm{Fr}\,\boldsymbol{K4.3} : \mathfrak{F} \text{ is the generated subframe generated by a state}\}$$

to show that ***K*4.3** is closed under extensions by a predecessor. Other trivial examples of normal modal logics invariant with respect to the box of its unique modality are $\mathrm{Log}\langle\omega, <\rangle$, $\mathrm{Log}\langle\omega, >\rangle$, $\mathrm{Log}\langle\mathbb{Z}, <\rangle$, $\mathrm{Log}\langle\mathbb{Q}, <\rangle$ and $\mathrm{Log}\langle\mathbb{R}, <\rangle$.

5.2.2. Proposition. *Let* $\mathbf{\Lambda}$ *be a normal modal logic invariant for* $[s]$*. Then,* B_s *is an embedding of* $\mathbf{\Lambda}$ *into* $\mathbf{\Lambda} \cap \mathcal{L}^{SW}(\vartheta)$*, i.e., for every* $\mathcal{L}^{MOD}(\tau_\vartheta)$*-formula* φ *it holds that*

$$\varphi \in \mathbf{\Lambda} \quad \textit{iff} \quad B_s(\varphi) \in \mathbf{\Lambda} \cap \mathcal{L}^{SW}(\vartheta).$$

Proof: It is trivial because $B_s(\varphi) \equiv [s]\varphi$. □

Thus, if we consider the map B_s for the case $\mathcal{L}^s$ (then $B_s : \mathcal{L}^{mod} \longrightarrow \mathcal{L}^s$), we have that this map is at the same time (i) an embedding of $\boldsymbol{K}$ into $\boldsymbol{K} \cap \mathcal{L}^s$, (ii) an embedding of $\boldsymbol{T}$ into $\boldsymbol{T} \cap \mathcal{L}^s$, (iii) an embedding of ***S*4** into $\boldsymbol{S4} \cap \mathcal{L}^s$, (iv) an embedding of ***S*5** into $\boldsymbol{S5} \cap \mathcal{L}^s$, and so on for all normal modal logics defined on page 32 except ***Verum*** and ***KD*45**.[12]

5.2.3. Remark. Analogously to the above we could have introduced the definition of invariance for $[s]$ in a consequence relation $\vdash$ between modal formulas whenever:

$$\Phi \vdash \varphi \quad \text{iff} \quad \{[s]\phi : \phi \in \Phi\} \vdash [s]\varphi.$$

[12] The map B_s goes from $\mathcal{L}^{mod}$ into $\mathcal{L}^s$. If we consider the restriction of B_s to Boolean formulas it is not hard to prove that we obtain an embedding from classical propositional into intuitionistic propositional logic (the same map is also an embedding of classical propositional logic into any superintuitionistic propositional logic). Hint: If φ is a Boolean formula and $tr(\varphi) = (\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_{k-1} \supset \pi_{k-1})$ then we can check that the strict implication symbol $\rightarrow$ does not appear in the ν's and π's.

This notion implies that B_s is also an embedding from the point of view of $\vdash$. Examples of consequence relations satisfying this condition are $\vdash_{l\boldsymbol{K}}$ and $\vdash_{l\boldsymbol{D}}$. However, in general the local consequences associated with normal modal logics that are invariant for $[s]$ do not satisfy this kind of invariance. For instance, $\vdash_{l\boldsymbol{S4}}$ and $\vdash_{l\boldsymbol{T}}$ are examples of normal modal logics where it fails. The first part follows from the fact that $\Box p_0 \vdash_{l\boldsymbol{S4}} \Box\Box p_0$ while $p_0 \nvdash_{l\boldsymbol{S4}} \Box p_0$; and the second part is a consequence[13] of $\Box(p_0 \wedge (\Box p_0 \supset \Box p_1)) \vdash_{l\boldsymbol{T}} \Box p_1$ and $p_0 \wedge (\Box p_0 \supset \Box p_1) \nvdash_{l\boldsymbol{T}} p_1$.

Is there any polynomial $p(x)$ such that for every $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula φ, it holds that $\mathrm{leng}\,(B_s(\varphi)) \leq p(\mathrm{leng}\,(\varphi))$? Unfortunately this seems to be false (cf. Question 5.1.1). Therefore, the map B_s does not seem to be a polynomial reduction from $\boldsymbol{K}$ to $\boldsymbol{K} \cap \mathcal{L}^s$ (from $\boldsymbol{T}$ to $\boldsymbol{T} \cap \mathcal{L}^s$, ...). So we will need to develop another argument (see Section 5.3) to determine the complexity class of $\boldsymbol{K} \cap \mathcal{L}^s$ (of $\boldsymbol{T} \cap \mathcal{L}^s$, ...).

Now let us restrict ourselves to the case of $\mathcal{L}^s$. We recall that Gödel's Translation is an embedding of $\boldsymbol{IPL}$ into $\boldsymbol{S}4$ and also of $\boldsymbol{IPL}$ into $\boldsymbol{Grz}$. On the other hand, in the literature there is no embedding of $\boldsymbol{S4}$ into $\boldsymbol{IPL}$[14], and the same holds for an embedding of $\boldsymbol{Grz}$ into $\boldsymbol{IPL}$. Now we present an embedding satisfying this last property. It is defined as a composition of B_s with another map.

Let $\mathsf{Prop} = \{p_n : n \in \omega\}$ and $\mathsf{Prop}' = \{q_n^k : n, k \in \omega\} \cup \{r_n^k : n, k \in \omega\}$. Given $\varphi \in \mathcal{L}^{mod}$ with propositions in $p_0, \ldots, p_{n-1}$ we define $e'(\varphi)$ as the $\mathcal{L}^{mod}$-formula with propositions in Prop' that is obtained by replacing each proposition p_i with

$$(q_i^0 \smallsetminus r_i^0) \vee \ldots \vee (q_i^{k-1} \smallsetminus r_i^{k-1})$$

where $k = 2^{\mathrm{leng}\,(\varphi)}$. Hence, we have just defined a map $e' : \mathcal{L}^{mod} \longrightarrow \mathcal{L}^{mod}$.

5.2.4. PROPOSITION. *The map $B_s \circ e' : \mathcal{L}^{mod} \longrightarrow \mathcal{L}^s$ is an embedding of $\boldsymbol{Grz}$ into $\boldsymbol{IPL}$, i.e., for every $\mathcal{L}^{mod}$-formula φ it holds that*

$$\varphi \in \boldsymbol{Grz} \quad \textit{iff} \quad (B_s \circ e')(\varphi) \in \boldsymbol{IPL}.$$

Proof: First of all, let us assume that $\varphi \in \boldsymbol{Grz}$. Using the fact that e' is a modal substitution it is clear that $e'(\varphi) \in \boldsymbol{Grz}$. Therefore, $e'(\varphi)$ holds in all finite partial orders. Using the fact that $\Box e'(\varphi) \equiv B_s(e'(\varphi))$ it easily follows that $B_s(e'(\varphi))$ holds in all finite partial orders. In particular $B_s(e'(\varphi))$ holds in all finite partial orders under a persistent valuation, i.e., $B_s(e'(\varphi)) \in \boldsymbol{IPL}$.

For the converse, let us assume that $\varphi \notin \boldsymbol{Grz}$ and let k be $2^{\mathrm{leng}\,(\varphi)}$. As a consequence of $\varphi \notin \boldsymbol{Grz}$ it is well known that there is a finite partial order

[13] This counterexample was given to the author by Dick de Jongh in a personal communication.

[14] We notice that in [FF86, Theorem 1.9] a map e is presented satisfying that $\varphi \in \boldsymbol{S4}$ implies $e(\varphi) \in \boldsymbol{IPL}$.

$\langle A, R\rangle$ and a valuation V in it such that $\langle A, R, V\rangle \not\Vdash \varphi$ and $|A| \leq k$ (see the proof[15] of [CZ97, Theorem 5.51]). Suppose that the propositions of φ are among $p_0, \ldots, p_{n-1}$.

Claim: There is a persistent valuation V' such that for every $i < n$,

$$V(p_i) \;=\; (V'(q_i^0) \setminus V'(r_i^0)) \cup \ldots \cup (V'(q_i^{k-1}) \setminus V'(r_i^{k-1})).$$

Proof of Claim: As $|A| \leq k$ we can suppose that for every $i < n$, $V(p_i) = \{a_i^0, \ldots, a_i^{k-1}\}$ (perhaps some of these elements are repeated). For every $i < n$ and $j < k$ we define

$$V'(q_i^j) = R[\{a_i^j\}] \quad \text{and} \quad V'(r_i^j) = R[\{a_i^j\}] \setminus \{a_i^j\}.$$

Using that we are in a partial order it is clear that V' is persistent and that $V'(q_i^j) \setminus V'(r_i^j) = \{a_i^j\}$ for every $i < n$ and $j < k$. Hence, this satisfies what we want. ⊣

From the claim together with the fact that $\langle A, R, V\rangle \not\Vdash \varphi$ it is not hard to prove that $\langle A, R, V'\rangle \not\Vdash e'(\varphi)$. Therefore, $\langle A, R, V'\rangle \not\Vdash \Box e'(\varphi)$. Using the fact that $\Box e'(\varphi) \equiv B_s(e'(\varphi))$ we obtain that $\langle A, R, V'\rangle \not\Vdash B_s(e'(\varphi))$, from where it follows that $B_s(e'(\varphi)) \notin \boldsymbol{IPL}$. □

To conclude the section, let us now introduce the diamond embeddings. For every $w \in \mathsf{WMod}$, we define the map $D_w : \mathcal{L}^{MOD}(\tau_\vartheta) \longrightarrow \mathcal{L}^{SW}(\vartheta)$ as any map satisfying that for every $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula φ, $D_w(\varphi)$ is a $\mathcal{L}^{SW}(\vartheta)$-formula equivalent to $\langle w\rangle\varphi$. We know how to define inductively a map D_w satisfying the previous condition: if $\mathrm{tr}(\sim\varphi) = (\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_{k-1} \supset \pi_{k-1})$ then take $D_w(\varphi) = (\nu_0 \leftharpoondown_w \pi_0) \wedge \ldots \wedge (\nu_{k-1} \leftharpoondown_w \pi_{k-1})$. We will call D_w the *diamond map associated with w*. We will say that a normal modal logic $\mathbf{\Lambda}$ is *invariant for $\langle w\rangle$* if it satisfies that for every $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula φ, it holds that

$$\varphi \in \mathbf{\Lambda} \quad \text{iff} \quad \langle w\rangle\varphi \in \mathbf{\Lambda}.$$

A necessary condition to be invariant for $\langle w\rangle$ is that $\langle w\rangle\top \in \mathbf{\Lambda}$. Hence, $\boldsymbol{K}$ is not invariant with respect to its unique modality. An interesting problem that we will not consider is to analyze which of the common normal modal logics satisfy the above invariance.

5.2.5. Proposition. *Let $\mathbf{\Lambda}$ be a normal modal logic is invariant for $\langle w\rangle$. Then, D_w is an embedding of $\mathbf{\Lambda}$ into $\mathbf{\Lambda} \cap \mathcal{L}^{SW}(\vartheta)$, i.e., for every $\mathcal{L}^{MOD}(\tau_\vartheta)$-formula φ it holds that*

$$\varphi \in \mathbf{\Lambda} \quad \textit{iff} \quad D_w(\varphi) \in \mathbf{\Lambda} \cap \mathcal{L}^{SW}(\vartheta).$$

Proof: It is trivial because $D_w(\varphi) \equiv \langle w\rangle\varphi$. □

[15] We recall that the number of subformulas of φ is bounded by $\mathrm{leng}(\varphi)$.

5.3 Complexity of some logics in $\mathcal{L}^s$

Given a normal modal logic $\mathbf{\Lambda}$ in $\mathcal{L}^{mod}$, we can ask about the problem of deciding its strict-weak fragment $\mathbf{\Lambda} \cap \mathcal{L}^s$. The problem $\mathbf{\Lambda} \cap \mathcal{L}^s$ is the problem of deciding, given an arbitrary $\varphi \in \mathcal{L}^s$, whether $\varphi \in \mathbf{\Lambda}$. The section is devoted to showing that for most common normal modal logics $\mathbf{\Lambda}$ in $\mathcal{L}^{mod}$, it holds that its strict-weak fragment $\mathbf{\Lambda} \cap \mathcal{L}^s$ is in the same complexity classes. We will assume throughout this section that Prop is countable.

5.3.1. EXAMPLE. Of course the above claim does not seem true for all normal modal logics. Let us consider $\mathbf{\Lambda}$ as the normal modal logic $\boldsymbol{Verum}$. Using the fact that it is a conservative expansion of classical propositional logic it is obvious that $\boldsymbol{Verum}$ is $\mathsf{co\,NP}$-hard. On the other hand, $\boldsymbol{Verum} \cap \mathcal{L}^s$ is in P. This follows from the fact that the model checking for modal formulas (in particular for strict-weak formulas) is in P together with the fact that for every $\varphi \in \mathcal{L}^s$,

$$\varphi \in \boldsymbol{Verum}^s \quad \text{iff} \quad \mathfrak{S}_{IrF} \Vdash \varphi.$$

The last equivalence is easily verified using quasi bisimilarity $\preceq_s$.

