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Real analysis in non-euclidean spaces: Trees and spaces of homogeneous type

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REAL ANALYSIS IN NON-EUCLIDEAN SPACES: TREES AND SPACES OF HOMOGENEOUS TYPE

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Certifico que la present memòria ha estat realitzada per en Josep Lluís Garcia Domingo i dirigida per mi.

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F. Javier Soria de Diego

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Resumen de la tesis

El contenido de esta tesis se enmarca dentro del Análisis Real. En particular, trata del estudio de ciertos problemas de la teoría de pesos, (una referencia clásica sobre esta teoría es el libro de J. García-Cuerva y J.L. Rubio de Francia [GR]).

Nosotros consideramos, por este orden, tres problemas clásicos diferentes, que abarcan buena parte de la teoría de pesos:

(i) Estudio de las inclusiones para espacios con pesos y acotación de operadores integrales entre estos espacios.

(ii) Estudio de propiedades funcionales de espacios con pesos asociados a una reordenada decreciente de funciones.

(iii) Estudio de la acotación de operadores maximales asociados a regiones de aproximación entre espacios con pesos.

Todos estos problemas han sido tratados extensamente en la literatura. Nuestro enfoque ha sido el de extender estos resultados a espacios con la mínima estructura necesaria. Concretamente, hemos trabajado respectivamente en cada capítulo en los siguientes contextos:

(i) Espacios de medida arbitrarios.

(ii) Árboles.

(iii) Espacios de tipo homogéneo.

Puesto que un árbol puede ser a su vez un espacio de medida, o puesto que su frontera puede ser un espacio de tipo homogéneo, algunos resultados para espacios de medida y espacios de tipo homogéneo han sido aplicados a los árboles (véanse los capítulos primero y tercero). En cambio, en el capítulo segundo trabajamos exclusivamente en árboles.

Los espacios donde hemos desarrollado nuestra teoría no poseen, en general, ningún tipo de estructura algebraica. Por tanto, todos los resultados persiguen un objetivo común: la extensión de la teoría de pesos a espacios no euclidianos.

Detallemos el contenido de cada capítulo.

1. Inclusiones y operadores en espacios con pesos de funciones monótonas. En este capítulo tratamos básicamente tres problemas clásicos:
(1.1) *Inclusiones de espacios de Lebesgue con pesos de funciones monótonas.* Sea un espacio de medida arbitrario (X, μ). Llamamos peso a toda función positiva $u : X \longrightarrow \mathbb{R}^+$ que sea localmente integrable. Para cierto $0 < p < \infty$, el espacio de Lebesgue con peso $L^p(u)$ es el conjunto de funciones f tales que el funcional

$$\|f\|_{L^p(u)} = \left(\int_X |f(x)|^p \; u(x) \, d\mu(x) \right)^{1/p}$$

es finito. Con esta notación, en el caso $u(x) = 1$, se tiene $L^p(1) = L^p(X, \mu)$. Por tanto, los espacios con pesos suponen una primera generalización de los espacios de Lebesgue.

Uno de los problemas resueltos en la literatura es la caracterización de las inclusiones entre espacios de Lebesgue con pesos: ¿bajo qué condiciones sobre los pesos u, v y los índice p, q se obtiene la inclusión $L^p(u) \hookrightarrow L^q(v)$? Es decir, ¿qué condiciones en u, v, p y q se requieren para asegurar la existencia de una constante positiva C tal que la desigualdad

$$\left(\int_X f(x)^q \, v(x) \, d\mu(x) \right)^{1/q} \leq C \left(\int_X f(x)^p \, u(x) \, d\mu(x) \right)^{1/p}, \tag{1}$$

sea satisfecha para cualquier función positiva $f \in L^p(u)$?

En nuestro estudio, damos respuesta a esta pregunta cuando nos restringimos a las funciones monótonas, es decir, funciones crecientes o bien decrecientes. Para poder considerar el concepto de función monótona, será necesario que nuestro espacio X esté ordenado. No obstante, no se requiere que el orden adoptado sea total, siendo suficiente que sea un orden parcial. La caracterización de la desigualdad (1) para funciones monótonas en espacios euclidianos como $\mathbb{R}_+$ o $\mathbb{R}_+^n$ ha despertado gran interés últimamente. Por ejemplo, se ha visto que dicha desigualdad es necesaria para la acotación de ciertos operadores integrales en la semirrecta ([S2]), o se ha usado para la caracterización de las inclusiones entre espacios de Lorentz definidos por reordenadas decrecientes en $\mathbb{R}_+^n$, para $n \geq 1$ ([S2] y [BPSo2]).

De esta manera, nuestro primer objetivo es caracterizar la desigualdad (1) para el conjunto de funciones decrecientes del espacio de Lebesgue, que denotamos $L^p_{\rm dec}(u)$, en el contexto del espacio de medida arbitrario (X,μ), donde X es un conjunto ordenado. Como se ha explicado anteriormente, esta desigualdad equivale a la inclusión $L^p_{\rm dec}(u) \hookrightarrow L^q_{\rm dec}(v)$.

Para nuestra demostración, utilizamos un argumento de discretización. Consecuentemente, nos es necesario conocer previamente la caracterización de la desigualdad (1) para el caso discreto $X = \mathbb{Z}$ y μ la medida contadora, que tiene interés por sí sola. Debemos caracterizar los pesos discretos $\{u_k : k \in \mathbb{Z}\}$ y $\{v_k : k \in \mathbb{Z}\}$ y los índices p y q tales que

$$\left(\sum_{k=-\infty}^{\infty} a_k^q\, v_k\right)^{1/q} \le C \left(\sum_{k=-\infty}^{\infty} a_k^p\, u_k\right)^{1/p}.$$

En el caso $0 < p \le q < \infty$ era ya conocida ([R], [CRSo]), y la podemos ver en el Teorema 1.1.4. En el rango $0 < q < p < \infty$, no se encuentra en la literatura, y queda resuelta en el Teorema 1.1.6. La demostración se basa en la Proposición 1.1.5, que manifiesta que la desigualdad discreta para $X = \mathbb{Z}$ con pesos u y v es equivalente a otra desigualdad no discreta del mismo tipo para $X = \mathbb{R}_+$ y para ciertos pesos $\tilde{u}$ y $\tilde{v}$ que se construyen en función de u y v. Esta última desigualdad en $\mathbb{R}_+$ se caracteriza gracias al conocido resultado sobre dualidad debido a E. Sawyer ([S2]) que citamos en el Teorema 1.1.1.

Una vez conocido el resultado para $\mathbb{Z}$, discretizamos el espacio X en términos de sucesiones de conjuntos decrecientes que lo recubren (véase la Definición 1.1.7), y obtenemos la caracterización pretendida en el Teorema 1.1.10, nuestro principal resultado en esta sección, gracias al resultado anterior en el ámbito discreto. Posteriormente, demostramos en el Teorema 1.1.13 la equivalencia con otro resultado ya conocido para el caso $X = \mathbb{R}^n_+$ ([BPSte]).

(1.2) *Pesos en la clase B_p y el operador de Hardy discreto.* La clase de pesos B_p fue introducida por M.A. Ariño y B. Muckenhoupt ([AM]). Son los pesos en la semirrecta $u : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ para los que el operador de Hardy

$$Af(x) = \frac{1}{x}\int_0^x f(t)\,dt, \qquad x > 0,$$

satisface la acotación $A : L^p_{\rm dec}(u) \longrightarrow L^p_{\rm dec}(u)$. En [R], J.A. Raposo estudió la clase $u \in B_p(\mathbb{N})$ de pesos discretos, es decir, de sucesiones positivas indexadas en $\mathbb{N}$, para los que se tiene la acotación $A_{\mathbb{N}} : \ell^p_{\text{dec}}(u) \longrightarrow \ell^p_{\text{dec}}(u)$, donde

$$A_{\mathbb{N}} f(n) = \frac{1}{n+1} \sum_{j=0}^{n} f_j \quad n = 0, 1, 2, \ldots,$$

es el operador de Hardy discreto definido sobre sucesiones $\{f_n : n \in \mathbb{N}\}$, y $\ell^p_{\text{dec}}(u)$ es el espacio de Lebesgue con peso de sucesiones decrecientes indexadas en $\mathbb{N}$. El citado autor demostró que, en cierto sentido, se tiene

$$B_p(\mathbb{N}) \subset B_p \cap \{f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ : \forall n \in \mathbb{N}, \exists\, a_n \geq 0 \,\text{tal que} f(x) = a_n \,\text{si}\, x \in [n, n+1)\} \,.$$

Nosotros vemos que, de hecho, se tiene la igualdad (Lema 1.2.3), es decir, los pesos discretos son restricción de pesos continuos. Nos interesa estudiar si existe un resultado a la *inversa*: ¿podemos reconstruir la clase B_p a partir de pesos discretos? Vemos que la respuesta es negativa. Sin embargo, observaremos que es afirmativa si en vez de considerar pesos discretos sobre $\mathbb{N}$ consideramos pesos discretos sobre $\mathbb{Z}$. Esto nos lleva a estudiar las acotaciones $A_{\mathbb{Z}} : \ell^p_{\text{dec}}(u) \longrightarrow \ell^p_{\text{dec}}(u)$, donde ahora el operador de Hardy es

$$A_{\mathbb{Z}} f(k) = \frac{1}{2^{k+1}} \sum_{j=-\infty}^{k} 2^j f_j, \quad k \in \mathbb{Z},$$

para sucesiones $\{f_j : j \in \mathbb{Z}\}$ con índices en $\mathbb{Z}$, y el espacio $\ell^p_{\text{dec}}(u)$ es el espacio de Lebesgue de sucesiones decrecientes sobre $\mathbb{Z}$. En los Teoremas 1.2.4, 1.2.6, 1.2.8, 1.2.9 y 1.2.10, se dan las caracterizaciones de la acotación de tipo fuerte y débil del operador $A_{\mathbb{Z}}$ en todo el rango de índices, que sirven para establecer las relaciones entre las clases B_p y $B_p(\mathbb{Z})$, y también entre las clases $B_{p,\infty}$ y $B_{p,\infty}(\mathbb{Z})$. Como corolario, podemos afirmar que efectivamente es posible reconstruir la clase B_p, para $0 < p < \infty$, a partir de la clase $B_p(\mathbb{Z})$, es decir

$$B_p = \{u \geq 0 : (u_k)_k \in B_p(\mathbb{Z})\} \,,$$

donde hemos definido

$$u_k = \int_{2^k}^{2^{k+1}} u(x)\, dx,$$

para todo $k \in \mathbb{Z}$ (véase Corolario 1.2.5). Con la misma notación, se obtiene análogamente (Corolario 1.2.7)

$$B_{p,\infty} = \{u \geq 0 : (u_k)_k \in B_{p,\infty}(\mathbb{Z})\} \,.$$

(1.3) *Operadores sobre funciones monótonas.* El operador de Hardy Af antes citado y el operador de Hardy-Volterra definido para una función $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ por la expresión

$$Vf(x) = \int_0^x f(t)\,dt,$$

son ejemplos de operadores que poseen propiedades monótonas: Vf es una función creciente para toda función f positiva y Af es una función decreciente, si f es positiva y decreciente. Es posible establecer acotaciones para estos operadores usando estas propiedades monótonas (véase por ejemplo [R], capítulo primero).

Nosotros obtenemos acotaciones de tipo débil para operadores con ciertas propiedades monótonas definidos sobre funciones en espacios de medida (X, μ), donde X es un conjunto ordenado. Para ello, necesitamos en primer lugar una definición de la *norma* débil que dependa en mayor grado de la geometría del espacio X (Lema 1.3.1 y Corolario 1.3.2):

$$\|f\|_{L^{q,\infty}(v)} = \sup_{E \subset X} V(E)^{1/q} \left(\inf_{x \in E} |f(x)| \right),$$

donde denotamos $U(E) = \int_E u(x)\,d\mu(x)$. Si f es decreciente, podemos restringir el supremo a conjuntos decrecientes

$$\|f\|_{L^{q,\infty}(v)} = \sup_{D\downarrow} V(D)^{1/q} \left(\inf_{x \in D} f(x) \right).$$

Esto permite obtener el resultado general que caracteriza la mencionada acotación débil para esta clase de operadores: los Teoremas 1.3.3 y 1.3.4 son los principales resultados en esta sección. En ellos se demuestra la equivalencia de la acotación débil del operador con cierta condición capacitaria sobre los conjuntos crecientes o decrecientes en X. En las posteriores subsecciones, se aplica esta caracterización a espacios y operadores concretos y se obtienen de esta manera nuevos resultados así como otros ya conocidos:

(1.3.1) *Operadores integrales en* $\mathbb{R}_+$. La aplicación de los Teoremas 1.3.3 y 1.3.4 al caso de operadores integrales monótonos en $X = \mathbb{R}_+$ da una nueva demostración de ciertos resultados debidos a J.A. Raposo ([R], [CRSo]), que demostramos en los Teoremas 1.3.5, 1.3.6 y 1.3.7.

(1.3.2) *Operadores integrales en árboles métricos.* El Teorema 1.3.8 es la aplicación del Teorema 1.3.4 al caso de los operadores integrales del tipo Hardy para árboles métricos. Este resultado, junto con la técnica de discretización usada en la Sección

1.1, dan una extensión de ciertos resultados en [EHP], que demostramos en el Teorema 1.3.10.

(1.3.3) *Operadores de Hardy-Volterra en árboles.* De nuevo, el Teorema 1.3.12 es la aplicación del Teorema 1.3.4 al caso del operador de Hardy-Volterra para árboles. Siguiendo con el estudio de las relaciones entre lo discreto y lo continuo, establecemos en el Teorema 1.3.14 la equivalencia de las acotaciones de los operadores de Hardy-Volterra en árboles regulares y en árboles métricos, y posteriormente caracterizamos en los Teoremas 1.3.15 y 1.3.16 la acotación del operador de Hardy-Volterra en árboles para un rango parcial de índices. Utilizamos estos resultados para obtener caracterizaciones de la acotación de este operador definido en $\mathbb{N}$ (Teorema 1.3.19 y Corolario 1.3.20).

(1.3.4) *Operador de Hardy en* $\mathbb{R}^2_+$. La aplicación del Teorema 1.3.3 en este contexto, nos establece la acotación débil del operador de Hardy definido para funciones en $\mathbb{R}^2_+$, enunciado en el Teorema 1.3.21.

2. Reordenadas no lineales en árboles. Para un espacio de medida (X, μ), la reordenada decreciente de una función $f : X \longrightarrow \mathbb{C}$ es la función

$$f^\star(t) = \inf \{\lambda : \mu(\{x \in X : |f(x)| > \lambda\}) \leq t\}, \qquad t > 0.$$

La reordenada decreciente conserva, por ejemplo, la p-norma de Lebesgue de la función original y, en consecuencia, es posible describir el espacio de funciones $L^p(\mu)$ en términos de la p-norma de las reordenadas decrecientes con respecto a la medida de Lebesgue en $\mathbb{R}_+$. También es posible considerar extensiones de estos espacios funcionales: si u es un peso definido en X y $0 < p < \infty$ es un índice, el espacio de Lorentz $\Lambda^p_X(u)$ consta de aquellas funciones f tales que el funcional

$$\|f\|_{\Lambda^p_X(u)} = \left(\int_0^\infty (f^\star(t))^p u(t)\, dt \right)^{1/p}, \tag{2}$$

es finito. Tenemos, por ejemplo, que $\Lambda^p(1) = L^p(\mu)$.

Sin embargo, esta reordenada no capta ninguna característica de la geometría del espacio X. Existen otras reordenadas, también llamadas simetrizaciones, que sí dependen en cierta manera de propiedades geométricas del espacio (véase [B]). Nuestra intención es definir una nueva reordenada decreciente en cierto espacio X que sí tenga en cuenta estas propiedades, y estudiar posteriormente la normabilidad de los espacios de Lorentz asociados a esta reordenada. Si queremos hablar de funciones decrecientes, nuestro espacio X debe estar ordenado, y debe poseer una geometría particular que lo haga interesante. Nuestra elección es la del árbol homogéneo. La estructura discreta del árbol conlleva que las técnicas utilizadas para las demostraciones sean de tipo combinatorio. Establecemos de esta manera un nuevo puente entre combinatoria y análisis.

Destacamos algunos de los contenidos de las secciones del capítulo:

(2.1) *Definiciones.* Esta sección está dedicada a dar las nociones básicas sobre árboles que utilizamos posteriormente. Como ejemplos importantes, se definen el orden en el árbol y la frontera de éste.

(2.2) *Reordenando conjuntos finitos.* El paso previo a la definición de la reordenada decreciente de funciones es describir la reordenada decreciente de conjuntos finitos, puesto que, si sabemos reordenar conjuntos, sabemos reordenar funciones gracias a la fórmula del "pastel de milhojas":

$$f(x) = \int_0^\infty \chi_{\{t\in T_o : f(t)>\lambda\}}(x)\, d\lambda.$$

Esta igualdad permite reconstruir una función en términos de sus conjuntos de nivel. La Sección 2.2 se dedica a dar la definición de reordenada decreciente de conjuntos finitos (Definición 2.2.7) y estudiar algunas propiedades básicas. Para dicha definición, es necesario dotar a la frontera del árbol de un orden. Para ello, vemos que la frontera del árbol está en biyección con un intervalo de la recta real, y consideramos entonces en la frontera el orden heredado de $\mathbb{R}$ (Definición 2.2.4). Nuestra reordenada depende del llamado origen de árbol y del orden en la frontera. El Teorema 2.2.15 es importante puesto que establece la canonicidad de la definición de la reordenada. En concreto, muestra que si se sabe reordenar respecto a un origen y a un orden determinado, entonces se sabe reordenar respecto a cualquier elección del origen o del orden.

Finalmente, demostramos en la Proposición 2.2.20 la monotonía de nuestra reordenada respecto a la inclusión de conjuntos. Cabe destacar que en el caso de la reordenada decreciente clásica, esta importante propiedad tiene una demostración trivial. En cambio, en nuestro contexto su demostración requiere de un fino análisis combinatorio previo.

(2.3) *La reordenada decreciente de funciones.* En la Definición 2.3.2 de la Sección 2.3 definimos la reordenada decreciente para funciones con conjuntos de nivel finitos a partir de la fórmula del "pastel de milhojas" antes mencionada:

$$f^*(x) = \int_0^\infty \chi_{\{y\in T : |f(y)|>\lambda\}^*}(x)\, d\lambda,$$

donde denotamos E^* el conjunto reordenado decreciente de E.

Esta definición depende de nuevo del origen y del orden elegidos en el árbol, pero vemos en la Proposición 2.3.3 que esta reordenada es también canónica en el sentido antes comentado. El Lema 2.3.4 es una nueva expresión de la reordenada de la función en términos de los valores que ésta toma: si

$$f(x) = \sum_{n=1}^{\infty} a_n \chi_{E_n}(x),$$

con

$$|a_1| > |a_2| > |a_3| > \ldots > |a_n| > |a_{n+1}| > \ldots \searrow 0,$$

y E_n son conjuntos disjuntos de vértices, entonces la reordenada de f es

$$f^*(x) = \sum_{n=1}^{\infty} a_n \chi_{F_n^* \setminus F_{n-1}^*}(x), \tag{3}$$

donde $F_n = \bigcup_{k=1}^{n} E_k$ y $F_0 = \emptyset$.

Esta expresión nos facilita la demostración la Proposición 2.3.6, que recoge todas las propiedades de la reordenada que serán usadas posteriormente. La Definición 2.3.7 es importante porque nos permite extender la reordenada a funciones con conjuntos de nivel arbitrarios: la reordenada de una función con conjuntos de nivel no finitos se define por paso al límite de la reordenada de la función truncada con soporte finito, es decir

$$f^*(x) = \lim_n \Big(|f(\cdot)| \cdot \chi_{\{y \in T : |y| \leq n\}}(\cdot) \Big)^*(x).$$

A su vez, nos permite restringir la totalidad de nuestro posterior estudio a funciones con soporte finito.

Ni la definición de la reordenada ni su expresión equivalente (3) facilitan en la práctica la comprensión del proceso de reordenamiento. Necesitamos una nueva expresión equivalente más manejable e intuitiva; esta expresión viene dada en (2.7) y (2.8), y la equivalencia se demuestra en el Teorema 2.3.10. La demostración se basa de nuevo en un minucioso análisis combinatorio sobre el proceso inductivo de reordenamiento. A partir de ella, observamos que reordenar una función en el árbol es equivalente a reordenarla por diferentes geodésicas consecutivamente. Este proceso es mucho más intuitivo y será de vital importancia para el desarrollo del capítulo.

(2.4) *La desigualdad de Hardy-Littlewood.* La Sección 2.4 está dedicada al estudio de la desigualdad de Hardy-Littlewood y cuestiones relacionadas. La desigualdad de Hardy-Littlewood se demuestra en el Teorema 2.4.2:

$$\sum_{x\in T} |f(x)g(x)| \leq \sum_{x\in T} f^*(x)g^*(x),$$

para todas las funciones f y g. La Proposición 2.4.3 y el Corolario 2.4.4 establecen las relaciones entre nuestra reordenada (f^*) y la reordenada clásica ($f^\star$)

$$(|f|^*)^\star(t) = f^\star(t), \qquad t > 0,$$

y entre la desigualdad clásica de Hardy-Littlewood y la nuestra.

En el resto de la sección, el posterior estudio persigue conocer bajo qué condiciones se tiene la saturación de la desigualdad de Hardy-Littlewood. Es decir, ¿cuándo se tiene

$$\sum_{x\in T} f^*(x)g^*(x) = \sup \sum_{x\in T} |f(x)h(x)|, \tag{4}$$

donde el supremo se toma sobre todas las funciones h tales que $h^* = g^*$? Veremos en la siguiente sección que esta igualdad está estrechamente relacionada con la normabilidad de los espacios de Lorentz. En el contexto de la reordenada decreciente clásica, la igualdad se obtiene si y sólo si la función g es decreciente. No ocurre así con nuestra reordenada en el árbol, como vemos en el Ejemplo 2.4.6.

Con el objetivo citado de obtener condiciones sobre la función g que aseguren la igualdad (4), vemos que es fundamental conocer las que llamamos *transformaciones reordenantes* (véase Definición 2.4.8). Se trata de las biyecciones φ_f entre los soportes de una función f y su reordenada f^* que nos permiten relacionar los valores de una y otra mediante la expresión

$$f(x) = f^*(\varphi_f(x)),$$

para todo x en el soporte de f. En el caso clásico, como apuntamos antes, para obtener la saturación de la desigualdad de Hardy-Littlewood es necesario y suficiente que g sea decreciente, y esto equivale a que g sea invariante respecto a todas las transformaciones reordenantes, en el sentido que se satisface que

$$g(\varphi_f(.))^* = g,$$

para toda función f.

En el caso de árboles, como hemos dicho, no es suficiente que g sea decreciente, aunque sí es necesario. Sin embargo, y al igual que en el caso clásico, se demuestra en el principal resultado de esta sección (Teorema 2.4.17) que, para obtener (4), debemos pedir que g sea invariante respecto a todas las transformaciones reordenantes. Veremos en la siguiente sección que la implicación inversa es cierta, completando así este resultado, y ligándolo con la normabilidad del espacio de Lorentz.

Previamente, en el Teorema 2.4.15 vemos que si g es linealmente decreciente, entonces es invariante respecto a transformaciones reordenantes. Debemos explicar que una función es linealmente decreciente si es decreciente respecto a un nuevo orden total en el árbol introducido en la Definición 2.4.10 que, a su vez, depende del orden en su frontera. Es sencillo comprobar que toda función linealmente decreciente es decreciente. A modo de resumen, el Corolario 2.4.18 dice que si la función g es linealmente decreciente, entonces se obtiene la saturación (4).

(2.5) *Los espacios de Lorentz.* En la Sección 2.5 definimos el espacio de Lorentz asociado a nuestra reordenada. Para un peso u en el árbol, $\Delta_T^p(u)$ es el conjunto de funciones f tales que el funcional

$$\|f\|_{\Delta_T^p(u)} = \left(\sum_{x\in T}(f^*(x))^p u(x)\right)^{1/p} \tag{5}$$

es finito.

Pretendemos saber cuándo este funcional es una cuasi-norma o una norma, pero antes vemos otras propiedades funcionales. Demostramos en la Proposición 2.5.5 que este espacio es completo y en la Proposición 2.5.6 que son una generalización de los espacios de Lebesgue, puesto que si $u \equiv 1$, tenemos que

$$\Delta_T^p(1) = L^p(T, |.|),$$

donde $|.|$ es la medida contadora. Estos espacios no pueden ser invariantes por reordenación salvo en el caso trivial que u sea constante, como vemos en la Proposición 2.5.7.

Volviendo a nuestro interés principal, el estudio de la normabilidad, se caracterizan en el Teorema 2.5.9 los pesos u para los cuales el funcional (5) es una cuasi-norma. Es más difícil saber cuándo $\Delta_T^p(u)$ equipado con el funcional (5) es un espacio de Banach, es decir, cuándo (5) es una norma, puesto que hemos comentado que $\Delta_T^p(u)$ es completo. En el contexto del espacio $X = (0, l)$ y de la reordenada clásica y para $p \geq 1$, G.G. Lorentz ([Lo]) probó que es necesario y suficiente que el peso u (definido en la semirecta) sea decreciente para que el espacio $\Lambda^p_{(0,l)}(u)$ sea de Banach, o equivalentemente, el funcional (2) es una norma; J.A. Raposo ([R], [CRSo]) obtuvo el mismo resultado para $\Lambda^p_X(u)$ con espacios de medida X más generales. Este último autor, en los citados trabajos, da una respuesta también para el rango $0 < p < 1$. Nuestra primera aproximación a la solución es en este sentido: en el Lema 2.5.10 demostramos que el peso u debe ser decreciente en el árbol para que (5) sea una norma. Pero un simple ejemplo muestra que no es condición suficiente (Ejemplo 2.5.11).

Se repite así el esquema anterior cuando estudiamos la saturación de la desigualdad de Hardy-Littlewood: es necesario que el peso sea decreciente pero no suficiente. Vimos entonces que si el peso u es linealmente decreciente, sí se garantiza la igualdad

$$\sum_{x \in T} f^*(x)u(x) = \sup \sum_{x \in T} |f(x)h(x)| \,, \tag{6}$$

donde el supremo se toma sobre todas las funciones h tales que $h^* = u$. La pregunta que surge es: ¿si el peso u es linealmente decreciente, es entonces el funcional (5) una norma? Damos respuesta afirmativa en el Teorema 2.5.12; pero aún mejor, se demuestra que esta condición en u también es necesaria, y que equivale a su vez a la igualdad (6), y que equivale a que u sea invariante con respecto a todas las transformaciones reordenantes. Este teorema es el más importante en la Sección 2.5.

Por último, usando los resultados del primer capítulo, nos es posible caracterizar las inclusiones

$$\Delta^p_T(u) \hookrightarrow \Delta^q_T(v)$$

en términos de u, v, p y q, en el Teorema 2.5.15.

(2.6) *Árboles finitos y árboles regulares.* Es posible extender nuestros resultados en árboles homogéneos a una clase más amplia como son los árboles regulares. También es posible adaptar los resultados para el caso de árboles finitos. En ambos casos, la idea principal es la de encajar los árboles regulares o finitos en un árbol homogéneo adecuado, donde ya sabemos reordenar. Para el caso finito, vemos en el ejemplo final que podemos enumerar los vértices en la frontera del árbol, y utilizar este orden para obtener la reordenada de un conjunto cualquiera (y por tanto, de cualquier función), en vez de usar el orden en la frontera del árbol de la Definición 2.2.4. Parece más natural elegir este orden sencillo, pero para nuestra sorpresa, con este nuevo orden el Teorema 2.5.12 es falso, como demostramos.

3. Desigualdades con pesos y la forma de las regiones de aproximación. En la historia del Análisis, se han estudiado muchos problemas relacionados con la existencia de valores en la frontera de un dominio para cierta clase de funciones. Uno de ellos es determinar el tipo de regiones contenidas en el dominio para las que existe el límite en la frontera de funciones pertenecientes a cierto espacio funcional. Estas regiones se denominan regiones de aproximación. Por ejemplo, si el dominio es el disco unidad

$$\mathbf{D} = \{z \in \mathbb{C} : |z| < 1\},$$

su frontera es $\partial\mathbf{D} = \{z \in \mathbb{C} : |z| = 1\}$, y las regiones son *conos* definidos por

$$\Gamma(\omega) = \{z \in \mathbf{D} : |z - \omega| < 2(1 - |z|)\},$$

para $\omega \in \partial\mathbf{D}$, es conocido que el límite

$$\lim_{\substack{z \to \omega \\ z \in \Gamma(\omega)}} f(z)$$

existe (a.e. ω) para toda función armónica en $\mathbf{D}$. En cambio, si la región contiene curvas que se acercan tangencialmente a ω, el límite no existe en general. Digamos que la forma geométrica de la región influye en la existencia o no del límite.

En 1930, Hardy y Littlewood ([HL]) introdujeron la idea de estudiar la convergencia de una sucesión de funciones a partir de la acotación de una función maximal. Más tarde, A. Nagel and E.M. Stein ([NS]) caracterizan las regions de aproximación

$$\Omega \subset \mathbb{R}^{n+1}_+ := \left\{(x,t) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n,\, t > 0\right\}$$

para las que se tiene convergencia hacia la frontera para la integral de Poisson de funciones en L^p de la frontera, es decir, $L^p(\mathbb{R}^n)$. Definen el operador maximal asociado a la región de aproximación al origen 0, que denotamos $\Omega(0)$,

$$M_\Omega f(x) = \sup_{(y,r)\in\Omega(0)} \frac{1}{|B(y,r)|} \int_{B(y,r)} |f(x+z)|\, dz,$$

y estudian las acotaciones

$$M_\Omega : L^p \longrightarrow L^p, \quad p > 1,$$
$$M_\Omega : L^1 \longrightarrow L^{1,\infty}.$$

Nosotros nos planteamos estudiar la acotación con pesos

$$M_\Omega : L^p(u) \longrightarrow L^{p,\infty}(v), \quad p \geq 1,$$

para un operador M_Ω asociado a una familia de aproximación (i.e., una familia de regiones de aproximación) en un contexto más general. Luego estudiamos bajo qué condiciones en la familia Ω se obtiene la estimación

$$M_\Omega f(x) \leq CMf(x),$$

para todo x y toda f, donde Mf es el operador maximal de Hardy-Littlewood. Esta desigualdad se traduce en una inclusión entre ciertas clases de pesos que serán también estudiadas. Estas cuestiones son introducidas por A. Sánchez-Colomer y J. Soria en [SS2] para el espacio euclídeo $\mathbb{R}^{n+1}_+$, y nosotros pretendemos extender el estudio a espacios más generales.

Hemos estructurado el capítulo siguiendo el orden cronológico en que efectuamos la investigación. Empezamos por estudiar en la Sección 3.2 las cuestiones antes mencionadas para familias de aproximación

$$\Omega \subset X_+ := X \times (0, \infty),$$

donde (X, μ, d) es un espacio de tipo homogéneo no simétrico, que denotamos por ns-espacio de tipo homogéneo, y que es una extensión de la noción del espacio de tipo homogéneo (véanse las Definiciones 3.2.1 y 3.2.2). En la Subsección 3.2.3 aplicamos los resultados obtenidos al caso (más sencillo) en que X es un grupo, puesto que existen muchos ejemplos con esta característica. Posteriormente, el estudio se extiende a un contexto abstracto en la Sección 3.3, dado que muchos ejemplos de dominios donde se pueden aplicar los resultados no poseen una estructura cartesiana como la de X_+. Para ilustrar este salto al caso abstracto, hacemos un estudio previo en la Subsección 3.3.1 del caso de los árboles homogéneos.

Debido a esta estructura, aparecen resultados semejantes a lo largo del capítulo, que corresponden a las mismas ideas aplicadas a cada contexto. Hemos creído oportuno incluirlos puesto que así se ve cuáles son las dificultades que debemos salvar en cada caso. Además, se facilita el seguimiento del estudio de esta manera.

Detallamos el contenido de cada sección:

(3.1) *Resultados preliminares en* $\mathbb{R}^{n+1}_+$. Con el fin de centrarnos en nuestro objeto de investigación, recogemos en esta sección todos aquellos resultados conocidos hasta el momento que nos son útiles. Algunos de ellos serán generalizados y completados en secciones posteriores.

(3.2) *El semiepacio $X \times (0, \infty)$ para un espacio de tipo homogéneo X.* Dedicamos esta sección al caso del producto cartesiano $X_+ := X \times (0, \infty)$, donde X es un espacio de tipo homogéneo. Hemos dividido la sección en tres partes:
(3.2.1) *Definiciones y resultados previos.* Empezamos por incluir las definiciones de los espacios de tipo homogéneo y los espacios de tipo homogéneo no simétrico, que llamamos ns-espacios de tipo homogéneo (Definiciones 3.2.1 y 3.2.2). Proseguimos con algunos resultados clásicos adaptados a los ns-espacios: lema de recubrimiento de tipo Vitali (Lema 3.2.5), estimaciones para el operador maximal de Hardy-Littlewood (Teorema 3.2.6), descomposición de tipo Whitney (Teorema 3.2.7) y extensión del concepto de par de Carleson a abiertos arbitrarios (Proposición 3.2.9). Introducimos entonces en la Definición 3.2.10 los conceptos básicos asociados a una familia de aproximación

$$\Omega = \{\Omega(x) : x \in X\},$$

donde cada $\Omega(x)$ es un conjunto medible en X_+. Asimismo, definimos para una función f el operador maximal asociado a la familia Ω como

$$M_\Omega f(x) = \sup_{(y,t)\in\Omega(x)} \frac{1}{\mu(B(y,t))} \int_{B(y,t)} |f(z)|\, d\mu(z).$$

El Teorema 3.2.13 es importante: se caracteriza la acotación

$$M_\Omega : L^p(\nu) \longrightarrow L^{p,\infty}(\rho),$$

para dos medidas ν y ρ, en términos de una condición de tipo Carleson, que es equivalente a una desigualdad entre las funciones de distribución de $M_\Omega f$ y Mf, para toda f.

La clase de pesos A_p fue introducida por B. Muckenhoupt ([M]). Es el conjunto de pesos u para los que se tiene la acotación del operador maximal de Hardy-Littlewood

$$M : L^p(u) \longrightarrow L^{p,\infty}(u).$$

Análogamente, definimos la clase de pesos A_p^Ω como los pesos u tales que se tiene la acotación

$$M_\Omega : L^p(u) \longrightarrow L^{p,\infty}(u).$$

Finalmente, se define la clase de pesos $W(\Omega)$ asociados a la familia Ω (Definición 3.2.16). Una consecuencia del Teorema 3.2.13 es el Teorema 3.2.17, también importante, donde se da la relación existente entre estas clases de pesos. Concretamente, obtenemos que

$$A_p^\Omega = A_p \cap W(\Omega). \tag{7}$$

(3.2.2) *La forma de las regiones de aproximación.* En esta subsección respondemos a una pregunta natural en vista de la igualdad (7): ¿cuándo se tiene $A_p = A_p^\Omega$? Para ello, necesitamos cierta propiedad de invariancia sobre el espacio de tipo homogéneo (X, μ, d): existe $C > 0$ tal que

$$\frac{1}{C}\mu(B(x,r)) \leq \mu(B(y,r)) \leq C\,\mu(B(x,r)),$$

para todo x, y y $r > 0$, donde $B(x,r)$ es la bola de centro x y radio $r > 0$. Con este fin, requerimos al espacio (X, μ, d) que cumpla la siguiente condición: existe una constante $M > 0$ tal que

$$B(x, Mr) \setminus B(x,r) \neq \phi, \tag{8}$$

para todo x y $r > 0$.

Si el espacio de tipo homogéneo (X, μ, d) satisface esta última condición, entonces podemos reemplazar la cuasi-distancia d por otra cuasi-distancia no simétrica δ, de tal manera que (X, μ, δ) es un ns-espacio de tipo homogéneo. Con este cambio, ni la topología ni las clases de pesos A_p y A_p^Ω varían, y se consigue que $\mu(B(x,r))$ sea comparable a r, con lo que obtenemos la invariancia deseada. Estos resultados se recogen en el Teorema 3.2.21 y el Corolario 3.2.25.

Antes de dar nuestro principal teorema, necesitamos que la familia de aproximación satisfaga cierta condición de regularidad descrita en la Definición 3.2.28. Con los ejemplos de familias de aproximación regulares que se presentan después, podemos ver que esta regularidad requerida no es demasiado restrictiva en el sentido que existen familias regulares, en $\mathbb{R}^2_+$ por ejemplo, no generadas por traslación.

Finalmente, probamos el principal teorema (Teorema 3.2.30) donde se dan diferentes caracterizaciones para la obtención de la igualdad $A_p = A_p^\Omega$. Una de ellas es la inclusión

$$\Omega(x) \subset \Gamma_\theta(x),$$

para cierta $\theta > 0$, donde $\Gamma_\theta(x)$ es el cono de vértice x y apertura θ. Así, vemos que condiciones analíticas en la clase de pesos se traducen en condiciones geométricas en la familia de regiones.

(3.2.3) *El caso de la estructura de grupo: algunos ejemplos.* En esta sección, damos algunos ejemplos de espacios de tipo homogéneo que poseen la estructura de grupo, y a los que se aplica nuestro resultado. Aprovechamos para obtener una nueva demostración del Teorema 3.2.30 más sencilla en este contexto.

Los ejemplos tratados son:

(i) $X = \mathbb{R}^n$ con la medida de Lebesgue y la distancia no isotrópica

$$d(x, y) = \sum_{k=1}^{n} |x_k - y_k|^{1/a_k},$$

es un espacio de tipo homogéneo.

(ii) $X = \mathbf{H}_\mathrm{n}$ el grupo de Heisenberg dotado de la distancia

$$d(x, y) = ||y^{-1} \cdot x||,$$

donde

$$||[\zeta, t]|| = \max(|\zeta|^2, |t|),$$

es la pseudo-norma en $\mathbf{H}_\mathrm{n}$, y de la medida de Lebesgue en $\mathbb{C}^n \times \mathbb{R}$, es un espacio de tipo homogéneo.

(iii) X es el conjunto de las matrices reales triangulares superior 3×3 con unos en la diagonal. La ley multiplicativa de grupo es el producto usual de matrices. La cuasi-distancia es $d'(x, y) = (d(x, y) + d(y, x))/2$, donde $d(x, y) = ||y^{-1} \cdot x||$ y $||x|| = \max\{|a|, |b|^{1/2}, |c|\}$. La medida es la medida de Lebesgue en $\mathbb{R}^3$.

(iv) Todo grupo de Lie nilpotente y unimodular, con la métrica Riemaniana y la medida inducida es un espacio de tipo homogéneo.

(3.3) *El caso general.* Muchos ejemplos interesantes de dominios donde se ha estudiado el fenómeno de la convergencia hacia la frontera no poseen una estructura de producto cartesiano como la estudiada en la sección anterior. Por este motivo, resolvemos aquí las mismas cuestiones, pero en el contexto más abstracto.

(3.3.1) *Regiones de aproximación en un árbol homogéneo isotrópico.* Desarrollamos la teoría en este nuevo marco. Ahora el árbol juega el papel del semiespacio X_+ y su frontera es un espacio de tipo homogéneo que juega el papel de X. En este caso, se tiene que la medida de todas las bolas en la frontera con el mismo radio coincide. El Teorema 3.3.8 es el resultado análogo al Teorema 3.2.30.

(3.3.2) *Regiones de aproximación en el contexto abstracto.* Usamos las técnicas introducidas por F. Di Biase ([DiB]) para la última extensión de nuestros resultados. Definimos la estructura general en la Definición 3.3.9, y suplimos la noción de cono por la de región supernatural (Definición 3.3.10). Básicamente, una familia supernatural Γ satisface que la sombra de ζ,

$$\Gamma^{\downarrow}(\zeta) = \{x \in X : \zeta \in \Gamma(x)\},$$

es comparable a una bola, y que para cada bola, existe una sombra comparable.

Definimos el operador maximal asociado a una familia de aproximación Ω en función de la familia de aproximación supernatural fijada Γ como sigue:

$$M_\Omega f(x) = \sup_{\zeta \in \Omega(x)} \frac{1}{\mu(\Gamma^\downarrow(\zeta))} \int_{\Gamma^\downarrow(\zeta)} |f(y)| \, d\mu(y).$$

Obtenemos también la igualdad $A_p^\Omega = A_p \cap W(\Omega)$ en el Teorema 3.3.15, con las definiciones pertinentes para cada clase de pesos.

Como hicimos anteriormente, podemos reemplazar el espacio de tipo homogéneo (X, μ, d) por el ns-espacio de tipo homogéneo (X, μ, δ) sin pérdida de generalidad (Teorema 3.3.16), siempre que (X, μ, d) cumpla la condición (8).

El análogo al Teorema 3.2.30 es el Teorema 3.3.18, que supone el mayor grado de generalización de nuestros resultados.

(3.4) *Vuelta a espacios euclídeos: dos aplicaciones.* En esta última sección, aplicamos las técnicas utilizadas en este capítulo para completar algunos resultados en $\mathbb{R}^n$ aparecidos en [FJR] y [RS].

(3.4.1) *Operadores integrales singulares.* En este apartado estudiamos la acotación del operador

$$N_\Omega f(x) = \sup_{(y,t) \in \Omega(x)} |(K_t * f)(y)|,$$

definido para una familia de aproximación $\Omega \subset \mathbb{R}^{n+1}_+$ y un núcleo de Calderón-Zygmund. Obtenemos la caracterización completa de la acotación

$$N_\Omega : L^p(m) \longrightarrow L^{p,\infty}(\rho),$$

si $p \geq 1$, m la medida de Lebesgue y ρ una medida arbitraria, cuando $K_j(x) = \omega_j(x)/|x|^n$ con $\omega_j(x) = x_j/|x_j|$, para cierto $1 \leq j \leq n$, es el núcleo de Riesz (Teorema 3.4.2). En el caso de un núcleo general de Calderón-Zygmund, obtenemos un resultado parcial en el Teorema 3.4.4.

(3.4.2) *Espacios potenciales.* Aquí damos estimaciones para el operador

$$N_\Omega f(x) = \sup_{(y,t) \in \Omega(x)} |(P_t * f)(y)|,$$

donde P es el núcleo de Poisson y Ω es una familia de aproximación en $\mathbb{R}^{n+1}_+$. Concretamente, se caracteriza la acotación

$$N_\Omega : L^p_k(m) \longrightarrow L^{p,\infty}(\rho),$$

donde

$$L^p_k(m) = \{f : \ f = F * k, \text{ para alguna } F \in L^p(m)\}$$

es el espacio potencial. El principal resultado es el Teorema 3.4.6.

Contents

Introduction

The main contents of this thesis belong to the area of Real Analysis. In particular, we deal with the study of certain problems in the theory of weights, (a classical reference for this theory is the book of J. García-Cuerva and J.L. Rubio de Francia [GR]).

We consider, in this order, three different classical problems, that cover a big part of the theory of weights:

(i) Study of embeddings of weighted spaces and boundedness of integral operators between weighted spaces.

(ii) Study of functional properties of weighted spaces related to a decreasing rearrangement of functions.

(iii) Study of the boundedness of maximal operators related to approach regions between weighted spaces.

All these problems have been intensively studied in the literature. Our point of view has been to extend these results to spaces with the minimum necessary structure. To be precise, to obtain our results, we have worked respectively in each chapter in:

(i) Arbitrary measure spaces.

(ii) Trees.

(iii) Spaces of homogeneous type.

Since a tree can also be seen as a measure space, and since its boundary can be a space of homogeneous type, some results about measure spaces and spaces of homogeneous type have been applied to trees (see the first and the third chapters). On the other hand, in the second chapter we work exclusively in trees.

The spaces where we work do not enjoy, in general, any type of algebraic structure. Therefore, our results go in one direction: the extension of the theory of weights to non-Euclidean spaces.

We detail the content of each chapter:

1. Embeddings and operators on weighted spaces of monotone functions. In this chapter, we deal basically with three classical problems:
(1.1) *Embeddings of weighted spaces of monotone functions.* Let (X,μ) be a measure space. A positive locally integrable function $u : X \longrightarrow \mathbb{R}^+$ is called a weight. For $0<p<\infty$, the weighted Lebesgue space $L^p(u)$ is the set of functions f such that the functional

$$\|f\|_{L^p(u)} = \left(\int_X |f(x)|^p \ u(x)\, d\mu(x) \right)^{1/p}$$

is finite. With this notation, we trivially have that if $u(x)=1$, then $L^p(1)=L^p(X,\mu)$. Then, these weighted spaces are a generalization of the Lebesgue spaces.

We consider the case of monotone functions. The space $L^p_{\text{dec}}(u)$ stands as the set of decreasing functions in $L^p(u)$. In our work, we characterize the embedding

$$L^p_{\text{dec}}(u) \hookrightarrow L^q_{\text{dec}}(v),$$

or equivalently, we find conditions on u, v, p and q such that there exists a constant $C>0$ such that the inequality

$$\left(\int_X f(x)^q \, v(x)\, d\mu(x) \right)^{1/q} \leq C \left(\int_X f(x)^p \, u(x)\, d\mu(x) \right)^{1/p}, \tag{9}$$

holds for all decreasing functions $f \in L^p(u)$.

To be able to consider this question, our space X has to be ordered, but we just need a partial order. The characterization of inequality (9) for the Euclidean cases $X=\mathbb{R}_+$ with the usual order, or $X=\mathbb{R}^n_+$ with the order

$$(a_1, a_2, \dots, a_n) \leq (b_1, b_2, \dots, b_n)$$

if and only if $a_i \leq b_i$ for $i=1,\dots,n$, have been of great interest in the recent years. For example, it has been proved that this inequality is necessary for the boundedness of certain integral operators defined in $\mathbb{R}_+$ ([S2]), or it has been used to characterize the embeddings of Lorentz spaces related to decreasing rearrangements in $\mathbb{R}^n_+$, $n \geq 1$ ([S2] and [BPSo2]).

To obtain the characterization of inequality (9), we use a discretization argument. Thus, we previously need to know the characterization of this inequality for the discrete case $X=\mathbb{Z}$ and μ the counting measure, which is of interest by itself. We have to give conditions on the discrete weights $\{u_k : k \in \mathbb{Z}\}$ and $\{v_k : k \in \mathbb{Z}\}$ and the indices p and q such that

$$\left(\sum_{k=-\infty}^{\infty} a_k^q\, v_k\right)^{1/q} \leq C \left(\sum_{k=-\infty}^{\infty} a_k^p\, u_k\right)^{1/p}.$$

In the range $0 < p \leq q < \infty$ this is known ([R], [CRSo]). In the range $0 < q < p < \infty$, it seems to be an open problem, and we solve it in Theorem 1.1.6. The proof is based in Proposition 1.1.5, that shows that the discrete inequality for $X = \mathbb{Z}$ and weights u and v is equivalent to the same non-discrete inequality in $X = \mathbb{R}_+$, for certain weights $\tilde{u}$ and $\tilde{v}$ that depend on u and v. This last inequality in $\mathbb{R}_+$ is characterized by using the well-known duality result due to E. Sawyer ([S2]), introduced in Theorem 1.1.1.

Once the result for $\mathbb{Z}$ is obtained, we discretize the space X in terms of covering sequences of decreasing sets (see Definition 1.1.7), and we obtain Theorem 1.1.10, our main result in this section, thanks to the discrete case. Then, we prove in Theorem 1.1.13 that our result is equivalent to the previously known case $X = \mathbb{R}_+^n$ ([BPSte]).

(1.2) *B_p weights and the discrete Hardy operator.* The B_p class of weights was introduced by M.A. Ariño and B. Muckenhoupt ([AM]). These are the weights $u : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that the Hardy operator

$$Af(x) = \frac{1}{x}\int_0^x f(t)\, dt, \qquad x > 0,$$

satisfies the boundedness $A : L^p_{\mathrm{dec}}(u) \longrightarrow L^p_{\mathrm{dec}}(u)$. In [R], J.A. Raposo studied the class $u \in B_p(\mathbb{N})$ of discrete weights, that is, of positive sequences indexed over $\mathbb{N}$, such that we have the boundedness $A_{\mathbb{N}} : \ell^p_{\mathrm{dec}}(u) \longrightarrow \ell^p_{\mathrm{dec}}(u)$, where

$$A_{\mathbb{N}} f(n) = \frac{1}{n+1}\sum_{j=0}^{n} f_j \quad n = 0, 1, 2, \dots,$$

is the discrete Hardy operator defined on sequences $\{f_n : n \in \mathbb{N}\}$, and $\ell^p_{\mathrm{dec}}(u)$ is the weighted Lebesgue space of sequences over $\mathbb{N}$. This author proved that, in some sense, we have

$$B_p(\mathbb{N}) \subset B_p \cap \{f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ : \forall n \in \mathbb{N}, \exists\, a_n \geq 0\, \text{such that} f(x) = a_n \,\text{if}\, x \in [n, n+1)\}\,.$$

We prove that the equality is true (Lemma 1.2.3), that is, the discrete weights come from the restriction of continuos weights. We are interested in the converse result: can we build the B_p class in terms of the discrete weights? The answer is negative, as a simple example shows. But we prove that the answer is affirmative if, instead of $B_p(\mathbb{N})$, we consider discrete weights indexed over $\mathbb{Z}$. This leads us to prove estimates $A_{\mathbb{Z}} : \ell^p_{\mathrm{dec}}(u) \longrightarrow \ell^p_{\mathrm{dec}}(u)$, where now the Hardy operator is

$$A_{\mathbb{Z}} f(k) = \frac{1}{2^{k+1}} \sum_{j=-\infty}^{k} 2^j f_j, \quad k \in \mathbb{Z},$$

defined for sequences $\{f_j : j \in \mathbb{Z}\}$ with indices in $\mathbb{Z}$, and $\ell^p_{\mathrm{dec}}(u)$ is the Lebesgue space of decreasing sequences over $\mathbb{Z}$. In Theorems 1.2.4, 1.2.6, 1.2.8, 1.2.9 and 1.2.10, we give several characterizations of the boundedness and the weak-boundedness of $A_{\mathbb{Z}}$ in all the range of indices $0 < p < \infty$. They are used to obtain relations between B_p and $B_p(\mathbb{Z})$, and also between $B_{p,\infty}$ and $B_{p,\infty}(\mathbb{Z})$. We mention, as a summary, that for $0 < p < \infty$,

$$B_p = \{u \geq 0 : (u_k)_k \in B_p(\mathbb{Z})\},$$

where

$$u_k = \int_{2^k}^{2^{k+1}} u(x)\, dx,$$

for all $k \in \mathbb{Z}$ (see Corollary 1.2.5). With the same notation (Corollary 1.2.7)

$$B_{p,\infty} = \{u \geq 0 : (u_k)_k \in B_{p,\infty}(\mathbb{Z})\}.$$

(1.3) *Operators on monotone functions.* We obtain estimates for the weak-boundedness of operators with certain monotone properties in measure spaces (X, μ), where X is an ordered set. For this, we need a more geometrical definition of the weak *norm* of a function (Lemma 1.3.1 and Corollary 1.3.2):

$$\|f\|_{L^{q,\infty}(v)} = \sup_{E \subset X} V(E)^{1/q} \left(\inf_{x \in E} |f(x)| \right),$$

where $U(E) = \int_E u(x)\, d\mu(x)$. If f is decreasing, then

$$\|f\|_{L^{q,\infty}(v)} = \sup_{D\downarrow} V(D)^{1/q} \left(\inf_{x \in D} f(x) \right).$$

This allows us to obtain the general result for the mentioned weak-boundedness of this class of operators: Theorems 1.3.3 and 1.3.4 are our main result in this section. In the forthcoming subsections, we apply it to different spaces and operators, obtaining new and known results:

(1.3.1) *Integral operators in* $\mathbb{R}_+$. The application of Theorems 1.3.3 and 1.3.4 to the

case of monotone integral operators in $X=\mathbb{R}_+$ give new proofs of certain results due to J.A. Raposo ([R], [CRSo]), that we prove in Theorems 1.3.5, 1.3.6 and 1.3.7.
(1.3.2) *Integral operators in metric trees.* Theorem 1.3.8 is the application of Theorem 1.3.4 to the case of Hardy type operators in metric trees. This result, with the use of the discretization technique of Section 1.1, gives an extension of certain results in [EHP], that we prove in Theorem 1.3.10.
(1.3.3) *Hardy-Volterra operators in trees.* Once more, Theorem 1.3.12 is the application of Theorem 1.3.4 to the Hardy-Volterra operator on trees. Following the study of the relations between discrete and continuous, we establish in Theorem 1.3.14 the equivalency of the boundedness of the Hardy-Volterra operators in regular trees and metric trees, and later we give the boundedness of the Hardy-Volterra operator in trees for a partial range of indices in Theorems 1.3.15 and 1.3.16. We use these results to obtain characterizations of this operator defined in $\mathbb{N}$ (Theorem 1.3.19 and Corollary 1.3.20).
(1.3.4) *Hardy operator in* $\mathbb{R}^2_+$. The application of Theorem 1.3.3 in this context, establishes the weak-boundedness of the Hardy operator defined for functions in $\mathbb{R}^2_+$, announced in Theorem 1.3.21.

2. Non-linear rearrangement on trees The classical decreasing rearrangement of a function $f : X \longrightarrow \mathbb{C}$ in a measure space (X,μ) is

$$f^\star(t)=\inf\left\{\lambda : \mu(\{x\in X : |f(x)|>\lambda\})\le t\right\}, \qquad t>0.$$

If u is a weight in X and $0<p<\infty$, the Lorentz space $\Lambda^p_X(u)$ contains the functions f such that the functional

$$\|f\|_{\Lambda^p_X(u)}=\left(\int_0^\infty (f^\star(t))^p u(t)\,dt\right)^{1/p},$$

is finite.

The classical rearrangement does not distinguish any geometric characteristic of X. There exist other rearrangements or symetrizations that depend on these characteristics of the space (see [B]). Our intention is to define a new decreasing rearrangement in certain space X that strongly depends on its geometric properties, and to study the normability of the related Lorentz spaces. Our space X has to be ordered, and must have an interesting geometry. We choose a homogeneous tree. Its discrete topology forces us to prove our results by using combinatorial techniques. Thus, we establish a new interplay between analysis and combinatorics.

Some of the contents of the chapter are:

(2.1) *Definitions.* We give the basic notions about trees. Two important examples are the order in the tree and its boundary.

(2.2) *Rearranging finite sets.* By using the "Layer-cake" formula

$$f(x) = \int_0^\infty \chi_{\{t\in T_o: f(t)>\lambda\}}(x)\, d\lambda,$$

to rearrange a function it is enough to rearrange its level sets. Section 2.2 is devoted to give the definition of the decreasing rearrangement for finite sets (Definition 2.2.7) and to study some basic properties. To this end, we need to adopt an order in the boundary of the tree. We do it by establishing a bijection between the boundary and an interval in $\mathbb{R}$, and then we transfer the usual order of the real numbers (Definition 2.2.4).

Our rearrangement depends on the so-called origin of the tree and the order in the boundary. Theorem 2.2.15 is important because it shows the canonicity of the definition.

Finally, Proposition 2.2.20 shows that our rearrangement is monotone with respect to the inclusion of sets. It turns out that, contrary to the classical rearrangement, the proof is not trivial and requires a sharp combinatorial analysis.

(2.3) *The decreasing rearrangement of functions.* Definition 2.3.2 in Section 2.3 is the decreasing rearrangement of functions with finite level sets:

$$f^*(x) = \int_0^\infty \chi_{\{y\in T: |f(y)|>\lambda\}^*}(x)\, d\lambda,$$

where E^* is the rearranged of E.

Proposition 2.3.3 shows its canonicity, and Proposition 2.3.6 collects all the main properties. Definition 2.3.7 is important because it allows us to extend the notion of the rearrangement to general functions. The idea is to cut the function in a sequence of functions with finite support, to rearrange them, and to define the rearrangement pointwise as the limit of the sequence of rearranged functions:

$$f^*(x) = \lim_n \Big(|f(\cdot)| \cdot \chi_{\{y\in T: |y|\le n\}}(\cdot) \Big)^*(x).$$

Therefore, we can restrict the study to finitely supported functions.

The definition of the decreasing rearrangement is elegant but not handy. So, we look for a more intuitive equivalent way of rearranging; this way is given in expressions

(2.7) and (2.8), and the equivalency is shown in Theorem 2.3.10. The proof is based in a detailed combinatorial analysis of the inductive process of rearrangement.

(2.4) *The Hardy-Littlewood inequality.* Section 2.4 is devoted to the Hardy-Littlewood inequality (Theorem 2.4.2) and related topics. Proposition 2.4.3 and Corollary 2.4.4 establish the relations between our rearrangement (f^*) and the classical one ($f^\star$),

$$(|f|^*)^\star(t) = f^\star(t), \qquad t > 0,$$

and between the Hardy-Littlewood classical inequality and ours.

The last part is devoted to study conditions on the function g such that we get the saturation of the Hardy-Littlewood inequality, that is: when do we have

$$\sum_{x\in T} f^*(x)g^*(x) = \sup \sum_{x\in T} |f(x)h(x)|\,, \tag{10}$$

where the supremum is taken over all h such that $h^* = g^*$? We will see in the next section that this equality is linked with normablility properties of the Lorentz spaces. In the context of the classical rearrangement, the equality is true if and only if g is decreasing. This is not the case in the tree, as Example 2.4.6 shows.

It turns out that to obtain the saturation, it is fundamental to know what we call the *rearranging transformations* (see Definition 2.4.8). They are the bijections φ_f between the supports of f and f^* that relate the values of the two functions, that is,

$$f(x) = f^*(\varphi_f(x)),$$

for all x in the support of f. In the classical context, a function g is decreasing if and only if it is invariant under the action of the rearranging transformations in the sense that

$$g(\varphi_f(.))^* = g,$$

for all functions f.

As in the classical context, we prove in Theorem 2.4.17, one of our main results in the section, that if g is invariant with respect to all the rearranging transformations, then (10) holds. We will see in the next section that the converse is also true.

Previously, we see in Theorem 2.4.15 that if g is linearly decreasing, then it is invariant with respect to all rearranging transformations. We shall explain that a function is linearly decreasing if it is decreasing with respect to a new total order, introduced in Definition 2.4.10, that depends also on the order of the boundary of the tree. It is trivial to prove that every linearly decreasing function is also a decreasing function in the tree. As a summary, Corollary 2.4.18 says that if g is linearly decreasing, then we obtain the saturation (10).

(2.5) *The Lorentz spaces.* For a weight in the tree u, the Lorentz space $\Delta_T^p(u)$ is the set of functions f such that the functional

$$\|f\|_{\Delta_T^p(u)} = \left(\sum_{x\in T}(f^*(x))^p u(x)\right)^{1/p} \tag{11}$$

is finite.

We prove in Proposition 2.5.5 its completeness, and in Proposition 2.5.6 that they are a generalization of the Lebesgue spaces, since if $u \equiv 1$ then $\Delta_T^p(1) = L^p(T, |.|)$, where $|.|$ is the counting measure. These spaces cannot be rearrangement invariant spaces in the classical sense except for the trivial case that u is constant (Proposition 2.5.7).

Our main interest are the normability properties. Theorem 2.5.9 gives the weights u for which the functional (11) is a quasi-norm. It is more difficult to know when $\Delta_T^p(u)$ equipped with the functional (11) is a Banach space, that is, when (11) is a norm. We show in Lemma 2.5.10 that the weight u must be decreasing in order to get this. But a simple example shows that it is not enough (Example 2.5.11).

Finally, we find in Theorem 2.5.12 that the saturation (10), with $g = u$, is equivalent to saying that (11) is a norm, also to the condition u is linearly decreasing and finally, to have that u is invariant with respect to all the rearranging transformations. This theorem completes the previous results and is the most important in this section.

In the last result, we characterize the embedding

$$\Delta_T^p(u) \hookrightarrow \Delta_T^q(v)$$

in terms of u, v, p and q in Theorem 2.5.15, by using some results of the first chapter.

(2.6) *Finite trees and regular trees.* It is possible to extend our results to a wider class of trees: the regular trees. It is also possible to adapt them to the finite trees. In both cases, the main idea is to embed the regular tree and the finite tree into a homogeneous tree. In the finite case, we see in the final example that we can list the vertices in the boundary of the tree and use it to rearrange, instead of using the order of Definition 2.2.4. Surprisingly, Theorem 2.5.12 is false with this order.

3. Weighted inequalities and the shape of approach regions. We study the weighted boundedness

$$M_\Omega : L^p(u) \longrightarrow L^{p,\infty}(v), \quad p \geq 1,$$

for a maximal operator M_Ω related to a family of approach regions in a general context, and the relations of this boundedness with the shape of the approach family. These are questions introduced by A. Sánchez-Colomer and J. Soria in [SS2] for the Euclidean space $\mathbb{R}^{n+1}_+$, and we want to extend them to more general spaces.

We have structured the chapter following the chronological order of our research. We start by studying in Section 3.2 the above questions for approach families

$$\Omega \subset X_+ := X \times (0, \infty),$$

where (X, μ, d) is a non symmetric space of homogeneous type, denoted by ns-space of homogeneous type (see Definitions 3.2.1 and 3.2.2). In Subsection 3.2.3 we apply the results to the (easier) case of a group structure in X. Later, the study extends to an abstract context in Section 3.3, since many examples of domains do not have a Cartesian structure as X_+. In order to illustrate the way of extension, we make a previous study in Subsection 3.3.1 for the case of the homogeneous trees.

Due to this structure, we obtain similar results, that correspond to the same ideas applied to the different contexts. We think that by writing things this way it is much easier to understand what are the main difficulties we have to overcome in each setting.

We detail the content of each section:
(3.1) *Preliminary results in* $\mathbb{R}^{n+1}_+$. We collect some known results in the Euclidean space. Some of them will be completed and generalized.

(3.2) *The half-space* $X \times (0, \infty)$ *for a space of homogeneous type* X. This section is devoted to the case $X_+ := X \times (0, \infty)$, where X is a space of homogeneous type. We have divided the section in three parts:
(3.2.1) *Definitions and previous results.* We begin by describing the spaces of homogeneous type and the non-symmetric spaces (ns-spaces) of homogeneous type (Definitions 3.2.1 and 3.2.2). We follow with some classic results adapted to the ns-spaces: covering lemma of Vitali type (Lemma 3.2.5), estimates for the Hardy-Littlewood maximal operator (Theorem 3.2.6), decomposition of Whitney type (Theorem 3.2.7) and extension to arbitrary open sets of the Carleson pair condition (Proposition 3.2.9).

We then introduce in Definition 3.2.10 the basic concepts related to a family of approach regions

$$\Omega = \{\Omega(x) : x \in X\},$$

where $\Omega(x)$ is a measurable set in X_+ for all x. We define the maximal operator related to Ω:

$$M_\Omega f(x) = \sup_{(y,t)\in\Omega(x)} \frac{1}{\mu(B(y,t))} \int_{B(y,t)} |f(z)|\, d\mu(z).$$

Theorem 3.2.13 is important: it determines the boundedness

$$M_\Omega : L^p(\nu) \longrightarrow L^{p,\infty}(\rho),$$

for two measures ν and ρ, in terms of a condition of Carleson type.

The A_p class of weights was introduced by B. Muckenhoupt ([M]). It is the set of weights u such that the boundedness

$$M : L^p(u) \longrightarrow L^{p,\infty}(u),$$

is satisfied for the Hardy-Littlewood maximal operator. We define A_p^Ω as the set of weights u such that we have the boundedness

$$M_\Omega : L^p(u) \longrightarrow L^{p,\infty}(u).$$

Finally, $W(\Omega)$ is a set of weights also related to Ω (Definition 3.2.16). A consequence of Theorem 3.2.13 is Theorem 3.2.17, also important, where the equality

$$A_p^\Omega = A_p \cap W(\Omega), \tag{12}$$

is proved.

(3.2.2) *The shape of approach regions.* We answer a natural question in view of (12): when do we have $A_p = A_p^\Omega$? To answer it, we need an invariancy property on the space (X, μ, d): there exists $C > 0$ such that

$$\frac{1}{C}\mu(B(x,r)) \leq \mu(B(y,r)) \leq C\,\mu(B(x,r)),$$

for all x, y and $r > 0$, where $B(x,r)$ is the ball of center x and radius $r > 0$. To this end, we require an extra condition on (X, μ, d): there exists $M > 0$ such that

$$B(x, Mr) \setminus B(x,r) \neq \phi, \tag{13}$$

for all x and $r > 0$.

If the space (X, μ, d) satisfies this condition, then we can replace the quasi-distance d by another non-symmetric quasi-distance δ, so that (X, μ, δ) is a ns-space of homogeneous type. The topology and the classes A_p and A_p^Ω are invariant under this change, and we get that $\mu(B(x, r))$ is comparable to r. These results are shown in Theorem 3.2.21 and Corollary 3.2.25.

Before giving our main result, we need that the approach family satisfies certain regularity condition described in Definition 3.2.28. This condition is not so restrictive as the given examples show.

Finally, we prove the main theorem (Theorem 3.2.30) where we give several characterizations so that equality $A_p = A_p^\Omega$ holds. One of them is that

$$\Omega(x) \subset \Gamma_\theta(x),$$

for certain $\theta > 0$, where $\Gamma_\theta(x)$ is the cone of vertex x and width θ. Thus, analytic conditions on the weights are equivalent to geometric properties of the regions.

(3.2.3) *The case of a group structure: some examples.* In this section we give some examples of spaces of homogeneous type that have a group structure, and we apply Theorem 3.2.30 to each case.

The examples are:

(i) $X = \mathbb{R}^n$ with the Lebesgue measure and the non-isotropic distance

$$d(x, y) = \sum_{k=1}^{n} |x_k - y_k|^{1/a_k}.$$

(ii) $X = \mathbf{H}_\mathrm{n}$ is the Heisenberg group, with the Lebesgue measure in $\mathbb{C}^n \times \mathbb{R}$, with the distance $d(x, y) = ||y^{-1} \cdot x||$, where $||[\zeta, t]|| = \max(|\zeta|^2, |t|)$.

(iii) X is the set of real upper triangular 3×3 matrices with ones along the diagonal. The quasi-distance is $d'(x, y) = (d(x, y) + d(y, x))/2$, where $d(x, y) = ||y^{-1} \cdot x||$ and $||x|| = \max\{|a|, |b|^{1/2}, |c|\}$. The measure is the Lebesgue measure in $\mathbb{R}^3$.

(iv) A nilpotent unimodular group, with the Riemannian metric and the induced measure.

(3.3) *The general case.* Many interesting examples of domains do not occur as the product space X_+. We solve here the same type of questions but in an abstract context.

(3.3.1) *Approach regions in an isotropic homogeneous tree.* We develop the theory in this new setting. Now, the tree plays the role of X_+ and its boundary the one of X. In this case, all the balls in the boundary with the same radius have the same measure. Theorem 3.3.8 is the analog of Theorem 3.2.30.

(3.3.2) *Approach regions in the abstract context.* We use the techniques introduced by F. Di Biase [DiB]. We first define the general structure in Definition 3.3.9, and we substitute the notion of a cone by the one of a supernatural family (Definition 3.3.10). Basically, Γ is supernatural if its shadow for ζ,

$$\Gamma^{\downarrow}(\zeta) = \{x \in X : \zeta \in \Gamma(x)\},$$

is comparable to a ball, and for every ball there exists a comparable shadow.

We define the maximal operator for an approach family Ω and a fixed supernatural family Γ:

$$M_\Omega f(x) = \sup_{\zeta \in \Omega(x)} \frac{1}{\mu(\Gamma^{\downarrow}(\zeta))} \int_{\Gamma^{\downarrow}(\zeta)} |f(y)| \, d\mu(y).$$

We also obtain that $A_p^{\Omega} = A_p \cap W(\Omega)$ in Theorem 3.3.15, with the adapted definitions for this case.

As we did before, we can replace the homogeneous type space (X, μ, d) by the ns-space of homogeneous type (X, μ, δ), without loss of generality (Theorem 3.3.16), under condition (13) on (X, μ, d).

The analog of Theorem 3.2.30 is Theorem 3.3.18, which is the most general result on this subject.

(3.4) *Back to Euclidean spaces: two applications.* In this last section, we apply the techniques used in this chapter to complete some results in $\mathbb{R}^n$ appeared in [FJR] and [RS].

(3.4.1) *Singular integral operators.* We study the boundedness of the operator

$$N_\Omega f(x) = \sup_{(y,t) \in \Omega(x)} |(K_t * f)(y)|,$$

defined for an approach family $\Omega \subset \mathbb{R}^{n+1}_+$ and a Calderón-Zygmund kernel. We obtain the complete characterization of

$$N_\Omega : L^p(m) \longrightarrow L^{p,\infty}(\rho),$$

if $p \geq 1$, m is the Lebesgue measure and ρ is an arbitrary measure, when $K_j(x) = \omega_j(x)/|x|^n$, with $\omega_j(x) = x_j/|x_j|$, for $1 \leq j \leq n$, the Riesz kernel (Theorem 3.4.2).

For the case of a general Calderón-Zygmund operator, we find a partial result in Theorem 3.4.4.

(3.4.2) *Potential spaces.* Here, we give estimates for the operator

$$N_\Omega f(x) = \sup_{(y,t)\in\Omega(x)} |(P_t * f)(y)|,$$

where P is the Poisson kernel and Ω is an approach family in $\mathbb{R}^{n+1}_+$. In particular, we characterize the boundedness

$$N_\Omega : L^p_k(m) \longrightarrow L^{p,\infty}(\rho),$$

where

$$L^p_k(m) = \{f : \ f = F * k, \text{ for some } F \in L^p(m)\}$$

is a potential space. The main result is Theorem 3.4.6.

Chapter 1

Embeddings and operators on weighted spaces of monotone functions

Let u and v be two weights in a measure space (X,μ), where X is an ordered set, non-necessarily totally ordered. We denote the order in X by $x \le y$ for $x, y \in X$. A positive function $f : X \longrightarrow [0,\infty)$ is said to be **decreasing (increasing)** if $f(x) \le f(y)$ whenever $x \ge y$ ($x \le y$). Recently, there is an incoming interest in obtaining necessary and sufficient conditions on the weights u and v so that there exists a constant $C > 0$ such that the inequality

$$\left(\int_X f(x)^q\, v(x)\, d\mu(x)\right)^{1/q} \le C \left(\int_X f(x)^p\, u(x)\, d\mu(x)\right)^{1/p}, \tag{1.1}$$

holds for all decreasing (increasing) functions f in X, for $0 < p, q < \infty$. If we denote by $L^p_{\mathrm{dec}}(u)$ the set of positive decreasing functions in the weighted Lebesgue space $L^p(u)$, inequality (1.1) is equivalent to the embedding

$$L^p_{\mathrm{dec}}(u) \hookrightarrow L^q_{\mathrm{dec}}(v).$$

If $X = [0,\infty)$ and μ is the Lebesgue measure, the embedding of decreasing functions is closely related to the boundedness of the Hardy operator as well as to the corresponding embeddings for Lorentz spaces. Some results on these questions are due to E. Sawyer ([S2]) and M.J. Carro, L. Pick, J. Soria and V.D. Stepanov ([CPSoSte]). Other results for monotone functions in this setting are in [CSo2] ($0 < p \le q < \infty$) and [Ste] ($0 < p, q < \infty$). If $X = \mathbb{R}^n_+ := \mathbb{R}_+ \times \overset{(n)}{\cdots} \times \mathbb{R}_+$ equipped with the order defined in (1.7) and μ is the Lebesgue measure, further results can be found in the work of S. Barza, L.E. Persson and J. Soria ([BPSo1], $0 < p \le q < \infty$) and S. Barza, L.E. Persson and V.D. Stepanov ([BPSte], $0 < p, q < \infty$). For the discrete case $X = \mathbb{Z}$ and μ the counting measure, see the paper of H.P. Heinig and A. Kufner ([HK]). For results in a general setting (X, μ) and $0 < p \leq q < \infty$, see the work of J.A. Raposo ([R]) and of M.J. Carro, J.A. Raposo and J. Soria ([CRSo]).

