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Invariant manifolds and bifurcations for one-dimensional and two-dimensional dissipative maps

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CERTIFICO :

Que la present memòria ha estat realitzada sota la meva direcció per En Joan Carles Tatjer i Montaña, i que constitueix la seva tesi per a aspirar al grau de Doctor en Ciències Matemàtiques.

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Introduction

It is known that for the study of continuous dynamical systems the discret case plays an important role because, with it we can study the continuous one by using the Poincaré return map. In the discret case we can distinguish between conservative maps (or area preserving maps, in the case of flows living on a 3-dimensional manifold) and non conservative maps. Among the last ones, there are the dissipative maps. Two of the main subjects of the study of dissipative maps are: the existence or not of attracting periodic orbits and the possible existence of strange attractors -that is, attractors that are neither periodic orbits nor invariant curves, which are minimal and contain a dense orbit. Moreover, these attractors can have sensitive dependece on the initial conditions, or have an absolutely continuous invariant measure. On the other hand there exists a transition between these two behaviours: the so-called flip or period doubling bifurcation cascade. After the final of this cascade (in a suitable set of parameters), strange attractors can appear, and also more attracting periodic orbits. When we restrict our attention to the two dimensional case, two facts are important: a) The behaviour of a dissipative two dimensional map is similar (but not in some details) to the behaviour corresponding to the one dimensional case. b) The creation and destruction of attracting periodic orbits and strange attractors seems very closely related to the Newhouse phenomenon: given a one parameter family of dissipative diffeomorphisms $\{f_a\}_{a\in I}$ having a non degenerate homoclinic tangency of the invariant manifolds of a saddle fixed point for $a = a_0$, there exist parameters a, close to a_0, for which there is an attracting periodic orbit close to the point of homoclinic tangency.

This work is divided in four chapters:

In the first one we study the dynamics of the so called logistic map, $f_a(x) = 1 - ax^2$. More specifically, we study first fold and flip bifurcations of this family, giving analytical expressions of the parameter values for which they occur. To do this, we use a parametric representation of the unstable invariant manifolds of the fixed points. Moreover, we compute the parameter values a for which there are homoclinic tangencies (that is $f_a^n(0) = x$ and $f_a^m(x) = x$). Also it is possible to do analytical estimates of the measure of the set of parameter values for which there is an attarcting periodic orbit. First we compute the width of the set of parameter values between fold and flip bifurcation. Then we estimate the measure of the set of parameter values for which there is an attracting periodic orbit when the parameter is close to 2. Although this estimate is not rigourous, it seems that can be reliable if we do some restrictions in the number of attracting periodic orbits considered, as it is precised in the text. It is remarkable that, in some sense, high iterates of the logistic map behave as f_a when a is close to 2. The last part of the chapter give some numerical estimates: the measure of parameter values for which there is not a strange attractor consisting of a unique interval, the measure of the parameter values for which there is an attracting periodic orbit, numerical evidence of the density of the set of parameter values for which there is an attracting periodic orbit (which it has not been proved yet) and the behaviour of the parameter set Δ_ϵ for which there is an attracting periodic orbit, when we consider this set restricted to intervals of the form $[2-\epsilon, 2]$ with $\epsilon \to 0$. Concerning the last point, it is known that $\lim_{\epsilon\to 0} \Delta_\epsilon/\epsilon = 0$. We see, numerically, that there exists a limit of $\Delta_\epsilon/\epsilon^2$, in a suitable sense, and that it is different from zero. Moreover, we verify that the analytical estimates of the first part of this chapter are reliable. Finally we compare our results with other ones of Farmer ([11]) and Ketoja ([12]).

In the second chapter, we consider the Hénon map $f_{a,b}$, with strong dissipation. We study first the invariant manifolds of the fixed points of $f_{a,b}$ when $b = 0$. Then we prove the differentiable dependence of the invariant manifolds on the parameters a and b, when b is close to 0. As an application of this, we show the existence and differentiability of a fan of homoclinic and heteroclinic bifurcation curves (that is, curves in the parameter plane consisting of parameters for which there are homoclinic or heteroclinic tangencies). We remark that the the existence of such a fan has been proved before (see [1]), but using other techniques. Moreover the differentiability of the invariant manifolds, and the homoclinic and heteroclinic bifurcation curves are not proved in [1]. Also the definition of the invariant manifolds in the case of non invertible maps has been used before (see for instance [2]).

In the third chapter we study the Newhouse phenomenon. To this end we prove a more complete version of the phenomenon than others proved before, in which we show the existence of generic saddle-node and flip bifurcations, for parameters close to the parameter of homoclinic tangency. Then, by using a quadratic model of the n-th iterate of the map, close to the homoclinic tangency, we compute the first bifurcation parameters related to this phenomenon, and their behaviour depending on the type of tangency. Moreover, we classify the possible behaviour of the basin of attraction of the attracting periodic orbits which appear due to this phenomenon. In particular, we regard the possible intersection of these basins with the unstable invariant manifold of the fixed point. This is important due to the fact that the closure of this unstable invariant manifold can be a strange attractor, and, of course, this attractor disappears if it has intersection with the basin of attraction of an attracting periodic orbit. Then we compute numerically some examples of periodic orbits related to this phenomenon, and verify the goodness of the analytical results concerning to the basin of attraction of such orbits. Moreover, for the Hénon map, we compute numerically the measure of the parameter values for which there are attracting periodic orbits, in a region of parameters a (fixed b) close to one having a homoclinic tangency. It seems that the measure is very small.

In chapter four we study the behaviour of the codimension one and two bifurcations in one and two dimensional families of maps. To do this, we consider one-parameter families of diffeomorphisms, to study saddle-node and flip bifurcations, and two-parameter families of dissipative diffeomorphisms, to study cusps and codimension two flips. By means of the normal form theory, we compute, for a general family having a fixed point, the conditions for which there exist such bifurcations, and also we prove that such conditions give a behaviour as the one of the one dimensional models of bifurcations. As the study is local, to see more global properties of codimension two bifurcations we study cubic models of cusp bifurcation and a quartic model of codimension three cusp. This give us four different types of interaction between the studied bifurcations, two of them having codimension two flips. It is remarkable that all these cases can be found in the Hénon map, as we see in the last section of this chapter, by using the conditions of existence of bifurcations computed before. Moreover we present a scheme of cusp cascades which seems to appear in all the cases, as it has been observed before for other maps. We remark that three of these cases have been studied before in some papers (se references in the text), but one case seems new in the literature.

Vull expressar el meu profund agraïment i admiració al director d'aquesta memòria Dr. Carles Simó que, al llarg de la meva tasca, amb la seva decisiva ajuda i consell ha fet possible l'elaboració i culminació d'aquest treball.

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Contents

Chapter 1

Windows of attraction for the logistic map

The objective of this chapter is to study the measure of the set of parameters of the logistic map, $f_a(x) = 1 - ax^2$, where $x \in \mathbb{R}$ and $a \in \mathbb{R}$ is the parameter, for which there is an attracting periodic orbit. To this end we give first some basic properties of f_a and we will do some analytic estimates to determine the location of this parameters. Later on we will give a numerical estimate of this set. Furthermore we will give numerical evidences concerning the density of this set in the interval $[-1/4, 2]$. Finally we study the Lebesgue point corresponding to $a = 2$.

1.1 Basic properties of the logistic map

Let $f_a(x) = 1 - ax^2$ be the logistic map. We summarize some well known facts about this map:

a) For $a = 2$ the 2^n periodic points of period n are real. All of them are repellors. The map has an absolutely continuous invariant measure μ such that $\mu(A) = \int_A (\pi\sqrt{1-y^2})^{-1} dy$ and f_2 is conjugated to $T_2(\bar{x}) = 1 - 2|\bar{x}|$.

b) There exist two fixed points $x_+ = (-1 + \sqrt{1+4a})/2a$ and $x_- = (-1 - \sqrt{1+4a})/2a$ if $a \geq -1/4$. The point x_- is a repellor if $a > -1/4$ and x_+ is a repellor if $a > 3/4$.

c) All the bifurcations of periodic points are of generic saddle-node or codimension one flip types.

d) Given a parameter of saddle-node bifurcation of period n, that we denote as $a^1_{n,j}$, where j is the order number of the saddle-node or flip bifurcations starting at the closest one to $a = 2$, there are infinitely many values of the parameter at flip bifurcations, $a^{2^i}_{n,j}$, where n is the period of the related saddle-node and $2^i n$ is the period of the bifurcated orbit, such that $a^{2^i}_{n,j} < a^{2^{i+1}}_{n,j}$, $a^{2^i}_{n,j} \to a^{2^\infty}_{n,j}$ for $i \to \infty$. Furthermore

$$\lim_{i\to\infty} \frac{a^{2^i}_{n,j} - a^{2^{i+1}}_{n,j}}{a^{2^{i+1}}_{n,j} - a^{2^{i+2}}_{n,j}} = \delta \approx 4.66920160910\,,$$

δ being the so called Feigenbaum constant.

e) For any given value of $a \in \mathbb{R}$ the map f_a has, at most, one attracting periodic orbit.

f) The set $J = \{a \in [-1/4, 2]$ such that there exists an absolutely continuous invariant measure $\}$ satisfies $\lambda(J) \neq 0$, where λ is the Lebesgue measure for $[-1/4, 2]$.

g) The value $a = 2$ is a Lebesgue or density point for J, that is:

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\lambda\{[2-\epsilon, 2] \cap J\} = 1\,.$$

The proof of a) can be found in [3]. b) is obvious. c) is a consequence of [4]. d) appears in [3], [5] and others in some particular cases. e) is shown in [3] and [6]. Finally the proofs of f) and g) are given in [7],[8] and [9].

A very interesting question is the study of invariant manifolds and the related homoclinic points. In this chapter we will be only interested in the unstable invariant manifolds of periodic points.

Definition 1.1.1 *Let $f : U \subset \mathbb{R} \longrightarrow \mathbb{R}$, U an open set and f a differentiable function with a repelling fixed point $p \in U$. The set $\mathcal{W}^u(p) = \{x \in \mathbb{R} \;:\; \exists (x_n)_n \to p \;,\; x_0 = x, f(x_{n+1}) = x_n\}$ is called the unstable invariant manifold of p.*

Definition 1.1.2 *Let $f_a : \mathbb{R} \longrightarrow \mathbb{R}$ be the logistic map. We say that f_a has a tangencial homoclinic point if there exists a repelling n-periodic point, $p \in \mathbb{R}$, and some $m \in \mathbb{N}$ such that $f^m_a(0) = p$.*

In chapter 2 we will see the equivalence of this definition with the usual for diffeomorphisms.

The next proposition give us a representation of the invariant manifold:

Proposition 1.1.3 *Let p one of the repelling fixed points of f_a ($p = x_+$ or $p = x_-$). Then there exists a (non unique) entire function $z(t)$ with $t \in \mathbb{C}$ such that:*

a) $z(0) = p$.

b) $\mathcal{W}^u(p) = z(\mathbb{R})$.

c) $f_a(z(t)) = z(\alpha t)$ *where* $\alpha = f_a'(p)$.

The proof of this proposition is postposed to chapter 2, proposition 2.2.2.

From now on we will denote $f_a(x)$ as $f(x,\epsilon)$ where $\epsilon = a - 2$. Furthermore we call $z_+(t,\epsilon)$ (respectively $z_-(t,\epsilon)$) the parametric representation of $\mathcal{W}^u(x_+)$ (resp. $\mathcal{W}^u(x_-)$) given by 1.1.3. We remark that $x_+ = x_+(\epsilon)$ (resp. $x_- = x_-(\epsilon)$).

Proposition 1.1.4 *a)* $z_+(t,0) = \sin\frac{\pi}{2}(\frac{1}{3}+t)$.

b) $z_-(t,0) = -\cos t^{1/2}$.

The proof is obtained by direct check.

1.2 Computation of the windows of attraction: Analytical estimates

The set of values of a for which there is an attracting periodic orbit is made of intervals going from a saddle-node bifurcation to the end of a bifurcation cascade. These intervals will be called windows of attraction. To find windows of attraction we will start by looking for the parameter values for which there is a superstable periodic orbit, that is, $\epsilon \in \mathbb{R}$ such that $f^n(0,\epsilon) = 0$ for some $n \in \mathbb{N}$. First we observe that we can know exactly the number of superstable periodic orbits associated to saddle-node bifurcations. Let m_k be the number of period k superstable periodic orbits associated to saddle-node bifurcations for $\epsilon \in (-9/4, 0)$. For $\epsilon = 0$ there are $p_k = 2^k - \sum_{j|k} p_j$ points of minimal period k, and therefore $O_k = p_k/k$ orbits of period k. Then $m_k = O_k/2$ if k is odd, and $m_k = (O_k - n_{k/2})/2$ if k is even, where $n_{k/2} = m_{k/2} + \alpha_k$ and $\alpha_k = 0$ if $k/2$ is odd or $\alpha_k = n_{k/4}$ if $k/2$ is even (see for exemple [10]). The values of m_k for $k = 1, ..., 30$ are given in table 1.1. To obtain the values of ϵ for which there appears a superstable periodic orbit, we will derive first an analytical approximation. We will also give an approximation of the width of the parameter interval from a saddle-node bifurcation to the corresponding flip bifurcation. We start with some properties concerning $\mathcal{W}^u(x_-)$.

Proposition 1.2.1 *Let* $z_-(t,\epsilon)$ *be the parametrization of* $\mathcal{W}^u(x_-)$ *given by 1.1.3. Then:*

$$z_-(t,\epsilon) = -\cos t^{1/2} + \varphi(t^{1/2})\epsilon + O(\epsilon^2)$$

where

$$\varphi(u) = \frac{1}{3} + \frac{\sin u}{2}\sum_{k\geq 1}\left[2^k\left(\frac{1}{2} - \frac{1}{6\cos(2^{-k}u)}\right)\tan\frac{u}{2^{k+1}} - \frac{u}{6}\right] .$$

Proof:

We look for $z_-(t,\epsilon) = -\cos t^{1/2} + \psi(t)\epsilon + O(\epsilon^2)$. As $1-(2+\epsilon)z_-(t,\epsilon)^2 = z_-(\alpha t,\epsilon)$, where $\alpha = -2ax_-$, by derivation with respect to ϵ we obtain:

$$-z_-(t,\epsilon)^2 - 2(2+\epsilon)D_2z_-(t,\epsilon)z_-(t,\epsilon) = \frac{d\alpha}{d\epsilon}tD_1z_-(\alpha t,\epsilon) + D_2z_-(\alpha t,\epsilon). \tag{1.1}$$

Putting $\epsilon = 0$ we have:

$$-\cos^2 t^{1/2} + 4\cos t^{1/2}D_2z_-(t,0) = \frac{1}{6}t^{1/2}\sin(2t^{1/2}) + D_2z_-(4t,0).$$

Therefore $\psi(t) = D_2z_-(t,0)$ satisfies the functional equation:

$$-\cos^2 t^{1/2} + 4\cos t^{1/2}\psi(t) = \frac{1}{6}t^{1/2}\sin(2t^{1/2}) + \psi(4t). \tag{1.2}$$

From 1.2 it follows $\lim_{t\to 0}\psi(t) = 1/3$. Then we can express $\psi(t)$ in terms of $\psi(t/4)$ and trigonometric functions, substitute $\psi(t/4)$ in terms of $\psi(t/16)$ and new trigonometric functions. By iteration of this procedure, collecting all the trigonometric functions and going to the limit, we obtain:

$$\psi(t) = \frac{1}{3} + \frac{\sin t^{1/2}}{2}\sum_{k\geq 1}\left[2^k\left(\frac{1}{2} - \frac{1}{6\cos(2^{-k}t^{1/2})}\right)\tan\frac{t^{1/2}}{2^{k+1}} - \frac{t^{1/2}}{6}\right] .$$

Furthermore the relation 1.2 assures that if $\psi(t)$ is defined in some neighbourhood of $t = 0$ then it is defined for all t. If

$$a_k = 2^k\left(\frac{1}{2} - \frac{1}{6\cos(2^{-k}u)}\right)\tan\frac{u}{2^{k+1}} - \frac{u}{6} .$$

one checks $a_k = -\frac{u^3}{36\cdot 2^{2k}} + O(\frac{u^4}{2^{4k}})$ and then the series $\sum_{k\geq 1} a_k$ is convergent for $u = t^{1/2}$ small enough. □

Remark 1.2.2 *As $z_(t,\epsilon)$ is not uniquely determined, if $\psi(t)$ is a solution of 1.2 then $\psi(t) + Kt^{1/2}\sin t^{1/2}$, K being a constant, is also a solution of 1.2.*

Proposition 1.2.3 *Let $\psi(t) = D_2z_-(t,0)$. Then for t large enough there is a constant $c_1 > 0$ such that $|\psi(t)| \leq c_1 t$, for $t > 0$. Furthermore $\psi(4^m\pi^2(2j+1)^2) = \frac{1}{3} - \frac{4^m}{3}$ for $m \geq 0$.*

Proof:

Dividing by $4t$ the formula 1.2 and rearranging we have

$$\frac{\psi(4t)}{4t} = -\frac{\cos^2 t^{1/2}}{4t} - \frac{\sin 2t^{1/2}}{24t^{1/2}} + \cos t^{1/2}\frac{\psi(t)}{t} \ .$$

Therefore

$$\left|\frac{\psi(4t)}{4t}\right| \leq \frac{1}{4t} + \frac{1}{24t^{1/2}} + \left|\frac{\psi(t)}{t}\right| \leq \frac{4}{4t} + \frac{2}{24t^{1/2}} + \left|\frac{\psi(t/4)}{t/4}\right| + \frac{1}{4t} + \frac{1}{24t^{1/2}} \leq \ldots$$

$$\leq \frac{1}{4t}(1+4+\ldots+4^k) + \frac{1}{24t^{1/2}}(1+2+\ldots+2^k) + \left|\frac{\psi(t/4^k)}{t/4^k}\right|$$

$$< \frac{4^{k+1}}{12t} + \frac{2^{k+1}}{24t^{1/2}} + \left|\frac{\psi(t/4^k)}{t/4^k}\right| \ .$$

If $t > 4$ there exists $k \in \mathbb{N}$ such that $4^k \leq t < 4^{k+1}$. Then

$$\left|\frac{\psi(4t)}{4t}\right| \leq \frac{1}{3} + \frac{1}{12} + \left|\frac{\psi(t/4^k)}{t/4^k}\right| \leq \frac{5}{12} + A \ ,$$

where $A = \max_{r\in[1,4)} |\psi(r)/r|$. Taking $c_1 = \frac{5}{12} + A$ we have $\psi(t) \leq c_1 t$ if $t > 16$.

The second part of this proposition is checked immediately. □

We can also find a bound of $D_2 z_-(t,\epsilon)$ independent of ϵ.

Proposition 1.2.4 *Let $\psi(t,\epsilon) = D_2 z_(t,\epsilon)$. Then for a fixed $r > 1$ and t large enough there is a constant A_r such that $|\psi(t,\epsilon)| \leq A_r t^r$ and A_r is independent of ϵ for ϵ small enough.*

Proof:

First we show $|D_1(z_-(t,\epsilon)| \leq A$ for all t and ϵ small enough and a suitable constant A.

From the recurrent relation we obtain

$$D_1 z_-(t,\epsilon) = -\frac{2a}{\alpha} D_1 z_-(\alpha^{-1}t,\epsilon) z_-(\alpha^{-1}t,\epsilon) =$$

$$-\frac{1}{x_-} D_1 z_-(\alpha^{-1}t,\epsilon) z_-(\alpha^{-1}t,\epsilon) \ .$$

Therefore

$$|D_1 z_-(t,\epsilon)| \leq |D_1 z_-(\alpha^{-1}t,\epsilon)| \ ,$$

that is, we can take $A = 1/2$ by using a determination of z_- with $D_1 z_-(0,\epsilon) = 1/2$.

Now we divide by $\alpha^r t^r$ the formula 1.1 of 1.2.1 and we obtain

$$\left|\frac{D_2 z_-(\alpha t,\epsilon)}{\alpha^r t^r}\right| \leq \frac{x_-^2}{\alpha^r t^r} + \left|\frac{D_2 z_-(t,\epsilon)}{t^r}\right| + \left|\frac{d\alpha}{d\epsilon}\right| \frac{1}{2\alpha^r t^{r-1}} \ .$$

If ϵ is small x_-^2 and $|\frac{d\alpha}{\epsilon}|$ are bounded and we can put

$$\left|\frac{D_2 z_-(\alpha t,\epsilon)}{\alpha^r t^r}\right| \leq \frac{B_1}{\alpha^r t^r} + \frac{B_2}{\alpha^r t^{r-1}} + \left|\frac{D_2 z_-(t,\epsilon)}{t^r}\right| \ .$$

By iteration of the last inequality we have

$$\left|\frac{D_2 z_-(\alpha t,\epsilon)}{\alpha^r t^r}\right| \le \frac{B_1}{\alpha^r t^r}(1+\alpha^r+\ldots+\alpha^{nr})+$$

$$\frac{B_2}{\alpha^r t^{r-1}}(1+\alpha^{r-1}+\ldots+\alpha^{n(r-1)})+\left|\frac{D_2 z_-(\alpha^{-n}t,\epsilon)}{t^r/\alpha^{nr}}\right| \le \frac{B_1}{\alpha^r t^r}\alpha^{(n+1)r}+$$

$$\frac{B_2}{\alpha^r t^{r-1}}\alpha^{(n+1)(r-1)}+\left|\frac{D_2 z_-(\alpha^{-n}t,\epsilon)}{t^r/\alpha^{nr}}\right| .$$

For fixed values of ϵ and t there is an $n \in \mathbb{N}\bigcup\{0\}$ such that $\alpha^n \le t < \alpha^{n+1}$. Using this value of n in the last bound, we obtain

$$\left|\frac{D_2 z_-(\alpha t,\epsilon)}{\alpha^r t^r}\right| \le B_1+\frac{B_2}{\alpha}+\max_{u\in[1,\alpha)}\left|\frac{D_2 z_-(u,\epsilon)}{u^r}\right| \le A_r \ .\square$$

Proposition 1.2.5 *Let $f(x,\epsilon)=1-(2+\epsilon)x^2$. For a fixed value of $j\in \mathbb{N}\cup\{0\}$ there are parameters $\epsilon_{n,j}$ for n large enough, such that $f(\cdot,\epsilon_{n,j})$ has a superstable periodic orbit of period n, and:*

$$\epsilon_{n,j}=-\frac{3}{32}\pi^2(2j+1)^2\alpha^{-(n-2)}+\frac{1}{2}\left[\frac{3}{512}\pi^4(2j+1)^4-\right.$$

$$\left.(-1)^j\frac{9}{128}\varphi\left(\frac{\pi}{2}(2j+1)\right)\pi^3(2j+1)^3)\alpha^{-2(n-2)}+O(\alpha^{-3(n-2)}\right],$$

where $\alpha=1+\sqrt{9+4\epsilon_{n,j}}$. and φ is the function introduced in proposition 1.2.1

Proof:

To prove 1.2.5 we will use the representation of $\mathcal{W}^u(x_-)$ given in 1.1.3, recalling the relations $f(z_-(t,\epsilon),\epsilon)=z_-(\alpha t,\epsilon)$, $z_-(0,\epsilon)=x_-$ and $D_1 z_-(0,0)=1/2$, because $z_-(t,0)=-\cos t^{1/2}$, and the equation $f^n(0,\epsilon_{n,j})=0$ defining $\epsilon_{n,j}$. First we state and prove an auxiliar lemma.

Lemma 1.2.6 *Given $j\ge 0$ there are functions $t_j=t_j(s)$, $\epsilon_j=\epsilon_j(s)$ defined in a neighbourhood of 0 such that $t_j(0)=(\frac{\pi}{2})^2(2j+1)^2$, $\epsilon_j(0)=0$ and satisfying*

$$z_-(t_j,\epsilon_j)=0 \quad , \quad z_-(st_j,\epsilon_j)=-1-\epsilon_j \quad , \qquad (1.3)$$

if s is small enough. Furthermore

$$\epsilon_j(s)=-\frac{3}{32}\pi^2(2j+1)^2 s+\frac{1}{2}\left[\frac{3}{512}\pi^4(2j+1)^4-\right.$$

$$\left.(-1)^j\frac{9}{128}\varphi\left(\frac{\pi}{2}(2j+1)\right)\pi^3(2j+1)^3\right]s^2+O(s^3).$$

Proof:

First of all it is easy to check that if $t_{j,0} = t_j(0) = (\frac{\pi}{2})^2(2j+1)^2$ and $\epsilon_{j,0} = \epsilon_j(0) = 0$, then

$$z_-(t_{j,0}, \epsilon_{j,0}) = 0 \ , \ z_-(0, \epsilon_{j,0}) = -1 - \epsilon_{j,0} \ ,$$

since $z_-(t,0) = -\cos t^{1/2} = 0$ if and only if $t = (\frac{\pi}{2})^2(2j+1)^2$, and the definition of $z_-(t,\epsilon)$. To show the existence of $t_j = t_j(s)$, $\epsilon_j = \epsilon_j(s)$ near $s = 0$ we apply the Implicit Function Theorem. If $F_1(t,\epsilon,s) = z_-(t,\epsilon)$ and $F_2(t,\epsilon,s) = z_-(st,\epsilon) + 1 + \epsilon$, as suggested by 1.3, we should check

$$\left(\frac{\partial F_1}{\partial t}\frac{\partial F_2}{\partial \epsilon} - \frac{\partial F_2}{\partial t}\frac{\partial F_1}{\partial \epsilon}\right)_{|(t,\epsilon,s)=(t_{j,0},0,0)} \neq 0 \ .$$

One obtains inmediately at the given point the values

$$\frac{\partial F_1}{\partial t} = (-1)^j \frac{(2j+1)^{-1}}{\pi} \ , \ \frac{\partial F_2}{\partial \epsilon} = \frac{4}{3} \ , \ \frac{\partial F_2}{\partial t} = 0 \ .$$

Hence the functions $t_j(s)$ and $\epsilon_j(s)$ exist in a neighbourhood of $s = 0$. To finish the proof of 1.2.6 we only need $\frac{d\epsilon_j}{s}(0)$, $\frac{d^2\epsilon_j}{ds^2}(0)$. The first one is obtained by derivation of $F_2(t,\epsilon,s) = 0$ with respect to s and substitution of s by 0.

From $\frac{d^2}{ds^2}(F_2(t,\epsilon,s)) = 0$ and substitution of s by 0 we obtain

$$\frac{d^2\epsilon_j}{ds^2}(0)$$

in terms of known quantities and $\frac{dt_j}{ds}(0)$. This value is given by computing $\frac{d}{ds}(F_1(t,\epsilon,s)) = 0$ and insertion of $s = 0$. □

To finish the proof of 1.2.5 we use 1.2.6 with $s = (\alpha(\epsilon_j))^{-(n-2)}$. From 1.3 we have

$$z_-(t_j(s), \epsilon_j(s)) = 0 \ , \ z_-(st_j(s), \epsilon_j(s)) = -1 - \epsilon_j(s) \ .$$

But

$$z_-(t,\epsilon) = f^{n-2}(z_-(\alpha^{-(n-2)}t,\epsilon),\epsilon) = f^{n-2}(f^2(0,\epsilon),\epsilon) = f^n(0,\epsilon) = 0 \ ,$$

showing that $\epsilon_{n,j}$ is the parameter corresponding to the superstable periodic orbit. □

Remark 1.2.7 *Notice that the function*

$$m_j(s) = \frac{1}{n_j(s)} = -\frac{\log(\alpha(\epsilon_j(s)))}{\log s}$$

exists in a neighbourhood of 0 *(depending on* j*) and in it, it satisfies* $m_j(0) = 0$, $m_j(s) \geq 0$. *Furthermore* $\frac{dm_j}{ds}(s) > 0$ *if* $s \neq 0$, *because*

$$\frac{dm_j}{ds}(s) = -\frac{d\alpha(\epsilon_j(s))/ds}{\alpha(\epsilon_j(s))\log s} + \frac{\log(\alpha(\epsilon_j(s)))}{s(\log s)^2}$$

and $\alpha(0) = 4$, $\frac{d}{ds}\alpha(\epsilon_j(s))_{|s=0} = -\frac{\pi^2}{16}(2j+1)^2$. *Therefore there exists a differentiable function defined on* $(0,\xi)$, $\xi > 0$, *such that* $s = s_j(m,)$ *and there also exist* $n = n_j(s)$ *and* $s = \tilde{s}_j(n)$, *in the same interval, such that* $\lim_{s\to 0} n_j(s) = +\infty$, $\lim_{n\to\infty} \tilde{s}_j(n) = 0$.

It is also interesting to give an explicit expression of the values of the parameters for which there is a superstable periodic orbit.

Proposition 1.2.8 *Let* $\epsilon_{n,j}$ *be the parameters for which there is the superstable periodic orbit of the logistic map given in proposition 1.2.5. Then:*

$$\epsilon_{n,j} = -\frac{6}{4^n}\frac{\pi^2}{4}(2j+1)^2 - \frac{6n}{4^{2n}}\frac{\pi^4}{16}(2j+1)^4 + \left[24\frac{\pi^4}{16}(2j+1)^4 - \right.$$

$$\left. (-1)^j 72\varphi\left(\frac{\pi}{2}(2j+1)\right)\frac{\pi^3}{8}(2j+1)^3\right] 4^{-2n} + O(n^2 4^{-3n})$$

Proof:

From proposition 1.2.5 we know:

$$\epsilon_{n,j} = -\frac{6}{4^2}\left(\frac{\pi}{2}\right)^2 (2j+1)^2 s(n-2) + \frac{1}{2}\left[\frac{3}{32}\left(\frac{\pi}{2}\right)^4 (2j+1)^4 - \right.$$

$$\left. (-1)^j \frac{9}{16}\varphi\left(\frac{\pi}{2}(2j+1)\right)\left(\frac{\pi}{2}\right)^3 (2j+1)^3\right] s(n-2)^2 + O(s(n-2)^3), \qquad (1.4)$$

where $s(n) = \alpha(\epsilon(s))^{-n}$.

It is easy to see that

$$\alpha(\epsilon) = 1 + \sqrt{9+4\epsilon} = 4 + \frac{2}{3}\epsilon - \frac{2}{27}\epsilon^2 + O(\epsilon^3) < 4 \text{ if } \epsilon < 4.$$

Let

$$\epsilon_{n,j} = c_1(j)s(n-2) + c_2(j)s(n-2)^2 + O(s(n-2)^3).$$

Then

$$s(n) = 4^{-n}\left\{1 + \frac{1}{6}\epsilon_{n+2,j} - \frac{1}{54}\epsilon_{n+2,j}^2 + O(\epsilon_{n+2,j}^3)\right\}^{-n} =$$

$$4^{-n}\left\{1 + \frac{1}{6}c_1 s(n) + \left(\frac{1}{6}c_2 - \frac{1}{54}c_1^2\right)s(n)^2 + O(s(n)^3)\right\}^{-n} =$$

$$4^{-n}\left\{1 - n\left[\frac{1}{6}c_1 s(n) + \left(\frac{1}{6}c_2 - \frac{1}{54}c_1^2\right)s(n)^2 + O(s(n)^3)\right] + \right.$$

$$\left.\binom{n+1}{2}\left[\frac{1}{6}c_1 s(n) + \left(\frac{1}{6}c_2 - \frac{1}{54}c_1^2\right)s(n)^2 + O(s(n)^3)\right]^2 + O(n^3 s(n)^3)\right\} =$$

$$4^{-n}\left\{1 - \frac{n}{6}c_1 s(n) + \frac{n^2}{72}c_1^2 s^2 + \left(-\frac{7}{216}c_1^2\right)ns(n)^2 + o(s(n)^2)\right\}.$$

Therefore

$$s(n) = 4^{-n} - \frac{n}{6}c_1 4^{-2n} + \frac{n^2}{24}c_1^2 4^{-3n} + \left(-\frac{1}{6}c_2 + \frac{7}{216}c_1^2\right) n4^{-3n} + O(4^{-3n}).$$

As

$$c_1 = -\frac{6}{4^2}\left(\frac{\pi}{2}\right)^2 (2j+1)^2,$$

we have

$$s(n-2) = 4^{-(n-2)} + \frac{1}{4^2}\left(\frac{\pi}{2}\right)^2 (2j+1)^2 4^{-2(n-2)}(n-2) + O(n^2 4^{-3n})$$

and by substituing in 1.4 we obtain the result. □

Another important question is to have analytical estimates of the parameters ϵ for which there is a homoclinic tangency, that is, such that $f^n(0,\epsilon) = x_+,\ n \geq 3\ \ (x_+ = \frac{-1+\sqrt{9+4\epsilon}}{4+2\epsilon})$.

Proposition 1.2.9 *Let $f(\cdot,\epsilon)$ the logistic map. Then, given $j \geq 0$ and for n large enough, there are parameters*

$$\tilde{\epsilon}^1_{n,j} = -\frac{3}{2}\pi^2\left(\frac{1}{6}+j\right)^2 \alpha^{-n+3} + O(\alpha^{-2n}),$$

and

$$\tilde{\epsilon}^2_{n,j} = -\frac{3}{2}\pi^2\left(\frac{5}{6}+j\right)^2 \alpha^{-n+3} + O(\alpha^{-2n}),$$

such that $f^n(0,\tilde{\epsilon}^k_{n,j}) = x_+$ for $k = 1,2$.

To prove proposition 1.2.9 we use the expression of $\mathcal{W}^u(x_-),\ z_-(t,\epsilon)$. To say $f^n(0,\epsilon) = x_+$ is equivalent to

$$z_-(\alpha^{n-3}t,\epsilon) = -x_+, \qquad z_-(t,\epsilon) = -1-\epsilon,$$

because

$$z_-(\alpha^{n-3}t,\epsilon) = f^{n-3}(z_-(t,\epsilon),\epsilon) = f^{n-1}(0,\epsilon) = -x_+,$$

and then $f^n(0,\epsilon) = x_+$. First we need a lemma.

Lemma 1.2.10 *There are functions $t^k_j = t^k_j(s)$ and $\tilde{\epsilon}^k_j = \tilde{\epsilon}^k_j(s),\ k = 1,2$, smooth for s small enough, such that $t^1_j(0) = 4\pi^2(\frac{1}{6}+j)^2,\ t^2_j(0) = 4\pi^2(\frac{5}{6}+j)^2,\ \tilde{\epsilon}^k_j(0) = 0$ for $k = 1,2$, satisfying the system:*

$$z_-(t,\epsilon) = -x_+, \qquad z_-(st,\epsilon) = -1-\epsilon \tag{1.5}$$

for $t = t^k_j(s)$, $\epsilon = \tilde{\epsilon}^k_j$. Furthermore

$$\tilde{\epsilon}^1_j(s) = -\frac{3}{2}\pi^2\left(\frac{1}{6}+j\right)^2 s + O(s^2), \quad \tilde{\epsilon}^2_j(s) = -\frac{3}{2}\pi^2\left(\frac{5}{6}+j\right)^2 s + O(s^2).$$

Proof:

Let

$$F_1(t,\epsilon,s) = z_-(t,\epsilon) + x_+, \quad F_2(t,\epsilon,s) = z_-(st,\epsilon) + 1 + \epsilon.$$

Then 1.5 becomes:

$$F_1(t,\epsilon,s) = 0, \quad F_2(t,\epsilon,s) = 0. \tag{1.6}$$

For $s = 0$ we have $z_-(0,\epsilon) = x_- = -(1+\sqrt{9+4\epsilon})/(4+2\epsilon)$, and, as $F_2(t,\epsilon,s) = 0$, we should have $z_-(0,\epsilon) = -1-\epsilon$. Then $\epsilon = 0$ or $\epsilon = -2$. We skip this last case. On the other hand, if $\epsilon = 0$, as $f_1(t,\epsilon,s) = 0$ we have $\cos t^{1/2} = x_+(0) = 1/2$. Therefore $t^{1/2} = \frac{\pi}{3} + 2j\pi$, $j \geq 0$ or $t^{1/2} = \frac{5\pi}{3} + 2j\pi$, $j \geq 0$. So we have the solutions of 1.6 with $s = 0$:

$$t_j^1(0) = 4\pi^2\left(\frac{1}{6}+j\right)^2, \quad t_j^2(0) = 4\pi^2\left(\frac{5}{6}+j\right)^2, \quad \tilde{\epsilon}_j^1(0) = \tilde{\epsilon}_j^2(0) = 0.$$

To see the existence of the functions $t_j^k(s)$, $\tilde{\epsilon}_j^k(s)$ we apply the implicit function theorem. One should see:

$$\frac{\partial F_1}{\partial t}\frac{\partial F_2}{\partial \epsilon} - \frac{\partial F_1}{\partial \epsilon}\frac{\partial F_2}{\partial t}_{|t=t_j^k(0),\epsilon=\tilde{\epsilon}_j^k(0),s=0} \neq 0.$$

This holds because

$$\begin{aligned}
\tfrac{\partial F_1}{\partial \epsilon}(t_j^k(0),0,0) &= D_2 z_-(t_j^k(0),0) + \tfrac{dx_+}{d\epsilon}(0),\\
\tfrac{\partial F_1}{\partial t}(t_j^k(0),0,0) &= D_1 z_-(t_j^k(0),0) = \pm\tfrac{1}{2}(t_j^k(0))^{-1/2}\tfrac{\sqrt{3}}{2} \neq 0,\\
\tfrac{\partial F_2}{\partial \epsilon}(t_j^k(0),0,0) &= D_2 z_-(0,0)1 = \tfrac{dx_-}{d\epsilon}(0) + 1 = \tfrac{4}{3} \neq 0,\\
\tfrac{\partial F_2}{\partial t}(t_j^k(0),0,0) &= D_1 z_-(0,0,s)_{|s=0} = 0.
\end{aligned}$$

Hence $t_j^k(s)$ and $\tilde{\epsilon}_j^k(s)$ exist in a neighbourhood of $s = 0$ and they are smooth. Furthermore

$$D_1 z_-(0,0)t_j^k(0) + D_2 z_-(0,0)\frac{d\tilde{\epsilon}_j^k}{ds}(0) = -\frac{d\tilde{\epsilon}_j^k}{ds}(0).$$

Then

$$2\pi^2\left(\frac{1}{6}+j\right)^2 + \frac{4}{3}\frac{d\tilde{\epsilon}_j^1}{ds}(0) = 0, \quad 2\pi^2\left(\frac{5}{6}+j\right)^2 + \frac{4}{3}\frac{d\tilde{\epsilon}_j^2}{ds}(0) = 0.$$

From this we obtain $\frac{d\tilde{\epsilon}_j^k}{ds}(0)$, $k = 1,2$ and the lemma follows. □

To prove the proposition let $s = \alpha^{-(n-3)}$ similar to proposition 1.2.5. Then it holds:

$$z_-(t_j^k, \tilde{\epsilon}_j^k) = -x_+, \quad z_-(\alpha^{-(n-3)}t_j^k, \tilde{\epsilon}_j^k) = -1 - \tilde{\epsilon}_j^k.$$

Therefore $f^n(0,\tilde{\epsilon}_j^k) = x_+$ and the expressions of $\tilde{\epsilon}_{n,j}^k = \tilde{\epsilon}_j^k(\alpha_2^{-(n-3)})$ are the ones given before. □

k	m_k
1	1
2	0
3	1
4	1
5	3
6	4
7	9
8	14
9	28
10	48
11	93
12	165
13	315
14	576
15	1091
16	2032
17	3855
18	7252
19	13797
20	26163
21	49929
22	95232
23	182361
24	349350
25	671088
26	1290240
27	2485504
28	4792905
29	9256395
30	17894588

Table 1.1: Number of superstable periodic orbits associated to saddle-node bifurcations.

From the last propositions the next result follows easily because $\alpha \to 4$ if $\epsilon \to 0$.

Corollary 1.2.11 *Let $\epsilon_{n,j}$, $\tilde{\epsilon}^1_{n,j}$, and $\tilde{\epsilon}^2_{n,j}$ be the values of the previous propositions. Then:*

$$\lim_{n\to\infty} \frac{\epsilon_{n,j}}{\epsilon_{n+1,j}} = \lim_{n\to\infty} \frac{\tilde{\epsilon}^1_{n,j}}{\tilde{\epsilon}^1_{n+1,j}} = \lim_{n\to\infty} \frac{\tilde{\epsilon}^2_{n,j}}{\tilde{\epsilon}^2_{n+1,j}} = 4.$$

Remark 1.2.12 *We have seen that the superstable periodic orbits accumulate at $\epsilon = 0$, for which we have a homoclinic tangency (because $f_2^2(0) = x_-(0) = -1$), with a geometrical ratio 4 which is the derivative of $f(\cdot, 0)$ at $x_-(0) = -1$. This is a general fact: for any homoclinic tangency associated to a periodic point, there is an accumulation of this type.*

The next proposition gives more information on the width of the windows of attraction, because it gives the asymptotic expression of the difference of values of ϵ between a saddle-node bifurcation and the corresponding flip.

Proposition 1.2.13 *Let $f(x,\epsilon) = 1 - (2+\epsilon)x^2$ be the logistic map. Then for a fixed $j \geq 0$ and n large enough there are parameters $\epsilon^+_{n,j}$, $\epsilon^-_{n,j}$ and points $x^+_{n,j}$, $x^-_{n,j}$ such that $f^n(x^+_{n,j}, \epsilon^+_{n,j}) = x^+_{n,j}$, $D_1 f^n(x^+_{n,j}, \epsilon^+_{n,j}) = 1$, $f^n(x^-_{n,j}, \epsilon^-_{n,j}) = x^-_{n,j}$, $D_1 f^n(x^-_{n,j}, \epsilon^-_{n,j}) = -1$, and*

$$\lim_{n\to\infty} \frac{\epsilon^-_{n,j} - \epsilon^+_{n,j}}{4^{-2n}} = 6\pi^(2j+1).$$

Proof:

We shall proceed as in the preceding propositions, that is, to display a sytem of equations for $\epsilon^\pm_{n,j}$, $x^\pm_{n,j}$ and to apply the implicit function theorem. We need first several lemmata.

Lemma 1.2.14 *If $\epsilon^+_{n,j}$ and $x^+_{n,j}$, $\epsilon^-_{n,j}$ and $x^-_{n,j}$ satisfy the equations studied in proposition 1.2.13, then there are $t^+_{n,j}$, $t^-_{n,j} \in \mathbb{R}$ such that $\epsilon^+_{n,j}$ and $t^+_{n,j}$ satisfy the system:*

$$z_-(t^+_{n,j}, \epsilon^+_{n,j}) - z_-(\alpha^{-n} t^+_{n,j}, \epsilon^+_{n,j}) = 0,$$

$$D_1 z_-(t^+_{n,j}, \epsilon^+_{n,j}) - \alpha^{-n} D_1(\alpha^{-n} t^+_{n,j}, \epsilon^+_{n,j}) = 0,$$

and $x^+_{n,j} = z_-(t^+_{n,j}, \epsilon^+_{n,j})$. Also $\epsilon^-_{n,j}$ and $t^-_{n,j}$ satisfy

$$z_-(t^-_{n,j}, \epsilon^-_{n,j}) - z_-(\alpha^{-n} t^-_{n,j}, \epsilon^-_{n,j}) = 0,$$

$$D_1 z_-(t^-_{n,j}, \epsilon^-_{n,j}) + \alpha^{-n} D_1(\alpha^{-n} t^-_{n,j}, \epsilon^-_{n,j}) = 0,$$

and $x^-_{n,j} = z_-(t^-_{n,j}, \epsilon^-_{n,j})$. Here $\alpha = -2(2+\epsilon)x_-(\epsilon)$ and $x_-(\epsilon) = (-1-\sqrt{9+4\epsilon})/(2\epsilon+4)$.

Proof:

We shall do it only for $\epsilon^+_{n,j}$, the other case being analogous. Let $t^+_{n,j} \in \mathbb{R}$ such that $D_1 z_-(\alpha^{-n} t^+_{n,j}, \epsilon^+_{n,j}) = x^+_{n,j}$. We remark that one can always take a $t^+_{n,j} \in \mathbb{R}$ such that $D_1 z_-(\alpha^{-n} t^+_{n,j}, \epsilon^+_{n,j}) \neq 0$. Otherwise one would have $x^+_{n,j} = 1$ (see theorem 1.1 in chapter 2), and this is not possible because then $D_1 f^n(x^+_{n,j}, \epsilon^+_{n,j}) = 0$.

On the other hand, as $f^n(z_-(\alpha^{-n}t, \epsilon), \epsilon) = z_-(t, \epsilon)$, by derivation with respect to t we have:

$$D_1 f^n(z_-(\alpha^{-n}t, \epsilon), \epsilon) D_1 z_-(\alpha^{-n}t, \epsilon)\alpha^{-n} = D_1 z_-(t, \epsilon),$$

and therefore

$$D_1 f^n(z_-(\alpha^{-n}t, \epsilon), \epsilon) = \frac{D_1 z_-(t, \epsilon)}{D_1 z_-(\alpha^{-n}t, \epsilon)\alpha^{-n}}.$$

Applying this to the point $(t^+_{n,j}, \epsilon^+_{n,j})$ and taking into account

$$D_1 f^n(x^+_{n,j}, \epsilon^+_{n,j}) = D_1 f^n(z_-(\alpha^{-n} t^+_{n,j}, \epsilon^+_{n,j}), \epsilon^+_{n,j}),$$

we obtain

$$D f^n(z_-(\alpha^{-n} t^+_{n,j}, \epsilon^+_{n,j}), \epsilon^+_{n,j}) = \frac{D_1 z_-(t^+_{n,j}, \epsilon^+_{n,j})}{\alpha^{-n} D_1 z_-(\alpha^{-n} t^+_{n,j}, \epsilon^+_{n,j})} = 1.$$

Hence

$$D_1 z_-(t^+_{n,j}, \epsilon^+_{n,j}) - \alpha^{-n} D_1 z_-(\alpha^{-n} t^+_{n,j}, \epsilon^+_{n,j}) = 0.$$

Furthermore we need that $x^+_{n,j}$ be an n-periodic point. Therefore:

$$z_-(t^+_{n,j}, \epsilon^+_{n,j}) - z_-(\alpha^{-n} t^+_{n,j}, \epsilon^+_{n,j}) = 0. \square$$

In the next lemma we use $s = \alpha^{-n}$ and the implicit function theorem to show the existence of ϵ^+_j :

Lemma 1.2.15 *There exist functions $t^+_j = t^+_j(s)$, $t^-_j = t^-_j(s)$, $\epsilon^+_j = \epsilon^+_j(s)$, $\epsilon^-_j = \epsilon^-_j(s)$, in a neighbourhood of $s = 0$ such that $t^+_j(0) = t^-_j(0) = 4(2j+1)^2\pi^2$, $\epsilon^+_j(0) = \epsilon^-_j(0) = 0$, and they satisfy the equations:*

$$\left.\begin{array}{rcl} z_-(t^+_j, \epsilon^+_j) - z_-(st^+_j, \epsilon^+_j) &=& 0 \\ D_1 z_-(t^+_j, \epsilon^+_j) - sD_1 z_-(st^+_j, \epsilon^+_j) &=& 0 \end{array}\right\} \qquad (1.7)$$

$$\left.\begin{array}{rcl} z_-(t^-_j, \epsilon^-_j) - z_-(st^-_j, \epsilon^-_j) &=& 0 \\ D_1 z_-(t^-_j, \epsilon^-_j) + sD_1 z_-(st^-_j, \epsilon^-_j) &=& 0 \end{array}\right\} \qquad (1.8)$$

Proof:

We show it for ϵ_j^+ and t_j^+. Putting $s = 0$ in 1.7 we have:

$$z_-(t,\epsilon) = x_-(\epsilon),$$

$$D_1 z_-(t,\epsilon) = 0.$$

This means $\epsilon = 0$ because if $\epsilon \neq 0$ and $z_-(t,\epsilon) = x_-(t,\epsilon)$ then $D_1 z_-(t,\epsilon) \neq 0$ (see theorem 1.1 in chapter 2). On the other hand, if $z_-(t,\epsilon) = x_-(\epsilon)$ one should have $t < 0$ (if $\epsilon < 0$). We skip this case. Therefore $z_-(t,0) = -1$ and $D_1 z_-(t,0) = 0$. Hence: $t(0) = (2j'\pi)^2 = 4^m(2j+1)^2\pi^2$, $m \geq 1$ and $\epsilon(0) = 0$.

At this point we can take $m = 1$ because if $(t(s), \epsilon(s))$ is a solution of 1.7 then $(\alpha^m t(s), \epsilon(s))$, where $\alpha = \alpha(\epsilon(s)) = 1 + \sqrt{9 + 4\epsilon}$, also is. Indeed:

$$z_-(\alpha^m t,\epsilon) - z_-(\alpha^m st,\epsilon) = f^m(z_-(t,\epsilon),\epsilon) - f^m(z_-(st,\epsilon),\epsilon) = 0.$$

Furthermore

$$\alpha^m D_1 z_-(\alpha^m t,\epsilon) = D_1 z_-(t,\epsilon) D_1 f^m(z_-(t,\epsilon),\epsilon),$$

$$\alpha^m D_1 z_-(\alpha^m st,\epsilon) = D_1 z_-(st,\epsilon) D_1 f^m(z_-(st,\epsilon),\epsilon) = D_1 z_-(st,\epsilon) D_1 f^m(z_-(t,\epsilon),\epsilon),$$

and the also $D_1 z_-(\alpha^m t,\epsilon) - sD_1 z(\alpha^m st,\epsilon) = 0$.

In particular, the only solutions which are of interest for us are $t_j^\pm(s)$ and $\epsilon_j^\pm(s)$ such that $t_{j0}^\pm = t_j^\pm(0) = 4(2j+1)^2\pi^2$ and $\epsilon_{j0}^\pm = \epsilon_j^\pm(0) = 0$.

To see the existence of $t_j^\pm, \epsilon_j^\pm$ we use the implicit function theorem. Let

$$F_1(t,\epsilon,s) = z_-(t,\epsilon) - z_-(st,\epsilon)$$

and

$$F_2^-(t,\epsilon,s) = D_1 z_-(t,\epsilon) - D_1 z_-(t,\epsilon)s,$$

and also

$$F_2^+(t,\epsilon,s) = D_1 z_-(t,\epsilon) + D_1 z_-(t,\epsilon)s.$$

Then the systems 1.7 and 1.8 can be written as:

$$\left.\begin{array}{rcl} F_1(t,\epsilon,s) & = & 0, \\ F_2^-(t,\epsilon,s) & = & 0 \end{array}\right\}. \tag{1.9}$$

$$\left.\begin{array}{rcl} F_1(t,\epsilon,s) & = & 0, \\ F_2^+(t,\epsilon,s) & = & 0 \end{array}\right\}. \tag{1.10}$$

To be allowed to apply the implicit function theorem, one should show

$$\frac{\partial F_1}{\partial t}\frac{\partial F_2^-}{\partial \epsilon} - \frac{\partial F_1}{\partial \epsilon}\frac{\partial F_2^-}{\partial t}_{|(t,\epsilon,s)=(t_{j0}^+,0,0)} \neq 0,$$

and the same for F_2^+ and t_{j0}^-. We have

a) $\frac{\partial F_1}{\partial t} = 0$. Indeed:

$$\frac{\partial F_1}{\partial t} = D_1 z_-(t,\epsilon) - D_1 z_-(st,\epsilon)s = F_2^-(t,\epsilon,s).$$

Then for $\epsilon = 0$ and $t = 4(2j+1)^2\pi^2$ we get $F_2^-(4(2j+1)^2\pi^2,0,0) = 0$.

b) $\frac{\partial F_1}{\partial \epsilon}(t_{j0}^{\pm},0,0) = -\frac{4}{3}$. Indeed:

$$\frac{\partial F_1}{\partial \epsilon}(t,\epsilon,s) = D_2 z_-(t,\epsilon) - D_2 z_-(st,\epsilon).$$

Putting $\epsilon = 0$ and $t = 4(2j+1)^2\pi^2$ we obtain

$$\frac{\partial F_1}{\partial \epsilon}(t_{j0}^{\pm},0,0) = D_2 z_-(t_{j0}^{\pm},0) - D_2 z_-(0,0) = D_2 z_-(t_{j0}^{\pm},0) - \frac{dx_-}{d\epsilon}(0).$$

Taking into account $x_-(0) = -1$ and $1 - (2+\epsilon)x_-^2(\epsilon) = x_-(\epsilon)$ we have $\frac{dx_-}{d\epsilon}(0) = \frac{1}{3}$.

In the proposition 1.2.3 we have seen that $\psi(t) = D_2 z_-(t,0)$ satisfies $\psi(4\pi^2(2j+1)^2) = -1$. Hence:

$$\frac{\partial F_1}{\partial \epsilon}(4(2j+1)^2\pi^2,0,0) = -\frac{4}{3}.$$

c) $\frac{\partial F_2^{\pm}}{\partial t}(4(2j+1)^2\pi^2,0,0) = \frac{1}{16(2j+1)^2\pi^2}$. Indeed:

$$\frac{\partial F_2^{\pm}}{\partial t}(4(2j+1)^2\pi^2,0,0) D_{11} z_-(4(2j+1)^2\pi^2,0).$$

As $z_-(t,0) = -\cos t^{1/2}$, c) holds.

This means:

$$\frac{\partial F_1}{\partial t}\frac{\partial F_2^-}{\partial \epsilon} - \frac{\partial F_1}{\partial \epsilon}\frac{\partial F_2^{\mp}}{\partial t}_{|(t,\epsilon,s)=(t_{j0}^{\pm},0,0)} = \frac{\partial F_1}{\partial \epsilon}\frac{\partial F_2^{\mp}}{\partial t}_{|(t,\epsilon,s)=(t_{j0}^{\pm},0,0)} = -\frac{1}{12(2j+1)^2\pi^2} \neq 0.$$

This shows the existence of $t_j^{\pm}(s)$ and $\epsilon_j^{\pm}(s)$ as desired. □

The next step is the computation of the parameters ϵ_j^+ and ϵ_j^- up to second order with respect to s.

Lemma 1.2.16 *Let ϵ_j^+ and ϵ_j^- be as in the previous lemma. Then*

$$\epsilon_j^-(s) - \epsilon_j^+(s) = 6\pi^2(2j+1)^2 s^2 + O(s^3).$$

Proof:

We know that 1.9 holds for $\epsilon = \epsilon_j^+$ and $t = t_j^+$, and 1.10 for $\epsilon = \epsilon_j^-$ and $t = t_j^-$.

To obtain expressions for $\epsilon_j^+(s)$ and $\epsilon_j^-(s)$ we compute derivatives of F_1, F_2^+ and F_2^- :

$$\frac{\partial F_1}{\partial s} = -tD_1 z_-(st,\epsilon),$$

$$\frac{\partial F_2^\pm}{\partial s} = \pm D_1 z_-(st,\epsilon) \pm stD_{11} z_-(st,\epsilon).$$

When $s=0$ we have:

$$\frac{\partial F_1}{\partial s}(t_{j0}^\pm,0,0) = -2(2j+1)^2\pi^2,$$

since $D_1 z_-(0,0) = \frac{1}{2}$.

Hence

$$\frac{d\epsilon_j^\pm}{ds}(0) = -\frac{\frac{\partial F_1}{\partial s}}{\frac{\partial F_1}{\partial \epsilon}} = -6(2j+1)^2\left(\frac{\pi}{2}\right)^2.$$

This means that:

$$\epsilon_j^\pm(s) = -6(2j+1)^2\left(\frac{\pi}{2}\right)^2 s + O(s^2).$$

Notice that, assimptotically, this is the same expression obtained for the superstable periodic orbit.

The next step is to compute $\frac{d^2\epsilon_j^\pm}{ds^2}(0)$. By derivation of 1.9 and 1.10 with respect to s we obtain:

$$\frac{\partial F_1}{\partial t}\frac{dt}{ds} + \frac{\partial F_1}{\partial \epsilon}\frac{d\epsilon}{ds} = -\frac{\partial F_1}{\partial s}, \tag{1.11}$$

$$\frac{\partial F_2^-}{\partial t}\frac{dt}{ds} + \frac{\partial F_2^-}{\partial \epsilon}\frac{d\epsilon}{ds} = -\frac{\partial F_2^-}{\partial s}, \tag{1.12}$$

$$\frac{\partial F_1}{\partial t}\frac{dt}{ds} + \frac{\partial F_1}{\partial \epsilon}\frac{d\epsilon}{ds} = -\frac{\partial F_1}{\partial s}, \tag{1.13}$$

$$\frac{\partial F_2^+}{\partial t}\frac{dt}{ds} + \frac{\partial F_2^+}{\partial \epsilon}\frac{d\epsilon}{ds} = -\frac{\partial F_2^+}{\partial s}, \tag{1.14}$$

Therefore

$$\frac{dt_j^+}{ds}(s) = \left(-\frac{\partial F_2^-}{\partial s} - \frac{\partial F_2^-}{\partial \epsilon}\frac{d\epsilon_j^+}{ds}\right)\left[\frac{\partial F_2^-}{\partial t}\right]^{-1}$$

and

$$\frac{dt_j^-}{ds}(s) = \left(-\frac{\partial F_2^+}{\partial s} - \frac{\partial F_2^+}{\partial \epsilon}\frac{d\epsilon_j^-}{ds}\right)\left[\frac{\partial F_2^+}{\partial t}\right]^{-1}$$

When $s=0$ the derivatives of F_2^- and F_2^+ are the same ones, except the derivatives with respect to s which satisfy:

$$\frac{\partial F_2^-}{\partial s}(t_j^+(0),0,0) = -\frac{\partial F_2^+}{\partial s}(t_j^-(0),0,0).$$

By derivation of 1.11 and 1.13 with respect to s and after simplification, we obtain in the two cases:

$$\frac{\partial^2 F_1}{\partial t^2}\left(\frac{dt}{ds}\right)^2 + 2\frac{\partial^2 F_1}{\partial t \partial \epsilon}\frac{d\epsilon}{ds}\frac{dt}{ds} + 2\frac{\partial^2 F_1}{\partial t \partial s}\frac{dt}{ds} + \frac{\partial^2 F_1}{\partial \epsilon^2}\left(\frac{d\epsilon}{ds}\right)^2 +$$

$$2\frac{\partial^2 F_1}{\partial \epsilon \partial s}\frac{d\epsilon}{ds} + \frac{\partial F_1}{\partial s}\frac{d^2 t}{ds^2} + \frac{\partial F_1}{\partial \epsilon}\frac{d^2 \epsilon}{ds^2} + \frac{\partial^2 F_1}{\partial s^2} = 0.$$

First we note that for $s = 0$ one has $\frac{\partial F_1}{\partial t}(t_j^{\pm}(0), 0, 0) = 0$. Now let

$$A \doteq \frac{\partial F_2^-}{\partial s}(t_j^+(0), \epsilon_j^+(0), 0) = -\frac{\partial F_2^+}{\partial s}(t_j^-(0), \epsilon_j^-(0), 0).$$

Then

$$\frac{dt_j^+}{ds}(0) - \frac{dt_j^-}{ds}(0) = -2A\left[\frac{\partial F_2^+}{\partial t}\right]^{-1},$$

$$\left(\frac{dt_j^+}{ds}(0)\right)^2 - \left(\frac{dt_j^-}{ds}(0)\right)^2 = \frac{4A\frac{\partial F_2^+}{\partial \epsilon}\frac{d\epsilon_j^+}{ds}}{\left(\frac{\partial F_2^+}{\partial t}\right)^2},$$

and therefore

$$\frac{\partial F_1}{\partial \epsilon}\frac{d^2\epsilon_j^+}{ds^2} - \frac{\partial F_1}{\partial \epsilon}\frac{d^2\epsilon_j^-}{ds^2} = -\frac{\partial^2 F_1}{\partial t^2}\frac{4A\frac{\partial F_2^+}{\partial \epsilon}\frac{d\epsilon_j^+}{ds}}{\left(\frac{\partial F_2^+}{\partial t}\right)^2} + 4\frac{\partial^2 F_1}{\partial t \partial \epsilon}\frac{d\epsilon_j^+}{ds}\frac{A}{\frac{\partial F_2}{\partial t}} + 4\frac{\partial^2 F_1}{\partial s \partial t}\frac{A}{\frac{\partial F_2}{\partial t}},$$

where all the derivatives have been calculated at $s = 0$.

Taking into account $\frac{\partial^2 F_1}{\partial t^2}(t, \epsilon, 0) = \frac{\partial F_2^{\pm}}{\partial t}(t, \epsilon, 0)$ and $\frac{\partial^2 F_1}{\partial t \partial \epsilon} = \frac{\partial F_2^{\pm}}{\partial \epsilon}(t, \epsilon, 0)$, one obtains:

$$4\frac{\partial^2 F_1}{\partial s \partial t}\frac{A}{\frac{\partial F_2^+}{\partial t}} = \frac{\partial F_1}{\partial \epsilon}\left(\frac{d^2\epsilon_j^+}{ds^2} - \frac{d^2\epsilon_j^-}{ds^2}\right),$$

Now, using

$$A = -D_1 z_-(0,0) = -\frac{1}{2}, \quad \frac{\partial F_1}{\partial s}(t_{j0}^{\pm}, 0, 0) = -t_j^{\pm}(0) D_1 z_-(0,0),$$

$$\frac{\partial^2 F_1}{\partial s \partial t}(t_j^{\pm}(0), 0, 0) = -\frac{1}{2}, \quad \frac{\partial F_2^+}{\partial t}\frac{\partial F_1}{\partial \epsilon} = -\frac{1}{12(2j+1)^2\pi^2},$$

we get

$$\frac{d^2\epsilon_j^+}{ds^2}(0) - \frac{d^2\epsilon_j^-}{ds^2}(0) = -12\pi^2(2j+1)^2,$$

and therefore the lemma holds. □

To end the proof of the proposition we note that the substitution $s = \alpha^{-n}$ (where $\alpha = 1 + \sqrt{4+9\epsilon}$) gives:

$$\epsilon^-_{n,j} - \epsilon^-_{n,j} = -6(2j+1)^2 \left(\frac{\pi}{2}\right)^2 s_1(n) + 6(2j+1)^2 \left(\frac{\pi}{2}\right)^2 s_2(n) +$$

$$\left(\frac{1}{2}\frac{d^2\epsilon^+_j}{ds^2}(0) + 6\pi^2(2j+1)^2\right) s_1(n)^2 - \frac{1}{2}\frac{d^2\epsilon^+_j}{ds^2}(0)s_2(n)^2 + O(4^{-3n}),$$

where $s_1(n) = \alpha(\epsilon^-_j(s_1(n))^{-n}$ and $s_2(n) = \alpha(\epsilon^+_j(s_2(n))^{-n}$

Lemma 1.2.17 *Let*

$$\epsilon^-_{n,j} = -6(2j+1)^2 \left(\frac{\pi}{2}\right)^2 s_1(n) + o(4^{-n})$$

and

$$\epsilon^+_{n,j} = -6(2j+1)^2 \left(\frac{\pi}{2}\right)^2 s_2(n) + o(4^{-n})$$

be as before. Then:

$$\lim_{n\to\infty} \frac{s_1(n) - s_2(n)}{4^{-2n}} = 0.$$

Proof:

To show that the assertion is true we remind

$$\alpha(\epsilon) = 4 + \frac{2}{3}\epsilon - \frac{2}{27}\epsilon^2 + O(\epsilon^3)$$

and

$$\epsilon^-_j = c_1 s + d_1 s^2 + O(s^3), \quad \epsilon^+_j = c_2 s + d_2 s^2 + O(s^3),$$

with $c_1 = c_2$. Hence

$$s_1(n) = 4^{-n}\left\{1 + \frac{1}{6}(c_1 s_1 + d_1 s_1^2) - \frac{2}{27}c_1^2 s_1^2 + O_3\right\}^{-n} =$$

$$4^{-n}\left\{1 - n\left[\frac{c_1 s_1}{6} + \left(\frac{d_1}{6} - \frac{2}{27}c_1^2\right)s_1^2\right] + \frac{n(n+1)}{2}\frac{c_1^2 s_1^2}{36} + O_3\right\},$$

$$s_2(n) = 4^{-n}\left\{1 - n\left[\frac{c_1 s_2}{6} + \left(\frac{d_2}{6} - \frac{2}{27}c_1^2\right)s_2^2\right] + \frac{n(n+1)}{2}\frac{c_1^2 s_2^2}{36} + O_3\right\},$$

$$s_1(n) - s_2(n) = 4^{-n}\left\{-\frac{c_1}{6}n(s_1 - s_2) + \frac{n(n+1)}{72}(s_1^2 - s_2^2) + \frac{2n}{27}c_1^2(s_1^2 - s_2^2) - \right.$$

$$\left. n\left(\frac{d_1}{6}(s_1^2 - s_2^2) + \left(\frac{d_1}{6} - \frac{d_2}{6}\right)s_2^2\right) + O_3\right\} =$$

$$-n4^{-n}\pi^2(2j+1)^2 s_2^2(1 + o(1)),$$

because $d_1 - d_2 = 6\pi^2(2j+1)^2$. Moreover:

$$s_1(n) = 4^{-n} + (2j+1)^2\left(\frac{\pi}{2}\right)^2 n4^{-2n} + O(n^2 4^{-3n}),$$

$$s_2(n) = 4^{-n} + (2j+1)^2 \left(\frac{\pi}{2}\right)^2 n4^{-2n} + O(n^2 4^{-3n}).$$

Here and in what follows $o(1)$ is understood when $n \to \infty$ for j fixed.

We conclude from this

$$s_1(n) - s_2(n) = -(2j+1)^2 \pi^2 n 4^{-3n}(1 + o(1))$$

and the lemma follows. □

To prove the proposition we should only use

$$\epsilon_{n,j}^- - \epsilon_{n,j}^+ = \left(\frac{1}{2}\frac{d^2\epsilon_{n,j}^+}{ds^2}(0) + 6\pi^2(2j+1)^2\right) s_1(n) - \frac{1}{2}\frac{d^2\epsilon_{n,j}^+}{ds^2}(0) s_2(n)^2 + o(4^{-2n}),$$

and from this it follows

$$\lim_{n\to\infty} \frac{\epsilon_{n,j}^- - \epsilon_{n,j}^+}{4^{-2n}} = 6\pi^2(2j+1)^2. \ \square$$

Corollary 1.2.18 *Let $\epsilon_{n,j}^+$ and $\epsilon_{n,j}^-$ be as in proposition 1.2.13. Then*

$$\epsilon_{n,j}^- - \epsilon_{n,j}^+ = 6\pi^2(2j+1)^2 4^{-2n} + \frac{9}{2}\pi^4(2j+1)^4 n 4^{-3n}(1 + o(1)).$$

Proof:

$$\epsilon_{n,j}^- - \epsilon_{n,j}^+ = -6(2j+1)^2\left(\frac{\pi}{2}\right)^2 (s_1(n) - s_2(n)) + \frac{1}{2}\frac{d^2\epsilon_j^+}{ds^2}(0)(s_1 - s_2)(s_1 + s_2) +$$

$$6\pi^2(2j+1)^2 s_1^2 + O(s_1^3) = -6(2j+1)^2\left(\frac{\pi}{2}\right)^2 (-(2j+1)^2\pi^2 n 4^{-3n}(1 + o(1))) +$$

$$6(2j+1)^2\pi^2 \left(4^{-2n} + 2(2j+1)^2\left(\frac{\pi}{2}\right)^2 n 4^{-3n}(1 + o(1))\right) =$$

$$6(2j+1)^2\pi^2 4^{-2n} + \frac{9}{2} n\pi^4 (2j+1)^4 4^{-3n}(1 + o(1)). \ \square$$

In the next proposition we compare the parameter values for which there is a superstable periodic orbit with the ones giving rise to a saddle-node bifurcation and a flip.

Proposition 1.2.19 *Let $\epsilon_{n,j}^+$ and $\epsilon_{n,j}$ as in propositions 1.2.13 and 1.2.5, respectively. Then*

$$\epsilon_{n,j}^+ - \epsilon_{n,j} = \frac{3}{2}\pi^2(2j+1)^2 4^{-2n}(1 + o(1)).$$

Proof:

We note that $\epsilon_{n,j}$ is a parameter corresponding to a superstable periodic orbit if and only if it exists $t_{n,j}$ such that

$$\left.\begin{aligned} z_-(t_{n,j},\epsilon_{n,j}) - z_-(\alpha^{-n}t_{n,j},\epsilon_{n,j}) &= 0, \\ D_1 z_-(t_{n,j},\epsilon_{n,j}) &= 0. \end{aligned}\right\} \tag{1.15}$$

Indeed:

If $\epsilon_{n,j}$ corresponds to a superstable periodic orbit then we take $t_{n,j}$ such that

$$z_-(\alpha^{-n}t_{n,j},\epsilon_{n,j}) = -1 - \epsilon_{n,j}.$$

Then the first equation of 1.15 holds because

$$f^n(z_-(\alpha^{-n}t_{n,j},\epsilon_{n,j}),\epsilon_{n,j}) = z_-(t_{n,j},\epsilon_{n,j}) = z_-(\alpha^{-n}t_{n,j},\epsilon_{n,j}).$$

Also the second holds because

$$D_1 f^n(z_-(\alpha^{-n}t_{n,j},\epsilon_{n,j}),\epsilon_{n,j})\alpha^{-n} D_1 z_-(\alpha^{-n}t_{n,j},\epsilon_{n,j}) = D_1 z_-(t_{n,j},\epsilon_{n,j}) = 0,$$

where we use $D_1 f^n(z_-(\alpha^{-n}t_{n,j},\epsilon_{n,j}),\epsilon_{n,j}) = 0$.

Reciprocally, if 1.15 holds then $z_-(t_{n,j},\epsilon_{n,j})$ gives an n-periodic orbit. The second equation assures that the point $x = 0$ belongs to the orbit.

As before, we put s instead of α^{-n} and we have:

$$\left.\begin{aligned} z_-(t,\epsilon) - z_-(st,\epsilon) &= 0, \\ D_1 z_-(t,\epsilon) &= 0. \end{aligned}\right\}$$

The system corresponding to a saddle-node bifurcation is

$$\left.\begin{aligned} z_-(t,\epsilon) - z_-(st,\epsilon) &= 0, \\ D_1 z_-(t,\epsilon) - sD_1 z_-(st,\epsilon) &= 0. \end{aligned}\right\}$$

Now let

$$F_1(t,\epsilon,s) = z_-(t,\epsilon) - z_-(st,\epsilon),$$

$$F_2^-(t,\epsilon,s) = D_1 z_-(t,\epsilon) - sD_1 z_-(st,\epsilon),$$

$$F_2(t,\epsilon,s) = D_1 z_-(t,\epsilon).$$

Hence the systems are:

$$\left.\begin{aligned} F_1(t,\epsilon,s) &= 0, \\ F_2(t,\epsilon,s) &= 0, \end{aligned}\right\} \tag{1.16}$$

$$\left.\begin{aligned} F_1(t,\epsilon,s) &= 0, \\ F_2^-(t,\epsilon,s) &= 0. \end{aligned}\right\} \tag{1.17}$$

Notice that the derivatives $\frac{\partial F_2}{\partial t}$, $\frac{\partial F_2}{\partial \epsilon}$ are respectively equal to $\frac{\partial F_2^-}{\partial t}$, $\frac{\partial F_2^-}{\partial \epsilon}$, when $s=0$. Furthermore, (from 1.16) for $s=0$, we get $t_{j,0}=t_j(0)=4\pi^2(2j+1)^2$, $\epsilon_{j,0}=\epsilon_j(0)=0$, and we know that for $s=0$ one obtains from 1.17 the values $t^+_{j,0}=t^+_j(0)=4\pi^2(2j+1)^2$, $\epsilon^+_{j,0}=\epsilon^+_j(0)=0$. From the preceeding considerations it can be deduced, as we did for 1.17, that one can apply the implicit function theorem and therefore there exists functions $t_j(s)$ and $\epsilon_j(s)$ verifying 1.16. Furthermore ϵ_j and ϵ^+_j are equal up to first order. In an analogous way to what we did in proposition 1.2.13, we have:

$$\frac{\partial F_1}{\partial \epsilon}\frac{d^2\epsilon^+_j}{ds^2}-\frac{\partial F_1}{\partial \epsilon}\frac{d^2\epsilon_j}{ds^2}=\frac{\partial^2 F_1}{\partial t^2}\left(\left(\frac{dt_j}{ds}\right)^2-\left(\frac{dt^+_j}{ds}\right)^2\right)+2\frac{\partial^2 F_1}{\partial t\partial\epsilon}\frac{d\epsilon^+_j}{ds}\left(\frac{dt_j}{ds}-\frac{dt^+_j}{ds}\right)+$$

$$2\frac{\partial^2 F_1}{\partial s\partial t}\left(\frac{dt_j}{ds}-\frac{dt^+_j}{ds}\right)=-\left(\frac{\partial F_2^-}{\partial s}\right)^2\left[\frac{\partial F_2^-}{\partial t}\right]^{-1}+2\frac{\partial^2 F_1^-}{\partial s\partial t}\frac{\partial F_2^-}{\partial s}\left[\frac{\partial F_2^-}{\partial t}\right]^{-1},$$

where all the derivatives have been evaluated at $s=0$. To see this one should take into account

$$\frac{dt^+_j}{ds}=\left(-\frac{\partial F_2^-}{\partial s}-\frac{\partial F_2^-}{\partial \epsilon}\frac{d\epsilon^+_j}{ds}\right)\left[\frac{\partial F_2^-}{\partial t}\right]^{-1},\quad \frac{dt_j}{ds}=-\frac{\partial F_2}{\partial \epsilon}\frac{d\epsilon_j}{ds}\left[\frac{\partial F_2}{\partial t}\right]^{-1}.$$

In proposition 1.2.13 we have shown

$$\frac{\partial F_2^-}{\partial s}=-\frac{1}{2},\quad \frac{\partial F_2^-}{\partial t}\frac{\partial F_1}{\partial \epsilon}=-\frac{1}{12\pi^2(2j+1)^2},\quad \frac{\partial^2 F_2^-}{\partial s\partial t}=-\frac{1}{2},$$

the derivatives being evaluated at $s=0$. Hence:

$$\frac{d^2\epsilon^+_j}{ds^2}(0)-\frac{d^2\epsilon_j}{ds^2}(0)=-3\pi^2(2j+1)^2,$$

and this means

$$\epsilon_j-\epsilon^+_j=\frac{3}{2}\pi^2(2j+1)^2s^2+O(s^3).$$

It remains to see that if

$$\epsilon_{n,j}=-6(2j+1)^2\left(\frac{\pi}{2}\right)^2 s_0(n)+\left(\frac{1}{2}\frac{d^2\epsilon^+_j}{ds^2}(0)+\frac{3}{2}\pi^2(2j+1)^2\right)s_0(n)^2+O(s_0(n)^3)$$

and

$$\epsilon^+_{n,j}=-6(2j+1)^2\left(\frac{\pi}{2}\right)^2 s_2(n)+\frac{1}{2}\frac{d^2\epsilon^+_j}{ds^2}(0)s_2(n)^2+O(s_2(n)^3),$$

where $s_0(n)=\alpha(\epsilon_{n,j}(s_0(n)))$ and $s_2(n)=\alpha(\epsilon^+_{n,j}(s_2(n)))$, then

$$\epsilon_{n,j}-\epsilon^+_{n,j}=\frac{3}{2}\pi^2(2j+1)^2 4^{-2n}+o(4^{-2n}).$$

Indeed:

Working as before one gets

$$s_0(n) = 4^{-n} + n(2j+1)^2 \left(\frac{\pi}{2}\right)^2 4^{-2n} + O(n^2 4^{-3n}),$$

$$s_2(n) = 4^{-n} + n(2j+1)^2 \left(\frac{\pi}{2}\right)^2 4^{-2n} + O(n^2 4^{-3n}),$$

and

$$s_0(n) - s_2(n) = -\frac{1}{4}\pi^2(2j+1)^2 n 4^{-3n}.$$

Hence

$$\epsilon_{n,j} - \epsilon^+_{n,j} = -6(2j+1)^2 \left(\frac{\pi}{2}\right)^2 (s_0(n) - s_2(n)) + \frac{1}{2}\frac{d^2\epsilon^+_j}{ds^2}(0)(s_0+s_2)(s_0-s_2)+$$

$$6\pi^2(2j+1)^2 s_0^2(1+o(1)) = \frac{27}{8}(2j+1)^4\pi^4 n 4^{-3n}(1+o(1)),$$

So ending the proof of the proposition. □

Corollary 1.2.20 *Let $\epsilon^+_{n,j}$, $\epsilon^-_{n,j}$, $\epsilon_{n,j}$ as in the previous propositions. Then*

$$\lim_{n\to\infty} \frac{\epsilon^-_{n,j} - \epsilon^+_{n,j}}{\epsilon_{n,j} - \epsilon^+_{n,j}} = 4.$$

Proof:

It follows immediately from

$$\epsilon^-_{n,j} - \epsilon^+_{n,j} = 6\pi^2(2j+1)^2 4^{-2n}(1+o(1))$$

and

$$\epsilon_{n,j} - \epsilon^+_{n,j} = \frac{3}{2}\pi^2(2j+1)^2 4^{-2n}(1+o(1)).$$

Remark 1.2.21 *If we consider the map $g_a(x) = a - x^2$ and compute $(a_1^- - a_1^+)/(a_1 - a_1^+)$, where a_1^+ is the value of a for which the fixed point has a saddle-node bifurcation, and a_1^- and a_1^+ are, respectively, the values corresponding to a flip bifurcation and a superstable periodic orbit, we also obtain 4. Hence, when $n \to \infty$ the behaviour of $f^n(\cdot, \epsilon)$ has some point in common with the logistic map.*

In the next proposition we look for the values, $\epsilon^f_{n,j}$, of the parameter for which there is an n-periodic point, $x^f_{n,j}$, which comes from the saddle-node bifurcation produced for $\epsilon = \epsilon^+_{n,j}$, such that $f^n(0, \epsilon^+_{n,j}) = -x^f_{n,j}$. That is, it is the end of the seemingly "mini"logistic map in the sense of the previous remark.

Proposition 1.2.22 *Let $f(\cdot,\epsilon)$ be the logistic map. Then, for a fixed $j \geq 0$ and n large enough, there are a parameter $\epsilon^f_{n,j} = -6(2j+1)^2\left(\frac{\pi}{2}\right)^2\alpha^{-n} + O(\alpha^{-2n})$ and an n-periodic point, $x^f_{n,j}$, such that $f^n(0,\epsilon^f_{n,j}) = -x^f_{n,j}$ and, furthermore*

$$\lim_{n\to\infty}\frac{\epsilon^f_{n,j} - \epsilon_{n,j}}{4^{-2n}} = 12\pi^2(2j+1)^2.$$

Proof:

As we did previously, we first set down the system to be satisfied by $\epsilon^f_{n,j}$ and $x^f_{n,j}$.

Lemma 1.2.23 *If $\epsilon^f_{n,j}$ and $x^f_{n,j}$ satisfy $f^n(0,\epsilon^f_{n,j}) = -x^f_{n,j}$ and $f^n(x^f_{n,j}) = x^f_{n,j}$ then there are $t^f_{n,j}$, $\bar{t}^f_{n,j} \in \mathbb{R}$ such that, for $\epsilon^f_{n,j}$ and $t^f_{n,j}$, $\bar{t}^f_{n,j}$ one has:*

$$\left.\begin{array}{rcl} z_-(\alpha^{-n+2}t^f_{n,j},\epsilon^f_{n,j}) &=& -1-\epsilon^f_{n,j} \\ z_-(t^f_{n,j},\epsilon^f_{n,j}) + z_-(\bar{t}^f_{n,j},\epsilon^f_{n,j}) &=& 0 \\ f^2(z_-(\bar{t}^f_{n,j},\epsilon^f_{n,j}),\epsilon^f_{n,j}) - z_-(\alpha^{-n+2}\bar{t}^f_{n,j},\epsilon^f_{n,j}) &=& 0 \end{array}\right\} \qquad (1.18)$$

Proof:

Let $\epsilon^f_{n,j}$ and $x^f_{n,j}$ as in the statement of the lemma, and $t^f_{n,j}$, $\bar{t}^f_{n,j}$ such that

$$z_-(\alpha^{-n+2}t^f_{n,j},\epsilon^f_{n,j}) = -1-\epsilon^f_{n,j}.$$

Then

$$f^n(0,\epsilon^f_{n,j}) = f^{n-2}(-1-\epsilon^f_{n,j},\epsilon^f_{n,j}) = z_-(t^f_{n,j},\epsilon^f_{n,j}) = -x^f_{n,j}.$$

Let $\bar{t}^f_{n,j} \in \mathbb{R}$ another value for which $z_-(\alpha^{-n+2}\bar{t}^f_{n,j},\epsilon^f_{n,j}) = x^f_{n,j}$. Then, as $x^f_{n,j}$ is an n-periodic point, we have:

$$x^f_{n,j} = z_-(\alpha^{-n+2}\bar{t}^f_{n,j},\epsilon^f_{n,j}) = f^n(z_-(\alpha^{-n+2}\bar{t}^f_{n,j},\epsilon^f_{n,j}),\epsilon^f_{n,j}) = f^2(z_-(\bar{t}^f_{n,j},\epsilon^f_{n,j}),\epsilon^f_{n,j}). \square$$

Lemma 1.2.24 *A solution of 1.18 give rise to a value of the parameter, $\epsilon^f_{n,j}$, such that $f^n(0,\epsilon^f_{n,j}) = -x^f_{n,j}$ and $x^f_{n,j}$ is an n-periodic point, or to a value $\epsilon_{n,j}$ such that $f^n(0,\epsilon_{n,j}) = 0$.*

Proof:

Consider the system

$$\left.\begin{array}{rcl} z_-(\alpha^{-n+2}t,\epsilon) &=& -1-\epsilon \\ z_-(t,\epsilon) + z_-(\bar{t},\epsilon) &=& 0 \\ f^2(z_-(\bar{t},\epsilon),\epsilon) - z_-(\alpha^{-n+2}\bar{t},\epsilon) &=& 0 \end{array}\right\}$$

If $t = \bar{t}$ this system gives as solution a parameter for which a superstable periodic orbit appears. Otherwise, let $x^f_{n,j} = z_-(\bar{t},\epsilon)$. We have

$$f^n(0,\epsilon) = f^{n-2}(-1-\epsilon,\epsilon) = f^{n-2}(z_-(\alpha^{-n+2}t,\epsilon),\epsilon) = z_-(t,\epsilon) = -z_-(\bar{t},\epsilon) = -x^f_{n,j}.$$

Furthermore

$$f^n(x^f_{n,j},\epsilon) = f^{n-2}(f^2(x^f_{n,j},\epsilon),\epsilon) = f^{n-2}(f^2(z_-(\bar{t},\epsilon),\epsilon) =$$

$$f^{n-2}(z_-(\alpha^{-n+2}\bar{t},\epsilon),\epsilon) = z_-(\bar{t},\epsilon) = x^f_{n,j}. \square$$

As before we put $s = \alpha^{-n+2}$. Then the system is:

$$\left.\begin{array}{rcl} z_-(st,\epsilon) &=& -1-\epsilon, \\ z_-(t,\epsilon)+z_-(\bar{t},\epsilon) &=& 0, \\ f^2(z_-(t,\epsilon),\epsilon) - -z_-(s\bar{t},\epsilon) &=& 0. \end{array}\right\} \quad (1.19)$$

First we remark that, if $s=0$, one can take $\epsilon = 0$, $t = \left(\frac{\pi}{2}\right)^2(2j+1)^2$ and $\bar{t} = \left(\frac{\pi}{2}\right)^2(2j+1)^2$ as solution. Furthermore, from the first equation of 1.19, it can obtained, as we did before, that if there exist $\epsilon(s)$, $t(s)$, $\bar{t}(s)$ satisfying 1.19 then

$$\epsilon(s) = -\frac{3}{8}t_{0j}s + O(s^2),$$

where $t_{0j} = \pi^2(2j+1)^2$.