Since for every $\varphi \in \mathcal{L}^s$,

$$\varphi \in \mathbf{\Lambda} \quad \text{iff} \quad \top \rhd \varphi \in \mathbf{\Lambda}^s,$$

it is obvious that if we know that $\mathbf{\Lambda}$ and $\mathbf{\Lambda} \cap \mathcal{L}^s$ are in the same complexity classes, then the strict-weak logic $\mathbf{\Lambda}^s$ is also in the same complexity classes. This explains why it is not necessary to consider the strict-weak logics $\mathbf{\Lambda}^s$.

Let us prove versions of Hemaspaandra's Theorem and Ladner's Theorem for the strict-weak fragments.

5.3.2. THEOREM. *(*Prop *is countable) Let* $\mathbf{\Lambda}$ *be a normal modal logic extending* $\boldsymbol{S4.3}$. *Then,* $\mathbf{\Lambda} \cap \mathcal{L}^s$ *is* $\mathsf{co\,NP}$*-complete.*

Proof: By Hemaspaandra's Theorem it is enough to show that $\mathbf{\Lambda} \cap \mathcal{L}^s$ is $\mathsf{co\,NP}$-hard. It is known that the equational logic of distributive lattices is $\mathsf{co\,NP}$-hard (see [Fre99]). Given two formulas φ_0 and φ_1 using only the connectives $\wedge$ and $\vee$, it is clear that

$$\begin{array}{cc}
\varphi_0 \approx \varphi_1 \text{ holds in all distributive lattices} & \text{iff} \\
\varphi_0 \approx \varphi_1 \text{ holds in all Boolean algebras} & \text{iff} \\
\varphi_0 \supset\subset \varphi_1 \in \boldsymbol{CPL} & \text{iff} \\
\varphi_0 \supset\subset \varphi_1 \in \mathbf{\Lambda} & \text{iff} \\
\Box(\varphi_0 \supset\subset \varphi_1) \in \mathbf{\Lambda} & \text{iff} \\
(\varphi_0 \rightarrow \varphi_1) \wedge (\varphi_1 \rightarrow \varphi_0) \in \mathbf{\Lambda} \cap \mathcal{L}^s. &
\end{array}$$

(i) q_0

(ii) $q_1 \vee q_2 \vee \ldots \vee q_n \vee q_{n+1} \vee p_0 \vee p_1 \vee \ldots \vee p_{n-1}$

(iii) $q_1 \rightarrow q_2$

(iv) $\gamma'_{1n} \wedge \Box^1 \gamma'_{2n} \wedge \ldots \wedge \Box^{n-1} \gamma'_{nn}$

(v) $\theta'_1 \wedge \Box^1 \theta'_2 \wedge \ldots \wedge \Box^{n-2} \theta'_{n-1}$ [16]

(vi) $\bigwedge_{i \in \{j<n : Q_j = \forall\}} \Box^{i-1} \delta'_i$ [17]

(vii) $\psi'_1 \wedge \Box^1(\psi'_1 \wedge \psi'_2) \wedge \Box^2(\psi'_1 \wedge \psi'_2 \wedge \psi'_3) \wedge \ldots \wedge \Box^{n-2}(\psi'_1 \wedge \psi'_2 \wedge \psi'_3 \wedge \ldots \wedge \psi'_{n-1})$

(viii) $\Box^{n-1}\Big(\big((q_n \wedge \nu_0) \rightarrow \pi_0\big) \wedge \big((q_n \wedge \nu_1) \rightarrow \pi_1\big) \wedge \ldots \wedge \big((q_n \wedge \nu_{k-1}) \rightarrow \pi_{k-1}\big)\Big)$

Figure 5.2: A list of $\mathcal{L}^s$-formulas

Hence, we have obtained a polynomial reduction that allow us to conclude that $\mathbf{\Lambda} \cap \mathcal{L}^s$ is co NP-hard. $\Box$

Now we consider our version of Ladner's Theorem. In the proof we will use the fact that it is known that if we restrict ourselves to quantified Boolean formulas β with matrix in conjunctive normal form, then the problem "$\beta \in \boldsymbol{QBF}$?" is also PSpace-complete [BDG95, Corollary 1.36].

5.3.3. THEOREM. *(*Prop *is countable) Let L be a set of $\mathcal{L}^s$-formulas such that either $\boldsymbol{K} \cap \mathcal{L}^s \subseteq L \subseteq \boldsymbol{FPL}$ or $\boldsymbol{K} \cap \mathcal{L}^s \subseteq L \subseteq \boldsymbol{IPL}$. Then, L is* PSpace*-hard.*

Proof: Let L be a set of $\mathcal{L}^s$-formulas such that either $\boldsymbol{K} \cap \mathcal{L}^s \subseteq L \subseteq \boldsymbol{FPL}$ or $\boldsymbol{K} \cap \mathcal{L}^s \subseteq L \subseteq \boldsymbol{IPL}$. The method of the proof is to show a polynomial time reduction from a known PSpace-complete problem to L^c. The PSpace-complete problem considered is the logic $\boldsymbol{QBF}$ of quantified Boolean formulas in conjunctive normal form.

Let β be a quantified Boolean formula

$$Q_0 p_0 Q_1 p_1 \ldots Q_{n-1} p_{n-1} \varphi(p_0, \ldots, p_{n-1})$$

such that $Q_0 = \exists$, $n \geq 2$ and φ is in conjunctive normal form. Therefore, φ is of the form $(\nu_0 \supset \pi_0) \wedge \ldots \wedge (\nu_{k-1} \supset \pi_{k-1})$ where the ν's are finite (maybe empty) conjunctions of propositions and the π's are finite (maybe empty) disjunctions

[16] The assumption $n \geq 2$ guarantees that it is a strict-weak formula.

[17] Since $Q_0 = \exists$ we know that $0 \notin \{j < n : Q_j = \forall\}$. Hence, $i - 1 \geq 0$ whenever $i \in \{j < n : Q_j = \forall\}$.

of propositions. Hence, the ν's and the π's are $\mathcal{L}^s$-formulas. We consider new propositions $q_0, \dots, q_{n+1}$, and we define the following $\mathcal{L}^s$-formulas:

$$\gamma'_{in} := \left(q_i \to (q_{i+1} \vee \bigwedge_{j\in\{0,\dots,i-1\}} q_j) \right) \wedge \bigwedge_{j\in\{i+2,\dots,n+1\}} ((q_i \wedge q_j) \to q_{i+1}) \wedge$$

$$\wedge \bigwedge_{j\in\{i,\dots,n-1\}} ((q_i \wedge p_j) \to q_{i+1}),$$

$$\theta'_i := (q_i \wedge (q_{i+1} \to q_{i+2})) \to q_{i+1},$$

$$\delta'_i := \left(\left(q_i \wedge ((q_{i+1} \wedge p_i) \to q_{i+2}) \right) \to q_{i+1} \right) \wedge$$

$$\wedge \left(\left(q_i \wedge \left(q_{i+1} \to (p_i \vee q_{i+2}) \right) \right) \to q_{i+1} \right),$$

and

$$\psi'_i := \left(q_i \wedge p_{i-1}) \to \Box p_{i-1} \right) \wedge \left(q_i \to (p_{i-1} \vee \neg p_{i-1}) \right).$$

It is easy to check that $\gamma'_{in} \equiv \Box\gamma_{in}$, $\theta'_i \equiv \Box\theta_i$, $\delta'_i \equiv \Box\delta_i$ and $\psi'_i \equiv \Box\psi_i$, where the formulas without $'$ are the ones considered in the proof of Ladner's Theorem (see p. 49). Now we consider the list of $\mathcal{L}^s$-formulas displayed in Figure 5.2. We notice that each one of them is either equivalent to the corresponding one in Figure 1.9 or equivalent to the classical negation $\sim$ of it. We define $f_0(\beta)$ as the conjunction of (i),(iv),(v),(vi),(vii),(viii); and we define $f_1(\beta)$ as the disjunction of (ii),(iii). Let $f(\beta)$ be the $\mathcal{L}^s$ formula $f_0(\beta) \to f_1(\beta)$. A moment of reflection shows that

$$\Diamond g(\beta) \equiv \sim f(\beta),$$

where $g(\beta)$ is the formula considered in the proof of Ladner's Theorem. Using this together with Lemma 1.5.3 it easily follows that the following statements are equivalent:

1. β is true.
2. $\sim f(\beta)$ is satisfiable, i.e., $f(\beta) \notin \boldsymbol{K}$.
3. $\sim f(\beta)$ is satisfiable in a finite strict order with a persistent valuation, i.e., $f(\beta) \notin \boldsymbol{FPL}$.
4. $\sim f(\beta)$ is satisfiable in a finite partial order with a persistent valuation, i.e., $f(\beta) \notin \boldsymbol{IPL}$.

Therefore,

$$\beta \in \boldsymbol{QBF} \quad \text{iff} \quad f(\beta) \notin \boldsymbol{K} \quad \text{iff} \quad f(\beta) \notin \boldsymbol{FPL} \quad \text{iff} \quad f(\beta) \notin \boldsymbol{IPL}.$$

Thus, using the fact that either $\boldsymbol{K} \cap \mathcal{L}^s \subseteq L \subseteq \boldsymbol{FPL}$ or $\boldsymbol{K} \cap \mathcal{L}^s \subseteq L \subseteq \boldsymbol{IPL}$ we obtain that

$$\beta \in \boldsymbol{QBF} \quad \text{iff} \quad f(\beta) \notin L.$$

Using the same method as in the proof of Ladner's theorem it is easily verified that there is a constant $K \in \omega$ such that for every quantified Boolean formula β with $Q_0 = \exists$, $n \geq 2$ and φ in conjunctive normal form,

$$\text{leng}\,(f(\beta)) \leq K\, \text{leng}\,(\beta)^2.$$

Therefore, it is clear that the map f is a polynomial time reduction that allows us to conclude that L is PSpace-hard. □

By the last theorem it trivially follows that $\boldsymbol{K}\cap\mathcal{L}^s$, $\boldsymbol{T}\cap\mathcal{L}^s$, $\boldsymbol{K4}\cap\mathcal{L}^s$, $\boldsymbol{S4}\cap\mathcal{L}^s$, $\boldsymbol{D}\cap\mathcal{L}^s$, $\boldsymbol{GL}\cap\mathcal{L}^s$ and $\boldsymbol{Grz}\cap\mathcal{L}^s$ are PSpace-complete. Another easy consequence of Theorem 5.3.3 is that the logics $\boldsymbol{BPL}$ and $\boldsymbol{FPL}$ are PSpace-complete: we know that they are in PSpace by the embedding into $\boldsymbol{K4}$ and $\boldsymbol{GL}$, respectively. As far as the author is aware this is the first time that the complexity classes for $\boldsymbol{BPL}$ and $\boldsymbol{FPL}$ have been calculated. We also obtain by this theorem that all subintuitionistic logics defined using Kripke structures are PSpace-hard, e.g., all logics considered in [Cor87, Došš93, Res94, Wan97, CJ01].

Up to now we have proved that most common normal modal logics are in the same complexity classes as their strict-weak fragment, but we have not given any polynomial time reduction from the normal modal logic into its strict-weak fragment. Now we present a reduction of this type that works well under certain (general) assumptions.