Section 1.1 will be devoted to the study of inequality (1.1) in the general setting of a measure space (X, μ), and in all the range $0 < p, q < \infty$. Our main idea is to begin with the characteritzation of this inequality for the particular case $X = \mathbb{Z}$ and μ the counting measure (which is done in Theorems 1.1.4 and 1.1.6), and to use it to give an answer in the general setting (see Theorem 1.1.10), by using a discretization technique. We finally apply our main result to get a new characterization in the case $X = \mathbb{R}^n_+$, equivalent to the ones given in [BPSte], but with a simpler proof (this is Theorem 1.1.13).

The Hardy operator is defined for any measurable function f on $[0, \infty)$ by

$$Af(x) = \frac{1}{x} \int_0^x f(t)\, dt, \qquad x > 0.$$

The boundedness of the Hardy operator

$$A : L^p_{\text{dec}}(u) \longrightarrow L^q(v), \tag{1.2}$$

is equivalent to the boundedness of the Hardy-Littlewood maximal operator

$$M : \Lambda^p(u) \longrightarrow \Lambda^q(v),$$

where $\Lambda^p(u)$ is the Lorentz space (see Chapter 2 for the definition of this space). In 1990, M.A. Ariño and B. Muckenhoupt ([AM]) characterized the weights for which (1.2) holds in the case $u = v$ and $1 < p = q$. These weights are called the B_p weights. For further results on this question, see [A], [S2], [CSo1], [CSo2], [CSo3], [HLa], [N], [Ste], [HK], [R] and [CRSo]. In [R], the author studied the boundedness (1.2) of the discrete Hardy operator defined for positive sequences indexed over $\mathbb{N}$ and showed that the discrete weights for which this holds form, in some sense, a subclass of the classical B_p weights (see Theorem 1.2.2). In Section 1.2, we will show that we can reverse this process, that is, we can construct a discrete weight for which (1.2) holds for the discrete Hardy operator, for every classical B_p weight (see Lemma 1.2.3). However, we will give an example to see that this correspondence between discrete and non-discrete weights is not one to one. Finally, we will consider the case of the discrete Hardy operator defined for sequences indexed in $\mathbb{Z}$. We will give results in two directions: Theorems 1.2.4 and 1.2.6 say that, for every B_p (or $B_{p,\infty}$) weight, we can construct a discrete weight satisfying the boundedness (1.2) (or the weak boundedness); Theorems 1.2.8 and 1.2.9 say that, for every discrete weight satisfying (1.2) (or the weak boudedness), we can give a classical B_p (or $B_{p,\infty}$) weight. As a consequence, we prove that in that case, B_p can be viewed as the set of weights for which the discretized weights satisfy (1.2) (see Corollary 1.2.5).

We use in Section 1.2 some results of embeddings of ℓ^p_{dec} spaces given in the first section.

The Hardy operator is an example of operator that preserves the monotonicity of the function, in the sense that if f is a positive decreasing function, then Af is also decreasing. On the other hand, the Hardy-Volterra operator (also called Hardy or Volterra operator) defined by

$$Vf(x) = \int_0^x f(t)\,dt, \tag{1.3}$$

for a measurable function f, satisfies that Vf is an increasing function whenever f is positive. We are interested in studying the weak boundedness of general operators

$$T : L^p(u) \longrightarrow L^{q,\infty}(v),$$

that satisfy some growth properties of the type described above, defined in a general measure space (X,μ). Here, $L^{q,\infty}(v)$ is the weak-$L^q(v)$ space, that is, the set of μ-measurable functions for which the functional

$$\|f\|_{L^{q,\infty}(v)} = \sup_{t>0} t\left(\int_{\{|f|>t\}} v(x)\,d\mu(x)\right)^q, \tag{1.4}$$

is finite. This is the aim of Section 3. In order to obtain our results, we begin by giving a more geometrical expression of the $L^{q,\infty}$-'norm' of a measurable function (Lemma 1.3.1), and a simple consequence of this is the characterization of the weak boundedness of general operators satisfying some growth conditions (Theorems 1.3.3 and 1.3.4). We then introduce some applications of this theorems to concrete settings like trees, metric trees, $\mathbb{R}_+$ and $\mathbb{R}^2_+$. In the case of trees and metric trees, we exploit the same ideas from Sections 1.1 and 1.2, giving results on equivalences between the discrete and the non-discrete context (see Theorems 1.3.14 and 1.3.15).

We will denote $f\downarrow$ ($f\uparrow$) whenever f is a decreasing (increasing) function defined in a ordered space X, and $D\downarrow$ ($D\uparrow$) for a set $D\subset X$ if χ_D is decreasing (increasing).

For a weight $u : X \longrightarrow \mathbb{R}_+$, we denote

$$U(E) = \int_E u(x)\,d\mu(x),$$

where μ is the ambient measure. In the case of $\mathbb{R}_+$, for a weight $u : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, we use the notation

$$U(x) = \int_0^x u(t)\,dt.$$

We will always write $r = pq/(p-q)$ in the case $0 < q < p < \infty$. Two positive quantities A and B are said to be equivalent ($A \approx B$) if there exists a constant $C > 1$ such that $C^{-1}A \le B \le CA$. If only $B \le CA$, we write $B \lesssim A$. The undetermined cases $0 \cdot \infty$, $\frac{\infty}{\infty}$, $\frac{0}{0}$, will always be taken equal to 0.

1.1 Embeddings of weighted spaces of monotone functions

We will first characterize the inequality (1.1) in the case $X = \mathbb{Z}$ and μ the counting measure, for all $0 < p, q < \infty$. We then call a powerful and useful idea to get the characterization of the same inequality in the general case of a measure space (X, μ): we take an appropriate discretization of the set X, and we reduce the problem to the discrete setting of $\mathbb{Z}$, where we have given an answer yet. We begin with some known results about the inequality.

In 1990, E. Sawyer characterizes the inequality (1.1) in the case $X = [0, \infty)$, μ the Lebesgue measure, and $0 < q < p < \infty$. The next theorem states his result.

Theorem 1.1.1 (Sawyer) *If $0 < q < p < \infty$, then:*

$$\sup_{0 \le f\downarrow} \frac{\left(\displaystyle\int_0^\infty f(x)^q\, v(x)\,dx\right)^{1/q}}{\left(\displaystyle\int_0^\infty f(x)^p\, u(x)\,dx\right)^{1/p}} \approx \left(\int_0^\infty V(x)^{r/p} U(x)^{-r/p} v(x)\,dx\right)^{1/r}$$

$$\approx \left(\int_0^\infty V(x)^{r/q} U(x)^{-r/q} u(x)\,dx\right)^{1/r} + \frac{V(\infty)^{1/q}}{U(\infty)^{1/p}},$$

where $U(\infty) = \displaystyle\int_0^\infty u(x)\,dx$, and analogously for $V(\infty)$.

For the range $0 < p \le q < \infty$, the solution is due to M.J. Carro and J. Soria ([CSo1]) or V. Stepanov ([Ste]). In this case, the result is sharp.

Theorem 1.1.2 (Carro-Soria, Stepanov) *If $0 < p \le q < \infty$, then:*

$$\sup_{0\le f\downarrow} \frac{\left(\int_0^\infty f(x)^q\, v(x)\, dx\right)^{1/q}}{\left(\int_0^\infty f(x)^p\, u(x)\, dx\right)^{1/p}} = \sup_{t>0} \frac{V(t)^{1/q}}{U(t)^{1/p}}.$$

We mention a more general result in this range, due to J.A. Raposo ([R]). A set L of measurable functions in a σ-finite measure space (X,μ) is a *regular class of functions* if for all $f \in L$, we have:

(i) $|kf| \in L$, if $k \in \mathbb{R}$.

(ii) $\chi_{\{|f|>\lambda\}} \in L$, for all $\lambda > 0$.

(iii) There exists a sequence of simple functions $\{f_n : n \in \mathbb{N}\} \subset L$ such that $0 \le f_n(x) \le f_{n+1}(x) \longrightarrow |f(x)|$ $\mu-$a.e. $x \in X$.

The set of decreasing (increasing) functions in X is a regular class.

Theorem 1.1.3 (Raposo) *If $0 < p \le q < \infty$ and L is a regular class of functions, then*

$$\sup_{0\le f\in L} \frac{\left(\int_X f(x)^q\, v(x)\, d\mu(x)\right)^{1/q}}{\left(\int_X f(x)^p\, u(x)\, d\mu(x)\right)^{1/p}} = \sup_{\chi_B\in L} \frac{\left(\int_B v(x)\, d\mu(x)\right)^{1/q}}{\left(\int_B u(x)\, d\mu(x)\right)^{1/p}}.$$

Our first goal is to consider the inequality (1.1) for decreasing sequences in $X = \mathbb{Z}$, that is,

$$\left(\sum_{k=-\infty}^{\infty} a_k^q\, v_k\right)^{1/q} \le C \left(\sum_{k=-\infty}^{\infty} a_k^p\, u_k\right)^{1/p},$$

for all decreasing sequences $\{a_k : k \in \mathbb{Z}\}$, where $\{u_k : k \in \mathbb{Z}\}$ and $\{v_k : k \in \mathbb{Z}\}$ are positive sequences in $\mathbb{Z}$.

We use the notation $(a_k)_k \downarrow$ for a decreasing sequence $\{a_k : k \in \mathbb{Z}\}$, and

$$U_k = \sum_{j=-\infty}^{k} u_j,$$

for a weight $\{u_k : k \in \mathbb{Z}\}$. The preceding theorem of J.A. Raposo solves the problem in the range $0 < p \le q < \infty$. We observe that χ_B is a decreasing function if and only if $B = (\ldots, 1, 1, 1, 0, 0, 0, \ldots)$.

Theorem 1.1.4 *If $0 < p \leq q < \infty$, then*

$$\sup_{0\leq (a_k)\downarrow} \frac{\left(\sum_{k\in\mathbb{Z}} a_k^q\, v_k\right)^{1/q}}{\left(\sum_{k\in\mathbb{Z}} a_k^p\, u_k\right)^{1/p}} = \sup_{k\in\mathbb{Z}} \frac{V_k^{1/q}}{U_k^{1/p}}.$$

In the range $0 < q < p < \infty$, the case of decreasing sequences in $\mathbb{Z}$ does not seem to be known in the literature, except for the result of H.P. Heinig and A. Kufner ([HK]), where extra hypothesis on the weights are needed. To solve the problem in its full generality, we consider the next proposition. We see that the required discrete embedding is equivalent to a continuous embedding.

Proposition 1.1.5 *If $0 < q < p < \infty$, then*

$$A = \sup_{0\leq (a_k)\downarrow} \frac{\left(\sum_{k=-\infty}^{\infty} a_k^q\, v_k\right)^{1/q}}{\left(\sum_{k=-\infty}^{\infty} a_k^p\, u_k\right)^{1/p}} \approx \sup_{0\leq f\downarrow} \frac{\left(\int_0^\infty f(x)^q\, \widetilde{v}(x)\, dx\right)^{1/q}}{\left(\int_0^\infty f(x)^p\, \widetilde{u}(x)\, dx\right)^{1/p}} = B,$$

where $\widetilde{v}(x) = \sum_{k=-\infty}^{\infty} \frac{v_k}{2^k} \chi_{[2^k,2^{k+1})}(x)$ *and* $\widetilde{u}(x) = \sum_{k=-\infty}^{\infty} \frac{u_k}{2^k} \chi_{[2^k,2^{k+1})}(x)$*. Moreover,*

$$A \leq B \leq \left(\frac{2^{r/q}-1}{r/q}\right)^{q/r} A.$$

Proof. For a given positive decreasing sequence $(a_k)_k$, we define the decreasing function

$$f(x) = \sum_{k\in\mathbb{Z}} a_k \chi_{[2^k,2^{k+1})}(x).$$

Then we have:

$$\int_0^\infty f(x)^q\, \widetilde{v}(x)\, dx = \sum_{k\in\mathbb{Z}} \frac{v_k}{2^k} \int_{2^k}^{2^{k+1}} f(x)^q\, dx = \sum_{k\in\mathbb{Z}} a_k^q\, v_k,$$

and analogously

$$\int_0^\infty f(x)^p\, \widetilde{u}(x)\, dx = \sum_{k\in\mathbb{Z}} a_k^p\, u_k,$$

and thus, $A \leq B$.

For a positive decreasing function f, we define the positive and decreasing sequence

$$a_k = \left(\int_{2^k}^{2^{k+1}} f(x)^p \frac{dx}{x} \right)^{1/p}.$$

Therefore,

$$\begin{aligned} \sum_{k\in\mathbb{Z}} a_k^p \, u_k &= \sum_{k\in\mathbb{Z}} u_k \left(\int_{2^k}^{2^{k+1}} f(x)^p \frac{dx}{x} \right) \\ &\leq \sum_{k\in\mathbb{Z}} \frac{u_k}{2^k} \left(\int_{2^k}^{2^{k+1}} f(x)^p \, dx \right) = \int_0^\infty f(x)^p \, \widetilde{u}(x) \, dx, \end{aligned}$$

and applying Hölder's inequality with $p/q > 1$ and since $r = pq/(p-q)$,

$$\begin{aligned} \int_0^\infty f(x)^q \, \widetilde{v}(x) \, dx &= \sum_{k\in\mathbb{Z}} \frac{v_k}{2^k} \left(\int_{2^k}^{2^{k+1}} f(x)^q \, dx \right) \\ &\leq \sum_{k\in\mathbb{Z}} \frac{v_k}{2^k} \left(\int_{2^k}^{2^{k+1}} f(x)^p \, \frac{dx}{x} \right)^{q/p} \left(\int_{2^k}^{2^{k+1}} x^{r/q-1} \, dx \right)^{q/r} \\ &= \left(\frac{2^{r/q}-1}{r/q} \right)^{q/r} \left(\sum_{k\in\mathbb{Z}} a_k^q \, v_k \right), \end{aligned}$$

and this is $B \leq \left(\frac{2^{r/q}-1}{r/q} \right)^{q/r} A$.

□

We now use this proposition and Theorem 1.1.1 to characterize the embedding in the desired range.

Theorem 1.1.6 *If $0 < q < p < \infty$, then*

$$\begin{aligned} \sup_{0\leq(a_k)\downarrow} \frac{\left(\sum_{k\in\mathbb{Z}} a_k^q \, v_k \right)^{1/q}}{\left(\sum_{k\in\mathbb{Z}} a_k^p \, u_k \right)^{1/p}} &\approx \left(\int_0^1 \left[\sum_{k\in\mathbb{Z}} \left(\frac{V_{k-1} + v_k t}{U_{k-1} + u_k t} \right)^{r/p} v_k \right] dt \right)^{1/r} \\ &\approx \left(\int_0^1 \left[\sum_{k\in\mathbb{Z}} \left(\frac{V_{k-1} + v_k t}{U_{k-1} + u_k t} \right)^{r/q} u_k \right] dt \right)^{1/r} + \frac{V_\infty^{1/q}}{U_\infty^{1/p}}, \end{aligned}$$

where $U_\infty = \sum_{k=-\infty}^{\infty} u_k$, and analogously for V_∞.

Proof. By the previous proposition and Theorem 1.1.1, we know that

$$\begin{aligned}
\sup_{0\le(a_k)\downarrow} \frac{\left(\sum_{k\in\mathbb{Z}} a_k^q\, v_k\right)^{1/q}}{\left(\sum_{k\in\mathbb{Z}} a_k^p\, u_k\right)^{1/p}} &\approx \left(\int_0^\infty \widetilde{V}(x)^{r/p}\, \widetilde{U}(x)^{-r/p}\, \widetilde{v}(x)\, dx\right)^{1/r} \\
&\approx \left(\int_0^\infty \widetilde{V}(x)^{r/q}\, \widetilde{U}(x)^{-r/q}\, \widetilde{u}(x)\, dx\right)^{1/r} + \frac{\widetilde{V}(\infty)^{1/q}}{\widetilde{U}(\infty)^{1/p}},
\end{aligned}$$

where $\widetilde{v}(x) = \sum_{k=-\infty}^{\infty} \frac{v_k}{2^k}\chi_{[2^k,2^{k+1})}(x)$ and $\widetilde{u}(x) = \sum_{k=-\infty}^{\infty} \frac{u_k}{2^k}\chi_{[2^k,2^{k+1})}(x)$. First, observe that if $2^k \le x < 2^{k+1}$ then

$$\begin{aligned}
\widetilde{V}(x) &= \sum_{j=-\infty}^{k-1} \int_{2^j}^{2^{j+1}} \widetilde{v}(t)\, dt + \int_{2^k}^{x} \widetilde{v}(t)\, dt = \sum_{j=-\infty}^{k-1} \frac{v_j}{2^j} \int_{2^j}^{2^{j+1}} dt + \frac{v_k}{2^k}\int_{2^k}^{x} dt \\
&= \sum_{j=-\infty}^{k-1} v_j + v_k \frac{x-2^k}{2^k} = V_{k-1} + v_k \frac{x-2^k}{2^k},
\end{aligned}$$

and the same for $\widetilde{U}$. Now, splitting the integral into dyadic intervals, we get:

$$\begin{aligned}
&\int_0^\infty \widetilde{V}(x)^{r/p}\, \widetilde{U}(x)^{-r/p}\, \widetilde{v}(x)\, dx \\
&= \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \widetilde{V}(x)^{r/p}\, \widetilde{U}(x)^{-r/p}\, \widetilde{v}(x)\, dx \\
&= \sum_{k\in\mathbb{Z}} \frac{v_k}{2^k} \int_{2^k}^{2^{k+1}} \left(V_{k-1} + v_k \frac{x-2^k}{2^k}\right)^{r/p} \left(U_{k-1} + u_k \frac{x-2^k}{2^k}\right)^{-r/p} dx \\
&= \int_0^1 \left(\sum_{k\in\mathbb{Z}} (V_{k-1} + v_k t)^{r/p}\, (U_{k-1} + u_k t)^{-r/p}\, v_k\right) dt,
\end{aligned}$$

where the last equality follows by the change of variable $t = \frac{x-2^k}{2^k}$ in each integral. The other equivalence is analogous.

$\square$

We now deal with the problem in the general setting of a measure space (X, μ), where X is an ordered set.

Our results are based on a discretization technique which shows that the embedding (1.1) is equivalent to a collection of embeddings of sequences in $\mathbb{Z}$. This technique was pointed out by E. Sawyer in [S1], and has been also used in [BP], [BPSte], [HLa] and [L]. In our case, the results are given in terms of covering sequences of decreasing sets in X. We say that set $D \subset X$ is **decreasing** if and only if the function χ_D is decreasing. In what follows, we will assume that every decreasing set is μ-measurable.

Definition 1.1.7 A collection of sets $\{D_k : k \in \mathbb{Z}\}$ is a **covering family of decreasing sets** for the set X if:

- $D_k \subset D_{k+1}$ for all $k \in \mathbb{Z}$.
- $\displaystyle\bigcup_{k\in\mathbb{Z}} D_k = X$.

The set of all covering families of decreasing sets in X is denoted by $\mathcal{D}(X)$ or simply $\mathcal{D}$ if there is no possible confusion. For a fixed family $\{D_k : k \in \mathbb{Z}\}$ in $\mathcal{D}(X)$, we denote

$$\Delta_k = D_{k+1} \setminus D_k.$$

We now present a lemma that can be found in a slightly different version in [HLa] for the case $X = [0,\infty)$ and in [BP] for $X = \mathbb{R}^n_+$. In our case, we do not require any additional condition neither on the covering family of decreasing sets nor in the modular functions. Recall that a **modular function** P is a positive and increasing function $P : [0,\infty) \longrightarrow [0,\infty)$ such that $P(0) = 0$ and $P(\infty) = \infty$. We will use the subsequent corollary for our purpose.

Lemma 1.1.8 *Let (X,μ) be a measure space, where X is an ordered set. Let Q and P be two modular functions. Let A be the infimum of the constants $C > 0$ such that the inequality*

$$Q^{-1}\Big(\int_X Q(f(x))\, v(x)\, d\mu(x)\Big) \le P^{-1}\Big(\int_X P(Cf(x))\, u(x)\, d\mu(x)\Big),$$

holds for all $0 \le f \downarrow$, and let B be the infimum of the constants $C > 0$ such that the inequality

$$Q^{-1}\Big(\sum_{k\in\mathbb{Z}} Q(\delta_k) \int_{\Delta_k} v(x)\, d\mu(x)\Big) \le P^{-1}\Big(\sum_{k\in\mathbb{Z}} P(C\delta_k) \int_{\Delta_k} u(x)\, d\mu(x)\Big),$$

holds for all $0 \le (\delta_k)_k \downarrow$ and for all $\{D_k\} \subset \mathcal{D}$. Then A=B.

Proof. For every positive decreasing sequence $(\delta_k)_k$ and every family $\{D_k : k \in \mathbb{Z}\}$, if we consider the decreasing function

$$f(x) = \sum_{k\in\mathbb{Z}} \delta_k \chi_{\Delta_k}(x),$$

we easily get that $A \leq B$.

Take a positive decreasing function f such that

$$Q^{-1}\left(\int_X Q(f(x))\, v(x)\, d\mu(x)\right) \leq P^{-1}\left(\int_X P((A+\varepsilon)f(x))\, u(x)\, d\mu(x)\right),$$

for a fixed $\varepsilon > 0$. Take $c > 1$ and define $\delta_k = c^{-k}$ and

$$\Delta_k = \left\{x \in X : c^{-k-1} < f(x) \leq c^{-k}\right\},$$

for all $k \in \mathbb{Z}$. Now, by using that P, Q, P^{-1} and Q^{-1} are increasing functions and our hypothesis, we have

$$\begin{aligned}
Q^{-1}\left(\sum_k Q(c^{-k-1}) \int_{\Delta_k} v(x)\, d\mu(x)\right) &\leq Q^{-1}\left(\int_X Q(f(x))\, v(x)\, d\mu(x)\right) \\
&\leq P^{-1}\left(\sum_k \int_{\Delta_k} P((A+\varepsilon)f(x))\, u(x)\, d\mu(x)\right) \\
&\leq P^{-1}\left(\sum_k P((A+\varepsilon)c^{-k}) \int_{\Delta_k} u(x)\, d\mu(x)\right),
\end{aligned}$$

and this is $B \leq c(A+\varepsilon)$. Letting $\varepsilon \to 0^+$ and $c \to 1^+$, we obtain $B \leq A$.

□

Corollary 1.1.9 *If $P(t) = t^p$ and $Q(t) = t^q$, then the previous lemma reads as*

$$\sup_{0\leq f\downarrow} \frac{\left(\int_X f(x)^q\, v(x)\, d\mu(x)\right)^{1/q}}{\left(\int_X f(x)^p\, u(x)\, d\mu(x)\right)^{1/p}} = \sup_{\{D_k\}\subset\mathcal{D}} \sup_{0\leq(\delta_k)\downarrow} \frac{\left(\sum_{k\in\mathbb{Z}} \delta_k^q\, v_k\right)^{1/q}}{\left(\sum_{k\in\mathbb{Z}} \delta_k^p\, u_k\right)^{1/p}}, \tag{1.5}$$

where we are using the notation

$$\begin{aligned}
u_k &:= \int_{\Delta_k} u(x)\, d\mu(x) = U(\Delta_k), \\
v_k &:= \int_{\Delta_k} v(x)\, d\mu(x) = V(\Delta_k).
\end{aligned}$$

We are ready to prove our main result of this section.

Theorem 1.1.10 *Let (X,μ) be a measure space, where X is an ordered set. For $0<p,q<\infty$, we have:*
(a) If $0<p\le q<\infty$, then

$$\sup_{0\le f\downarrow}\frac{\left(\displaystyle\int_X f(x)^q\, v(x)\, d\mu(x)\right)^{1/q}}{\left(\displaystyle\int_X f(x)^p\, u(x)\, d\mu(x)\right)^{1/p}}=\sup_{D\downarrow}\frac{V(D)^{1/q}}{U(D)^{1/p}}.$$

(b) If $0<q<p<\infty$, then the following conditions are equivalent:

(i) There exists $C>0$ such that

$$\left(\int_X f(x)^q\, v(x)\, d\mu(x)\right)^{1/q}\le C\left(\int_X f(x)^p\, u(x)\, d\mu(x)\right)^{1/p},$$

for all positive decreasing functions f.

(ii) There exists $C>0$ such that

$$\left(\int_0^1\left[\sum_{k\in\mathbb{Z}}\left(\frac{V(D_k)+V(\Delta_k)t}{U(D_k)+U(\Delta_k)t}\right)^{r/p}V(\Delta_k)\right]dt\right)^{1/r}\le C,$$

for all $\{D_k\}\subset\mathcal{D}$.

(iii) There exists $C>0$ such that

$$\left(\int_0^1\left[\sum_{k\in\mathbb{Z}}\left(\frac{V(D_k)+V(\Delta_k)t}{U(D_k)+U(\Delta_k)t}\right)^{r/q}U(\Delta_k)\right]dt\right)^{1/r}+\frac{V(X)^{1/q}}{U(X)^{1/p}}\le C,$$

for all $\{D_k\}\subset\mathcal{D}$.

Proof. We first observe that, in the case (a), we can apply Theorem 1.1.4 and (1.5) to characterize the embedding for sequences in $\mathbb{Z}$, and thus we get

$$\sup_{0\le f\downarrow}\frac{\left(\displaystyle\int_X f(x)^q\, v(x)\, d\mu(x)\right)^{1/q}}{\left(\displaystyle\int_X f(x)^p\, u(x)\, d\mu(x)\right)^{1/p}}=\sup_{\{D_k\}\subset\mathcal{D}}\sup_{k\in\mathbb{Z}}\frac{V_k^{1/q}}{U_k^{1/p}}.$$

Now, we observe that

$$U_k=\sum_{j=-\infty}^{k}u_j=\sum_{j=-\infty}^{k}U(\Delta_j)=U(D_{k+1}), \tag{1.6}$$

and similarly for V_k, and therefore we get

$$\sup_{0\le f\downarrow} \frac{\left(\int_X f(x)^q\, v(x)\, d\mu(x)\right)^{1/q}}{\left(\int_X f(x)^p\, u(x)\, d\mu(x)\right)^{1/p}} = \sup_{\{D_k\}\subset\mathcal{D}} \sup_{k\in\mathbb{Z}} \frac{V(D_k)^{1/q}}{U(D_k)^{1/p}},$$

and the right hand side of this equality is trivially equal to

$$\sup_{D\downarrow} \frac{V(D)^{1/q}}{U(D)^{1/p}}.$$

For the case (b), we return to (1.5) and we apply Theorem 1.1.6 getting

$$\begin{aligned} &\sup_{0\le f\downarrow} \quad \frac{\left(\int_X f(x)^q\, v(x)\, d\mu(x)\right)^{1/q}}{\left(\int_X f(x)^p\, u(x)\, d\mu(x)\right)^{1/p}} \\ \approx\ &\sup_{\{D_k\}\subset\mathcal{D}} \quad \left(\int_0^1 \left[\sum_{k\in\mathbb{Z}} \left(\frac{V_{k-1}+v_k t}{U_{k-1}+u_k t}\right)^{r/p} v_k\right] dt\right)^{1/r} \\ \approx\ &\sup_{\{D_k\}\subset\mathcal{D}} \quad \left(\left(\int_0^1 \left[\sum_{k\in\mathbb{Z}} \left(\frac{V_{k-1}+v_k t}{U_{k-1}+u_k t}\right)^{r/q} u_k\right] dt\right)^{1/r} + \frac{V_\infty^{1/q}}{U_\infty^{1/p}}\right), \end{aligned}$$

and we finally get the characterizations (ii) and (iii) if we observe that $U_{k-1} = U(D_k)$ and $V_{k-1} = V(D_k)$ by considering (1.6).

□

Remark 1.1.11 Part (a) was already proved by J.A. Raposo (see Theorem 1.1.3).

Consider the particular case $X = \mathbb{R}^n_+$ equipped with the order defined by

$$(a_1, a_2, \ldots, a_n) \le (b_1, b_2, \ldots, b_n) \tag{1.7}$$

if and only if $a_i \le b_i$ for $i = 1, \ldots, n$. Let μ be the Lebesgue measure. The following result is the characterization of the inequality (1.1) for $0 < q < p < \infty$ in this setting, due to S. Barza, L.E. Persson and V.D. Stepanov ([BPSte]) (we observe that our notation is different from the one used in that paper):

Theorem 1.1.12 (Barza-Persson-Stepanov) *For $0 < q < p < \infty$, the following conditions are equivalent:*

(i) There exists a constant $C \geq 0$ such that

$$\left(\int_{\mathbb{R}^n_+} f(x)^q\, v(x)\, dx\right)^{1/q} \leq C \left(\int_{\mathbb{R}^n_+} f(x)^p\, u(x)\, dx\right)^{1/p},$$

for all $0 \leq f \downarrow$.

(ii) There exists a constant $C \geq 0$ such that

$$\left(\int_0^\infty U(D_{f,t})^{-r/p}\, d\Big[-V(D_{f,t})^{r/q}\Big]\right)^{1/r} \leq C,$$

for all $0 \leq f \downarrow$, where $D_{f,t} = \{x : f(x) > t\}$.

(iii) There exists a constant $C \geq 0$ such that

$$\left(\sum_{k\in\mathbb{Z}} V(\Delta_k)^{r/q}\, U(D_{k+1})^{-r/p}\right)^{1/r} \leq C,$$

for all families $\{D_k : k \in \mathbb{Z}\}$ in $\mathcal{D}(\mathbb{R}^n_+)$.

We now relate our characterization with this theorem. We point out that our proof is simpler than the one in [BPSte].

Theorem 1.1.13 *For $0 < q < p < \infty$, the following conditions are equivalent:*

(i) There exists a constant $C > 0$ such that

$$\left(\int_{\mathbb{R}^n_+} f(x)^q\, v(x)\, dx\right)^{1/q} \leq C \left(\int_{\mathbb{R}^n_+} f(x)^p\, u(x)\, dx\right)^{1/p},$$

for all $0 \leq f \downarrow$.

(ii) There exists a constant $C > 0$ such that

$$\left(\int_0^1 \sum_{k\in\mathbb{Z}} \left[\frac{V(D_k) + V(\Delta_k)t}{U(D_k) + U(\Delta_k)t}\right]^{r/p} V(\Delta_k)\, dt\right)^{1/r} \leq C,$$

for all families $\{D_k : k \in \mathbb{Z}\}$ in $\mathcal{D}(\mathbb{R}^n_+)$.

(iii) There exists a constant $C > 0$ such that

$$\left(\int_0^\infty U(D_{f,t})^{-r/p}\, d\Big[-V(D_{f,t})^{r/q}\Big]\right)^{1/r} \leq C,$$

for all $0 \leq f \downarrow$, where $D_{f,t} = \{x : f(x) > t\}$.

(iv) There exists a constant $C>0$ such that

$$\left(\sum_{k\in\mathbb{Z}} V(\Delta_k)^{r/q}\, U(D_{k+1})^{-r/p}\right)^{1/r} \leq C,$$

for all families $\{D_k : k\in\mathbb{Z}\}$ in $\mathcal{D}(\mathbb{R}^n_+)$.

Proof. The equivalence between (i) and (ii) is Theorem 1.1.6. Let us see that (ii) implies (iii). Given a function $0\leq f\downarrow$, we assume that $U(D_{f,t})$ is continuous and strictly decreasing in t. The general case follows by a standard limiting argument. We define a positive decreasing sequence $(t_k)_k$ as follows: fix $t_0=1$ and

$$\begin{aligned} t_k &= \sup\left\{t : U(D_{f,t}) = 2\,U(D_{f,t_{k-1}})\right\} \quad \text{if} \quad k\geq 1,\\ t_k &= \inf\left\{t : 2\,U(D_{f,t}) = U(D_{f,t_{k+1}})\right\} \quad \text{if} \quad k\leq -1. \end{aligned}$$

We denote $D_k = D_{f,t_k}$ and we observe that this is a decreasing set for all $k\in\mathbb{Z}$. Using that $U(D_k)+U(\Delta_k)t\leq U(D_{k+1})$, if $0\leq t\leq 1$ we have:

$$\begin{aligned} &\int_0^1 \sum_{k\in\mathbb{Z}} \left[\frac{V(D_k)+V(\Delta_k)t}{U(D_k)+U(\Delta_k)t}\right]^{r/p} V(\Delta_k)\,dt\\ \geq\;& \sum_{k\in\mathbb{Z}} V(\Delta_k)\,U(D_{k+1})^{-r/p}\int_0^1 (V(D_k)+V(\Delta_k)t)^{r/p}\,dt\\ =\;& (q/r)\sum_{k\in\mathbb{Z}} V(\Delta_k)\,U(D_{k+1})^{-r/p}\left(\frac{(V(D_k)+V(\Delta_k)t)^{r/q}}{V(\Delta_k)}\right)\Bigg]_0^1\\ =\;& (q/r)\sum_k U(D_{k+1})^{-r/p}\,(V(D_{k+1})^{r/q}-V(D_k)^{r/q})\\ =\;& 2^{-r/p}(q/r)\sum_k U(D_k)^{-r/p}\,(V(D_{k+1})^{r/q}-V(D_k)^{r/q}). \end{aligned}$$

Last equality follows from the definition of the sequence $(t_k)_k$. It is now enough to see that the expression in (iii) is smaller than this last quantity. This is done using that $D_k\subset D_{f,t}\subset D_{k+1}$, if $t_{k+1}\leq t\leq t_k$:

$$\begin{aligned} \int_0^\infty U(D_{f,t})^{-r/p}\,d\Big[-V(D_{f,t})^{r/q}\Big] &= \sum_{k\in\mathbb{Z}}\int_{t_{k+1}}^{t_k} U(D_{f,t})^{-r/p}\,d\Big[-V(D_{f,t})^{r/q}\Big]\\ &\leq \sum_{k\in\mathbb{Z}} U(D_k)^{-r/p}\int_{t_{k+1}}^{t_k} d\Big[-V(D_{f,t})^{r/q}\Big]\\ &= \sum_{k\in\mathbb{Z}} U(D_k)^{-r/p}\,(V(D_{k+1})^{r/q}-V(D_k)^{r/q}). \end{aligned}$$

Let us see that (iii) implies (iv). For a fixed family $\{D_k : k \in \mathbb{Z}\}$, define a decreasing function $f(x) = \sum_{k\in\mathbb{Z}} 2^{-k}\chi_{\Delta_k}(x)$. Then $D_k = \left\{x : f(x) > 2^{-k}\right\}$ and $D_{f,t} = D_{k+1}$ if $2^{-k-1} < t \leq 2^{-k}$ and thus, we have:

$$
\begin{aligned}
\int_0^\infty U(D_{f,t})^{-r/p}\, d\Big[-V(D_{f,t})^{r/q}\Big] &= \int_0^\infty U(D_{f,t})^{-r/p}\, V(D_{f,t})^{r/p}\, d\Big[-V(D_{f,t})\Big] \\
&= \sum_{k\in\mathbb{Z}} \int_{2^{-k-1}}^{2^{-k}} U(D_{f,t})^{-r/p}\, V(D_{f,t})^{r/p}\, d\Big[-V(D_{f,t})\Big] \\
&= \sum_{k\in\mathbb{Z}} U(D_{k+1})^{-r/p}\, V(D_{k+1})^{r/p} \int_{2^{-k-1}}^{2^{-k}} d\Big[-V(D_{f,t})\Big] \\
&= \sum_{k\in\mathbb{Z}} U(D_{k+1})^{-r/p}\, V(D_{k+1})^{r/p}\, (V(D_{k+1}) - V(D_k)) \\
&= \sum_{k\in\mathbb{Z}} U(D_{k+1})^{-r/p}\, V(D_{k+1})^{r/p}\, V(\Delta_k) \\
&\geq \sum_{k\in\mathbb{Z}} U(D_{k+1})^{-r/p}\, V(\Delta_k)^{r/p}\, V(\Delta_k) \\
&= \sum_{k\in\mathbb{Z}} U(D_{k+1})^{-r/p}\, V(\Delta_k)^{r/q}.
\end{aligned}
$$

Finally we prove that (iv) implies (i). For a fixed decreasing function f, let $(t_k)_k$ be a decreasing sequence constructed in the same way as in the implication $(ii) \Rightarrow (iii)$. Also denote $D_k = D_{f,t_k}$ and $\Delta_k = D_{k+1} \setminus D_k = \{x : t_{k+1} < t \leq t_k\}$. Then, applying Hölder's inequality, we have:

$$
\begin{aligned}
\left(\int_{\mathbb{R}^n_+} f(x)^q\, v(x)\, dx\right)^{1/q} &= \left(\sum_{k\in\mathbb{Z}} \int_{\Delta_k} f(x)^q\, v(x)\, dx\right)^{1/q} \\
&\leq \left(\sum_{k\in\mathbb{Z}} t_k^q\, V(\Delta_k)\right)^{1/q} \\
&\leq \left(\sum_{k\in\mathbb{Z}} t_k^p\, U(D_{k+1})\right)^{1/p} \left(\sum_{k\in\mathbb{Z}} V(\Delta_k)^{r/q}\, U(D_{k+1})^{-r/p}\right)^{1/r}.
\end{aligned}
$$

By construction, we have

$$
U(\Delta_k) = U(D_{k+1}) - U(D_k) = U(D_{k+1}) - \frac{1}{2}U(D_{k+1}) = \frac{1}{2}U(D_{k+1}),
$$

and thus $U(\Delta_{k-1}) = \frac{1}{4}U(D_{k+1})$. Now, the hypothesis and this equality give:

$$\begin{aligned}\left(\int_{\mathbb{R}^n_+} f(x)^q\, v(x)\, dx\right)^{1/q} &\leq C\left(\sum_{k\in\mathbb{Z}} t_k^p\, U(D_{k+1})\right)^{1/p} \\ &= C4^{1/p}\left(\sum_{k\in\mathbb{Z}} t_k^p\, U(\Delta_{k-1})\right)^{1/p} \\ &\leq C4^{1/p}\left(\sum_{k\in\mathbb{Z}} \int_{\Delta_{k-1}} f(x)^p\, u(x)\, dx\right)^{1/p} \\ &= C4^{1/p}\left(\int_{\mathbb{R}^n_+} f(x)^p\, u(x)\, dx\right)^{1/p}.\end{aligned}$$

□

Remark 1.1.14 We observe that the preceding theorem is also true in every space X where we can assume that the function $h(t) = U(D_{f,t})$ is continuous and strictly decreasing for a decreasing function f. If $X = \mathbb{N}$, this is not true in general.

1.2 B_p weights and the discrete Hardy operator

We present a new characterization for a weight to be in the B_p class in terms of the boundedness of the discrete Hardy operator defined for sequences indexed in $\mathbb{Z}$.

We recall that the Hardy operator is defined for any measurable function f on $[0,\infty)$ by

$$Af(x) = \frac{1}{x}\int_0^x f(t)\, dt, \qquad x > 0.$$

The set of weights u for which the boundedness

$$A : L^p_{\text{dec}}(u) \longrightarrow L^p(u), \tag{1.8}$$

holds, is called the B_p class, and the set of weights for which the boundedness

$$A : L^p_{\text{dec}}(u) \longrightarrow L^{p,\infty}(u), \tag{1.9}$$

holds, is called the $B_{p,\infty}$ class. Since $L^p(v) \subset L^{p,\infty}(v)$, we always have $B_p \subset B_{p,\infty}$. It is proved in [N], that $B_p = B_{p,\infty}$ if $1 < p < \infty$. This is not true if $0 < p \leq 1$; for example, the weight $u(x) = x^{p-1}$ is a $B_{p,\infty}$ weight not in B_p.

Further characterizations of these classes are known, and we collect some of them in the next theorem, that will be used later.

Theorem 1.2.1 (Ariño-Muckenhoupt [AM], Soria [So], Carro-Soria [CSo2]) *(a) For $0 < p < \infty$, the following conditions are equivalent:*

(i) $u \in B_p$.

(ii) There exists a constant $C > 0$ such that

$$\int_r^\infty \frac{u(x)}{x^p}\,dx \leq C\frac{1}{r^p}\int_0^r u(x)\,dx, \quad \forall\, r > 0. \tag{1.10}$$

(iii) There exists a constant $C > 0$ such that

$$\int_0^r \frac{1}{U(x)^{1/p}}\,dx \leq C\frac{r}{U(r)^{1/p}}, \quad \forall\, r > 0. \tag{1.11}$$

(b) For $0 < p \leq 1$, the following conditions are equivalent:

(i) $u \in B_{p,\infty}$.

(ii) There exists a constant $C > 0$ such that

$$\frac{U(r)^{1/p}}{r} \leq C\frac{U(s)^{1/p}}{s}, \quad \forall\, 0 < s < r. \tag{1.12}$$

In [R], J.A. Raposo studied a discrete Hardy operator defined for sequences indexed in $\mathbb{N}$, namely

$$A_{\mathbb{N}}f(n) = \frac{1}{n+1}\sum_{j=0}^{n} f_j \quad n = 0, 1, 2, \ldots,$$

where $(f_n)_{n\in\mathbb{N}} \subset \mathbb{C}$. For a weight $(u_n)_{n\in\mathbb{N}}$, that is, a positive sequence, $\ell^p_{\mathrm{dec}}((u_n)_n)$ is the set of positive decreasing sequences in $\ell^p((u_n)_n)$, and $\ell^{p,\infty}((u_n)_n)$ is the set of sequences $(f_n)_{n\in\mathbb{N}}$ such that

$$\|(f_n)_n\|_{\ell^{p,\infty}((u_n)_n)} = \sup_{n\in\mathbb{N}} n^{1/p} f_n^\star < \infty, \tag{1.13}$$

where $(f_n^\star)_{n\in\mathbb{N}}$ is the decreasing rearrangement of $(f_n)_{n\in\mathbb{N}}$. The result proved in [R] is the following:

Theorem 1.2.2 (Raposo) *For $1 < p < \infty$, the following conditions are equivalent for a weight $(u_n)_n$:*

(i) $A_{\mathbb{N}} : \ell^p_{\mathrm{dec}}((u_n)_n) \longrightarrow \ell^{p,\infty}((u_n)_n)$.

(ii) $A_{\mathbb{N}} : \ell^{p,\infty}_{\text{dec}}((u_n)_n) \longrightarrow \ell^{p,\infty}((u_n)_n)$.

(iii) $A_{\mathbb{N}} : \ell^{p}_{\text{dec}}((u_n)_n) \longrightarrow \ell^{p}((u_n)_n)$.

(iv) $\widetilde{u}(x) = \sum_{n=0}^{\infty} u_n \, \chi_{[n,n+1)}(x) \in B_p$.

(v) $\displaystyle\sum_{k=0}^{n} \frac{1}{U_k^{1/p}} \leq C \frac{n+1}{U_n^{1/p}}, \quad \forall\, n \geq 0.$

We see that the boundedness $A_{\mathbb{N}} : \ell^{p}_{\text{dec}}((u_n)_n) \longrightarrow \ell^{p}((u_n)_n)$ for a discrete weight $(u_n)_n$ is equivalent to $\widetilde{u} \in B_p$ for an extended weight $\widetilde{u} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$. Can this process be reversed in some sense? That is, if $u \in B_p$, is there a discrete weight $(u_n)_{n\in\mathbb{N}}$ related to u such that $A_{\mathbb{N}}$ is bounded from $\ell^{p}_{\text{dec}}((u_n)_n)$ to $\ell^{p}((u_n)_n)$? The answer is affirmative, as next lemma shows. We denote $B_p(\mathbb{N})$ the class of discrete weights $(u_n)_{n\in\mathbb{N}}$ such that $A_{\mathbb{N}} : \ell^{p}_{\text{dec}}((u_n)_n) \longrightarrow \ell^{p}((u_n)_n)$ is bounded.

Lemma 1.2.3 *If $u : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a B_p weight, then $(u_n)_{n\in\mathbb{N}}$ defined by*

$$u_n = \int_n^{n+1} u(x)\, dx,$$

is a $B_p(\mathbb{N})$ weight.

Proof. For a given positive decreasing sequence $(f_n)_{n\in\mathbb{N}}$, we consider the extended decreasing function

$$\widetilde{f}(x) = \sum_{n=0}^{\infty} f_n \, \chi_{[n,n+1)}(x).$$

We then have that

$$\sum_{l=0}^{n} f_l = \int_0^{n+1} \widetilde{f}(x)\, dx.$$

Using this equality and that $A\widetilde{f}$ is a decreasing function, we get:

$$\begin{aligned}
\sum_{n=0}^{\infty}\Big(\frac{1}{n+1}\sum_{j=0}^{n} f_j\Big)^p u_n &= \sum_{n=0}^{\infty}\Big(\frac{1}{n+1}\int_0^{n+1}\widetilde{f}(x)\,dx\Big)^p u_n \\
&= \sum_{n=0}^{\infty}\Big(\frac{1}{n+1}\int_0^{n+1}\widetilde{f}(x)\,dx\Big)^p \Big(\int_n^{n+1} u(x)\,dx\Big) \\
&\leq \sum_{n=0}^{\infty}\int_n^{n+1}\Big(\frac{1}{x}\int_0^{x}\widetilde{f}(y)\,dy\Big)^p u(x)\,dx \\
&\leq C\int_0^{\infty}\widetilde{f}^p(x)\,u(x)\,dx \\
&= \sum_{n=0}^{\infty} f_n^p\,u_n.
\end{aligned}$$

□

Theorem 1.2.2 says that $B_p(\mathbb{N})$ can be viewed as a subset of the B_p weights that are constant at each interval $[n, n+1)$, and Lemma 1.2.3 says that every B_p weight which is constant at each interval $[n, n+1)$ can be viewed as a $B_p(\mathbb{N})$ weight. In other words,

$$B_p(\mathbb{N}) \equiv \{u \in B_p : \ u(x) = c_n \quad \forall x \in [n, n+1), \text{ for some positive } (c_n)_n\}.$$

Now, another question arises. Can we characterize B_p as the class of weights such that the boundedness of $A_{\mathbb{N}}$ holds for the discretized weights? That is, can we characterize B_p in terms of $B_p(\mathbb{N})$? Now the answer is negative. There are weights which are not in B_p but their discretized ones are in $B_p(\mathbb{N})$. For example, take the weight

$$u(x) = \begin{cases} 0, & 0 < x < 1/2 \\ 2, & 1/2 \leq x < 1 \\ 1, & x \geq 1, \end{cases}$$

which is not a B_p weight because it equals zero in a neighborhood of 0, and this contradicts condition (1.10). But,

$$u_n = \int_n^{n+1} u(x)\,dx = 1,$$

for all $n \geq 0$, and thus, if

$$\widetilde{u}(x) = \sum_{n=0}^{\infty} u_n\,\chi_{[n,n+1)}(x) = 1,$$

then $\widetilde{u}$ is a B_p weight for all $1 < p < \infty$, and therefore $A_{\mathbb{N}}$ is bounded from $\ell^p_{\text{dec}}((u_n)_n)$ to $\ell^p((u_n)_n)$, by Theorem 1.2.2, that is, $(u_n)_{n\in\mathbb{N}}$ is a $B_p(\mathbb{N})$ weight.

In order to get complete results in both directions (extension and discretization of weights), we work with discrete weights indexed in $\mathbb{Z}$ rather than in $\mathbb{N}$. The Hardy operator, defined for sequences $(f_j)_{j\in\mathbb{Z}}$ over $\mathbb{Z}$, is defined by

$$A_{\mathbb{Z}}f(k) = \frac{1}{2^{k+1}} \sum_{j=-\infty}^{k} 2^j f_j, \quad k \in \mathbb{Z}.$$

The function $A_{\mathbb{Z}}f$ is decreasing if $(f_k)_{k\in\mathbb{Z}}$ is decreasing.

A weight $(u_k)_{k\in\mathbb{Z}}$ is in the $B_p(\mathbb{Z})$ class if and only if

$$A_{\mathbb{Z}} : \ell^p_{\text{dec}}((u_k)_k) \longrightarrow \ell^p((u_k)_k),$$

and it belongs to the $B_{p,\infty}(\mathbb{Z})$ class if and only if

$$A_{\mathbb{Z}} : \ell^p_{\text{dec}}((u_k)_k) \longrightarrow \ell^{p,\infty}((u_k)_k).$$

We will use another expression for the functional defined in (1.13), which is a particular case of (1.4). In the case of a positive decreasing sequence $(f_k)_{k\in\mathbb{Z}}$, it is easy to see that we can write

$$\|(f_k)_k\|_{\ell^{p,\infty}((u_k)_k)} = \sup_{k\in\mathbb{Z}} U_k^{1/p} f_k, \tag{1.14}$$

and in the case of a positive decreasing function f we also have

$$\|f\|_{L^{p,\infty}(u)} = \sup_{t>0} U(t)^{1/p} f(t). \tag{1.15}$$

See Corollary 1.3.2 for a general proof of these expressions.

We first present the case of the **discretization** of a weight $u : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$. For such a weight, we denote

$$u_k = \int_{2^k}^{2^{k+1}} u(x)\, dx$$

and

$$\widetilde{u}(x) = \sum_{k=-\infty}^{\infty} \frac{u_k}{2^k} \chi_{[2^k, 2^{k+1})}(x).$$

Theorem 1.2.4 *If $0 < p < \infty$, for a weight $u : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, the following conditions are equivalent :*

(i) $u \in B_p$.

(ii) $(u_k)_k \in B_p(\mathbb{Z})$.

(iii) $\displaystyle\sum_{j=k+1}^{\infty} \frac{u_j}{(2^{j+1})^p} \le C \frac{U_k}{(2^{k+1})^p}, \quad \forall\, k \in \mathbb{Z}.$

(iv) $\widetilde{u} \in B_p$.

Proof. Suppose that (i) holds. For a positive decreasing sequence $(f_k)_k$, consider the decreasing function $\widetilde{f}(x) = \sum_{k\in\mathbb{Z}} f_k \chi_{[2^k, 2^{k+1})}(x)$. Using that

$$\int_0^{2^{n+1}} \widetilde{f}(x)\, dx = \sum_{k\le n} \int_{2^k}^{2^{k+1}} \widetilde{f}(x)\, dx = \sum_{k\le n} 2^k f_k,$$

and that $\frac{1}{x}\int_0^x \widetilde{f}(y)\, dy$ is decreasing, we have:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} A_{\mathbb{Z}} f(k)^p\, u_k &= \sum_{k\in\mathbb{Z}} \left(\frac{1}{2^{k+1}} \int_0^{2^{k+1}} \widetilde{f}(x)\, dx\right)^p \left(\int_{2^k}^{2^{k+1}} u(x)\, dx\right) \\
&\le \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left(\frac{1}{x}\int_0^x \widetilde{f}(y)\, dy\right)^p u(x)\, dx \\
&\le C \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \widetilde{f}(x)^p\, \widetilde{u}(x)\, dx \\
&= C \sum_{k\in\mathbb{Z}} f_k^p\, u_k,
\end{aligned}$$

and this is condition (ii). To see that (ii) implies (iii), it is enough to consider the boundedness of the operator on the functions $f(l) = \chi_{\{j: j\le k\}}(l)$, for all $k \in \mathbb{Z}$. Let us see that (iii) implies condition (1.10) for the weight $\widetilde{u}$, which is (iv). For $r > 0$, take $2^k \le r < 2^{k+1}$. Using that $\int_{2^j}^{2^{j+1}} \widetilde{u}(x)\, dx = u_j$, we have:

$$\begin{aligned}
\int_r^\infty \frac{\widetilde{u}(x)}{x^p}\, dx &\le \sum_{j=k}^{\infty} \int_{2^j}^{2^{j+1}} \frac{\widetilde{u}(x)}{x^p}\, dx \le \sum_{j=k}^{\infty} \frac{1}{2^{jp}} \int_{2^j}^{2^{j+1}} \widetilde{u}(x)\, dx \\
&= \sum_{j=k}^{\infty} \frac{u_j}{2^{jp}} \le 2^p\, C\, \frac{U_{k-1}}{2^{kp}} = 2^p\, C \frac{1}{2^{kp}} \int_0^{2^k} \widetilde{u}(x)\, dx \\
&\le 4^p\, C\, \frac{1}{r} \int_0^r u(x)\, dx.
\end{aligned}$$

Finally, that (iv) implies (i) is easy, if we use the characterization (1.10) of the B_p weights, and observe that for all k

$$\int_{2^k}^{2^{k+1}} \frac{u(x)}{x^p}\, dx \approx \int_{2^k}^{2^{k+1}} \frac{\widetilde{u}(x)}{x^p}\, dx,$$

and $\int_0^{2^k} u(x)\,dx = \int_0^{2^k} \widetilde{u}(x)\,dx$.

$\square$

Corollary 1.2.5 *If $0 < p < \infty$ and $u_k = \displaystyle\int_{2^k}^{2^{k+1}} u(x)\,dx$ for a weight u in $\mathbb{R}_+$, we have*

$$B_p = \{u \geq 0 : (u_k)_k \in B_p(\mathbb{Z})\}.$$

In the following theorem, we complete the results by considering the case of the $B_{p,\infty}$ weights.

Theorem 1.2.6 *For a weight $u : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, we have:*
(a) If $0 < p \leq 1$, the following conditions are equivalent:

(i) $u \in B_{p,\infty}$.

(ii) $(u_k)_k \in B_{p,\infty}(\mathbb{Z})$.

(iii) $\dfrac{U_n^{1/p}}{2^{n+1}} \leq C \dfrac{U_k^{1/p}}{2^{k+1}}, \quad \forall\, k \leq n.$

(iv) $\widetilde{u} \in B_{p,\infty}$.

(b) If $1 < p < \infty$ the following conditions are equivalent:

(i) $u \in B_p$.

(ii) $(u_k)_k \in B_p(\mathbb{Z})$.

(iii) $\displaystyle\sum_{j=k+1}^{\infty} \frac{u_j}{(2^{j+1})^p} \leq C \frac{U_k}{(2^{k+1})^p}, \quad \forall\, k \in \mathbb{Z}.$

(iv) $(u_k)_k \in B_{p,\infty}(\mathbb{Z})$.

(v) $\widetilde{u} \in B_p$.

Proof. (a) Let us see that condition (ii) holds if and only if (iii) holds. The boundedness of $A_{\mathbb{Z}}$ is equivalent, by (1.14), to

$$\frac{U_n^{1/p}}{2^{n+1}} \sum_{j \leq n} 2^j f_j \leq C \left(\sum_{k \in \mathbb{Z}} f_k^p\, u_k \right)^{1/p},$$

for all decreasing sequences $(f_k)_k$, and therefore also to

$$\frac{U_n^{1/p}}{2^{n+1}} \sup_{0\le (f_k)\downarrow} \frac{\displaystyle\sum_{j\le n} 2^j f_j}{\left(\displaystyle\sum_{k\in\mathbb{Z}} f_k^p\, u_k\right)^{1/p}} \le C.$$

Using Theorem 1.1.4 with $v_k = 2^k$, if $k \le n$, and 0 otherwise, and $q = 1$, the last expression is equivalent to

$$\frac{U_n^{1/p}}{2^{n+1}} \frac{2^{k+1}}{U_k^{1/p}} \le C, \quad \forall\, k \le n, \tag{1.16}$$

which is (iii). Let us see that (iii) holds if and only if (iv) holds. Using that

$$\widetilde{U}(2^{n+1}) = \int_0^{2^{n+1}} \widetilde{u}(x)\, dx = \sum_{k\le n} u_k = U_n,$$

it is not difficult to see that (1.16) is equivalent to

$$\frac{\widetilde{U}(r)^{1/p}}{r} \le C \frac{\widetilde{U}(s)^{1/p}}{s}, \quad \forall\, 0 < s < r,$$

and this condition is actually equivalent to (iv) by (1.12). Finally, let us see that (i) is equivalent to (iii). As before, using that $U(2^{n+1}) = \int_0^{2^{n+1}} u(x)\, dx = U_n$, it is easy to see that condition (1.16) is equivalent to

$$\frac{U(r)^{1/p}}{r} \le C \frac{U(s)^{1/p}}{s}, \quad \forall\, 0 < s < r,$$

which is in fact equivalent to (i) by (1.12).

(b) The equivalence between $(i), (ii), (iii)$ and (v) is already proved in the previous theorem. That (ii) implies (iv) is well-known. Let us see that (iv) implies (v). As is shown in the proof of (a), (iv) is equivalent to

$$\frac{U_n^{1/p}}{2^{n+1}} \sup_{0\le (f_k)\downarrow} \frac{\displaystyle\sum_{j\le n} 2^j f_j}{\left(\displaystyle\sum_{k\in\mathbb{Z}} f_k^p\, u_k\right)^{1/p}} \le C.$$

We use Proposition 1.1.5 with weights $v_k = 2^k$, if $k \le n$, and 0 otherwise, and u_k to obtain that the boundedness of $A_{\mathbb{Z}}$ is equivalent to

$$\frac{U_n^{1/p}}{2^{n+1}} \sup_{0\le f\downarrow} \frac{\displaystyle\int_0^{2^{n+1}} f}{\left(\displaystyle\int_0^{\infty} f(x)^p\, \widetilde{u}(x)\, dx\right)^{1/p}} \le C,$$

and using $\widetilde{U}(2^{n+1}) = \displaystyle\int_0^{2^{n+1}} \widetilde{u}(x)\,dx = \sum_{k\leq n} u_k = U_n$, this is

$$\widetilde{U}(2^{n+1})^{1/p} Af(2^{n+1}) \leq C\left(\int_0^\infty f(x)^p\,\widetilde{u}(x)\,dx\right)^{1/p},$$

for all positive decreasing f. We claim that this condition also holds for all $t > 0$ instead of 2^{n+1}. Observe that the hypothesis on $A_{\mathbb{Z}}$ implies the necessity of condition (1.16) (simply by taking $f = \chi_{\{j:j\leq k\}}$ for every $k \in \mathbb{Z}$), and therefore

$$\widetilde{U}(2^{n+1})^{1/p} = U_n^{1/p} \leq CU_{n-1}^{1/p} = C\widetilde{U}(2^n)^{1/p}.$$

Then, if $2^n < t \leq 2^{n+1}$ we have

$$\begin{aligned}
\widetilde{U}(t)^{1/p} Af(t) &\leq \widetilde{U}(2^{n+1})^{1/p} Af(2^n) \\
&\leq C\widetilde{U}(2^n)^{1/p} Af(2^n) \\
&\leq C\left(\int_0^\infty f(x)^p \widetilde{u}(x) dx\right)^{1/p}.
\end{aligned}$$

This last condition is equivalent to $A : L^p_{\mathrm{dec}}(\widetilde{u}) \to L^{p,\infty}(\widetilde{u})$ by (1.15), and this is equivalent to $\widetilde{u} \in B_p$, since $B_p = B_{p,\infty}$ if $1 < p < \infty$.

□

Corollary 1.2.7 *If $0 < p < \infty$ and $u_k = \displaystyle\int_{2^k}^{2^{k+1}} u(x)\,dx$ for a weight u in $\mathbb{R}_+$, we have*

$$B_{p,\infty} = \{u \geq 0 : (u_k)_k \in B_{p,\infty}(\mathbb{Z})\}.$$

We now present the result in the other direction, that is, the **extension** result. For a discrete weight $(u_k)_{k\in\mathbb{Z}}$, we denote

$$\widetilde{u}(x) = \sum_{k\in\mathbb{Z}} \frac{u_k}{2^k}\,\chi_{[2^k,2^{k+1})}(x).$$

Theorem 1.2.8 *If $0 < p < \infty$, for a weight $(u_k)_k$, the following conditions are equivalent :*

(i) $(u_k)_k \in B_p(\mathbb{Z})$.

(ii) $\displaystyle\sum_{j=k+1}^{\infty} \frac{u_j}{(2^{j+1})^p} \leq C\frac{U_k}{(2^{k+1})^p}, \quad \forall\, k \in \mathbb{Z}.$

(iii) $\widetilde{u} \in B_p$.

Proof. That (i) implies (ii) is easy if we consider the boundedness of $A_{\mathbb{Z}}$ on the functions $f(l) = \chi_{\{j:j\leq k\}}(l)$, for all $k \in \mathbb{Z}$. Let us see that (ii) implies condition (1.10) for the weight $\widetilde{u}$, which is (iii). For $r > 0$, take $2^k \leq r < 2^{k+1}$. Using that $\int_{2^j}^{2^{j+1}} \widetilde{u}(x)\,dx = u_j$, we have:

$$\begin{aligned}
\int_r^\infty \frac{\widetilde{u}(x)}{x^p}\,dx &\leq \sum_{j=k}^\infty \int_{2^j}^{2^{j+1}} \frac{\widetilde{u}(x)}{x^p}\,dx \leq \sum_{j=k}^\infty \frac{1}{2^{jp}} \int_{2^j}^{2^{j+1}} \widetilde{u}(x)\,dx \\
&= \sum_{j=k}^\infty \frac{u_j}{2^{jp}} \leq 2^p\,C\,\frac{U_{k-1}}{2^{kp}} = 2^p\,C\frac{1}{2^{kp}} \int_0^{2^k} \widetilde{u}(x)\,dx \\
&\leq 4^p\,C\,\frac{1}{r} \int_0^r \widetilde{u}(x)\,dx.
\end{aligned}$$

Let us see that (iii) implies (i). For a positive decreasing sequence $(f_k)_k$, consider the decreasing function $\widetilde{f}(x) = \sum_{k\in\mathbb{Z}} f_k \chi_{[2^k,2^{k+1})}(x)$. Using that

$$\int_0^{2^{n+1}} \widetilde{f}(x)\,dx = \sum_{k\leq n} \int_{2^k}^{2^{k+1}} \widetilde{f}(x)\,dx = \sum_{k\leq n} 2^k f_k,$$

the equality $\int_{2^k}^{2^{k+1}} \widetilde{u}(x)\,dx = u_k$ and the fact that $\frac{1}{x}\int_0^x \widetilde{f}(y)\,dy$ is decreasing, we have:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} A_{\mathbb{Z}} f(k)^p\,u_k &= \sum_{k\in\mathbb{Z}} \left(\frac{1}{2^{k+1}} \int_0^{2^{k+1}} \widetilde{f}(x)\,dx\right)^p \left(\int_{2^k}^{2^{k+1}} \widetilde{u}(x)\,dx\right) \\
&\leq \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left(\frac{1}{x}\int_0^x \widetilde{f}(y)\,dy\right)^p \widetilde{u}(x)\,dx \\
&\leq C \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \widetilde{f}(x)^p\,\widetilde{u}(x)\,dx = C\sum_{k\in\mathbb{Z}} f_k^p\,u_k,
\end{aligned}$$

and this is condition (i).

□

In the case of the $B_{p,\infty}$ weights, we have:

Theorem 1.2.9 *For a weight $(u_k)_k$, we have:*
(a) If $0 < p \leq 1$, the following conditions are equivalent:

(i) $(u_k)_k \in B_{p,\infty}(\mathbb{Z})$.

(ii) $\dfrac{U_n^{1/p}}{2^{n+1}} \leq C\dfrac{U_k^{1/p}}{2^{k+1}}, \quad \forall\, k \leq n.$

(iii) $\widetilde{u} \in B_{p,\infty}$.

(b) If $1 < p < \infty$, *the following conditions are equivalent:*

(i) $(u_k)_k \in B_{p,\infty}(\mathbb{Z})$.

(ii) $\widetilde{u} \in B_p$.

(iii) $\displaystyle\sum_{j \le k} \frac{2^{j+1}}{U_j^{1/p}} \le C \frac{2^{k+1}}{U_k^{1/p}}, \quad \forall\, k \in \mathbb{Z}$.

Proof. (a) The equivalence between (i) and (ii) has been proved in (a) of Theorem 1.2.6. To prove that (ii) and (iii) are equivalent, we proceed as in Theorem 1.2.6: using that

$$\widetilde{U}(2^{n+1}) = \int_0^{2^{n+1}} \widetilde{u}(x)\, dx = \sum_{k \le n} u_k = U_n,$$

it is not difficult to see that (ii) is equivalent to

$$\frac{\widetilde{U}(r)^{1/p}}{r} \le C \frac{\widetilde{U}(s)^{1/p}}{s}, \quad \forall\, 0 < s < r,$$

and this condition is actually equivalent to (iii) by (1.12).

(b) The equivalence between (i) and (ii) is already proved in Theorem 1.2.6, when proving the equivalence between conditions (iv) and (v) in the case (b) (we observe that in that proof, we are not using the 'continuous' weight u). The equivalence with (iii) is an easy consequence of the characterization (1.11) of $\widetilde{u} \in B_p$, once we show that for all $k \in \mathbb{Z}$,

$$\int_{2^k}^{2^{k+1}} \frac{1}{\widetilde{U}(x)^{1/p}}\, dx \approx \frac{2^{k+1}}{U_k^{1/p}}.$$

In one direction we have:

$$\frac{2^{k+1}}{U_k^{1/p}} = \frac{2}{\widetilde{U}(2^{k+1})^{1/p}} \int_{2^k}^{2^{k+1}} dx \le 2 \int_{2^k}^{2^{k+1}} \frac{1}{\widetilde{U}(x)^{1/p}}\, dx.$$

In the other direction, we use again that $U_{n+1}^{1/p} \le C U_n^{1/p}$, and thus:

$$\int_{2^k}^{2^{k+1}} \frac{1}{\widetilde{U}(x)^{1/p}}\, dx \le \frac{2^k}{\widetilde{U}(2^k)^{1/p}} = \frac{2^k}{U_{k-1}^{1/p}} \le C\, \frac{2^{k+1}}{U_k^{1/p}}.$$

$\square$

Theorem 1.2.10 *For $0 < p < \infty$, we have*

$$A_{\mathbb{Z}} : \ell^{p,\infty}_{\rm dec}((u_k)_k) \longrightarrow \ell^{p,\infty}((u_k)_k)$$

if and only if

$$\sum_{j \le k} \frac{2^{j+1}}{U_j^{1/p}} \le C \frac{2^{k+1}}{U_k^{1/p}}, \quad \forall\, k \in \mathbb{Z}.$$

Proof. The boundedness of $A_{\mathbb{Z}}$ is equivalent to

$$\|A_{\mathbb{Z}} f\|_{\ell^{p,\infty}((u_k)_k)} \le C \|f\|_{\ell^{p,\infty}((u_k)_k)},$$

for all decreasing $f = (f_k)_{k \in \mathbb{Z}}$. We observe that the sequence $f_k = U_k^{-1/p}$ is decreasing, and that $\|f\|_{\ell^{p,\infty}((u_k)_k)} = 1$, by using (1.14), and therefore the boundedness of $A_{\mathbb{Z}}$ implies

$$\|A_{\mathbb{Z}} f\|_{\ell^{p,\infty}((u_k)_k)} \le C. \tag{1.17}$$

On the other hand, we observe that for every $(f_k)_k \in \ell^{p,\infty}((u_k)_k)$, we have that $f_k \le C U_k^{-1/p}$ for all $k \in \mathbb{Z}$, and this implies that $A_{\mathbb{Z}} f(k) \le A_{\mathbb{Z}}(U^{-1/p})(k)$ for all $k \in \mathbb{Z}$, and thus, (1.17) is also sufficient for the boundedness of $A_{\mathbb{Z}}$. Now, if we write condition (1.17) by using (1.14), we find the desired condition.

□

By considering Theorems 1.2.8, 1.2.9 and 1.2.10, we can state the following result:

Corollary 1.2.11 *If $1 < p < \infty$, the following conditions are equivalent for a weight $(u_k)_k$:*

(i) $\widetilde{u}(x) = \sum_{k \in \mathbb{Z}} \frac{u_k}{2^k} \chi_{[2^k, 2^{k+1})}(x) \in B_p$.

(ii) $\sum_{j \le k} \frac{2^{j+1}}{U_j^{1/p}} \le C \frac{2^{k+1}}{U_k^{1/p}}, \quad \forall\, k \in \mathbb{Z}$.

(iii) $(u_k)_k \in B_{p,\infty}(\mathbb{Z})$.

(iv) $A_{\mathbb{Z}} : \ell^{p,\infty}_{\rm dec}((u_k)_k) \longrightarrow \ell^{p,\infty}((u_k)_k)$.

(v) $(u_k)_k \in B_p(\mathbb{Z})$.

1.3 Operators on monotone functions

In this section, we study the boundedness of operators with some growth properties. The classical example is the Hardy operator Af defined in (1.2), which is a decreasing function whenever f is decreasing, or the Hardy-Volterra operator Vf defined in (1.3), which is always an increasing function if f is positive. The purpose of this section is to consider general operators defined on functions in a general measure space. The key idea is that it is possible to characterize the boundedness of the operator in terms of capacity conditions on the level sets of the operator, that are increasing or decreasing sets.

So, we go back to the general case of a measure space (X,μ), where X is an ordered set. Recall that we assume that every decreasing set is μ-measurable, and that we denote the order by $x \leq y$. In some occasions, we will need to consider the case where X is a topological connected space. The connectedness guarantees that every non-empty open set has non-empty boundary. In that case, we avoid, for example, the spaces with the discrete topology.

For a weight v, the weak $L^q(v)$-'norm' of a measurable function f is given in (1.4). We need to express this quantity in a more geometrical way.

Lemma 1.3.1 *For every measurable function f in (X,μ), we have that*

$$\|f\|_{L^{q,\infty}(v)} = \sup_{E\subset X} V(E)^{1/q}\left(\inf_{x\in E}|f(x)|\right),$$

where the supremum is taken over all measurable sets E in X.

Proof. By a density argument, it is enough to prove it for a simple positive function $f(x)=\sum_{k=1}^n a_k\chi_{E_k}(x)$, where $0<a_1<a_2<\ldots<a_n$. In this case, we know that

$$\|f\|_{L^{q,\infty}(v)} = \max_{k=1,\ldots,n} a_k\, V(F_k)^{1/q},$$

where $F_k=\cup_{i=k}^n E_i$. Now take $E\subset X$ such that $E\subset \cup_{k=1}^n E_k$ (if not, $\inf_{x\in E}|f(x)|=0$ and it has no contribution in the supremum). If $i=\min\{k:\ 1\leq k\leq n,\ E\subset F_k\}$, then $\inf_{x\in E}|f(x)|=a_i$ and so

$$\sup_{E\subset X} V(E)^{1/q}\left(\inf_{x\in E}|f(x)|\right) = \max_{k=1,\ldots,n}\sup_{E\subset F_k} a_k\, V(E)^{1/q} = \max_{k=1,\ldots,n} a_k\, V(F_k)^{1/q}.$$

□

The next result shows that if the function is monotone, we can restrict the supremum in the weak norm to monotone sets.

