Communication Rates for Fading Channels with Imperfect Channel-State Information

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A Ju e Leo
À ma mère
A mio padre

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Summary

An important specificity of wireless communication channels are the rapid fluctuations of propagation coefficients. This effect is called *fading* and is caused by the motion of obstacles, scatterers and reflectors standing along the different paths of electromagnetic wave propagation between the transmitting and the receiving terminal. These changes in the geometry of the wireless channel prompt the attenuation coefficients and the relative phase shifts between the multiple propagation paths to vary. This suggests to model the channel coefficients (the transfer matrix) as random variables.

The present thesis studies information rates for reliable transmission of information over fading channels under the realistic assumption that the receiver has only *imperfect* knowledge of the random fading state. While the over-idealized assumption of *perfect* channel-state information at the receiver (CSIR) gives rise to many simple expressions and is fairly well understood, the settings with *imperfect* CSIR or downright *absence* of CSIR are significantly more complex to treat, and less is known about theoretical limits of communication in these circumstances.

Of particular interest are analytical expressions of achievable transmission rates under imperfect and no CSI, that is, lower bounds on the mutual information and on the Shannon capacity. A well-known mutual information lower bound for Gaussian codebooks is based on the notion that the Gaussian distribution is the "worst-case" additive noise distribution in that it minimizes the input-output mutual information. By conflating the additive noise (induced by thermal noise in amplifiers at the receiver) with the multiplicative noise term due to the imperfections of the CSIR into a single effective noise term, one can exploit the extremal property of the Gaussian noise distribution to construct a "worse" channel by assuming that the effective noise is Gaussian. This worst-case-noise approach allows to derive a strikingly simple lower bound on the mutual information of the channel. This lower bound is well-known in literature and is

frequently used to provide simple expressions of achievable rates.

A first part of this thesis proposes a simple way to improve this worst-case-noise bound by means of a rate-splitting approach: by expressing the Gaussian channel input as a sum of several independent Gaussian inputs, and by assuming that the receiver performs successive decoding of the corresponding information streams (as if multiple virtual users were transmitting over the same physical link in a multiple-access fashion), we show how to derive a larger lower bound on the channel's mutual information. On channels with a single transmit antenna, the optimal allocation of transmit power across the different inputs is found to be approached as the number of inputs (so-called *layers*) tends to infinity, and the power assigned to each layer tends to zero (i.e., becomes infinitesimally small). This infinite-layering limit gives rise to a mutual information bound expressible as an integral. On channels with multiple transmit antennas, an analogous result is derived. However, since multiple transmit antennas open up more possibilities for spatial multiplexing, this leads to a higher-dimensional allocation problem, thus giving rise to a whole family of infinite-layering mutual information bounds.

This family of bounds is closely studied for independent and identically zero-mean Gaussian distributed fading coefficients (so-called i.i.d. Rayleigh fading). Several properties of the family of bounds are derived. Most notably, it is shown that for asymptotically perfect CSIR, any member of the family of bounds is asymptotically tight at high signal-to-noise ratios (SNR). Specifically, this means that the difference between the mutual information and its lower bound tends to zero as the SNR tends to infinity, provided that the CSIR tends to be exact as the SNR tends to infinity.

A second part of this thesis proposes a framework for the optimization of a class of utility functions in block-Rayleigh fading multiple-antenna channels with transmit-side antenna correlation, and no CSI at the receiver. A fraction of each fading block is reserved for transmitting a sequence of training symbols, while the remaining time instants are used for transmission of data. The receiver estimates the channel matrix based on the noisy training observation and then decodes the data signal using this channel estimate. The class of utility functions under study consists of symmetric functions of the eigenvalues of the matrix-valued effective SNR. Most notably, a simple achievable rate expression based on the worst-case-noise bound belongs to this class.

The problems consisting in optimizing the pilot sequence and the linear precoder are cast into convex (or quasi-convex) problems for concave (or quasi-concave) utility functions. We also study an important subproblem of the joint optimization, which consists in computing jointly Pareto-optimal pilot sequences and precoders. By wrapping these optimization procedures into a cyclic iteration, we obtain an algorithm which converges to a local joint optimum for any utility.

Resum

Una de les característiques específiques importants dels canals de comunicació sense fils és la ràpida fluctuació dels coeficients de propagació. Aquest efecte, anomenat esvaïment, està causat pel moviment d'obstacles, dispersors i reflectors situats en els diferents camins de propagació de les ones electromagnètiques entre el terminal transmissor i receptor. Aquests canvis en la geometria del canal sense fil produeixen una variació dels coeficients d'atenuació i dels retards relatius de fase entre els diferents camins. Aquest efecte suggereix modelar els coeficients del canal (matriu de transferència) com a variables aleatòries.

Aquesta tesi estudia les taxes d'informació per la transmissió fiable d'informació en canals amb esvaïments sota la hipòtesi realista de que el receptor té un coneixement tan sols imperfecte de l'esvaïment aleatori. Mentre la suposició ideal de coneixement perfecte de l'estat del canal en el receptor proporciona expressions simples i és bastant ben conegut, les configuracions amb coneixement imperfecte o sense cap coneixement del mateix són significativament més complexes de tractar i es coneix molt menys sobre els límits teòrics de la comunicació en aquestes circumstàncies.

De particular interès són les expressions analítiques de les taxes de transmissió assolibles amb coneixement imperfecte i sense coneixement de l'estat del canal, és a dir, cotes inferiors de la informació mútua i de la capacitat de Shannon. Una cota inferior de la informació mútua per a codis gaussians ben coneguda es basa en la noció de que la distribució gaussiana és la pitjor distribució possible de soroll additiu en el sentit que minimitza la informació mútua entre entrada i sortida. Combinant el soroll additiu (induït pel soroll tèrmic dels amplificadors del receptor) amb el terme de soroll multiplicatiu causat per les imperfeccions del coneixement de l'estat del canal en un únic soroll efectiu, es pot explotar la propietat de la distribució gaussiana de definir el pitjor canal suposant que el soroll efectiu és efectivament gaussià. Aquesta aproximació del pitjor

soroll permet obtenir una expressió de la informació mútua del canal sorprenentment simple. Aquesta expressió és ben coneguda en la literatura i s'utilitza sovint per obtenir expressions simples de les taxes d'informació assolibles.

Una primera part d'aquesta tesi proposa un procediment senzill per a millorar aquesta cota associada al pitjor cas mitjançant una estratègia de repartiment de taxa: expressant l'entrada gaussiana del canal com a la suma de diverses entrades gaussianes independents i suposant que el receptor realitza una descodificació seqüencial dels fluxos d'informació (com si diversos usuaris virtuals transmetessin sobre el mateix enllaç físic en un canal de múltiple accés), es mostra com obtenir una major cota inferior de la informació mútua del canal. En canals amb una única antena en transmissió, la distribució òptima de potència als diferents fluxos s'obté quan el seu nombre (capes) tendeix a infinit, i la potència associada a cada capa tendeix a zero (és infinitesimalment petita). El límit associat a un nombre infinit de capes dóna lloc a una expressió integral de la cota de la informació mútua. En canals amb múltiples antenes s'obté un resultat similar. No obstant això, atès que la utilització de múltiples antenes proporciona més possibilitats de multiplexat espacial, el problema augmenta la seva dimensionalitat i dóna lloc a tota una família de cotes inferiors de la informació mútua associades a una combinació de capes infinita.