Let us put, up to second order,

$$\epsilon = \epsilon_{0j}s + \epsilon_{1j}s^2, \quad t = t_{0j} + t_{1j}s, \quad \bar{t} = t_{0j} + \bar{t}_{1j}s,$$

where $\epsilon_{0j} = -\frac{3}{8}t_{0j}$. Then, by substituing in 1.19, we have:

$$\frac{1}{s^2}[z_-(st_{0j}+s^2t_{1j}, s\epsilon_{0j}+s^2\epsilon_{1j}) + 1 + s\epsilon_{0j} + s^2\epsilon_{1j}] = \frac{1}{2}D_{11}z_-(0,0)t_{0j}^2 +$$

$$D_{12}z_-(0,0)t_{0j}\epsilon_{0j} + \frac{1}{2}D_{22}z_-(0,0)\epsilon_{0j}^2 + D_1z_-(0,0)t_{1j} + (D_2z_-(0,0)+1)\epsilon_{1j} + O(s) = 0.$$

As

$$D_{12}z_-(0,0) = 0, \quad D_{22}z_-(0,0) = -\frac{8}{27}, \quad D_{11}z_-(0,0) = -\frac{1}{12},$$

for $s=0$ one obtains

$$\frac{1}{2}t_{1j}(0) + \frac{4}{3}\epsilon_{1j}(0) = \frac{1}{6}t_{0j}^2. \quad (1.20)$$

From the second equation of 1.19 we get

$$\frac{1}{s}(z_-(t_{0j}+st_{1j}, s\epsilon_{0j}+s^2\epsilon_{1j}) + z_-(t_{0j}+s\bar{t}_{1j}, s\epsilon_{0j}+s^2\epsilon_{1j})) =$$

$$D_1z_-(t_{0j},0)(t_{1j}+\bar{t}_{1j}) + 2D_2z_-(t_{0j},0)\epsilon_{0j} + O(s) = 0,$$

and therefore, for $s=0$ we obtain

$$D_1z_-(t_{0j},0)(t_{1j}+\bar{t}_{1j}) + 2D_2z_-(t_{0j},0)\epsilon_{0j} = 0. \quad (1.21)$$

From the third equation of 1.19, by derivation with respect to s and setting

$$a = (z_-(t_{0j} + s\bar{t}_{1j}, s\epsilon_{0j} + s^2\epsilon_{1j}), \epsilon_{0j} + s^2\epsilon_{1j}),$$

$$b = (z_-(st_{0j} + s^2\bar{t}_{1j}, s\epsilon_{0j} + s^2\epsilon_{1j}), \epsilon_{0j} + s^2\epsilon_{1j}),$$

$$a_1 = (t_{0j} + s\bar{t}_{1j}, s\epsilon_{0j} + s^2\epsilon_{1j}),$$

$$b_1 = (st_{0j} + s^2\bar{t}_{1j}, s\epsilon_{0j} + s^2\epsilon_{1j}),$$

we obtain

$$D_1f^2(a)[D_1z_-(a_1)\bar{t}_{1j} + D_2z_-(a_1)(\epsilon_{0j} + 2s\epsilon_{1j})] +$$
$$D_2f^2(a)(\epsilon_{0j} + 2s\epsilon_{1j}) - D_1z_-(t_{0j} + 2st_{1j}) - D_2z_-(b_1)(\epsilon_{0j} + 2s\epsilon_{1j}) = 0.$$

As

$$D_2f(0,0) = 0,\ D_1f^2(0,0) = 0,\ D_2f^2(0,0) = -1,$$

and $z_-(t_{0j}, 0) = 0$, $z_-(0,0) = -1$, the derivative is zero for $s = 0$. This means that the third equation of 1.19 has no terms independents of s. To see the terms in s^2 we derive again. As

$$D_{11}f^2(0,0) = 16,\ D_2f^2(0,0) = -1,\ D_{12}f^2(0,0) = 0,\ D_1f^2(0,0) = 0,\ D_{22}f^2(0,0) = 0,$$

it remains, when $s = 0$ and taking into account that the third equation of 1.19 holds:

$$16[D_1z_-(a_1)\bar{t}_{1j} + D_2z_-(a_1)\epsilon_{0j}]^2 - 2\epsilon_{1j} - D_{11}z_-(b_1)t_{0j}^2 - 2D_{12}z_-(b_1)t_{0j}\epsilon_{0j} -$$
$$2D_1z_-(b_1)\bar{t}_{1j} - D_{22}z_-(b_1)\epsilon_{0j}^2 - 2D_2z_-(b_1)\epsilon_{1j} = 0,$$

where $\epsilon_{1j} = \epsilon_{1j}(0)$, $\bar{t}_{1j} = \bar{t}_{1j}(0)$, a_1 and b_1 are computed for $s = 0$.

As $D_{12}z_-(0,0) = 0$, $D_2z_-(0,0) = \frac{1}{3}$, $D_1z_-(0,0) = \frac{1}{2}$, $D_{22}z_-(0,0) = -\frac{8}{27}$, $D_{11}z(0,0) = -\frac{1}{12}$, we have

$$16[D_1z_-(a_1)\bar{t}_{1j}(0) + D_2z_-(a_1)\epsilon_{0j}]^2 - 2\bar{t}_{1j}(0) - \frac{2D_2z_-(t_{0j},0)\epsilon_{0j}}{D_1z_-(t_{0j},0)} = 0. \tag{1.22}$$

From equations 1.21 and 1.22 we obtain two possible values for $\bar{t}_{1j}(0)$:

$$\bar{t}_{1j}^1 = -\frac{D_2z_-(t_{0j},0)\epsilon_{0j}}{D_1z_-(t_{0j},0)}$$

and

$$\bar{t}_{1j}^2 = -\frac{D_2z_-(t_{0j},0)\epsilon_{0j}}{D_1z_-(t_{0j},0)} + \frac{1}{8D_1z_-(t_{0j},0)^2}.$$

Comparing with 1.21 we see that in the case $\bar{t}_{1j}(0) = \bar{t}_{1j}^1$ one has $t_{1j}(0) = \bar{t}_{1j}(0)$ and it corresponds to a superstable periodic orbit. The other case corresponds to a value for which there is an n-periodic orbit and $f^n(0,\epsilon)$ falls in this orbit.

From 1.20 we have

$$\epsilon_{1j}(0) = \frac{3}{64}t_{0j}^2 + \frac{3}{8}\bar{t}_{1j}(0) + \frac{9}{32}\frac{D_2 z_-(t_{0j},0)}{D_1 z_-(t_{0j},0)}.$$

Therefore, if we denote by $\epsilon_{1j}^1 = \epsilon_{1j}^1(s)$ and $\epsilon_{1j}^2 = \epsilon_{1j}^2(s)$, respectively, the solutions corresponding to $\bar{t}_{1j}(0) = \bar{t}_{1j}^1$ and $\bar{t}_{1j}(0) = \bar{t}_{1j}^2$, one has

$$\epsilon_{1j}^1(0) - \epsilon_{1j}^2(0) = -\frac{3}{64 D_1 z_-(t_{0j},0)^2} = -\frac{3}{16}t_{0j},$$

because $D_1 z_-(t_{0j},0)^2 = \frac{1}{4}t_{0j}^{-1}$.

As $t_{0j} = \left(\frac{\pi}{2}\right)^2(2j+1)^2$, it follows

$$\epsilon_{1j}^1(0) - \epsilon_{1j}^2(0) = -\frac{3}{16}\left(\frac{\pi}{2}\right)^2(2j+1)^2.$$

Hence, if $\epsilon_j^f = s\epsilon_{0j} + s^2\epsilon_{1j}^2$ and $\epsilon_j = s\epsilon_{0j} + s^2\epsilon_{1j}^1$ then

$$\epsilon_j^f(s) - \epsilon_j(s) = \frac{3}{16}\left(\frac{\pi}{2}\right)^2(2j+1)^2 s^2 + O(s^3).$$

It remains to see that

$$s_f = s_f(n) = \alpha(\epsilon_j^f(s_f(n)))^{-n+2}, \quad s_0 = s_0(n) = \alpha(\epsilon_j(s_0(n)))^{-n+2},$$

verify $s_f(n) = s_0(n) = o(4^{-2n})$. But this is done as in lemma 1.2.17. Perfoming again the change $s = \alpha^{-n+2}$ we obtain

$$\epsilon_{n,j}^f - \epsilon_{n,j} = \frac{3}{16}\left(\frac{\pi}{2}\right)^2(2j+1)^2 4^{-2n+4}(1+o(1)),$$

and hence

$$\lim_{n\to\infty}\frac{\epsilon_{n,j}^f - \epsilon_{n,j}}{4^{-2n}} = 12\pi^2(2j+1)^2,$$

ending th proof of the proposition. □

Corollary 1.2.25 *Using the previous notation one has*

$$\epsilon_{n,j}^f - \epsilon_{n,j}^+ = \frac{27}{2}\pi^2(2j+1)^2 4^{-2n}(1+o(1)), \quad \lim_{n\to\infty}\frac{\epsilon_{n,j}^f - \epsilon_{n,j}^+}{\epsilon_{n,j}^- - \epsilon_{n,j}^+} = \frac{9}{4}.$$

The proof follows from the preceding propositions.

Remark 1.2.26 *a) It is seen again that this limit coincides with the quocient corresponding to the parameters* $\epsilon_1^f = 0$, $\epsilon_1^+ = -1/4$, $\epsilon_1^- = 3/4$ *in the case of the fixed point of the logistic map.*

b) It is possible to obtain equations as the ones used in the last proposition, to have values of the parameters for which there is a flip bifurcation to period $n2^m$. *One can hope that taking the limits of the quocients between successive flip bifurcations, they also coincide with the corresponding ones in the case of the logistic map* ($n = 1$).

Estimates on the width of the windows of attraction

Given a fixed period, n_1, we know that the "first" periodic orbit of this period (that is the one which appears for ϵ as close as possible to 0) is found for $\epsilon_{n_1,0} \approx -\frac{3}{2}\pi^2 4^{-n_1}$. Let $\chi(n)$, for $n \geq n_1$, be the sum of the widths of the windows of attraction for orbits of period n and parameters between $\epsilon_{n_1,0}$ and 0. Then

$$\chi(n_1) \approx 6\pi^2 4^{-2n_1} \approx 6\pi^2 \left(\frac{2\epsilon}{3\pi^2}\right)^2 = \frac{8}{3\pi^2}\epsilon^2,$$

if $\epsilon = \epsilon_{n_1,0}$.

Given $n = n_1 + m$ let us obtain the index j_n corresponding to the maximum value of j in order to the window be located on $(\epsilon, 0)$. We have

$$\epsilon \approx -\frac{3}{2}\pi^2(2j_n + 1)^2 4^{-n} \approx -\frac{3}{2}\pi^2 4^{-n_1}$$

Hence

$$2_{j_{n_1+m}} + 1 \approx 2^m, \qquad 2_{j_{n_1+m}} + 1 < 2^m,$$

and this implies $j_{n_1+m} = (2^m - 2)/2$ if $m \geq 1$.

Therefore:

$$\chi(n_1 + m) = \sum_{j=0}^{j_{n_1+m}} 6\pi^2 16^{-n_1} 16^{-m}(2j+1)^2 = \chi(n_1)\left[\sum_{j=0}^{j_{n_1+m}} (2j+1)^2\right] 16^{-m} =$$

$$\chi(n_1)\frac{16^{-m}}{3}\left[\frac{1}{2}(2^{3m} - 6\cdot 2^m + 12\cdot 2^m - 8) + 3(2^{2m} + 4) + \frac{11}{2}(2^m - 2) + 3\right],$$

because

$$\sum_{j=0}^{j_{n_1+m}} (2j+1)^2 = \frac{1}{3}(4j_{n_1+m}^3 + 12j_{n_1+m}^2 + 11j_{n_1+m} + 3).$$

Hence

$$\chi(n_1 + m) = \chi(n_1)\frac{16^{-m}}{3}\left(\frac{1}{2}2^{3m} - \frac{1}{2}2^m\right) \tag{1.23}$$

If we add 1.23 for all $n \geq n_1$, we obtain

$$\sum_{n\geq n_1} \chi(n) = \chi(n_1)\left[1 + \sum_{m\geq 1}\left(\frac{1}{6}2^{-m} - \frac{1}{6}2^{-3m}\right)\right] =$$

$$\chi(n_1)\left(1 + \frac{1}{6} - \frac{1}{42}\right) = \frac{64}{21\pi^2}\epsilon^2.$$

However this estimate of the width of the windows of attraction is not correct by two reasons. First because there are windows counted several times. Second because, due to the lack of uniformity in j of the bounds depending on (n, j), the formula 1.23 only holds for m not too large (and depending on n_1).

Taking into account what we have seen in propositions 1.2.1, 1.2.3, 1.2.4 and 1.2.5 one cannot expect that the approximation of the value of the parameter for which a superstable periodic orbit occurs (and in the same way a flip, fold, etc.) be reliable if $j > 2^{4n/5}$. Even if we assume

$$\epsilon \approx A_1 4^{-n} u_0^2 + A_2 u_0^5 4^{-2n} + A_3 u_0^9 4^{-3n} + A_4 u_0^{13} 4^{-4n} + \dots,$$

(where ϵ is the same that before and $u_0 = 2j + 1$,) as one can see by working at higher order, then it is reliable if $j < 2^{n/2}$. In this case we can rely on the formula 1.23 up to period at most $2n_1$.

1.3 Numerical results

The objective of this section is to give numerical estimates on the measure of the parameters of the logistic map, $f_a(x) = 1 - ax^2$ or $f(x, \epsilon) = 1 - (2+\epsilon)x^2$, for which there is an attracting periodic orbit, and also numerical evidence on the density of such set of parameters. More concretely, we shall see:

a) Estimate of the measure of the set of parameters a or ϵ such that there is not a strange attractor consisting of a unique interval.

b) Estimate of the measure of the set of parameters for which there is an attracting periodic orbit.

c) Numerical evidence of the density of the set of parameters for which there is an attracting periodic orbit. Prediction of the largest gap between windows of attraction as a function of the maximal period.

d) Behaviour of the measure of the set of parameters for which there is an attracting periodic orbit, when we consider this set restricted to intervals of the form $[2 - \epsilon, 2]$ with $\epsilon \to 0$. If Δ_ϵ is this measure on $[2 - \epsilon, 2]$, we shall see that $\frac{\Delta_\epsilon}{\epsilon^2}$ tends to $\bar{\alpha}$ in a suitable sense, and we shall give an estimate of $\bar{\alpha}$.

First we give the notation to be used.

1.3.1 Notation

In our study there appear parameters a and ϵ associated to the logistic map $f_a(x) = 1 - ax^2$ or $f(x, \epsilon) = 1 - (2 + \epsilon)x^2$. We have $a = 2 + \epsilon$, and we shall use as parameter a or ϵ. The parameters for the saddle-node bifurcation will be denoted by $\epsilon^1_{n,j}$ or ϵ^1_n, where j is the order number, the first one being $j = 0$, in the set of all fold and flip bifurcations of period n (strict or not), for decreasing ϵ from $\epsilon = 0$. If it is understood we can skip j. The flip bifurcations will be denoted by $\epsilon^{2^i}_{n,j}$, where n is the period of the associated saddle-node and $2^i n$ is the associated period. Here j has the same meaning as before but for the associated saddle-node. The values of the parameters for which there is a superstable periodic orbit will be denoted by $\epsilon_{n,j}$, where j is the number of order of the appearance of the superstable periodic orbit ,of period n (non strict) , starting at $\epsilon = 0$ for decreasing ϵ. Furthermore, if it comes from a flip bifurcation of period $n2^i$ we shall use $\epsilon^{(2^i)_0}_{n,j}$, j being the order number of the associated saddle-node bifurcation which occurs for $\epsilon = \epsilon^1_{n,j}$. The union of the intervals (also denoted as windows) corresponding to n-periodic attracting orbits and the associated cascade of period doubling, will be denoted by Δ_n. A particular window will be denoted by $\Delta_{n,j}$, where j is the order number of the corresponding saddle-node bifurcation. The window corresponding to the interval between saddle-node and flip bifurcation will be denoted by $\Delta^1_{n,j}$. A window between the saddle-node bifurcation $\epsilon^1_{n,j}$ and the final tangency $\epsilon^f_{n,j}$ (in the sense of proposition 1.2.22) will be called $\Delta^t_{n,j}$, and the union for all j will be denoted as Δ^t_n. In a similar way the window between the flip $\epsilon^{2^i}_{n,j}$ and the final tangency $\epsilon^{(2^i)_f}_{n,j}$, asociated to the flip bifurcation, will be called $\Delta^{(2^i)_t}_{n,j}$. $\Delta_{n,j}$ is called window of attraction and $\Delta^t_{n,j}$ a sadlle-node-homoclinic window (or snh window). The final parameter of a cascade of flips is denoted by $\epsilon^{2^\infty}_{n,j}$. Given a set $A \subset \mathbb{R}$, $|A|$ denotes the Lebesgue measure of A. The union of all the windows $\Delta_{n,j}$ for $j \in J$, where J is the set of j corresponding to windows $\Delta^t_{n,j}$ mutually disjoint and such that $\bigcup_{j\in J} \Delta^t_{n,j} = \Delta^t_n$, will be called union of simple attraction windows, and denoted by Δ'_n. For the values $\epsilon = \tilde{\epsilon}^k_{n,j}$ $(k = 1, 2)$ it holds $f^n(0, \tilde{\epsilon}^k_{n,j}) = -x_+(\tilde{\epsilon}^k_{n,k})$, where $x_+(\epsilon) = (-1 + \sqrt{9 + 4\epsilon})/(4 + 2\epsilon)$ (see proposition 1.2.9). The intervals $\tilde{\Delta}_n = (\tilde{\epsilon}^1_{n,0}, 0)$ are called windows of principal tangencies. The windows $\tilde{\Delta}^c_n = (\epsilon^{(2^{n+1})_f}_{1,0}, \epsilon^{(2^n)_f}_{1,0})$ will be called n-th cascade windows.

1.3.2 Computation of the first cascade of attraction associated to a saddle-node bifurcation

The saddle-node bifurcation for the fixed point x_+ occurs for $a^1_1 = -1/4$. It is not difficult to compute exactly the first flip bifurcations associated to period 1. However we should compute the end of the cascade $a^{2^\infty}_1$. To obtain this, we shall not use the values of the parameters for which there is a flip bifurcation, but the values $a^{(2^i)_0}_1$, for which there is a superstable periodic orbit of period 2^i (that is $f^{2^i}_a(0) = 0$, with $a = a^{(2^i)_0}_1$). Table 1.2 gives these values (computed in quadruple precision) together with

$$\delta_{in} = \frac{a_1^{(2^{i-2})_0} - a_1^{(2^{i-1})_0}}{a_1^{(2^{i-1})_0} - a_1^{(2^i)_0}}$$

for the first values of i in the case of the logistic map. Hence $\delta = 4.669201691029\ldots$ is the Feigenbaum's constant, $a^{2^\infty}_1 = 1.4011551890920\ldots$ and $|\Delta_1| = 1.6511551890920\cdots$, where Δ_1 is the window of attraction associated to period 1.

1.3.3 Computation of superstable periodic orbits. Comparison with analytical results

When the measure of the first window of attraction, corresponding to period 1, is available, to obtain other windows of attraction we shall obtain first the value of the parameter for which it appears the superstable periodic orbit associated to the window under consideration. Given n we know the number of saddle-node bifurcations giving rise to n-periodic orbits, and therefore, the number of values of the parameter for which there is a superstable periodic orbit asociated to a saddle-node bifurcation. Furthermore, to compute these parameters is equivalent to compute real zeros of a degree $2^{n-1} - 1$ polynomial: $P_{2^{n-1}-1}(a) = f_a^n(0)$, taking into account that we should skip the zeros comming from flip bifurcation or the ones not giving rise to an orbit of period n strict, in the case of n not being a prime number. This computation has been carried out in two different ways:

a) Using, for every n, the approximate expression of the values of the parameters for which there is a superstable periodic orbit and refining the zero by means of the secant method.

b) If we assume that all the windows of attraction of a period n have been computed, then the ones of period $n + 1$ are found in the gaps.

The first method has the advantage that it does not require information on the attracting periodic orbits of lower period, but it has the problem that the approximate values for the superstable periodic orbit get worse when we go away from $\epsilon = 0$. The second method is more reliable but it requires an amount of information that increases by a factor of 2 (roughly) when n goes to $n + 1$. However this process can be splitted by using the parameters $\tilde{a}_{n,j}^1$ and $\tilde{a}_{n,j}^2$ in the following way: It is known (see for example [3]) that for this parameters of homoclinic tangency ther is not attracting periodic orbits. Then to compute all the parameters for which there are superstable periodic orbit we can split the initial interval of parameters on the union of these intervals.

An illustration of the first method is given on table 1.3, where we present the approximate values of the parameter, $\epsilon_{n,j}$, for the superstable periodic orbit of period n ($n \leq 10$) and the first values (the closer ones to $\epsilon = 0$) of j. There

$$\bar{\epsilon}_{n,j}^{(1)} = -\frac{6}{4^2}\alpha(\bar{\epsilon}_{n,j}^{(1)})^{-(n-2)}\left(\frac{\pi}{2}\right)^2(2j+1)^2$$

and

$$\bar{\epsilon}_{n,j}^{(2)} = \bar{\epsilon}_{n,j}^{(1)} + \left[\frac{3}{64}\left(\frac{\pi}{2}\right)^4(2j+1)^4 - (-1)^j\frac{9}{32}\varphi\left(\frac{\pi}{2}(2j+1)\right)\left(\frac{\pi}{2}\right)^3(2j+1)^3\right]\alpha^{-2(n-2)},$$

where φ is the function appearing in proposition 1.2.5. We compare the value of $\bar{\epsilon}_{n,j}^{(1,2)}$ with the one obtained numerically. We give also the differences $|\bar{\epsilon}_{n,j}^{(2)} - \epsilon_{n,j}|$ and the differences relatives to the size of the window $|\Delta_{n,j}|$. The agreement is good only for the initial values of j.

1.3.4 Computation of the windows of attraction. Comparison with the analytical results

Given a superstable periodic orbit of period n with parameter $a_{n,j}$ or $\epsilon_{n,j}$, to compute the associated window of attraction we should compute the value of the parameter for the saddle-node bifurcation , and the associated cascade of bifurcation.

a) Saddle-node bifurcation. We look for a zero of

$$\left.\begin{array}{rcl} f^n(x,\epsilon) & = & 0, \\ D_1 f^n(x,\epsilon) - 1 & = & 0. \end{array}\right\}$$

To do this we consider a quadratic approximation of $f^n(\cdot,\epsilon)$ in a neighbourhood of $x = 0$, $\epsilon = \epsilon_{n,j}$:

$$f^n(x,\epsilon) \approx f^n(0,\epsilon_{n,j}) + D_1 f^n(0,\epsilon_{n,j})x + D_2 f^n(0,\epsilon_{n,j})(\epsilon - \epsilon_{n,j}) +$$
$$\frac{1}{2} D_{11} f^n(0,\epsilon_{n,j})x^2 + D_{12} f(0,\epsilon_{n,j})x(\epsilon - \epsilon_{n,j}).$$

Taking into account

$$f^n(0,\epsilon_{n,j}) = D_{12} f^n(0,\epsilon_{n,j}) = D_1 f^n(0,\epsilon_{n,j}) = 0,$$

we can obtain an approximate value for ϵ and x. Then we apply Newton's method to refine $\epsilon^1_{n,j}$.

b) Cascades of bifurcations: We shall find the values of ϵ to have superstable periodic orbit of period $2^m \cdot n$, $\epsilon_n^{(2^m)_0}$, because this is cheaper than to compute flip orbits. To this end we take into account the limit relation

$$\lim_{n\to\infty} \frac{\epsilon_{n,j}^{(2^m)_0} - \epsilon_{n,j}^{(2^{m+1})_0}}{\epsilon_{n,j}^{(2^{m+1})_0} - \epsilon_{n,j}^{(2^{m+2})_0}} = \delta = 4.6692016\ldots.$$

With this we can compute $\epsilon_{n,j}^{(2^m)_0}$ for several m and, when the previous relation comes close to δ, we extrapolate and estimate $\epsilon_{n,j}^{2^\infty}$.

Table 1.4 gives an analytical approximation of the windows $\Delta^1_{n,j}$ between saddle-node and the first flip. Moreover $\Delta^{(1)}_{n,j} = 6\pi^2(2j+1)^2 4^{-2n}$. Columns 2 and 3 give the absolute and relative errors:

$$\frac{|\Delta^{(1)}_{n,j} - \Delta^1_{n,j}|}{\Delta^1_{n,j}},$$

for the first values of j and $n \leq 10$.

2^n	$a_1^{(2^n)_0}$	δ_{n1}
4	1.310702641336833E+00	
8	1.381547484432061E+00	
16	1.396945359704561E+00	4.600949276538075E+00
32	1.400253081214783E+00	4.655130495391980E+00
64	1.400961962944841E+00	4.666111947828571E+00
128	1.401113804939776E+00	4.668548581446841E+00
256	1.401146325826946E+00	4.669060660648268E+00
512	1.401153290849924E+00	4.669171555379511E+00
1024	1.401154782546618E+00	4.669195156030017E+00
2048	1.401155102022464E+00	4.669200229086856E+00
4096	1.401155170444411E+00	4.669201313294204E+00
8192	1.401155185098297E+00	4.669201545780907E+00
16384	1.401155188236711E+00	4.669201595537494E+00
32768	1.401155188908863E+00	4.669201606198152E+00
65536	1.401155189052817E+00	4.669201608480804E+00
131072	1.401155189083648E+00	4.669201608969745E+00
262144	1.401155189090251E+00	4.669201609074453E+00
524288	1.401155189091665E+00	4.669201609096879E+00
1048576	1.401155189091968E+00	4.669201609101682E+00
2097152	1.401155189092033E+00	4.669201609102710E+00
4194304	1.401155189092047E+00	4.669201609102931E+00
8388608	1.401155189092050E+00	4.669201609102985E+00

Table 1.2: Cascade of superstable periodic orbits corresponding to the fixed point.

1.3.5 Windows saddle-node to homoclinic (snh). Cascade of tangencies and associated windows.

In general a parameter ϵ for which there is a homoclinic tangency satisfies $f^n(0,\epsilon) = -x$, where x is an n-periodic point. First we consider homoclinic tangencies of the following type: let ϵ such that $f^{2^i n}(0,\epsilon) = -x_{2^i n}$, where $x_{2^i n}$ is an n periodic point which comes from a saddle-node or flip bifurcation $\epsilon = \epsilon^{2^i}_{n,j}$. We denote such a parameter as $\epsilon^f_{n,j}$ or $\epsilon^{(2^i)_f}_{n,j}$, if it comes from a flip bifurcation. The window $\Delta^t_{n,j} = (\epsilon^1_{n,j}, \epsilon^f_{n,j})$ or $\Delta^{(2^i)_t}_{n,j}$ will be denoted "saddle-node to homoclinic" window, and the union of all of them for a given period n will be called $\Delta^t_n = \cup \Delta^t_{n,j}$. We remark that the union is not a disjoint union when j ranges between 0 and the number of saddle-node bifurcations of period n. If $J(n)$ is the set of all possible j then we shall denote by $J'(n)$ the subset of $J(n)$ such that $\Delta_n^t = \cup_{j\in J'(n)}\Delta_{n,j}^t$ is a disjoint union.

A particular case of these parameters are the values associated to the bifurcation cascade of the fixed point, that is to the parameters $\epsilon_{1,0}^{(2^i)_f}$. These ones are given in table 1.5 (notice that $\epsilon_{1,0}^f = 0$). Those parameters are in an inverse cascade having $\epsilon_{1,0}^{2^\infty}$ as limit. The windows $\tilde{\Delta}_n^c = \left(\epsilon_{1,0}^{(2^n)_f}, \epsilon_{1,0}^{(2^{n+1})_f}\right)$ will be denoted "n-th cascade windows".

Other interesting tangency parameters are the ones satisfying $f^n(0,\epsilon) = -x_+(\epsilon)$, where x_+ is a fixed point of $f(\cdot,\epsilon)$ such that $x_+ > 0$. We denote them as $\tilde{\epsilon}_{n,j}^1$ or $\tilde{\epsilon}_{n,j}^2$, according to the notation at the begining of the chapter. It holds always: $\tilde{\epsilon}_{n,j}^2 < \tilde{\epsilon}_{n,j}^1 < \tilde{\epsilon}_{n,j-1}^2$. Table 1.6 give some values of $\tilde{\epsilon}_{n,0}^1$ toghether with the quotients $\tilde{\epsilon}_{n,0}^1/\tilde{\epsilon}_{n+1,0}^1$ which tend to four as expected. Table 1.7 gives the approximate values $\tilde{\epsilon}_{n,j}^{(1)}$ and $\tilde{\epsilon}_{n,j}^{(2)}$ computed in proposition 1.2.9, and compared with the real values.

To compute $\tilde{\epsilon}_{n,j}^k$ one has to solve the system

$$f^n(x,\epsilon) = x, \quad f^m(0,\epsilon) = -x.$$

As before we compute first an approximate solution using the quadratic approximation of f^n and then we refine using the Newton method.

1.3.6 Estimates of the measure of the snh windows

The value $\tilde{\epsilon}_{2,0}^1$ such that $f^2(0,\tilde{\epsilon}_{2,0}^1) = -x_+$ is $\tilde{\epsilon}_{2,0}^1 \approx -4.563109873 \times 10^{-1}$. Let $\tilde{\Delta}_2$ be the window $(\tilde{\epsilon}_{2,0}^1, 0)$. We have $(\epsilon_{1,0}^{2^\infty}, 0) = \cup_{n\geq 1}\tilde{\Delta}_n^c \cup \tilde{\Delta}_2$. According to our notation $\tilde{\epsilon}_{2,0}^1 = \epsilon_{1,0}^{(2)_f}$. We shall study the measure of the snh windows on each of the sets $\tilde{\Delta}_n^c$ and $\tilde{\Delta}_2$. Table 1.8 give the measures of the snh windows for the first periods in $\tilde{\Delta}_2$ and the total measure.

We plot in figure 1.1 $\log_{10}(\Delta_n^t \cap \tilde{\Delta}_2)$ against the period. We remark that, adding the size of the windows by blocks of 4 elements, we obtain (starting at $n = 3$, $n = 4$, or $n = 6$) a decreasing ratio between 3.7796 and 4.1448 (table 1.9).

Assuming that the limit of the ratio exists and belongs to the given interval, we have that the remainder, $\sum_{j=25}^{\infty}|\Delta_j^t \cap \tilde{\Delta}_2|$ is in $[5.0746 \times 10^{-6}, 5.7413 \times 10^{-6}]$ and therefore for the total measure we have

$$5.937831 \times 10^{-2} \leq \sum_{j=3}^{\infty}|\Delta_j^t \cap \tilde{\Delta}_2| \leq 5.937898 \times 10^{-2}.$$

It remains to compute the contribution of the snh windows in the intervals $\tilde{\Delta}_n^c$, $n \geq 1$:

Table 1.10 give the values of the sum of windows of snh type up to the period shown. Notice that in a Δ_n^c window we can only have periods $2^n k$, $k \in \mathbb{N}$, and the first period is always $2^n 3$. Finally in table 1.11 we present the quotients $|\Delta^t_{2^i n} \cap \tilde{\Delta}^c_i| / |\Delta^t_{2^{i+1} n} \cap \tilde{\Delta}^c_{i+1}|$ for $i \geq 1$ and $|\Delta^t_n \cap \tilde{\Delta}_2| / |\Delta_{2n} \cap \tilde{\Delta}^c_1|$. It seems that there exist the limit of the quotients when $n \to \infty$, and it is close to δ for i large.

With these data we can estimate the remaining measure in each one of the windows $\tilde{\Delta}^c_i$, $i \geq 1$:

a) Measure in $\tilde{\Delta}^c_1$:

Looking at table 1.11, the values of the 6 last quotients vary in the range

$$[3.8969103993, 4.0890273833].$$

Assuming that the ratio remains in this interval, we have to add to the obtained measure in $\tilde{\Delta}^c_1$ the value $\sum_{j=22}^{\infty} |\Delta^t_{2j} \cap \tilde{\Delta}^c_1|$ which belongs to $[3.8687 \times 10^{-6}, 4.2305 \times 10^{-6}]$.

b) Measure in $\tilde{\Delta}^c_2$:

Looking also at table 1.11, the ratio is in $[4.8593321111, 4.8904005914]$. Proceeding as before, we had to add a value in $[1.0526 \times 10^{-6}, 1.1338 \times 10^{-6}]$. Therefore the total measure in this window is in $[3.2361769 \times 10^{-3}, 3.2362581 \times 10^{-3}]$.

c) Measure in $\tilde{\Delta}^c_3$:

The ratio is in $[4.6681632188, 4.6700494406]$. We had to add a value in $[3.1987 \times 10^{-7}, 3.3739 \times 10^{-7}]$, and the total measure is in $[6.9351386 \times 10^{-4}, 6.9353138 \times 10^{-4}]$.

d) Measure in $\tilde{\Delta}^c_4$.

The ratio is in $[4.6750125620, 4.6739535273]$. We add a value in $[1.3299 \times 10^{-7}, 1.3674 \times 10^{-7}]$, and the total measure is in $[1.4831347 \times 10^{-4}, 1.4831722 \times 10^{-4}]$.

e) Measure in $\tilde{\Delta}^c_5$.

The ratio is close to 4.66968. The value to add is in $[4.3410 \times 10^{-8}, 4.4213 \times 10^{-8}]$, and the total measure is in $[3.1760351 \times 10^{-5}, 3.1761154 \times 10^{-5}]$.

From this point on we do not compute the measure of the windows because we had to add a value small compared with the initial error. Therefore the total measure is the initial one plus an amount varying in $[1.0492 \times 10^{-5}, 1.1623 \times 10^{-5}]$. That is:

$$\text{Total measure} \in [7.9542763 \times 10^{-2}, 7.9543895 \times 10^{-2}],$$

giving 4 correct figures.

Table 1.12 gives the values of the computed snh windows.

1.3.7 Estimates of the measure of the simple attraction windows

By simple attraction windows we understand the windows $\Delta_{n,j}$ with $j \in J'(n)$ in the sense of section 1.3.6. Table 1.13 gives the quotients between the sum of the snh windows of a given period and the sum of the measures corresponding to simple attraction windows. The union of the simple windows for a period n will be denoted by $\Delta'_n = \cup_{j\in J'(n)}\Delta_{n,j}$. In table 1.14 we give the values of Δ'_n inside the cascade windows $\tilde{\Delta}^c_i$ and $\tilde{\Delta}_2$. This allows us, using the results of the last sections, to make a prediction of the total measure of the simple attraction windows:

First we note that

$$\frac{\epsilon^{f}_{1,0} - \epsilon^{1}_{1,0}}{\epsilon^{2\infty}_{1,0} - \epsilon^{1}_{1,0}} = \frac{|\Delta^{t}_{1,0}|}{|\Delta_{1,0}|} = 1.3626824\ldots$$

and this is a value very close to the one appearing in table 1.13 when the period is high. This allows to estimate in each window $\tilde{\Delta}^c_n$ the measure of the simple windows corresponding to the periods for which we have computed the snh windows. Finally, using the same estimates done for the snh windows and dividing by 1.3626824, we should add:

In the window $\tilde{\Delta}_2$ an amount in $[8.24 \times 10^{-6}, 8.73 \times 10^{-6}]$; in $\tilde{\Delta}^c_1$ between 8.00×10^{-6} and 8.27×10^{-6}; in $\tilde{\Delta}^c_2$ between 4.49×10^{-6} and 4.55×10^{-6}; in $\tilde{\Delta}^c_3$ between 3.95×10^{-6} and 3.96×10^{-6}; in $\tilde{\Delta}^c_4$, the value 1.81×10^{-6}; in $\tilde{\Delta}^c_5$, the value 2.20×10^{-6}; in $\tilde{\Delta}^c_6$, the value 1.59×10^{-6}; in $\tilde{\Delta}^c_7$, the value 1.07×10^{-6}; in $\tilde{\Delta}^c_8$, the value 2.3×10^{-7}; in $\tilde{\Delta}^c_9$ the value 4×10^{-8}, etc. Therefore we should add, in total, an amount between 3.166×10^{-5} and 3.248×10^{-5}. So the total amount is in

$$[5.8802 \times 10^{-2}, 5.8803 \times 10^{-2}].$$

1.3.8 Estimate of the total measure of the attraction windows

By definition, we call simple windows as 1-simple windows. Then the simple attraction windows inside the snf windows, corresponding to the simple windows, are denoted as 2-simple attraction windows. In an analogous way one can define the n-simple attraction windows. To obtain the total measure of the set of parameters for which there is an attracting periodic orbit, it remains to compute the measure of the n-simple windows. To this end we consider first the largest simple attraction windows (see table 1.15). Then we shall compute the measure of the 2-simple windows corresponding to those periods, and we shall extrapolate . We note that given the sum of the 2-simple windows of period kn, corresponding to a 1-simple window of period n, it seems that the quotients $|\Delta^{+}_{kn}|/|/|\Delta_k|$ tend to a limit, and we can estimate the measure of the 2-simple windows. Here Δ^{+}_{kn} represents the union of the 2-simple windows of period kn corresponding to a simple window of period n (see table 1.16). Therefore, if the set of parameters corresponding to 2-simple windows is denoted by Δ^{2s} and the one of 1-simple windows by Δ^{1s}, we have $|\Delta^{2s}|/|\Delta^{1s}| \in [3.492 \times 10^{-2}, 3.506 \times 10^{-2}]$. Note that the quotient $|\Delta^{1s}|/|\Delta_{1,0}|$ equals 3.5612×10^{-2}. Furthermore $|\Delta^{2s} \in [2.0534 \times 10^{-3}, 2.0614 \times 10^{-3}]$. . Finally, we shall also give an estimate of the 3-simple windows: To this end we compute the measure inside the largest 2-simple windows, that is the one of period 9 corresponding to the 1-simple of period 3. In this case we obtain, by extrapolation, that the measure $|\Delta^{3s}|$ of the 3-simple windows is in $[1.8995 \times 10^{-5}, 1.9138 \times 10^{-5}]$, and therefore, taking into account that the measure of the n-simple windows for $n \geq 4$ can be neglected, compared to the errors that we have, we obtain:

Total measure of windows of attraction: Between 6.0874×10^{-2} and 6.0883×10^{-2}, without including the one corresponding to period 1. The ratio between the total measure of the attraction windows (including the one corresponding to period 1) and the interval of parameters where the logistic map is defined is, therefore, 7.6090×10^{-1}. If we only consider windows of period larger than one, we obtain $-|\bigcup_{n=3}^{\infty} \Delta_n/\epsilon_{1,0}^{2^\infty} = 1.0165 \times 10^{-1}$.

1.3.9 Density of the set of parameters for which there is an attracting periodic orbit. Estimate of the largest gap for a given period.

In table 1.17 we have the largest interval where there is no attracting periodic orbit of period $\leq n$ (gap of period n). We shall denote these intervals by I_n. The quotients $|I_n|/|I_{n+1}|$ give

n	$\|I_n\|/\|I_{2n}\|$
4	4.7435942
5	4.8932843
6	4.9804862
7	5.0979310
8	4.7275989
9	4.7275989

It seems that the process of creation of the gaps is as follows. Let $(\epsilon_n^1, \epsilon_n^{2^\infty})$ the last window (the one closest to $\epsilon_1^{2^\infty}$) of period n. Then, if we consider the largest gap which remains when we take all the windows of period $n \geq 4$, it holds:

Let $n = 2^{m_1} + a_{m_1-1}2^{m_1-1} + a_{m_1-2}2^{m_1-2} + k$ such that $k < 2^{m_1-2}$ and a_{n_1-1}, a_{m_1-2} are 0 or 1. Then the following cases can happen:

a) $a_{m_1-1} = a_{m_1-2} = 0$. The largest gap is $I_n = (\epsilon_{n-k-2^{m_1-2}}^1, \epsilon_1^{2^\infty})$.

b) $a_{m_1-1} = 0$, $a_{m_1-2} = 1$. The largest gap is $I_n = (\epsilon^1_{n-k}, \epsilon^{2^\infty}_1)$.

c) $a_{m_1-1} = 1$, $a_{m_1-2} = 0$. The largest gap is $I_n = (\epsilon^1_{n-k-2^{m_1-2}}, \epsilon^{2^\infty}_{n-k})$.

d) $a_{m_1-1} = a_{m_1-2} = 1$. The largest gap is $I_n = (\epsilon^1_{n-k}, \epsilon^{2^\infty}_{n-k-2^{m_1-2}})$.

This is due to the following: Up to period 7 we have the following scheme concerning the closets windows to $\epsilon^{2^\infty}_1$ (the scale is not real):

$$\epsilon^{2^\infty}_1 \; \epsilon^1_6 \quad \epsilon^{2^\infty}_6 \; \epsilon^1_7 \quad \epsilon^{2^\infty}_7 \; \epsilon^1_5 \quad \epsilon^{2^\infty}_5 \; \epsilon^1_3 \quad \epsilon^{2^\infty}_3 \; \epsilon^1_4 \quad \epsilon^{2^\infty}_4 \; \epsilon^{(2^0)_f}_{1,0} = 0$$

Taking into account that, when ϵ tends to $\epsilon^{2^\infty}_1$, this structure repeats itself with the same order but for double period, and a scaling close to the one given by Feigenbaum's constant, we have the following situation:

$$\epsilon^{2^\infty}_1 \quad \epsilon^1_{12} \quad \epsilon^{2^\infty}_{12} \quad \epsilon^1_{14} \quad \epsilon^{2^\infty}_{14} \quad \epsilon^1_{10} \quad \epsilon^{2^\infty}_{10} \quad \epsilon^1_6 \quad \epsilon^{2^\infty}_6 \quad \epsilon^1_8 \quad \epsilon^{2^\infty}_8 \quad \epsilon^{(2)_f}_{1,0} \quad \epsilon^1_7 \quad \epsilon^\infty_7$$

and in this case odd periods do not imply a decrease of the largest gap. We conjecture that this behaviour repeats itself, taking at each step a shorter interval $(\epsilon^{2^\infty}_1, \epsilon^{(2^i)_f}_{1,0})$, $i \geq 2$. If this conjecture is true, we can know the largest gap for any period (see table 1.18)

1.3.10 Comparison with the papers of Farmer and Ketoja

The papers [12] and [11] try to obtain the measure of the set of parameters for which there is an attracting periodic orbit. In [5] and for the map $\tilde{f}_r(x) = r(1 - 2x^2)$, one considers attracting periodic orbits with a window of size larger than or equal to ϵ. This allows to say that the measure of the set of parameters for which there is no attracting periodic orbits, starting at the end of the first bifurcation cascade, divided by the full interval (this ratio is named fraction of chaotic parameters) equals 0.89795 ± 0.00005. To compare this with our result we note that one should do a non linear change of parameters, $r = \sqrt{a/2}$, where a is the parameter of the family $f_a(x) = 1 - ax^2$. This means that this fraction has not to be equal in both cases, but it should be close to. In fact, in our case this value is ≈ 0.89835.

On the other hand, the paper of Ketoja tries to see the limit value of the fraction of chaotic parameters (that is the fraction of parameters for which there is no attarcting periodic orbit) inside the intervals Δ^c_n when $n \to \infty$. In his paper this fraction is 0.982... If we redo the computation with our data, we know that the measure of the simple windows inside $\tilde{\Delta}_5^c$ is 2.3547×10^{-5} as computed before. It remains to obtain the measure of the 2-simple, 3-simple, etc. windows. For this we obtain the measure of the 2-simple windows inside the window corresponding to period 96 (table 1.19). Comparing with the measure that we have in the case of 1-simple attraction windows, one can deduce that the missing measure is between 2.46×10^{-8} and 2.50×10^{-8}. Therefore the total measure of 2-simple windows inside the window of period 96 is between 5.5065×10^{-7} and 5.5105×10^{-7}. Finally, as $|\Delta_{96}| = 1.6063548 \times 10^{-5}$, we have that the quotient between the measure of 2-simple windows in the window corresponding to period 96, and $|\Delta_{96}|$ is between 3.42×10^{-2} and 3.43×10^{-2}. Extrapolating, we can assume that the ratio between n-simple windows and n+1-simple windows, is between 3.3×10^{-2} and 3.5×10^{-2}. Then the total measure of the attraction windows will be in $[2.4350 \times 10^{-5}, 2.4383 \times 10^{-5}]$. Therefore the function of chaotic parameters is in $[8.915 \times 10^{-1}, 8.917 \times 10^{-1}]$. Taking into account that when passing from $\tilde{\Delta}_5^c$ to $\tilde{\Delta}_6^c$ we had to multiply by a number between 4.6696/4.6699 and 4.6693/4.6699 we have that in $\tilde{\Delta}_6^c$ the function of chaotic parameters is in $[8.913 \times 10^{-1}, 8.916 \times 10^{-1}]$. Then the limit seems to be $8.91 \ldots \times 10^{-1}$.

1.3.11 Estimate of the behaviour of the Lebesgue point $\epsilon = 0$. Comparison with analytical results

In table 1.6 we have the values $\tilde{\epsilon}_{n,0}^1$ such that $f^n(0, \tilde{\epsilon}_{n,0}^1) = -x_+$ holds, where x_+ is the fixed point with $x_+ > 0$. One can see $\tilde{\epsilon}_{n,0}^1 / \tilde{\epsilon}_{n+1,0}^1 \to 4$ when $n \to \infty$, as predicted analytically. On the other hand, we can compute the measure of the snh windows inside the windows $\tilde{\Delta}_n = (\tilde{\epsilon}_{n,0}^1, 0)$ (table 1.20). It seems that

$$\frac{\left|\cup_{i=3}^{\infty} \Delta_i^t \cap \tilde{\Delta}_n\right|}{\left|\cup_{i=3}^{\infty} \Delta_i^t \cap \tilde{\Delta}_{n+1}\right|} \to 16,$$

for $n \to \infty$. Furthermore, if we plot

$$\log_{10}(\text{measure of the set of parameters in a snh window inside } \tilde{\Delta}_{14})$$

against the period, and we consider, up to period 26 (figure 1.2), one can note that those measures tend to scale with a factor of 2. Assuming this to be true, we can give an estimate of

$$\frac{\left|\cup_{i=3}^{\infty} \Delta_i^t \cap \tilde{\Delta}_n\right|}{|\tilde{\Delta}_n|^2}$$

when $n \to \infty$. We have:

$$\lim_{n \to \infty} \frac{\left|\cup_{i=3}^{\infty} \Delta_i^t \cap \tilde{\Delta}_n\right|}{|\tilde{\Delta}_n|^2} = 0.24823 \ldots.$$

Finally, taking into account that to find the measure of the attraction windows one has to multiply the measure of the snh by a factor $D \approx 7.609 \times 10^{-1}$, one has

$$\lim_{n\to\infty} \frac{\left|\cup_{i=3}^{\infty}\Delta_i \cap \tilde{\Delta}_n\right|}{|\tilde{\Delta}_n|^2} = 0.188\ldots.$$

Now, we would like to know if the analytical estimates made in the section 1.2 are good enough. For simplicity reasons we shall consider snh windows instead of attraction ones. In such a case we have that the measure of the snh windows, for $\epsilon \in (\tilde{\epsilon}, 0)$, where $\tilde{\epsilon} = -\frac{3\pi^2}{2}4^{-n_1}$, is equal to $\frac{48}{7\pi^2}\tilde{\epsilon}^2$ (without taking into account the first window of period n_1 which has a measure $\frac{6}{\pi^2}\tilde{\epsilon}^2$). This value equals $0.86846\ldots\tilde{\epsilon}^2$. The table 1.21 give the measure of th snh windows from periods n_1+1 to $2n_1$ divided by $\tilde{\epsilon}^2$, showing a reasonable agreement with the analytical predictions.

Remark 1.3.1 *Some of the results of this chapter have appeared in [13].*

n	$\bar{\epsilon}^{(2)}_{n,j}$	$D_n = \lvert\bar{\epsilon}^{(2)}_{n,j} - \bar{\epsilon}_{n,j}\rvert$	$D_n/\Delta_{n,j}$
3	-2.448101258E-01	3.122078788E-04	1.047042339E-02
4	-5.919610575E-02	4.087713588E-06	2.518603656E-03
5	-1.457568737E-02	5.956811979E-08	6.211658436E-04
5	-1.391071730E-01	1.103046986E-04	8.609440053E-02
5	-3.646863112E-01	9.899963669E-03	1.494820036E+00
6	-3.623861380E-03	9.085508614E-10	1.545325610E-04
6	-3.322549286E-02	1.290741161E-06	2.196548847E-02
6	-9.262176479E-02	9.814414407E-05	5.057559789E-01
7	-9.043176588E-04	1.408606419E-11	3.856947216E-05
7	-8.185808863E-03	1.858700035E-08	5.492228215E-03
7	-2.281910213E-02	1.310857934E-06	1.327938021E-01
7	-4.623384140E-02	6.026430861E-05	2.326780867E+00
7	-7.266427156E-02	1.880190680E-04	4.010592954E+00
7	-1.148256865E-01	3.707418658E-04	5.127885710E+00
7	-1.667441101E-01	9.406870699E-04	5.787563524E+00
7	-3.000080080E-01	2.592590044E-02	3.518082761E+01
7	-1.577086804E-01	2.674021797E-01	2.333204244E+02
8	-2.259513060E-04	2.195690838E-13	9.637325901E-06
8	-2.037084118E-03	2.835612308E-10	1.371677569E-03
8	-5.667013786E-03	1.949813814E-08	3.348748333E-02
8	-1.120596401E-02	7.616794678E-07	6.236219568E-01
8	-1.834151266E-02	2.701057601E-06	1.303533022E+00
8	-2.779561305E-02	4.548581063E-06	1.450926361E+00
8	-3.923115493E-02	9.857863869E-06	2.035171919E+00
9	-5.647823432E-05	3.428326977E-15	2.408945897E-06
9	-5.085619792E-04	4.399983055E-12	3.427461848E-04
9	-1.413410857E-03	3.001289370E-10	8.383740879E-03
9	-2.776964030E-03	1.127307543E-08	1.579232638E-01
9	-4.580926505E-03	4.083270698E-08	3.435012739E-01
9	-6.869678866E-03	6.625441595E-08	3.715427519E-01
9	-9.623480501E-03	1.384423013E-07	5.419823458E-01
9	-1.298342881E-02	1.222367389E-05	2.770328964E+01
9	-1.619613469E-02	6.384690590E-06	1.109339360E+01
9	-2.053600456E-02	6.490577284E-06	1.127366540E+01
9	-2.521043816E-02	8.702937444E-06	1.221225874E+01

10	-1.411885960E-05	5.355816390E-17	6.022262464E-07
10	-1.270882336E-04	6.861327110E-14	8.567022564E-05
10	-3.530822620E-04	4.668761719E-12	2.096302633E-03
10	-6.924955458E-04	1.735779474E-10	3.958790512E-02
10	-1.144233387E-03	6.312541768E-10	8.691371933E-02
10	-1.711091635E-03	1.014118555E-09	9.334669245E-02
10	-2.391904038E-03	2.098144462E-09	1.373703988E-01
10	-3.195283070E-03	1.641868792E-07	7.617198421E+00
10	-4.075922506E-03	7.443694314E-08	2.677870918E+00
10	-5.110704525E-03	9.612396276E-08	2.903489990E+00
10	-6.252008793E-03	1.269608645E-07	3.127592333E+00
10	-7.520424646E-03	1.931747443E-07	3.838756684E+00
10	-8.878923519E-03	2.003576178E-07	3.329006494E+00
10	-1.039899202E-02	4.210608577E-07	5.979955403E+00
10	-1.205725401E-02	2.171700739E-06	2.468710383E+01

n	$\bar{\epsilon}^{(1)}_{n,j}$	$D_n = \lvert\bar{\epsilon}^{(1)}_{n,j} - \bar{\epsilon}_{n,j}\rvert$	$D_n/\Delta_{n,j}$
3	-2.412981229E-01	3.824210804E-03	1.282514279E-01
4	-5.899164416E-02	2.085493048E-04	1.284955587E-01
5	-1.456338859E-02	1.235834922E-05	1.288706852E-01
5	-1.398143647E-01	5.968869758E-04	4.658788521E-01
5	-4.680405159E-01	9.345424107E-02	1.411088734E+01
6	-3.623103931E-03	7.583576624E-07	1.289866718E-01
6	-3.326344103E-02	3.665742959E-05	6.238263494E-01
6	-2.648135555E-02	3.761446623E-03	1.938347047E+01
7	-9.042705542E-04	4.711876034E-08	1.290172818E-01
7	-8.188072233E-03	2.244782136E-06	6.633052966E-01
7	-2.302943373E-02	2.090207462E-04	2.117442241E+01
7	-4.602293749E-02	2.711682220E-04	1.046969666E+01
7	-7.819816651E-02	5.345875876E-03	1.140316901E+02
7	-1.212582065E-01	6.061778114E-03	8.384298682E+01
7	-1.780981079E-01	1.041331073E-02	6.406774295E+01
7	-2.540944938E-01	7.183941464E-02	9.748436965E+01
7	-3.612431293E-01	6.386773084E-02	5.572746671E+01
8	-2.259483666E-04	2.939603059E-09	1.290250531E-01
8	-2.037223554E-03	1.391522322E-07	6.731244435E-01
8	-5.679635237E-03	1.260195351E-05	2.164348744E+01
8	-1.119384248E-02	1.288321026E-05	1.054807584E+01
8	-1.864327753E-02	2.990638146E-04	1.443284875E+02
8	-2.811738598E-02	3.172243518E-04	1.011896167E+02
8	-3.973652827E-02	4.955154727E-04	1.022999697E+02
9	-5.647805069E-05	1.836266925E-10	1.290270066E-01
9	-5.085706562E-04	8.672629080E-09	6.755731765E-01
9	-1.414190207E-03	7.790500768E-07	2.176182690E+01
9	-2.776224427E-03	7.508757668E-07	1.051893536E+01
9	-4.599053076E-03	1.808573830E-05	1.521445577E+02
9	-6.888607415E-03	1.886229418E-05	1.057763258E+02
9	-9.652452314E-03	2.883337088E-05	1.128786349E+02
9	-1.289989366E-02	9.575881638E-05	2.170242963E+02
9	-1.664211465E-02	4.523646512E-04	7.859831356E+02
9	-2.089234555E-02	3.498504140E-04	6.076649791E+02
9	-2.566607337E-02	4.469322774E-04	6.271506199E+02

10	-1.411884813E-05	1.147488717E-11	1.290275418E-01
10	-1.270887752E-04	5.415563162E-10	6.761848118E-01
10	-3.531307993E-04	4.853267080E-08	2.179146671E+01
10	-6.924496412E-04	4.607813884E-08	1.050903651E+01
10	-1.145353336E-03	1.119317742E-06	1.541123554E+02
10	-1.712254390E-03	1.161741373E-06	1.069349477E+02
10	-2.393671334E-03	1.765198464E-06	1.155716498E+02
10	-3.190230692E-03	5.216564903E-06	2.420145272E+02
10	-4.102669380E-03	2.682131097E-05	9.648973431E+02
10	-5.131837556E-03	2.103690760E-05	6.354341717E+02
10	-6.278701951E-03	2.656619778E-05	6.544397504E+02
10	-7.544349698E-03	2.373187720E-05	4.715984099E+02
10	-8.929992712E-03	5.086883536E-05	8.452021198E+02
10	-1.043697264E-02	3.755955983E-05	5.334252489E+02
10	-1.206676649E-02	7.340771422E-06	8.344721858E+01

Table 1.3:

n	j	$\Delta^{(1)}_{n,j}$	$D_{n,j} = \lvert\Delta^{(1)}_{n,j} - \Delta^{1}_{n,j}\rvert$	$D_{n,j}/\Delta^{1}_{n,j}$
3	0	1.445742832E-02	4.071724145E-03	2.197469178E-01
4	0	9.035892701E-04	8.337502219E-05	8.447622963E-02
5	0	5.647432938E-05	1.653905899E-06	2.845271134E-02
5	1	5.082689644E-04	2.698538208E-04	3.468010781E-01
5	2	1.411858234E-03	2.626903208E-03	6.504229688E-01
6	0	3.529645586E-06	3.181458252E-08	8.933016520E-03
6	1	3.176681027E-05	3.832160981E-06	1.076480821E-01
6	2	8.824113966E-05	2.933118364E-05	2.494735395E-01
7	0	2.206028491E-07	5.942140984E-10	2.686356182E-03
7	1	1.985425642E-06	6.430414508E-08	3.137201082E-02
7	2	5.515071228E-06	4.637553777E-07	7.756628653E-02
7	3	1.080953960E-05	4.880342688E-06	3.110503059E-01
7	4	1.786883078E-05	1.052309991E-05	3.706369963E-01
7	5	2.669294474E-05	1.710545938E-05	3.905498321E-01
7	6	3.728188150E-05	6.120104710E-05	6.214381311E-01
7	7	4.963564105E-05	3.975629087E-04	8.890075983E-01
7	8	6.375422340E-05	6.308550695E-04	9.082157062E-01
8	0	1.378767807E-08	1.082713765E-11	7.846601855E-04
8	1	1.240891026E-07	1.113217566E-09	8.891349330E-03
8	2	3.446919517E-07	7.945427569E-09	2.253143890E-02
8	3	6.755962255E-07	6.413153947E-08	8.669613675E-02
8	4	1.116801923E-06	1.381528628E-07	1.100859284E-01
8	5	1.668309046E-06	2.303723946E-07	1.213328310E-01
8	6	2.330117594E-06	6.035029083E-07	2.057194882E-01
9	0	8.617298794E-10	1.934267486E-13	2.244129463E-04
9	1	7.755568915E-09	1.927739652E-11	2.479456924E-03
9	2	2.154324698E-08	1.379608054E-10	6.363151297E-03
9	3	4.222476409E-08	1.007695593E-09	2.330877310E-02
9	4	6.980012023E-08	2.193363871E-09	3.046614424E-02
9	5	1.042693154E-07	3.729678698E-09	3.453438366E-02
9	6	1.456323496E-07	9.070160306E-09	5.862969068E-02
9	7	1.938892228E-07	7.334540574E-08	2.744607093E-01
9	8	2.490399351E-07	9.952338030E-08	2.855245399E-01
9	9	3.110844864E-07	3.759981267E-08	1.078333976E-01
9	10	3.800228768E-07	5.157909634E-08	1.195061643E-01

10	0	5.385811746E-11	3.402078021E-15	6.316342446E-05
10	1	4.847230572E-10	3.315770600E-13	6.835870365E-04
10	2	1.346452936E-09	2.386227868E-12	1.769097406E-03
10	3	2.639047755E-09	1.644106831E-11	6.191352854E-03
10	4	4.362507514E-09	3.623320059E-11	8.237175805E-03
10	5	6.516832213E-09	6.280952553E-11	9.546040351E-03
10	6	9.102021852E-09	1.482539282E-10	1.602697387E-02
10	7	1.211807643E-08	9.362964196E-10	7.172281889E-02
10	8	1.556499594E-08	1.269922477E-09	7.543383613E-02
10	9	1.944278040E-08	6.076554160E-10	3.030634453E-02
10	10	2.375142980E-08	8.336814100E-10	3.391001174E-02
10	11	2.849094414E-08	1.986078173E-09	6.516641138E-02
10	12	3.366132341E-08	2.789173861E-09	7.651950095E-02
10	13	3.926256763E-08	3.381594543E-09	7.929794772E-02
10	14	4.529467679E-08	7.982689985E-09	1.498326675E-01

Table 1.4:

n	$\epsilon_{1,0}^{(2^{n-1})_f}$	$(\epsilon_{1,0}^{(2^{n-3})_f} - \epsilon_{1,0}^{(2^{n-2})_f})/(\epsilon_{1,0}^{(2^{n-2})_f} - \epsilon_{1,0}^{(2^{n-1})_f})$
1	0.000000000000000E+00	
2	-4.563109873079239E-01	
4	-5.696423675486926E-01	4.026342804071622E+00
8	-5.925948818352980E-01	4.937645559245185E+00
16	-5.975078236414357E-01	4.671847376236172E+00
32	-5.985585057464117E-01	4.675954584997647E+00
64	-5.987834956905848E-01	4.669906954453772E+00
128	-5.988316791606988E-01	4.669442521282707E+00
256	-5.988419984947881E-01	4.669242191101691E+00
512	-5.988442085753859E-01	4.669211656573828E+00
1024	-5.988446819067696E-01	4.669203593609498E+00
2048	-5.988447832798475E-01	4.669202054798063E+00
4096	-5.988448049908561E-01	4.669201702001840E+00
8192	-5.988448096406892E-01	4.669201629314997E+00
16384	-5.988448106365411E-01	4.669201613392723E+00
32768	-5.988448108498220E-01	4.669201610026550E+00
65536	-5.988448108955003E-01	4.669201609300192E+00
131072	-5.988448109052832E-01	4.669201609145299E+00
262144	-5.988448109073784E-01	4.669201609112043E+00
524288	-5.988448109078271E-01	4.669201609104930E+00

Table 1.5:

n	$\tilde{\epsilon}^1_{n,0}$	$\tilde{\epsilon}^1_{n-1,0}/\tilde{\epsilon}^1_{n,0}$
2	-4.563109873079239E-01	0.000000000000000E+00
3	-1.070890120921806E-01	4.261043952064273E+00
4	-2.606795540508039E-02	4.108071017771880E+00
5	-6.454913394094307E-03	4.038467104598233E+00
6	-1.608638467934874E-03	4.012656369197080E+00
7	-4.017629842503633E-04	4.003948922612623E+00
8	-1.004109674019055E-04	4.001186271239320E+00
9	-2.510056681856938E-05	4.000346610803289E+00
10	-6.274986101706795E-06	4.000099189341954E+00
11	-1.568735570269421E-06	4.000027933725696E+00
12	-3.921831309758896E-07	4.000007767712638E+00
13	-9.804573033820877E-08	4.000002138013086E+00
14	-2.451142900879048E-08	4.000000583525622E+00
15	-6.127857009937270E-09	4.000000158137077E+00
16	-1.531964236169590E-09	4.000000042598195E+00
17	-3.829910579493860E-10	4.000000011415529E+00
18	-9.574776441444951E-11	4.000000003045376E+00
19	-2.393694109876984E-11	4.000000000809216E+00
20	-5.984235274371897E-12	4.000000000214271E+00
21	-1.496058818571821E-12	4.000000000056558E+00
22	-3.740147046415632E-13	4.000000000014886E+00
23	-9.350367616029948E-14	4.000000000003907E+00
24	-2.337591904006890E-14	4.000000000001022E+00
25	-5.843979760016839E-15	4.000000000000264E+00
26	-1.460994940004187E-15	4.000000000000063E+00
27	-3.652487350010460E-16	4.000000000000008E+00
28	-9.131218375026149E-17	4.000000000000000E+00
29	-2.282804593756537E-17	4.000000000000000E+00
30	-5.707011484391343E-18	4.000000000000000E+00
31	-1.426752871097836E-18	4.000000000000000E+00
32	-3.566882177744589E-19	4.000000000000000E+00

Table 1.6:

n	i	$\bar{\epsilon}^{(i)}_{n,j}$	$\lvert\bar{\epsilon}^{(i)}_{n,j} - \bar{\epsilon}^{i}_{n,j}\rvert$
2	1	-4.112335167120566E-01	4.507747059586690E-02
3	1	-1.046555799960426E-01	2.433432096138039E-03
4	1	-2.592631502893058E-02	1.416403761498133E-04
5	1	-6.446293605645235E-03	8.619788449017523E-06
5	2	-1.759675115158956E-01	5.412354259774303E-03
5	1	-3.887422345037810E-01	8.629762774132855E-02
6	1	-1.608104539731448E-03	5.339282034258696E-07
6	2	-4.128955069379783E-02	2.008621276899157E-04
6	1	-8.328313146693868E-02	3.821868232715702E-03
7	1	-4.017297081233063E-04	3.327612705695235E-08
7	2	-1.012512032757154E-02	1.110749130551572E-05
7	1	-2.001036219622088E-02	2.159495774785713E-04
7	2	-5.071199451865429E-02	1.424853972689181E-03
7	1	-7.213663062678545E-02	6.394675712145410E-03
7	2	-1.296708120275776E-01	5.432845003546691E-03
7	1	-1.674862718093427E-01	1.225501785755246E-02
7	2	-2.692827767437570E-01	8.639875775768156E-02
7	1	-3.402441220564580E-01	5.027670268901714E-02
8	1	-1.004088894267888E-04	2.077975116675192E-09
8	2	-2.516297078468580E-03	6.716533046694005E-07
8	1	-4.943947287040688E-03	1.308004614738287E-05
8	2	-1.229895207705487E-02	6.187947524896626E-05
8	1	-1.726386971977237E-02	3.438321558997327E-04
8	2	-2.990118746354666E-02	2.860849951279868E-04
8	1	-3.764485349830130E-02	5.769280384046214E-04
8	2	-5.621731512105809E-02	3.029034979884692E-02
8	1	-6.716881102607978E-02	3.191011408059093E-02
8	2	-9.280593283041615E-02	1.086406029099092E-01
8	1	-1.077052311817013E-01	1.643406269970178E-01
9	1	-2.510043697784171E-05	1.298407275112491E-10
9	2	-6.279524813974406E-04	4.159795173642213E-08
9	1	-1.231654349062288E-03	8.097185842806135E-07
9	2	-3.047888993442175E-03	3.571155851340705E-06
9	1	-4.263016696117401E-03	2.067554470857655E-05
9	2	-7.316082273868321E-03	1.702207436016439E-05
9	1	-9.158502154661080E-03	3.349879346369007E-05
9	2	-1.348890652391262E-02	2.552571657672123E-04
9	1	-1.598351668532116E-02	6.102274733498006E-04
9	2	-2.165115941992472E-02	3.422119129222830E-04
9	1	-2.483336735656422E-02	4.523349134562297E-04
:	:	:	:

:	:	:	:
9	2	-3.192123364413629E-02	2.932458828388456E-04
9	1	-3.583923294891286E-02	9.999567973512030E-04
9	2	-4.445900439233321E-02	5.480417505691683E-04
9	1	-4.917721217829987E-02	3.064218532366411E-04
9	2	-5.947774647367815E-02	1.019396556154158E-02
9	1	-6.508202586732556E-02	1.085266686690205E-02
9	2	-7.726237266309548E-02	1.411491548251046E-02
9	1	-8.386815344064590E-02	1.008952496233917E-02
9	2	-9.819815041169867E-02	1.329161773178373E-02
9	1	-1.059635329000924E-01	1.307641065244162E-02
9	2	-1.228178714432088E-01	3.959639344546205E-02
9	1	-1.319658862006278E-01	4.112142457225160E-02
9	2	-1.518839385279482E-01	1.668708336449478E-01
9	1	-1.627430588011630E-01	1.703461234764050E-01
9	2	-1.865427964190559E-01	2.337100141655251E-01
9	1	-1.996279181024988E-01	2.296748780352343E-01
10	1	-6.274977987248759E-06	8.114458034390814E-12
10	2	-1.569059607153430E-04	2.593372068896822E-09
10	1	-3.075974854873092E-04	5.046378911714420E-08
10	2	-7.600359136480467E-04	2.187052679105458E-07
10	1	-1.061965347231620E-03	1.278117395180230E-06
10	2	-1.817855817695290E-03	1.048413554264789E-06
10	1	-2.272124115771400E-03	2.050059748287854E-06
10	2	-3.334235090307513E-03	1.234038988214952E-05
10	1	-3.942514434448218E-03	3.385592937249817E-05
10	2	-5.314780427236825E-03	2.055347082571378E-05
10	1	-6.079339920968768E-03	2.693167572279484E-05
10	2	-7.766922597623811E-03	1.830647898386397E-05
10	1	-8.690663911384225E-03	5.685738668861649E-05
10	2	-1.070003428788051E-02	3.237321475605842E-05
10	1	-1.178653854990756E-02	2.152761706781740E-05
10	2	-1.412558333883191E-02	3.516635820677338E-03
10	1	-1.537917104522356E-02	3.673196414288764E-03
10	2	-1.805732726314929E-02	4.066918761176489E-03
10	1	-1.948313358013557E-02	4.038971000709897E-03
10	2	-2.251155689110556E-02	4.433725941533052E-03
10	1	-2.411562549734639E-02	4.557271049744852E-03
10	2	-2.750739954523549E-02	1.067648072883212E-02
10	1	-2.929679918405455E-02	1.101732767229457E-02

Table 1.7:

n	$\|\Delta_n^t \cap \tilde{\Delta}_2\|$
3	4.032749199934570E-02
4	2.211222072139026E-03
5	1.083651924439175E-02
6	3.524725302645279E-04
7	2.998347708613134E-03
8	7.047941697216196E-04
9	8.888947230156415E-04
10	1.807600946204198E-04
11	3.774862094579465E-04
12	1.311040764076555E-04
13	1.371718655566157E-04
14	6.509665136970219E-05
15	4.658519635942487E-05
16	3.319111422491411E-05
17	3.023970269448826E-05
18	1.312578941001657E-05
19	1.437151524771879E-05
20	8.408345724910689E-06
21	5.213785756058523E-06
22	4.586967125556409E-06
23	3.918486684957282E-06
24	2.239383968877338E-06
total	5.9373241632100E-02

Table 1.8:

i	$\sum_{j=3}^{6} \lvert\Delta^t_{4i+j} \cap \tilde{\Delta}_2\rvert$	$\sum_{j=3}^{6} \lvert\Delta^t_{4(i-1)+j} \cap \tilde{\Delta}_2\rvert / \sum_{j=3}^{6} \lvert\Delta^t_{4i+j} \cap \tilde{\Delta}_2\rvert$
0	5.372770584614101E-02	
1	4.772796695970815E-03	1.125706986251852E+01
2	7.108588027919200E-04	6.714127583741677E+00
3	1.231418026888438E-04	5.772684720136235E+00
4	3.258061385424442E-05	3.779603516365350E+00

i	$\sum_{j=0}^{3} \lvert\Delta^t_{4(i+1)+j} \cap \tilde{\Delta}_2\rvert$	$\sum_{j=0}^{3} \lvert\Delta^t_{4i+j} \cap \tilde{\Delta}_2\rvert / \sum_{j=0}^{3} \lvert\Delta^t_{4(i+1)+j} \cap \tilde{\Delta}_2\rvert$
0	1.639856155540844E-02	
1	2.151935196815628E-03	7.620378894157486E+00
2	3.799577896933983E-04	5.663616473166932E+00
3	9.092812157713776E-05	4.178660936826522E+00
4	2.212758529148291E-05	4.109265443081887E+00

i	$\sum_{j=1}^{4} \lvert\Delta^t_{4(i+1)+j} \cap \tilde{\Delta}_2\rvert$	$\sum_{j=1}^{4} \lvert\Delta^t_{4i+j} \cap \tilde{\Delta}_2\rvert / \sum_{j=1}^{4} \lvert\Delta^t_{4(i+1)+j} \cap \tilde{\Delta}_2\rvert$
0	1.489213365299104E-02	
1	1.578245103501663E-03	9.435881422948663E+00
2	2.820448275106569E-04	5.595724330175956E+00
3	6.614535307713433E-05	4.264015753030374E+00
4	1.595862353544956E-05	4.144803148605072E+00

i	$\sum_{j=2}^{5} \lvert\Delta^t_{4(i+1)+j} \cap \tilde{\Delta}_2\rvert$	$\sum_{j=2}^{5} \lvert\Delta^t_{4i+j} \cap \tilde{\Delta}_2\rvert / \sum_{j=2}^{5} \lvert\Delta^t_{4(i+1)+j} \cap \tilde{\Delta}_2\rvert$
0	4.944509131614923E-03	
1	8.265222460426376E-04	5.982306169360930E+00
2	1.751126646485295E-04	4.719945571621330E+00
3	4.111943613870458E-05	4.258634871787572E+00

Table 1.9:

$\tilde{\Delta}_0^c$	5.937324163210067E-02	(UP TO PERIOD 24)
$\tilde{\Delta}_1^c$	1.604219582708864E-02	(UP TO PERIOD 42)
$\tilde{\Delta}_2^c$	3.235124313141708E-03	(UP TO PERIOD 80)
$\tilde{\Delta}_3^c$	6.931939933174490E-04	(UP TO PERIOD 152)
$\tilde{\Delta}_4^c$	1.481804800495428E-04	(UP TO PERIOD 272)
$\tilde{\Delta}_5^c$	3.171694100629041E-05	(UP TO PERIOD 512)
$\tilde{\Delta}_6^c$	6.788902025180304E-06	(UP TO PERIOD 960)
$\tilde{\Delta}_7^c$	1.452954826769992E-06	(UP TO PERIOD 1792)
$\tilde{\Delta}_8^c$	3.108437826206848E-07	(UP TO PERIOD 3328)
$\tilde{\Delta}_9^c$	6.591294558106544E-08	(UP TO PERIOD 5120)
TOTAL	7.953227180028445E-02	

Table 1.10: Totals of measure of snh windows in the window $\tilde{\Delta}_n^c$, up to the considered period.

$n \mid i$	0	1	2	3	4	5	6	7
3	3.698326463	4.952402208	4.667605401	4.676038622	4.669803363	4.669435437	4.669238804	4.669211162
4	2.237347273	5.388888200	4.637025167	4.683081014	4.669683072	4.669606887	4.669251367	4.669216826
5	4.080110738	4.870059835	4.669822719	4.674150322	4.669683621	4.669375055	4.669230141	4.669208781
6	2.671976075	5.309003892	4.645598694	4.681850402	4.669790001	4.669584543	4.669252143	4.669216307
7	4.247362678	4.841029984	4.671828117	4.673640250	4.669687347	4.669362099	4.669229055	4.669208340
8	3.636355277	4.976313300	4.666243606	4.676409633	4.669804648	4.669445066	4.669239727	4.669211500
9	4.383319784	4.812887306	4.673398300	4.673114466	4.669678047	4.669347563	4.669227480	
10	3.611528797	4.990647012	4.665249965	4.676644877	4.669800511	4.669450823	4.669240144	
11	4.167836308	4.853741495	4.670972109	4.673887508	4.669688245	4.669368662	4.669229677	
12	4.111727291	4.864081362	4.671744209	4.674236968	4.669741239	4.669382675	4.669232351	
13	4.169017327	4.852715140	4.670901744	4.673861623	4.669683787	4.669367591	4.669229461	
14	3.823846310	4.913564407	4.667234700	4.675012562	4.669697305	4.669398878	4.669232674	
15	4.302617552	4.827426570	4.672248922	4.673364476	4.669671757	4.669353514		
16	4.017620842	4.874150928	4.669311556	4.674251451	4.669679171			
17	4.089027333	4.859332111	4.670049440	4.673953527				
18	4.019673543	4.875467257	4.669118121					
19	3.986382947	4.873695421	4.668163218					
20	3.896910399	4.890400591						
21	4.075203076							

Table 1.11: $|\Delta^t_{2^i n} \cap \tilde{\Delta}^c_i| / |\Delta^t_{2^{i+1} n} \cap \tilde{\Delta}^c_{i+1}|$, with $\tilde{\Delta}^c_0 = \tilde{\Delta}_2$

n	$\Delta_n^t \cap \tilde{\Delta}_2$	n	$\Delta_n^t \cap \tilde{\Delta}_1^c$
3	4.032749199934570E-02	6	1.090425423585436E-02
4	2.211222072139026E-03	8	9.883231353174611E-04
5	1.083651924439175E-02	10	2.655937531974641E-03
6	3.524725302645279E-04	12	1.319145532255458E-04
7	2.998347708613134E-03	14	7.059316417632419E-04
8	7.047941697216196E-04	16	1.938188421828452E-04
9	8.888947230156415E-04	18	2.027902974629083E-04
10	1.807600946204198E-04	20	5.005085235336782E-05
11	3.774862094579465E-04	22	9.057126563454615E-05
12	1.311040764076555E-04	24	3.188540170979963E-05
13	1.371718655566157E-04	26	3.290268539711320E-05
14	6.509665136970219E-05	28	1.702386709065356E-05
15	4.658519635942487E-05	30	1.082717573437303E-05
16	3.319111422491411E-05	32	8.261385414183356E-06
17	3.023970269448826E-05	34	7.395329067275081E-06
18	1.312578941001657E-05	36	3.265386919334568E-06
19	1.437151524771879E-05	38	3.605151696869051E-06
20	8.408345724910689E-06	40	2.157695421095906E-06
21	5.213785756058523E-06	42	1.279392869016790E-06
22	4.586967125556409E-06	TOTAL	1.604219582708864E-02
23	3.918486684957282E-06		
24	2.239383968877338E-06		
TOTAL	5.937324163210067E-02		

n	$\Delta_n^t \cap \tilde{\Delta}_2^c$	n	$\Delta_n^t \cap \Delta_3^c$
12	2.201811116572237E-03	24	4.717217774963033E-04
16	1.834001928842342E-04	32	3.955126105022537E-05
20	5.453603490594009E-04	40	1.167839513039938E-04
24	2.484732652089640E-05	48	5.348573596847062E-06
28	1.458226129771564E-04	56	3.121318021966766E-05
32	3.894827967848187E-05	64	8.346816618937755E-06
36	4.213485264547539E-05	72	9.015891635903474E-06
40	1.002893056378502E-05	80	2.149709155461460E-06
44	1.866009257560142E-05	88	3.994905586397921E-06
48	6.555277211393404E-06	96	1.403175541674934E-06
52	6.780263099568493E-06	104	1.451596173745436E-06
56	3.464667536440667E-06	112	7.423383992927738E-07
60	2.242846281673588E-06	120	4.800356998709198E-07
64	1.694938366684179E-06	128	3.629953465701413E-07
68	1.521881793245574E-06	136	3.258813022417426E-07
72	6.697587630121034E-07	144	1.434443819173216E-07
76	7.397162492273396E-07	152	1.584598083978682E-07
80	4.412103631931052E-07	TOTAL	6.931939933174489E-04
TOTAL	3.235124313141707E-03		

n	$\Delta_n^t \cap \tilde{\Delta}_4^c$
48	1.008806418391758E-04
64	8.445564133664806E-06
80	2.498506536038107E-05
96	1.142405915747046E-06
112	6.678558585723493E-06
128	1.784877132861109E-06
144	1.929311105203431E-06
160	4.596691029235339E-07
176	8.547286555242763E-07
192	3.001934970565052E-07
208	3.105774818838012E-07
224	1.587885357402044E-07
240	1.027173682524362E-07
256	7.765849791082405E-08
272	6.972283749433762E-08
TOTAL	1.481804800495427E-04

n	$\Delta_n^t \cap \tilde{\Delta}_5^c$
96	2.160276011392635E-05
128	1.808594716677887E-06
160	5.350483541005753E-06
192	2.446375351916136E-07
224	1.430193948603028E-06
256	3.822166594384026E-07
288	4.131571996324689E-07
320	9.843441958019042E-08
352	1.830376270724982E-07
384	6.428482471383487E-08
416	6.650931755707915E-08
448	3.400403181749074E-08
480	2.199669989308730E-08
512	1.663037118072310E-08
TOTAL	3.171694100629041E-05

n	$\Delta_n^t \cap \tilde{\Delta}_6^c$
192	4.6264179908427288E-06
256	3.8731198581310976E-07
320	1.1458671615578771E-06
384	5.2389571898658846E-08
448	3.0629321911129584E-07
512	8.1854835858156943E-08
576	8.8482854184384502E-08
640	2.1080513172168475E-08
704	3.9199652097159730E-08
768	1.3767306981705843E-08
832	1.4243752769209700E-08
896	7.2823146413359194E-09
960	4.7108662525121524E-09
TOTAL	6.7889020251803036E-06

n	$\Delta_n^t \cap \tilde{\Delta}_7^c$
384	9.9082916600421990E-07
512	8.2949482758508111E-08
640	2.4540815654080329E-07
768	1.1220120542733931E-08
896	6.5598242336081594E-08
1024	1.7530656089327260E-08
1152	1.8950212759735452E-08
1280	4.5147631138606347E-09
1408	8.3953146035636040E-09
1536	2.9485161468817747E-09
1664	3.0505574607724492E-09
1792	1.5596384135039806E-09
TOTAL	1.4529548267699919E-06

n	$\Delta_n^t \cap \tilde{\Delta}_8^c$	n	$\Delta_n^t \cap \tilde{\Delta}_9^c$
768	2.1220483107060248E-07	1536	4.5447758471903085E-08
1024	1.7765181151336782E-08	2048	3.8047555834991872E-09
1280	5.2558831281820062E-08	2560	1.1256488767521822E-08
1536	2.4029986629379448E-09	3072	5.1464843246545596E-10
1792	1.4049114444095710E-08	3584	3.0088892276236928E-09
2048	3.7545217407334889E-09	4096	8.0410325428621003E-10
2304	4.0585498689188107E-09	4608	8.6921686507974489E-10
2560	9.6692191607835606E-10	5120	2.0708497868623932E-10
2816	1.7980166183293125E-09	TOTAL	6.5912945581065438E-08
3072	6.3148084208811378E-10		
3328	6.5333502374376908E-10		
TOTAL	3.1084378262068483E-07		

TOTALS OF MEASURE OF SNH WINDOWS IN nth-CASCADE WINDOWS

$\tilde{\Delta}_2$	5.937324163210067E-02	(UP TO PERIOD 24)
$\tilde{\Delta}_1^c$	1.604219582708864E-02	(UP TO PERIOD 42)
$\tilde{\Delta}_2^c$	3.235124313141708E-03	(UP TO PERIOD 80)
$\tilde{\Delta}_3^c$	6.931939933174490E-04	(UP TO PERIOD 152)
$\tilde{\Delta}_4^c$	1.481804800495428E-04	(UP TO PERIOD 272)
$\tilde{\Delta}_5^c$	3.171694100629041E-05	(UP TO PERIOD 512)
$\tilde{\Delta}_6^c$	6.788902025180304E-06	(UP TO PERIOD 960)
$\tilde{\Delta}_7^c$	1.452954826769992E-06	(UP TO PERIOD 1792)
$\tilde{\Delta}_8^c$	3.108437826206848E-07	(UP TO PERIOD 3328)
$\tilde{\Delta}_9^c$	6.591294558106544E-08	(UP TO PERIOD 5120)
TOTAL	7.953227180028445E-02	

Table 1.12: Measure of snh windows.