Corollary 1.3.2 *For a positive measurable function f, we have:*
(a) $\|f\|_{L^{q,\infty}(v)} = \sup_{D\downarrow} V(D)^{1/q}\left(\inf_{x\in D} f(x)\right)$, *if f is decreasing.*
(b) $\|f\|_{L^{q,\infty}(v)} = \sup_{I\uparrow} V(I)^{1/q}\left(\inf_{x\in I} f(x)\right)$, *if f is increasing.*

Proof. (a) We trivially have that

$$\sup_{D\downarrow} V(D)^{1/q}\left(\inf_{x\in D} f(x)\right) \leq \|f\|_{L^{q,\infty}(v)},$$

for every positive decreasing f in view of the previous lemma. Let us see the reverse inequality for a fixed decreasing $f \geq 0$. For a measurable set E, set

$$E_d = \bigcap_{D\supset E, D\downarrow} D,$$

which is a decreasing set, and hence, measurable. It is clear that

$$V(E) \leq V(E_d). \tag{1.18}$$

We claim that

$$\inf_{x\in E} f(x) = \inf_{x\in E_d} f(x).$$

Thus, using (1.18) and the claim, we get

$$\|f\|_{L^{q,\infty}(v)} = \sup_{E\subset X} V(E)^{1/q}\left(\inf_{x\in E} f(x)\right) \leq \sup_{D\downarrow} V(D)^{1/q}\left(\inf_{x\in D} f(x)\right).$$

It is now enough to prove the inequality

$$\inf_{x\in E} f(x) \leq \inf_{x\in E_d} f(x),$$

because the reverse inequality is trivially true. Suppose that

$$\inf_{x\in E} f(x) > \inf_{x\in E_d} f(x).$$

Then, there exists $y \in E_d$ such that $f(y) < \inf_{x\in E} f(x)$ and thus, since f is decreasing, we have

$$F(y) \cap E = \emptyset, \tag{1.19}$$

where $F(y) = \{x \in X : x \geq y\}$. For every decreasing $D \supset E$, define $D' = D \setminus F(y)$, which is also a decreasing set. By (1.19), $E \subset D'$ for all decreasing $D \supset E$. But observe that $y \notin D'$, getting a contradiction with the fact that $y \in E_d$.
(b) The proof is analogous, but considering

$$E_i = \bigcap_{I \supset E, I\uparrow} D,$$

instead of E_d.

$\square$

Let L be a subclass of the set $\mathcal{M}(X)$ of all the measurable functions in (X, μ). We will consider operators

$$S : \mathcal{M}(X) \longrightarrow \mathcal{M}(X),$$

that are (positively) homogeneous, that is,

$$|S(\lambda f)(x)| = |\lambda| \; |Sf(x)| \, ,$$

for all $\lambda > 0$, all measurable f and all $x \in X$.

For such L and $S : \mathcal{M}(X) \longrightarrow \mathcal{M}(X)$, the $L^p(u)$-**capacity** of a measurable set E is defined as

$$\mathrm{Cap}_{p,u,L,S}(E) = \inf \left\{ \|f\|_{L^p(u)} : \; f \in L, \quad |Sf(x)| \geq 1 \quad \forall x \in E \right\}.$$

If $L = L^p(u)$, we will simply write $\mathrm{Cap}_{p,u,S}(E)$.

Theorem 1.3.3 *Let $S : \mathcal{M}(X) \longrightarrow \mathcal{M}(X)$ be a homogeneous operator such that $|Sf|$ is a decreasing function for every $f \in L$. Then,*

$$A := \sup_{f \in L} \frac{\|Sf\|_{L^{q,\infty}(v)}}{\|f\|_{L^p(u)}} = \sup_{D\downarrow} \frac{V(D)^{1/q}}{\mathrm{Cap}_{p,u,L,S}(D)} := B.$$

Moreover, if X is a topological connected space, and Sf is a continuous function for every $f \in L$, then

$$A = B = \sup_{D\downarrow} \frac{V(D)^{1/q}}{\mathrm{Cap}'_{p,u,L,S}(D)} := C,$$

where

$$\mathrm{Cap}'_{p,u,L,S}(E) = \inf \left\{ \|f\|_{L^p(u)} : \; f \in L, \quad |Sf(x)| = 1 \quad \forall x \in \partial E \right\},$$

for every measurable set E.

Proof. We first prove that $A = B$. For a decreasing set D, take $f \in L$ such that $|Sf(x)| \geq 1$, if $x \in D$. Then,

$$D \subset \{x \in X : |Sf(x)| \geq 1\},$$

and therefore

$$V(D)^{1/q} \leq V(\{x \in X : |Sf(x)| \geq 1\})^{1/q} \leq A \|f\|_{L^p(u)}.$$

Taking the infimum, we get

$$V(D)^{1/q} \leq A \operatorname{Cap}_{p,u,L,S}(D),$$

for all decreasing sets D, which shows $B \leq A$. Now, for a function $f \in L$, and for a decreasing set D, set $\lambda_D = \inf\limits_{x \in D} |Sf(x)|$. We can assume that $\lambda_D > 0$, and then, by the homogeneity

$$|S(f/\lambda_D)(x)| \geq 1,$$

for all $x \in D$. By definition, we have

$$\operatorname{Cap}_{p,u,L,S}(D) \leq \frac{\|f\|_{L^p(u)}}{\lambda_D}.$$

Using this last inequality and Corollary 1.3.2, we get

$$\begin{aligned} \|Sf\|_{L^{q,\infty}(v)} &= \sup_{D\downarrow} V(D)^{1/q} \lambda_D \\ &\leq B \sup_{D\downarrow} \operatorname{Cap}_{p,u,L,S}(D) \lambda_D \\ &\leq B \|f\|_{L^p(u)}, \end{aligned}$$

and this is $A \leq B$.
We now prove $A = B = C$, if Sf is a continuous function for every $f \in L$. If $|Sf|$ is a decreasing function such that $|Sf(x)| = 1$ for all $x \in \partial D$, where D is a decreasing set, then $|Sf(x)| \geq 1$ for all $x \in D$, and therefore

$$\operatorname{Cap}_{p,u,L,S}(D) \leq \operatorname{Cap}'_{p,u,L,S}(D),$$

for all decreasing D, and that is $B \geq C$. For a fixed $f \in L$, set as before $\lambda_D = \inf\limits_{x \in D} |Sf(x)|$. We can assume that $\lambda_D > 0$. We have

$$D \subset D' := \{x \in X : |Sf(x)| \geq \lambda_D\},$$

and thus,

$$V(D)^{1/q} \leq V(D')^{1/q} \leq C\,\mathrm{Cap}'_{p,u,L,S}(D'). \tag{1.20}$$

By continuity we have that $|Sf(x)| = \lambda_D$ for all $x \in \partial D'$, and using the homogeneity of S, we get

$$\mathrm{Cap}'_{p,u,L,S}(D') \leq \|f/\lambda_D\|_{L^p(u)} = \frac{\|f\|_{L^p(u)}}{\lambda_D}.$$

By using this inequality, Corollary 1.3.2 and (1.20), we finally get

$$\begin{aligned} \|Sf\|_{L^{q,\infty}(v)} &= \sup_{D\downarrow} V(D)^{1/q}\,\lambda_D \\ &\leq C \sup_{D\downarrow} \mathrm{Cap}_{p,u,L,S}(D')\,\lambda_D \\ &\leq C\,\|f\|_{L^p(u)}\,, \end{aligned}$$

which gives $A \leq C$, and thus $A = B = C$.

□

Analogously, we have the next result.

Theorem 1.3.4 *Let $S : \mathcal{M}(X) \longrightarrow \mathcal{M}(X)$ be a homogeneous operator such that $|Sf|$ is an increasing function for every $f \in L$. Then,*

$$A := \sup_{f\in L} \frac{\|Sf\|_{L^{q,\infty}(v)}}{\|f\|_{L^p(u)}} = \sup_{I\uparrow} \frac{V(I)^{1/q}}{\mathrm{Cap}_{p,u,L,S}(I)} := B.$$

Moreover, if X is a topological space, and Sf is a continuous function for every $f \in L$, then

$$A = B = \sup_{I\uparrow} \frac{V(I)^{1/q}}{\mathrm{Cap}'_{p,u,L,S}(I)} := C.$$

We give some examples of application of our results. We collect known results and new ones.

1.3.1 Integral operators in $\mathbb{R}_+$

Let X be $\mathbb{R}_+$ with the usual topology, and μ the Lebesgue measure. A decreasing set is an interval $[0, x)$ or $[0, x]$, and the increasing sets are intervals of the form (x, ∞) or $[x, \infty)$.

If S is a homogeneous operator, it is easy to see in this context that

$$\mathrm{Cap}'_{p,u,L,S}([0, x]) = \left(\sup_{f\in L} \frac{|Sf(x)|}{\|f\|_{L^p(u)}}\right)^{-1} = \inf_{f\in L} \frac{\|f\|_{L^p(u)}}{|Sf(x)|}. \tag{1.21}$$

In fact, the inequality

$$\mathrm{Cap}'_{p,u,L,S}([0,x]) \geq \inf_{f\in L} \frac{\|f\|_{L^p(u)}}{|Sf(x)|},$$

is trivial. Take $f \in L$ such that $|Sf(x)| > 0$, and consider $g(y) = f(y)/|Sf(x)|$. We then have $|S(g)(x)| = 1$ and $\|g\|_{L^p(u)} = \|f\|_{L^p(u)} / |Sf(x)|$, and hence we have the reverse inequality.

We consider integral operators of the form

$$Sf(x) = \int_0^\infty K(x,y)\, f(y)\, dy, \tag{1.22}$$

where $K(x,y)$ is a measurable function.

By using our previous results, we can prove the following theorems due to J.A. Raposo (see [R]):

Theorem 1.3.5 *Suppose that the operator defined in (1.22) satisfies that $|Sf|$ is a decreasing function for all positive measurable f. Then,*

(i) If $0 < p < 1$, $\displaystyle\sup_{f\geq 0} \frac{\|Sf\|_{L^{q,\infty}(v)}}{\|f\|_{L^p(u)}} = \infty.$

(ii) If $p = 1$, $\displaystyle\sup_{f\geq 0} \frac{\|Sf\|_{L^{q,\infty}(v)}}{\|f\|_{L^p(u)}} = \sup_{x>0}\sup_{y>0} \frac{V([0,x])^{1/q} K(x,y)}{u(y)}.$

(iii) If $1 < p < \infty$, $\displaystyle\sup_{f\geq 0} \frac{\|Sf\|_{L^{q,\infty}(v)}}{\|f\|_{L^p(u)}} = \sup_{x>0} \frac{V([0,x])^{1/q}}{\left(\int_0^\infty K(x,s)^{p'} u(s)^{1-p'} ds\right)^{-1/p'}}.$

Proof. We apply Theorem 1.3.3 and observation (1.21) to our operator and to the class $L = \{f : f \geq 0\}$, and we have

$$\begin{aligned} \sup_{f\geq 0} \frac{\|Sf\|_{L^{q,\infty}(v)}}{\|f\|_{L^p(u)}} &= \sup_{x>0} V([0,x])^{1/q} \sup_{f\geq 0} \frac{|Sf(x)|}{\|f\|_{L^p(u)}} \\ &= \sup_{x>0} V([0,x])^{1/q} \sup_{f\geq 0} \frac{\|K(x,.)f(.)u(.)^{-1}\|_{L^1(u)}}{\|f\|_{L^p(u)}}. \end{aligned}$$

By duality we have

$$\sup_{f\geq 0} \frac{\|Sf\|_{L^{q,\infty}(v)}}{\|f\|_{L^p(u)}} = \left\|K(x,.)u(.)^{-1}\right\|_{(L^p(u))'},$$

and then, the result is obtained if we use the well-known identities

$$(L^p(u))' = \begin{cases} \{0\} & \text{if } 0 < p < 1, \\ L^\infty(u) & \text{if } \quad p = 1, \\ L^{p'}(u) & \text{if } \quad p > 1. \end{cases}$$

□

The *conjugate Hardy operator*

$$Qf(x) = \int_x^\infty f(y)\,\frac{dy}{y}$$

is an example of an operator satisfying the requirements of the previous theorem.

Analogously, using Theorem 1.3.4, we have:

Theorem 1.3.6 *Suppose that the operator defined in (1.22) satisfies that $|Sf|$ is an increasing function for all positive measurable f. Then,*

(i) *If* $0 < p < 1$, $\displaystyle\sup_{f\geq 0} \frac{\|Sf\|_{L^{q,\infty}(v)}}{\|f\|_{L^p(u)}} = \infty.$

(ii) *If* $p = 1$, $\displaystyle\sup_{f\geq 0} \frac{\|Sf\|_{L^{q,\infty}(v)}}{\|f\|_{L^p(u)}} = \sup_{x>0}\sup_{y>0} \frac{V([x,\infty])^{1/q}K(x,y)}{u(y)}.$

(iii) *If* $1 < p < \infty$, $\displaystyle\sup_{f\geq 0} \frac{\|Sf\|_{L^{q,\infty}(v)}}{\|f\|_{L^p(u)}} = \sup_{x>0} \frac{V([x,\infty])^{1/q}}{\left(\int_0^\infty K(x,s)^{p'}u(s)^{1-p'}ds\right)^{-1/p'}}.$

Some examples of operators satisfying the conditions in the last theorem are the *Hardy-type operators*. These are operators of the form

$$Sf(x) = \int_0^x K(x,y)\,f(y)\,dy,$$

where the kernel $K(x,y)$ satisfies

(i) $K(x,y) > 0$, for all $x > y > 0$, and K is increasing in x and decreasing in y.

(ii) There exists a constant $C > 0$ such that $K(x,y) \leq C(K(x,z) + K(z,y))$ if $0 < y < z < x$.

Theorem 1.3.7 *Suppose that the operator defined in (1.22) satisfies that $|Sf|$ is a decreasing function for all positive and decreasing measurable f. Then,* *(i) If* $0 < p \leq 1$, $\displaystyle\sup_{0\leq f\downarrow} \frac{\|Sf\|_{L^{q,\infty}(v)}}{\|f\|_{L^p(u)}} = \sup_{x>0} V([0,x])^{1/q} \sup_{y>0} \frac{\int_0^y K(x,s)\,ds}{U([0,y])^{1/p}}$.

(ii) If $1 < p < \infty$,

$$\begin{aligned}
\sup_{0\leq f\downarrow} \frac{\|Sf\|_{L^{q,\infty}(v)}}{\|f\|_{L^p(u)}} &\approx \sup_{x>0} V([0,x])^{1/q} \times \\
&\quad \left(\int_0^\infty \left(\int_0^y K(x,s)\,ds\right)^{r/p} U(y)^{-r/p} K(x,y)\,dy\right)^{1/r} \\
&\approx \sup_{x>0} V([0,x])^{1/q} \times \\
&\quad \left(\int_0^\infty \left(\int_0^y K(x,s)\,ds\right)^{r/q} U(y)^{-r/q} u(y)\,dy\right)^{1/r} \\
&\quad + \frac{\left(\int_0^\infty K(x,s)\,ds\right)^{1/q}}{U(\infty)^{1/p}}.
\end{aligned}$$

Proof. If we apply Theorem 1.3.3 and observation (1.21) to our operator and to the class $L = \{f : 0 \leq f \downarrow\}$, we have

$$\begin{aligned}
\sup_{0\leq f\downarrow} \frac{\|Sf\|_{L^{q,\infty}(v)}}{\|f\|_{L^p(u)}} &= \sup_{x>0} V([0,x])^{1/q} \sup_{0\leq f\downarrow} \frac{|Sf(x)|}{\|f\|_{L^p(u)}} \\
&= \sup_{x>0} V([0,x])^{1/q} \sup_{0\leq f\downarrow} \frac{\|f\|_{L^1(K(x,.))}}{\|f\|_{L^p(u)}}.
\end{aligned}$$

If $0 < p \leq 1$, by Theorem 1.1.3 we know that

$$\sup_{0\leq f\downarrow} \frac{\|f\|_{L^1(K(x,.))}}{\|f\|_{L^p(u)}} = \sup_{y>0} \frac{\int_0^y K(x,s)\,ds}{U([0,y])^{1/p}},$$

and thus obtaining the result. For $p > 1$, we apply Theorem 1.1.1 getting

$$\begin{aligned}
\sup_{0\leq f\downarrow} \frac{\|f\|_{L^1(K(x,.))}}{\|f\|_{L^p(u)}} &\approx \left(\int_0^\infty \left(\int_0^y K(x,s)\,ds\right)^{r/p} U(y)^{-r/p} K(x,y)\,dy\right)^{1/r} \\
&\approx \left(\int_0^\infty \left(\int_0^y K(x,s)\,ds\right)^{r/q} U(y)^{-r/q} u(y)\,dy\right)^{1/r} \\
&\quad + \frac{\left(\int_0^\infty K(x,s)\,ds\right)^{1/q}}{U(\infty)^{1/p}}.
\end{aligned}$$

□

If the kernel $K(x, y)$ takes the special form

$$K(x, y) = \frac{1}{x}\, a(y/x),$$

for a positive measurable function a, we obtain in last theorem the results of K. Andersen ([A]).

1.3.2 Integral operators on metric trees

A *metric tree* $\widetilde{T}$ is a connected graph without loops or cycles, where the edges are non-degenerate closed line segments whose endpoints are vertices that meet a finite number of edges. For every pair of points x and y in $\widetilde{T}$, there is a unique polygonal path in $\widetilde{T}$ joining x and y denoted $[x, y]$. If we fix a point o in $\widetilde{T}$, we can define the partial order $x \leq y$ if and only if $x \in [o, y]$.

$\widetilde{T}$ is endowed with the metric topology generated by the distance d between x and y, that is, the length of $[x, y]$, and with the one-dimensional Lebesgue measure.

If we parameterize $[x, y]$ by $s(t) = d(x, t)$, we define

$$\int_x^y f(t)\, dt = \int_0^{d(x,y)} f(t(s))\, ds,$$

for every $f \in L^1_{\mathrm{loc}}(\widetilde{T})$.

We consider integral operators of the form

$$Sf(x) = \int_o^x K(x, y)\, f(y)\, dy, \tag{1.23}$$

where $K : \widetilde{T} \times \widetilde{T} \longrightarrow \mathbb{C}$ is a measurable function.

Assume that the operator S is of Hardy-type, that is, the kernel K satisfies:

(i) $K(x, y) > 0$, for all $x > y > o$ in $\widetilde{T}$ and K is increasing in x and decreasing in y.

(ii) There exists a constant $C > 0$ such that $K(x, y) \leq C(K(x, z) + K(z, y))$, if $0 < y < z < x$.

In this context, Theorem 1.3.4 applies:

Theorem 1.3.8 *For a Hardy-type operator (1.23), we have:*

$$\sup_{f \geq 0} \frac{\|Sf\|_{L^{q,\infty}(v)}}{\|f\|_{L^p(u)}} = \sup_{I\uparrow} \frac{V(I)^{1/q}}{\mathrm{Cap}'_{p,u,L,S}(I)}.$$

We introduce now a theorem about the Hardy-Volterra operator on metric trees that extends a result of W.D. Evans, D.J. Harris and L. Pick ([EHP]). Our proof is somehow easier and follows the discretization technique used in Section 1. We need a proposition as a first step.

We recall that $\mathcal{D}(\widetilde{T})$ stands as the set of all covering families of decreasing sets in $\widetilde{T}$, and we set

$$\Delta_k = D_{k+1} \setminus D_k.$$

For such a family $\{D_k : k \in \mathbb{Z}\}$, we denote

$$\alpha_k := \inf \left\{ \|f\|_{L^p(u)} : \operatorname{supp}(f) \subset \Delta_{k-1}, \quad \int_o^t f(x)\,dx = 1 \quad \forall t \in \partial D_k \right\}.$$

Proposition 1.3.9 *Consider the Hardy-Volterra operator*

$$Sf(x) = \int_o^x f(y)\,dy, \tag{1.24}$$

defined for a function f in the metric tree $\widetilde{T}$. For $0 < p, q < \infty$, we have:

$$A := \sup_{f \geq 0} \frac{\|Sf\|_{L^q(v)}}{\|f\|_{L^p(u)}} \approx \sup_{0 \leq \{a_k\}} \sup_{\{D_k\} \subset \mathcal{D}(\widetilde{T})} \frac{\left(\displaystyle\sum_{k \in \mathbb{Z}} a_k^q \, V(\Delta_k) \right)^{1/q}}{\left(\displaystyle\sum_{k \in \mathbb{Z}} a_k^p \, \alpha_k^p \right)^{1/p}} := B.$$

Proof. For a fixed positive sequence $\{a_k : k \in \mathbb{Z}\}$ and a family $\{D_k\} \subset \mathcal{D}(\widetilde{T})$, consider a sequence of positive functions $\{f_k : k \in \mathbb{Z}\}$ such that:

- $\displaystyle\int_o^t f_k(x)\,dx = 1$ for all $t \in \partial D_k$,
- $\operatorname{supp} f_k \subset \Delta_{k-1}$.

Define $f(x) = \displaystyle\sum_{k \in \mathbb{Z}} a_k f_k(x)$. Then

$$||f||_{L^p(u)} = \left(\sum_{k \in \mathbb{Z}} a_k^p \, ||f_k||^p_{L^p(u)} \right)^{1/p}, \tag{1.25}$$

and by construction

$$\begin{aligned}
\|Sf\|^q_{L^q(v)} &= \int_{\widetilde{T}} \left(\int_o^x f(y)\,dy \right)^q v(x)\,dx \\
&= \sum_{k \in \mathbb{Z}} \int_{\Delta_k} \left(\int_o^x f(y)\,dy \right)^q v(x)\,dx \\
&\geq \sum_{k \in \mathbb{Z}} a_k^q \int_{\Delta_k} \left(\int_o^x f_k(y)\,dy \right)^q v(x)\,dx \geq \sum_{k \in \mathbb{Z}} a_k^q \, V(\Delta_k).
\end{aligned}$$

Combining this inequality and (1.25), we get

$$\Big(\sum_{k\in\mathbb{Z}} a_k^q\, V(\Delta_k)\Big)^{1/q} \le A\Big(\sum_{k\in\mathbb{Z}} a_k^p\, ||f_k||^p_{L^p(u)}\Big)^{1/p},$$

and taking the infimum in the right hand side of this expression over all positive sequences $\{f_k : k \in \mathbb{Z}\}$ satisfying our requirements, we have

$$\Big(\sum_{k\in\mathbb{Z}} a_k^q\, V(\Delta_k)\Big)^{1/q} \le A\Big(\sum_{k\in\mathbb{Z}} a_k^p\, \alpha_k^p\Big)^{1/p},$$

and this is $B \le A$ if we take supremum. Let us see that $A \lesssim B$. For a given positive f, consider the decreasing sets $D_k = \Big\{x \in \widetilde{T} : \int_o^x f(y)\, dy < 2^k\Big\}$. Then:

$$\begin{aligned}
\|Sf\|_{L^q(v)} &= \Big(\int_{\widetilde{T}}\Big(\int_o^x f(y)\, dy\Big)^q v(x)\, dx\Big)^{1/q} \\
&= \Big(\sum_{k\in\mathbb{Z}}\int_{\Delta_k}\Big(\int_o^x f(y)\, dy\Big)^q v(x)\, dx\Big)^{1/q} \\
&\le \Big(\sum_{k\in\mathbb{Z}} 2^{(k+1)q}\, V(\Delta_k)\Big)^{1/q} \\
&\le B\Big(\sum_{k\in\mathbb{Z}} 2^{(k+1)p}\, \alpha_k^p\Big)^{1/p}.
\end{aligned}$$

Now, if $t \in \partial D_k$, then $\displaystyle\int_o^t f(y)\, dy = 2^k$ and there is $t' \in \partial D_{k-1}$ such that

$$\int_o^t f(y)\chi_{\Delta_{k-1}}(y)\, dy = \int_{t'}^t f(y)\, dy = \int_o^t f(y)\, dy - \int_o^{t'} f(y)\, dy = 2^{k-1},$$

which implies that

$$\alpha_k \le 2^{1-k}||f\chi_{\Delta_{k-1}}||_{L^p(u)}.$$

Then,

$$\begin{aligned}
\|Sf\|_{L^q(v)} &\le 4B\Big(\sum_k ||f\chi_{\Delta_{k-1}}||^p_{L^p(u)}\Big)^{1/p} \\
&= 4B\Big(\int_{\widetilde{T}} f(x)^p u(x)\, dx\Big)^{1/p},
\end{aligned}$$

and taking the supremum over all positive functions, we have that $A \le 4B$.

□

As a direct consequence, we have:

Theorem 1.3.10 *For the Hardy-Volterra operator*

$$Sf(x) = \int_o^x f(y)\, dy,$$

defined in the metric tree $\widetilde{T}$*, and for* $0 < p, q < \infty$*, we have:*
(a) If $0 < p \le q < \infty$*, the following conditions are equivalent:*

(i) $S : L^p(u) \longrightarrow L^q(v)$.

(ii) There exists a constant $C > 0$ *such that*

$$V(\Delta_k)^{1/q} \le C\alpha_k,$$

for all $k \in \mathbb{Z}$ *and all* $\{D_k\} \subset \mathcal{D}(\widetilde{T})$.

(iii) There exists a constant $C > 0$ *such that*

$$V(I)^{1/q} \le C\, \mathrm{Cap}'_{p,u,S}(I),$$

for all increasing set $I \subset \widetilde{T}$.

(iv) $S : L^p(u) \longrightarrow L^{q,\infty}(v)$.

(b) If $0 < q < p < \infty$*, the following conditions are equivalent:*

(i) $S : L^p(u) \longrightarrow L^q(v)$.

(ii) There exists a constant $C > 0$ *such that*

$$\left(\sum_{k\in\mathbb{Z}} V(\Delta_k)^{r/q} \alpha_k^{-r} \right)^{1/r} \le C,$$

for all $\{D_k\} \subset \mathcal{D}(\widetilde{T})$.

Proof. For two positive sequences $\{u_k : k \in \mathbb{Z}\}$ and $\{v_k : k \in \mathbb{Z}\}$, the well-known embedding characterization between $\ell^q(\{v_k\})$ and $\ell^p(\{u_k\})$ is

$$\sup_{(a_k)\ge 0} \frac{\left(\sum_{k\in\mathbb{Z}} a_k^q\, v_k \right)^{1/q}}{\left(\sum_{k\in\mathbb{Z}} a_k^p\, u_k \right)^{1/p}} = \begin{cases} \displaystyle\sup_{k\in\mathbb{Z}} \frac{v_k^{1/q}}{u_k^{1/p}} & \text{if} \quad 0 < p \le q < \infty, \\ \displaystyle\left(\sum_{k\in\mathbb{Z}} v_k^{r/q} u_k^{-r/p} \right)^{1/r} & \text{if} \quad 0 < q < p < \infty. \end{cases}$$

Now, we use these embeddings, with $u_k = \alpha_k^p$ and $v_k = V(\Delta_k)$, and the previous proposition to get

$$\sup_{f\geq 0} \frac{\|Sf\|_{L^q(v)}}{\|f\|_{L^p(u)}} \approx \begin{cases} \displaystyle\sup_{\{D_k\}\subset\mathcal{D}(\widetilde{T})} \sup_{k\in\mathbb{Z}} \frac{V(\Delta_k)^{1/q}}{\alpha_k} & \text{if} \quad 0<p\leq q<\infty, \\ \displaystyle\sup_{\{D_k\}\subset\mathcal{D}(\widetilde{T})} \left(\sum_{k\in\mathbb{Z}} V(\Delta_k)^{r/q}\alpha_k^{-r}\right)^{1/r} & \text{if} \quad 0<q<p<\infty, \end{cases}$$

and this proves part (b), and also the equivalence between (i) and (ii) in part (a). By Theorem 1.3.8 with $K(x,y)=1$, (iii) and (iv) are equivalent. It is trivial that (i) implies (iv). It is now enough to see that (iii) implies (ii). Take a family $\{D_k : k\in\mathbb{Z}\}$ in $\mathcal{D}(\widetilde{T})$, and for each $k\in\mathbb{Z}$, consider an increasing set $I_k = \widetilde{T}\setminus D_k$. Observe that we trivially have that

$$\mathrm{Cap}'_{p,u,S}(I_k) \leq \alpha_k.$$

Using this and (iii), we have

$$V(\Delta_k) \leq V(I_k) \leq C\,\mathrm{Cap}'_{p,u,S}(I_k) \leq C\alpha_k.$$

□

Remarks 1.3.11

(i) Part (b) and $(i),(ii)$ of (a) of the preceding theorem are contained in [EHP], and the family of decreasing sets $\mathcal{D}(\widetilde{T})$ can be replaced by a smaller family of decreasing sets (called *maximal subtrees* there).

(ii) In the case $\widetilde{T}=\mathbb{R}_+$ and the range $0<p<1$, it is proved in Theorem 1.3.6 that the boundedness $S : L^p(u) \longrightarrow L^q(v)$ of the Hardy operator is not possible.

1.3.3 Hardy-Volterra operators on trees

A *tree* $T=(G,A)$ is a connected graph without circuits or cycles, consisting of a set of *vertices* G and a family A of two-elements subsets of G called *edges*. We identify a tree with the set of its vertices. We are interested in *locally finite* trees, that is, trees such that every vertex belongs to a finite number of edges.

A *path* in the tree $T=(G,A)$ is a finite sequence $x_0,x_1,\dots,x_n$ of vertices such that $\{x_i,x_{i+1}\}\in A$. In a tree, there exists a unique path $x_0,x_1,\dots,x_n$ joining two vertices x and y, that is, with $x_0=x$ and $x_n=y$, and such that $x_i\neq x_{i+2}$ for all $0\leq i\leq n-2$. We call this path a *geodesic* and we denote it by $[x,y]$ (or $[y,x]$).

Then the tree becomes a geodesic space, and also a metric space if we define the *distance* between x and y as the number of edges in the path $[x, y]$, that is, the *length* of $[x, y]$. As usual, we denote it by $d(x, y)$. Now, the vertices x and y are *neighbors* if $d(x, y) = 1$.

For a vertex x, we denote by $\deg(x)$ the number of its neighbor vertices. A tree is called *regular* if there exists $M \geq 1$ such that

$$2 \leq \deg(x) \leq M + 1, \tag{1.26}$$

for all $x \in T$.

We consider *rooted* trees, that is, trees with a fixed reference vertex o called *origin* of the tree. In a rooted tree, we can define a partial *order* structure: the vertex x is grater than or equal to the vertex y if y belongs to $[o, x]$. We denote it by $y \leq x$.

A function defined on a tree is a discrete function evaluated on each vertex and, if we endow T with the counting measure, a function is measurable if it is finite at each vertex. We refer to Chapter 2 for a complete introduction to trees.

The Hardy-Volterra operator in a tree is defined by

$$S_d f(x) = \sum_{y \in [o,x]} f(y), \tag{1.27}$$

for a function f.

In this new setting, Theorem 1.3.4 applies.

Theorem 1.3.12 *For the Hardy-Volterra operator (1.27), we have:*

$$\sup_{f \geq 0} \frac{\|S_d f\|_{L^{q,\infty}(v)}}{\|f\|_{L^p(u)}} = \sup_{I\uparrow} \frac{V(I)^{1/q}}{\mathrm{Cap}_{p,u,S}(I)}.$$

We introduce now some results in the spirit of Section 1.2 of this chapter. We will prove in Theorem 1.3.14 that the boundedness of the *discrete* Hardy-Volterra operator (1.27) in a tree T is equivalent to the boundedness of the *continuous* Hardy-Volterra operator (1.24) in a metric tree $\widetilde{T}$.

For a tree T, we consider a metric tree $\widetilde{T}$ such that there exists an embedding

$$i : T \hookrightarrow \widetilde{T},$$

satisfying:

(i) x is a vertex in T if and only if $i(x)$ is a vertex in $\widetilde{T}$.

(ii) i is an isometry, that is, for two vertices x, y in T

$$d(x,y) = \widetilde{d}(x,y),$$

where $\widetilde{d}$ is the metric in $\widetilde{T}$. As a consequence, the length of all the non-degenerate edges in $\widetilde{T}$ equals 1.

We can transfer the partial order from T to $\widetilde{T}$ so that if $o \in T$ is the origin in the tree, then $i(o) \in \widetilde{T}$ is the origin in the metric tree. We write the order in $\widetilde{T}$ also by $\xi \leq \zeta$. For a point $\xi \in \widetilde{T} \setminus i(T)$, there exist two unique vertices $\mathrm{up}(\xi)$ and $\mathrm{down}(\xi)$ in $i(T)$ such that

$$\xi \in [\mathrm{up}(\xi), \mathrm{down}(\xi)].$$

Choose $\mathrm{up}(\xi)$ to be the first vertex in the geodesic path from ξ to $i(o)$ (in this order, see Figure 1).

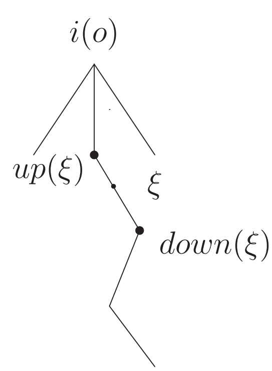


Figure 1: The vertices $\mathrm{up}(\xi)$ and $\mathrm{down}(\xi)$.

We then can define for $\xi, \zeta \in \widetilde{T}$:

(i) If ξ, ζ are vertices of $\widetilde{T}$, $\xi \leq \zeta$ in $\widetilde{T}$ if and only if $i^{-1}(\xi) \leq i^{-1}(\zeta)$.

(ii) If ξ is a vertex of $\widetilde{T}$ but ζ is not, $\xi \leq \zeta$ in $\widetilde{T}$ if and only if $i^{-1}(\xi) \leq i^{-1}(\mathrm{up}(\zeta))$.

(iii) If neither ξ nor ζ are vertices of $\widetilde{T}$, then (see Figure 2):

(a) If $\mathrm{up}(\xi) = \mathrm{up}(\zeta)$, $\xi \leq \zeta$ in $\widetilde{T}$ if and only if $i^{-1}(\mathrm{down}(\xi)) = i^{-1}(\mathrm{down}(\zeta))$ and $\widetilde{d}(i(o), \xi) \leq \widetilde{d}(i(o), \zeta)$.

(b) If $\mathrm{up}(\xi) \neq \mathrm{up}(\zeta)$, $\xi \leq \zeta$ in $\widetilde{T}$ if and only if $i^{-1}(\mathrm{down}(\xi)) \leq i^{-1}(\mathrm{up}(\zeta))$.

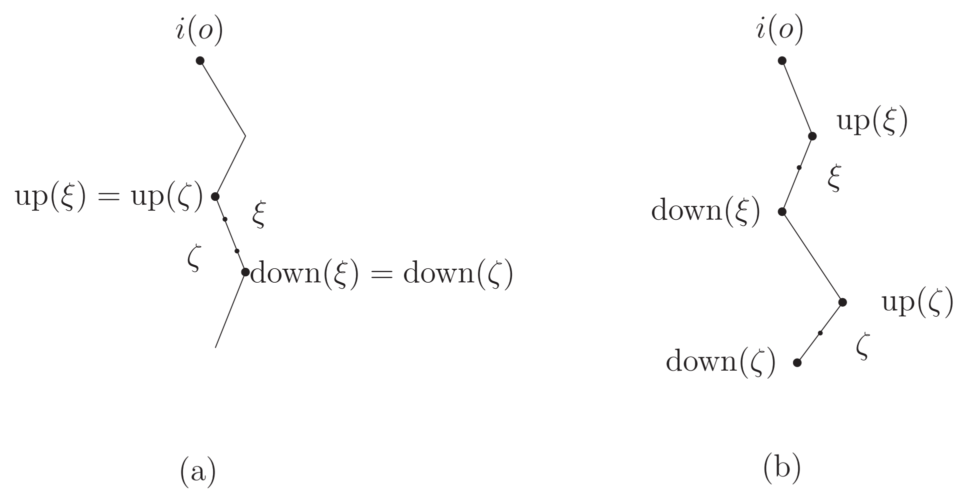


Figure 2: The two cases in (iii).

A simple example is $T = \mathbb{N} \cup \{0\}$ and $\widetilde{T} = \mathbb{R}_+$.

We set for every vertex $x \in T$,

$$\Omega(x) = \left\{ \xi \in \widetilde{T} : \widetilde{d}(i(x), \xi) < 1,\, i(x) \leq \xi \right\}.$$

We can extend every function f defined in T to a function $\widetilde{f}$ in $\widetilde{T}$ by using the expression

$$\widetilde{f}(\xi) = \sum_{x \in T} f(x)\, \chi_{\Omega(x)}(\xi),$$

i.e, f is constant on edges $[x, y)$, with $d(o, y) = d(o, x) + 1$. Analogously, $\widetilde{u}$ and $\widetilde{v}$ are the extended weights of u and v. In order to prove the result of the equivalence of the boundedness, we need a lemma:

Lemma 1.3.13 *Let T be a regular tree. Suppose that $S : L^p(\widetilde{u}) \longrightarrow L^q(\widetilde{v})$ for $0 < p, q < \infty$. Then we have:*

(i) If $0 < p \leq q < \infty$, there exists a constant $C > 0$ such that

$$v(x)^{1/q} \leq C u(x)^{1/p},$$

for all $x \in T$.

(ii) If $0 < q < p < \infty$, there exists a constant such that

$$\left(\sum_{x\in T} v(x)^{r/q}\, u(x)^{-r/p} \right)^{1/r} \leq C.$$

Proof. (i) For a fixed $x \in T$, we consider the function $g(\xi) = \chi_{[i(x),i(x')]}(\xi)$, where x' is a neighbor vertex of x and such that $x' \geq x$ (it exists by the assumption of regularity of the tree). Then, with $C_q = \int_x^{x'} \left(\int_x^{\xi} d\zeta \right)^q d\xi$, using the hypothesis, we have:

$$\begin{aligned} C_q\, v(x) &= \int_x^{x'} \left(\int_o^{\xi} g(\zeta)\, d\zeta \right)^q \widetilde{v}(\xi)\, d\xi \\ &\leq \int_{\widetilde{T}} \left(\int_o^{\xi} g(\zeta)\, d\zeta \right)^q \widetilde{v}(\xi)\, d\xi \\ &\leq C \left(\int_{\widetilde{T}} g(\xi)^p\, \widetilde{u}(\xi)\, d\xi \right)^{q/p} = C\, u(x)^{q/p}. \end{aligned}$$

(ii) Consider the positive function $g(\xi) = \sum_{x\in T} (v(x)^{r/q}\, u(x)^{-r/q})^{1/p} \chi_{\Omega(x)}(\xi)$. We obtain a lower bound for its $L^p(u)$-norm:

$$\begin{aligned} \int_{\widetilde{T}} g(\xi)^p\, \widetilde{u}(\xi)\, d\xi &= \sum_{x\in T} \int_{\Omega(x)} g(\xi)^p\, \widetilde{u}(\xi)\, d\xi \\ &= \sum_{x\in T} u(x)\, v(x)^{r/q}\, u(x)^{-r/q} \int_{\Omega(x)} d\xi \\ &\geq \sum_{x\in T} v(x)^{r/q}\, u(x)^{-r/p}, \end{aligned} \tag{1.28}$$

where we have used that, by the regularity of the tree (1.26),

$$1 \leq |\Omega(x)|_\ell \leq M,$$

where $|E|_\ell$ is the one-dimensional Lebesgue measure of a set E in $\widetilde{T}$. On the other hand,

$$\begin{aligned} \int_{\widetilde{T}} \left(\int_o^{\xi} g(\zeta)\, d\zeta \right)^q \widetilde{v}(\xi)\, d\xi &= \sum_{x\in T} \int_{\Omega(x)} \left(\int_o^{\xi} g(\zeta)\, d\zeta \right)^q \widetilde{v}(\xi)\, d\xi \\ &= \sum_{x\in T} v(x) \int_{\Omega(x)} \left(\int_o^{\xi} g(\zeta)\, d\zeta \right)^q d\xi \\ &\geq \sum_{x\in T} v(x) \int_{\Omega'(x)} \left(\int_o^{\xi} g(\zeta)\, d\zeta \right)^q d\xi, \end{aligned} \tag{1.29}$$

where we have denoted $\Omega'(x) = \left\{\xi \in \Omega(x) : \widetilde{d}(i(x),\xi) \geq \frac{1}{2}\right\}$. Now, for $\xi \in \Omega(x)$, we have

$$\begin{aligned}
\int_o^\xi g(\zeta)\,d\zeta &= \int_o^x g(\zeta)\,d\zeta + \int_x^\xi g(\zeta)\,d\zeta \\
&\geq \int_x^\xi g(\zeta)\,d\zeta \\
&= \Big(v(x)^{r/q}\,u(x)^{-r/q}\Big)^{1/p}\widetilde{d}(x,\xi).
\end{aligned}$$

We use this inequality in (1.29):

$$\begin{aligned}
\int_{\widetilde{T}}\bigg(\int_o^\xi g(\zeta)\,d\zeta\bigg)^q \widetilde{v}(\xi)\,d\xi &\geq \sum_{x\in T} v(x)\Big(v(x)^{r/q}\,u(x)^{-r/q}\Big)^{q/p}\int_{\Omega'(x)}\widetilde{d}(x,\xi)\,d\xi \\
&\geq \sum_{x\in T} v(x)^{1+r/p}\,u(x)^{-r/p}\frac{1}{4}\,|\Omega(x)| \\
&\geq \frac{1}{4}\sum_{x\in T} v(x)^{r/q}\,u(x)^{-r/p}.
\end{aligned}$$

Finally, this last inequality, (1.29) and the boundedness of the operator imply

$$\bigg(\sum_{x\in T} v(x)^{r/q}\,u(x)^{-r/p}\bigg)^{1/q} \leq C\bigg(\sum_{x\in T} v(x)^{r/q}\,u(x)^{-r/p}\bigg)^{1/p},$$

that is

$$\bigg(\sum_{x\in T} v(x)^{r/q}\,u(x)^{-r/p}\bigg)^{1/r} \leq C.$$

□

We are ready to prove the theorem. Recall that S is defined in (1.24) and S_d in (1.27).

Theorem 1.3.14 *Let T be a regular tree. If $1 \leq p < \infty$ and $0 < q < \infty$, for two weights u and v in T, the following conditions are equivalent:*

(i) $S_d : L^p(u) \longrightarrow L^q(v)$.

(ii) $S : L^p(\widetilde{u}) \longrightarrow L^q(\widetilde{v})$.

Proof. Suppose that S_d is bounded. For a positive $g : \widetilde{T} \longrightarrow \mathbb{R}_+$ in $L^p(\widetilde{u})$, consider the discrete function

$$f(x) := \max_{\left\{y\in T:\,\widetilde{d}(i(x),i(y))=1,\,y\geq x\right\}} \int_x^y g(\xi)\,d\xi.$$

It is not difficult to see, by Jensen's inequality, that

$$\begin{aligned} f(x)^p &= \max_{\left\{y\in T:\, \widetilde{d}(i(x),i(y))=1,\, y\geq x\right\}} \left(\int_x^y g(\xi)\,d\xi\right)^p \\ &\leq \max_{\left\{y\in T:\, \widetilde{d}(i(x),i(y))=1,\, y\geq x\right\}} \int_x^y g(\xi)^p\,d\xi \\ &\leq \int_{\Omega(x)} g(\xi)^p\,d\xi, \end{aligned} \tag{1.30}$$

and this easily implies $f \in L^p(u)$. By definition,

$$Sg(\xi) = \int_o^\xi g(\zeta)\,d\zeta \leq \sum_{y\in[o,x]} f(y) = S_d f(x), \tag{1.31}$$

if $\xi \in \Omega(x)$. This fact, the regularity and the hypothesis give:

$$\begin{aligned} \|Sg\|^q_{L^q(\widetilde{v})} &= \sum_{x\in T} \int_{\Omega(x)} \left(\int_o^\xi g(\zeta)\,d\zeta\right)^q \widetilde{v}(\xi)\,d\xi \\ &\leq \sum_{x\in T} (S_d f(x))^q\, v(x)\ |\Omega(x)| \\ &\leq M \sum_{x\in T} (S_d f(x))^q\, v(x) \\ &\leq MC\left(\sum_{x\in T} f(x)^p\, u(x)\right)^{q/p} \\ &\leq MC\left(\sum_{x\in T} u(x)\left(\int_{\Omega(x)} g(\xi)^p\,d\xi\right)\right)^{q/p} \\ &= MC\left(\int_{\widetilde{T}} g(\xi)^p\,\widetilde{u}(\xi)\,d\xi\right)^{q/p} = MC\,\|g\|_{L^p(\widetilde{u})}\,, \end{aligned}$$

where in the last inequality, (1.30) is used.

Suppose now that S is bounded. For a positive $f : T \longrightarrow \mathbb{R}_+$, consider the extended function $\widetilde{f}$ in $\widetilde{T}$. For a vertex $x \in T$, let us denote x_- the unique neighbor vertex of x such that $x_- \in [o,x]$. Therefore,

$$\begin{aligned} \|S_d f\|^q_{L^q(v)} &= \sum_{x\in T}\left(\sum_{y\in[o,x]} f(y)\right)^q v(x) \\ &\leq C_q\left(\sum_{x\in T}\left(\sum_{y\in[o,x_-]} f(y)\right)^q v(x) + \sum_{x\in T} f(x)^q\, v(x)\right) \\ &= C_q(I + II). \end{aligned}$$

We study I and II separately. By definition, we have that

$$\sum_{y\in[o,x_-]} f(y) = \int_o^x \widetilde{f}(\zeta)\,d\zeta.$$

This equality, the regularity and the hypothesis give the desired bound for I:

$$\begin{aligned}
I = \sum_{x\in T}\Bigg(\sum_{y\in[o,x_-]} f(y)\Bigg)^q v(x) &= \sum_{x\in T}\bigg(\int_o^x \widetilde{f}(\zeta)\,d\zeta\bigg)\,v(x)\\
&\le \frac{1}{2}\sum_{x\in T}\bigg(\int_o^x \widetilde{f}(\zeta)\,d\zeta\bigg)\bigg(\int_{\Omega(x)} \widetilde{v}(\xi)\,d\xi\bigg)\\
&\le \frac{1}{2}\sum_{x\in T}\int_{\Omega(x)}\bigg(\int_o^\xi \widetilde{f}(\zeta)\,d\zeta\bigg)\int_{\Omega(x)} \widetilde{v}(\xi)\,d\xi\\
&\le \frac{C}{2}\bigg(\sum_{x\in T}\int_{\Omega(x)} \widetilde{f}(\xi)^p\,\widetilde{u}(\xi)\,d\xi\bigg)^{q/p}\\
&= \frac{C}{2}\bigg(\sum_{x\in T} f(x)^p\,u(x)\,|\Omega(x)|_\ell\bigg)^{q/p}\\
&\le \frac{C}{2}M^{q/p}\,\|f\|^q_{L^p(u)}\,.
\end{aligned}$$

To proceed with II, we need to consider two cases:

(i) If $0 < p \le q < \infty$, we apply part (i) of Lemma 1.3.13 and the fact that $q/p \ge 1$, to get:

$$\begin{aligned}
II = \sum_{x\in T} f(x)^q\,v(x) &\le C\sum_{x\in T} f(x)^q\,u(x)^{q/p}\\
&\le C\bigg(\sum_{x\in T} f(x)^p\,u(x)\bigg)^{q/p} = C\,\|f\|^q_{L^p(u)}\,.
\end{aligned}$$

(ii) If $0 < q < p < \infty$, we apply Hölder inequality for $p/q > 1$ and part (ii) of Lemma 1.3.13, obtaining:

$$\begin{aligned}
II = \sum_{x\in T} f(x)^q\,v(x) &\le \bigg(\sum_{x\in T} f(x)^p\,u(x)\bigg)^{q/p}\bigg(\sum_{x\in T} v(x)^{r/q}\,u(x)^{-r/p}\bigg)^{1-q/p}\\
&\le C^{1-q/p}\bigg(\sum_{x\in T} f(x)^p\,u(x)\bigg)^{q/p} = C^{1-q/p}\,\|f\|^q_{L^p(u)}\,.
\end{aligned}$$

□

As a consequence of this result we can characterize the boundedness of S_d in the range $1 \leq p \leq q < \infty$.

Theorem 1.3.15 *Let T be a regular tree. If $1 \leq p \leq q < \infty$, the following conditions are equivalent:*

(i) $S_d : L^p(u) \longrightarrow L^{q,\infty}(v)$.

(ii) There exists a constant $C > 0$ such that

$$V(I)^{1/q} \leq C \,\mathrm{Cap}_{p,u,S_d}(I),$$

for all increasing sets of vertices $I \subset T$.

(iii) $S_d : L^p(u) \longrightarrow L^q(v)$.

(iv) $S : L^p(\widetilde{u}) \longrightarrow L^q(\widetilde{v})$.

Proof. That (i) and (ii) are equivalent is proved in Theorem 1.3.4. The equivalence between (iii) and (iv) is stated in the previous theorem. It is trivial that (iii) implies (i). So, it is enough to see the implication $(ii) \Rightarrow (iv)$.

By Theorem 1.3.10, this is equivalent to showing that (ii) implies the following condition: there exists a constant $C > 0$ such that

$$\widetilde{V}(\widetilde{I})^{1/q} \leq C \,\mathrm{Cap}_{p,\widetilde{u},S}(\widetilde{I}),$$

for all increasing sets $\widetilde{I} \subset \widetilde{T}$. Fix an increasing set $\widetilde{I} \subset \widetilde{T}$, and define

$$I = \left\{ x \in T : \, \widetilde{d}(i(x), \widetilde{I}) < 1 \right\},$$

which is an increasing set of vertices in T (see Figure 3). We define

$$\partial I = \{ x \in I : [o,x] \cap I = \{x\} \}.$$

Take $g : \widetilde{T} \longrightarrow \mathbb{R}_+$ such that $Sg(\xi) = 1$ for all $\xi \in \partial \widetilde{I}$. As we did in Theorem 1.3.14, we consider the discrete function

$$f(x) := \max_{\left\{ y \in T : \, \widetilde{d}(i(x), i(y)) = 1, \, y \geq x \right\}} \int_x^y g(\xi)\, d\xi.$$

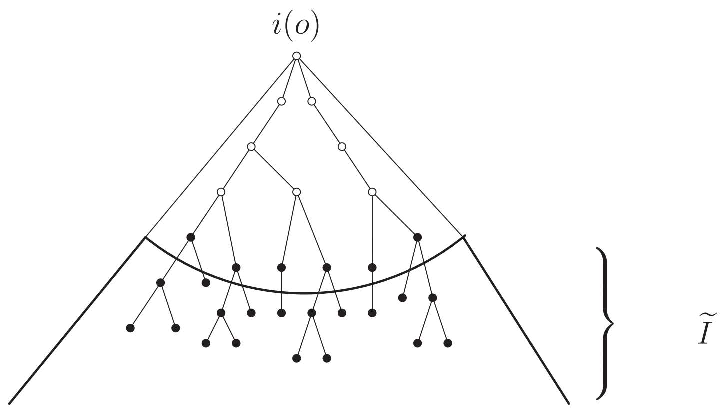


Figure 3: The sets $\widetilde{I}$ and I. The thick dots are the vertices of I.

By construction, for all $x \in \partial I$, there exists $\xi \in \partial\widetilde{I}$ such that $\xi \in \Omega(x)$ and this observation and inequality (1.31) lead to

$$1 = Sg(\xi) \leq S_d f(x)$$

for all $x \in \partial I$, and since $S_d f$ is an increasing function, we have

$$S_d f(x) \geq 1,$$

for all $x \in I$. But also, by using (1.30), we have:

$$\begin{aligned} \|f\|^p_{L^p(u)} &= \sum_{x\in T} f(x)^p\, u(x) \\ &\leq \sum_{x\in T} \left(\int_{\Omega(x)} g(\xi)^p\, d\xi \right) u(x) \\ &= \sum_{x\in T} \int_{\Omega(x)} g(\xi)^p\, \widetilde{u}(\xi)\, d\xi = \|g\|^p_{L^p(\widetilde{u})}\,. \end{aligned}$$

Consequently,

$$\mathrm{Cap}_{p,u,S_d}(I) = \inf \left\{ \|f\|_{L^p(u)} : \, S_d f(x) \geq 1 \, \forall x \in I \right\} \leq \|g\|_{L^p(\widetilde{u})}\,,$$

for all g with $Sg(\xi) = 1$ if $\xi \in \partial\widetilde{I}$. Taking infimum on the right hand side, we arrive at

$$\mathrm{Cap}_{p,u,S_d}(I) \leq \mathrm{Cap}'_{p,\widetilde{u},S}(\widetilde{I}).$$

Finally, this last inequality and the hypothesis give the desired implication:

$$\begin{aligned}
\widetilde{V}(\widetilde{I})^{1/q} &\leq \left(\sum_{x\in I} \widetilde{V}(\Omega(x))\right)^{1/q} \\
&= \left(\sum_{x\in I} v(x)\,|\Omega(x)|_\ell\right)^{1/q} \\
&\leq M^{1/q} V(I)^{1/q} \\
&\leq M^{1/q} C\,\mathrm{Cap}_{p,u,S_d}(I) \leq M^{1/q} C\,\mathrm{Cap}'_{p,\widetilde{u},S}(\widetilde{I}).
\end{aligned}$$

□

We can easily complete this result for the range $0 < p \leq 1$ and $p \leq q < \infty$.

Theorem 1.3.16 *Let T be a tree. If $0 < p \leq 1$ and $p \leq q < \infty$, we have*

$$A := \sup_{f\geq 0} \frac{\|S_d f\|_{L^q(v)}}{\|f\|_{L^p(u)}} = \sup_{x\in T} \frac{V(T(x))^{1/q}}{u(x)^{1/p}} := B,$$

where $T(x) = \{y \in T : y \geq x\}$.

Proof. Inequality $A \geq B$ is obtained by evaluating A for the function $f(y) = \chi_{\{x\}}(y)$, for $x \in T$. We observe that

$$S_d(\chi_{\{x\}})(y) = \chi_{T(x)}(y).$$

A function in the tree can be expressed in the following way

$$f(x) = \sum_{y\in T} f(y)\,\chi_{\{y\}}(x),$$

and by linearity of S_d, we obtain:

$$S_d f(x) = \sum_{y\in T} f(y)\,\chi_{T(y)}(x).$$

Thus, if $0 < p \leq 1$,

$$S_d f(x)^p \leq \sum_{y\in T} f(y)^p\,\chi_{T(y)}(x),$$

and therefore, by Fubini's theorem and the hypothesis,

$$\begin{aligned}
\|S_d f\|^p_{L^q(v)} &= \Big(\sum_{x\in T} S_d f(x)^q\, v(x)\Big)^{p/q} \\
&\leq \bigg(\sum_{x\in T}\Big(\sum_{y\in T} f(y)^p \chi_{T(y)}(x)\Big)\, v(x)\bigg)^{p/q} \\
&\leq \sum_{y\in T} f(y)^p \Big(\sum_{x\in T} \chi_{T(y)}(x)\, v(x)\Big)^{p/q} \\
&= \sum_{y\in T} f(y)^p\, V(T(y))^{p/q} \\
&\leq B \sum_{y\in T} f(y)^p\, u(y) = B\, \|f\|^p_{L^p(u)}\,,
\end{aligned}$$

and this leads to $A \leq B$.

□

Corollary 1.3.17 *Let T be a tree. If $0 < p \leq 1$ and $p \leq q < \infty$, the following conditions are equivalent:*

(i) $S_d : L^p(u) \longrightarrow L^{q,\infty}(v)$.

(ii) $S_d : L^p(u) \longrightarrow L^q(v)$.

(iii) There exists a constant $C > 0$ such that $V(T(x))^{1/q} \leq C\, u(x)^{1/p}$, for all $x \in T$.

Proof. The equivalence between (ii) and (iii) is the previous theorem. It is enough to see that (i) implies (iii). By Lemma 1.3.2, the boundedness (i) is equivalent to the existence of a constant $C > 0$ such that

$$\sup_{I\uparrow} V(I)^{1/q}\Big(\inf_{y\in I} S_d f(y)\Big) \leq C\Big(\sum_{y\in T} f(y)^p\, u(y)\Big)^{1/p},$$

for all positive f. For $x \in T$, take $I = T(x)$ and $f(y) = \chi_{\{x\}}(y)$. Then

$$\inf_{y\in T(x)} S_d f(y) = S_d(\chi_{\{x\}})(x) = \chi_{T(x)}(x) = 1,$$

and the last inequality becomes

$$V(T(x))^{1/q} \leq C\, u(x)^{1/p}.$$

□

Example 1.3.18 Set $T_k = \{x \in T : d(x,o) = k\}$ and let $|T_k|$ be its cardinal. The weights $v(x) = (2^k\,|T_k|)^{-1}$, if $x \in T_k$, and $u(x) = (2^{1-k})^{p/q}$, if $x \in T_k$, satisfy condition (iii) in the last corollary.

We can apply our results to the special case of $T = \mathbb{N} \cup \{0\}$ and $\widetilde{T} = \mathbb{R}_+$. Now the Hardy-Volterra operator is

$$S_d f(n) = \sum_{j=0}^{n} f_j,$$

for the sequence $\{f_j : j \geq 0\}$. We restrict our attention to the diagonal case $u = v$ and $p = q$, although our last results give an answer to more general cases. See [BSte] and [HK] for some results about the discrete Hardy-Volterra operator.

Theorem 1.3.19 *If $0 < p < \infty$, and $u := \{u_k : k \geq 0\}$ is a positive sequence, we have:*

(a) If $0 < p \leq 1$, the following conditions are equivalent:

(i) $S_d : \ell^p(u) \longrightarrow \ell^p(u)$.

(ii) $S_d : \ell^p(u) \longrightarrow \ell^{p,\infty}(u)$.

(iii) There exists a constant $C > 0$ such that $\displaystyle\sum_{j=n}^{\infty} u_j \leq C u_n$ *for all $n \geq 0$.*

(b) If $1 < p \leq \infty$, the following conditions are equivalent:

(i) $S_d : \ell^p(u) \longrightarrow \ell^p(u)$.

(ii) $S_d : \ell^p(u) \longrightarrow \ell^{p,\infty}(u)$.

(iii) There exists a constant $C > 0$ such that $\displaystyle\left(\sum_{j=n}^{\infty} u_j\right)^{1/p} \leq C\left(\sum_{j=0}^{n} u_j^{1-p'}\right)^{-1/p'}$.

Proof. Part (a) is directly obtained from the last corollary observing that in this case $U(T(n)) = \displaystyle\sum_{j=n}^{\infty} u_j$. In case (b), by Theorem 1.3.15, it is enough to see the equality

$$\mathrm{Cap}_{p,u,S_d}(I_n) = \left(\sum_{j=0}^{n} u_j^{1-p'}\right)^{-1/p'},$$

where we have denoted, for each $n \geq 0$, the increasing set $I_n = \{n, n+1, n+2, \dots\}$. The sequence $f := \{f_j : j \geq 0\}$ with

$$f_j = \frac{1}{\displaystyle\sum_{k=0}^{n} u_k^{1-p'}} \, u_j^{1-p'}$$

satisfies that $S_d f(k) \geq 1$, for all $k \geq n$, and thus

$$\mathrm{Cap}_{p,u,S_d}(I_n) \leq \|f\|_{\ell^p(u)} = \left(\sum_{j=0}^{n} u_j^{1-p'} \right)^{-1/p'}.$$

If $g := \{g_j : j \geq 0\}$ is a positive sequence such that $S_d(g)(k) \geq 1$ for all $k \geq n$, we then have

$$\begin{aligned} \left(\sum_{j=0}^{n} u_j^{1-p'} \right)^{-1/p'} &\leq \left(\sum_{j=0}^{n} u_j^{1-p'} \right)^{-1/p'} \left(\sum_{j=0}^{n} g_j \right) \\ &\leq \left(\sum_{j=0}^{\infty} g_j^p \, u_j \right)^{1/p} = \|g\|_{\ell^p(u)}, \end{aligned}$$

where the last inequality follows from Hölder's inequality. Taking the infimum in g, we finally obtain that

$$\left(\sum_{j=0}^{n} u_j^{1-p'} \right)^{-1/p'} \leq \mathrm{Cap}_{p,u,S_d}(I_n).$$

$\square$

Corollary 1.3.20 *If $u_j > 0$ for all $j \geq 0$, then for all $0 < p < \infty$, the following conditions are equivalent:*

(i) $S_d : \ell^p(u) \longrightarrow \ell^p(u)$.

(ii) $S_d : \ell^p(u) \longrightarrow \ell^{p,\infty}(u)$.

(iii) There exists a constant $C > 0$ such that $\displaystyle\sum_{j=n}^{\infty} u_j \leq C u_n$ for all $n \geq 0$.

Proof. By the preceding theorem, we only have to prove the result for the case $1 < p < \infty$, and, in fact, it is enough to prove the equivalence between:

$$(1) \quad \sum_{j=n}^{\infty} u_j \leq C u_n \quad \text{for all} \quad n \geq 0.$$

$$(2) \quad \left(\sum_{j=n}^{\infty} u_j \right)^{1/p} \leq C \left(\sum_{j=0}^{n} u_j^{1-p'} \right)^{-1/p'}.$$

One implication is trivial because clearly

$$\left(\sum_{j=0}^{n} u_j^{1-p'}\right)^{-1/p'} \le u_n^{1/p}.$$

Conversely, we write $0 < S_n = \sum_{j=n}^{\infty} u_j$. Then (1) reads as

$$S_n \le C(S_n - S_{n+1}),$$

or equivalently

$$S_{n+1} \le \kappa\, S_n, \tag{1.32}$$

where $0 < \kappa < 1$. With $\beta = p' - 1$, (2) reads as

$$S_n\left(\sum_{j=0}^{n}(S_j - S_{j+1})^{-\beta}\right)^{1/\beta} \le C.$$

Using (1.32), this is equivalent to

$$S_n\left(\sum_{j=0}^{n} S_j^{-\beta}\right)^{1/\beta} \le C,$$

and this expression is also equivalent to

$$S_n\left(\sum_{j=0}^{n-1} S_j^{-\beta}\right)^{1/\beta} \le C.$$

If we write $T_n = S_n^{\beta}$, this inequality is

$$T_n\left(\sum_{j=0}^{n-1} \frac{1}{T_j}\right) \le C.$$

We know, by (1.32), that $T_{n+1} \le \rho\, T_n$ with $0 < \rho < 1$, which implies that $T_n \le \rho^{n-j}\, T_j$ for all $0 \le j \le n-1$. Thus,

$$\begin{aligned} T_n\left(\sum_{j=0}^{n-1} \frac{1}{T_j}\right) &\le T_n\left(\sum_{j=0}^{n-1} \frac{\rho^{n-j}}{T_n}\right) \\ &= \rho^n \frac{1-\rho^{-n}}{1-\rho^{-1}} \\ &= \rho \frac{\rho^n - 1}{\rho - 1} \overset{n\to\infty}{\longrightarrow} \frac{\rho}{1-\rho} < \infty. \end{aligned}$$

□

1.3.4 Hardy operator in $\mathbb{R}^2_+$

Consider the case $X = \mathbb{R}^2_+$ equipped with the order defined by

$$(a_1, a_2) \leq (b_1, b_2)$$

if and only if $a_i \leq b_i$ for $i = 1, 2$. Let μ be the Lebesgue measure.

The Hardy operator is defined for a positive function $f : \mathbb{R}^2_+ \longrightarrow [0, \infty)$ by

$$Sf(x, y) = \frac{1}{xy} \int_0^x \int_0^y f(s, t)\, dtds,$$

for $(x, y) \in \mathbb{R}^2_+$. It is easy to see that Sf is decreasing if f is decreasing. In this new context, Theorem 1.3.3 applies with L being the class of decreasing functions.

Theorem 1.3.21 *For $0 < p, q < \infty$ we have*

$$\sup_{0 \leq f\downarrow} \frac{\|Sf\|_{L^{q,\infty}(v)}}{\|f\|_{L^p(u)}} = \sup_{D\downarrow} \frac{V(D)^{1/q}}{\mathrm{Cap}'_{p,u,L,S}(D)}.$$

It is possible to give a similar expression to (1.21) in this context. However, the computation of the capacity is not easy.

Lemma 1.3.22 *If $D \subset \mathbb{R}^2_+$ is a decreasing set, then*

$$A := \mathrm{Cap}'_{p,u,L,S}(D) = \left(\sup_{\|f\|_{L^p(u)}=1, f\downarrow} \left(\inf_{(x,y)\in\partial D} |Sf(x, y)| \right) \right)^{-1} := B.$$

Proof. We take $\varepsilon > 0$ and a decreasing f such that $Sf(x, y) = 1$ for all $(x, y) \in \partial D$ and such that $A - \varepsilon \geq \|f\|_{L^p(u)}$. Thus,

$$\begin{aligned} \frac{1}{A - \varepsilon} &\leq \frac{\inf_{(x,y)\in\partial D} Sf(x, y)}{\|f\|_{L^p(u)}} \\ &= \inf_{(x,y)\in\partial D} S(f/\|f\|_{L^p(u)})(x, y) \\ &\leq B. \end{aligned}$$

On the other hand, for a $\varepsilon > 0$, take a decreasing f such that $\|f\|_{L^p(u)} = 1$ and $\inf_{(x,y)\in\partial D} Sf(x, y) \geq B - \varepsilon$. Consider a decreasing g such that $0 \leq g \leq f$ and $Sg(x, y) = B - \varepsilon$ for all $(x, y) \in \partial D$. Therefore,

$$A \leq \frac{\|g\|_{L^p(u)}}{B - \varepsilon},$$

and finally

$$B - \varepsilon \quad \leq \quad \frac{\|g\|_{L^p(u)}}{A} \leq \frac{1}{A},$$

using that $\|g\|_{L^p(u)} \leq \|f\|_{L^p(u)} = 1$.

$\square$

Chapter 2

Non-linear rearrangement on trees

The classical decreasing rearrangement of a function f defined on a measure space (X, μ) is the function

$$f^\star(t) = \inf \{\lambda : \mu(\{x \in X : |f(x)| > \lambda\}) \leq t\}, \qquad t > 0.$$

It appears in the literature in the final part of the XIX$^{\text{th}}$ Century, in works of, for example, H.A. Schwarz ([Sc2]). The first systematic treatment is done by G.H. Hardy, J.E. Littlewood and G. Pólya in their book *Inequalities* ([HLP]). The paper [HL] by G.H. Hardy and J.E. Littlewood on the maximal function is an example of an important result where the decreasing rearrangement plays a fundamental role.

Let (X, μ) be a measure space. For every $0 < p < \infty$ and every weight in the positive real line, the Lorentz space $\Lambda^p_X(u)$ is defined as the set of μ-measurable functions f such that the functional

$$\|f\|_{\Lambda^p_X(u)} = \left(\int_0^\infty (f^\star(t))^p u(t)\, dt \right)^{1/p} \tag{2.1}$$

is finite. The Lorentz spaces were introduced in 1951 by G.G. Lorentz in [Lo] in the case $X = (0, l)$ and μ the Lebesgue measure, and they are generalitzations of the L^p and $L^{p,q}$ spaces. In this paper, G.G. Lorentz proved that in the case $p \geq 1$, the functional defined in (2.1) is a norm if and only if the weight u is decreasing. A. Haaker ([H]) studied these spaces in the case $X = \mathbb{R}^+$ with the Lebesgue measure and characterized the normability of the weak version of these spaces for all $0 < p < \infty$. In 1990, E. Sawyer proved that, in the case $X = \mathbb{R}^n$, the Lebesgue measure and $p > 1$, the Lorentz space $\Lambda^p_X(u)$ is normable if and only if the Hardy-Littlewood maximal operator is bounded from $\Lambda^p_X(u)$ to $\Lambda^p_X(u)$, which, in turn, is equivalent to the B_p condition on u found by M.A. Ariño and B. Muckenhoupt ([AM]). In

1993, M.J. Carro and J. Soria studied in [CSo1] the embedding of the Lorentz spaces with $X = \mathbb{R}^n$, as well as the characteritzation of their quasi-normability in terms of a doubling condition on the primitive of the weight. Later, these authors and A. García del Amo ([CGSo]) solved the normability for the case $p = 1$ and $X = \mathbb{R}^n$. In 1998, J. Soria studied the normability of the weak version of the Lorentz spaces in [So]. Recently, J.A. Raposo ([R] or the book [CRSo]) has studied the Lorentz spaces in the general context of a resonant measure space (see Definition 2.3 in Chapter 2 of [BS]). The discrete context of $X = \mathbb{N}^*$ has been studied by many different authors (see [R] or [CRSo] and the references therein).

The classical decreasing rearrangement of functions can be seen as a particular case of the theory of symmetrization, which has applications in potential theory or in PDE's, as the comparison theorems for solutions of PDE's. We refer to the work of A. Baernstein [B] for an introduction. Recently, S. Barza, L.E. Persson and J. Soria have introduced a decreasing rearrangement for functions defined in $\mathbb{R}^2_+$ ([BPSo2]). This new decreasing rearrangement is strongly linked with the geometry of $\mathbb{R}^2_+$, while the classical decreasing rearrangement introduced above does not take into account the geometry of the space X.

In the recent years, the study of trees has had a wide development. The study of Harmonic Analysis in trees begun in 1972 with the work of P. Cartier ([C]), and follows with M.H. Taibleson ([T]). A tree is an example of a discrete domain with a very rich geometric structure. In some occasions, it is taken as model for other non-discrete spaces where we are not able to solve a problem (see for example [FPR]). Sometimes, we can use results on trees to solve problems in other non-discrete contexts (see [DiB]). Examples of references about real and harmonic analysis on trees are [RT], [KP], [BCPT], [PW], [KPT], [ADiBU], [CP], [EHL], [NaS1], [NaS2] and [Le].

Our intention is to give a new decreasing rearrangement of functions defined in a homogeneous tree, which takes strongly into account the geometric structure of the tree. Then, we introduce the weighted Lorentz space related to our decreasing rearrangement and we characterize some normability properties of these spaces in terms of the weight. It is important to remark that the classical techniques do not work in our context due to the lack of algebraic structure, and trivial facts for the classical rearrangement of functions become difficult in the tree (see for example the monotonic condition proved in Proposition 2.2.20). Instead, we use combinatorial techniques.

In this context, we mention the works of A.R. Pruss [Pr1] and [Pr2], where a decreasing rearrangement on homogeneous trees is given by means of a ‘spiral-like’ ordering. We point out that this rearrangement is not useful for our purpose because it does no satisfy point (*iii*) of Definition 2.2.1.