S'estudia en detall aquesta família de cotes per al cas de coeficients d'esvaïments gaussians de mitjana zero, independents i idènticament distribuïts (conegut com esvaïment i.i.d. Rayleigh). S'obtenen diverses propietats de la família de cotes. És important destacar que per a coneixement asimptòtic perfecte del canal en recepció, qualsevol membre de la família de cotes és asimptòticament ajustat per alta relació senyal a soroll (SNR). En concret, la diferència entre la informació mútua i la seva cota inferior tendeix a zero quan la SNR tendeix a infinit sempre que el coneixement del canal tendeixi a ser exacte a mesura que la SNR tendeix a infinit.

Una segona part d'aquesta tesi proposa un marc per a l'optimització d'una classe de funcions d'utilitat en canals amb múltiples antenes i esvaïments Rayleigh per blocs amb correlació en transmissió i sense informació sobre el canal a recepció. Una fracció temporal de cada bloc d'esvaïment es reserva per transmetre una seqüència de símbols d'entrenament mentre que la resta de mostres temporals s'utilitzen per transmetre informació. El receptor estima la matriu del canal partint de la seva observació sorollosa i descodifica la informació mitjançant la seva estimació del canal. La classe de funcions d'utilitat considerades són funcions simètriques dels autovalors de la SNR matricial efectiva. De fet, una simple expressió de la taxa d'informació assolible basada en el pitjor soroll possible pertany a aquesta classe.

Els problemes consistent en optimitzar la seqüència pilot i el precodificador lineal es transforman en problemes convexos (o quasi-convexos) per a funcions d'utilitat còncaves (o quasi-còncaves). També s'estudia un subproblema important de l'optimització conjunta, que consisteix en el càlcul de les seqüències d'entrenament i dels precodificadors conjuntament Pareto-òptims. Integrant aquests procediments d'optimització en una iteració cíclica, s'obté un algoritme que convergeix a un òptim local conjunt per a qualsevol utilitat quasi-còncava.

Resumen

Una de las características específicas importantes de los canales de comunicación inalámbricos es la fluctuación rápida de los coeficientes de propagación. Este efecto, denominado desvanecimiento, está causado por el movimiento de obstáculos, dispersores y reflectores situados en los diferentes caminos de propagación de las ondas electromagnéticas entre el terminal transmisor y receptor. Estos cambios en la geometría del canal inalámbrico producen una variación de los coeficientes de atenuación y en los retardos relativos de fase entre los diferentes caminos. Este efecto sugiere modelar los coeficientes del canal (matriz de transferencia) como variables aleatorias.

Esta tesis estudia las tasas de información para transmisión fiable de información en canales con desvanecimientos bajo la hipótesis realista que el receptor tiene un conocimiento tan sólo imperfecto del desvanecimiento aleatorio. Mientras la suposición ideal de conocimiento perfecto del estado del canal en el receptor proporciona expresiones simples y es bastante bien conocido, las configuraciones con conocimiento imperfecto o sin ningún conocimiento del mismo son significativamente más complejas de tratar y se conoce mucho menos sobre los límites teóricos de la comunicación en estas circunstancias.

De particular interés son las expresiones analíticas de las tasas de transmisión alcanzables bajo conocimiento imperfecto y sin conocimiento del estado del canal, es decir, cotas inferiores de la información mutua y de la capacidad de Shannon. Una cota inferior de la información mutua para códigos gaussianos bien conocida se basa en la noción de la distribución gaussiana es la peor distribución posible de ruido aditivo en el sentido de que minimiza la información mutua entre entrada y salida. Combinando el ruido aditivo (inducido por el ruido térmico de los amplificadores del receptor) con el término de ruido multiplicativo causado por las imperfecciones del conocimiento del estado del canal en un único ruido efectivo, se puede explotar la propiedad de la distribución gaussiana de definir el peor canal suponiendo que el ruido efectivo

es efectivamente gaussiano. Esta aproximación del peor ruido permite obtener una sorprendentemente simple expresión de la información mutua del canal. Esta expresión es bien conocida en la literatura y se utiliza a menudo para obtener expresiones simples de las tasas de información alcanzables.

Una primera parte de esta tesis propone un procedimiento sencillo de mejorar esta cota asociada al peor caso mediante una estrategia de repartición de tasa: expresando la entrada gaussiana del canal como la suma de varias entradas gaussianas independientes y suponiendo que el receptor realiza una decodificación secuencial de los flujos de información (como si varios usuarios virtuales transmitieran sobre el mismo enlace físico en un canal de múltiple acceso), se muestra como obtener una mayor cota inferior de la información mutua del canal. En canales con una única antena en transmisión, la distribución de potencia óptima a los diferentes flujos se obtiene cuando su número (capas) tiende a infinito, y la potencia asociada a cada capa tiende a cero (es infinitesimamente pequeña). El límite asociado a un número infinito de capas da lugar a una expresión integral de la cota de la información mutua. En canales con múltiples antenas se obtiene un resultado similar. Sin embargo, dado que la utilización de múltiples antenas proporciona más posibilidades de multiplexado espacial, el problema aumenta su dimensionalidad y da lugar a toda una familia de cotas inferiores de la información mutua asociadas a una combinación de capas infinita.

Se estudia en detalle esta familia de cotas para el caso de coeficientes de desvanecimientos gaussianos de media cero, independientes e idénticamente distribuidos, (conocido como desvanecimiento i.i.d. Rayleigh). Se obtienen diversas propiedades de la familia de cotas. Es importante destacar que para conocimiento asintótico perfecto del canal en recepción, cualquier miembro de la familia de cotas es asintóticamente ajustado para alta relación señal a ruido (SNR). En concreto, la diferencia entre la información mutua y su cota inferior tiende a cero cuando la SNR tiende a infinito siempre que el conocimiento del canal tienda a ser exacto a medida que la SNR tienda a infinito.

Una segunda parte de esta tesis propone un marco para la optimización de una clase de funciones de utilidad en canales con múltiples antenas y con desvanecimientos Rayleigh por bloques con correlación en transmisión y sin información sobre el canal en recepción. Una fracción temporal de cada bloque de desvanecimiento se reserva para transmitir una secuencia de símbolos de entrenamiento mientras que el resto de muestras temporales se utilizan para transmitir información. El receptor estima la matriz del canal a partir de su observación ruidosa y decodifica la información mediante su estimación del canal. La clase funciones de utilidad bajo estudio son funciones simétricas de los autovalores de la SNR matricial efectiva. De hecho, una simple expresión de la tasas de información alcanzable basada en el peor ruido posible pertenece a esta clase.

Los problemas consistentes en optimizar la secuencia piloto y el precodificador lineal se transforman en problemas convexos (o casi-convexos) para funciones de utilidad cóncavas (o casi-cóncavas). También se estudia un subproblema importante de la optimización conjunta, que consiste en calcular las secuencias de entrenamiento y los precodificadores conjuntamente Pareto-óptimos. Integrando estos procedimientos de optimización en una iteración cíclica, se obtiene un algoritmo que converge a un óptimo local conjunto para cualquier utilidad cuasi-cóncava.