Let Prop be the set $\{p_n : n \in \omega\}$, and let Prop$'$ be the disjoint union $\{p_n : n \in \omega\} \cup \{q_n : n \in \omega\} \cup \{r_\varphi : \varphi \in \mathcal{L}^{mod}(\mathsf{Prop})\}$. Then, we simultaneously define two translations $^+$ and $^-$ from $\mathcal{L}^{mod}(\mathsf{Prop})$ into $\mathcal{L}^s(\mathsf{Prop}')$:

$$\begin{array}{lcllcl}
\bot^+ & := & \bot & \bot^- & := & \top \\
\top^+ & := & \top & \top^- & := & \bot \\
p_n^+ & := & p_n & p_n^- & := & q_n \\
(\sim\varphi)^+ & := & \varphi^- & (\sim\varphi)^- & := & \varphi^+ \\
(\varphi_0 \wedge \varphi_1)^+ & := & \varphi_0^+ \wedge \varphi_1^+ & (\varphi_0 \wedge \varphi_1)^- & := & \varphi_0^- \vee \varphi_1^- \\
(\Box\varphi)^+ & := & \Box\varphi^+ & (\Box\varphi)^- & := & r_\varphi.
\end{array}$$

By a straightforward (simultaneous) induction it is easily verified that for every $\mathcal{L}^{mod}(\mathsf{Prop})$-formula φ with propositions among $p_0, \ldots, p_{n-1}$ and with modal degree k, it holds that[18]

$$\left(\Box^{(k)}\Big(\bigwedge_{0\leq i<n}(p_i \supset\subset \sim q_i) \wedge \bigwedge_{\phi\in Sub(\varphi)}(r_\phi \supset\subset \sim\Box\phi^+)\Big)\right) \supset (\varphi \supset\subset \varphi^+) \in \boldsymbol{K} \tag{5.5}$$

[18] $Sub(\varphi)$ refers to the set of subformulas of φ.

and

$$\left(\Box^{(k)}\Big(\bigwedge_{0\le i<n}(p_i \supset\subset \sim q_i) \;\wedge \bigwedge_{\phi\in Sub(\varphi)}(r_\phi \supset\subset \sim \Box\phi^+)\Big)\right) \supset (\sim\varphi \supset\subset \varphi^-) \in \boldsymbol{K}. \quad (5.6)$$

It is clear that the box of the modal formula

$$\bigwedge_{0\le i<n}(p_i \supset\subset \sim q_i) \;\wedge \bigwedge_{\phi\in Sub(\varphi)}(r_\phi \supset\subset \sim \Box\phi^+)$$

is equivalent to the formula

$$\bigwedge_{0\le i\le n}\Big(\big((p_i\wedge q_i)\to\bot\big)\wedge\Box(p_i\vee q_i)\Big) \wedge \bigwedge_{\phi\in Sub(\varphi)}\Big(\big((r_\phi\wedge\Box\phi^+)\to\bot\big)\wedge\Box(r_\phi\vee\Box\phi^+)\Big).$$

Let us call this $\mathcal{L}^s(\mathsf{Prop}')$-formula $g(\varphi)$. We define $f(\varphi)$ as the $\mathcal{L}^s(\mathsf{Prop}')$-formula

$$\Box^{(k)}g(\varphi)\to\Box\varphi^+.$$

It is easy to obtain a polynomial $p(x)$ such that for every $\mathcal{L}^{mod}(\mathsf{Prop})$-formula φ,

$$\text{leng}\,(f(\varphi)) \le p(\text{leng}\,(\varphi)).$$

Now we prove the following theorem.

5.3.4. Theorem. *Let $\mathbf{\Lambda}$ be a normal modal logic such that is closed under extensions by a predecessor. For every modal formula φ, it holds that*

$$\varphi\in\mathbf{\Lambda} \quad \textit{iff} \quad f(\varphi)\in\mathbf{\Lambda}\cap\mathcal{L}^s.$$

Proof: $(\Rightarrow)$: Suppose that $\varphi\in\mathbf{\Lambda}$. By (5.5) we obtain that

$$\Box^{(k-1)}g(\varphi) \supset \varphi^+ \in \mathbf{\Lambda}.$$

From here it follows that $\Box^{(k)}g(\varphi)\to\Box\varphi^+\in\mathbf{\Lambda}$, i.e., $f(\varphi)\in\mathbf{\Lambda}$.

$(\Leftarrow)$: Let us assume that $f(\varphi)\in\mathbf{\Lambda}$, and let C be the class of frames given by the closure under extensions by an s-predecessor condition. Let $\mathfrak{F}\in\mathsf{C}$, $a\in F$, and V a valuation in Prop for $\mathfrak{F}$. We want to prove that $\mathfrak{F},V,a\Vdash\varphi$. Applying twice the property that C satisfies we can assume that there are states a_0 and a_1 such that $\langle a_0,a_1\rangle\in R^{\mathfrak{F}}$ and $\langle a_1,a\rangle\in R^{\mathfrak{F}}$. Now we extend the valuation V to a valuation V' in Prop' for $\mathfrak{F}$ according to the following conditions:

- for every $n\in\omega$, $V'(q_n):=\{x\in F: x\notin V(p_n)\}$,
- for every $\phi\in\mathcal{L}^{mod}(\mathsf{Prop})$, $V'(r_\phi):=\{x\in F:\mathfrak{F},V,x\nVdash\Box\phi\}$.

By a straightforward induction it is easily verified that for every $\phi \in \mathcal{L}^{mod}(\mathsf{Prop})$ and every $x \in F$,

$$\mathfrak{F}, V, x \Vdash \phi \quad \text{iff} \quad \mathfrak{F}, V', x \Vdash \phi \quad \text{iff} \quad \mathfrak{F}, V', x \Vdash \phi^{+} \quad \text{iff} \quad \mathfrak{F}, V', x \nVdash \phi^{-}. \tag{5.7}$$

By (5.7) and the definition of V' it is not hard to verify that $\mathfrak{F}, V' \Vdash g(\varphi)$. Using now that $\mathfrak{F}, V', a_0 \Vdash f(\varphi)$ (because $f(\varphi) \in \mathbf{\Lambda}$) and that $\mathfrak{F}, V', a_1 \Vdash \Box^{(k)} g(\varphi)$ we deduce that $\mathfrak{F}, V', a_1 \Vdash \Box\varphi^{+}$. Therefore, $\mathfrak{F}, V', a \Vdash \varphi^{+}$. By (5.7) we know that $\mathfrak{F}, V', a \Vdash \varphi$. As the propositions in φ are among Prop we conclude that $\mathfrak{F}, V, a \Vdash \varphi$. □

We have already seen that f is a polynomial time reduction for normal modal logics that are closed under extensions by a predecessor. We recall that after Lemma 5.2.1 we remarked that all normal modal logics defined on page 32 except ***Verum*** and ***KD*45** are closed under extensions by a predecessor.

5.4 Complexity of $\boldsymbol{K} \cap \mathcal{L}^s$ without propositions

5.4.1. Remark. Using the fact that $\bot \supset\subset \Box\bot \in \boldsymbol{D}$ and that $\top \supset\subset \Box\top \in \boldsymbol{K}$ it is clear that all extensions of $\boldsymbol{D}$ have the same modal formulas without propositions. It is easily verified that for every normal modal logic $\mathbf{\Lambda}$ extending $\boldsymbol{D}$, it holds that $\mathbf{\Lambda}$ without propositions is in P.

We devote the section to proving that $\boldsymbol{K} \cap \mathcal{L}^s$ without propositions is PSpace-complete. First of all, we prove that the normal modal logic $\boldsymbol{K}$ without propositions is PSpace-complete, which is an improvement for $\boldsymbol{K}$ of Halpern's Theorem. We prove this by presenting a polynomial time reduction from $\boldsymbol{K}$ with a countable number of propositions to $\boldsymbol{K}$ without propositions. We define $\mathsf{t}_1 : \mathcal{L}^{mod} \longrightarrow \mathcal{L}^{mod}$ according to the following clauses:

$$\begin{array}{lcl}
\mathsf{t}_1(\bot) & := & \bot \\
\mathsf{t}_1(\top) & := & \top \\
\mathsf{t}_1(p_n) & := & \Diamond(\Box\Diamond\top \;\wedge\; \Diamond^{n+2}\Box\bot) \\
\mathsf{t}_1(\sim\varphi) & := & \sim\mathsf{t}_1(\varphi) \\
\mathsf{t}_1(\varphi_0 \wedge \varphi_1) & := & \mathsf{t}_1(\varphi_0) \wedge \mathsf{t}_1(\varphi_1) \\
\mathsf{t}_1(\Box\varphi) & := & \Box(\Diamond\Box\bot \supset \mathsf{t}_1(\varphi)).
\end{array}$$

The map t_1 associates a modal formula $\mathsf{t}_1(\varphi)$ without propositions with a modal formula φ using propositions among $\{p_n : n \in \omega\}$.

5.4.2. Proposition. *For every $\mathcal{L}^{mod}$-formula φ, it holds that*

$$\varphi \textit{ is satisfiable} \quad \textit{iff} \quad \mathsf{t}_1(\varphi) \textit{ is satisfiable.}$$

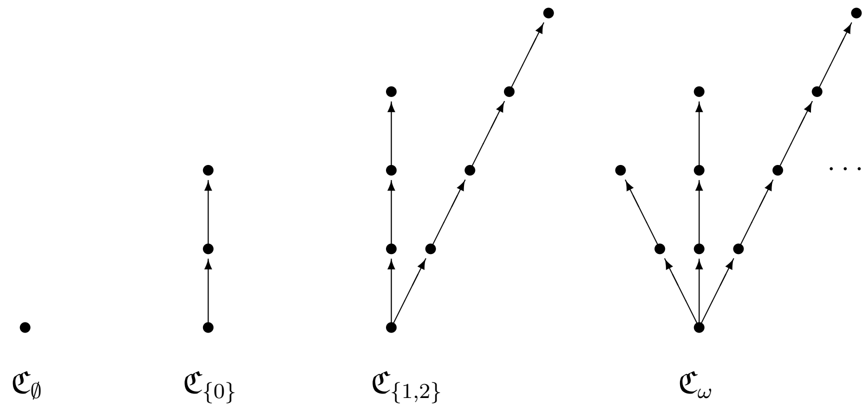


Figure 5.3: Some examples of $\mathfrak{C}_X$

Proof: $(\Leftarrow)$: Let us assume that $\mathtt{t}_1(\varphi)$ is satisfiable in a pointed structure $\langle \mathfrak{A}, a\rangle$. Replacing it with its unravelling we can assume that $\mathfrak{A}$ is a tree with root a. In particular there is no state $x \in A$ such that $\langle x, a\rangle \in R^{\mathfrak{A}}$. Now we define $\mathfrak{B}$ as the structure such that (i) its universe is $\{a\} \cup \{x \in A : \mathfrak{A}, x \Vdash \Diamond\Box\bot\}$, (ii) its accessibility relation is $R^{\mathfrak{A}} \cap B$, and (iii) its valuation is the map $p_n \longmapsto \{b \in B : \mathfrak{A}, b \Vdash \Diamond(\Box\Diamond\top \wedge \Diamond^{n+2}\Box\bot)\}$.

Claim: For every $\mathcal{L}^{mod}$-formula ϕ and every $b \in B$ it holds that

$$\mathfrak{B}, b \Vdash \phi \quad \text{iff} \quad \mathfrak{A}, b \Vdash \mathtt{t}_1(\phi).$$

Proof of Claim: The proof is by induction. We show only the case of $\Box$. So, let us assume by inductive hypothesis that for every $b \in B$, it holds that $\mathfrak{B}, b \Vdash \phi$ iff $\mathfrak{A}, b \Vdash \mathtt{t}_1(\phi)$. We want to prove that for every $b \in B$,

$$\mathfrak{B}, b \Vdash \Box\phi \quad \text{iff} \quad \mathfrak{A}, b \Vdash \Box(\Diamond\Box\bot \supset \mathtt{t}_1(\phi)).$$

Fist of all we check the implication to the right. Thus, we suppose that $b \in B$, $\mathfrak{B}, b \Vdash \Box\phi$, $\langle b, x\rangle \in R^{\mathfrak{A}}$ and $\mathfrak{A}, x \Vdash \Diamond\Box\bot$. By definition it is clear that $x \in B$ and $\langle b, x\rangle \in R^{\mathfrak{B}}$. Therefore, using that $\mathfrak{B}, b \Vdash \Box\phi$ we obtain that $\mathfrak{B}, x \Vdash \phi$. By the inductive hypothesis we conclude that $\mathfrak{A}, x \Vdash \mathtt{t}_1(\phi)$.

Let us now prove the converse implication. We assume that $b \in B$, $\mathfrak{A}, b \Vdash \Box(\Diamond\Box\bot \supset \mathtt{t}_1(\phi))$, $x \in B$ and $\langle b, x\rangle \in R^{\mathfrak{B}}$. Then, $\langle b, x\rangle \in R^{\mathfrak{A}}$, from where, using the fact that $\mathfrak{A}, b \Vdash \Box(\Diamond\Box\bot \supset \mathtt{t}_1(\phi))$ we deduce that $\mathfrak{A}, x \Vdash \Diamond\Box\bot \supset \mathtt{t}_1(\phi)$. As $\langle b, x\rangle \in R^{\mathfrak{A}}$ we know that $x \in B \setminus \{a\}$, which implies that $\mathfrak{A}, x \Vdash \Diamond\Box\bot$. By the last two sentences we obtain that $\mathfrak{A}, x \Vdash \mathtt{t}_1(\phi)$. Finally, using the inductive hypothesis we conclude that $\mathfrak{B}, x \Vdash \phi$. $\dashv$

By the claim we obtain that $\mathfrak{B}, a \Vdash \varphi$. Therefore φ is satisfiable.