$n \mid i$	0	1	2	3	4	5	6
3	1.3524511851	1.3448765953	1.3450126186	1.3448495268	1.3448365967	1.3448311547	1.3448303181
4	1.3624222625	1.3607733393	1.3609687632	1.3609345091	1.3609357345	1.3609349720	
5	1.3545733342	1.3529805572	1.3529626404	1.3529310804	1.3529277052	1.3529265737	
6	1.3624974183	1.3623341795	1.3623481375	1.3623454586	1.3623455534		
7	1.3602623414	1.3600436661	1.3600323055	1.3600275142	1.3600267478		
8	1.3612738183	1.3609566327	1.3609763075	1.3609712040	1.3609716855		
9	1.3619938401	1.3619319658	1.3619317437	1.3619306121	1.3619308086		
10	1.3623786609	1.3622918998	1.3622988734	1.3622978254			
11	1.3623738583	1.3622881204	1.3622983402	1.3622972445			
12	1.3624659062	1.3624508222	1.3624506956				
13	1.3626047487	1.3625795411	1.3625829842				
14	1.3624745775	1.3623959167	1.3624058307				
15	1.3626671133	1.3626651614	1.3626658485				
16	1.3626305597	1.3626119001					
17	1.3625759459	1.3625200900					
18	1.3626676507	1.3626655487					
19	1.3626445160						
20	1.3625755187						
21	1.3626766080						
22	1.3626431833						

Table 1.13: $|\Delta^t_{2^i n} \cap \tilde{\Delta}^c_i| / |\Delta'_{2^{i+1} n} \cap \tilde{\Delta}^c_i|$, with $\tilde{\Delta}^c_0 = \tilde{\Delta}_2$

MEASURE OF n-PERIODIC SIMPLE ATTRACTION WINDOWS IN $\tilde{\Delta}_i^c$, $i = 0 \ldots 6$.

n	$\tilde{\Delta}_2$		$\tilde{\Delta}_1^c$		$\tilde{\Delta}_2^c$
3	2.981807583499140E-02	6	8.1079961342530176E-03	12	1.6370189290784847E-03
4	1.623007883083403E-03	8	7.2629519317035651E-04	16	1.3475709203614183E-04
5	7.999950221091194E-03	10	1.9630271239646918E-03	20	4.0308603709385271E-04
6	2.586959252353693E-04	12	9.6829805205339569E-05	24	1.8238602775375034E-05
7	2.204242238617134E-03	14	5.1905071827619553E-04	28	1.0721996263142463E-04
8	5.177460700566267E-04	16	1.4241367985792785E-04	32	2.8617896919755358E-05
9	6.526422490675849E-04	18	1.4889899242873094E-04	36	3.0937565586475201E-05
10	1.326797753117149E-04	20	3.6740182014077804E-05	40	7.3617696966649104E-06
11	2.770797510259652E-04	22	6.6484662297333629E-05	44	1.3697508118873380E-05
12	9.622558319577115E-05	24	2.3402974397762621E-05	48	4.8113867404386834E-06
13	1.006688591752759E-04	26	2.4147350230749320E-05	52	4.9760368200507665E-06
14	4.777825028524133E-05	28	1.2495535902545817E-05	56	2.5430510193213962E-06
15	3.418677673032233E-05	30	7.9455878387313383E-06	60	1.6459253632666870E-06
16	2.435811672339753E-05	32	6.0629042015203360E-06		
17	2.219304016428393E-05	34	5.4276844221462505E-06		
18	9.632421671678953E-06	36	2.3963230907086963E-06		
19	1.054678243583049E-05				
20	6.170920884335372E-06				
21	3.826135801865000E-06				
22	3.366227624114151E-06				

n	$\tilde{\Delta}_3^c$	n	$\tilde{\Delta}_4^c$	n	$\tilde{\Delta}_5^c$
24	3.5076175296583356E-04	48	7.5013308001040113E-05	96	1.6063548229028334E-05
32	2.9061840071772926E-05	64	6.2057038547613796E-06	128	1.3289354406365028E-06
40	8.6319216846458823E-05	80	1.8467406103422900E-05	160	3.9547479109836025E-06
48	3.9260039096794541E-06	96	8.3855811242039557E-07		
56	2.2950403498089855E-05	112	4.9106082631034109E-06	n	$\tilde{\Delta}_6^c$
64	6.1329854699199159E-06	128	1.3114726425678124E-06	192	3.4401499793001999E-06
72	6.6199346397399772E-06	144	1.4165999425025388E-06		
80	1.5780023394501574E-06				
88	2.9324771832622831E-06				

TOTALS OF MEASURE FOR SIMPLE ATTRACTION WINDOWS

$\tilde{\Delta}_2$	4.3843073063172518E-02	(UPTO 22)
$\tilde{\Delta}_1^c$	1.1889614851551835E-02	(UPTO 36)
$\tilde{\Delta}_2^c$	2.3949117638801253E-03	(UPTO 60)
$\tilde{\Delta}_3^c$	5.1028261692420695E-04	(UPTO 88)
$\tilde{\Delta}_4^c$	1.0816365691981855E-04	(UPTO 144)
$\tilde{\Delta}_5^c$	2.1347231580648440E-05	(UPTO 160)
$\tilde{\Delta}_6^c$	3.4401499793001999E-06	(UPTO 192)
TOTAL	5.8770833334008453E-02	

n	$\|\Delta_n\|$	ϵ_n^f	ϵ_n^1
3	2.981807583411301E-02	-2.096725080006543E-01	-2.201819241658870E-01
6	8.107996134024227E-03	-5.144003679616797E-01	-5.171966260635098E-01
5	6.622846513950202E-03	-3.666412963087239E-01	-3.689801643186392E-01
12	1.637018928072165E-03	-5.810874943646800E-01	-5.816522865531801E-01
4	1.623007881829238E-03	-5.723798895171137E-02	-5.782620314202115E-02
10	1.515311818918605E-03	-5.511761040848819E-01	-5.517074202293009E-01
5	1.281206418890707E-03	-1.376689094410995E-01	-1.381318203966798E-01
7	1.146072746725499E-03	-4.237261647947689E-01	-4.241382226100451E-01
7	7.369326477080078E-04	-3.250434264434559E-01	-3.253090570981257E-01
8	7.262951926135934E-04	-4.773057104173951E-01	-4.775677383600991E-01
10	3.688660056715976E-04	-4.978377204540769E-01	-4.979707318543069E-01
24	3.507617529658336E-04	-5.950470120208444E-01	-5.951679720453749E-01
20	3.165683299826694E-04	-5.886800473342604E-01	-5.887910933996004E-01
8	2.859158382328142E-04	-2.885747448358570E-01	-2.886777845588647E-01
14	2.340685456913016E-04	-5.633056351482539E-01	-5.633897078588289E-01
9	2.090100394914890E-04	-4.040661026928790E-01	-4.041416911389079E-01
6	1.940543425922314E-04	-9.248493382874270E-02	-9.255526770868568E-02
14	1.810306538455042E-04	-5.421008649125095E-01	-5.421660125940350E-01
9	1.716254784234067E-04	-4.445095006926015E-01	-4.445716155334447E-01
7	1.625359385166031E-04	-1.674880202384408E-01	-1.675468982349900E-01
16	1.347570919328772E-04	-5.735893698769146E-01	-5.736380129778659E-01
9	1.290097530846079E-04	-3.437112706052193E-01	-3.437579835310577E-01
SUM	5.595296788727617E-02		

Table 1.15: Windows of attraction greater than 10^{-4}

	$n=3$		$n=6$		$n=5$
k	$\lvert\Delta_{kn}^{+j}\rvert/\lvert\Delta_k\rvert$	k	$\lvert\Delta_{kn}^{+j}\rvert/\lvert\Delta_k\rvert$	k	$\lvert\Delta_{kn}^{+j}\rvert/\lvert\Delta_k\rvert$
3	1.820870630E-02	3	4.778980980E-03	3	3.892172023E-03
4	2.052968993E-02	4	5.148050137E-03	4	3.853190250E-03
5	1.767763426E-02	5	4.696916717E-03	5	3.907018395E-03
6	1.696852882E-02	6	4.592270071E-03	6	3.929118993E-03
7	1.750130073E-02	7	4.667569163E-03	7	3.904160336E-03
8	1.725750591E-02	8	4.626567170E-03	8	3.909533397E-03
9	1.737300000E-02	9	4.646676275E-03	9	3.906897850E-03
10	1.703370961E-02	10	4.605481871E-03	10	3.931329678E-03
11	1.758504153E-02	11	4.681163581E-03	11	3.906722895E-03
12	1.712260583E-02	12	4.626922837E-03		
13	1.758419225E-02	13	4.681203200E-03		
14	1.706153475E-02				

	$n=12$		$n=4$		$n=10$
k	$\lvert\Delta_{kn}^{+j}\rvert/\lvert\Delta_k\rvert$	k	$\lvert\Delta_{kn}^{+j}\rvert/\lvert\Delta_k\rvert$	k	$\lvert\Delta_{kn}^{+j}\rvert/\lvert\Delta_k\rvert$
3	9.671215781E-04	3	9.910053778E-04	3	8.780910408E-04
4	1.048431631E-03	4	1.019113492E-03	4	8.513205273E-04
5	9.489781166E-04	5	9.837984552E-04	5	8.866511862E-04
6	9.256910320E-04	6	9.739437290E-04	6	8.997609504E-04
7	9.425553491E-04	7	9.814532291E-04	7	8.891145217E-04
8	9.336188802E-04	8	9.781776565E-04	8	8.925987063E-04
9	9.379978915E-04	9	9.797648164E-04	9	8.909055730E-04
10	9.285169978E-04	10	9.747599599E-04	10	8.993584441E-04
		11	9.825441741E-04		

	$n = 5$		$n = 7$		$n = 7$
k	$\lvert\Delta_{kn}^{+j}\rvert / \lvert\Delta_k\rvert$	k	$\lvert\Delta_{kn}^{+j}\rvert / \lvert\Delta_k\rvert$	k	$\lvert\Delta_{kn}^{+j}\rvert / \lvert\Delta_k\rvert$
3	7.759035169E-04	3	6.848481606E-04	3	4.436193568E-04
4	7.850923786E-04	4	6.750438854E-04	4	4.438301925E-04
5	7.736278827E-04	5	6.877304347E-04	5	4.436826162E-04
6	7.706167521E-04	6	6.919515075E-04	6	4.438803881E-04
7	7.728516882E-04	7	6.886011545E-04	7	4.436700152E-04
8	7.717650114E-04	8	6.898267271E-04	8	4.436534013E-04
9	7.723001777E-04	9	6.892310110E-04	9	4.436653168E-04
10	7.709293657E-04	10	6.917392473E-04		

	$n = 8$		$n = 10$		$n = 24$
k	$\lvert\Delta_{kn}^{+j}\rvert / \lvert\Delta_k\rvert$	k	$\lvert\Delta_{kn}^{+j}\rvert / \lvert\Delta_k\rvert$	k	$\lvert\Delta_{kn}^{+j}\rvert / \lvert\Delta_k\rvert$
3	4.400059413E-04	3	2.231066267E-04	3	2.070757884E-04
4	4.478239022E-04	4	2.262689145E-04	4	2.243077880E-04
5	4.380728554E-04	5	2.223365238E-04	5	2.032354824E-04
6	4.355088155E-04	6	2.213289116E-04	6	1.983135169E-04
7	4.374163305E-04	7	2.220701759E-04	7	2.018731188E-04
8	4.364983816E-04	8	2.216975697E-04	8	1.999772153E-04
9	4.369489765E-04	9	2.218813597E-04	9	2.009066533E-04

	$n = 20$		$n = 8$		$n = 14$
k	$\lvert\Delta_{kn}^{+j}\rvert / \lvert\Delta_k\rvert$	k	$\lvert\Delta_{kn}^{+j}\rvert / \lvert\Delta_k\rvert$	k	$\lvert\Delta_{kn}^{+j}\rvert / \lvert\Delta_k\rvert$
3	1.836746536E-04	3	1.711803794E-04	3	1.394319540E-04
4	1.785204691E-04	4	1.684985668E-04	4	1.365390140E-04
5	1.853374075E-04	5	1.719463254E-04	5	1.402693821E-04
6	1.878976346E-04	6	1.730475961E-04	6	1.414814753E-04
7	1.858110670E-04	7	1.721835632E-04	7	1.405270857E-04
8	1.864809129E-04	8	1.725175338E-04	8	1.408902274E-04
9	1.861561774E-04	9	1.723539161E-04	9	1.407123257E-04

	$n=9$		$n=6$		$n=14$
k	$\lvert\Delta_{kn}^{+j}\rvert/\lvert\Delta_k\rvert$	k	$\lvert\Delta_{kn}^{+j}\rvert/\lvert\Delta_k\rvert$	k	$\lvert\Delta_{kn}^{+j}\rvert/\lvert\Delta_k\rvert$
3	1.260971209E-04	3	1.174998457E-04	3	1.088130060E-04
4	1.257109983E-04	4	1.176587055E-04	4	1.089071502E-04
5	1.262147577E-04	5	1.174610810E-04	5	1.088248169E-04
6	1.263912777E-04	6	1.174109025E-04	6	1.088753232E-04
7	1.262485422E-04	7	1.174472958E-04	7	1.088181767E-04
8	1.262961564E-04	8	1.174280970E-04	8	1.088092733E-04
9	1.262729125E-04	9	1.174371846E-04	9	1.088146494E-04

	$n=9$		$n=7$		$n=16$
k	$\lvert\Delta_{kn}^{+j}\rvert/\lvert\Delta_k\rvert$	k	$\lvert\Delta_{kn}^{+j}\rvert/\lvert\Delta_k\rvert$	k	$\lvert\Delta_{kn}^{+j}\rvert/\lvert\Delta_k\rvert$
3	1.034916757E-04	3	9.833794327E-05	3	8.174230201E-05
4	1.027753081E-04	4	9.839800860E-05	4	8.338407458E-05
5	1.036911786E-04	5	9.832703003E-05	5	8.133279316E-05
6	1.039746954E-04	6	9.831680431E-05	6	8.078565799E-05
7	1.037533907E-04	7	9.832177656E-05	7	8.119487681E-05
8	1.038408810E-04	8	9.831455603E-05	8	8.100214915E-05
9	1.037979001E-04	9	9.831795929E-05	9	8.109614616E-05

	$n=9$
k	$\lvert\Delta_{kn}^{+j}\rvert/\lvert\Delta_k\rvert$
3	7.800513021E-05
4	7.800753488E-05
5	7.800987684E-05
6	7.802157075E-05
7	7.800966516E-05
8	7.800948920E-05
9	7.800937702E-05

n	(s_{1n}, s_{2n})	(r_{1n}, r_{2n})
3	1.0487176E-03 1.0512265E-03	3.5170534E-02 3.5254673E-02
6	2.7771006E-04 2.7819361E-04	3.4251381E-02 3.4311019E-02
5	2.2937585E-04 2.2955286E-04	3.4634028E-02 3.4660756E-02
12	5.6114357E-05 5.6313126E-05	3.4278380E-02 3.4399801E-02
4	5.7960445E-05 5.8037573E-05	3.5711746E-02 3.5759267E-02
10	5.1935126E-05 5.2039091E-05	3.4273557E-02 3.4342166E-02
5	4.5527139E-05 4.5552503E-05	3.5534585E-02 3.5554382E-02
7	4.0368786E-05 4.0402866E-05	3.5223581E-02 3.5253317E-02
7	2.6088921E-05 2.6090721E-05	3.5402043E-02 3.5404485E-02
8	2.5789760E-05 2.5820760E-05	3.5508647E-02 3.5551330E-02
10	1.3086122E-05 1.3098376E-05	3.5476628E-02 3.5509850E-02
24	1.2015678E-05 1.2076079E-05	3.4255953E-02 3.4428153E-02
20	1.0850525E-05 1.0879636E-05	3.4275462E-02 3.4367418E-02
8	1.0087786E-05 1.0100658E-05	3.5282363E-02 3.5327381E-02
14	8.2230933E-06 8.2372213E-06	3.5131133E-02 3.5191492E-02
9	7.4182345E-06 7.4202622E-06	3.5492240E-02 3.5501941E-02
6	6.9075703E-06 6.9081834E-06	3.5596061E-02 3.5599221E-02
14	6.3991731E-06 6.3996284E-06	3.5348561E-02 3.5351076E-02
9	6.0912033E-06 6.0945329E-06	3.5491253E-02 3.5510653E-02
7	5.7820620E-06 5.7822232E-06	3.5574052E-02 3.5575044E-02
16	4.7888518E-06 4.7954463E-06	3.5536918E-02 3.5585854E-02
9	4.5870552E-06 4.5871686E-06	3.5555879E-02 3.5556758E-02
TS	2.0534087E-03 2.0614883E-03	3.4920730E-02 3.5058133E-02

The total measure of 2-simple windows, Δ_n^{+j}, is in the interval (s_{1n}, s_{2n}), and $|\Delta_n^{+j}|/|\Delta_{n,j}| \in (r_{1n}, r_{2n})$. Moreover the total measure of 2-simple windows is in the last row (corresponding to TS), and also the ratio between 2-simple windows and 1-simple windows.

Table 1.16: 2-simple windows of attraction: Δ_{kn}^{+j} means the union of 2-simple windows of period kn, corresponding to a window $\Delta_{n,j}$.

n	Maximal interval width	Maximal interval extrema
3	3.488448109079493E-01	-2.500000000000000E-01 -5.988448109079493E-01
4	3.488448109079493E-01	-2.500000000000000E-01 -5.988448109079493E-01
5	2.232418000753599E-01	-3.756030108325894E-01 -5.988448109079493E-01
6	1.417467687732761E-01	-3.756030108325894E-01 -5.173497796058655E-01
7	9.206548424909488E-02	-4.252842953567706E-01 -5.173497796058655E-01
8	7.354018871041500E-02	-5.253046221975343E-01 -5.988448109079493E-01
9	7.354018871041500E-02	-5.253046221975343E-01 -5.988448109079493E-01
10	4.562207885972950E-02	-5.532227320482198E-01 -5.988448109079493E-01
11	4.562207885972950E-02	-5.532227320482198E-01 -5.988448109079493E-01
12	2.846042682629192E-02	-5.532227320482198E-01 -5.816831588745117E-01
13	2.846042682629192E-02	-5.532227320482198E-01 -5.816831588745117E-01
14	1.805938246999122E-02	-5.636237764045205E-01 -5.816831588745117E-01
15	1.805938246999122E-02	-5.636237764045205E-01 -5.816831588745117E-01
16	1.555550542669680E-02	-5.832893054812525E-01 -5.988448109079493E-01
17	1.555550542669680E-02	-5.832893054812525E-01 -5.988448109079493E-01
18	1.555550542669680E-02	-5.832893054812525E-01 -5.988448109079493E-01

Table 1.17: Width of largest interval in which there is no periodic orbits of period smaller than or equal to n.

n	$\|I_n\|$	$\|I_{n-1}\|/\|I_n\|$
4	3.488448109794998E-01	
8	7.354018878196551E-02	4.743594172891872E+00
16	1.555550549824730E-02	4.727598777825098E+00
32	3.326077181158884E-03	4.676832391732831E+00
64	7.120099674375821E-04	4.671391319322314E+00
128	1.524776406734371E-04	4.669602469535197E+00
256	3.265543821424322E-05	4.669287843362377E+00
512	6.993822323380864E-06	4.669183274083712E+00

Table 1.18: Width of largest hole up to period n

n	$\|\Delta_n\|$
288	2.827637234774411E-07
384	1.667270187045180E-08
480	7.445452805876459E-08
576	7.598050966885335E-08
672	2.037661816798496E-08
768	1.139175102275308E-08
864	6.003904733941709E-09
960	1.908920696060860E-08
1056	2.569048309446478E-09
TOTAL	5.260526116937182E-07

Table 1.19: Measure of n-periodic 2-simple windows inside the largest snh window of period 96.

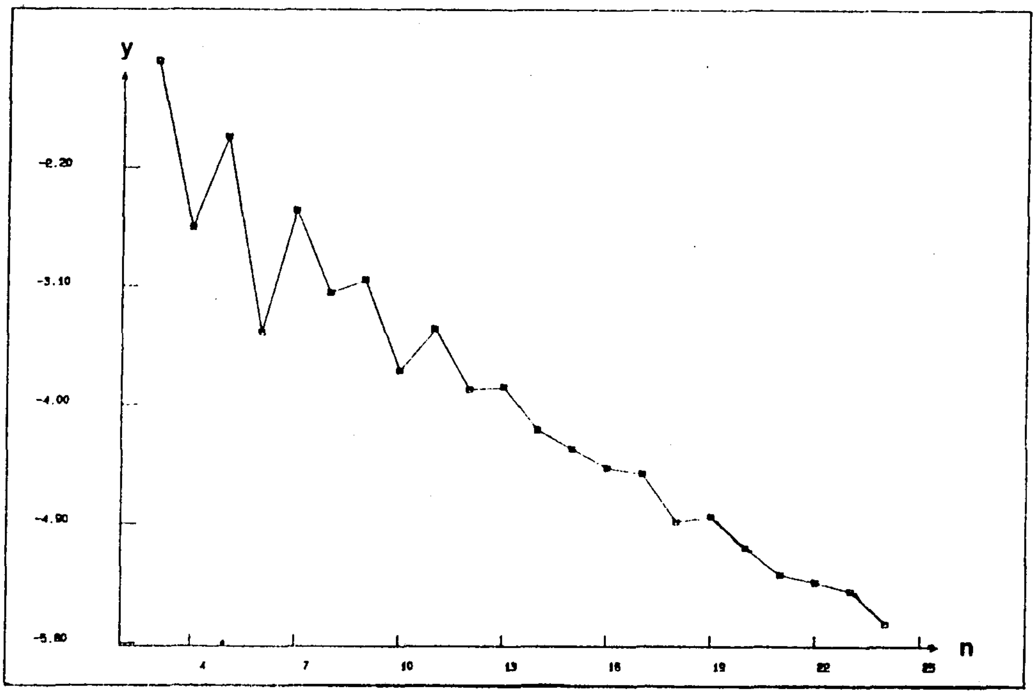
n	$M_n = \| \cup_{i=3}^{24} \Delta_i^t \cap \tilde{\Delta}_n \|$	M_{n-1}/M_n
2	5.937324163210067E-02	
3	2.948389290601315E-03	2.013751773599465E+01
4	1.703340517368363E-04	1.730945316299136E+01
5	1.037195112266315E-05	1.642256598805688E+01
6	6.429362060578866E-07	1.613216214133898E+01
7	4.008325355082139E-08	1.604002043503556E+01
8	2.503350651645803E-09	1.601184137926065E+01
9	1.564248493831158E-10	1.600353563719659E+01
10	9.775785651406443E-12	1.600125605869959E+01
11	6.109536079143389E-13	1.600086409961441E+01
12	3.818157154535050E-14	1.600126928218953E+01
13	2.385988848508091E-15	1.600240988939434E+01
14	1.490796036872275E-16	1.600479736660658E+01
15	9.311917519291988E-18	1.600954941647319E+01
16	5.812940916136393E-19	1.601928809123560E+01
17	3.624481792529490E-20	1.603799177062385E+01
18	2.254160241311570E-21	1.607907781400938E+01
19	1.395878826393845E-22	1.614868138042494E+01
20	8.538097179190348E-24	1.634882804796342E+01
21	5.162883287798387E-25	1.653745921270139E+01
22	2.858925651571642E-26	1.805882319800862E+01
23	1.681721318103705E-27	1.699999649641917E+01

Table 1.20: Measures of snh windows (up to period 24) within principal tangency window

n	$m_n/\tilde{\epsilon}_n^2$	$-\epsilon^f_{n,0}$
3	1.009608917500249E-01	2.096725080006543E-01
4	8.824015463667236E-02	5.723798895171137E-02
5	8.609790747390427E-02	1.445962151916002E-02
6	8.601297328606208E-02	3.616741186812089E-03
7	8.628152751356520E-02	9.038753062155362E-04
8	8.651761908420589E-02	2.259237096849571E-04
9	8.666806453859027E-02	5.647651048596137E-05
10	8.675323408371579E-02	1.411875188495575E-05
11	8.679878024548109E-02	3.529658258929121E-06
12	8.682241388191965E-02	8.824123184064164E-07

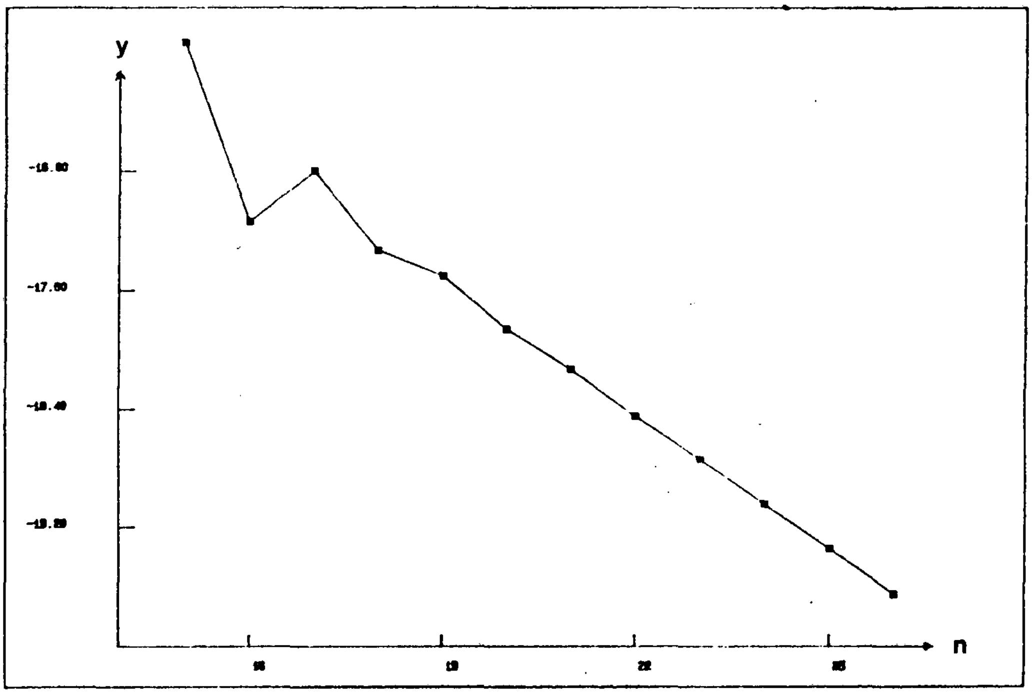
m_n is the measure of snh windows for $\epsilon \in (\tilde{\epsilon}_n, 0)$, without taking into account the first window of period n, from period $n+1$ to period $2n$; $\tilde{\epsilon}_n = -\frac{3\pi^2}{2}4^{-n}$.

Table 1.21:



$n = period,\ y = \log_{10} n.$

Figure 1.1:



n =period, $y = \log_{10}$(measure of the set of parameters in a n-periodic snh window inside $\tilde{\Delta}_{14}$.

Figure 1.2:

Chapter 2

On the strongly dissipative Hénon map

One of the standard forms of the Hénon map is $f_{a,b}(x,y) = (1+y-ax^2, bx)$, where a and b are real parameters, and $f_{a,b}$ is a map of the plain into itself. The goal of this chapter is to analyze the invariant manifolds of the fixed points of $f_{a,b}$ when b is close to 0. The results obtained for the invariant manifolds will be applied to obtain the values of the parameter for which there are homoclinic or heteroclinic tangencies.

2.1 Basic properties

Proposition 2.1.1 *The Hénon map $f_{a,b}$ satisfies the following properties:*

a) $\det Df_{a,b}(x,y) = -b$ *and therefore it is constant when b is fixed.*

b) If $a \geq -\frac{1}{4}(1-b)^2$ then there are two fixed points $p_+ = (x_+, y_+)$ and $p_- = (x_-, y_-)$ of $f_{a,b}$ such that

$$x_+ = \frac{-(1-b)+\sqrt{(1-b)^2+4a}}{2a}$$

and

$$x_- = \frac{-(1-b)-\sqrt{(1-b)^2+4a}}{2a}.$$

c) p_- is a saddle fixed point if $a > -\frac{1}{4}(1-b)^2$ and p_+ is a saddle fixed point if $a > \frac{3}{4}(1-b)^2$.

d) For $b=0$, $f_{a,0}$ is not a diffeomorphism because all the plane collapses to the x axis: $f_{a,0}(x,y) = (1+f_a(x), 0)$, where $f_a(x) = 1-ax^2$ is the so called logistic map.

e) For $b \neq 0$, $f_{a,b}$ is a diffeomorphism in all $\mathbb{R}^2$ with inverse map: $f_{a,b}^{-1} = (b^{-1}y, x + ab^{-2}y^2 - 1)$.

f) All the quadratic maps with constant jacobian are conjugated to this map.

Theorem 2.1.2 *Let $a, b \in \mathbb{R}$, $b \neq 0$ small enough, and $p = (x_0, y_0)$ one of the fixed points of the Hénon map. Assume that p is a saddle fixed point . Then there exist open sets U and V of x_0, and analytical functions $g : U \longrightarrow \mathbb{R}$, $h : V \longrightarrow \mathbb{R}$ such that*

a) $\mathcal{W}^u_{loc}(p) = \{(x, y) \in \mathbb{R}^2 \ : \ y = g(x),\ x \in U\}$,

b) $\mathcal{W}^s_{loc}(p) = \{(x, y) \in \mathbb{R}^2 \ : \ y = h(x),\ x \in V\}$,

where $\mathcal{W}^u_{loc}(p)$ and $\mathcal{W}^s_{loc}(p)$ are, respectively, the unstable and stable local invariant manifolds.

The proof is a consequence of the stable manifold theorem.

Remark 2.1.3 *Notice that the functions g and h as well as the neighbouroods U and V depend on a and b.*

2.2 Invariant manifolds for $b = 0$

For $b = 0$ we can give an explicit expression of the invariant manifolds. First we shall give a general definition of unstable manifold that can be used when the map is not a diffeomorphism. We will use a generalization of the definition 1.1.1 of chapter 1.

Definition 2.2.1 *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a differentaible map with a saddle fixed point. Then $\mathcal{W}^u(p) = \{(x, y) \in \mathbb{R}^2 \ : \ \exists (x_n, y_n)_{n \in \mathbb{N}} \rightarrow p, \text{ with } (x_0, y_0) = (x, y) \text{ and } f(x_{n+1}, y_{n+1}) = (x_n, y_n)\}$ is called the unstable invariant manifold of p.*

The next proposition allows to use a parametrization of the unstable manifold similar to the one used for diffeomorphisms (see chapter 3, proposition 3.1.2).

Proposition 2.2.2 *Let $p = (x_0, 0)$ be one of the fixed points of $f_{a,0}$, such that p is a saddle. Then there exists an entire function $\vec{x}(t) = (x(t), y(t))$, $t \in \mathbb{R}$, such that:*

a) $\vec{x}(0) = p$,

b) $\mathcal{W}^u(p) = \vec{x}(\mathbb{R})$,

c) $f_{a,0}(\vec{x}(t)) = \vec{x}(\alpha t)$, where $\alpha = f'_a(x_0)$.

Proof:

Assume, for instance, that $p = (x_+, 0)$. The equation for $\vec{x}(t)$ is:

$$\left.\begin{array}{rcl} 1 - ax(t)^2 + y(t) &=& x(\alpha t), \\ 0 &=& y(\alpha t), \end{array}\right\} \tag{2.1}$$

because $f_{a,0}(x,y) = (1 - ax^2 + y, 0)$.

In this case $\alpha = f_a'(x_+) = -2ax_+$ and $x_+ = \frac{1}{2a}(-1 + \sqrt{1+4a})$. From 2.1 one has $y(t) = 0 \forall x \in \mathbb{R}$ and

$$1 - ax(t)^2 = x(\alpha t). \tag{2.2}$$

This equation can be transformed in the equivalent one (when it has sense)

$$x(\alpha^{-1}t) = \pm\sqrt{\frac{1 - x(t)}{a}}.$$

But as we require $x(0) = x_+ > 0$, one should have

$$x(\alpha^{-1}t) = \sqrt{\frac{1 - x(t)}{a}}. \tag{2.3}$$

Therefore, we have transformed our initial problem in the following one: To find a function $x(t)$ such that, $x(0) = x_+$, satisfying 2.3. This is equivalent to say that the function $\tilde{f}(x) = \sqrt{(1-x)a^{-1}}$, defined in a neighbourhood of x_+, is conjugated to $F(x) = \alpha^{-1}x$, by means of an analytical function. This is true by Sternberg theorem ([14]) in dimension 1, because $|\alpha| \neq 1$. Therefore, it exists locally the function $x(t)$ satisfying 2.2. To extend it to the real line $\mathbb{R}$, one uses 2.2. This proves a) and c).

It remains to see $\mathcal{W}^u(p_+) = \vec{x}(\mathbb{R})$:

a) $\mathcal{W}^u(p_+) \supset \vec{x}(\mathbb{R})$:

Given $(x(t), 0)$, let $(x_n, y_n) = (x(\alpha^{-n}t), 0)$. Then $(x_0, y_0) = (x(t), 0)$ and

$$f(x_{n+1}, y_{n+1}) = (x(\alpha^{-n}t), 0) = (x_n, y_n).$$

Furthermore

$$\lim_{n\to\infty}(x_n, y_n) = \lim_{n\to\infty}(x(\alpha^{-n}t), 0) = (x_+, 0),$$

because $|\alpha| > 1$. Hence $(x(t), 0) \in \mathcal{W}^u(p_+)$.

b) $\mathcal{W}^u(p_+) \subset \vec{x}(\mathbb{R})$:

From the definition , it follows that $\mathcal{W}^u(p_+) = \cup_{n\in\mathbb{N}} f_a^n(U) \times \{0\}$, where $U \subset \mathbb{R}$ is a small enough neighbourhood of x_+. Let $U = (x(t_0), x(t_1))$, for $t_0 < 0 < t_1$ and $\max(|t_0|, |t_1|)$ small enough. Then $\cup_{n\in\mathbb{N}} f_a^n(U) = \vec{x}(\mathbb{R}) = \mathcal{W}^u(p_+)$. □

Remark 2.2.3 *a) The case* $p = (x_-, 0)$ *is shown in an analogous way, but then* $\tilde{f}(x) = -\sqrt{(1-x)a^{-1}}$.

b) The function $x(t)$ *is not injective and goes an infinite number of times through the same points. It gives a parametrization close to the one obtained for* $b \neq 0$.

c) One can always normalize $x(t)$ *taking* $x'(0) = 1$ *and therefore, locally, for* $t > 0$ $x(t)$ *is to the right of* x_0*, and for* $t < 0$ $x(t)$ *is to the left of* x_0*. Indeed, if* $x(t)$ *is a parametrization as in proposition 2.2.2, then* $\bar{x}(t) = x(\lambda t)$*, for* $\lambda \in \mathbb{R}$, $\lambda \neq 0$*, also is.*

From now on, we shall distinguish between the invariant manifolds of p_+ and p_-, denoting by $\vec{x}_1$ and $\vec{x}_2$, respectively, the parametrizations of $\mathcal{W}^u(p_+)$ and $\mathcal{W}^u(p_-)$, and $\alpha_1 = f'_a(x_+)$, $\alpha_2 = f'_a(x_-)$. We need two lemmas before describing the invariant manifolds.

Lemma 2.2.4 *Assume* $g : [a,b] \longrightarrow [c,d]$ *is a continuous non decreasing function such that* $[c,d] = g([a,b]) \subset [a,b]$ *and having one fixed point* $x_0 \in [c,d]$*. Then*

$$\forall x \in [a,b] \quad \lim_{n\to\infty} g^n(x) = x_0.$$

Proof:

Let $x \in [a,b]$, $x \neq x_0$. There are two possible cases: $x < g(x)$ or $x > g(x)$. In the first case we have that $(g^n(x))_{n\in\mathbb{N}}$ is a non increasing sequence, and in the second case, it is non decreasing. In both cases it is bounded. Therefore $\exists\, \bar{x} = \lim_{n\to\infty} g^n(x)$. But $g(\bar{x}) = g(\lim_{n\to\infty} g^n(x)) = \lim_{n\to\infty} g(g^n(x)) = \bar{x}$. By uniqueness $x_0 = \bar{x}$. □

Lemma 2.2.5 *Let* $g : [a,b] \longrightarrow [c,d]$ *be a differentiable non decreasing function such that* $[c,d] = g([a,b]) \subset [a,b]$ *and having exactly three fixed points* $x_0 < x_1 < x_2$ *with* $f'(x_1) > 1$*. Then* $\forall x \in (x_0, x_1)$ *one has* $\lim_{n\to\infty} g^n(x) = x_0$ *and* $\forall x \in (x_1, x_2)$ *one has* $\lim_{n\to\infty} g^n(x) = x_2$.

Proof:

Let $x \in (x_0, x_1)$. As $f'(x_0) > 1$ and there are no fixed points in (x_0, y_0), we have $g(x) < x$. Therefore $(g^n(x))_{n\in\mathbb{N}}$ is a non increasing sequence. As in lemma 2.2.4 this means $\lim_{n\to\infty} g^n(x) = x_0$. The proof is similar for $x \in (x_1, x_2)$. □

The main result of this section is the following:

Theorem 2.2.6 *Let* p_+ *and* p_- *be the two fixed points of* $f_{a,0}$*. Then:*

a) If $a > 0$ *we have* $\mathcal{W}^u(p_-) = (-\infty, 1] \times \{0\}$ *and there is an interval* $I = (-\infty, t_0)$ *such that* $x'_2(t) \neq 0\, \forall t \in I$ *and* $x'_2(t_0) = 0$, $x_2(t_0) = 1$.

b) If $\frac{3}{4} < a \leq 1$ then $\mathcal{W}^u(p_+) = (x', x'') \times \{0\}$, where x' and x'' are the two periodic points of f_a. Furthermore, $\forall t \in \mathbb{R}$ one has $x_1'(t) \neq 0$.

c) If $1 < a \leq 2$ then $\mathcal{W}^u(p_+) = [1-a, 1] \times \{0\}$. Furthermore $\exists t_0, t_1 \in \mathbb{R}$ such that $t_0 < 0 < t_1$, $x_1'(t_0) = x_1'(t_1) = 0$, $x_1(t_0) = 1-a$, $x_1(t_1) = 1$ and $x_1'(t) \neq 0$, $\forall t \in (t_0, t_1)$.

d) If $a > 2$ then $\mathcal{W}^u(p_+) = (-\infty, 1] \times \{0\}$. Furthermore $\exists\, t_0, t_1 \in \mathbb{R}$ such that $t_0 < 0 < t_1$, $x_1'(t_0) = x_1'(t_1) = 0$, $x_1(t_0) = 1-a$, $x_1(t_1) = 1$ and $x_1'(t) \neq 0$, $\forall t \in (t_0, t_1)$.

e) $\mathcal{W}^u(p_+)$ (resp. $\mathcal{W}^s(p_-)$) is a set of parabolas: $y = ax^2 + x_0 - 1$, where $f_a^n(x_0) = x_+$ (resp. $f_a^n(x_0) = x_-$) for some $n \in \mathbb{N}$.

This theorem is consequence of propositions 2.2.7 to 2.2.11. In all this propositions we suppose $b = 0$.

Proposition 2.2.7 *Let $a > 0$. Then $\mathcal{W}^u(p_-) = (-\infty, 1] \times \{0\}$ and there exists an interval $I = (-\infty, t_0)$ such that $x_2'(t) \neq 0\, \forall t \in I$ and $x_2'(t_0) = 0$, $x_2(t_0) = 1$.*

Proof:

We can take $[a, b]$ of 2.2.4 as $J = [x_-(1+2a) - a, 0]$, $[c, d]$ as $K = [x_- - 1, -\sqrt{a^{-1}}]$ and g as $f_-^{-1} = -\sqrt{(1-x)a^{-1}}$. As $K \subset J$, and f_-^{-1} is not decreasing, from 2.2.4 it follows $J \times \{0\} \subset \mathcal{W}^u(p_-)$. As $\mathcal{W}^u(p_-) = \cup_{n \in \mathbb{N}} f_a^n(J) \times \{0\}$, we have $\mathcal{W}^u(p_-) = (-\infty, 1] \times \{0\}$.

To show the second part of the proposition we observe that, by derivation of $1 - ax_2(t)^2$ one has

$$-2ax_2(t)x_2'(t) = \alpha_2 x_2'(\alpha_2 t). \tag{2.4}$$

Let $t_0 = \min\{t > 0 : x_2(t) = 1\}$. This value is well defined because there are $t \in \mathbb{R}$ with $x_2(t) = 1$ due to the fact that $(1, 0) \in \mathcal{W}^u(p_+)$. Furthermore $t_0 \neq 0$ because $x_2(0) = x_- \neq 1$. Then one has $x_2'(t_0) = 0$ because as $1 - ax_2(\alpha_2^{-1}t_0)^2 = x(t_0) = 1$, then $x_2(\alpha_2^{-1}t_0) = 0$ and, by 2.4, we obtain $x_2'(t_0) = 0$.

Let us see now that $\forall t \in (-\infty, t_0)$ $x_2'(t) \neq 0$:

Assume that there is $0 < \bar{t} < t_0$ such that $x_2'(\bar{t}) = 0$. We can choose $\bar{t}$ in such a way that $\forall t \in (0, \bar{t})$ $x_2'(t) \neq 0$, because $x_2'(0) = 1$. We have:

$$-2ax_2(\alpha_2^{-1}\bar{t})x_2'(\alpha_2^{-1}\bar{t}) = \alpha_2 x_2'(\bar{t}) = 0.$$

Therefore two cases can occur:

a) $x_2(\alpha_2^{-1}\bar{t}) = 0$. Then $x_2(\bar{t}) = 1 - ax_2(\alpha_2^{-1}\bar{t})^2 = 1$. This is an absurdity because $x_2(\bar{t}) < 1$.

b) $x_2'(\alpha_2^{-1}\bar{t}) = 0$. This is again an absurdity because $0 < \alpha_2^{-1}\bar{t} < \bar{t}$.

Therefore $\forall t \in [0, t_0)$ $x_2'(t) \neq 0$.

Suppose now that $\exists \bar{t}$ such that $\bar{t} < 0$, $x_2'(\bar{t}) = 0$ and $\forall t \in [\bar{t}, 0]$ one has $x_2'(t) \neq 0$. One reaches also a contradiction because in one case $x_2(\bar{t}) = 1$ is not possible because $f_a((-\infty, x_-]) = (-\infty, x_-]$, and in the other $x_2'(\alpha_2^{-1}\bar{t}) = 0$ and $\bar{t} < \alpha_2^{-1}\bar{t} < 0$, also not possible. Hence $\forall t \in (-\infty, t_0)$ one has $x_2'(t) \neq 0$. □

Proposition 2.2.8 *Let $\frac{3}{4} < a < 1$. Then $W^u(p_+) = (x', x'') \times \{0\}$, where x' and x'' are the two 2-periodic points of f_a. Furthermore $x_1'(t) \neq 0 \, \forall t \in \mathbb{R}$.*

Proof:

We note first that, for these values of a, the point p_+ is a saddle and the two 2-periodic points $p_1 = (x', 0)$ and $p_2 = (x'', 0)$ are attractors, such that $f_a(x') = x''$. It is easy to see that x' and x'' are attractors for $a \in [3/4, 5/4]$. Furthermore $0 < x' < x'' < 1$. Indeed: x' and x'' satisfy:

$$\begin{aligned} 1 - ax'^2 &= x'', \\ 1 - ax''^2 &= x'. \end{aligned}$$

Hence $x' + x'' = \frac{1}{a}$ and $x'x'' = \frac{1-a}{a^2}$.

This means that x' and x'' are roots of the polynomial

$$P_2(x) = x^2 - \frac{1}{a}x - \frac{a-1}{a^2}.$$

As $0 < a < 1$: $1/a > 0$ and $(a-1)/a^2 > 0$. Therefore x' and $x'' > 0$. Also

$$x' = \frac{1 - \sqrt{4a-3}}{2a}, \quad x'' = \frac{1 + \sqrt{4a-3}}{2a}, \quad x_+ = \frac{-1 + \sqrt{1+4a}}{2a},$$

and one can check $x' < x_+ < x''$ if $a > 3/4$, $x' > 1 - a$ for $a \in (\frac{3}{4}, 1) \cup (1, \infty)$ and $x'' < 1$ for $a \in (\frac{3}{4}, 1) \cup (1, \infty)$.

We want to apply 2.2.5 to $g = f_+^{-2}$, $[a, b] = [c, d] = [x', x'']$. To this end we should check that the following holds

a) $f^{-2}_{+|[x',x'']}$ is non decreasing: if $x, y \in [x', x'']$ and $x \leq y$ then $1 - y \leq 1 - x$. As $x < 1$ and $y < 1$ we have $\sqrt{(1-y)a^{-1}} \leq \sqrt{(1-x)a^{-1}}$. Furthermore $x' > x'x'' = (1-a)/a^2 > 1 - a$ because $3/4 < a < 1$. In particular $x > 1 - a$, $y > 1 - a$. Hence

$$\sqrt{(1 - \sqrt{(1-x)a^{-1}})a^{-1}} = f_+^{-2}(x) \leq \sqrt{(1 - \sqrt{(1-y)a^{-1}})a^{-1}} = f_+^{-2}(y).$$

b) $f_+^{-2}([x', x'']) = [f_+^{-2}(x'), f_+^{-2}(x'')] = [x', x'']$.

c) x_+ is a saddle fixed point.

Then 2.2.5 assures $\mathcal{W}^u(p_+) = (x', x'') \times \{0\}$.

Let us prove the second part of 2.2.8: Assume that $\exists \bar{t} \in \mathbb{R}$ such that $x_1'(\bar{t}) = 0$. Let $\bar{t} > 0$ (if $\bar{t} < 0$ the proof is similar). We can always take $\bar{t}$ such that $x_1'(t) \neq 0\, \forall t \in (0, \bar{t})$. We have:

$$-2ax_1(\alpha_1^{-1}\bar{t})x_1'(\alpha_1^{-1}\bar{t}) = \alpha_1 x_1'(\bar{t}) = 0.$$

Then either $x_1(\alpha_1^{-1}\bar{t}) = 0$, which is an absurdity because $0 \notin \mathcal{W}^u(p_+)$, or $x_1'(\alpha_1^{-1}\bar{t}) = 0$. In this last case, applying the same reasoning, it follows $x_1'(\alpha_1^{-2}\bar{t}) = 0$ and $0 < \alpha_1^{-2}\bar{t} < \bar{t}$, again an absurdity. Therefore $\forall t \in \mathbb{R}\ x_1'(t) \neq 0$. □

Proposition 2.2.9 *Let $1 < a \leq 2$. Then $\mathcal{W}^u(p_+) = [1-a, 1] \times \{0\}$. Furthermore $\exists t_0, t_1 \in \mathbb{R}$ such that $t_0 < 0 < t_1$, $x_1'(t_0) = x_1'(t_1) = 0$, $x_1(t_0) = 1 - a$, $x_1(t_1) = 1$ and $x_1'(t) \neq 0\, \forall t \in (t_0, t_1)$.*

Proof:

We note that now $1 - a < x' < x_+ < x'' < 1$. To see $\mathcal{W}^u(p_+) = [1 - a, 1] \times \{0\}$ we shall use 2.2.4 for $g = f_+^{-2}$, $[a, b] = [0, 1]$, $[c, d] = [\sqrt{(1 - \sqrt{a^{-1}})a^{-1}}, \sqrt{a^{-1}}]$. Let us see that the hypothesis of 2.2.4 hold:

a) g is non decreasing: Let $x, y \in [0, 1]$, $x \leq y$. As $x < 1$ and $y < 1$ one has $\sqrt{(1-y)a^{-1}} \leq \sqrt{(1-x)a^{-1}}$. As $a > 1$ and $x > 0$ then $x > 1 - a$ and hence $\sqrt{(1-x)a^{-1}} < 1$. Therefore $f_+^{-2}(x) \leq f_+^{-2}(y)$. So we have g non decreasing and $g([0,1]) = [g(0), g(1)] = [\sqrt{(1 - \sqrt{a^{-1}})a^{-1}}, \sqrt{a^{-1}}]$.

b) $[\sqrt{(1 - \sqrt{a^{-1}})a^{-1}}, \sqrt{a^{-1}}] \subset [0, 1]$ because $a > 1$ implies $1 - \sqrt{a^{-1}} > 0$ and then $0 < \sqrt{(1 - \sqrt{a^{-1}})a^{-1}} < \sqrt{a^{-1}} < 1$.

c) g has only one fixed point in $[0, 1]$ and it is x_+. To see this it is only necessary to note that one of the 2-periodic points, x', of f_a, is negative. Then $f_+^{-1}(x'') \neq x'$.

Then lemma 2.2.4 assures $[0, 1] \subset \mathcal{W}^u(p_+)$ and, as $f_a(1) = 1 - a$, we have $[1 - a, 1] \subset \mathcal{W}^u(p_+)$. Furthermore $f_a([1-a, 1]) = [1-a, 1]$, because $a \leq 2$. From this it follows $\mathcal{W}^u(p_+) = [1 - a, 1] \times \{0\}$.

To prove the second part of 2.2.9 we proceed as follows: Let $t_1 > 0$ be such that $x_1(t_1) = 1$ and $x_1(t) \neq 1\, \forall t \in (0, t_1)$. We note that $x_1(\alpha_1^{-1}t_2) = 0$ and hence $x_1'(t_1) = 0$ because

$$\alpha_1 x_1'(t_1) = -2ax_1(\alpha_1^{-1}t_1)x_1'(\alpha_1^{-1}t_1).$$

Furthermore $x_1(\alpha_1 t_1) = 1-a$ and $\alpha_1 x_1'(\alpha_1 t_1) = -2ax_1(t_1)x_1'(t_1) = 0$, that is $x_1'(\alpha_1 t_1) = 0$. Let $\bar{t} = \min\{t \in [0,t_1] : x_1'(t) = 0\} \neq 0$. Then

$$-2ax_1(\alpha_1^{-1}\bar{t})x_1'(\alpha_1^{-1}\bar{t}) = \alpha_1 x_1'(\bar{t}) = 0.$$

Two cases can occur:

a) $x_1(\alpha_1^{-1}\bar{t}) = 0$. Then $x_1(\bar{t}) = 1 - ax_1(\alpha_1^{-1}\bar{t})^2 = 1$ imply $\bar{t} = t_1$.

b) $x_1'(\alpha_1^{-1}\bar{t}) = 0$. Then $-2ax_1(\alpha_1^{-2}\bar{t})x_1'(\alpha_1^{-2}\bar{t}) = \alpha_1 x_1'(\alpha_1^{-1}\bar{t}) = 0$. As $0 < \alpha_1^{-2}\bar{t} < \bar{t}$ we have $x_1'(\alpha_1^{-2}\bar{t}) \neq 0$. This means $x_1(\alpha_1^{-2}\bar{t}) = 0$. This is an absurdity because in such case $x_1(\bar{t}) = 1 - a$ and hence $\bar{t} > t_1$.

Therefore $\bar{t} = t_1$. Let us see now that $\forall t \in (\alpha_1 t_1, 0)$ one has $x_1'(t) \neq 0$:

If there is $t' \in (\alpha_1 t_1, 0)$ such that $x_1'(t') = 0$ then $0 < \alpha_1^{-1}t' < t_1$, $x_1'(\alpha_1^{-1}t') = 0$ or $x_1(\alpha_1^{-1}t') = 0$. In both cases we reach a contradiction.

Hence, if $t_0 = \alpha_1 t_1$ then 2.2.9 holds for $[t_0, t_1]$. □

Proposition 2.2.10 *Let $a > 2$. Then $\mathcal{W}^u(p_+) = (-\infty, 1] \times \{0\}$. Furthermore $\exists t_0, t_1 \in \mathbb{R}$ such that $t_0 < 0 < t_1$, $x_1'(t_0) = x_1'(t_1) = 0$, $x_1(t_0) = 1 - a$, $x_1(t_1) = 1$ and $x_1'(t) \neq 0 \, \forall t \in (t_0, t_1)$.*

Proof:

Proceeding as in 2.2.9 we have $[1 - a, 1] \subset \mathcal{W}^u(p_+)$. But now we obtain $\mathcal{W}^u(p_+) = \cup_{n \in \mathbb{N}} f_{a,b}^n([1 - a, 1]) = (-\infty, 1]$ because $1 - a < x_-$. The second part is as in 2.2.9. □

Proposition 2.2.11 *$\mathcal{W}^s(p_+)$ (respectively $\mathcal{W}^s(p_-)$) is a set of parabolas $y = ax^2 + x_0 - 1$, where $f_a^n(x_0) = x_+$ (resp. $f_a^n(x_0) = x_-$) for some $n \in \mathbb{N}$ if p_+ or p_- are saddles.*

Proof:

We do it for p_+ (for p_- it is done in the same way). By definition $\mathcal{W}^s(p_+) = \{(x, y) \in \mathbb{R}^2 \text{ s. t. } \lim_{n\to\infty} f_{a,0}^n(x, y) = p_+\}$. Hence all the points $(x_0, 0)$ such that $f_a^n(x_0) = x_+$ are in $\mathcal{W}^s(p_+)$. Furthermore, given one of these x_0, let $y = g(x) = ax^2 + x_0 - 1$. Then $f_{a,0}(x, y) = (x_0, 0)$. Hence $f_{a,0}^{n+1}(x, y) = (x_+, 0)$. This implies $(x, y) \in \mathcal{W}^s(p_+)$. These ones are the only points belonging to $\mathcal{W}^s(p_+)$ because in the interval there is no other point on the manifold, due the fact that p_+ is a repellor. □

2.3 Invariant manifolds for b close to 0

In this section we shall prove the differentiable dependence of the invariant manifolds with respect to a and b, and we shall give local expressions for b close to 0.

Proposition 2.3.1 *The map* $\bar{f}_{a,b}(x,y) = (1 + by - ax^2, x)$ *is conjugated to* $f_{a,b}$ *and its inverse is* $\bar{f}_{a,b}^{-1}(x,y) = (y, b^{-1}(x+ay^2-1))$ *if* $b \neq 0$. *If* $\bar{g} : U \longrightarrow \mathbb{R}$, *being* U *a neighbourhood of* x_+ *or* x_-, *represents the local unstable manifold of* $\bar{p}_+$ *or* $\bar{p}_-$ *under* $\bar{f}$, *then* $g : U \longrightarrow \mathbb{R}$, *given by* $g(x) = b\bar{g}(x)$, *represents the local unstable manifold of* $p_+ = (x_+, y_+)$ *or* $p_- = (x_-, y_-)$ *under* f.

Proof:

By means of the change $\bar{x} = x$, $\bar{y} = by$ we pass from $f_{a,b}$ to $\bar{f}_{a,b}$. Furthermore, let $\bar{g} : U \longrightarrow \mathbb{R}$ as in the statement. Assume, for instance, that $\bar{g}(x_+) = x_+$ and $\bar{g}(1 + b\bar{g}(x) - ax^2) = x$. If $g(x) = b\bar{g}(x)$ then $g(1 + g(x) - ax^2) = bx$ and $g(x_+) = bx_+ = y_+$. Concerning the inverse map, it is easy to make the check. □

Proposition 2.3.2 *Let* (x_0, y_0) *be one of the fixed points of* $\bar{f}_{a,b}$. *Translating* (x_0, y_0) *to the origin, and taking as axes the eigendirections at the fixed point, we have*

$$\bar{f}_{a,b}(x,y) = \begin{pmatrix} \bar{\alpha}_1^{-1} & 0 \\ 0 & \bar{\alpha}_2^{-1} \end{pmatrix} \begin{pmatrix} x + \frac{a}{\bar{\alpha}_1^{-1} - \bar{\alpha}_2^{-1}}(x+y)^2 \\ y - \frac{a}{\bar{\alpha}_1^{-1} - \bar{\alpha}_2^{-1}}(x+y)^2 \end{pmatrix}, \tag{2.5}$$

where $\bar{\alpha}_1^{-1}$, $\bar{\alpha}_2^{-1}$ *are the eigenvalues associated to the fixed point with* $|\bar{\alpha}_1^{-1}| > 1$ *and* $|\bar{\alpha}_2^{-1}| < 1$. *We denote also by* $\bar{f}_{a,b}$ *the map after the change of coordinates.*

Proof:

By translating $\xi = x + x_0$, $\eta = y + x_0$, we have:

$$\bar{f}_{a,b}^{-1}(x,y) = \begin{pmatrix} 0 & 1 \\ b^{-1} & 2ax_0b^{-1} \end{pmatrix} \begin{pmatrix} x + ay^2 \\ y \end{pmatrix}$$

because $x_0 + ax_0^2 - 1 = bx_0$, (x_0, y_0) being the fixed point.

The eigenvalues of $(0,0)$ (fixed point after the translation) satisfy $\bar{\alpha}^2 + 2ax_0\bar{\alpha} - b = 0$. Therefore $\bar{\alpha}_{1,2} = -ax_0 \pm \sqrt{a^2x_0^2 + b}$. The eigenvectors $(\beta_{1,2}, 1)$ satisfy:

$$\begin{pmatrix} -2ax_0 - \bar{\alpha}_{1,2} & b \\ 1 & -\bar{\alpha}_{1,2} \end{pmatrix} \begin{pmatrix} \beta_{1,2} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence $\beta_1 = \bar{\alpha}_1$ and $\beta_2 = \bar{\alpha}_2$. Perfoming the linear change by means of the matrix C, we have

$$C^{-1} \circ \bar{f}_{a,b}^{-1} \circ C = C^{-1} \circ A \circ C \circ (I + C^{-1} \circ \tilde{f} \circ C),$$

where

$$C = \begin{pmatrix} \bar{\alpha}_1 & \bar{\alpha}_2 \\ 1 & 1 \end{pmatrix}, \; A = \begin{pmatrix} 0 & 1 \\ b^{-1} & 2ax_0b^{-1} \end{pmatrix}, \; \tilde{f}(x,y) = (ay^2, 0),$$

and I is the identity. Hence we obtain 2.5. □

Remark 2.3.3 *Despite* $\bar{\alpha}_2 = 0$ *when* $b = 0$, *one has* $\bar{\alpha}_1 \neq \bar{\alpha}_2$ *if* $(0,0)$ *is a saddle. Therefore the nonlinear part of* $\bar{f}_{a,b}^{-1}$ *is well defined even for* $b = 0$.

The main theorem of this section gives the (at least) C^2 dependence of the invariant manifolds of the fixed points with respect to a and b, for b close to zero.

Theorem 2.3.4 *Let* $p = (x_0, y_0)$ *be one of the two fixed points of* $f_{a,b}$. *Let* $a = a_0$ *such that, under* $f_{a_0,0}$, p *is a saddle (that is* x_0 *is a repellor under* $f_{a_0,0|y=0}$*). Then there is a neighbourhood* $U = I_1 \times I_2 \times I_3$ *of* $(x_0(a_0,0), a_0, 0)$, *(where* I_1, I_2, I_3 *are open intervals), and* C^2 *functions* $g_1 : U \longrightarrow \mathbb{R}$ *and* $g_2 : U \longrightarrow \mathbb{R}$ *such that*

a) $g_1(x_0, a, b) = g_2(x_0, a, b) = y_0 \ \forall (a,b) \in I_2 \times I_3$,

b) $V_1 = \{(x,y) \in \mathbb{R}^2 \ : \ y = g_1(x,a,b),\ x \in I_1\} \subset \mathcal{W}^u(p) \ \forall (a,b) \in I_2 \times I_3$,

c) $V_2 = \{(x,y) \in \mathbb{R}^2 \ : \ y = g_2(x,a,b),\ x \in I_1\} \subset \mathcal{W}^s(p) \ \forall (a,b) \in I_2 \times I_3$,

d) For $b = 0$ *the invariant manifolds are the ones given in theorem 2.2.6.*

Proof:

We shall do it separately for the unstable and stable manifolds.

Unstable manifold.

As seen before, by means of a change of coordinates $\bar{f}_{a,b}^{-1}(x,y) = A \circ (I + \Phi)$ where

$$A = \begin{pmatrix} \bar{\alpha}_1^{-1} & 0 \\ 0 & \bar{\alpha}_2^{-1} \end{pmatrix}$$

and

$$\Phi(x,y) = (f_1(x,y), f_2(x,y)) = \left(\frac{a}{\bar{\alpha}_1 - \bar{\alpha}_2}(x+y)^2, -\frac{a}{\bar{\alpha}_1 - \bar{\alpha}_2}(x+y)^2 \right),$$

and $\bar{\alpha}_1 = \bar{\alpha}_1(a,b)$, $\bar{\alpha}_2 = \bar{\alpha}_2(a,b)$, are the eigenvalues of p with $|\bar{\alpha}_1| > 1$ and $|\bar{\alpha}_2| < 1$. We will denote also f_1 and f_2 as $f_{1(a,b)}$ and $f_2(a,b)$, if we need to use the parameter dependence of these maps. Let α an upper bound of $|\bar{\alpha}_1^{-1}(a_0,0)|$ and $\bar{\alpha}_2(a_0,0) = 0$, such that $\alpha < 1$. We consider the space K of sequences γ, where $\gamma(n) \in \mathbb{R}^2$ and $\gamma(n) \to 0$ when $n \to \infty$, with the norm $\|\gamma\| = \sup_{n \in \mathbb{N}} \|\gamma(n)\|$. Let $G \subset K$ defined by $G = \{\gamma \in K \ : \ \gamma(n) \in B_\beta \, \forall n \geq 0\}$, where B_β is the ball centered at $(0,0)$ and with radius β small enough. We can define the map

$$F : B_\beta^u \times G \times A_1 \times A_2 \times \Gamma_1 \times \Gamma_2 \longrightarrow K,$$

where B_β^u is the ball of radius β centered at $0 \in \mathbb{R}$, A_1, A_2, Γ_1 Γ_2 are neighbourhoods, respectively, of $\lambda_1(a_0,0) = \bar{\alpha}_1^{-1}(a_0,0)$, $\lambda_2(a_0,0) = \bar{\alpha}_2(a_0,0)$, $f_{1(a_0,0)}$, $f_{2(a_0,0)}$, the last two with respect to the C^2 norm ($f_i : B_\beta \longrightarrow \mathbb{R}$, $i = 1, 2$), given by

$$F(x,\gamma,\lambda_1,\lambda_2,f_1,f_2)(n) = \gamma(n) - \left(\lambda_1^n x + \sum_{i=0}^{n-1} \lambda_1^{n-i} f_1(\gamma(i)), -\sum_{i=n}^{\infty} \lambda_2^{-(n-i)} f_2(\gamma(i))\right).$$

We note that this map is defined even for $b = 0$ because, despite $\lambda_2 = \tilde{\alpha}_2 = 0$, we always have λ_2 raised to positive exponents:

$$F(x,\gamma,\lambda_1,0,f_1,f_2)(n) = \gamma(n) - \left(\lambda_1^n x + \sum_{i=0}^{n-1} \lambda_1^{n-i} f_1(\gamma(i)), -f_2(\gamma(n))\right).$$

We want to see that F is of class C^2. The proof is split in several lemmas.

Lemma 2.3.5 *F is well defined.*

Proof:

We shall see $F(x,\gamma,\lambda_1,\lambda_2,f_1,f_2) \in K$. Let $\bar{b} > 0$ such that $\|\Phi(z)\| < \bar{b}$ for $z \in B_\beta$. As $0 < \alpha < 1$: $\sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha}$. We know that $\gamma(n) \to 0$ and $\lambda_1^n x \to 0$, when $n \to \infty$, because $|\lambda_1| < \alpha$. And also

$$\left\|\sum_{i=n}^{\infty} \lambda_2^{-(n-i)} f_2(\gamma(i))\right\| \leq \frac{1}{1-\alpha} \sup_{i\geq n} \|f_2(\gamma(i))\|,$$

holds. Therefore given $\epsilon > 0$ we choose n large enough to have $\|f_2(\gamma(i))\| < (1-\alpha)\epsilon \; \forall i \geq n$. This says that the second component of F goes to zero. Let us look for the first one. For $0 \leq m \leq n$ one has:

$$\left|\sum_{i=0}^{n-1} \lambda_1^{n-i} f_1(\gamma(i))\right| = \left|\lambda_1^{n-m} \sum_{i=0}^{m-1} \lambda_1^{m-i} f_1(\gamma(i)) + \sum_{i=m}^{n-1} \lambda_1^{n-i} f_1(\gamma(i))\right| \leq$$

$$\frac{\alpha^{n-m}\bar{b}}{1-\alpha} + \sup_{i\geq m} f_1(\gamma(i)) \frac{1}{1-\alpha}.$$

Given $\epsilon > 0$ let us take m large enough to make the second term less than $\epsilon/2$. Then we choose n large enough to make the first one less than $\epsilon/2$. This means that the first component of F goes to zero and hence $F(x,\gamma,\lambda_1,\lambda_2,f_1,f_2) \in K$. □

Lemma 2.3.6 *F is continuous.*

Proof:

We rewrite F as

$$F = (x,\gamma,\lambda_1,\lambda_2,f_1,f_2)(n) = \gamma(n) - \left(\lambda_1^n x + \sum_{i=0}^{n-1} \lambda_1^{n-i} f_1(\gamma(i)), -\sum_{i=0}^{\infty} \lambda_2^{i} f_2(\gamma(i+n))\right).$$

It is enough to see that F_1, F_2, F_3, defined by

$$\begin{aligned} F_1(x,\lambda_1)(n) &= \lambda_1^n x, \\ F_2(\gamma,\lambda_1,f_1)(n) &= \textstyle\sum_{i=0}^{n-1} \lambda_1^{n-i} f_1(\gamma(i)), \\ F_3(\gamma,\lambda_2,f_2)(n) &= \textstyle\sum_{i=0}^{\infty} \lambda_2^i f_2(\gamma(i+n)), \end{aligned}$$

are continuous.

a) F_1 is continuous.

$$|(F_1(x,\lambda_1) - F_1(\bar{x},\bar{\lambda}_1))(n)| = |\lambda_1^n x - \bar{\lambda}_1^n \bar{x}| \le |\lambda_1^n - \bar{\lambda}_1^n||x| + |\bar{\lambda}_1^n||x-\bar{x}| \le$$
$$n\alpha^{n-1}|\lambda_1 - \bar{\lambda}_1|\beta + |x-\bar{x}|,$$

because $|x| < \beta$, $|\bar{\lambda}_1| < 1$. Then F_1 is continuous since $n\alpha^{n-1} \to 0$ when $n \to \infty$.

b) F_2 is continuous.

$$\left|\sum_{i=0}^{n-1} \lambda_1^{n-i} f_1(\gamma(i)) - \sum_{i=0}^{n-1} \bar{\lambda}_1^{n-i} \bar{f}_1(\bar{\gamma}(i))\right| \le$$
$$\sum_{i=0}^{n-1} \left(|\lambda_1^{n-i}|\,\|f_1 - \bar{f}_1\| + |\lambda_1^{n-i} - \bar{\lambda}_1^{n-i}|\,\|\bar{f}_1\| + |\bar{\lambda}_1^{n-i}|\,|\bar{f}_1(\gamma(i)) - \bar{f}_1(\bar{\gamma}(i))|\right) \le$$
$$\sum_{i=0}^{n-1}(\alpha^{n-i}\|f_1 - \bar{f}_1\| + n\alpha^{n-i-1}\|D\bar{f}_1\|\,|\lambda_1 - \bar{\lambda}_1| + \alpha^{n-i}\|\bar{f}_1\|\,\|\gamma - \bar{\gamma}\| \le$$
$$\frac{1}{1-\alpha}\|f_1 - \bar{f}_1\| + \frac{1}{(1-\alpha)^2}\|\bar{f}_1\|\,|\lambda_1 - \bar{\lambda}_1| + \frac{1}{1-\alpha}\|D\bar{f}_1\|\,\|\gamma - \bar{\gamma}\|.$$

Hence F_2 is continuous.

c) F_3 is continuous.

$$\left|\sum_{i=0}^{\infty} \lambda_2^i f_2(\gamma(i+n)) - \sum_{i=0}^{\infty} \bar{\lambda}_2^i \bar{f}_2(\bar{\gamma}(i+n))\right| \le \sum_{i=0}^{\infty} |\lambda_2^i f_2(\gamma(i+n)) - \bar{\lambda}_2^i \bar{f}_2(\bar{\gamma}(i+n))|.$$

For every i we have

$$|\lambda_2^i f_2(\gamma(i+n)) - \bar{\lambda}_2^i \bar{f}_2(\bar{\gamma}(i+n))| \le |\lambda_2^i|\,\|f_2 - \bar{f}_2\| + |\lambda_2^i - \bar{\lambda}_2^i|\,\|\bar{f}_2\| +$$
$$|\bar{\lambda}_2^i|\,|\bar{f}_2(\gamma(i+n) - \bar{f}_2(\bar{\gamma}(i+n)| \le \alpha^i\|f_2 - \bar{f}_2\| + i\alpha^{i-1}|\lambda_2 - \bar{\lambda}_2|\,\|\bar{f}_2\| + \alpha^i\|D\bar{f}_2\|\,\|\gamma - \bar{\gamma}\|.$$

Hence

$$\left|\sum_{i=0}^{\infty} \lambda_2^i f_2(\gamma(i+n)) - \sum_{i=0}^{\infty} \bar{\lambda}_2^i \bar{f}_2(\bar{\gamma}(i+n))\right| \le \sum_{i=0}^{\infty} \alpha^i\|f_2 - \bar{f}_2\| + i\alpha^{i-1}|\lambda_2 - \bar{\lambda}_2|\,\|f_2\| +$$

$$\alpha^i\|\bar{f}_2\|\,\|\gamma-\bar{\gamma}\| \le \frac{1}{1-\alpha}\|f_2-\bar{f}_2\| + \frac{1}{(1-\alpha)^2}|\lambda_2-\bar{\lambda}_2|\,\|\bar{f}_2\| + \frac{1}{1-\alpha}\|D\bar{f}_2\|\,\|\gamma-\bar{\gamma}\|.$$

As a conclusion F_3 is continuous. □

Lemma 2.3.7 *F is of class C^1.*

Proof:

a) The map $x \longrightarrow F(x,\gamma,\lambda_1,\lambda_2,f_1,f_2)$ is affine and continuous, and the derivative is $D_1F(x,\gamma,\lambda_1,\lambda_2,f_1,f_2)(y)(n) = (-\lambda_1^n y, 0)$, also continuous.

b) We want to see that $\forall u \in K$:

$$D_2F(x,\gamma,\lambda_1,\lambda_2,f_1,f_2)(u)(n) = u(n) -$$

$$\left(\sum_{i=0}^{\infty}\lambda_1^{n-i}Df_1(\gamma(i))(u(i)), -\sum_{i=0}^{\infty}\lambda_2^{i}Df_2(\gamma(i+n))(u(i+n))\right).$$

To simplify we denote by λ the right hand part of this expression. We want to see that, given $\delta > 0$:

$$\|F(x,\gamma+u)(n) - F(x,\gamma)(n) - \lambda\| \le \delta\|u\|$$

if $\|u\|$ is small enough.

The left hand part of this expression is less than

$$\left|\sum_{i=0}^{n-1}\lambda_1^{n-i}[f_1(\gamma(i)+u(i)) - f_1(\gamma(i)) - Df_1(\gamma(i))(u(i))]\right| +$$

$$\left|\sum_{i=0}^{\infty}\lambda_2^{i}[f_2(\gamma(i+n)+u(i+n)) - f_2(\gamma(i+n) - Df_2(\gamma(i+n))(u(i+n))]\right| \qquad (2.6)$$

Let $\Phi(z) = F(x,z,\lambda_1,\lambda_2,f_1,f_2)$. As $D\Phi$ is continuous in $\overline{B_\beta}$ (compact), it is uniformly continuous in B_β. Hence, given $\delta' > 0$ there exists $\rho > 0$ such that $\|D\Phi(z+u) - D\Phi(z)\| < \delta'$ for $z \in B_\beta$ and u such that, $z+u \in B_\beta$ and $\|u\| < \rho$. Applying the mean value theorem to $\Gamma(u) = \Phi(z+u) - \Phi(z) - D\Phi(z)u$, we obtain:

$$\|\Gamma(u) - \Gamma(0)\| = \|\Phi(z+u) - \Phi(z) - D\Phi(z)u\| \le \delta'\|u\|.$$

Hence every term in 2.6 is less than $(1-\alpha)^{-1}\delta'\|u\|$. It is enough to consider δ' such that $2(1-\alpha)^{-1}\delta' < \delta$ and we conclude $\lambda = D_2F(x,\gamma)(u)(n)$.

We recall

$$D_2F : B_\beta^u \times G \times A_1 \times A_2 \times \Gamma_1 \times \Gamma_2 \longrightarrow K',$$

where K' is the set of linear continuous maps of K into K. It is easy to see that D_2F is continuous making a reasoning similar to the one of lemma 2.3.6.

c) Let $g(\lambda_1) = F(x, \gamma, \lambda_1, \lambda_2, f_1, f_2)$. We want to obtain $Dg = D_3F$. We have

$$g(\lambda_1)(n) = \gamma(n) - \left(\lambda_1^n x + \sum_{i=0}^{n-1} \lambda_1^{n-i} R_i, Q_n \right),$$

where $R_n = f_1(\gamma(n)) \to 0$, $Q_n \to 0$, when $n \to \infty$, and are independent of λ_1. Hence:

$$Dg(\lambda_1) = Dg_1(\lambda_1) + Dg_2(\lambda_1),$$

where $g_1(\lambda_1)(n) = (-\lambda_1^n x, 0)$ and $g_2(\lambda_1)(n) = \left(-\sum_{i=0}^{n-1} \lambda_1^{n-i} R_i, 0\right)$.

First we see $D\tilde{g}_1(\lambda_1)(\mu)(n) = n\lambda_1^{n-1}\mu$, $n \geq 0$, where $\tilde{g}_1(\lambda_1) = \lambda_1^n$. For this we observe first that it is well defined because $n\lambda_1^{n-1} \to 0$ when $n \to \infty$, since $|\lambda_1| < 1$. Furthermore

$$|\tilde{g}_1(\lambda_1 + \mu)(n) - \tilde{g}_1(\lambda_1)(n) - n\lambda_1^{n-1}\mu| = |(\lambda_1 + \mu)^n - \lambda_1^n - n\lambda_1^{n-1}\mu| =$$

$$|\mu|^2 \left| \binom{n}{2} \lambda_1^{n-2} + \binom{n}{3} \lambda_1^{n-3}\mu + \ldots + \binom{n}{n-1} \lambda_1 \mu^{n-3} + \mu^{n-2} \right| \leq$$

$$|\mu|^2 \binom{n}{2} (|\lambda_1| + |\mu|)^{n-2},$$

if $n \geq 2$ because

$$\binom{n}{p} \left[\binom{n}{2} \right]^{-1} = \frac{n!2!(n-2)!}{p!(n-p)!n!} = \frac{2}{p(p-1)} \binom{n-2}{p-2} \leq \binom{n-2}{p-2}$$

if $n \geq p \geq 2$.

As

$$\lim_{n \to \infty} \binom{n}{2} (|\lambda_1| + |\mu|)^{n-2} = 0,$$

if $|\mu|$ is small enough, we have:

$$|\tilde{g}_1(\lambda_1 + \mu)(n) - \tilde{g}_1(\lambda_1)(n) - n\lambda_1^{n-1}\mu| \leq |\mu|^2 M,$$

where M is an upper bound of $\binom{n}{2} (|\lambda_1| + |\mu|)^{n-2}$ independent of n $(n \geq 2)$.

For $n = 0$:

$$|\tilde{g}_1(\lambda_1 + \mu)(0) - \tilde{g}_1(\lambda_1)(0)| = 0 \leq M|\mu|^2.$$

For $n = 1$,

$$|\tilde{g}_1(\lambda_1 + \mu)(1) - \tilde{g}_1(\lambda_1)(1)| = 0 \leq M|\mu|^2.$$

Hence $\tilde{g}_1$ is differentiable and it is easy to see that $D\tilde{g}_1$ is continuous.

Let now

$$\tilde{g}_2(\lambda_1)(n) = \sum_{i=0}^{n-1} \lambda_1^{n-i} R_i, \quad R_i \to 0 \text{ when } i \to \infty.$$

Then

$$(D\tilde{g}_2(\lambda)\mu)(n) = \sum_{i=0}^{n-1} (n-i)\lambda_1^{n-i-1} R_i \mu.$$

To see this we show first that it is well defined, that is $\sum_{i=0}^{n-i}(n-i)\lambda_1^{n-i-1}R_i \to 0$ when $n \to \infty$:

$$\left|\sum_{i=0}^{n-1}(n-i)\lambda_1^{n-i-1}R_i\right| \leq \left|\lambda_1^{n-m}\sum_{i=0}^{m-1}(n-i)\lambda_1^{m-i-1}\right|\bar{b}+$$

$$\left|\sum_{i=m}^{n-1}(n-i)\lambda_1^{n-i-1}R_i\right| \leq \left|n\lambda_1^{n-m}\sum_{i=0}^{m-1}\frac{n-i}{n}\lambda_1^{n-i-1}\right|\bar{b} + \sup_{i\geq m}|R_i|\frac{1}{(1-\alpha)^2} \leq$$

$$n|\lambda_1|^{n-m}\frac{\bar{b}}{1-\alpha} + \frac{1}{(1-\alpha)^2}\sup_{i\geq m}|R_i| \quad (m<n),$$

where $|R_i| \leq \bar{b}$ $\forall i \in \mathbb{N}$. This goes to zero when $n \to \infty$.

Let us now identify the derivative:

$$\left|\sum_{i=0}^{n-1}(\lambda_1+\mu)^{n-i}R_i - \sum_{i=0}^{n-1}\lambda_1^{n-i}R_i - \sum_{i=0}^{n-1}(n-i)\lambda_1^{n-i-1}R_i\right| \leq$$

$$|\mu|^2\sum_{i=0}^{n-1}\binom{n-i}{2}(|\lambda_1|+|\mu|)^{n-i-2}|R_i| \leq$$

$$|\mu|^2\left(\sum_{i=1}^{n}\binom{i}{2}(|\lambda_1|+|\mu|)^{i-1}\right)\bar{b} \leq |\mu|^2\bar{M},$$

where $\bar{M}$ is a constant which does not depend on n (if $|\mu|$ small enough).

The continuity of $D\tilde{g}_2(\lambda_1)$ is done as in lemma 2.3.6.

d) Differentiabilty with respect to λ_2 :

Let

$$\bar{g}(\lambda_2)(n) = \sum_{i=0}^{\infty}\lambda_2^i f_2(\gamma(i+n)) = \sum_{i=0}^{\infty}\lambda_2^i \bar{R}_{i+n},$$

where $\bar{R}_i \to 0$ when $i \to \infty$.

We shall see

$$D\bar{g}(\lambda_2)(\mu)(n) = \sum_{i=1}^{\infty} i\lambda_2^{i-1}\bar{R}_{i+n}\mu,$$

which goes to zero when $n \to \infty$:

$$\left| \bar{g}(\lambda_2 + \mu) - \bar{g}(\lambda_2) - \sum_{i=1}^{\infty} i \lambda_2^{i-1} \bar{R}_{i+n} \right| \leq$$

$$|\mu|^2 \sum_{i=2}^{\infty} \binom{i}{2} (|\lambda_2| + |\mu|)^{i-2} \bar{R}_{i+n}.$$

Hence, if $|\mu|$ is small enough, then

$$\sum_{i=2}^{\infty} \binom{i}{2} (\lambda_2| + |\mu|)^{i-2}$$

is convergent, and this means that $\bar{g}$ is differentiable.

e) Differentiability with respect to f_1.

Let $h(f_1)(n) = \sum_{i=0}^{n-1} \lambda_1^{n-i} f_1(a_i)$, where $a_n \to 0$ for $n \to \infty$. It is a continuous linear map because

$$|h(f_1)(n)| \leq \sum_{i=0}^{n-1} |\lambda_1|^{n-i} \|f_1\| \leq \frac{1}{1-\alpha} \|f_1\|.$$

Hence it is differentiable.

f) Differentibility with respect to f_2. Let $\bar{h}(f_2)(n) = \sum_{i=0}^{\infty} \lambda_2^i f_2(a_{i+n})$, where $a_i \to 0$ for $i \to \infty$. It is a continuous linear map:

$$|\bar{h}(f_2)(n)| \leq \sum_{i=0}^{\infty} |\lambda_2|^i |f_2(a_{i+n})| \leq \sum_{i=0}^{\infty} |\lambda_2|^i \|f_2\| \leq \frac{1}{1-\alpha} \|f_2\|.$$

Hence it is differentiable .□

Lemma 2.3.8 *F is of class C^2.*

Proof:

We shall proof the existence of the partial derivatives of D_1F, D_2F, D_3F, D_4F, D_5F, and D_6F :

a) First we note that $D_{11}F = D_{12}F = D_{14}F = D_{15}F = D_{16}F = 0$. It remains to obtain $D_{13}F$.

Let $g(x, \lambda_1)(n) = \lambda_1^n x$. Then $D_1 g(x, \lambda_1)(n) = \lambda_1^n$ and this map is differentiable with respect to λ_1. Hence, it follows that $D_{13}F$ exists and it is continuous.

b) One has $D_{21}F = 0$ easily.

We know that

$$D_2F(x,\gamma,\lambda_1,\lambda_2,f_1,f_2)(n) = u(n) -$$
$$\left(\sum_{i=0}^{n-1} \lambda_1^{n-i} Df_1(\gamma(i))(u(i)), -\sum_{i=0}^{\infty} \lambda_2^i Df_2(\gamma(i+n)u(i+n)\right).$$

We want to see

$$D_{22}F(x,\gamma,\lambda_1,\lambda_2,f_1,f_2)(u,v)(n) =$$
$$\left(-\sum_{i=0}^{n-1} \lambda_1^{n-i} D^2f_1(\gamma(i))(u(i),v(i)), \sum_{i=0}^{\infty} \lambda_2^i D^2f_2(\gamma(i+n))(u(i+n),v(i+n))\right).$$

Indeed: one has to have, if $\Phi(\gamma) = F(x,\gamma,\lambda_1,\lambda_2,f_1,f_2)$,

$$\left\| D\Phi(\gamma+u) - D\Phi(\gamma) - D^2\Phi(\gamma)u \right\| \leq \|u\|\epsilon,$$

for ϵ small enough.

The left hand part of the above expression is a linear map, when applied to $v \in K$:

$$|[D\Phi(\gamma+u)v - D\Phi(\gamma)v - D^2\Phi(u,v)](n)| \leq$$
$$\left|\sum_{i=0}^{n-1} \lambda_1^{n-i} Df_1(\gamma(i)+u(i))v(i) - \sum_{i=0}^{n-1} \lambda_1^{n-i} Df_1(\gamma(i))v(i) - \right.$$
$$\left.\sum_{i=0}^{n-1} \lambda_1^{n-i} D^2f_1(\gamma(i))(u(i),v(i))\right| +$$
$$\left|\sum_{i=0}^{\infty} \lambda_2^i Df_2(\gamma(i+n)+u(i+n))v(i+n) - \right.$$
$$\left.\sum_{i=0}^{\infty} \lambda_2^i Df_2(\gamma(i+n))v(i+n) - \sum_{i=0}^{\infty} \lambda_2^i D^2f_2(\gamma(i+n))(u(i+n),v(i+n))\right| \leq$$
$$\left|\sum_{i=0}^{n-1} \lambda_1^{n-i} Df_1(\gamma(i)+u(i)) - \sum_{i=0}^{n-1} \lambda_1^{n-i} Df_1(\gamma(i)) - \sum_{i=0}^{n-1} \lambda_1^{n-i} D^2f_1(\gamma(i))u(i)\right| \|v\| +$$
$$\left|\sum_{i=0}^{\infty} \lambda_2^i Df_2(\gamma(i+n)+u(i+n)) - \sum_{i=0}^{\infty} \lambda_2^i Df_2(\gamma(i+n)) - \right.$$
$$\left.\sum_{i=0}^{\infty} \lambda_2^i D^2f_2(\gamma(i+n))u(i+n)\right| \|v\|.$$

Hence we have only to show

$$\left\|\sum_{i=0}^{n-1} \{\lambda_1^{n-i}[(Df_1(\gamma(i)+u(i)) - Df_1(\gamma(i)) - D^2f_1(\gamma(i))(u(i))]\}\right\| +$$

$$\left\| \sum_{i=0}^{\infty} \{ \lambda_2^i [Df_2(\gamma(i+n)+u(i+n)) - Df_2(\gamma(i+n)) -$$
$$Df_2(\gamma(i+n)) - D^2 f_2(\gamma(i+n))u(i+n)] \} \right\| \le \|u\|\epsilon.$$

The proof is analogous to the one of the existence of D_2F, by substituting f_1 and f_2 by Df_1 and Df_2. Furthermore it follows that it is continuous.

c) As D_2F is an affine continuos map with respect to f_1, it exists $D_{25}F$. The map D_2F is also affine and continuous with respect to f_2 and hence $D_{26}F$ exists and it is continuous.

d) $D_{23}F$ and $D_{24}F$ are obtained in a similar way to D_3F and D_4F.

e) To obtain the derivatives of D_3F and D_4F we have only to take into account that F is linear with respect to f_1 and f_2, and differentiable with respect to all the remaining variables.

f) The derivatives of D_5F and D_6F are also obtained as the ones of F.

It follows finally that $F(x, \gamma, \lambda_1, \lambda_2, f_1, f_2)$ is of class C^2 (in fact one can see that it is of class C^r if f_1 and f_2 are of class C^r.) □

The next step in the proof of 2.3.4 is to see that F, defined as before, as a function of x, γ, a and b is at least C^2. This is the objective of the next lemma.

Lemma 2.3.9 *$\bar{F}(x,\gamma,a,b) = F(x,\gamma,\lambda_1(a,b),\lambda_2(a,b),f_1(a,b),f_2(a,b))$ is a C^2 function, where $f_1(a,b)(x,y) = \frac{a}{\bar{\alpha}_1-\bar{\alpha}_2}(x+y)^2$, $f_2(a,b)(x,y) = -\frac{a}{\bar{\alpha}_1-\bar{\alpha}_2}(x+y)^2$, $\lambda_1 = \bar{\alpha}_1^{-1}$ and $\lambda_2 = \bar{\alpha}_2$.*

Proof:

First we observe that $\lambda_1 = [-ax_0 \pm \sqrt{a^2x_0^2+b}]^{-1}$ and $\lambda_2 = -ax_0 \mp \sqrt{a^2x_0^2+b}$ are C^ω functions with respect to a and b. Also $\mathcal{F} = \mathcal{F}_2 \circ \mathcal{F}_1$ defined by

$$\begin{array}{ccccc} U_1 \subset \mathbb{R}^2 & \xrightarrow{\mathcal{F}_1} & \mathbb{R} & \xrightarrow{\mathcal{F}_2} & C^2[B_\beta] \\ (a,b) & \longrightarrow & \frac{a}{\bar{\alpha}_1-\bar{\alpha}_2} & \longrightarrow & \left(\bar{f}_1\left(\frac{a}{\bar{\alpha}_1-\bar{\alpha}_2}\right), \bar{f}_2\left(\frac{a}{\bar{\alpha}_1-\bar{\alpha}_2}\right)\right), \end{array}$$

where $C^2[B_\beta] = \{\Phi : B_\beta \longrightarrow \mathbb{R}^2,\ \Phi \text{ of class } C^2\}$, U_1 small enough neighbourhood of $(a_0, 0)$ and $\bar{f}_1(\bar{a})(x,y) = \bar{a}(x+y)^2$, $\bar{f}_2(\bar{a})(x,y) = -\bar{a}(x+y)^2$, for $\bar{a} \in \mathbb{R}$ and $(x,y) \in B_\beta$, is a C^∞ function. Indeed:

$\mathcal{F}_1$ is C^∞ because $\bar{\alpha}_1 - \bar{\alpha}_2 \neq 0$, and $\mathcal{F}_2$ is a linear continuous function because

$$|\mathcal{F}_2(\bar{a})(x,y)| \le 2|a(x+y)|^2 \le 8\beta^2|\bar{a}|.$$

As $\mathcal{F} = \mathcal{F}_2 \circ \mathcal{F}_\infty$, we have that it is C^∞.

From this it follows that $\bar{F}$ is also C^2 because it is composition of C^2 functions. $\square$

The last step in the proof of 2.3.4 consists of to apply the implicit function theorem to $\bar{F}$, and to see that it give rise to a C^2 function representing the local unstable invariant manifold.