Contents

Notation

Vectors and Matrices

$\mathbf{a}$	a vector (bold lowercase)
$\mathbf{A}$	a matrix (bold uppercase)
$\mathbf{A}^{\mathrm{T}}$	transpose of the matrix $\mathbf{A}$
$\mathbf{A}^{*}$	complex conjugate of the matrix $\mathbf{A}$
$\mathbf{A}^{\dagger}$	conjugate transpose (Hermitian adjoint) of the matrix $\mathbf{A}$
$\mathbf{A}^{\frac{1}{2}}$	Hermitian square root of the Hermitian matrix $\mathbf{A}$, uniquely defined via $\mathbf{A}^{\frac{1}{2}}\mathbf{A}^{\frac{1}{2}} = \mathbf{A}$
$\mathbf{I}_n$	$n \times n$ identity matrix. The subscript 'n' may be omitted at times, especially if the dimension is clear from context.
$\mathbf{1}_n$	vector of dimension $n \times 1$ whose entries are all equal to 1 (so-called *all-ones* vector)
$\mathbf{0}$	zero matrix or zero vector (the dimension is always clear from context)
$\mathbf{a} \leq \mathbf{b}$	$\mathbf{a} = \left[a_1 \ \dots \ a_n\right]^{\mathrm{T}} \in \mathbb{R}_{+}^{n}$ is not larger than $\mathbf{b} = \left[b_1 \ \dots \ b_n\right]^{\mathrm{T}} \in \mathbb{R}_{+}^{n}$, in the sense that $a_i \leq b_i$ for all $i = 1, \dots, n$
$\mathbf{A} \preceq \mathbf{B}$	$\mathbf{B} - \mathbf{A}$ is positive semidefinite, the matrices $\mathbf{A}$ and $\mathbf{B}$ being Hermitian and of equal size (alternative notation: $\mathbf{B} \succeq \mathbf{A}$)
$\mathrm{vec}(\mathbf{A})$	vector resulting from stacking the columns of the matrix $\mathbf{A} = \left[\mathbf{a}_1 \, \mathbf{a}_2 \ \dots\right]$ on top of each other, i.e., $\mathrm{vec}(\mathbf{A}) = [\mathbf{a}_1^{\mathrm{T}} \ \mathbf{a}_2^{\mathrm{T}} \ \dots]^{\mathrm{T}}$
$\mathrm{rank}(\mathbf{A})$	rank of the square matrix $\mathbf{A}$
$\mathrm{range}(\mathbf{A})$	range of the matrix $\mathbf{A}$, defined as the linear space spanned by the columns of $\mathbf{A}$
$\mathrm{tr}(\mathbf{A})$	trace of the square matrix $\mathbf{A}$, defined as the sum of its diagonal entries

$\det(\mathbf{A})$	determinant of the square matrix $\mathbf{A}$, defined as the product its eigenvalues
$\mathbf{U}_{\mathbf{A}}$	the reduced left singular basis of a matrix $\mathbf{A}$. For positive semidefinite $\mathbf{A} \in \mathbb{C}_+^{n\times n}$, this coincides with the reduced eigenbasis, of size $n\times\text{rank}(\mathbf{A})$.
$\mathbf{V}_{\mathbf{A}}$	the reduced right singular basis of a matrix $\mathbf{A}$

Sets

$\mathbb{N}$ — the set of positive integers (excluding 0)

$\mathbb{Z}$ — the set of integers

$\mathbb{R}$ — the set of real numbers

$\mathbb{C}$ — the set of complex numbers

$\mathbb{U}^{m\times n}$ — the set of (sub-)unitary matrices of the $\mathbb{C}^{m\times n}$, defined such that for $\mathbf{U} \in \mathbb{U}^{m\times n}$, we have $\mathbf{U}\mathbf{U}^\dagger = \mathbf{I}_m$ if $m \leq n$ and $\mathbf{U}^\dagger\mathbf{U} = \mathbf{I}_n$ if $m \geq n$

$\mathbb{P}^n$ — the symmetric group of permutation matrices, defined as $\mathbb{P}^n = \mathbb{U}^{n\times n} \cap \{0,1\}^{n\times n}$

$\mathbb{C}_+^{n\times n}$ — the cone of positive semidefinite matrices from $\mathbb{C}^{n\times n}$, defined as

$$\mathbb{C}_+^{n\times n} = \{\mathbf{A} \in \mathbb{C}^{n\times n} \,|\, \forall \mathbf{x} \in \mathbb{C}^n \colon \mathbf{x}^\dagger\mathbf{A}\mathbf{x} \geq 0\}$$

$\mathbb{C}_{++}^{n\times n}$ — the cone of positive definite matrices from $\mathbb{C}^{n\times n}$, defined as

$$\mathbb{C}_{++}^{n\times n} = \{\mathbf{A} \in \mathbb{C}^{n\times n} \,|\, \forall \mathbf{x} \in \mathbb{C}^n \colon \mathbf{x}^\dagger\mathbf{A}\mathbf{x} > 0\}$$

$\{A_1, \ldots, A_n\}$ — the (unordered) set containing elements $A_1, \ldots, A_n$

$(A_1, \ldots, A_n)$ — the ordered collection (also called *tuple*) containing elements $A_1, \ldots, A_n$

A^n — shorthand for the ordered collection $(A_1, \ldots, A_n)$. By convention, A^0 denotes the empty collection.

$\text{int}(\mathcal{A})$ — the set of inner points of $\mathcal{A} \subset \mathbb{R}^n$ (with respect to the Euclidian distance)

$\partial\mathbf{A}$ — the boundary of $\mathcal{A}$, defined as $\mathbf{A} \setminus \text{int}(\mathcal{A})$

$\partial^+\mathcal{A}$ — the so-called *Pareto border* of $\mathcal{A}$. For $\mathcal{A} \subset \mathbb{C}_+^{n\times n}$, it is defined as

$$\partial^+\mathcal{A} = \{\mathbf{A} \in \mathcal{A} \mid \nexists \mathbf{A}' \in \mathcal{A} \colon \mathbf{A}' \succeq \mathbf{A} \text{ and } \mathbf{A}' \neq \mathbf{A}\}.$$

For $\mathcal{B} \subset \mathbb{R}_+^n$, it is defined as

$$\partial^+\mathcal{B} = \{\mathbf{b} \in \mathcal{B} \mid \nexists \mathbf{b}' \in \mathcal{B} \colon \mathbf{b}' \geq \mathbf{b} \text{ and } \mathbf{b} \neq \mathbf{b}'\}$$

$\mathbf{A}(\mathcal{B})$	a concise notation for a feasible set: if $\mathbf{A}(\mathbf{B})$ denotes a function of $\mathbf{B} \in \mathcal{B}$, then $\mathbf{A}(\mathcal{B}) = \cup_{\mathbf{B}\in\mathcal{B}} \mathbf{A}(\mathbf{B})$.
$\mathcal{D}(a)$	Subset of the non-negative orthant, whose elements sum up to a value not larger than a, defined as $$\mathcal{D}(a) = \{\mathbf{x} \in \mathbb{R}_+^n \mid \mathbf{1}^{\mathrm{T}}\mathbf{x} \le a\}$$ The dimension of $\mathcal{D}(a)$ (n in the above case) will be clear from the context.
$\mathrm{col}(\mathbf{A})$	the set of columns of the matrix $\mathbf{A}$

Functions and Operations

$\mathrm{Ei}(\cdot)$	exponential integral function, defined as $\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}x$
$\Gamma(\cdot)$	gamma function, defined as $\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t$
$\Gamma(\cdot,\cdot)$	incomplete gamma function, defined as $\Gamma(x,y) = \int_y^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t$
$\jmath$	imaginary unit, defined via $\jmath^2 = -1$
$\Re\{\cdot\}$	real part of a complex-valued matrix, vector or scalar
$\Im\{\cdot\}$	imaginary part of a complex-valued matrix, vector or scalar
$\overline{\lim}$	limit superior
$\underline{\lim}$	limit inferior
$n!$	factorial of n, defined as $n! = \prod_{k=1}^n k = \Gamma(n+1)$
$\mathrm{I}\{\cdot\}$	indicator function (it is 1 if the statement in the curly brackets is true and is 0 otherwise)
$\delta(x)$	Dirac delta-function, defined such that $f(0) = \int_{\mathcal{A}} f(x)\delta(x)\,\mathrm{d}x$
$(A)^+$ or $[A]^+$	non-negative part of A, i.e., $(A)^+ = \max(0, A)$