$(\Rightarrow)$: Let us assume that $\mathfrak{A}, a \Vdash \varphi$. Now for every $X \subseteq \omega$, we take $\langle \mathfrak{C}_X, c_X\rangle$ as any structure $\mathfrak{C}_X$ that is a tree with root c_X, and that satisfies that (i) there is no branch of length 1, (ii) if $n \in X$ then there is a single branch of length $n+2$, and (iii) if $n \notin X$ then there is no branch of length $n+2$ (In Figure 5.3 we have depicted some examples). Let $\mathfrak{C}$ be the disjoint union of $\{\mathfrak{C}_X : X \subseteq \omega\}$. It is clear that we can assume that $\mathfrak{A}$ and $\mathfrak{C}$ are disjoint. We take a new point $\bullet$. Now we take $\mathfrak{B}$ as a structure such that (i) its universe is $A \cup C \cup \{\bullet\}$, (ii) its accessibility relation is

$$R^{\mathfrak{A}} \cup R^{\mathfrak{C}} \cup \{\langle x, \bullet\rangle : x \in A\} \cup \{\langle x, c_X\rangle : x \in A \text{ and } X = \{n \in \omega : \mathfrak{A}, x \Vdash p_n\}\},$$

and (iii) its valuation is arbitrary. By our construction we have the following properties:

(1): For every $x \in A$, it holds that $\mathfrak{B}, x \Vdash \Diamond\Box\bot$ (because $\mathfrak{B}, \bullet \Vdash \Box\bot$).

(2): For every $x \in A$ and $n \in \omega$, it holds that $\mathfrak{B}, x \nVdash \Box\Diamond\top \wedge \Diamond^{n+2}\Box\bot$ (because $\mathfrak{B}, \bullet \nVdash \Diamond\bot$).

(3): For every $n \in \omega$, it holds that $\mathfrak{B}, \bullet \nVdash \Box\Diamond\top \wedge \Diamond^{n+2}\Box\bot$.

(4): For every $X \subseteq \omega$, it holds that $\mathfrak{B}, c_X \nVdash \Diamond\Box\bot$ (because in $\mathfrak{C}_X$ there is no branch of length 1); and $\mathfrak{B}, \bullet \nVdash \Diamond\Box\bot$.

(5): For every $X \subseteq \omega$ and $n \in \omega$, it holds that $n \in X$ iff $\mathfrak{B}, c_X \Vdash \Diamond^{n+2}\Box\bot$ (because in $\mathfrak{C}_X$ there is a branch of length $n+2$ iff $n \in X$), and also iff $\mathfrak{B}, c_X \Vdash \Box\Diamond\top \wedge \Diamond^{n+2}\Box\bot$.

Using these properties it is not hard to prove the following claim.

CLAIM: For every $\mathcal{L}^{mod}$-formula ϕ and every $x \in A$ it holds that

$$\mathfrak{A}, x \Vdash \phi \quad \text{iff} \quad \mathfrak{B}, x \Vdash \mathsf{t}_1(\phi).$$

Proof of Claim: The proof is by induction. Conditions (2), (3) and (5) take care of propositions. Now we prove the case of $\Box$. So, let us assume by inductive hypothesis that for every $x \in A$, it holds that $\mathfrak{A}, x \Vdash \phi$ iff $\mathfrak{B}, x \Vdash \mathsf{t}_1(\phi)$. We want to prove that for every $x \in A$,

$$\mathfrak{A}, x \Vdash \Box\phi \quad \text{iff} \quad \mathfrak{B}, x \Vdash \Box(\Diamond\Box\bot \supset \mathsf{t}_1(\phi)).$$

Let us start by checking the implication to the right. So, we suppose that $x \in A$, $\mathfrak{A}, x \Vdash \Box\phi$, $\langle x, y\rangle \in R^{\mathfrak{B}}$ and $\mathfrak{B}, y \Vdash \Diamond\Box\bot$. By (4) it follows that $y \in A$. Hence, $\langle x, y\rangle \in R^{\mathfrak{A}}$, from where we deduce that $\mathfrak{A}, y \Vdash \phi$. Finally, using the inductive hypothesis we conclude that $\mathfrak{B}, y \Vdash \mathsf{t}_1(\phi)$.

Now we prove the implication to the left. We assume that $x \in A$, $\mathfrak{B}, x \Vdash \Box(\Diamond\Box\bot \supset \mathsf{t}_1(\phi))$, $y \in A$ and $\langle x, y\rangle \in R^{\mathfrak{A}}$. Then, $\langle x, y\rangle \in R^{\mathfrak{B}}$, from where, using the fact that $\mathfrak{B}, x \Vdash \Box(\Diamond\Box\bot \supset \mathsf{t}_1(\phi))$ we deduce that $\mathfrak{B}, y \Vdash \Diamond\Box\bot \supset \mathsf{t}_1(\phi)$. By (1) we know that $\mathfrak{B}, y \Vdash \mathsf{t}_1(\phi)$. By the inductive hypothesis it follows that $\mathfrak{A}, y \Vdash \phi$. $\dashv$

By the claim we obtain that $\mathfrak{B}, a \Vdash \mathsf{t}_1(\varphi)$. Hence $\mathsf{t}_1(\varphi)$ is satisfiable. □

5.4.3. THEOREM. *For every $\mathcal{L}^{mod}$-formula φ, it holds that*

$$\varphi \in \boldsymbol{K} \quad \textit{iff} \quad \mathsf{t}_1(\varphi) \in \boldsymbol{K}.$$

Proof: It follows from Proposition 5.4.2 together with the fact that t_1 is a homomorphism with respect to material negation (i.e., $\mathsf{t}_1(\sim\varphi) = \sim\mathsf{t}_1(\varphi)$). □

5.4.4. REMARK. It is not hard to prove that the map t_1 also behaves well with respect to the consequence relation $\vdash_{l\boldsymbol{K}}$, i.e.,

$$\Phi \vdash_{l\boldsymbol{K}} \varphi \quad \text{iff} \quad \mathsf{t}_1[\Phi] \vdash_{l\boldsymbol{K}} \mathsf{t}_1(\varphi).$$

The reason is that in our definition of t_1 we have a homomorphism with respect to all Boolean connectives[19]. As a particular case of the previous equivalence we have that Φ is satisfiable iff $\mathsf{t}_1[\Phi]$ is satisfiable.

5.4.5. THEOREM. *(*Prop *is empty)* $\boldsymbol{K}$ *is* PSpace-*complete.*

Proof: It is enough to prove that $\boldsymbol{K}$ without propositions is PSpace-hard. By Theorem 5.4.3 the map t_1 is a reduction from $\boldsymbol{K}$ with a countable number of propositions to $\boldsymbol{K}$ without propositions, and we recall that $\boldsymbol{K}$ with a countable number of propositions is PSpace-hard by Ladner's Theorem. Unfortunately, t_1 is not a polynomial time reduction since there is no polynomial $p(x)$ such that for every $\mathcal{L}^{mod}$-formula φ, $\text{leng}\,(\mathsf{t}_1(\varphi)) \leq p(\text{leng}\,(\varphi))$. The non-existence of this polynomial is a consequence of the fact that the set

$$\{\text{leng}\,(\mathsf{t}_1(p_n)) : n \in \omega\}$$

does not have a maximum.

However, it is possible to introduce some slight changes for t_1 to obtain a polynomial reduction. Let φ be a $\mathcal{L}^{mod}$-formula. We rewrite the formula φ replacing their propositions by propositions in $\{p_n : n < \text{leng}\,(\varphi)\}$. If φ' is the result of this rewriting, we define $\mathsf{t}'_1(\varphi)$ as the formula $\mathsf{t}_1(\varphi')$. Now it is easily verified that t'_1 is a polynomial reduction from $\boldsymbol{K}$ with a countable number of propositions to $\boldsymbol{K}$ without propositions: indeed, we can take a polynomial of degree 2. □

Now it is time to prove that $\boldsymbol{K} \cap \mathcal{L}^s$ without propositions is PSpace-complete. The proof that we present can be considered an easy consequence of the properties that we have verified for t_1. We define $\mathsf{t}_2 : \mathcal{L}^s \longrightarrow \mathcal{L}^s$ according to the following clauses:

[19] We notice that t_1 is not a homomorphism with respect to $\Box$ because $\mathsf{t}_1(\Box\varphi) \neq \Box\mathsf{t}_1(\varphi)$. If we replace the last clause in the definition of t_1 in such a way that we have a homomorphism condition with respect to $\Box$ (indeed, this implies that t_1 is a modal substitution) then Proposition 5.4.2 becomes false: as a counterexample we can take the formula $\Diamond(\sim p_0 \wedge p_1 \wedge \Box \sim p_0)$.

$$\begin{array}{lll}
\mathsf{t}_2(\bot) & := & \bot \\
\mathsf{t}_2(\top) & := & \top \\
\mathsf{t}_2(p_n) & := & \neg\neg\top \to \Box^{n+1}\neg\neg\top \\
\mathsf{t}_2(\varphi_0 \wedge \varphi_1) & := & \mathsf{t}_2(\varphi_0) \wedge \mathsf{t}_2(\varphi_1) \\
\mathsf{t}_2(\varphi_0 \vee \varphi_1) & := & \mathsf{t}_2(\varphi_0) \vee \mathsf{t}_2(\varphi_1) \\
\mathsf{t}_2(\varphi_0 \to \varphi_1) & := & \mathsf{t}_2(\varphi_0) \to (\mathsf{t}_2(\varphi_1) \vee \neg\neg\top).
\end{array}$$

The map t_2 associates a $\mathcal{L}^s$-formula $\mathsf{t}_2(\varphi)$ without propositions with a $\mathcal{L}^s$-formula φ using propositions among $\{p_n : n \in \omega\}$. The connection of t_2 with t_1 is given by the following lemma.

5.4.6. LEMMA. *Let e be the modal substitution such that $e(p_n) = \sim p_n$ for every $n \in \omega$. For every $\mathcal{L}^s$-formula φ, it holds that*

$$\mathsf{t}_2(\varphi) \equiv e(\mathsf{t}_1(\varphi)).$$

Proof: It is straightforward by induction. The case of propositions works well thanks to the following chain of equivalences:

$$\begin{array}{ll}
\neg\neg\top \to \Box^{n+1}\neg\neg\top & \equiv \\
\Box(\Box\Diamond\top \supset \Box^{n+1}\Box\Diamond\top) & \equiv \\
\sim\Diamond(\Box\Diamond\top \wedge \Diamond^{n+2}\Box\bot). &
\end{array}$$

For the strict implication $\to$ we must use the fact that

$$\begin{array}{ll}
\Box\big(\Diamond\Box\bot \supset (\varphi_0 \supset \varphi_1)\big) & \equiv \\
\Box\big((\varphi_0 \wedge \Diamond\Box\bot) \supset \varphi_1\big) & \equiv \\
\Box\big(\varphi_0 \supset (\varphi_1 \vee \Box\Diamond\top)\big) & \equiv \\
\Box\big(\varphi_0 \supset (\varphi_1 \vee \neg\neg\top)\big) & \equiv \\
\varphi_0 \to (\varphi_1 \vee \neg\neg\top). &
\end{array}$$

The rest of cases are trivial. □

5.4.7. PROPOSITION. *For every $\mathcal{L}^s$-formula φ, it holds that*

$$\varphi \textit{ is satisfiable} \quad \textit{iff} \quad \mathsf{t}_2(\varphi) \textit{ is satisfiable.}$$

Proof: It is a consequence of Lemma 5.4.6 and Proposition 5.4.2. □

5.4.8. THEOREM. *For every $\mathcal{L}^s$-formula φ, it holds that*

$$\varphi \in \boldsymbol{K} \quad \textit{iff} \quad \mathsf{t}_2(\varphi) \in \boldsymbol{K}.$$

Proof: It is a consequence of Lemma 5.4.6 and Theorem 5.4.3. □

5.4.9. THEOREM. *(*Prop *is empty) $\boldsymbol{K} \cap \mathcal{L}^s$ is* PSpace*-complete.*

Proof: By Theorem 5.3.3 we know that $\boldsymbol{K} \cap \mathcal{L}^s$ with a countable number of propositions is PSpace-complete. Reasoning as in the proof of Theorem 5.4.5, but replacing t_1 with t_2, it is easily verified that $\boldsymbol{K} \cap \mathcal{L}^s$ without propositions is PSpace-complete. □

5.5 Open questions

Our aim in this last section is to state several open questions that need further research.

- First of all, we recall that Question 5.1.1 remains open. Another open question related to standard form representations is the characterization of ϱ_∞ for all SW-vocabularies: we recall that Theorem 5.1.15 does not give a concrete answer for all cases.

- In Theorem 5.3.3 we have generalized Ladner's Theorem to the $\mathcal{L}^s$-fragment. In this fragment, besides strict implication $\rightarrow$, there are also the symbols $\bot, \top, \wedge$ and $\vee$. Is it possible to extend Ladner's Theorem if we restrict the language to the fragment that only considers strict implication? Hardly anything is known of this fragment. One of the few known results is that the strict implication fragment of $\boldsymbol{IPL}$ is indeed PSpace-complete. This was proved in the paper [Sta79] in which the PSpace-completeness of $\boldsymbol{IPL}$ was first obtained. To approach this problem perhaps it may be interesting to analyze the goal-directed calculi developed by Gabbay and Olivetti [GO00, GO02].