Lemma 2.3.10 *There exist a neighbourhood $\bar{U}$ of $(0, a_0, 0) \in B^u_\beta \times U_1 \subset \mathbb{R}^3$ and a function $\varphi : \bar{U} \longrightarrow K$ of class C^2 such that: $\varphi(0, a_0, 0) = 0$ and $\bar{F}(x, \varphi(x, a, b), a, b) = 0, \ \forall (x, a, b) \in \bar{U}$.*

Proof:

First we observe that $D_2\bar{F}(0, 0, a, b) = Id$ because $D(f_1(a, b))(0, 0) = D(f_2(a, b))(0, 0) = 0 \ \forall a, b$. Furthermore $\bar{F}(0, 0, a_0, 0) = 0$. Hence we can apply the implicit function theorem: it exists $\varphi : \bar{U} \longrightarrow K$ of class C^2 such that $\varphi(0, a_0, 0) = 0$ and $\bar{F}(x, \varphi(x, a, b), a, b) = 0$ $\forall (x, a, b) \in \bar{U}$. $\square$

Lemma 2.3.11 *There exist a neighbourhood $U = I_1 \times I_2 \times I_3$ and a C^2 function $\bar{g}_1 : U \longrightarrow \mathbb{R}$ such that $\bar{g}_1(0, a, b) = 0, \ \forall (a, b) \in I_2 \times I_3$ and $V_1 = \{(x, y) \in \mathbb{R}^2 \ : \ y = \bar{g}_1(x, a, b), \ x \in I_1\} \subset \mathcal{W}^u_{\bar{f}_{a,b}}(0, 0), \ \forall (a, b) \in I_2 \times I_3$. Here $\bar{f}_{a,b}$ is the function of proposition 2.3.2.*

Proof:

We see that for $n = 0$, $\varphi(x, a, b)(0)$ can be written as

$$\varphi(x, a, b)(0) = \left(x, -\sum_{i=0}^{\infty} \lambda_2^i f_2(a, b)(\varphi(x, a, b)(0)) \right).$$

Then we define $\bar{g}_1(x, a, b)$ as the second component of this expression. Trivially $\bar{g}_1$ is also C^2 because φ so is. For all $n \geq 0$ one has:

$$\varphi(x, a, b)(n) = \left(\lambda_1^n x + \sum_{i=0}^{n-1} \lambda_1^{n-i} f_1(a, b)(\varphi(x, a, b)(i)), -\sum_{i=n}^{\infty} \lambda_2^{-(n-i)} f_2(a, b)(\varphi(x, a, b)(i)) \right),$$

because $\bar{F}(x, \varphi(x, a, b))(n) = 0 \ \forall n \in \mathbb{N}$. From this it follows

$$\varphi(x, a, b)(n + 1) = \bar{f}_{a,b}^{-1}(\varphi(x, a, b)(n))$$

and, therefore,

$$\varphi(x, a, b)(n) = \bar{f}_{a,b}^{-n}(x, \bar{g}_1(x, a, b)), \ \ \forall n > 0.$$

Hence $\bar{f}_{a,b}^{-n}(x, \bar{g}_1(x, a, b)) \rightarrow 0$ when $n \rightarrow \infty$, because $\varphi(x, a, b) \in K$. As a consequence $(x, \bar{g}_1(x, a, b)) \in \mathcal{W}^u_{\bar{f}_{a,b}}(0, 0)$. This is true if $b \neq 0$.

For $b = 0$ we have

$$\varphi(x,a,0) = (x, -f_2(a,0)(\varphi(x,a,0)(0)).$$

Then $\bar{g}_1(x,a,0)$ satisfies

$$\bar{g}_1(x,a,0) = -f_2(a,0)(x,\bar{g}_1(x,a,0))$$

which is the equation corresponding to the unstable manifold because

$$\bar{f}_{a,b}^{-1}(x,y) = \begin{pmatrix} \bar{\alpha}_1^{-1} + \bar{\alpha}_1^{-1} f_1(a,b)(x,y) \\ \bar{\alpha}_2^{-1} + \bar{\alpha}_2^{-1} f_2(a,b)(x,y) \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix},$$

or,

$$x + f_1(a,b)(x,y) = \bar{\alpha}_1 \bar{x},$$

$$y + f_2(a,b)(x,y) = \bar{\alpha}_2 \bar{y},$$

which holds $\forall a,\ b$. Therefore, for $b = 0$ one has $\bar{\alpha}_2 = 0$ and $y = -f_2(a,0)(x,y)$. This means that the image under $\bar{f}_{a,0}$ of any point is on the curve $y = -f_2(a,0)(x,y)$ and, in particular, this applies to the unstable manifold. □

To finish the proof of the theorem for the unstable manifold, we go back through the change of variables.

Stable manifold.

For the stable manifold we shall use the map $f_{a,b}(x,y) = (1+y-ax^2, bx)$. Then, moving one of the two fixed points to the origin, we have

$$\bar{f}_{a,b}(x,y) = \begin{pmatrix} -2ax_0 & 1 \\ b & 0 \end{pmatrix} \begin{pmatrix} x \\ y - ax^2 \end{pmatrix}.$$

Taking as axes the eigendirections, one obtains

$$\bar{f}_{a,b}(x,y) = \begin{pmatrix} \bar{\alpha}_1 & 0 \\ 0 & \bar{\alpha}_2 \end{pmatrix} \begin{pmatrix} x + \frac{a}{\bar{\alpha}_2 - \bar{\alpha}_1}(x+y)^2 \\ y - \frac{a}{\bar{\alpha}_2 - \bar{\alpha}_1}(x+y)^2 \end{pmatrix}.$$

This map is analogous to the one used for the unstable manifold, with the adventage that $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are always defined (they are the eigenvalues). We can use the same reasoning as before to end the theorem. □

As a consequence of this theorem we can give the invariant manifolds to the first order in b.

Remark 2.3.12 *a) We have only used that f_1 and f_2 are C^2. Hence the result can be applied to maps more general than the Hénon map.*

b) This theorem generalizes a result of [15] (see also [16]) only true for diffeomorphisms.

c) Considering $f^n_{a,b}(x, g_1(a,b))$ one can extend the domain of definition of g_1. Indeed: For $b = 0$, $f^n_{a,0}(x,0) = (f^n_a(x), 0)$ and $g_1(x,a,0) = 0$. Furthermore $\mathcal{W}^u(p_0) \subset \mathbb{R} \times \{0\}$ and $\bigcup_{n\in\mathbb{N}} f^n_a(V) \times \{0\} = \mathcal{W}^u(p_0)$, where V is a neighbourhood of x_0 in $\mathbb{R}$. Let $n = 1$. Then the local unstable manifold of p_0 is written as $(1 - ax^2, 0)$ $\forall x \in I_1$. If $0 \notin I_1$ we can extend the domain of definition of $g_1(x,a,0)$ to $I_{11} = f_a(I_1)$. This can be done for successive $n \in \mathbb{N}$. As $(0,0) \in \mathcal{W}^u(p_0)$, there is some n such that we cannot apply the inverse function theorem or, equivalently, one can extend the domain of definition of $g_1(\cdot,\cdot,0)$ in such a way that $f_{a,0}(x, g_1(x,a,0))$ contains a turning point. On the other hand the implicit function theorem assure that this can be done for (a,b) close to $(a_0, 0)$. Hence we can assume that $f_{a,b}(x, g_1(x,a,b))$ has a turning point.
One can see for the stable manifold that, if (a,b) is close to $(a_0,0)$, one can extend the domain of definition as far as one wants. Now we shall use $f^{-1}_{a,b}$ for $b \neq 0$.

Proposition 2.3.13 *Let a fixed such that $p_+(a,0)$ and $p_-(a,0)$ are saddles. Then there is $b_0 > 0$ and there are neighbourhoods U_1, U_2 of $x_+(a,0)$ and V_1, V_2 of $x_-(a,0)$ such that $\forall b$ with $|b| < b_0$ it holds:*

a)

$$g_1(x) = \sqrt{(1-x)a^{-1}}\, b + O(b^2), \quad \forall x \in U_1,$$
$$g_2(x) = -\sqrt{(1-x)a^{-1}}\, b + O(b^2), \quad \forall x \in V_1,$$

$(x, g_1(x))$ being the local unstable manifold of p_+ and $(x, g_2(x))$ the one of p_-.

b)

$$h_1(x) = ax^2 + x_0^+ - 1 + \frac{x - x_+^0(1 + 2ax_0^+)^{-1}}{2ax_0^+} b + O(b^2), \quad \forall x \in U_2,$$
$$h_2(x) = ax^2 + x_0^- - 1 + \frac{x - x_-^0(1 + 2ax_0^-)^{-1}}{2ax_0^-} b + O(b^2), \quad \forall x \in V_2,$$

where $x_+^0 = x_+(a,0)$, $x_-^0 = x_-(a,0)$ and $(x, h_1(x))$ (respectively $(x, h_2(x))$ is the local stable manifold of p_+ (resp. p_-).

Proof:

From the last theorem we know that the stable and unstable manifolds can be written as $(x, g(x,a,b))$ and $(x, h(x,a,b))$ for a fixed point $p_0 = (x_0, y_0)$ ($p_0 = p_+$ or $p_0 = p_-$). These curves satisfy the following functional equations:

$$bx = g(1 + g(x,a,b) - ax^2, a, b), \quad g(x_0) = y_0,$$

$$g'(x_0) = \text{slope of the eigenvector associated to } \mathcal{W}^u(p_0). \tag{2.7}$$

$$bx = h(1 + h(x,a,b) - ax^2, a, b),\ h(x_0) = y_0,$$

$$g'(x_0) = \text{slope of the eigenvector associated to } \mathcal{W}^s(p_0). \tag{2.8}$$

By derivation of 2.7 with respect to b we have:

$$x = \frac{\partial g}{\partial x}(1 + g(x,a,b) - ax^2, a, b)\frac{\partial g}{\partial b}(x,a,b) + \frac{\partial g}{\partial b}(1 + g(x,a,b) - ax^2, a, b).$$

For $b = 0$:

$$x = \frac{\partial g}{\partial x}(1 - ax^2, a, 0)\frac{\partial g}{\partial b}(x,a,0) + \frac{\partial g}{\partial b}(1 - ax^2, a, 0),$$

because $g(x,a,0) = 0$. Hence $\frac{\partial g}{\partial b}(1 - ax^2, a, 0)\frac{\partial g}{\partial b}(x,a,0) = 0$. From this it follows

$$x = \frac{\partial g}{\partial b}(1 - ax^2, a, 0). \tag{2.9}$$

Setting $x_0 = x_+$ we have

$$x_+ = \frac{\partial g_1}{\partial b}(x_+, a, 0) > 0,$$

putting g_1 because it corresponds to $\mathcal{W}^u(p_+)$. Therefore, for x close to x_+ : $\frac{\partial g_1}{\partial b}(x,a,0) > 0$.

We can write 2.9 as

$$1 - ax^2 = 1 - a\left(\frac{\partial g_1}{\partial b}(1 - ax^2, a, 0)\right)^2,$$

and setting $y = 1 - ax^2$, we have

$$y = 1 - a\left(\frac{\partial g_1}{\partial b}(y, a, 0)\right)^2,$$

or

$$\frac{\partial g_1}{\partial b}(y,a,0) = \pm\sqrt{\frac{1-y}{a}}.$$

As $\frac{\partial g_1}{\partial b}(x_+,a,0) = \sqrt{(1 - x_+)a^{-1}} = x_+$ then

$$\frac{\partial g_1}{\partial b}(x,a,0) = \sqrt{\frac{1-x}{a}}.$$

Therefore

$$g_1(x,a,b) = \sqrt{\frac{1-x}{a}}b + O(b^2).$$

If $x_0 = x_-$ then:

$$x_- = \frac{\partial g_1}{\partial b}(x_-, a, 0) < 0.$$

In this case we have

$$g_2(x,a,b) = -\sqrt{\frac{1-x}{a}}b + O(b^2).$$

For $\mathcal{W}^s(p_0)$ we do the same with 2.8:

$$x = \frac{\partial h}{\partial x}(1 + h(x,a,b) - ax^2, a, b)\frac{\partial h}{\partial b}(x,a,b) + \frac{\partial h}{\partial b}(1 + h(x,a,b) - ax^2, a, b).$$

For $b = 0$:

$$x = \frac{\partial h}{\partial x}(x_0, a, 0)\frac{\partial h}{\partial b}(x, a, 0) + \frac{\partial h}{\partial b}(x_0, a, 0).$$

If $x = x_0$ then

$$x_0 = \frac{\partial h}{\partial b}(x_0, a, 0)\left(1 + \frac{\partial h}{\partial b}(x_0, a, 0)\right) = \frac{\partial h}{\partial b}(x_0, a, 0)(1 + 2ax_0),$$

because $h(x, a, 0) = ax^2 + x_0 - 1$. Therefore

$$\frac{\partial h}{\partial b}(x_0, a, 0) = \frac{x_0}{1 + 2ax_0},$$

and

$$\frac{\partial h}{\partial b}(x, a, 0) = \frac{x - x_0(1 + 2ax_0)^{-1}}{2ax_0}.$$

So we have

$$h_1(x) = ax^2 + x_0^+ - 1 + \frac{x - x_+^0(1 + 2ax_0^+)^{-1}}{2ax_0^+}b + O(b^2),$$

$$h_2(x) = ax^2 + x_0^- - 1 + \frac{x - x_-^0(1 + 2ax_0^-)^{-1}}{2ax_0^-}b + O(b^2). \square$$

2.4 Homoclinic and heteroclinic tangencies of the fixed points

We want to describe the values of the parameters a and b for which there are homoclinic or heteroclinic tangencies of the invariant manifolds of the fixed points. First we give a definition of homoclinic tangency that can be applied to all b.

Definition 2.4.1 *Let $f_{a,b}$ be the Hénon map and let $p_0 = (\bar{x}_0, \bar{y}_0)$ and $p_1 = (\bar{x}_1, \bar{y}_1)$ be two fixed points of $f_{a,b}$. Let $\vec{x}_0(t) = (x_0(t), y_0(t))$ and $\vec{x}_1(t) = (x_1(t), y_1(t))$ be parametrizations of $\mathcal{W}^u(p_0)$ and $\mathcal{W}^s(p_1)$, respectively, such that $f(\vec{x}_i(t)) = \vec{x}_i(\alpha_i t)$, $i = 0, 1$, where α_0 is the eigenvalue associated to $\mathcal{W}^u(p_0)$ and α_1 the one associated to $\mathcal{W}^s(p_1)$. Then we say that $\mathcal{W}^u(p_0)$ and $\mathcal{W}^s(p_0)$ have a homoclinic tangency (if $p_0 = p_1$) or heteroclinic tangency (if $p_0 \neq p_1$) at $\bar{p} \in \mathbb{R}^2$ if:*

a) $\exists t_0, t_1 \in \mathbb{R}$ such that $\vec{x}_0(t_0) = \vec{x}_1(t_1) = \bar{p}$.

b) $\vec{x}_0'(t_0) \wedge \vec{x}_1'(t_0) = 0$.

Remark 2.4.2 *When $b = 0$ we shall take $\vec{x}_1(t) = (t, at^2 + \bar{x}_1 - 1)$.*

Proposition 2.4.3 *Let $b = 0$, and $\bar{p} = (\bar{x}, 0)$ a tangencial "clinic" point. Then there is $n \in \mathbb{N}$ such that $f_a^n(0) = x$.*

Proof:

In this case the condition of tangencial "clinic" point is

a) $\vec{x}_0(t_0) = \vec{x}_1(t_1) = \bar{p} = (\bar{x}, 0)$.

b) $\vec{x}_0'(t_0) \wedge \vec{x}_1'(t_1) = 0$.

As $\vec{x}_0(t) = (x_0(t), 0)$ and $\vec{x}_1(t) = (t, at^2 + \bar{x}_1 - 1)$ then one should have $x_0'(t_0) = 0$ or $2at_1 = 0$. But if $2at_1 = 0$ one obtains $t_1 = 0$ and therefore $\bar{p} = (0, \bar{x}_1 - 1) = (0, 0)$ reaching an absurdity. Hence $x_0'(t_0) = 0$. Let $n_0 = \max\{n \in \mathbb{N} : x_0'(\alpha_0^{-n} t) = 0\}$. Then

$$1 - a x_0^2(\alpha_0^{-(n_0+1)} t_0) = x_0(\alpha_0^{-n_0} t_0).$$

By derivation with respect to t:

$$-2a x_0(\alpha_0^{-(n_0+1)} t_0) x_0'(\alpha_0^{-(n_0+1)} t_0) = \alpha_0 x_0'(\alpha_0^{-n_0} t_0) = 0.$$

As $x_0'(\alpha^{-(n_0+1)} t_0) \neq 0$ we have $x_0(\alpha^{-(n_0+1)} t_0) = 0$ and $f_a^{n_0+1}(x_0(\alpha^{-(n_0+1)} t_0)) = x_0(t_0) = x$. Therefore $n = n_0 + 1$. □

This proposition gives an equivalence between the usual definition of "clinic" tangency for interval maps and 2.4.1

2.4.1 Computation of homoclinic and heteroclinic tangencies for $a \approx 2$ and $b \approx 0$.

Theorem 2.4.4 *There are four countable sets of C^1 functions defined in a neighbourhood of zero: $a_n^{1,1} = a_n^{1,1}(b)$, $a_n^{1,2} = a_n^{1,2}(b)$, $a_n^{2,1} = a_n^{2,1}(b)$, $a_n^{2,2} = a_n^{2,2}(b)$, $n \in \mathbb{N}$, such that*

a) $f_{a_n^{1,1}(b),b}$, $f_{a_n^{1,2}(b),b}$ have a homoclinic tangency between $W^u(p_-)$ and $W^s(p_-)$, and $f_{a_n^{2,1}(b),b}$, $f_{a_n^{2,2}(b),b}$ have a heteroclinic tangency between $W^u(p_+)$ and $W^s(p_-)$ for b small enough.

b) $a_n^{1,1}(0) = a_n^{1,2}(0) = a_n^{2,1}(0) = a_n^{2,2}(0) = 2$, $\dot{a}_n^{1,1}(0) = \dot{a}_n^{2,1}(0) \neq \dot{a}_n^{1,2}(0) = \dot{a}_n^{2,2}(0)$, and $\dot{a}_n^{i,j} = \dot{a}_m^{i,j}$, for i, $j \in \{1, 2\}$ and m, $n \in \mathbb{N}$.

Proof:

The homoclinic and heteroclinic tangencies can be obtained from the system

$$\begin{aligned} h(k_1(x,a,b),a,b) - k_2(x,a,b) &= 0, \\ \frac{\partial k_2}{\partial x}(x,a,b) - \frac{\partial h}{\partial x}(k_1(x,a,b),a,b)\frac{\partial k_1}{\partial x}(x,a,b) &= 0, \end{aligned} \tag{2.10}$$

where $(x, h(x,a,b))$ is a local unstable invariant manifold, and $(k_1(x,a,b), k_2(x,a,b))$ is a local unstable invariant manifold.

For the heteroclinic tangency we consider, following the notation of 2.3.13:

$$h(x,a,b) = h_2(x,a,b) \text{ and } (k_{1n}(x,a,b), k_{2n}(x,a,b)) = f^n_{a,b}(x, g_1(x,a,b)).$$

For the homoclinic one

$$h(x,a,b) = h_2(x,a,b) \text{ and } (k_{1n}(x,a,b), k_{2n}(x,a,b)) = f^n_{a,b}(x, g_2(x,a,b)).$$

Then we define

$$\begin{aligned} F_{1n}(x,a,b) &= h_2(k_{1n}(x,a,b),a,b) - k_{2n}(x,a,b), \\ F_{2n}(x,a,b) &= \frac{\partial k_{2n}}{\partial x}(x,a,b) - \frac{\partial h_2}{\partial x}(k_{1n}(x,a,b)\frac{\partial k_{1n}}{\partial x}(x,a,b). \end{aligned}$$

To have a "clinic" tangency at $(k_{1n}(x,a,b), k_{2n}(x,a,b))$, one should have

$$\begin{aligned} F_{1n}(x,a,b) &= 0, \\ F_{2n}(x,a,b) &= 0. \end{aligned} \tag{2.11}$$

By 2.3.4 the system 2.11 is made of (at least) C^1 functions. We shall apply the implicit function theorem to show the existence of $a^{i,j}_n = a^{i,j}_n(b)$. To this end we need some lemmas.

Lemma 2.4.5 *For all $n \in \mathbb{N}$ there are points $x^m_n \in \mathbb{R}$ $m = 1, \ldots, 2^{n-1}$, such that $(x^m_n, 0) \in \mathcal{W}^u(p_+)$ and $(x^m_n, 0) \in \mathcal{W}^u(p_-)$ for $a = 2$ and $b = 0$, and $F_{1n}(x^m_n, 2, 0) = 0$, $F_{2n}(x^m_n, 2, 0) = 0$.*

Proof:

Let x^m_n be a preimage of 0 under f_2^{n-1}, that is $f_2^{n-1}(x^m_n) = 0$. As f_2 is conjugated to the Chebyshev polynomial P_2 and $P_2 \circ \overset{n}{\cdots} \circ P_2 = P_{2^n}$, there are 2^n points x such that $f_2^n(x) = 0$.

Let $f_{a,b}(x,y) = (f^{1n}_{a,b}(x,y), f^{2n}_{a,b}(x,y))$. Then:

$$k_{1n}(x,a,0) = f^{1n}_{a,0}(x, g_i(x,a,0)) = f^{1n}_{a,0}(x,0) = f^n_a(x),$$

$$k_{2n}(x,a,0) = f^{2n}_{a,0}(x, g_i(x,a,0)) = 0.$$

Hence:

$$F_{1n}(x^m_n, 2, 0) = h_2(f^n_2(x^m_n), 2, 0) = h_2(1,2,0) = 0,$$

because $f_2^n(x_n^m) = f_2(0) = 1$, $h_2(x,a,0) = ax^2 + x_- - 1$ and $x_-(2) = -1$. Furthermore

$$\frac{\partial k_{1n}}{\partial x}(x,a,0) = \frac{\partial f_a^n}{\partial x}(x) = f_a'(f_a^{n-1}(x)) \cdots f_a'(x),$$

and, as $f_2^{n-1}(x_n^m) = 0$, one obtains $\frac{\partial k_{1n}}{\partial x}(x_n^m, 2, 0) = 0$, and from this

$$F_{2n}(x_n^m, 2, 0) = \frac{\partial k_{2n}}{\partial x}(x_n^m, 2, 0) - \frac{\partial h_2}{\partial x}(1,2,0)\frac{\partial k_{1n}}{\partial x}(x_n^m, 2, 0) = 0. \square$$

Lemma 2.4.6 *For all $n > 0$ and all m such that $1 \leq m \leq 2^{n-1}$ one has:*

$$\left| \begin{array}{cc} \frac{\partial F_{1n}}{\partial x} & \frac{\partial F_{1n}}{\partial a} \\ \frac{\partial F_{2n}}{\partial x} & \frac{\partial F_{2n}}{\partial a} \end{array} \right|_{|(x_n^m,2,0)} \neq 0.$$

Proof:

We need the values of the partial derivatives at $(x_n^m, 2, 0)$:

a) $\frac{\partial F_{1n}}{\partial x}(x_n^m, 2, 0) = 0.$

It is enough to use

$$\frac{\partial F_{1n}}{\partial x} = \frac{\partial h_2}{\partial x}(k_{1n}(x,a,b),a,b)\frac{\partial k_{1n}}{\partial x}(x,a,b) - \frac{\partial k_{2n}}{\partial x}(x,a,b) = -F_{2n}(x,a,b).$$

As $F_{2n}(x_n^m, 2, 0) = 0$ then $\frac{\partial F_{1n}}{\partial x}(x_n^m, 2, 0) = 0$.

b) $\frac{\partial F_{1n}}{\partial a}(x_n^m, 2, 0) = 4/3.$

In general for any (x,a,b) :

$$\frac{\partial F_{1n}}{\partial a}(x,a,b) = \frac{\partial h_2}{\partial x}(k_{1n}(x,a,b),a,b)\frac{\partial k_{1n}}{\partial a}(x,a,b) +$$

$$\frac{\partial h_2}{\partial a}(k_{1n}(x,a,b),a,b) - \frac{\partial k_{2n}}{\partial a}(x,a,b).$$

As $k_{1n}(x,a,0) = f_a^n(x)$ and $k_{2n}(x,a,0) = 0$ then

$$\frac{\partial k_{1n}}{\partial a}(x_n^m, 2, 0) = \frac{\partial f_a^n}{\partial a}(x_n^m)_{|a=2} = 0$$

and

$$\frac{\partial k_{2n}}{\partial a}(x_n^m, 2, 0) = 0.$$

The second equality is evident. The first one follows by derivation of $f_a^n(x) = 1 - a(f_a^{n-1}(x))^2$ with respect to a :

$$\frac{\partial f_a^n}{\partial a}(x) = -f_a^{n-1}(x) - 2af_a^{n-1}(x)\frac{\partial f_a^{n-1}}{\partial a}(x).$$

If $a = 2$ and $x = x_n^m$ then

$$\frac{\partial f_a^n}{\partial a}(x_n^m)_{|a=2} = -f_2^{n-1}(x_n^m) - 4f_2^{n-1}(x_n^m)\frac{\partial f_a^{n-1}}{\partial a}(x)_{|a=2} = 0.$$

With this we already have

$$\frac{\partial F_{1n}}{\partial a}(x_n^m, 2, 0) = \frac{\partial h_2}{\partial a}(k_{1n}(x_n^m, 2, 0), 2, 0) = \frac{\partial h_2}{\partial a}(1, 2, 0).$$

When $b = 0$ one obtains $h_2(x, a, 0) = ax^2 + x_- - 1$. Therefore:

$$\frac{\partial h_2}{\partial a} = x^2 + x'_-.$$

As $1 - ax_-^2 = x_-$, by derivation with respect to a we obtain

$$x'_-(a) = -\frac{x_-^2(a)}{1 + 2ax_-(a)},$$

where $x_-(a) = x_-(a, 0)$. Hence $x'_-(2) = \frac{1}{3}$ because $x_-(2) = -1$.

So we have

$$\frac{\partial h_2}{\partial a}(1, 2, 0) = \frac{4}{3} = \frac{\partial F_{1n}}{\partial a}(x_n^m, 2, 0).$$

c)

$$\frac{\partial F_{2n}}{\partial x}(x_n^m, 2, 0) = 16\left(\frac{\partial f_2^{n-1}}{\partial x}(x_n^m)\right)^2.$$

Indeed:

$$\frac{\partial F_{2n}}{\partial x}(x, a, b) = \frac{\partial^2 k_{2n}}{\partial x^2}(x, a, b) - \frac{\partial^2 h_2}{\partial x^2}(k_{1n}(x, a, b), a, b)\left(\frac{\partial k_{1n}}{\partial x}(x, a, b)\right)^2 -$$

$$\frac{\partial h_2}{\partial x}(k_{1n}(x, a, b), a, b)\frac{\partial^2 k_{1n}}{\partial x^2}(x, a, b).$$

First we note that $\frac{\partial^2 k_{2n}}{\partial x^2}(x, a, 0) = 0$, $\frac{\partial k_{1n}}{\partial x}(x_n^m, 2, 0) = 0$ and $\frac{\partial h_2}{\partial x}(1, 2, 0) = 4$ because $\frac{\partial h_2}{\partial x}(x, a, 0) = 2ax$, $k_{2n}(x, a, 0) = 0$ and $f_2^{n-1}(x_n^m) = 0$.

Therefore

$$\frac{\partial F_{2n}}{\partial x}(x_n^m, 2, 0) = -\frac{\partial h_2}{\partial x}(1, 2, 0)\frac{\partial^2 k_{1n}}{\partial x^2}(x_n^m, 2, 0) = -4\frac{\partial^2 k_{1n}}{\partial x^2}(x_n^m, 2, 0).$$

As $f_a^n(x) = 1 - a(f_a^{n-1}(x))^2$ by derivation with respect to x :

$$\frac{\partial f_a^n}{\partial x}(x) = -2af_a^{n-1}(x)\frac{\partial f_a^{n-1}}{\partial x}(x)$$

and

$$\frac{\partial^2 f_a^n}{\partial x^2} = -2a\left(\frac{\partial f_a^{n-1}}{\partial x}(x)\right)^2 - 2af_a^{n-1}(x)\frac{\partial^2 f_a^{n-1}}{\partial x^2}(x).$$

Hence, for $x = x_n^m$ and $a = 2$:

$$\frac{\partial^2 f_2^n}{\partial x^2}(x_n^m) = -4\left(\frac{\partial f_2^{n-1}}{\partial x}(x_n^m)\right)^2,$$

always different from zero because, for $a = 2$ any preimage of 0 is different from zero. So we have

$$\frac{\partial^2 k_{1n}}{\partial x^2}(x_n^m, 2, 0) = \frac{\partial^2 f_2^n}{\partial x^2}(x_n^m) = -4\left(\frac{\partial f_2^{n-1}}{\partial x}(x_n^m)\right)^2.$$

and

$$\frac{\partial F_{2n}}{\partial x}(x_n^m, 2, 0) = 16\left(\frac{\partial f_a^{n-1}}{\partial x}(x_n^m)\right)^2.$$

Finally with this we have that:

$$\left|\begin{array}{cc} \frac{\partial F_{1n}}{\partial x} & \frac{\partial F_{1n}}{\partial a} \\ \frac{\partial F_{2n}}{\partial x} & \frac{\partial F_{2n}}{\partial a} \end{array}\right|_{(x_n^m,2,0)} = -\frac{\partial F_{1n}}{\partial a}(x_n^m, 2, 0)\frac{\partial F_{2n}}{\partial x}(x_n^m, 2, 0) = -\frac{64}{3}\left(\frac{\partial f_a^{n-1}}{\partial x}(x_n^m)\right)^2 \neq 0. \square$$

Last two lemmas allow to apply the implicit function theorem. There are C^1 functions $a_n^{i,j}$, $i = 1, 2$, $n \in \mathbb{N}$ in a neighbourhood of $b = 0$ which satisfy the thesis of the part a) of the theorem.

To obtain the values of $\dot{a}_n^{i,j}(0)$ one should compute $\frac{\partial F_{1n}}{\partial b}(x_n^m, 2, 0)$.

Lemma 2.4.7 *If $f_2^{n-2}(x_n^m) = \frac{1}{\sqrt{2}}$ then*

$$\frac{\partial F_{1n}}{\partial b}(x_n^m, 2, 0) = 2\sqrt{2} - \frac{1}{6},$$

and if

$$f_2^{n-2}(x_n^m) = -\frac{1}{\sqrt{2}}$$

then

$$\frac{\partial F_{1n}}{\partial b}(x_n^m, 2, 0) = -2\sqrt{2} - \frac{1}{6},$$

for $n > 1$. For $n = 1$:

$$\frac{\partial F_{1n}}{\partial b}(x_n^m, 2, 0) = 2\sqrt{2} - \frac{1}{6},$$

if we consider $(0, 0) \in \mathcal{W}^u(p_+)$, and

$$\frac{\partial F_{1n}}{\partial b}(x_n^m, 2, 0) = -2\sqrt{2} - \frac{1}{6},$$

if we consider $(0, 0) \in \mathcal{W}^u(p_-)$, where $x_1^1 = 0$.

Proof:

$$\frac{\partial F_{1n}}{\partial b}(x,a,b) = \frac{\partial h_2}{\partial x}(k_{1n}(x,a,b),a,b)\frac{\partial k_{1n}}{\partial b}(x,a,b) + \frac{\partial h_2}{\partial b}(k_{1n}(x,a,b),a,b) - \frac{\partial k_{2n}}{\partial b}(x,a,b).$$

Let us compute the required partial derivatives.

a)

$$\frac{\partial k_{1n}}{\partial b}(x_n^m,2,0) = \pm\frac{1}{\sqrt{2}} :$$

First we suppose $n > 1$. Then:

$$\begin{aligned} f_{a,b}^{1n}(x,y) &= 1 + f_{a,b}^{2(n-1)}(x,y) - a\left(f_{a,b}^{1(n-1)}(x,y)\right)^2, \\ f_{a,b}^{2n}(x,y) &= b f_{a,b}^{1(n-1)}(x,y). \end{aligned} \tag{2.12}$$

By derivation with respect to b we obtain

$$\frac{\partial f_{a,b}^{1n}}{\partial b}(x,y) = \frac{\partial f_{a,b}^{2(n-1)}}{\partial b}(x,y) - 2a f_{a,b}^{1(n-1)}(x,y)\frac{\partial f_{a,b}^{1(n-1)}}{\partial b}(x,y).$$

Taking into account $f_{a,b}^{2(n-1)} = b f_{a,b}^{1(n-2)}$ and putting $x = x_n^m,\ a = 2,\ b = 0,$ we have

$$\frac{\partial f_{a,b}^{1n}}{\partial b}(x_n^m,0)_{|(a,b)=(2,0)} = \frac{\partial f_{a,b}^{2(n-1)}}{\partial b}(x_n^m,0)_{|(a,b)=(2,0)} = f_{2,0}^{1(n-2)}(x_n^m,0) = \pm\frac{1}{\sqrt{2}}.$$

As, by definition, $k_{1n}(x,a,b) = f_{a,b}^{1n}(x,g_i(x,a,b))$, where $i = 1$ or $i = 2$, according to the case, one has

$$\frac{\partial k_{1n}}{\partial b} = \frac{\partial f_{a,b}^{1n}}{\partial y}(x,g_i(x,a,b))\frac{\partial g_i}{\partial b}(x,a,b) + \frac{\partial f_{a,b}^{1n}}{\partial b}(x,g_i(x,a,b)).$$

For $x = x_n^m,\ a = 2$ and $b = 0$ it follows:

$$\frac{\partial k_{1n}}{\partial b}(x_n^m,2,0) = \frac{\partial f_{2,0}^{1n}}{\partial y}(x_n^m,0)\frac{\partial g_i}{\partial b}(x_n^m,2,0) + \frac{\partial f_{a,b}^{1n}}{\partial b}(x_n^m,0)_{|(a,b)=(2,0)} =$$

$$\frac{\partial f_{2,0}^{1n}}{\partial y}(x_n^m,0)\frac{\partial g_i}{\partial b}(x_n^m,2,0) \pm \frac{1}{\sqrt{2}}.$$

By derivation with respect to y of the first equality in 2.12 we have

$$\frac{\partial f_{a,b}^{1n}}{\partial y}(x,g(x,a,b)) = \frac{\partial f_{a,b}^{2(n-1)}}{\partial y}(x,g_i(x,a,b)) -$$

$$2a f_{a,b}^{1(n-1)}(x,g_i(x,a,b))\frac{\partial f_{a,b}^{1(n-1)}}{\partial y}(x,g_i(x,a,b))$$

and, from the second equality,

$$\frac{\partial f_{a,b}^{2(n-1)}}{\partial y}(x,y) = b\frac{\partial f_{a,b}^{1(n-2)}}{\partial y}(x,y).$$

For $x = x_n^m$, $a = 2$ and $b = 0$ it follows

$$\frac{\partial f_{a,b}^{1n}}{\partial y}(x_n^m, 0)_{|(a,b)=(2,0)} = 0.$$

Hence

$$\frac{\partial k_{1n}}{\partial b}(x_n^m, 2, 0) = \pm\frac{1}{\sqrt{2}}.$$

For $n = 1$:

$$k_{11}(x,a,b) = 1 + g_i(x,a,b) - ax^2, \quad \frac{\partial k_{11}}{\partial b} = \frac{\partial g_i}{\partial b}.$$

Therefore

$$\frac{\partial k_{11}}{\partial b}(0,2,0) = \pm\frac{1}{\sqrt{2}},$$

from proposition 2.3.13, depending on whether $i = 1$ or $i = 2$.

b)

$$\frac{\partial k_{2n}}{\partial b}(x_n^m, 2, 0) = 0 :$$

By definition $k_{2n}(x,a,b) = f_{a,b}^{2n}(x, g_i(x,a,b))$, where $i = 1, 2$. Hence:

$$\frac{\partial k_{22}}{\partial b} = \frac{\partial f_{a,b}^{2n}}{\partial y}(x, g_i(x,a,b))\frac{\partial g_i}{\partial b}(x,a,b) + \frac{\partial f_{a,b}^{2n}}{\partial b}(x, g_i(x,a,b)).$$

As:

$$\frac{\partial f_{a,b}^{2n}}{\partial y}(x,y) = b\frac{\partial f_{a,b}^{1(n-1)}}{\partial y}(x,y),$$

and

$$\frac{\partial f_{a,b}^{2n}}{\partial b} = b\frac{\partial f_{a,b}^{1(n-1)}}{\partial b}(x,y) + f_{a,b}^{1(n-1)}(x,y),$$

for $x = x_n^m$, $a = 2$, and $b = 0$ we have

$$\frac{\partial f_{a,b}^{2n}}{\partial y}(x_n^m, 0)_{|(a,b)=(2,0)} = 0, \quad \text{and} \quad \frac{\partial f_{a,b}^{2n}}{\partial b}(x_n^m, 0)_{|(a,b)=(2,0)} = 0,$$

and therefore

$$\frac{\partial k_{2n}}{\partial b}(x_n^m, 2, 0) = 0.$$

c)

$$\frac{\partial h_2}{\partial b}(1,2,0) = -\frac{1}{6} \ :$$

From proposition 2.3.13 it follows

$$\frac{\partial h_2}{\partial b}(1,2,0) = \frac{x - x_-(1+2ax_-)^{-1}}{2ax_-}\Big|_{(x,a)=(1,2)} = -\frac{1}{6}.$$

d)

$$\frac{\partial h_2}{\partial x}(1,2,0) = 4\,.$$

Indeed:

$$\frac{\partial h_2}{\partial x}(x,a,0) = 2ax,$$

and then $\frac{\partial h_2}{\partial x}(1,2,0) = 4$.

For all this values one has

$$\frac{\partial F_{1n}}{\partial b}(x_n^m,2,0) = \frac{\partial h_2}{\partial x}(1,2,0)\frac{\partial k_{1n}}{\partial b}(x_n^m,2,0) + \frac{\partial h_2}{\partial b}(1,2,0) - \frac{\partial k_{2n}}{\partial b}(x_n^m,2,0) = \pm 2\sqrt{2} - \frac{1}{6}. \ \square$$

Part b) of the theorem follows from the last lemma because

$$\dot{a}_n^{i,j}(0) = -\frac{\partial F_{1n}/\partial b}{\partial F_{1n}/\partial a}(x_n^m,2,0) = \pm\frac{3\sqrt{2}}{2} + \frac{1}{8}.$$

These two values are, approximately, 2.2463204 and -1.9963204. □

Remark 2.4.8

a) In [1] the theorem 4.1 (page 71) is more general than this theorem but it is proved using other techniques. In fact the proof of the smoothness of the homoclinic and heteroclinic bifurcation curves is not given explicitely.

b) Some of the results of this chapter have appeared in [17].

Chapter 3

The Newhouse Phenomenon

The purpose of this chapter is to study the behaviour of a two dimensional dissipative diffeomorphism which is a perturbation of one having a non degenerate homoclinic tangency. It is known ([18], [19], [20], [21], [22]) that in this case periodic sinks of large period can appear near the homoclinic tangency if the perturbation is small enough. Also it is possible to show that such a map can have infinitely many attracting periodic orbits. We want to study the behaviour of the bifurcation of periodic points when we consider a one-parameter family of dissipative diffeomorphisms having a parameter a_0 such that there exists a homoclinic tangency. Moreover we will see the possible different behaviours of the basin of attraction of the sink which appear for parameters near to a_0. In the second part of this chapter we give numerical results that concide with the analytical ones.

3.1 Existence of attracting periodic orbits near a homoclinic tangency.

First we give some definitions concerning a two-dimensional diffeomorphism:

Definition 3.1.1 *Let $f : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, where U is an open set, be a C^r diffeomorphism having a hyperbolic fixed point p. Then we say that this point is dissipative if $|\det Df(p)| < 1$.*

In particular any attracting periodic orbit is dissipative. So this definition has only real sense when p is a saddle point. Obviously there exist saddle points which are not dissipative.

If a map f has a saddle point p it is known that it can be defined its invariant manifolds (see chapter 2). Let $\mathcal{W}^u(p)$ denotes the unstable invariant manifold and $\mathcal{W}^s(p)$ the stable one. It is known that $\mathcal{W}^s(p)$ and $\mathcal{W}^u(p)$ are smooth manifolds but are not, in general, submanifolds of $\mathbb{R}^2$ ([16]). If this manifolds have an intersection point p we call this point a homoclinic point. If this intersection is transversal (respectively tangencial) we say that p has a transversal (resp. tangencial) homoclinic point of its invariant manifolds.

It is possible to find a parametric representation of the invariant manifolds:

Proposition 3.1.2 *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a C^r diffeomorphism having a fixed point $p_0 = (x_0, y_0)$. Let α_1 and α_2 be the eigenvalues of $Df(p_0)$ such that $|\alpha_1| < 1$ and $|\alpha_2| > 1$. Then the invariant manifolds of p_0 are characterized as the images of two immersions: $\gamma_i : \mathbb{R} \longrightarrow \mathbb{R}^2$, $i = 1, 2$ such that:*

$$(f \circ \gamma_i) = \gamma_i(\alpha_i t) \quad , \tag{3.1}$$

and γ_i are of class C^r .

Proof:

By the stable manifold theorem we know that there exists a function $g : U \longrightarrow \mathbb{R}$ of class C^r, defined in a neighbourhood U of x_0 , such that $(x, g(x))$ or $(g(y), y)$ represents locally the stable invariant manifold of p_0. Suppose that $(x, g(x))$ represents the stable invariant manifold. In the other case we proceed in a similar way:

We look for a function $x : V \longrightarrow \mathbb{R}$ where $V \subset U$ is an open set, such that if $f = (f_1, f_2)$ and $\varphi(x) = f_2(x, g(x))$ then:

$$\varphi(x(s)) = x(\alpha_1 s) \tag{3.2}$$

It holds if and only if $x^{-1} \circ \varphi \circ x(s) = \alpha_1 s$. Then, by the Sternberg theorem ([14]), there exists a function x which verifies 3.2 and it is of class C^r. Thus the function γ_1 can be defined as:

$$\gamma_1(s) = (x(s), g(x(s))) \ .$$

This function exists for all $s \in \mathbb{R}$ because we can use 3.1 for extending $\gamma_1(s)$ to the whole $\mathbb{R}$.

For the unstable invariant manifold we proceed in the same way. □

This type of parametric representation can be used to prove the existence of homoclinic intersections in the Hénon map ([23]).

Now we want to study the behaviour of a diffeomorphism having homoclinic intersections. To do this we define first, in general, transversal and tangencial intersections of submanifolds of a given manifold:

Let M be a two dimensional manifold of class C^k:

Definition 3.1.3 *Let N_1 and N_2 be two submanifolds of M of dimension 1. We say that N_1 and N_2 have a transversal intersection in $x \in N_1 \cap N_2$ if $T_xN_1 + T_xN_2 = T_xM$.*

Definition 3.1.4 *Let N_1 and N_2 be two submanifolds of M of dimension 1. Then N_1 and N_2 have a tangency of order n if and only if:*

a) $x \in N_1 \cap N_2$.

b) $\dim(T_xN_1 + T_xN_2) = 1$.

c) There exist coordinates (x_1, x_2) in a neighbourhood of x such that N_1 and N_2 have the representation:

$$N_1 = \{(x_1, x_2) \quad : \quad x_2 = 0\}$$

$$N_2 = \{(x_1, x_2) \quad : \quad x_2 = \varphi(x_1)\} \ ,$$

where φ is a function such that $\varphi(0) = \varphi'(0) = ... = \varphi^{(n)}(0) = 0$ and $\varphi^{(n+1)}(0) \neq 0$.

This definitions can be applied to the invariant manifolds. So we can have transversal or tangencial homoclinic points with a tangency of order n. If $n = 1$ we say that the tangency is non-degenerate.

The next theorem establishes when a map having a hyperbolic fixed point is locally C^r conjugate to its linear part ([14]).

Theorem 3.1.5 *Let $f : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a diffeomorphism, defined on the open set U, having a hyperbolic fixed point $p \in U$. Then for all $n \geq 0$ there exists $N = N(n)$ such that if f is C^N and the point p has not resonances of order i for $2 \leq i \leq N$ then $f_{|V}$ is C^n conjugated to $Df(p)$ in a neighbourhood V of p. In particular $N(1) = 2$.*

The following proposition is a generalization of proposition 1.1.3 in chapter 1:

Proposition 3.1.6 *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a diffeomorphism having a saddle fixed point $p \in \mathbb{R}^2$, and suppose that there is a neighbourhood V of p such that $f_{|V}$ is C^n conjugated to $Df(p)$. Then there exists a C^n map $\vec{x} : V_1 \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined in a neighbourhood V_1 of the set $A = \{(t_1, t_2) \quad : \quad t_1 t_2 = 0\}$ such that:*

a) $\vec{x}(\alpha_1 t_1, \alpha_2 t_2) = f(\vec{x}(t_1, t_2))$

b) $\vec{x}(0, 0) = p$

where α_1 and α_2 are the eigenvalues of $Df(p)$.

Proof:

By hypothesis we know that there exists a map $g : \tilde{V}_1 \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\tilde{V}_1$ being a neighbourhood of $(0,0)$, such that: $g^{-1} \circ f \circ g = Df(p)$. As $\alpha_1 \neq \alpha_2$ we can conjugate f to the linear map

$$\bar{f} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} ,$$

with a C^n conjugation. We call this conjugation $\vec{x}$. Then $\vec{x} : \tilde{V}_1 \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ verifies a) and b). To extend this map we use a). Then we can define the map $\vec{x}$ in a neighbourhood $V_1 = \cup_{i \in \mathbb{N}} U_{1,i} \cup_{i \in \mathbb{N}} U_{2,i}$ where $U_{1,1} = U_{2,1} = \tilde{V}_1$ and $U_{1,i} = \bar{f}(U_{1,i-1})$, $U_{2,i} = \bar{f}^{-1}(U_{2,i-1})$ if $i \geq 2$. So $\vec{x}$ is of class C^n and is defined in a neighbourhood V_1 of A .□

By applying the theorem 3.1.5 we have:

Corollary 3.1.7 *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a diffeomorphism such that there exists a saddle fixed point $p \in U$. Then if f is of class C^N (N=N(n) of the theorem 3.1.5) and there are not resonances of orders $2 \leq i \leq N$, there exists a C^n map $\vec{x} : V_1 \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined on a neighbourhood V_1 of the set $A = \{(t_1,t_2) \; : \; t_1 t_2 = 0\}$ such that:*

a) $\vec{x}(\alpha_1 t_1, \alpha_2 t_2) = f(\vec{x}(t_1,t_2))$,

b) $\vec{x}(0,0) = p$,

where α_1 and α_2 are the eigenvalues of $Df(p)$.

Remark 3.1.8 *a) When $t_1 = 0$ or $t_2 = 0$, $\vec{x}(0,t_2)$ and $\vec{x}(t_1,0)$ represent the invariant manifolds of p. This parametrization is the same of the proposition 3.1.2.*

b) The map $\vec{x}$ is not unique. For example we can define $\vec{y}(t_1,t_2) = \vec{x}(\lambda t_1, \lambda t_2)$, with $\lambda \neq 0$. Then $\vec{y}$ verifies the thesis of the proposition 3.1.6.

By using the results of the proposition 3.1.6 it is possible to prove directly the following proposition, which is a corollary of the Smale horseshoe theorem ([18],[22]):

Proposition 3.1.9 *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a C^2 diffeomorphism having a saddle fixed point p, such that its invariant manifolds have a transversal homoclinic point $\bar{p}$. Then this point is in the clousure of the hyperbolic periodic points of f.*

Proof:

We want to find points $(x,y) \in \mathbb{R}^2$ such that $f^n(x,y) = (x,y)$. By corollary 3.1.7, there exists a C^1 map $\vec{x} : V \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $\vec{x}(0,0) = p$ and $\vec{x}(\alpha_1 t_1, \alpha_2 t_2) = f(\vec{x}(t_1,t_2))$; α_1 , α_2 being the eigenvalues of $Df(p)$ such that $|\alpha_1| < 1$ and $|\alpha_2| > 1$. We can always suppose that $\alpha_1 > 0$ and $\alpha_2 > 0$ since if not, we can consider f^2 instead of f. Then let $s_1 = \alpha_1^n$ and $s_2 = \alpha_2^{-n}$. The equation $f^n(x,y) = (x,y)$ is equivalent to $\vec{x}(t_1, \alpha_2^{-n} t_2) = \vec{x}(\alpha_1^n t_1, t_2)$ or

$$\vec{x}(t_1, s_2 t_2) = \vec{x}(s_1 t_1, t_2) \ . \tag{3.3}$$

Observe that, when $s_1 = s_2 = 0$, we have:

$$\vec{x}(t_1, 0) = \vec{x}(0, t_2) \ ,$$

so $t_1(0,0) = \bar{t_1}$ and $\vec{x}(\bar{t_1}, 0) = \vec{x}(0, \bar{t_2}) = \bar{p}$ is the transversal homoclinic point. Moreover, if $\vec{x}(t_1, t_2) = (x(t_1,t_2), y(t_1,t_2))$:

$$\begin{vmatrix} D_1 x(\bar{t}_1, 0) & -D_2 x(0, \bar{t}_2) \\ D_1 y(\bar{t}_1, 0) & -D_2 y(0, \bar{t}_2) \end{vmatrix} \neq 0 \ ,$$

since the intersection of the invariant manifolds is transversal. Therefore, we can apply the implicit function theorem: there exist C^1 maps $t_1 = t_1(s_1, s_2)$ and $t_2 = t_2(s_1, s_2)$ such that $t_1(0,0) = \bar{t}_1$, $t_2(0,0) = \bar{t}_2$ and, t_1 and t_2 verifies the equation 3.3. If we recall that $s_1 = \alpha_1^n$ and $s_2 = \alpha_2^{-n}$, we see that $\vec{x}(t_1(\alpha_1^n, \alpha_2^{-n}), \alpha_2^{-n} t_2(\alpha_1^n, \alpha_2^{-n})) = p_n$ is an n-periodic point. Moreover, if $n \to \infty$ then $\alpha_1^n \to 0$ and $\alpha_2^{-n} \to 0$. Therefore $p_n \to \vec{x}(\bar{t}_1, 0) = \bar{p}$. Thus we have proved that the point $\bar{p}$ is in the clousure of the periodic points of f.

It remains to see that this periodic points are hyperbolic if n is large enough (in fact they are saddle points).

It is enough to show that $|\mathrm{tr}\, Df^n(p_n)| \neq |1 + \det Df^n(p_n)|$. Otherwise one eigenvalue is 1 or -1. To compute $Df^n(p_n)$ we consider the following equality:

$$f^n(\vec{x}(t_1, \alpha_2^{-n} t_2)) = \vec{x}(\alpha_1^n t_1, t_2) \ .$$

Then if we compute the differential, we obtain:

$$Df^n(\vec{x}(t_1, \alpha_2^{-n} t_2)) D\vec{x}(t_1, \alpha_2^{-n} t_2) \begin{pmatrix} 1 & 0 \\ 0 & \alpha_2^{-n} \end{pmatrix} =$$

$$D\vec{x}(\alpha_1^n t_1, t_2) \begin{pmatrix} \alpha_1^n & 0 \\ 0 & 1 \end{pmatrix} \ , \tag{3.4}$$

Therefore:

$$\det Df^n(p_n) = \det Df^n(\vec{x}(t_1, \alpha_2^{-n} t_2)) = \alpha_1^n \alpha_2^n \frac{\det D\vec{x}(\alpha_1^n t_1, t_2)}{\det D\vec{x}(t_1, \alpha_2^{-n} t_2)} \ .$$

We can always suppose that $\det D\vec{x}(t_1, \alpha_2^{-n} t_2)) \neq 0$ because we can take the homoclinic point $\bar{p}$ such that it is in a neighbourhood of p, where the map $\vec{x}$ is invertible.

By using 3.4 we obtain:

$$\operatorname{tr} Df^n(p_n) = \operatorname{tr} Df^n(\vec{x}(t_1, \alpha_2^{-n} t_2)) = \frac{1}{\det D\vec{x}(t_1, \alpha_2^{-n} t_2)}$$

$$(\alpha_1^n D_1 x(\alpha_1^n t_1, t_2) D_2 y(t_1, \alpha_2^{-n} t_2) - \alpha_2^n D_1 y(t_1, \alpha_2^{-n} t_2) D_2 x(\alpha_1^n t_1, t_2) -$$

$$\alpha_1^n D_1 y(\alpha_1^n t_1, t_2) D_2 x(t_1, \alpha_2^{-n} t_2) + \alpha_2^n D_2 y(\alpha_1^n t_1, t_2) D_1 x(t_1, \alpha_2^{-n} t_2)) \ .$$

Then we have:

$$\lim_{n\to\infty} \alpha_2^{-n}(|\operatorname{tr} Df^n(p_n)| - |1 + \det Df^n(p_n)|) =$$

$$(-D_2 x(0, \bar{t}_2) D_1 y(\bar{t}_1, 0) + D_2 y(0, \bar{t}_2) D_1 x(\bar{t}_1, 0))(\det D\vec{x}(\bar{t}_1, 0))^{-1} \neq 0 \ ,$$

since the homoclinic intersection is transversal. So we have seen that the n-periodic points p_n are hyperbolic and $p_n \to \bar{p}$ when $n \to \infty$.□

Remark 3.1.10 *The points p_n have eigenvalues λ_1 , λ_2 such that $|\lambda_1| < 1$ and $|\lambda_2| > 1$. So p_n are n-periodic saddle points. In fact:*

$$\lambda_1 = \frac{\det D\vec{x}(0, \bar{t}_2)}{D_2 y(0, \bar{t}_2) D_1 x(\bar{t}_1, 0) - D_1 y(\bar{t}_1, 0) D_2 x(0, \bar{t}_2)} \alpha_1^n + h.o.t.$$

and

$$\lambda_2 = \frac{D_2 y(0, \bar{t}_2) D_1 x(\bar{t}_1, 0) - D_1 y(\bar{t}_1, 0) D_2 x(0, \bar{t}_2)}{\det D\vec{x}(\bar{t}_1, 0)} \alpha_2^n + h.o.t. \ .$$

The next step is to prove the Newhouse phenomenon. For this, let $\{f_a\}_{a\in I}$, where I is an interval, be a two dimensional smooth one-parameter family of diffeomorphisms, defined in a open set U of $\mathbb{R}^2$, having a hyperbolic fixed point p_0, which is a saddle, for $a = a_0$. In that case, by the implicit function theorem, there exists a map $p = p(a)$, defined on a neighbourhood I' of$a = a_0$, of saddle fixed points of f_a such that $p(a_0) = p_0$. Then we have the following proposition:

Proposition 3.1.11 *The family $\{f_a\}_{a\in I}$ is conjugated to a family $\{g_a\}_{a\in I}$, with a smooth conjugation, such that the fixed point of g_a, corresponding to p_0, is $(0,0)$, and its stable invariant manifold is locally the straight line $y = 0$, in coordinates (x, y).*

Proof:

By using a translation we can move the fixed point to the origin. With a linear change of coordinates it is possible to have the differential of the map in the fixed point in diagonal form. So the family $\{f_a\}_{a\in I}$ is conjugated to the family $\{\bar{g}_a\}_{a\in I}$ such that $\bar{g}_a(0,0) = (0,0)\ \forall a \in I$ and

$$D\bar{g}_a(0,0) = \begin{pmatrix} \alpha_1(a) & 0 \\ 0 & \alpha_2(a) \end{pmatrix} \ .$$

Suppose that $|\alpha_1(a)| < 1$ and $|\alpha_2(a)| > 1$ and consider the following change of coordinates:

$$h_a(x,y) = (x, y - \bar{h}(x,a)) \quad ,$$

where $y = \bar{h}(x,a)$ is the local expression of the stable invariant manifold of $(0,0)$. This map is a diffeomorphism and it is differentiable with respect to a of the same class that $\{f_a\}_{a\in I}$ (see [16]). Then we can extend the diffeomorphism h_a to the whole neighbourhood where f_a is defined. Moreover $\{f_a\}_{a\in I}$ is conjugated to $\{g_a\}_{a\in I}$ where $g_a = h_a \circ \bar{g}_a \circ h_a^{-1}$ and the stable invariant manifold of $(0,0)$ for g_a is, locally, $y = 0$.□

Remark 3.1.12 *In the previous proposition we consider the interval I smaller if it is necessary.*

Now we will define a generic unfolding of a homoclinic tangency:

Definition 3.1.13 *Let $\{f_a\}_{a\in I}$ be a smooth one-parameter family of C^r, $(r \geq 2)$ diffeomorphisms in the plane, having a fixed point p_0 for $a = a_0$ such that the invariant manifolds of p_0 have a non-degenerate homoclinic tangency. Let $\{g_a\}_{a\in I}$ be the family of proposition 3.1.11 conjugated to $\{f_a\}_{a\in I}$. Suppose that $\vec{x}(t,a) = (x(t,a), y(t,a))$ is a parametrization of the unstable invariant manifold of $(0,0)$ for g_a. Then we say that the homoclinic tangency unfolds generically with $\{f_a\}_{a\in I}$ in $a = a_0$ if $D_2y(t_1,a_0) \neq 0$, where $(x(t_1,a_0), y(t_1,a_0))$ is a tangencial homoclinic point.*

Proposition 3.1.14 *Let $\{f_a\}_{a\in I}$ be a smooth family of class C^r $(r \geq 2)$ like the one of the previous definition. Let $\bar{p}$ denote a tangencial homoclinic point and $p = p(a)$ the fixed point of f_a such that $p(a_0) = a_0$. Then there exists a neighbourhood U of $\bar{p}$ such that: for $a > a_0$ (or for $a < a_0$), there exist two transversal homoclinic points of p in U, and for $a < a_0$ (respectively for $a > a_0$). there are no homoclinic points of p in U.*

Proof:

We consider a family $\{g_a\}_{a\in I}$ conjugated to $\{f_a\}_{a\in I}$ such that the stable invariant manifold of the fixed point $(0,0)$ is $y = 0$. If we have a non degenerate homoclinic tangency which unfolds generically with $\{g_a\}_{a\in I}$ for $a = a_0$, then: $y(t_1,a_0) = 0$, $D_1y(t_1,a_0) = 0$, $D_{11}y(t_1,a_0) \neq 0$ and $D_2y(t_1,a_0) \neq 0$, where $(x(t,a), y(t,a))$ is a parametric representation of the invariant manifold $\mathcal{W}^u((0,0))$ and $(x(t_1,a_0), y(t_1,a_0))$ is the point of homoclinic tangency. Then we look for solutions of the equation $y(t,a) = 0$. By the Implicit Function Theorem there exists a map $a = a(t)$ such that $a_0 = a(t_1)$ and $y(t,a(t)) = 0$ in a neighbourhood of (t_1,a_0). Moreover

$$a(t) = a_0 - \frac{1}{2}\frac{D_{11}y(t_1,a_0)}{D_2y(t_1,a_0)}(t-t_1)^2 + o((t-t_1)^2).$$

So $a(t) - a_0 > 0$ or $a(t) - a_0 < 0$ for $|t - t_1|$ small enough. It means that the homoclinic points exist only for $a > a_0$ if $D_{11}y(t_1, a_0)D_2y(t_1, a_0) < 0$ or for $a < a_0$ if the product is negative.

It remains to prove that the points are transversal. This is equivalent to see that $D_1y(t, a(t)) \neq 0$. We have:

$$D_1y(t, a(t)) = -D_2y(t, a(t))a(t) = D_{11}y(t_1, a_0)(t - t_1) + h.o.t.,$$

where *h.o.t.* means higher order terms. This is different from zero if $|t - t_1|$ is small enough. □

We consider now a smooth (C^r) one-parameter family of planar diffeomorphisms $\{f_a\}_{a\in V}$ such that $f_a : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $V \subset \mathbb{R}$ is an open neighbourhood of $a_0 \in V$. Suppose that there exists a hyperbolic fixed point $p_0 = p(a_0)$ of f_{a_0}. Then, by the implicit function theorem, there exists a hyperbolic fixed point $p(a)$ of f_a, for $|a - a_0|$ small, such that $p(a_0) = p_0$ and $p(a)$ is a C^r map. We can always take the neighbourhood V such that there exists a fixed point for all $a \in V$. We make the following assumption on the family $\{f_a\}_{a\in V}$:

(A): There exists a map $\vec{x} : U_1 \times V \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ where U_1 is an open neighbourhood of $\{(t_1, t_2) \ : \ t_1t_2 = 0\}$, such that:

a) $\vec{x}(\alpha_1 t_1, \alpha_2 t_2, a) = f_a(\vec{x}(t_1, t_2, a))$, where $\alpha_1 = \alpha_1(a)$ and $\alpha_2 = \alpha_2(a)$ are the eigenvalues of $Df_a(p_a)$).

b) $\vec{x}$ is of class C^s, $s \geq 2$.

c) $\vec{x}$ is, in a neighbourhood of $p(a)$, a local C^s conjugation of f_a with its diagonalized linear part.

By using the so-called Hartmann-Grobman theorem (see for instance [24]) and the proposition 3.1.6, it is possible to prove that such a map exists and it is continuos. Also, if we use the theorem 3.1.5, we see that $\vec{x}$ can be chosen of class C^s under some generic assumptions. The differentiability of $\vec{x}$ with respect to a is a consequence of the theorem 26 of [25].

Then we will prove the so-called Newhouse phenomenon:

Theorem 3.1.15 *Let $\{f_a\}_{a\in V}$ be a smooth family of planar diffeomorphisms such that $f_a : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is of class C^r , $r \geq 3$, for all $a \in V$, and V is an open neighbourhood of $a_0 \in V$. Suppose that:*

a) For $a = a_0$, f_{a_0} has a dissipative saddle fixed point p_0 with a non-degenerate homoclinic tangency of its invariant manifolds, which unfolds generically with $\{f_a\}_{a\in V}$.

b) The family $\{f_a\}_{a\in V}$ verifies the assumption (A).

Then, for n large enough, there exist parameters a_n^+ and a_n^- such that:

1) $f_{a_n^+}$ has an n-periodic saddle-node point p_n^+ which unfolds generically with $\{f_a\}_{a\in V}$.

2) $f_{a_n^-}$ has an n-periodic flip point p_n^- which unfolds generically with $\{f_a\}_{a\in V}$.

3) $\lim_{n\to\infty} a_n^+ = \lim_{n\to\infty} a_n^- = a_0$ and

$$\lim_{n\to\infty} \frac{a_n^+ - a_0}{a_{n+1}^+ - a_0} = \lim_{n\to\infty} \frac{a_n^- - a_0}{a_{n+1}^- - a_0} = \bar{\alpha_2} = \alpha_2(a_0) \ ,$$

where $\bar{\alpha_1}$ and $\bar{\alpha_2}$ are the eigenvalues of $Df_{a_0}(p_0)$ such that $|\bar{\alpha_1}| < 1$ and $|\bar{\alpha_2}| > 1$.

4) The parameter $a = a_0$ is in the closure of the set of parameters for which there exist attracting periodic orbits.

Proof:

For our proof we need α_1 , $\alpha_2 > 0$. If not, we will consider f_a^2 instead of f_a. However if any of the eigenvalues is negative, the bahaviour of the bifurcation parameters is slightly different (see remark at the end).

First we observe that, by using the implicit function theorem, there exists a saddle fixed point $p(a)$ of f_a, for $|a - a_0|$ small enough, such that $p(a_0) = p_0$. To prove the theorem we will consider first a suitable change of coordinates in order to simplify the computations:

Lemma 3.1.16 *The family $\{f_a\}_{a\in J}$, where J is a suitable open interval containing a_0, and $J \subset V$, is conjugated via a change of coordinates and parameter, to the family $\{g_\epsilon\}_{\epsilon\in I}$, $I = (-r, r)$, $r > 0$, such that:*

a) $\epsilon = a - a_0$.

b) $g_\epsilon(0,0) = (0,0)$ for all $\epsilon \in I$.

c) $g_\epsilon(x,y) = (\alpha_1 x, \alpha_2 y)$ in a neighbourhood U of $(0,0)$, where $\alpha_1 = \alpha_1(\epsilon)$ and $\alpha_2 = \alpha_2(\epsilon)$ are the eigenvalues of $Dg_\epsilon(0,0)$ such that $|\alpha_1| < 1$ and $|\alpha_2| > 1$ (and therefore also of $Df_a(p(a))$ for $a = a_0 + \epsilon$).

d) The point of homoclinic tangency is $(1,0)$ for $\epsilon = 0$.

e) There exists a smooth map $\vec{x}_1(t_1,t_2,\epsilon) = (x_1(t_1,t_2,\epsilon), y_1(t_1,t_2,\epsilon))$ *such that*

$$\vec{x}_{1|U} = Id,\ g_\epsilon(\vec{x}_1(t_1,t_2,\epsilon)) = \vec{x}_1(\alpha_1 t_1, \alpha_2 t_2, \epsilon),\ x_1(0,1,\epsilon) = 1,$$

$$y_1(0,1,0) = 0,\ D_2 y_1(0,1,\epsilon) = 0,\ D_{22} y_1(0,1,0) \neq 0,\ D_3 y(0,1,0) \neq 0$$

for $\epsilon \in I$.

Proof:

If we move $p(a)$ to the origin and use the local smooth conjugation of the map f_a to its diagonalized linear part, we can suppose that, for δ small, the family $\{f_a\}_{a\in V}$ verifies

a) $f_a(0,0) = (0,0)$ for all $a \in (a_0 - \delta, a_0 + \delta)$.

b) $f_a(x,y) = (\alpha_1 x, \alpha_2 y)$ in a neighbourhood $\bar{U}$ of $(0,0)$ and for all $a \in (a_0 - \delta, a_0 + \delta)$.

c) The point of homoclinic tangency is $\bar{p} = (\bar{t}_1, 0)$ and it is into U.

We note that it is possible to assume c) because the point of homoclinic tangency $\bar{p}$ can be taken as close as we want of the fixed point p_0.

Let $\epsilon = a - a_0$ and $f(x,y,\epsilon) = f_{a_0+\epsilon}(x,y)$. By the assumption (A) we know that there exists a map $\vec{x} = \vec{x}(t_1,t_2,\epsilon) = (x(t_1,t_2,\epsilon), y(t_1,t_2,\epsilon))$ of class C^s, $s \geq 3$, such that:

$$\vec{x}(\alpha_1(\epsilon)t_1, \alpha_2(\epsilon)t_2, \epsilon) = f(\vec{x}(t_1,t_2,\epsilon),\epsilon)$$

and $\vec{x}_{|U} = Id$ for all $\epsilon \in (-\delta,\delta)$.

As the map has a homoclinic tangency for $\epsilon = 0$, (then $a = a_0$), there exists a parameter $\bar{t}_2 \in \mathbb{R}$ such that $\vec{x}(0,\bar{t}_2,0) = (\bar{t}_1, 0)$, $D_2 y(0,\bar{t}_2,0) = 0$ and $D_{22} y(0,\bar{t}_2,0) \neq 0$ since the tangency is non-degenerate. Then there exists a map $\tilde{t}_2 = \tilde{t}_2(\epsilon)$, for ϵ small enough, such that $\tilde{t}_2(0) = \bar{t}_2$ and $D_2 y(0,\tilde{t}_2(\epsilon),\epsilon) = 0$. Let

$$g_\epsilon(x,y) = (\lambda_1^{-1} f_1(\lambda_1 x, \lambda_2 y, \epsilon), \lambda_2^{-1} f_2(\lambda_1 x, \lambda_2 y, \epsilon))$$

be a map conjugated to $f(x,y,\epsilon) = (f_1(x,y,\epsilon), f_2(x,y,\epsilon))$. Then

$$\vec{x}_1(t_1,t_2,\epsilon) = (\lambda_1^{-1} x(\lambda_1 t_1, \lambda_2 t_2, \epsilon), \lambda_2^{-1} x(\lambda_1 t_1, \lambda_2 t_2, \epsilon))$$

satisfies $g_\epsilon(\vec{x}_1(t_1,t_2,\epsilon)) = \vec{x}_1(\alpha_1 t_1, \alpha_2 t_2, \epsilon)$ and $\vec{x}_{1|U_1} = Id$, where U_1 is a neighbourhood of $(0,0)$. Let $\lambda_1 = x(0,\tilde{t}_2(\epsilon),\epsilon)$. Then for g_ϵ the point of homoclinic tangency is $(1,0)$, for $\epsilon = 0$, and $x_1(0,1,\epsilon) = 1$, $D_2 y_1(0,1,\epsilon) = 0$, $D_{22} y(0,1,0) \neq 0$, $y_1(0,1,0) = 0$ and $D_3 y_1(0,1,0) \neq 0$. So the lemma is proved.□

Then we will study the maps g_ϵ instead of $f_{a_0+\epsilon}$. A possible behaviour of the family $\{g_\epsilon\}_{\epsilon\in I}$ is in the figure 3.1. We will denote again the map g_ϵ as f_ϵ and the map $\vec{x}_1$ as

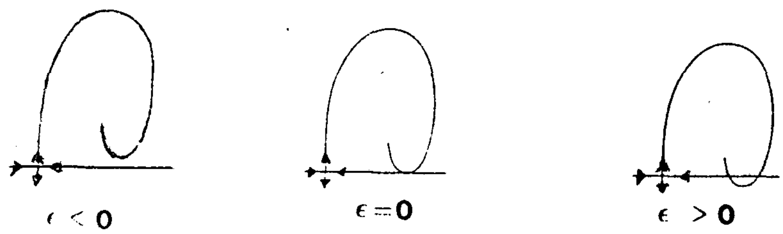


Figure 3.1: A typical homoclinic tangency unfolding generically.

$\vec{x} = (x, y)$. Then we want to find parameters $a_n^+ = a_0 + \epsilon_n^+$, $a_n^- = a_0 + \epsilon_n^-$. These parameters satisfy, respectively, the systems of equations:

$$\left.\begin{array}{rcl} f^n(x,y,\epsilon) & = & (x,y) \\ \mathrm{tr}\, Df_\epsilon^n(x,y) & = & 1 + \det Df_\epsilon^n(x,y) \end{array}\right\} \tag{3.5}$$

$$\left.\begin{array}{rcl} f^n(x,y,\epsilon) & = & (x,y) \\ \mathrm{tr}\, Df_\epsilon^n(x,y) & = & -1 - \det Df_\epsilon^n(x,y) \end{array}\right\} \tag{3.6}$$

where $f^2(x,y,\epsilon) = f(f(x,y,\epsilon),\epsilon)$ and $f^n(x,y,\epsilon) = f^{n-1}(f(x,y,\epsilon),\epsilon)$.

Lemma 3.1.17 *The systems 3.5 and 3.6 are equivalent to the systems:*

$$\left.\begin{array}{rcl} \vec{F}(\alpha_1^n, \alpha_2^{-n}, t_1, t_2, \epsilon) & = & 0 \\ F_3^+(\alpha_1^n, \alpha_2^{-n}, t_1, t_2, \epsilon) & = & 0 \end{array}\right\} \tag{3.7}$$

and

$$\left.\begin{array}{rcl} \vec{F}(\alpha_1^n, \alpha_2^{-n}, t_1, t_2, \epsilon) & = & 0 \\ F_3^-(\alpha_1^n, \alpha_2^{-n}, t_1, t_2, \epsilon) & = & 0 \end{array}\right\} \tag{3.8}$$

where

$$\vec{F}(\alpha_1^n, \alpha_2^{-n}, t_1, t_2, \epsilon) = (t_1, \alpha_2^{-n} t_2) - \vec{x}(\alpha_1^n t_1, t_2, \epsilon)$$

and

$$F_3^\pm(\alpha_1^n, \alpha_2^{-n}, t_1, t_2, \epsilon) = D_2 y(\alpha_1^n t_1, t_2, \epsilon) + \alpha_1^n \alpha_2^{-n} D_1 x(\alpha_1^n t_1, t_2, \epsilon) \mp \alpha_2^{-n} \mp$$
$$\alpha_1^n (D_1 x(\alpha_1^n t_1, t_2, \epsilon) D_2 y(\alpha_1^n t_1, t_2, \epsilon) - D_1 y(\alpha_1^n t_1, t_2, \epsilon) D_2 x(\alpha_1^n t_1, t_2, \epsilon)) = 0 \ .$$

Proof:

As we have seen in proposition 3.1.9, the equation $f^n(x,y,\epsilon) = (x,y)$ is equivalent to

$$\vec{x}(t_1, \alpha_2^{-n} t_2, \epsilon) - \vec{x}(\alpha_1^n t_1, t_2, \epsilon) = 0.$$

Moreover, in this case $\vec{x}(t_1, \alpha_2^{-n} t_2, \epsilon) = (t_1, \alpha_2^{-n} t_2)$. So we have obtained $\vec{F}$. To compute $\mathrm{tr}\, Df_\epsilon^n(x,y)$ and $\det Df_\epsilon^n(x,y)$ we observe that:

$$f^n(t_1, t_2, \epsilon) = f^n(\vec{x}(t_1, t_2, \epsilon, \epsilon) = \vec{x}(\alpha_1^n t_1, \alpha_2^n t_2, \epsilon)$$

if t_1 and t_2 are small enough. Hence:

$$Df_\epsilon^n(t_1, \alpha_2^{-n} t_2) = \begin{pmatrix} \alpha_1^n D_1 x(\alpha_1^n t_1, t_2, \epsilon) & \alpha_2^n D_2 x(\alpha_1^n t_1, t_2, \epsilon) \\ \alpha_1^n D_1 y(\alpha_1^n t_1, t_2, \epsilon) & \alpha_2^n D_2 y(\alpha_1^n t_1, t_2, \epsilon) \end{pmatrix} .$$

From this, we obtain that:

$$F_3^{\pm} = \alpha_2^{-n}(\mp 1 \mp \det Df_\epsilon^n(t_1, \alpha_2^{-n} t_2) + \operatorname{tr} Df_\epsilon^n(t_1, \alpha_2^{-n} t_2)) \, .\square$$

Then, in the systems 3.7 and 3.8, we put $\alpha_1^n = s_1$ and $\alpha_2^{-n} = s_2$. We have:

$$\left.\begin{array}{rcl} F_1(s_1, s_2, t_1, t_2, \epsilon) & = & 0 \\ F_2(s_1, s_2, t_1, t_2, \epsilon) & = & 0 \\ F_3^+(s_1, s_2, t_1, t_2, \epsilon) & = & 0 \end{array}\right\} \tag{3.9}$$

and

$$\left.\begin{array}{rcl} F_1(s_1, s_2, t_1, t_2, \epsilon) & = & 0 \\ F_2(s_1, s_2, t_1, t_2, \epsilon) & = & 0 \\ F_3^-(s_1, s_2, t_1, t_2, \epsilon) & = & 0 \end{array}\right\} \tag{3.10}$$

where $(F_1, F_2) = \vec{F}$ and F_1, F_2, $F_3^{\pm}$ are C^s maps with $s \geq 2$. When $s_1 = s_2 = \epsilon = 0$ one has:

$$\vec{F}(0,0,t_1,t_2,0) = (t_1, 0) - \vec{x}(0, t_2, 0) \ ,$$

$$F_3^{\pm}(0,0,t_1,t_2,0) = D_2 y(0, t_2, 0) \ .$$

By hypothesis, f_0 has a homoclinic point $\bar{p} = (1,0)$ and it is a tangencial one. This means that, for $t_1 = 1$ and $t_2 = 1$ we have:

$$\vec{F}(0,0,1,1,0) = (0,0) \ ,$$

$$F_3^{\pm}(0,1,1,0) = 0 \ ,$$

if we take into account the results of the lemma 3.1.16. Then the folowing lemma holds:

Lemma 3.1.18 *There exist differentiable maps*

$$t_1^+(s_1,s_2),\ t_2^+(s_1,s_2),\ \epsilon^+(s_1,s_2),\ t_1^-(s_1,s_2),\ t_2^-(s_1,s_2),\ \epsilon^-(s_1,s_2)$$

defined in a neighbourhood of $(0,0)$ *such that:*

a) $t_1^+(0,0) = t_1^-(0,0) = 1$, $t_2^+(0,0) = t_2^-(0,0) = 1$, $\epsilon^+(0,0) = \epsilon^-(0,0) = 0$.

b) Let $\vec{F}_+ = (F_1, F_2, F_3^+)$ *and* $\vec{F}_- = (F_1, F_2, F_3^-)$. *Then:*

$$\vec{F}_+(s_1, s_2, t_1^+(s_1,s_2), t_2^+(s_1,s_2), \epsilon(s_1,s_2)) = 0 \ ,$$

$$\vec{F}_-(s_1, s_2, t_1^-(s_1,s_2), t_2^-(s_1,s_2), \epsilon(s_1,s_2)) = 0 \ ,$$

for (s_1, s_2) *near* $(0,0)$.

c)

$$\epsilon^{+}(s_1,s_2)=[D_3y(0,1,0)]^{-1}s_2-D_1y(0,1,0)[D_3y(0,1,0)]^{-1}s_1+O(2)\ ,$$

d)

$$\epsilon^{-}(s_1,s_2)=[D_3y(0,1,0)]^{-1}s_2-D_1y(0,1,0)[D_3y(0,1,0)]^{-1}s_1+O(2)\ .$$

Proof:

We know that $\vec{F}_{+}(0,0,1,1,0)=\vec{F}_{-}(0,0,1,1,0)=0$. To show there exist the maps t_1^{+} , t_1^{-} , t_2^{+} , t_2^{-} , ϵ^{+} and ϵ^{-} we use the implicit function theorem. One has:

$$\begin{aligned}
\frac{\partial F_1}{\partial t_1} &= 1-s_1D_1x(s_1,t_1,\epsilon)\\
\frac{\partial F_1}{\partial t_2} &= -D_2x(s_1t_1,t_2,\epsilon)\\
\frac{\partial F_1}{\partial \epsilon} &= -D_3x(s_1t_1,t_2,\epsilon)\\
\frac{\partial F_2}{\partial t_1} &= -s_1D_1y(s_1t_1,t_2,\epsilon)\\
\frac{\partial F_2}{\partial t_2} &= s_2-D_2y(s_1t_1,t_2,\epsilon)\\
\frac{\partial F_2}{\partial \epsilon} &= -D_3y(s_1,t_1,t_2,\epsilon)
\end{aligned}$$

$$\frac{\partial F_3^{+}}{\partial t_1}=s_1D_{21}y(s_1t_1,t_2,\epsilon)+s_1^2s_2D_{11}x(s_1t_1,t_2,\epsilon)-$$

$$s_1\frac{\partial}{\partial t_1}(D_1x(s_1t_1,t_2,\epsilon)D_2y(s_1t_1,t_2,\epsilon)-D_1y(s_1t_1,t_2,\epsilon)D_2x(s_1t_1,t_2,\epsilon))\ ,$$

$$\frac{\partial F_3^{-}}{\partial t_1}=s_1D_{21}y(s_1t_1,t_2,\epsilon)+s_1^2s_2D_{11}x(s_1t_1,t_2,\epsilon)+$$

$$s_1\frac{\partial}{\partial t_1}(D_1x(s_1t_1,t_2,\epsilon)D_2y(s_1t_1,t_2,\epsilon)-D_1y(s_1t_1,t_2,\epsilon)D_2x(s_1t_1,t_2,\epsilon))\ ,$$

$$\frac{\partial F_3^{+}}{\partial t_2}=D_{22}y(s_1t_1,t_2,\epsilon)+s_1s_2D_{12}x(s_1t_1,t_2,\epsilon)-$$

$$s_1\frac{\partial}{\partial t_2}(D_1x(s_1t_1,t_2,\epsilon)D_2y(s_1t_1,t_2,\epsilon)-D_1y(s_1t_1,t_2,\epsilon)D_2x(s_1t_1,t_2,\epsilon))\ ,$$

$$\frac{\partial F_3^{-}}{\partial t_2}=D_{22}y(s_1t_1,t_2,\epsilon)+s_1s_2D_{12}x(s_1t_1,t_2,\epsilon)+$$

$$s_1\frac{\partial}{\partial t_2}(D_1x(s_1t_1,t_2,\epsilon)D_2y(s_1t_1,t_2,\epsilon)-D_1y(s_1t_1,t_2,\epsilon)D_2x(s_1t_1,t_2,\epsilon))\ .$$

Then if we take $s_1=s_2=\epsilon$ and $t_1=t_2=1$, we obtain:

$$\frac{\partial F_1}{\partial t_1}=1\ ,\ \frac{\partial F_1}{\partial t_2}=-D_2x(0,1,0)\ ,$$

$$\frac{\partial F_1}{\partial \epsilon}=-D_3x(0,1,0)=0\ ,\ \frac{\partial F_2}{\partial t_1}=0\ ,$$

$$\frac{\partial F_2}{\partial t_2}=-D_2y(0,1,0)=0\ ,\ \frac{\partial F_2}{\partial \epsilon}=-D_3y(0,1,0)\ ,$$

$$\frac{\partial F_3^+}{\partial t_1}=\frac{\partial F_3^-}{\partial t_1}=0\ ,\ \frac{\partial F_3^+}{\partial t_2}=\frac{\partial F_3^-}{\partial t_2}=D_{22}y(0,1,0)\ ,$$

where all the derivatives are taken for $s_1=s_2=\epsilon=0$ and $t_1=t_2=1$. Then:

$$\det D\vec{F}_\pm(0,0,1,1,0)=-\frac{\partial F_1}{\partial t_1}\frac{\partial F_2}{\partial \epsilon}\frac{\partial F_3^\pm}{\partial t_2}=D_3y(0,1,0)D_{22}y(0,1,0)\neq 0\ ,$$

since the homoclinic tangency is non-degenerate ($D_{22}y(0,1,0)\neq 0$) and unfolds generically ($D_3y(0,1,0)\neq 0$). Then, by the implicit function theorem, there are functions $t_1^+(s_1,s_2)$, $t_2^+(s_1,s_2)$, $\epsilon^+(s_1,s_2)$ and $t_1^-(s_1,s_2)$, $t_2^-(s_1,s_2)$, $\epsilon^-(s_1,s_2)$ which satisfy a) and b) in a neighbourhoodof $(0,0)$.

In order to see c) we have to compute $D_1\epsilon^\pm$ and $D_2\epsilon^\pm$. We get:

$$D_1F_2+D_3F_2D_1t_1^\pm+D_4F_2D_1t_2^\pm+D_5F_2D_1\epsilon^\pm=0$$

$$D_2F_2+D_3F_2D_2t_1^\pm+D_4F_2D_2t_2^\pm+D_5F_2D_2\epsilon^\pm=0$$

where all the derivatives are computed for $s_1=s_2=\epsilon=0$, and $t_1=t_2=1$. We know that $D_3F_2=D_4F_2=0$ and $D_1F_2=-D_1y(0,1,0)$, $D_2F_2=1$. Therefore:

$$D_1\epsilon^\pm(0,0)=-\frac{D_1F_2}{D_5F_2}=-\frac{D_1(0,1,0)}{D_3y(0,0,1)}\ ,$$

$$D_2\epsilon^\pm(0,0)=-\frac{D_2F_2}{D_5F_2}=-\frac{1}{D_3y(0,0,1)}\ ,$$

and this proves the part c) of the lemma.□

Lemma 3.1.19 *The maps $g_\pm:[0,r_0]\times[-\epsilon_0,\epsilon_0]\longrightarrow[-\epsilon_0,\epsilon_0]$, with r_0 and ϵ_0 small enough, such that $g_\pm(r,\eta)=\epsilon^\pm(\alpha_1^{1/r}(\eta),\alpha_2^{-1/r}(\eta))$, are uniform contractions (that is, there exists a $0<K<1$ such that $|g_\pm(r,\eta_1)-g_\pm(r,\eta_2)|<K|\eta_1-\eta_2|\ \forall r\in[0,r_0]$) and are continuous with respect to r and η .*

Proof:

If η is small enough then $0<\alpha_1(\eta)<1$ and $0<\alpha_2^{-1}(\eta)<1$. If $|\eta|\leq\epsilon_0$ then $|\alpha_1(\eta)-\bar{\alpha}_1|\leq A\epsilon_0$, $|\alpha_2(\eta)^{-1}-\bar{\alpha}^{-1}|\leq B\epsilon_0$ with A and B suitable constants. Therefore $0<\alpha_1(\eta)\leq A_1<1$ and $0<\alpha_2(\eta)^{-1}\leq B_1<1$ if ϵ_0 is small enough for suitable constants A_1 and B_1. Hence $0\leq\alpha_1(\eta)^{1/r}\leq A_1^{1/r}$ and $0\leq\alpha_2(\eta)^{-1/r}\leq B_1^{1/r}$. Therefore, if $0\leq r\leq r_0$ with r_0 small enough then $|\epsilon^\pm(\alpha_1(\eta)^{1/r},\alpha_2(\eta)^{-1/r})|\leq\epsilon_0$ for $|\eta|\leq\epsilon_0$, since $\epsilon^\pm$ is a continuous map. This means that the map $g_\pm$ is well defined.

To see that $g_\pm$ is a contraction we can apply the mean value theorem:

$$|g_\pm(r,\eta_1)-g_\pm(r,\eta_2)|\leq M[|\alpha_1(\eta_1)^{1/r}-\alpha_1(\eta_2)^{1/r})|+$$

$$|\alpha_2(\eta_1)^{-1/r}-\alpha_2(\eta_2)^{-1/r})|]\leq$$

$$M[\frac{1}{r}A_1^{\frac{1}{r}-1}|\alpha_1(\eta_1)-\alpha_1(\eta_2)|+\frac{1}{r}B_1^{\frac{1}{r}-1}|\alpha_2(\eta_1)^{-1}-\alpha_2(\eta_2)^{-1}|\leq K|\eta_1-\eta_2|$$

with $0<K<1$ if r is small enough, since $r^{-1}A_1^{\frac{1}{r}-1}\to 0$ and $r^{-1}B_1^{\frac{1}{r}-1}\to 0$ when r goes to 0.