Probability

$\mathsf{Pr}\{A\}$	probability of the event A
$\mathsf{Pr}\{A\|B\}$	probability of the event A conditioned on event B
$\mathsf{Pr}\{A,B\}$	probability of the event "A and B", defined as $\mathsf{Pr}\{A,B\} = \mathsf{Pr}\{A\}\,\mathsf{Pr}\{B\|A\}$
$\mathsf{E}[\mathbf{A}]$	expectation of the random variable $\mathbf{A}$
$\mathsf{E}[\mathbf{A}\|\mathsf{\mathbf{B}}]$	expectation of the random variable $\mathbf{A}$ conditioned on $\mathbf{B} = \mathsf{\mathbf{B}}$
$\mathrm{cov}(\mathbf{a}\|\mathsf{\mathbf{B}})$	covariance of the random vector $\mathbf{a}$ conditioned on $\mathbf{B} = \mathsf{\mathbf{B}}$, defined as $\mathrm{cov}(\mathbf{a}\|\mathsf{\mathbf{B}}) = \mathsf{E}[\mathbf{a}\mathbf{a}^\dagger\|\mathsf{\mathbf{B}}] - \mathsf{E}[\mathbf{a}\|\mathsf{\mathbf{B}}]\,\mathsf{E}[\mathbf{a}^\dagger\|\mathsf{\mathbf{B}}]$
A	realization of the random scalar A (sans-serif typesetting convention)

a	realization of the random vector $\mathbf{a}$ (sans-serif typesetting convention)
A	realization of the random matrix $\mathbf{A}$ (sans-serif typesetting convention)
$\mathsf{E}[\mathbf{A}\vert\mathbf{B}]$	random variable defined as the expectation of the random variable $\mathbf{A}$ conditioned on the random variable $\mathbf{B}$
$\mathbf{x} \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{\mu}, \mathbf{C})$	$\mathbf{x}$ is a complex circularly-symmetric Gaussian random vector with mean $\boldsymbol{\mu}$ and covariance $\mathbf{C}$

Mathematical constants

e	Euler's number, defined as $e = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.718\ldots$
π	the number *pi*, approximately equal to $\pi \approx 3.14159\ldots$
γ	the Euler-Mascheroni constant, defined as $\gamma = \lim_{n\to\infty} \left\{ -\log(n) + \sum_{k=1}^{n} \frac{1}{k} \right\} \approx 0.577\ldots$

Acronyms and Abbreviations

AWGN	Additive White Gaussian Noise
CSI	Channel-State Information
CSIR	Channel-State Information at the Receiver
CSIT	Channel-State Information at the Transmitter
MIMO	Multiple-Input Multiple-Output
MISO	Multiple-Input Single-Output
MMSE	Minimum Mean-Square Error
MSE	Mean-Square Error
SIMO	Single-Input Multiple-Output
SISO	Single-Input Single-Output
SNR	Signal-to-Noise Ratio
e.g.	for example (from Latin *exempli gratia*)
i.e.	that is (from Latin *id est*)
i.i.d.	independent and identically distributed
vs.	versus

Introduction

Seminal work from the end of the nineties [Fos98] [Tel99] sparked a strong interest in multiple-input multiple-output (MIMO) systems for modern communication systems, by showing that the use of multiple antennas at both link ends can strongly potentiate spectral efficiency. These results, much like many earlier results on achievable rates and capacity expressions, rely on the assumption that the channel is static (constant in time). When extending these results to the wireless setting (in which the channel coefficients are random variables), and under the critical assumption that the decoder is cognizant of the exact values of the channel coefficients, capacity results can be derived in some relevant cases of interest by simple *averaging* (over the fading distribution) of the capacity expression from the static-channel setup. In fact, a code with long codewords allows the receiver to exploit the diversity created by the alternation of strong channel states (constructive interference) and deep fades (destructive interference) in order to partly compensate for the occasionally poor channel quality in deep fades, so long as he is informed of the sequence of channel states.

Such situation is commonly referred to as *coherent transmission*. The coherent assumption is legitimate in circumstances where the channel parameters change slowly over time (compared to the duration of a symbol), since one can then assume that the receiver has enough resources to accurately and timely sense the current channel state. In practical situations such as cellular radio, however, this assumption does not hold. Therefore, modifications of the channel model and of capacity results are required.

With the inception of research on MIMO systems, it soon became important to extend their study to the *noncoherent* setting, in which the receiver has no *a priori* knowledge of the random channel state: structural properties of the noncoherent capacity-achieving distribution in block-fading channels were established in [Mar99], the diversity techniques of space-time codes

[Ala98] [Tar98] [Tar99] [Hoc00] were adapted to the case of imperfect channel knowledge [Vis01] with blind differential space-time detection [Dig02] [Swi02]. The capacity pre-log of block-fading noncoherent channels was found in [Zhe02], while the noncoherent capacity of memoryless fading channels was shown to have a double-logarithmic growth in the signal-to-noise ratio [Lap03].

The realistic assumption that the receiver has no *a priori* knowledge of the channel state entails significant changes to the capacity behavior, the optimal transmit schemes, the coupling of estimation and detection problems, and resource allocation strategies. Roughly speaking, detection and decoding schemes can be divided into two categories: those that prescind from computing an estimate of the channel state (or do so implicitly) as an intermediate step, and those that explicitly compute an estimate of the channel state prior to detection. In the former case, the symbol or codeword detection is said to be done *blindly*, while in the latter case, it is done *coherently*. A simple and popular method for acquiring channel-state information at the receiver (CSIR) is by inserting so-called *pilot* or *training* symbols, which are not information-bearing and are known to the receiver. Depending on the channel properties, the insertion can be done in reserved time slots (time-duplexed training) or in reserved frequency slots (frequency-duplexed training). A good survey of general principles is found in [Ton04]. In practical systems such as cellular systems, the training pattern is generally a mixed time-frequency pattern.

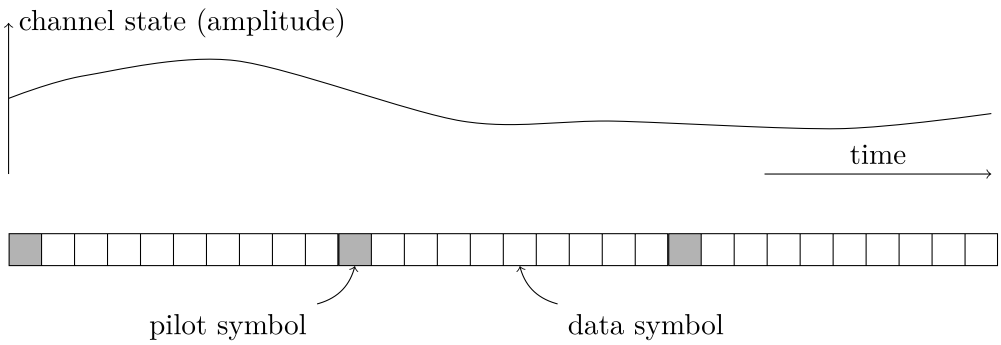


Figure 1 Time-duplexed training

The present thesis is grounded on the idea of explicit training schemes, and more specifically, on time-duplexed training (esp. Chapter 4). Nonetheless, many findings that hold for time-duplexed training can be easily transferred to frequency-duplexed training by swapping the roles of time and frequency.