The chapter is organized as follows: the first section is devoted to introduce the general facts about trees that will be useful; in the second section, we give the decreasing rearrangement for finite sets of vertices, and we prove that this definition is canonical (see Theorem 2.2.15) and that we have a monotonic property on this rearrangement (Proposition 2.2.20); in the third section, we introduce the decreasing rearrangement of functions defined in the homogeneous tree, we see some related properties, and we give an easier alternative way for rearranging a function (Theorem 2.3.10); in the fourth section, we study the Hardy-Littlewood inequality for our decreasing rearrangement and we find conditions on the functions in order to get the saturation of this inequality (Theorems 2.4.15 and 2.4.17); in section five, we introduce the Lorentz spaces related to our decreasing rearrangement and we characterize when these spaces become Banach spaces (Theorem 2.5.12); in section six, we apply ours results to finite trees and regular trees.

In order to illustrate the results, we include some figures. In all the figures, the thick vertices are the vertices of a set or of the support of a function that we are considering. The hollow vertices are the vertices out of the set or the support of the function.

2.1 Definitions

We give the basic definitions and facts about trees that we will need.

Many different definitions of trees have been given. We will use the one in [FTN], as well as some notation from there.

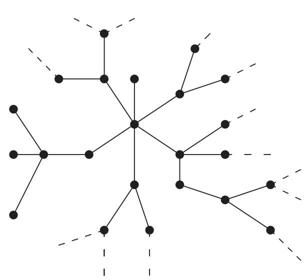
A **graph** is a pair (G, A) consisting of a set of **vertices** G and a family A of two-elements subsets of G called **edges**. When for a given two vertices x and y in G, we have that $\{x, y\} \in A$, we say that x and y are **adjacent** vertices or simply **neighbor** vertices.

A **path** in the graph (G, A) is a finite sequence $x_0, x_1, \ldots, x_n$ of vertices such that $\{x_i, x_{i+1}\} \in A$. A graph is **connected** if for every two vertices x and y in G, there exists a path joining x and y, that is, there exists a path $x_0, x_1, \ldots, x_n$ with

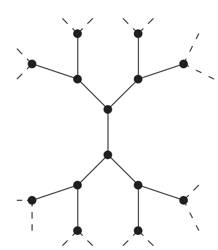
$x_0 = x$ and $x_n = y$. A **chain** is a path $x_0, x_1, \ldots, x_n$ such that $x_i \neq x_{i+2}$, and a chain $x_0, x_1, \ldots, x_n$ with $x_0 = x_n$ is a **circuit** or a **cycle**.

A **tree** $T = (G, A)$ is a connected graph without circuits or cycles. In the sequel, we will identify a tree with the set of its vertices. We are interested in **nonfinite** and **locally finite** trees, that is trees with an infinite family of vertices, but such that every vertex belongs to a finite number of edges. A tree may be graphically represented as shown in Figure 4.

The **degree** of a vertex is the number of edges to which it belongs or equivalently, is the number of neighbors it has. A tree is called **homogeneous** if the degree is independent of the choice of the vertex. Then the tree is called homogeneous of degree $q+1$ if the number of neighbors is $q+1$, $q \geq 1$. An example of a homogeneous tree of degree $q+1 = 3$ is shown in Figure 4 (b).



(a)



(b)

Figure 4: Examples of trees.

In a tree, there exists a unique chain joining two vertices x and y. We call this chain a **geodesic** and we denote it by $[x, y]$ (or $[y, x]$). Then the tree becomes a geodesic space, and also a metric space if we define the **distance** between x and y as the number of edges in the path $[x, y]$, that is, the **length** of $[x, y]$. As usual, we denote it by $d(x, y)$. Now, the vertices x and y are neighbors if $d(x, y) = 1$.

An **infinite chain** is an infinite sequence $x_0, x_1, x_2, \ldots$ of vertices such that x_i and x_{i+1} are neighbors and $x_i \neq x_{i+2}$ for all $i \geq 0$. We define an equivalent relation on the set of infinite chains: $x_0, x_1, x_2, \ldots$ and $y_0, y_1, y_2, \ldots$ are equivalent if they share infinite vertices (see Figure 5). This means that there is an integer $n \in \mathbb{Z}^+$ such that $x_k = y_{n+k}$ for every k large enough. The **boundary** of the tree ∂T is the set of equivalent classes of infinite chains.

A **rooted** tree is a tree with a fixed reference vertex o called **origin** of the tree. In the sequel, every tree will be a rooted tree. The boundary of a rooted tree is the set of all infinite chains starting at o. The boundary can be viewed as the set of points at infinity. We can represent it graphically as shown in Figure 6. Every point of the boundary is an infinite chain starting at o, in other words, the boundary is the set of end points.

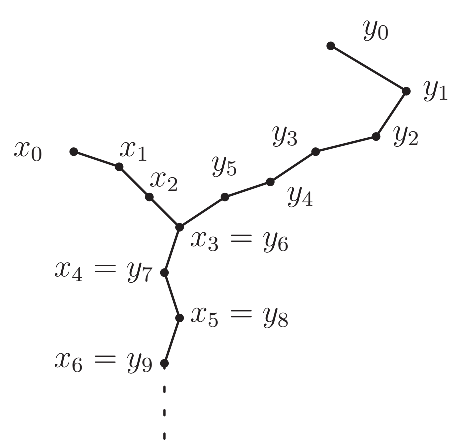


Figure 5: Two equivalent infinite chains.

If x is a vertex and ω is a point at the boundary of the tree, there exists a unique infinite chain in the equivalent class of ω starting at x. Then we say that this infinite chain is the **infinite geodesic** joining x and ω. We denote it by $[x, \omega)$. A **doubly infinite chain** is a sequence of vertices indexed by the integers $\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$ such that x_i and x_{i+1} are neighbors and $x_i \neq x_{i+2}$ for all $i \in \mathbb{Z}$. A doubly infinite chain identifies two boundary points, and we call it the infinite geodesic joining these points. If ω and ν are boundary points, we denote it by (ω, ν).

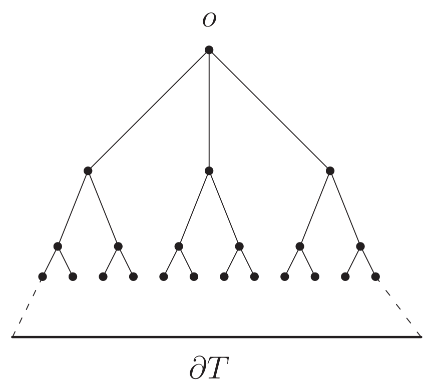


Figure 6: A rooted homogeneous tree of degree 3 and its boundary.

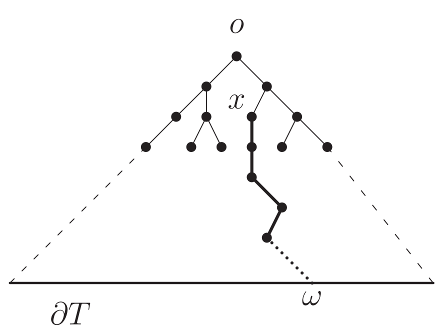


Figure 7: The infinite geodesic between x and ω.

For every x in T, we write the geodesic joining o to x by

$$\{x(0) = o, x(1), \dots, x(n) = x\} := [o, x],$$

where $k = d(o, x(k))$ and $n = d(o, x)$. Analogously, for a point ω in the boundary, we write the geodesic joining o to ω by

$$\{\omega(0) = o, \omega(1), \dots, \omega(n), \dots\} := [o, \omega).$$

The **confluent** vertex of the vertices x and y is the unique vertex $c(x, y)$ such that the geodesics $[o, c(x, y)]$, $[c(x, y), x]$ and $[c(x, y), y]$ meet only at $c(x, y)$. If ω and ν are two boundary points, we can also define their confluent vertex $c(\omega, \nu)$ as the unique vertex $c(\omega, \nu)$ such that the geodesics $[o, c(\omega, \nu)]$, $[c(\omega, \nu), \omega)$ and $[c(\omega, \nu), \nu)$ meet only at $c(\omega, \nu)$ (see next figure).

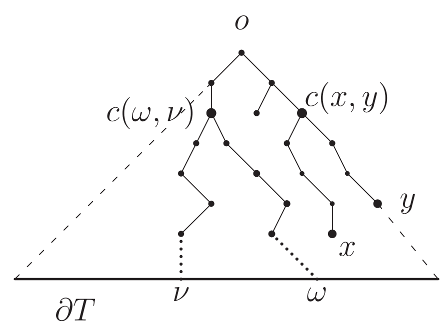


Figure 8: Confluent vertices.

The **tent** of x, $T(x)$, is the set of vertices such that x belongs to the geodesic between o and those vertices and the **shadow** of x, $I(x)$, is the set of end points in ∂T such that x belongs to the geodesic between o and these boundary points (see Figure 9). We can write it as:

$$\begin{aligned} T(x) &= \{y \in T : x \in [o,y]\}, \\ I(x) &= \{\omega \in \partial T : x \in [o,\omega)\}. \end{aligned}$$

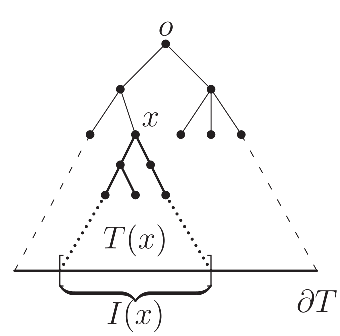
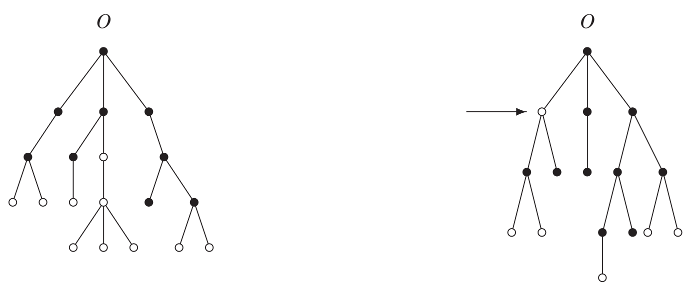


Figure 9: The tent $T(x)$ and the shadow $I(x)$ of x.

Finally, we can define a partial **order** structure: the vertex x is grater than or equal to the vertex y if y belongs to $[o,x]$. We denote it by $y \leq_o x$. In other words:

$$y \leq_o x \Leftrightarrow y \in [o,x] \Leftrightarrow x \in T(y).$$



(a) (b)

Figure 10: A decreasing set (a) and a non-decreasing set (b).

A function defined on a tree is a discrete function evaluated on each vertex. We are interested in monotone functions. A function is **decreasing** if $f(x) \leq f(y)$ whenever $y \leq_o x$. A set of vertices E in T is a **decreasing set** if whenever $x \in E$, then we have that $y \in E$ for all y such that $y \leq_o x$, that is, χ_E is a decreasing function.

2.2 Rearranging finite sets

We will define the decreasing rearrangement for finite sets of vertices, which is the first step to introduce a decreasing rearrangement of functions. Then, we show the canonicity of this definition in the sense that it is mostly independent on the choice of the parameters.

In the sequel, T will be a homogeneous tree of degree $q+1$. We choose a reference vertex o as its origin, and we then write the tree as T_o. The underlying measure is the counting measure and if E is a finite set of vertices in T_o, we note by $|E|$ its cardinal.

The so-called "Layer cake" formula allows us to reconstruct a positive measurable function by means of its level sets (see [LL]). That is:

$$f(x) = \int_0^\infty \chi_{\{t \in T_o : f(t) > \lambda\}}(x)\, d\lambda.$$

In order to introduce a decreasing rearrangement of functions, it is enough to define a rearrangement for finite sets and then use this formula to get a decreasing rearrangement of any function with finite level sets.

Definition 2.2.1 A map between finite sets of vertices in T_o

$$E \longrightarrow E^*,$$

is a **decreasing rearrangement of finite sets** if the following conditions are satisfied:

(i) E^* is decreasing.

(ii) $|E| = |E^*|$.

(iii) If E is decreasing, then $E^* = E$.

(iv) If $D \subset E$, then $D^* \subset E^*$.

To this aim, it will be necessary to introduce an order structure in the boundary of the tree.

Let T_k be the set of vertices at distance k from o. Observe that $\{T_k : k \geq 0\}$ is a disjoint partition of T_o and that

$$\begin{aligned} |T_0| &= 1 \\ |T_k| &= (q+1)q^{k-1},\ k \geq 1. \end{aligned}$$

The tree T_o is then a countable union of vertices. On the other hand, the boundary of the tree is uncountable, as we will see later.

Let $\mathcal{F}_0$ be the interval $[0, (q+1)q^{-1}]$. For every $k \geq 1$, let $\mathcal{F}_k$ be the set of all q-adic intervals of the form $(j\,q^{-k}, (j+1)q^{-k})$, with $j \in \mathbb{Z}^+$ and $0 \leq j \leq (q+1)q^{k-1} - 1$, contained in the interval $[0, (q+1)q^{-1}]$. Set $\mathcal{F} = \bigcup_k \mathcal{F}_k$.

Definition 2.2.2 An **admissible map** σ is a bijection between the tree T_o and $\mathcal{F}$ satisfying :

$$\begin{array}{llcll} (a) & \sigma(T_k) & = & \mathcal{F}_k, & \\ (b) & \sigma(x) & \subset & \sigma(y) & \text{if} \quad y \leq_o x. \end{array}$$

Using an admissible map σ, we can define another bijection between ∂T_o and a subset of the interval $[0, (q+1)q^{-1}]$ as follows: with the exception of the q-adic numbers, every point λ in the interval $[0, (q+1)q^{-1}]$ is uniquely identified with the sequence $\{I_k(\lambda) : k \geq 0\}$ of q-adic intervals with length q^{-k} containing it. Then by the definition of σ,

$$\{\sigma^{-1}(I_k(\lambda)) : k \geq 0\}$$

is an infinite geodesic in T_o starting at o, that is, it is a point $\omega(\lambda)$ in ∂T_o. Conversely, a point ω in the boundary of T_o can be viewed as an infinite geodesic

$$[o, \omega) = \{\omega(0) = o, \omega(1), \omega(2), \ldots\}.$$

Then $\{\sigma(\omega(k)) : k \geq 0\}$ is a sequence of q-adic intervals satisfying that for every $k \geq 0$ that

$$\sigma(\omega(k+1)) \subset \sigma(\omega(k)),$$

and therefore it determines a unique point $\lambda(\omega)$ in $[0, (q+1)q^{-1}]$.

It is natural to also denote by σ this new bijection, and we also call it an **admissible map**. We have that

$$\sigma : \partial T_o \longrightarrow [0, (q+1)q^{-1}] \setminus N(q) \tag{2.2}$$

is a one-to-one correspondence between ∂T_o onto the interval $[0, (q+1)q^{-1}]$ minus the set of q-adic numbers $N(q)$, and hence ∂T_o is uncountable.

Examples 2.2.3 We give two examples of possible admissible maps that will be used in what follows. Suppose that T_o is a homogeneous tree of degree $q+1 = 3$. We need first to label the vertices. Recall that T_k is the set of vertices at distance k from o. Then denote

$$T_k = \{x_{0,k}, x_{2,k}, \ldots, x_{n_k,k}\},$$

where $n_k + 1 = 3{\cdot}2^{k-1}$ is the total number of edges in T_k, and hence, for all k and j, the vertices $x_{2j,k+1}$ and $x_{2j+1,k+1}$ are the adjacent vertices of $x_{j,k}$ in T_{k+1}. Denote $I_{0,0} = [0, 3{\cdot}2^{-1}]$ and

$$I_{j,k} = (j{\cdot}2^{-k}, (j+1){\cdot}2^{-k}),$$

for $k \geq 1$ and $0 \leq j \leq 3{\cdot}2^{k-1} - 1$. Observe that $I_{2j,k+1}$ and $I_{2j+1,k+1}$ are the dyadic intervals contained in $I_{j,k}$. Then define two admissible maps σ and σ' as follows:

(i) $\sigma(x_{j,k}) = I_{j,k}$ for all j and k.

(ii) $\sigma'(x_{0,0}) = I_{0,0}$.
For T_1, set $\sigma'(x_{0,1}) = I_{1,1}$, $\sigma'(x_{1,1}) = I_{0,1}$ and $\sigma'(x_{2,1}) = I_{2,1}$.
For T_2 set $\sigma'(x_{0,2}) = I_{2,2}$, $\sigma'(x_{1,2}) = I_{3,2}$, $\sigma'(x_{2,2}) = I_{0,2}$, $\sigma'(x_{3,2}) = I_{1,2}$,

$\sigma'(x_{4,2}) = I_{5,2}$ and $\sigma'(x_{5,2}) = I_{4,2}$.

Now, we can proceed by choosing the dyadic intervals so that if we have $\sigma'(x_{2j,k+1}) = I_{2i,k+1}$, then $\sigma'(x_{2j+1,k+1}) = I_{2i+1,k+1}$.

We give now our two examples graphically. We draw the tree in an ordered way following the labels, from left to right. Then we draw the images of the vertices in the same way.

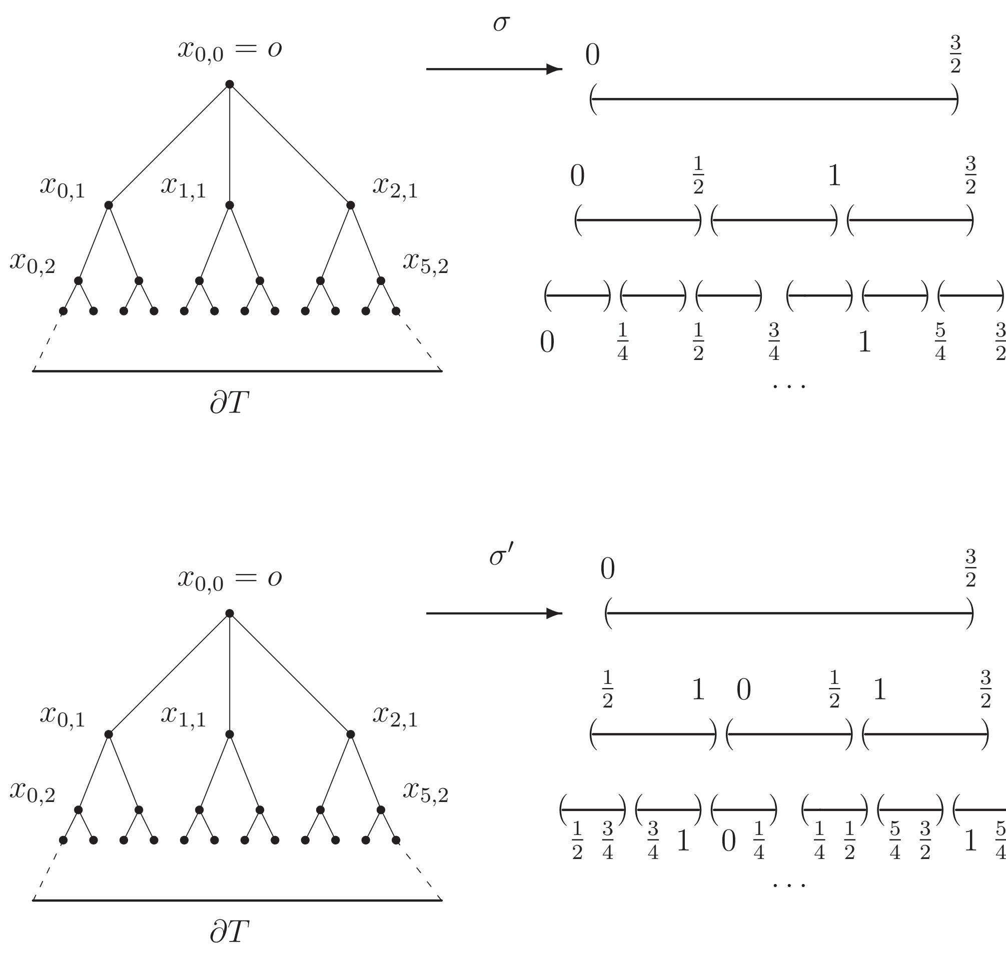


Figure 11: The two maps σ and σ'.

Now, we can introduce an order relation in ∂T_o by using an admissible map.

Definition 2.2.4 Let σ be an admissible map as in (2.2). Given ω and ω' in ∂T_o, we define

$$\omega \leq_\sigma \omega'$$

if and only if

$$\sigma(\omega) \leq \sigma(\omega').$$

In the sequel, every admissible map σ will be called an **order** in ∂T_o. Observe that for both maps in Examples 2.2.3, the largest points at the boundary are those in $I(x_{2,1})$. In the case of σ, the smallest are in $I(x_{0,1})$, but for σ', they are in $I(x_{1,1})$.

For two given disjoint sets A and B in $[0,(q+1)q^{-1}]$, we will write $A < B$, if $x < y$ for all $x \in A$ and all $y \in B$. Analogously, for two given disjoint sets A and B in ∂T_o, we will write $A <_\sigma B$, if $x <_\sigma y$ for all $x \in A$ and all $y \in B$.

Lemma 2.2.5 *Let σ be an order in ∂T_o and x a vertex in T_o. Then the shadow of x, $I(x)$, is an interval of the boundary, in the sense that if $\omega, \omega' \in I(x)$ with $\omega <_\sigma \omega'$, then for all $\nu \in \partial T_o$ satisfying*

$$\omega <_\sigma \nu <_\sigma \omega',$$

we have $\nu \in I(x)$.

Proof. Set $n = d(o,x)$. Take $\omega, \omega' \in I(x)$ and $\nu \in \partial T_o$ satisfying $\omega <_\sigma \nu <_\sigma \omega'$. We write

$$\begin{aligned}
[o,\omega) &= \{\omega(0), \omega(1), \omega(2), \ldots\}, \\
[o,\omega') &= \{\omega'(0), \omega'(1), \omega'(2), \ldots\}, \\
[o,\nu) &= \{\nu(0), \nu(1), \nu(2), \ldots\},
\end{aligned}$$

where $\omega(k), \omega'(k), \nu(k) \in T_k$, and by hypothesis $\omega(n) = \omega'(n) = x$. We want to see that $\nu(n) = x$ or equivalently, that there exists an integer $k \geq n$ such that $x \leq_o \nu(k)$. Take

$$k = \min\{j : \omega(j) \neq \omega'(j), \omega(j) \neq \nu(j), \omega'(j) \neq \nu(j)\}.$$

The fact that $\omega, \omega' \in I(x)$ implies that $k \geq n$. By the definition of σ, there exist three different q-adic intervals $J_k(\omega)$, $J_k(\omega')$ and $J_k(\nu)$ in $\mathcal{F}_k$ such that

$$\begin{aligned}
\sigma(\omega(k)) &= J_k(\omega), \\
\sigma(\omega'(k)) &= J_k(\omega'), \\
\sigma(\nu(k)) &= J_k(\nu),
\end{aligned}$$

and by hypothesis $J_k(\omega) < J_k(\nu) < J_k(\omega')$. By the properties of σ, we know that $I(\omega)$ and $I(\omega')$ are subintervals of $\sigma(x)$, and then we also have that $I(\nu) \subset \sigma(x)$, that is, $x \leq_o \nu(k)$.

□

We need to define some new concepts:

Definition 2.2.6 For a finite set of vertices E, the **boundary** of E, ∂E, is the set of vertices x of E such that no bigger vertices than x belong to E. Explicitly,

$$\partial E = \{x \in E : T(x) \cap E = \{x\}\}.$$

Observe that by this definition, if x and y are different boundary points in ∂E, then $I(x) \cap I(y) = \emptyset$. See Figure 12.

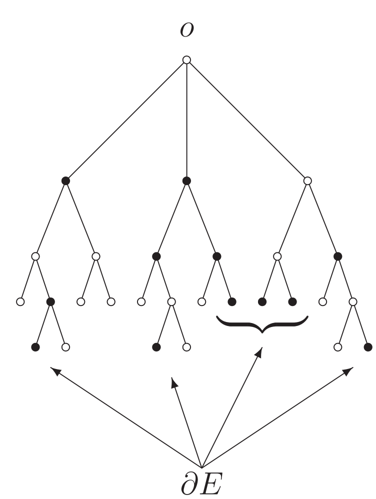


Figure 12: The boundary of a finite set E, ∂E.

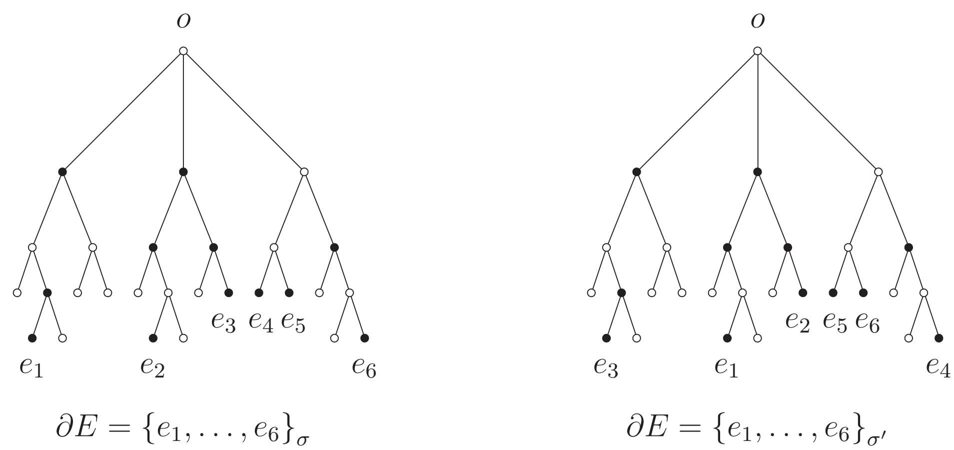


Figure 13: The set E and its boundary ordered by σ and σ'.

In view of the previous lemma, using an order σ in ∂T_o, it makes sense to introduce the following notation on the boundary of every finite set E. We write:

$$\partial E = \{e_1, e_2, \cdots, e_n\}_\sigma \,, \tag{2.3}$$

if $n = n(E) = |\partial E|$, and for all $1 \le k \le n$, $e_k \in \partial E$ and $I(e_k) <_\sigma I(e_{k+1})$ if $k \ne n$. See Figure 13, where the boundary of the set E of the previous figure is ordered by using the two maps of Examples 2.2.3, supposing the tree is drawn with the labels from left to right.

Recall that for a vertex e in T_o, we write $[o, e] = \{e(0) = o, e(1), \ldots, e(n)\}$ with $n = d(o, e)$ and $k = d(o, e(k))$. Now we are able to define the **rearrangement of finite sets**:

Definition 2.2.7 Let σ be an order in ∂T_o, and let E be a finite set of vertices in T_o with boundary $\partial E = \{e_1, e_2, \cdots, e_n\}_\sigma$. Set

$$\mathcal{R}_{(o,\sigma,0)}(E) := E,$$

and then recursively define, for every $0 \le k \le n-1$, the sets

$$\mathcal{R}_{(o,\sigma,k+1)}(E) := \big(\mathcal{R}_{(o,\sigma,k)}(E) \setminus [o, e_{k+1}]\big) \cup [o, e_{k+1}(s)],$$

where $s + 1 = s(k) + 1 = \big|\mathcal{R}_{(o,\sigma,k)}(E) \cap [o, e_{k+1}]\big|$. Finally, the **decreasing rearrangement** of E is

$$\mathcal{R}_{(o,\sigma)}(E) := \mathcal{R}_{(o,\sigma,n)}(E).$$

This definition needs a practical explanation. What do we do in every step of the construction of the rearranged set? We count the number of vertices we have in a fixed geodesic from o to a boundary point e_k, then we erase them, and finally we fill in the same geodesic with the same number of vertices we had, but we now impose they are adjacent vertices starting from o.

We can give another easy explanation of the rearrangement by using marbles: think that every vertex in E is a marble and only those. Then suppose that we can lift up one by one every geodesic leaving o at the bottom, following a fixed order, so that marbles can go down until they fill up the empty vertices near o. The process stops when we have proceeded with all the geodesics. We observe that what we get is a decreasing set. See Figures 14 and 15 for more details.

We now want to study the dependence of the defined decreasing rearrangement in terms of the origin o and the order σ chosen in T.

An **automorphism** of the tree is a bijective map of the set of vertices onto itself which preserves the edges. In fact, a map is an automorphism if and only if it is an **isometry** of the tree, with respect to the natural metric defined in the previous section, and then we trivially have the following lemma.

Lemma 2.2.8 *If ϱ is an automorphism of the tree, then*

$$\varrho([x,y]) = [\varrho(x), \varrho(y)],$$

for all x and y in T.

As a consequence, we have that:

(i) Every automorphism ϱ takes the rooted tree T_o into the rooted tree $T_{\varrho(o)}$, and we can extend the automorphism to the boundary in a natural way: if

$$\omega = \{\omega(0) = o, \omega(1), \ldots, \omega(n), \ldots\} \in \partial T_o,$$

then define

$$\varrho(\omega) := \{\varrho(\omega(0)) = \varrho(o), \varrho(\omega(1)), \ldots, \varrho(\omega(n)), \ldots\} \in \partial T_{\varrho(o)}.$$

(ii) For all x and y in T, $x \leq_o y$ if and only if $\varrho(x) \leq_{\varrho(o)} \varrho(y)$.

(iii) For all $x \in T_o$, $\varrho(T(x)) = T(\varrho(x))$, where $T(\varrho(x))$ is taken with respect to the induced order in $T_{\varrho(o)}$.

(iv) For all $x \in T_o$, $\varrho(I(x)) = I(\varrho(x))$, where $I(\varrho(x))$ is taken with respect to the induced order in $T_{\varrho(o)}$.

Let us see the effect of an automorphism over an order in ∂T_o:

Lemma 2.2.9 *Let σ be an order in ∂T_o and ϱ an automorphism of the tree. Then there exists a unique admissible map σ' in $T_{\varrho(o)}$ such that*

$$\omega \leq_{\sigma'} \nu \Longleftrightarrow \varrho^{-1}(\omega) \leq_\sigma \varrho^{-1}(\nu),$$

for all ν and ω of ∂T.

Proof. Take $\sigma' = \sigma \cdot \varrho^{-1}$, which satisfies the required condition. By Definition 2.2.2, we need to prove that

$$\begin{array}{lccccc} (a) & \sigma'(T_k) & = & \mathcal{F}_k, & & \\ (b) & \sigma'(y) & \subset & \sigma'(x) & \text{if} & x \leq_{\varrho(o)} y, \end{array}$$

where now

$$T_k = \{x \in T : d(x, \varrho(o)) = k\}.$$

Since ϱ is an isometry, we have that

$$\varrho(\{x \in T : d(x,o) = k\}) = \{x \in T : d(x, \varrho(o)) = k\},$$

and then (a) is easily derived. By the previous lemma, we have that if x and y are vertices in T, then,

$$x \leq_{\varrho(o)} y \Longleftrightarrow \varrho^{-1}(x) \leq_o \varrho^{-1}(y),$$

and by definition, $\sigma(\varrho^{-1}(y)) \subset \sigma(\varrho^{-1}(x))$, that is, $\sigma'(y) \subset \sigma'(x)$, which is (b). To see the uniqueness, suppose there exists an admissible map μ satisfying

$$\omega \leq_\mu \nu \Longleftrightarrow \varrho^{-1}(\omega) \leq_\sigma \varrho^{-1}(\nu).$$

Fix $k \geq 0$ and consider as before T_k the set of vertices at distance k from $\varrho(o)$. Using the notation introduced in (2.3), we denote

$$T_k = \left\{x_1, x_2, \ldots, x_{n(k)}\right\}_{\sigma'},$$

where $n(k) = (q+1)q^{k-1}$ and $I(x_i) <_{\sigma'} I(x_{i+1})$ for all $1 \leq i \leq n(k)-1$. By hypothesis, we have that

$$T_k = \left\{x_1, x_2, \ldots, x_{n(k)}\right\}_{\sigma'} = \left\{\varrho^{-1}(x_1), \varrho^{-1}(x_2), \ldots, \varrho^{-1}(x_{n(k)})\right\}_\sigma,$$

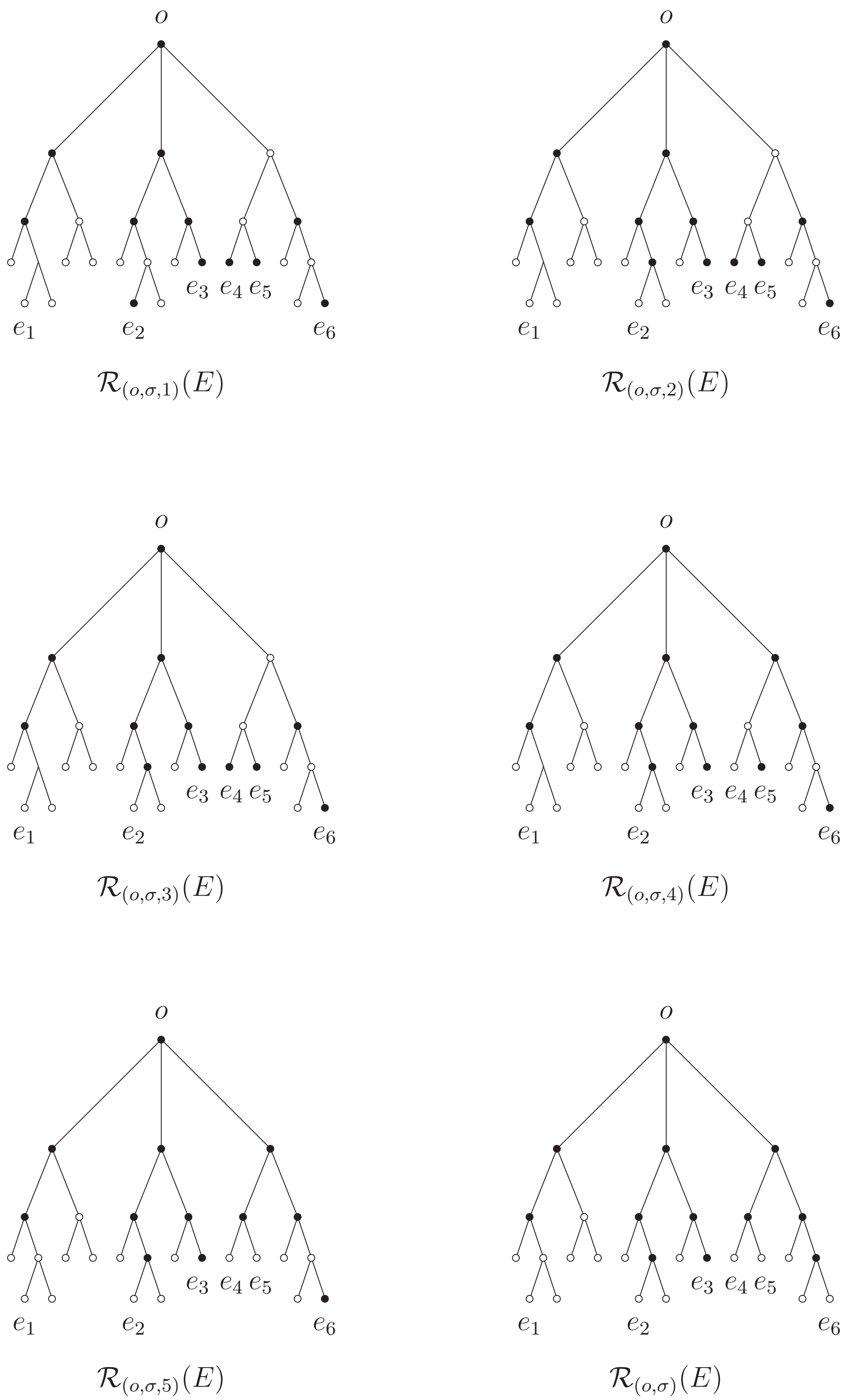


Figure 14: Rearranging the set E of Figure 13, using the order σ.

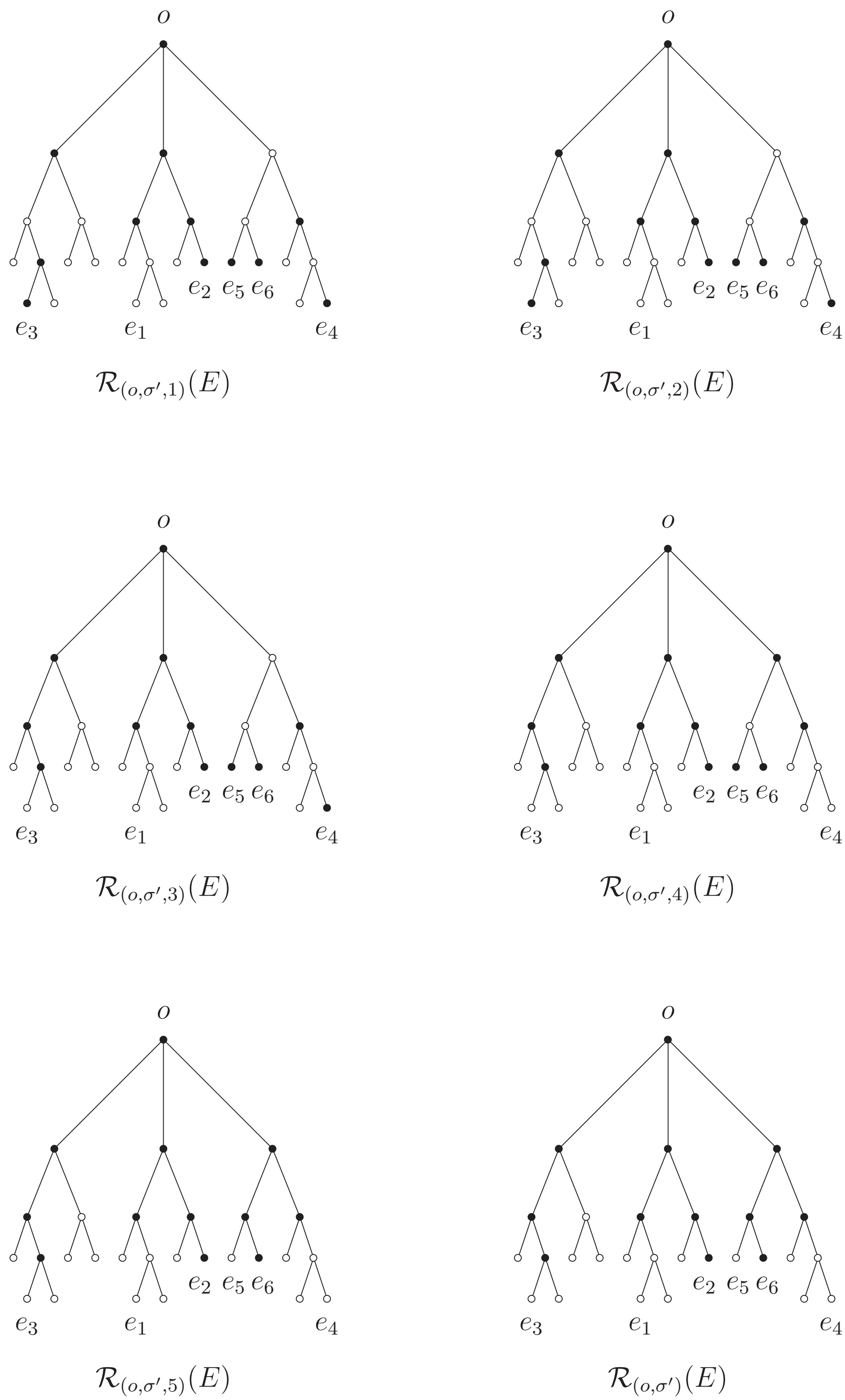


Figure 15: Rearranging the set E of Figure 13, using the order σ'.

and therefore we have

$$T_k = \left\{x_1, x_2, \ldots, x_{n(k)}\right\}_{\sigma'} = \left\{x_1, x_2, \ldots, x_{n(k)}\right\}_{\mu},$$

and this is only possible if for all $1 \leq i \leq n(k) - 1$

$$\sigma'(x_i) = \mu(x_i),$$

that is $\sigma' \equiv \mu$.

$\square$

We describe the action of an automorphism over the boundary of a finite set in the tree:

Lemma 2.2.10 *Let E be a finite set of vertices and ϱ an automorphism of the tree. If σ is an order in ∂T_o and $\partial E = \{e_1, e_2, \ldots, e_n\}_\sigma$ then*

$$\partial(\varrho(E)) = \{\varrho(e_1), \varrho(e_2), \ldots, \varrho(e_n)\}_{\sigma'},$$

where $\sigma' = \sigma \cdot \varrho^{-1}$ is an order in $\partial T_{\varrho(o)}$.

Proof. Let us see first that $\varrho(\partial E) = \partial(\varrho(E))$ by using the consequences of Lemma 2.2.8:

$$\begin{aligned} e \in \partial E \quad &\Leftrightarrow \quad E \cap T(e) = \{e\} \\ &\Leftrightarrow \quad \varrho(E) \cap \varrho(T(e)) = \{\varrho(e)\} \\ &\Leftrightarrow \quad \varrho(E) \cap T(\varrho(e)) = \{\varrho(e)\} \\ &\Leftrightarrow \quad \varrho(e) \in \partial(\varrho(E)). \end{aligned}$$

Finally, by using the previous lemma and the consequences of Lemma 2.2.8, we get for all $1 \leq i \leq n - 1$:

$$\begin{aligned} I(e_i) <_\sigma I(e_{i+1}) \quad &\Leftrightarrow \quad \varrho(I(e_i)) <_{\sigma'} \varrho(I(e_{i+1})) \\ &\Leftrightarrow \quad I(\varrho(e_i)) <_{\sigma'} I(\varrho(e_{i+1})). \end{aligned}$$

$\square$

This lemma says that it is equivalent to order the boundary of any finite set with respect to σ and to order the boundary of the image of the set given by the automorphism ϱ by means of the order $\sigma' = \sigma \cdot \varrho^{-1}$. We can now explain the action of an automorphism over the decreasing rearrangement of a set:

Theorem 2.2.11 *Let σ be an order in ∂T_o and ϱ an automorphism of the tree. Then*

$$\varrho(\mathcal{R}_{(o,\sigma)}(E)) = \mathcal{R}_{(\varrho(o),\sigma\cdot\varrho^{-1})}(\varrho(E)).$$

Proof. Set $\sigma' = \sigma \cdot \varrho^{-1}$. It is enough to see that

$$\varrho(\mathcal{R}_{(o,\sigma,k)}(E)) = \mathcal{R}_{(\varrho(o),\sigma',k)}(\varrho(E)),$$

for all $0 \leq k \leq n$, where $n = |\partial E| = |\partial(\varrho(E))|$. Let us show it by induction. If $k = 0$ it follows because

$$\varrho(E) = \varrho(\mathcal{R}_{(o,\sigma,0)}(E)) = \mathcal{R}_{(\varrho(o),\sigma',0)}(\varrho(E)).$$

Suppose it is true for $k \geq 1$. By definition we have

$$\varrho(\mathcal{R}_{(o,\sigma,k+1)}(E)) = \varrho\left(\mathcal{R}_{(o,\sigma,k)}(E) \setminus [o, e_{k+1}]\right) \cup \varrho([o, e_{k+1}(s)]),$$

where $s{+}1 = s(k){+}1 = \left|\mathcal{R}_{(o,\sigma,k)}(E) \cap [o, e_{k+1}]\right|$. Using the consequences of Lemma 2.2.8, we have

$$\varrho(\mathcal{R}_{(o,\sigma,k+1)}(E)) = \left(\varrho(\mathcal{R}_{(o,\sigma,k)}(E)) \setminus [\varrho(o), \varrho(e_{k+1})]\right) \cup [\varrho(o), \varrho(e_{k+1}(s))].$$

By the hypothesis of induction, we then have that

$$\varrho(\mathcal{R}_{(o,\sigma,k+1)}(E)) = \left(\mathcal{R}_{(\varrho(o),\sigma',k)}(\varrho(E)) \setminus [\varrho(o), \varrho(e_{k+1})]\right) \cup [\varrho(o), \varrho(e_{k+1}(s))],$$

and we also observe that

$$\begin{aligned} s+1 &= s(k)+1 = \left|\mathcal{R}_{(o,\sigma,k)}(E) \cap [o, e_{k+1}]\right| = \left|\varrho(\mathcal{R}_{(o,\sigma,k)}(E)) \cap [\varrho(o), \varrho(e_{k+1})]\right| \\ &= \left|\mathcal{R}_{(\varrho(o),\sigma',k)}(\varrho(E)) \cap [\varrho(o), \varrho(e_{k+1})]\right|. \end{aligned}$$

Now, using this equality and Lemma 2.2.10 which says, roughly speaking, that both rearrangements are compatible in some sense, we then have by recalling the definition of the decreasing rearrangement that

$$\begin{aligned} \varrho(\mathcal{R}_{(o,\sigma,k+1)}(E)) &= \left(\mathcal{R}_{(\varrho(o),\sigma',k)}(\varrho(E)) \setminus [\varrho(o), \varrho(e_{k+1})]\right) \cup [\varrho(o), \varrho(e_{k+1}(s))] \\ &= \mathcal{R}_{(\varrho(o),\sigma',k+1)}(\varrho(E)). \end{aligned}$$

□

In [FTN], it is shown that in a homogeneous tree, there are only three kind of isometries:

- An isometry ϱ is a **rotation**, if there exists a vertex x such that $\varrho(x) = x$.
- An isometry ϱ is an **inversion**, if there exist two neighbor vertices x and y such that $\varrho(x) = y$ and $\varrho(y) = x$.
- An isometry ϱ is a **translation**, if there exist a doubly infinite chain

$$\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$$

and an integer k such that $\varrho(x_j) = x_{j+k}$ for all $j \in \mathbb{Z}$.

Corollary 2.2.12 *Let σ and σ' be two orders in T_o. Then, there exists a unique rotation ϱ of center o such that*

$$\omega \leq_{\sigma'} \nu \Longleftrightarrow \varrho^{-1}(\omega) \leq_{\sigma} \varrho^{-1}(\nu),$$

for all ω and ν in ∂T_o.

Proof. Define $\varrho^{-1} = \sigma^{-1} \cdot \sigma' : T \longrightarrow T$, which is clearly a bijection. Trivially we have that $\varrho(o) = o$. Let us see that it preserves the edges. Take $x \in T_n$ and $y \in T_{n+1}$ with $d(x, y) = 1$. Then,

$$\begin{aligned}
x \in T_n,\ y \in T_{n+1},\ d(x,y) = 1 \quad &\Leftrightarrow \quad \sigma'(x) \in \mathcal{F}_n,\ \sigma'(y) \in \mathcal{F}_{n+1},\ \sigma'(y) \subset \sigma'(x) \\
&\Leftrightarrow \quad \sigma^{-1}(\sigma'(x)) \in T_n,\ \sigma^{-1}(\sigma'(y)) \in T_{n+1}, \\
&\qquad\qquad d(\sigma^{-1}(\sigma'(x)), \sigma^{-1}(\sigma'(y)) = 1 \\
&\Leftrightarrow \quad \varrho(x) \in T_n,\ \varrho(y) \in T_{n+1},\ d(\varrho(x), \varrho(y)) = 1.
\end{aligned}$$

The rest of the result is a consequence of Lemma 2.2.9.

□

Corollary 2.2.13 *Let o and o' be two vertices in T. Then, there exists a translation τ in the tree such that*

$$\tau(T_o) = T_{o'}.$$

Proof. Simply take a doubly infinite chain passing through o and o', and consider the translation τ along this infinite chain such that $\tau(o) = o'$.

□

Corollary 2.2.14 *Let o and o' be two vertices, and σ and σ' be two orders in ∂T_o and $\partial T_{o'}$ respectively. Then there exists an automorphism ϱ such that:*

- $\varrho(T_o) = T_{o'}$,
- $\omega \leq_{\sigma'} \nu \Longleftrightarrow \varrho^{-1}(\omega) \leq_\sigma \varrho^{-1}(\nu)$, *for all* ν *and* ω *in* ∂T.

Proof. By Lemma 2.2.9 and Corollary 2.2.13, there exists a translation τ such that:

- $\tau(T_o) = T_{o'}$,
- $\omega \leq_\eta \nu \Longleftrightarrow \tau^{-1}(\omega) \leq_\sigma \tau^{-1}(\nu)$, for all ν and ω in ∂T,

where $\eta = \sigma \cdot \tau^{-1}$. By Corollary 2.2.12, there exists a unique rotation δ of center o' such that

$$\omega \leq_{\sigma'} \nu \Longleftrightarrow \delta^{-1}(\omega) \leq_\eta \delta^{-1}(\nu),$$

for all ω and ν in ∂T. Then, the automorphism $\varrho = \delta^{-1} \cdot \tau$, satisfies:

- $\varrho(T_o) = \delta^{-1}(\tau(T_o)) = \delta^{-1}(T_{o'}) = T_{o'}$,
- For all ν and ω in ∂T:

$$\begin{aligned} \omega \leq_{\sigma'} \nu &\Longleftrightarrow \delta^{-1}(\omega) \leq_\eta \delta^{-1}(\nu) \\ &\Longleftrightarrow \tau^{-1}(\delta(\omega)) \leq_\sigma \tau^{-1}(\delta(\nu)) \\ &\Longleftrightarrow \varrho^{-1}(\omega) \leq_\sigma \varrho^{-1}(\nu). \end{aligned}$$

□

As a final consequence of this corollary and Theorem 2.2.11, we have the following theorem, which says that our rearrangement is canonical in the sense that **if we know how to rearrange a set with respect to an origin o and an order σ, then we know how to rearrange it with respect to any origin and any order**.

Theorem 2.2.15 *Let o and o' be two vertices, and σ and σ' be two orders in ∂T_o and $\partial T_{o'}$ respectively. Then, there exists an automorphism ϱ such that:*

$$\varrho(\mathcal{R}_{(o,\sigma)}(E)) = \mathcal{R}_{(o',\sigma')}(\varrho(E)).$$

Remark 2.2.16 From now on, and as a consequence of this theorem, we will not need to specify the origin and the order that we are using to rearrange a set. So, we will always assume that we have chosen an origin o and an order σ, and we will denote the **decreasing rearrangement** of any set E as E^*, that is

$$E^* := \mathcal{R}_{(o,\sigma)}(E). \tag{2.4}$$

We will also use the notation

$$\mathcal{R}_k(E) = \mathcal{R}_{(o,\sigma,k)}(E),$$

for all $k \geq 0$. Moreover, we denote

$$x \leq y \Longleftrightarrow x \in [o, y],$$

for $x, y \in T$, and also

$$\omega \leq \nu \Longleftrightarrow \sigma(\omega) \leq \sigma(\nu),$$

for $\omega, \nu \in \partial T$. But we will keep the notation of the boundary of a finite set given in (2.3), in order not to forget the meaning of this notation. Furthermore, in all figures that we will draw, we will always use the order σ given in (i) of Examples 2.2.3, that is, we will always order the boundary of any set from left to right, and all the trees will be homogeneous of degree 3 for simplicity.

It is easy to see that conditions (i), (ii) and (iii) of Definition 2.2.1 are trivially satisfied by our transformation. To see condition (iv), we need some new facts. First, as we are working with finite sets, it is enough to see this condition with D and $E = D \cup \{x\}$, where x is a vertex in $T \setminus D$. Now, we need to understand how the boundary of the set D can change when we add a new vertex (see Figure 16).

Lemma 2.2.17 *Let D be a finite set of vertices and $x \in T \backslash D$. Consider $E = D \cup \{x\}$. Then we have one of the following situations:*

(i) $\partial E = \partial D$ *if and only if* $T(x) \cap D \neq \emptyset$.

(ii) There exists a unique $y \in \partial D$ *such that* $\partial E = (\partial D \setminus \{y\}) \cup \{x\}$ *if and only if there exists a unique* $y \in \partial D$ *such that* $x \in T(y)$.

(iii) $\partial E = \partial D \cup \{x\}$ *if and only if* $T(x) \cap D = \emptyset$ *and* $x \notin T(y)$ *for all* $y \in \partial D$.

Proof. (i) If $\partial E = \partial D$, then $x \notin \partial D$ and by definition $T(x) \cap D \neq \emptyset$.

Conversely, if $T(x) \cap D \neq \emptyset$, there exists $z \in D$, with $z \neq x$ because $x \notin D$, such that $z \in T(x) \cap D$, and then $x \notin \partial E$. Now, if $y \in \partial D$, then $T(y) \cap E = \{y\}$ and hence $\partial D \subset \partial E$ (if $T(y) \cap E \neq \{y\}$, then $T(y) \cap E = \{y, x\}$ and we get a contradiction because $z \in T(x) \cap D \subset T(y) \cap E = \{y, x\}$). We also have that if $y \in \partial E$, then $T(y) \cap E = \{y\}$ and $y \neq x$ by hypothesis. Therefore $T(y) \cap D = \{y\}$ and thus $y \in \partial D$, in other words, $\partial E \subset \partial D$.

(ii) If $x \in \partial E$ and there exists a unique $y \in \partial D$ with $T(y) \cap E = \{y, x\}$, then $x \in T(y)$.

Conversely, if there exists (a necessarily unique) $y \in \partial D$ with $x \in T(y)$, then by definition $y \notin \partial E$ and also $T(x) \cap D = \emptyset$, that is, $x \in \partial E$. If $z \in \partial D \setminus \{y\}$, then $T(z) \cap D = \{z\}$ and $z \neq y$. Therefore $T(z) \cap E = \{z\}$, that is $z \in \partial E$ (if $T(z) \cap E = \{z, x\}$, then $x \in T(z)$ and hence $z = y$ getting a contradiction). On the other hand, if $z \in \partial E$ and $z \neq x$ and $z \neq y$, then $T(z) \cap E = \{z\}$ and consequently $T(z) \cap D = \{z\}$, that is $z \in \partial D$.

(iii) If $\partial E = \partial D \cup \{x\}$, then clearly $T(x) \cap D = \emptyset$ and if there exists $y \in \partial D$ such that $x \in T(y)$, then $y \notin \partial E$ contradicting the fact that $\partial D \subset \partial E$.
Conversely, if $y \in \partial D$ with $y \neq x$, then $T(y) \cap E = \{y\}$,that is $y \in \partial E$ (if $T(y) \cap E = \{y, x\}$, then $x \in T(y)$ getting into a contradiction). We have also that if $T(x) \cap D = \emptyset$, then $T(x) \cap E = \{x\}$ and therefore $x \in \partial E$.

$\square$

Next lemma will be crucial to prove the monotonic condition on the rearrangement.

Lemma 2.2.18 *Let D and E be two finite sets in T. Write $\partial D = \{d_1, d_2, \dots, d_n\}_\sigma$ and $\partial E = \{e_1, e_2, \dots, e_m\}_\sigma$. Suppose that there exist $0 \leq k < n$ and $0 \leq l < m$ such that:*

- $\mathcal{R}_k(D) \subset \mathcal{R}_l(E)$,
- $e_{l+1} \in T(d_{m+1})$.

Then

$$\mathcal{R}_{k+1}(D) \subset \mathcal{R}_{l+1}(E).$$

Proof. By hypothesis we have that

$$s + 1 := |\mathcal{R}_k(D) \cap [o, d_{k+1}]| \leq |\mathcal{R}_l(E) \cap [o, e_{l+1}]| =: t + 1,$$

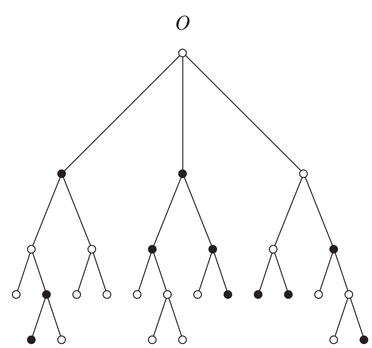
and therefore

$$[o, d_{k+1}(s)] \subset [o, e_{l+1}(t)].$$

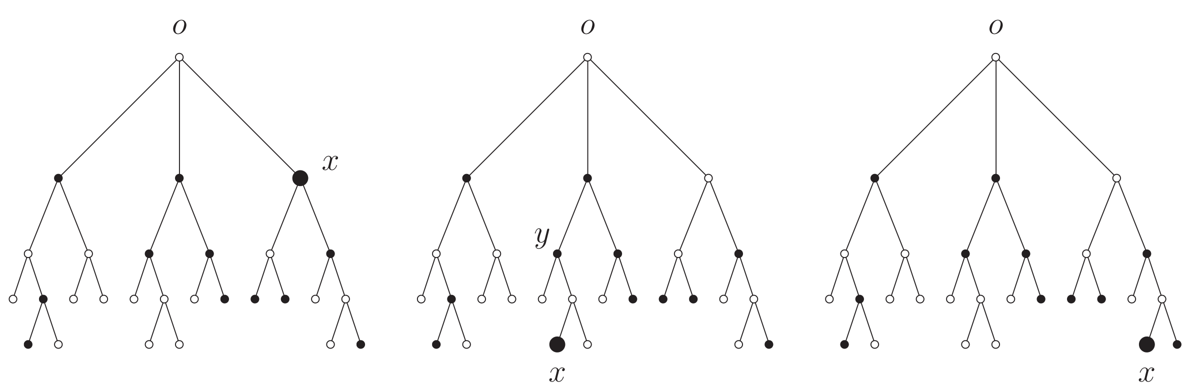
Then we have:

$$
\begin{aligned}
\mathcal{R}_{k+1}(D) &= (\mathcal{R}_k(D) \setminus [o, d_{k+1}]) \cup [o, d_{k+1}(s)] \\
&= (\mathcal{R}_k(D) \setminus [o, e_{l+1}]) \cup [o, d_{k+1}(s)] \\
&\subset (\mathcal{R}_l(E) \setminus [o, e_{l+1}]) \cup [o, e_{l+1}(t)] \\
&= \mathcal{R}_{l+1}(E).
\end{aligned}
$$

□



The set D.



$\partial E = \partial D$ $\quad$ $\partial E = (\partial D \setminus \{y\}) \cup \{x\}$ $\quad$ $\partial E = \partial D \cup \{x\}$

Figure 16: The three cases of Lemma 2.2.17, where $E = D \cup \{x\}$.

Finally, we need a technical lemma of the decomposition at each step of the rearrangement. The set E_k is the part of the set $\mathcal{R}_k(E)$ that is not rearranged at step k, meanwhile E'_k is the part that is rearranged at this step. See Figure 17 for details.

Lemma 2.2.19 *Let E be a finite set of vertices. For each $k \geq 0$, define*

$$E_k := E \setminus \left(\bigcup_{j=1}^{k} [o, e_j] \right), \qquad E'_k := \mathcal{R}_k(E) \cap \left(\bigcup_{j=1}^{k} [o, e_j] \right),$$

where we denote $\partial E = \{e_1, e_2, \dots, e_n\}_\sigma$. Then we have:

(i) $\mathcal{R}_k(E) = E'_k \cup E_k$ for all $k \geq 0$, and E_k and E'_k are disjoint sets.

(ii) $E'_{k+1} = (E'_k \setminus [o, e_{k+1}]) \cup [o, e_{k+1}(s)]$ for all k, where $s + 1 = |\mathcal{R}_k(E) \cap [o, e_{k+1}]|$.

(iii) $E'_k \subset E'_{k+1}$ for all $k \geq 0$, and $E^* = E'_n$.

Proof. (i) The disjointness follows by definition. It is enough to prove that

$$\mathcal{R}_k(E) \setminus \left(\bigcup_{j=1}^{k} [o, e_j] \right) = E \setminus \left(\bigcup_{j=1}^{k} [o, e_j] \right).$$

We prove it by induction. If $k = 0$, it is true since $\mathcal{R}_0(E) = E$. Suppose it is true for $k > 0$. Then, by the definition of the rearrangement, the fact that

$$[o, e_{k+1}(s)] \subset [o, e_{k+1}],$$

and using the hypothesis of induction, we have that:

$$\begin{aligned}
\mathcal{R}_{k+1}(E) \setminus \left(\bigcup_{j=1}^{k+1} [o, e_j] \right) &= \left(\left(\mathcal{R}_k(E) \setminus [o, e_{k+1}] \right) \cup [o, e_{k+1}(s)] \right) \setminus \left(\bigcup_{j=1}^{k+1} [o, e_j] \right) \\
&= \mathcal{R}_k(E) \setminus \left(\bigcup_{j=1}^{k+1} [o, e_j] \right) \\
&= \left(\mathcal{R}_k(E) \setminus \left(\bigcup_{j=1}^{k} [o, e_j] \right) \right) \setminus [o, e_{k+1}] \\
&= \left(E \setminus \left(\bigcup_{j=1}^{k} [o, e_j] \right) \right) \setminus [o, e_{k+1}] \\
&= E \setminus \left(\bigcup_{j=1}^{k+1} [o, e_j] \right).
\end{aligned}$$

(ii) By the definition of the rearrangement and the definition of E'_k, we have:

$$\begin{aligned}
E'_{k+1} &= \mathcal{R}_{k+1}(E) \cap \left(\bigcup_{j=1}^{k+1} [o, e_j] \right) \\
&= \left(\left(\mathcal{R}_k(E) \setminus [o, e_{k+1}] \right) \cup [o, e_{k+1}(s)] \right) \cap \left(\bigcup_{j=1}^{k+1} [o, e_j] \right) \\
&= \left(\left(\mathcal{R}_k(E) \setminus [o, e_{k+1}] \right) \cap \left(\bigcup_{j=1}^{k+1} [o, e_j] \right) \right) \cup [o, e_{k+1}(s)] \\
&= \left(\left(\mathcal{R}_k(E) \cap \left(\bigcup_{j=1}^{k} [o, e_j] \right) \right) \setminus [o, e_{k+1}] \right) \cup [o, e_{k+1}(s)] \\
&= (E'_k \setminus [o, e_{k+1}]) \cup [o, e_{k+1}(s)].
\end{aligned}$$

(iii) It is clear that $E_n = \emptyset$ and

$$E'_n = \mathcal{R}_n(E) \cap \left(\bigcup_{j=1}^{n} [o, e_j] \right) = E^*.$$

To see the inclusion, it is enough to observe that

$$E'_k \cap [o, e_{k+1}] \subset [o, e_{k+1}(s)],$$

and this is a consequence of $s + 1 = |\mathcal{R}_k(E) \cap [o, e_{k+1}]| \geq |E'_k \cap [o, e_{k+1}]|$.

□

Proposition 2.2.20 *Let D and E be finite sets of vertices such that $D \subset E$. Then $D^* \subset E^*$.*

Proof. It is enough to prove it when $E = D \cup \{x\}$, where $x \notin D$. We distinguish the three cases of Lemma 2.2.17:

(i) $\partial E = \partial D$. Since we have

$$D = \mathcal{R}_0(D) \subset \mathcal{R}_0(E) = E,$$

we can apply Lemma 2.2.18 recursively to obtain the result.

(ii) $\partial E = (\partial D \setminus \{y\}) \cup \{x\}$. Write $\partial D = \{d_1, d_2, \ldots, d_n\}_\sigma$ and $\partial E = \{e_1, e_2, \ldots, e_n\}_\sigma$. There exists $0 \leq k \leq n$ such that $d_j = e_j$ for all $j \neq k$, and $d_k = y$ and $e_k = x$. Applying Lemma 2.2.18 recursively we get

$$\mathcal{R}_{k-1}(D) \subset \mathcal{R}_{k-1}(E).$$

But now, Lemma 2.2.17 gives

$$e_k = x \in T(y) = T(d_k),$$

so applying Lemma 2.2.18 recursively, we get the result.

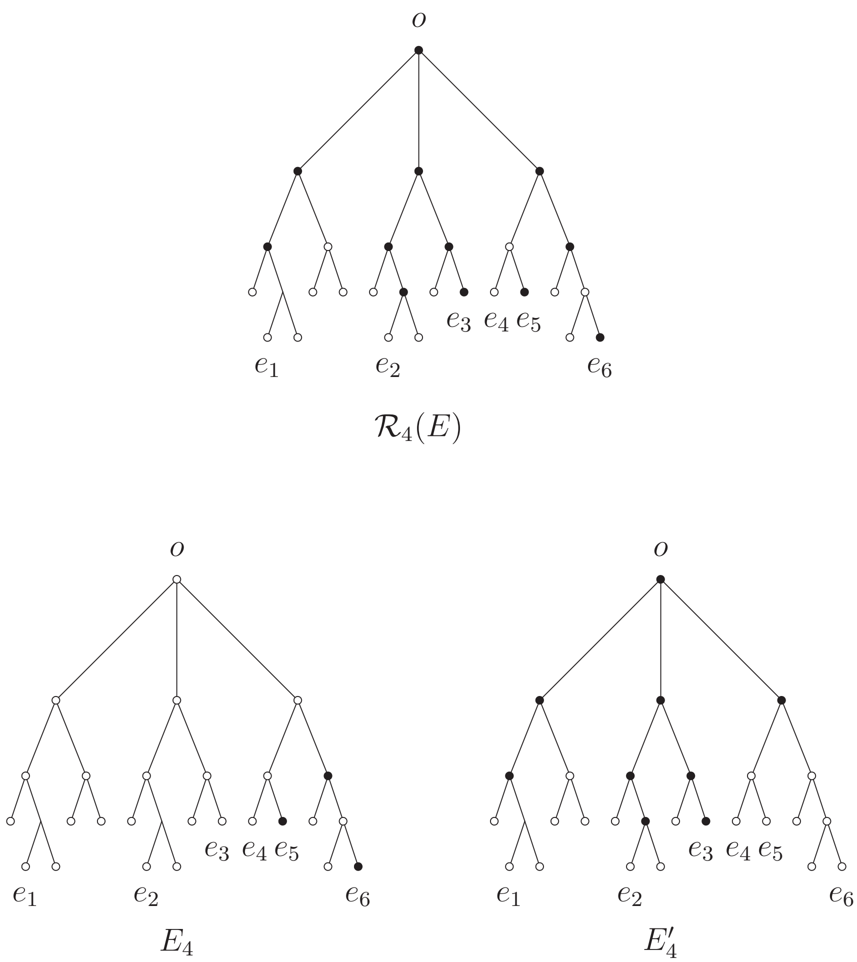


Figure 17: The sets E_4 and E'_4 of Figure 14, at step 4.

(iii) $\partial E = \partial D \cup \{x\}$. Write $\partial E = \{d_1, d_2, \cdots, d_k, x, d_{k+1}, \cdots, d_n\}$. By Lemma 2.2.18, we have that

$$\mathcal{R}_k(D) \subset \mathcal{R}_k(E). \tag{2.5}$$

Using the notation of Lemma 2.2.19, we claim that

$$D'_k \subset E'_k \qquad \text{and} \qquad D_k \subset E_k \setminus [o, x].$$

Therefore, using Lemma 2.2.19, we get

$$\begin{aligned} D'_k &\subset E'_k \subset E'_{k+1} \\ D_k &\subset E_k \setminus [o,x] = E_{k+1}, \end{aligned}$$

(where here $s+1 = |\mathcal{R}_k(E) \cap [o,x]|$) and as a consequence, using (i) of Lemma 2.2.19, we get:

$$\mathcal{R}_k(D) = D'_k \cup D_k \subset E'_{k+1} \cup E_{k+1} = \mathcal{R}_{k+1}(E).$$

To finish, we call recursively Lemma 2.2.18 to obtain the result.

We now prove the claim. Take $y \in D'_k$. By (2.5), we know that $y \in E'_k \cup E_k$. Suppose that $y \in E_k$. Then by the Lemma 2.2.19,

$$y \in E \setminus \left(\bigcup_{j=1}^{k} [o, d_j] \right) \subset E \setminus D'_k,$$

getting a contradiction. Take now $y \in D_k$. Then

$$y \in D \setminus \left(\bigcup_{j=1}^{k} [o, d_j] \right),$$

and $y \notin [o,x]$ (if $y \in [o,x]$, there exists $z \in \partial D$ such that $x \in T(z)$, and this contradicts the hypothesis (iii)) and therefore

$$y \in \left(D \setminus \bigcup_{j=1}^{k} [o, d_j] \right) \setminus [o,x] = E_k \setminus [o,x].$$

□

2.3 The decreasing rearrangement of functions

We will define the decreasing rearrangement for functions in the tree and we will see useful properties of this rearrangement.

Let $\mathcal{M}_0$ be the set of functions f defined on the tree such that the level sets

$$\{x \in T : |f(x)| > \lambda\}$$

are finite for all $\lambda > 0$.

Lemma 2.3.1 *If $f \in \mathcal{M}_0$, then*

$$\lim_{n\to\infty} |f(x_n)| = 0,$$

for all infinite sequences $\{x_n : n \in \mathbb{N}\}$ of different vertices (being not necessarily an infinite chain) in the tree.

Proof. Suppose that there exists an infinite sequence of vertices $\{x_n : n \in \mathbb{N}\}$ such that

$$\lim_{n\to\infty} |f(x_n)| \neq 0.$$

There exists $\varepsilon > 0$ such that for all $n \geq 1$, there exists $m(n) \geq n$ satisfying

$$\left|f(x_{m(n)})\right| \geq \varepsilon.$$

Thus, we have that

$$\left\{x_{m(n)} : n \in \mathbb{N}\right\} \subset \{x \in T : |f(x)| > \varepsilon/2\},$$

contradicting the fact that $f \in \mathcal{M}_0$.

□

As a consequence of this lemma, if $f \in \mathcal{M}_0$ we have that

$$\lim_{n\to\infty} |f(\omega(n))| = 0,$$

for all $\omega \in \partial T$. We have also that, if $\{a_n : n \in \mathbb{N}\}$ is the set of images of all the vertices by f, we then can assume that

$$|a_1| > |a_2| > |a_3| > \ldots > |a_n| > |a_{n+1}| > \ldots \searrow 0.$$

Consider the rooted tree T_o and an order σ in ∂T_o. Recall that we denote

$$E^* = \mathcal{R}_{(o,\sigma)}(E)$$

as the decreasing rearrangement of the set E. We define the **decreasing rearrangement of functions** in $\mathcal{M}_0$:

Definition 2.3.2 For every $f \in \mathcal{M}_0$, the **decreasing rearrangement** of f is the function

$$f^*(x) = \int_0^\infty \chi_{\{y \in T : |f(y)| > \lambda\}^*}(x)\, d\lambda,$$

defined for all $x \in T$.

Observe that this definition strongly depends on the choice of o and σ, and it would be more correct to write this dependence by denoting $f^*_{(o,\sigma)}$, but we will avoid it by simplicity. However, we recall that we have shown in Theorem 2.2.15 the canonicity of the rearrangement, and as a consequence we have the following proposition. We keep for a moment the long notation $f^*_{(o,\sigma)}$.

Proposition 2.3.3 *Let o and o' be two vertices in T, and σ and σ' be two orders in ∂T_o and $\partial T_{o'}$ respectively. Then, there exists an automorphism ϱ such that*

$$(f \circ \varrho)^*_{(o,\sigma)}(x) = f^*_{(o',\sigma')}(\varrho(x)),$$

for all $x \in T$ and all $f \in \mathcal{M}_0$.