The continuity of $g_\pm$ is due to the continuity of the maps $\epsilon^\pm$, $\alpha_1(\eta)^{1/r}$ and $\alpha_2(\eta)^{-1/r}$.□

Then let $n=1/r$ in lemma 3.1.19. The uniform contraction principle (see for instance [26] page 25) implies that there exist continuous functions $\eta^\pm=\eta^\pm(\eta)$ such that:

$$\eta^\pm(n)=\epsilon^\pm(\alpha_1^n(\eta^\pm(n)),\alpha_2^{-n}(\eta^\pm(n)))\ . \tag{3.11}$$

Let $s_1=\alpha_1^n(\eta^\pm(n))$ and $s_2=\alpha_2^{-n}(\eta^\pm(n))$. If n is large enough, s_1 and s_2 are small. So by lemma 3.1.18:

$$\vec{F}_\pm(\alpha_1^n,\alpha_2^{-n},t_1^\pm(\alpha_1^n,\alpha_2^{-n}),t_2^\pm(\alpha_1^n,\alpha_2^{-n}),\epsilon^\pm(\alpha_1^n,\alpha_2^{-n}))=0$$

where $\alpha_1^n=\alpha_1^n(\eta^\pm(n))$ and $\alpha_2^{-n}=\alpha_2^{-n}(\eta^\pm(n))$.

Taking into account 3.11 we obtain:

$$\vec{F}_\pm(\alpha_1^n,\alpha_2^{-n},t_1^\pm(\alpha_1^n,\alpha_2^{-n}),t_2^\pm(\alpha_1^n,\alpha_2^{-n}),\eta^\pm(n))=0$$

where $\alpha_1^n=\alpha_1^n(\eta^\pm(n))$ and $\alpha_2^{-n}=\alpha_2^{-n}(\eta^\pm(n))$.

By lemma 3.1.17 it follows that $f^n_{\epsilon_n^+}$ (respectively $f^n_{\epsilon_n^-}$) has a saddle-node (flip) bifurcation point of period n, where $\epsilon_n^\pm=\eta^\pm(n)$ and the periodic point is $p_n^\pm=(t_{1n}^\pm,t_{2n}^\pm)$, with $t_{1n}^\pm=t_{1n}^\pm(\alpha_1^n,\alpha_2^{-n})$ and $t_{2n}^\pm=\alpha_2^{-n}t_{2n}^\pm(\alpha_1^n,\alpha_2^{-n})$.

To finish the proof of parts a) and b) we have to show that these saddle-node and flip are generic and unfolds generically. First we will make another change of parameter and coordinates:

Lemma 3.1.20 *There exists a parameter $\tilde{\epsilon}=H(\epsilon)$ and an interval $J=(-\delta,\delta)$ such that, if $\bar{f}_{\tilde{\epsilon}}(x,y)=f_{H(\epsilon)}(x,y)$ then the family $\{\bar{f}_{\tilde{\epsilon}}\}_{\tilde{\epsilon}\in J}$ has a map*

$$\vec{x}_1(t_1,t_2,\tilde{\epsilon})=(x_1(t_1,t_2,\tilde{\epsilon}),y_1(t_1,t_2,\tilde{\epsilon}))$$

satisfying:

a) There exist a neighbourhood U_1 of $(0,0)$ such that $\vec{x}_{1|U_1}=Id$.

b) $x_1(0,1,\tilde{\epsilon})=1$, $y_1(0,1,0)=0$, $D_2y_1(0,1,\tilde{\epsilon})=0$, $D_{22}y_1(0,1,0)\neq 0$, $y_1(0,1,\tilde{\epsilon})=\tilde{\epsilon}$.

c) H is a local diffeomorphism in a neighbourhood of $\epsilon=0$

Proof:

Consider the map $\tilde{x}$ corresponding to f_ϵ. Let $\tilde{\epsilon} = y(0,1,\epsilon) = H(\epsilon)$. This map is invertible in a neighbourhood of 0 because $D_3y(0,1,0) \neq 0$. Moreover $H(0) = 0$ and $\tilde{x}_1(t_1,t_2,\tilde{\epsilon}) = \tilde{x}(t_1,t_2,H(\epsilon))$. Then the lemma holds. $\square$

As before, we denote the new family and parameter as $\{f_\epsilon\}_{\epsilon\in J}$. We know that if a bifurcation unfolds generically with respect to a parameter, also it unfolds generically with respect to a new parameter, provided that the change of parameters is a diffeomorphism.

To simplify the computations we consider the following transformation $z_1 = \xi$, $z_2 = \alpha_2^n\eta$ for the map

$$f_\epsilon^n(\xi,\eta) = (x(\alpha_1^n\xi,\alpha_2^n\eta,\epsilon),y(\alpha_1^n\xi,\alpha_2^n\eta,\epsilon)) \ .$$

Then we obtain the following map:

$$g_\epsilon^n(z_1,z_2) = (x(\alpha_1^n z_1,z_2,\epsilon),\alpha_2^n y(\alpha_1^n z_1,z_2,\epsilon))$$

The fixed points of this map corresponding to the n-periodic points $p_n^\pm$ are

$$q_n^\pm = (t_{1n}^\pm,\alpha_2^n t_{2n}^\pm) = (t_1^\pm(\alpha_1^n,\alpha_2^{-n}),t_2^\pm(\alpha_1^n,\alpha_2^{-n})) \ .$$

Lemma 3.1.21 *Let $g_n(z_1,z_2,\epsilon) = (g_1^n(z_1,z_2,\epsilon),g_2^n(z_1,z_2,\epsilon)) = g_\epsilon^n(z_1,z_2)$ be as before. Then:*

a) $D_1g_1^n = \alpha_1^n D_1x$, $D_1g_2^n = \alpha_1^n\alpha_2^n D_1y$, $D_2g_1^n = D_2x$, $D_2g_2^n = \alpha_2^n D_2y$.

b) $v_1 = -D_2x$, $v_2 = \alpha_1^n D_1x - 1$, $w_1 = \alpha_2^n D_2y - 1$, $w_2 = -D_2x$, *where* $v = (v_1,v_2)$ *and* $w = (w_1,w_2)$ *are, respectively, right and left eigenvectors of eigenvalue* 1 *of* $Dg^n_{\epsilon_n^\pm}(q_n^\pm)$.

c) $D_{11}g_1^n = \alpha_1^{2n}D_{11}x,\ D_{12}g_1^n = \alpha_1^n D_{12}x,\ D_{22}g_1^n = D_{22}x,$

$$D_{11}g_2^n = \alpha_1^{2n}\alpha_2^n D_{11}y,\ D_{12}g_2^n = \alpha_1^n\alpha_2^n D_{12}y,\ D_{22}g_2^n = \alpha_2^n D_{22}y.$$

d) $w^T D^2 g^n_{\epsilon_n^\pm}(q_n^+)(v,v) = -\alpha_2^n(D_{22}y(0,1,0)D_2x(0,1,0) + o(1))$.

e)

$$D_3g_1^n = D_1xn\alpha_1^{n-1}\frac{d\alpha_1}{d\epsilon}t_1^+ + D_3x \ ,$$

$$D_3g_2^n = n\alpha_2^{n-1}\frac{d\alpha_2}{d\epsilon}y + \alpha_2^n D_1y\alpha_1^{n-1}\frac{d\alpha_1}{d\epsilon}t_1^+ + \alpha_2^n D_3y \, ,$$

f) $w^T D_3g^n = \alpha_2^n(-D_2x(0,1,0)D_3y(0,1,0) + o(1)) = \alpha_2^n(-D_2x(0,1,0) + o(1))$.

The derivatives of g_1^n and g_2^n are taken in $(q_n^\pm,\epsilon_n^\pm)$, the derivatives of x and y are taken in $(\alpha_1^n t_1^\pm,t_2^\pm,\epsilon_n^\pm)$, in the items a) and c). In the rest of the items the derivatives of g_1^n and g_2^n are taken in (q_n^+,ϵ_n^+), the derivatives of x and y are taken in $(\alpha_1^n t_1^+,t_2^+,\epsilon_n^+)$ and the derivatives of α_1 and α_2 are taken in ϵ_n^+ .

Proof:

The proof of a), b), c) and e) is a simple check. To show d) notice that

$$w^T D^2 g^n_{\epsilon_n^\pm}(q_n^+)(v,v) = w_1 D_{11} g_1^n v_1^2 + 2w_1 D_{12} g_1^n v_1 v_2 + w_1 D_{22} g_1^n v_2^2 +$$

$$w_2 D_{11} g_1^n v_1^2 + 2w_2 D_{12} g_1^n v_1 v_2 + w_2 D_{22} g_1^n v_2^2 =$$

$$-\alpha_2^n (D_{22}(0,1,0) D_2 x(0,1,0) + o(1))$$

since $\alpha_2^n D_2 y = o(\alpha_2^n)$.

All the derivatives of x and y are taken in $(\alpha_1^n t_1^+, t_2^+, \epsilon_n^+)$.

For f) we have:

$$w^T D_3 g^n = w_1 D_3 g_1^n + w_2 D_3 g_2^n =$$

$$\alpha_2^n(-D_2 x(0,1,0) D_3(0,1,0) + o(1)) = \alpha_2^n(-D_2 x(0,1,0) + o(1)) ,$$

since $\alpha_2^n D_2 y = o(\alpha_2^n)$.□

By using the previous lemma and taking into account that, by hypothesis, $D_{22}y(0,1,0) \neq 0$, $D_3 y(0,1,0) = 1$ and $D_2 x(0,1,0) \neq 0$ because $\vec{x}$ is a local diffeomorphism, we have finish the proof of the part 1) of the theorem (see theorem 4.1.22 of chapter 4, which characterizes the saddle-node bifurcation).

To prove 2) we need the following

Lemma 3.1.22 *Let $g^n(z_1, z_2, \epsilon)$ as in the previous lemma. Then*

a) $D_{111} g_1^n = \alpha_1^{3n} D_{111} x$, $D_{112} g_1^n = \alpha_1^{2n} D_{112} x$, $D_{122} g_1^n = \alpha_1^n D_{122} x$,

$$D_{222} g_1^n = D_{222} x,\ D_{111} g_2^n = \alpha_1^{3n} \alpha_2^n D_{111} y,\ D_{112} g_2^n = \alpha_1^{2n} \alpha_2^n D_{112} y,$$

$$D_{122} g_2^n = \alpha_1^n \alpha_2^n D_{122} y,\ D_{222} g_2^n = \alpha_2^n D_{222} y.$$

b) Let $v = (v_1, v_2)$ and $\bar{v} = (\bar{v}_1, \bar{v}_2)$ (respectively $w = (w_1, w_2)$ and $\bar{w} = (\bar{w}_1, \bar{w}_2)$) be right (resp. left) eigenvectors of eigenvalues -1 and λ, where λ is the other eigenvalue of $Dg^n_{\epsilon_n^-}(q_n^-)$. Then: $v_1 = -D_2 x$, $v_2 = \alpha_1^n D_1 x + 1$, $w_1 = (\alpha_2^n D_2 y + 1)/A$, $w_2 = -(D_2 x)/A$, $\lambda = \alpha_2^n \alpha_1^n D_1 y D_2 x - \alpha_1^n \alpha_2^n D_1 x D_2 y$, $\bar{v}_1 = -D_2 x$, $\bar{v}_2 = \alpha_1^n D_1 x - \lambda$, $\bar{w}_1 = (\alpha_2^n D_2 y - \lambda)/B$, $w_2 = -(D_2 x)/B$. Here $A = -D_2 x(\alpha_2^n D_2 y + 1) - D_2 x(\alpha_1^n D_1 x + 1)$ and $B = -D_2 x(\alpha_2^n D_2 y - \lambda) - D_2 x(\alpha_1^n D_1 x - \lambda)$.

c)

$$\frac{1}{2(1-\lambda)}(w^T D^2 g^n_{\epsilon_n^-}(v,\bar{v}))(\bar{w}^T D^2 g^n_{\epsilon_n^-}(v,v)) + \frac{1}{4}(w^T D^2 g^n_{\epsilon_n^-}(v,v))^2 +$$

$$\frac{1}{6} w^T D^3 g^n_{\epsilon_n^-}(v,v,v) = \frac{1}{4}\alpha_2^{2n}(D_{22}y(0,1,0)^2 + o(1)).$$

d) Let $q_n = (z_1, z_2)$ be the fixed point of g_ϵ^n for ϵ near ϵ_n^- such that $q_n(\epsilon_n^-) = q_n^-$. Then:

$$\frac{dz_1}{d\epsilon} = \frac{1}{2}\alpha_2^n D_3 y D_2 x + o(\alpha_2^n) \ ,$$

$$\frac{dz_2}{d\epsilon} = \frac{1}{2}\alpha_2^n D_3 y + o(\alpha_2^n) \ .$$

e) Let $\mu = \mu(\epsilon)$ be the eigenvalue of $Dg_\epsilon^n(q_n)$ such that $\mu(\epsilon_n^-) = -1$. Then:

$$\frac{d}{d\epsilon}\mu(\epsilon_n^-) = -\frac{1}{2(1+\lambda)} D_{22} y(0,1,0)\alpha_2^{2n} + o(\alpha_2^{2n}) \ .$$

Here, derivatives of x and y are taken in $(\alpha_1^n t_1^-, t_2^-, \epsilon_n^-)$ and derivatives of g_1^n and g_2^n are taken in (q_n^-, ϵ_n^-), if the point is not given explicitely.

Proof:

a) and b) are obtained by using the expression

$$g_\epsilon^n(z_1, z_2) = (x(\alpha_1^n z_1, z_2, \epsilon), \alpha_2^n y(\alpha_1^n z_1, z_2, \epsilon)).$$

The eigenvalues are computed from:

$$Dg_\epsilon^n(z_1, z_2) = \begin{pmatrix} \alpha_1^n D_1 x(\alpha_1^n z_1, z_2) & D_2 x(\alpha_1^n z_1, z_2) \\ \alpha_2^n \alpha_1^n D_1 y(\alpha_1^n z_1, z_2) & \alpha_2^n D_2 y(\alpha_1^n z_1, z_2) \end{pmatrix} .$$

We note that $\lambda = -\det Dg_\epsilon^n(q_n^-)$ because the other eigenvalue is -1. So $\lambda = O(\alpha_1^n \alpha_2^n)$. Moreover the eigenvectors satisfy $v \cdot w = 1$ and $\bar{v} \cdot \bar{w} = 1$, and $A = -D_2 x + o(1)$ due to -1 is an eigenvalue of $Dg_\epsilon^n(q_n^-)$. We have:

$$1 + \operatorname{tr} Dg_\epsilon^n(q_n^-) + \det Dg_\epsilon^n(q_n^-) = 0 \ ,$$

but

$$\det Dg_\epsilon^n(q_n^-) = \alpha_1^n \alpha_2^n D_1 y D_2 x + o(\alpha_1^n \alpha_2^n) \ ,$$

$$\operatorname{tr} Dg_\epsilon^n(q_n^-) = \alpha_2^n D_2 y + o(1) \ .$$

This means that $\alpha_2^n D_2 y = -1 + o(1)$ and, therefore, $A = -D_2 x + o(1)$. By the same reason: $v_1 = O(1)$, $v_2 = O(1)$ and $w_1 = O(1)$, $w_2 = O(1)$.

On the other hand $B = -\alpha_2^n D_2 y D_2 x + o(1) = O(1)$. So $\bar{v}_1 = O(1)$, $\bar{v}_2 = O(\alpha_1^n \alpha_2^n)$ and $\bar{w}_1 = O(1)$, $\bar{w}_2 = O(1)$.

It is easy to see that $\frac{1}{6} w^T D^3 g_{\epsilon_n^-}^n(v, v, v) = O(\alpha_2^n)$ and, using the values of the second derivatives computed in lemma 3.1.21, that:

$$\frac{1}{2(1-\lambda)}(w^T D^2 g_{\epsilon_n^-}^n(v, \bar{v}))(\bar{w}^T D^2 g_{\epsilon_n^-}^n(v, v)) = (O(1) + O(\alpha_1^n \alpha_2^{2n}))O(\alpha_2^n) = o(\alpha_2^{2n}) \ ,$$

and

$$\frac{1}{4}(w^T D^2 g^n_{\epsilon_n^-}(v,v))^2 = \frac{1}{4}\frac{(D_{22}yD_2x)^2}{A^2}\alpha_2^{2n} = \frac{1}{4}\alpha_2^{2n}(D_{22}y(0,1,0)+o(1)) \; .$$

With these computations the proof of c) is finished.

To prove d) observe that $q_n = (z_1, z_2)$ verifies:

$$\left.\begin{array}{rcl} x(\alpha_1^n z_1, z_2, \epsilon) & = & z_1 \\ \alpha_2^n y(\alpha_1^n z_1, z_2, \epsilon) & = & z_2 \end{array}\right\}$$

Then, if we compute the derivative with respect to ϵ, we get:

$$\frac{dz_1}{d\epsilon} = D_1 x \left(\alpha_1^n \frac{dz_1}{d\epsilon} + n\alpha_1^{n-1}\frac{d\alpha_1}{d\epsilon} z_1\right) + D_2 x \frac{dz_2}{d\epsilon} + D_3 x \; ,$$

$$\frac{dz_2}{d\epsilon} = n\alpha_2^{n-1}\frac{d\alpha_2}{d\epsilon} y + \alpha_2^n \left(D_1 y \left(\alpha_1^n \frac{dz_1}{d\epsilon} + n\alpha_1^{n-1}\frac{d\alpha_1}{d\epsilon} z_1\right) + D_2 y \frac{dz_2}{d\epsilon} + D_3 y\right) \; ,$$

where the derivatives of x and y are computed at the point $(\alpha_1^n z_1, z_2, \epsilon)$.

So for $\epsilon = \epsilon_n^-$ we have:

$$\begin{gathered} \frac{dz_1}{d\epsilon} = \left[\left(-D_1 x n \alpha_1^{n-1} t_{1n}^- - D_3 x\right)\left(\alpha_2^n D_2 y - 1\right) + \right. \\ \left. D_2 x \left(n\alpha_2^{n-1}\frac{d\alpha_2}{d\epsilon} y + \alpha_2^n \left(D_1 y \frac{d\alpha_1}{d\epsilon} n\alpha_1^{n-1} t_{1n}^- + D_3 y\right)\right)\right] \\ \left[(D_1 x \alpha_1^n - 1)(D_2 y \alpha_2^n - 1) - D_2 x D_1 y \alpha_1^n \alpha_2^n\right]^{-1} \end{gathered}$$

and

$$\begin{gathered} \frac{dz_2}{d\epsilon} = \left[(D_1 x \alpha_1^n - 1)\left(-n\alpha_1^{n-1}\frac{d\alpha_2}{d\epsilon} y - \alpha_2^n \left(D_1 y \frac{d\alpha_1}{d\epsilon} t_{1n}^- + D_3 y\right) + \right.\right. \\ \left.\left. D_1 y \alpha_1^n \alpha_2^n \left(D_1 x n \alpha_1^{n-1}\frac{d\alpha_1}{d\epsilon} t_{1n}^- + D_3 x\right)\right)\right] \\ \left[(D_1 x \alpha_1^n - 1)(D_2 y \alpha_2^n - 1) - D_2 x D_1 y \alpha_1^n \alpha_2^n\right]^{-1} \; . \end{gathered}$$

Since -1 is an eigenvalue of $Dg^n_{\epsilon_n^-}(q_n^+)$, we have:

$$1 + \operatorname{tr} Dg^n_{\epsilon_n^-}(q_n^-) + \det Dg^n_{\epsilon_n^-}(q_n^-) = 0 \; ,$$

and therefore:

$$\begin{gathered} (D_1 x \alpha_1^n - 1)(D_2 y \alpha_2^n - 1) - D_2 x D_1 y \alpha_1^n \alpha_2^n = 1 - \operatorname{tr} Dg^n_{\epsilon_n^-}(q_n^-) + \det Dg^n_{\epsilon_n^-}(q_n^-) = \\ 2(1 + \det Dg^n_{\epsilon_n^-}(q_n^-)) = 2 + o(1) \; . \end{gathered}$$

We also see that

$$y(\alpha_1^n t_{1n}^-, \alpha_2^n t_{2n}^-, \epsilon_n^-) = D_1 y(0,1,0)\alpha_1^n t_{1n}^- + D_3 y(0,1,0)\epsilon_n^- + h.o.t. = O(\alpha_2^{-n}) \; .$$

Then:

$$\frac{dz_1}{d\epsilon} = \frac{1}{2}\alpha_2^n D_3 y D_2 x + o(\alpha_2^n)$$

and

$$\frac{dz_2}{d\epsilon} = \frac{1}{2}\alpha_2^n D_3 y + o(\alpha_2^n) \ .$$

Here all the derivatives of x and y are taken at $(\alpha_1^n t_{1n}^-, \alpha_2^n t_{2n}^-, \epsilon_n^-)$.

To see e) let $\mu = \mu(\epsilon)$ be the eigenvalue of $Dg_\epsilon^n(q_n)$ such that $\mu(\epsilon_n^-) = -1$. If $\lambda = \lambda(\epsilon)$ is the other eigenvalue, then:

$$\text{tr}\, Dg_\epsilon^n(q_n) = \alpha_1^n D_1 x(\alpha_1^n z_1, z_2, \epsilon) + \alpha_2^n D_2 y(\alpha_1^n z_1, z_2, \epsilon) = \mu + \lambda \ ,$$

$$\det Dg_\epsilon^n(q_n) = \alpha_1^n \alpha_2^n D_1 x(\alpha_1^n z_1, z_2, \epsilon) D_2 y(\alpha_1^n z_1, z_2, \epsilon) -$$

$$\alpha_1^n \alpha_2^n D_2 x(\alpha_1^n z_1, z_2, \epsilon) D_1 y(\alpha_1^n z_1, z_2, \epsilon) = \lambda\mu \ .$$

By derivation, and isolating $\frac{d\mu}{d\epsilon}(\epsilon_n^-)$, we obtain for $\epsilon = \epsilon_n^-$:

$$\frac{d\mu}{d\epsilon_n^-} = \frac{\frac{d}{d\epsilon}\text{tr}\, Dg_\epsilon^n(q_n) + \frac{d}{d\epsilon}\det Dg_\epsilon^n(q_n)}{1+\lambda} \ .$$

This is well defined because $|\lambda| \neq 1$.

Then we have:

$$\frac{d}{d\epsilon}\text{tr}\, Dg_\epsilon^n(q_n)_{|\epsilon=\epsilon_n^-} = n\alpha_1^{n-1} D_1 x + \alpha_1^n \left(D_{11} x \left[n\alpha_1^{n-1}\frac{d\alpha_1}{d\epsilon} t_{1n}^- + \alpha_1^n \frac{dz_1}{d\epsilon}\right] + \right.$$

$$\left. D_{12} x \frac{dz_2}{d\epsilon} + D_{13} x \right) + n\alpha_2^{n-1} \left(D_{21} y \left[n\alpha_1^{n-1}\frac{d\alpha_1}{d\epsilon} + \alpha_1^n \frac{dz_1}{d\epsilon}\right] + \right.$$

$$\left. D_{22} y \frac{dz_2}{d\epsilon} + D_{23} y \right) = \frac{1}{2} D_{22} y(0,1,0) D_3 y(0,1,0) \alpha_2^{2n} + o(\alpha_2^{2n}),$$

$$\frac{d}{d\epsilon}\det Dg_\epsilon^n(q_n) = \alpha_1^n \alpha_2^n \frac{d}{d\epsilon}(D_1 x D_2 y - D_2 x D_1 y) +$$

$$\left(n\alpha_1^{n-1}\frac{d\alpha_1^n}{d\epsilon}\alpha_2^n + n\alpha_2^{n-1}\frac{d\alpha_2}{d\epsilon}\alpha_1^n \right)(D_1 x D_2 y - D_2 x D_1 y) =$$

$$\alpha_1^n \alpha_2^n \left(D_1 x \frac{d}{d\epsilon} D_2 y + \frac{d}{d\epsilon} D_1 x D_2 y - D_2 x \frac{d}{d\epsilon} D_1 y - \frac{d}{d\epsilon} D_2 x D_1 y \right) + O(n\alpha_1^n \alpha_2^n) \ .$$

It is easy to see that $\frac{d}{d\epsilon} D_i x = O(\alpha_2^n)$ and $\frac{d}{d\epsilon} D_i y = O(\alpha_2^n)$. So:

$$\frac{d}{d\epsilon}\det Dg_\epsilon^n(q_n)_{|\epsilon=\epsilon_n^-} = O(\alpha_1^n \alpha_2^{2n})$$

and

$$\frac{d}{d\epsilon}\mu(\epsilon_n^-) = -\frac{1}{2(1+\lambda)} D_{22} y(0,1,0) \alpha_2^{2n} + o(\alpha_2^{2n}) \ . \square$$

Following with the proof of the theorem 3.1.15, if we take into account that

$$D_{22}y(0,1,0)D_3y(0,1,0) \neq 0$$

then:

$$\frac{1}{2(1-\lambda)}(w^T D^2 g^n_{\epsilon_n^-}(v,\bar{v}))(\bar{w}^T D^2 g^n_{\epsilon_n^-}(v,v))+$$

$$\frac{1}{4}(w^T D^2 g^n_{\epsilon_n^-}(v,v))^2 + \frac{1}{6}w^T D^3 g^n_{\epsilon_n^-}(v,v,v) \neq 0$$

and

$$\frac{d}{d\epsilon}\mu(\epsilon_n^-) \neq 0 \ ,$$

if n is large enough.

Then, by theorem 4.1.24 of chapter 4, we have that g^n_ϵ has a flip fixed point q_n^- which unfolds generically with ϵ. Hence we have proved the part 2) of the theorem.

It remains to prove 3) and 4):

$$\lim_{n\to\infty}\frac{a_n^+ - a_0}{a_{n+1}^+ - a_0} = \lim_{n\to\infty}\frac{\epsilon_n^+}{\epsilon_{n+1}^+} =$$

$$\lim_{n\to\infty}\frac{[D_3y(0,1,0)]^{-1}\alpha_2^{-n} + o(\alpha_2^{-n})}{[D_3y(0,1,0)]^{-1}\alpha_2^{-(n+1)} + o(\alpha_2^{-(n+1)})} = \bar{\alpha}_2$$

and also

$$\lim_{n\to\infty}\frac{a_n^- - a_0}{a_{n+1}^- - a_0} = \bar{\alpha}_2 \ .$$

Moreover

$$\lim_{n\to\infty} a_n^+ = \lim_{n\to\infty}(a_0 + (D_3y(0,1,0))^{-1}\alpha_2^{-n} + o(\alpha_2^{-n})) = \lim_{n\to\infty} a_n^- = a_0 \ .$$

Finally, 4) is evident since a_n^+ has a saddle-node which unfolds generically. Then if we move the parameter slightly, we obtain an attracting periodic orbit. Of course these parameters tend to a_0 when n tends to ∞. This finishes the proof of the theorem. □

Remark 3.1.23 *a) There exist several versions of this theorem ([18], [19], [20], [21], [22],[27]). In [22] it is proved also that the map f_a^n near the homoclinic tangency, for n large enough, tends (in a suitable sense) to the logistic map. We shall use this fact later. In our version we use the changes of coordinates and parameter (lemmas 3.1.16 and 3.1.20) of [22]. The existence of generic saddle-node and flip bifurcations, however, is not explicitly proved in these versions.*

b) The hypothesis (A) can be removed as we can see in [21], at least for the last part of the theorem. However the techniques used there are different.

c) *For the case $\alpha_1 < 0$ or $\alpha_2 < 0$ we have to distinguish n even and odd. For n even one has the same results that we have seen before. If n is odd there exist also saddle-node and flip periodic orbits and the expression of the values $\epsilon_n^{\pm}$ is the same that the previous case. We shall see this later.*

d) *This theorem can be generalized to situations in which the homoclinic tangency is of order n. In this case it should appear codimension n bifurcations. For example, for cubic tangencieswe should have cusps and codimension two flips. Of course we have to consider, in general, n-parameter families of diffeomorphisms.*

3.2 Possible cases of homoclinic tangencies

To study all possible cases of homoclinic tangencies,we will introduce a quadratic model of the map f_a^n for $a \approx a_0$ near the homoclinic tangency point. First one observes that the possible types of homoclinic tangencies depend on:

a) The geometry of the tangency: in which of the two semispaces determined by the stable invariant manifold of the fixed point, the unstable invariant one stays, locally.

b) In which one of the two possible senses, the orbits follow the unstable invariant manifold.

c) The signs of the eigenvalues of $Df_{a_0}(p_0)$.

Therefore, we have 16 possible cases of homoclinic tangencies essentially different. In the figure 3.2 there is a picture of these possible cases.

Now consider a family of diffeomorphisms $\{f_a\}_{a\in V}$ which satisfies the hypothesis of theorem 3.1.15. We have seen (lemmas 3.1.16 and 3.1.20) that we can suppose that the family verifies, if $a = a_0 + \epsilon$,:

a) There exists a smooth map $\vec{x} = (x, y)$ such that $f_{a_0+\epsilon}(\vec{x}(t_1, t_2, \epsilon)) = \vec{x}(\alpha_1 t_1, \alpha_2 t_2, \epsilon)$ and $\vec{x}_{|U} = Id$, where U is a neighbourhood of $(0,0)$ and $p_\epsilon = (0,0)$ is the dissipative saddle fixed point.

b) The point of homoclinic tangency for $\epsilon = 0$ is $(1,0)$, and $x(0,1,\epsilon) = 1$, $D_2(0,1,\epsilon) = 0$, $y(0,1,\epsilon) = \epsilon$, $y(0,1,0) = 0$, $D_2y(0,1,\epsilon) = 0$.

For this family of diffeomorphisms we have:

$$\epsilon_n^+ = \alpha_2^{-n} - \alpha_1^n D_1(0,1,0) + O(\alpha_2^{-2n}) \ ,$$

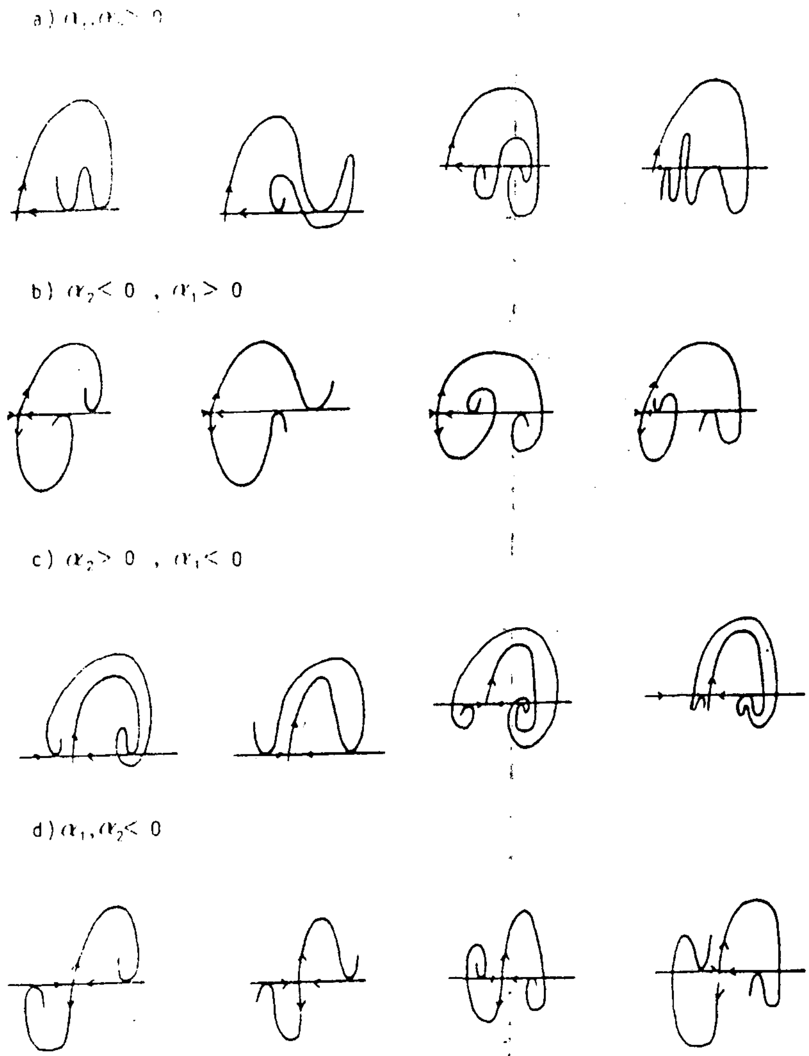


Figure 3.2: Possible cases of homoclinic tangencies

$$\epsilon_n^- = \alpha_2^{-n} - \alpha_1^n D_1(0,1,0) + O(\alpha_2^{-2n}) \ ,$$
$$p_n^+ = (1 + O(\alpha_2^{-n}), \alpha_2^{-n} + O(\alpha_2^{-2n})) \ ,$$
$$p_n^- = (1 + O(\alpha_2^{-n}), \alpha_2^{-n} + O(\alpha_2^{-2n})) \ ,$$

where p_n^+ (resp. p_n^-) is te saddle-node (flip) bifurcation point for $\epsilon = \epsilon_n^+$ ($\epsilon = \epsilon_n^-$).

Now we consider the map $f_{a_0+\epsilon}^n$ for n large enough and ϵ near 0. We know that:

$$f_{a_0+\epsilon}^n(t_1, t_2) = \begin{pmatrix} x(\alpha_1^n t_1, \alpha_2^n t_2, \epsilon) \\ y(\alpha_1^n t_1, \alpha_2^n t_2, \epsilon) \end{pmatrix}$$

if (t_1, t_2) is near to $(1,0)$. Consider the following rectangle:

$$R = [1 - 2\alpha_2^{-n/2}, 1 + 2\alpha_2^{-n/2}] \times [\alpha_2^{-n} - 2\alpha_2^{-3n/2}, \alpha_2^{-n} + 2\alpha_2^{-3n/2}]$$

such that $(t_1, t_2) \in R$ and $\epsilon \in [-2\alpha_2^{-n}, 2\alpha_2^{-n}]$. Then $f_{a_0+\epsilon}^n = (f_{a_0+\epsilon}^{1n}, f_{a_0+\epsilon}^{2n})$ verifies:

$$f_{a_0+\epsilon}^{1n} = x(0,1,\epsilon) + D_1 x(0,1,\epsilon)\alpha_1^n t_1 + D_2 x(0,1,\epsilon)(\alpha_2^n t_2 - 1) + O(\alpha_2^{-n})$$
$$f_{a_0+\epsilon}^{2n} = y(0,1,\epsilon) + D_1 y(0,1,\epsilon)\alpha_1^n t_1 + D_2 y(0,1,\epsilon)(\alpha_2^n t_2 - 1) +$$
$$\frac{1}{2} D_{22} y(0,1,\epsilon)(\alpha_2^n t_2 - 1)^2 + O(\alpha_2^{-3n/2})$$

Then, taking into account that $x(0,1,\epsilon) = 0$, $y(0,1,\epsilon) = \epsilon$ and $D_2 y(0,1,\epsilon) = 0$, we obtain:

$$f_{a_0+\epsilon}^{1n} = 1 + D_2 x(0,1,\epsilon)(\alpha_2^n t_2 - 1) + O(\alpha_2^{-n})$$
$$f_{a_0+\epsilon}^{2n} = \epsilon + D_1 x(0,1,\epsilon)\alpha_1^n t_1 + \frac{1}{2} D_{22} y(0,1,\epsilon)(\alpha_2^n t_2 - 1)^2 + O(\alpha_2^{-3n/2})$$

If we do not take into account $O(\alpha_2^{-n})$ and $O(\alpha_2^{-3n/2})$, we have a quadratic model of the map $f_{a_0+\epsilon}^n$ near the homoclinic tangency. Moreover, as $\vec{x} = (x, y)$ is the identity near $(0,0)$, we have that:

$$\begin{vmatrix} D_1 x(0,1,0) & D_2 x(0,1,0) \\ D_1 y(0,1,0) & D_2 y(0,1,0) \end{vmatrix} = -D_2 x(0,1,0) D_1 y(0,1,0) > 0 \ .$$

This means that there are 4 possible types of maps depending on the signs of the derivatives $D_1 y(0,1,0)$, $D_2 x(0,1,0)$ and $D_{22} y(0,1,0)$:

a) $D_2 x(0,1,0) > 0$, $D_1 y(0,1,0) < 0$, $D_{22} y(0,1,0) > 0$.

b) $D_2 x(0,1,0) > 0$, $D_1 y(0,1,0) < 0$, $D_{22} y(0,1,0) < 0$.

c) $D_2 x(0,1,0) < 0$, $D_1 y(0,1,0) > 0$, $D_{22} y(0,1,0) > 0$.

d) $D_2x(0,1,0) < 0$, $D_1y(0,1,0) > 0$, $D_{22}y(0,1,0) < 0$.

These cases have been studied in [28]. Observe that in all the cases the stable invariant manifold is the straight line $t_2 = 0$ (in coordinates (t_1, t_2)) and the unstable invariant manifold is $(x(0,t_2,\epsilon), y(0,t_2,\epsilon))$, with

$$x(0,t_2,\epsilon) = 1 + D_2x(0,1,\epsilon)(t_2-1) + O((t_2-1)^2) \ ,$$

$$y(0,t_2,\epsilon) = \epsilon + \frac{1}{2}D_{22}y(0,1,\epsilon)(t_2-1)^2 + O((t_2-1)^3) \ .$$

If we take $t_2 \in [1 - 2\alpha_2^{-n/2}, 1 + 2\alpha_2^{-n/2}]$ and $\epsilon \in [-2\alpha_2^{-n}, 2\alpha_2^{-n}]$ then:

$$x(0,t_2,\epsilon) = 1 + D_2x(0,1,\epsilon)(t_2-1) + O(\alpha_2^{-n}) \ ,$$

$$y(0,t_2,\epsilon) = \epsilon + \frac{1}{2}D_{22}y(0,1,\epsilon)(t_2-1)^2 + O(\alpha_2^{-3n/2}) \ .$$

Hence:

$$y(0,t_2,\epsilon) - \frac{1}{2}(x(0,t_2,\epsilon)-1)^2 \frac{D_{22}y(0,1,0)}{[D_2x(0,1,0)]^2} - \epsilon = O(\alpha_2^{-3n/2}) \ .$$

To study the different cases with easier computations, we will suppose that $D_2x(0,1,0) = \pm 1$, $D_{22}y(0,1,0) = \pm 2$ and $D_1y(0,1,0) = \pm 1$. Then we have the following models:

$$f^n_{a_0+\epsilon}(\xi,\eta) = (1-\alpha_2^{-n}\eta, (1-\alpha_2^{-n}\eta)^2 + \epsilon + \alpha_1^n(1+\xi)) \tag{3.12}$$

$$f^n_{a_0+\epsilon}(\xi,\eta) = (\alpha_2^{-n}\eta - 1, (1-\alpha_2^{-n}\eta)^2 + \epsilon - \alpha_1^n(1+\xi)) \tag{3.13}$$

$$f^n_{a_0+\epsilon}(\xi,\eta) = (\alpha_2^{-n}\eta - 1, -(1-\alpha_2^{-n}\eta)^2 + \epsilon - \alpha_1^n(1+\xi)) \tag{3.14}$$

$$f^n_{a_0+\epsilon}(\xi,\eta) = (1-\alpha_2^{-n}\eta, -(1-\alpha_2^{-n}\eta)^2 + \epsilon + \alpha_1^n(1+\xi)) \tag{3.15}$$

where $\xi = t_1 - 1$, $\eta = t_2$ are new coordinates with origin in the homoclinic point.

All this maps are considered for

$$(\xi,\eta) \in \bar{R} = [-2|\alpha_2|^{-n/2}, 2|\alpha_2|^{-n/2}] \times [\alpha_2^{-n} - 2|\alpha_2|^{-3n/2}, \alpha_2^{-n} + 2|\alpha_2|^{-3n/2}]$$

and $\epsilon \in [-2\alpha_2^{-n}, 2\alpha_2^{-n}]$.

The stable invariant manifold is always $\eta = 0$, and the unstable one is $\eta = \xi^2 + \epsilon$ in the first two cases and $\eta = -\xi^2 + \epsilon$ in the other ones. We also suppose that α_1 and α_2 can have negative sign.

Let $c(\xi,\eta) = (f^{2n}_{a_0+\epsilon}(\xi,\eta) - (f^{1n}_{a_0+\epsilon}(\xi,\eta))^2 - \epsilon$. Then we have the following propositions:

Proposition 3.2.1 *For n large enough the map 3.12 verifies:*

a) If α_1 , $\alpha_2 > 0$ *or n is even, then* $f^n_{a_0}(\bar{R}) \cap \bar{R}$ *has two connected components and for all* $(\xi,\eta) \in \bar{R}$: $c(\xi,\eta) > 0$.

b) If $\alpha_1 > 0$, $\alpha_2 < 0$ *and n is odd then* $f^n_{a_0}(\bar{R}) \cap \bar{R} = \emptyset$ *and* $c(\xi,\eta) > 0$.

c) If $\alpha_1 < 0$, $\alpha_2 > 0$ *and n is odd then* $f^n_{a_0}(\bar{R}) \cap \bar{R}$ *has two connected components and* $c(\xi,\eta) < 0$.

d) If α_1 , $\alpha_2 < 0$ *and n is odd then* $f^n_{a_0}(\bar{R}) \cap \bar{R} = \emptyset$ *and* $c(\xi,\eta) < 0$.

Proposition 3.2.2 *For n large enough the map 3.13 verifies:*

a) If α_1 , $\alpha_2 > 0$ *or n is even, then* $f^n_{a_0}(\bar{R}) \cap \bar{R}$ *has two connected components and for all* $(\xi,\eta) \in \bar{R}$: $c(\xi,\eta) < 0$.

b) If $\alpha_1 > 0$, $\alpha_2 < 0$ *and n is odd then* $f^n_{a_0}(\bar{R}) \cap \bar{R} = \emptyset$ *and* $c(\xi,\eta) < 0$.

c) If $\alpha_1 < 0$, $\alpha_2 > 0$ *and n is odd then* $f^n_{a_0}(\bar{R}) \cap \bar{R}$ *has two connected components and* $c(\xi,\eta) > 0$.

d) If α_1 , $\alpha_2 < 0$ *and n is odd then* $f^n_{a_0}(\bar{R}) \cap \bar{R} = \emptyset$ *and* $c(\xi,\eta) > 0$.

Let $c(\xi,\eta) = (f^{1n}_{a_0+\epsilon}(\xi,\eta) + (f^{2n}_{a_0+\epsilon}(\xi,\eta))^2 - \epsilon$.

Proposition 3.2.3 *For n large enough the map 3.14 verifies:*

a) If α_1 , $\alpha_2 > 0$ *or n is even, then* $f^n_{a_0}(\bar{R}) \cap \bar{R} = \emptyset$ *and for all* $(\xi,\eta) \in \bar{R}$: $c(\xi,\eta) < 0$.

b) If $\alpha_1 > 0$, $\alpha_2 < 0$ *and n is odd then* $f^n_{a_0}(\bar{R}) \cap \bar{R}$ *has two connected components and* $c(\xi,\eta) < 0$.

c) If $\alpha_1 < 0$, $\alpha_2 > 0$ *and n is odd then* $f^n_{a_0}(\bar{R}) \cap \bar{R} = \emptyset$ *and* $c(\xi,\eta) > 0$.

d) If α_1 , $\alpha_2 < 0$ *and n is odd then* $f^n_{a_0}(\bar{R}) \cap \bar{R}$ *has two connected components and* $c(\xi,\eta) > 0$.

Proposition 3.2.4 *For n large enough the map 3.15 verifies:*

a) If α_1 , $\alpha_2 > 0$ *or n is even, then* $f^n_{a_0}(\bar{R}) \cap \bar{R} = \emptyset$ *and for all* $(\xi,\eta) \in \bar{R}$: $c(\xi,\eta) > 0$.

b) If $\alpha_1 > 0$, $\alpha_2 < 0$ *and n is odd then* $f^n_{a_0}(\bar{R}) \cap \bar{R}$ *has two connected components and* $c(\xi,\eta) > 0$.

c) If $\alpha_1 < 0$, $\alpha_2 > 0$ *and n is odd then* $f^n_{a_0}(\bar{R}) \cap \bar{R} = \emptyset$ *and* $c(\xi,\eta) < 0$.

d) If α_1 , $\alpha_2 < 0$ *and n is odd then* $f^n_{a_0}(\bar{R}) \cap \bar{R}$ *has two connected components and* $c(\xi,\eta) < 0$.

These four propositions are easy to prove (see [28]). Pictures of the rectangles $\bar{R}$ and their images, for the four types of tangency, are shown in figures 3.3, 3.4, 3.5 and 3.6.

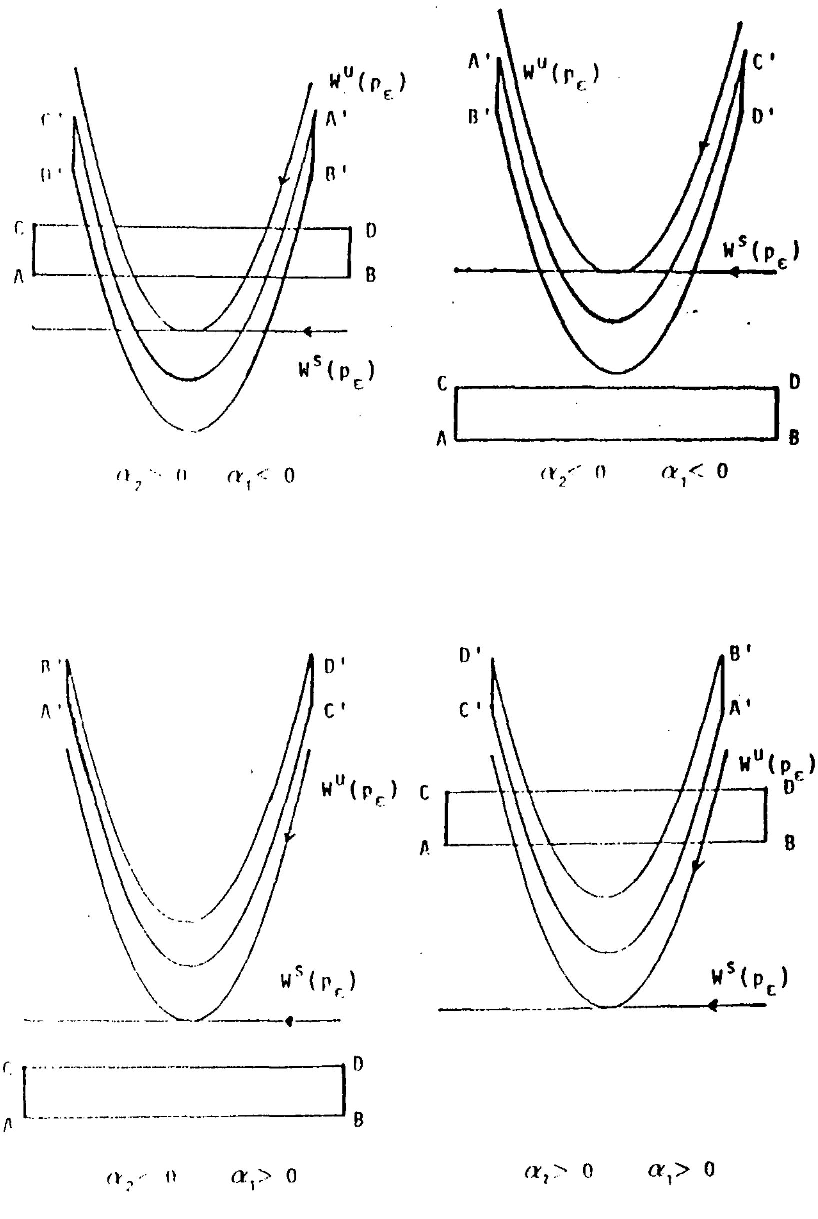


Figure 3.3: Case 1 (A', B', C', D' are the images of A, B, C, D).

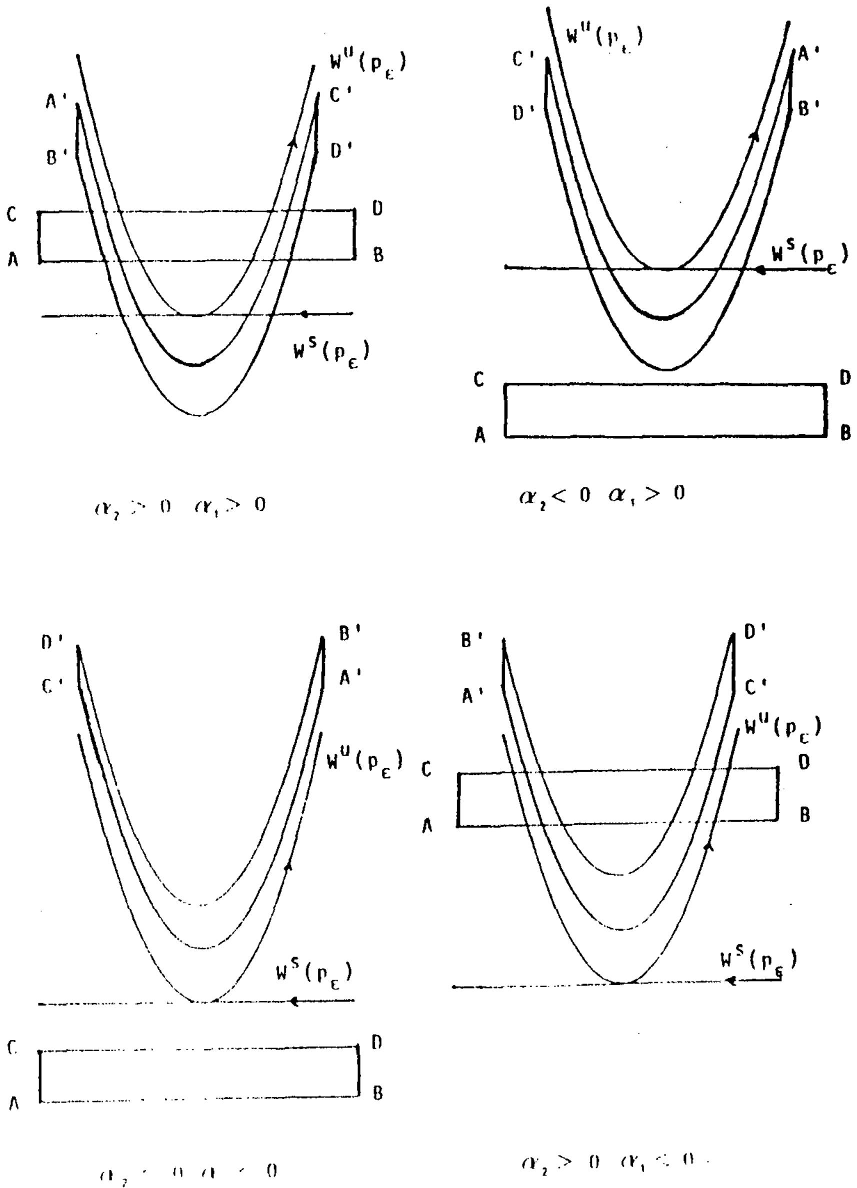


Figure 3.4: Case 2.

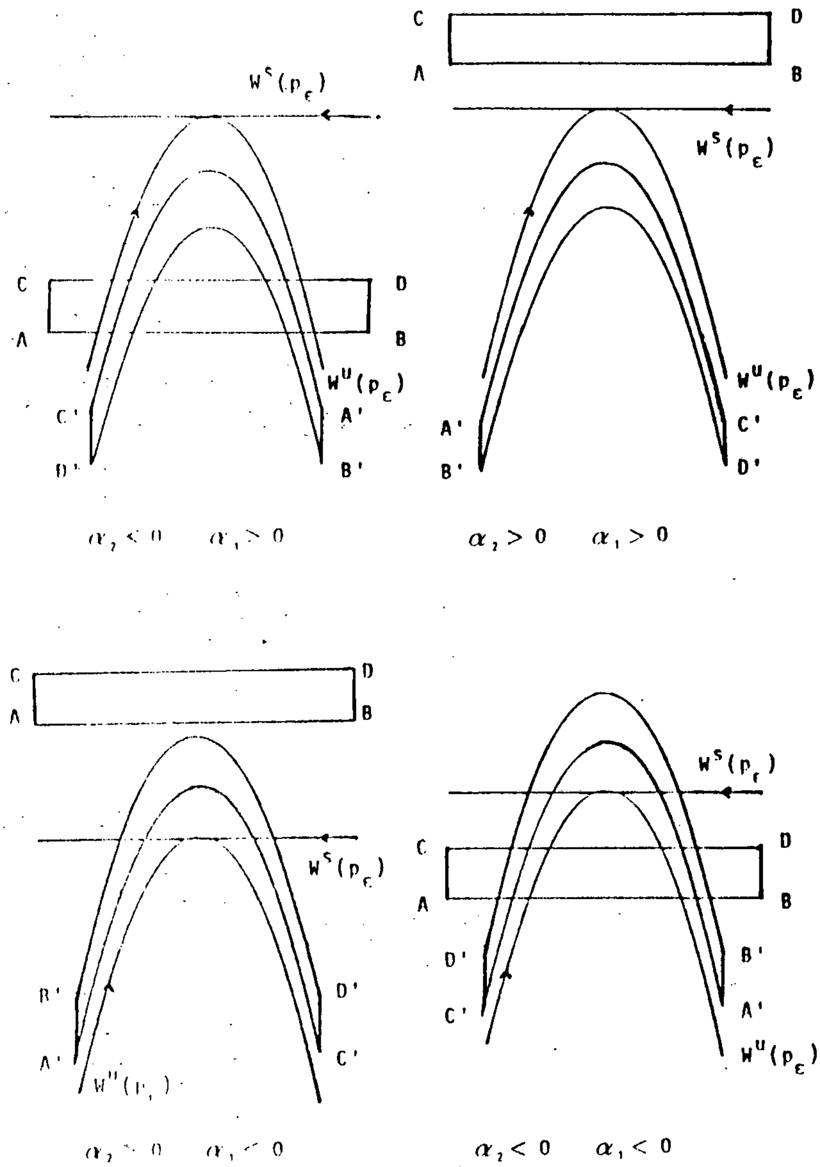


Figure 3.5: Case 3.

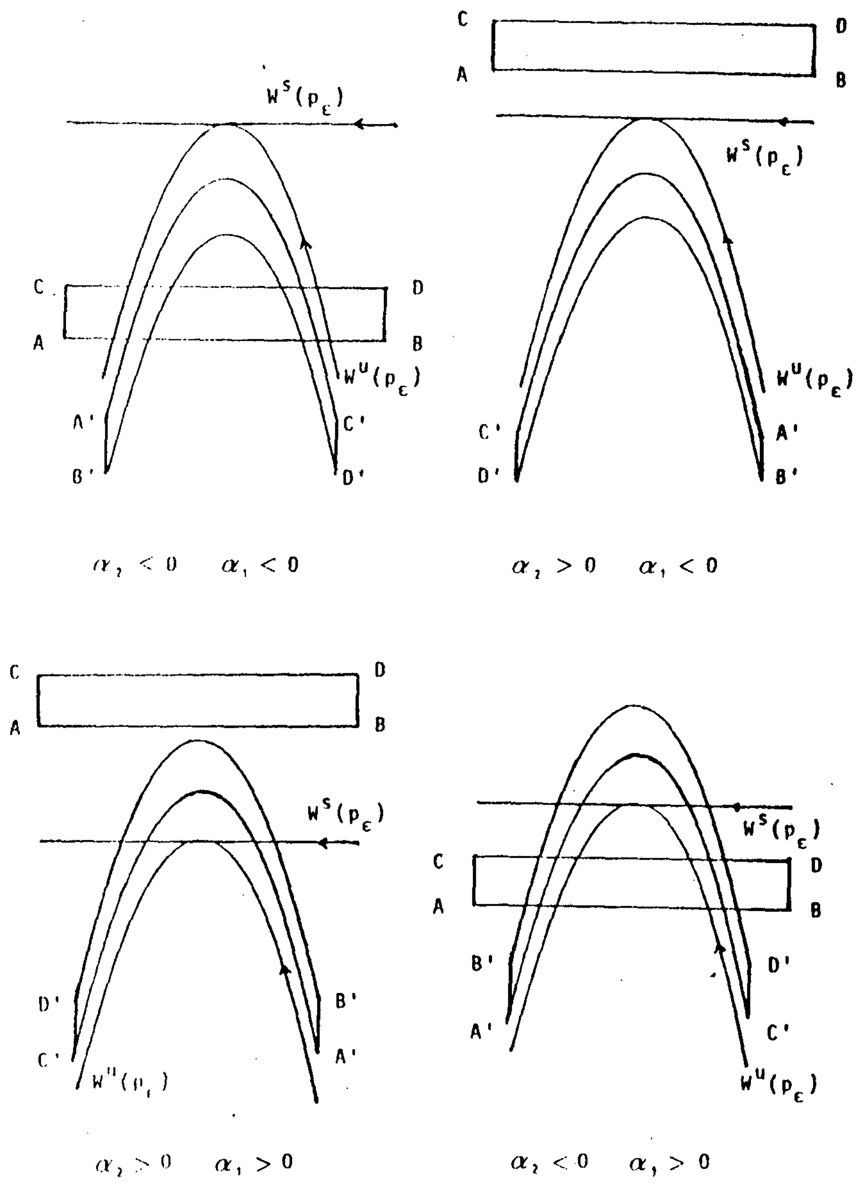


Figure 3.6: Case 4.

3.3 Periodic points and bifurcations

We analize in this section the fixed points of maps 3.12, 3.13, 3.14 and 3.15, and the corresponding bifurcations of this points.

From now on, when we put $x_n \approx y_n$ it means $\lim_{n\to\infty} \frac{x_n}{y_n} = 1$, where x_n and y_n belong to $\mathbb{R}$ for all $n \in \mathbb{N}$. The following properties are supposed to hold for n large enough.

We will study the first model (3.12). After we give the results corresponding to the other models:

Fixed points

The equations of the fixed points of $f^n_{a_0+\epsilon}$ are:

$$\xi = 1 - \alpha_2^n \eta$$

$$\eta = (1 - \alpha_2^n \eta)^2 + \epsilon + \alpha_1^n (1 + \xi)$$

Hence:

$$\alpha_2^{2n}\eta^2 - (2\alpha_2^n + \alpha_2^n\alpha_1^n + 1)\eta + 1 + \epsilon + 2\alpha_1^n = 0$$

and, therefore,

$$\eta = \frac{1 + 2\alpha_2^n + \alpha_1^n\alpha_2^n \pm \sqrt{4\alpha_2^n(1 - \alpha_1^n\alpha_2^n) + (\alpha_1^n\alpha_2^n + 1)^2 - 4\epsilon\alpha_2^{2n}}}{2\alpha_2^{2n}} \tag{3.16}$$

So, there exist periodic points if

$$(1 + 2\alpha_2^n + \alpha_1^n\alpha_2^n)^2 \geq 4\alpha_2^{2n}(1 + \epsilon + 2\alpha_1^n) \ .$$

Parameter of saddle-node bifurcation

The parameter for which the fixed points appear is:

$$\epsilon_{1n} = \alpha_2^{-n} - \alpha_1^n + \frac{1}{4}(\alpha_2^{-n} + \alpha_1^n)^2 \approx \alpha_2^{-n} \ .$$

For this parameter the fixed point is:

$$(\xi_0, \eta_0) = \left(-\frac{1}{2}\alpha_2^{-n} - \frac{1}{2}\alpha_1^n, \alpha_2^{-n} + \frac{1}{2}\alpha_2^{-n}\alpha_1^n + \frac{1}{2}\alpha_2^{-2n}\right) \approx \left(-\frac{1}{2}\alpha_2^{-n}, \alpha_2^{-n}\right) \ .$$

Proposition 3.3.1 *Suppose that p_ϵ is the original dissipative saddle fixed point. Let W^u be the subset of $\mathcal{W}^u(p_\epsilon)$ such that it is, locally, the parabola $\eta = \xi^2 + \epsilon$ in a neighbourhood of $(0,0)$, and let W^s be the subset of $\mathcal{W}^s(p_\epsilon)$ which is, locally, the straight line $\eta = 0$. Then:*

a) $\lim_{n\to\infty} \epsilon_{1n} = 0$.

b) For n large enough $\operatorname{sign}\epsilon_{1n} = \operatorname{sign}\alpha_2^{-n}$.

c) If α_1 , $\alpha_2 > 0$ *or* n *is even then* $\epsilon_{1n} > 0$ *and when the bifurcation occurs one has* $W^u \cap W^s = \emptyset$.

d) If $\alpha_1 > 0$, $\alpha_2 < 0$ *and* n *is odd, then* $\epsilon_{1n} < 0$ *and when the bifurcation occurs one has* $W^u \cap W^s \neq \emptyset$.

e) If $\alpha_1 < 0$, $\alpha_2 > 0$ *and* n *is odd, then* $\epsilon_{1n} > 0$ *and when the bifurcation occurs one has* $W^u \cap W^s = \emptyset$.

f) If α_1 , $\alpha_2 < 0$ *and* n *is odd, then* $\epsilon_{1n} < 0$ *and when the bifurcation occurs one has* $W^u \cap W^s \neq \emptyset$.

g) The fixed points of $f^n_{a_0+\epsilon}$, *for* ϵ *between* 0 *and* $2\epsilon_{1n}$ *are in the rectangle*

$$\bar{R} = [-2|\alpha_2|^{-n/2}, 2|\alpha_2|^{-n/2}] \times [\alpha_2^{-n} - 2|\alpha_2|^{-3n/2}, \alpha_2^{-n} + 2|\alpha_2|^{-3n/2}] \ .$$

The proof is a simple check.

Parameter of flip bifurcation

The two periodic points born due to a saddle-node bifurcation are transformed into one saddle and one node. The node is an attracting periodic point. When the parameter ϵ is suitable changed, this point becomes a saddle periodic point of double period through a flip (also called period doubling) bifurcation.

The differential of the map $f^n_{a_0+\epsilon}$ is:

$$Df^n_{a_0+\epsilon}(\xi,\eta) = \begin{pmatrix} 0 & -\alpha_2^n \\ \alpha_1^n & 2\alpha_2^n(\alpha_2^n\eta - 1) \end{pmatrix} .$$

Thus, when the flip bifurcation takes place, the eigenvalues are -1 and $-\alpha_1^n\alpha_2^n$. Therefore $1 + \alpha_1^n\alpha_2^n = 2\alpha_2^n(1 - \alpha_2^n\eta)$, where η has the value given in 3.16 with negative sign.

Then the parameter of flip bifurcation is:

$$\epsilon_{2n} = -\frac{3}{4}(\alpha_2^{-n} + \alpha_1^n)^2 + \alpha_2^{-n} - \alpha_1^n \ .$$

Proposition 3.3.2 *For n large enough:*

a) $\lim_{n\to\infty} \epsilon_{2n} = 0$.

b) $\epsilon_{1n} - \epsilon_{2n} = -(\alpha_2^{-n} + \alpha_1^n)^2 \approx -\alpha_2^{-2n}$. *Then the parameter of flip bifurcation is closer to the parameter of homoclinic tangency than* ϵ_{1n} *if* $\alpha_2^n > 0$ *and further if* $\alpha_2^n < 0$.

Second flip bifurcation

After the flip bifurcation, the attracting periodic orbit which appears, has again a flip bifurcation and the same occurs for this new orbit, and so on. This is the so-called cascade of period doubling bifurcations.

Let $(\xi_{i+1},\eta_{i+1}) = f^n_{a_0+\epsilon}(\xi_i,\eta_i)$. Then:

$$\xi_{i+1} = 1 - \alpha_2^n \eta_i \ ,$$

$$\eta_{i+1} = (1-\alpha_2^n\eta_i)^2 + \epsilon + \alpha_1^n(1+\xi_i) \ .$$

Thus a two-periodic orbit $\{(\xi_0,\eta_0)\ ,\ (\xi_1,\eta_1)\}$ satisfies:

$$\begin{array}{rcl} x_1 & = & 1-\alpha_2^n x_0^2 - \alpha_2^n\epsilon - \alpha_1^n\alpha_2^n(1+x_1) \\ x_0 & = & 1-\alpha_2^n x_1^2 - \alpha_2^n\epsilon - \alpha_1^n\alpha_2^n(1+x_0) \end{array} \tag{3.17}$$

where $x_i = 1-\alpha_2^n\eta_i = \xi_{i+1}$. By substracting the two expressions we get:

$$x_1 - x_0 = (\alpha_2^n(x_1+x_0) - \alpha_1^n\alpha_2^n)(x_1-x_0) \ .$$

As the orbit has to be of strict period two, one has $x_1 \neq x_0$ and, therefore,

$$x_1 + x_0 = \alpha_2^{-n} + \alpha_1^n \tag{3.18}$$

By adding the two equations in 3.17 we obtain:

$$x_1 + x_0 = 2 - \alpha_2^n((x_0+x_1)^2 - 2x_0x_1) - 2\alpha_2^n\epsilon - 2\alpha_1^n\alpha_2^n - \alpha_1^n\alpha_2^n(x_0+x_1) \ ,$$

and taking into account 3.18, one has:

$$x_0x_1 = (\alpha_2^{-n} + \alpha_1^n)^2 + \alpha_1^n - \alpha_2^{-n} + \epsilon \ .$$

We conclude that x_0 , x_1 satisfy the equation in x:

$$x^2 - (\alpha_1^n + \alpha_2^{-n})x + (\alpha_1^n+\alpha_2^{-n})^2 + \alpha_1^n - \alpha_2^{-n} + \epsilon = 0 \ .$$

The differential of $f^n_{a_0+\epsilon}$ at the point (ξ_i,η_i) is:

$$Df^n_{a_0+\epsilon}(\xi_i,\eta_i) = \begin{pmatrix} -\alpha_1^n\alpha_2^n & 2\alpha_2^{2n}x_i \\ -2\alpha_1^n\alpha_2^n x_j & -\alpha_1^n\alpha_2^n + 4\alpha_2^{2n}x_ix_j \end{pmatrix} \ ,$$

where $j=1$ if $i=0$ and $j=0$ if $i=1$. Therefore, when ϵ is the parameter corresponding to flip bifurcation, it satisfies:

$$\mathrm{tr}\, Df^{2n}_{a_0+\epsilon}(\xi_i,\eta_i) = -2\alpha_1^n\alpha_2^n + 4\alpha_2^{2n}x_0x_1 = -1 - \alpha_1^{2n}\alpha_2^{2n} \ ,$$

because at flip bifurcation the eigenvalues are -1 and $-\alpha_1^{2n}\alpha_2^{2n}$.

Taking into account the value of x_0x_1 we obtain the parameter of the second flip bifurcation:

$$\epsilon_{4n} = -(\alpha_2^{-n} + \alpha_1^n)^2 - \frac{1}{4}(\alpha_2^{-n} - \alpha_1^n)^2 + \alpha_2^{-n} - \alpha_1^n \ .$$

Proposition 3.3.3 *If n is large enough:*

a) $\epsilon_{2n} - \epsilon_{4n} \approx \frac{1}{2}\alpha_2^{-2n}$.

b) $\lim_{n\to\infty}(\epsilon_{1n} - \epsilon_{2n})/(\epsilon_{2n} - \epsilon_{4n}) = 2$.

This means that ϵ_{4n} is closer to the parameter of tangency than ϵ_{2n} if $\alpha_2^n > 0$ and further if $\alpha_2^n < 0$.

Third flip bifurcation

To study the third flip bifurcation we consider first the following change of coordinates and parameter.

$$x = \alpha_2^n - \alpha_2^{2n}\eta \ ,$$
$$y = \alpha_2^n\xi \ ,$$
$$\epsilon = \alpha_2^{-n} - \alpha_1^n + \alpha_2^{-2n}\bar{\epsilon} \ .$$

Then the map 3.12 is transformed into:

$$g_{\bar{\epsilon}}^n(x,y) = \begin{pmatrix} -\bar{\epsilon} - x^2 - \alpha_1^n\alpha_2^n y \\ x \end{pmatrix} \ .$$

The 4-periodic orbit $(\vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x}_3)$ $(\vec{x}_i = (x_i, y_i))$ satisfies:

$$\begin{array}{rcl} x_2 & = & -\bar{\epsilon} - x_1^2 - \alpha_1^n\alpha_2^n x_0 \\ x_3 & = & -\bar{\epsilon} - x_2^2 - \alpha_1^n\alpha_2^n x_1 \\ x_0 & = & -\bar{\epsilon} - x_3^2 - \alpha_1^n\alpha_2^n x_2 \\ x_1 & = & -\bar{\epsilon} - x_0^2 - \alpha_1^n\alpha_2^n x_3 \end{array} \qquad (3.19)$$

Let $S = x_0 + x_1 + x_2 + x_3$, $P = x_0x_1x_2x_3$ and $C = (x_0 + x_2)(x_1 + x_3)$. If in the system 3.19 one substracts the third row from the first one, and the second from the forth one, one obtains:

$$x_2 - x_0 = (x_1 + x_3)(x_3 - x_1) + \alpha_1^n\alpha_2^n(x_2 - x_0) \ ,$$
$$x_3 - x_1 = (x_0 + x_2)(x_2 - x_0) + \alpha_1^n\alpha_2^n(x_3 - x_1) \ .$$

For solutions of strict period four: $x_2 \neq x_0$ or $x_1 \neq x_3$. Therefore:

$$(\alpha_1^n\alpha_2^n - 1)^2 + (x_0 + x_2)(x_1 + x_3) = 0 \ .$$

Then we have:

Proposition 3.3.4 *If x_0 , x_1 , x_2 , x_3 satisfy the equation 3.19 and $x_2 \neq x_0$ or $x_1 \neq x_3$, then $C = -(\alpha_1^n \alpha_2^n - 1)^2$.*

There exist other relations between S , C and P :

Proposition 3.3.5 *Under the conditions of the previous proposition, the following relation holds:*

$$S^3 + (3(\alpha_1^n \alpha_2^n - 1)^2 + 4\bar{\epsilon})S - 4(1 + \alpha_1^n \alpha_2^n)(\alpha_1^n \alpha_2^n - 1)^2 = 0 . \tag{3.20}$$

Proof:

It is easy to see that:

$$S^3 - 3CS = x_0^3 + x_1^3 + x_2^3 + x_3^3 + 3x_0^2 x_2 + 3x_1^2 x_3 + 3x_0 x_2^2 + 3x_1 x_3^2 . \tag{3.21}$$

From the system 3.19 we obtain:

$$x_2 x_1 = -\bar{\epsilon} x_1 - x_1^3 - \alpha_1^n \alpha_2^n x_0 x_1$$

$$x_3 x_2 = -\bar{\epsilon} x_2 - x_2^3 - \alpha_1^n \alpha_2^n x_1 x_2$$

$$x_2 x_1 = -\bar{\epsilon} x_3 - x_3^3 - \alpha_1^n \alpha_2^n x_2 x_3$$

$$x_2 x_1 = -\bar{\epsilon} x_0 - x_0^3 - \alpha_1^n \alpha_2^n x_3 x_0$$

Hence:

$$(1 + \alpha_1^n \alpha_2^n)C = -\bar{\epsilon} S - (x_0^3 + x_1^3 + x_2^3 + x_3^3) \tag{3.22}$$

It also holds:

$$x_2 x_3 = -\bar{\epsilon} x_3 - x_1^2 x_3 - \alpha_1^n \alpha_2^n x_0 x_3$$

$$x_3 x_0 = -\bar{\epsilon} x_0 - x_2^2 x_0 - \alpha_1^n \alpha_2^n x_1 x_0$$

$$x_0 x_1 = -\bar{\epsilon} x_1 - x_3^2 x_1 - \alpha_1^n \alpha_2^n x_2 x_1$$

$$x_1 x_2 = -\bar{\epsilon} x_2 - x_1^2 x_2 - \alpha_1^n \alpha_2^n x_3 x_2$$

Therefore:

$$x_1^2 x_3 + x_2^2 x_0 + x_3^2 x_1 + x_0^2 x_2 = -\bar{\epsilon} S - (1 + \alpha_1^n \alpha_2^n)C . \tag{3.23}$$

Then, taking into account that $C = -(\alpha_1^n \alpha_2^n - 1)^2$ and by using 3.21, 3.22, and 3.23, we obtain 3.20. □

Proposition 3.3.6 *Under the hypothesis of the previous propositions, P and S satisfy the equation:*

$$\begin{aligned} 0 = 4P &- 4\bar{\epsilon}^2 - 4\bar{\epsilon}\alpha_1^{2n}\alpha_2^{2n} + 8\bar{\epsilon}\alpha_1^n \alpha_2^n - 4\bar{\epsilon} - 4\alpha_1^{4n}\alpha_2^{4n} \\ &+ 4\alpha_1^{3n}\alpha_2^{3n} + 4\alpha_1^n \alpha_2^n - 4 + 2\bar{\epsilon}(1 + \alpha_1^n \alpha_2^n)S - 2\bar{\epsilon} S^2 \end{aligned} \tag{3.24}$$

Proof:

$$C^2 = 4P + x_0^2x_1^2 + x_0^2x_3^2 + x_1^2x_2^2 + x_2^2x_3^2 + 2x_0^2x_1x_3 + 2x_0x_1^2x_2 + 2x_0x_2x_3^2 + 2x_1x_2^2x_3 \ . \quad (3.25)$$

From the system 3.19 we have:

$$x_2^3 = -\bar{\epsilon}x_2^2 - x_1^2x_2^2 - \alpha_1^n\alpha_2^n x_0x_2^2$$

$$x_3^3 = -\bar{\epsilon}x_3^2 - x_2^2x_3^2 - \alpha_1^n\alpha_2^n x_1x_3^2$$

$$x_0^3 = -\bar{\epsilon}x_0^2 - x_3^2x_0^2 - \alpha_1^n\alpha_2^n x_2x_0^2$$

$$x_1^3 = -\bar{\epsilon}x_1^2 - x_0^2x_1^2 - \alpha_1^n\alpha_2^n x_3x_1^2$$

Hence:

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = -\bar{\epsilon}(x_0^2 + x_1^2 + x_2^2 + x_3^2) - (x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_0^2 + x_0^2x_1^2) -$$

$$- \alpha_1^n\alpha_2^n(x_0x_2^2 + x_1x_3^2 + x_0x_2^2 + x_3x_1^2) \ . \quad (3.26)$$

From the system 3.19 we obtain:

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = -4\bar{\epsilon} - (1 + \alpha_1^n\alpha_2^n)S \quad (3.27)$$

and from 3.22, 3.23, 3.27 and 3.26:

$$x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_0^2 + x_0^2x_1^2 = \bar{\epsilon}^2 + (1 + \alpha_1^n\alpha_2^n)^2C + 2\bar{\epsilon}(1 + \alpha_1^n\alpha_2^n)S \ .$$

From the system 3.19 we have:

$$x_2^2x_0 = -\bar{\epsilon}x_0x_2 - x_1^2x_0x_2 - \alpha_1^n\alpha_2^n x_0^2x_2 \ ,$$

$$x_3^2x_1 = -\bar{\epsilon}x_1x_3 - x_2^2x_1x_3 - \alpha_1^n\alpha_2^n x_1^2x_3 \ ,$$

$$x_0^2x_2 = -\bar{\epsilon}x_2x_0 - x_3^2x_2x_0 - \alpha_1^n\alpha_2^n x_2^2x_0 \ ,$$

$$x_1^2x_3 = -\bar{\epsilon}x_3x_1 - x_0^2x_3x_1 - \alpha_1^n\alpha_2^n x_3^2x_1 \ .$$

Therefore:

$$(1 + \alpha_1^n\alpha_2^n)(x_2^2x_0 + x_3^2x_1 + x_0^2x_2 + x_1^2x_3) =$$

$$- 2\bar{\epsilon}(x_0x_2 + x_1x_3) - (x_1^2x_0x_2 + x_2^2x_1x_3 + x_3^2x_0x_2 + x_0^2x_1x_3) \ . \quad (3.28)$$

Taking into account 3.27:

$$S^2 = -4\bar{\epsilon} - (1 + \alpha_1^n\alpha_2^n)S + 2C + 2(x_0x_2 + x_1x_3) \ .$$

From this equation and 3.23, we have, by substitution in 3.28,

$$x_1^2x_0x_2 + x_2^2x_1x_3 + x_3^2x_0x_2 + x_0^2x_1x_3 =$$

$$-\bar{\epsilon}S^2 - 4\bar{\epsilon}^2 + 2\bar{\epsilon}C + (1+\alpha_1^n\alpha_2^n)^2C \ .$$

Finally, taking into account the expression of C^2 in 3.25 and the value of C, we get 3.24. □

Proposition 3.3.7 *For the parameter of the third flip bifurcation ϵ_{8n} corresponding to 4-periodic points, it is satisfied:*

$$P = -\frac{1}{16} - \frac{1}{4}\alpha_1^n\alpha_2^n + \frac{3}{8}\alpha_1^{2n}\alpha_2^{2n} - \frac{1}{4}\alpha_1^{3n}\alpha_2^{3n} - \frac{1}{16}\alpha_1^{4n}\alpha_2^{4n} \ . \tag{3.29}$$

The jacobian matrix of the map g_ϵ^{4n} is:

$$Dg_\epsilon^{4n}(x_0,y_0) = \begin{pmatrix} 4x_3x_2 - \alpha_1^n\alpha_2^n & 2\alpha_1^n\alpha_2^n x_3 \\ -2x_2 & -\alpha_1^n\alpha_2^n \end{pmatrix} \begin{pmatrix} 4x_1x_0 - \alpha_1^n\alpha_2^n & 2\alpha_1^n\alpha_2^n x_1 \\ -2x_0 & -\alpha_1^n\alpha_2^n \end{pmatrix}$$

For ϵ_{8n} the trace of the jacobian has to be $-1-\alpha_1^{4n}\alpha_2^{4n}$. Therefore:

$$\operatorname{tr} Dg_\epsilon^{4n}(x_0,y_0) = 16P - 4\alpha_1^n\alpha_2^nC + 2\alpha_1^n\alpha_2^n = -1 - \alpha_1^{4n}\alpha_2^{4n} \ .$$

If we take into account that $C = -(\alpha_1^n\alpha_2^n - 1)^2$, we obtain 3.29. □

Corollary 3.3.8 *Under the condition of the previous propositions if $S = x_0+x_1+x_2+x_3$ and $\bar{\epsilon} = \epsilon_{8n}\alpha_2^{2n} - \alpha_2^n + \alpha_1^n\alpha_2^{2n}$, then S and $\bar{\epsilon}$ satisfy:*

$$S^3 + (3(\alpha_1^n\alpha_2^n-1)^2 + 4\bar{\epsilon})S - 4(1+\alpha_1^n\alpha_2^n)(\alpha_1^n\alpha_2^n-1)^2 = 0 \ ,$$

$$2\bar{\epsilon}S^2 = 2\bar{\epsilon}(1+\alpha_1^n\alpha_2^n)S - 4\bar{\epsilon}^2 - 4\bar{\epsilon}\alpha_1^{2n}\alpha_2^{2n} + 8\bar{\epsilon}\alpha_1^n\alpha_2^n -$$

$$4\bar{\epsilon} - \frac{17}{4} + 3\alpha_1^n\alpha_2^n + \frac{3}{2}\alpha_1^{2n}\alpha_2^{2n} + 3\alpha_1^{3n}\alpha_2^{3n} - \frac{17}{4}\alpha_1^{4n}\alpha_2^{4n} \ .$$

This system is derived from the previous propositions.

Proposition 3.3.9 *There exists* $\lim_{n\to\infty} \frac{\epsilon_{2n}-\epsilon_{4n}}{\epsilon_{4n}-\epsilon_{8n}} = 4.23373...$

Proof:

Recall that $\epsilon = \alpha_2^{-n} - \alpha_1^n + \bar{\epsilon}\alpha_2^{-2n}$. From corollary 3.3.8, if we take $\alpha_1^n\alpha_2^n = t = 0$, we obtain the system:

$$S^3 + (3+4\bar{\epsilon})S - 4 = 0 \ ,$$

$$-2\bar{\epsilon}S^2 + 2\bar{\epsilon}S - 4\bar{\epsilon}^2 - 4\bar{\epsilon} - \frac{17}{4} = 0 \ .$$

Then $\bar{\epsilon}$ and S satisfy:

$$\bar{\epsilon} = -\frac{S^3 + 3S - 4}{4S} ,$$

$$S^6 - 2S^5 + 4S^4 - 6S^3 - 6S^2 + 8S - 16 = 0 .$$

The polynomial on S has only two real roots, which are: $S_{10} = -1.2965435...$ and $S_{20} = 2.0935646...$ Associated to these values we obtain two values of $\bar{\epsilon}$: $\bar{\epsilon}_{10} = -1.9415376...$ and $\bar{\epsilon}_{20} = -1.3680989...$

Now we want to get solutions $\bar{\epsilon}(t)$, $S(t)$ of:

$$F_1(t, \bar{\epsilon}, S) = 0 \ , F_2(t, \bar{\epsilon}, S) = 0 ,$$

where

$$F_1(t, \bar{\epsilon}, S) = S^3 + (3(t-1)^2 + 4\bar{\epsilon})S - 4(1+t)(t-1)^2 ,$$

$$F_2(t, \bar{\epsilon}, S) = -2\bar{\epsilon}S^2 + 2\bar{\epsilon}(1+t)S - 4\bar{\epsilon}^2 - 4\bar{\epsilon}t^2 + 8\bar{\epsilon}t - 4\bar{\epsilon} - \frac{17}{4} + 3t + \frac{3}{2}t^2 + 3t^3 - \frac{17}{4}t^4 ,$$

such that $\bar{\epsilon}(0) = \bar{\epsilon}_{10}$ and $S(0) = S_{10}$, or $\bar{\epsilon}(0) = \bar{\epsilon}_{20}$ and $S(0) = S_{20}$. We denote these solutions as $\bar{\epsilon}_1 = \bar{\epsilon}_1(t)$, $S_1 = S_1(t)$ and $\bar{\epsilon}_2 = \bar{\epsilon}_2(t)$, $S_2 = S_2(t)$. Let $F = (F_1, F_2)$. Then:

$$D_{\bar{\epsilon},S}F(0, \bar{\epsilon}_{i0}, S_{i0}) = \begin{pmatrix} 3S_{i0}^2 + 3 + 4\bar{\epsilon}_{i0} & -4S_{i0} \\ 2\bar{\epsilon}_{i0} - 4\bar{\epsilon}_{i0}S_{i0} & 8\bar{\epsilon}_{i0} + 4 - 2S_{i0} + 2S_{i0}^2 \end{pmatrix}$$

where $i = 1, 2$. We claim that $\det D_{\bar{\epsilon},S}F(0, \bar{\epsilon}_{i0}, S_{i0}) \neq 0$ for $i = 1, 2$. Then, by the implicit function theorem, there exist functions $\bar{\epsilon}_i = \bar{\epsilon}_i(t)$ and $S_i = S_i(t)$ such that $\bar{\epsilon}_i(0) = \bar{\epsilon}_{i0}$, $S_i(0) = S_{i0}$, and

$$F_1(t, \bar{\epsilon}_i(t), S_i(t)) = 0 ,$$

$$F_2(t, \bar{\epsilon}_i(t), S_i(t)) = 0 ,$$

in a neighbourhood of $t = 0$. Let us see that the claim holds:

$$\det D_{\bar{\epsilon},S}F(0, \bar{\epsilon}_{i0}, S_{i0}) =$$

$$6S_{i0}^4 - 6S_{i0}^3 + 16\bar{\epsilon}_{i0}S_{i0}^2 + 18S_{i0}^2 + 32\bar{\epsilon}S_{i0}^2 - 6S_{i0} + 40\bar{\epsilon}_{i0} + 12 .$$

By setting $\bar{\epsilon}_{i0} = -\frac{S_{i0}^3 + 3S_{i0} - 4}{4S_{i0}}$ we obtain the polynomial:

$$6S_{i0}^6 - 3S_{i0}^3 + 26S_{i0}^4 - 19S_{i0}^3 + 30S_{i0}^2 - 44S_{i0} + 16 .$$

This expression is different from zero if $i = 1, 2$ because the roots of this polynomials are all possitive and less than 2 . Then we have that:

$$\bar{\epsilon}_i = \bar{\epsilon}_{i0} + O(\alpha_1^n \alpha_2^n) \ , \ S_i = S_{i0} + O(\alpha_1^n \alpha_2^n) .$$

Therefore there exist two possible parameter of bifurcation. But it is known for the logistic map $g_a(x) = a + x^2$ that the parameter of the third flip bifurcation is $a \approx 1.3680989$ (see [3]). Then:

$$\epsilon_{8n} = \alpha_2^{-n} - \alpha_1^n - 1.3680989\ldots\alpha_2^{-2n} + O(\alpha_1^n\alpha_2^{-n}) \ .$$

Hence:

$$\lim_{n\to\infty} \frac{\epsilon_{2n} - \epsilon_{4n}}{\epsilon_{4n} - \epsilon_{8n}} = \lim_{n\to\infty} \frac{-\frac{1}{2}\alpha_2^{-2n} + O(\alpha_1^n\alpha_2^{-n})}{-\frac{5}{4} + \bar{\epsilon}_{20})\alpha_2^{-2n} + O(\alpha_1^n\alpha_2^{-n})} = 4.23373\ldots \square$$

Other flip and saddle-node bifurcations related to the Newhouse phenomenon

We have seen that, in the space of parameters, thre exists a family of open intervals I_n, of lenght going to zero, which tends to the parameter of homoclinic tangency and such that, for parameters belonging to these intervals, we have an attracting periodic orbit of period n. These periodic orbits bifurcate to periodic orbits of period $2n$, $4n$, $8n$. It is possible also to compute the bifurcations of attracting periodic orbits of period $2^j n$ with $j \in \mathbb{N}$. The limit of these bifurcation parameters, if it exists, $\epsilon_{2^\infty n} = \lim_{j\to\infty} \epsilon_{2^j n}$ must be far from zero because $\epsilon_{2^j n} = \alpha_2^{-n} - \alpha_1^n + \bar{\epsilon}_{2^j n}\alpha_2^{-2n} + h.o.t.$ Also the parameters $\epsilon_{2^j n}$ seem to satisfy, for n large enough:

$$\lim_{j\to\infty} \frac{\epsilon_{2^j n} - \epsilon_{2^{j-1} n}}{\epsilon_{2^{j+1} n} - \epsilon_{2^j n}} = \delta \ ,$$

where $\delta = 4.66920...$ is the Feigenbaum constant (see [3]).