A communication system that employs an explicit training scheme (e.g., time-duplexed training) must be designed in such way that the overhead due to the training time slots is well calibrated. In fact, if training symbols are too far apart, then the channel knowledge gets outdated before the next training symbol is sent, so that the average quality of channel knowledge gained from interpolation is poor. On the other hand, if the training symbols are too frequent, then an excessive fraction of time is being spent on training instead of transmitting data. It is thus clear that, under a maximum distortion constraint (e.g., maximum bit error rate), an optimal balance must be stricken in order to maximize the net data throughput. The authors of [Has03] proposed an elegant analysis of this trade-off for a capacity lower bound on block-fading MIMO channels (i.e., in the asymptotic regime of infinite blocklength). The present thesis continues along a similar line of research.

It should be noted that this trade-off problem only emerges because the channel estimation error is non-negligible. In the training-based system under consideration, the receiver first exploits the training observation to generate an estimate of the fading gains, and then uses this channel estimate to decode the transmitted message coherently (i.e., using the channel estimate as if it were the true channel). In terms of achievable rates, this two-stage approach is suboptimal compared to full-blown maximum-likelihood decoding, but it drastically reduces decoding complexity while maintaining near-optimal performance results. Hence, the receiver needs to decode the data signal under genuinely *imperfect CSIR* conditions. This is the essential link between the *noncoherent* and *imperfect CSIR* settings: though the transmission of data under imperfect CSIR will be treated as a problem in its own right in Chapters 2 and 3, we actually regard this as a subproblem related to the noncoherent setting, which is treated in Chapter 4.

In Chapter 2, we consider capacity lower bounds for imperfect CSIR. The starting point is a well-known capacity lower bound for imperfect CSIR, due to [MÓ0] and [Bal01]. Hassibi and Hochwald point out in [Has03] that this bound corresponds to a worst-case-noise scenario, in the sense that said bound is equal to the capacity of a “worst-case” channel in which the effective noise distribution is made to be Gaussian and independent of the channel input. In numerous variations and different settings, the bounding technique proposed in [MÓ0, Has03] has been extensively used in subsequent work on transmission with imperfect channel-state information (e.g. in [Yoo04, Mus05, Yoo06, Loz08, Soy10, Din10, Aub13], to cite only a few) as a performance metric for system design. We show that this bound can be improved by simple means via a rate-splitting approach. The improved bound is first derived for the SISO setup, and then extended to the MIMO setup. As a consequence of the large number of possible rate-splitting allocations in MIMO systems (due to the spatial degrees of freedom for beamforming), the MIMO generalization requires to consider an entire family of improved bounds, parametrized by so-called *layering functions*.

In Chapter 3, the MIMO rate-splitting bounds from Chapter 2 are studied in closer detail for the special case of i.i.d. Rayleigh fading. In this highly symmetric correlation model, the general results from Chapter 2 take a simpler form and several additional properties of the family of bounds can be formulated which offer additional insight.

In Chapter 4, we will consider a frequency-flat block-fading MIMO channel with time-duplexed training in which we seek to optimize the linear precoder and the training sequence under an overall average power constraint. The assumption is, of course, that neither terminal has *a priori* knowledge of the channel-state. A frequent yet generally suboptimal choice for the pilot sequence is that of orthonormal pilot symbols. Furthermore, some publications such as [Soy10, Aub10, Aub13] focus on minimizing distortion measures like the mean-square error when designing the pilot sequence (as in [Ton04], [Big06]), while focusing on other measures such as bit-error rate or mutual information when designing the precoder. For instance in [Aub10, Aub13], the authors propose two alternative figures of merit for optimizing the training sequence: the variance of the channel estimation error norm, and the average volume in which said error vector is concentrated. In contrast to these works, we propose to examine how the pilot sequence and the precoder can be *jointly* designed with respect to the *same* performance metric.

A first successful attempt in this direction was made in [Has03], where the problem was considered of finding the optimal time share between training and transmission, as well as of optimally balancing the power levels of pilot and data symbols. Assuming i.i.d. coefficients for the channel estimate and for the channel estimation error, the authors of [Yoo04] proved that the optimal transmit covariance is diagonal, and that its eigenvalues are solutions to a convex optimization problem. However, in both [Has03] and [Yoo04], all results were derived exclusively for i.i.d. fading. Instead, when facing spatially *correlated* fading, the question of joint optimality of pilot sequence and precoder is significantly more involved.

Below, we shall give a brief overview of all novel results presented in this thesis, with forward references to the relevant lemmata, theorems and corollaries.

Summary of Main Contributions

Chapter 2

(1) We demonstrate how the worst-case-noise bound can be improved by a rate-splitting approach: a Gaussian input signal is split into two independent Gaussian input signals, and using the chain rule, we derive a modified capacity lower bound for imperfect CSIR, which we observe by a simple convexity argument improves over the worst-case-noise bound.

(2) We argue that the rate-splitting approach is extensible to any arbitrary number of layers, and prove that the bound increases monotonically (i.e., becomes sharper) as we introduce additional splits [Lemma 2.2, Theorem 2.1]

(3) The supremum over all rate-splitting bounds with respect to the per-layer power allocation is shown to be approached in the limit as the number of layers tends to infinity while the power of each layer tends to zero. An integral expression for this

infinite-layering bound is computed [Theorem 2.2].

(4) Assuming that the variance of the channel estimation error tends to zero as the SNR tends to infinity (so-called asymptotically perfect CSI), we prove that under mild conditions, the difference between the infinite-layering bound and the exact mutual information is asymptotically upper-bounded by a constant gap which only depends on the limit of the ratio between the entropy power of the channel estimation error, and the channel estimation error variance [Theorem 2.3]. In particular, if the conditional channel estimation error (conditioned on the channel estimate) is Gaussian, then the infinite-layering bound is asymptotically tight [Corollary 2.1]. For the proof of these two results, we make use of a mutual information upper bound for Gaussian i.i.d. codebooks, which is a generalization of a conventional upper bound well known in the literature, and whose derivation is given in Appendix 1.1.

(5) The rate-splitting approach is extended to MIMO channels. Results analogous to the above-mentioned are derived for the MIMO setting: from the observation that splitting yields an improved bound, it follows that the best of all rate-splitting bounds is approached for an infinite number of layers [Theorem 2.5]. The layering structure is specified by a combination of so-called *layering functions* and *indexings* [Definitions 2.1 and 2.2]. Layering functions allow to parametrize a family of MIMO infinite-layering bounds expressible in terms of a Riemann-Stieltjes integral [Theorem 2.6].

Chapter 3

This Chapter particularizes the MIMO infinite-layering bounds to the case where there is an i.i.d. Rayleigh-distributed channel estimate and an i.i.d. Rayleigh-fading channel estimation error, these two being mutually independent. Additional properties of this family of rate-splitting bounds are derived especially for this setup.

(1) The asymptotic tightness of MIMO infinite-layering bounds under asymptotically perfect CSI is established [Theorem 3.2]. It appears that this tightness holds irrespective of the layering function and of the speed at which the CSI tends to perfect CSI.

(2) For SNR-independent CSI, the difference between any infinite-layering bound and the worst-case-noise bound is shown to vanish as the number of transmit antennas tends to infinity [Theorem 3.3]. In contrast, this bound difference tends to a positive constant is the number of receive antennas is taken to infinity [Theorem 3.4].

(3) Again for SNR-independent CSI, a large random-matrix limit of an infinite-layering bound is derived, for which both the number of transmit and receive antennas are taken to infinity (at a constant ratio) [Theorem 3.6]. This asymptotic limit is shown in simulations to be a highly accurate approximation for systems with a finite number

of antennas.