Proof. By Theorem 2.2.15, there exists an automorphism ϱ such that

$$\varrho(\mathcal{R}_{(o,\sigma)}(E)) = \mathcal{R}_{(o',\sigma')}(\varrho(E)),$$

for all finite set E. Taking $E = \{x \in T : |f(x)| > \lambda\}$ for $\lambda > 0$, it is easy to see that

$$x \in \mathcal{R}_{(o,\sigma)}(E) \Longleftrightarrow \varrho(x) \in \mathcal{R}_{(o',\sigma')}\Big(\big\{y : \big|f(\varrho^{-1}(y))\big| > \lambda\big\}\Big),$$

and that is

$$\chi_{\mathcal{R}_{(o,\sigma)}(E)}(x) = \chi_{\mathcal{R}_{(o',\sigma')}(\{|f(\varrho^{-1}(\cdot))|>\lambda\})}(\varrho(x)).$$

Finally, by Definition 2.3.2, the last equality takes to the result.

□

We trivially have that

$$f^*(x) = (|f|)^*(x),$$

for all $x \in T$. So, in the sequel, we will always work with positive functions on the tree.

By Lemma 2.3.1, for every positive function f in $\mathcal{M}_0$, there exists a positive strictly decreasing sequence of real numbers $\{a_n : n \in \mathbb{N}\}$, with $\lim_{n\to\infty} a_n = 0$, and a collection of disjoint finite sets of vertices $\{E_n : n \in \mathbb{N}\}$, such that

$$f(x) = \sum_{n=1}^{\infty} a_n \chi_{E_n}(x), \tag{2.6}$$

for all $x \in T$.

Lemma 2.3.4 *Take a positive $f \in \mathcal{M}_0$, and consider the representation (2.6) of f. Then*

$$f^*(x) = \sum_{n=1}^{\infty} a_n \chi_{F_n^* \setminus F_{n-1}^*}(x),$$

for all $x \in T$, where $F_n = \bigcup_{k=1}^{n} E_k$ and $F_0 = \emptyset$.

Proof. If $a_{n+1} < \lambda \leq a_n$, then

$$\{x \in T : f(x) > \lambda\} = E_1 \cup E_2 \cup \ldots \cup E_n = F_n,$$

and for $a_1 < \lambda$, we have

$$\{x \in T : f(x) > \lambda\} = \emptyset.$$

Therefore,

$$\begin{aligned} f^*(x) &= \int_0^\infty \chi_{\{y \in T : f(y) > \lambda\}^*}(x)\, d\lambda \\ &= \sum_{n=1}^{\infty} \int_{a_{n+1}}^{a_n} \chi_{\{y \in T : f(y) > \lambda\}^*}(x)\, d\lambda + \int_{a_1}^{\infty} \chi_{\{y \in T : f(y) > \lambda\}^*}(x)\, d\lambda \\ &= \sum_{n=1}^{\infty} \chi_{F_n^*}(x)(a_n - a_{n+1}) \\ &= \sum_{n=1}^{\infty} a_n \; \chi_{F_n^* \setminus F_{n-1}^*}(x), \end{aligned}$$

where in the last equality we have used Proposition 2.2.20, that is, $F_n^* \subset F_{n+1}^*$ because $F_n \subset F_{n+1}$.

$\square$

Remark 2.3.5 We can give another formulation of this lemma. If we write

$$f(x) = \sum_{n=1}^{\infty} b_n \chi_{F_n}(x),$$

where $F_n \subset F_{n+1}$, $b_n > 0$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

$$f^*(x) = \sum_{n=1}^{\infty} b_n \chi_{F_n^*}(x).$$

It suffices to consider $E_n = F_n \setminus F_{n-1}$ and $a_n = \sum_{k=n}^{\infty} b_k$.

See Figure 18 as an example of the rearrangement of a positive function. Observe that the function takes the values $1, 2, 3, 4, 5$. Following Lemma 2.3.4, we rearrange consecutively, for $n = 1, 2, 3, 4, 5$, the sets

$$F_n = \bigcup_{k=1}^{n} \{x : f(x) = 6 - k\},$$

and we put the value $6 - n$ at the new layer. Recall that we use the order σ from left to right fixed in Remarks 2.2.16.

We give the basic properties of the decreasing rearrangement:

Proposition 2.3.6 *The decreasing rearrangement f^* of a function $f \in \mathcal{M}_0$ is a nonnegative decreasing function on T. Furthermore, for f, g, f_k, $k \geq 1$, functions in $\mathcal{M}_0$ we have:*

(i) $(\chi_E)^*(x) = \chi_{E^*}(x)$ *for all finite sets E.*

(ii) $\operatorname{supp}(f^*) = (\operatorname{supp}(f))^*$.

(iii) $(kf)^*(x) = |k|\, f^*(x)$ *for all $k \in \mathbb{C}$.*

(iv) If $|g(x)| \leq |f(x)|$ for all $x \in T$, then $g^*(x) \leq f^*(x)$ for all $x \in T$.

(v) If f is positive and decreasing, then $f^*(x) = f(x)$ for all $x \in T$.

(vi) $\{x \in T : |f(x)| > \lambda\}^* = \{x \in T : f^*(x) > \lambda\}$ *for all $\lambda > 0$.*

(vii) $(|f|^p)^* = (f^*)^p$.

(viii) If $|f(x)| \leq \liminf_{n\to\infty} |f_n(x)|$, then $f^*(x) \leq \liminf_{n\to\infty} f_n^*(x)$. In particular, whenever $|f_n(x)| \nearrow |f(x)|$ for all $x \in T$, we have $f_n^*(x) \nearrow f^*(x)$ for all $x \in T$.

Proof. Lemma 2.3.4 directly gives (ii) and that f^* is decreasing. Let us now prove the other statements:
(i) $(\chi_E)^*(x) = \int_0^\infty \chi_{\{\chi_E > \lambda\}^*}(x)\, d\lambda = \int_0^1 \chi_{E^*}(x)\, d\lambda = \chi_{E^*}(x)$.
(iii) $(kf)^*(x) = \int_0^\infty \chi_{\{|kf|>\lambda\}^*}(x)\, d\lambda = \int_0^\infty |k|\, \chi_{\{|f|>\zeta\}^*}(x)\, d\zeta = |k|\, f^*(x)$.
(iv) If $|g(x)| \leq |f(x)|$ for all $x \in T$, then

$$\{x \in T : |g(x)| > \lambda\} \subset \{x \in T : |f(x)| > \lambda\},$$

for all $\lambda > 0$ and hence

$$\{x \in T : |g(x)| > \lambda\}^* \subset \{x \in T : |f(x)| > \lambda\}^*.$$

Using this inclusion, we have:

$$g^*(x) = \int_0^\infty \chi_{\{y\in T:|g(y)|>\lambda\}^*}(x)\,d\lambda \le \int_0^\infty \chi_{\{y\in T:|f(y)|>\lambda\}^*}(x)\,d\lambda = f^*(x).$$

(v) If f is positive and decreasing, we have

$$\{x\in T : f(x)>\lambda\}^* = \{x\in T : f(x)>\lambda\},$$

and therefore

$$f^*(x) = \int_0^\infty \chi_{\{y\in T:f(y)>\lambda\}^*}(x)\,d\lambda = \int_0^\infty \chi_{\{y\in T:f(y)>\lambda\}}(x)\,d\lambda = f(x).$$

(vi) We use the representation (2.6) of $|f|$. Then,

$$\{x\in T : |f(x)|>\lambda\} = \bigcup_{k=1}^n E_k = F_n,$$

if $a_{n+1} < \lambda \le a_n$, and also

$$\{x\in T : f^*(x)>\lambda\} = \bigcup_{k=1}^n (F_k^*\setminus F_{k-1}^*) = F_n^*.$$

(vii) Using twice a change of variable and (vi), we get:

$$\begin{aligned}(|f|^p)^*(x) &= \int_0^\infty \chi_{\{y\in T:|f(y)|^p>\lambda\}^*}(x)\,d\lambda = \int_0^\infty p\,\zeta^{p-1}\chi_{\{y\in T:|f(y)|>\zeta\}^*}(x)\,d\zeta\\ &= \int_0^\infty p\,\zeta^{p-1}\chi_{\{y\in T:f^*(y)>\zeta\}}(x)\,d\zeta = \int_0^\infty \chi_{\{y\in T:(f^*(y))^p>\lambda\}}(x)\,d\lambda\\ &= (f^*(x))^p.\end{aligned}$$

$(viii)$ Recall that

$$\liminf_{n\to\infty}|f_n(x)| = \sup_k \inf_{n\ge k}|f_n(x)|.$$

Fix $\lambda>0$ and take $x\in\{y\in T : |f(y)|>\lambda\}$. Then, by hypothesis, there exists $n=n(\lambda,x)$ such that $x\in\{y\in T : |f_k(y)|>\lambda\}$ for all $k\ge n$. Since $f\in\mathcal{M}_0$, the set $E_\lambda = \{y\in T : |f(y)|>\lambda\}$ is finite. Thus, set

$$n_0(\lambda) = \max_{x\in E_\lambda} n(\lambda,x).$$

We then have that $x\in\{y\in T : |f_k(y)|>\lambda\}$ for all $k\ge n_0(\lambda)$, that is,

$$\{y\in T : |f(y)|>\lambda\} \subset \bigcap_{k\ge n_0(\lambda)} \{y\in T : |f_k(y)|>\lambda\}.$$

By Proposition 2.2.20, we obtain

$$\{y\in T : |f(y)|>\lambda\}^* \subset \bigcap_{k\ge n_0(\lambda)} \{y\in T : |f_k(y)|>\lambda\}^*,$$

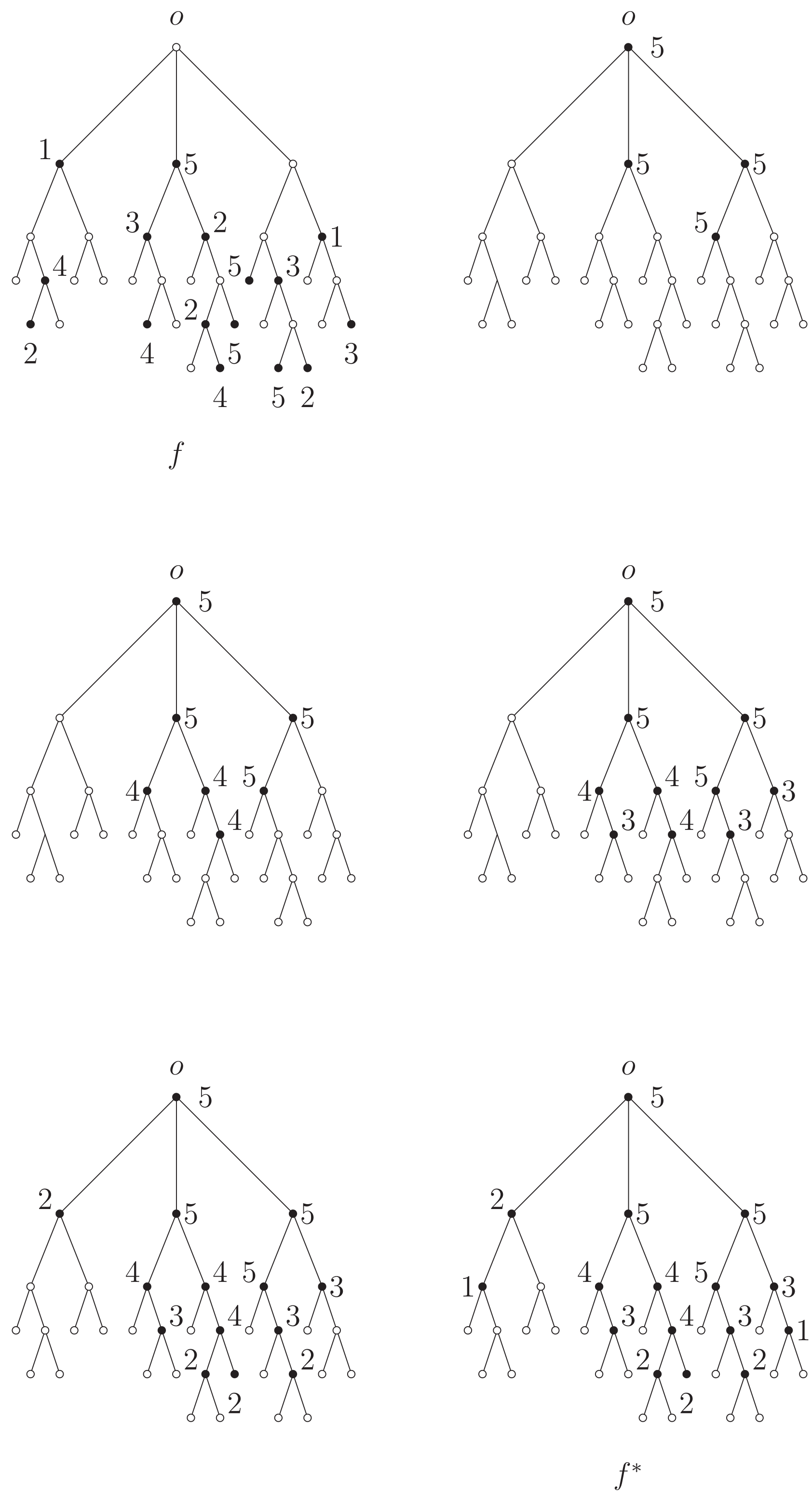


Figure 18: The rearrangement of a positive function.

and therefore

$$\chi_{\{y\in T:|f(y)|>\lambda\}^*}(x) \le \chi_{\{y\in T:|f_k(y)|>\lambda\}^*}(x),$$

for all $x \in T$ and all $k \ge n_0(\lambda)$, and hence:

$$\chi_{\{y\in T:|f(y)|>\lambda\}^*}(x) \le \sup_n \inf_{k\ge n} \chi_{\{y\in T:|f_k(y)|>\lambda\}^*}(x) = \liminf_n \chi_{\{y\in T:|f_n(y)|>\lambda\}^*}(x).$$

Finally, using Fatou's lemma, we have

$$\begin{aligned} f^*(x) &= \int_0^\infty \chi_{\{y\in T:|f(y)|>\lambda\}^*}(x)\,d\lambda \le \int_0^\infty \liminf_n \chi_{\{y\in T:|f_n(y)|>\lambda\}^*}(x)\,d\lambda \\ &\le \liminf_n \int_0^\infty \chi_{\{y\in T:|f_n(y)|>\lambda\}^*}(x)\,d\lambda = \liminf_n f_n^*(x). \end{aligned}$$

□

How can we extend the definition of the decreasing rearrangement to any function defined in the tree? Take $f : T \longrightarrow \mathbb{C}$ a function in the tree, and suppose that there exist two sequences $\{f_n : n \in \mathbb{N}\}$ and $\{g_n : n \in \mathbb{N}\}$ of functions in $\mathcal{M}_0$ such that

$$|f_n| \nearrow |f| \quad \text{and} \quad |g_n| \nearrow |f|,$$

pointwise. Define $f^*(x) = \lim_n f_n^*(x)$ and $g^*(x) = \lim_n g_n^*(x)$. Observe that these limits exist by $(viii)$ of Proposition 2.3.6, and they can be infinite. We claim that $f^* = g^*$. In fact, we have

$$\{y \in T : |f(y)| > \lambda\} = \bigcup_{k=1}^{\infty} \{y \in T : |f_n(y)| > \lambda\} = \bigcup_{k=1}^{\infty} \{y \in T : |g_n(y)| > \lambda\},$$

for all $\lambda > 0$, and since f_n and g_n are in $\mathcal{M}_0$ for all $n \ge 1$, their level sets are finite sets, and thus, for all $n \ge 1$ there exists $m(n) \ge 1$ such that

$$\{y \in T : |f_n(y)| > \lambda\} \subset \left\{y \in T : \left|g_{m(n)}(y)\right| > \lambda\right\},$$

and by Proposition 2.2.20,

$$\{y \in T : |f_n(y)| > \lambda\}^* \subset \left\{y \in T : \left|g_{m(n)}(y)\right| > \lambda\right\}^*.$$

Using this inclusion, we get:

$$f^*(x) = \lim_n \int_0^\infty \chi_{\{y\in T:|f_n(y)|>\lambda\}^*}(x)\,d\lambda \le \lim_n \int_0^\infty \chi_{\left\{y\in T:\left|g_{m(n)}(y)\right|>\lambda\right\}^*}(x)\,d\lambda = g^*(x).$$

Analogously, we have the converse inequality, and therefore $f^* = g^*$. Thanks to this equality, the following definition makes sense:

Definition 2.3.7 For a function $f : T \longrightarrow \mathbb{C}$ defined on the tree, the decreasing rearrangement of f is the function

$$f^*(x) = \lim_n \Big(|f(\cdot)| \cdot \chi_{\{y \in T : |y| \le n\}}(\cdot) \Big)^*(x).$$

We observe that from now on, it is enough to consider functions with finite support.

In view of the definition of the decreasing rearrangement and looking at Figure 18, we can ask if the defined rearrangement is equivalent to rearrange recursively the function restricted to each geodesic from o to a boundary point in the support of the function, following the order given by σ. The answer is positive, as we will see in Theorem 2.3.10, and it will be helpful in the following sections. We need first two lemmas.

Lemma 2.3.8 *Let $S : \mathcal{M}_0 \longrightarrow \mathcal{M}_0$ be a positive operator defined on positive functions in $\mathcal{M}_0$ such that for all $f \in \mathcal{M}_0$ and all $\lambda > 0$,*

$$|\{x \in T : Sf(x) = \lambda\}| = |\{x \in T : f(x) = \lambda\}|.$$

For a positive $f \in \mathcal{M}_0$, let x be a minimum of f. Let $E = \operatorname{supp}(f)$ be the support of f, and set $E_0 = E \setminus \{x\}$.

Then there exists a unique x' in the support of Sf such that

- $\operatorname{supp}(Sf) = \operatorname{supp}(S(f \cdot \chi_{E_0})) \cup \{x'\}$.
- $Sf(x') = f(x)$.

Proof. By hypothesis, $|\operatorname{supp}(f)| = |\operatorname{supp}(Sf)|$, and therefore

$$|\operatorname{supp}(f \cdot \chi_{E_0})| = |\operatorname{supp}(S(f \cdot \chi_{E_0}))|.$$

Thus, we have

$$\begin{aligned} |\operatorname{supp}(Sf)| = |\operatorname{supp}(f)| &= |\operatorname{supp}(f \cdot \chi_{E_0})| + 1 \\ &= |\operatorname{supp}(S(f \cdot \chi_{E_0}))| + 1. \end{aligned}$$

Set $x' = \operatorname{supp}(Sf) \setminus \operatorname{supp}(S(f \cdot \chi_{E_0}))$. By hypothesis, we get that

$$\begin{aligned} |\{y \in T : f(y) = f(x)\}| &= |\{y \in T : (f \cdot \chi_{E_0})(y) = f(x)\}| + 1 \\ &= |\{y \in T : S(f \cdot \chi_{E_0})(y) = f(x)\}| + 1, \end{aligned}$$

and also that for all $\lambda \neq f(x)$:

$$\begin{aligned}|\{y \in T : f(y) = \lambda\}| &= |\{y \in T : (f \cdot \chi_{E_0})(y) = \lambda\}| \\ &= |\{y \in T : S(f \cdot \chi_{E_0})(y) = \lambda\}| .\end{aligned}$$

Consequently, we have that $x' \in \{y \in T : Sf(y) = f(x)\}$, that is $Sf(x') = f(x)$.

□

As an easy consequence of this lemma with $Sf(y) = f^*(y)$, we get the following result. It is important to remark that this lemma is not true in general if the vertex x is not the minimum of the function.

Lemma 2.3.9 *Suppose that* $\mathrm{supp}(g) \subset [o, e]$ *for a positive* $g \in \mathcal{M}_0$*, and* $e \in T$*. If* g *attains its minimum at* x *and* $A_0 = \mathrm{supp}(g) \setminus \{x\}$*, then*

$$(g \cdot \chi_{A_0})^*(y) = g^*(y) \cdot \chi_{A_0^*}(y),$$

for all $y \in T$*.*

We now give a decreasing rearrangement of a function by rearranging recursively the restriction of the function to each geodesic from o to the boundary vertices of its support, ordered by using σ. Specifically, take a positive $f \in \mathcal{M}_0$ with finite support and set $E = \mathrm{supp}(f)$ and $\partial E = \{e_1, e_2, \dots, e_n\}_\sigma$. Define for each $1 \leq k \leq n$:

$$f_k^\triangle(y) = \begin{cases} f_{k-1}^\triangle(y) & \text{if} \quad y \in \mathcal{R}_k(E) \cap [o, e_k]^c, \\ \\ (f_{k-1}^\triangle \cdot \chi_{[o,e_k]})^*(y) & \text{if} \quad y \in \mathcal{R}_k(E) \cap [o, e_k], \end{cases} \tag{2.7}$$

where $f_0^\triangle = f$. Observe that $\mathrm{supp}(f_k^\triangle) = \mathcal{R}_k(E)$. Finally, set

$$f^\triangle = f_n^\triangle. \tag{2.8}$$

See Figures 19 and 20 for an example. Observe that we rearrange the same function than in Figure 18, and that $f^\triangle$ and f^* coincide.

Theorem 2.3.10 *For every function* $f \in \mathcal{M}_0$ *with finite support, we have*

$$f^* = f^\triangle.$$

Proof. We prove it by induction on the cardinal of the support of the function. If $|\mathrm{supp}(f)| = 1$, then clearly $f^* = f^\triangle$. Suppose that it is true for $|\mathrm{supp}(f)| = n$. Take a positive $f \in \mathcal{M}_0$ with $|\mathrm{supp}(f)| = n+1$. Write $E = \mathrm{supp}(f)$, and let x be a minimum of f. Set $E_0 = E \setminus \{x\}$. By Lemma 2.3.8, there exists a unique $x' \in E^*$ such that $E^* = E_0^* \cup \{x'\}$ and $f^*(x') = f(x)$. As the support of g^* and $g^\triangle$ coincide for all $g \in \mathcal{M}_0$ with finite support, we then also have that

$$f^*(x') = f^\triangle(x') = f(x). \tag{2.9}$$

By induction, we have

$$(f \cdot \chi_{E_0})^* = (f \cdot \chi_{E_0})^\triangle. \tag{2.10}$$

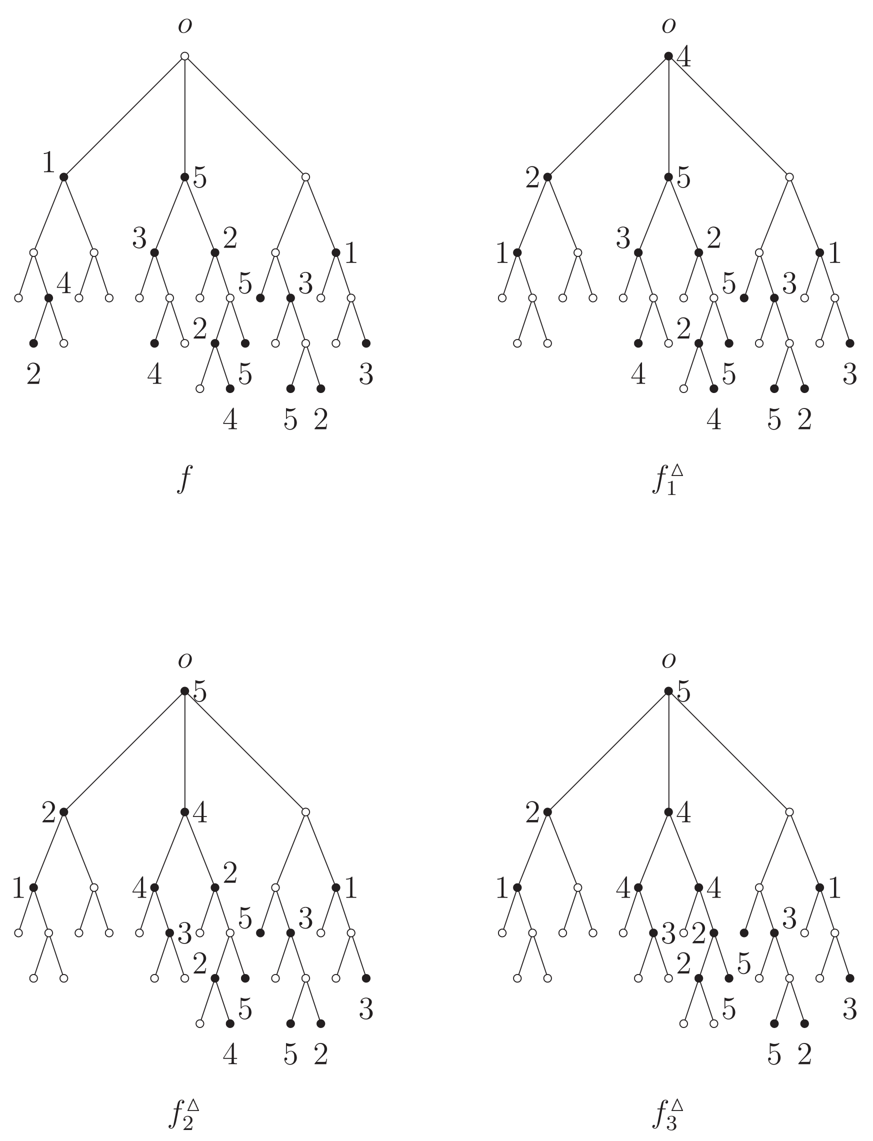


Figure 19: The first steps of the definition of $f^\triangle$.

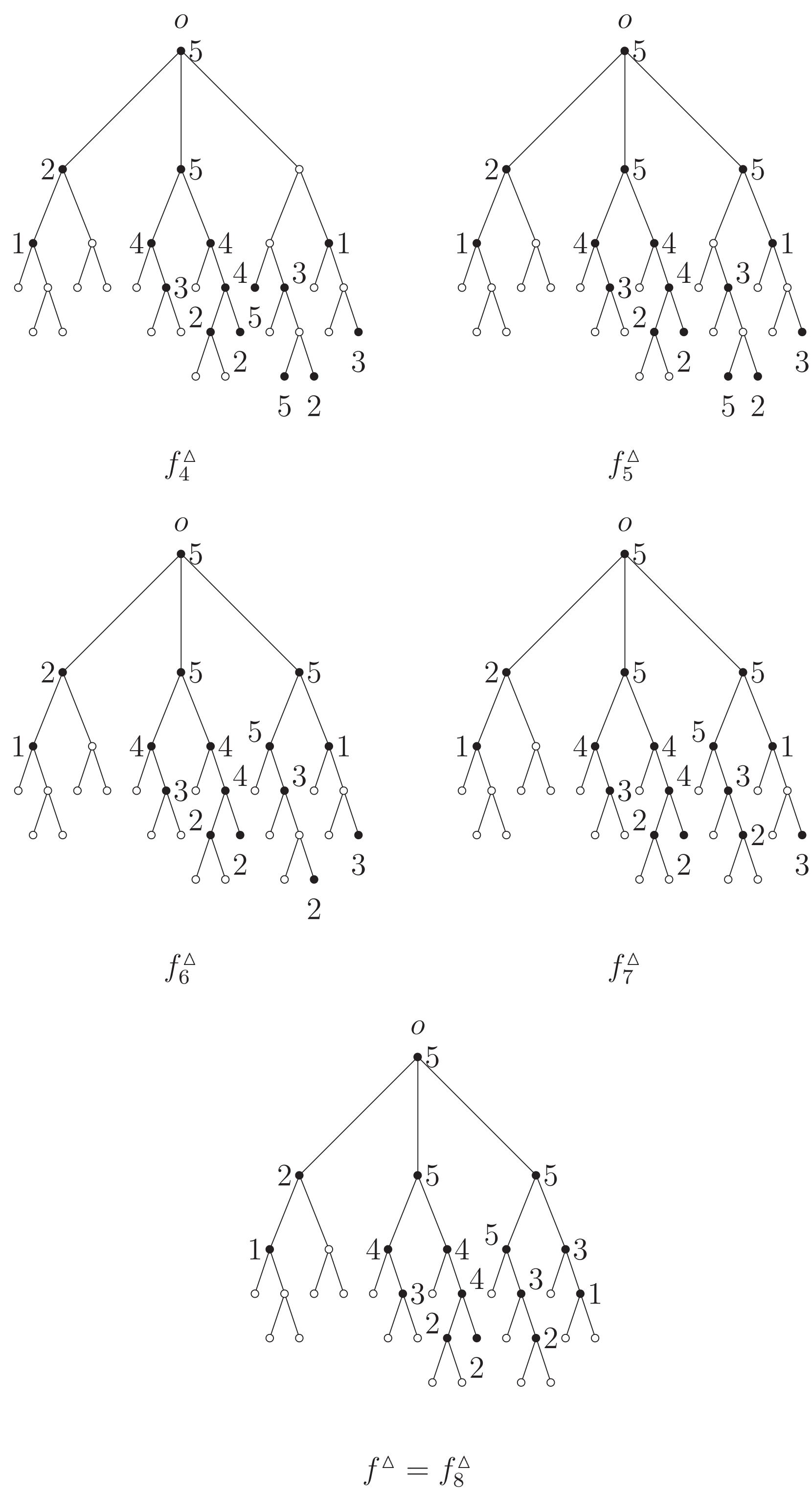


Figure 20: The last steps of the definition of $f^{\triangle}$.

Let us show that

$$(f \cdot \chi_{E_0})^*(y) = f^*(y) \cdot \chi_{E_0^*}(y), \tag{2.11}$$

for all $y \in T$. Write

$$f(y) = \sum_{k=1}^{N} a_k \chi_{E_k}(y),$$

with $E_k \cap E_j = \emptyset$ if $j \neq k$, and $a_k > a_j > 0$ if $j > k$ (and so, $f(x) = a_N$). By Lemma 2.3.4, we know that

$$f^*(y) = \sum_{k=1}^{N} a_k \chi_{F_k^* \setminus F_{k-1}^*}(y),$$

for all $y \in T$, where $F_k = \bigcup_{j=1}^{k} E_j$ and $F_0 = \emptyset$. Thus,

$$(f \cdot \chi_{E_0})^*(y) = \sum_{k=1}^{N-1} a_k \chi_{F_k^* \setminus F_{k-1}^*}(x) + a_N \chi_{E_0^* \setminus F_{N-1}^*}(y).$$

On the other hand,

$$\begin{aligned} f^*(y) \cdot \chi_{E_0^*}(y) &= \sum_{k=1}^{N-1} a_k \chi_{F_k^* \setminus F_{k-1}^*}(y) + a_N \chi_{(E^* \setminus \{x'\}) \setminus F_{N-1}^*}(y) \\ &= \sum_{k=1}^{N-1} a_k \chi_{F_k^* \setminus F_{k-1}^*}(x) + a_N \chi_{E_0^* \setminus F_{N-1}^*}(y). \end{aligned}$$

Now, we will show that

$$(f \cdot \chi_{E_0})^{\triangle}(y) = f^{\triangle}(y) \cdot \chi_{E_0^*}(y), \tag{2.12}$$

for all $y \in T$. To this end, we need to know what vertex in E^* is x', that is, we need to know where the minimum value $f(x)$ of f is going in the construction of $f^{\triangle}$. Let e_k be the first vertex (with respect to σ) in ∂E such that $x \in [o, e_k]$. Recall the definition of confluent vertex $c(x, y)$ of two vertices x and y in page 76. Two things can happen when we construct $f_k^{\triangle}$, taking into account that x is a minimum of f (see Figure 21):

- (1_k) If $|\mathcal{R}_{k-1}(E) \cap [o, e_k]| > |[o, c(e_k, e_{k+1})]|$, then

  $$x' \in [o, e_k] \setminus [o, c(e_k, e_{k+1})]$$

  and $f_k^{\triangle}(x') = f(x)$, and we get x'.

- (2_k) If $|\mathcal{R}_{k-1}(E) \cap [o, e_k]| \le |[o, c(e_k, e_{k+1})]|$ then there exists
$$x_k \in [o, c(e_k, e_{k+1})]$$
such that $f_k^{\triangle}(x_k) = f(x)$, that is, the vertex x_k is now the vertex with minimum value $f(x)$ for $f_k^{\triangle}$. In this case, we proceed now by constructing $f_{k+1}^{\triangle}$, and two things can happen again:
  - ► (1_{k+1}) If $|\mathcal{R}_k(E) \cap [o, e_{k+1}]| > |[o, c(e_{k+1}, e_{k+2})]|$, then
$$x' \in [o, e_{k+1}] \setminus [o, c(e_{k+1}, e_{k+2})]$$
and $f_{k+1}^{\triangle}(x') = f(x)$, and we get x'.
  - ► (2_{k+1}) If $|\mathcal{R}_k(E) \cap [o, e_{k+1}]| \le |[o, c(e_{k+1}, e_{k+2})]|$ then there exists
$$x_{k+1} \in [o, c(e_{k+1}, e_{k+2})]$$
such that $f_{k+1}^{\triangle}(x_{k+1}) = f(x)$, and we follow repeating this process at each step.

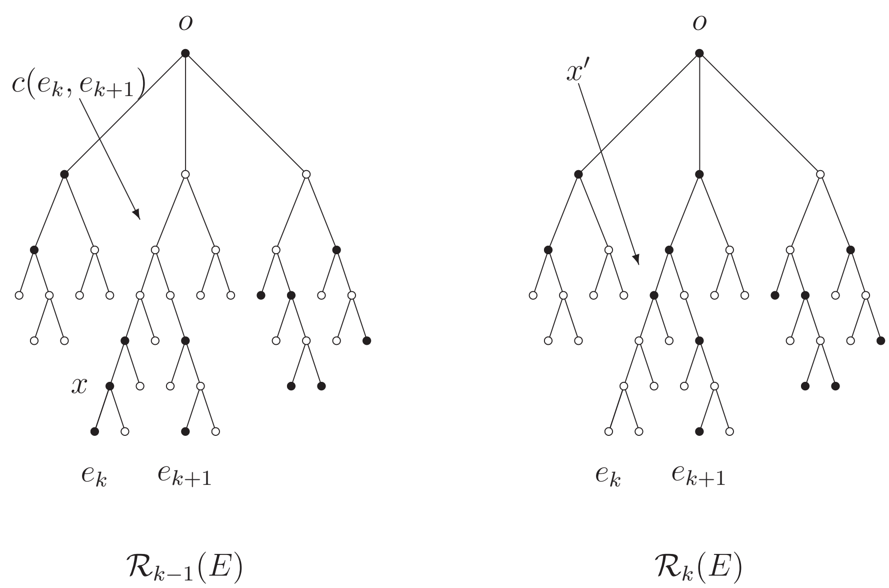
In view of this, the search stops if:

- (1_m) There exists $m < n$ such that $|\mathcal{R}_{m-1}(E) \cap [o, e_m]| > |[o, c(e_m, e_{m+1})]|$, and then
$$x' \in [o, e_m] \setminus [o, c(e_m, e_{m+1})]$$
and $f_m^{\triangle}(x') = f(x)$.

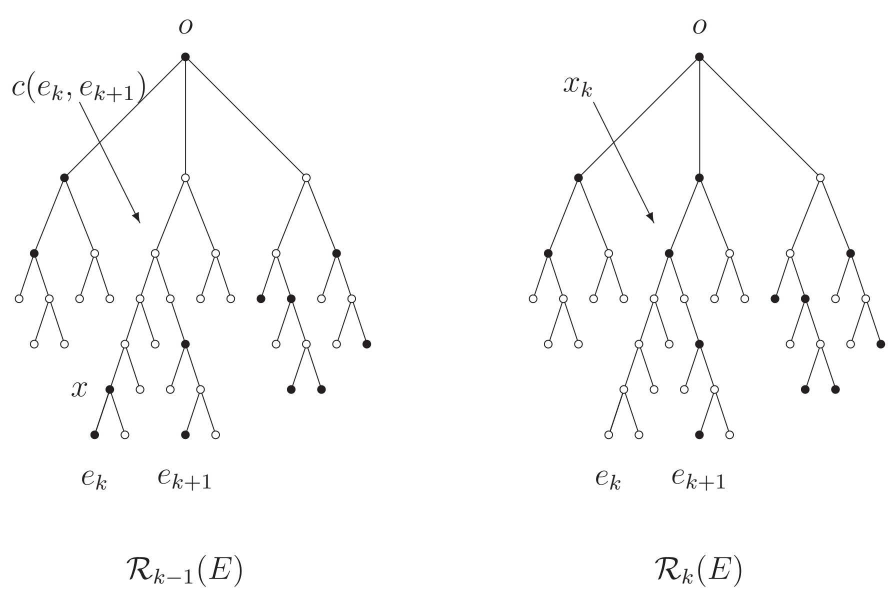
- (2_{n-1}) We arrive at the end of the rearrangement, that is, $|\mathcal{R}_{n-2}(E) \cap [o, e_{n-1}]| \le |[o, c(e_{n-1}, e_n)]|$ and then there exists
$$x_{n-1} \in [o, c(e_{n-1}, e_n)]$$
such that $f_{n-1}^{\triangle}(x_{n-1}) = f(x)$. Rearranging now with respect to e_n, we have that
$$x' \in E^* \cap [o, e_n].$$

In both cases, there exists a family of vertices $S(x) = \{x_k, x_{k+1}, \cdots, x_{n-j}\}$ for certain $1 \le j \le n-k+1$, with $S(x) = \emptyset$ if $j = n-k+1$, such that

$$x_i \in [o, c(e_i, e_{i+1})] \tag{2.13}$$
$$f_i^{\triangle}(x_i) = f(x), \quad \forall i = k, \dots, n-j. \tag{2.14}$$
$$\begin{cases} x' \in [o, e_{n-j+1}] \setminus [o, c(e_{n-j+1}, e_{n-j+2})] & \text{if} \quad j > 1 \\ \\ x' \in E^* \cap [o, e_n] & \text{if} \quad j = 1 \end{cases} \tag{2.15}$$



The case (1_k).



The case (2_k).

Figure 21: The two possibilities in the definition of $f_k^{\triangle}$.

This family of vertices $S(x)$ can be seen as the path where the minimum value $f(x)$ is moving during the process of constructing $f^\vartriangle$ (see Figure 22). As e_k is the first vertex in ∂E (with respect to σ) with $x \in [o, e_k]$, we trivially have that

$$(f \cdot \chi_{E_0})^\vartriangle_{k-1} = f^\vartriangle_{k-1} \cdot \chi_{\mathcal{R}_{k-1}(E)\setminus\{x\}}. \tag{2.16}$$

By definition, we have:

$$(f \cdot \chi_{E_0})^\vartriangle_k(y) = \begin{cases} (f \cdot \chi_{E_0})^\vartriangle_{k-1}(y) & \text{if} \quad y \in \mathcal{R}_k(E_0) \cap [o, e_k]^c, \\ ((f \cdot \chi_{E_0})^\vartriangle_{k-1} \chi_{[o,e_k]})^*(y) & \text{if} \quad y \in \mathcal{R}_k(E_0) \cap [o, e_k]. \end{cases}$$

We observe that, by construction, we have that

$$\begin{aligned} \mathcal{R}_k(E_0) \cap [o, e_k]^c &= \mathcal{R}_k(E) \cap [o, e_k]^c \\ \mathcal{R}_k(E_0) \cap [o, e_k] &= (\mathcal{R}_k(E) \cap [o, e_k]) \setminus \{x_k\}\,. \end{aligned}$$

Thus, using (2.16) and observing that $\mathcal{R}_k(E) \cap [o, e_k]^c \subset \mathcal{R}_{k-1}(E) \setminus \{x\}$, this is equivalent to:

$$(f \cdot \chi_{E_0})^\vartriangle_k(y) = \begin{cases} f^\vartriangle_{k-1}(y) & \text{if} \quad y \in \mathcal{R}_k(E) \cap [o, e_k]^c, \\ (f^\vartriangle_{k-1} \cdot \chi_{\mathcal{R}_{k-1}(E)\setminus\{x\}} \chi_{[o,e_k]})^*(y) & \text{if} \quad y \in (\mathcal{R}_k(E) \cap [o, e_k]) \setminus \{x_k\}\,. \end{cases}$$

Now, we apply Lemma 2.3.9 with $A = \mathcal{R}_{k-1}(E) \cap [o, e_k]$, $A_0 = (\mathcal{R}_{k-1}(E) \setminus \{x\}) \cap [o, e_k]$ and $g = f^\vartriangle_{k-1} \cdot \chi_{[o,e_k]}$, getting:

$$(f \cdot \chi_{E_0})^\vartriangle_k(y) = \begin{cases} f^\vartriangle_{k-1}(y) & \text{if} \quad y \in \mathcal{R}_k(E) \cap [o, e_k]^c, \\ (f^\vartriangle_{k-1} \cdot \chi_{[o,e_k]})^*(y) & \text{if} \quad y \in (\mathcal{R}_k(E) \cap [o, e_k]) \setminus \{x_k\}\,. \end{cases}$$

where we have used that $((\mathcal{R}_{k-1}(E) \setminus \{x\}) \cap [o, e_k])^* = (\mathcal{R}_k(E) \cap [o, e_k]) \setminus \{x_k\}$, and that is:

$$(f \cdot \chi_{E_0})^\vartriangle_k(y) = f^\vartriangle_k(y) \cdot \chi_{\mathcal{R}_k(E)\setminus\{x_k\}}(y).$$

Repeating the same argument and using that the minimum is attained at x_i, $i = k+1, \dots, n-j$, by (2.14), we arrive at

$$(f \cdot \chi_{E_0})^\vartriangle_{n-j}(y) = f^\vartriangle_{n-j}(y) \cdot \chi_{\mathcal{R}_{n-j}(E)\setminus\{x_{n-j}\}}(y),$$

and applying the argument once more, we get

$$(f \cdot \chi_{E_0})^\vartriangle_{n-j+1}(y) = f^\vartriangle_{n-j+1}(y) \cdot \chi_{\mathcal{R}_{n-j+1}(E)\setminus\{x'\}}(y).$$

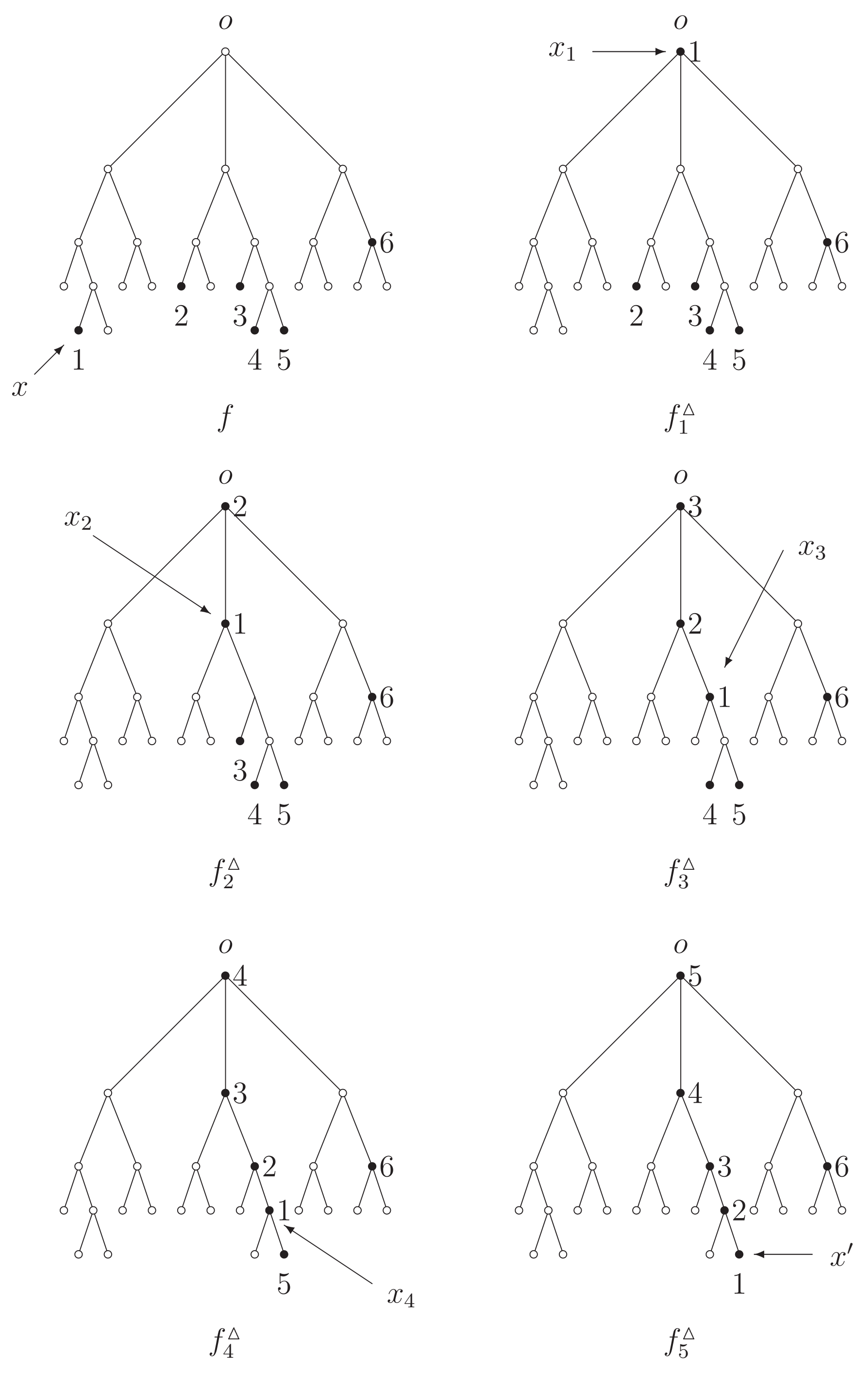


Figure 22: The "path" $S(x) = \{x_1, x_2, x_3, x_4\}$ of the minimum value $f(x) = 1$ and the final vertex x', in the process of constructing $f^\triangle$.

If $j=1$ this is exactly (2.12). If $j>1$, by (2.15), $x' \in [o, e_{n-j+1}] \backslash [o, c(e_{n-j+1}, e_{n-j+2})]$ and so, in the following geodesic rearrangements, the lack of x' does not imply changes with respect to the case of the presence of x', and therefore, (2.12) holds. Finally, using Lemma 2.3.8, (2.9), (2.10), (2.11) and (2.12), we have:

$$\begin{aligned} f^*(y) &= f^*(y) \cdot \chi_{E_0^*}(y) + f^*(x') \cdot \chi_{\{x'\}}(y) \\ &= (f \cdot \chi_{E_0})^*(y) + f(x) \cdot \chi_{\{x'\}}(y) \\ &= (f \cdot \chi_{E_0})^{\triangle}(y) + f^{\triangle}(x') \cdot \chi_{\{x'\}}(y) \\ &= f^{\triangle}(y) \cdot \chi_{E_0^*}(y) + f^{\triangle}(x') \cdot \chi_{\{x'\}}(y) \\ &= f^{\triangle}(y), \end{aligned}$$

for all $y \in T$.

$\square$

2.4 The Hardy-Littlewood inequality

In this section we will study the Hardy-Littlewood inequality in our context. We then consider what kind of conditions on the functions are needed in order to get the saturation of this inequality.

We now recall the **classical decreasing rearrangement** for measurable functions. For a function f we denote it by $f^\star$ to distinguish it from our rearrangement f^*. Thus,

$$f^\star(t) = \inf\left\{\lambda : |\{|f| > \lambda\}| \leq t\right\}, \quad t \geq 0,$$

where $|\{|f| > \lambda\}|$ is the counting measure of the λ-level set of $|f|$. This is a decreasing function defined in $[0, \infty)$. See [BS] (Chapter 2) for details.

The classical Hardy-Littlewood inequality for measurable functions f and g defined in T is

$$\sum_{x \in T} |f(x) g(x)| \leq \int_0^\infty f^\star(t) g^\star(t)\, dt. \tag{2.17}$$

Consequently, the inequality

$$\sum_{x \in T} |f(x) h(x)| \leq \int_0^\infty f^\star(t) g^\star(t)\, dt \tag{2.18}$$

holds for all measurable functions h such that $h^\star = g^\star$. In some measure spaces, the so-called *strongly resonant spaces* (see [BS]), the equality is attained in (2.18) for a suitable h.

A weaker requirement than the attainment of equality in (2.18) holds in our measure space. In particular, the measure space $(T, |.|)$ is *resonant* (with respect to the classical rearrangement), that is, for each measurable f and g in $(T, |.|)$, the identity

$$\int_0^\infty f^\star(t)g^\star(t)\,dt = \sup \sum_{x\in T} |f(x)h(x)|\,, \tag{2.19}$$

holds, where the supremum is taken over all measurable functions h on T such that $h^\star = g^\star$. We will see in the next section that this identity is strongly linked with normability properties of some functional spaces, and that is the reason for our study.

Our purpose is to give the same type of results but using our decreasing rearrangement on the tree.

Proposition 2.4.1 *For all $f \in \mathcal{M}_0$ and for all finite set of vertices $E \subset T$ the inequality*

$$\sum_{x\in E} |f(x)| \leq \sum_{x\in E^*} f^*(x)$$

holds.

Proof. We use the notation from Remark 2.3.5 for the function $|f|$ and Proposition 2.2.20:

$$\begin{aligned}
\sum_{x\in E} |f(x)| &= \sum_{n=1}^{\infty} b_n\,|E\cap F_n| \\
&= \sum_{n=1}^{\infty} b_n\,|(E\cap F_n)^*| \\
&\leq \sum_{n=1}^{\infty} b_n\,|E^*\cap F_n^*| = \sum_{x\in E^*} f^*(x).
\end{aligned}$$

□

We obtain for our rearrangement the Hardy-Littlewood inequality.

Theorem 2.4.2 (Hardy-Littlewood inequality) *For all f and g in $\mathcal{M}_0$, the inequality*

$$\sum_{x\in T} |f(x)g(x)| \leq \sum_{x\in T} f^*(x)g^*(x)$$

holds.

Proof. We use the notation from Remark 2.3.5 for the function $|f|$ and last proposition:

$$\sum_{x\in T}|f(x)g(x)| = \sum_{n=1}^{\infty} b_n\Big(\sum_{x\in F_n}|g(x)|\Big) \le \sum_{n=1}^{\infty} b_n\Big(\sum_{x\in F_n^*} g^*(x)\Big) = \sum_{x\in T} f^*(x)g^*(x).$$

□

It is natural to ask if there is any relationship between both rearrangements. The following two results give some information about this.

Proposition 2.4.3 *For all $f \in \mathcal{M}_0$, the identity*

$$(|f|^*)^\star(t) = f^\star(t), \qquad t > 0,$$

holds.

Proof. We use the notation of Remark 2.3.5 for the function $|f|$, and this notation also works for the classical rearrangement, that is

$$f^\star(t) = \sum_{n=1}^{\infty} b_n \chi_{[0,|F_n|)}(t).$$

Applying this to the function $f^*(x) = \sum_{n=1}^{\infty} b_n \chi_{F_n^*}(x)$ and recalling that $|E| = |E^*|$ for finite sets E, we get:

$$(|f|^*)^\star(t) = \sum_{n=1}^{\infty} b_n \chi_{[0,|F_n^*|)}(t) = \sum_{n=1}^{\infty} b_n \chi_{[0,|F_n|)}(t) = f^\star(t).$$

□

Using this proposition, Theorem 2.4.2, (2.17) and (2.19) we have the following result.

Corollary 2.4.4 *For all $f, g \in \mathcal{M}_0$, we have:*

(i) For all measurable functions f and g in T,

$$\sum_{x\in T}|f(x)g(x)| \le \sum_{x\in T} f^*(x)g^*(x) \le \int_0^\infty f^\star(t)g^\star(t)\,dt.$$

(ii) For all measurable functions f and g in T,

$$\int_0^\infty f^\star(t)g^\star(t)\,dt = \sup \sum_{x\in T} f^*(x)h^*(x),$$

where the supremum is taken over all measurable functions h on T such that $h^\star = g^\star$.

By Proposition 2.4.3, if $f^* = g^*$ then $f^\star = g^\star$. The converse is not true in general as we can see in the next example.

Example 2.4.5 Consider the functions defined below on the pictures. Observe that they differ only at two vertices and that their rearrangements on the tree also differ at two vertices.

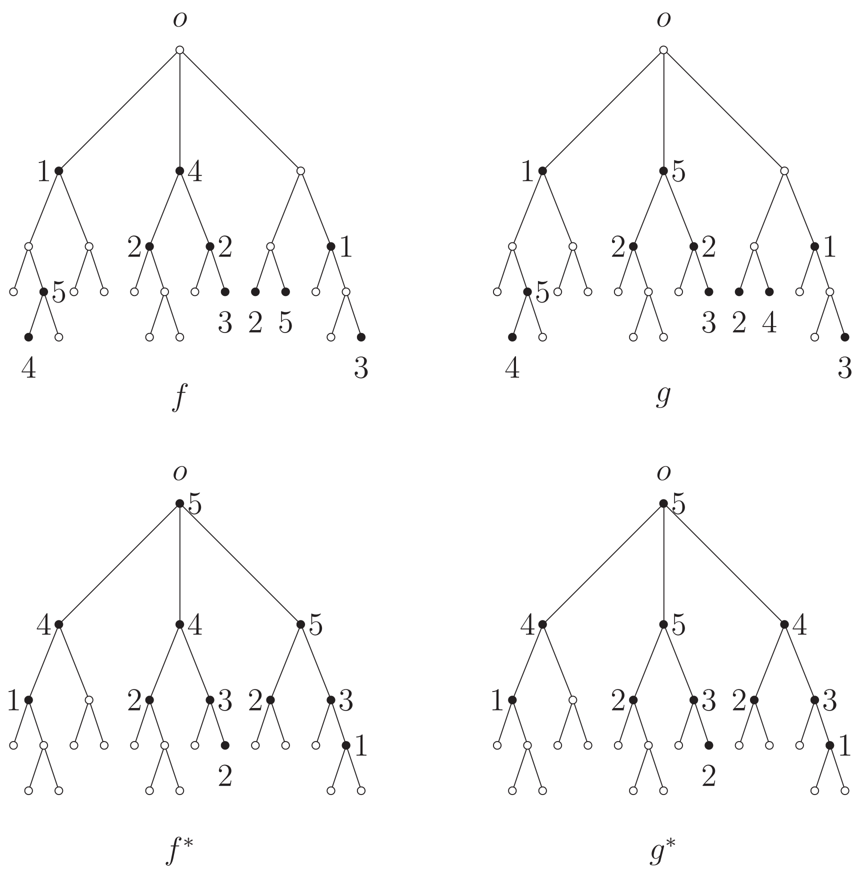


Figure 23: Two functions on T with the same classical rearrangement, but different rearrangement on the tree.

But their classical rearrangements coincide. In particular,

$$f^\star(t) = g^\star(t) = 5\chi_{[0,2)}(t) + 4\chi_{[2,4)}(t) + 3\chi_{[4,6)}(t) + 2\chi_{[6,9)}(t) + 1\chi_{[9,11)}(t).$$

Returning to the Hardy-Littlewood inequality of Theorem 2.4.2, we observe that as a consequence, we have that the inequality

$$\sum_{x \in T} |f(x)h(x)| \leq \sum_{x \in T} f^*(x)g^*(x)$$

holds for all functions $h \in \mathcal{M}_0$ such that $h^* = g^*$. Is it possible to saturate this inequality in order to obtain an identity such as (2.19) but with our rearrangement on the tree? That is, is it possible to get the equality

$$\sum_{x\in T} f^*(x)g^*(x) = \sup \sum_{x\in T} |f(x)h(x)|\,, \tag{2.20}$$

where the supremum is taken over all functions $h \in \mathcal{M}_0$ such that $h^* = g^*$? The answer is negative in general, as the following example shows.

Example 2.4.6 Consider the four vertices in the tree of the following figure.

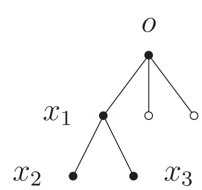


Figure 24: The four vertices.

Take $E = \{o, x_1, x_2\}$ and $g = \chi_E$. Since E is a decreasing set, we have that $g^* = g$. For a fixed $f \in \mathcal{M}_0$, we would like to have the equality

$$\sum_{x\in E} f^*(x) = \sup_{\{h\in\mathcal{M}_0 :\, h^*=\chi_E\}} \sum_{x\in T} |f(x)h(x)|\,,$$

or equivalently

$$\sum_{x\in E} f^*(x) = \sup_{\{D\subset T :\, D^*=E\}} \sum_{x\in D} |f(x)|\,.$$

To see that this equality does not hold it is enough to take the function f to be large enough at x_3 with respect to its values at the rest of the vertices. In particular, take

$$f(x) = 4\chi_{\{x_3\}}(x) + \chi_{\{o\}}(x) + \chi_{\{x_1\}}(x) + \chi_{\{x_2\}}(x).$$

Therefore,

$$f^*(x) = 4\chi_{\{o\}}(x) + \chi_{\{x_3\}}(x) + \chi_{\{x_1\}}(x) + \chi_{\{x_2\}}(x).$$

and thus

$$\sum_{x\in E} f^*(x) = 6,$$

but

$$\sup_{\{D \subset T \colon D^* = E\}} \sum_{x \in D} |f(x)| = 3,$$

because any set D such that $D^* = E$ cannot contain x_3 (recall that we take the order σ fixed in Remarks 2.2.16).

Last example says that identity (2.20) fails to be true for a general decreasing function g, even for a general characteristic function of a decreasing set. Now, we choose four new vertices given in the next figure.

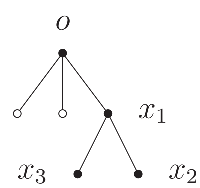


Figure 25: The new four vertices.

As before set $E = \{o, x_1, x_2\}$ and $g = \chi_E = g^*$, and

$$f(x) = 4\chi_{\{x_3\}}(x) + \chi_{\{o\}}(x) + \chi_{\{x_1\}}(x) + \chi_{\{x_2\}}(x),$$

and thus

$$f^*(x) = \chi_{\{x_3\}}(x) + 4\chi_{\{o\}}(x) + \chi_{\{x_1\}}(x) + \chi_{\{x_2\}}(x).$$

In this case, we have

$$\sum_{x \in E} f^*(x) = \sup_{\{D \subset T \colon D^* = E\}} \sum_{x \in D} |f(x)| = 6,$$

simply taking $D = \{x_1, x_2, x_3\}$.

We see that there exist decreasing functions g in the tree such that (2.20) holds. Our purpose now is to identify the class of decreasing functions in the tree such that this equality holds.

We first fix our attention to the functions with finite support, and we begin by looking at the case of functions with support in one geodesic from the origin o to a fixed vertex e. In this case, our rearrangement is equivalent to the classical rearrangement of discrete functions defined on $\mathbb{N}$, with support in the interval $[0, N]$ for a fixed N, simply by considering the bijection

$$[o, e] = \{o = e(0), e(1), \ldots, e(N) = e\} \equiv [0, N],$$

such that $e(i) \longleftrightarrow i$. The measure space $([0, N], |.|)$ is *strongly resonant* and this means that, for fixed f and g, the equality

$$\sum_{x\in[o,e]} |f(x)h(x)| = \sum_{x\in[o,e]} f^*(x)g^*(x), \tag{2.21}$$

holds for a suitable h. In fact, a suitable h can be constructed by permuting the values of g^*, by using the permutation that takes f into f^*. To be precise, consider the permutation

$$\varphi_f : [o, e] \longrightarrow [o, e]$$

such that

$$|f(x)| = f^*(\varphi_f(x)),$$

for all $x \in [o, e]$. Then, $h(x) = g^*(\varphi_f(x))$ satisfies equality (2.21) (by considering a change of variable $x = \varphi_f(y)$), and trivially $h^* = g^*$.

Example 2.4.7 Consider the function

$$f(x) = 2\chi_{\{e(0)\}}(x) + 3\chi_{\{e(1)\}}(x) + 5\chi_{\{e(2)\}}(x) + 4\chi_{\{e(3)\}}(x) + 2\chi_{\{e(4)\}}(x) + \chi_{\{e(5)\}}(x),$$

with rearrangement

$$f^*(x) = 5\chi_{\{e(0)\}}(x) + 4\chi_{\{e(1)\}}(x) + 3\chi_{\{e(2)\}}(x) + 2\chi_{\{e(3)\}}(x) + 2\chi_{\{e(4)\}}(x) + \chi_{\{e(5)\}}(x).$$

Now, set

$$\begin{array}{ll} \varphi_f(e(0)) = e(3), & \varphi_f(e(1)) = e(2), \\ \varphi_f(e(2)) = e(0), & \varphi_f(e(3)) = e(1), \\ \varphi_f(e(4)) = e(4), & \varphi_f(e(5)) = e(5), \end{array}$$

that satisfies $f(x) = f(\varphi_f(x))$. For all decreasing functions g^* in $[e(0), e(5)]$, the function $h(x) = g^*(\varphi_f(x))$ satisfies equality (2.21) and trivially $h^* = g^*$. For instance, take

$$g^*(x) = 6\chi_{\{e(0)\}}(x) + 6\chi_{\{e(1)\}}(x) + 4\chi_{\{e(2)\}}(x) + 3\chi_{\{e(3)\}}(x) + 2\chi_{\{e(4)\}}(x) + \chi_{\{e(5)\}}(x),$$

then

$$h(x) = 3\chi_{\{e(0)\}}(x) + 4\chi_{\{e(1)\}}(x) + 6\chi_{\{e(2)\}}(x) + 6\chi_{\{e(3)\}}(x) + 2\chi_{\{e(4)\}}(x) + \chi_{\{e(5)\}}(x).$$

It is easy to compute that

$$\sum_{x\in[o,e]} |f(x)h(x)| = \sum_{x\in[o,e]} f^*(x)g^*(x) = 77.$$

Observe that the permutation φ_f is not unique in general. In our example, we could have also taken $\varphi_f(e(0)) = e(4)$ and $\varphi_f(e(4)) = e(3)$. If we require the permutation to satisfy that $\varphi_f(e(i)) < \varphi_f(e(j))$ if $e(i) < e(j)$, whenever $f(e(i)) = f(e(j))$, then φ_f is unique.

We observe that the permutation φ_f, for measurable f, plays an important role in order to obtain equality (2.21). In the "linear" case of $[o, e]$, we trivially have that if g is decreasing, then $g(\varphi_f(.))^* = g$ for all f. In fact, the reverse implication is also true, and we can trivially state that g is decreasing if and only if $g(\varphi_f(.))^* = g$ for all f. This is not true in the case of general finite support.

We fix now our attention on functions with general finite support not necessarily contained in a geodesic.

Definition 2.4.8 Let f be a positive function with finite support $E \subset T$. A **rearranging transformation** for f is a bijection

$$\varphi_f : E \longrightarrow E^*$$

such that

$$f(x) = f^*(\varphi_f(x)),$$

for all $x \in E$.

In view of Theorem 2.3.10 and definitions (2.7) and (2.8), we can decompose φ_f into the composition of rearranging transformations for every geodesic from o to each vertex in the boundary of E. To be precise, if $n = |\partial E|$,

$$\varphi_f = \varphi_{f,n} \circ \varphi_{f,n-1} \circ \cdots \circ \varphi_{f,1},$$

where each $\varphi_{f,k}$ is a mapping

$$\varphi_{f,k} : \mathcal{R}_{k-1}(E) \longrightarrow \mathcal{R}_k(E)$$

such that

- $\varphi_{f,k}$ is the identity out of $[o, e_k]$, that is

$$\varphi_{f,k} \cdot \chi_{\mathcal{R}_{k-1}(E)\setminus[o,e_k]} = \mathrm{Id}.$$

- Each $\varphi_{f,k}$ is the rearranging transformation for $f_k^{\triangle}$ restricted to $[o, e_k]$, that is:

  (i) $\varphi_{f,k} : \mathcal{R}_{k-1}(E) \cap [o, e_k] \longleftrightarrow \mathcal{R}_k(E) \cap [o, e_k]$.

  (ii) $f_k^{\triangle}(y) = \begin{cases} f_{k-1}^{\triangle}(y) & \text{if} \quad y \in \mathcal{R}_k(E) \setminus [o, e_k], \\ \\ f_{k-1}^{\triangle}(\varphi_{f,k}^{-1}(y)) & \text{if} \quad y \in \mathcal{R}_k(E) \cap [o, e_k]. \end{cases}$

In other words, each $\varphi_{f,k}$ is the rearranging transformation for $f_k^{\triangle} \cdot \chi_{[o,e_k]}$, extended to all $\mathcal{R}_{k-1}(E)$ as the identity. We have seen in Example 2.4.7 that these bijections are not unique, unless we require that $\varphi_{f,k}(x) \leq \varphi_{f,k}(y)$ if $x \leq y$ whenever $f_{k-1}^{\triangle}(x) = f_{k-1}^{\triangle}(y)$. We keep this condition for granted, so that φ_f is also unique for every f.

For a finite set of vertices E, we define

$$\Phi(E) = \{\varphi : E \longleftrightarrow E^* : \exists f \text{ s.t.}\, \varphi = \varphi_f\},$$

and

$$\Phi = \bigcup_{\{E \subset T : |E| < \infty\}} \Phi(E).$$

Thus, Φ **is the set of all the rearranging transformations** in the tree.

In general, for a decreasing positive function g, the equality

$$g(\varphi_f(.))^* = g \tag{2.22}$$

does not hold.

Example 2.4.9 Consider the vertices of the next figure,

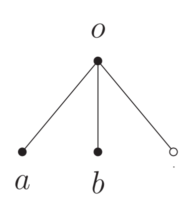


Figure 26: The support of f and g.

and the functions

$$f(x) = \chi_{\{o\}}(x) + 2\chi_{\{a\}}(x) + 3\chi_{\{b\}}(x),$$

and

$$g(x) = 2\chi_{\{o\}}(x) + \chi_{\{a\}}(x),$$

and thus,

$$f^*(x) = 3\chi_{\{o\}}(x) + \chi_{\{a\}}(x) + 2\chi_{\{b\}}(x),$$

and $g = g^*$ is decreasing. By definition, we have

$$\varphi_f(b) = o, \quad \varphi_f(o) = a, \quad \varphi_f(a) = b.$$

Therefore,

$$g(\varphi_f(x)) = \chi_{\{o\}}(x) + 2\chi_{\{b\}}(x),$$

and

$$(g(\varphi_f(.))^*(x) = 2\chi_{\{o\}}(x) + \chi_{\{b\}}(x),$$

and so, (2.22) does not hold.

Examples 2.4.6 and 2.4.9 say that g must be something better than decreasing in order to have (2.20) or (2.22). We need to consider a new order structure in the tree.

Definition 2.4.10 Given two vertices x and y in T, we define

$$x \trianglelefteq y$$

if and only if

$$x \leq y \quad \text{or} \quad I(x) \geq_\sigma I(y).$$

We illustrate the definition below. Recall that we use the order σ (from left to right) fixed in Remarks 2.2.16.

It is very important to observe that this new order is a total order, compatible with the natural partial order, and that it depends on the choice of σ.

We give now some results that will lead to the final result of this section. In what follows, we will use the notation

$$(x, y] = [x, y] \setminus \{x\},$$

or

$$[x, y) = [x, y] \setminus \{y\},$$

for two vertices x and y in T.

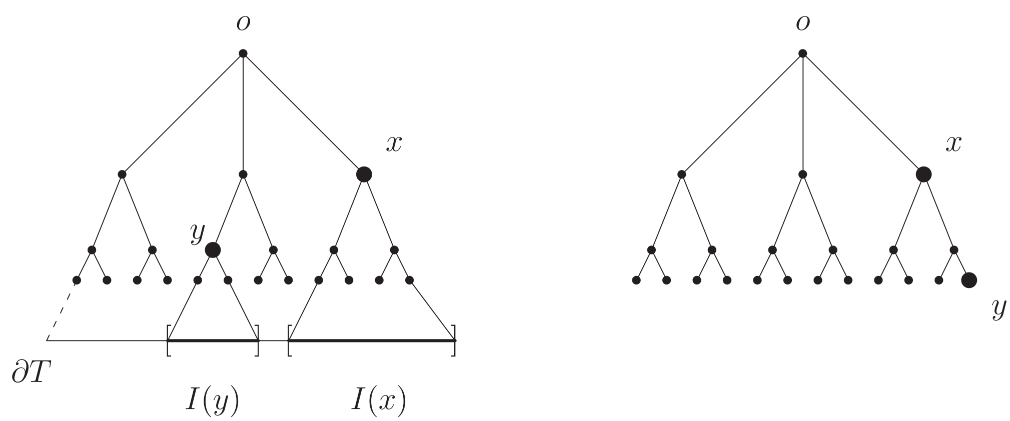


Figure 27: The two possibilities for $x \trianglelefteq y$.

Lemma 2.4.11 *Let f be a positive function in T with finite support E, and $\partial E = \{e_1, e_2, \ldots, e_n\}_\sigma$ its boundary. If $x, y \in E \cap [o, e_1]$ satisfy that*

$$f(x) \geq f(y),$$

then

$$\varphi_f(x) \trianglelefteq \varphi_f(y).$$

Proof. If $\varphi_f(x)$ and $\varphi_f(y)$ lie in the same geodesic, we trivially have that

$$\varphi_f(x) \leq \varphi_f(y),$$

because of the hypothesis and that f^* is decreasing. Suppose that $\varphi_f(x)$ and $\varphi_f(y)$ do not lie in the same geodesic. Take the decomposition

$$\varphi_f = \varphi_{f,n} \circ \varphi_{f,n-1} \circ \cdots \circ \varphi_{f,1},$$

and write $y_j = \varphi_{f,j} \circ \cdots \circ \varphi_{f,1}(y)$ and $x_j = \varphi_{f,j} \circ \cdots \circ \varphi_{f,1}(x)$. By hypothesis, there exists $1 \leq k \leq n$ such that x_k and y_k lie in the same geodesic and

$$x_k \leq y_k,$$

but x_{k+1} and y_{k+1} do not lie in the same geodesic. This can only happen if (see Figure 28)

$$y_k \in \mathcal{R}_k(E) \cap (c(e_k, e_{k+1}), e_k],$$

and

$$x_{k+1} \in \mathcal{R}_{k+1}(E) \cap (c(e_k, e_{k+1}), e_{k+1}],$$

since then, by definition of $\varphi_{f,k+1}$ we have

$$y_{k+1} = \varphi_{f,k+1}(y_k) = y_k \in (c(e_k, e_{k+1}), e_k],$$

and then x_{k+1} and y_{k+1} do not lie in the same geodesic, and by construction

$$x_{k+1} \trianglelefteq y_{k+1}.$$

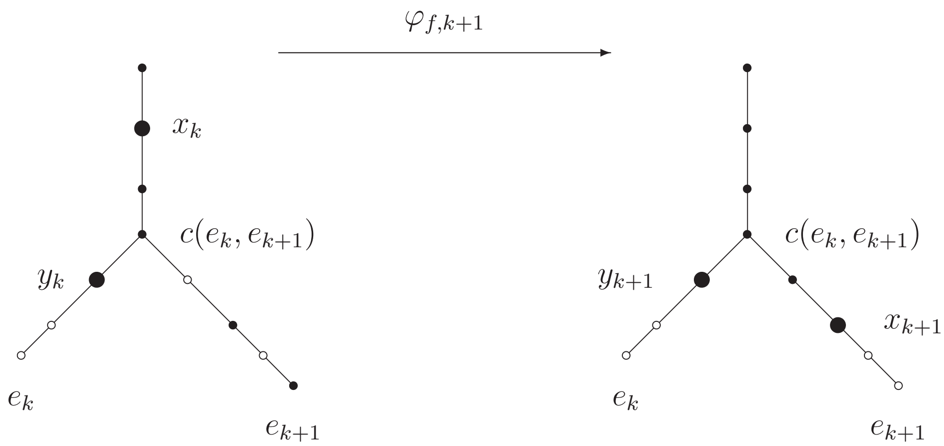


Figure 28: The situation of x_k and y_k, and the action of $\varphi_{f,k+1}$.