- We have proved that Ladner's Theorem holds for the $\mathcal{L}^s$-fragment, but what happens if we restrict the number of propositions to 1? Of course we cannot give the same version as in Theorem 5.3.3 because it is known that intuitionistic propositional logic with one variable is decidable in linear time [Nis60], but it may be possible to give the same version when there are only two propositions. We stress that it is unknown the complexity of $\boldsymbol{IPL}$ with a finite number $k \geq 2$ of propositions (see [CZ97, p. 564]). A related question is whether Halpern's Theorem holds when we restrict to the $\mathcal{L}^s$-fragment.

- We have seen that the complexity of $\boldsymbol{K} \cap \mathcal{L}^s$ without propositions is PSpace-complete. Now we should determine what happens for the rest of fragments of normal modal logics when we restrict them to no propositions. By Lemma 5.4.1 the answer to this question is trivial for all extensions of $\boldsymbol{D}$, but it remains open for normal modal logics like $\boldsymbol{K4}$ and $\boldsymbol{GL}$.

- Another unexplored question is the presence of two or more strict implications (corresponding to different accessibility relations). We recall that in the modal case it is known that the situation is essentially the same one than with a single modality, with some famous exceptions like $\boldsymbol{K45}_2$, $\boldsymbol{KD45}_2$, $\boldsymbol{S5}_2$, ... (see [HM92]).

- Finally we ask for what happens when there is a single weak difference. At first glance it seems that these problems can be efficiently solved, but this is because they are not really interesting. For instance, now we sketch an argument showing that $\boldsymbol{K} \cap \mathcal{L}^w$ is in P (indeed, it is linear).

Given a normal modal logic $\mathbf{\Lambda}$, we define the *satisfiability problem for* $\mathbf{\Lambda} \cap \mathcal{L}^s$ as the problem of deciding, given a formula $\varphi \in \mathcal{L}^s$, whether φ is satisfiable in a model validating $\mathbf{\Lambda}$. Since for every $\mathcal{L}^w$-formula φ, it holds that

$$\varphi \in \mathbf{\Lambda} \cap \mathcal{L}^w \quad \text{iff} \quad \varphi^d \text{ is not satisfiable in a model validating } \mathbf{\Lambda},$$

it follows that the complexity class of $\mathbf{\Lambda} \cap \mathcal{L}^w$ coincides with the complexity class of the satisfiability problem for $\mathbf{\Lambda} \cap \mathcal{L}^s$. Then, using the fact that every satisfiable $\mathcal{L}^s$-formula is satisfiable in the model $\mathfrak{S}_{IrT}$ (see p. 73) we conclude that $\boldsymbol{K} \cap \mathcal{L}^w$ is in P.

Appendix A

Agradecimientos

Mucha gente ha contribuido de una forma u otra a esta tesis doctoral.

En primer lugar, quiero expresar mi agradecimiento a mi director Ramon Jansana. Me dejó libertad para que me adentrara por aquellos caminos que quisiera recorrer, y supo guiarme por ellos. Finalmente, sus rápidas lecturas de los borradores permitieron concluir la redacción en un breve plazo de tiempo. Gracias Ramon.

Igualmente quiero agradecer sus enseñanzas a todos los profesores del doctorado *Lògica i Fonaments de la Matemàtica* del *Departament de Lògica, Historia i Filosofia de la Ciència (UB)*; especialmente a los de la facultad de Matemáticas, con los que me he visto la cara a diario durante más de cuatro años. A todos ellos les agradezco su presencia. A Ventura por su exquisito trato. A Josep Maria por su inicial supervisión y por encaminarme hacia el tema final de la tesis. A Toni por esas entretenidas conversaciones a primera hora de la mañana. A Joan por sus consejos. A Josep por esas enérgicas charlas que sacan a uno de la rutina. A Juan Carlos por prestarse a todas las preguntas con que le asaltaba.

También quiero recordar a los compañeros de doctorado con los que he compartido buenos momentos, entre los cuales se encuentran Jorge Castro, Ángel García-Cerdanya, Andrés Stisman, Pere Pardo, Saverio Trioni, Joan Pons, Alessandra Palmigiano, Roberto Arpaia, José Gil, Carles Noguera, Neus Castells y Rodrigo Peláez. Entre todos ellos quiero destacar a mi compañero de promoción Ángel, con el cual he pasado multitud de horas discutiendo sobre lógica y otros temas.

Mi más sincera gratitud a mis amigos Joan Pons, Quim Puig i Lluís Quer, con los cuales he compartido momentos de distensión tanto en la facultad de Matemáticas como fuera de ella.

En el pasado tuve la oportunidad de realizar visitas científicas al King's College (Londres) y al ILLC (Amsterdam) gracias a las invitaciones de Misha Zakharyaschev e Yde Venema. A ambos les expreso mi agradecimiento. Especialmente interesante para el desarrollo de esta tesis fue la estancia en Amsterdam, donde tuve oportunidad de adentrarme en el terreno computacional. A más a más de Misha e Yde, quiero agradecer a Nick Bezhanishvili, Ulle Endriss, Clemens Kupke Darko Sarenac y Stefan Schlobach por haber contribuido a que estas estancias fueran tan agradables.

Mi agradecimiento también va para aquellos que me han urgido a redactar esta tesis. El principal responsable de esto es Hiroakira Ono, a quien le agradezco su confianza. A banda de él, también es de justicia mencionar a Francesc Esteva, Josep Maria Font, Ramon Jansana y Yde Venema.

Por último quiero mencionar a todos esos amigos con los que he contado estos años. Los integrantes del Mogauto (la banda del fútbol); amigos como Sergio, Iván, Pedro, Javi y Frank; amigos del viaje de fin de carrera y de las tortuosas travesías pre-pirinaicas; excompañeros como Oscar, Xavi, Maite, Manuel y Ana; Josep Maria Mondelo por esas agradables conversaciones en el campus de la UAB; etc. Y por supuesto mis padres Félix y M. Carmen, y mi hermana Laura.

No quiero acabar sin expresar un agradecimiento general a todos los lectores; siempre es sorprendente que alguien tenga interés por lo que he escrito.

Barcelona, julio de 2004

Bibliography

[Acz88] P. Aczel. *Non-Well-Founded Sets*. Number 14 in CSLI Lecture Notes. CSLI Publications, Stanford, CA, 1988.

[Aie02] M. Aiello. *Spatial Reasoning: Theory and Practice*. Ph. D. Dissertation DS-2002-02, Institute for Logic, Language and Computation, University of Amsterdam, 2002.

[ANvB98] H. Andréka, I. Németi, and J. F. A. K. van Benthem. Modal Logics and Bounded Fragments of Predicate Logic. *Journal of Philosophical Logic*, 27(3):217–274, 1998.

[AR98] M. Ardeshir and W. Ruitenburg. Basic propositional calculus I. *Mathematical Logic Quarterly*, 44:317–343, 1998.

[Ard95] M. Ardeshir. *Aspects of Basic Logic*. Ph. D. Dissertation, Marquette University, 1995.

[AS92] M. Abramowitz and I. A. Stegun, editors. *Handbook of mathematical functions*. Dover Publications Inc., New York, 1992. Reprint of the 1972 edition.

[Bac59] J. W. Backus. The syntax and Semantics of the Proposed International Algebraic Language of the Zürich ACM-GRAMM conference. In *Proceedings of the International Conference on Information Processing, UNESCO*, pages 125–132, 1959.

[Bal98] A. Baltag. *A Structural Theory of Sets*. Ph. D. Dissertation, Indiana University, 1998.

[Bar73] J. Barwise. Back and forth through infinitary logic. In *Studies in Model Theory*, volume 8 of *MAA Studies in Mathematics*, pages 5–34. 1973.

[Bar93] J. Barwise, editor. *Handbook of Mathematical Logic*, volume 90 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, Amsterdam, 1993. Reprint of the 1977 edition.

[BD84] M. Božić and K. Došen. Models for normal intuitionistic modal logics. *Studia Logica*, 43:217–245, 1984.

[BDG90] J. L. Balcázar, J. Díaz, and J. Gabarró. *Structural complexity. II*, volume 22 of *EATCS Monographs on Theoretical Computer Science*. Springer-Verlag, Berlin, 1990.

[BDG95] J. L. Balcázar, J. Díaz, and J. Gabarró. *Structural complexity. I*. Texts in Theoretical Computer Science. An EATCS Series. Springer-Verlag, Berlin, second edition, 1995.

[BdRV01] P. Blackburn, M. de Rijke, and Y. Venema. *Modal logic*. Number 53 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, Cambridge, 2001.

[BE87] J. Barwise and J. Etchemendy. *The Liar: an essay on truth and circularity*. Oxford University Press, New York, 1987.

[BFGL04] F. Bou, J. M. Font, and J. L. García Lapresta. On weakening the Deduction Theorem and strengthening Modus Ponens. *Mathematical Logic Quarterly*, 50(3):303–324, 2004.

[BGG97] E. Börger, E. Grädel, and Y. Gurevich. *The Classical Decision Problem*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1997.

[Blo78] W. J. Blok. On the degree of incompleteness of modal logics. *Bulletin of the Section of Logic*, 7:167–175, 1978.

[BM91] J. Barwise and L. Moss. Hypersets. *The Mathematical Intelligencer*, pages 31–41, 1991.

[BM97] J. Barwise and L. Moss. *Vicious Circles*. CSLI Publications, Stanford, CA, 1997.

[BM98] J. Barwise and L. Moss. Modal correspondence for models. *Journal of Philosophical Logic*, 27:275–294, 1998.

[Boo93] G. Boolos. *The Logic of Provability*. Cambridge University Press, 1993.

[Bou01] F. Bou. Implicación estricta y lógicas subintuicionistas. Trabajo de investigación (doctorado), Universitat de Barcelona, 2001.

[Bou02] F. Bou. Hilbert-style calculi for $\vdash_{\mathbf{BPL}}$ and $\vdash_{\mathbf{FPL}}$. In *1st North American Summer School in Logic, Language and Information. Student Session*, pages 20–30. Stanford University, 2002.

[Bou03] F. Bou. Quasi-bisimulation: a new paradigm for the strict implication language. In Álvarez, E. and Bosch, R. and Villamil, L., editor, *Volume of abstracts. 12th International Congress of Logic, Methodology and Philosophy of Science (Oviedo, August 7–13, 2003)*, pages 82–83. University of Oviedo, 2003.

[Bož84] M. Božić. Positive logic with double negation. *Publications de l'Institut Mathématique de Beograd*, 35:21–31, 1984.

[BS84] R. A. Bull and K. Segerberg. Basic modal logic. In D. M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume II, pages 1–88. Reidel, Dordrecht, 1984.

[Bur84] J. P. Burgess. Basic tense logic. In D. M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume II, pages 89–133. Reidel, Dordrecht, 1984.

[Car46] R. Carnap. Modalities and quantification. *The Journal of Symbolic Logic*, 11:33–64, 1946.

[CCF82] J. M. V. Castilho, M. A. Casanova, and A. L. Furtado. A temporal framework for database specification. In *Proc. 8th Int. Conf. on Very Large Data Bases*, pages 280–291, 1982.

[Cha73] C. C. Chang. What's so special about saturated models? In *Studies in Model Theory*, volume 8 of *MAA Studies in Mathematics*, pages 59–95. 1973.

[Che80] B.F. Chellas. *Modal Logic: an introduction.* Cambridge University Press, Cambridge, Cambridge, 1980.

[Chu40] R. Church. Numerical analysis of certain free distributive structures. *Duke Math. J.*, 6:732–734, 1940.

[Chu65] R. Church. Enumeration by rank of the elements of the free distributive lattice with seven generators. *Notices of the AMS*, 12:724, 1965.

[CJ00] S. Celani and R. Jansana. Bounded distributive lattices with strict implication. Manuscript, to appear, 2000.

[CJ01] S. Celani and R. Jansana. A closer look at some subintuitionistic logics. *Notre Dame Journal of Formal Logic*, 42(4):225–255, 2001.

[CJ02] S. Celani and R. Jansana. Teoría de modelos para la lógica subintuicionista. Manuscript, 2002.

[CK90] C. C. Chang and H. J. Keisler. *Model Theory*, volume 73 of *Studies in Logic*. North-Holland, Amsterdam, 3rd. edition, 1990.

[Coo71] S. A. Cook. The complexity of theorem-proving procedures. In *ACM, Proc. 3rd ann. ACM Sympos. Theory Computing, Shaker Heights, Ohio 1971, 151-158*. 1971.

[Cop02] B. J. Copeland. The genesis of possible worlds semantics. *Journal of Philosophical Logic*, 31:99–137, 2002.

[Cor87] G. Corsi. Weak logics with strict implication. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 33:389–406, 1987.