Chapter 4

(1) Regarding the two *separate* problems that consist in optimizing the precoder for a prescribed pilot sequence, and the pilot sequence for a prescribed precoder, it is shown that for the former problem, the precoded streams should not outnumber the pilot symbols [Theorem 4.1 and Equation (4.21)], while for the latter problem, the pilot symbols should not outnumber the precoded streams [Theorem 4.3]. Furthermore, it is shown that both problems can be cast into quasi-convex problems, provided that the utility is quasi-concave [Theorem 4.2 and Section 4.4]. The main utilities of interest exhibit this quasi-concavity [Table D.1 in Appendix D.1].

(2) Regarding the *joint* optimization problem, we prove that for optimality, a necessary condition is that the number of streams should equal the number of pilot symbols [Section 4.5.2] and that the eigenbasis of the transmit covariance and the eigenbasis of the Gram form[1] of the pilot sequence should both be equal to the eigenbasis of the channel's transmit correlation matrix [Theorem 4.4]. Loosely speaking, the pilot symbols and the transmit beamforming vectors should all be aligned in direction of the channel eigenmodes. This alignment significantly reduces the dimensionality of the residual problem of eigenvalue optimization.

(3) To further reduce the remaining optimization of the eigenvalues of the transmit covariance and pilot Gram, a method is proposed in Section 4.5 for computing eigenvalue vectors that are *jointly Pareto optimal* (in a sense to be explained in detail in Subsection 4.5.1). This amounts to a further reduction of dimensionality of the optimization problem.

(4) All the above procedures for solving subproblems of the full-fledged optimization problem are combined constructively to yield an iterative procedure which converges towards a local optimum of the joint optimization problem.

[1] The Gram form of a matrix $\mathbf{A}$ shall be defined as $\mathbf{A}\mathbf{A}^{\dagger}$.

1

Communication Model

1.1 The Discrete-Time Channel

In wireless communications, it is generally assumed that the channel is linear, in the sense that the channel's output is a linear function of the channel's input. In addition, the propagation environment fulfills the superposition principle, in that the waveform resulting from superimposing waveforms generated by different sources is the sum of the individual waveforms generated by each source. These properties are consistently verified in practice, and are a consequence of the fact that air as a propagation medium fufills the superposition principle, and that sinusoidal waves reflected or scattered by static obstacles maintain their sinusoidal shape. A more exhaustive discussion based on first principles (far-field approximation, radiation patterns, ray tracing) can be found, e.g., in [Tse05].

All signals shall be represented in the baseband. The input signal is assumed to occupy a frequency band of half-bandwidth $B/2$, centered around the zero frequency. This means that the support of the input signal's Fourier transform (for an energy-limited signal) or power spectral density (for a power-limited signal) belongs to the interval $\left[-\frac{B}{2}; \frac{B}{2}\right]$.

Consider a power-limited, continuous-time, complex-valued narrowband input signal of the form

$$x(t) = \sum_i a_i \mathrm{e}^{\jmath 2\pi f_i t} \tag{1.1}$$

where the i-th sinusoid component has complex amplitude $a_i \in \mathbb{C}$ and frequency $f_i \in \left[-\frac{B}{2}; \frac{B}{2}\right]$. To be power-limited, the sum of squared amplitudes $\sum_i |a_i|^2$ must be finite. An actual physical quantity such as the voltage or current feeding the transmit antenna, or the electric field or magnetic field measured in the vicinity of the transmit antenna, is represented by the real part (the in-phase component) or the imaginary part (the quadrature component) of the complex baseband representation of the transmit signal up-converted to the carrier frequency f_c, i.e.,

$$x_\mathsf{I}(t) = \Re\left\{x(t)\mathrm{e}^{\jmath 2\pi f_\mathsf{c} t}\right\} \qquad\qquad x_\mathsf{Q}(t) = \Im\left\{x(t)\mathrm{e}^{\jmath 2\pi f_\mathsf{c} t}\right\}. \tag{1.2}$$

Since the signal processing at the terminals is performed at baseband frequencies, and since the up-converted signals $x_\mathsf{I}(t)$ or $x_\mathsf{Q}(t)$ carry the same information as the baseband signal $x(t)$, we will limit our analysis to the baseband model.

Assuming that the channel's response (passband-filtered and down-converted to the baseband) is linear and independent of the frequency f_i, the output signal will be of the form

$$\begin{aligned} y(t) &= \sum_j h_j x(t-\tau_j) \\ &= \sum_{i,j} h_j a_i \mathrm{e}^{\jmath 2\pi f_i \cdot (t-\tau_j)} \end{aligned} \tag{1.3}$$

where $\tau_j \in \mathbb{R}$ and $h_j \in \mathbb{R}$ respectively denote the propagation delay and the attenuation factor on the j-th propagation path.

The above assumption that the channel response to sinusoids is independent of the frequency f_i is justified as long as we assume that the bandwidth B is small compared to the channel's *coherence bandwidth*, or equivalently, that the reciprocal $1/B$ is large compared to the standard deviation of the delays τ_j (or any similar measure of the delay spread). This allows to approximate the channel's frequency response as being constant over the communication bandwidth, so that we circumvent the complications of a frequency-selective channel behavior. We say, in our case, that the channel is *frequency-flat*.

The above relationship (1.3) linking the input with the ouput of the channel can be represented as a convolution operation. In fact, if we denote the Dirac delta-function as $\delta(x)$ and define the so-called channel impulse response as

$$h(\tau) = \sum_j h_j \delta(\tau - \tau_j) \tag{1.4}$$

then the ouput $y(t)$ is simply the convolution of the input with the impulse response:

$$y(t) = \int_{-\infty}^{+\infty} h(\tau) x(t-\tau)\,\mathrm{d}\tau. \tag{1.5}$$

This is, in fact, the general system equation governing any linear time-invariant system.

In wireless communications, the system equation (1.5) needs to be extended so as to capture the temporal variations of the channel geometry (moving terminals, reflectors, scatterers and shadowers). These motions should occur at speeds slow enough to not cause too important Doppler shifts, so that the convolution operation will not broaden the communication band too significantly. As a countermeasure, the sampling frequency can be stepped up. However, in practice this effect can be neglected since the typical velocities of moving objects are too low to cause shifts of more than a few hundred Hertz.

As a consequence of the channel variations—and this, in turn, cannot be neglected—the impulse response becomes itself a function of time. That is, the positive path delays $\tau_i(t)$ and the complex-valued attenuation factors $h_j(t)$ become functions of time. Accordingly, the channel response at time instant t to an impulse sent at time instant $t-\tau$ is denoted as $h(\tau;t)$ and is expressed as

$$h(\tau;t) = \sum_j h_j(t)\delta(\tau - \tau_j(t)). \tag{1.6}$$

The system equation (1.5) now becomes

$$y(t) = \int_{-\infty}^{+\infty} h(\tau;t)x(t-\tau)\,\mathrm{d}\tau. \tag{1.7}$$

Let us define the sampled input signal $X[k]$, obtained from sampling $x(t)$ at multiples of the sampling period $1/B$, as

$$X[k] = x\left(\frac{k}{B}\right), \qquad k \in \mathbb{Z}. \tag{1.8}$$

By virtue of the Whittaker-Shannon interpolation formula, the continuous-time input signal $x(t)$ can be reconstructed from the sequence of samples $X[k]$ via

$$x(t) = \sum_{k=-\infty}^{+\infty} X[k]\phi_k(t) \tag{1.9}$$

where the functions $\{\phi_k(t)\}_{k\in\mathbb{Z}}$ are shifted cardinal sine functions, defined as

$$\phi_k(t) = \frac{\sin(\pi(Bt-k))}{\pi(Bt-k)}. \tag{1.10}$$

These functions constitute an orthogonal basis, since

$$\int_{-\infty}^{\infty} \phi_i(t)\phi_j(t)\,\mathrm{d}t = \begin{cases} \frac{1}{B} & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \tag{1.11}$$