Now, we claim that

$$\varphi_f(x) \leq x_{k+1}. \tag{2.23}$$

Thus,

$$\varphi_f(x) \leq x_{k+1} \trianglelefteq y_{k+1} = \varphi_f(y),$$

where the last equality is due to the fact that $\varphi_{f,j} = \mathrm{Id}$ in $(c(e_k, e_{k+1}), e_k]$ for all $j \geq k+1$. We now prove the claim. Two possibilities can happen: first, if $x_{k+1} \in (c(e_{k+1}, e_{k+2}), e_{k+1}]$ then

$$\varphi_f(x) = x_{k+1},$$

because $\varphi_{f,j} = \mathrm{Id}$ in $(c(e_{k+1}, e_{k+2}), e_{k+1}]$ for all $j \geq k+2$, and so we have an equality in (2.23). Second, if $x_{k+1} \in [o, c(e_{k+1}, e_{k+2})]$ and if it does not exist $y \in (c(e_{k+1}, e_{k+2}), e_{k+2}]$ such that

$$f^{\vartriangle}_{k+1}(x_{k+1}) \leq f^{\vartriangle}_{k+1}(y),$$

then

$$x_{k+2} = \varphi_{f,k+2}(x_{k+1}) = x_{k+1},$$

and nothing changes in this rearrangement. If there exists $y \in (c(e_{k+1}, e_{k+2}), e_{k+2}]$ such that

$$f^{\triangle}_{k+1}(x_{k+1}) \leq f^{\triangle}_{k+1}(y),$$

then necessarily $x_{k+2} = \varphi_{f,k+2}(x_{k+1}) \in T(x_{k+1})$, and then

$$x_{k+2} \leq x_{k+1}.$$

Repeating this argument, we obtain

$$\varphi_f(x) = x_n \leq x_{n-1} \leq \ldots x_{k+1}.$$

□

Lemma 2.4.12 *Let E be a finite set in T, and $n = |\partial E|$. If*

$$\varphi = \varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_2 \circ \varphi_1 \in \Phi(E),$$

then

$$\varphi' := \varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_2 \in \Phi(D),$$

where $D = \mathcal{R}_1(E) \setminus (c(e_1, e_2), e_1]$.

Proof. Observe that $|D| = n - 1$. Moreover,

$$\partial D = \partial E \setminus \{e_1\}.$$

There exists f supported in E such that $\varphi = \varphi_f$. Set

$$g = f^{\triangle}_1 \cdot \chi_D,$$

and then we get

$$g^* = f^* \cdot \chi_{E^* \setminus (c(e_1,e_2),e_1]},$$

and

$$g(x) = g^*(\varphi'(x)),$$

for all $x \in D$.

□

We illustrate this lemma with a graphic example.

Example 2.4.13 We give a function with finite support and its decreasing rearrangement:

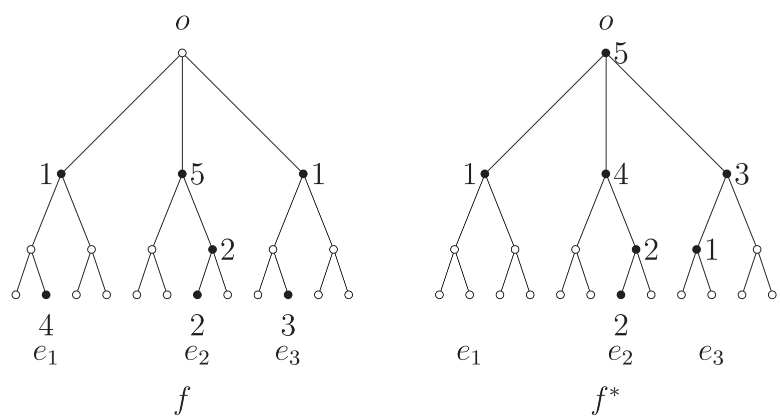


Figure 29: A function f and its rearrangement f^*.

We label the vertices in order to explicit the rearranging transformations:

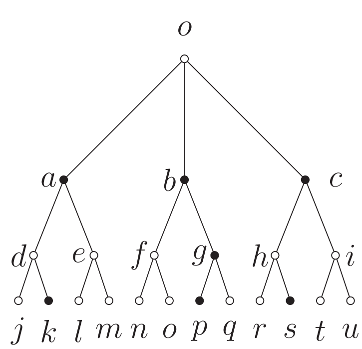


Figure 30: Putting labels at the vertices.

The geodesic rearranging transformations are:

$$\varphi_{f,1}(k) = o, \quad \varphi_{f,1}(a) = a.$$

$$\varphi_{f,2}(b) = o, \quad \varphi_{f,2}(o) = b, \quad \varphi_{f,2}(g) = g, \quad \varphi_{f,2}(p) = p.$$

$$\varphi_{f,3}(o) = o, \quad \varphi_{f,3}(s) = c, \quad \varphi_{f,3}(c) = h.$$

It is easy to check that $\varphi_f = \varphi_{f,3} \circ \varphi_{f,2} \circ \varphi_{f,1}$. The function g defined in last lemma and g^* are in our case:

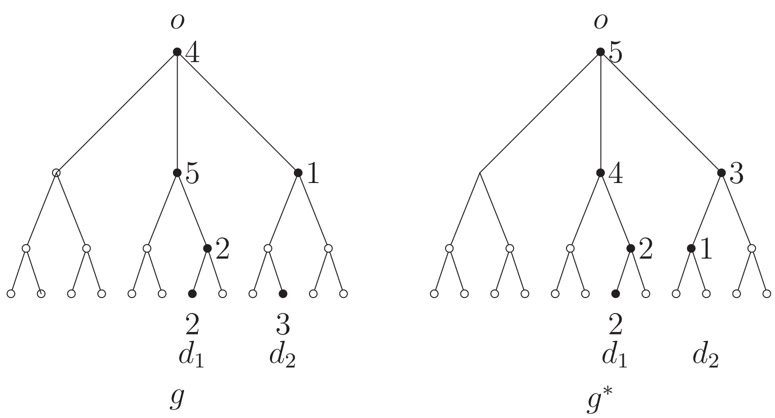


Figure 31: The function g and its rearrangement g^*.

It is straightforward to see that

$$\varphi_{g,1} = \varphi_{f,2} \quad \text{and} \quad \varphi_{g,2} = \varphi_{f,3},$$

and hence $\varphi_g = \varphi_{f,3} \circ \varphi_{f,2}$.

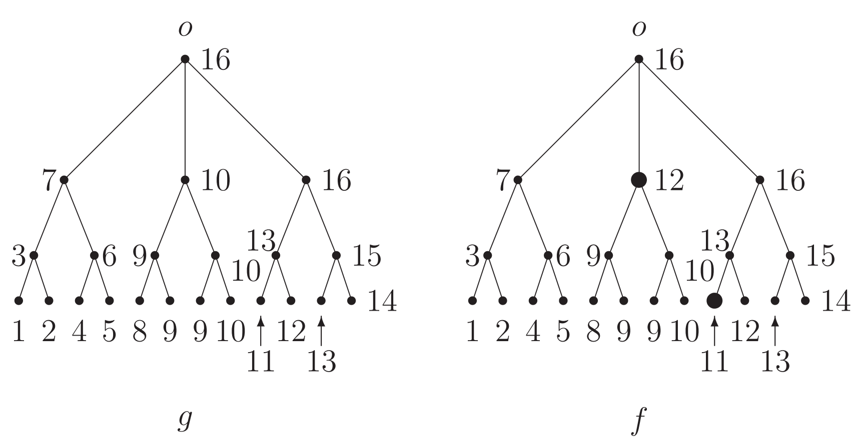


Figure 32: A linearly decreasing function g and a function f not linearly decreasing.

Before proving the next important result, we define a new decreasing property for functions in the tree.

Definition 2.4.14 A function g is **linearly decreasing** if

$$g(x) \geq g(y)$$

if and only if

$$x \trianglelefteq y.$$

We observe that if g is linearly decreasing, then g is decreasing. We give a graphic example in Figure 32; we agree that we extend the function to each tent of the vertices at the bottom of the figure, with the same value at these bottom vertices. In the case of the function not linearly decreasing, we remark what vertices do not satisfy the definition.

The basic result of this section is the following:

Theorem 2.4.15 *If g is a linearly decreasing positive function, then*

$$(g \circ \varphi)^*(y) = g(y), \tag{2.24}$$

for all $\varphi \in \Phi$, and for all y in the support of φ.

Proof. Take $\varphi \in \Phi(E)$ for a certain finite set E. We prove the proposition by induction on $|\partial E|$. If $|\partial E| = 1$, then E is contained into a geodesic $[o, e]$, and we are done because if g is linearly decreasing, it is decreasing. Suppose it is true for $|\partial E| = n-1$. Fix E such that $\partial E = \{e_1, e_2, \dots, e_n\}_\sigma$. If $\varphi \in \Phi(E)$, we know that

$$\varphi = \varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_1,$$

where $\varphi_i = \varphi_{f,i}$ for a certain f supported in E. Set $h = g \circ \varphi$, which is supported in E. There exists $\varphi_h \in \Phi(E)$ such that

$$h(x) = h^*(\varphi_h(x)),$$

for all $x \in E$. Its decomposition is

$$\varphi_h = \varphi_{h,n} \circ \varphi_{h,n-1} \circ \cdots \circ \varphi_{h,1}.$$

We claim that

$$\varphi_1 = \varphi_{h,1}.$$

Take $x, y \in E \cap [o, e_1]$ and suppose that $\varphi_1(x) < \varphi_1(y)$. Necessarily then, $f(x) \geq f(y)$. By Lemma 2.4.11, we have that $\varphi(x) \trianglelefteq \varphi(y)$, and then

$$h(x) = g(\varphi(x)) \geq g(\varphi(y)) = h(y),$$

because g is linearly decreasing. Therefore,

$$\varphi_{h,1}(x) < \varphi_{h,1}(y).$$

Since this is true for all x and y in $E \cap [o, e_1]$ which is a finite set, we have proved the claim. Using the claim and that $\varphi_1 = \mathrm{Id}$ in $\mathcal{R}_1(E) \setminus [o, e_1] \subset E \setminus [o, e_1]$, we have:

$$\begin{aligned} h_1^{\triangle}(y) &= \begin{cases} h(y) & \text{if} \quad y \in \mathcal{R}_1(E) \setminus [o, e_1], \\ h(\varphi_{h,1}^{-1}(y)) & \text{if} \quad y \in \mathcal{R}_1(E) \cap [o, e_1]. \end{cases} \\ &= g(\varphi_n \circ \cdots \circ \varphi_2(y)), \end{aligned} \tag{2.25}$$

for all $y \in \mathcal{R}_1(E)$. So, for $y \in E * \setminus (c(e_1, e_2), e_1]$, we get

$$h^*(y) = (h_1^{\triangle})^*(y) = (g(\varphi_n \circ \cdots \circ \varphi_2(.))^*(y). \tag{2.26}$$

If $\mathcal{R}_1(E) \cap (c(e_1, e_2), e_1] \neq \emptyset$, using (2.25) and that $\varphi_j = \mathrm{Id}$ in $\mathcal{R}_1(E) \cap (c(e_1, e_2), e_1]$ for $j \geq 2$, we get for $y \in \mathcal{R}_1(E) \cap (c(e_1, e_2), e_1]$:

$$h^*(y) = h_1^{\triangle}(y) = g(\varphi_n \circ \cdots \circ \varphi_2(y)) = g(y).$$

This equality, (2.26) and the fact that $E^* \cap (c(e_1, e_2), e_1] = \mathcal{R}_1(E) \cap (c(e_1, e_2), e_1]$ finally takes to:

$$\begin{aligned} h^*(y) &= h^*(y) \cdot \chi_{E^* \cap (c(e_1,e_2),e_1]}(y) + h^*(y) \cdot \chi_{E^* \setminus (c(e_1,e_2),e_1]}(y) \\ &= g(y) \cdot \chi_{E^* \cap (c(e_1,e_2),e_1]}(y) + (g(\varphi_n \circ \cdots \circ \varphi_2(.)))^*(y) \cdot \chi_{E^* \setminus (c(e_1,e_2),e_1]}(y) \\ &= g(y) \cdot \chi_{E^* \cap (c(e_1,e_2),e_1]}(y) + g(y) \cdot \chi_{E^* \setminus (c(e_1,e_2),e_1]}(y) \\ &= g(y), \end{aligned}$$

where we have used in the third equality Lemma 2.4.12 and the hypothesis of induction.

□

We give a graphic example of this last proposition.

Example 2.4.16 In the figures below, we give a function f and its rearrangement. If we label the vertices as in Figure 30, we can give explicitly the rearranging transformation $\varphi = \varphi_f$:

$\varphi(o) = \mathrm{e}$, $\varphi(\mathrm{a}) = \mathrm{j}$, $\varphi(\mathrm{b}) = \mathrm{n}$, $\varphi(\mathrm{c}) = \mathrm{c}$, $\varphi(\mathrm{d}) = \mathrm{o}$, $\varphi(\mathrm{e}) = \mathrm{a}$, $\varphi(\mathrm{f}) = o$,
$\varphi(\mathrm{g}) = \mathrm{p}$, $\varphi(\mathrm{h}) = \mathrm{r}$, $\varphi(\mathrm{i}) = \mathrm{i}$, $\varphi(\mathrm{j}) = \mathrm{d}$, $\varphi(\mathrm{k}) = \mathrm{k}$, $\varphi(\mathrm{l}) = \mathrm{l}$, $\varphi(\mathrm{m}) = \mathrm{m}$,
$\varphi(\mathrm{n}) = \mathrm{b}$, $\varphi(\mathrm{o}) = \mathrm{f}$, $\varphi(\mathrm{p}) = \mathrm{g}$, $\varphi(\mathrm{q}) = \mathrm{q}$, $\varphi(\mathrm{r}) = \mathrm{h}$, $\varphi(\mathrm{s}) = \mathrm{s}$, $\varphi(\mathrm{t}) = \mathrm{t}$,
$\varphi(\mathrm{u}) = \mathrm{u}$.

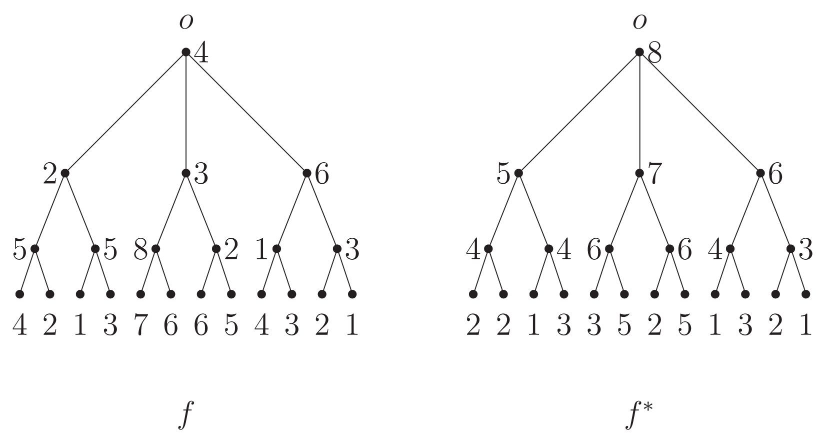


Figure 33: A function f and its rearrangement f^*.

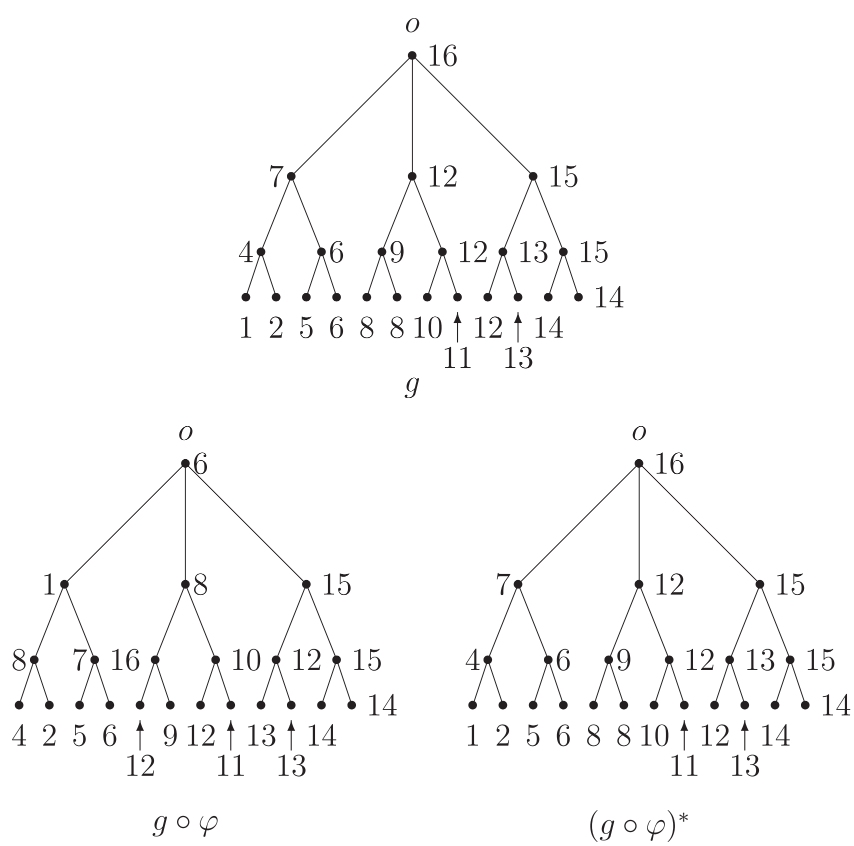


Figure 34: A linearly decreasing function g, the function $g \circ \varphi$ and its rearrangement.

Then, we have chosen a positive linearly decreasing function g, we have found $g \circ \varphi$ and we have computed $(g \circ \varphi)^*$. We observe that $g(y) = (g \circ \varphi)^*(y)$ for all y in the support of f.

The last result of this section is the equality (2.20).

Theorem 2.4.17 *If g is a positive function in T such that*

$$(g \circ \varphi)^* = g,$$

in the support of φ, for all $\varphi \in \Phi$, then

$$\sum_{x \in T} f^*(x) g(x) = \sup_{\{h\colon h^* = g\}} \sum_{x \in T} |f(x) h(x)|,$$

for all measurable functions f.

Proof. Set $C = \sum_{x \in T} f^*(x) g(x)$. It is enough to see that for all $\varepsilon > 0$, there exists a function h such that $h^* = g$ and

$$C - \varepsilon < \sum_{x \in T} |f(x) h(x)|.$$

Set $E_k = \{x \in T : |x| \leq k\}$ and $f_k = |f| \cdot \chi_{E_k}$. By Proposition 2.3.6 $(viii)$, we know that

$$f_k^* \nearrow f^*,$$

because $f_k \nearrow |f|$. By the Monotone Convergence Theorem, and observing that $E_k^* = E_k$ for all $k \geq 0$, there exists n such that

$$C - \varepsilon < \sum_{x \in E_n} f_n^*(x) g(x). \tag{2.27}$$

By hypothesis, we know that $h_n = g \circ \varphi_{f_n}$ satisfies

$$h_n^* = g \cdot \chi_{E_n},$$

and

$$\sum_{x \in E_n} f_n(x) h_n(x) = \sum_{x \in E_n} f_n^*(x) g(x),$$

because trivially

$$\sum_{x \in E_n} f_n^*(x) g(x) = \sum_{y \in E_n} f_n^*(\varphi_{f_n}(y)) g(\varphi_{f_n}(y)) = \sum_{y \in E_n} f_n(y) h_n(y).$$

This equality and (2.27) lead to

$$C - \varepsilon < \sum_{x \in E_n} f_n(x) h_n(x).$$

Define $h = h_n \cdot \chi_{E_n} + g \cdot \chi_{T \setminus E_n}$. We claim that

$$h^* = g,$$

and using the last inequality, we get

$$C - \varepsilon < \sum_{x \in E_n} f_n(x) h_n(x) \leq \sum_{x \in T} f(x) h(x),$$

as we wanted to prove. We now prove the claim. Since we know that

$$h^* = \lim_k (h \cdot \chi_{E_k})^*,$$

it is enough to prove that

$$(h \cdot \chi_{E_m})^* = g \cdot \chi_{E_m},$$

for all $m \geq n$. We denote $\partial E_n = T_n = \{e_1, e_2, \ldots, e_r\}_\sigma$, with $r = (q+1)q^{n-1}$. If we decompose

$$\varphi_{f_n} = \varphi_r \circ \varphi_{r-1} \circ \cdots \circ \varphi_1,$$

then

$$(g \circ \varphi_{f_n})_k^{\triangle}(e_k) = g(e_k),$$

because we know that $(g \circ \varphi_{f_n})^* = g$ and $\varphi_j = \mathrm{Id}$ in $\mathcal{R}_{j-1}(E) \setminus [o, e_j]$ for all j, and in our case $e_k \in \mathcal{R}_{j-1}(E) \setminus [o, e_j]$ for all $j \geq k+1$. Now, to finish the proof, we only need to observe that g is a decreasing function because of the hypothesis, and to consider the following trivial fact at each geodesic rearrangement: let $\{a_1, \ldots, a_k\}$ be a sequence of positive real numbers and $\{a_1^*, \ldots, a_k^*\}$ its decreasing rearrangement; let us add some new values $\{b_1, \ldots, b_m\}$ to the sequence satisfying $b_i \geq b_{i+1}$ for all $1 \leq i \leq m$ and $b_1 \leq a_k^*$. Then, the rearrangement of

$$\{a_1, \ldots, a_k, b_1, \ldots, b_m\}$$

is

$$\{a_1^*, \ldots, a_k^*, b_1, \ldots, b_m\}.$$

□

As a consequence of the last two results, we can state the following corollary:

Corollary 2.4.18 *If g is a positive linearly decreasing function, then*

$$\sum_{x\in T} f^*(x)g(x) = \sup_{\{h\colon h^*=g\}} \sum_{x\in T} |f(x)h(x)|\,,$$

for all measurable functions f.

We will show in Theorem 2.5.12 that the converse also holds.

2.5 The Lorentz spaces

We give the definition of the Lorentz spaces related to our decreasing rearrangement, and we study normability properties of these new spaces.

In 1951, G.G. Lorentz (see [Lo]) introduced the so-called **Lorentz spaces** for an interval $(0,l)$. For a positive real number p and a weight u in the positive real line, the Lorentz space $\Lambda^p(u)$ is defined as the set of Lebesgue measurable functions defined in $(0,l)$ such that the functional

$$\|f\|_{\Lambda^p(u)} = \left(\int_0^\infty (f^\star(t))^p u(t)\,dt\right)^{1/p} \tag{2.28}$$

is finite, where $f^\star$ stands as the classical decreasing rearrangement. For a general measure space (X,μ), the Lorentz space $\Lambda^p_X(u)$ is the set of μ-measurable functions f defined in X such that the functional (2.28) is finite, where now the classical decreasing rearrangement is defined with respect to the measure μ, that is

$$f^\star(t) = \inf\left\{\lambda : \mu(\{|f|>\lambda\}) \le t\right\}, \quad t\ge 0.$$

These spaces are generalizations of the Lebesgue spaces $L^p(X,\mu)$.

We give now our version of these spaces in the tree, using our decreasing rearrangement.

Definition 2.5.1 Let $0<p<\infty$ be a real number and u a positive function defined in T, that is, a weight. The **Lorentz space** $\Delta^p_T(u)$ is the set of measurable functions f defined in T such that the functional

$$\|f\|_{\Delta^p_T(u)} = \left(\sum_{x\in T}(f^*(x))^p u(x)\right)^{1/p}$$

is finite.

Remarks 2.5.2

(i) The simple functions with finite support are in $\Delta_T^p(u)$. If $u \in L^1(T)$, then $L^\infty(T) \subset \Delta_T^p(u)$ and all simple functions are in $\Delta_T^p(u)$.

(ii) Once more, we observe that this space depends on the choice of the origin o and the order σ. But we do not complicate the notation by doing explicit this dependence.

We give some basic properties, with trivial proof derived from Proposition 2.3.6.

Proposition 2.5.3 *For measurable functions f, g and f_k, $k \geq 1$, defined in T, we have:*

(i) If $|f| \leq |g|$, then $\|f\|_{\Delta_T^p(u)} \leq \|g\|_{\Delta_T^p(u)}$.

(ii) $\|\lambda f\|_{\Delta_T^p(u)} = |\lambda|\,\|f\|_{\Delta_T^p(u)}$.

(iii) If $0 \leq f_k \nearrow f$ pointwise, then $\|f_k\|_{\Delta_T^p(u)} \nearrow \|f\|_{\Delta_T^p(u)}$.

(iv) $\|\liminf_k |f_k|\|_{\Delta_T^p(u)} \leq \liminf_k \|f_k\|_{\Delta_T^p(u)}$.

We will use the notation

$$U(E) = \sum_{x\in E} u(x),$$

for every set $E \subset T$ and for every weight u in T. We describe the functional in a new way that will be useful later on.

Lemma 2.5.4 *For all $f \in \Delta_T^p(u)$, we have*

$$\|f\|_{\Delta_T^p(u)} = \left(\int_0^\infty p\,\lambda^{p-1} U(\{|f| > \lambda\}^*)\,d\lambda\right)^{1/p}.$$

Proof. By Proposition 2.3.6 (vii), we have:

$$\|f\|_{\Delta_T^p(u)} = \left(\sum_{x\in T} (|f|^p)^*(x))u(x)\right)^{1/p}.$$

We use Definition 2.3.2 of the decreasing rearrangement and then we apply Fubini's Theorem obtaining:

$$
\begin{aligned}
\|f\|_{\Delta_T^p(u)} &= \left(\sum_{x\in T}\left(\int_0^\infty \chi_{\{|f|^p>\lambda\}^*}(x)\,d\lambda\right)u(x)\right)^{1/p}\\
&= \left(\sum_{x\in T}\left(\int_0^\infty p\,\xi^{p-1}\chi_{\{|f|>\xi\}^*}(x)\,d\xi\right)u(x)\right)^{1/p}\\
&= \left(\int_0^\infty p\,\xi^{p-1}\left(\sum_{x\in\{|f|>\xi\}^*} u(x)\right)d\xi\right)^{1/p}.
\end{aligned}
$$

□

We show that our Lorentz spaces have the property of completeness.

Proposition 2.5.5 *Suppose $u(o)\neq 0$, and let $\{f_k : k\geq 0\}$ be a sequence of measurable functions defined in T. If*

$$\lim_{m,n}\|f_m-f_n\|_{\Delta_T^p(u)}=0,$$

then there exists a function $f\in\Delta_T^p(u)$ defined in T such that

$$\lim_{n}\|f-f_n\|_{\Delta_T^p(u)}=0.$$

Proof. Using the expression given in the previous lemma for the functional, we get for all $t>0$:

$$
\begin{aligned}
\|f\|^p_{\Delta_T^p(u)} &\geq \int_0^t p\,\lambda^{p-1}U(\{|f|>\lambda\}^*)\,d\lambda \geq U(\{|f|>t\}^*)\int_0^t p\,\lambda^{p-1}\,d\lambda\\
&= U(\{|f|>t\}^*)\,t^p,
\end{aligned}
$$

and thus, for all $t>0$ we have

$$U(\{|f|>t\}^*)\leq\frac{\|f\|^p_{\Delta_T^p(u)}}{t^p}.$$

This inequality and the hypothesis lead to

$$\lim_{m,n}U(\{|f_m-f_n|>t\}^*)=0,$$

for all $t>0$. Since $u(o)\neq 0$, the last equality implies that for all $t>0$, there exists p such that

$$\{|f_m-f_n|>t\}=\emptyset,$$

for all $m, n \geq p$, that is, $\{f_k : k \geq 0\}$ is a Cauchy sequence in measure. Then, we know that the sequence is converging in measure to certain measurable f, and there exists a subsequence $\{f_{n_k} : k \geq 0\}$ that converges pointwise to f. By Proposition 2.5.3 (iv), we have that $f \in \Delta_T^p(u)$ and also that

$$\|f - f_n\|_{\Delta_T^p(u)} \leq \liminf_k \|f_{n_k} - f_n\|_{\Delta_T^p(u)},$$

and therefore

$$\lim_n \|f - f_n\|_{\Delta_T^p(u)} = 0.$$

□

The classical Lorentz spaces are generalizations of the classical Lebesgue spaces, in the sense that

$$\Lambda_X^p(1) = L^p(X, \mu).$$

In view of this, it is logical to ask if this relation holds true in the case of our Lorentz spaces. Next proposition gives an answer to this question.

Proposition 2.5.6 *For $0 < p < \infty$, we have*

$$\Delta_T^p(1) = L^p(T, |.|).$$

Proof. We will use Proposition 2.3.6 (vi) and (vii), and that $|E| = |E^*|$ for all $E \subset T$, and by Fubini's Theorem, we get

$$\begin{aligned} \|f\|_{L^p(T,|.|)} &= \sum_{x \in T} |f(x)|^p = \int_0^\infty |\{|f|^p > \lambda\}| \, d\lambda = \int_0^\infty |\{(f^*)^p > \lambda\}| \, d\lambda \\ &= \sum_{x \in T} (f^*(x))^p = \|f\|_{\Delta_T^p(1)}. \end{aligned}$$

□

As a consequence we have $\Lambda_T^p(1) = \Delta_T^p(1)$. However, the spaces $\Lambda_T^p(v)$ (v a weight in $[0, \infty)$) and $\Delta_T^p(u)$ (u a weight in T) are not equal in general. The classical Lorentz spaces are *rearrangement invariant spaces*, that is

$$\|f\|_{\Lambda_X^p(u)} = \|g\|_{\Lambda_X^p(u)},$$

whenever f and g are *equimeasurable* functions, in the sense that

$$\mu(\{|f| > \lambda\}) = \mu(\{|g| > \lambda\}),$$

for all $\lambda > 0$. In fact, two functions f and g are equimeasurable if and only if $f^\star = g^\star$. The Lorentz spaces $\Delta^p_T(u)$ are not rearrangement invariant spaces in this sense in general, as the functions f and g in Example 2.4.5 show ($f^\star = g^\star$ but $f^* \neq g^*$). Furthermore:

Proposition 2.5.7 *The space $\Delta^p_T(u)$ is a rearrangement invariant space if and only if the weight u is a constant in $T \setminus \{o\}$.*

Proof. Suppose first that u is constant in $T \setminus \{o\}$. Take two equimeasurable functions f and g in $(T, |.|)$, that is

$$|\{|f| = \lambda\}| = |\{|g| = \lambda\}| . \tag{2.29}$$

We can assume that their supports are finite. We have:

$$\|f\|_{\Delta^p_T(u)} = (f^*(o))^p u(o) + C \sum_{x \neq o} (f^*(x))^p,$$

$$\|g\|_{\Delta^p_T(u)} = (g^*(o))^p u(o) + C \sum_{x \neq o} (g^*(x))^p.$$

By (2.29), $f^*(o) = g^*(o)$ and using also the last proposition, we get :

$$\begin{aligned} \sum_{x \neq o} (f^*(x))^p &= \|f\|_{L^p(T,|.|)} - (f^*(o))^p \\ &= \|g\|_{L^p(T,|.|)} - (g^*(o))^p \\ &= \sum_{x \neq o} (g^*(x))^p, \end{aligned}$$

and thus

$$\|f\|_{\Delta^p_T(u)} = \|g\|_{\Delta^p_T(u)} .$$

Let us see the converse implication. Suppose that $\Delta^p_T(u)$ is a rearrangement invariant space. We first show that necessarily, u is radial. Take x and y two different vertices such that

$$d(o, x) = d(o, y).$$

Then $f = \chi_{[o,x]}$ and $g = \chi_{[o,y]}$ are equimeasurable functions and thus

$$U([o, x]) = \|f\|_{\Delta^p_T(u)} = \|g\|_{\Delta^p_T(u)} = U([o, y]).$$

This equality implies that u is radial. Now take $x \in T$, and y such that $y \notin [o, x]$ and $d(o, y) = 1$. Set $E = ([o, x] \setminus \{x\}) \cup \{y\}$ and $f = \chi_E$ and $g = \chi_{[o,x]}$ are equimeasurable functions satisfying $f^* = f$ and $g^* = g$, and thus

$$U([o, x]) = \|g\|_{\Delta^p_T(u)} = \|f\|_{\Delta^p_T(u)} = U([o, x]) - u(x) + u(y),$$

that is, $u(x) = u(y)$. This equality and the fact that u is radial leads to $u = C$ in $T \setminus \{o\}$.

□

In any case, we always have an inclusion between these two spaces.

Proposition 2.5.8 *If u is a weight in T, then $\Lambda^p_T(u^\star)$ is a subspace of $\Delta^p_T(u^*)$.*

Proof. We simply apply Corollary 2.4.4 (i) and Proposition 2.4.3:

$$\begin{aligned}
\|f\|^p_{\Delta^p_T(u^*)} &= \sum_{x\in T}(f^*(x))^p u^*(x) \\
&= \sum_{x\in T}(|f|^p)^*(x) u^*(x) \\
&\le \int_0^\infty (|f|^p)^\star(t)(u^*)^\star(t)\,dt \\
&= \int_0^\infty (f^\star(t))^p u^\star(t)\,dt \\
&= \|f\|_{\Lambda^p_T(u^\star)}\,.
\end{aligned}$$

We have used the well-known fact $(|f|^p)^\star(x) = (f^\star(x))^p$.

□

We focus our attention on the functional $\|.\|_{\Delta^p_T(u)}$, and we study what kind of conditions are required on the weight u such that it becomes a quasi-norm or a norm. We observe that we trivially have

$$\|f\|_{\Delta^p_T(u)} = 0 \Leftrightarrow f \equiv 0.$$

In the classical context, M.J. Carro and J. Soria ([CSo1]) characterized the weights u such that the functional $\|.\|_{\Lambda^p_X(u)}$ is a quasi-norm, if X is non-atomic. Later, J.A. Raposo ([R]) completed this result for all X. In our case, we have the following characteritzation.

Theorem 2.5.9 *The functional $\|.\|_{\Delta^p_T(u)}$ is a quasi-norm if and only if there exists a constant $C > 0$ such that*

$$0 < U((E \cup D)^*) \le C(U(E^*) + U(D^*)), \tag{2.30}$$

for all sets E and D such that $E \cup D \neq \emptyset$.

Proof. Suppose that condition (2.30) holds. By Lemma 2.5.4, if $\|f\|_{\Delta^p_T(u)} = 0$, then

$$U(\{|f| > \lambda\}^*) = 0,$$

for all $\lambda > 0$, and by hypothesis,

$$\{|f| > \lambda\} = \emptyset,$$

for all λ, that is, $f \equiv 0$. Also by Lemma 2.5.4 and applying our hypothesis, we have:

$$\begin{aligned}
\|f+g\|^p_{\Delta^p_T(u)} &= \int_0^\infty p\,\lambda^{p-1} U(\{|f+g| > \lambda\}^*)\,d\lambda \\
&\leq \int_0^\infty p\,\lambda^{p-1} U((\{|f| > \lambda/2\} \cup \{|g| > \lambda/2\})^*)\,d\lambda \\
&\leq C\Big(\int_0^\infty p\,\lambda^{p-1} U(\{|f| > \lambda/2\}^*)\,d\lambda + \int_0^\infty p\,\lambda^{p-1} U(\{|g| > \lambda/2\}^*)\,d\lambda\Big) \\
&= 2C\Big(\int_0^\infty p\,\lambda^{p-1} U(\{|f| > \lambda\}^*)\,d\lambda + \int_0^\infty p\,\lambda^{p-1} U(\{|g| > \lambda\}^*)\,d\lambda\Big) \\
&= 2C(\|f\|^p_{\Delta^p_T(u)} + \|g\|_{\Delta^p_T(u)})^p,
\end{aligned}$$

where we have used the monotonic property $E^* \subset D^*$ if $E \subset D$. Now, suppose that the functional is a quasi-norm. Take E and D such that $E \cup D \neq \emptyset$, an then:

$$\begin{aligned}
U((E\cup D)^*)^{1/p} &= \|\chi_{E\cup D}\|_{\Delta^p_T(u)} \leq C(\|\chi_E\|_{\Delta^p_T(u)} + \|\chi_D\|_{\Delta^p_T(u)}) \\
&= C(U(E^*)^{1/p} + U(D^*)^{1/p}).
\end{aligned}$$

□

We study now when the functional $\|.\|_{\Delta^p_T(u)}$ is a norm. In the classical context, this problem was solved by G.G. Lorentz ([Lo]) in the case $X = (0, l)$ and $p \geq 1$. The general case is completely characterized by J.A. Raposo in [R]. In all cases, the necessary and sufficient condition on the weight so that $\|.\|_{\Lambda^p_X(u)}$ becomes a norm is that u must be a decreasing function. With these results in mind, we obtain our first positive result:

Lemma 2.5.10 *If $\|.\|_{\Delta^p_T(u)}$ is a norm, then u is a decreasing function in T.*

Proof. We take two neighbor vertices x and y, with $x \leq y$. Set $f = \chi_{[o,x]} + \lambda\chi_{\{y\}}$ and $g = \chi_{[o,y]\setminus\{x\}} + \lambda\chi_{\{x\}}$, with $0 < \lambda < 1$. We then have $g^* = f^* = f$ and

$$\|f\|^p_{\Delta^p_T(u)} = \|g\|^p_{\Delta^p_T(u)} = U([o,x]) + \lambda^p u(y).$$

Moreover, $f+g = 2\chi_{[o,y]\setminus\{x\}} + (1+\lambda)(\chi_{\{x\}} + \chi_{\{y\}})$, and thus

$$\|f+g\|^p_{\Delta^p_T(u)} = 2^p U([o,y]\setminus\{x\}) + (1+\lambda)^p(u(x)+u(y)).$$

By the triangle inequality, we have

$$\Big(2^p U([o,y]\setminus\{x\}) + (1+\lambda)^p(u(x)+u(y))\Big)^{1/p} \leq 2\Big(U([o,x]) + \lambda^p u(y)\Big)^{1/p}.$$

We derive from here that

$$u(y) \leq \frac{2^p - (1+\lambda)^p}{(1+\lambda)^p - 2^p\lambda^p} u(x).$$

We obtain $u(y) \leq u(x)$ by letting $\lambda \nearrow 1$.

$\square$

Unfortunately, it is easy to see that this condition on u is not sufficient to get a norm.

Example 2.5.11 Consider the decreasing weight u and the functions f and g in the figure.

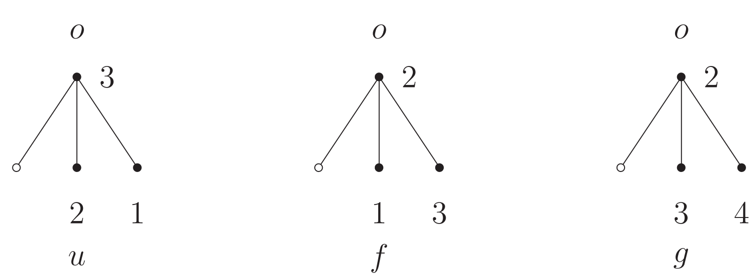


Figure 35: A decreasing weight u and the functions f and g.

It is easy to compute that $\|f+g\|_{\Delta^1_T(u)} = 33$ and $\|f\|_{\Delta^1_T(u)} + \|g\|_{\Delta^1_T(u)} = 32$, that is $\|.\|_{\Delta^1_T(u)}$ is not a norm.

As Examples 2.4.6 and 2.4.9 in the previous section, this example says that u must be something better than decreasing. If we observe the proof of G.G. Lorentz in [Lo] of the characteritzation as a norm of the functional, we can complete this result by stating the following: if $p \geq 1$, the following are equivalent

(i) The weight u is a decreasing function in $[0,\infty)$.

(ii) For all Lebesgue measurable functions f in $(0,l)$, the equality

$$\sup_{\{h:\, h^\star=u\}} \int_0^l |f(t)h(t)|\, dt = \int_0^\infty f^\star(t)u(t)\, dt,$$

holds.

(iii) The functional $\|.\|_{\Lambda^p_{(0,l)}(u)}$ in (2.28) is a norm.

In view of this, maybe the required condition on the weight u is to be a linearly decreasing function, since in the previous section we saw that this condition leads to an equality of the type (ii) above. The answer is next theorem. Recall that Φ is the set of all the rearranging transformations in the tree.

Theorem 2.5.12 *Let u be a weight in T.*

(i) If $0<p<1$, the functional $\|.\|_{\Delta^p_T(u)}$ is a norm if and only if $\operatorname{supp}(u)=\{o\}$.

(ii) If $p\geq 1$, the following are equivalent:

(a) u is linearly decreasing in T.

(b) For all $\varphi\in\Phi$, the equality

$$(u\circ\varphi)^*(y) = u(y)$$

holds for all y in the support of φ.

(c) For all measurable functions f in T, the equality

$$\sup_{\{h:\, h^*=u\}} \sum_{x\in T} |f(x)h(x)| = \sum_{x\in T} f^*(x)u(x)$$

holds.

(d) The functional $\|.\|_{\Delta^p_T(u)}$ is a norm.

Proof. We first proof that if $\|.\|_{\Delta^p_T(u)}$ is a norm, then u is linearly decreasing, for all $0<p<\infty$. If $x\geq y$, then Lemma 2.5.10 shows that $u(x)\leq u(y)$. Take $x\trianglerighteq y$ and $0<a<b<c<d$, set $2m=a+b$ and consider the functions f and g of Figure 36. Observe that $f^*=g^*$. The triangle inequality gives, after cancelations

$$((a+b)^p-(2a)^p)\, u(x) \leq ((2b)^p-(a+b)^p)\, u(y).$$

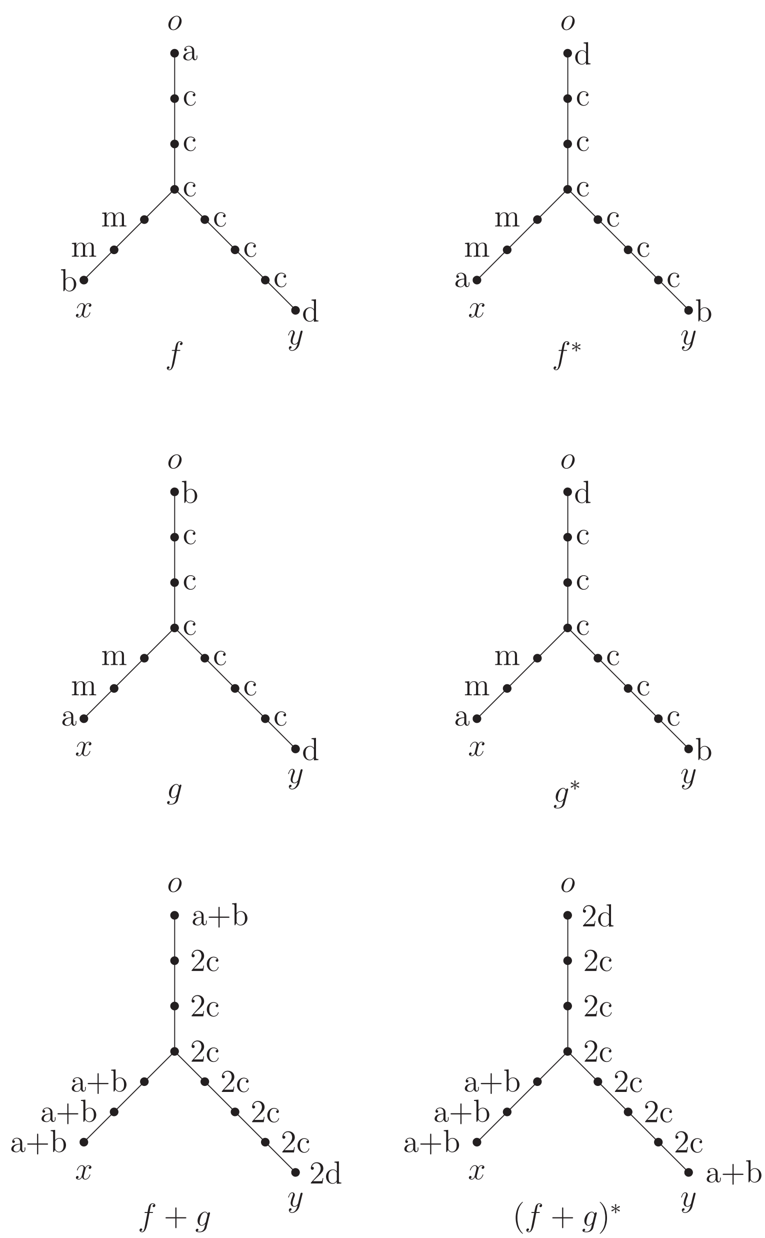


Figure 36: The functions f, g, $f+g$ and their rearrangements.

If we set $1 < \lambda = b/a$, the inequality becomes

$$((1+\lambda)^p - 2^p)\, u(x) \le ((2\lambda)^p - (1+\lambda)^p)\, u(y).$$

Now, observe that

$$\lim_{\lambda\to 1} \frac{(2\lambda)^p - (1+\lambda)^p}{(1+\lambda)^p - 2^p} = 1,$$

and thus, $u(x) \le u(y)$. This proves $(d) \Rightarrow (a)$ in part (ii).

(i) The sufficiency is obvious. Suppose that the functional is a norm. Set $f = \chi_{\{o\}}$ and $g = \lambda\chi_{\{x\}}$ with $0 < \lambda < 1$ and x a neighbor vertex of o such that $x \trianglelefteq y$ for all $y \neq 0$ in T. Then $f^* = f$, $g^* = \lambda\chi_{\{o\}}$ and $(f+g)^* = f + g = \chi_{\{o\}} + \lambda\chi_{\{x\}}$. The triangle inequality gives

$$\|f+g\|_{\Delta_T^p(u)} = \Big(u(o) + \lambda^p u(x)\Big)^{1/p} \le u(o)^{1/p} + (\lambda^p u(o))^{1/p} = \|f\|_{\Delta_T^p(u)} + \|g\|_{\Delta_T^p(u)}\,.$$

From this, we have

$$\frac{\Big(u(o) + \lambda^p u(x)\Big)^{1/p} - u(o)^{1/p}}{\lambda} \le u(o)^{1/p} < \infty.$$

If $u(x) \neq 0$, then

$$\lim_{\lambda\to 0} \frac{\Big(u(o) + \lambda^p u(x)\Big)^{1/p} - u(o)^{1/p}}{\lambda} = \infty,$$

because $p < 1$, getting a contradiction. Thus, $u(x) = 0$ and since u is linearly decreasing, $u = u(o)\chi_{\{o\}}$.

(ii)$(a) \Rightarrow (b)$ This is Theorem 2.4.15.

$(b) \Rightarrow (c)$ This is Theorem 2.4.17. $(c) \Rightarrow (d)$ We apply Proposition 2.3.6 (vii), and the hypothesis (twice):

$$
\begin{aligned}
\|f+g\|_{\Delta_T^p(u)} &= \left(\sum_{x\in T}(f+g)^*(x)^p u(x)\right)^{1/p} \\
&= \left(\sum_{x\in T}(|f+g|^p)^*(x)u(x)\right)^{1/p} \\
&= \sup_{\{h:\, h^*=u\}}\left(\sum_{x\in T}|f(x)+g(x)|^p\, h(x)\right)^{1/p} \\
&\le \sup_{\{h:\, h^*=u\}}\left(\sum_{x\in T}|f(x)|^p\, h(x)\right)^{1/p} + \sup_{\{h:\, h^*=u\}}\left(\sum_{x\in T}|g(x)|^p\, h(x)\right)^{1/p} \\
&= \left(\sum_{x\in T}(f^*(x))^p u(x)\right)^{1/p} + \left(\sum_{x\in T}(g^*(x))^p u(x)\right)^{1/p} \\
&= \|f\|_{\Delta_T^p(u)} + \|g\|_{\Delta_T^p(u)}\,.
\end{aligned}
$$

□

Taking into account Proposition 2.5.5, we obtain:

Corollary 2.5.13 *For* $1 \le p < \infty$, $\Delta_T^p(u)$ *is a Banach space if and only if* u *is linearly decreasing in* T.

By considering Proposition 2.5.8, we can state:

Corollary 2.5.14 *For* $1 \le p < \infty$, *if* u *is a linearly decreasing weight in* T, $\Lambda_T^p(u^\star)$ *is a Banach proper subspace of the Banach space* $\Delta_T^p(u)$.

In view of Theorem 2.5.12, it is now clear that the property of being a Banach space is not invariant under the choice of the origin and the order of the rearrangement, because the linearly decreasing property on a weight is not invariant by the change of the origin and the order in T.

Finally, we use Theorem 1.1.10 to completely characterize the embeddings between these spaces.

Theorem 2.5.15 *For weights* u *and* v *in* T *and* $0 < p, q < \infty$, *we have:*
(a) If $0 < p \le q < \infty$, *then* $\Delta_T^p(u) \hookrightarrow \Delta_T^q(v)$ *if and only if there exists a constant* $C > 0$ *such that*

$$\sup_{D\downarrow}\frac{V(D)^{1/q}}{U(D)^{1/p}} \le C.$$

(b) If $0 < q < p < \infty$, *then the following are equivalent:*

(i) $\Delta_T^p(u) \hookrightarrow \Delta_T^q(v)$.

(ii) There exists $C > 0$ *such that*

$$\left(\int_0^1 \left[\sum_{k \in \mathbb{Z}} \left(\frac{V(D_k) + V(\Delta_k)t}{U(D_k) + U(\Delta_k)t} \right)^{r/p} V(\Delta_k) \right] dt \right)^{1/r} \leq C,$$

for all $\{D_k\} \subset \mathcal{D}(T)$.

(iii) There exists $C > 0$ *such that*

$$\left(\int_0^1 \left[\sum_{k \in \mathbb{Z}} \left(\frac{V(D_k) + V(\Delta_k)t}{U(D_k) + U(\Delta_k)t} \right)^{r/q} U(\Delta_k) \right] dt \right)^{1/r} + \frac{V(X)^{1/q}}{U(X)^{1/p}} \leq C,$$

for all $\{D_k\} \subset \mathcal{D}(T)$.

2.6 Finite trees and regular trees

2.6.1 Regular trees

Recall that a **regular** tree T is a tree with the property that

$$2 \leq \deg(x) \leq M + 1,$$

for all $x \in T$, for some $M \geq 1$, where $\deg(x)$ is the number of neighbour vertices of x. For a rooted regular tree T_o, consider the homogeneous tree T^M of degree $M+1$ and an injective map

$$i : T_o \hookrightarrow T^M,$$

such that $d(x, y) = d_M(i(x), i(y))$, where d and d_M are the distances defined in T_o and T^M respectively, that is, i is an isometry. Define a partial order in T^M as follows: $x \leq y$ if and only if $x \in [i(o), y]$. With this embedding i, we can think of T_o as a subtree of $T^M_{i(o)}$.

Extending, in the natural way, the map i to the boundary of T_o, we easily obtain that

$$i(\partial T_o) \subset \partial T^M_{i(o)}.$$

Hence, if we consider an order σ in $\partial T^M_{i(o)}$ (recall Definition 2.2.4), it is also an order in $i(\partial T_o)$.

It is possible to rearrange a set $E \subset T_o$ by rearranging its inclusion $i(E)$ in $T^M_{i(o)}$. More explicitly, for a finite set E in T_o, define its rearrangement by

$$E^*_{(o,\sigma)} := \mathcal{R}_{(i(o),\sigma)}(i(E)),$$

where o is the origin in T and σ is an order in ∂T^M.

In this case, we loose the canonicity of the rearrangement expressed in Theorem 2.2.15 for the case of a homogeneous tree, because the group of automorphisms of a regular tree T is not as reach as the one of a homogeneous tree T^M. Moreover, we observe that, in general, an automorphism of T^M is not an automorphism of T. As a consequence, we need to be clear in specifying the origin of the regular tree and the order in the boundary.

However, the rest of the results for homogeneous trees given in the previous sections can be applied to this special case, because the group of automorphisms is not playing any role (except in Proposition 2.3.3). We observe that we have only needed to prove this results for finite sets of vertices, and that every finite set of vertices E in T can be viewed as a finite set of vertices $i(E)$ in T^M.

Definition 2.6.1 For every $f \in \mathcal{M}_0(T)$, its **decreasing rearrangement** with respect to the origin o in T and the order σ in ∂T_o is the function

$$f^*_{(o,\sigma)}(x) = \int_0^\infty \chi_{\{y \in T^M : |f(i^{-1}(y))| > \lambda\}^*_{(o,\sigma)}}(i(x))\, d\lambda,$$

defined for all $x \in T$.

Definition 2.6.2 Let $0 < p < \infty$ be a real number and u a positive function defined in T. The **Lorentz space** $\Delta^p_{(o,\sigma)}(u)$ is the set of measurable functions f defined in T such that the functional

$$\|f\|_{\Delta^p_{(o,\sigma)}(u)} = \left(\sum_{x \in T} (f^*_{(o,\sigma)}(x))^p u(x) \right)^{1/p}$$

is finite.

Theorem 2.6.3 *The functional $\|.\|_{\Delta^p_{(o,\sigma)}(u)}$ is a quasi-norm if and only if there exists a constant $C > 0$ such that*

$$0 < U((E \cup D)^*_{(o,\sigma)}) \leq C(U(E^*_{(o,\sigma)}) + U(D^*_{(o,\sigma)})), \tag{2.31}$$

for all sets E and D such that $E \cup D \neq \emptyset$.

Definition 2.6.4 Given two vertices x and y in T, we define

$$x \trianglelefteq_{(o,\sigma)} y$$

if and only if

$$x \leq_o y \quad \text{or} \quad i(I(x)) \geq_\sigma i(I(y)).$$

Definition 2.6.5 A function g is (o,σ)**-linearly decreasing** if

$$g(x) \geq g(y)$$

if and only if

$$x \trianglelefteq_{(o,\sigma)} y.$$

We denote by $\Phi(o,\sigma)$ the set of all the rearranging transformations (recall Definition 2.4.8), for the given pair (o,σ).

Theorem 2.6.6 *Let u be a weight in a regular tree T_o, and σ an order defined in the boundary of the associated homogeneous tree T^M.*

(i) If $0 < p < 1$, the functional $\|.\|_{\Delta^p_{(o,\sigma)}(u)}$ is a norm if and only if $\operatorname{supp}(u) = \{o\}$.

(ii) If $p \geq 1$, the following are equivalent:

(a) u is (o,σ)-linearly decreasing in T.

(b) For all $\varphi \in \Phi(o,\sigma)$, the equality

$$(u \circ \varphi)^*_{(o,\sigma)}(y) = u(y)$$

holds for all y in the support of φ.

(c) For all measurable functions f in T, the equality

$$\sup_{\left\{h:\, h^*_{(o,\sigma)}=u\right\}} \sum_{x\in T} |f(x)h(x)| = \sum_{x\in T} f^*_{(o,\sigma)}(x)u(x)$$

holds.

(d) The functional $\|.\|_{\Delta^p_{(o,\sigma)}(u)}$ is a norm.

2.6.2 Finite trees

The same results of the previous subsection can be given for finite trees, using the same idea. A **finite tree** T is a tree with a finite number of vertices. We fix an origin vertex o and we define a partial order in T_o as usual:

$$x \leq_o y \Leftrightarrow x \in [o,y].$$

We define the boundary of T_o as in Definition 2.2.6, that is,

$$\partial T_o = \{x \in T : T(x) \cap T = \{x\}\},$$

where $T(x) = \{y \in T : y \geq_o x\}$.

Set

$$M + 1 = \max_{x \in T_o} \deg(x).$$

The finite tree can be seen as a subtree of the homogeneous tree T^M of degree $M+1$, if we consider an injective isometric map

$$i : T_o \hookrightarrow T^M.$$

Define a partial order in T^M as before: $x \leq y$ if and only if $x \in [i(o), y]$. Therefore, every order in $\partial T^M_{i(o)}$ is also an order in $\partial i(T_o)$, and we can rearrange every finite set E in T_o by rearranging its image $i(E)$ in $T^M_{i(o)}$. We define

$$E^*_{(o,\sigma)} := \mathcal{R}_{(i(o),\sigma)}(i(E)).$$

In this context, and with the same definitions of the previous subsection, we can state the main result.

Theorem 2.6.7 *Let u be a weight in a finite tree T_o, and σ an order defined in the boundary of the associated homogeneous tree T^M.*

(i) If $0 < p < 1$, the functional $\|.\|_{\Delta^p_{(o,\sigma)}(u)}$ is a norm if and only if $\mathrm{supp}(u) = \{o\}$.

(ii) If $p \geq 1$, the following are equivalent:

(a) u is (o,σ)-linearly decreasing in T.

(b) For all $\varphi \in \Phi(o,\sigma)$, the equality

$$(u \circ \varphi)^*_{(o,\sigma)}(y) = u(y)$$

holds for all y in the support of φ.

(c) For all measurable functions f in T, the equality

$$\sup_{\left\{h:\, h^*_{(o,\sigma)} = u\right\}} \sum_{x \in T} |f(x)h(x)| = \sum_{x \in T} f^*_{(o,\sigma)}(x)u(x)$$

holds.

(d) The functional $\|.\|_{\Delta^p_{(o,\sigma)}(u)}$ is a norm.

We now point out a surprising fact. For a rooted finite tree T_o, it is easy to choose an order in ∂T_o simply by listing these vertices. So, we can use this listing order to rearrange sets and, therefore, functions. A natural question arises: can we have the same previous theorem for this rearrangement? The answer is negative in general, as we are going to show.

By the previous theorem, if there exists an order in ∂T^M such that its restriction to $\partial i(T_o)$ coincides with our listing order, the theorem will be true. But it is easy to see that there are listing orders that are not restriction of any order in ∂T^M. And for this listing orders, the theorem fails to be true, as stated. Let us see an example that shows that we can have a linearly decreasing weight ω not satisfying the condition $(\omega \circ \varphi)^* = \omega$, for all rearranging transformations φ. We consider the finite tree T of Figure 37:

We observe that $\partial T_o = \{x_1, x_3, x_4\}$. We choose the listing order $x_3 < x_1 < x_4$. With this choice, the linearly decreasing order is

$$o \lhd x_2 \lhd x_4 \lhd x_1 \lhd x_3.$$

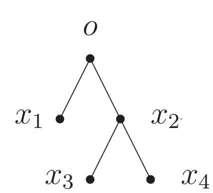


Figure 37: The finite tree T.

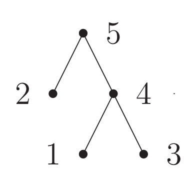


Figure 38: The linearly decreasing weight ω.

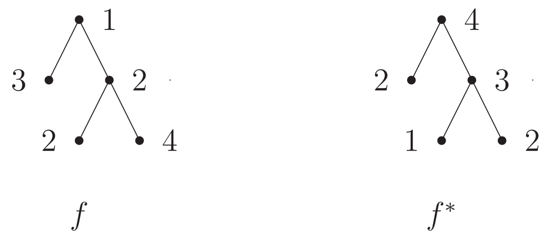


Figure 39: The function f and its rearrangement.

In the two preceding figures, a linearly decreasing weight ω is given, and we have considered a positive function f and its rearrangement.

Thus, the rearranging transformation for f is

$$\varphi_f(o) = x_3, \quad \varphi_f(x_1) = x_2, \quad \varphi_f(x_2) = x_1, \quad \varphi_f(x_3) = x_4, \quad \varphi_f(x_4) = o.$$

Finally, we compute $\omega \circ \varphi_f$ and we rearrange it. We observe that $2 = \omega(x_1) \neq (\omega \circ \varphi_f)^*(x_1) = 3$.

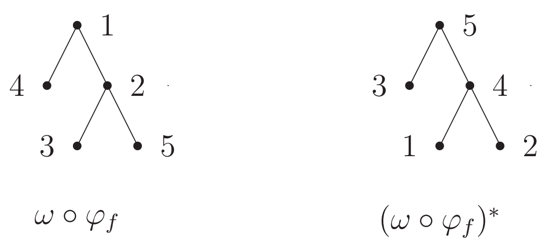


Figure 40: $\omega \circ \varphi_f$ and its rearrangement.

We can see in this example that the crucial result in Lemma 2.4.11 is not satisfied by a general listing order. In our case, $f(x_2) \geq f(x_3)$ but $\varphi(x_2) = x_1 \rhd x_4 = \varphi(x_3)$.

We observe that the listing order in this example cannot come from an order defined in the boundary of the homogenous tree of degree 3 (an admissible map), because the assumptions in Definition 2.2.2 oblige to satisfy

$$x_1 \lhd x_3, \quad x_1 \lhd x_4,$$

or, on the other hand,

$$x_3 \lhd x_1, \quad x_4 \lhd x_1.$$

Chapter 3

Weighted inequalities and the shape of approach regions

In the history of analysis, many problems are concerned with the existence of boundary limits for certain classes of functions defined on a general domain. The beginning of the study of these questions goes back to 1872, when H.A. Schwarz ([Sc1]) proved the existence of radial limits for all boundary points in the unit disc D for the Poisson integral of a continuous 2π-periodic function. P. Fatou proved in 1906 ([F]) the existence of boundary limits, for almost every point in the boundary of D, for bounded holomorphic functions if we approach the boundary point within certain regions called *nontangential* approach regions. The nontangential condition became strongly linked with boundary convergence when J. Littlewood ([Li], 1927) proved the failure of convergence for bounded holomorphic functions on D along tangential curves. In 1930, Hardy and Littlewood ([HL]) introduced the idea of studying the convergence of a sequence of operators by means of estimates on a maximal function. It turns out that a natural setting to study estimates on maximal functions is the spaces of homogeneous type.

In [K], A. Korányi gave the boundary convergence for H^p functions on the generalized half-plane of $\mathbb{C}^{n+1}$ within the so-called admissible domains. These regions allow parabolic tangential approach to the boundary along certain directions. In 1984, A. Nagel and E.M. Stein ([NS]) completely characterized the approach regions in the half space $\mathbb{R}^{n+1}_+ := \{(x,t) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n,\, t > 0\}$ where we have boundary convergence for the Poisson integral of L^p functions on the boundary. They studied estimates on a maximal operator M_Ω related to the approach region, for the origin 0,

$\Omega(0)$ defined by

$$M_\Omega f(x) = \sup_{(y,r)\in\Omega(0)} \frac{1}{|B(y,r)|} \int_{B(y,r)} |f(x+z)|\, dz, \tag{3.1}$$

and the result is quite surprising: there exist nontangential regions (without tangential directions) not contained in any cone for which convergence is allowed. These are the so-called *non nontangential* approach regions. J. Sueiro ([Su], 1986) generalizes this result by studying the problem in the general setting of the spaces of homogeneous type, and he applies it in the case of the generalized half-plane in $\mathbb{C}^{n+1}$ which can be seen as the product $\mathbf{H}_{\mathrm{n}} \times (0,\infty)$, where $\mathbf{H}_{\mathrm{n}}$ is the Heisenberg group. Later, M. Andersson and H. Carlsson ([AC]) gave an easy proof of the Nagel-Stein result using the key concept of Carleson measures. A complete and original overview of this subject can be found in [DiB], where new interesting results are given.

Following the ideas of [Su], Pan (see [P]) studied the weak-type weighted norm estimates for M_Ω also in spaces of homogeneous type. Later, in [SS1], A. Sánchez-Colomer and J. Soria gave and answer to the strong-type weighted estimates for M_Ω in the Euclidean space, and they also studied the relationship between weighted inequalities for this operator and the geometry of the region Ω (see [SS2]).

Our intention is to follow these researches by extending the results in the Euclidean spaces of [SS2] to the more general setting of the spaces of homogeneous type. In this new context, we find two new difficulties: the lack of a group structure and the lack of the invariance of the measure with respect to the metric (i.e., balls of the same radii but with different centres can have very different measures). Thus, we will characterize geometric properties of a family of approach regions by means of analytic properties on the class of weights related to the boundedness of the maximal operator associated to this family, in the framework of the spaces of homogeneous type. In order to obtain our results, we will need to go deeply in the description of this class of weights giving the relationship with the classical Muckenhoupt A_p weights.

The chapter is organized as follows. The first section is devoted to collect some of the known results for the Euclidean spaces in order to present the current state of the theory. The second section is our main section, and we work in the product space $X\times(0,\infty)$ for a space of homogeneous type X. There, we find another (easier) characterization of the weak–type inequalities for M_Ω, in terms of the classical A_p condition plus an extra property related to being a Carleson measure (see Theorem 3.2.17). For this, we use some of the techniques given in [AC]. This result allows us to prove that the equivalence of weighted inequalities for M_Ω and the classical Hardy–Littlewood maximal function M completely determines the geometry of the family of approach regions Ω (see Theorem 3.2.30). To this end, we observe that there exists a class of "power" weights which are the key to establish the correspondence between analytic properties (boundedness of maximal operators) and geometric properties of the domains Ω. The main idea behind this technique is to find an equivalent metric in the given space which enjoys some extra invariance properties (see Theorem 3.2.21 and Corollary 3.2.25). This new quasi-metric is described in [ST]. Some particular examples of spaces with a group structure where our results apply are given in the final subsection. The contents of the second section have been published in [GS1] and [GS2]. In the third section we go further into the generalization of the results by considering an abstract context. We first show that our study makes sense in a more general setting by considering the case of homogeneous trees. In the fourth section, we go back to the case of Euclidean spaces to obtain two applications of our results that allow us to extend two results dealing with approach regions and singular integral operators of [FJR] and with approach regions and potential spaces of [RS].

We now introduce some notation. For a measure space (X, μ) equipped with a quasi-metric function d, M will always denote the non-centered Hardy-Littlewood maximal operator defined for a measurable function f by

$$Mf(x) = \sup_{x\in B} \frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y),$$

where the supremum is taken over all the balls containing x, and M_{cen} stands as the centered Hardy-Littlewood maximal operator defined for a measurable function f by

$$M_{\text{cen}}f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\, d\mu(y).$$

For a weight u defined in X, we write $U(E) = \int_E u(z)\, d\mu(z)$ for a measurable set $E \subset X$. Two positive functions f and g are said to be equivalent if there exists a positive constant $C \geq 1$ such that $C^{-1}g(x) \leq f(x) \leq C\, g(x)$ for all x. We then write $f \sim g$.

3.1 Preliminary results in $\mathbb{R}^{n+1}_{+}$

This section is devoted to give the classical definitions of approach regions in the particular case of $\mathbb{R}^{n+1}_{+}$, and to collect the known results in this setting that will be extended in the forthcoming sections.

We consider the Euclidean half-space $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty)$. Its boundary, $\mathbb{R}^n$, equipped with the Lebesgue measure and the Euclidean metric is a space of homogeneous type (see Section 2 for a definition). The measure of a set $E \subset \mathbb{R}^n$ is denoted by $|E|$, and the balls of the metric by $B(x,r)$, for $x \in \mathbb{R}^n$ and $r > 0$.

As usual the existence of boundary limits for certain classes of functions follows from estimates for the corresponding maximal operator. Thus, for a given family of measurable sets $\Omega = \{\Omega(x) : x \in \mathbb{R}^n\}$ in $\mathbb{R}^{n+1}_+$, we consider the operator M_Ω defined by

$$M_\Omega f(x) = \sup_{(y,r)\in\Omega(x)} \frac{1}{|B(y,r)|} \int_{B(y,r)} |f(z)|\, dz,$$

which is a generalization of the operator (3.1) of [NS], where A. Nagel and E.M. Stein characterized the boundedness of M_Ω when $\Omega(x) = (x,0) + \Omega_0$, for a measurable set Ω_0. We will always assume that Ω is chosen in such a way that $M_\Omega f$ is a well-defined measurable function.

We define some concepts that will be used.

Definition 3.1.1 For a family of sets Ω, the *shadow of a point* (x,t) *by* Ω is the set

$$\Omega^{\downarrow}(x,t) = \{y \in \mathbb{R}^n : (x,t) \in \Omega(y)\},$$

the *cross–section at height* t *of* $\Omega(x)$ is the set

$$\Omega^t(x) = \{y \in \mathbb{R}^n : (y,t) \in \Omega(x)\},$$

and the Ω*–neighborhood at height* t *of a point* $x \in \mathbb{R}^n$ is the set

$$S_\Omega(x,t) = \left\{y \in \mathbb{R}^n : \Omega^t(y) \cap B(x,t) \neq \emptyset\right\}.$$

The set $\Omega(x)$ is said to be **full on the vertical direction** if

$$\Omega^{\downarrow}(x,s) \subset \Omega^{\downarrow}(x,t),$$

whenever $0 < s \leq t$. This means that if $(y,s) \in \Omega(x)$, then $(y,t) \in \Omega(x)$ for all $t \geq s$. It is proved in [Su] that in order to get estimates on M_Ω, we can assume with no loss of generality, that $\Omega(x)$ is full on the vertical direction for all $x \in \mathbb{R}^n$, that is, the approach family is full on the vertical direction.

We recall that a weight $u : \mathbb{R}^n \longrightarrow [0,\infty)$ is a locally integrable positive function. The weight u is doubling if there exists a constant $K_u > 0$ such that

$$U(B(x,2r)) \leq K_u U(B(x,r)),$$

for all $x \in \mathbb{R}^n$ and $r > 0$.

The main result in [Su] applies to the case of $\mathbb{R}^{n+1}_+$ with the measure $u(x)\,dx$, for a doubling weight u, instead of the Lebesgue measure:

Theorem 3.1.2 (Sueiro) *For a family of sets $\Omega \subset \mathbb{R}^{n+1}_+$ full on the vertical direction and a doubling weight u, the following conditions are equivalent:*

(i) $M_\Omega : L^1(u) \longrightarrow L^{1,\infty}(u)$ *is bounded.*

(ii) There exists a constant $C > 0$ such that $U(S_\Omega(x,t)) \leq C\,U(B(x,t))$ for all $x \in \mathbb{R}^n$ and $t > 0$.

Since M_Ω is bounded on $L^\infty(u)$, the weak type (1,1) implies the strong type (p,p) for $p > 1$, by the Marcinkiewicz interpolation theorem.

The set $\Omega(x) \subset \mathbb{R}^{n+1}_+$ is an **approach region** for $x \in \mathbb{R}^n$ if and only if $(x,0) \in \overline{\Omega(x)}$. If every set $\Omega(x)$ is an approach region for x, then the family Ω is said to be an **approach family**.

The following proposition appears in [SS1]. It makes explicit the relation between the geometric condition of being an approach region and a pointwise estimate for the maximal operator related to the approach family.