[Cra53] W. Craig. On axiomatizability within a system. *The Journal of Symbolic Logic*, 18:30–32, 1953.

[Cra57] W. Craig. Three uses of the Herbrand-Gentzen theorem in relating model theory and proof theory. *The Journal of Symbolic Logic*, 22:269–285, 1957.

[Cra63] W. Craig. An extension of the Craig-Lyndon interpolation theorem. *The Journal of Symbolic Logic*, 28(3):201–216, 1963.

[CZ97] A. Chagrov and M. Zakharyaschev. *Modal Logic*, volume 35 of *Oxford Logic Guides*. Oxford University Press, 1997.

[Cze01] J. Czelakowski. *Protoalgebraic logics*, volume 10 of *Trends in Logic—Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 2001.

[D'A98] G. D'Agostino. *Modal Logic and non-well-founded Set Theory: translation, bisimulation, interpolation*. Ph. D. Dissertation, Institute for Logic, Language and Computation, 1998.

[Ded97] R. Dedekind. Ueber Zerlegungen von Zahlen durch ihre grössten gemeinsamen Teiler. *Festschrift Hoch. Braunschweig u. ges. Werke, II*, pages 103–148, 1897.

[Dev91] K. Devlin. *The joy of sets. Fundamentals of Contemporary Set Theory*. Undergraduate Texts in Mathematics. Springer-Verlag, second edition, 1991.

[DG79] B. Dreben and W. D. Goldfarb. *The decision problem.* Addison-Wesley Publishing Co., Reading, Mass., 1979. Solvable classes of quantificational formulas, Advanced Book Program.

[DH98] G. D'Agostino and M. Hollenberg. Uniform interpolation, automata and the modal μ-calculus. In *Advances in modal logic, Vol. 1 (Berlin, 1996)*, volume 87 of *CSLI Lecture Notes*, pages 73–84. CSLI Publ., Stanford, CA, 1998.

[DL59] M. Dummett and E. J. Lemmon. Modal logics between S4 and S5. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 3:250–264, 1959.

[DLNS96] F. Donnini, M. Lenzerini, D. Nardi, and A. Schaerf. Reasoning in description logics. In G. Brewka, editor, *Principles of Knowledge Representation*, pages 193–238. CSLI Publications, 1996.

[Doe] K. Doets. Introduction to Ehrenfeucht's Game. Manuscript. September 31th 1999. Amsterdam.

[Doš85] K. Došen. Models for stronger normal intuitionistic modal logics. *Studia Logica*, 44:39–70, 1985.

[Doš93] K. Došen. Modal translations in K and D. In M. de Rijke, editor, *Diamond and Defaults*, pages 103–127. Kluwer Academic Publishers, 1993.

[dR93] M. de Rijke. *Extending Modal Logic*. Ph. D. Dissertation, Institute for Logic, Language and Computation, 1993.

[dR95] M. de Rijke. A Lindström theorem for modal logic. In A. Ponse, M. de Rijke, and Y. Venema, editors, *Modal Logic and Process Algebra. A Bisimulation Perspective*, volume 53 of *CSLI Lecture Notes*, pages 217–230. CSLI Publications, Stanford, 1995.

[dRS01] M. de Rijke and H. Sturm. Global definability in basic modal logic. In H. Wansing, editor, *Essays on Non-Classical Logic*, volume 1 of *Advances in Logic*, pages 111–135. World Scientific Publishers, 2001.

[dRV95] M. de Rijke and Y. Venema. Sahlqvist's Theorem for Boolean Algebras with Operators with an Application to Cylindric Algebras. *Studia Logica*, 54:61–78, 1995.

[DSH93] K. Došen and P. Schroeder-Heister, editors. *Substructural Logics*, volume 2 of *Studies in Logic and Computation*. Oxford University Press, 1993.

[Dum00] M. Dummett. *Elements of intuitionism*. Oxford Logic Guides. Oxford University Press, 2ª edition, 2000.

[Dyc92] R. Dyckhoff. Contraction-free sequent calculi for intuitionistic logic. *The Journal of Symbolic Logic*, 57:795–807, 1992.

[EFT94] H.-D. Ebbinghaus, J. Flum, and W. Thomas. *Mathematical Logic.* Undergraduate Texts in Mathematics. Springer, second edition, 1994.

[EH76] G. Epstein and A. Horn. Logics which are characterized by subresiduated lattices. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 22:199–210, 1976.

[Ehr61] A. Ehrenfeucht. An application of games to the completeness problem for formalized theories. *Fundamenta Mathematicae*, 49:128–141, 1961.

[Eme90] E. A. Emerson. Temporal and modal logic. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B, pages 995–1072. Elsevier, 1990.

[End77] H. B. Enderton. *Elements of set theory.* Academic Press, New York, 1977.

[End00] H. B. Enderton. *A mathematical introduction to logic.* Academic Press, New York, 2nd edition, 2000. 1st edition in 1972.

[Fag94] R. Fagin. A quantitative analysis of modal logic. *The Journal of Symbolic Logic*, 59(1):209–252, March 1994.

[Fel86] W. Felscher. Dialogues as a foundation for intuitionistic logic. In D. M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic, vol. III : Alternatives to classical logic*, pages 341–372. Reidel, Dordrecht, 1986.

[FF86] R. C. Flagg and H. Friedman. Epistemic and intuitionistic formal systems. *Annals of Pure and Applied Logic*, 32(1):53–60, 1986.

[FH83] M. Forti and F. Honsell. Set theory with free construction principles. *Annali Scuola Normale Superiore—Pisa Classe di Scienza*, 10:493–522, 1983. Serie IV.

[FHMV95] R. Fagin, J. Y. Halpern, Y. Moses, and M. Y. Vardi. *Reasoning about Knowledge.* MIT Press, Cambridge, Mass., 1995.

[Fin74a] K. Fine. An Incomplete Logic Containing S4. *Theoria*, 40:23–29, 1974.

[Fin74b] K. Fine. Logics containing $K4$, Part I. *The Journal of Symbolic Logic*, 34:31–42, 1974.

[Fin85] K. Fine. Logics containing $K4$, Part II. *The Journal of Symbolic Logic*, 50:619–651, 1985.

[Fit93] M. Fitting. Basic modal logic. In D. M. Gabbay, C. J. Hooger, and J. A. Robinson, editors, *Handbook of Logic in Artificial Intelligence and Logic Programming*, volume 1, pages 368–448. Oxford University Press, 1993.

[Fit02] M. Fitting. Bisimulations and Boolean Vectors. volume 4 of *Advances in Modal Logic*, pages 1–29. World Scientific Publishers, 2002.

[FJ96] J. M. Font and R. Jansana. *A general algebraic semantics for sentential logics*, volume 7 of *Lecture Notes in Logic*. Springer-Verlag, 1996. 135 pp. Presently distributed by the Association for Symbolic Logic.

[FJP99] J. M. Font, R. Jansana, and D. Pigozzi. Fully adequate Gentzen systems and closure properties of the class of full models. Manuscript, 1999.

[Fon84] J. M. Font. Implication and deduction in some intuitionistic modal logics. *Reports on Mathematical Logic*, 17:27–38, 1984.

[Fon86] J. M. Font. Modality and possibility in some intuitionistic modal logics. *Notre Dame Journal of Formal Logic*, 27:533–546, 1986.

[Fre99] R. Freese. Algorithms in finite, finitely presented and free lattices. Preprint, July 1999.

[FS77] G. Fischer-Servi. On modal logics with an intuitionistic base. *Studia Logica*, 36:141–149, 1977.

[FS84] G. Fischer-Servi. Axiomatizations for some intuitionistic modal logics. *Rend. Sem. Mat. Univ. Politec. Torino*, 42:179–194, 1984.

[Gab81] D. M. Gabbay. Expressive functional completeness in tense logic (preliminary report). In U. Mönnich, editor, *Aspects of Philosophical Logic: Some Logical Forays into Central Notions of Linguistics and Philosophy*, pages 91–117. Reidel, Dordrecht, 1981.

[Gar84] J. W. Garson. Quantification in modal logic. In D. M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic, Volume II: Extensions of Classical Logic*, volume 165 of *Synthese Library*, chapter II.5, pages 249–307. D. Reidel Publishing Co., Dordrecht, 1984.

[Gen35] G. Gentzen. Untersuchungen über das logische Schliessen. *Mathematische Zeitschrift*, 39:176–210, 405–431, 1935. English traslation in *The Collected Papers of Gerhard Gentzen*, Szabo (ed.), North-Holland Public Co., 1969, 68–131.

[Ger97] J. Gerbrandy. Bisimulation and Bounded Bisimulation. Research Report LP-1997-05, Institute for Logic, Language and Computation, University of Amsterdam, 1997.

[Ger99] J. Gerbrandy. *Bisimulations on Planet Kripke*. Ph. D. Dissertation, Institute for Logic, Language and Computation, 1999.

[Ghi95] S. Ghilardi. An algebraic theory of normal forms. *Annals of Pure and Applied Logic*, 71(3):189–245, 1995.

[Ghi99] S. Ghilardi. Unification in intuitionistic logic. *The Journal of Symbolic Logic*, 64(2):859–880, 1999.

[GHR94] D. M. Gabbay, I. Hodkinson, and M. Reynolds. *Temporal Logic: Mathematical Foundations and Computational Aspects, Vol. 1*, volume 28 of *Oxford Logic Guides*. Clarendon Press, Oxford, 1994.

[GHV03] R. Goldblatt, I. Hodkinson, and Y. Venema. Erdös graphs resolve Fine's canonicity problem. Research Report PP-2003-26, Institute for Logic, Language and Computation, University of Amsterdam, Amsterdam, 2003.

[Gil54] E. N. Gilbert. Lattice theoretic properties of frontal switching functions. *J. Math. Physics*, 33:57–67, 1954.

[GJ79] M. R. Garey and D. S. Johnson. *Computers and Intractability. A Guide to the Theory of NP-Completeness*. W. H. Freeman and Co., 1979.

[GKV97] E. Grädel, P. G. Kolaitis, and M. Y. Vardi. On the decision problem for two-variable first-order logic. *The Bulletin of Symbolic Logic*, 3(1):53–69, 1997.

[GKWZ03] D. M. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-dimensional modal logics: theory and applications*, volume 148 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, Amsterdam, 2003.

[GM97] S. Ghilardi and G. Meloni. Constructive canonicity in non-classical logics. *Annals of Pure and Applied Logic*, 86(1):1–32, 1997.

[GO00] D. M. Gabbay and N. Olivetti. *Goal-Directed Proof Theory*, volume 21 of *Applied Logic Series*. Kluwer Academic Publishers, Dordrecht, 2000.

[GO02] D. M. Gabbay and N. Olivetti. Goal-oriented deductions. In D. M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic, second edition*, volume 9, pages 199–285. Kluwer, Dordrecht, 2002.

[Göd33] K. Gödel. Eine Interpretation des intuitionistischen Aussagenkalküls. 1933. English traslation, with an introduction note by A. S. Troelstra, in *Kurt Gödel, Collected Works, Volume 1*, Solomon Feferman et. al. (eds.), Oxford University Press, 1986, 296–303.

[Gol75] R. Goldblatt. First-order definability in modal logic. *The Journal of Symbolic Logic*, 40:35–40, 1975.

[Gol82] R. Goldblatt. In *Axiomatising the Logic of Computer Programming*, volume 130 of *Lecture Notes in Computer Science*. Springer-Verlag, 1982.

[Gol89] R. Goldblatt. Varieties of Complex Algebras. *Annals of Pure and Applied Logic*, 44(3):173–242, 1989.

[Gol92] R. Goldblatt. *Logics of Time and Computation*. Number 7 in CSLI Lecture Notes. CSLI Publications, Stanford, CA, second edition, 1992. Distributed by Cambridge University Press.

[Gol93] R. Goldblatt. *Mathematics of Modality*. Number 43 in CSLI Lecture Notes. CSLI Publications, Stanford University, 1993. Distributed by Cambridge University Press.

[Gol95] R. Goldblatt. Saturation and the Hennessy-Milner property. In A. Ponse, M. de Rijke, and Y. Venema, editors, *Modal Logic and Process Algebra. A Bisimulation Perspective*, volume 53 of *CSLI Lecture Notes*, pages 107–129. CSLI Publications, Stanford, 1995.

[Gol00] R. Goldblatt. Algebraic Polymodal Logic: a survey. *Logic Journal of the I.G.P.L.*, 8:393–450, 2000.

[Gol03] R. Goldblatt. Mathematical modal logic: a view of its evolution. *Journal of Applied Logic*, 1(5-6):309–392, 2003.

[Grä98] G. Grätzer. *General lattice theory, Second edition*. Birkhäuser Verlag, Basel, 1998.

[Grä01] E. Grädel. Why are modal logics so robustly decidable? In G. Paun, G. Rozenberg, and A. Salomaa, editors, *Current Trends in Theoretical Computer Science. Entering the 21st Century*, pages 393–408. World Scientific, 2001.