Equation (1.9) thus represents an expansion of the input signal $x(t)$ on the orthogonal basis $\{\phi_k(t)\}_{k\in\mathbb{Z}}$. Upon inserting this expansion (1.9) into the system equation (1.7), we get

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{+\infty} X[k] \int_{-\infty}^{\infty} h(\tau;t)\phi_k(t-\tau)\,\mathrm{d}\tau \\ &= \sum_{\ell=-\infty}^{+\infty} X[k-\ell] \int_{-\infty}^{\infty} h(\tau;t)\phi_{k-\ell}(t-\tau)\,\mathrm{d}\tau \end{aligned} \tag{1.12}$$

where in the last line we have made a substitution of indices. By sampling the output signal at the same rate as the input signal, we obtain a time-discrete output $Y[k]$ which reads as

$$\begin{aligned} Y[k] &= \sum_{\ell=-\infty}^{+\infty} X[k-\ell] \int_{-\infty}^{\infty} h\left(\tau; \frac{k}{B}\right) \phi_{k-\ell}\left(\frac{k}{B} - \tau\right) \mathrm{d}\tau \\ &\triangleq \sum_{\ell=-\infty}^{+\infty} X[k-\ell] H[\ell; k]. \end{aligned} \tag{1.13}$$

where we have introduced the time-varying discrete-time channel impulse response $H[\ell; k]$. By plugging (1.6) into its definition, this new discretized impulse response is related to its continuous-time counterpart via

$$H[\ell; k] = \sum_j h_j\left(\frac{k}{B}\right) \phi_{k-\ell}\left(\frac{k}{B} - \tau_j\left(\frac{k}{B}\right)\right). \tag{1.14}$$

To summarize, we have thus converted a time-varying channel model (1.7) from a continuous-time setting to an entirely discrete-time setting (1.13). The values of $H[0; k], H[1; k], \ldots$ are commonly called *taps* or *tap gains* at time k.

1.2 Rayleigh and Rician Fading

Consider the radio signal up-converted to the carrier frequency f_c as in (1.2). In typical macroscopic wireless communication systems (e.g., cellular radio), the distance travelled by a wave along a propagation path between transmitter and receiver is of the order of several meters or tens of meters at least, which is larger than the typical carrier wavelength (for cellular radio, in the sub-meter domain). Consequently, at any given point in time t, the phases of the taps $h_j(t), j = 1, 2, \ldots$, i.e., the phase delays

$$\varphi_j(t) = (2\pi f_\mathsf{c} \tau_j(t) \bmod 2\pi), \qquad j = 1, 2, \ldots \tag{1.15}$$

can be modeled as a collection of mutually independent and uniformly distributed random variables on the interval $[0; 2\pi)$. In contrast, the tap gains $|h_j(t)|, j = 1, 2, \ldots$ can too be modeled as random samples, albeit from some distribution which is more difficult to determine, as it may depend to a larger extent on the channel geometry. However, in a richly scattering environment, it is reasonable to assume a large number of tap gains $h_j(t)$ which, to a large degree, are statistically independent of each other, and also independent of the uniformly distributed phase delays $\varphi_j(t)$. In the discrete-time impulse response (1.14), let us write out the tap gains $h_j(k/B)$ in polar representation:

$$H[\ell; k] = \sum_j \left| h_j\left(\frac{k}{B}\right) \right| \mathrm{e}^{\jmath \varphi_j(k/B)} \phi_{k-\ell}\left(\frac{k}{B} - \tau_j\left(\frac{k}{B}\right)\right). \tag{1.16}$$

We see that, due to the circular symmetry of the complex exponential factor, the real and imaginary parts of $H[\ell;k]$ are sums of a large number of independent random contributions, and are therefore approximately Gaussian as a consequence of the central-limit theorem. The decay of the cardinal sine functions has been neglected in this reasoning, since we expect a large number of taps within the first dominant lobes of $\phi_{k-\ell}$. As a result, the channel gain $H[\ell;k]$ can be modeled as a Gaussian random variable.

This fading distribution is generally called *Rayleigh fading* in reference to the fact that the amplitude of the channel gains follow a Rayleigh distribution.

In environments where, in addition to a large number of scatterers and reflectors, there is a single dominant path due to a line-of-sight connection between the transmitter and the receiver, the same reasoning as above leads to a Gaussian fading distribution with a non-zero mean. We refer to this situation as *Rician fading*.

1.3 Additive Noise

Thus far, we have only applied sampling and convolution operations to the input signal, both of which are invertible. Indeed, by observing that a convolution translates to a multiplication in the frequency domain, the inverse of a convolution translates to a division in the frequency domain. As to the sampling operation, it can be inverted by the interpolation formula (1.9) as mentioned earlier.

Since the continuous-time transmit signal $x(t)$ can be losslessly recovered from the sampled output sequence $Y[k]$, the transition from a continuous-time to a discrete-time model incurs no loss of information, so long as the sampling rate is at least B, and the function $h(\tau;t)$ is precisely known for the task of reconstruction. This precise knowledge of the channel is one of the assumptions that are not always legitimate, and the present thesis is concerned with a back-off from this over-idealization.

But most important still, the amplifier at the receiving terminal is affected by thermal and shot noise, which are a main source of impairment when it comes to reconstructing the transmitted messages or symbols intended for the receiver. These physical effects are best modeled by a continuous-time additive white Gaussian noise (AWGN) denoted as $z(t)$:

- The *white* color refers to frequency-flatness over the communication band (in analogy to the white light spectrum), meaning that the power spectral density of the noise process is equal to a constant value N_0. Hence the noise power is BN_0.
- The *Gaussian* distribution is a consequence of the fact that it is generated by a large number of statistically independent noise sources, which by the central-limit theorem

add up to an approximately Gaussian variable.

To incorporate the AWGN into our model, we redefine the channel output signal $y(t)$ as the signal *after* down-conversion and amplification and *prior* to sampling and processing. This is done by simply adding $z(t)$ to the right-hand side of the continuous-time baseband system equation (1.12). Upon including the sampling operation, we obtain a full discrete-time system equation [cf. (1.13)]

$$Y[k] = \sum_{\ell=-\infty}^{+\infty} X[k-\ell]H[\ell;k] + Z[k]. \tag{1.17}$$

where $Z[k] = z(k/B)$. Obviously, the noise has exogenous causes and therefore, the AWGN sequence is known by neither of the communication parties. We do assume, though, that the transmitter and receiver know the noise statistics. In the case of AWGN, this amounts to knowing the value of N_0.

1.4 Frequency-Flat Fading

Finally, let us recall that we initially assumed that the communication band was narrow enough to justify a frequency-flat channel response. A further consequence is that the energy of the impulse response is concentrated at a single tap $H[\ell_0;k]$. In other words, the taps $H[\ell;k]$ are approximately zero for $\ell > \ell_0$, and the system equation (1.17) further simplifies to

$$Y[k] = X[k-\ell_0]H[\ell_0;k] + Z[k]. \tag{1.18}$$

This situation is sometimes referred to as *frequency-flat fading*. Upon redefining the output signal by an appropriate shift, and removing ℓ_0 from the notation of the fading sequence, we obtain

$$Y[k] = H[k]X[k] + Z[k]. \tag{1.19}$$

This discrete-time system equation describing a narrowband fading channel will be the starting point for this thesis.

1.5 Channel Dynamics

As a consequence of the above derivation of the fading channel model, the fading sequence $\{H[k]\}_{k\in\mathbb{Z}}$ from (1.19) is a stationary random sequence. When this sequence exhibits a temporal correlation, we say that the channel is *time-selective*.