On the other hand, the behaviour of the bifurcation cascades due to the Newhouse phenomenon is similar to the behaviour of the bifurcations cascades for the logistic map, $f_a(x) = 1 - ax^2$ or $g_a(x) = a - x^2$, in the following sense: If a_{2^j} is the parameter corresponding to bifurcation of periodic points of period 2^j, which come from the saddle-node bifurcation of the fixedpoint of g_a, then:

$$\lim_{n\to\infty} \frac{\epsilon_{2^j n} - \epsilon_{2^{j-1} n}}{\epsilon_{2^{j+1} n} - \epsilon_{2^j n}} = \frac{a_{2^j} - a_{2^{j-1}}}{a_{2^{j+1}} - a_{2^j}} \ .$$

This seems true since the return map g_ϵ^n tends to the logistic map when $n \to \infty$.

Moreover, there exist other periodic orbits related to the Newhouse phenomenon. We can findd periodic orbits of period $n_1 + n_2 + ... + n_m$ with $\tilde{n} < n < \bar{n}$, such that n_i is associated to a horseshoe map in the following sense: Suppose, for simplicity, that $m = 3$. Then for all n_i we can define a horshoe map by using $f_{a_0+\epsilon}^{n_i}$, as we see in the figure 3.7 (recall that $f_{a_0+\epsilon}^{n_i}$ is the original quadratic model). It can exist a $q \in R_1$ such that

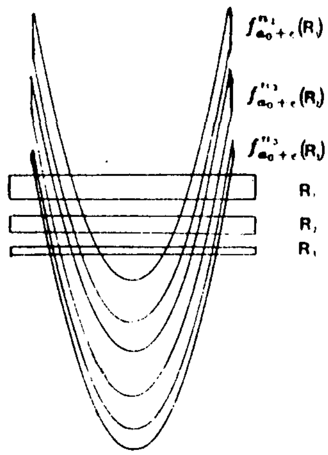


Figure 3.7: Other periodic points related to the Newhouse phenomenon.

$f^{n_1}_{a_0+\epsilon}(q) \in R_3$, $f^{n_1+n_2}_{a_0+\epsilon}(q) \in R_2$ and $f^{n_1+n_2+n_3}_{a_0+\epsilon}(q) = q$. These periodic points are not considered in our study of bifurcations. In this case, the width of the parameter values ϵ for which these periodic orbits are attracting, is smaller than the widths corresponding to periods n_1 , n_2 and n_3 . Also, inside R_i, there are periodic orbits of period pn_i for all p because there exists, in a horseshoe map, periodic orbits of any period. We have to note, too, that if one or two of the eigenvalues α_1 or α_2 are negative, there are two different types of periodic orbits. One of these types corresponds to the case of positive eigenvalues.

Finally, by using a theorem of [18], one can see that there exist a parameter ϵ_1 near 0, an open neighbourhood A_{ϵ_1} of ϵ_1 and an open and dense subset B of A_{ϵ_1}, such that: if $\epsilon \in B$ then $f_{a_0+\epsilon}$ has an attracting periodic orbit.

For the maps corresponding to formulae 3.13, 3.14, 3.15, propositions similar to 3.3.1, 3.3.2, 3.3.3 and 3.3.1 hold ([28]). We list the differences:

Assertion	cases 3.14 and 3.15
3.3.1	The parabola should be $\eta = -\xi + \epsilon$
3.3.1 c)	$W^u \cap W^s \neq \emptyset$
3.3.1 d)	$W^u \cap W^s = \emptyset$
3.3.1 e)	$W^u \cap W^s \neq \emptyset$
3.3.1 f)	$W^u \cap W^s = \emptyset$
3.3.2 b)	$\epsilon_{2n} - \epsilon_{1n} \approx \alpha_2^{-2n}$
3.3.3 a)	$\epsilon_{2n} - \epsilon_{4n} \approx \frac{1}{2}\alpha_2^{-2n}$

3.4 Behaviour of the basin of attraction of the attracting periodic orbits

Given an attracting periodic orbit created by the Newhouse phenomenon, we want to see if a strange attractor may exist. If p is a saddle fixed point and $\mathcal{W}^u(p)$ is the unstable invariant manifold of p, then $\overline{\mathcal{W}^u(p)}$ can be a strange attractor (see [29] in the case of the Hénon map). To show that this attractor cannot exist, where there are attracting periodic orbits, it is enough to prove that the basin of attraction of the periodic attractor has non empty intersection with $\mathcal{W}^u(p)$.

Proposition 3.4.1 *Let N be an attracting m-periodic point of a diffeomorphism f over a two-dimensional manifold. Suppose that Λ is a closed attracting set such that $N \notin \Lambda$. Then, for the basin of attraction of N, $\mathcal{W}^s(N)$, one has $\mathcal{W}^s(N) \cap \Lambda = \emptyset$.*

Proof:

If $x \in \mathcal{W}^s(N) \cap \Lambda$, as Λ is closed an invariant, then $f^{mn}(x) \in \Lambda$ for all n and $\lim_{n\to\infty} f^{nm}(x) = N \in \Lambda$, reaching an absurdity. □

Proposition 3.4.2 *Let $\{f_a\}_{a\in I}$ denotes a one-parameter family of dissipative diffeomorphisms in the plane. Let $p_{\bar{a}}$ a saddle-node fixed point which unfolds generically in $a = \bar{a}$. Then, for $a - \bar{a}$ small enough, there exists a branch $\mathcal{W}_1^u(S_a)$ of $\mathcal{W}^u(S_a)$ such that $\lim_{n\to\infty} f_a^n(x) = N_a$ if $x \in \mathcal{W}_1^u(S_a)$. S_a and N_a denote , respectively, the saddle and node which appear for a near $\bar{a}$ ($a > \bar{a}$ or $a < \bar{a}$ depending on the region of the parameters for which there exist periodic orbits).*

Proof:

By the center manifold theorem (see for example [30]) we know that there exists a two dimensional invariant surface, tangent to the eigenspace of $Dg(p_{\bar{a}}, \bar{a})$ belonging to the part of spectrum on the unit circle, with respect to $g(x, y, a) = f_a(x, y)$. Then, by a change of coordinates we can transform the map f_a to the map $\bar{f}_a$ such that: $\bar{f}_a(x, 0) = (\bar{f}_1(x, a), 0)$. Moreover the map $\bar{f}_1$ has a saddle-node in 0 which unfolds generically. It means that when $a \approx \bar{a}$ we have two fixed points which are in the invariant line $y = 0$. This straight line is the unstable manifold of one of the points and the other is an attractor. Then the proposition is proved. □

Remark 3.4.3 *If f_a is C^k then the center manifold is C^{k-1} .*

If we want to show that the basin of attraction of an attracting periodic orbit has non empty intersection with an unstable invariant manifold, it is enough to prove that, the stable invariant manifold of the saddle born by saddle-node bifurcation intersects the first unstable invariant manifold.

Proposition 3.4.4 *Let $\{f_a\}_{a\in I}$ denote a one-parameter family of diffeomorphisms with a saddle-node periodic point, which unfolds generically in $a = \bar{a}$. Let S_a and N_a be the saddle and the node which are born near $\bar{a}$, and p_a a saddle fixed point such that $\mathcal{W}^s(S_a)\cap\mathcal{W}^u(p_a) \neq \emptyset$. If $\mathcal{W}^s(N_a)$ is the basin of attraction of N_a then $\mathcal{W}^s(N_a)\cap \mathcal{W}^u(p_a) \neq \emptyset$.*

Proof:

Let $x \in \mathcal{W}^s(S_a)\cap \mathcal{W}^u(p_a)$. Consider $y \in \mathcal{W}^u(p_a)$ such that y is near to x . Let V be a neighbourhood of S_a, where the Hartmann theorem can be applied for the saddle S_a . There exists an $n_0 \in \mathbb{N}$ such that $f_{a_0}^{n_0}(x) \in V$, and, if y is close enough to x, $f_{a_0}^{n_0}(y) \in V$. By the Hartmann theorem, if $f_{a_0}^{n_0}(y)$ is close enough to the stable invariant manifold of S_a, then there exists $n_1 \in \mathbb{N}$ such that: $f_a^{n_0+n_1}(y)$ is close to $\mathcal{W}^u(S_a)$. As one of the branches of $\mathcal{W}^u(S_a)$ belongs to the basin of attraction of N_a, if we choose a suitable y (in one of the sides of $\mathcal{W}^s(S_a)$),then we shall have that $f_a^{n_0+n_1}(y)$ is close to the branch of $\mathcal{W}^u(S_a)$ which belongs to $\mathcal{W}^s(N_a)$. Therefore, if y is close enough to x then $f_a^{n_0+n_1}(y) \in \mathcal{W}^s(N_a)$. Then $y \in \mathcal{W}^s(N_a)$. □

To study the basin of attraction of the periodic attractors, first we shall study the map $f_{a_0+\epsilon}^n$ near the homoclinic tangency point:

Proposition 3.4.5 *Let $\{f_{a_0+\epsilon}\}_{a_0+\epsilon\in(-1,1)}$ be a C^r $(r \geq 3)$ one-parameter family of diffeomorphisms on the plane, having a dissipative saddle fixed point p_0 for $\epsilon = 0$ such that, the eigenvalues $\alpha_1(0)$, $\alpha_2(0)$ of Df_{a_0} satisfy: $0 < \alpha_1(0) < 1$ and $\alpha_2(0) > 1$. Suppose that the invariant manifolds of p_0 have a non degenerate homoclinic tangency in q which unfolds generically, and that the family verifies the hypothesis (A). Then there are, for each positive integer n, reparametrizations $\epsilon = M_n(\bar{\epsilon})$ of the ϵ variable and $\bar{\epsilon}$-dependent coordinate transformations $(\bar{x},\bar{y}) \longrightarrow \psi_{n,\bar{\epsilon}}(\bar{x},\bar{y})$, such that:*

a) For each compact set K in the $\bar{\epsilon},\bar{x},\bar{y}$ space, the images of K under the maps

$$(\bar{\epsilon},\bar{x},\bar{y}) \longrightarrow (M_n(\bar{\epsilon}),\psi_{n,\bar{\epsilon}}(\bar{x},\bar{y}))$$

converge, for $n \to \infty$, to $(0,q)$.

b) The domains of the maps

$$(\bar{\epsilon},\bar{x},\bar{y}) \longrightarrow (\bar{\epsilon},(\psi_{n,\bar{\epsilon}}^{-1}\circ f_{a_0+M_n(\bar{\epsilon})}\circ\psi_{n,\bar{\epsilon}}))$$

can be chosen as a fixed compact set.

c) The previous maps converge, for $n \to \infty$, to the map

$$(\bar{\epsilon},\bar{x},\bar{y}) \longrightarrow (\bar{\epsilon},\bar{f}_{\bar{\epsilon}}(\bar{x},\bar{y}))$$

with $\bar{f}_{\bar{\epsilon}}(\bar{x},\bar{y}) = (\bar{y},\bar{y}^2+\bar{\epsilon})$.

Suppose, moreover, that $\alpha_1(0)\alpha_2(0)^2 > 1$. *Then there exist other transformations* $\epsilon = M_n(\bar{\epsilon})$, $(\bar{x},\bar{y}) \longrightarrow \bar{\psi}_{n,\bar{\epsilon}}(\bar{x},\bar{y})$ *which satisfy a) and b), and the maps*

$$(\bar{\epsilon},\bar{x},\bar{y}) \longrightarrow (\bar{\epsilon},(\bar{\psi}_{n,\bar{\epsilon}}^{-1} \circ f_{a_0+M_n(\bar{\epsilon})} \circ \bar{\psi}_{n,\bar{\epsilon}}))$$

converge, for $n \to \infty$ *to the map*

$$(\bar{\epsilon},\bar{x},\bar{y}) \longrightarrow (\bar{\epsilon},\tilde{f}_{\bar{\epsilon}}(\bar{x},\bar{y}))$$

with $\tilde{f}_{\bar{\epsilon}}(\bar{x},\bar{y}) = (0, 1+\bar{x}-\bar{\epsilon}\bar{y}^2)$.

Proof:

As before (see begining of section 3.2), we can suppose that $(1,0) = q$ is the tangent homoclinic point, and that there exists a map $\vec{x}(t_1,t_2,\epsilon) = (x(t_1,t_2,\epsilon), y(t_1,t_2,\epsilon))$ in a neighbourhood of $B = \{(t_1,t_2,\epsilon) \ : \ \epsilon = 0, t_1t_2 = 0\}$ such that $\vec{x}_{|U} = Id$, being U a neighbourhood of $(0,0)$. We know also that $x(0,1,\epsilon) = 1$, $y(0,1,\epsilon) = \epsilon$ and $D_2y(0,1,\epsilon) = 0$. As the homoclinic tangency is non degenerate, we have $D_{22}y(0,1,0) \neq 0$. Then we have seen that the parameters of the saddle-node bifurcation, ϵ_n^+, and the filp bifurcation, ϵ_n^-, and their associated periodic points of period n, p_n^+ and p_n^-, verify:

$$\begin{aligned} \epsilon_n^+ &= \alpha_2^{-n} - \gamma\alpha_1^n + O(\alpha_2^{-2n}) \\ \epsilon_n^- &= \alpha_2^{-n} - \gamma\alpha_1^n + O(\alpha_2^{-2n}) \\ p_n^+ &= (1+O(\alpha_2^{-n}), \alpha_2^{-n} + O(\alpha_2^{-2n})) \\ p_n^- &= (1+O(\alpha_2^{-n}), \alpha_2^{-n} + O(\alpha_2^{-2n})) \end{aligned}$$

where $\gamma = D_1y(0,1,0)$.

Then, following [22], we consider the change of parameters and coordinate transformation:

$$\left.\begin{array}{rcl} \epsilon &=& \alpha_2^{-2n}\bar{\epsilon} - \gamma\alpha_1^n + \alpha_2^{-n} \\ t_1 &=& 1 + \alpha_2^{-n}\bar{t}_1 \\ t_2 &=& \alpha_2^{-n} + \alpha_2^{-2n}\bar{t}_2 \end{array}\right\} \tag{3.30}$$

for the map $f_{a_0+\epsilon}^n(t_1,t_2) = (x(\alpha_1^n t_1, \alpha_2^n t_2, \epsilon), y(\alpha_1^n t_1, \alpha_2^n t_2, \epsilon))$. By the properties of $\vec{x}$ we have:

$$x(t_1,t_2,\epsilon) = 1 + D_1x(0,1,0) + D_2x(0,1,0)(t_2-1) + x_1(t_1,t_2,\epsilon)$$
$$y(t_1,t_2,\epsilon) = \epsilon + \gamma t_1 + \frac{1}{2}D_{22}y(0,1,0)(t_2-1)^2 + y_1(t_1,t_2,\epsilon) \ ,$$

where for $\epsilon = t_1 = 0$ and $t_2 = 1$

$$x_1 = D_1x_1 = D_2x_1 = D_3x_1 = 0 \ ,$$

$$y_1 = D_1y_1 = D_2y_1 = D_3y_1 = D_{22}y_1 = D_{23}y_1 = D_{33}y_1 = 0 \; .$$

Then, let $\alpha = D_2x(0,1,0) \neq 0$ and $\beta = \frac{1}{2}D_{22}y(0,1,0) \neq 0$. By using the coordinate transformation 3.30 we obtain the new map:

$$g_\epsilon^n(\bar{t}_1,\bar{t}_2) = \begin{pmatrix} \alpha\bar{t}_2 + D_1x(0,1,0)\alpha_1^n\alpha_2^n\bar{t}_1 + \alpha_2^n x_1 \\ \beta\bar{t}_2^2 + \bar{\epsilon} + \gamma\alpha_1^n\alpha_2^n\bar{t}_1 + \alpha_2^{2n}y_1 \end{pmatrix} ,$$

where

$$x_1 = x_1(\alpha_1^n(1+\alpha_2^{-n}\bar{t}_1), 1+\alpha_2^{-n}\bar{t}_2, \alpha_2^{-2n}\bar{\epsilon} - \gamma\alpha_1^n + \alpha_2^{-n})$$

and

$$y_1 = y_1(\alpha_1^n(1+\alpha_2^{-n}\bar{t}_1), 1+\alpha_2^{-n}\bar{t}_2, \alpha_2^{-2n}\bar{\epsilon} - \gamma\alpha_1^n + \alpha_2^{-n}) \; .$$

As the fixed point is dissipative, we have: $0 < \alpha_1\alpha_2 < 1$. Moreover:

$$\alpha_2^n x_1 = O(\alpha_2^n\alpha_1^{2n}) + O(\alpha_2^{-n}) \; , \; \alpha_2^{2n}y_1 = O(\alpha_1^{2n}\alpha_2^{2n}) + O(\alpha_2^{-n}) + O(\alpha_1^n\alpha_2^n) \; .$$

Then the map g_ϵ^n, when $n \to \infty$, tends to the map

$$\begin{pmatrix} \bar{t}_1 \\ \bar{t}_2 \end{pmatrix} \longmapsto \begin{pmatrix} \alpha\bar{t}_2 \\ \beta\bar{t}_2^2 + \bar{\epsilon} \end{pmatrix} .$$

By the substitution:

$$\bar{\epsilon} = \beta^{-1}\tilde{\epsilon} \; , \; \bar{t}_1 = \alpha\beta^{-1}x \; , \; \bar{t}_2 = \beta^{-1}y \; ,$$

this limiting transformation becomes:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} y \\ y^2 + \tilde{\epsilon} \end{pmatrix} .$$

Then the parts a), b) and c) of the proposition are proved. Note that the limiting transformation is essentially the logistic map.

It remains to prove the last part: Consider now (if $\alpha_1\alpha_2^2 > 1$) the coordinate transformation:

$$\left.\begin{array}{rcl} \epsilon & = & \alpha_2^{-2n}\bar{\epsilon} - \gamma\alpha_1^n + \alpha_2^{-n} \\ t_1 & = & 1 + \alpha_2^{-2n}\alpha_1^{-n}\bar{t}_1 \\ t_2 & = & \alpha_2^{-n} + \alpha_2^{-2n}\bar{t}_2 \end{array}\right\}$$

for the map $f_{a_0+\epsilon}^n(t_1,t_2)$. As $\alpha_1\alpha_2^2 > 1$, when $\bar{t}_1$, $\bar{t}_2$, $\bar{\epsilon}$ are bounded, (t_1,t_2,ϵ) tends to $(1,0,0)$. By using this transformation we obtain:

$$g_\epsilon^n(\bar{t}_1,\bar{t}_2) = \begin{pmatrix} \alpha\alpha_1^n\alpha_2^n\bar{t}_2 + D_1x(0,1,0)\alpha_1^{2n}\alpha_2^{2n}\bar{t}_1 + \alpha_2^{2n}\alpha_1^n x_1 \\ \beta\bar{t}_2^2 + \bar{\epsilon} + \alpha_2^{2n}y_1 \end{pmatrix} ,$$

where

$$x_1 = x_1(\alpha_1^n(1+\alpha_2^{-2n}\alpha_1^{-n}\bar{t}_1), 1+\alpha_2^{-n}\bar{t}_2, \alpha_2^{-2n}\bar{\epsilon} - \gamma\alpha_1^n + \alpha_2^{-n})$$

and

$$y_1 = y_1(\alpha_1^n(1+\alpha_2^{-2n}\alpha_1^{-n}\bar{t}_1), 1+\alpha_2^{-n}\bar{t}_2, \alpha_2^{-2n}\bar{\epsilon} - \gamma\alpha_1^n + \alpha_2^{-n}) \ .$$

Then we have:

$$\alpha_2^{2n}\alpha_1^n x_1 = O(\alpha_1^{2n}\alpha_2^n) \ ,$$

$$\alpha_2^{2n} y_1 = O(\alpha_1^n\alpha_2^n) \ .$$

Hence the map g_ϵ^n tends, when $n \to \infty$, to the map:

$$\begin{pmatrix} \bar{t}_1 \\ \bar{t}_2 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ \beta\bar{t}_2^2 + \bar{\epsilon} + \gamma\bar{t}_1 \end{pmatrix} .$$

By the substitution:

$$\bar{\epsilon} = -\beta^{-1}\tilde{\epsilon} \ , \ \bar{t}_1 = \gamma^{-1}\bar{\epsilon}x \ , \ \bar{t}_2 = \bar{\epsilon}y \ ,$$

this limiting transformation becomes:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 1 + x - \tilde{\epsilon}y^2 \end{pmatrix} .$$

With this, the proposition is proved. □

Remark 3.4.6 *This proposition in the parts a), b) and c) is proved in [22]. The last part is necessary to study the basin of attraction of the periodic points near the homoclinic tangency, for the case* $\alpha_1\alpha_2^2 > 1$.

Now we shall study the different types of behaviour of the basin of attraction of the attarcting periodic point, which exist for parameters close to the parameter of saddle-node bifurcation. To do this we only need to study the stable invariant manifold of the periodic saddle point which exists at the same time that the attractor. Then we consider again the quadratic models of the n-th iterate of the initial map near the homoclinic tangency: 3.12, 3.13, 3.14, 3.15. We shall consider this cases separately:

Case 1

In this case the map is:

$$f^n_{a_0+\epsilon}(\xi,\eta) = (1-\alpha_2^{-n}\eta, (1-\alpha_2^{-n}\eta)^2 + \epsilon + \alpha_1^n(1+\xi)).$$

If p_ϵ is the initial fixed point of $f_{a_0+\epsilon}$, then $\mathcal{W}^u(p_\epsilon)$ is locally the parabola $\eta = \xi^2 + \epsilon$. The saddle periodic point is (if it exists) $S_\epsilon = (\xi_0, \eta_0)$, such that:

$$\xi_0 = \frac{-\alpha_1^n\alpha_2^n - 1 - \sqrt{\Delta}}{2\alpha_2^n} \quad , \eta_0 = \frac{2\alpha_2^n + \alpha_1^n\alpha_2^n + 1 + \sqrt{\Delta}}{2\alpha_2^{2n}} \, ,$$

$$\Delta = 4\alpha_2^n(1 - \alpha_1^n\alpha_2^n + (1 + \alpha_1^n\alpha_2^n)^2 - 4\epsilon\alpha_2^{2n} \, .$$

The first bifurcations, as we have seen before, are:

$$\epsilon_{1n} = \alpha_2^{-n} - \alpha_1^n + \frac{1}{4}(\alpha_2^{-n} + \alpha_1^n)^2 \, ,$$

$$\epsilon_{2n} = -\frac{3}{4}(\alpha_2^{-n} + \alpha_1^n)^2 + \alpha_2^{-n} - \alpha_1^n \, .$$

The eigenvalue α of $Df^n_{a_0+\epsilon}(S_\epsilon)$ such that $|\alpha| < 1$ verifies:

$$\alpha = -\alpha_2^n\xi_0 - \sqrt{\alpha_2^{2n}\xi_0^2 - \alpha_1^n\alpha_2^n} \, ,$$

and it is a root of the polynomial:

$$\alpha^2 + 2\alpha_2^n\xi_0\alpha + \alpha_1^n\alpha_2^n = 0 \, .$$

If the stable invariant manifold of S_ϵ is given by $(\xi(t), \eta(t))$, then it satisfies (see proposition 3.1.2):

$$\left.\begin{array}{rcl} \xi(\alpha t) & = & 1 - \alpha_2^n\eta(t) \\ \eta(\alpha t) & = & (1 - \alpha_2^n\eta(t))^2 + \epsilon + \alpha_1^n(1 + \xi(t)) \end{array}\right\}$$

or

$$\xi(\alpha t)^2 + \alpha_2^{-n}\xi(\alpha^2 t) + \alpha_1^n\xi(t) + \epsilon + \alpha_1^n - \alpha_2^{-n} = 0 \tag{3.31}$$

We can write $\xi(t) = \xi_0 + \sum_{k=1}^{\infty} c_n t^n$ because ξ is an analytic function (since $f_{a_0+\epsilon}$ is analytic). If we substitute ξ in the equation 3.31, we obtain:

$$c_k = \beta_k \sum_{i=1}^{k-1} c_i c_{k-i} \, ,$$

for all $k > 1$, where

$$\beta_k = -\alpha^k(2\xi_0\alpha^k + \alpha^{2k}\alpha_2^{-n} + \alpha_1^n)^{-1}$$

In th case $k = 1$ we have that c_1 can be arbitrarily fixed because $2\xi_0\alpha + \alpha_2^{-n}\alpha^2 + \alpha_1^n = 0$. We fix $c_1 = 1$. Also it is easy to see that $d_k = -\alpha_2^{-n}\alpha^k c_k$, where $\eta(t) = \eta_0 + \sum_{k=1}^{\infty} d_k t^k$.

Proposition 3.4.7 *The series $\xi(t) = \xi_0 + \sum_{k=1}^{\infty} c_k t^k$ has an infinite radius of convergence.*

Proof:

We know that this series has a radius of convergence $\rho \neq 0$ (by the stable manifold theorem). Suppose that $\rho \neq \infty$. Then:

$$\xi(t) = \alpha_1^{-n}[-\xi(\alpha t)^2 - \alpha_2^{-n}\xi(\alpha^2 t) - \epsilon - \alpha_1^n + \alpha_2^{-n}].$$

If $|\alpha t| < \rho$ and $|\alpha^2 t| < \rho$ then $\xi(t)$ is defined by a convergent series. Hence, if $|t| < |\alpha^{-1}\rho| < \alpha^{-2}\rho$ then $\xi(t)$ is well defined and its radius of convergence is $\rho \geq |\alpha^{-1}\rho| > \rho$. This is an absurdity. Then $\rho = \infty$.□

Now, to see if there is intersection between $\mathcal{W}^u(p_\epsilon)$ and $\mathcal{W}^s(S_\epsilon)$ we will compute $\eta(t) - \xi(t) - \epsilon$: In the foolowing propositions we will always suppose that n is great enough and ϵ is between ϵ_{1n} and ϵ_{2n} :

Proposition 3.4.8 *Let $t \in \mathbb{R}$ be a parameter. Then the point $(\xi(t), \eta(t))$ verifies:*

$$\eta(t) - \xi(t)^2 - \epsilon = \alpha_1^n(\xi(\alpha^{-1}t) + 1).$$

Proof:

The proposition holds because

$$\eta(t) = \alpha_2^{-n} - \alpha_2^{-n}\xi(\alpha t),\ \xi(t)^2 = -\alpha_2^{-n}\xi(\alpha t) - \alpha_1^n\xi(\alpha^{-1}t) - \epsilon - \alpha_1^n + \alpha_2^{-n}.\square$$

Proposition 3.4.9 *Let (ξ_0, η_0) be the coordinates of the periodic saddle point S_ϵ. Suppose that ϵ is between ϵ_{1n} and ϵ_{2n}. Then the following relations hold:*

a) If $\alpha_1^n > 0$, $\alpha_2^n > 0$ then $\eta_0 - \xi_0^2 - \epsilon > 0$.

b) If $\alpha_1^n > 0$, $\alpha_2^n < 0$ then $\eta_0 - \xi_0^2 - \epsilon > 0$.

c) If $\alpha_1^n < 0$, $\alpha_2^n > 0$ then $\eta_0 - \xi_0^2 - \epsilon < 0$.

d) If $\alpha_1^n < 0$, $\alpha_2^n < 0$ then $\eta_0 - \xi_0^2 - \epsilon < 0$.

The proof is an easy check ([28]).

Proposition 3.4.10 *There exist two points $(\xi(t_1), \eta(t_1))$ and $(\xi(t_2), \eta(t_2))$ of the stable manifold $\mathcal{W}^s(S_\epsilon)$ such that, if n is large enough:*

a) $t_1 < 0$ and $t_2 > 0$.

b) $(\xi(t), \eta(t))$ belongs to

$$\tilde{R} = [-2|\alpha_2|^{-n}, 2|\alpha_2|^{-n}] \times [\alpha_2^{-n} - 2\alpha_2^{-2n}, \alpha_2^{-n} + 2\alpha_2^{-2n}]\ \forall t \in [t_1, t_2].$$

c) $\xi(t_1) = -2|\alpha_2|^{-n}$, $\xi(t_2) = 2|\alpha_2|^{-n}$.

Proof: Consider the following change of coordinates and parameters:

$$\left.\begin{array}{rcl}\epsilon & = & \alpha_2^{-2n}\bar{\epsilon} - \alpha_1^n + \alpha_2^{-n} \\ \xi & = & \alpha_2^{-n}x \\ \eta & = & \alpha_2^{-n} + \alpha_2^{-2n}y\end{array}\right\}$$

Then the map $f^n_{a_0+\epsilon}$ is conjugated to the map:

$$g^n_{\bar{\epsilon}}(x,y) = \begin{pmatrix} -y \\ y^2 + \bar{\epsilon} + \alpha_1^n\alpha_2^n x \end{pmatrix}.$$

When $n \to \infty$ this map tends to the map:

$$g_\epsilon(x,y) = \begin{pmatrix} -y \\ y^2 + \bar{\epsilon} \end{pmatrix}.$$

The stable invariant manifold of the repellor point of $g_{\bar{\epsilon}}$, (x_0, y_0) , such that $y_0 = \frac{1+\sqrt{1-4\bar{\epsilon}}}{2}$ and $x_0 = -y_0$, verifies the equation $y = y_0$. When ϵ is between ϵ_{1n} and ϵ_{2n}, $\bar{\epsilon}$ is between $\frac{1}{4}$ and $-\frac{3}{4}$. So the point (x_0, y_0) belongs to $[-2,2] \times [-2,2]$ for all $\bar{\epsilon} \in [-\frac{3}{4}, \frac{1}{4}]$. Moreover there exist two points p_1 , $p_2 \in \mathcal{W}^s((x_0,y_0), g_\epsilon)$ such that $p_1 \in \{-2\} \times [-2,2]$ and $p_2 \in \{2\} \times [-2,2]$. Then this is also true if we take n large enough for the map g^n_ϵ and the corresponding fixed point (x_{0n}, y_{0n}), by the stable manifold theorem (see [16] and chapter 2). Undoing the change of parameter and coordinates the proposition is proved. □

Remark 3.4.11 *In fact, we can consider rectangles smaller than the previous one. For example* $\tilde{R} = [-2|\alpha_2|^{-n}, 2|\alpha_2|^{-n}] \times [\alpha_2^{-n} + (y_0 - \delta)\alpha_2^{-2n}, \alpha_2^{-n} + (y_0+\delta)\alpha_2^{-2n}]$ *with* $\delta > 0$ *as small as we want, if* n *is large enough. Moreover we can take* $\delta = 1/8$ *and* n *large enough such that* $y_0 - \delta > \bar{\epsilon}$ *because* $y_0 = y_0^2 + \bar{\epsilon}$ *and* $y_0 \neq 0$.

Proposition 3.4.12 *There exists* $t_0 \in \mathbb{R}^+$ *such that, for all* t *with* $|t| \leq t_0$ *the point* $(\xi(t), \eta(t))$ *belonging to* $\mathcal{W}^s(S_\epsilon)$*, is into the rectangle:*

$$R_1 = [-2|\alpha_2|^{-n/2}, |\alpha_1|^{-n}] \times [\alpha_2^{-n} - 2|\alpha_2|^{-3n/2}, \alpha_2^{-n} + 2|\alpha_2|^{-3n/2}].$$

Moreover, if $\mathcal{W}^s_\pm(S_\epsilon)$ *has some intersection with the boundary of* R_1 *and* $\alpha_1 > 0$*, then the first intersection is in the segment*

$$\{-|\alpha_2|^{-n/2}\} \times [\alpha_2^{-n} - 2|\alpha_2|^{-3n/2}, \alpha_2^{-n} + 2|\alpha_2|^{-3n/2}].$$

Here $\mathcal{W}^s_\pm(S_\epsilon)$ *denotes the two branches of* $\mathcal{W}^s(S_\epsilon)$.

Proof:

We know that the saddle fixed point, $S_\epsilon = (\xi_0, \eta_0)$, of $f^n_{a_0+\epsilon}$ is inside R_1 (proposition 3.3.1). Therefore there exists $t_0 \in \mathbb{R}^+$ such that, if $|t| \leq t_0$ then $(\xi(t), \eta(t)) \in \mathcal{W}^s(S_\epsilon)$.

On the other hand, the images of the segments:

$$s_1 = [-2|\alpha_2|^{-n/2}, 2|\alpha_1|^{-n}] \times \{\alpha_2^{-n} - 2|\alpha_2|^{-3n/2}\} ,$$

$$s_2 = [-2|\alpha_2|^{-n/2}, 2|\alpha_1|^{-n}] \times \{\alpha_2^{-n} + 2|\alpha_2|^{-3n/2}\}.$$

are vertical segments:

$$\{\mp 2|\alpha_2|^{-n/2}\} \times [4|\alpha_2|^{-n} + \epsilon + \alpha_1^n(1 - 2|\alpha_2|^{-n/2}), 4|\alpha_2|^{-n} + \epsilon + \alpha_1^n(1 + 2|\alpha_2|^{-n/2})],$$

where the sign plus corresponds to $f^n_{a_0+\epsilon}(s_2)$ and the sign minus corresponds to $f^n_{a_0+\epsilon}(s_1)$. These two segments are outside R. The image of the segment

$$\{|\alpha_1|^{-n}\} \times [\alpha_2^{-n} - 2|\alpha_2|^{-3n/2}, \alpha_2^{-n} + 2|\alpha_2|^{-3n/2}].$$

is the parabola: $\eta = \xi^2 + \epsilon + \alpha_1^n + 1$ and hence it is outside R_1.

Now suppose that $\mathcal{W}^s_\pm(S_\epsilon)$ cuts, for the first time, the boundary of R_1 in $(\xi(\bar{t}), \eta(\bar{t}))$. Then $f^n_{a_0+\epsilon}(\xi(\bar{t}), \eta(\bar{t}))$ is contained in the rectangle R_1 because $\mathcal{W}^s(S_\epsilon)$ is the stable invariant manifold. Therefore

$$(\xi(\bar{t}), \eta(\bar{t})) \in \{-2|\alpha_2|^{-n/2} \times [\alpha_2^{-n} - 2|\alpha_2|^{-3n/2}, \alpha_2^{-n} + 2|\alpha_2|^{-3n/2}]$$

since in the other cases

$$f^n_{a_0+\epsilon}(\xi(\bar{t}), \eta(\bar{t})) \notin R_1,$$

which is an absurdity. □

Corollary 3.4.13 *Let $\alpha_1 > 0$. Then the point of the first intersection of both branches of $\mathcal{W}^s(S_\epsilon)$ and $\mathcal{W}^u(p_\epsilon)$, if it exists, is contained in the strip defined by the stright lines: $\eta = \alpha_2^{-n} + 2|\alpha_2|^{-3n/2}$ and $\eta = \alpha_2^{-n} - 2|\alpha_2|^{-3n/2}$.*

Proof:

It is a consequence of the fact that the segment $\{-|\alpha_2|^{-n/2}\} \times [\alpha_2^{-n} - 2|\alpha_2|^{-3n/2}, \alpha_2^{-n} + 2|\alpha_2|^{-3n/2}]$, in the previous proposition, does not intersect the parabola $\eta = \xi^2 + \epsilon$ but the segment s_2 does. Then, as the point S_ϵ is inside the parabola (see proposition 3.4.9), the corollary holds. □

Proposition 3.4.14 *Suppose that $\alpha_1 > 0$. Then there exist $t_1 < 0$ and $t_2 > 0$ such that:*

a) $\xi(t_1) = -2|\alpha_2|^{-n}$, $\xi(t_2) = 2|\alpha_2|^{-n}$.

b) $\xi(\alpha^{-k}t_1) \leq -A_k^2 \alpha_2^{-2^k n} \alpha_1^{-(2^k-1)n}$, $|\xi(\alpha^{-k}t_1)| \leq B_k^2 \alpha_2^{-2^k n} \alpha_1^{-(2^k-1)n}$.

c) $\xi(\alpha^{-k}t_2) \leq -C_k^2\alpha_2^{-2^k n}\alpha_1^{-(2^k-1)n}$, $|\xi(\alpha^{-k}t_2)| \leq D_k^2\alpha_2^{-2^k n}\alpha_1^{-(2^k-1)n}$.

for n large enough. Here α is the eigenvalue of $Df^n_{a_0+\epsilon}(S_\epsilon)$ such that $|\alpha| < 1$ and A_k, B_k, C_k and D_k do not depend on n.

Proof:

We know that $\xi(t)$ satisfies the functional equation:

$$\xi(t) = (-\bar{\epsilon}\alpha_2^{-2n} - \xi(\alpha t)^2 - \alpha_2^{-n}\xi(\alpha^2 t))\alpha_1^{-n},$$

where $\epsilon = \alpha_2^{-n} - \alpha_1^n + \bar{\epsilon}\alpha_2^{-2n}$. Then, by proposition 3.4.10, there exist two parameters t_1 and t_2, $t_1 < 0$ and $t_2 > 0$, such that $\xi(t_1) = -2|\alpha_2|^{-n}$, $\xi(t_2) = 2|\alpha_2|^{-n}$. Moreover $\xi(t) \in [-2|\alpha_2|^{-n}, 2|\alpha_2|^{-n}]$ if $t \in [t_1, t_2]$. Then we have:

$$\xi(\alpha^{-1}t_1) = (-\bar{\epsilon}\alpha_2^{-2n} - \xi(t_1)^2 - \alpha_2^{-n}\xi(\alpha t_1))\alpha_1^{-n} \leq -\frac{5}{4}\alpha_2^{-2n}\alpha_1^{-n},$$

$$|\xi(\alpha^{-1}t_1)| = -\xi(\alpha^{-1}t_1) = (\bar{\epsilon}\alpha_2^{-2n} + \xi(t_1)^2 + \alpha_2^{-n}\xi(\alpha t_1))\alpha_1^{-n} \leq \frac{25}{4}\alpha_2^{-2n}\alpha_1^{-n}.$$

So the proposition is true for $t = t_1$ and $k = 1$. Also for $k = 2$ we have:

$$\xi(\alpha^{-2}t_1) = (-\bar{\epsilon}\alpha_2^{-2n} - \xi(\alpha^{-1}t_1)^2 - \alpha_2^{-n}\xi(t_1))\alpha_1^{-n} \leq$$

$$(-\bar{\epsilon}\alpha_2^{-2n} - A_1^4\alpha_2^{-4n}\alpha_1^{-2n} - 2\alpha_2^{-2n})\alpha_1^{-n} \leq -A_2^2\alpha_2^{-4n}\alpha_1^{-3n},$$

for a suitable A_2 and n large enough.

$$|\xi(\alpha^{-2}t_1)| = -\xi(\alpha^{-2}t_1) = (\bar{\epsilon}\alpha_2^{-2n} + \xi(\alpha^{-1}t_1)^2 + \alpha_2^{-n}\xi(t_1))\alpha_1^{-n} \leq B_2^2\alpha_2^{-4n}\alpha_1^{-3n}.$$

Now suppose that the assertion is true for $k \leq m - 1$. Then:

$$\xi(\alpha^{-m}t_1) = (-\bar{\epsilon}\alpha_2^{-2n} - \xi(\alpha^{-m+1}t_1)^2 - \alpha_2^{-n}\xi(\alpha^{-m+2}t_1))\alpha_1^{-n} \leq$$

$$(-\bar{\epsilon}\alpha_2^{-2n} - A_{m-1}^4\alpha_2^{-2^m n}\alpha_1^{-(2^m-2)n} + B_{n-2}^2\alpha_2^{(-2^{m-2}-1)n}\alpha_1^{(-2^{m-2}+1)n})\alpha_1^{-n} \leq$$

$$-A_m^2\alpha_2^{-2^m n}\alpha_1^{-(2^m-1)n},$$

$$|\xi(\alpha^{-m}t_1)| = -\xi(\alpha^{-m}t_1) \leq B_m^2\alpha_2^{-2^m n}\alpha_1^{-(2^m-1)n}.$$

The proof for $t = t_2$ is analogous.□

Proposition 3.4.15 *Both branches of the stable invariant manifold $\mathcal{W}^s(S_\epsilon)$ intersect the unstable invariant manifold $\mathcal{W}^u(p_\epsilon)$ if $\alpha_1^n > 0$ and $\alpha_2^n > 0$ or $\alpha_1^n > 0$ and $\alpha_2^n < 0$, for n large enough.*

Proof:

The condition of non empty intersection is the following: there exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) - \xi(t_0)^2 - \bar{\epsilon} < 0$. This is equivalent to the existence of a point $\bar{t}_0$ such that $\xi(\bar{t}_0) < -1$. Then the parameter $t_0 = \alpha\bar{t}_0$ verifies the previous inequality.

Then let $k_0 \in \mathbb{N}$ be such that $\alpha_2^{-2^{k_0}} \alpha_1^{-(2^{k_0}-1)} > 1$. It exists since $|\alpha_1\alpha_2| < 1$. By the previous proposition we have that, for n large enugh there exist $t_1 < 0$ and $t_2 > 0$, such that

$$\xi(\alpha^{-k_0}t_1) \leq -A_k^2 \alpha_2^{-2^{k_0}n} \alpha_1^{-(2^{k_0}-1)n} \leq -1,$$

and

$$\xi(\alpha^{-k_0}t_2) \leq -C_k^2 \alpha_2^{-2^{k_0}n} \alpha_1^{-(2^{k_0}-1)n} \leq -1.$$

This prove the proposition. □

Proposition 3.4.16 *If* $|\alpha_1\alpha_2^2| < 1$ *then* $\mathcal{W}^s(S_\epsilon) \cap \mathcal{W}^u(p_\epsilon) \neq \emptyset$ *for* n *large enough.*

Proof:

In this case it is easy to see that the parabola $\eta = \xi^2 + \epsilon$ has non empty intersection with the rectangle of the remark 3.4.11:

$$\tilde{R} = [-2|\alpha_2|^{-n}, 2|\alpha_2|^{-n}] \times \left[\alpha_2^{-n} + \left(y_0 + \frac{1}{8}\right)\alpha_2^{-2n}, \alpha_2^{-n}\left(y_0 - \frac{1}{8}\right)\alpha_2^{-2n}\right],$$

since the minimum of the parabola $\eta = \xi^2 + \epsilon = \xi^2 + \alpha_2^{-n} - \alpha_1^n + \bar{\epsilon}\alpha_2^{-2n}$ is below the straight line $\eta = \alpha_2^{-n} + (y_0 - \frac{1}{8})\alpha_2^{-2n}$ and, for example, the points $p_\pm = (\pm 2\alpha_2^{-n}, 4\alpha_2^{-2n} + \alpha_2^{-n} - \alpha_1^n + \bar{\epsilon}\alpha_2^{-2n}) \in \mathcal{W}^u(p_\epsilon)$ but $p_\pm \notin \tilde{R}$. Therefore, by the proposition 3.4.10 and the remark 3.4.11, there exist points of intersection of both branches of $\mathcal{W}^s(S_\epsilon)$ with $\mathcal{W}^u(p_\epsilon)$. □

Proposition 3.4.17 *If* $\alpha_2^4\alpha_1^3 < 1$ *and* $\alpha_1 > 0$ *then there exist points* $(\xi, \eta) \in \mathcal{W}^s(S_\epsilon)$ *such that* $\eta - \xi^2 - \epsilon < 0$ *and* $\xi > 0$.

Proof:

We observe first that the case $\alpha_2^2\alpha_1 < 1$ has been proved in proposition 3.4.16. So we will suppose that $\alpha_2^2\alpha_1 > 1$ Consider the following change of parameters and coordinates:

$$\left.\begin{array}{rcl} \epsilon & = & \alpha_2^{-2n}\bar{\epsilon} - \gamma\alpha_1^n + \alpha_2^{-n} \\ \xi & = & \alpha_2^{-2n}\alpha_1^{-n}x \\ \eta & = & \alpha_2^{-n} + \alpha_2^{-2n}y \end{array}\right\}$$

Then the map $f^n_{a_0+\epsilon}$ is conjugated to the map:

$$g_{\bar{\epsilon}}^n = \begin{pmatrix} -\alpha_1^n\alpha_2^n y \\ y^2 + \bar{\epsilon} + x \end{pmatrix}.$$

When $n \to \infty$ this map tends to $g_{\bar\epsilon}(x,y) = (0, y^2 + \bar\epsilon + x)$. The stable invariant manifold of the repellor point of $g_{\bar\epsilon}$ corresponding to S_ϵ when n goes to infinity, which is $(x_0, y_0) = (0, (1+\frac{\sqrt{1-4\bar\epsilon}}{2}))$, verifies the equation: $x = -y^2 - \bar\epsilon + y_0 = -y^2 + y_0^2$. When ϵ is between ϵ_{1n} and ϵ_{2n}, $\bar\epsilon$ is between $\frac{1}{4}$ and $-\frac{3}{4}$. So $(x_0, y_0) \in [-\frac{1}{5}, \frac{1}{5}] \times [-2, 2]$ and the first intersections of the stable invariant manifold with this rectangle are in the segments $\{-\frac{1}{5}\} \times [-2, 2]$ and $\{\frac{1}{5}\} \times [-2, 2]$. Let $(x_0(n), y_0(n))$ the saddle fixed point such that $(x_0(n), y_0(n)) \to (x_0, y_0)$ when $n \to \infty$. By the stable manifold theorem, if n is large enough, the stable invariant manifold of $(x_0(n), y_0(n))$ is close to the parabola $x = -y^2 + y_0^2$. Undoing the change of coordinates and parameter, we have that for n large enough the rectangle

$$R_2 = \left[-\frac{1}{5}\alpha_1^{-n}\alpha_2^{-2n}, \frac{1}{5}\alpha_1^{-n}\alpha_2^{-2n}\right] \times [\alpha_2^{-n} - 2\alpha_2^{-2n}, \alpha_2^{-n} + 2\alpha_2^{-2n}]$$

contains, locally, $\mathcal{W}^s(S_\epsilon)$. Moreover the first points of intersection of $\mathcal{W}^s(S_\epsilon)$ with the boundary of R_2 are in the segments

$$\left\{\mp\frac{1}{5}\alpha_1^{-n}\alpha_2^{-2n}\right\} \times [\alpha_2^{-n} - 2\alpha_2^{-2n}, \alpha_2^{-n} + 2\alpha_2^{-2n}].$$

On the other hand the parabola $\eta = \xi^2 + \epsilon$, which represents $\mathcal{W}^u(p_\epsilon)$, has its minimum in $(0, \alpha_2^{-n} - \alpha_1^n + \bar\epsilon\alpha_2^{-2n})$. As $\alpha_1 > \alpha_2^{-2}$, this minimum is below the rectangle R_2. Moreover, the points

$$(x_1^{\pm}, y_1^{\pm}) = \left(\pm\frac{1}{5}\alpha_2^{-2n}\alpha_1^{-n}, \frac{1}{25}\alpha_2^{-4n}\alpha_1^{-2n} + \alpha_2^{-n} - \alpha_1^n + \bar\epsilon\alpha_2^{-2n}\right)$$

belong to $\mathcal{W}^u(p_\epsilon)$ and $y_1^{\pm} > \alpha_2^{-n} + 2\alpha_2^{-2n}$ since $\alpha_2^4\alpha_1^3 < 1$. Then the parabola crosses the rectangle R_2. □

Proposition 3.4.18 *Suppose that $\alpha_1 < 0$ and $|\alpha_2^2\alpha_1| > 1$. Then $\mathcal{W}^s(S_\epsilon)$ does not intersect, locally, $\mathcal{W}^u(p_\epsilon)$ for n odd and large enough.*

Proof:

We consider again the following change of coordinates and parameters:

$$\left.\begin{array}{rcl} \epsilon & = & \alpha_2^{-2n}\bar\epsilon - \gamma\alpha_1^n + \alpha_2^{-n} \\ \xi & = & \alpha_2^{-2n}\alpha_1^{-n}x \\ \eta & = & \alpha_2^{-n} + \alpha_2^{-2n}y \end{array}\right\}$$

Then, in the same way of the previous proposition, we can see that if n is large enough the stable invariant manifold of S_ϵ is locally contained in

$$R_3 = [-A\alpha_2^{-2n}\alpha_1^{-n}, A\alpha_2^{-2n}\alpha_1^{-n}] \times [\alpha_2^{-n} - B\alpha_2^{-2n}, \alpha_2^{-n} + B\alpha_2^{-2n}],$$

with suitable $A > 0$, $B > 0$. But $\mathcal{W}^u(p_\epsilon)$ has the local expression $\eta = \xi^2 + \epsilon = \xi^2 + \alpha_2^{-n} - \alpha_1^n + \bar\epsilon\alpha_2^{-2n}$ and this parabola does not intersect the rectangle R_3 if $\alpha_1 < 0$ and n is odd and large enough, since $|\alpha_2^2\alpha_1| > 1$. □

Proposition 3.4.19 *Suppose that $\alpha_1 > 0$, $\alpha_2^4\alpha_1^3 > 1$ and $|\alpha_2^5\alpha_1^4| < 1$. Then the first point of intersection of both branches of $\mathcal{W}^s(S_\epsilon)$ and $\mathcal{W}^u(p_\epsilon)$ has negative abscisa.*

Proof:

For simplicity we consider the case $\alpha_2 > 0$.

By the corollary 3.4.13, we know that the first intersection of both branches of $\mathcal{W}^s(S_\epsilon)$ with $\mathcal{W}^u(p_\epsilon)$, if it exists, is contained in the strip defined by the lines $\eta = \alpha_2^{-n} + 2|\alpha_2|^{-3n/2}$ and $\eta = \alpha_2^{-n} - 2|\alpha_2|^{-3n/2}$. Now we consider the following set:

$$R_4 = ([-4\alpha_2^{-2n}\alpha_1^{-n}, 4\alpha_2^{-2n}\alpha_1^{-n}] \times [\alpha_2^{-n} - 3\alpha_2^{-2n}, \alpha_2^{-n} + 3\alpha_2^{-2n}]) \cup$$

$$[-2\alpha_2^{-n/2}, -4\alpha_2^{-2n}\alpha_1^{-n}] \times [\alpha_2^{-n} - 2\alpha_2^{-3n/2}, \alpha_2^{-n} + 2\alpha_2^{-3n/2}].$$

Observe that $\alpha_2^{-n/2} > \alpha_2^{-2n}\alpha_1^{-n}$ since $\alpha_2^3\alpha_1^2 > 1$ It is easy to see that $S_\epsilon \in R_4$. Moreover the points

$$\begin{aligned}
p_1 &= (-2\alpha_2^{-n/2}, \alpha_2^{-n} + 2\alpha_2^{-3n/2}),\\
p_2 &= (-2\alpha_2^{-n/2}, \alpha_2^{-n} - 2\alpha_2^{-3n/2}),\\
p_3 &= (-4\alpha_2^{-2n}\alpha_1^{-n}, \alpha_2^{-n} + 2\alpha_2^{-3n/2}),\\
p_4 &= (-4\alpha_2^{-2n}\alpha_1^{-n}, \alpha_2^{-n} - 2\alpha_2^{-3n/2}),\\
p_5 &= (-4\alpha_2^{-2n}\alpha_1^{-n}, \alpha_2^{-n} + 3\alpha_2^{-2n}),\\
p_6 &= (-4\alpha_2^{-2n}\alpha_1^{-n}, \alpha_2^{-n} - 3\alpha_2^{-2n}),\\
p_7 &= (4\alpha_2^{-2n}\alpha_1^{-n}, \alpha_2^{-n} + 3\alpha_2^{-2n}),\\
p_8 &= (4\alpha_2^{-2n}\alpha_1^{-n}, \alpha_2^{-n} - 3\alpha_2^{-2n}),
\end{aligned}$$

(see figure 3.8), have their image outside R_4 :

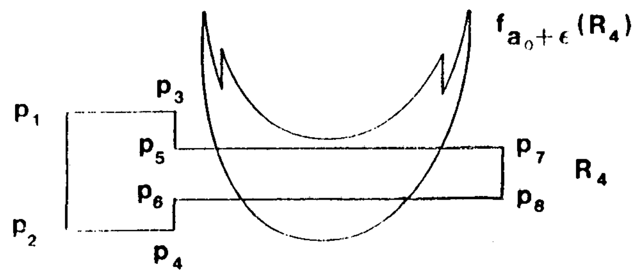


Figure 3.8:

$$f^n_{a_0+\epsilon}(p_1) \approx (-2\alpha_2^{-n/2}, 5\alpha_2^{-n}),$$

$$f^n_{a_0+\epsilon}(p_2) \approx (2\alpha_2^{-n/2}, 5\alpha_2^{-n}),$$

$$f^n_{a_0+\epsilon}(p_3) \approx (-2\alpha_2^{-n/2}, 5\alpha_2^{-n}),$$

$$f^n_{a_0+\epsilon}(p_4) \approx (2\alpha_2^{-n/2}, 5\alpha_2^{-n}),$$

$$f^n_{a_0+\epsilon}(p_5) \approx (-3\alpha_2^{-n}, \alpha_2^{-n} + (5+\bar\epsilon)\alpha_2^{-2n}),$$

(this point is outside R_4 since $\bar\epsilon \in [-\frac{3}{4}, \frac{1}{4}]$),

$$f^n_{a_0+\epsilon}(p_6) \approx (3\alpha_2^{-n}, \alpha_2^{-n} + (5+\bar\epsilon)\alpha_2^{-2n}),$$

$$f^n_{a_0+\epsilon}(p_5) \approx (-3\alpha_2^{-n}, \alpha_2^{-n} + (13+\bar\epsilon)\alpha_2^{-2n}),$$

$$f^n_{a_0+\epsilon}(p_5) \approx (3\alpha_2^{-n}, \alpha_2^{-n} + (13+\bar\epsilon)\alpha_2^{-2n}).$$

On the other hand, the image of the segment

$$\{4\alpha_2^{-2n}\alpha_1^{-n}\} \times [\alpha_2^{-n} - 3\alpha_2^{-2n}, \alpha_2^{-n} + 3\alpha_2^{-2n}]$$

is in the parabola $\eta = \xi^2 + \alpha_2^{-n} + (4+\bar\epsilon)\alpha_2^{-2n}$ and it is ouside the region R_4 if $\bar\epsilon \in [-\frac{3}{4}, \frac{1}{2}]$. Moreover, the images of the segments $\overline{p_3p_5}$ and $\overline{p_4p_6}$ do not intersect R_4, since the intersection of the line $\xi = -4\alpha_2^{-2n}\alpha_1^{-n}$ with $f^n_{a_0+\epsilon}(\bar{p_3p_5})$ is in the point:

$$(-4\alpha_2^{-2n}\alpha_1^{-n}, \alpha_2^{-n} + 16\alpha_2^{-4n}\alpha_1^{-2n} + (\bar\epsilon - 4)\alpha_2^{-2n}) \approx$$

$$(-4\alpha_2^{-2n}\alpha_1^{-n}, \alpha_2^{-n} + 16\alpha_2^{-4n}\alpha_1^{-2n}),$$

and this point is outside R_4 since $\alpha_2^{-4}\alpha_1^{-2} > \alpha_2^{-3/2}$ if and only if $\alpha_2^5\alpha_1^4 < 1$. By the same argument $f^n_{a_0+\epsilon}(\bar{p_4p_6})$ is outside R_4. Then the first time that the stable manifold $\mathcal{W}^s(S_\epsilon)$ touches the boundary of R_4 is in a point of the segment $\overline{p_1p_2}$.

Finally, we observe that the unstable invariant manifold $\mathcal{W}^u(p_\epsilon)$ is, locally, the parabola $\eta = \xi^2 + \alpha_2^{-n} - \alpha_1^n + \bar\epsilon\alpha_2^{-2n}$. This parabola has intersection with the segment $\overline{p_1p_3}$ in a point approximately equal to $(-\alpha_1^{n/2}, \alpha_2^{-n} + 2\alpha_2^{-3n/2})$, because $\alpha_1^2\alpha_2^3 > 1$. However it has no intersection, with positive abscisa, with the region R_4 since the intersection of the line $\xi = 4\alpha_2^{-2n}\alpha_1^{-n}$ with $\mathcal{W}^u(p_\epsilon)$ is approximately equal to $(4\alpha_2^{-2n}\alpha_1^{-n}, \alpha_2^{-n} - \alpha_1^n)$. This point is below the line $\eta = \alpha_2^{-n} - 3\alpha_2^{-2n}$ since $\alpha_2^4\alpha_1^3 > 1$. With this fact we have that the first intersection between both branches of $\mathcal{W}^s(S_\epsilon)$ and $\mathcal{W}^u(p_\epsilon)$ has negative abscisa. □

Proposition 3.4.20 *Let $|\alpha_1^4\alpha_2^5| > 1$ and $\alpha_1 > 0$. Then there exists $t_0 \in \mathbb{R}^+$ such that: $\forall t \in \mathbb{R}$, $|t| \le t_0$ the point $(\xi(t), \eta(t))$ belongs to*

$$R_5 = [-2|\alpha_2|^{-n/2}, \frac{1}{2}\alpha_1^{n/2}] \times [\alpha_2^{-n} - 2|\alpha_2|^{-3n/2}, \alpha_2^{-n} + \frac{1}{4}\alpha_1^{3n/2}],$$

and $\mathcal{W}^s(S_\epsilon)$ intersects for the first time the boundary of R_5 in the segment

$$\{-2|\alpha_2|^{-n/2}\} \times [\alpha_2^{-n} - 2|\alpha_2|^{-3n/2}, \alpha_2^{-n} + \frac{1}{4}\alpha_1^{3n/2}].$$

Proof:

First we denote the segments of R_5 as:

$$s_1 = \{-2|\alpha_2|^{-n/2}\} \times [\alpha_2^{-n} - 2|\alpha_2|^{-3n/2}, \alpha_2^{-n} + \frac{1}{4}\alpha_1^{3n/2}],$$

$$s_2 = \left\{\frac{1}{2}\alpha_1^{n/2}\right\} \times \left[\alpha_2^{-n} - 2|\alpha_2|^{-3n/2}, \alpha_2^{-n} + \frac{1}{4}\alpha_1^{3n/2}\right],$$

$$s_3 = \left[-2|\alpha_2|^{-n/2}, \frac{1}{2}\alpha_1^{n/2}\right] \times \left\{\alpha_2^{-n} - 2|\alpha_2|^{-3n/2}\right\},$$

$$s_4 = \left[-2|\alpha_2|^{-n/2}, \frac{1}{2}\alpha_1^{n/2}\right] \times \left\{\alpha_2^{-n} + \frac{1}{4}\alpha_1^{3n/2}\right\}.$$

In order to prove the proposition we require:

a) $S_\epsilon = (\xi_0, \eta_0) \in R_5$,

b) $f^n_{a_0+\epsilon}(s_4)$ does not intersect R_5,

c) $f^n_{a_0+\epsilon}(s_2)$ is outside the rectangle R_5.

Then, taking into account the proposition 3.4.12, the image of s_3 does not intersect R_5 and hence the proposition follows.

We will prove a), b), c):

a) Suppose that $\alpha_2 > 0$. If $\alpha_2 < 0$ the arguments are similar. We know by the proposition 3.4.9 that $\xi_0(\epsilon)$ is increasing when $\epsilon \in [\epsilon_{2n}, \epsilon_{1n}]$. So $\xi_0(\epsilon)$ is maximum when $\epsilon = \epsilon_{1n}$ and has the value:

$$\xi_0(\epsilon_{1n}) = \frac{1}{2}\alpha_2^{-n} + \frac{1}{2}\alpha_1^n < \frac{1}{2}\alpha_1^{n/2},$$

since $\alpha_2^2\alpha_1 > \alpha_2^5\alpha_1^4 > 1$.

Therefore the abscisa of the point S_ϵ, ξ_0, is between the straight lines $\xi = -2|\alpha_2|^{-n/2}$ and $\xi = \frac{1}{2}\alpha_1^{n/2}$. As ξ_0 is increasing, $\eta_0 = \alpha_2^{-n} - \alpha_2^{-n}\xi_0$ is decreasing. Therefore $\eta_0(\epsilon)$ is maximum when $\epsilon = \epsilon_{2n}$ and:

$$\eta_0(\epsilon_{2n}) = \alpha_2^{-n} + \frac{3}{2}\alpha_2^{-2n} + \frac{3}{2}\alpha_1^n\alpha_2^{-n} < \alpha_2^{-n} + \frac{1}{4}\alpha_1^{3n/2},$$

since $\alpha_2^4\alpha_1^3 > \alpha_2^5\alpha_1^4 > 1$.

Hence the point (ξ_0, η_0) belongs to R_5.

b) $f^n_{a_0+\epsilon}(s_4)$ is the segment

$$\left\{-\frac{1}{4}\alpha_1^{3n/2}\alpha_2^n\right\} \times \left[\frac{1}{16}\alpha_2^{2n}\alpha_1^{3n} + \alpha_2^{-n} - 2\alpha_2^{-n/2}\alpha_1^n, \frac{1}{16}\alpha_2^{2n}\alpha_1^{3n} + \alpha_2^{-n} + \frac{1}{2}\alpha_1^{3n/2}\right],$$

skipping higher order terms. We have to see that $\frac{1}{16}\alpha_2^{2n}\alpha_1^{3n} + \alpha_2^{-n} - 2\alpha_2^{-n/2}\alpha_1^n > \alpha_2^{-n} + \frac{1}{4}\alpha_1^{3n/2}$. this is true because $\alpha_2^5\alpha_1^4 > 1$.

c) $f^n_{a_0+\epsilon}(s_2)$ is contained in the parabola:

$$\eta = \xi^2 + \epsilon + \alpha_1^n(1 + \frac{1}{2}\alpha_1^{n/2}).$$

This parabola is outside the rectangle R_5 since:

$$\epsilon + \alpha_1^n\left(1 + \frac{1}{2}\alpha_1^{n/2}\right) > \alpha_2^{-n} + \frac{1}{4}\alpha_1^{3n/2}.$$

As $\epsilon = \alpha_2^{-n} - \alpha_1^n + O(\alpha_2^{-2n})$ and $\alpha_2^4\alpha_1^3 > 1$, this is equivalent to say:

$$\frac{1}{2}\alpha_1^{3n/2} - \frac{1}{4}\alpha_1^{3n/2} = \frac{1}{4}\alpha_1^{3n/2} > 0. \ \Box$$

Corollary 3.4.21 *Under the hypothesis of the previous proposition the first point of intersection of both branches of $\mathcal{W}^s(S_\epsilon)$ and $\mathcal{W}^u(p_\epsilon)$ is a point of negative abscisa.*

Proof:

We consider only the case $\alpha_2 > 0$. We want to see that the boundary of the rectangle R_5 has two intersections with negative abscisa. This is true because s_2 has no intersection with $\mathcal{W}^u(p_\epsilon)$. The intersection of the line $\xi = \frac{1}{2}\alpha_1^{n/2}$ with the parabola $\eta = \xi^2 + \epsilon$ is the point $(\frac{1}{2}\alpha_1^{n/2}, \alpha_2^{-n} - \frac{3}{4}\alpha_1^n)$ (skipping higher order terms). This point is outside the rectangle R_5 since $\alpha_2^3\alpha_1^2 > 1$. So the corollary is proved. $\Box$

By using the previous propositions we have the following:

Theorem 3.4.22 *Let $\{f_\epsilon\}_{\epsilon\in[-1,1]}$ a one parameter family of diffeomorphisms having a dissipative saddle fixed point p_0 for $\epsilon = 0$. Let p_ϵ denote the saddle point which exists for $\epsilon \approx 0$. Suppose that, for $\epsilon \approx 0$, the invariant manifolds of the fixed point have, locally, the following expression in the coordinates (ξ, η) : $\eta = 0$ for $\mathcal{W}^s(p_\epsilon)$ and $\eta = \xi^2 + \epsilon$ for $\mathcal{W}^u(p_\epsilon)$. Suppose also that the map f_ϵ^n for n large enough and (ξ, η) near the point of homoclinic tangency $(0,0)$ is:*

$$f_\epsilon^n(\xi, \eta) = (1 - \alpha_2^{-n}\eta, (1 - \alpha_2^{-n}\eta)^2 + \epsilon + \alpha_1^n(1 + \xi)),$$

where $|\alpha_1| < 1$ and $|\alpha_2| > 1$ are the absolute values of the eigenvalues of $Df_\epsilon^n(p_\epsilon$. Let ϵ_{1n} and ϵ_{2n} be, respectively, the saddle-node and flip bifurcation parameters associated to the n-periodic points of f_ϵ^n which appear by the Newhouse phenomenon. Let S_ϵ denote the n-periodic saddle which appear by saddle-node bifurcation. If n is large enough and ϵ is between ϵ_{1n} and ϵ_{2n} then:

a) Both branches of $\mathcal{W}^s(S_\epsilon)$ intersect $\mathcal{W}^u(p_\epsilon)$ if $\alpha_1 > 0$.

b) If $\alpha_1 < 0$, n odd and $|\alpha_1\alpha_2^2| < 1$ then $\mathcal{W}^s(\S_\epsilon) \cap \mathcal{W}^u(p_\epsilon) \neq \emptyset$.

c) If $|\alpha_2^4\alpha_1^3| < 1$ and $\alpha_1 > 0$ then there exists $(\xi,\eta) \in \mathcal{W}^s(S_\epsilon) \cap \mathcal{W}^u(p_\epsilon)$ such that $\xi > 0$.

d) If $\alpha_1 > 0$ and $|\alpha_2^4\alpha_1^3| > 1$ then the first point of intersection of both branches of $\mathcal{W}^s(S_\epsilon)$ with $\mathcal{W}^u(p_\epsilon)$ has negative abscisa.

e) If $\alpha_1 < 0$, n odd and $|\alpha_1\alpha_2^2| > 1$ then $\mathcal{W}^s(S_\epsilon)$ does not intersect, locally, $\mathcal{W}^u(p_\epsilon)$.

f) All the points of intersection of the previous items are in the rectangle:

$$[-2|\alpha_2|^{-n/2}, 2|\alpha_2|^{-n/2}] \times [\alpha_2^{-n} - 2|\alpha_2|^{-3n/2}, \alpha_2^{-n} + 2|\alpha_2|^{-3n/2}].$$

Other cases of homoclinic tangency

All the other cases, corresponding to the maps 3.13, 3.14, 3.15, are anologous to the first case. We only state the following theorems:

Theorem 3.4.23 *Let $\{f_\epsilon\}_{\epsilon\in[-1,1]}$ be as in the previous theorem, with the following change: f_ϵ^n is, for (ξ,η) near $(0,0)$,*

$$f_\epsilon^n(\xi,\eta) = (\alpha_2^{-n}\eta - 1, (1-\alpha_2^{-n}\eta)^2 + \epsilon - \alpha_1^n(1+\xi)).$$

Then if n is large enough and ϵ is between ϵ_{1n} and ϵ_{2n} one has:

a) Both branches of $\mathcal{W}^s(S_\epsilon)$ intersect $\mathcal{W}^u(p_\epsilon)$ if $\alpha_1 < 0$ and n odd.

b) If $\alpha_1 > 0$, and $|\alpha_1\alpha_2^2| < 1$ then $\mathcal{W}^s(S_\epsilon) \cap \mathcal{W}^u(p_\epsilon) \neq \emptyset$.

c) If $|\alpha_2^4\alpha_1^3| < 1$, $\alpha_1 < 0$ and n odd then there exists $(\xi,\eta) \in \mathcal{W}^s(S_\epsilon) \cap \mathcal{W}^u(p_\epsilon)$ such that $\xi > 0$.

d) If $\alpha_1 < 0$, $|\alpha_2^4\alpha_1^3| > 1$ and n odd then the first point of intersection of both branches of $\mathcal{W}^s(S_\epsilon)$ with $\mathcal{W}^u(p_\epsilon)$ has negative abscisa.

e) If $\alpha_1 > 0$ and $|\alpha_1\alpha_2^2| > 1$ then $\mathcal{W}^s(S_\epsilon)$ does not intersect, locally, $\mathcal{W}^u(p_\epsilon)$.

f) All the points of intersection of the previous items are in the rectangle:

$$[-2|\alpha_2|^{-n/2}, 2|\alpha_2|^{-n/2}] \times [\alpha_2^{-n} - 2|\alpha_2|^{-3n/2}, \alpha_2^{-n} + 2|\alpha_2|^{-3n/2}].$$

Theorem 3.4.24 *Let $\{f_\epsilon\}_{\epsilon\in[-1,1]}$ be as in the previous theorem, but suppose now that $\mathcal{W}^u(p_\epsilon)$ is $\eta = -\xi^2 + \epsilon$ and f_ϵ^n is, for (ξ,η) near $(0,0)$,*

$$f_\epsilon^n(\xi,\eta) = (\alpha_2^{-n}\eta - 1, -(1-\alpha_2^{-n}\eta)^2 + \epsilon - \alpha_1^n(1+\xi)).$$

Then if n is large enough and ϵ is between ϵ_{1n} and ϵ_{2n} one has:

a) Both branches of $\mathcal{W}^s(S_\epsilon)$ intersect $\mathcal{W}^u(p_\epsilon)$ if $\alpha_1 > 0$.

b) If $\alpha_1 < 0$, n odd and $|\alpha_1\alpha_2^2| < 1$ then $\mathcal{W}^s(S_\epsilon) \cap \mathcal{W}^u(p_\epsilon) \neq \emptyset$.

c) If $|\alpha_2^4\alpha_1^3| < 1$ and $\alpha_1 > 0$ then there exists $(\xi,\eta) \in \mathcal{W}^s(S_\epsilon) \cap \mathcal{W}^u(p_\epsilon)$ such that $\xi > 0$.

d) If $\alpha_1 > 0$ and $|\alpha_2^4\alpha_1^3| > 1$ then the first point of intersection of both branches of $\mathcal{W}^s(S_\epsilon)$ with $\mathcal{W}^u(p_\epsilon)$ has negative abscisa.

e) If $\alpha_1 < 0$, n odd and $|\alpha_1\alpha_2^2| > 1$ then $\mathcal{W}^s(S_\epsilon)$ does not intersect, locally, $\mathcal{W}^u(p_\epsilon)$.

f) All the points of intersection of the previous items are in the rectangle:

$$[-2|\alpha_2|^{-n/2}, 2|\alpha_2|^{-n/2}] \times [\alpha_2^{-n} - 2|\alpha_2|^{-3n/2}, \alpha_2^{-n} + 2|\alpha_2|^{-3n/2}].$$

Theorem 3.4.25 *Let $\{f_\epsilon\}_{\epsilon\in[-1,1]}$ be as in the previous theorem, but suppose now that $\mathcal{W}^u(p_\epsilon)$ is $\eta = -\xi^2 + \epsilon$ and f_ϵ^n is, for (ξ,η) near $(0,0)$,*

$$f_\epsilon^n(\xi,\eta) = (1 - \alpha_2^{-n}\eta, -(1-\alpha_2^{-n}\eta)^2 + \epsilon + \alpha_1^n(1+\xi)).$$

Then if n is large enough and ϵ is between ϵ_{1n} and ϵ_{2n} one has:

a) Both branches of $\mathcal{W}^s(S_\epsilon)$ intersect $\mathcal{W}^u(p_\epsilon)$ if $\alpha_1 < 0$ and n odd.

b) If $\alpha_1 > 0$, and $|\alpha_1\alpha_2^2| < 1$ then $\mathcal{W}^s(S_\epsilon) \cap \mathcal{W}^u(p_\epsilon) \neq \emptyset$.

c) If $|\alpha_2^4\alpha_1^3| < 1$, $\alpha_1 < 0$ and n odd then there exists $(\xi,\eta) \in \mathcal{W}^s(S_\epsilon) \cap \mathcal{W}^u(p_\epsilon)$ such that $\xi > 0$.

d) If $\alpha_1 < 0$, $|\alpha_2^4\alpha_1^3| > 1$ and n odd then the first point of intersection of both branches of $\mathcal{W}^s(S_\epsilon)$ with $\mathcal{W}^u(p_\epsilon)$ has negative abscisa.

e) If $\alpha_1 > 0$ and $|\alpha_1\alpha_2^2| > 1$ then $\mathcal{W}^s(S_\epsilon)$ does not intersect, locally, $\mathcal{W}^u(p_\epsilon)$.

f) All the points of intersection of the previous items are in the rectangle:

$$[-2|\alpha_2|^{-n/2}, 2|\alpha_2|^{-n/2}] \times [\alpha_2^{-n} - 2|\alpha_2|^{-3n/2}, \alpha_2^{-n} + 2|\alpha_2|^{-3n/2}].$$

Remark 3.4.26 *a) We have seen that in 8 of the 16 possible cases of homoclinic tangency always $\mathcal{W}^s(S_\epsilon) \cap \mathcal{W}^u(p_\epsilon) \neq \emptyset$. In other cases it depends on the relation between the eigenvalues of p_ϵ. Of course, in the cases in which there is non empty intersection, $\overline{\mathcal{W}^u(p_\epsilon)}$ cannot be a strange attractor.*

b) It seems possible to extend this theorems to the general case. Recall that, in general, the map f_ϵ^n near the homoclinic tangency is not a quadratic map, but it is close to such a map.

c) All the cases of homoclinic tangency are possible: Let $f_{a,b}(x,y)$ the Hénon map, and suppose that $|b| < 1$. Then the following is suggested by numerical simulations:

(a) If $b > 0$ and for suitable values of a and b there are the four types of homoclinic tangencies with $\alpha_1 > 0,\ \alpha_2 > 0$, for the fixed point p_+, if we take $f^2_{a,b}$. If we take $f_{a,b}$, we obtain the four types of homoclinic tangencies with $\alpha_1 > 0,\ \alpha_2 < 0$.

(b) If $b > 0$ there exist the four types of homoclinic tangencies for p_-, corresponding to $\alpha_2 > 0$ and $\alpha_1 < 0$.

(c) If $b < 0$ then there exist the four types of homoclinic tangencies for p_+ corresponding to $\alpha_1 < 0$ and $\alpha_2 < 0$.

3.5 Numerical results

In this section we shall display some examples of the Newhouse phenomenon. For this, we use the Hénon map (see chapter 2). The standard form of this quadratic mapping is $f_{a,b}(x,y) = (1+y-ax^2, bx)$, where a and b are real parameters. This map has been largely studied. Hénon found, numerically, a strange attractor for $a = 1.4$ and $b = 0.3$ (see [31], and other numerical examples in [29]). Recently it seems that there is a rigorous proof of the existence of a strange attractor for a wide set of parameters a if b is small (see [32]). We will also show that, numerically, it is posible to see that the measure of the parameters for which there exists attracting periodic orbits is very small.

First we recall (see chapter 2) that the Hénon map has two fixed points p_+ and p_- if $a \geq -\frac{1}{4}(1-b)^2$. Moreover p_- is a saddle point if $a > -\frac{1}{4}(1-b)^2$ and p_+ is a saddle point if $a > \frac{3}{4}(1-b)^2$. The map $f_{a,b}$ is globally dissipative if $|b| < 1$, and it is a diffeomorphism with quadratic inverse function if $b \neq 0$.

Let $|b| < 1$. By the Newhouse phenomenon, if there is a non degenerate tangency for $a = a_0$ which unfolds generically, there exist attracting periodic orbits of large period for a close to a_0. These orbits can look like strange attractors when doing a numerical simulations.

We shall compute, first, some parameters for which there are homoclinic tangencies. For this, we shall compute the invariant manifolds of the fixed points:

3.5.1 Invariant manifolds of the fixed points and homoclinic tangencies

Supose that $y = \Phi(x)$ represents one of the invariant manifolds near the fixed point, and let $f_{a,b} = (f_1, f_2)$. The map Φ satisfies the functional equation:

$$f_2(x, \Phi(x)) = \Phi(f_1(x, \Phi(x))).$$

If (x_0, y_0) is the fixed point, then

$$y = \Phi(x) = y_0 + \sum_{i=1}^{\infty} \beta_i (x - x_0)^i.$$

As $f_{a,b}$ is analytic then Φ is also analytic, provided that (x_0, y_0) is a hyperbolic point. The terms of this series can be sequentially computed. The first terms are:

$$\beta_1 = ax_0 \pm \sqrt{a^2 x_0 + b},$$

$$\beta_2 = \frac{a\beta_1}{\beta_1 + \gamma_1^3},$$

$$\beta_3 = -\frac{2\beta_2\gamma_1\gamma_2}{\beta_1 + \gamma_1^3},$$

$$\beta_4 = -\frac{\beta_2(\gamma_2^2 + 2\gamma_1\beta_3) + 3\beta_3) + 3\beta_3\gamma_1^2\gamma_2}{\beta_1 + \gamma_1^4},$$

$$\beta_5 = -(\beta_2(2\gamma_1\beta_4 + 2\gamma_2\beta_3) + \beta_3(3\gamma_1\gamma_2^2 + 3\gamma_1^2\beta_3) + 4\beta_4\gamma_1^2\gamma_2)(\beta_1 + \gamma_1^5)^{-1},$$

$$\beta_6 = -(\beta_2(2\gamma_1\beta_5 + 2\gamma_2\beta_4 + \beta_3^2) + \beta_3(3\gamma_1^2\beta_4 + 6\gamma_1\gamma_2\beta_3 + \gamma_2^3) +$$
$$\beta_4(4\gamma_1^3\beta_3 + 6\gamma_1^2\gamma_2^2) + 5\beta_5\gamma_1^4\gamma_2)(\beta_1 + \gamma_1^6)^{-1}.$$

where $\gamma_1 = \beta_1 - 2ax_0$ and $\gamma_2 = \beta_2 - a$.

Then to continue the invariant manifolds, locally approximated by the analytical expression, we use the map $f_{a,b}^n$ (unstable case) or $f_{a,b}^{-n}$ (stable case).

Another way to compute the invariant manifolds is to consider the parametrized form of the invariant manifolds used in the previous section: $(x(t), y(t))$ such that $f(x(t), y(t)) = (x(\alpha t), y(\alpha t))$, where α is the eigenvalue of $Df_{a,b}(x_0, y_0)$ associated to this manifold.

A picture of the invariant manifolds of p_+ is in figure 3.9 for the parameters $a = 0.5734437995$ and $b = 0.8$

To compute the homoclinic tangencies we observe that parameters a and b for which they exist, verify the following system of equations:

$$\left.\begin{array}{rcl} f_1^n(q) - f_1^{-m}(\bar{q}) & = & 0 \\ f_2^n(q) - f_2^{-m}(\bar{q}) & = & 0 \\ D_1 f_1^n(q) D_2 f_2^{-m}(\bar{q}) - D_1 f_2^n(q) D_2 f_1^{-m}(\bar{q}) & = & 0 \end{array}\right\} \quad (3.32)$$

where $q = (x_1, \Phi(x_1))$, $\bar{q} = (x_2, \bar{\Phi}(x_2))$, $y = \Phi(x)$ and $y = \bar{\Phi}(x)$ are, respectively, the local stable and unstable invariant manifolds of (x_0, y_0), and $f_{a,b}^m = (f_1^m, f_2^m)$. More precisely,

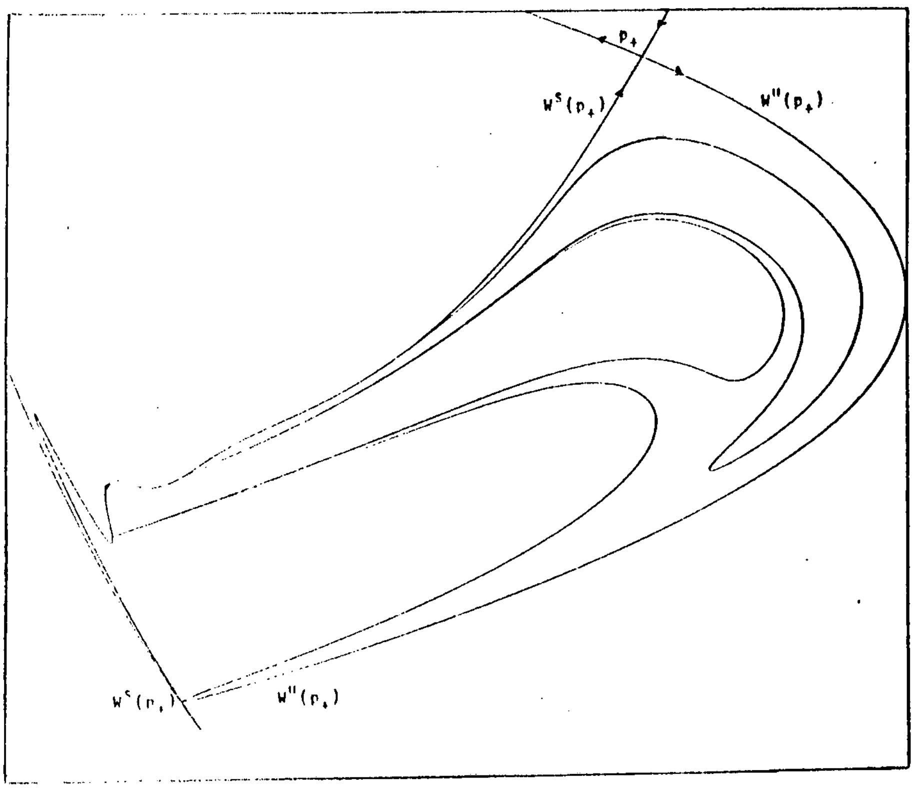


Figure 3.9: Invariant manifolds of p_+, for $a = 0.5734437995$ and $b = 0.8$

for all $\epsilon > 0$ there exist $n, m \in \mathbb{N}$ and $x_1, x_2 \in \mathbb{R}$ such that $|x_1 - x_0| < \epsilon$, $|x_2 - x_0| < \epsilon$ and x_1, x_2, a, b satisfy the system 3.32.

Numerically, we obtain for $b = 0.8$ a value $a = 0.5734437995744...$ A tangent homoclinic point is $(x, y) \approx (-0.981306583, -2.07665027)$.

We observe finally that the equation 3.32 can be expressed as $H(b, a, x_1, x_2) = 0$ where $H : U \subset \mathbb{R}^4 \longrightarrow \mathbb{R}^3$. Then we can compute $a = a(b)$, $x_1 = x_1(b)$, $x_2 = x_2(b)$ by using a continuation method (see for example [33]).

3.5.2 Periodic orbits related to the Newhouse phenomenon

We look for attracting periodic points for values of the parameter near a homoclinic tangency. Numerically, one see that, for the parameter of homoclinic tangency of the previous section, the behaviour of the map $f^n_{a,b}$ near the homoclinic tangency point is similar to the horeshoe map for n large (see figure 3.10). We can find in this case couples of hyperbolic periodic points that, in all the cases, are saddle points, as we can see in the table 3.1.

Now we fix the parameter $b = 0.8$. To obtain the evolution of these periodic points when we move the parameter a, we use a continuation method. Then we see that the saddle periodic points with negative eigenvalues are transformed into sink if the parameter a is suitably decreased. If the parameter is decreased again, one finds a saddle-node bifurcation. We can find also a cascade of flip bifurcations associated to this saddle-node bifurcation. Let a_{1n} be the parameter of saddle-node bifurcation and a_{2n}, a_{4n}, ..., $a_{2^k n}$ be the parameters of flip bifurcation corresponding to periods $2, ..., 2^k n$. In table 3.2 we have the values of $\Delta_{1n} = (a_{1n} - a_{2n})/(a_{2n} - a_{4n})$ and $\Delta_{2n} = (a_{2n} - a_{4n})/(a_{4n} - a_{8n})$ for some periods n (compare with the analytic results of section 3.2).