[GT74] R. I. Goldblatt and S. K. Thomason. Axiomatic classes in propositional modal logic. In J. Crossley, editor, *Algebra and logic*, volume 450 of *Lecture Notes in Mathematics*, pages 163–173. Springer, 1974.

[GZ95] S. Ghilardi and M. Zawadowski. Undefinability of propositional quantifiers in the modal system S4. *Studia Logica*, 55(2):259–271, 1995.

[Hac63] I. Hacking. What is strict implication? *The Journal of Symbolic Logic*, 28:51–71, 1963.

[Hal95] J. Y. Halpern. The effect of bounding the number of primitive propositions and the depth of nesting on the complexity of modal logic. *Artificial Intelligence*, 75(2):361–372, 1995.

[Han65] W. Hanf. Model-theoretic methods in the study of elementary logic. In J. Addison, L. Henkin, and A. Tarski, editors, *The Theory of Models*, pages 132–145. North-Holland, 1965.

[Han66] G. Hansel. Sur le nombre des fonctions booléennes monotones de n variables. *C. R. Acad. Sci. Paris Sér. A-B*, 262:A1088–A1090, 1966.

[HC96] G. E. Hughes and M. J. Cresswell. *A new introduction to modal logic*. Routledge, London, 1996.

[Hem96] E. Hemaspaandra. The price of universality. *Notre Dame Journal of Formal Logic*, 37:174–203, 1996.

[Hen49] L. Henkin. The completeness of the first order functional calculus. *The Journal of Symbolic Logic*, 14:159–166, 1949.

[Hen67] L. Henkin. *Logical systems containing only a finite number of symbols*. Séminaire de Mathématiques Supérieures, No. 21. Les Presses de l'Université de Montréal, Montreal, Que., 1967.

[Hen96] L. Henkin. The discovery of my completeness proofs. *The Bulletin of Symbolic Logic*, 2(2):127–158, 1996.

[HKT00] D. Harel, D. Kozen, and J. Tiuryn. *Dynamic logic*. Foundations of Computing Series. MIT Press, Cambridge, MA, 2000.

[HM85] M. Hennessy and R. Milner. Algebraic Laws for Nondeterminism and Concurrency. *Journal of the Association for Computing Machinery*, 32:137–161, 1985.

[HM90] J. Y. Halpern and Y. Moses. Knowledge and common knowledge in a distributed environment. *Journal of the ACM*, 37(3):549–587, 1990.

[HM92] J. Y. Halpern and Y. Moses. A guide to completeness and complexity for modal logics of knowledge and belief. *Artificial Intelligence*, 54:319–379, 1992.

[HMU01] J. E. Hopcroft, R. Motwani, and J. D. Ullman. *Introduction to Automata Theory, Languages and Computation.* Addison-Wesley, second edition, 2001.

[Hol95] M. Hollenberg. Hennessy-Milner classes and process algebras. In A. Ponse, M. de Rijke, and Y. Venema, editors, *Modal Logic and Process Algebra. A Bisimulation Perspective*, volume 53 of *CSLI Lecture Notes*, pages 187–216. CSLI Publications, Stanford, 1995.

[Hol98] M. Hollenberg. *Logic and Bisimulation.* Ph. D. Dissertation, Utrecht University, 1998.

[Imm99] N. Immerman. *Descriptive complexity.* Graduate Texts in Computer Science. Springer-Verlag, New York, 1999.

[Jan90] R. Jansana. *Una introducción a la lógica modal.* Tecnos, Madrid, 1990.

[Joh36] I. Johansson. Der minimalkalkül, ein reduzierter intuitionistischer formalismus. *Compositio Mathematica*, 4:119–136, 1936.

[JR97] B. Jacobs and J. Rutten. A Tutorial on (Co)Algebras and (Co)Induction. *Bulletin of the EATCS*, 62:222–259, 1997.

[JW96] D. Janin and I. Walukiewicz. On the expressive completeness of the propositional mu-calculus with respect to the monadic second order logic. In *Proceedings CONCUR'96*, volume 1119 of *Lecture Notes in Computer Science*, pages 263–277. Springer-Verlag, 1996.

[Kah02] J. Kahn. Entropy, independent sets and antichains: a new approach to Dedekind's problem. *Proceedings of the American Mathematical Society*, 130(2):371–378 (electronic), 2002.

[Kam68] H. Kamp. *Tense Logic and the Theory of Linear Order*. Ph. D. Thesis, University of California, Los Angeles, 1968.

[Kar65] C. R. Karp. Finite-quantifier equivalence. In J. Addison, L. Henkin, and A. Tarski, editors, *The Theory of Models*, pages 407–412. North-Holland, 1965.

[KdR97] N. Kurtonina and M. de Rijke. Simulating without negation. *Journal of Logic and Computation*, 7(4):501–522, August 1997.

[Kei71] H. J. Keisler. *Model Theory for Infinitary Logic*, volume 62 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, 1971.

[Kis88] A. Kisielewicz. A solution of Dedekind's problem on the number of isotone Boolean functions. *J. Reine Angew. Math.*, 386:139–144, 1988.

[Kle69] D. Kleitman. On Dedekind's problem: The number of monotone Boolean functions. *Proc. Amer. Math. Soc.*, 21:677–682, 1969.

[KM75] D. Kleitman and G. Markowsky. On Dedekind's problem: the number of isotone Boolean functions. II. *Transactions of the Americal Mathematical Society*, 213:373–390, 1975.

[Knu76] D. E. Knuth. Big omicron and big omega and big theta. *SIGACT News*, 8(2):18–24, 1976. Reprinted in *Selected papers on analysis of algorithms*, Knuth, D. E., CSLI Publications, 2000, pp. 35–41.

[Koo03] B. Kooi. *Knowledge, chance, and change.* Ph. D. Dissertation DS-2003-01, Institute for Logic, Language and Computation, 2003.

[Kor77] A. D. Koršunov. Solution of Dedekind's problem on the number of monotone Boolean functions. *Dokl. Akad. Nauk SSSR*, 233(4):543–546, 1977.

[Koz83] D. Kozen. Results on the propositional μ-calculus. *Theoretical Computer Science*, 27:333–354, 1983.

[KP81] D. Kozen and R. Parikh. An elementary proof of the completeness of PDL. *Theoretical Computer Science*, 14:113–118, 1981.

[Kra99] M. Kracht. *Tools and techniques in modal logic.* Studies in Logic and the Foundations of Mathematics. Elsevier, Amsterdam, 1999.

[Kri63] S. Kripke. Semantical analysis of modal logic I: Normal modal propositional calculi. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 9:67–96, 1963.

[Kur95] N. Kurtonina. *Frames and Labels. A Modal Analysis of Categorical Inference.* Ph. D. Dissertation DS-1995-08, Institute for Logic, Language and Computation, University of Amsterdam, 1995.

[Lad77] R. E. Ladner. The computational complexity of provability in systems of modal propositional logic. *SIAM J. Comput.*, 6(3):467–480, 1977.

[Lam58] J. Lambek. The mathematics of sentence structure. *American Mathematical Monthly*, 65:154–170, 1958.

[Lem77] E. J. Lemmon. *An Introduction to Modal Logic (The "Lemmon Notes")*, volume 11 of *Americal Philosophical Quarterly Monograph Series*. Basil Blackwell, Oxford, 1977. In collaboration with D. Scott.

[Lew79] H. R. Lewis. *Unsolvable classes of quantificational formulas*. Addison-Wesley Publishing Co., Reading, Mass., 1979.

[Lip77] W. Lipski. On the logic of incomplete information. In *Proc. 6th International Symposium on Mathematical Foundations of Computer Science*, volume 53 of *Lecture Notes in Computer Science*, pages 374–381. Springer, 1977.

[Mak03] L. L. Maksimova. Implicit decidability and positive logics. *Algebra and Logic*, 42(1):37–53, 2003.

[Mar98] M. Marx. Interpolation in modal logic. In *Algebraic methodology and software technology (Amazonia, 1998)*, volume 1548 of *Lecture Notes in Comput. Sci.*, pages 154–163. Springer, Berlin, 1998.

[Men97] E. Mendelson. *Introduction to mathematical logic*. Chapman & Hall, London, etc., 1997. 4ª edición; la 1ª es de 1964.

[MH69] J. McCarthy and P. Hayes. Some philosophical problems from the standpoint of artificial intelligence. In *Machine Intelligence*, volume 4, pages 463–502. Edinburg University Press, 1969. Available as http://www-formal.stanford.edu/jmc/mcchay69.html and reprinted in James Allen, James Hendler, and Austin Tate, eds., *Readings in Planning*, Morgan Kaufmann Publishers, Inc., San Mateo, CA, 1990, 393–435.

[Mil80] R. Milner. *A calculus of communicating systems*, volume 92 of *Lecture Notes in Computer Science*. Springer-Verlag, Berlin, 1980.

[Mor75] M. Mortimer. On languages with two variables. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 21:135–140, 1975.

[Mos99] L. Moss. Coalgebraic Logic. *Annals of Pure and Applied Logic*, 96:227–317, 1999. The last sections appear with lots of blank spaces due to a bug in the printing process, and they re-appear in the same journal: 99:241-259, 1999.

[Nau60] P. Naur. Report on the Algorithmic Language ALGOL 60. *Communications of the ACM*, 3(5):299–314, 1960.

[Nis60] I. Nishimura. On formulas of one variable in intuitionistic propositional calculus. *The Journal of Symbolic Logic*, 25:327–331 (1962), 1960.

[Ono77] H. Ono. On some intuitionistic modal logics. *Publ. Res. Inst. Math. Sci.*, 13:687–722, 1977.

[Ono98] H. Ono. Proof-theoretic methods for nonclassical logic - an introduction. In M. Takahashi, M. Okada, and M. Dezani-Ciancaglini, editors, *Theories of Types and Proofs*, MSJ Memoirs 2, pages 207–254. Mathematical Society of Japan, 1998.

[Ott02] M. Otto. Bisimulation Invariance and Finite Models. (23 pages), preprint, 2002.

[Pal03] A. Palmigiano. Coalgebraic semantics for positive modal logic. In H. P. Gumm, editor, *Electronic Notes in Theoretical Computer Science*, volume 82. Elsevier, 2003.

[Pap94] C. H. Papadimitriou. *Computational complexity*. Addison-Wesley Publishing Company, Reading, MA, 1994.

[Par81] D. Park. Concurrency and Automata on Infinite Sequences. In P. Deussen, editor, *Proceedings of the 5th G.I. Conference*, volume 104 of *Lecture Notes in Computer Science*, pages 167–183. Springer-Verlag, New York, 1981.

[PdRV95] A. Ponse, M. de Rijke, and Y. Venema, editors. *Modal Logic and Process Algebra. A Bisimulation Perspective*, volume 53 of *CSLI Lecture Notes*. Stanford, 1995.

[Pit92] A. M. Pitts. On an interpretation of second-order quantification in first-order intuitionistic propositional logic. *The Journal of Symbolic Logic*, 57(1):33–52, 1992.

[Pnu76] A. Pnuelli. The temporal logic of programs. In *Proc. 18th IEEE Symposium on Foundation of Computer Science*, pages 109–121, 1976.

[Poi00] B. Poizat. *A Course in Model Theory: An Introduction to Contemporary Mathematical Logic*. Universitext. Springer-Verlag, 2000.

[Pop92] S. Popkorn. *First Steps in Modal Logic*. Cambridge University Press, 1992.

[Rau74a] C. Rauszer. A formalization of the propositional calculus of $H - B$ logic. *Studia Logica*, 33:23–34, 1974.

[Rau74b] C. Rauszer. Semi-Boolean algebras and their applications to intuitionistic logic with dual operations. *Fundamenta Mathematicae*, 83(3):219–249, 1974.

[Rau77] C. Rauszer. Applications of Kripke models to Heyting-Brouwer logic. *Studia Logica*, 36(1-2):61–71, 1977.

[Rau80] C. Rauszer. An algebraic and Kripke-style approach to a certain extension of intuitionistic logic. *Dissertationes Mathematicae. (Rozprawy Mat.)*, 167:62, 1980.

[RdLKV] G. R. Renardel de Lavalette, B. Kooi, and R. Verbrugge. Strong completeness and limited canonicity for PDL and similar modal logics. Preprint.

[Res94] G. Restall. Subintuitionistic logics. *Notre Dame Journal of Formal Logic*, 35(1):116–129, 1994.

[Ros97] E. Rosen. Modal logic over finite structures. *Journal of Logic, Language and Information*, 6(4):427–439, 1997.

[Rui91] W. Ruitenburg. Constructive logic and the paradoxes. *Modern Logic*, 1:207–301, 1991.

[Rui93] W. Ruitenburg. Basic logic and Fregean set theory. In H. Barendregt, M. Bezem, and J. W. Klop, editors, *Dirk van Dalen Festschrift, Questiones Infinitae*, volume 5, pages 121–142. Department of Philosophy, Utrecht University, 1993.