Lemma 3.1.3 (Sánchez-Colomer, Soria) *Suppose that $\Omega(x)$ is full on the vertical direction, for certain $x \in \mathbb{R}^n$. Then $\Omega(x)$ is an approach region for x if and only if $M_{\text{cen}}f(x) \leq M_\Omega f(x)$, for all measurable functions f.*

A weight u is in the A_p class of Muckenhoupt (see [M]) if and only if

$$M_{\text{cen}} : L^p(u) \longrightarrow L^{p,\infty}(u)$$

is bounded. It is well-known that the non-centered and the centered Hardy–Littlewood maximal operators are equivalent in a space of homogeneous type as $\mathbb{R}^n$, and that every A_p weight is doubling. An A_p^Ω weight is a weight u such that

$$M_\Omega : L^p(u) \longrightarrow L^{p,\infty}(u)$$

is bounded.

A consequence of Theorem 3.1.2 and Lemma 3.1.3 is the following result in [SS2]:

Proposition 3.1.4 (Sánchez-Colomer, Soria) *If Ω is an approach family full on the vertical direction, the following conditions are equivalent for $p \geq 1$:*

(i) $u \in A_p^\Omega$.

(ii) $u \in A_p$ *and there exists* $C > 0$ *such that* $U(S_\Omega(x,t)) \le C\,U(B(x,t))$ *for all* $(x,t) \in \mathbb{R}_+^{n+1}$.

With this result in mind, the last two authors ask the following natural question: how must a region Ω be to ensure that $A_p^\Omega = A_p$? They could answer it in the particular case of a translation invariant family of regions, obtaining a nice result which relates weighted inequalities for maximal operators and the shape of the approach regions:

Proposition 3.1.5 *Let* $\Omega(0) \subset \mathbb{R}_+^{n+1}$ *be an approach region for* $0 = (0,\dots,0) \in \mathbb{R}^n$ *full on the vertical direction, and set* $\Omega(x) = (x,0) + \Omega(0)$ *for all* $x \in \mathbb{R}^n$. *Then,* $A_p = A_p^\Omega$ *for some* $p \ge 1$, *if and only if* $\Omega(0)$ *is contained in a cone with vertex at* $0 \in \mathbb{R}^n$.

We will extend this result to a more general setting.

3.2 The half space $X \times (0,\infty)$ for a space of homogeneous type X

We are interested in understanding more deeply the relationship between weighted inequalities and the shape of approach regions. So, our main purpose is to give a new and clarifying proof of Proposition 3.1.5, which allows us to extend the results to the framework of the spaces of homogeneous type. To be precise, we will work with the half-space

$$X_+ := X \times (0,\infty),$$

where (X,μ,d) is a space of homogeneous type with the measure μ and the metric d (see the definition below). The space X can be viewed as the boundary of X_+. In this new setting, two difficulties appear:

(i) The lack of a group or pseudo-group transformation in X.

(ii) The lack of the invariance of the measure with respect to the collection of balls, that is, in general, we have neither

$$\mu(B(x,r)) = \mu(B(y,r)) \quad \text{nor} \quad \mu(B(x,r)) \approx \mu(B(y,r)),$$

for all $x,y \in X$ and $r > 0$.

3.2.1 Definitions and previous results

Let X be a topological space with a nonnegative Borel measure μ. Suppose we have a nonnegative real-valued function d defined in $X \times X$ that satisfies the following properties:

(i) $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$ for all x, y in X.

(iii) There is a constant $A \geq 1$ such that $d(x, y) \leq A\,(d(x, z) + d(y, z))$, for all x, y, z in X.

(iv) The balls $B(x, r) = \{y \in X : d(x, y) < r\}$ are measurable sets for all x in X and $r > 0$. Moreover, $\{B(x, r) : r > 0\}$ is a basis of open neighborhoods for all x in X.

(v) There is a constant $K_\mu > 1$, such that $0 < \mu(B(x, 2r)) \leq K_\mu\, \mu(B(x, r))$, for all x in X and $r > 0$. .

Definition 3.2.1 A trio (X, μ, d) satisfying the above conditions is called a **space of homogeneous type** (see [CW] for a more general definition), a measure satisfying (v) is called a **doubling measure** and d is called a **quasi-distance**.

Although our purpose is to work in the setting of spaces of homogeneous type, we need to assume our quasi-distance not symmetric in general (we still call it a quasi-distance), and in order to distinguish it from the symmetric case, we will denote it by δ.

Let us introduce the precise definition.

Definition 3.2.2 A **ns-space of homogeneous type** is a trio (X, μ, δ) satisfying:

(i) $\delta(x, y) = 0$ if and only if $x = y$.

(ii) There exists a constant $D \geq 1$ such that $\delta(x, y) \leq D\,\delta(y, x)$ for all x, y in X.

(iii) There is a constant $A \geq 1$ such that $\delta(x, y) \leq A\,(\delta(x, z) + \delta(y, z))$, for all x, y, z in X.

(iv) The balls $B(x, r) = \{y \in X : \delta(x, y) < r\}$ are measurable sets for all x in X and $r > 0$. Moreover, $\{B(x, r) : r > 0\}$ is a basis of open neighborhoods for all x in X.

(v) There is a constant $K_\mu > 1$, such that $0 < \mu(B(x, 2r)) \le K_\mu \, \mu(B(x, r))$, for all x in X and $r > 0$.

Obviously, every space of homogeneous type is a ns-space of homogeneous type, with $D = 1$.

In the sequel, it will be important in every computation to be careful with constants, since we are trying to have uniform results in $x \in X$. We will explicit the dependence of the constants appearing at each result if it is needed. If not, a constant $C > 0$ may change from one occurrence to the next. The constant K_ν will always be the doubling constant of a measure ν. We give some technical results involving the metric and the measure of our space that are important for later purposes. The first one is well-known.

Lemma 3.2.3 *A measure μ is doubling if and only if there exists $\alpha > 0$ such that $\mu(B(x, tr)) \le K_\mu \, t^\alpha \, \mu(B(x, r))$, for all $x \in X$ and $r > 0$.*

Lemma 3.2.4 ([ST]) *Let $a > 0$. If $B(x, r) \cap B(y, r') \ne \phi$ and $r \le ar'$, then $B(x, r) \subset B(y, c_0 r')$, with $c_0 = A^2(1 + a) + ADa$.*

The next result is a Vitali-type covering lemma:

Lemma 3.2.5 ([ST]) *Let $\mathcal{F}$ be a family of balls with bounded radii. Then there is a countable subfamily of pairwise disjoint balls $B(x_k, r_k)$ such that each ball in $\mathcal{F}$ is contained in one of the balls $B(x_k, c_0 r_k)$, where c_0 is the constant of the previous lemma in the case a=2.*

Given a nonnegative measure ν, the Hardy-Littlewood maximal function of ν with respect to the measure μ is:

$$M_\mu \nu(x) = \sup_{B \ni x} \frac{\nu(B)}{\mu(B)}.$$

We write $M = M_\mu$ if there is no possible confusion.

As a consequence of Lemma 3.2.5, it is now easy to prove the following well-known estimates (see also [ST]):

Theorem 3.2.6 *For a doubling measure μ, we have:*
(a) The Hardy-Littlewood maximal operator is of weak-type (1,1) and strong-type (p,p)

for $1 < p \leq \infty$ *on* $L^p(\mu)$.
(b) There is a constant $C_\mu > 0$ *such that for every* $\lambda > 0$,

$$\mu(\{x \in X : M\nu(x) > \lambda\}) \leq \frac{C_\mu}{\lambda}\nu(X),$$

for all nonnegative measures ν *of finite total variation.*

The following theorem, a Whitney-type decomposition, is proved in [CW] in the case of spaces of homogeneous type, and works in the setting of the ns-spaces of homogeneous type with slight modifications.

Theorem 3.2.7 *Let* μ *be a doubling measure. Let* $f \in L^1(\mu)$ *be a positive function with bounded support. Then, there exists a countable family of balls* $\{B(x_k, r_k)\}_k$ *such that:*

(i) $O = \{x \in X : Mf(x) > 1/2\} = \bigcup_k B(x_k, r_k)$.

(ii) There is a constant $C_\mu > 0$ *such that* $\sum_k \mu(B(x_k, r_k)) \leq C_\mu \, ||f||_1$.

(iii) $B(x_k, 3Ar_k) \cap O^c \neq \phi$, *for all* k.

We will follow the ideas of M. Andersson and H. Carlsson in [AC], involving the concept of Carleson measures to obtaining estimates for maximal operators. We now introduce a general notion of pairs of Carleson measures. The **tent** of an open set $O \subset X$ is the set $T(O) = \{(y,t) \in X_+ : B(y,t) \subset O\}$.

Definition 3.2.8 We say that two measures, ρ defined in X_+ and ν defined in X, are a **Carleson pair** if there exists a constant $C_{\rho,\nu} > 0$ such that

$$\rho(T(B)) \leq C_{\rho,\nu} \, \nu(B),$$

for all balls $B \subset X$. In this case, we use the notation $(\rho,\nu) \in \mathcal{C}(X)$.

By using the Whitney-type decomposition of Theorem 3.2.7, it is now a classical computation to extend the definition of a Carleson pair to every open set.

Proposition 3.2.9 *Let* $(\rho,\nu) \in \mathcal{C}(X)$ *so that* ν *is doubling. Then, there exists a constant* $C'_{\rho,\nu} > 0$ *such that*

$$\rho(T(O)) \leq C'_{\rho,\nu} \, \nu(O),$$

for all $O \subset X$ *open.*

Proof. We can assume that $\nu(O) < \infty$. Let $f = \chi_O \in L^1(\nu)$. Then

$$O \subset \{x \in X : M_\nu f(x) > 1/2\}.$$

We use Theorem 3.2.7 to obtain a family of balls $\{B(x_k, r_k)\}_k$ satisfying:

(i) $O \subset \bigcup_k B(x_k, r_k)$.

(ii) There is a constant $C_\nu > 0$ such that $\sum_k \nu(B(x_k, r_k)) \leq C_\nu\, \nu(O)$.

(iii) $B(x_k, 3Ar_k) \cap O^c \neq \phi$, for all k.

Take $(x, s) \in T(O)$. By definition, $B(x, s) \subset O$ and (i) implies that there is k_0 so that $x \in B(x_{k_0}, r_{k_0})$. Now, (iii) implies there exists $y \in B(x_{k_0}, 3Ar_{k_0}) \setminus B(x, s)$ and hence:

$$\begin{aligned} s \leq \delta(x, y) &\leq A(\delta(x, x_{k_0}) + \delta(y, x_{k_0})) \\ &\leq AD(\delta(x_{k_0}, x) + \delta(x_{k_0}, y)) \\ &\leq AD(1 + 3A) r_{k_0}. \end{aligned}$$

Using Lemma 3.2.4, there exists $C > 0$ independent of x, s, x_{k_0} and r_{k_0} such that $B(x, s) \subset B(x_{k_0}, Cr_{k_0})$, that is $(x, s) \in T(B(x_{k_0}, Cr_{k_0}))$. Therefore $T(O) \subset \cup_k T(B(x_k, Cr_k))$. Now, using (ii) and the hypothesis on the measures, we have:

$$\begin{aligned} \rho(T(O)) &\leq \sum_k \rho(T(B(x_k, Cr_k))) \\ &\leq C_{\rho,\nu} \sum_k \nu(B(x_k, Cr_k)) \\ &\leq C_{\rho,\nu}\, K_\nu\, C^{\alpha(\nu)} \sum_k \nu(B(x_k, r_k)) \\ &\leq C_{\rho,\nu}\, K_\nu\, C^{\alpha(\nu)}\, C_\nu\, \nu(O) \\ &= C'_{\rho,\nu}\, \nu(O), \end{aligned}$$

where K_ν and $\alpha(\nu)$ are the constants appearing in Lemma 3.2.3 for the measure ν.

$\square$

We consider a family of measurable sets $\Omega = \{\Omega(x) : x \in X\}$, with $\Omega(x) \subset X_+$ for all $x \in X$. For such a family, let us introduce the following definitions:

Definition 3.2.10

(i) The *shadow of a point* $(x, t) \in X_+$ *by the family* Ω is the set

$$\Omega^{\downarrow}(x, t) = \{y \in X : (x, t) \in \Omega(y)\}.$$

(ii) The *cross-section of* $\Omega(x)$ *at height* $t > 0$ is the set

$$\Omega^t(x) = \{y \in X : (y, t) \in \Omega(x)\}.$$

(iii) The Ω*-neighborhood at height* $t > 0$ *of a point* $x \in X$ is the set

$$S_\Omega(x, t) = \{y \in X : \Omega^t(y) \cap B(x, t) \neq \phi\}.$$

(iv) Given a nonnegative measure σ in X, we define the outer measure in X_+

$$\sigma_\Omega(E) = \sigma(\{x \in X : \Omega(x) \cap E \neq \phi\}),$$

for $E \subset X_+$ (see [AC]).

As we did in the first section, we define the maximal operator related to the family Ω.

Definition 3.2.11 For a measurable function f, the maximal operator related to a family of sets Ω is the function

$$M_\Omega f(x) = \sup_{(y,t) \in \Omega(x)} \frac{1}{\mu(B(y, t))} \int_{B(y,t)} |f(z)| \, d\mu(z).$$

We will always assume that $M_\Omega f$ is a measurable function.

In the next proposition, we find a necessary condition for the boundedness of this operator.

Proposition 3.2.12 *Let* ρ *and* ν *be two nonnegative measures on* X *so that* ν *is doubling. If* $M_\Omega : L^p(\nu) \longrightarrow L^{p,\infty}(\rho)$ *is bounded for some* $p \geq 1$*, then there exists a constant* $C = C(K_\nu, ||M_\Omega||_{L^p(\nu) \to L^{p,\infty}(\rho)}) > 0$ *such that:*

$$\rho(S_\Omega(x, t)) \leq C \, \nu(B(x, t)),$$

for all $(x, t) \in X_+$.

Proof. Take $y \in S_\Omega(x,t)$. Then, there exists $z \in B(x,t)$ so that $z \in \Omega^t(y))$. Now, by condition (iii) on δ, we have $B(z,t) \subset B(x, A(D+1)t)$. Let $f = \chi_{B(x,A(D+1)t)}$. Then, $M_\Omega f(y) > 1/2$ for all $y \in S_\Omega(x,t)$, and therefore:

$$\begin{aligned} \rho(S_\Omega(x,t)) &\leq \rho(\{x \in X : M_\Omega f(x) > 1/2\}) \\ &\leq C\,||f||^p_{L^p(\nu)} \\ &= C\,\nu(B(x, A(D+1)t)) \\ &\leq C\,\nu(B(x,t)). \end{aligned}$$

□

It is proved in [Su] that with no loss of generality, we can always assume that Ω is **full on the vertical direction**, that is $(y,s) \in \Omega(x)$ implies $(y,t) \in \Omega(x)$ for all $t \geq s$. This is equivalent to the fact that $\Omega^\downarrow(y,s) \subset \Omega^\downarrow(y,t)$ whenever $s \leq t$. We will also take this condition for granted.

We say that a family of measurable sets $\Omega = \{\Omega(x) : x \in X\}$ in X_+ is an **approach family** if $(x,0) \in \overline{\Omega(x)}$ for all $x \in X$, with respect to the product topology in X_+. The natural example of approach family is the cone of width $\theta > 0$, $\Gamma_\theta(x) = \{(y,t) \in X_+ : x \in B(y, \theta t)\}$. We denote $\Gamma(x) = \Gamma_1(x)$. In the particular case of $\Omega(x) = \Gamma_\theta(x)$, it is known that $Mf \sim M_{\Gamma_\theta} f$ for all measurable f, that is, there is a positive constant C such that

$$C^{-1} Mf(x) \leq M_{\Gamma_\theta} f(x) \leq C\,Mf(x), \tag{3.2}$$

for all measurable f and $x \in X$. So, the operator M_{Γ_θ} has the same estimates of M in Theorem 3.2.6.

We now characterize the boundedness of $M_\Omega : L^p(\nu) \longrightarrow L^{p,\infty}(\rho)$ for some $p \geq 1$.

Theorem 3.2.13 *Let (X, μ, δ) be a ns-space of homogeneous type, and consider a family Ω. Let ρ and ν be two nonnegative measures on X. If $M : L^p(\nu) \longrightarrow L^{p,\infty}(\nu)$ is bounded for some $p \geq 1$, then the following conditions are equivalent:*

(i) There exists $C > 0$ such that

$$\rho(\{x \in X : M_\Omega f(x) > \lambda\}) \leq C\,\nu(\{x \in X : M_\Gamma f(x) > \lambda\}),$$

for all $\lambda > 0$ and measurable f.

(ii) $M_\Omega : L^p(\nu) \longrightarrow L^{p,\infty}(\rho)$ is bounded.

(iii) There exists $C > 0$ such that $\rho(S_\Omega(x,t)) \le C\,\nu(B(x,t))$ for all $(x,t) \in X_+$.

(iv) $(\rho_\Omega, \nu) \in \mathcal{C}(X)$.

Proof. It is known that ν is necessarily a doubling measure. That (i) implies (ii) follows trivially by using (3.2). That (ii) implies (iii) is Proposition 3.2.12. Let us see that (iii) implies (iv). Take $y \in X$ so that $\Omega(y) \cap T(B(x,t)) \neq \phi$. There is $(z,s) \in \Omega(y)$ with $B(z,s) \subset B(x,t)$. Since $\Omega(y)$ is full on the vertical direction, $(z,t) \in \Omega(y)$. Therefore $z \in \Omega^t(y) \cap B(x,t)$, and hence $\{y \in X : \Omega(y) \cap T(B(x,t)) \neq \phi\} \subset S_\Omega(x,t)$. So, using the definition of ρ_Ω, we have:

$$\rho_\Omega(T(B(x,t)) \le \rho(S_\Omega(x,t)) \le C\,\nu(B(x,t)).$$

Now, suppose $(\rho_\Omega, \nu) \in \mathcal{C}(X)$. Observe that

$$\left\{(y,t) : \frac{1}{\mu(B(y,t))}\int_{B(y,t)} |f(z)|d\mu(z) > \lambda\right\} \subset T(O),$$

where $O = \{x \in X : M_\Gamma f(x) > \lambda\}$, for all functions f. Then, applying Proposition 3.2.9, we obtain:

$$\begin{aligned}
\rho(\{x \in X : M_\Omega f(x) > \lambda\}) &= \rho_\Omega\left(\left\{(y,t) : \frac{1}{\mu(B(y,t))}\int_{B(y,t)} |f|d\mu > \lambda\right\}\right) \\
&\le \rho_\Omega(T(O)) \\
&\le C\,\nu(O) \\
&= C\,\nu(\{x \in X : M_\Gamma f(x) > \lambda\}).
\end{aligned}$$

$\square$

Remarks 3.2.14

(i) The hypothesis of the boundedness of M is only needed in the implication $(i) \Rightarrow (ii)$.

(ii) We observe that Ω need not be an approach family.

A weight u in (X, μ, δ) is a positive and locally integrable function. We say that u is doubling if the measure $u(z)\,d\mu(z)$ is doubling.

A weight u is in the A_p class of Muckenhoupt (see [M]), $1 \le p < \infty$, if the maximal operator $M : L^p(u) \longrightarrow L^{p,\infty}(u)$ is bounded. By the A_p constant $||u||_{A_p}$ of a weight u in A_p we mean the norm of the maximal operator. It is a well-known fact that every weight u in A_p is doubling, but we need to be more explicit for later purposes:

Lemma 3.2.15 *If u is an A_p weight for some $p \geq 1$ in a ns-space of homogeneous type (X, μ, δ), then u is doubling and*

$$K_u \leq K_\mu^p \, ||u||_{A_p}, \tag{3.3}$$

where K_u is the doubling constant of the measure $u(z)\, d\mu(z)$.

Proof. We recall that $u \in A_p$ if and only if there exists $C > 0$ such that

$$U(\{x \in X : Mf(x) > \lambda\}) \leq \frac{C}{\lambda^p} \, \|f\|^p_{L^p(u)} \,,$$

for all $f \in L^p(u)$ and all $\lambda > 0$. For a ball $B = B(y, 2r)$, it is easy to see that for all $x \in X$

$$f_B := \frac{1}{\mu(B)} \int_B |f(z)| \, d\mu(z) \leq M(f\chi_B)(x).$$

Thus $B \subset \{x \in X : M(f\chi_B)(x) > t\}$, if $t < f_B$. Using this and the definition above of an A_p weight, we have

$$U(B) \leq U(\{x \in X : M(f\chi_B)(x) > t\}) \leq \frac{C}{t^p} \int_B |f(z)|^p \, u(z) \, d\mu(z),$$

for all $t < f_B$. Letting $t \longrightarrow f_B$, we obtain

$$f_B^p \, U(B) \leq C \int_B |f(z)|^p \, u(z) \, d\mu(z).$$

Now, we choose $f = f\chi_{B'}$ whith $B' = B(y, r)$, and we substitute this in the last expression getting

$$\left(\frac{1}{\mu(B)} \int_{B'} |f(z)| \; d\mu(z) \right)^p U(B) \leq C \int_{B'} |f(z)|^p \; u(z) \, d\mu(z).$$

Finally, if $f \equiv 1$, this is

$$\left(\frac{\mu(B)}{\mu(B)} \right)^p U(B) \leq C \, U(B'),$$

that is,

$$U(B) \leq C \left(\frac{\mu(B)}{\mu(B')} \right)^p U(B') \leq C \, K_\mu^p \, U(B').$$

□

We introduce two new classes of weights related to a family Ω.

Definition 3.2.16 For a family Ω, we consider:
(a) For $1 \leq p < \infty$, a weight u belongs to the A_p^Ω class of weights if and only if $M_\Omega : L^p(u) \longrightarrow L^{p,\infty}(u)$ is bounded, and the A_p^Ω-constant $||u||_{A_p^\Omega}$ is the norm of M_Ω.
(b) Set

$$W(\Omega) = \left\{u \in L^1_{\text{loc}}(\mu), u \geq 0 : \exists C > 0,\ U(S_\Omega(x,t)) \leq C\, U(B(x,t)),\ \forall\ (x,t) \in X_+\right\},$$

and let the $W(\Omega)$-constant $||u||_{W(\Omega)}$ be the infimum of the constants appearing in this expression.

A basic result to explain the interplay between weighted inequalities and approach regions is the following theorem, that is our version of Proposition 3.1.4.

Theorem 3.2.17 *Let Ω be a family of approach regions in X_+. For $1 \leq p < \infty$, we have*

$$A_p^\Omega = A_p \cap W(\Omega),$$

and

$$||u||_{A_p} \leq ||u||_{A_p^\Omega},$$

for all u in A_p^Ω.

Proof. Theorem 3.2.13 with $d\rho(z) = d\nu(z) = u(z)\, d\mu(z)$ says that $A_p \cap W(\Omega) \subset A_p^\Omega$ and $A_p^\Omega \subset W(\Omega)$. Now, let us see that if Ω is a family of approach regions full on the vertical direction, then

$$M_{\text{cen}} f(x) \leq M_\Omega f(x),$$

for all $x \in X$ and all measurable f. Therefore, we will have that $A_p^\Omega \subset A_p$, and $||u||_{A_p} \leq ||u||_{A_p^\Omega}$.

Suppose $\{(x_k, r_k) : k \geq 0\} \subset \Omega(x)$, with $x_k \longrightarrow x$ and $r_k \longrightarrow 0$. Given $r > 0$, we can find $k_0 \geq 0$ such that if $k \geq k_0$, then $r_k \leq r$ and hence $(x_k, r) \in \Omega(x)$ since Ω is full on the vertical direction. Since

$$M_\Omega f(x) \geq \frac{1}{\mu(B(x_k, r))} \int_{B(x_k,r)} |f(y)|\, d\mu(y) \longrightarrow \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\, d\mu(y),$$

by the Lebesgue's differentiation theorem. Hence, taking the supremum over $r > 0$, we obtain

$$M_\Omega f(x) \geq M_{\text{cen}} f(x).$$

□

Remark 3.2.18 The argument to show the estimate between the maximal operators comes from Proposition 2.1 in [SS1].

3.2.2 The shape of approach regions

In this subsection we show our main theorem that answers a natural question in view of Theorem 3.2.17: when is it true that $A_p = A_p^{\Omega}$? To this end, we will need some kind of invariance property on the measure of the balls of comparable radius. This will be a consequence of a new requirement on the metric of our space of homogeneous type. In fact, this new requirement on the initial quasi-distance will allow us to replace it by a second quasi-distance, changing neither the topology nor the maximal operators.

Let (X, μ, d) be a space of homogeneous type (and hence, d is symmetric). To be precise, we denote the balls with respect the quasi-distance d by

$$B^d(x,t) = \{y \in X : d(x,y) < t\}, \quad (x,t) \in X_+.$$

We impose on (X, μ, d) the following extra condition: there is a constant $M > 1$ such that

$$B^d(x, Mr) \setminus B^d(x,r) \neq \phi, \tag{3.4}$$

for all x in X and $r > 0$. (If X is bounded, this should be considered for r small enough.)

As a consequence of Lemma 3.2.4 and the hypothesis in (X, μ, d), we obtain (see also [ST]):

Proposition 3.2.19 *There exist $\alpha > 0$ and $\beta > 0$, and $0 < K_1 < 1$ such that*

$$K_1\, t^{\beta} \mu(B(x,r)) \leq \mu(B(x,tr)) \leq K_\mu\, t^{\alpha} \mu(B(x,r)), \tag{3.5}$$

for all x in X, $r > 0$ and $t \geq 1$.

Proof. The right inequality is Lemma 3.2.3, and condition (3.4) is not needed. To see the left inequality, we claim that there exist two constants $A_0 > 1$ and $0 < K_1 < 1$ such that

$$\mu(B(x,r)) \leq K_1\, \mu(B(x, A_0 r)),$$

for all $x \in X$ and $r > 0$, and therefore, letting $n \geq 1$ so that $A_0^{n-1} \leq t < A_0^n$, we obtain:

$$\mu(B(x,tr)) \geq \mu(B(x, A_0^{n-1} r)) \geq K_1^{1-n}\, \mu(B(x,r)) \geq K_1\, t^{\beta} \mu(B(x,r)),$$

where $\beta = \log_{A_0}(K_1^{-1})$.

We now prove the claim. Let c_0 be the constant in Lemma 3.2.4 when $a = 1$. Choose $L > Dc_0$, and take $y \in B(x, MLr) \setminus B(x, Lr)$ which is not empty by condition (3.4). Then $Lr \leq d(x, y) < MLr$, and by condition (ii) in Definition 3.2.1, we have $Lr/D \leq d(y, x) < DMLr$. By the choice on L, we have $c_0 r < d(y, x) < DMLr$. Now, using Lemma 3.2.4, we obtain:

$$B(x, r) \cap B(y, r) = \phi \;\text{ and }\; B(y, r) \subset B(x, c_0 MLr).$$

Let $A_0 = c_0 ML$. By construction, we have:

$$\mu(B(x, r)) \leq \mu(B(x, A_0 r)) - \mu(B(y, r)).$$

The right inequality in (3.5), gives us the existence of a constant $C > 1$ such that $\mu(B(x, A_0 r)) \leq C\, \mu(B(y, r))$, and hence:

$$\mu(B(x, r)) \leq \mu(B(x, A_0 r)) - \frac{1}{C}\, \mu(B(x, A_0 r)) = K_1\, \mu(B(x, A_0 r)).$$

□

This proposition says that our measure is *invariant* in some sense. We observe that we avoid measures with atoms.

In [ST], under condition (3.4), J.O. Strömberg and A. Torchinsky introduce a non-symmetric quasi-distance δ in $X \times X$ having the property that the measure of a δ-ball is comparable to its radius. Specifically, for a fixed $x \in X$, consider the function:

$$\begin{aligned} r_x(t) &= \exp\left(\frac{-1}{1+t}\right) \int_{1/2}^{1} \mu(B(x, st))\, ds, \qquad \text{if } t \neq 0, \\ r_x(0) &= 0. \end{aligned}$$

This function is strictly increasing, continuous (continuity at 0 is given by Proposition 3.2.19), $r_x(t) \to \infty$ if $t \to \infty$ when X is not a compact space, and the measure of a ball $B^d(x, t)$ is comparable to $r_x(t)$:

Lemma 3.2.20 *For all $x \in X$ and $t > 0$*

$$2\, r_x(t) \leq \mu(B^d(x, t)) \leq K_\mu\, 2^{\alpha - 1}\, e\, r_x(t). \tag{3.6}$$

Proof. In one direction,

$$r_x(t) \leq \int_{1/2}^{1} \mu(B(x, st))\, ds \leq \mu(B^d(x, t)) \int_{1/2}^{1} ds = \frac{1}{2}\, \mu(B^d(x, t)).$$

Using Proposition 3.2.19, we have

$$\mu(B^d(x,t)) = \mu(B^d(x,(1/s)st)) \le K_\mu\,(1/s)^\alpha\,\mu(B^d(x,st)) \le K_\mu\,2^\alpha\,\mu(B^d(x,st)),$$

for all $1/2 \le s \le 1$. From this inequality it follows:

$$\begin{aligned}
\mu(B^d(x,t)) &= \frac{1}{2}\int_{1/2}^{1} \mu(B(x,t))\,ds \\
&\le K_\mu\,2^{\alpha-1}\int_{1/2}^{1} \mu(B(x,st))\,ds \\
&\le K_\mu\,2^{\alpha-1}\,e\,\exp\!\left(\frac{-1}{1+t}\right)\int_{1/2}^{1} \mu(B(x,st))\,ds \\
&= K_\mu\,2^{\alpha-1}\,e\,r_x(t).
\end{aligned}$$

□

The function r_x has an inverse r_x^{-1} for all x in X. If X is a compact space, there exists a constant $c_x > 0$ such that r_x^{-1} is defined in $[0, c_x)$.

We define the normalized quasi-distance

$$\delta(x,y) := r_x(d(x,y)),$$

for all $x, y \in X$. This new quasi-distance is non-symmetric in general. The δ-balls are $B^\delta(x,t) = \{y \in X : \delta(x,y) < t\}$. By (3.6), we trivially have that

$$2\,t \le \mu(B^\delta(x,t)) \le K_\mu\,2^{\alpha-1}\,e\,t.$$

Theorem 3.2.21 ([ST]) *If (X,μ,d) is a space of homogeneous type satisfying property (3.4), then (X,μ,δ) is a ns-space of homogeneous type such that:*

(i) $B^d(x,t) = B^\delta(x, r_x(t))$ for all x in X and $t > 0$.

(ii) $\mu(B^\delta(x,t))$ is comparable to t, for all $x \in X$ and $t > 0$.

Since the collection of balls for both quasi-distances coincide, the topology does not change if we adopt the new quasi-distance. But, what is the effect on the A_p and A_p^Ω weights?

Definition 3.2.22 Given an approach family $\Omega = \{\Omega(x) : x \in X\}$ in X_+ with respect to (X,μ,d), we define the corresponding family Ω^δ with respect to (X,μ,δ) as follows. Set

$$\Omega^\delta(x) := \left\{(y,s) : (y, r_y^{-1}(s)) \in \Omega(x)\right\} = \{(y, r_y(t)) : (y,t) \in \Omega(x)\}\,.$$

The next lemma ensures that we can still assume that our approach regions are full on the vertical direction.

Lemma 3.2.23 *For all $x \in X$ and $t > 0$, we have*

$$\Omega^{\downarrow}(x,t) = (\Omega^{\delta})^{\downarrow}(x, r_x(t)),$$

and therefore, Ω is full on the vertical direction if and only if Ω^{δ} is.

Proof. By definition,

$$y \in \Omega^{\downarrow}(x,t) \Leftrightarrow (x,t) \in \Omega(y) \Leftrightarrow (x, r_x(t)) \in \Omega^{\delta}(y) \Leftrightarrow y \in (\Omega^{\delta})^{\downarrow}(x, r_x(t)).$$

□

Consider the Hardy-Littlewood maximal operator with respect to δ:

$$M^{\delta} f(x) = \sup_{B^{\delta} \ni x} \frac{1}{\mu(B^{\delta})} \int_{B^{\delta}} |f(z)| d\mu(z),$$

for a measurable function f. The maximal operator associated to Ω^{δ} in (X, μ, δ) is:

$$M^{\delta}_{\Omega^{\delta}} f(x) = \sup_{(y,s) \in \Omega^{\delta}(x)} \frac{1}{\mu(B^{\delta}(y,s))} \int_{B^{\delta}(y,s)} |f(z)|\, d\mu(z),$$

for a measurable function f.

Lemma 3.2.24 *Let Ω be a family in X_+ with respect to (X, μ, d), and Ω^{δ} the corresponding family with respect to (X, μ, δ). Then,*

(i) For all x in X and measurable f, $M^{\delta} f(x) = M f(x)$, where M is the maximal operator with respect to d.

(ii) For all x in X and measurable f, $M^{\delta}_{\Omega^{\delta}} f(x) = M_{\Omega} f(x)$, where M_{Ω} is the maximal operator related to Ω in (X, μ, d).

Proof. We simply need to observe that, by definition, $(y,t) \in \Omega(x) \Leftrightarrow (y, r_y(t)) \in \Omega^{\delta}(x)$, and to recall that $B^{d}(y,t) = B^{\delta}(y, r_y(t))$ for all $t > 0$ and r_x is one to one.

□

Corollary 3.2.25 *Let Ω be a family in X_+ with respect to (X, μ, d), and Ω^{δ} be the corresponding family with respect to (X, μ, δ). Then,* *(i) If A_p^δ is the class of weights u such that $M^\delta : L^p(u) \longrightarrow L^{p,\infty}(u)$, then $A_p^\delta = A_p$ with the same constants.*

(ii) Define $A_p^{\Omega^\delta}$ as the class of weights u such that $M_{\Omega^\delta}^\delta : L^p(u) \longrightarrow L^{p,\infty}(u)$ is bounded. We have $A_p^{\Omega^\delta} = A_p^\Omega$ with the same constant.

In the sequel, we take δ as the ambient quasi-distance, and we simplify the notation putting $B(x,t) = B^\delta(x,t)$ and $\Omega = \Omega^\delta$. So, we are working in a ns-space of homogeneous type (X, μ, δ), where $\mu(B(x,t))$ is comparable to t, for all $x \in X$ and $t > 0$. The next result is essentially proved in [CR], but we give the proof in our general setting of the ns-space of homogeneous type for the sake of completeness.

Proposition 3.2.26 *Let ν be a Borel measure in X such that $M\nu \not\equiv \infty$. Then $M\nu^\epsilon \in A_1$, for all $0 \leq \epsilon < 1$, with A_1 constant depending only on ϵ.*

Proof. We need to see that $M : L^1(M\nu^\epsilon) \longrightarrow L^{1,\infty}(M\nu^\epsilon)$ with norm depending only on ϵ. It is a well-known fact that this is equivalent to proving that there exists a constant $C = C(\epsilon)$ such that

$$\frac{1}{\mu(B)} \int_B M\nu(x)^\epsilon \, d\mu(x) \leq C \operatorname{ess\,inf}_{x \in B} M\nu(x)^\epsilon,$$

for all balls $B \subset X$ (see [ST] for a proof of the equivalence in this context).

For a fixed B_0, take $x \in B_0$ and consider $\mathcal{Q}_1 = \{B \ni x : \mu(B) \leq \mu(2B_0)\}$ and $\mathcal{Q}_2 = \{B \ni x : \mu(B) > \mu(2B_0)\}$, where $2B_0$ is the ball with the same center than B_0 but with double radius. We have

$$M\nu(x) \leq \sup_{B \in \mathcal{Q}_1} \frac{\nu(B)}{\mu(B)} + \sup_{B \in \mathcal{Q}_2} \frac{\nu(B)}{\mu(B)} = A(x) + B(x),$$

and then $M\nu(x)^\epsilon \leq A(x)^\epsilon + B(x)^\epsilon$.

If $B \in \mathcal{Q}_2$, by Lemma 3.2.4, $2B_0 \subset c_0 B$, and then:

$$\frac{\nu(B)}{\mu(B)} \leq C \frac{\nu(c_0 B)}{\mu(c_0 B)} \leq C \operatorname{ess\,inf}_{y \in c_0 B} M\nu(y) \leq C \operatorname{ess\,inf}_{y \in B_0} M\nu(y),$$

and so $B(x) \leq C \operatorname{ess\,inf}_{y \in B_0} M\nu(y)$. If $B \in \mathcal{Q}_1$, by Lemma 3.2.4, $B \subset 2c_0 B_0$. Consider the measure ν_0 so that $d\nu_0(y) = \chi_{2c_0 B_0}(y) d\nu(y)$. Then we have $A(x) \leq M\nu_0(x)$, and therefore, it is enough to prove

$$\frac{1}{\mu(B_0)} \int_{B_0} M\nu_0(y)^\epsilon d\mu(y) \leq C \operatorname{ess\,inf}_{y \in B_0} M\nu(y)^\epsilon,$$

with C depending only on ϵ. Applying Fubini's theorem, we get:

$$\begin{aligned}\frac{1}{\mu(B_0)}\int_{B_0} M\nu_0(y)^\epsilon d\mu(y) &= \frac{1}{\mu(B_0)}\int_0^\infty \epsilon t^{\epsilon-1}\mu(\{y\in B_0 : M\nu_0(y)>t\})dt\\ &= \frac{\epsilon}{\mu(B_0)}\left(\int_0^R+\int_R^\infty\right)t^{\epsilon-1}\mu(\{M\nu_0>t\}\cap B_0)dt\\ &= I_1+I_2,\end{aligned}$$

for $R>0$ to be chosen later. We obtain $I_1 \le R^\epsilon$ if we get the distribution function bounded by the total mass of B_0. In the second integral, we use the boundedness of the maximal operator (Theorem 3.2.6 (b)) obtaining:

$$I_2 \le \frac{C_\mu}{\mu(B_0)}\int_R^\infty \epsilon t^{\epsilon-2}\nu_0(X)dt \le \frac{C_\mu}{\mu(B_0)}\nu_0(X)\frac{\epsilon}{1-\epsilon}R^{\epsilon-1}.$$

Since $\nu_0(X)=\nu(2c_0B_0)$, taking $R=\frac{\nu(2c_0B_0)}{\mu(B_0)}$ and using the doubling condition on μ, we finally have:

$$I_1+I_2 \le C(\epsilon)\left(\frac{\nu(2c_0B_0)}{\mu(B_0)}\right)^\epsilon \le C(\epsilon)\left(\frac{\nu(2c_0B_0)}{\mu(2c_0B_0)}\right)^\epsilon \le C(\epsilon)\,\mathrm{ess\,inf}_{y\in B_0} M\nu(y)^\epsilon.$$

□

Corollary 3.2.27 *For all $x \in X$, the weight $u(\xi)=\delta(x,\xi)^{-\epsilon}$ is in A_1, for all $0 \le \epsilon < 1$, with A_1 constant independent of x.*

Proof. The result follows from the previous theorem taking $\nu=\delta_x$, the Dirac delta at x, since $M\delta_x(\xi)$ is pointwise equivalent to $\delta(x,\xi)^{-1}$.

□

If u_1 and u_2 are two A_1 weights, Hölder's inequality gives us that $u_1u_2^{1-p}$ is an A_p weight for $1<p<\infty$. Using the last result, there exists $0<\gamma=\gamma(p)\le 1$ such that $u(\xi)=\delta(x,\xi)^\gamma$ is an A_p weight for all $x\in X$, with A_p constant independent of x.

We will need some kind of regularity on the approach family to prove our main result. However, in the case of the existence of a group or pseudo-group structure in X, this additional condition allows us to work with a larger class of regions that those generated by translation of a fixed one, as we can see in the examples below.

Definition 3.2.28 We say that an approach family Ω is **regular** if there is a constant $C>0$ such that for all $(x,t)\in X_+$ the next condition is satisfied:

$$\forall y \in \Omega^t(x),\ \exists\, y^* \in X \ \text{ with } \delta(y,x)=\delta(y^*,x) \ \text{ such that } B(y^*,t)\subset S_\Omega(x,Ct).$$

Some examples of regular approach regions are:

Examples 3.2.29

(i) Assume that (X, μ, δ) is a space of homogeneous type and X is a group. Denote e the identity element of the group. Let $\Omega(e)$ be an approach region of e, and for each $x \in X$ set $\Omega(x) = \{(yx, t) : (y, t) \in \Omega(e)\}$. Then (see [Su])

$$S_\Omega(x, t) = [\Omega^t(e)]^{-1} B(x, t),$$

and since δ is left-invariant, also

$$S_\Omega(x, t) = \bigcup_{z \in \Omega^t(e)} B(z^{-1}x, t).$$

Now, $y \in \Omega^t(x)$ if and only if $yx^{-1} \in \Omega^t(e)$. Then, take $y^* = xy^{-1}x$ which satisfies $\delta(y, x) = \delta(y^*, x)$ because of the left-invariance, and $B(xy^{-1}x, t) \subset S_\Omega(x, t)$, and consequently, Ω is regular with $C = 1$. This is the case of Euclidean spaces.

(ii) Let (X, μ, δ) be a space of homogeneous type. Consider a family of approach regions given by cones of width bounded by a constant $R \geq 1$, that is, $\Omega(x) = \Gamma_{\theta(x)}(x)$ and $0 < \theta(x) \leq R$. Then $\Omega^t(x) = B(x, \theta(x)t)$ and $B(x, t) \subset S_\Omega(x, t)$ for all x in X and $t > 0$. Take $y \in \Omega^t(x) \subset B(x, Rt)$ and use Lemma 3.2.4 to obtain $B(y, t) \subset B(x, c_0 Rt)$ and hence $B(y, t) \subset S_\Omega(x, c_0 Rt)$. The family is regular with $C = c_0 R$ and $y^* = y$.

(iii) Set $X = \mathbb{R}$, and let μ be the Lebesgue measure and δ the usual distance $\delta(x, y) = |x - y|$. We consider the family of cones $\Gamma = \left\{\Gamma_{\theta(x)}(x) : x \in \mathbb{R}\right\}$ with $\theta(x) = |x|$. By symmetry, we can take $0 < x$, without loss of generality. Assume that $(y, t) \in \Gamma(x)$ and that $y > x$, that is

$$0 < y - x < tx. \tag{3.7}$$

We claim that $B(y, t) \subset S_\Gamma(x, t)$, and thus Γ is regular with $C = 1$ and the choice

$$y^* = \begin{cases} y & \text{if} \quad |y| > |x| \\ 2x - y & \text{if} \quad |y| < |x| \, . \end{cases}$$

We now prove the claim. Take $z \in B(y, t) = (y - t, y + t)$. If $z < x + t$, then $z \in B(x, t)$ but also $z \in \Gamma^t(z)$ and thus $z \in S_\Gamma(x, t)$ as claimed. Therefore, we can restrict our attention to $z \in [x + t, y + t)$, and thus $0 < z$. To see that

$\Gamma^t(z) \cap B(x,t) \neq \emptyset$, it is now enough to show that $z - tz < x + t$. This is trivially true if $t \geq 1$. If $t < 1$, then we must show that $z < (x+t)/(1-t)$. We know by (3.7) that $z < y + t < x + tx + t$, so all is reduced to see that $x + tx + t < (x+t)/(1-t)$ or equivalently $x(1-t^2) + t(1-t) < x + t$, which is true from the assumption $t < 1$.

We can see that regularity holds even when the regions are not the translated of a fixed one. We now prove our main result:

Theorem 3.2.30 *Let (X,μ,δ) be a ns-space of homogeneous type, and assume $\Omega \subset X_+$ to be a regular approach family. Then the following conditions are equivalent:*

(i) There exists $C > 0$ and $\theta > 0$ such that $M_\Omega f(x) \leq C\, M_{\Gamma_\theta} f(x)$, for all x in X and all measurable functions f.

(ii) $A_p^\Omega = A_p$ for all $1 \leq p < \infty$, with equivalent constants.

(iii) There is $p \geq 1$ such that $A_p = A_p^\Omega$, with equivalent constants.

(iv) There exists $0 < \gamma \leq 1$ such that the family of weights $\{\delta(x,.)^\gamma : x \in X\}$ is in $W(\Omega)$ uniformly in x.

(v) There exists $\theta > 0$ such that $\Omega(x) \subset \Gamma_\theta(x)$ for all x in X.

Proof. It is obvious that (v) implies (i) and (ii) implies (iii). The implication $(i) \Rightarrow (ii)$ is easy if we recall Theorem 3.2.17. Now, suppose $A_p^\Omega = A_p$ for some $p \geq 1$, with equivalent constants. We can assume that $p > 1$ by the extrapolation theorem of Rubio de Francia, as proved in [J]. We have seen that there is $0 < \gamma = \gamma(p) \leq 1$ such that the family of weights $\{\delta(x,.)^\gamma : x \in X\}$ is in A_p, uniformly in x. So, by hypothesis, this family is in A_p^Ω uniformly in x. Then, we can see by Proposition 3.2.12 and (3.3) that this family of weights is in $W(\Omega)$ uniformly in x, that is, there exists a constant $C > 0$ such that $\|\delta(x,.)^\gamma\|_{W(\Omega)} \leq C$ for all x in X.

Suppose the family of weights is uniformly in $W(\Omega)$. Take $(y,t) \in \Omega(x)$ for a fixed $x \in X$. Then, using the assumption of the regularity on Ω and conditions (ii) and

(iii) of Definition 3.2.2 on δ, we have:

$$\begin{aligned}
\delta(y,x)^\gamma &= \delta(y^*,x)^\gamma \\
&\le D^\gamma\, \delta(x,y^*)^\gamma \\
&= \frac{D^\gamma}{\mu(B(y^*,t))}\int_{B(y^*,t)} \delta(x,y^*)^\gamma d\mu(\xi) \\
&\le \frac{(AD)^\gamma}{\mu(B(y^*,t))}\int_{B(y^*,t)} (\delta(x,\xi)+\delta(y^*,\xi))^\gamma d\mu(\xi) \\
&\le (AD)^\gamma\left(\frac{1}{\mu(B(y^*,t))}\int_{S_\Omega(x,Ct)} \delta(x,\xi)^\gamma d\mu(\xi) + t^\gamma\right).
\end{aligned}$$

Set $u_x(\xi)=\delta(x,\xi)^\gamma$. Using the hypothesis on the family $\{u_x : x\in X\}$, and the fact that $\mu(B(y^*,t))$ is comparable to the radius t, we have:

$$\begin{aligned}
\delta(y,x)^\gamma &\le (AD)^\gamma\left(\frac{U_x(S_\Omega(x,Ct))}{\mu(B(y^*,t))}+t^\gamma\right) \\
&\le C(AD)^\gamma\left(\frac{U_x(B(x,Ct))}{t}+t^\gamma\right). \qquad (3.8)
\end{aligned}$$

Let us now see that $U_x(B(x,t))$ is comparable to $t^{\gamma+1}$ for all $t>0$ uniformly in x:

$$\begin{aligned}
U_x(B(x,t)) &= \int_{B(x,t)} \delta(x,\xi)^\gamma d\mu(\xi) = \sum_{k\ge 0}\int_{2^{-k-1}t\le\delta(x,\xi)<2^{-k}t} \delta(x,\xi)^\gamma d\mu(\xi) \\
&\approx t^\gamma \sum_{k\ge 0} 2^{-k\gamma}\mu(\{\xi : 2^{-k-1}t\le\delta(x,\xi)<2^{-k}t\}) \\
&\approx t^\gamma \sum_{k\ge 0} 2^{-k\gamma}(\mu(B(x,2^{-k}t))-\mu(B(x,2^{-k-1}t))) \\
&\approx t^{\gamma+1} \sum_{k\ge 0} 2^{-k\gamma}(2^{-k}-2^{-k-1}) \\
&\approx t^{\gamma+1}.
\end{aligned}$$

Finally, returning to (3.8), we get $\delta(y,x)\le C^{\frac{1}{\gamma}}\,A\,D\,t$, that is, $(y,t)\in\Gamma_\theta(x)$, where $\theta=C^{\frac{1}{\gamma}}\,A\,D$, and the proof is completed.

□

Remarks 3.2.31

(i) The implication $(ii)\Rightarrow(iii)$ can be viewed as a local result: Fix $x\in X$; if $\Omega(x)$ satisfies the condition in the Definition 3.2.28 for all $t>0$, then $u_x=(x,.)^\gamma\in W(\Omega)$ implies there exists $\theta_x>0$ such that $\Omega(x)\subset\Gamma_{\theta_x}(x)$.

(ii) We observe that in the definition of a regular approach family, we could have just assumed that the point y^* satisfies that $\delta(y,x) \approx \delta(y^*,x)$. Also in statement (iv), the restriction $\gamma \leq 1$ is not really needed.

Next corollary allows us to obtain the result for an initial approach family defined with respect to a space of homogeneous type.

Corollary 3.2.32 *Let (X,μ,d) be a space of homogeneous type. Suppose Ω is an approach family in X_+ with respect to (X,μ,d) such that the related family Ω^δ with respect to (X,μ,δ) is regular. Then the following conditions are equivalent:*

(i) $A_p = A_p^\Omega$ for all $p \geq 1$, with equivalent constants.

(ii) There is $p \geq 1$ such that $A_p = A_p^\Omega$ with equivalent constants.

(iii) There is $\theta > 0$ such that $\Omega(x) \subset \Gamma_\theta(x)$ for all x in X.

Proof. We only need to see that (ii) implies (iii). We saw before that $A_p^\Omega = A_p^{\Omega^\delta}$ with the same constants. We can apply the previous theorem to obtain that $\Omega^\delta(x)$ is contained in a cone $\Gamma_\theta^\delta(x)$ with respect to δ, and then $\Omega(x)$ is also contained in a cone. In fact, we will see that there exists $\theta > 0$ such that $\Omega(x) \subset \Gamma_\theta(x)$, if and only if there exists $\theta' > 0$ such that $\Omega^\delta(x) \subset \Gamma_{\theta'}^\delta(x)$. First, we observe that using Proposition 3.2.19 and the fact that

$$\exp\left(\frac{-1}{1+t}\right) \leq \exp\left(\frac{-1}{1+\theta t}\right) \leq e\,\exp\left(\frac{-1}{1+t}\right),$$

for all $\theta \geq 1$ and $t > 0$, we have that

$$K_1\,\theta^\beta\,r_y(t) \leq r_y(\theta t) \leq e\,K_\mu\,\theta^\alpha\,r_y(t). \tag{3.9}$$

Suppose first that $\Omega(x) \subset \Gamma_\theta(x)$. We can assume that $\theta \geq 1$. Take $(y,s) \in \Omega^\delta(x)$, that is $(y, r_y^{-1}(s)) \in \Omega(x)$. By hypothesis, $d(y,x) < \theta\, r_y^{-1}(s)$ and by definition $\delta(y,x) < r_y(\theta r_y^{-1}(s))$; using (3.9) we get $\delta(y,x) < e\,K\,\theta^\alpha s$, and so, $(y,s) \in \Gamma_{\theta'}^\delta(x)$ with $\theta' = e\,K\,\theta^\alpha$.

On the other hand, assume that $\Omega^\delta(x) \subset \Gamma_{\theta'}^\delta(x)$ with $\theta' \geq 1$. Take $(y,s) \in \Omega(x)$. Then $(y, r_y(s)) \in \Omega^\delta(x)$ and therefore $\delta(y,x) < \theta'\,r_y(s)$. Using the definition of δ and (3.9), we get $d(y,x) < (\theta'/K_1)^{1/\beta}s$, and so $(y,s) \in \Gamma_\theta(x)$ with $\theta = (\theta'/K_1)^{1/\beta}$.

□

We now give a version of our result in the case of a group structure in X.

Corollary 3.2.33 *Let (X, μ, d) be a space of homogeneous type. Suppose that X is a group and that d and μ are left-invariant, that is*

(i) $yB(x,t) = B(yx,t)$ *for all x and y in X and $t > 0$.*

(ii) $\mu(xE) = \mu(E)$ *for all measurable sets E and $x \in X$.*

Given an approach region $\Omega(e)$ for the identity element e of X, set $\Omega(x) = \{(yx,t) : (y,t) \in \Omega_e\}$. Then, the following conditions are equivalent:

(i) $A_p = A_p^\Omega$ *for all $p \geq 1$.*

(ii) There is $p \geq 1$ such that $A_p = A_p^\Omega$.

(iii) There is $\theta > 0$ such that $\Omega(x) \subset \Gamma_\theta(x)$ for all x in X.

Proof. The assumptions on d and μ show that $r_x(t) = r_e(t)$ for all x in X (and therefore, δ is symmetric). Then, it is easy to see that $\Omega^\delta(x) = \big\{(yx,t) : (y,t) \in \Omega^\delta(e)\big\}$ using the definition of Ω^δ, and we saw in (i) of Examples 3.2.29 that this kind of family of approach regions is regular. So, we are in the hypothesis of Theorem 3.2.30, but we do not need the equivalence on the constants because in the case of translated approach regions, if one region is contained in a cone, so are all the rest.

□

Corollary 3.2.34 *Let (X, μ, d) be a space of homogeneous type as in the previous corollary. There exists a family of approach regions Ω for which A_p^Ω is not A_p, but $u \equiv 1 \in A_p^\Omega$ for $p \geq 1$; i.e., $\phi \neq A_p^\Omega \neq A_p$.*

Proof. In [Su], the author gives a family of translated regions Ω which is not non-tangential, and so, not contained in a cone, for which the operator M_Ω is of weak-type (p,p) on $L^p(\mu)$, for $p \geq 1$, that is, $u \equiv 1 \in A_p^\Omega$ for all $p \geq 1$. Using the previous corollary, we have that $A_p^\Omega \neq A_p$ for this family.

□

3.2.3 The case of a group structure: some examples

We will give some examples of spaces of homogeneous type with a group structure where we can apply our results. First, we give a simpler proof of Theorem 3.2.30 in this setting.

Consider (X, μ, d) a space of homogeneous type. Suppose that there exist two constants $C, C' > 0$ such that

$$C \leq \frac{\mu(B(x,r))}{\mu(B(y,r))} \leq C', \tag{3.10}$$

for all $x, y \in X$ and $r > 0$. (We recall that this can occur under the assumption of condition (3.4), as is proved in the last section.) Assume that X is a group with a multiplicative law (not necessary commutative) such that d is left-invariant and μ is left-invariant and invariant under inversion, that is respectively,

$$\begin{aligned} x \cdot B(y,r) &= B(x \cdot y, r) \quad \forall\, x, y \in X,\ r > 0, \\ \mu(x \cdot E) &= \mu(E) \quad \forall\, x \in X,\ E \subset X \text{ measurable}, \\ \mu(E) &= \mu(E^{-1}) \quad \forall\, E \subset X \text{ measurable}, \end{aligned}$$

e.g. this is the case if X is unimodular and μ is the Haar measure ([Fo], Proposition 2.9). We denote by e the identity element of the group.

For a measurable set $\Omega(e)$ in X_+ full on the vertical direction, consider the translated family of sets given by

$$\Omega(x) := x \cdot \Omega(e) = \{(x \cdot y, t) : (y,t) \in \Omega(e)\}.$$

Recall that this family is regular in the sense of Definition 3.2.28.

With this hypothesis, the proof of Theorem 3.2.30 is much simpler:

Theorem 3.2.35 *If $\Omega(x) = \{(x \cdot y, t) : (y,t) \in \Omega(e)\}$ for a given approach region $\Omega(e)$ of e, then the following conditions are equivalent:*

(i) There exist $C > 0$ and $\theta > 0$ such that $M_\Omega f(x) \leq C M_{\Gamma_\theta} f(x)$, for all $x \in X$ and all measurable functions f.

(ii) $A_p^\Omega = A_p$ for all $1 \leq p < \infty$.

(iii) There is $p \geq 1$ such that $A_p = A_p^\Omega$.

(iv) There exists $0 < \gamma \leq 1$ such that $u(y) = d(e,y)^\gamma \in W(\Omega)$.

(v) There exists $\theta > 0$ such that $\Omega(e) \subset \Gamma_\theta(e)$.

Proof. It is obvious that (ii) implies (iii). Because of the left-invariance of d, if $\Omega(e) \subset \Gamma_\theta(e)$ then $\Omega(x) \subset \Gamma_\theta(x)$ for all x, and so, (v) implies (i). The implication $(i) \Rightarrow (ii)$ is easy if we recall Theorem 3.2.17 (one can also show that, since Ω is full on the vertical direction, then $Mf(x) \leq CM_\Omega f(x)$). Now, suppose $A_p^\Omega = A_p$ for some $p \geq 1$. We can assume that $p > 1$ by the extrapolation theorem of Rubio de Francia, as proved in [J]. We have seen that there is $0 < \gamma = \gamma(p) \leq 1$ such that $u(y) = d(e,y)^\gamma$ is in A_p, and so, by hypothesis and Theorem 3.2.17, $u \in W(\Omega)$.

Suppose that $u(y) = d(e,y)^\gamma \in W(\Omega)$. Take $(y,t) \in \Omega(e)$. Using the triangle inequality, we have:

$$\begin{aligned} d(e,y)^\gamma &= \frac{1}{\mu(B(y,t)^{-1})} \int_{B(y,t)^{-1}} d(e,y)^\gamma \, d\mu(z) \\ &\leq \frac{A^\gamma}{\mu(B(y,t)^{-1})} \int_{B(y,t)^{-1}} (d(e,z^{-1})^\gamma + d(z^{-1},y)^\gamma) \, d\mu(z) \\ &= \frac{A^\gamma}{\mu(B(y,t))} \int_{B(y,t)^{-1}} (d(e,z)^\gamma + d(z^{-1},y)^\gamma) \, d\mu(z) \\ &\leq A^\gamma \left(\frac{1}{\mu(B(y,t))} U(B(y,t)^{-1}) + t^\gamma \right). \end{aligned}$$

We now use that $S_\Omega(e,t) = B(e,t) \cdot [\Omega^t(e)]^{-1}$ (see [Su]), that is

$$S_\Omega(e,t) = \bigcup_{y \in \Omega^t(e)} B(y,t)^{-1},$$

and so, by hypothesis, we have:

$$\begin{aligned} d(e,y)^\gamma &\leq A^\gamma \left(\frac{U(S_\Omega(e,t))}{\mu(B(y,t))} + t^\gamma \right) \\ &\leq C\, A^\gamma \left(\frac{U(B(e,t))}{\mu(B(y,t))} + t^\gamma \right). \end{aligned}$$

Finally, observe that $U(B(e,t)) = \int_{B(e,t)} d(e,z)^\gamma \, d\mu(z) \leq t^\gamma \, \mu(B(e,t))$, and then using (3.10),

$$d(e,y)^\gamma \leq 2\, C\, A^\gamma \, t^\gamma.$$

Hence, $\Omega(e) \subset \Gamma_\theta(e)$, with $\theta = A\,(2C)^{1/\gamma}$.

□

We consider now some examples that satisfy our assumptions:

Examples 3.2.36

(i) Take $X = \mathbb{R}^n$ and μ the Lebesgue measure. Consider the non-isotropic quasi-distance

$$d(x, y) = \sum_{k=1}^{n} |x_k - y_k|^{1/a_k},$$

where $a_1, \ldots, a_n$ are strictly positive constants, or the equivalent

$$d'(x, y) = \sup_k |x_k - y_k|^{1/a_k}.$$

It is easy to see that all the required conditions are satisfied. We write $|x| = d(x, 0) \approx d'(x, 0)$. Our theorem completes now the result proved in [SS2] (Theorem 3.1.5 in Section 1 of this chapter):

Theorem 3.2.37 *If $\Omega(x) = \{(x + y, t) : (y, t) \in \Omega(0)\}$ for a given approach region $\Omega(0)$ of 0, then, the following conditions are equivalent:*
(a) There exist $C > 0$ and $\theta > 0$ such that $M_\Omega f(x) \leq C M_{\Gamma_\theta} f(x)$, for all $x \in X$ and all measurable functions f.
(b) $A_p^\Omega = A_p$ for all $1 \leq p < \infty$.
(c) There is $p \geq 1$ such that $A_p = A_p^\Omega$.
(d) There exists $0 < \gamma \leq 1$ such that $u(x) = |x|^\gamma \in W(\Omega)$.
(e) There exists $\theta > 0$ such that $\Omega(0) \subset \Gamma_\theta(0)$.

If $a_k = 2$ for all k, Nagel and Stein proved in [NS] that if the weight $u(y) = |y|^0 = 1$ is in A_p^Ω, then Ω_0 need not be contained in a cone: in fact, it can contain a sequence of points approaching 0 tangentially. Our theorem states that this is the extremal situation, because if A_p^Ω contains a power weight with positive exponent, Ω_0 is necessarily a subset of a cone.

(ii) Take $X = \mathbf{H}_\mathrm{n}$ the Heisenberg group (see [St2, XII.1.4] or [Su] for the details). This is the set

$$\mathbb{C}^n \times \mathbb{R} = \{[\zeta, t] : \zeta \in \mathbb{C}^n,\ t \in \mathbb{R}\},$$

with the (noncommutative) multiplicative law

$$[\zeta, t] \cdot [\eta, s] = [\zeta + \eta, t + s + 2\ \mathrm{Im}(\zeta\overline{\eta})].$$

The identity element is $e = [0, 0]$, and we have $[\zeta, t]^{-1} = [-\zeta, -t]$.

Consider the generalized half-plane $D = \{z \in \mathbb{C}^{n+1} : h(z) > 0\}$, where

$$h(z) = \operatorname{Im} z_{n+1} - \sum_{k=1}^{n} |z_k|^2,$$

for $z = (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1}$. Then, we can see $\mathbf{H}_\mathrm{n}$ as the boundary $\partial D = \{z \in \mathbb{C}^{n+1} : h(z) = 0\}$, by considering the map $\psi : \mathbb{C}^{n+1} \longrightarrow \mathbf{H}_\mathrm{n}$ defined as $\psi(z) = (z_1, \ldots, z_n, \operatorname{Re} z_{n+1})$, where the restriction to ∂D gives the desired bijection. Now, we are able to think of D as the product $\mathbf{H}_\mathrm{n} \times (0, \infty)$ using the identification

$$\Phi : D \longrightarrow \mathbf{H}_\mathrm{n} \times (0, \infty),$$

defined by $\Phi(z) = (\psi(z), h(z))$.

The group $\mathbf{H}_\mathrm{n}$ acts on D (and ∂D) associating to each element $[\zeta, t] \in \mathbf{H}_\mathrm{n}$ the following affine self-mapping:

$$[\zeta, t] : (z', z_{n+1}) \longmapsto (z' + \zeta, z_{n+1} + t + 2iz'\overline{\zeta} + i|\zeta|^2).$$

It is not difficult to see that $[\eta, s]([\zeta, t]z) = ([\eta, s] \cdot [\zeta, t])z$, and that this action is simply transitive on ∂D: for every two points in ∂D, there is exactly one element in $\mathbf{H}_\mathrm{n}$ mapping the first to the second. So, we can also identify $\mathbf{H}_\mathrm{n}$ as the *translations* on ∂D.

We consider the pseudo-norm function defined in $\mathbf{H}_\mathrm{n}$

$$||[\zeta, t]|| = \max(|\zeta|^2, |t|),$$

satisfying the quasi-triangle inequality

$$||x \cdot y|| \le c(||x|| + ||y||),$$

for all $x, y \in \mathbf{H}_\mathrm{n}$. Observe the different homogeneity in ζ and t. We can define the quasi-distance in $\mathbf{H}_\mathrm{n}$ by $d(x, y) = ||y^{-1} \cdot x||$, symmetric because $||x|| = ||x^{-1}||$, and left-invariant with respect to the group action.

We take as the underlying measure $d\mu$ in $\mathbf{H}_\mathrm{n}$ to be the Euclidean Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$, which is left-invariant and invariant under inversion.

With these definitions, $(\mathbf{H}_\mathrm{n}, \mu, d)$ becomes a space of homogeneous type, with the conditions required in Theorem 3.2.35. For the following result we will use the trivial fact that $d(e, [\zeta, t]) = \max(|\zeta|^2, |t|) \approx (|\zeta|^2 + |t|)$.

Theorem 3.2.38 *If* $\Omega(x) = \{(x \cdot z, t) : (z,t) \in \Omega(e)\} \subset \boldsymbol{H}_{\mathrm{n}} \times (0,\infty)$ *for a given approach region* $\Omega(e)$ *of* $e \in \boldsymbol{H}_{\mathrm{n}}$*, then, the following conditions are equivalent:*
(a) There exist $C > 0$ *and* $\theta > 0$ *such that* $M_\Omega f(x) \leq C M_{\Gamma_\theta} f(x)$*, for all* $x \in \boldsymbol{H}_{\mathrm{n}}$ *and all measurable functions* f*.*
(b) $A_p^\Omega = A_p$ *for all* $1 \leq p < \infty$*.*
(c) There is $p \geq 1$ *such that* $A_p = A_p^\Omega$*.*
(d) There exists $0 < \gamma \leq 1$ *such that* $u([\zeta, t]) = (|\zeta|^2 + |t|)^\gamma \in W(\Omega)$*.*
(e) There exists $\theta > 0$ *such that* $\Omega(e) \subset \Gamma_\theta(e)$*.*

The Korányi admissible regions for D are defined by

$$\tilde{\Gamma}_\theta(0) = \{z \in D : ||\psi(z)|| < \theta h(z)\},$$

and $\tilde{\Gamma}_\theta(g \cdot 0) = g \cdot \tilde{\Gamma}_\theta(0)$ if $g \in \mathbf{H}_{\mathrm{n}}$. We have

$$\Phi(\tilde{\Gamma}_\theta(\zeta)) = \Gamma_\theta(\psi(\zeta)),$$

where $\Gamma_\theta(x) = \{(y,s) \in \mathbf{H}_{\mathrm{n}} \times (0,\infty) : d(y,x) < \theta s\}$ is a cone in $\mathbf{H}_{\mathrm{n}} \times (0,\infty)$. If $\tilde{\Omega}(0)$ is an approach region of $0 \in \partial D$, we can translate it by the action of the Heisenberg group: for $\zeta \in \partial D$, there exists a unique element $x \in \mathbf{H}_{\mathrm{n}}$ such that $\zeta = x \cdot 0$, and then consider $\tilde{\Omega}(\zeta) = x \cdot \tilde{\Omega}(0)$. Then, taking $\Phi(\tilde{\Omega}(0)) = \Omega(e)$ we have

$$\Phi(\tilde{\Omega}(\zeta)) = \Omega(x),$$

with $\Omega(x) = \{(x \cdot y, t) : (y,t) \in \Omega(e)\}$. We have that $\Omega(e)$ is contained in a cone if and only if $\Phi^{-1}(\Omega(e))$ is contained in an admissible region (see [Su]). With this notations, we can give the next result:

Corollary 3.2.39 *For a given approach region* $\tilde{\Omega}(0) \subset D$ *of* $0 \in \partial D$*, the following conditions are equivalent:*
(a) There exist $C > 0$ *and* $\theta > 0$ *such that* $M_\Omega f(x) \leq C M_{\Gamma_\theta} f(x)$*, for all* $x \in \boldsymbol{H}_{\mathrm{n}}$ *and all measurable functions* f*.*
(b) $A_p^\Omega = A_p$ *for all* $1 \leq p < \infty$*.*
(c) There is $p \geq 1$ *such that* $A_p = A_p^\Omega$*.*
(d) There exists $0 < \gamma \leq 1$ *such that* $u([\zeta, t]) = (|\zeta|^2 + |t|)^\gamma \in W(\Omega)$*.*
(e) There exists $\theta > 0$ *such that* $\tilde{\Omega}(0) \subset \tilde{\Gamma}_\theta(0)$*.*

(iii) Consider X as the set of all real 3×3 upper-triangular matrices having ones along the diagonal. The group multiplicative law is the usual matrix product (see [St, XIII.5.2.3]). The norm function is defined by

$$||x|| = \max\{|a|, |b|^{1/2}, |c|\},$$

where

$$x = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

The function $d(x, y) = ||y^{-1} \cdot x||$ is left-invariant but nonsymmetric. We can consider the equivalent quasi-distance $d'(x, y) = (d(x, y) + d(y, x))/2$, since there exists a constant C such that $d(x, y) \leq C\, d(y, x)$ for all $x, y \in X$. Notice that we can realize X as $\mathbb{R}^3$ with the inner product

$$(a, b, c) \cdot (d, e, f) = (a + d, b + e + af, c + f).$$

We write $x = (a, b, c) \in X$ and observe that $||x|| \approx (a^2 + |b| + c^2)^{1/2} \approx d'(x, 0)$. Then we take as the underlying measure $d\mu$ to be the Lebesgue measure, which is left-invariant and invariant under inversion. Also, (X, μ, d') is a space of homogeneous type. We can state the theorem:

Theorem 3.2.40 *If $\Omega(x) = \{(x \cdot y, t) : (y, t) \in \Omega(0)\} \subset X \times (0, \infty)$ for a given approach region Ω_0 of 0, then, the following conditions are equivalent:*
(a) There exist $C > 0$ and $\theta > 0$ such that $M_\Omega f(x) \leq C M_{\Gamma_\theta} f(x)$, for all $x \in X$ and all measurable functions f.
(b) $A_p^\Omega = A_p$ for all $1 \leq p < \infty$.
(c) There is $p \geq 1$ such that $A_p = A_p^\Omega$.
(d) There exists $0 < \gamma \leq 1$ such that $u(x) = (a^2 + |b| + c^2)^{\gamma/2} \in W(\Omega)$.
(e) There exists $\theta > 0$ such that $\Omega(0) \subset \Gamma_\theta(0)$.

(iv) Every connected nilpotent unimodular Lie group with a left-invariant Riemannian metric and the induced measure is a space of homogeneous type satisfying our conditions (see [Ch, Example VI.7]).

3.3 The general case

The boundary convergence phenomena has been studied for classes of functions defined in general sets that have not a product structure of the type $X \times (0, \infty)$ as studied in the previous section, but X is just the boundary of a more general set. Some examples are: the unit ball in $\mathbb{C}^n$, non-tangentially accessible domains (NTA domains), strongly pseudoconvex domains in $\mathbb{C}^n$ and trees (see [DiB] and the references therein.)

In this section we extend the results of Section 3.2 to an abstract setting, that allows us to extend this results to a wider class of sets. Without the product structure, we find some new difficulties: the notion of being full on the vertical direction has no sense, in general; we are not able to work with cones; and we do not have the natural assignment between points and balls $(x, t) \in X \times (0, \infty) \leftrightarrow B(x, t) \subset X$. We will replace the notion of a cone by the definition of a so-called supernatural approach region, which is a slight modification of the definition of a natural region given in page 30 of [DiB]. Then, we also replace the notion of an approach region completed in the vertical direction by the approach region completed with respect the supernatural approach region. This is a notion introduced in [DiB], and it is a generalization of the *cone condition* appearing in [NS]. Finally, we replace the maximal operator related to an approach family by a new maximal operator that depends on the supernatural family.

Before to go into the abstract case, we begin with the case of a homogeneous tree in order to illustrate the way of generalization. Here, the tree T_o takes the place of $X \times (0, \infty)$ and the ∂T_o is just the boundary. The tree has not a product structure but its geometry has a vertical direction. Moreover, there are cones, and to every vertex in the tree corresponds a ball in the boundary. Boundary convergence on trees are considered in, for example, [ADiBU], [C], [DiB], [FPR], [KP], [KPT] and [SV].

3.3.1 Approach regions in an isotropic homogeneous tree

We briefly recall some notions about homogeneous trees. We refer to Chapter 2 for a complete introduction.