Now we want to know the basin of attraction of the attracting periodic points. We are specially interested in knowing if this basin of attraction contains points of the unstable invariant manifold of p_+. The coexistence of the strange attractor $\overline{W^u(p_+)}$ (if it exists) and the attracting periodic orbit takes place if the basin of attraction does not intersect $\overline{W^u(p_+)}$. We have found numerically, for $n = 20$ and $b = 0.8$, the stable invariant manifold of the n-periodic saddle S which is born by saddle-node bifurcation when the node N appears. This manifold intersect $W^u(p_+)$ on two places. If we consider the parameter value $a \approx 0.5728841$ then $\alpha_2 \approx -1.7721$ and $|\alpha_2^4 \alpha_1^3| \approx 0.9093 < 1$, where α_1 and α_2 are the eigenvalues of $Df_{a,b}(p_+)$. This agrees with the analytical results (see figure 3.11).

When $b = 0.916$ we have a homoclinic tangency of the invariant manifolds of p_+ at $a \approx 0.71539869$. There is a 20-periodic attracting point for $a \approx 0.6681773996$. In this case

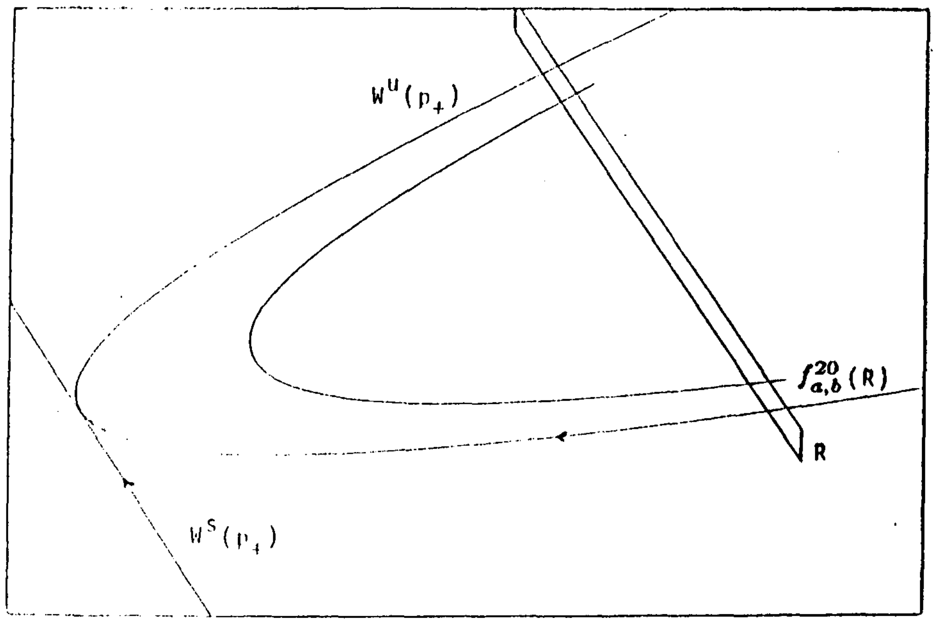


Figure 3.10: Behaviour of $f^{20}_{a,b}$ close to a homoclinic tangency.

$|\alpha_1^3\alpha_2^4| \approx 1.542993 > 1$ and $\alpha_2 \approx -2.007602$. Moreover, the 20 periodic saddle associated to the attractor intersects only at one place, the unstable invariant manifold (see figure 3.12).

We have studied also the Hénon map for $b = 0.3$, and from now on this will be the adopted value of b. By simple iteration we can find attracting periodic orbits which are related to homoclinic tangencies. In the figure 3.13 we can see the structure of the unstable invariant manifold of p_+ for different values of a and some periodic attractors.

Let a_{1n} be the saddle-node bifurcation parameter and a_{2n} the flip bifurcation parameter of an n peridic point. In our case the greatest eigenvalue in absolute value of p_+ is $\alpha_2 \approx -1.737365$. Then we have:

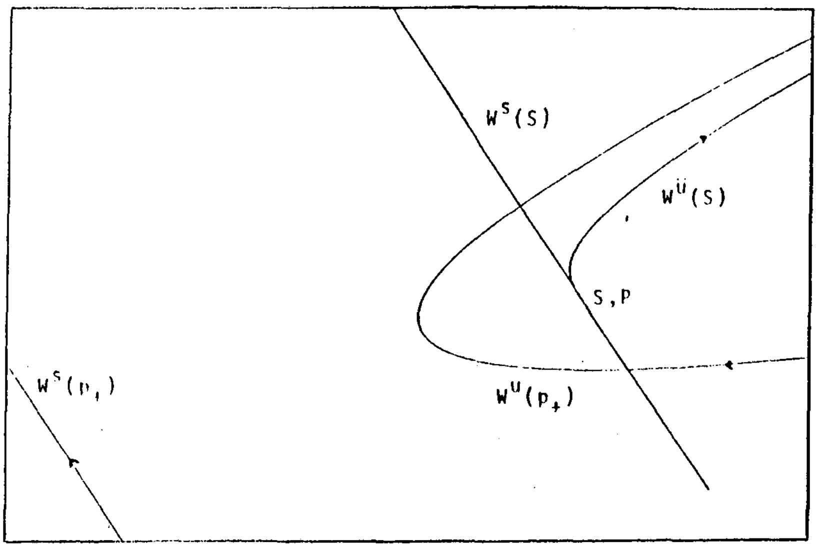
$$a_c = \frac{a_{1(n+2)} - \alpha_2^{-2} a_{1n}}{1 - \alpha_2^{-2}} ,$$

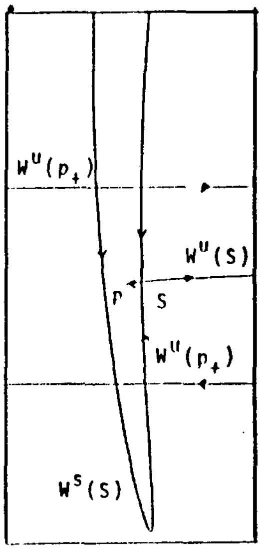
$$\frac{a_{2n} - a_{1n}}{a_{2(n-2)} - a_{1(n-2)}} = \alpha_2^{-4} \approx 0.10975 ,$$

where a_c is a parameter of homoclinic tangency related to the periodic point which appears at $a = a_{1n}$, $n \in \mathbb{N}$. These formulae are deduce from the estimates of the parameters a_{1n} in section 3.2. We have found three families of periodic attractors related to homoclinic tangencies (see tables 3.3, 3.4, 3.5).

3.5.3 Frecuency of aperiodic behaviour

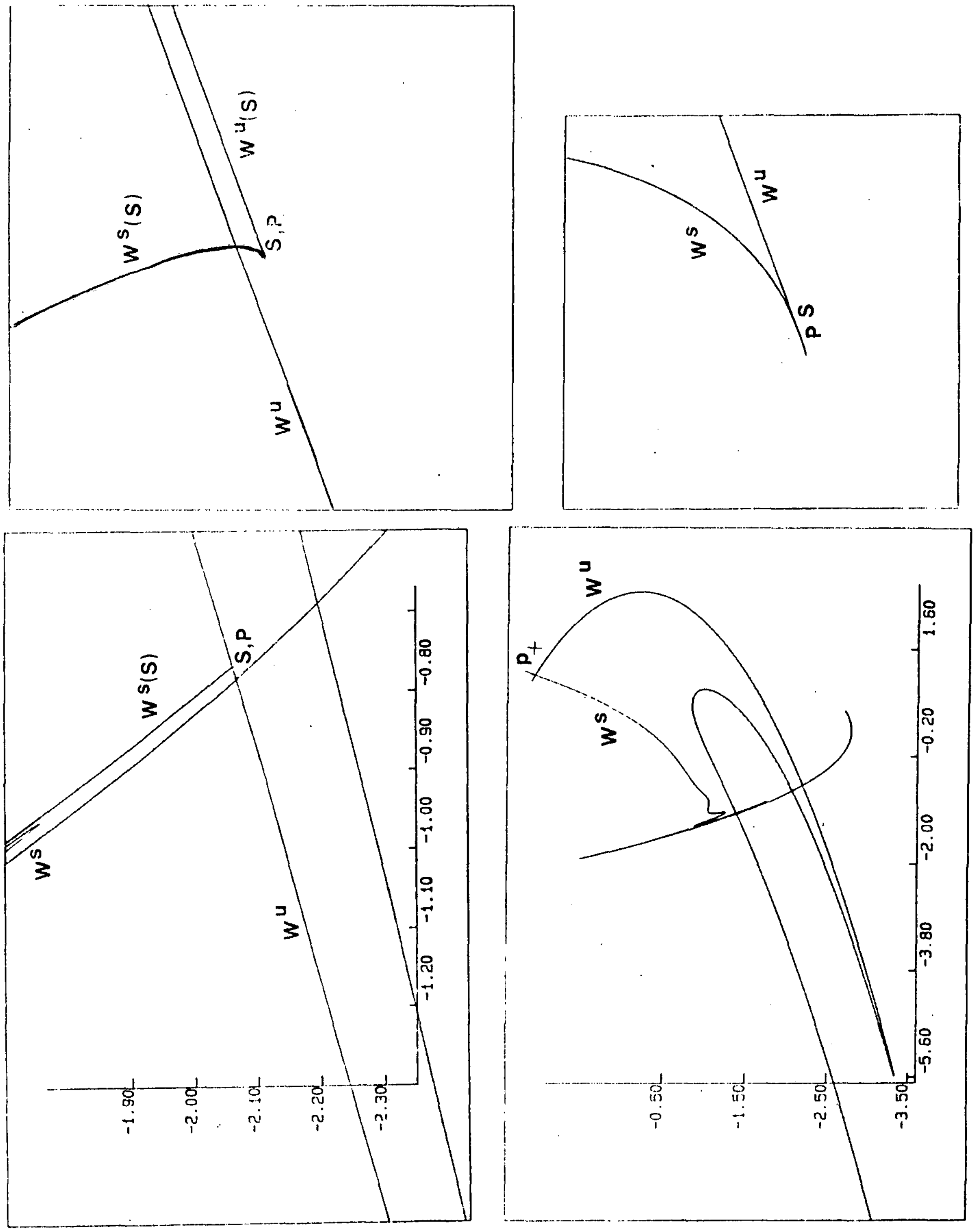
We have studied numerically the measure of the set of parameters $a \in [1.15, 1.16]$ and $b = 0.3$, for which there exist attracting periodic orbits. To do this we have consider increments of the parameter a, $\Delta a = 10^{-8}$ and periods less or equal to 100. The resullts of this computation are in the table 3.6. We see that the total measure of parameters a for which there exist attracting periodic orbits is $S \approx 1.0018 \times 10^{-4}$, and the ratio of the attracting periodic orbits in the considered interval is $r \approx 1.0018 \times 10^{-2}$. Then we observe that the frecuency of aperiodic behaviour is very large. Also one can see that, associated to a period n, there exist periods nm which appear in the same order (with respect to the parameter a) that the the corresponding one in the logistic map, if m is not very large. For example one observe this fact for $n = 11$.





A blow up of the above picture.

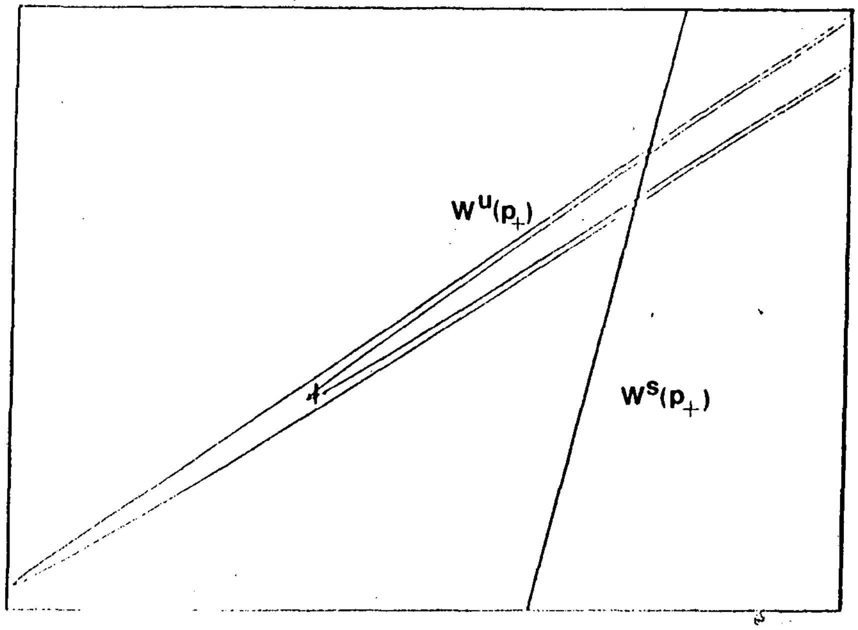
Figure 3.11: Basin of attraction of a periodic point related to the Newhouse phenomenon. Case 1.



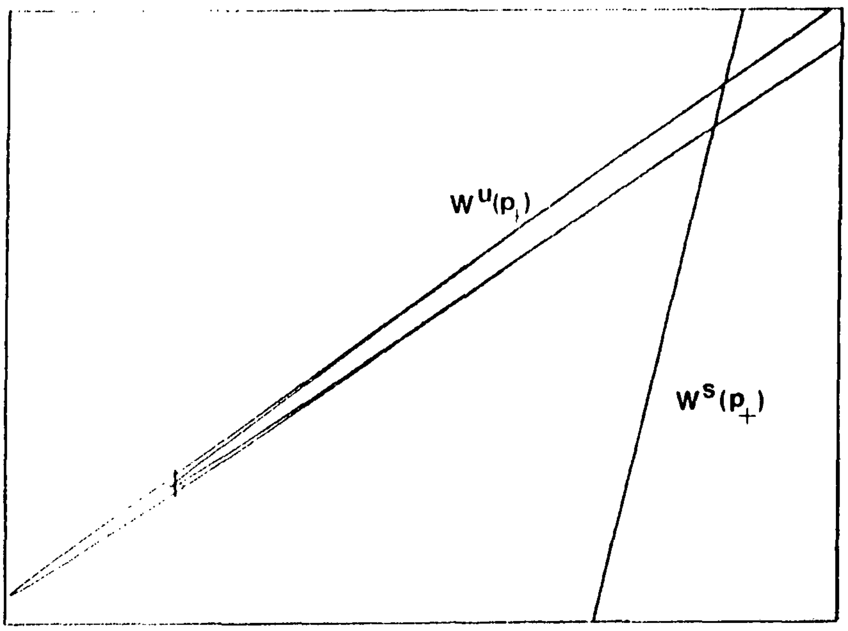
A general picture and three successive details.

Figure 3.12: Basin of attraction of a periodic point related to the Newhouse phenomenon. Case 2.

a)



b)



a) $a = 1.161842301$. Attracting periodic orbit of period 17 corresponding to table 3.4.
b) $a = 1.170689$. Attracting periodic orbit of period 15 corresponding to 3.4.

Figure 3.13: Invariant manifolds of p_+ for $b = 0.3$, and some attracting periodic points

n	20	20
x_n	$-9.732329972 \times 10^{-1}$	$-9.704099188 \times 10^{-1}$
y_n	-2.071365538	-2.076188468
α_{1n}	$8.171751331 \times 10^{1}$	$-1.434178526 \times 10^{-4}$
α_{2n}	$1.410862198 \times 10^{-4}$	$-8.038898112 \times 10^{1}$
n	22	22
x_n	$-9.783872780 \times 10^{-1}$	$-9.762804181 \times 10^{-1}$
y_n	-2.073775079	-2.077137292
α_{1n}	$1.293906060 \times 10^{2}$	$-5.790743443 \times 10^{-5}$
α_{2n}	$5.702653238 \times 10^{-5}$	$-1.274222852 \times 10^{2}$
n	24	24
x_n	$-9.804761360 \times 10^{-1}$	$-9.790857076 \times 10^{-1}$
y_n	-2.075120302	-2.077281023
α_{1n}	$1.948976709 \times 10^{2}$	$-2.455142524 \times 10^{-5}$
α_{2n}	$2.422997904 \times 10^{-5}$	$-1.931032083 \times 10^{2}$
n	26	26
x_n	$-9.811971017 \times 10^{-1}$	$-9.803517938 \times 10^{-1}$
y_n	-2.075853208	-2.077152751
α_{1n}	$3.113150874 \times 10^{2}$	$-9.762282189 \times 10^{-6}$
α_{2n}	$9.708217419 \times 10^{-6}$	$-3.095908427 \times 10^{2}$
n	28	28
x_n	$-9.813774373 \times 10^{-1}$	$-9.808858015 \times 10^{-1}$
y_n	-2.076235348	-2.076988138
α_{1n}	$5.293268554 \times 10^{2}$	$-3.665816649 \times 10^{-6}$
α_{2n}	$3.654228578 \times 10^{-6}$	$-5.276535896 \times 10^{2}$

Table 3.1: Saddle n-periodic orbits (x_n, y_n) for $a = 0.57344379957443$ and $b = 0.8$. α_{1n} and α_{2n} are their associated eigenvalues.

n	20	22	24
a_{1n}	0.5728840634	0.57316692302	0.573328857003
a_{2n}	0.5728844173	0.57316699183	0.573328869423
a_{4n}	0.5728845902	0.57316702573	0.573328875575
a_{8n}	0.5728846311	0.57316703373	0.573328877028
Δ_{1n}	2.0462359	2.0295525	2.0189025
Δ_{2n}	4.2376348	4.2359850	4.2350788

Table 3.2:

n	a_{1n}	a_{2n}
12	1.150007353	1.150018943
14	1.152384725	1.152386017
15	1.154251157	1.154251585
13	1.155612217	1.155561608
11	1.159704578	1.159740282
9	1.172384176	1.172761143

Table 3.3: Family associated to the parameter $a_c \approx 1.1535702$ (first homoclinic tangency).

n	a_{1n}	a_{2n}
17	1.161842300	1.161842673
15	1.170687847	1.170691059

Table 3.4: Family associated to the parameter $a_c \approx 1.1574228$.

n	a_{1n}	a_{2n}
21	1.155428802	1.155423820
19	1.158801236	1.158801381
17	1.168860523	1.168861538

Table 3.5: Family associated to the parameter $a_c \approx 1.1537336$.

n	a_{1n}	a_{2n}	$\lvert a_{1n} - a_{2n} \rvert$
12	1.150007353042388E+00	1.150018943597931E+00	1.159055554370351E-05
24	1.150018943597931E+00	1.150024733513397E+00	5.789915465644665E-06
48	1.150024733513397E+00	1.150026100930045E+00	1.367416647985963E-06
96	1.150026100930045E+00	1.150026401347071E+00	3.004170257922537E-07
72	1.150027335064324E+00	1.150027393412677E+00	5.834835276555041E-08
84	1.150028505540453E+00	1.150028513581571E+00	8.041118521651515E-09
60	1.150029067903544E+00	1.150029114685026E+00	4.678148130389459E-08
36	1.150030522614228E+00	1.150030737435759E+00	2.148215306960673E-07
72	1.150030737435759E+00	1.150030838214531E+00	1.007787722972990E-07
84	1.150031489064327E+00	1.150031490206137E+00	1.141809412243273E-09
60	1.150031804119162E+00	1.150031813143402E+00	9.024239377070810E-09
48	1.150032730876397E+00	1.150032742346934E+00	1.147053655275386E-08
96	1.150032755191483E+00	1.150032760869139E+00	5.677655548095048E-09
20	1.151020170250909E+00	1.151020719536835E+00	5.492859256140886E-07
40	1.151020719536835E+00	1.151020995076497E+00	2.755396621384657E-07
80	1.151020995076498E+00	1.151021060163799E+00	6.508730132098711E-08
60	1.151021270602230E+00	1.151021280746132E+00	1.014390145996121E-08
20	1.151318345024216E+00	1.151318843536116E+00	4.985119004042119E-07
40	1.151318843536117E+00	1.151319091984270E+00	2.484481539608620E-07
80	1.151319091984270E+00	1.151319150659140E+00	5.867486998653112E-08
60	1.151319340420900E+00	1.151319349684394E+00	9.263493948090197E-09
18	1.151493924920060E+00	1.151497858171563E+00	3.933251503057538E-06
36	1.151497858171563E+00	1.151499823478173E+00	1.965306610147889E-06
72	1.151499823478173E+00	1.151500287556142E+00	4.640779694759333E-07
90	1.151501294529930E+00	1.151501310396511E+00	1.586658095497582E-08
54	1.151501787943636E+00	1.151501860731222E+00	7.278758599959489E-08
90	1.151502222334296E+00	1.151502225390751E+00	3.056455800652128E-09
14	1.152384725426882E+00	1.152386017545705E+00	1.292118822564156E-06
28	1.152386004811036E+00	1.152386650685550E+00	6.458745136988065E-07
56	1.152386650685550E+00	1.152386803236696E+00	1.525511465708724E-07
84	1.152386940926747E+00	1.152386947437455E+00	6.510708251589167E-09
70	1.152387134282972E+00	1.152387139500781E+00	5.217809175610255E-09
42	1.152387296548257E+00	1.152387320494734E+00	2.394647649793597E-08
84	1.152387320494734E+00	1.152387331728641E+00	1.123390764761703E-08
18	1.152991329676382E+00	1.152991517608698E+00	1.879323161840746E-07
36	1.152991517608698E+00	1.152991611582366E+00	9.397366776051479E-08
72	1.152991611582366E+00	1.152991633778744E+00	2.219637806149789E-08
54	1.152991705555558E+00	1.152991709037441E+00	3.481883724675679E-09
15	1.154251157099296E+00	1.154251585909257E+00	4.288099606326022E-07
30	1.154251585909257E+00	1.154251800346386E+00	2.144371285385668E-07
60	1.154251813001730E+00	1.154251863651396E+00	5.064966545962989E-08
⋮	⋮	⋮	⋮

:	:	:	:
90	1.154251909368371E+00	1.154251911530356E+00	2.161984710419393E-09
45	1.154252014782159E+00	1.154252022726285E+00	7.944125725480344E-09
21	1.155423802406139E+00	1.155423820820576E+00	1.841443648265529E-08
13	1.155612217280488E+00	1.155616089442237E+00	3.872161749703662E-06
26	1.155616089442238E+00	1.155618026349523E+00	1.936907285460356E-06
52	1.155618026349523E+00	1.155618483836246E+00	4.574867224532602E-07
78	1.155618896774811E+00	1.155618916304646E+00	1.952983511207045E-08
65	1.155619476756644E+00	1.155619492396040E+00	1.563939616195788E-08
39	1.155619963159472E+00	1.155620034861823E+00	7.170235102670370E-08
78	1.155620034861822E+00	1.155620068498765E+00	3.363694275385101E-08
18	1.156951101668626E+00	1.156952258049464E+00	1.156380837904244E-06
36	1.156952258049464E+00	1.156952836144176E+00	5.780947119124497E-07
72	1.156952836144176E+00	1.156952972684174E+00	1.365399979998637E-07
90	1.156953276844380E+00	1.156953281513916E+00	4.669536835019733E-09
54	1.156953414209211E+00	1.156953435634450E+00	2.142523867142601E-08
90	1.156953686405551E+00	1.156953542972390E+00	1.434331612615807E-07
20	1.157987554298489E+00	1.157987846663137E+00	2.923646483224268E-07
40	1.157987846663137E+00	1.157987992861049E+00	1.461979118088584E-07
80	1.157988000767341E+00	1.157988035298845E+00	3.453150438624721E-08
60	1.157988139056989E+00	1.157988144473385E+00	5.416395580819471E-09
19	1.158801233670830E+00	1.158801381766007E+00	1.480951760563027E-07
38	1.158801381766007E+00	1.158801455820859E+00	7.405485255595802E-08
76	1.158801455820859E+00	1.158801473312449E+00	1.749158966702873E-08
11	1.159704578470662E+00	1.159740282480347E+00	3.570400968444009E-05
22	1.159740282480347E+00	1.159758155103120E+00	1.787262277280760E-05
44	1.159758155103119E+00	1.159762375725751E+00	4.220622632346334E-06
88	1.159762375725751E+00	1.159763303037892E+00	9.273121411725837E-07
66	1.159766185383398E+00	1.159766365615102E+00	1.802317037489376E-07
88	1.159767866083663E+00	1.159767881888981E+00	1.580531825645606E-08
99	1.159769066601450E+00	1.159769070320182E+00	3.718731942787214E-09
77	1.159769762274772E+00	1.159769787099837E+00	2.482506435450555E-08
55	1.159771537226679E+00	1.159771681352666E+00	1.441259865419478E-07
77	1.159773306453470E+00	1.159773322395701E+00	1.594223078013913E-08
33	1.159776020360530E+00	1.159776679949389E+00	6.595888593343052E-07
66	1.159776679949389E+00	1.159776989365282E+00	3.094158934509610E-07
99	1.159777293911626E+00	1.159777305924217E+00	1.201259120614306E-08
88	1.159778169432825E+00	1.159778171245103E+00	1.812277804189986E-09
55	1.159779961271760E+00	1.159779988944504E+00	2.767274419062929E-08
66	1.159781637322853E+00	1.159781641499937E+00	4.177084075554607E-09
44	1.159782809682642E+00	1.159782844593603E+00	3.491096156822548E-08
88	1.159782844593603E+00	1.159782861873854E+00	1.728025083032503E-08
TOTAL	1.001829938000431E-04		
RATIO	1.001829938000431E-02		

Table 3.6: Some attracting periodic orbits of the Hénon map

Chapter 4

Bifurcation curves of periodic points in one- and two-parameter families of dissipative diffeomorphisms

In this chapter we will consider diffeomorphisms $f : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which are dissipative (that is, $\det Df(p) < 1 \; \forall p \in U$) and smooth enough. In this case, f has a hyperbolic n-periodic point p if and only if $\pm 1 \notin \operatorname{Spec} Df^n(p)$. In order to study bifurcations, we consider non hyperbolic points and families of diffeomorphisms depending on one or two parameters. It is known that the hyperbolicity condition implies that the behaviour of the considered map, near the hyperbolic point, does not change when the map is perturbed slightly. But this is not the case of the non hyperbolic points.

Since the maps we consider are dissipative, the non hyperbolicity condition means that one and only one eigenvalue of $Df^n(p)$ is 1 (corresponding to saddle-node -also called fold- bifurcation) or -1 (corresponding to flip -also called subharmonic or period doubling-bifurcation). The degenerate cases of this bifurcations are called cusps (eigenvalue equal to 1) and codimension two flips (eigenvalue equal to -1). We will see that these bifurcations appear, in a natural way, in families of diffeomorphisms depending on two parameters. Also we will see that the behaviour of this diffeomorphisms can be studied using one dimensional maps

First we will study the different types of bifurcations, by using normal forms:

4.1 Normal forms, models and existence conditions of generic bifurcations

In this section we use the theory of normal forms for mappings given in [34]. From now on, when we apply local theorems (say, the implicit function theorem) we will suppose that, if it is necessary, the neighbourhoods which appear are small enough. Also, when we say *differentiable* or *smooth mapping*, it means that it is differentiable enough. Moreover, all the indexes which appear are larger or equal to zero.

First we give the following definition:

Definition 4.1.1 *Let*

$$f(x,y) = \left(\begin{array}{c} \lambda x + \sum_{j=2}^{\infty} \sum_{i=0}^{j} a_{i,j-i} x^i y^{j-i} \\ \mu y + \sum_{j=2}^{\infty} \sum_{i=0}^{j} b_{i,j-i} x^i y^{j-i} \end{array} \right),$$

be a formal series. Then, the terms $a_{i,j}x^iy^j$ such that $\lambda = \lambda^i\mu^j$, and the terms $b_{i,j}x^iy^j$ such that $\mu = \lambda^i\mu^j$, with $i+j \geq 2$, are called resonant terms of the formal map f.

Then we have the following theorem([34]):

Theorem 4.1.2 *Let $f : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a map, defined on a neighbourhood U of $(0,0)$, of the form:*

$$f(x,y) \left(\begin{array}{c} \lambda x + \sum_{i \in I_1^{n-1}} a_{i,j} x^i y^j + \sum_{i=0}^{n} a_{i,n-i} x^i y^{n-i} + O_{n+1} \\ \mu y + \sum_{i \in I_2^{n-1}} b_{i,j} x^i y^j + \sum_{i=0}^{n} b_{i,n-i} x^i y^{n-i} + O_{n+1} \end{array} \right),$$

where $I_1^n = \{(i,j) \in \mathbb{N}_0^2 \; : \; \lambda = \lambda^i\mu^j \, , \; 2 \leq i+j \leq n\}$ and $I_2^n = \{(i,j) \in \mathbb{N}_0^2 \; : \; \mu = \lambda^i\mu^j \, , \; 2 \leq i+j \leq n\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and O_n denotes terms of order equal or larger than n. Let $J_1^n = \{i \in \mathbb{N}_0 \; : \; (i,n-i) \notin I_1^n\}$ and $J_2^n = \{i \in \mathbb{N}_0 \; : \; (i,n-i) \notin I_2^n\}$.

Then, the map f is conjugated (in a neighbourhood of $(0,0)$), via the following polynomial change of coordinates:

$$F : \left. \begin{array}{rcl} \bar{x} & = & x + \sum_{i \in J_1^n} \alpha_{i,n-i} x^i y^{n-i} \\ \bar{y} & = & y + \sum_{i \in J_2^n} \beta_{i,n-i} x^i y^{n-i} \end{array} \right\},$$

to the map:

$$g(\bar{x},\bar{y}) = \left(\begin{array}{c} \lambda \bar{x} + \sum_{(i,j) \in I_1^n} a_{i,j} \bar{x}^i \bar{y}^j + O_{n+1} \\ \mu \bar{y} + \sum_{(i,j) \in I_2^n} b_{i,j} \bar{x}^i \bar{y}^j + O_{n+1} \end{array} \right).$$

Moreover $\alpha_{i,j} = \frac{a_{i,j}}{\lambda^i\mu^j - \lambda}$ for $(i,j) \in J_1^n$ and $\beta_{i,j} = \frac{b_{i,j}}{\lambda^i\mu^j - \mu}$ for $(i,j) \in J_2^n$.

Let $f : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a dissipative diffeomorphism, having a non hyperbolic fixed point $p = (x_0, y_0)$. If λ and μ are the eigenvalues of $Df(p)$, there are two possibilities: $|\lambda| < 1$ and $\mu = 1$, or $|\lambda| < 1$ and $\mu = -1$. We will study these cases separately.

Suppose first that $\mu = 1$. By means of an affine change of coordinates, it is possible to write f as:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \lambda x + a_{20}x^2 + a_{11}xy + a_{02}y^2 + O_3 \\ y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + O_3 \end{pmatrix}.$$

By the theorem 4.1.2, it is possible, using a polynomial change of coordinates, to remove some terms of increasing degree, until we reach the normal form:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \lambda x + a_1 xy + a_2 xy^2 + \cdots \\ y + b_2 y^2 + b_3 y^3 + \cdots \end{pmatrix}. \tag{4.1}$$

Only the resonant terms, corresponding to resonances $\lambda = \lambda\mu^k,\ k \geq 1$, and $\mu = \mu^k,\ k \geq 2$ remain. Then we have the following definitions:

Definition 4.1.3 *The map f has a saddle-node point in $p = (x_0, y_0)$ if, and only if, in the normal form 4.1, one has $b_2 \neq 0$.*

Definition 4.1.4 *The map f has a cusp point in $p = (x_0, y_0)$ if, and only if, in the normal form 4.1, one has $b_2 = 0$ and $b_3 \neq 0$.*

Cosider now the case $\mu = -1$. By means of an affine change of coordinates, we can write the map f as:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \lambda x + a_{20}x^2 + a_{11}xy + a_{02}y^2 + O_3 \\ -y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + O_3 \end{pmatrix}.$$

By using polynomial changes of coordinates, as in the previous case, the non resonant terms can be removed. Then we have the following map, conjugated to f:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \lambda x + a_1 xy^2 + a_2 xy^4 + \cdots \\ -y + b_3 y^3 + b_5 y^5 + \cdots \end{pmatrix}, \tag{4.2}$$

since the resonances are, in this case: $\lambda = \lambda\mu^k,\ k \geq 2$ even, and $\mu = \mu^k,\ k \geq 3$ odd. Then we define:

Definition 4.1.5 *The map f has a flip point in $p = (x_0, y_0)$ if, and only if, in the normal form 4.2, one has $b_3 \neq 0$.*

Definition 4.1.6 *The map f has a codimension two flip point in $p = (x_0, y_0)$ if, and only if, in the normal form 4.2, one has $b_3 = 0$ and $b_5 \neq 0$.*

Now , in order to study the behavoiur of maps, we will introduce families of diffeomorphisms near a map having a bifurcation point.

Consider a one-parameter family of diffeomorphisms on the plane: $\{f_a\}_{a\in I}$, where I is an open interval, and suppose that, for $a = a_0 \in I$, there exists a non hyperbolic fixed point $p = (x_0, y_0)$. If $\lambda = \lambda(a_0)$ and $\mu = \mu(a_0)$ are the eigenvalues of $Df_{a_0}(p)$ and $|\lambda| \neq 1$, we can have $\mu = 1$ or $\mu = -1$.

Let $\mu = 1$. Then, the family $\{f_a\}_{a\in I}$ has the general form:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \sum_{n=0}^{2} \sum_{i+j+k=n} a_{ijk} a^i x^j y^k + O_3 \\ \sum_{n=0}^{2} \sum_{i+j+k=n} b_{ijk} a^i x^j y^k + O_3 \end{pmatrix},$$

where $i, j, k \in \mathbb{N}_0$.

By means of an affine change of coordinates, it is possible to conjugate this family to the following one:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \bar{a}_{100} a + \lambda x + \sum_{i+j+k=2} \bar{a}_{ijk} a^i x^j y^k + O_3 \\ \bar{b}_{100} a + y + \sum_{i+j+k=2} \bar{b}_{ijk} a^i x^j y^k + O_3 \end{pmatrix},$$

where $i, j, k \in \mathbb{N}_0$.

Finally, we can make zero one linear term in a, with the following transformation:

$$\left.\begin{array}{rcl} \bar{x} & = & x - \frac{\bar{a}_{100}}{\lambda - 1} \\ \bar{y} & = & y \end{array}\right\}.$$

So, the family $\{f_a\}_{a\in I}$ is conjugated to the following one:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \lambda x + \sum_{i+j+k=2} \tilde{a}_{ijk} a^i x^j y^k + O_3 \\ \tilde{b}_{100} a + y + \sum_{i+j+k=2} \tilde{b}_{ijk} a^i x^j y^k + O_3 \end{pmatrix}, \tag{4.3}$$

where $i, j, k \in \mathbb{N}_0$.

It is important to remark that $\tilde{b}_{100} = \bar{b}_{100}$, $\tilde{a}_{020} = \bar{a}_{020}$, $\tilde{a}_{011} = \bar{a}_{011}$, $\tilde{a}_{002} = \bar{a}_{002}$, $\tilde{b}_{020} = \bar{b}_{020}$, $\tilde{b}_{011} = \bar{b}_{011}$, $\tilde{b}_{002} = \bar{b}_{002}$.

Now, we define:

Definition 4.1.7 *The family $\{f_a\}_{a\in I}$ has a saddle-node $p = (x_0, y_0)$, in $a = a_0$, which unfolds generically with $\{f_a\}_{a\in I}$ if, and only if, in the normal form 4.3, we have:*

a) $\tilde{b}_{002} \neq 0$.

b) $\tilde{b}_{100} \neq 0$.

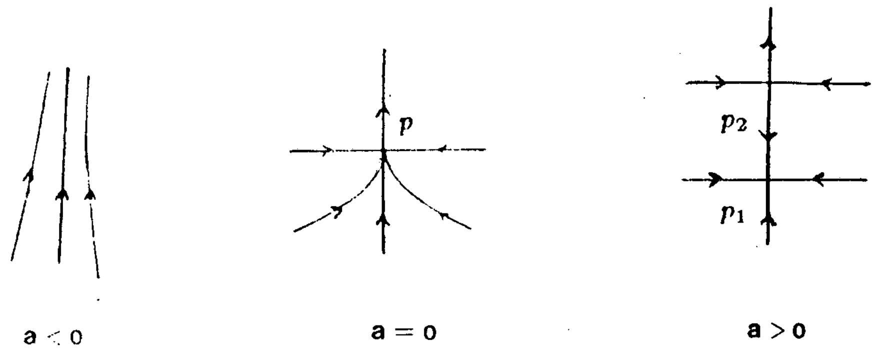


Figure — 4.1: Qualitative behaviour of a saddle-node bifurcation, $g_a(x,y) = (\lambda x, y + a + y^2)$, $|\lambda| < 1$.

There is a theorem in [35] which give, under mild conditions, the local conjugation of such a family, with the family:

$$g_a(x,y) = (\lambda x, y \pm a + y^2), \ |\lambda| \neq 1.$$

For this family, the local behaviour is represented in figure 4.1 That is, for $a < 0$ there are not fixed points, for $a = 0$ there is a saddle-node point,p, and for $a > 0$ there are two fixed points, one is a saddle and the other is a node (p_1 and p_2 in the figure).

Now assume that, in the family $\{f_a\}_{a\in I}$, we have $\mu = -1$. In this case, we obtain the following proposition:

Proposition 4.1.8 *Let $\{f_a\}_{a\in I}$ be a smooth one-parameter family of diffeomorphisms, having a fixed point $p = (\bar{x}_0, \bar{y}_0)$ for $a = a_0$. Let $\lambda(a_0)$ and $\mu(a_0)$ be the eigenvalues of $Df_{a_0}(p)$, such that $|\lambda(a_0)| \neq 1$ and $\mu(a_0) = -1$. Then there exists, for a near a_0, a fixed point $p(a) = (x_0(a), y_0(a))$ such that: $x_0(a_0) = \bar{x}_0$ and $y_0(a_0) = \bar{y}_0$. Moreover, the map $p(a)$ is of the same class of differentiability as f_a, with respect to a.*

Proof:

We only have to apply the implicit function theorem to the equation:

$$f(x,y,a) = f_a(x,y) = (x,y).$$

This is possible because $\det(Df_{a_0}(x,y) - Id) \neq 0$. □

By means of the proposition 4.1.8, we see that the family $\{f_a\}_{a\in I}$ can be conjugated to the following one:

$$g_a(x,y) = \begin{pmatrix} \lambda(a)x + a_{20}x^2 + a_{11}xy + a_{02}y^2 + O_3 \\ \mu(a)y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + O_3 \end{pmatrix},$$

where $\lambda(a)$ and $\mu(a)$ are the eigenvalues of $Df_a(p(a))$. Then, by using polynomial changes of coordinates, we can remove the non resonant terms of the map, obtaining:

$$\bar{g}_a(x,y) = \begin{pmatrix} \lambda(a)x + \bar{a}_{30}x^3 + \bar{a}_{21}x^2y + \bar{a}_{12}xy^2 + \bar{a}_{03}y^3 + O_4 \\ \mu(a)y + \bar{b}_{30}x^3 + \bar{b}_{21}x^2y + \bar{b}_{12}xy^2 + \bar{b}_{03}y^3 + O_4 \end{pmatrix}, \tag{4.4}$$

where $\bar{a}_{ij} = \bar{a}_{ij}(a)$ and $\bar{b}_{ij} = \bar{b}_{ij}(a)$.

Then we define:

Definition 4.1.9 *The family $\{f_a\}_{a\in I}$ has a flip point, $p = (x_0, y_0)$, which unfolds generically with $\{f_a\}_{a\in I}$ if, and only if, in the normal form 4.4, one has:*

a) $\bar{b}_{03}(a_0) \neq 0$.

b) $\frac{d}{da}\mu(a_0) \neq 0$.

Also in this case, there is a theorem in [35] which give, under mild conditions, the local conjugation of a such a family to the following family:

$$g_a(x,y) = (\lambda x, (-1 \pm a)y \pm y^3).$$

The behaviour of this family is represented in figure 4.2 That is: for $a < 0$ there is a

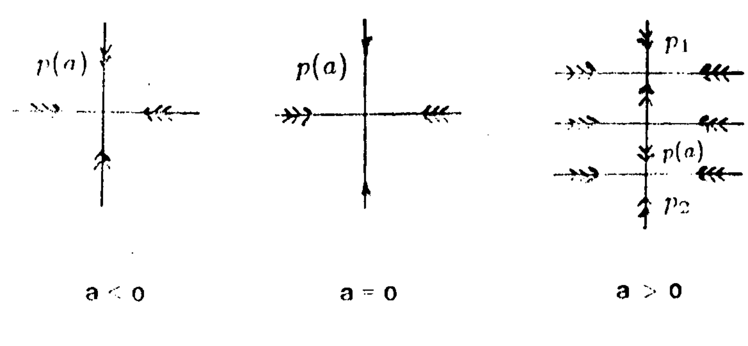


Figure 4.2: Qualitative behaviour of a flip bifurcation. $g_a(x,y) = (\lambda x, (-1+a)y + y^3)$, $|\lambda| < 1$.

hyperbolic fixed point $p(a)$, for $a = 0$ there is a flip, and for $a > 0$ there are one hyperbolic fixed point $p(a)$, and two hyperbolic periodic points of period two (p_1 and p_2) such that, $g_a(p_1) = p_2$. The double (respectively triple) arrows in the figure, denote relatively strong (resp. very strong) attraction.

Now, in order to study codimension two bifurcations, we introduce two-parameter families of diffeomorphisms on the plane: Let $\{f_{a,b}\}_{a,b\in A}$, where A is an open set, be a smooth family of diffeomorphisms such that, there is a fixed point $p = (x_0, y_0)$ for $(a, b) = (a_0, b_0)$. We will consider two cases:

First, suppose that p is a cusp point. In this case, one has eigenvalues λ and μ of $Df_{a_0,b_0}(p)$, verifying $|\lambda| \neq 1$ and $\mu = 1$. Hence, by means of an affine change of coordinates, the family $\{f_{a,b}\}_{a,b\in A}$ is conjugated to the family:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \sum_{i+j+k+l\geq 1} a_{ijkl} a^i b^j x^k y^l \\ \sum_{i+j+k+l\geq 1} b_{ijkl} a^i b^j x^k y^l \end{pmatrix},$$

where $a_{0010} = \lambda$, $a_{0001} = b_{0010} = 0$ and $b_{0001} = 1$. Also, by using the change of coordinates:

$$\left.\begin{array}{rcl} \bar{x} & = & x + \alpha a + \beta b \\ \bar{y} & = & y \end{array}\right\},$$

it is possible to eliminate a_{1000} and a_{0100}, without changing the value of b_{1000} and b_{0100}. It is enough to take $\alpha = \frac{a_{1000}}{1-\lambda}$ and $\beta = \frac{a_{0100}}{1-\lambda}$. By using normal forms theory, we can prove the following proposition:

Proposition 4.1.10 *The family of diffeomorphisms $\{\bar{f}_{a,b}\}_{a,b\in A}$ such that,*

$$\bar{f}_{a,b} = (\bar{f}^1_{a,b}, \bar{f}^2_{a,b}) \tag{4.5}$$

and

$$\begin{aligned} \bar{f}^1_{a,b}(\bar{x},\bar{y}) &= \lambda\bar{x} + \sum_{i+j=2} \bar{a}_{ij00} a^i b^j + (\bar{a}_{1010}\bar{x} + \bar{a}_{1001}\bar{y})a + \\ &\quad (\bar{a}_{0110}\bar{x} + \bar{a}_{0101}\bar{y})b + \bar{a}_{0011}\bar{x}\bar{y} + O_3, \\ \bar{f}^2_{a,b}(\bar{x},\bar{y}) &= \bar{b}_{1000}a + \bar{b}_{0100}b + \bar{y} + \sum_{i+j=2} \bar{b}_{ij00} a^i b^j + (\bar{b}_{1010}\bar{x} + \bar{b}_{1001}\bar{y})a + \\ &\quad (\bar{b}_{0110}\bar{x} + \bar{b}_{0101}\bar{y})b + \bar{b}_{0002}\bar{y}^2 + O_3, \end{aligned}$$

is locally conjugated to the family:

$$f_{a,b}(x,y) = \begin{pmatrix} \lambda x + \sum_{i+j+k+l=2} a_{ijkl} a^i b^j x^k y^l + O_3 \\ b_{1000}a + b_{0100}b + y + \sum_{i+j+k+l=2} a_{ijkl} a^i b^j x^k y^l + O_3 \end{pmatrix},$$

where $\bar{b}_{1000} = b_{1000}$, $\bar{b}_{0100} = b_{0100}$, $\bar{b}_{0110} = b_{0110}$, $\bar{b}_{0101} = b_{0101}$, $\bar{b}_{0002} = b_{0002}$.

Proof:

Let F be the following diffeomorphism:

$$F(\vec{z}) = F(\bar{x}, \bar{y}, a, b) = (\bar{f}_{a,b}(\bar{x}, \bar{y}), a, b).$$

The linear part of F is:

$$M = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & b_{1000} & b_{0100} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

So F can be written as: $F(\vec{z}) = M\vec{z} + \bar{F}(\vec{z})$, where $\bar{F}$ is of order two.

Then we consider the change of coordinates $\vec{w} = H(\vec{z}) = \vec{z} + h(\vec{z})$, where $\vec{w} = (x, y, a, b)$ and h is a homogeneous quadratic polynomial. Then:

$$H \circ F \circ H^{-1}(\vec{w}) = M\vec{w} + \bar{F}(\vec{w}) + h(M\vec{w}) - Mh(\vec{w}) + O_3,$$

since: $H^{-1}(\vec{w}) = \vec{w} - h(\vec{w}) + O_3$ (see [34]).

Assume that:

$$h(\vec{w}) = \bar{h}(x, y) = \frac{a_{0020}}{\lambda^2 - \lambda} x^2 \vec{e}_1 + \frac{b_{0011}}{\lambda - 1} xy\vec{e}_2 +$$

$$\frac{b_{0020}}{\lambda^2 - 1} x^2 \vec{e}_2 + \frac{a_{0002}}{1 - \lambda} y^2 \vec{e}_1,$$

where $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, $\vec{e}_4$ is the canonical basis of $\mathbb{R}^4$. Then:

$$h(M\vec{w}) - Mh(\vec{w}) = \alpha\vec{e}_1 + \beta\vec{e}_2,$$

where

$$\alpha = b_{1000}^2 a_{0002} a^2 + b_{0100}^2 a_{0002} b^2 + 2b_{1000} b_{0100} a_{0002} ab +$$

$$2b_{1000} a_{0002} ay + 2b_{0100} a_{0002} by + a_{0020} x^2 + a_{0002} y^2,$$

and

$$\beta = \frac{\lambda}{1 - \lambda} b_{1000} ax + \frac{\lambda}{1 - \lambda} b_{0100} bx + b_{0011} xy + b_{0020} x^2.$$

Then, if we consider for $\bar{f}_{a,b}$ the change of coordinates $(\bar{x}, \bar{y}) = (x, y) + \bar{h}(x, y)$, we obtain $f_{a,b}$ with the conditions of the proposition. □

By means of the previous proposition, we see that, any family of diffeomorphisms having a cusp point, can be written as the family 4.5 with $b_{0020} = 0$, by using a polynomial change of coordinates. Then we define:

Definition 4.1.11 *The family $\{f_{a,b}\}_{a,b\in A}$ has a cusp point, $p = (x_0, y_0)$, in $(a,b) = (a_0, b_0)$, which unfolds generically with $\{f_{a,b}\}_{a,b\in A}$, if and only if, in the normal form 4.5, one has:*

a) $b_{0002} = 0$ *and* $b_{0003} \neq 0$.

b)

$$\begin{vmatrix} b_{1000} & b_{0100} \\ b_{1001} & b_{0101} \end{vmatrix} \neq 0.$$

A model of this family is the following one:

$$g_{a,b}(x,y) = (\lambda x, y \pm y^3 + ay + b) \quad |\lambda| \neq 1.$$

The fixed points of $g_{a,b}$ verify the equations:

$$\left.\begin{array}{rcl} x & = & 0 \\ \pm y^3 + ay + b & = & 0 \end{array}\right\}.$$

Therefore, there are one, two or three fixed points, depending on the values of the parameters. In order to have two fixed points, the parameters must verify the equations:

$$\left.\begin{array}{rcl} \pm y^3 + ay + b & = & 0 \\ \pm 3y^2 + a & = & 0 \end{array}\right\}.$$

Then, if we remove y, we obtain:

$$\pm 27b^2 + 4a^3 = 0.$$

For example, if we draw the curve $-27b^2 + 4a^3 = 0$ in the plane of parameters, we get figure 4.3. In each one of the regions 0.,1.,2.,3.,and 4. there is a different behaviour of the map $f_{a,b}$. This can be seen in the same figure.

Now we consider the case in which the fixed point p is a codimension-two flip. In this case, we have eigenvalues $\bar{\lambda} = \lambda(a_0, b_0)$ and $\bar{\mu} = \mu(a_0, y_0)$, with $|\bar{\lambda}| < 1$ and $\bar{\mu} = -1$. As in the case of the flip we have the following

Proposition 4.1.12 *Let $\{f_{a,b}\}_{a,b\in A}$ a smooth family of diffeomorphism on the plane, having a fixed point $\bar{p} = (\bar{x}_0, \bar{y}_0)$ for $(a,b) = (a_0, b_0)$. If the eigenvalues, $\lambda(a_0, b_0)$ and $\mu(a_0, b_0)$, of $Df_{a_0,b_0}(\bar{p})$ satisfy $|\lambda| < 1$ and $\mu = -1$, then there exists, for (a,b) near (a_0, b_0), a fixed point $p(a) = (x_0(a,b), y_0(a,b))$ such that, $p(a_0, b_0) = \bar{p}$ and the map $p(a,b)$ is of the same class of differentiabilty that $f_{a,b}$, with respect to a and b.*

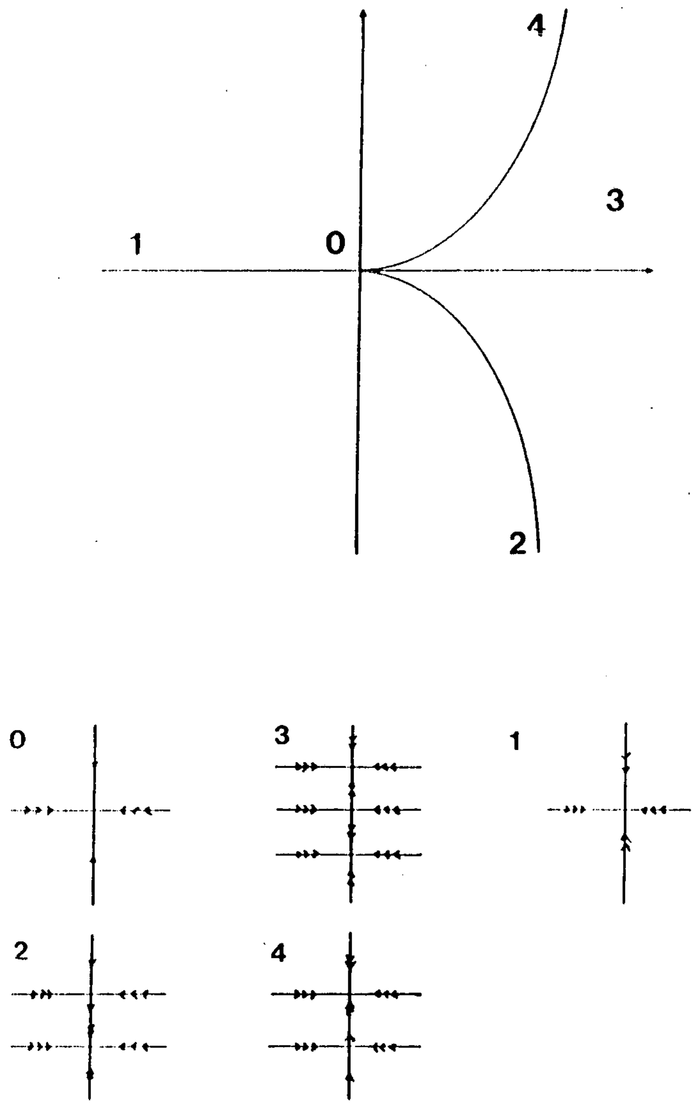


Figure 4.3: Cusp Bifurcation: In the region 0. there is a cusp fixed point, in 1. there is one hyperbolic point, in the curve 2. there are two fixed points: one hyperbolic and one saddle-node, in 3. there are three hyperbolic fixed points, and in 4. there are two fixed points: one hyperbolic and one saddle-node.

The proof is similar to the one of proposition 4.1.8.

By using the previous proposition and a polynomial change of coordinates, we can write the family as:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \lambda x + a_{11}xy + a_{12}xy^2 + O_5 \\ \mu y + b_{03}y^3 + O_5 \end{pmatrix}, \tag{4.6}$$

where $\lambda = \lambda(a,b)$, $\mu = \mu(a,b)$, $\mu(a_0,b_0) = -1$ and $a_{ij} = a_{ij}(a,b)$, $b_{ij} = b_{ij}(a,b)$.

Now, we define:

Definition 4.1.13 *The family $\{f_{(a,b)}\}_{(a,b)\in A}$ has a codimension two flip $\bar{p} = (\bar{x}_0, \bar{y}_0)$ in $(a,b) = (a_0,b_0)$ which unfolds generically with $\{f_{(a,b)}\}_{(a,b)\in A}$ if, and only if, in the normal form 4.6, one has:*

a) $b_{03}(a_0,b_0) = 0$.

b) $b_{05}(a_0,b_0) \neq 0$.

c)

$$\begin{vmatrix} \frac{\partial \mu}{\partial a} & \frac{\partial \mu}{\partial b} \\ \frac{\partial b_{03}}{\partial a} & \frac{\partial b_{03}}{\partial b} \end{vmatrix} \neq 0,$$

where the derivatives of μ and b_{03} are taken in $(a,b) = (a_0,b_0)$.

A model of this family is the following one:

$$g_{a,b}(x,y) = (\lambda x, -y \pm y^5 + ay^3 + by), \quad |\lambda| \neq 1.$$

In order to find the bifurcation curves on the parameter plane, we have to study flip bifurcations of period 1 and saddle-node bifurcations of period 2. First, suppose there is a flip point (x,y) for (a,b). Then:

$$\left.\begin{array}{rcl} \lambda x & = & x \\ -y + y^5 + ay^3 + by & = & y \\ -1 + 5y^4 + 3ay^2 + b & = & -1 \end{array}\right\}.$$

So $x = y = b = 0$ because the other points are far from the origin. Therefore, $b = 0$ is a flip bifurcation curve.

Now, suppose that there exist a saddle-node point of period two: Let

$$T(y,a,b) = -y + y^5 + ay^3 + by, \; T^2(y,a,b) = T(T(y,a,b),a,b).$$

Then:

$$\left.\begin{array}{rcl} T^2(y,a,b) & = & y \\ D_1T^2(y,a,b) & = & 1 \end{array}\right\}. \tag{4.7}$$

If we compute $T^2(y,a,b)$ and $D_1T^2(y,a,b)$, up to order 5, we obtain:

$$T^2(y,a,b) - y = (-2b + b^2)y + [a(b-1) + a(b-1)^3]y^3 +$$

$$[(b-1) + 3a^2(b-1)^2 + (b-1)^5]y^5 + O(y^7),$$

$$D_1T^2(y,a,b) - 1 = -2b + b^2 + 3[a(b-1) + a(b-1)^3]y^2 +$$

$$5[(b-1) + 3a^2(b-1)^2 + (b-1)^5]y^4 + O(y^6).$$

We notice that, for the system 4.7, we have also the solution $y = 0$ and $b = 0$, found before. Therefore, we can divide the first equation by y. Then, we obtain the system:

$$\left.\begin{array}{rcl} \Delta_1 + \Delta_2 y^2 + \Delta_3 y^4 + O(y^6) & = & 0 \\ \Delta_1 + 3\Delta_2 y^2 + 5\Delta_3 y^4 + O(y^6) & = 0 & \end{array}\right\},$$

where $\Delta_1 = -2 + b^2$, $\Delta_2 = a(b-1) + a(b-1)^3$ and $\Delta_3 = (b-1) + 3a^2(b-1)^2 + (b-1)^5$.

In order to study the solutions of this system near $a = 0$, $b = 0$, $y = 0$, we consider the change of variables:

$$y = \epsilon,\ a = \alpha\epsilon^2,\ b = \beta\epsilon^4.$$

Then, we get:

$$\left.\begin{array}{rcl} -2(\alpha + \beta + 1)\epsilon^4 + O(\epsilon^8) & = & 0 \\ -2(3\alpha + \beta + 5)\epsilon^4 + O(\epsilon^8) & = & 0 \end{array}\right\}.$$

Dividing by ϵ^4 and applying the implicit function theorem, we obtain:

$$\alpha(\epsilon) = -2 + O(\epsilon^4),\ \beta = 1 + O(\epsilon^4).$$

Therefore:

$$\left.\begin{array}{rcl} a & = & -2\epsilon^2 + O(\epsilon^6) \\ b & = & \epsilon^4 + O(\epsilon^8) \end{array}\right\}.$$

That is, the curve of saddle-node bifurcations consists, at first orders, of a branch of parabola corresponding to $a < 0$.

In figure 4.4, we have the behaviour of the family in the plane of parameters. It is easy to see that, if $b < 0$ is small, the fixed point $(0,0)$ is an attractor, because $|D_1T(0,0,0)| = |-1+b| < 1$.

If $b = 0$, the equation of 2-periodic points is:

$$-2ay^2 + (-2 + 3a^2)y^4 + O(y^6) = 0.$$

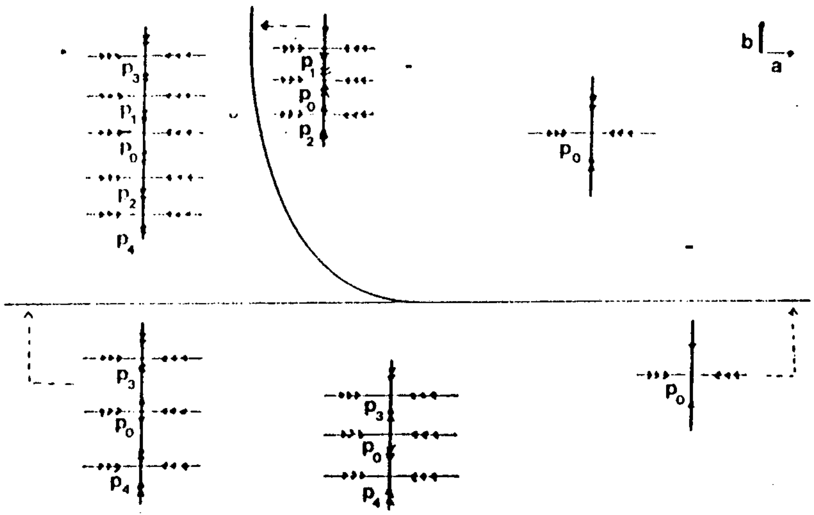


Figure 4.4: Codimension two flip bifurcation. p_0 denotes a fixed point, p_1, p_2, p_3 and p_4 denote two periodic points.

By setting $a = \gamma y^2$, one obtains:

$$-2(\gamma+1)y^4 + O(y^6) = 0.$$

Therefore, $a = -y^2 + O(y^3)$. Then, if $b = 0$ and $a < 0$, one has two periodic points of period two and, if $a > 0$, no periodic points of period two. Moreover, these points are attractors because:

$$0 < D_{11}T(y,0,0) = 1 - 4y^4 + o(y^4) < 1.$$

This means that the flip bifurcation occurs for $b > 0$, if $a < 0$, and for $b < 0$, if $a > 0$. Also, it is easy to see that, on the left of the curve $a = \alpha(\epsilon)\epsilon^2$, $b = \beta(\epsilon)\epsilon^4$, there are not two-periodic points. Then, the behaviour is as in the figure 4.4.

Once we have defined the different types of bifurcation, we want to show that the behaviour is similar to the models we have studied. First we will prove the auxiliar proposition:

Proposition 4.1.14 *Let A be a linear map such that:*

$$A = \begin{pmatrix} \lambda(\epsilon) & O(\epsilon^n) \\ O(\epsilon^n) & \mu(\epsilon) \end{pmatrix},$$

with $\lambda(\epsilon)$, $\mu(\epsilon)$ continuous in ϵ, and $\lambda(0) \neq \mu(0)$. Then, the eigenvalues of A, $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$, verify:

$$\lambda_1 = \lambda(\epsilon) + O(\epsilon^{2n}),\ \lambda_2 = \mu(\epsilon) + O(\epsilon^{2n}).$$

Proof: Let $\bar\lambda = \lambda_1 - \lambda(\epsilon)$ and $\bar\mu = \lambda_2 - \mu(\epsilon)$. We know that $\bar\lambda(0) = 0$ and $\bar\mu(0) = 0$. Moreover, $\bar\lambda + \bar\mu = 0$ and

$$\bar\lambda\bar\mu = (\lambda_1\lambda_2 - \lambda_2\lambda(\epsilon) - \lambda_1\mu(\epsilon) + \lambda\mu =$$

$$2\lambda\mu - \lambda_2\lambda - \lambda_1\mu + O(\epsilon^{2n}) = -\lambda\bar{\mu} - \mu\bar{\lambda} + O(\epsilon^{2n}).$$

Therefore, if we put $\bar{\lambda} = -\bar{\mu}$ in this equation, we obtain:

$$-\bar{\mu} = (\mu - \lambda)\bar{\mu} + O(\epsilon^{2n}).$$

Hence $\bar{\mu} = O(\epsilon^{2n})$, and therefore $\bar{\lambda} = O(\epsilon^{2n})$. So $\lambda_1 = \lambda + O(\epsilon^{2n})$ and $\lambda_1 = \lambda + O(\epsilon^{2n})$. □

Now, let $\{f_a\}_{a\in I}$, where I is an open interval, be a one-parameter family of diffeomorphisms defined on an open set of $\mathbb{R}^2$.

Theorem 4.1.15 *Suppose that the family $\{f_a\}_{a\in I}$ has a saddle-node fixed point $\bar{p} = (\bar{x}_0, \bar{y}_0)$ in $a = a_0$, which unfolds generically with $\{f_a\}_{a\in I}$. Then, there exists a curve $(p(\epsilon), a(\epsilon))$, for ϵ small enough, such that: $p(0) = \bar{p}$, $a(0) = a_0$, $p(\epsilon)$ is a hyperbolic fixed point for $\epsilon \neq 0$ and $(p(\epsilon), a(\epsilon))$ has a quadratic tangency with the plane $a = a_0$ in the space (x, y, a).*

Proof: We know that, by means of a polynomial change of coordinates and a change of the parameter $a = \bar{a} + a_0$, we can write the family as:

$$f_a(x,y) = \begin{pmatrix} \lambda x + \sum_{i+j+k=2} a_{ijk} a^i x^j y^k + O_3 \\ b_{100}a + y + \sum_{i+j+k=2} a_{ijk} a^i x^j y^k + O_3 \end{pmatrix},$$

where we denote the new parameter $\bar{a}$ as a.

Then, the following system gives the fixed points of f_a :

$$\left.\begin{array}{rcl} \lambda x + O_2 & = & x \\ b_{100}a + b_{200}a^2 + b_{110}ax + b_{101}ay + b_{220}x^2 + b_{011}xy + b_{002}y^2 + O_3 & = & 0 \end{array}\right\}.$$

From the first equation, we obtain $x = x(y,a) = O_2$. Then, in the second equation, the terms in x are negligible up to third order. So, we have:

$$b_{100}a + b_{002}y^2 + b_{200}a^2 + b_{101}ay + O_3 = 0. \tag{4.8}$$

By hypothesis, we know that $b_{100} \neq 0$ and $b_{002} \neq 0$. Let $y = \epsilon$ and $a = \alpha(\epsilon)\epsilon^2$. Substituing in 4.8, we have:

$$\epsilon^2(b_{100}\alpha(\epsilon) + b_{002} + b_{101}\alpha(\epsilon)\epsilon + O(\epsilon^2)) = 0,$$

and, therefore:

$$\alpha(\epsilon) = -\frac{b_{002}}{b_{100}} + O(\epsilon).$$

So, the curve of fixed points of the theorem is:

$$x = O(\epsilon^2),\ y = \epsilon,\ a = -\frac{b_{002}}{b_{100}}\epsilon^2 + O(\epsilon^3).$$

This curve has a quadratic tangency with the plane $a = 0$.

Now, we have to show that $p(\epsilon) = (O(\epsilon^2), \epsilon)$ is hyperbolic if $\epsilon \neq 0$. It is easy to see that:

$$Df_a(p(\epsilon)) = \begin{pmatrix} \lambda + O(\epsilon) & O(\epsilon) \\ O(\epsilon) & 1 + 2b_{002}\epsilon + O(\epsilon^2) \end{pmatrix}.$$

Then, by the proposition 4.1.14, we have that, if λ_1 and λ_2 are the eigenvalues of $Df_a(p(\epsilon))$, $\lambda_1 = \lambda + O(\epsilon)$ and $\lambda_2 = 1 + 2b_{002}\epsilon + O(\epsilon^2)$. Hence, if $\epsilon \neq 0$ is small enough then $p(\epsilon)$ is a hyperbolic point. □

Theorem 4.1.16 *Suppose that $\{f_a\}_{a\in I}$ has a flip fixed point $\bar{p}_0 = (x_0, y_0)$, in $a = a_0$, which unfolds generically with $\{f_a\}_{a\in I}$. Then there exist two curves, $(p_1(\epsilon), a_1(\epsilon))$ and $(p_2(\epsilon), a_2(\epsilon))$, in $\mathbb{R}^4$, and for ϵ small enough, such that:*

a) $p_1(0) = p_2(0) = \bar{p}_0$, $a_1(0) = a_2(0) = a_0$.

b) $p_1(\epsilon)$ is a fixed point of $f_{a_1(\epsilon)}$, and $p_2(\epsilon)$ is a two periodic point of $f_{a_2(\epsilon)}$. Moreover, p_1 and p_2 are hyperbolic if $\epsilon \neq 0$.

c) The curve $(p_1(\epsilon), a_1(\epsilon))$ has a transversal intersection with $a = a_0$, and $(p_2(\epsilon), a_2(\epsilon))$ has a quadratic tangency with $a = a_0$.

Proof:

By means of a polynomial change of coordinates and a translation of the parameter, we can suppose that $a_0 = 0$ and the family is (see 4.4):

$$f_a(x,y) = \begin{pmatrix} \lambda(a)x + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + O_4 \\ \mu(a)y + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + O_4 \end{pmatrix},$$

where $a_{ij} = a_{ij}(a)$, $b_{ij} = b_{ij}(a)$, and $|\lambda(0)| = |\bar{\lambda}| \neq 1$, $\mu(0) = -1$.

Then the curve $(p_1(\epsilon), a_1(\epsilon))$ is equal to $(0, 0, \epsilon)$, and, therefore, it has a transversal intersection with $a = 0$. Now we want to see that $p_1(\epsilon)$ is a hyperbolic fixed point. We have that:

$$Df_a(0,0) = \begin{pmatrix} \lambda(a) & 0 \\ 0 & \mu(a) \end{pmatrix},$$

and $\mu(a) = -1 + \frac{d\mu}{da}(0)a + O(a^2)$, $|\lambda(a)| \neq 1$. Then, since $\bar{p}_0$ unfolds generically, we have that $\frac{d\mu}{da}(0) \neq 0$ and, therefore, $|\mu(a)| \neq 1$ if $a \neq 0$ small enough.

Finally, we want to get the curve $(p_2(\epsilon), a_2(\epsilon))$. It is easy to see that:

$$f_a^2(x,y) = \begin{pmatrix} \lambda^2 x + c_{30}x^3 + c_{21}x^2y + c_{12}xy^2 + c_{03}y^3 + \dots \\ \mu^2 y + d_{30}x^3 + d_{21}x^2y + d_{12}xy^2 + d_{03}y^3 + \dots \end{pmatrix},$$

where $c_{30} = a_{30}(\lambda + \lambda^3)$, $c_{21} = a_{21}(\lambda + \lambda^2\mu)$, $c_{12} = a_{12}(\lambda + \lambda\mu^2)$, $c_{03} = a_{03}(\lambda + \mu^3)$, $d_{30} = b_{30}(\mu + \lambda^3)$, $d_{21} = b_{21}(\mu + \lambda^2\mu)$, $d_{12} = b_{12}(\mu + \lambda\mu^2)$, $d_{03} = b_{03}(\mu + \mu^3)$.

Suppose that $x = \alpha(\epsilon)\epsilon^3$, $y = \epsilon$, and $a = \beta(\epsilon)\epsilon^2$. Then, by substituing in the equation $f_a^2(x,y) = (x,y)$ and taking into account that

$$\mu^2 = 1 - 2\frac{d\mu}{da}(0) + O(a^2),$$

we obtain:

$$\alpha(0) = -\frac{a_{03}(0)(\bar{\lambda}-1)}{\bar{\lambda}^2 - 1}\,,\ \beta(0) = -b_{03}(0)\left[\frac{d\mu}{da}(0)\right]^{-1}.$$

Then:

$$(p_2(\epsilon), a_2(\epsilon)) = \left(-\frac{a_{03}(0)(\bar{\lambda}-1)}{\bar{\lambda}^2-1}\epsilon^3 + O(\epsilon^4), \epsilon, -b_{03}(0)\left[\frac{d\mu}{da}(0)\right]^{-1}\right),$$

and this curve has a quadratic tangency with $a = 0$.

In order to see that these points are hyperbolic, we must compute the eigenvalues λ_1 and λ_2 of $Df^2_{a_2(\epsilon)}(p_2(\epsilon))$. We have:

$$Df^2_{a_2(\epsilon)}(p_2(\epsilon)) = \begin{pmatrix} \bar{\lambda}^2 + 2a_{12}(0)\epsilon^2 + O(\epsilon^3) & O(\epsilon^2) \\ O(\epsilon^2) & 1 - 2b_{03}(0)\epsilon^2 + O(\epsilon^3) \end{pmatrix}.$$

Then the eigenvalues satisfy:

$$\lambda_1 = \bar{\lambda}^2 + 2a_{12}(0)\epsilon^2 + O(\epsilon^3)$$

and

$$\lambda_2 = 1 - 2b_{03}(0)\epsilon^2 + O(\epsilon^3).$$

This means that if $\epsilon \neq 0$ is small enough then $|\lambda_1(\epsilon)| \neq 1$ and $|\lambda_2(\epsilon)| \neq 1$. So, the point $p_2(\epsilon)$ is hyperbolic. □

Remark 4.1.17 *a) As we have seen in the proof of the theorem, the two 2-periodic points appear for $a > a_0$ or $a < a_0$. It is easy to see that this points form a 2-periodic orbit.*

b) The type of the flip bifurcation depends on the sign of $b_{03}(0)$. Suppose that $\frac{d\mu}{da}(0) > 0$ (the other case is similar). Then:

(a) If $b_{03}(0) > 0$ then for $a > a_0$ one has one attracting fixed point and no 2-periodic points, and for $a < a_0$ one has one saddle fixed point and two 2-periodic points which are attractors.

(b) If $b_{03}(0) < 0$ then for $a > a_0$ one has one saddle fixed point and no 2-periodic points, and for $a < a_0$ one has one attractting fixed point and two 2 periodic points which are saddles.

These are (respectively) the so-called supercritical and subcritical flip bifurcations, according to [1].

The next theorems correspond to bifurcations of codimension two. Then we consider two-parameter families of diffeomorphisms on the plane $\{f_{(a,b)}\}_{(a,b)\in U}$, where U is an open set.

Theorem 4.1.18 *Suppose that $\{f_{(a,b)}\}_{(a,b)\in U}$ has a cusp point $\bar{p}_0 = (x_0, y_0)$ in $(a,b) = (a_0, b_0)$ which unfolds generically with $\{f_{(a,b)}\}_{(a,b)\in U}$. Then there exist a curve in the plane of parameters $(a(\epsilon), b(\epsilon))$ such that, for ϵ small enough:*

a) For $\epsilon \neq 0$, $f_{a(\epsilon),b(\epsilon)}$ has a saddle-node fixed point and a hyperbolic fixed point.

b) $(a(0), b(0)) = (a_0, b_0)$.

c) $(\frac{da}{d\epsilon}(0), \frac{db}{d\epsilon}(0)) = (0,0)$.

d) Let

$$\vec{n}(\epsilon) = \frac{(\frac{da}{d\epsilon}, \frac{db}{d\epsilon})}{||(\frac{da}{d\epsilon}, \frac{db}{d\epsilon})||}$$

be the normal vector of the curve. Then:

$$\lim_{\epsilon\to 0^+} \vec{n} = -\lim_{\epsilon\to 0^-} \vec{n}.$$

Proof:

As we have seen before, by means of a polynomial change of coordinates and a translation of the parameter, we can suppose that the family verifies (see 4.5):

$$f_{a,b} = (f^1_{a,b}, f^2_{a,b}),$$

where

$$f^1_{a,b}(x,y) = \lambda x + \sum_{i+j=2} a_{ij00} a^i b^j + (a_{1010}x + a_{1001}y)a +$$
$$(a_{0110}x + a_{0101}y)b + a_{0011}xy + O_3,$$
$$f^2_{a,b}(x,y) = b_{1000}a + b_{0100}b + y + \sum_{i+j=2} b_{ij00} a^i b^j + (b_{1010}x + b_{1001}y)a +$$
$$(b_{0110}x + b_{0101}y)b + b_{0002}y^2 + O_3.$$

As $\bar{p}_0$ unfolds generically, by definition 4.1.11 we have: $b_{0002} = 0$, $b_{0003} \neq 0$ and

$$\begin{vmatrix} b_{1000} & b_{0100} \\ b_{1001} & b_{0101} \end{vmatrix} \neq 0.$$

Consider now another change of parameters:

$$\left.\begin{array}{rcl} \bar{a} & = & b_{1001}a + b_{0101}b \\ \bar{b} & = & b_{1000}a + b_{0100}b \end{array}\right\} \tag{4.9}$$

This linear change can be done since its determinant is different from zero. Now, we take:

$$x = \alpha(\epsilon)\epsilon^3,\ y = \epsilon,\ \bar{a} = \beta(\epsilon)\epsilon^2,\ \bar{b} = \gamma(\epsilon)\epsilon^3.$$

If we impose the conditions of existence of a fixed point $f_{a,b}(x,y) = (x,y)$, we have:

$$x = \frac{a_{0003}}{1-\lambda}\epsilon^3 + O(\epsilon^4),$$

and

$$(\gamma(\epsilon) + b_{0003} + \beta(\epsilon) + O(\epsilon))\epsilon^3 = 0. \tag{4.10}$$

If we impose $1 \in \operatorname{Spec} Df_{a,b}(x,y)$, and taking into account that

$$Df_{a,b}(\alpha\epsilon^3,\epsilon) = \begin{pmatrix} \lambda + a_{0011}\epsilon + O(\epsilon^2) & O(\epsilon^2) \\ O(\epsilon^2) & 1 + \beta\epsilon^2 + 3b_{0003}\epsilon^2 + O(\epsilon^3) \end{pmatrix},$$

and using proposition 4.1.14, we have:

$$1 + \beta\epsilon^2 + 3b_{0003}\epsilon^2 + O(\epsilon^3) = 1. \tag{4.11}$$

Thus, taking into account 4.10 and 4.11 we see that $\alpha(0)$ and $\gamma(0)$ satisfy the system:

$$\left.\begin{array}{rcl} \gamma(0) + \beta(0) + b_{0003} & = & 0 \\ \beta(0) + 3b_{0003} & = & 0 \end{array}\right\}.$$

Then, $\beta(0) = -3b_{0003}$ and $\gamma = -2b_{0003}$. Therefore:

$$\bar{a} = -3b_{0003}\epsilon^2 + O_3,\ \bar{b} = -2b_{0003}\epsilon^3 + O_4,\ y = \epsilon.$$

As $\bar{a}$ and $\bar{b}$ verify 4.9, it is easy to see that b), c) and d) hold. Also a) holds due to the way we have obtained the curve $a(\epsilon), b(\epsilon)$. □

Theorem 4.1.19 *Suppose that $\{f_{(a,b)}\}_{(a,b)\in U}$ has a codimension two flip point $\bar{p}_0 = (x_0, y_0)$ in $(a,b) = (a_0, b_0)$ which unfolds generically with $\{f_{(a,b)}\}_{(a,b)\in U}$. Then there exist two curves $(a_1(\epsilon), b_1(\epsilon))$ and $(a_2(\epsilon), b_2(\epsilon))$, in the plane of parameters, for ϵ small enough, such that:*

a) For $\epsilon \neq 0$ there exists a flip fixed point for $f_{a_1(\epsilon),b_1(\epsilon)}$ and two two-periodic saddle-node points for $f_{a_2(\epsilon),b_2(\epsilon)}$.

b) $(a_1(0), b_1(0)) = (a_2(0), b_2(0)) = (a_0, b_0)$.

c) $(\frac{da_1}{d\epsilon}(0), \frac{db_1}{d\epsilon}(0)) \neq (0,0)$ and $(\frac{da_2}{d\epsilon}(0), \frac{db_2}{d\epsilon}(0)) = (0,0)$

d) Let

$$\vec{n}_1(\epsilon) = \frac{(\frac{da_1}{d\epsilon}, \frac{db_1}{d\epsilon})}{||(\frac{da_1}{d\epsilon}, \frac{db_1}{d\epsilon})||}$$

and

$$\vec{n}_2(\epsilon) = \frac{(\frac{da_2}{d\epsilon}, \frac{db_2}{d\epsilon})}{||(\frac{da_2}{d\epsilon}, \frac{db_2}{d\epsilon})||}$$

be the normal vectors of the curves. Then:

$$\lim_{\epsilon \to 0^+} \vec{n}_2 = \pm \vec{n}_1,$$

$$\lim_{\epsilon \to 0^-} \vec{n}_2 = \mp \vec{n}_1.$$

and the tangency of the two curves in $\epsilon = 0$ is quadratic.

e) The geometric curve corresponding to the parametrized curve $(a_2(\epsilon), b_2(\epsilon))$, for $|\epsilon| \leq \epsilon_0$ (with ϵ_0 small enough), is $C_2 = \{(a_2(\epsilon), b_2(\epsilon)) \, : \, 0 \leq \epsilon \leq \epsilon_0\}$.

Proof:

We know that the family $\{f_{(a,b)}\}_{(a,b)\in U}$ can be taken, by means of polynomial change of coordinates and a translation of the parameters, as (see 4.6):

$$f_{a,b}(x,y) = \begin{pmatrix} \lambda x + a_{11}xy + a_{12}xy^2 + O_5 \\ \mu y + b_{03}y^3 + O_5 \end{pmatrix},$$

where $\lambda = \lambda(a,b)$, $\mu = \mu(a,b)$, $\mu(a_0,b_0) = -1$ and $a_{ij} = a_{ij}(a,b)$, $b_{ij} = b_{ij}(a,b)$. Moreover, $|\bar{\lambda}| = |\lambda(0,0)| \neq 0$ and $\mu(0,0) = -1$.

Since the codimension two flip unfolds generically, we know that:

$$b_{03}(0,0) = 0, \; b_{05}(0,0) \neq 0,$$

and

$$\begin{vmatrix} \frac{\partial \mu}{\partial a} & \frac{\partial \mu}{\partial b} \\ \frac{\partial b_{03}}{\partial a} & \frac{\partial b_{03}}{\partial b} \end{vmatrix} \neq 0, \tag{4.12}$$

where the derivatives of μ and b_{03} are taken in $(0,0)$.

To obtain the curve of flip bifurcations, we have to impose:

$$\left.\begin{array}{rcl} f_{a,b}(x,y) & = & (x,y) \\ -1 & \in & \operatorname{Spec} Df_{a,b}(x,y) \end{array}\right\}.$$

Then $x = y = 0$ and $\mu(a,b) = 1$. By using 4.12, one see that $\frac{\partial \mu}{\partial a}(0,0) \neq 0$ or $\frac{\partial \mu}{\partial b} \neq 0$. So we can write $a = a_1(\epsilon)$, $b = b_1(\epsilon)$ such that, $(a_1(0), b_1(0)) = (0,0)$ and $(\frac{da_1}{d\epsilon}(0), \frac{db_1}{d\epsilon}(0)) \neq (0,0)$.

Now, we want to get the curve of saddle-node bifurcations of period two. Such a curve must verify:

$$\left.\begin{array}{rcl} f^2_{a,b}(x,y) & = & (x,y) \\ 1 & \in & \operatorname{Spec} Df^2_{a,b}(x,y) \end{array}\right\}. \tag{4.13}$$

Let $f^2_{a,b} = (f^{12}_{a,b}, f^{22}_{a,b})$. By the first equation we obtain $f^{12}_{a,b} = x$. So we have $x = \frac{a_{05}}{1-\lambda^2}y^5 + O(y^6)$. In order to find $a_2(\epsilon)$ and $b_2(\epsilon)$ we use the following lemma:

Lemma 4.1.20 *Let $f(x,y) = (f_1(x,y), f_2(x,y))$ be a differentiable map such that $f_1(0,0) = 0$ and $\frac{\partial f_1}{\partial x}(0,0) \neq 1$. Then:*

a) There exists a differentiable map $x = x(y)$ such that

$$x(0) = 0, \ f_1(x(y), y) = x,$$

in a neighbourhood of 0.

b) If

$$\frac{\partial f_1}{\partial y}(x(y), y) = O(y^n), \ \frac{\partial f_2}{\partial x}(x(y), y) = O(y^n),$$
$$\frac{\partial f_1}{\partial x}(x(y), y) - \frac{\partial f_2}{\partial y}(x(y), y) = O(1),$$

then the eigenvalues λ_1 and λ_2 of $Df(x(y), y)$ verify:

$$\lambda_1 = \frac{\partial f_1}{\partial x} + O(y^{2n})$$

and

$$\lambda_2 = \frac{\partial f_2}{\partial y} + O(y^{2n})$$

c) $\frac{d}{dy} f_2(x(y), y) - \lambda_2 = O(y^{2n})$.

The proof is a simple application of the implicit function theorem and the proposition 4.1.14. □

Following with the proof of the theorem, we have that:

$$Df^2_{a,b}(x(y), y) = \begin{pmatrix} \lambda^2 + O(y) & O(y^4) \\ O(y^4) & \mu^2 + O(y^2) \end{pmatrix}, \tag{4.14}$$

By using the lemma 4.1.20, if λ_1 and λ_2 are the eigenvalues of $Df^2_{a,b}(x(y), y)$ and $|\lambda_1| \neq 1$, then

$$\lambda_2 = \frac{d}{dy} f^{22}_{a,b}(x(y), y) + O(y^8).$$

Now the system 4.13 can be substituted by:

$$\left.\begin{array}{rcl} f^{22}_{a,b}(x(y), y) & = & y \\ \frac{d}{dy}(f^{22}_{a,b}(x(y), y)) + O(y^8) & = & 1 \end{array}\right\}.$$

If we compute these expressions, we obtain, dividing the first equation by y (we can do this because we are looking for periodic points of period two which are not fixed points):

$$\left.\begin{array}{rcl} \mu^2 + \Delta_1 y^2 + \Delta_2 y^4 + O(y^5) & = & 1 \\ \mu^2 + 3\Delta_1 y^2 + 5\Delta_2 y^4 + O(y^6) & = & 1 \end{array}\right\},$$

where $\Delta_1 = b_{03}\mu + b_{03}\mu^3$ and $\Delta_2 = b_{05}\mu + 3b_{03}^2\mu^2 + b_{05}\mu^5$.

Now, we consider the change of parameters $\mu = -1 + \bar{b}$ and $b_{03} = \bar{a}$, which is well defined because 4.12 holds. Proceeding as in the study of the model of codimension two flip, we obtain:

$$y(\epsilon) = \epsilon,\ \bar{a}_2(\epsilon) = -2b_{05}(0,0)\epsilon^2 + O(\epsilon^5),\ \bar{b}_2(\epsilon) = b_{05}(0,0)\epsilon^4 + O(\epsilon^5),$$

where $\bar{a}_1$ and $\bar{a}_2$ are the functions of the theorem corresponding to the new parameters, and $b_{05} \neq 0$.

So, $\frac{d\bar{a}_2}{d\epsilon}(0) = 0$, $\frac{d\bar{b}_2}{d\epsilon}(0) = 0$, and $(\bar{a}_2(0), \bar{b}_2(0)) = (0,0)$. This finishes the prove of a), b) and c), taking into account the change of parameters.

In order to prove d) we observe that, in these new parameters, the curve of flip bifurcation is $\bar{b} = 0$ or $\bar{a}_1(\epsilon) = \epsilon$, $\bar{b}_1(\epsilon) = 0$. Then

$$\lim_{\epsilon\to 0^+} \frac{(\frac{d\bar{a}_2}{d\epsilon}, \frac{d\bar{b}_2}{d\epsilon})}{||(\frac{d\bar{a}_2}{d\epsilon}, \frac{d\bar{b}_2}{d\epsilon})||} = (-1,0) = -\left(\frac{d\bar{a}_1}{d\epsilon}(0), \frac{d\bar{a}_2}{d\epsilon}(0)\right),$$

$$\lim_{\epsilon\to 0^-} \frac{(\frac{d\bar{a}_2}{d\epsilon}, \frac{d\bar{b}_2}{d\epsilon})}{||(\frac{d\bar{a}_2}{d\epsilon}, \frac{d\bar{b}_2}{d\epsilon})||} = (1,0) = \left(\frac{d\bar{a}_1}{d\epsilon}(0), \frac{d\bar{a}_2}{d\epsilon}(0)\right).$$

Since $b_{05}(0,0) \neq 0$, the tangency of these curve is quadratic.