[Rui99] W. Ruitenburg. Basic logic, $K4$ and persistence. *Studia Logica*, 63:343–352, 1999.

[Rut00] J. Rutten. Universal coalgebra: a theory of systems. *Theoretical Computer Science*, 249(1):3–80, 2000.

[RV93] J. Rebagliato and V. Verdú. On the algebraization of some Gentzen systems. *Fundamenta Informaticae, Special Issue on Algebraic Logic and its Applications*, 18:319–338, 1993.

[RV95] J. Rebagliato and V. Verdú. Algebraizable Gentzen systems and the deduction theorem for Gentzen systems. Mathematics Preprint Series 175, University of Barcelona, June 1995.

[Sah75] H. Sahlqvist. Completeness and Correspondence in First and Second Order Semantics for Modal Logic. In S. Kanger, editor, *Proceedings of the third Scandinavian logic symposium*, pages 110–143. North-Holland, Amsterdam, 1975.

[Sap00] A. A. Sapozhenko. The Dedekind problem and the method of boundary functionals. In *Mathematical problems in cybernetics, No. 9 (Russian)*, volume 9 of *Mat. Vopr. Kibern.*, pages 161–220. FizMatLit, Moscow, 2000.

[Sav70] W. J. Savitch. Relationships between nondeterministic and deterministic tape complexities. *Journal of Computer and System Sciences*, 4:177–192, 1970.

[SB01] C. Stirling and J. Bradfield. Modal logics and mu-calculi. In J. Bergstra, A. Ponse, and S. Smolka, editors, *Handbook of Process Algebra*, pages 293–332. Elsevier, North-Holland, 2001.

[Sch52] J. Schmidt. Über die Rolle den transfinite Schusswessen in einer allgemeinen Idealtheorie. *Mathematische Nachrichten*, 7:165–182, 1952.

[Sco62] D. Scott. A decision method for validity of sentences in two variables. *The Journal of Symbolic Logic*, 27:377, 1962.

[Sco65] D. Scott. Logic with denumerably long formulas and finite strings of quantifiers. In J. Addison, L. Henkin, and A. Tarski, editors, *The Theory of Models*, pages 329–341. North-Holland, 1965.

[Seg71] K. Segerberg. *An essay in Classical Modal Logic*. Philosophical Studies. Uppsala University, Uppsala, 1971. 3 vols.

[Seg94] K. Segerberg. A model existence theorem in infinitary propositional modal logic. *Journal of Philosophical Logic*, 23:337–367, 1994.

[Sha70] H. N. Shapiro. On the counting problem for monotone boolean functions. *Comm. Pure Appl. Math.*, 23:299–312, 1970.

[SM73] L. J. Stockmeyer and A. R. Meyer. Word problems requiring exponential time: preliminary report. In *Fifth Annual ACM Symposium on Theory of Computing (Austin, Tex., 1973)*, pages 1–9. Assoc. Comput. Mach., New York, 1973.

[Smu68] R. Smullyan. Analytic cut. *The Journal of Symbolic Logic*, 33:560–564, 1968.

[Spa93] E. Spaan. *Complexity of modal logics*. Ph. D. Dissertation, Institute for Logic, Language and Computation, 1993.

[Sta79] R. Statman. Intuitionistic propositional logic is polynomial-space complete. *Theoretical Computer Science*, 9(1):67–72, 1979.

[Sto36] M. H. Stone. The theory of representations for Boolean algebras. *Transactions of the Americal Mathematical Society*, 39:37–111, 1936.

[Sun77] G. Sundholm. A completeness proof for an infinitary tense-logic. *Theoria*, 43:47–51, 1977.

[SWZ98] Y. Suzuki, F. Wolter, and M. Zakharyashev. Speaking about transitive frames in propositional languages. *Journal of Logic, Language and Information*, 7:317–339, 1998.

[Tho74] S. K. Thomason. An incompleteness theorem in modal logic. *Theoria*, 40:30–34, 1974.

[Var97] M. Y. Vardi. Why is modal logic so robustly decidable? In N. Immerman and P. Kolaitis, editors, *Descriptive Complexity and Finite Models*, volume 31 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, pages 149–184. AMS, Providence, 1997.

[vB76] J. F. A. K. van Benthem. *Modal Correspondence Theory*. Ph. D. Thesis, University of Amsterdam, 1976.

[vB82] J. F. A. K. van Benthem. *Modal logic and classical logic*. Bibliopolis, Napoli, 1982.

[vB83] J. F. A. K. van Benthem. *The Logic of Time: a model-theoretic investigation into the varieties of temporal ontology and temporal discourse.* Dordrecht, 1983.

[vB84] J. F. A. K. van Benthem. Correspondence theory. In D. M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume II, pages 167–247. Reidel, Dordrecht, 1984.

[vB96] J. F. A. K. van Benthem. *Exploring Logical Dynamics*. Number 7 in Studies in Logic, Language and Information. CSLI Publications, 1996.

[vB97] J. F. A. K. van Benthem. Modal Foundations for Predicate Logic. *Logic Journal of the I.G.P.L.*, 5(2):259–286, 1997.

[vB98] J. F. A. K. van Benthem. Program operations that are safe for bisimulation. *Studia Logica*, 60(2):311–330, 1998.

[vB99] J. F. A. K. van Benthem. Modality, Bisimulation and Interpolation in Infinitary Logic. *Annals of Pure and Applied Logic*, 96:29–41, 1999.

[vBB95] J. F. A. K. van Benthem and J. A. Bergstra. Logic of Transition Systems. *Journal of Logic, Language and Information*, 3(4):247–283, 1995.

[vBvES94] J. F. A. K. van Benthem, J. van Eijck, and V. Stebletsova. Modal logic, transition systems and processes. *Journal of Logic and Computation*, 4(5):811–855, 1994.

[Ven99] Y. Venema. Modal definability, purely modal. In J. Gerbrandy, M. Marx, M. de Rijke, and Y. Venema, editors, *JFAK. Essays dedicated to Johan van Benthem on the Ocasion of his 50th Birthday (CD-Rom)*. Amsterdam, 1999.

[Ven01] Y. Venema. Temporal logic. In L. Goble, editor, *The Blackwell guide to philosophical logic*, Blackwell Philos. Guides, pages 203–223. Blackwell, Oxford, 2001.

[Vis81] A. Visser. A propositional logic with explicit fixed points. *Studia Logica*, 40:155–175, 1981.

[Vis96a] A. Visser. Bisimulations, model descriptions and propositional quantifiers. Logic Group Preprint Series 161, Department of Philosophy, Utrecht University, April 1996.

[Vis96b] A. Visser. Uniform interpolation and layered bisimulation. In P. Hájek, editor, *Gödel '96 (Brno, 1996)*, volume 6 of *Lecture Notes Logic*, pages 139–164. Springer, Berlin, 1996. Logical foundations of mathematics, computer science and physics—Kurt Gödel's legacy.

[Vis98] A. Visser. An overview of Interpretability Logic. In *Advances in modal logic, Vol. 1 (Berlin, 1996)*, volume 87 of *CSLI Lecture Notes*, pages 307–360. CSLI Publ., Stanford, CA, 1998.

[VvBJL95] A. Visser, J. F. A. K. van Benthem, D. de Jongh, and G. R. Renardel de Lavalette. NNIL, a study in intuitionistic propositional logic. In A. Ponse, M. de Rijke, and Y. Venema, editors, *Modal Logic and Process Algebra. A Bisimulation Perspective*, volume 53 of *CSLI Lecture Notes*, pages 289–326. CSLI Publications, Stanford, 1995.

[Wal95] I. Walukiewicz. *Notes on the Propositional μ-Calculus*. Number NS-95-1 in BRICS Notes Series. 1995.

[Wal00] I. Walukiewicz. Completeness of Kozen's axiomatisation of the propositional Mu-calculus. *Information and Computation*, 157:142–182, 2000.

[Wan97] H. Wansing. Displaying as temporalizing. Sequent systems for subintuitionistic logic. In S. Akama, editor, *Logic, Language and Computation*, pages 159–178. Kluwer, 1997.

[War46] M. Ward. Note on the order of free distributive lattices (abstract). *Bull. Amer. Math. Soc.*, 52:423, 1946.

[Wie91] D. Wiedemann. A computation of the eighth Dedekind number. *Order*, 8(1):5–6, 1991.

[Wol98] F. Wolter. Fusions of modal logic revisited. In *Advances in modal logic, Vol. 1 (Berlin, 1996)*, volume 87 of *CSLI Lecture Notes*, pages 361–379. CSLI Publ., Stanford, CA, 1998.

[WZ97] F. Wolter and M. Zakharyaschev. On the relation between intuitionistic and classical modal logics. *Algebra and Logic*, 36(2):73–92, 1997.

[WZ99a] F. Wolter and M. Zakharyaschev. Intuitionistic modal logic. In A. Cantani, E. Casari, and P. Minari, editors, *Logic and Foundations of Mathematics*, pages 227–238. Kluwer Academic Publishers, 1999.

[WZ99b] F. Wolter and M. Zakharyaschev. Intuitionistic modal logics as fragments of classical bimodal logics. In E. Orłowska, editor, *Logic at work: Essays dedicated to the memory of Helena Rasiowa*, pages 168–186. Physica-Verlag, Heidelberg, 1999.

[ZWC01] M. Zakharyaschev, F. Wolter, and A. Chagrov. Advanced modal logic. In D. M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic, second edition*, volume 3, pages 83–266. Kluwer, Dordrecht, 2001.

Index

List of symbols

Dissertations on *Mathematical Logic* in the *Department of Logic, History and Philosophy of Science (University of Barcelona)*

2004: **Jorge Ernesto Castro**
Estructuras casi-modales e implicativas generalizadas

2003: **Joan Rosselló Moya**
Un estudi comparatiu de les Contribucions del corrent algèbric i logicista a la lògica contemporànea

2002: **Roger Bosch Bastardes**
Projective Forcing

2001: **Román Adillón**
Contribució a l'estudi de les lógiques intuïcionistes i de Gödel sense contracció i dels seus fragments

2001: **Carles Dorce Polo**
L'astronomia pre-copernicana de Marga en el Magrib: Tècniques de càlcul en el Tay al-azyay de Muhdi al-Din al Magribi

2000: **Jordi López Abad**
Weakly-Ramsey sets in Banach spaces

2000: **David Asperó Herrando**
Bounded Forcing Axioms and the Continuum

1998: **Joan Gispert**
Estudi algebraic de les extensions dels càlculs multivalorats de Łukasiewicz

1996: **Pilar Dellunde**
Contributions to the model theory of equality-free logic

1996: **Angel Javier Gil**
Sistemes de Gentzen multidimensionals i lògiques finitament valorades. Teoria i Aplicacions

1995: **Sergio A. Celani**
Lógicas modales distributivas y de De Morgan

1994: **Jordi Rebagliato**
Sistemes de Gentzen algebraitzables i el Teorema de la Deducció

1994: **Raimon Elgueta**
Algebraic Model Theory for Languages without Equality

1994: **M. Antonia Huertas Sánchez**
Modal logic (of predicates) and (partial and heterogeneous) non-classical logic

1993: **Carlos Gómez**
La teoría de conjuntos en Georg Cantor

1993: **Rafael Farré Cirera**
Model theory for valued and ordered fields, and applications

1992: **Miquel Rius Font**
Lògiques Modals Tetravalents

1991: **José Luis García Lapresta**
Lógicas finitamente deductivas. Restricciones de cardinalidad en la propiedad de la deducción

1991: **Calixto Badesa**
El teorema de Löwenheim en el marco de la teoría de relativos

1990: **Gonzalo Rodríguez Pérez**
Àlgebres i lògiques abstractes associades al càlcul R de la rellevància

1990: **Enrique Droeven**
La teoría de los números reales en Georg Cantor

1987: **Enrique Casanovas**
Contribuciones a la teoría abstracta de modelos

1985: **Carles Rafels Pallarola**
Estructures lògic-algebraiques d'ideals d'un anell abelià unitari

1982: **Ramon Jansana Ferrer**
Modelos Booleanos e independencia de la hipótesis del continuo

1982: **Ignasi Jané**
Estructuras generales de segundo orden

1981: **Josep Maria Font Llovet**
Estudi algebraic de certes lògiques intuïcionistes modals

1980: **Antonio Jesús Rodríguez Salas**
Un estudio algebraico del cálculo proposicional de Łukasiewicz

1980: **Antoni Torrens Torrell**
Estudi i algebrització de certes lògiques: àlgebres d-completes

1978: **Ventura Verdú Solans**
Contribució a l'estudi de certs tipus de lògiques abstractes

1977: **Maria Gracia Manzano**
Sistemas generales de la lógica de segundo orden

1975: **Josep Pla Carrera**
Contribució a l'estudi de les estructures algebraiques dels sistemes lògics deductius