A homogeneous tree of degree q is a tree where every vertex has $q + 1$ neighbor vertices. When a origin vertex o has been chosen, the tree T_o is *rooted* and it is endowed of a partial order structure, namely, $x \leq y$ if and only if x belongs to the unique shortest path joining y and o, that is, the *geodesic* from o to y (or from y to o), which is written as $[o, y]$. Then, the boundary of the tree ∂T_o is seen as the set of geodesics of infinite length (and actually, it does not depend on the choice of o).

The tree is a metric space endowed with the so-called *hyperbolic distance* $d(x, y)$, which counts the number of edges between x and y. We denote $|x| = d(o,x)$ for every $x \in T$.

For a vertex $x \in T_o$, the *tent* under x is the set

$$T(x) = \{y \in T : y \geq x\},$$

and its *shadow* in ∂T_o is

$$I(x) = \{\omega \in \partial T : x \in [o,\omega)\},$$

where $[o,\omega)$ is the infinite geodesic joining the origin o and the boundary point ω, also denoted as

$$[o,\omega) = \{o = \omega(0), \omega(1), \omega(2), \dots, \omega(n), \dots\},$$

with the convention that $\omega(n)$ is the unique vertex in the path from o to ω at distance n from o. Equivalently, we can write $T(x) = \{y \in T : I(y) \subset I(x)\}$.

The **Euclidean distance** in $T_o \cup \partial T_o$ is given by

$$d_e(x,y) = q^{-|c(x,y)|},$$

for all $x \neq y \in T_o \cup \partial T_o$, where $c(x,y)$ is the confluent vertex of x and y (with the convention $c(x,y) = x$ if $x \in [o,y]$), and $d_e(x,y) = 0$ if $x = y$ (see [T]). The balls in ∂T_o with respect to d_e are exactly the shadows $I(x)$ for some $x \in T_o$. Specifically, if $0 < r \leq q$, we have

$$B(\omega,r) := \{\zeta \in \partial T_o : d_e(\omega,\zeta) < r\} = I(\omega(k+1)), \tag{3.11}$$

whenever $q^{-k-1} < r \leq q^{-k}$ for $k \geq -1$. Conversely,

$$I(x) = B(\omega,r), \quad \forall\, \omega \in I(x),$$

if $q^{-|x|} < r \leq q^{1-|x|}$. The space $T_o \cup \partial T_o$ is compact with respect this distance. The set of tents $\{T(x) : x \in T\}$ forms a basis of open sets of the topology generated by d_e, and the set of shadows $\{I(x) : x \in T\}$ is a basis of open sets in the boundary.

We endow ∂T_o with a **probability measure** μ defined on the sets $\{I(x) : x \in T\}$ by

$$\mu(I(x)) = \frac{q^{1-|x|}}{q+1},$$

if $x \neq o$, and $\mu(\partial T_o) = 1$ (see again [T]).

We observe that from (3.11) and the equality

$$\mu(I(x)) = q\,\mu(I(y)),$$

if x and y are neighbor vertices and $|y| = |x|+1$ (that is, y is a *son* of x), the measure μ is doubling with respect to the Euclidean balls. Hence, $(\partial T_o, \mu, d_e)$ is a space of homogeneous type.

A pair of measures (ρ, ν), ρ defined in T_o and ν in ∂T_o, form a Carleson pair if there exists a constant $C > 0$ such that

$$\rho(T(x)) \le C\,\nu(I(x)),$$

for all $x \in T$. Set $T(O) := \{x \in T : I(x) \subset O\}$ for an open set $O \subset \partial T_o$. The proof of Proposition 3.2.9 also holds in this setting, and therefore we have:

Lemma 3.3.1 *If (ρ, ν) is a Carleson pair and ν is doubling, then for all open sets $O \subset \partial T_o$,*

$$\rho(T(O)) \le C\,\nu(O).$$

A family of sets $\Omega = \{\Omega(\omega) : \omega \in \partial T_o\}$, with $\Omega(\omega) \subset T_o$ for all ω, is an *approach family* in T_o if

$$\omega \in \overline{\Omega(\omega)},$$

where the closure is taken with respect to the Euclidean distance. An example is the family of cones of width $0 \le \theta \in \mathbb{Z}$

$$\Gamma_\theta(\omega) = \left\{x \in T : d_e(\omega, x) \le q^{\theta - |x|}\right\}.$$

An approach family Ω is *full on the vertical direction* if $x \in \Omega(\omega)$, then $y \in \Omega(\omega)$ for all $y \le x$, for all $\omega \in \partial T_o$. For every Ω, it is possible to give a new approach region full in the vertical direction by considering

$$\widehat{\Omega}(\omega) = \bigcup_{x \in \Omega(\omega)} \{y \in T : y \le x\}.$$

$\widehat{\Omega}(\omega)$ is called the *vertical completion* of Ω.

For a measure ρ in ∂T_o and a family Ω, we define a (outer) measure in T_o by

$$\rho_\Omega(E) = \rho(\{\omega \in \partial T_o : \Omega(\omega) \cap E \ne \emptyset\}),$$

for a set $E \subset T_o$.

We can define a maximal operator related to an approach family by mimicking the Definition 3.2.11:

$$M_\Omega f(\omega) = \sup_{x\in\Omega(\omega)} \frac{1}{I(x)} \int_{I(x)} |f(z)|\, d\mu(z).$$

In the special case of the cone $\Gamma_0(\omega) = [o,\omega)$, the operator becomes the centered Hardy-Littlewood maximal operator M_{cen}, which is known that satisfies the estimates of Theorem 3.2.6.

The analog of Theorem 3.2.13 is:

Theorem 3.3.2 *Let ρ and ν be two nonnegative measures on T_o. If $M : L^p(\nu) \longrightarrow L^{p,\infty}(\nu)$ is bounded for some $p \geq 1$, then the following conditions are equivalent:*

(i) There exists $C > 0$ such that

$$\rho(\{\omega \in \partial T_o : M_\Omega f(\omega) > \lambda\}) \leq C\,\nu(\{\omega \in \partial T_o : M_{\text{cen}} f(\omega) > \lambda\}),$$

for all $\lambda > 0$ and measurable f.

(ii) $M_\Omega : L^p(\nu) \longrightarrow L^{p,\infty}(\rho)$ is bounded.

(iii) There exists $C > 0$ such that $\rho_\Omega(T(x)) \leq C\,\nu(I(x))$ for all $x \in T_o$.

Proof. It is known that ν is necessarily doubling. That (i) implies (ii) is trivial. To see that (ii) implies (iii), we take $f = \chi_{I(x)}$ for a fixed $x \in T_o$. If there exists $y \in \Omega(\omega) \cap T(x)$, then $I(y) \subset I(x)$, and therefore $M_\Omega f(\omega) > 1/2$ for all $\omega \in I(x)$. Thus, by hypothesis we have

$$\rho_\Omega(T(x)) \leq \rho(\{\omega \in \partial T_o : M_\Omega f(\omega) > 1/2\}) \leq C\, \|f\|_{L^p(\nu)} = C\,\nu(I(x)).$$

Suppose that (iii) holds and let us see (i). We take ω such that $M_\Omega f(\omega) > \lambda$ for certain $\lambda > 0$. There exists $x \in \Omega(\omega)$ such that

$$\frac{1}{I(x)} \int_{I(x)} |f(z)|\, d\mu(z) > \lambda,$$

and hence

$$\Omega(\omega) \cap \left\{ x \in T_o : \frac{1}{I(x)} \int_{I(x)} |f(z)|\, d\mu(z) > \lambda \right\} \neq \emptyset.$$

We observe that

$$\left\{ x \in T_o : \frac{1}{I(x)} \int_{I(x)} |f(z)|\, d\mu(z) > \lambda \right\} \subset T(O),$$

where $O = \{\omega \in \partial T_o : M_{\text{cen}} f(\omega) > \lambda\}$, for all functions f. Thus, by Lemma 3.3.1,

$$\begin{aligned}
\rho(\{\omega \in \partial T_o : M_\Omega f(\omega) > \lambda\}) &\leq \rho_\Omega\left(\left\{x \in T_o : \frac{1}{I(x)} \int_{I(x)} |f(z)|\, d\mu(z) > \lambda\right\}\right) \\
&\leq \rho_\Omega(T(O)) \\
&\leq C\,\nu(O) \\
&= C\,\nu(\{x \in T_o : M_{\text{cen}} f(\omega) > \lambda\})
\end{aligned}$$

□

Corollary 3.3.3 *Let ρ and ν be two nonnegative measures on T_o. If $M : L^p(\nu) \longrightarrow L^{p,\infty}(\nu)$ is bounded for some $p \geq 1$, then the following conditions are equivalent:*

(i) $M_\Omega : L^p(\nu) \longrightarrow L^{p,\infty}(\rho)$ *is bounded.*

(ii) $M_{\widehat{\Omega}} : L^p(\nu) \longrightarrow L^{p,\infty}(\rho)$ *is bounded.*

(iii) There exists $C > 0$ such that $\rho_{\widehat{\Omega}}(T(x)) \leq C\,\nu(I(x))$ for all $x \in T_o$.

Proof. Since $M_{\widehat{\Omega}} f(\omega) \geq M_\Omega f(\omega)$, simply because $\Omega(\omega) \subset \widehat{\Omega}(\omega)$ for all $\omega \in \partial T_o$, we have that (ii) implies (i). That (ii) is equivalent to (iii) is proved in the last theorem. So, it is enough to see $(i) \Longrightarrow (iii)$. Trivially,

$$\{\omega \in \partial T_o : \Omega(\omega) \cap T(x) \neq \emptyset\} \subset \left\{\omega \in \partial T_o : \widehat{\Omega}(\omega) \cap T(x) \neq \emptyset\right\}.$$

On the other hand, if $\widehat{\Omega}(\omega) \cap T(x) \neq \emptyset$, there exists $y \in \widehat{\Omega}(\omega) \cap T(x)$. By definition of $\widehat{\Omega}$, this means that there exists $z \in \Omega(\omega)$ such that $z \geq y$, with $y \in T(x)$. Consequently, $z \in \Omega(\omega)$ and $I(z) \subset I(y)$ for $y \in T(x)$, and hence $z \in \Omega(\omega)$ and $z \in T(x)$, that is $\Omega(\omega) \cap T(x) \neq \emptyset$. Finally,

$$\{\omega \in \partial T_o : \Omega(\omega) \cap T(x) \neq \emptyset\} = \left\{\omega \in \partial T_o : \widehat{\Omega}(\omega) \cap T(x) \neq \emptyset\right\},$$

and it follows

$$\rho_\Omega(T(x)) = \rho_{\widehat{\Omega}}(T(x)).$$

Now, the implication is derived from the last theorem.

□

This corollary says that we can always assume that our approach family is full on the vertical direction. From now on, we take this condition for granted.

Remark 3.3.4 If Ω is full on the vertical direction, it is easy to see that condition (iii) is equivalent to the existence of $C > 0$ such that

$$\rho(\Omega^{\downarrow}(x)) \leq C\,\nu(I(x)),$$

for all $x \in T_o$, since in the context of our tree we have

$$\rho_{\Omega}(T(x)) = \rho(\Omega^{\downarrow}(x)).$$

Moreover, it is easy to see the equality

$$\Omega^{\downarrow}(x) = \left\{\omega \in \partial T_o : \Omega^{|x|}(\omega) \cap I(x) \neq \emptyset\right\} := S_{\Omega}(x),$$

where $\Omega^k(\omega) = \{\zeta \in \partial T_o : \zeta(k) \in \Omega(\omega)\}$ is the cross-section at height $0 \leq k \in \mathbb{Z}$. The set $S_{\Omega}(x)$ is the substitute of $S_{\Omega}(x,t)$ of the last section. Thus, if Ω is full on the vertical direction, condition (iii) also reads as follows: there exists $C > 0$ such that $\rho(S_{\Omega}(x)) \leq C\,\nu(I(x))$ for all $x \in T_o$, which is the same condition appearing in Theorem 3.2.13.

Lemma 3.3.5 *If Ω is an approach family in T_o full on the vertical direction, then*

$$M_{\text{cen}}f(\omega) \leq M_{\Omega}f(\omega),$$

for all $\omega \in \partial T_o$ and measurable f.

Proof. If $\omega \in \overline{\Omega(\omega)}$, there exists $\{x_k : k \geq 0\} \subset \Omega(\omega) \subset T_o$ such that

$$d_e(x_k, \omega) \longrightarrow 0.$$

For a fixed $j \geq 0$, there exists $n(j) \geq 0$ such that $d_e(x_k, \omega) < \frac{q^{-j}}{q+1}$ for all $k \geq n(j)$. Thus, $x_k \in T(\omega(j))$, and since Ω is full on the vertical direction, $\omega(j) \in \Omega(\omega)$. Then,

$$M_{\Omega}f(\omega) \geq \frac{1}{I(\omega(j))}\int_{I(\omega(j))} |f(z)|\,d\mu(z),$$

for all $j \geq 0$ and measurable f, and therefore

$$M_{\Omega}f(\omega) \geq M_{\text{cen}}f(\omega),$$

for all $\omega \in \partial T_o$ and measurable f.

□

Remark 3.3.6 In fact, we have proved that $[o,\omega) \subset \Omega(\omega)$ for all $\omega \in \partial T_o$ if Ω is full in the vertical direction.

As usual, a weight is a positive function $u \in L^1_{\text{loc}}(\mu)$. For $p \geq 1$, A_p is the class of weights $u : \partial T_o \longrightarrow [0,\infty)$ such that $M_{\text{cen}} : L^p(u) \longrightarrow L^{p,\infty}(u)$ is bounded. We denote A^p_Ω the class of weights u such that $M_\Omega : L^p(u) \longrightarrow L^{p,\infty}(u)$ is bounded. Finally, we define

$$W(\Omega) = \{u : \exists\, C > 0 \,\text{such that}\, U_\Omega(T(x)) \leq C\, U(I(x)),\, \forall x \in T_o\}\,.$$

Combining Theorem 3.3.2 and Lemma 3.3.5, we easily get:

Proposition 3.3.7 *Let Ω be a family of approach regions in T_o. For $1 \leq p < \infty$, we have*

$$A^\Omega_p = A_p \cap W(\Omega),$$

and

$$\|u\|_{A^p} \leq \|u\|_{A^p_\Omega}\,.$$

The last result is the natural analog of Theorem 3.2.30. We observe that in our actual context we have that

$$\mu(B(\omega,r)) = \mu(B(\zeta,r)),$$

for all $\omega, \zeta \in \partial T_o$ and $r > 0$, or equivalently

$$\mu(I(x)) = \mu(I(y)),$$

if $|x| = |y|$. An approach family Ω is *regular* if for all $\omega \in \partial T_o$ we have

$$\forall x \in \Omega(\omega),\ \exists\, x^* \in T_o \ \text{ with } d_e(\omega,x) = d_e(\omega,x^*) \ \text{ such that } I(x^*) \subset S_\Omega(\omega(|x|)).$$

Theorem 3.3.8 *Assume $\Omega \subset T_o$ to be a regular approach family, then the following conditions are equivalent:*

(i) There exists $C > 0$ and $\theta > 0$ such that $M_\Omega f(\omega) \leq C\, M_{\Gamma_\theta} f(\omega)$, for all ω in ∂T_o and all measurable functions f.

(ii) $A^\Omega_p = A_p$ for all $1 \leq p < \infty$.

(iii) There is $p \geq 1$ such that $A_p = A^\Omega_p$.

(iv) *There exists* $0 < \gamma \leq 1$ *such that the family of weights* $\{\delta(\omega, .)^\gamma : \omega \in \partial T_o\}$ *is in* $W(\Omega)$ *uniformly in* ω.

(v) *There exists* $\theta > 0$ *such that* $\Omega(\omega) \subset \Gamma_\theta(\omega)$ *for all* ω *in* ∂T_o.

Proof. In view of the proof of Theorem 3.2.30, it is enough to prove that (iv) implies (v), because by using Proposition 3.3.7 the same argument holds.

Suppose the family of weights is uniformly in $W(\Omega)$. Take $x \in \Omega(\omega)$ for a fixed $\omega \in \partial T_o$. Then, using the assumption of the regularity on Ω, we have:

$$\begin{aligned} d_e(\omega, x)^\gamma &= d_e(\omega, x^*)^\gamma \\ &= \frac{1}{\mu(I(x^*))} \int_{I(x^*)} d_e(\omega, x^*)^\gamma d\mu(\xi) \\ &\leq \frac{1}{\mu(I(x^*))} \int_{I(x^*)} (d_e(\omega, \xi) + d_e(x^*, \xi))^\gamma d\mu(\xi) \\ &\leq \frac{1}{\mu(I(x^*))} \int_{S_\Omega(\omega(|x|))} d_e(\omega, \xi)^\gamma d\mu(\xi) + q^{-|x|\gamma}. \end{aligned}$$

Set $u_\omega(\xi) = d_e(\omega, \xi)^\gamma$. Using the hypothesis on the family $\{u_\omega : \omega \in \partial T_o\}$ and taking into account Remark 3.3.4, we have:

$$\begin{aligned} d_e(\omega, x)^\gamma &\leq \frac{U_\omega(S_\Omega(\omega(|x|)))}{\mu(I(x^*))} + q^{-|x|\gamma} \\ &\leq \frac{C\, U_\omega(I(\omega(|x|)))}{\mu(I(x^*))} + q^{-|x|\gamma}. \end{aligned} \tag{3.12}$$

But,

$$\begin{aligned} U_\omega(I(\omega(|x|))) &= \int_{I(\omega(|x|))} d_e(\omega, \xi)^\gamma \, d\mu(\xi) \\ &= q^{-|\omega(|x|)|\gamma} \mu(I(\omega(|x|))) \\ &= q^{-|x|\gamma} \mu(I(\omega(|x|))), \end{aligned}$$

and hence, returning to (3.12), we get $d_e(\omega, x) \leq (C+1)^{\frac{1}{\gamma}} q^{-|x|}$, that is, $x \in \Gamma_\theta(\omega)$, where θ is the integer part of $\log(C+1)/(\gamma \log(q))$.

□

3.3.2 Approach regions in the abstract context

We present in this subsection the more general context where our results apply. We use some of the terminology and the techniques introduced in [DiB].

Let (X, δ) be a quasi-metric space.

Definition 3.3.9 A set D is a **space of approach** to (X,δ) for the **approach function**

$$a : D \times X \longrightarrow [0,\infty),$$

if the following conditions are satisfied:

(i) For each $x \in X$ there exists a sequence $\{\zeta_n : n \in \mathbb{N}\}$ in D such that

$$\lim_{n\to\infty} a(\zeta_n, x) = 0.$$

(ii) Whenever $\lim_{n\to\infty} \delta(x_n, y_n) = \lim_{n\to\infty} a(\zeta_n, x_n) = 0$, then

$$\lim_{n\to\infty} a(\zeta_n, y_n) = 0,$$

for sequences $\{x_n : n \in \mathbb{N}\}$ and $\{y_n : n \in \mathbb{N}\}$ in X and $\{\zeta_n : n \in \mathbb{N}\}$ in D.

(iii) Whenever $\lim_{n\to\infty} a(\zeta_n, x_n) = \lim_{n\to\infty} a(\zeta_n, y_n) = 0$, then

$$\lim_{n\to\infty} \delta(x_n, y_n) = 0,$$

for sequences $\{x_n : n \in \mathbb{N}\}$ and $\{y_n : n \in \mathbb{N}\}$ in X and $\{\zeta_n : n \in \mathbb{N}\}$ in D.

If D is a space of approach to (X,δ), a subset $\Omega(x) \subset D$ is an **approach region to** $x \in X$ if there exists a sequence $\{\zeta_n : n \in \mathbb{N}\}$ in $\Omega(x)$ such that

$$\lim_{n\to\infty} a(\zeta_n, x) = 0.$$

The family of sets $\Omega = \{\Omega(x) : x \in X\}$ is an **approach family for** $(D; X, \delta)$ if every $\Omega(x) \subset D$ is an approach region to x.

For an approach family Ω, the **shadow of a set** $E \subset D$ is the set

$$\Omega^{\downarrow}(E) = \{x \in X : \Omega(x) \cap E \neq \emptyset\},$$

and, in particular, the **shadow of a point** $\zeta \in D$ is the set

$$\Omega^{\downarrow}(\zeta) = \{x \in X : \zeta \in \Omega(x)\}.$$

Definition 3.3.10 The approach family $\Gamma = \{\Gamma(x) : x \in X\}$ for $(D; X, \delta)$ is **supernatural** if the following conditions are satisfied:

(i) For all $\zeta \in D$, $\Gamma^{\downarrow}(\zeta)$ is open.

(ii) There exist two constants $0 < L_1 \leq L_2$ such that:

(a) For every $\zeta \in D$, there is a ball $B(x, r) = B(x(\Gamma, \zeta), r(\Gamma, \zeta))$ satisfying

$$B(x, L_1 r) \subset \Gamma^{\downarrow}(\zeta) \subset B(x, L_2 r).$$

(b) Conversely, for every ball $B(x, r)$ in X, there exists $\zeta = \zeta(x, r, \Gamma) \in D$ such that

$$B(x, L_1 r) \subset \Gamma^{\downarrow}(\zeta) \subset B(x, L_2 r).$$

(iii) If $\lim_{n\to\infty} a(\zeta_n, x) = 0$, then

(a) $\lim_{n\to\infty} \operatorname{diam}(\Gamma^{\downarrow}(\zeta_n)) = 0$ and

(b) $\lim_{n\to\infty} \sup_{y\in\Gamma^{\downarrow}(\zeta_n)} a(\zeta_n, y) = 0.$

Examples 3.3.11

(i) If (X, μ, d) is a space of homogeneous type, then $D = X_+$ is a space of approach to X for the induced metric in X_+. The set of cones of a given width $\theta > 0$

$$\Gamma_\theta(x) = \{(y, t) \in X_+ : d(x, y) < \theta t\}, \quad x \in X,$$

is a supernatural approach family, because $(\Gamma_\theta)^{\downarrow}(y, t) = B(y, \theta t)$.

(ii) The tree T_o is a space of approach to its boundary ∂T_o for the Euclidean distance d_e, and again the set of cones of a certain width $0 < \theta \in \mathbb{Z}$

$$\Gamma_\theta(\omega) = \left\{x \in T_o : d_e(\omega, x) \leq q^{\theta-|x|}\right\}, \quad \omega \in \partial T_o,$$

is a supernatural approach family, since $(\Gamma_\theta)^{\downarrow}(x) = I(x(|x| - \theta))$, where we recall that $[o, x] = \{o = x(0), x(1), \ldots, x(|x|) = x\}$ is the geodesic from o to x, and we use the convention $x(|x| - \theta) = o$ if $\theta > |x|$.

(iii) The unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ is a space of approach to its boundary $\partial D = \{z \in \mathbb{C} : |z| = 1\}$ for the Euclidean distance. For $\alpha > 0$ and $\omega \in \partial D$, the sets

$$\Gamma_\alpha(\omega) = \{z \in D : |z - \omega| < (1 + \alpha)(1 - |z|)\}$$

form a supernatural approach family.

In the sequel, we fix the notation Γ for a chosen supernatural approach family in the space of approach D to (X, δ), where (X, μ, δ) is a ns-space of homogeneous type (see Definition 3.2.2).

Given an approach family Ω for $(D; X, \delta)$, the related **maximal operator** is defined by

$$M_\Omega f(x) = \sup_{\zeta \in \Omega(x)} \frac{1}{\mu(\Gamma^\downarrow(\zeta))} \int_{\Gamma^\downarrow(\zeta)} |f(y)|\, d\mu(y), \tag{3.13}$$

for a measurable f. If we choose $\Omega = \Gamma$, the operator M_Γ is easily seen to be pointwise equivalent to the Hardy-Littlewood maximal operator M since μ is doubling, in view of the definition of a supernatural family Γ.

Remark 3.3.12 For the particular setting $D = X_+$ for a space of homogeneous type X, this operator and the operator in Definition 3.2.11 are pointwise equivalent if Γ is a supernatural family, but they are not equivalent in general if Γ is not supernatural.

We give, for this abstract setting, some known definitions.

Definition 3.3.13

(i) The **tent** of a ball B is the set

$$T(B) = \left\{\zeta \in D : \Gamma^\downarrow(\zeta) \subset B\right\}.$$

(ii) For a measure σ in X and an approach family Ω for $(D; X, \delta)$, the **outer measure** σ_Ω in D is given by

$$\sigma_\Omega(E) = \sigma(\Omega^\downarrow(E)) = \sigma(\{x \in X : \Omega(x) \cap E \neq \emptyset\}),$$

for a measurable set E.

(iii) For $p \geq 1$ and an approach family Ω for $(D; X, \delta)$, the class of A_p^Ω-weights is the set of weights u defined in X such that $M_\Omega : L^p(u) \longrightarrow L^{p,\infty}(u)$ is bounded, and the A_p^Ω-constant $||u||_{A_p^\Omega}$ is the norm of M_Ω.

(iv) A weight u is in the $W(\Omega)$-class if there is a constant $C > 0$ such that

$$U_\Omega(T(B)) \leq CU(B),$$

for all balls B in X. Let $||u||_{W(\Omega)}$ be the infimum of the constants satisfying this inequality for all balls.

The following theorem is essentially proved in [DiB] (Theorem 2.14, page 37):

Theorem 3.3.14 *Let (X, μ, δ) be a ns-space of homogeneous type. Consider an approach family Ω for $(D; X, \delta)$, with a fixed supernatural approach family Γ. Let ρ and ν be two nonnegative measures on X. If $M : L^p(\nu) \longrightarrow L^{p,\infty}(\nu)$ is bounded for some $p \geq 1$, then the following conditions are equivalent:*

(i) There exists $C > 0$ such that

$$\rho(\{x \in X : M_\Omega f(x) > \lambda\}) \leq C\, \nu(\{x \in X : M_\Gamma f(x) > \lambda\}),$$

for all $\lambda > 0$ and measurable f.

(ii) $M_\Omega : L^p(\nu) \longrightarrow L^{p,\infty}(\rho)$ is bounded.

(iii) There exists $C > 0$ such that $\rho_\Omega(T(B)) \leq C\, \nu(B)$ for all ball B in X.

In our new abstract context, the notion of a region full in the vertical direction has no meaning. We now define its natural substitute. For $x \in X$, an approach region $\Omega(x)$ is Γ**-complete** if for all $\zeta \in \Omega(x)$, we also have $\xi \in \Omega(x)$ whenever $\Gamma^{\downarrow}(\zeta) \subset \Gamma^{\downarrow}(\xi)$. It is proved in [DiB] that we can assume, without loss of generality, that the region $\Omega(x)$ is Γ-complete for every $x \in X$, that is, the family Ω is Γ-complete. This notion appears for the first time in [NS] in the Euclidean context $D = \mathbb{R}^{n+1}_+$ and $X = \mathbb{R}^n$.

Theorem 3.3.15 *Let Ω be a Γ-complete family of approach regions for $(D; X, \delta)$. For $1 \leq p < \infty$, we have*

$$A_p^\Omega = A_p \cap W(\Omega),$$

and there exists $C > 0$ such that

$$||u||_{A_p} \leq C||u||_{A_p^\Omega},$$

for all u in A_p^Ω.

Proof. The fact that M_Γ and M are equivalent and Theorem 3.3.14 with $d\rho(z) = d\nu(z) = u(z)\, d\mu(z)$ say that $A_p \cap W(\Omega) \subset A_p^\Omega$ and $A_p^\Omega \subset W(\Omega)$. Now, let us see that if Ω is a family of Γ-complete approach regions, there exists a constant $C > 0$ such that

$$M_{\text{cen}} f(x) \leq C\, M_\Omega f(x),$$

for all $x \in X$ and all measurable f, where M_{cen} is the centered Hardy-Littlewood maximal operator. Therefore, we will have that $A_p^\Omega \subset A_p$, and $||u||_{A_p} \leq C||u||_{A_p^\Omega}$.

Fix $x \in X$ and $r > 0$. Since Γ is supernatural, there exists $\zeta = \zeta(x, r) \in D$ such that

$$B(x, L_1 r) \subset \Gamma^{\downarrow}(\zeta) \subset B(x, L_2 r). \tag{3.14}$$

Suppose $\{\zeta_n : n \geq 0\} \subset \Omega(x)$, with $\lim_{n\to\infty} a(\zeta_n, x) = 0$. By definition, $x \in \Gamma^{\downarrow}(\zeta_n)$ for all $n \geq 0$ and $\lim_{n\to\infty} \operatorname{diam}(\Gamma^{\downarrow}(\zeta_n)) = 0$. Thus, there exists n_0 such that $\Gamma^{\downarrow}(\zeta_n) \subset B(x, L_1 r)$ for all $n \geq n_0$, and hence, by (3.14), $\Gamma^{\downarrow}(\zeta_n) \subset \Gamma^{\downarrow}(\zeta)$. Since $\Omega(x)$ is Γ-complete, we have that $\zeta \in \Omega(x)$. Consequently,

$$M_{\Omega} f(x) \geq \frac{1}{\mu(\Gamma^{\downarrow}(\zeta))} \int_{\Gamma^{\downarrow}(\zeta)} |f(y)|\, d\mu(y) \geq \frac{C}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\, d\mu(y),$$

where the last inequality follows from (3.14) and the doubling property of μ. Hence, taking the supremum over $r > 0$, we obtain

$$M_{\Omega} f(x) \geq C\, M_{\mathrm{cen}} f(x),$$

with C independent of x.

$\square$

Suppose that (X, μ, d) is a space of homogeneous type satisfying condition (3.4). We proved in Theorem 3.2.21 of the previous section the existence of a non-symmetric quasi-distance δ with a good invariance property, namely, the measure of a ball is comparable to its radius, uniformly on the center of the ball. Moreover, (X, μ, δ) is a ns-space of homogeneous type.

Theorem 3.3.16 *If (X, μ, d) is a space of homogeneous type satisfying property (3.4), then (X, μ, δ) is a ns-space of homogeneous type such that:*

(i) If D is a space of approach to (X, d) for the approach function a, then it is also a space of approach to (X, δ) for the same approach function a.

(ii) If Γ is a supernatural family for $(D; X, d)$, then it is also a supernatural family for $(D; X, \delta)$.

(iii) The A_p and A_p^{Ω} class of weights do not change with the change of quasi-distance.

Proof. Condition (i) holds simply because by definition we have that

$$\lim_{n\to\infty} d(x_n, y_n) = 0 \Longleftrightarrow \lim_{n\to\infty} \delta(x_n, y_n) = 0,$$

for all sequences $\{x_n : n \geq 0\}$ and $\{y_n : n \geq 0\}$ in X. (ii) is due to (i) of Theorem 3.2.21 and to condition (3.9). Condition (iii) follows from (i) of Corollary 3.2.25 and from the fact that the operator M_Ω defined in (3.13) is invariant under the change of quasi-distance.

$\square$

As a consequence, we assume that the ambient ns-space of homogeneous type (X, μ, δ) satisfies that $\mu(B(x,r)) \sim r$, for all $x \in X$.

Definition 3.3.17 An **approach system of dilates of** Γ is the collection $\{\Gamma_\theta : \theta > 0\}$ of approach regions Γ_θ satisfying:

(i) Γ_θ is a supernatural approach family for all $\theta > 0$.

(ii) $\Gamma_{\theta_0} = \Gamma$ for some $\theta_0 > 0$.

(iii) $\Gamma_\theta(x) \subset \Gamma_\vartheta(x)$ whenever $\theta \leq \vartheta$, for all $x \in X$.

(iv) There exist two positive increasing functions $L_1(\theta) \leq L_2(\theta)$ satisfying that

$$\lim_{\theta \to 0} L_i(\theta) = 0 \quad \text{and} \quad \lim_{\theta \to \infty} L_i(\theta) = \infty,$$

for $i = 1, 2$, such that for all $\theta > 0$:

(a) For every $\zeta \in D$, there is a ball $B(x,r) = B(x(\Gamma_\theta, \zeta), r(\Gamma_\theta, \zeta))$ satisfying

$$B(x, L_1(\theta)r) \subset (\Gamma_\theta)^\downarrow(\zeta) \subset B(x, L_2(\theta)r),$$

(b) and conversely, for every ball $B(x,r)$ in X, there exists $\zeta = \zeta(x, r, \Gamma_\theta) \in D$ such that

$$B(x, L_1(\theta)r) \subset (\Gamma_\theta)^\downarrow(\zeta) \subset B(x, L_2(\theta)r).$$

We say that a family of approach regions Ω is **regular** if there is a constant $C > 0$ such that for all $\zeta \in D$ the next condition is satisfied:

$$\forall y \in \Omega^\downarrow(\zeta),\ \exists\, y^* \in X \text{ with } \delta(y,x) = \delta(y^*,x) \text{ such that } B(y^*,t) \subset \Omega^\downarrow(T_\Gamma(B(x,Cr))),$$

where we recall that $(x,r) = (x(\Gamma,\zeta), r(\Gamma,\zeta))$ comes from the definition of Γ being a supernatural family.

Our last result is the next theorem.

Theorem 3.3.18 *Let (X, μ, δ) be a ns-space of homogeneous type, and assume $\Omega \subset D$ to be a regular family of approach regions. Then the following conditions are equivalent:*

(i) *There exists $C > 0$ and $\theta > 0$ such that $M_\Omega f(x) \le C\, M_{\Gamma_\theta} f(x)$, for all x in X and all measurable functions f.*

(ii) *$A_p^\Omega = A_p$ for all $1 \le p < \infty$, with equivalent constants.*

(iii) *There is $p \ge 1$ such that $A_p = A_p^\Omega$, with equivalent constants.*

(iv) *There exists $0 < \gamma \le 1$ such that the family of weights $\{\delta(x, .)^\gamma : x \in X\}$ is in $W(\Omega)$ uniformly in x.*

(v) *There exists $\theta > 0$ such that $\Omega(x) \subset \Gamma_\theta(x)$ for all x in X.*

Proof. In view of the proof of the preceding results, the proof of equivalency between (i), (ii) and (iii) is standard, and the completed proof follows once we get proved that (iv) implies (v). We observe that (v) is equivalent to $\Omega^\downarrow(\zeta) \subset (\Gamma_\theta)^\downarrow(\zeta)$ for all $\zeta \in D$.

Suppose the family of weights is uniformly in $W(\Omega)$. Fix $\zeta \in D$. We know that there exist $x \in X$ and $r > 0$ such that

$$B(x, L_1(\theta)r) \subset (\Gamma_\theta)^\downarrow(\zeta) \subset B(x, L_2(\theta)r),$$

for all $\theta > 0$. Take $y \in \Omega^\downarrow(\zeta)$. Then, using the assumption of the regularity on Ω, we have:

$$\begin{aligned}
\delta(x,y)^\gamma &= \delta(y^*, x)^\gamma \\
&\le D^\gamma\, \delta(x, y^*)^\gamma \\
&= \frac{D^\gamma}{\mu(B(y^*, r))} \int_{B(y^*, r)} \delta(x, y^*)^\gamma d\mu(z) \\
&\le \frac{(AD)^\gamma}{\mu(B(y^*, r))} \int_{B(y^*, r)} (\delta(x, z) + \delta(y^*, z))^\gamma d\mu(z) \\
&\le (AD)^\gamma \left(\frac{1}{\mu(B(y^*, r))} \int_{\Omega^\downarrow(T_\Gamma(B(x, Cr)))} \delta(x, z)^\gamma d\mu(z) + r^\gamma \right).
\end{aligned}$$

Set $u_x(z) = \delta(x, z)^\gamma$. Using the hypothesis on the family $\{u_x : x \in X\}$, and the fact that $\mu(B(y^*,r))$ is comparable to the radius r, we have:

$$\begin{aligned}
\delta(y,x)^\gamma &\leq (AD)^\gamma\left(\frac{U_x(\Omega^\downarrow(T_\Gamma(B(x,Cr))))}{\mu(B(y^*,r))}+r^\gamma\right)\\
&= (AD)^\gamma\left(\frac{(U_x)_\Omega(T_\Gamma(B(x,Cr)))}{\mu(B(y^*,r))}+r^\gamma\right)\\
&\leq C(AD)^\gamma\left(\frac{U_x(B(x,Cr))}{r}+r^\gamma\right). \qquad (3.15)
\end{aligned}$$

Now, we saw in Theorem 3.2.30 that $U_x(B(x,t))$ is comparable to $t^{\gamma+1}$ for all $t>0$ uniformly in x, and thus returning to (3.15), we get that for all $y\in\Omega^\downarrow(\zeta)$ and for all $\theta>0$ we have

$$\delta(y,x(\Gamma_\theta,\zeta))\leq C^{\frac{1}{\gamma}}\,A\,D\,r(\Gamma_\theta,\zeta),$$

that is,

$$\Omega^\downarrow(\zeta)\subset B(x(\Gamma_\theta,\zeta),ADr(\Gamma_\theta,\zeta)).$$

To complete the proof, choose θ such that $C^{\frac{1}{\gamma}}\,A\,D\leq L_1(\theta)$, and thus $\Omega^\downarrow(\zeta)\subset(\Gamma_\theta)^\downarrow(\zeta)$.

□

3.4 Back to Euclidean spaces: two applications

The ideas and techniques used in the previous sections can be applied to extend some known results and to simplify the proofs. We consider two cases.

3.4.1 Singular integral operators

We work in $\mathbb{R}^n$ equipped with the Lebesgue measure m and the Euclidean distance, and we think of $\mathbb{R}^{n+1}_+$ as a space of approach to $\mathbb{R}^n$. We denote the the unit sphere by S^{n-1} and $|E|=m(E)$ is the Lebesgue measure of a set E. We will easily proof an extended version of the results obtained in [FJR]. Moreover, we will not use the group structure of $\mathbb{R}^n$ to construct an approach family by translating a fixed one.

Let $K:\mathbb{R}^n\longrightarrow\mathbb{R}$ be a Calderón-Zygmund kernel, that is, $K(x)=\omega(x)/|x|^n$ such that:

(i) $\omega\in L^\infty(S^{n-1})$ is homogeneous of degree 0.

(ii) $\displaystyle\int_{S^{n-1}}\omega(x)\,dx=0.$

(iii) There exists a constant $C > 0$ such that $|K(x+y) - K(x)| \leq C \dfrac{|y|}{|x|^{n+1}}$, if $|x| > 2\,|y|$.

The Riesz kernel $K_j(x) = \dfrac{\omega_j(x)}{|x|^n}$, where $\omega_j(x) = \dfrac{x_j}{|x_j|}$, is an example of a Calderón-Zygmund kernel.

We consider the truncated singular integral operator defined by

$$T_\varepsilon f(x) = \int_{|y|>\varepsilon} \frac{\omega(y)}{|y|^n} f(x-y)\,dy = (K_\varepsilon * f)(x),$$

where $K_\varepsilon(x) = K(x) \cdot \chi_{\{|y|>\varepsilon\}}(x)$.

The maximal operator related to this family of truncated singular integral operators is

$$T^* f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)| = \sup_{\varepsilon>0} |(K_\varepsilon * f)(x)|\,.$$

It is well-known that this operator satisfies the boundedness $T^* : L^p(m) \longrightarrow L^p(m)$ if $p > 1$ and $T^* : L^1(m) \longrightarrow L^{1,\infty}(m)$ (see Theorem 4 in page 42 of [St1]).

Let $\Omega = \{\Omega(x) : x \in \mathbb{R}^n\}$ be an approach family in $\mathbb{R}^{n+1}_+$. For such a family, we define the maximal operator for a measurable function f by

$$N_\Omega f(x) = \sup_{(y,t)\in\Omega(x)} |(K_t * f)(y)|\,.$$

In the particular case of cones $\Gamma(x) = \left\{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\right\}$, the next lemma is proved in [FJR].

Lemma 3.4.1 *For a Calderón-Zygmund kernel K, there exists a constant $C > 0$ depending on the dimension n such that:*

$$N_\Gamma f(x) = \sup_{(y,t)\in\Gamma(x)} |(K_t * f)(y)| \leq T^* f(x) + C\, Mf(x),$$

for all measurable f. Thus, we have the estimates $N_\Gamma : L^p(m) \longrightarrow L^p(m)$ if $p > 1$ and $N_\Gamma : L^1(m) \longrightarrow L^{1,\infty}(m)$

We recall some concepts defined in the previous sections. The set

$$\Omega^{\downarrow}(y,t) = \{x \in \mathbb{R}^n : (y,t) \in \Omega(x)\}$$

is the **shadow** under (y,t) by the approach family Ω. We define the $\Gamma-$**completion** of Ω by

$$\widehat{\Omega}(x) = \left\{(y,t) \in \mathbb{R}^{n+1}_+ : \exists\ (z,s) \in \Omega(x) \text{ such that } B(z,s) \subset B(y,t)\right\}.$$

The approach family Ω is $\Gamma-$**complete** (it is also said that it satisfies the so-called *cone condition*) if and only if $\widehat{\Omega}(x) = \Omega(x)$ for all $x \in \mathbb{R}^n$. For a measure σ in $\mathbb{R}^n$ and an approach family Ω, we define the (outer) measure σ_Ω in $\mathbb{R}^{n+1}_+$ by

$$\sigma_\Omega(E) = \sigma(\{x \in \mathbb{R}^n : \Omega(x) \cap E \neq \emptyset\}),$$

for a measurable set E.

The **tent** under a point $(y,t) \in \mathbb{R}^{n+1}_+$ is the set

$$T(y,t) = \{(z,s) : B(z,s) \subset B(y,t)\}.$$

A measure ρ defined in $\mathbb{R}^{n+1}_+$ is a **Carleson measure** if there exists a constant $C > 0$ such that

$$\rho(T(y,t)) \leq C\ |B(y,t)|,$$

for all $(y,t) \in \mathbb{R}^{n+1}_+$. By Proposition 3.2.9 this is equivalent to the existence of a constant $C > 0$ such that

$$\rho(T(O)) \leq C\ |O|, \tag{3.16}$$

for all open sets $O \subset \mathbb{R}^n$, where $T(O) = \{(z,s) : B(z,s) \subset O\}$.

For the special case of the Riesz kernel, we have:

Theorem 3.4.2 *Let Ω be an approach family, and consider the Riesz kernel $K_j(x) = \omega_j(x)/|x|^n$ where $\omega_j(x) = x_j/|x_j|$ for $1 \leq j \leq n$. For a measure ρ defined in $\mathbb{R}^n$, the following conditions are equivalent:*

(i) There exists $C > 0$ such that

$$\rho(\{x \in \mathbb{R}^n : N_\Omega f(x) > \lambda\}) \leq C\ |\{x \in \mathbb{R}^n : N_\Gamma f(x) > \lambda\}|,$$

for all $\lambda > 0$ and measurable f.

(ii) $N_\Omega : L^p(m) \longrightarrow L^{p,\infty}(\rho)$ is bounded for $p \geq 1$.

(iii) There exists $C > 0$ such that $\rho(\widehat{\Omega}^{\downarrow}(x,t)) \leq C\ |B(x,t)|$ for all $(x,t) \in \mathbb{R}^{n+1}_+$.

(iv) $\rho_{\widehat{\Omega}}$ is a Carleson measure.

(v) ρ_Ω is a Carleson measure.

Proof. That (i) implies (ii) follows trivially by using Lema 3.4.1. Let us see that (iii) implies (iv). Take $x \in \mathbb{R}^n$ so that $\widehat{\Omega}(x) \cap T(y,t)) \neq \phi$. There is $(z,s) \in \widehat{\Omega}(x)$ with $B(z,s) \subset B(y,t)$. Since $\widehat{\Omega}(x)$ is $\Gamma-$complete, $(y,t) \in \widehat{\Omega}(x)$. Therefore $x \in \widehat{\Omega}^{\downarrow}(y,t)$, and hence

$$\left\{ x \in \mathbb{R}^n : \widehat{\Omega}(x) \cap T(y,t) \neq \phi \right\} \subset \widehat{\Omega}^{\downarrow}(y,t).$$

So, using the definition of $\rho_{\widehat{\Omega}}$, we have:

$$\rho_{\widehat{\Omega}}(T(y,t)) \leq \rho(\widehat{\Omega}^{\downarrow}(y,t)) \leq C \left| B(y,t) \right|.$$

That (iv) and (v) are equivalent conditions follows once we have proved that

$$\rho_{\Omega}(T(y,t)) = \rho_{\widehat{\Omega}}(T(y,t)).$$

That $\rho_{\Omega}(T(y,t)) \leq \rho_{\widehat{\Omega}}(T(y,t))$ is due to $\Omega(x) \subset \widehat{\Omega}(x)$ for all $x \in \mathbb{R}^n$. Now, if $\widehat{\Omega}(x) \cap T(y,t) \neq \emptyset$, there exists $(z,s) \in \widehat{\Omega}(x)$ with $B(z,s) \subset B(y,t)$. By definition of the $\Gamma-$completion, there exists $(u,r) \in \Omega(x)$ such that $B(u,r) \subset B(z,s)$. Thus, $(u,r) \in \Omega(x)$ and $B(u,r) \subset B(y,t)$, that is, $\Omega(x) \cap T(y,t) \neq \emptyset$, which leads to $\rho_{\widehat{\Omega}}(T(y,t)) \leq \rho_{\Omega}(T(y,t))$. Now, suppose that ρ_{Ω} is a Carleson measure. Observe that

$$\{(y,t) : |(K_t * f)(y)| > \lambda\} \subset T(O),$$

where $O = \{x \in \mathbb{R}^n : N_{\Gamma}f(x) > \lambda\}$, for all functions f. Then, applying (3.16), we obtain:

$$\begin{aligned}
\rho(\{x \in \mathbb{R}^n : N_{\Omega}f(x) > \lambda\}) &= \rho_{\Omega}(\{(y,t) : |(K_t * f)(y)| > \lambda\}) \\
&\leq \rho_{\Omega}(T(O)) \\
&\leq C \, |O| \\
&= C \left| \{x \in \mathbb{R}^n : N_{\Gamma}f(x) > \lambda\} \right|.
\end{aligned}$$

Finally, let us see that (ii) implies (iii). Without loss of generality, we proof it in the case of $\omega_1(x) = x_1/|x_1|$. Take $x \in \widehat{\Omega}^{\downarrow}(y,t)$. By the definition of $\widehat{\Omega}$ being the $\Gamma-$completion of Ω, there exists (z,s) such that $x \in \Omega^{\downarrow}(z,s)$ and $B(z,s) \subset B(y,t)$. Let us write $y = (y_1, y') \in \mathbb{R}^n$ with $y_1 \in \mathbb{R}$ and $y' \in \mathbb{R}^{n-1}$. We consider $B' = B(y',t) \subset \mathbb{R}^{n-1}$, and $A(y,t) = (y_1 - 3t, y_1) \times B'$. We observe that $|A(y,t)| = 3\,|B(y,t)|$. Set $f(w) = \chi_{A(y,t)}(w)$. We claim that for this choice, there exists a positive constant C_n depending on the dimension n such that $|(K_s * f)(z)| \geq C_n$, and thus $N_{\Omega}f(x) > C_n/2$. As a consequence, we have

$$\widehat{\Omega}^{\downarrow}(y,t) \subset \{x \in \mathbb{R}^n : N_{\Omega}f(x) > C_n/2\},$$

and by hypothesis

$$\begin{aligned}\rho(\widehat{\Omega}^{\downarrow}(y,t)) &\leq \mu(\{x \in \mathbb{R}^n : N_\Omega f(x) > C_n/2\}) \\ &\leq C_n \, \|f\|_{L^p(m)} \\ &= C_n \, |A(y,t)| \\ &= C_n \, |B(y,t)| ,\end{aligned}$$

which is (iii). Let us proof the claim:

$$\begin{aligned}|(K_s * f)(z)| &= \left| \int_{\{y_1-3t<z_1-\xi_1<y_1,\ |z_i-\xi_i-y_i|<t,\ i\neq 1\}} \frac{\omega_1(\xi)}{|\xi|^n} \chi_{\{|\xi|\}>s}(\xi)\, d\xi \right| \\ &= \left| \int_{\{z_1-y_1<\xi_1<3t+z_1-y_1,\ |z_i-\xi_i-y_i|<t,\ i\neq 1\}} \frac{\omega_1(\xi)}{|\xi|^n} \chi_{\{|\xi|>s\}}(\xi)\, d\xi \right| \\ &= \left| \int_{\{z_1-y_1<\xi_1<0,\ |z_i-\xi_i-y_i|<t,\ i\neq 1\}} \frac{-1}{|\xi|^n} \chi_{\{|\xi|>s\}}(\xi)\, d\xi \right. \\ &\qquad \left. + \int_{\{0<\xi_1<3t+z_1-y_1,\ |z_i-\xi_i-y_i|<t,\ i\neq 1\}} \frac{1}{|\xi|^n} \chi_{\{|\xi|>s\}}(\xi)\, d\xi \right| \\ &= \int_{\{|z_1-y_1|<\xi_1<3t+|z_1-y_1|,\ |z_i-\xi_i-y_i|<t,\ i\neq 1\}} \frac{1}{|\xi|^n} \chi_{\{|\xi|>s\}}(\xi)\, d\xi,\end{aligned}$$

where the last equality follows by symmetry. Now, since $B(z,s) \subset B(y,t)$ we have that

$$\begin{aligned}|z_1 - y_1| &< |z-y| < t-s < t \\ 3t - |z_1 - y_1| &> 2t,\end{aligned}$$

and therefore

$$\begin{aligned}|(K_s * f)(z)| &\geq \int_{\{t<\xi_1<2t,\ |z_i-\xi_i-y_i|<t,\ i\neq 1\}} \frac{1}{|\xi|^n} \chi_{\{|\xi|>s\}}(\xi)\, d\xi \\ &= \int_{\{t<\xi_1<2t|z_i-\xi_i-y_i|<t,\ i\neq 1\}} \frac{1}{|\xi|^n}\, d\xi,\end{aligned}$$

where the equality is due to the fact that if $t < \xi_1$, then $s < t < |\xi|$. We observe that $\xi_1 < 2t$ but also $|\xi_i| \leq |\xi - z_i + y_i| + |y_i - z_i| < 2t$ if $i \neq 1$, and thus $|\xi| < \sqrt{n}2t$, that is,

$$\frac{1}{|\xi|^n} > \frac{C_n}{t^n}.$$

Using this estimate, we finally have

$$|(K_s * f)(z)| \ \geq\ \frac{C_n}{t^n} \left|\{\xi \in \mathbb{R}^n : t < \xi_1 < 2t,\ |z_i - \xi_i - y_1| < t,\ i \neq 1\}\right| = C_n$$

□

An easy consequence of Theorem 3.4.2 is that, without loss of generality, we can always assume our approach family Ω to be $\Gamma-$complete :

Corollary 3.4.3 *Let Ω be an approach family, and consider the Riesz kernel $K_j(x) = \omega_j(x)/|x|^n$ where $\omega_j(x) = x_j/|x_j|$ for $1 \leq j \leq n$. For a measure ρ defined in $\mathbb{R}^n$, the following conditions are equivalent:*

(i) $N_\Omega : L^p(m) \longrightarrow L^{p,\infty}(\rho)$ *is bounded for* $p \geq 1$.

(ii) $N_{\widehat{\Omega}} : L^p(m) \longrightarrow L^{p,\infty}(\rho)$ *is bounded for* $p \geq 1$.

(iii) There exists $C > 0$ *such that* $\rho(\widehat{\Omega}^{\downarrow}(x,t)) \leq C\,|B(x,t)|$ *for all* $(x,t) \in \mathbb{R}^{n+1}_+$.

In Theorem 3.4.2, all the implications remain true for a general Calderón-Zygmund kernel, maybe except implication $(ii) \Longrightarrow (iii)$, and with the same proof. In particular, we have:

Theorem 3.4.4 *Let Ω be an approach family, and K a Calderón-Zygmund kernel. If, for a measure ρ in $\mathbb{R}^n$, there exists $C > 0$ such that $\rho(\widehat{\Omega}^{\downarrow}(x,t)) \leq C\,|B(x,t)|$ for all $(x,t) \in \mathbb{R}^{n+1}_+$, then $N_\Omega : L^p(m) \longrightarrow L^{p,\infty}(\rho)$ is bounded for $p \geq 1$.*

3.4.2 Potential spaces

We extend the result of [RS] relaying potential spaces and approach regions. As before, we do not need to consider translated regions of an initial approach region.

For $1 \leq p < \infty$, let k be a positive radially decreasing function in $L^1(m)$. We define

$$L^p_k(m) = \{f :\ f = F * k,\ \text{for some } F \in L^p(m)\},$$

endowed with the quotient norm $\|f\|_{L^p_k(m)} = \inf\left\{\|F\|_{L^p(m)} : f = F * k\right\}$. We denote $V_t(x) = (P_t * k)(x)$ the harmonic extension of k, where P is the Poisson kernel and $P_t(y) = t^{-n}P(t^{-1}y)$, that is

$$P_t(y) = \frac{c_n t}{(|x|^2 + t^2)^{(n+1)/2}},$$

for $(y,t) \in \mathbb{R}^{n+1}_+$, where $c_n = \Gamma(n + 1/2)$. We set $r(t) = \|V_t\|_{p'}^{-p/n}$.

For an approach family in $\mathbb{R}^{n+1}_+$, we consider the operator

$$N_\Omega f(x) = \sup_{(y,t)\in\Omega(x)} |(P_t * f)(y)|,$$

for a measurable f.

The **r-cones** are the sets $\Gamma_r(x) = \big\{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < r(t)\big\}$. The following lemma is proved in [NRS].

Lemma 3.4.5 *For a radially decreasing function $k \in L^1(m) \setminus L^{p'}(m)$, there exists a positive constant C such that*

$$N_{\Gamma_r}(F * k)(x) \le C\, M_p F(x),$$

for all $x \in \mathbb{R}^n$, where $M_pF(x) = \sup_{B(y,t)\ni x} \left(\frac{1}{|B(y,t)|}\int_{B(y,t)} |F(z)|^p\, dz\right)^{1/p}$. As a consequence $N_{\Gamma_r} : L^p_k(m) \longrightarrow L^{p,\infty}(m)$ is bounded.

The Γ_r**-completion** of an approach family Ω is the approach family

$$\Omega_r(x) = \{(y,t) :\ \exists\ (z,s) \in \Omega(x) \text{ such that } B(z, r(s)) \subset B(y, r(t))\}\,.$$

An approach family Ω is Γ_r**-complete** if $\Omega(x) = \Omega_r(x)$ for all $x \in \mathbb{R}^n$.

The **r-tents** are the sets $T_r(y,t) = \big\{(z,s) \in \mathbb{R}^{n+1}_+ : B(z,r(s)) \subset B(y,r(t))\big\}$. A measure ρ defined in $\mathbb{R}^{n+1}_+$ is a **r-Carleson measure** if there exists a constant $C > 0$ such that

$$\rho(T_r(y,t)) \le C\ |B(y,t)|\,,$$

for all $(y,t) \in \mathbb{R}^{n+1}_+$. It is proved in [RS] that this is equivalent to the existence of a constant $C > 0$ such that

$$\rho(T_r(O)) \le C\ |O|\,, \tag{3.17}$$

for all open sets $O \subset \mathbb{R}^n$, where $T_r(O) = \big\{(y,t) \in \mathbb{R}^{n+1}_+ : B(y, r(t)) \subset O\big\}$.

Theorem 3.4.6 *Let Ω be an Γ_r-complete approach family, and consider a radially decreasing function $k \in L^1(m) \setminus L^{p'}(m)$ and $r(t) = \|V_t\|_{p'}^{-p/n}$. For a measure ρ defined in $\mathbb{R}^n$, the following conditions are equivalent:*

(i) There exists $C > 0$ such that

$$\rho(\{x \in \mathbb{R}^n : N_\Omega f(x) > \lambda\}) \le C\ |\{x \in \mathbb{R}^n : N_{\Gamma_r} f(x) > \lambda\}|\,,$$

for all $\lambda > 0$ and measurable f.

(ii) $N_\Omega : L^p_k(m) \longrightarrow L^{p,\infty}(\rho)$ is bounded for $p \ge 1$.

(iii) There exists $C > 0$ such that $\rho(\Omega^{\downarrow}(x,t)) \leq C\ |B(x,t)|$ for all $(x,t) \in \mathbb{R}^{n+1}_+$.

(iv) ρ_Ω is a r-Carleson measure.

Proof. That (i) implies (ii) is Lemma 3.4.5. Let us see that (iii) implies (iv). Take $x \in \mathbb{R}^n$ so that $\Omega_r(x) \cap T_r(y,t)) \neq \phi$. There is $(z,s) \in \Omega_r(x)$ with $B(z, r(s)) \subset B(y, r(t))$. Since $\Omega_r(x)$ is Γ_r-complete, $(y,t) \in \Omega_r(x)$. Therefore $x \in \Omega_r^{\downarrow}(y,t)$, and hence

$$\{x \in \mathbb{R}^n : \Omega_r(x) \cap T_r(y,t) \neq \phi\} \subset \Omega_r^{\downarrow}(y,t).$$

So, using the definition of ρ_{Ω_r}, we have:

$$\rho_{\Omega_r}(T_r(y,t)) \leq \rho(\Omega_r^{\downarrow}(y,t)) \leq C\ |B(y,t)|\,.$$

That (iv) implies (v) is easy because $\rho_\Omega(E) \leq \rho_{\Omega_r}(E)$, since $\Omega(x) \subset \Omega_r(x)$ for all $x \in \mathbb{R}^n$. Now, suppose that (ρ_Ω, m) is a Carleson pair. Observe that

$$\left\{(y,t) \in \mathbb{R}^{n+1}_+ : |(P_t * f)(y)| > \lambda\right\} \subset T(O),$$

where $O = \{x \in \mathbb{R}^n : N_{\Gamma_r} f(x) > \lambda\}$, for all functions f. Then, applying (3.17), we obtain:

$$\begin{aligned}
\rho(\{x \in \mathbb{R}^n : N_\Omega f(x) > \lambda\}) &= \rho_\Omega(\left\{(y,t) \in \mathbb{R}^{n+1}_+ : |(P_t * f)(y)| > \lambda\right\}) \\
&\leq \rho_\Omega(T(O)) \\
&\leq C\ |O| \\
&= C\ |\{x \in \mathbb{R}^n : N_{\Gamma_r} f(x) > \lambda\}|\,.
\end{aligned}$$

Finally, let us proof that (ii) implies (iii). We fix $(y,t) \in \mathbb{R}^{n+1}_+$ and take $x \in \Omega^{\downarrow}(y,t)$. Since the application

$$F \in L^p(m) \mapsto\ H_y(F)(.) = F(y + .) \in L^p(m)$$

is an isomorphic and isometric map of L^p for all $y \in \mathbb{R}^n$, we have that

$$\|V_t\|_{p'} = \sup_{\|F\|_p = 1} \left|\int_{\mathbb{R}^n} V_t(w) H_y(F)(-w)\, dw\right| = \sup_{\|F\|_p = 1} |(V_t * F)(y)|\,.$$

We choose $F \in L^p$ with $\|F\|_p = 1$ satisfying that $|V_t * F(y)| \geq \frac{\|V_t\|_{p'}}{2}$. Now, we take $f = F * k$, and therefore:

$$N_\Omega f(x) = \sup_{(z,s) \in \Omega(x)} |(V_s * F)(z)| \geq |(V_t * F)(y)| > \frac{\|V_t\|_{p'}}{4}.$$

Then,

$$\Omega^{\downarrow}(y,t) \subset \left\{ x \in \mathbb{R}^n : N_{\Omega} f(x) > \frac{\|V_t\|_{p'}}{4} \right\},$$

and by hypothesis, we obtain:

$$\begin{aligned} \rho(\Omega_r^{\downarrow}(y,t)) &\leq \rho\Big(\Big\{ x \in \mathbb{R}^n : N_{\Omega} f(x) > \|V_t\|_{p'} /4 \Big\} \Big) \\ &\leq C \, \|V_t\|_{p'}^{-p} \, \|f\|_{L^p_\nu}^p \\ &\leq C \, |B(y, r(t))| \, . \end{aligned}$$

$\square$

Bibliography

[A] K.F. Andersen, *Weighted generalized Hardy inequalities for nonincreasing functions*, Can. J. Math. **43** (1991), 1121-1135.

[AC] M. Andersson and H. Carlsson, *Boundary convergence in non-tangential and non-admissible approach regions*, Math. Scand. **70** (1992), 293-301.

[ADiBU] N. Arcozzi, F. Di Biase and R. Urbanke, *Approach regions for trees and the unit disc*, J. Reine Angew. Math. **472** (1996), 157-175.

[AM] M.A. Ariño and B. Muckenhoupt, *Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions*, Trans. Amer. Math. Soc. **320** (1990), 727-735.

[B] A. Baernstein, *A unified approach to symmetrizations*, Symp. Math. **35** (1994), 47-91.

[BP] S. Barza and L.E. Persson, *Weighted multidimensional inequalities for monotone functions*, Math. Bohemica **124** (1999), 325-335.

[BPSo1] S. Barza, L.E. Persson and J. Soria, *Sharp weighted multidimensional integral inequalities for monotone functions*, Math. Nachr. **210** (2000), 43-58.

[BPSo2] S. Barza, L.E. Persson and J. Soria, *Multidimensional rearrangement and Lorentz spaces*, Preprint.

[BPSte] S. Barza, L.E. Persson and V.D. Stepanov, *On weighted multidimensional embeddings for monotone functions*, Math. Scand. **88** (2001), 303-319.

[BS] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure and Applied Mathematics **129**, Academic Press, 1988.

[BCPT] C.A. Berenstein, J.M. Cohen, M.A. Picardello and E.C. Tarabusi, *Integral geometry on trees*, Amer. J. Math. **113** (1991), 441-470.

[BSte] M.S. Braverman and V.D. Stepanov, *On the discrete Hardy inequality*, Bull. London Math. Soc. **26** (1994), 283-287.

[CFPR] A. Cantón, J.L. Fernández, D. Pestana and J.M. Rodríguez, *On harmonic functions on trees*, Potential Anal. **15** (2001), 199-244.

[CGSo] M.J. Carro, A. García del Amo and J. Soria, *Weak-type weights and normable Lorentz spaces*, Proc. Amer. Math. Soc. **124** (1996), 849-857.

[CPSoSte] M.J. Carro, L. Pick, J. Soria and V.D. Stepanov, *On embeddings between classical Lorentz spaces*, Math. Ineq. App. **4** (2001), 397-428.

[CRSo] M.J. Carro, J.A. Raposo and J. Soria, *Recent Developments in the Theory of Lorentz Spaces and Weighted Inequalities*, Submitted.

[CSo1] M.J. Carro and J. Soria, *Weighted Lorentz spaces and the Hardy operator*, J. Funct. Anal. **112** (1993), 480-494.

[CSo2] M.J. Carro and J. Soria, *Boundedness of some integral operators*, Can. J. Math. **45** (1993), 1155-1166.

[CSo3] M.J. Carro and J. Soria, *The Hardy-Littlewood maximal function and weighted Lorentz spaces*, J. London Math. Soc. **55** (1997), 146-158.

[C] P. Cartier, *Fonctions harmoniques sur un arbre*, Symp. Math. **9** (1972), 203-270.

[Ch] M. Christ, *Lectures on Singular Integral Operators*, CBMS vol. **77**, Amer. Math. Soc., 1990.

[CR] R.R. Coifman and R. Rochberg, *Another characterization of BMO*, Proc. Amer. Math. Soc. **79** (1980), 249-254.

[CW] R.R. Coifman and G. Weiss, *Analyse Harmonique Non-commutative sur Certain Espaces Homogènes*, Lecture Notes in Mathematics, vol. **242**, 1971.

[CP] M. Coornaert and A. Papadopoulos, *Positive eigenfunctions of the Laplacian and conformal densities on homogeneous trees*, J. London Math. Soc. **55** (1997), 609-624.

[DiB] F. Di Biase, *Fatou Type Theorems. Maximal Functions and Approach Regions*, Progress in Mathematics **147**, Birkhäuser, 1998.

[EHL] W.D. Evans, D.J. Harris and J. Lang, *The approximation numbers of Hardy-type operators on trees*, Proc. London Math. Soc. **83** (2001), 390-418.

[EHP] W.D. Evans, D.J. Harris and L. Pick, *Weighted Hardy and Poincaré inequalities on trees*, J. London Math. Soc. **52** (1995), 121-136.

[F] P. Fatou, *Series trigonometriques et series de Taylor*, Acta Math. **30** (1906), 335-400.

[FJR] S. Ferrando, R.L. Jones and K. Reinhold, *On approach regions for the conjugate Poisson integral and singular integrals*, Studia Math. **120** (1996), 169-182.

[FTN] A. Figà-Talamanca and C. Nebbia, *Harmonic Analysis and Representation Theory for Groups Acting on Homogeneous Trees*, Lecture Note Series **162**, Cambridge University Press, 1991.

[Fo] G. Folland, *A Course in Abstract Harmonic Analysis*, CRC Press, 1995.

[GR] J. García-Cuerva and J.L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, Mathematical Studies **116**, Noth-Holland, 1985.

[GS1] J.L. García and J. Soria, *Weighted inequalities and the shape of approach regions*, Studia Math. **133** (1999), 261-274.

[GS2] J.L. García and J. Soria, *Nontangential approach regions on groups*, Function spaces and applications, Narosa Publishing House (2000), 82-90.

[H] A. Haaker, *On the Conjugate Space of Lorentz Space*, University of Lund, Lund Institute of Technology, Dept. of Math., 1970.

[HL] G.H. Hardy and J.E. Littlewood, *A maximal theorem with function-theoretic applications*, Acta Math. **54** (1930), 81-116.

[HLP] G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, 1934.

[HK] H.P. Heinig and A. Kufner, *Hardy operators of monotone functions and sequences in Orlicz spaces*, J. London Math. Soc. **53** (1996), 256-270.

[HLa] H.P. Heinig and Q. Lai, *Weighted modular inequalities for Hardy-type operators on monotone functions*, J. Inequal. Pure Appli. Math. **1** (2000), Article 10, 25 pp. (electronic).

[J] B. Jawerth, *Weighted inequalities for maximal operators: linearization, localization and factorization*, Amer. J. Math. **108** (1986), 361-414.

[K] A. Korányi, *Harmonic functions on hermitian hyperbolic space*, Trans. Amer. Math. Soc. **135** (1969), 507-516.

[KP] A. Korányi and M.A. Picardello, *Boundary bahaviour of eigenfunctions of the Laplace operator on trees*, Ann. Scuola Norm. Sup. Pisa **13** (1986), 389-399.

[KPT] A. Korányi, M.A. Picardello and M. Taibleson, *Hardy spaces on non-homogeneous trees*, Symp. Math. **29** (1988), 206-265.

[L] Q. Lai, *Weighted modular inequalities for Hardy type operators*, Proc. London Math. Soc. **79** (1999), 649-672.

[Le] J. Leydold, *A Faber-Krahn type inequality for regular trees*, Geom. Funct. Anal. **7** (1997), 364-378.

[LL] E.H. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics **14**, American Mathematical Society, Providence, RI, 2001.

[Li] J. Littlewood, *On a theorem of Fatou*, J. London Math. Soc. **2** (1927), 172-176.

[Lo] G.G. Lorentz, *On the theory of spaces* Λ, Pacific J. Math **1** (1951), 411-429.

[M] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207-227.

[NRS] A. Nagel, W. Rudin and J. Shapiro, *Tangential boundary behaviour of functions in Dirichlet-type spaces*, Ann. of Math. **116** (1982), 331-360.

[NS] A. Nagel and E.M. Stein, *On certain maximal functions and approach regions*, Adv. Math. **54** (1984), 83-106.

[NaS1] K. Naimark and M. Solomyak, *Eigenvalue estimates for the weighted Laplacian on metric trees*, Proc. London Math. Soc. **80** (2000), 690-724.

[NaS2] K. Naimark and M. Solomyak, *Geometry of Sobolev spaces on regular trees and the Hardy inequalities*, Russ. J. Math. Phys. **8** (2001), 322-335.

[N] C.J. Neugebauer, *Weighted norm inequalities for average operators of monotone functions*, Publ. Mat. **35** (1991), 429-447.

[P] Pan Wenjie, *Weighted norm estimates for certain maximal operators with approach regions*, Lecture Notes in Mathematics, vol. **1494**, 1992, 169-175.

[PW] M.A. Picardello and W. Woess, *Martin boundaries of random walks: ends of trees and groups*, Trans. Amer. Math. Soc. **302** (1987), 185-205.

[Pr1] A.R. Pruss, *Discrete convolution-rearrangement inequalities and the Faber-Krahn inequality on regular trees*, Duke Math. J. **91** (1998), 463-514.

[Pr2] A.R. Pruss, *Discrete harmonic measure, Green's functions and symmetrization: a unified probabilistic approach*, Ann. Univ. Mariae Curie-Sklodowska **55** (2001), 139-174.

[R] J.A. Raposo, *Acotación de Operadores Maximales en Análisis Armónico*, Ph.D. Thesis, Universitat de Barcelona, 1998.

[RS] J.A. Raposo and J. Soria, *Best approach regions for potential spaces*, Proc. Am. Math. Soc. **125** (1997), 1105-1109.

[RT] R. Rochberg and M. Taibleson, *Factorization of the Green's operator and weak-type estimates for a random walk on a tree*, Publ. Mat. **35** (1991), 187-207.

[SV] M. Salvatori and M. Vignati, *Tangential boundary behaviour of harmonic functions on trees*, Potential Anal. **6** (1997), 269-287.

[SS1] A. Sánchez-Colomer and J. Soria, *Weighted norm inequalities for general maximal operators and approach regions*, Math. Nachr. **172** (1995), 249-260.

[SS2] A. Sánchez-Colomer and J. Soria, *A_p and approach regions*, Fourier Analysis and Partial Differential Equations, CRC Press (1995), 311-315.

[S1] E. Sawyer, *Weighted Lebesgue and Lorentz norm inequalities for the Hardy operator*, Trans. Amer. Math. Soc. **281** (1984), 329-337.

[S2] E. Sawyer, *Boundedness of classical operators on classical Lorentz spaces*, Studia Math. **96** (1990), 145-158.

[Sc1] H.A. Schwarz, *Zur integration der partiellen differentialgleichung* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, J. Reine Angew. Math. **74** (1872), 218-253.

[Sc2] H.A. Schwarz, *Beweis des Satzes dass die Kugel kleinere Oberfläche besitzt, als jeder andere Körper gleichen Volumens*, Göttinger Nachrichten (1884), 1-13.

[So] J. Soria, *Lorentz spaces of weak type*, Quart. J. Math. Oxford **49** (1998), 93-103.

[St1] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[St2] E.M. Stein, *Harmonic Analysis*, Princeton University Press, 1993.

[Ste] V.D. Stepanov, *The weighted Hardy's inequality for nonincreasing functions*, Trans. Amer. Math. Soc. **338** (1993), 173-186.

[ST] J.O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Mathematics, vol. **1381**, 1989.

[Su] J. Sueiro, *On maximal functions and Poisson-Szegö integrals*, Trans. Amer. Math Soc. **298** (1986), 653-669.

[T] M.H. Taibleson, *Fourier Analysis on Local Fields*, Math. Notes **15**, Princeton University Press, 1975.

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