Finally, to prove e) we only have to take into account that:

$$\bar{a}_2(\epsilon) = \bar{a}_2(f_{H(\bar{a}_2,\bar{b}_2)}(x(y),y))\,\bar{b}_2(\epsilon) = \bar{b}_2(f_{H(\bar{a}_2,\bar{b}_2)}(x(y),y)) \qquad (4.15)$$

where H is the map which transform the new parameters to the old ones, and $(x(y), y)$ is the two periodic point corresponding to the saddle-node bifurcation. It holds because $f_{H(\bar{a}_2,\bar{b}_2)}(x(y), y)$ is also a periodic point of period two.

Using this fact, the geometric curve C_2 corresponding to the parametrized curve

$$(\bar{a}_2(\epsilon), \bar{b}_2(\epsilon)),\ \text{for}\ |\epsilon| \leq \epsilon_0,\ \ \text{is}\ \ C_2 = \{(\bar{a}_2(\epsilon), \bar{b}_2(\epsilon))\,,\ 0 \leq \epsilon \leq \epsilon_0\},$$

because if $\epsilon < 0$, then $f_{H(\bar{a}_2,\bar{b}_2)}(x(y(\epsilon)), y(\epsilon)) = -\epsilon + O(\epsilon^3)$. So, 4.15 holds and e) is proved.
□

Remark 4.1.21 *a) The type of flip bifurcation in the curve* $(a_1(\epsilon), b_1(\epsilon))$ *depends on the sign of* ϵ*. In one case it is supercritical and in the other is subcritical.*

b) The sign of $b_{05}(0,0)$ *gives two different types of codimension two flips. With a suitable change of parameters we can see this types in figure 4.5.*

In the last part of this section we will give conditions, over a general family, of existence of bifurcation which unfolds generically:

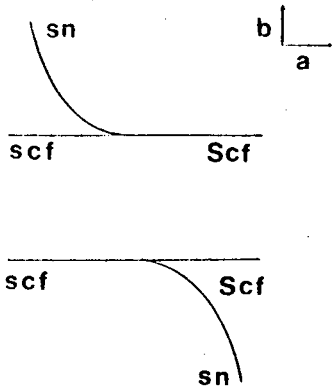


Figure 4.5: Types of codimension two flip: scf (resp. Scf) means subcritical (resp. supercritical) flip, and sn means saddle-node.

Theorem 4.1.22 *Let $\{f_a\}_{a\in I}$, where I is an open interval, be a one-parameter family of diffeomorphisms having a fixed point $\bar{p}=(x_0,y_0)$ for $a=a_0\in I$. Suppose that λ and μ are the eigenvalues of $Df_{a_0}(\bar{p})$, and v and w are (respectively) the right and left eigenvectors of eigenvalue μ. If this family verifies:*

a) $|\lambda|\neq 1$ *and* $\mu=1$,

b) $w^T D^2 f_{a_0}(\bar{p})(v,v)\neq 0$,

c) $w^T D_a f(\bar{p},a_0)\neq 0$,

where $f(\vec{x},a)=f(x,y,a)=f_a(x,y)$, then $\{f_a\}_{a\in I}$ has a saddle-node fixed point $\bar{p}$ for $a=a_0$, which unfolds generically with $\{f_a\}_{a\in I}$.

Proof:

We write $\vec{z}=(x,y,a)$ and $\vec{z}_0=(x_0,y_0,a_0)$. Then:

$$f(\vec{z})=f(\vec{z}_0)+Df(\vec{z}_0)(\vec{z}-\vec{z}_0)+\frac{1}{2}D^2f(\vec{z}_0)(\vec{z}-\vec{z}_0)^2+\dots$$

By means of a translation of the fixed point into the origin and taking into account that $f(x_0,y_0,a_0)=(x_0,y_0)$, we obtain the following family of diffeomorphisms conjugated to $\{f_a\}_{a\in I}$:

$$g_1(\vec{z})=g_1(x,y,a)=Df(\vec{z}_0)\vec{z}+\frac{1}{2}D^2f(\vec{z}_0)\vec{z}^2+\dots$$

Let M a matrix such that:

$$M^{-1}Df_{a_0}(x_0,y_0)M=\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}.$$

If we consider the transformation $\bar{x} = Mx$ and the change of parameters $\bar{a} = a - a_0$, we have another family conjugated to the first one:

$$g_2(x, y, \bar{a}) = M^{-1}Df(\vec{z}_0)(M\vec{x}, \bar{a}) + \frac{1}{2}M^{-1}D^2f(\vec{z}_0)(M\vec{x}, \bar{a})^2 + \ldots =$$

$$M^{-1}(D_{\bar{x}}f(\vec{z}_0)M\vec{x} + \bar{a}D_af(\vec{z}_0)) + \frac{1}{2}M^{-1}(D_{\bar{x}\bar{x}}f(\vec{z}_0)(M\vec{x})^2 +$$

$$M^{-1}(2\bar{a}D_{\bar{x}a}f(\vec{z}_0)M\vec{x} + \bar{a}^2D_{aa}f(\vec{z}_0)) + \ldots =$$

$$\begin{pmatrix} a_{100}\bar{a} + \lambda x + \sum_{i+j+k=2} a_{ijk}\bar{a}^i x^j y^k + O_3 \\ b_{100}\bar{a} + y + \sum_{i+j+k=2} b_{ijk}\bar{a}^i x^j y^k + O_3 \end{pmatrix}.$$

Then, we have $b_{100} = w^T D_af(\vec{z}_0)$ and $b_{002} = \frac{1}{2}w^T D^2 f_{a_0}(x_0, y_0)(v, v)$ since

$$b_{100} = \vec{e}_2^T M^{-1}D_af(\vec{z}_0), \; b_{002} = \frac{1}{2}\vec{e}_2^T M^{-1}D^2f_{a_0}(x_0, y_0)(M\vec{e}_2)^2,$$

where $\vec{e}_1$ and $\vec{e}_2$ is the canonical basis of $\mathbb{R}^2$, and, $\vec{e}_2^{\,T}M^{-1} = w^T$ and $M\vec{e}_2 = v$.

Since the conditions of the definition 4.1.11 are $b_{100} \neq 0$ and $b_{002} \neq 0$, the theorem is proved. □

Remark 4.1.23 *The condition b) of the theorem is also the condition of existence of a saddle-node fixed point for $a = a_0$.*

Theorem 4.1.24 *Let $\{f_a\}_{a \in I}$, where I is an open interval, be a one-parameter family of diffeomorphisms having a fixed point $\bar{p} = (x_0, y_0)$ for $a = a_0$. Suppose that $Df_{a_0}(\bar{p})$ has two eigenvalues $\bar{\lambda}$, $\bar{\mu} \in \mathbb{R}$ such that $|\bar{\lambda}| \neq 1$ and $\bar{\mu} = -1$. If this family verifies:*

a)

$$\frac{1}{2(1-\bar{\lambda})}(w^T D^2 f_{a_0}(\bar{p})(v, v_1))(w_1^T D^2 f_{a_0}(\bar{p})(v, v)) +$$

$$\frac{1}{4}(w^T D^2 f_{a_0}(\bar{p})(v, v))^2 + \frac{1}{6}w^T D^3 f_{a_0}(\bar{p})(v, v, v) \neq 0,$$

where v and w are, respectively, right and left eigenvectors of eigenvalue $\bar{\mu} = -1$, and v_1 and w_1 are, respectively, right and left eigenvectors of eigenvalue $\bar{\lambda}$.

b) If $p(a)$ is the fixed point of f_a, obtained using prosition 4.1.8, such that $p(a_0) = \bar{p}$, and $\mu(a)$ is the eigenvalue of $Df_a(p(a))$ such that $\mu(a_0) = -1$, then $\frac{d\mu}{da}(a_0) \neq 0$.

Then $\{f_a\}_{a \in I}$ has a flip fixed point $\bar{p}$ in $a = a_0$ which unfolds generically with $\{f_a\}_{a \in I}$.

Proof:

Suppose that $\vec{x} = (x, y)$ and $\vec{x}_0(a) = p(a)$. Then:

$$f_a(\vec{x}) = f_a(\vec{x}_0(a)) + Df_a(\vec{x}_0(a))(\vec{x} - \vec{x}_0) +$$

$$\frac{1}{2}D^2 f_a(\vec{x}_0(a))(\vec{x}-\vec{x}_0)^2 + \frac{1}{6}D^3 f_a(\vec{x}_0(a))(\vec{x}-\vec{x}_0)^3 + \dots$$

If we move the fixed point to the origin, we have the following family conjugated to $\{f_a\}_{a\in I}$:

$$g_a(\vec{x}) = Df_a(\vec{x}_0)\vec{x} + \frac{1}{2}D^2 f_a(\vec{x}_0)\vec{x}^2 + \frac{1}{6}D^3 f(\vec{x}_0)\vec{x}^3 + \dots$$

Then we can digonalize the linear part, by means of a linear change of coordinates, $\vec{z} = M\vec{x}$. If we call again this new maps as g_a, we have:

$$g_a(\vec{x}) = \begin{pmatrix} \lambda x \\ \mu y \end{pmatrix} + \frac{1}{2}M^{-1}D^2 f_a(\vec{x}_0)(M\vec{x})^2 + \frac{1}{6}M^{-1}D^3 f_a(\vec{x}_0)(M\vec{x})^3 + \dots =$$

$$\begin{pmatrix} \lambda(a)x + a_{20}x^2 + a_{11}xy + a_{02}y^2 + O_3 \\ \mu(a)y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + O_3 \end{pmatrix}.$$

To obtain the conditions of definition 4.1.9, we have to transform $\tilde{g}_a$ to a normal form, removing the non resonant terms of second order:

Lemma 4.1.25 *Let $\bar{b}_{03}(a_0)$ the term on y^3 of the second component of the map $\bar{g}_{a_0}$, which is the normal form of $g_{a_0}(\vec{x})$. Then:*

$$\bar{b}_{03}(a_0) = \frac{b_{11}(a_0)a_{02}(a_0)}{1-\bar{\lambda}} + b_{02}^2(a_0) + b_{03}(a_0).$$

Proof:

We want to conjugate the map g_{a_0} to the map:

$$\bar{g}_{a_0}(x,y) = \begin{pmatrix} \bar{\lambda}x + \bar{a}_{30}x^3 + \bar{a}_{21}x^2y + \bar{a}_{12}xy^2 + \bar{a}_{03}y^3 + \dots \\ -y + \bar{b}_{30}x^3 + \bar{b}_{21}x^2y + \bar{b}_{12}xy^2 + \bar{b}_{03}y^3 + \dots \end{pmatrix}.$$

Consider the change of coordinates:

$$h(x,y) = \begin{pmatrix} x + \alpha_{20}x^2 + \alpha_{11}xy + \alpha_{02}y^2 \\ y + \beta_{20}x^2 + \beta_{11}xy + \beta_{02}y^2 \end{pmatrix}.$$

Then:

$$h \circ \bar{g}_{a_0}(0,y) = \begin{pmatrix} \alpha_{02}y^2 + \bar{a}_{03}y^3 + \dots \\ -y + \beta_{02}y^2 + \bar{b}_{03}y^3 + \dots \end{pmatrix}$$

and:

$$g_{a_0} \circ h(0,y) = \begin{pmatrix} (\bar{\lambda}\alpha_{02} + a_{02})y^2 + (a_{11}\alpha_{02} + 2a_{02}\beta_{02} + a_{03})y^3 + .. \\ -y + (-\beta_{02} + b_{02})y^2 + (b_{11}\alpha_{02} + 2b_{02}\beta_{02} + b_{03})y^3 + .. \end{pmatrix}.$$

Therefore:

$$\bar{b}_{03} = b_{11}\alpha_{02} + 2b_{02}\beta_{02} + b_{03}, \tag{4.16}$$

and

$$\alpha_{02} = \bar{\lambda}\alpha_{02} + a_{02},\ \beta_{02} = -\beta_{02} + b_{02}.$$

Substituing these values in 4.16, we obtain the formula of the lemma. □

To finish the proof of the theorem, we have to take into account that:

$$b_{11} = w^T D^2 f_a(p(a))(v, v_1),\ b_{02} = \frac{1}{2} w^T D^2 f_a(p(a))(v, v),$$

$$b_{03} = \frac{1}{6} w^T D^3 f(p(a))(v, v, v),$$

where $w^T = \vec{e}_2^T M^{-1}$, $v = M\vec{e}_2$, $w_1^T = \vec{e}_1^T M^{-1}$, $v_1 = M\vec{e}_1$ are the eigenvectors of the theorem, and $\vec{e}_1$, $\vec{e}_2$ is the canonical basis of $\mathbb{R}^2$. Hence, we have obtained the first condition, since in the definition of flip we have $\bar{b}_{03} \neq 0$. The second condition is the same that the definition of a flip which unfolds generically. □

Remark 4.1.26 *When the condition of the previous theorem is computed, one has to take into account that $w^T v = \vec{e}_2 M^{-1} M \vec{e}_2 = 1$ and $w_1^T v_1 = \vec{e}_1 M^{-1} M \vec{e}_1 = 1$. So the choice of the eigenvectors is not free at all.*

Theorem 4.1.27 *Let $\{f_{(a,b)}\}_{(a,b)\in U}$, where U is an open set of $\mathbb{R}^2$, be a two parameter family of diffeomorphisms having a fixed point $\bar{p} = (x_0, y_0)$ for $(a, b) = (a_0, b_0)$. Suppose that λ and μ are the eigenvalues of $Df_{a_0,b_0}(\bar{p})$, and v, w are, respectively, right and left eigenvectors of eigenvalue μ, and v_1, w_1 are (resp.) right and left eigenvectors of eigenvalue λ. Suppose that this family verifies:*

a) $|\lambda| \neq 1,\ \mu = -1,$

b) $w^T D^2 f_{a_0,b_0}(\bar{p})(v, v) = 0,$

c)

$$\frac{1}{2(1-\lambda)}[w^T D^2 f_{a_0,b_0}(\bar{p})(v, v_1)][w_1^T D^2 f_{a_0,b_0}(\bar{p})(v, v)] +$$
$$\frac{1}{6} w^T D^3 f_{a_0,b_0}(\bar{p})(v, v, v) \neq 0,$$

d)

$$\begin{vmatrix} w^T D_a f(\bar{p}, a_0, b_0) & w^T D_{\vec{x}a} f(\bar{p}, a_0, b_0) v \\ w^T D_b f(\bar{p}, a_0, b_0) & w^T D_{\vec{x}b} f(\bar{p}, a_0, b_0) v \end{vmatrix} \neq 0,$$

where $f(\vec{x}, a, b) = f(x, y, a, b) = f_{a,b}(x, y)$, D_a (resp. D_b) means partial derivative with respect to a (resp. b,) and $D_{\vec{x}a}$ (resp. $D_{\vec{x}b}$) means second partial derivative with respect a and $\vec{x}$ (resp. b and $\vec{x}$).

Then the family $\{f_{(a,b)}\}_{(a,b)\in U}$ has a cusp fixed point $\bar{p}$, for $(a, b) = (a_0, b_0)$, which unfolds generically with $\{f_{(a,b)}\}_{(a,b)\in U}$.

Proof:

First, we write $\vec{z} = (x, y, a, b)$, $\vec{x} = (x, y)$, $\bar{a} = a - a_0$, $\bar{b} = b - b_0$, and $\vec{z}_0 = (x_0, y_0, a_0, b_0)$, $\vec{x}_0 = (x_0, y_0)$. Then:

$$f(\vec{z}) = f_{a,b}(x,y) = \vec{z}_0 + Df(\vec{z}_0)(\vec{z} - \vec{z}_0) + \frac{1}{2}D^2 f(\vec{z}_0)(\vec{z} - \vec{z}_0)^2 +$$

$$\frac{1}{6}D^3 f(\vec{z}_0)(\vec{z} - \vec{z}_0)^3 + \ldots = \vec{z}_0 + D_{\vec{x}} f(\vec{z}_0)(\vec{x} - \vec{x}_0) + \bar{a} D_a f(\vec{z}_0) + \bar{b} D_b f(\vec{z}_0) +$$

$$\frac{1}{2}[D_{\vec{x}\vec{x}} f(\vec{z}_0)(\vec{x} - \vec{x}_0)^2 + 2\bar{a} D^2_{a\vec{x}} f(\vec{z}_0)(\vec{x} - \vec{x}_0) + 2\bar{b} D_{b\vec{x}} f(\vec{z}_0)(\vec{x} - \vec{x}_0) +$$

$$\bar{a}^2 D_{aa} f(\vec{z}_0) + \bar{b}^2 D_{bb} f(\vec{z}_0) + 2ab D_{ab} f(\vec{z}_0)] +$$

$$\frac{1}{6} D_{\vec{x}\vec{x}\vec{x}} f(\vec{z}_0)(\vec{x} - \vec{x}_0)^3 + \ldots$$

If we translate the fixed point $\bar{p}$ to the origin, and diagonalize the linear part with respect to x and y, by means of a linear change of coordinates of matrix M, we obtain the following family, conjugated to $f_{a,b}$:

$$g_{\bar{a},\bar{b}}(\vec{x}) = M^{-1} D f_{a_0,b_0}(\vec{x}_0) M\vec{x} + M^{-1}\bar{a} D_a f(\vec{z}_0) + M^{-1}\bar{b} D_b f(\vec{z}_0) +$$

$$\frac{1}{2}[M^{-1} D_{\vec{x}\vec{x}} f(\vec{z}_0)(M\vec{x})^2 + 2\bar{a} M^{-1} D^2_{a\vec{x}} f(\vec{z}_0) M\vec{x} + 2\bar{b} M^{-1} D_{b\vec{x}} f(\vec{z}_0) M\vec{x} +$$

$$\bar{a}^2 M^{-1} D_{aa} f(\vec{z}_0) + \bar{b}^2 M^{-1} D_{bb} f(\vec{z}_0) + 2ab M^{-1} D_{ab} f(\vec{z}_0)] +$$

$$\frac{1}{6} M^{-1} D_{\vec{x}\vec{x}\vec{x}} f(\vec{z}_0)(M\vec{x})^3 + \ldots = \begin{pmatrix} \sum_{i+j+k+l \le 3} a_{ijkl} \bar{a}^i \bar{b}^j x^k y^l + O_4 \\ \sum_{i+j+k+l \le 3} b_{ijkl} \bar{a}^i \bar{b}^j x^k y^l + O_4 \end{pmatrix},$$

with $a_{0010} = \lambda$, $a_{0001} = b_{0010} = 0$, $b_{0001} = 1$.

We know, by the proposition 4.1.10 and the definition 4.1.11, that $g_{\bar{a},\bar{b}}$ is conjugated to $\tilde{g}_{\bar{a},\bar{b}} = (\tilde{g}_1, \tilde{g}_2)$, where:

$$\tilde{g}_1 = \lambda x + \sum_{i+j=2} \bar{a}_{ij00} \bar{a}^i \bar{b}^j + (\bar{a}_{1010} x + \bar{a}_{1001} y)\bar{a} +$$

$$(\bar{a}_{0110} x + \bar{a}_{0101} y)\bar{b} + \bar{a}_{0011} xy + O_3,$$

$$\tilde{g}_2 = b_{1000}\bar{a} + b_{0100}\bar{b} + y + \sum_{i+j=2} \bar{b}_{ij00} \bar{a}^i \bar{b}^j + (\bar{b}_{1010} x + \bar{b}_{1001} y)\bar{a} +$$

$$(b_{0110} x + b_{0101} y)\bar{b} + b_{0002} y^2 + O_3,$$

and that there is a cusp which unfolds generically if:

a) $|\lambda| \ne 0$, $\mu = 1$.

b) $b_{0002} = 0$.

c) $\bar{b}_{0003} \neq 0$.

d)

$$\begin{vmatrix} b_{1000} & b_{0100} \\ b_{1001} & b_{0101} \end{vmatrix} \neq 0.$$

The conditions a), b) and d) can be transformed into the condition a), b) and d) of the theorem, taking into account the expansion of $g_{\bar{a},\bar{b}}$ and using a similar argument that in the previous theorem. It is necessary to compute $\bar{b}_{0003}$ in order to obtain the condition c) of the theorem. Then we suppose $\bar{a} = 0$ and $\bar{b} = 0$. We know that: $h \circ \tilde{g}_{0,0} = g_{0,0} \circ h$, where

$$h(x,y) = \begin{pmatrix} x + \alpha_{20}x^2 + \alpha_{02}y^2 \\ y + \beta_{20}x^2 + \beta_{11}xy \end{pmatrix}.$$

We have:

$$h \circ \tilde{g}_{0,0}(0,y) = \begin{pmatrix} \bar{a}_{0003}y^3 + \alpha_{02}(y^2 + 2\bar{b}_{0002}y^3) + O_4 \\ y + b_{0002}y^2 + \bar{b}_{0003}y^3 + O_4 \end{pmatrix},$$

$$g_{0,0} \circ h(0,y) = \begin{pmatrix} \lambda\alpha_{02}y^2 + a_{0011}\alpha_{02}y^3 + a_{0002}y^2 + a_{0003}y^3 + O_4 \\ y + b_{0011}\alpha_{02}y^3 + b_{0002}y^2 + b_{0003}y^3 + O_4 \end{pmatrix}.$$

Then:

$$\bar{b}_{0003} = b_{0011}\alpha_{02} + b_{0003}, \quad \alpha_{02} = \frac{a_{0002}}{1-\lambda},$$

and, therefore,

$$\bar{b}_{0003} = \frac{b_{0011}a_{0002}}{1-\lambda} + b_{0003}.$$

Taking into account that:

$$b_{0011} = w^T D^2 f_{a_0,b_0}(\bar{p})(v, v_1),$$

$$a_{0002} = \frac{1}{2} w_1^T D^2 f_{a_0,b_0}(\bar{p})(v,v),$$

$$b_{0003} = \frac{1}{6} w^T D^3 f_{a_0,b_0}(\bar{p})(v,v,v),$$

we obtain the condition c) of the theorem. □

Theorem 4.1.28 *Let $\{f_{(a,b)}\}_{(a,b)\in U}$, where U is an open set, denote a two-parameter family of diffeomorphisms having a fixed point $\bar{p} = (x_0, y_0)$, for $(a,b) = (a_0, b_0)$. Suppose that $Df_{a_0,b_0}(\bar{p})$ has two eigenvalues, $\bar{\lambda}$, $\bar{\mu}$, such that $|\bar{\lambda}| \neq 1$ and $\bar{\mu} = -1$. Let $p(a,b)$ be the fixed point of $f_{a,b}$, obtained by using proposition 4.1.12, such that $p(a_0,b_0) = \bar{p}$. Let $\lambda(a,b)$, $\mu(a,b)$ be the eigenvalues of $Df_{a,b}(p(a,b))$ such that, $\lambda(a_0,b_0) = \bar{\lambda}$ and $\mu(a_0,b_0) = -1$. Finally, let v and w denote, respectively, right and left eigenvectors of μ, and v_1 and w_1 right and left eigenvectors of λ. Suppose, moreover, that the family $\{f_{(a,b)}\}_{(a,b)\in U}$ satisfies:*

a) $c(a_0, b_0) = 0$, *where*

$$c(a,b) = \frac{1}{2(1-\lambda)}(w^T D^2 f_{a,b}(p)(v,v_1))(w_1^T D^2 f_{a,b}(p)(v,v)) +$$

$$\frac{1}{4}(w^T D^2 f_{a,b}(p)(v,v))^2 + \frac{1}{6} w^T D^3 f_{a,b}(p)(v,v,v).$$

b)

$$20e_{13}\alpha_{02} + 120e_{12}\beta_{02}\alpha_{02} + 60e_{12}\alpha_{03} + 120e_{11}\beta_{02}\alpha_{03} +$$

$$120e_{11}\alpha_{11}\alpha_{03} + 120e_{11}\alpha_{04} + 60e_{12}\alpha_{02}^2 + 120e_{20}\alpha_{02}\alpha_{03} +$$

$$e_{05} + 20e_{04}\beta_{02} + 60e_{03}\beta_{02}^2 + 120e_{02}\beta_{11}\alpha_{03} + 120e_{02}\beta_{04} \neq 0,$$

where:

$$\alpha_{04} = -\frac{1}{24(\bar{\lambda}-1)}(12d_{12}\alpha_{02} + 24d_{11}\beta_{02}\alpha_{02} + 24d_{11}\alpha_{03} +$$

$$12d_{20}\alpha_{02} + d_{04} + 12d_{03}\beta_{02} + 12d_{02}\beta_{02}^2 + 24\bar{\lambda}\alpha_{11}\alpha 03 - 24\alpha_{11}\alpha_{03}),$$

$$\beta_{04} = \frac{1}{48}(12e_{12}\alpha_{02} + 24e_{11}\beta_{02}\alpha_{02} + 24e_{11}\alpha_{03} +$$

$$12e_{20}\alpha_{02}^2 + e_{04} + 12e_{03}\beta_{02} + 12e_{02}\beta_{02}^2 - 48\beta_{11}\alpha_{03}),$$

$$\alpha_{03} = -\frac{1}{6(\bar{\lambda}+1)}(6d_{11}\alpha_{02} + d_{03} + 6d_{02}\beta_{02}),$$

$$\alpha_{02} = -\frac{1}{2(\bar{\lambda}-1)}d_{02},\ \beta_{02} = \frac{1}{4}e_{02},$$

$$\beta_{11} = -\frac{1}{\bar{\lambda}-1}e_{11},\ \alpha_{11} = -\frac{1}{2\bar{\lambda}}d_{11},$$

$$d_{ij} = w_1(a_0,b_0)D^{i+j}(\bar{p})v_1(a_0,b_0)^i v(a_0,b_0)^j,$$

$$e_{ij} = w(a_0,b_0)D^{i+j}(\bar{p})v_1(a_0,b_0)^i v(a_0,b_0)^j.$$

c)

$$\begin{vmatrix} \frac{\partial c}{\partial a}(a_0,b_0) & \frac{\partial c}{\partial b}(a_0,b_0) \\ \frac{\partial \mu}{\partial a}(a_0,b_0) & \frac{\partial \mu}{\partial b}(a_0,b_0) \end{vmatrix} \neq 0.$$

Then the family $\{f_{(a,b)}\}_{(a,b)\in U}$ has a codimension two flip point $\bar{p}$, in (a_0, b_0), which unfolds generically with $\{f_{(a,b)}\}_{(a,b)\in U}$.

Proof:

We know that a family like $\{f_{(a,b)}\}_{(a,b)\in U}$ can be written, by using a polynomial change of coordinates, as:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \lambda x + \bar{a}_{11}xy + \bar{a}_{12}xy^2 + O_5 \\ \mu y + \bar{b}_{03}y^3 + O_5 \end{pmatrix}, \tag{4.17}$$

where $\lambda = \lambda(a,b)$, $\mu = \mu(a,b)$, $\mu(a_0,b_0) = -1$, $\lambda(a_0,b_0) = \bar{\lambda}$, and $\bar{a}_{ij} = \bar{a}_{ij}(a,b)$, $\bar{b}_{i,j} = \bar{b}_{i,j}(a,b)$. By the definition of codimension two flip which unfolds generically, we have

a) $\bar{b}_{03}(a_0, b_0) = 0$.

b) $\bar{b}_{05}(a_0, b_0) \neq 0$.

c)

$$\left| \begin{array}{cc} \frac{\partial \mu}{\partial a} & \frac{\partial \mu}{\partial b} \\ \frac{\partial \bar{b}_{03}}{\partial a} & \frac{\partial \bar{b}_{03}}{\partial b} \end{array} \right| \neq 0,$$

where the derivatives of μ and b_{03} are taken in $(a, b) = (a_0, b_0)$.

Using the proof of theorem 4.1.24 we have that:

$$\bar{b}_{03}(a,b) = \frac{1}{2(1-\lambda)}(w^T D^2 f_{a,b}(p)(v, v_1))(w_1^T D^2 f_{a,b}(p)(v,v)) +$$

$$\frac{1}{4}(w^T D^2 f_{a,b}(p)(v,v))^2 + \frac{1}{6} w^T D^3 f_{a,b}(p)(v,v,v) = c(a,b).$$

So the conditions a) and c) of the theorem are the same that the conditions a) and c) of the generic unfolding.

We have to see, to finish the proof, that condition b) of the theorem is the same that condition b) of generic unfolding. For this, we must compute the term $\bar{b}_{05}(a_0, b_0)$ of the normal form 4.17.

We know that there exist a map $\bar{f}$ defined by:

$$\bar{f} = \left(\begin{array}{c} \bar{\lambda} x + \sum_{n=2}^{5} \sum_{i+j=n} a_{ij} x^i y^j + O_6 \\ -y + \sum_{n=2}^{5} \sum_{i+j=n} b_{ij} x^i y^j + O_6 \end{array} \right),$$

such that it is conjugated to f_{a_0,b_0}, by means of an affine change of coordinates. Moreover

$$a_{ij} = \frac{1}{(i+j)!} \left(\begin{array}{c} i+j \\ i \end{array} \right) \bar{w}_1^T D f_{a_0,b_0}(\bar{p}) \bar{v}_1^i \bar{v}^j,$$

and

$$b_{ij} = \frac{1}{(i+j)!} \left(\begin{array}{c} i+j \\ i \end{array} \right) \bar{w}^T D f_{a_0,b_0}(\bar{p}) \bar{v}_1^i \bar{v}^j,$$

where $\bar{v}$, $\bar{v}_1, \bar{w}$, $\bar{w}_1$ are the eigenvectors of the theorem taken in (a_0, b_0).

Then, let $g = g(x, y)$ denote the following map:

$$g(x,y) = \left(\begin{array}{c} \bar{\lambda} x + \bar{a}_{12} x y^2 + \bar{a}_{50} x^5 + \ldots \\ -y + \bar{a}_{50} x^5 + \ldots \end{array} \right),$$

and:

$$f_4(x,y) = \left(\begin{array}{c} x + \alpha_{40} x^4 + \alpha_{31} x^3 y + \alpha_{22} x^2 y^2 + \alpha_{13} x y^3 + \alpha_{04} y^4 \\ y + \beta_{40} x^4 + \beta_{31} x^3 y + \beta_{22} x^2 y^2 + \beta_{13} x y^3 + \beta_{04} y^4 \end{array} \right),$$

$$f_3(x,y) = \begin{pmatrix} x + \alpha_{30}x^3 + \alpha_{21}x^2y + \alpha_{03}y^3 \\ y + \beta_{30}x^3 + \beta_{21}x^2y + \beta_{12}xy^2 \end{pmatrix},$$

$$f_2(x,y) = \begin{pmatrix} x + \alpha_{02}x^2 + \alpha_{11}xy + \alpha_{02}y^2 \\ y + \beta_{02}x^2 + \beta_{11}xy + \beta_{02}y^2 \end{pmatrix}.$$

By the theorem 4.1.2, we know that there exist α_{ij}, βij, $\bar{a}_{ij}$, $\bar{b}_{ij}$, such that:

$$\bar{f} \circ f_2 \circ f_3 \circ f_4 = f_2 \circ f_3 \circ f_4 \circ g.$$

Since we are interested in the term $\bar{b}_{05}$, we can take $x = 0$:

$$\bar{f} \circ f_2 \circ f_3 \circ f_4(0,y) = f_2 \circ f_3 \circ f_4 \circ g(0,y).$$

From this equation we find the value of $\bar{b}_{05}$. □

Remark 4.1.29 *a) Also in the previous theorems we have to impose the following conditions to the eigenvectors $w^T v = 1$ and $w_1^T v_1 = 1$, when we compute the conditions a), b) and c).*

b) It is possible, when we study bifurcations of dissipative families of diffeomorphisms, to consider only a one dimensional map, by using the center manifold theorem ([36], [30], [37], [38]). But in that case we do not obtain the general conditions of existence of generic bifurcations for a general two dimensional family.

4.2 One dimensional models of cusps

In this section we will study the cubic models of cusps and a quartic model of a codimension three cusp. Our goal is to obtain a semiglobal behaviour of a family of maps having a cusp point.

First, we will see the following:

Proposition 4.2.1 *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a cubic map. Then, it is conjugated to one of the following maps*

$$g_1(y) = y + y^3 + ay + b, \; g_2(y) = y - y^3 + ay + b,$$

for suitable a and b.

Proof:

Let $f = a_0 + a_1y + a_2y^2 + a_3y^3$. It is enough to consider the following change of coordinates: $\bar{y} = \lambda(y + \alpha)$, where $\alpha = -\frac{a_2}{3a_3}$ and $\lambda = \sqrt{|a_3^{-1}|}$. Then f is conjugated to

$$g(y) = y + \text{sign}\,(a_3)y^3 + ay + b,$$

where

$$a = a_1 + 2a_2\alpha + 3a_3\alpha^2 - 1,\ b = \sqrt{|a_3|}(a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3) - \alpha. \ \square$$

Proposition 4.2.2 *Let $g_1(y) = (1+a)y + y^3 + b$. Then:*

a) The fold bifurcation curve corresponding to fixed points, verifies the equation:

$$27b^2 + 4a^3 = 0. \quad (fo1),$$

and it has a cusp at $a = b = 0$.

b) The flip bifurcation curve corresponding to fixed points, verifies the equation:

$$27b^2 + 4(a+2)(a-1)^2 = 0. \quad (fl1).$$

The curve $(fl1)$ is smooth and has no codimension two flips.

c) The flip and fold bifurcation curves, $(fl0)$ and $(fl1)$ have no intersections.

d) The fold bifurcation curve corresponding to period two points, verifies:

$$4(a+3)^3 + 27b^2 = 0 \ . \quad (fo2),$$

and this curve has a cusp point at $a = -3$ and $b = 0$.

e) The flip bifurcation curves corresponding to period two, verify:

$$16a^6 + 144a^5 + 360a^4 - 160a^3 - 1440a^2 - 384a + 1664$$

$$+216a^3b^2 + 972a^2b^2 + 2430ab^2 + 3294b^2 + 729b^4 = 0 \ . \quad (fl2).$$

This equation represents two smooth curves which intersect at the point: $(a,b) = (-1-\sqrt{5}, 0)$.

f) The curves corresponding to $(fl2)$ have no intersection with $(fo2)$ and two intersections with $(fl1)$ (one on each curve).

Proof:

a) We have to solve the system:

$$\left.\begin{array}{rcl} g_1(y) & = & y, \\ g_1'(y) & = & 1. \end{array}\right\}$$

By removing y, we get the fold bifurcation curve $(fo1)$. As we have seen before, there is a cusp point in $y = 0$ for $(a,b) = (0,0)$.

b) In this case we have to remove y from the system:

$$\left.\begin{array}{rcl} g_1(y) & = & y, \\ g_1'(y) & = & -1, \end{array}\right\} \qquad (4.18)$$

and so we obtain the curve $(fl1)$. To see that it is a smooth curve, we write $a = a(y)$ and $b = b(y)$. We have, using the system 4.18:

$$\left.\begin{array}{rcl} a(y) & = & -2 - 3y^2, \\ b(y) & = & 2y + 2y^2. \end{array}\right\}$$

It is a smooth curve because $(a'(y), b'(y)) \neq (0,0)\, \forall y \in \mathbb{R}$.

It is easy to see that the existence condition of codimension two flips is: $2g_1'''(y) + 3(g_1'')^2 = 0$. In this case this equation has no solution. Then, there are not codimension two flips.

c) To see that $(fo1)$ and $(fl1)$ do not intersect each other, we consider the system

$$\left.\begin{array}{rcl} 27b^2 + 4a^3 & = & 0, \\ 27b^2 + 4(a+2)(a-1)^2 & = & 0. \end{array}\right\}$$

From this we have: $4a^3 = 4(a+2)(a-1)^3$. Then $a = 1/3$, but from the first equation we obtain $a < -2$. So there are not intersections between both curves.

d) The conditions for the existence of fold bifurcations are:

$$\left.\begin{array}{rcl} (g_1 \circ g_1)(y) & = & y, \\ (g_1 \circ g_1)'(y) & = & 1, \end{array}\right\}$$

We can write

$$g_1^2(y) - y = (g_1(y) - y)\bar{g}(y) = 0,$$

because the fixed points are also two periodic points. Then, computing the differential of this expression, we obtain:

$$(g_1^2(y))' - 1 = (g_1' - 1)\bar{g} + (g_1(y) - y)\bar{g}'(y).$$

If we want to avoid the bifurcations of fixed points, we must solve:

$$\left.\begin{array}{rcl} \bar{g}(y) & = & 0, \\ \bar{g}'(y) & = & 0, \end{array}\right\}$$

with

$$\bar{g}(y) = a^2y^2 + 2aby + 2ay^4 + 3ay^2 + b^2 + 2by^3 + 3by + y^6 + 3y^4 + 3y^2 + 2 + a.$$

Removing y one obtains:

$$[27b^2 + 4(a+2)(a-1)^2][4(a+3)^3 + 27b^2]^2 = 0.$$

Then the condition, on the parameters a and b, of existence of a fold bifurcation is $(fo2)$, because $27b^2 + 4(a+2)(a-1)^2 = 0$ represents the flip bifurcation curves. It has a cusp point in $(a,b) = (-3,0)$ because, in this case, $g_1(y) = -2y + y^3$. Then:

$$\bar{g}(y) = y^6 - 3y^4 + 3y^2 - 1 = (y^2-1)^3 = 0,$$

$$\bar{g}'(y) = 6y^5 - 12y^3 + 6y = 6y(y^2-1) = 0.$$

Thus $y = \pm 1$. Finally

$$g_1^2(y) = 4y - 10y^3 + 12y^5 - 6y^7 + y^9,$$

and

$$(g_1^2(y))'' = -60y + 240y^3 - 252y^5 + 72y^7 = 0,$$

if $y = \pm 1$. This is a necessary condition of existence of a cusp bifurcation. Also $g_1^2(\pm 1)''' = -96 < 0$. Then it is a cusp. Observe that in the first cusp we have found: $g_1'''(0) = 6 > 0$. So these two cusps are of different type.

e) Proceeding in the same way as before, in order to find the curve of two-periodic flip bifurcation, we have to solve:

$$\left.\begin{array}{rcl} (g_1 \circ g_1)(y) & = & y, \\ (g_1 \circ g_1)'(y) & = & -1, \end{array}\right\}$$

Since $g_1 \circ g_1(y) - y = (g_1(y) - y)\bar{g}(y)$, then

$$(g_1 \circ g_1)'(y) = 1 + (g_1(y) - y)\bar{g}'(y) + (g_1' - 1)\bar{g}(y).$$

As we are interested in bifurcations of periodic points of strict period two, we can consider the system of equations:

$$\left.\begin{array}{rcl} \bar{g}(y) & = & 0, \\ (g_1(y) - y)\bar{g}'(y) & = & -2, \end{array}\right\}$$

Then, removing y, we obtain the equation $(fl2)$.

To see that this equation represents two smooth curves, consider first $b = 0$. If $(a,0)$ is a point that satisfies $(fl2)$ then:

$$P_6(a) = 16a^6 + 144a^5 + 360a^4 - 160a^3 - 1440a^2 - 384a + 1664 = 0.$$

This equation has two real roots $a_1 = -1 - \sqrt{5}$ and $a_2 = -1 + \sqrt{5}$, since

$$P_6(a) = 8(2a^2 + 10a + 13)(a^2 + 2a - 4)^2,$$

and $2a^2+10a+13=0$ has no real roots.

If $a=a_2$ and $b=0$, the 2-periodic point y has to verify the equation:

$$a^2y^2+2ay^4+3ay^2+y^6+3y^4+3y^2+2+a=0.$$

But this polynomial in y has no real roots, because all its coefficients are positive.

On the other hand if we put $a=a_1+\alpha b$, by substitution in $(fl2)$, we obtain:

$$\begin{aligned}&b^2(-96\sqrt{5}b^3\alpha^5-240b^2\alpha^4-648\sqrt{5}b^2\alpha^2-1120\sqrt{5}b\alpha^3-648\sqrt{5}b\alpha-\\&96\sqrt{5}\alpha^2-2214\sqrt{5}+16b^4\alpha^6+48b^3\alpha^5+216b^3\alpha^3+1080b^2\alpha^4+\\&324b^2\alpha^2+729b^2+1920b\alpha^3+4374b\alpha+2400\alpha^2+3240)=0.\end{aligned}\tag{4.19}$$

Then, when $b=0$, we have:

$$(-96\sqrt{5}+2400)\alpha(0)^2-2214\sqrt{5}+3240=0,$$

and therefore:

$$\alpha(0)=\alpha_\pm^0=\pm\frac{3}{4\sqrt{5}}\sqrt{17\sqrt{5}+22}.$$

Then, due to the implicit function theorem, there exist two functions α_+ and α_- such that they satisfy the equation 4.19 and $\alpha_\pm(0)=\alpha_\pm^0$.

To show the existence of a real two-periodic orbit on the curve $(fl2)$, we see first that for $a=-1-\sqrt{5}$, $b=0$ the corresponding values of y are:

$$y_1=\sqrt{\frac{1}{2}(\sqrt{5}+1)},\, y_2=-\sqrt{\frac{1}{2}(\sqrt{5}+1)},\; y_3=\sqrt{\frac{1}{2}(\sqrt{5}-1)},\, y_4=-\sqrt{\frac{1}{2}(\sqrt{5}-1)},$$

where (y_1,y_2) and (y_3,y_4) are two-periodic orbits.

To see that y exists on the curve $(fl2)$ it is enough to observe that the system:

$$\left.\begin{array}{rcl}\bar{g}(y)&=&0,\\(g_1(y)-y)\bar{g}'(y)&=&-2,\end{array}\right\}$$

has solutions $(a_1(y),b_1(y))$ and $(a_2(y),b_2(y))$ such that $(a_1(y_1),b_1(y_1))=(-1-\sqrt{5},0)$ and $(a_2(y_3),b_2(y_3))=(-1-\sqrt{5},0)$. This holds by the implicit function theorem.

f) To see that $(fl1)$ and $(fl2)$ do not intersect, we substitute $b^2=-\frac{4}{27}(a+2)(a-1)^2$ in $(fl2)$. Then we obtain:

$$8(-27a^3-63a^2+21a+94)=0.$$

It is easy to see that all the real roots of this polynomial are larger than -2. Therefore there is not intersection because, using the first equation, we obtain $a\le-2$.

If we intersect $(fo2)$ with $(fl2)$ then we have: $8(6a + 19) = 0$. So $a = -19/6$ and $b = \pm[27\sqrt{2}]^{-1}$, using the equation $(fo2)$. $\square$

The properties of the curves $(fo1)$, $(fo2)$, $(fl1)$ and $(fl2)$ can be seen in the figure 4.6.

Proposition 4.2.3 *Let $g_2(y) = (1+a)y - y^3 + b$. Then:*

a) The fold bifurcation curve corresponding to fixed points, satisfies the equation:

$$4a^3 - 27b^2 = 0 \ , \quad (fo1)$$

and it has a cusp for $a = b = 0$.

b) The flip bifurcation curve corresponding to fixed points satisfy:

$$4(a+2)(a-1)^2 - 27b^2 = 0 \ , \quad (fl1)$$

This curve has two intersections with the fold one and one self-intersection (it forms a loop).

c) $(fl1)$ has two codimension two flips.

d) The fold bifurcation curve corresponding to two-periodic point is:

$$4(a+3)^3 - 27b^2 = 0 \ , \quad (fo2)$$

This curve has also a cusp. However the true bifurcation curves (corresponding to real periodic orbits) only reach the two tangent points with $(fl1)$, in which there are the codimension two flips.

e) The flip bifurcation curve corresponding to two-periodic points, satisfies:

$$16a^6 + 144a^5 + 360a^4 - 160a^3 - 1440a^2 - 384a + 1664$$

$$-216a^3b^2 - 972a^2b^2 - 2430ab^2 - 3294b^2 + 729b^4 = 0 \ . \quad (fl2)$$

It represents two smooth curves which have a intersection at the point $(a,b) = (-1+\sqrt{5}, 0)$. It has intersections with the other curves except with $(fo2)$.

Proof:

To obtain the curves $(fo1)$, $(fl1)$, $(fo2)$, $(fl2)$ we proceed as in the previous proposition.

a) It has been proved before.

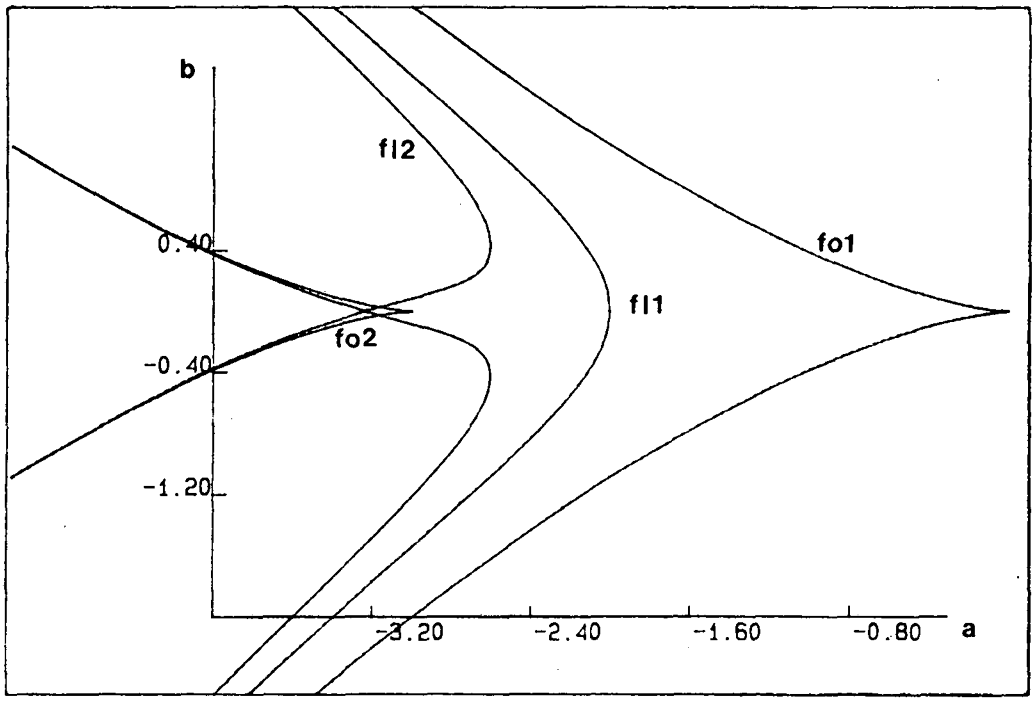


Figure 4.6: Bifurcation curves corresponding to $g_1(y) = (1 + a)y + y^3 + b$.

b) The curves $4(a+2)(a-1)^2 - 27b^2 = 0$ can have a self intersection if $b = 0$. Then we obtain $a = -2$ or $a = 1$. But if $(a,b) = (-2,0)$, by the implicit function theorem, we see that in a neighbourhood of this point the curve is smooth. On the case $a = 1$ we have the system:

$$\left.\begin{array}{rcccl} g_2(y) & = & 2y - y^3 & = & y, \\ g_2'(y) & = & 2 - 3y^2 & = & -1 \end{array}\right\}$$

So $y = \pm 1$. This tell us that, locally, the equation $(fl1)$ represents two smooth curves which have a intersection in $(a,b) = (1,0)$. To see this, take into account that (a,b) has to verify:

$$\left.\begin{array}{rcl} g_2(y) & = & y, \\ g_2'(y) & = & -1, \end{array}\right\}$$

or

$$\left.\begin{array}{rcl} a & = & 3y^2 - 2, \\ b & = & -2y^3 + 2y. \end{array}\right\} \tag{4.20}$$

For $y = \pm 1$: $b'(y) = -4$ and $a'(y) = \pm 6$. Hence, this curves forms a loop.

c) If there are codimension two flips, they have to verify: $2g_2'''(y) + 3g_2''(y)^2 = 0$. Then, by using 4.20, we obtain $a = -5/3$ and $b = \pm\frac{16}{27}$.

d) We observe first that the points

$$(a,b,y) = (-5/3, 16/27, 1/3) \text{ and } (a,b,y) = (-5/3, -16/27, -1/3)$$

belong to the fold bifurcation curve of period 2. Moreover, it is easy to see that, when $(a,b) = (-3,0)$ (where there is a cusp in the curve $(fo2)$), we have $y = \pm i$. Taking into account the behaviour of a codimension two flip studied before, we have, necessarily, that the true curves stop at the intersections with the flip curve, and do not reach the cusp.

e) This case is analogous to the case e) of the previous proposition. $\square$.

The properties of the curves $(fo1)$, $(fl1)$, $(fo2)$, $(fl2)$ can be seen in figure 4.7

Now we will study a three-parameter quartic family of maps, in order to see if there are other types of interaction of bifurcation curves. Let

$$f(y) = c + (1+a)y + by^2 + y^4.$$

As in the cubic case, we have:

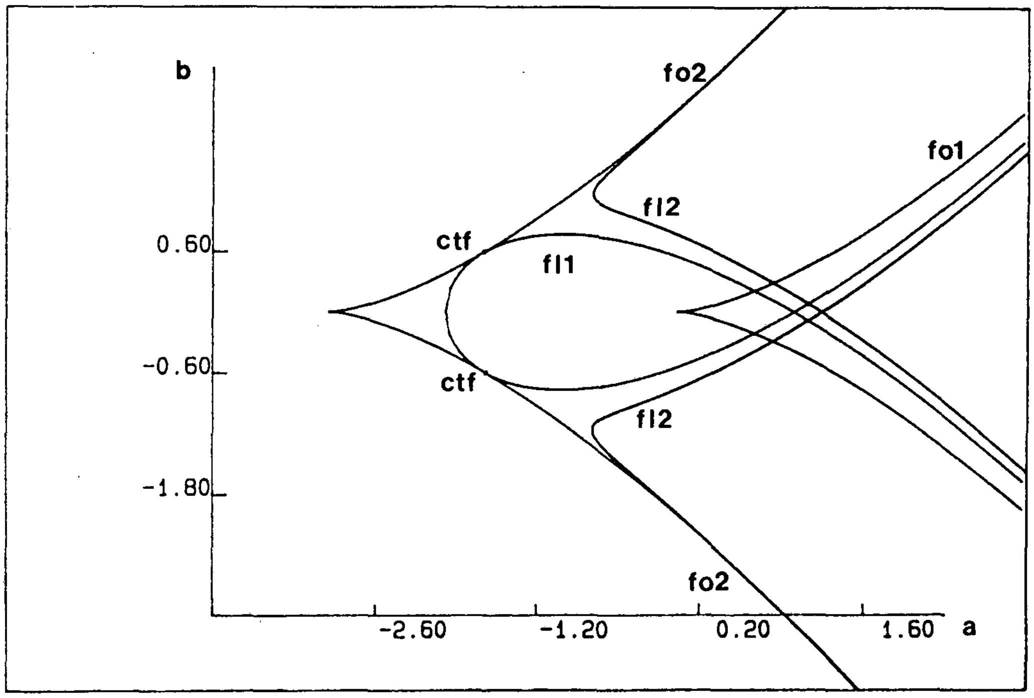


Figure 4.7: Bifurcation curves corresponding to $g_2(y) = (1 + a)y - y^3 + b$ (ctf means codimension two flip).

Proposition 4.2.4 *Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a quartic map. Then this map is conjugate to the map $f(y)$ for suitable a, b, and c.*

Proof:

By means of a translation and a change of scale, it is possible to remove the cubic term of the map g and to make the coefficient of y^4 equal to 1. □

The map f is a model of a codimension three cusp because, when $a = b = 0$, we have $f(y) = y + y^4$.

Proposition 4.2.5 *Let $f(y)$ be as before. Then:*

a) The surface of fold bifurcation corresponding to fixed points, in the parameter space (a, b, c), is:

$$-27a^4 - 4a^2b^3 + 144a^2bc + 16b^4c - 128b^2c^2 + 256c^3 = 0. \tag{4.21}$$

b) The curve of cusp bifurcations corresponding to fixed points verifies:

$$\left.\begin{aligned} 27a^2 + 8b^3 &= 0, \\ 12c + b^2 &= 0, \end{aligned}\right\} \tag{4.22}$$

and there is a unique cusp for a fixed $a = a_0$.

c) The surface of flip bifurcations corresponding to fixed points verifies:

$$\begin{gathered} -27a^4 - 4a^2b^3 + 144a^2bc + 72a^2 - 64a + 16b^4c + \\ 16b^3 - 128b^2c^2 - 64bc + 256c^3 + 16 = 0. \end{gathered} \tag{4.23}$$

d) The curve of codimension two flip bifurcations corresponding to fixed points is:

$$\left.\begin{aligned} a &= -8t^6 + 4t^3 - 2, \\ b &= -6t^4 + 2t, \\ c &= t^8 + 2t^5 - 2t^2, \\ y &= -t^2. \end{aligned}\right\} \tag{4.24}$$

If $a > -3/2$ there are not codimension two flips and if $a < -3/2$ there are two codimension two flips.

Proof:

a) The surface of fold bifurcations satisfies:

$$\left.\begin{aligned} f(y) &= y, \\ f'(y) &= 1. \end{aligned}\right\}$$

So, removing y we obtain 4.21.

b) The curve of cusp bifurcations verifies:

$$\left.\begin{aligned} f(y) &= y, \\ f'(y) &= 1, \\ f''(y) &= 0. \end{aligned}\right\}$$

From this we obtain

$$a = 8y^3,\ b = -6y^2,\ c = -3y^4,$$

and, therefore, the equation 4.22 and that there is a unique cusp for each a.

c)The surface of flip bifurcation 4.23 is obtained by removing y from the system of equations:

$$\left.\begin{aligned} f(y) &= y, \\ f'(y) &= -1. \end{aligned}\right\}$$

d) In this case this curve has to verify the system:

$$\left.\begin{aligned} f(y) &= y, \\ f'(y) &= -1, \\ 2f'''(y) + 3f''(y)^2 &= 0, \end{aligned}\right\}$$

or, by substitution,

$$\left.\begin{aligned} c + ay + by^2 + y^4 &= 0, \\ (a+2) + 2by + 4y^3 &= 0, \\ b^2 + 4y + 12by^2 + 36y^4 &= 0. \end{aligned}\right\} \qquad (4.25)$$

From the last equation we obtain: $b = -6y^2 \pm \sqrt{-y}$. Now, consider a parameter t such that $y = -t^2$. Then $b = -6t^4 + 2t$ and, using 4.25, we obtain the system 4.24.

The existence of two codimension two flips if $a < -3/2$ and no one if $a > -3/2$, follows from the first equation of 4.24. This equation has two solutions in t if $a < -3/2$ and no one if $a > -3/2$. □

Now we are interested in the study of the behaviour, in the parameter plane, of a two-parameter family of diffeomorphisms having two fold bifurcation curves, only one of them having a cusp. For this, we shall use, as a model of such a family, the map $f(y)$ with a fixed parameter a.

In order to see how the bifurcation curves are, we have studied numerically the function f for different values of a and draw the bifurcation curves in the parameters plane (b, c).

To obtain the cusps and codimension two flips we use the conditions of existence of such bifurcations. Remarkable facts are:

a) For all a there are two fold bifurcation curves corresponding to fixed points. They always look similar. One has a cusp and the other is smooth.

b) When $a > 0$ we have a global behaviour like in the first cubic model, for the cusp. The smooth fold bifurcartion curve and the successive flip bifurcation curves have no relation with the cusp. This behaviour is called a saddle area by C. Mira ([2]). See figure 4.8.

c) When $a < -2$ we have a cusp like in the second cubic model (spring area by C. Mira). See figure 4.9.

d) If $-2 < a < 0$ then the fold bifurcation curves corresponding to period one exchange their flip curves (cross-road area by C. Mira). But we distinguish two different cases:

   (a) When $-1.5 < a < 0$ there are not codimension two flips. See figure 4.10.

   (b) When $-2 < a < -1.5$ there are two codimension two flips, both in the same branch of the flip bifurcation curves. See figure 4.11.

e) It seems that, as it has been noted in [39] and [40] in other cases, cascades of cusps of doubling periods can exist, and that they are in communication through their flip bifurcation curves. A scheme of this behaviour is shown in figure 4.12. This diagram is not always complete, since it depends on the type of the successive cusps. It seems complete if $-1.5 < a < 0$. However, in the other cases the existence of codimension two flips breaks part of this structure, in the following sense:

   When there are two codimension two flips, the fold bifurcation curves of double period associated to these bifurcations of codimension two, are similar to one fold bifurcation curve with a cusp. However the periodic points associated to this cusp are not real. Only in this sense is possible to say that the cusps of doubling period always exist. For the first case, in which codimension two flips appear ($a < -2$), we have the behaviour in figure 4.13. And for the second case ($-2 < a < -1.5$) we can see the behaviour in figure 4.14

The notation used in the pictures is the following:

a) fon means n-periodic fold bifurcation curve.

b) fln means n-periodic flip bifurcation curve

c) cn_i means n-periodic cusp point.

d) ctf means codimension two flip point.

4.3 Bifurcations of codimension two in two-dimensional maps: The Hénon map

The most simple non linear diffeomorphism, the Hénon map, give examples of all the behaviours found before. Consider:

$$f_{a,b}(x,y) = (1 + y - ax^2, bx).$$

Then we can obtain saddle-node and flip bifurcation curves corresponding to period n. To do this we must solve the system:

$$\left.\begin{array}{rcl} f^n_{a,b}(x,y) & = & (x,y), \\ \operatorname{tr} Df^n_{a,b}(x,y) & = & 1 + (-b)^n, \end{array}\right\}$$

in the case of saddle-node bifurcation, or

$$\left.\begin{array}{rcl} f^n_{a,b}(x,y) & = & (x,y), \\ \operatorname{tr} Df^n_{a,b}(x,y) & = & -1 + (-b)^n, \end{array}\right\},$$

in the case of flip bifurcation.

To solve these systems we fix one variable x, y, a or b and obtain the other ones by means of the Newton method. Then we use a continuation method. We have no problem on the continuation of the curve because it is smooth in the space (x, y, a, b).

To see that the saddle-node bifurcation curve or the flip one becomes degenerate (that is, there is a cusp or a codimension two flip), we use the conditions of existence of cusps and codimension two flips found before.

In the following pictures (4.15, 4.16, 4.17, 4.18) we see the different behaviours. The notation is the same as the one in the figures corresponding to the quartic map. In figure 4.15 we have a behaviour like b) in the quartic case (saddle area). In figure 4.16 there is a behaviour like d(a) (cross road-area without codimension two flips. In figure 4.17 we have a cross-road area with two codimension two flips (d(b)). Finally, in figure 4.18 we have a spring area with two codimension two flips (c)).

The lowest periods for the different cases are the following: behaviour like b) appear for period 8, c) for period 6, d(a) for period 5 and d(b) for period 6.

Remark 4.3.1 *a) Other authors have given examples like these in the Hénon map ([2], [41]). However the case d(b) seems new in the literature.*

b) Some of the results of this chapter have appear in [42].

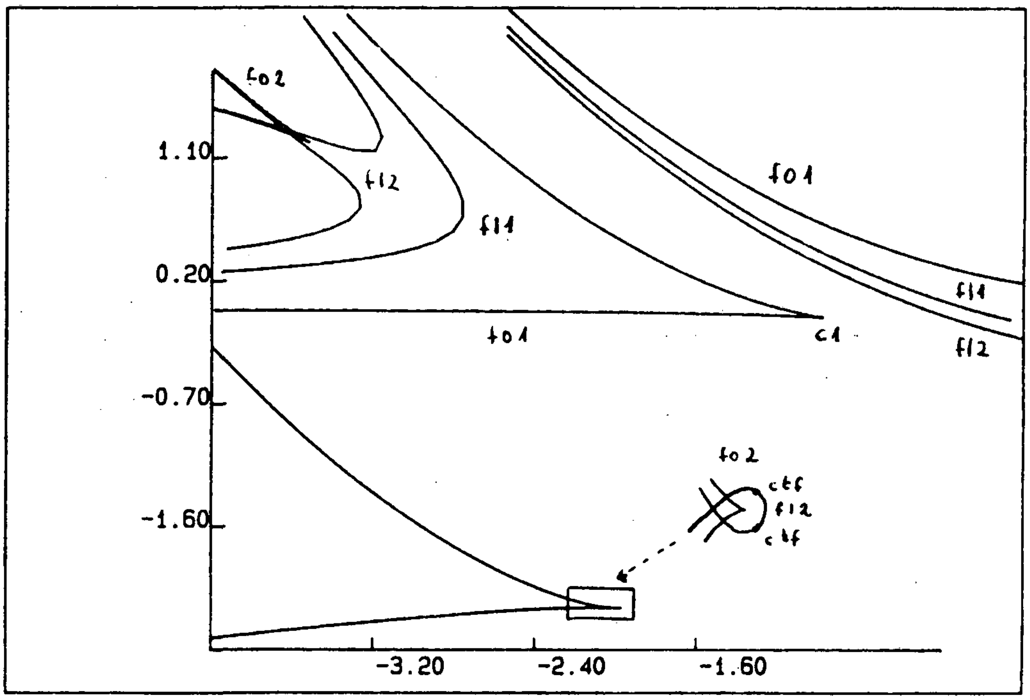


Figure 4.8: Bifurcation curves. Case $a > 0$.

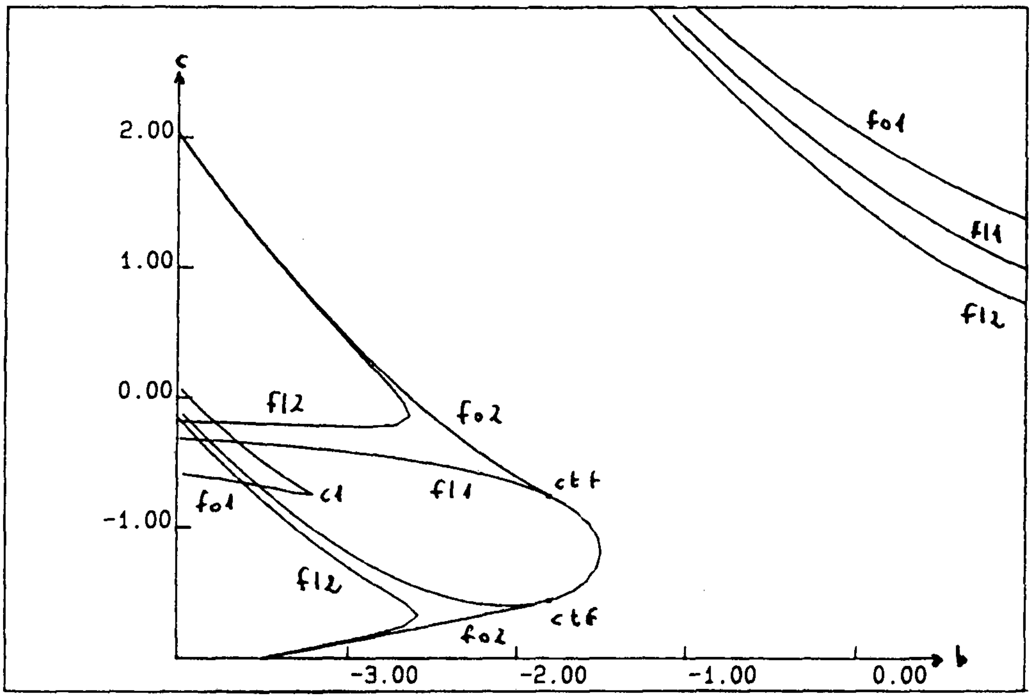


Figure 4.9: Bifurcation curves. Case $a < -2$.

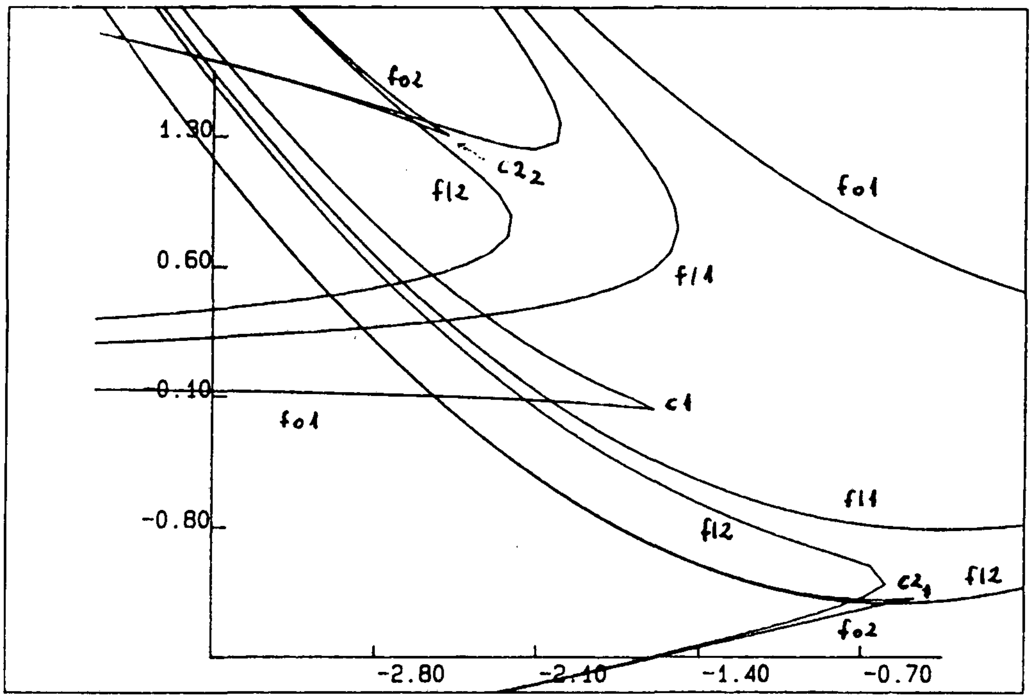


Figure 4.10: Bifurcation curves. Case $-1.5 < a < 0$.

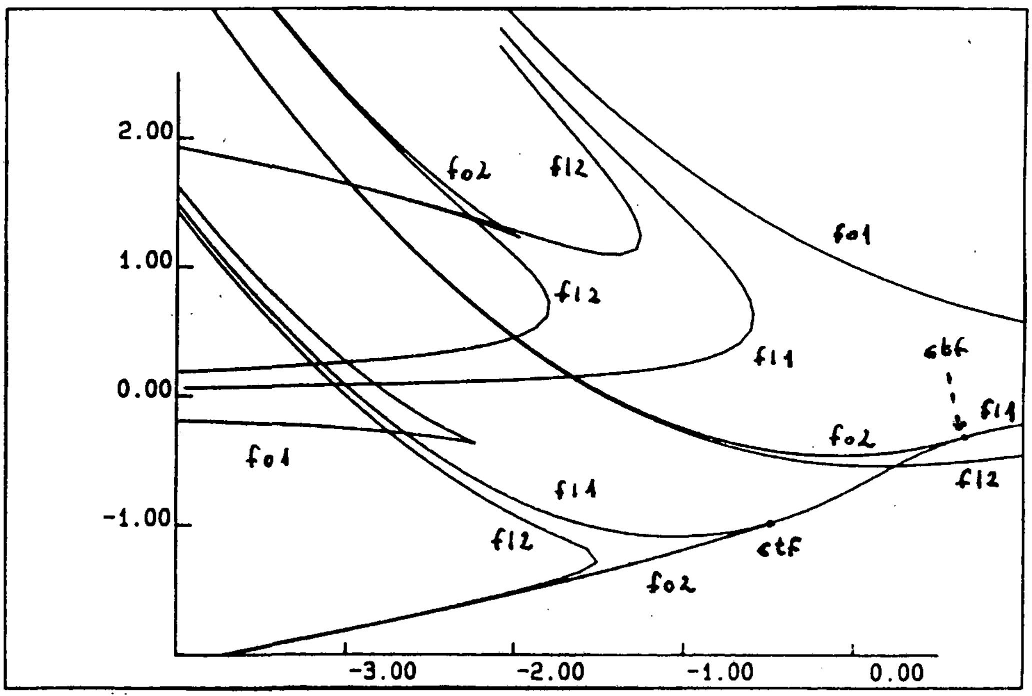


Figure 4.11: Bifurcation curves. Case $-2 < a - 1.5$.

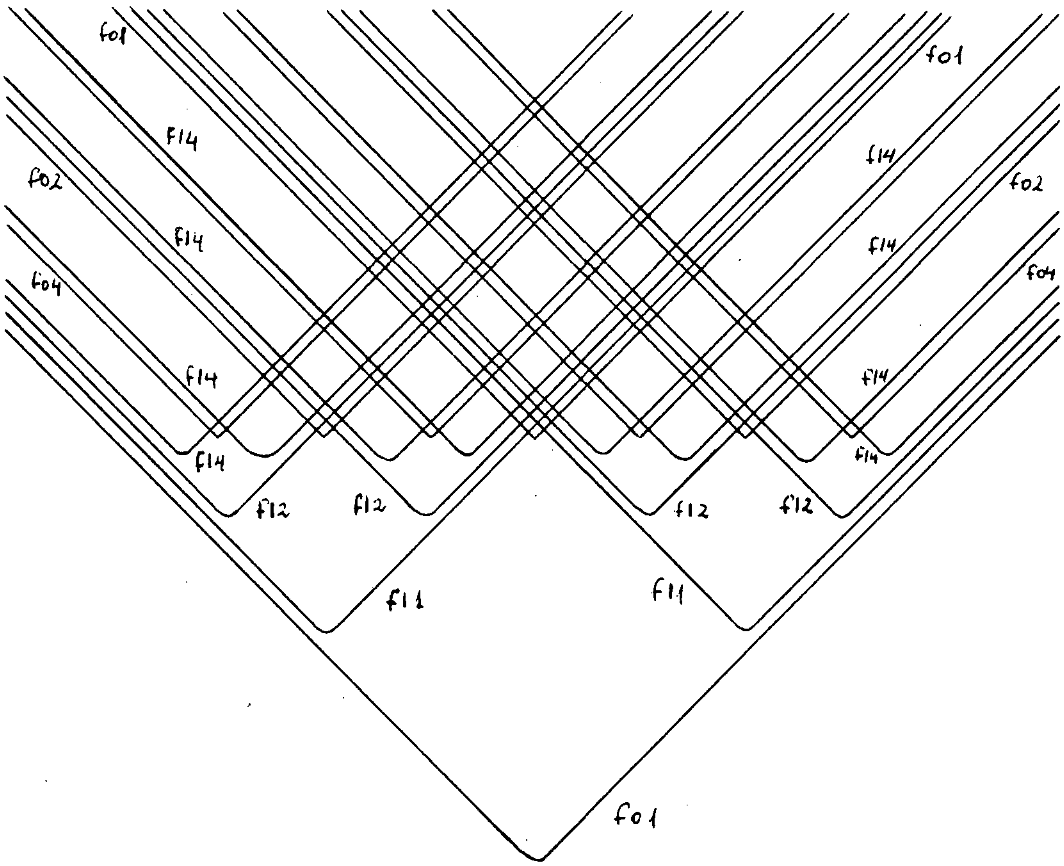


Figure 4.12: General scheme of cascades of cusps

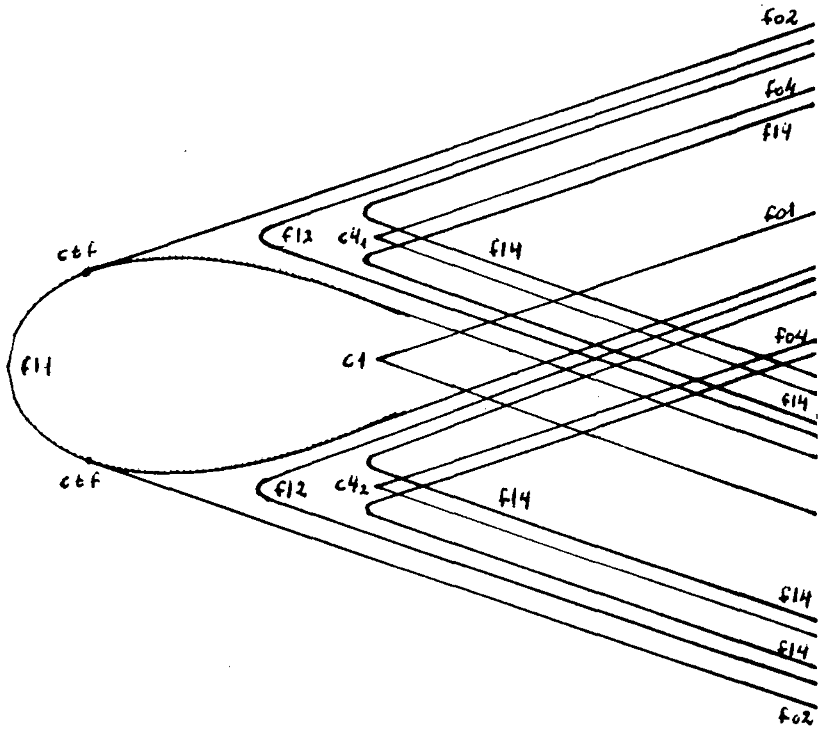


Figure 4.13: Cascades of cusps: Case $a < -2$. The cusps $c4_1$ and $c4_2$ are, probably, the initial steps of a cascade as in fig. 4.12.

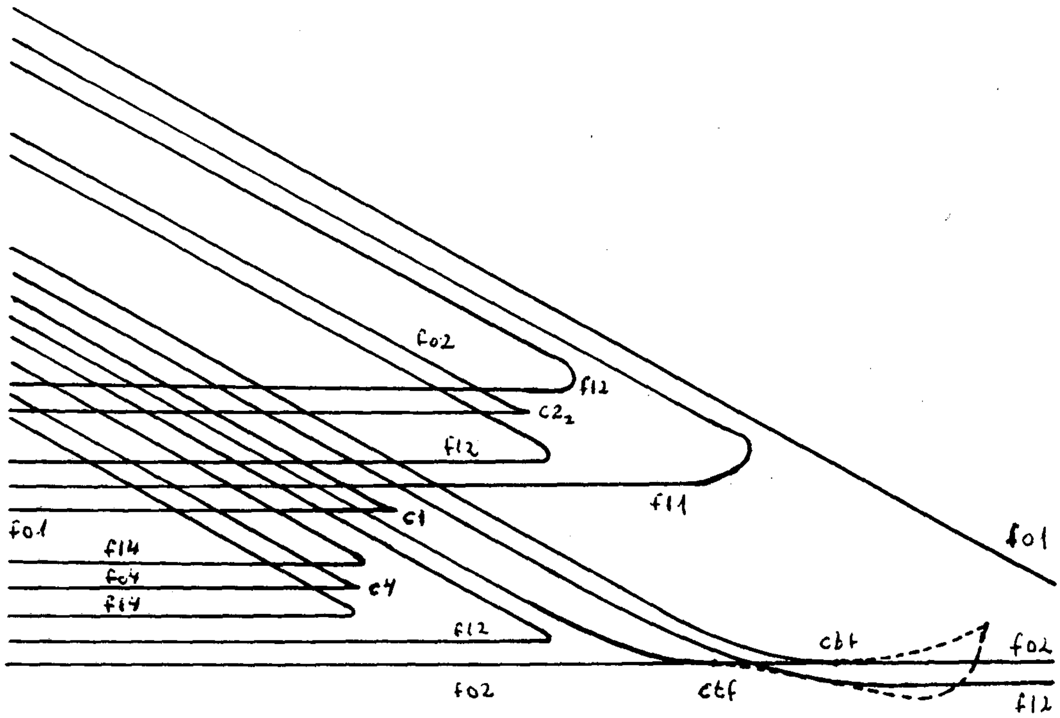


Figure 4.14: Cascades of cusps: Case $-2 < a < -1.5$. $c2_2$ and $c4$ seem the initial steps of a cascade as in fig. 4.12

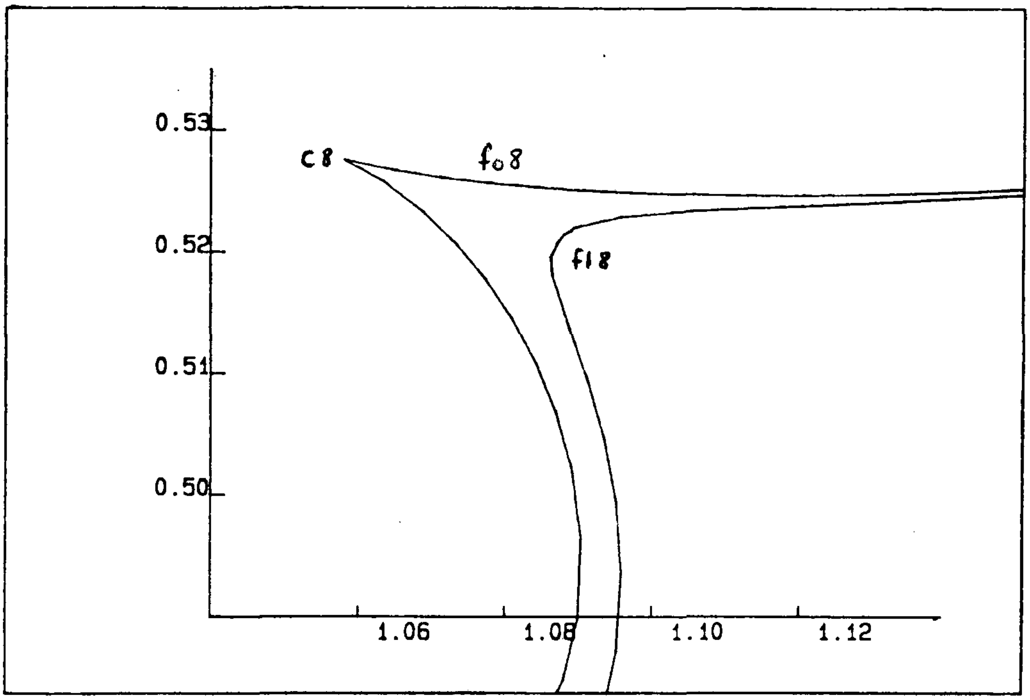


Figure 4.15: Henon map. Case b).

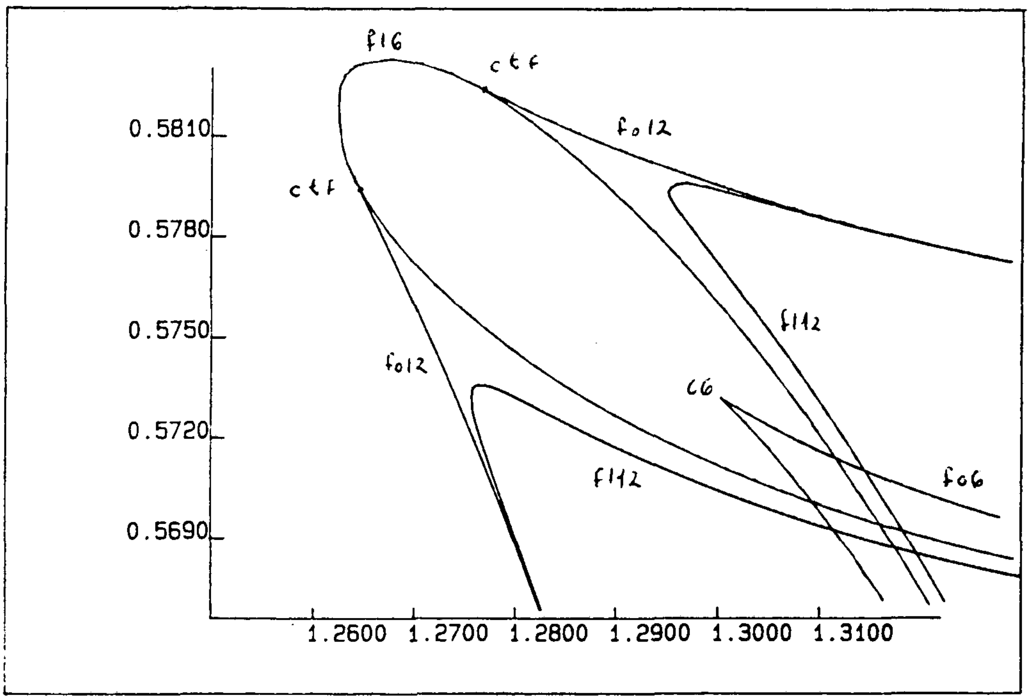


Figure 4.16: Henon map. Case c).

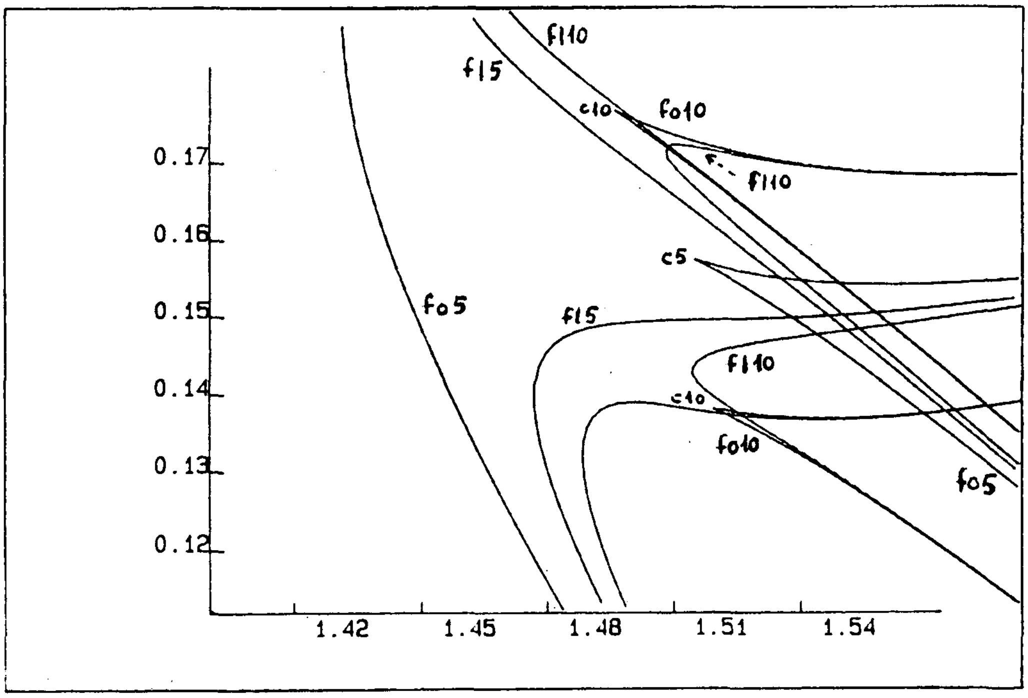


Figure 4.17: Henon map. Case d(a).

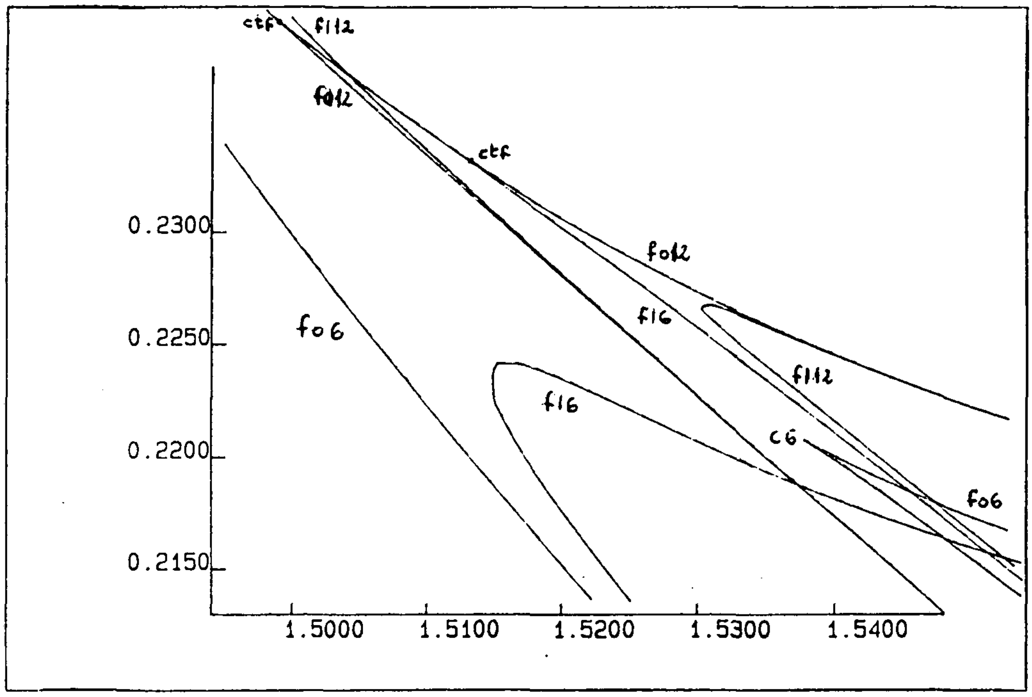


Figure 4.18: Henon map. Case d(b).

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