To add some accuracy to this notion, we consider the autocorrelation function of the stationary process $\{H[k]\}_{k\in\mathbb{Z}}$, which is defined as

$$r_H[k] = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{m=-M}^{M} H[m+k]H[m]^* \\ = \mathsf{E}\Big[H[m+k]H[m]^*\Big]. \tag{1.20}$$

at any time instant m. The last equality reflects the stationarity of the process.

A channel is *memoryless* if the autocorrelation function r_H reduces to a centered Dirac delta-function. Otherwise, we say that the channel has *memory*, in reference to the fact that the present channel gain is correlated with some channel gains from the past. The slower the values of $r_H[k]$ decay with increasing k, the lower the time-selectivity.

A quantitative measure for the amount of time-selectivity is the *Doppler spread*. It is a measure of how fast the channel impulse response varies over time, and may be defined as the standard deviation of the Doppler shifts $2\pi f_{\mathsf{c}} \frac{\mathrm{d}\tau_j(t)}{\mathrm{d}t}$ (or any similar measure). The *coherence time* (in discrete time units) is the smallest duration over which the fading $H[k]$ makes a significant change, and can be defined as the reciprocal of the Doppler spread.

The concept of time-selectivity should not be confused with the concept of *slow fading* vs. *fast fading* often invoked in the literature of wireless communications. A channel is generally said to be *slow fading* if the coherence time spans multiple codeword lengths, that is, if every codeword sees only a single fading realization. In this acception, all channels considered in this thesis are *fast fading*, because we will only be concerned with the asymptotic regime of information theory, where codeword lengths are infinite.

To summarize, a channel is time-selective if its autocorrelation is wider than one *symbol duration* (discrete time unit). In contrast, a channel is said to be slow-fading if its autocorrelation is wider than one *codeword*.

1.6 Imperfect Channel-State Information

We consider a fading channel governed by the system equation (1.19). For simplicity, we will assume that the fading is memoryless [cf. (1.5)]. This means that the channel gains $H[k]$ are independent and identically distributed (i.i.d.).

To reflect the fact that the receiver relies only on *imperfect* channel-state information (CSI), we use an additive noise model:

$$H[k] = \hat{H}[k] + \tilde{H}[k]. \tag{1.21}$$

The term $\hat{H}[k]$ can be thought of as a channel estimate, and $\tilde{H}[k]$ can be thought of as the channel estimation error. We assume that the receiver is cognizant of the sequence of estimates

$\{\hat{H}[k]\}_{k\in\mathbb{Z}}$, but not of the exact values $\{H[k]\}_{k\in\mathbb{Z}}$. The transmitter knows none.

However, we do assume that both the transmitter and receiver are fully aware of the statistics. That is, they are informed of the joint distribution of $\hat{H}[k]$, $\tilde{H}[k]$, $X[k]$ and $Z[k]$.

By combining the discrete-time system equation (1.19) with the representation (1.21), the system equation thus reads as

$$Y[k] = \left(\hat{H}[k] + \tilde{H}[k]\right) X[k] + Z[k], \quad k \in \mathbb{Z} \tag{1.22}$$

Note that $\hat{H}[k]$ and $\tilde{H}[k]$ need not be mutually independent, though we do assume that the pair $(\hat{H}[k], \tilde{H}[k])$, the noise $Z[k]$ and the input $X[k]$, are mutually independent. The fading pair $(\hat{H}[k], \tilde{H}[k])$ is an arbitrary sequence of i.i.d. complex-valued random variables whose means and variances shall satisfy the following conditions:

- the random variable $\hat{H}[k]$ has mean μ and variance $\hat{V}$;
- the random variable $\tilde{H}[k]$ has, conditioned on any realization $\hat{H}[k] = \hat{\mathsf{H}}$, mean zero and variance $\tilde{V}(\hat{\mathsf{H}})$, i.e.,

$$\mathsf{E}\left[\tilde{H}[k] \mid \hat{H}[k] = \hat{\mathsf{H}}\right] = 0 \tag{1.23a}$$

$$\mathsf{E}\left[|\tilde{H}[k]|^2 \mid \hat{H}[k] = \hat{\mathsf{H}}\right] \triangleq \tilde{V}(\hat{\mathsf{H}}). \tag{1.23b}$$

The condition (1.23a) is, for example, satisfied when $\hat{H}[k]$ is the minimum mean-square error (MMSE) estimate of $H[k]$ from some receiver side information that is independent of the input $X[k]$.

1.7 Reliable Transmission of Information

For the receiver, the communication task consists in an attempt to recover the information that the transmitter seeks to convey across the channel. Figure 1.7 depicts the basic communication problem: The channel input is the transmit signal $X[k]$, and the channel output is the pair $(\hat{H}[k], Y[k])$. The receiver then seeks to reconstruct the sequence $\{X[k]\}_{k\in\mathbb{Z}}$ from the observation of the sequence $\{(\hat{H}[k], Y[k])\}_{k\in\mathbb{Z}}$.

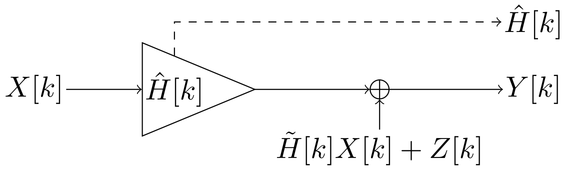


Figure 1.1 Sketch of a communication systems with imperfect CSI

To forearm against transmission errors, we use random coding, which is known to achieve the classical Shannon capacity. A block code of rate R (measured in nats per channel use) and blocklength n_{b} is a collection of $\lfloor e^{n_{\mathsf{b}}R} \rfloor$ codewords, where a codeword is an input sequence of length n_{b}. In random coding, rather than being constructed by a deterministic algorithm, the code is drawn from a random ensemble. We will restrict the analysis to i.i.d. ensembles for simplicity, that is, each codeword is sampled as an i.i.d. sequence from some distribution P_X, and the codewords are i.i.d. among them.

We say that a communication rate R is *achievable* when there exists a sequence of block codes of rate R and of increasing blocklength n_{b} such that the sequence of decoding block-error probabilities (associated to each code in the sequence) tends to zero as the blocklength n_{b} tends to infinity.

In this context, the mutual information plays a central role. For a pair (U, V) of random variables with a continuous probability distribution, the mutual information between U and V is

$$I(U; V) = h(U) - h(U|V) = h(V) - h(V|U) \tag{1.24}$$

where $h(\cdot)$ denotes differential entropy.

By the channel coding theorem, if the codewords are drawn i.i.d. from a distribution P_X, then the mutual information between input and output $I(X; Y, \hat{H})$ is an achievable rate. Using the chain rule for mutual information and the fact that X and $\hat{H}$ are independent (hence $I(X; \hat{H}) = 0$), this mutual information is equal to

$$\begin{aligned} I(X; Y, \hat{H}) &= I(X; Y|\hat{H}) + I(X; \hat{H}) \\ &= I(X; Y|\hat{H}) \end{aligned} \tag{1.25}$$

which is the *conditional* mutual information between X and Y conditioned on $\hat{H}$. This mutual information is the key figure of interest in this thesis.

Subject to the average-power constraint

$$\lim_{M\to\infty} \frac{1}{2M+1} \sum_{m=-M}^{M} |X[m]|^2 = \mathsf{E}\Big[|X[k]|^2\Big] \le P \tag{1.26}$$

the *Shannon capacity* of the channel (1.22) is given by [Big98]

$$C(P) = \sup_{P_X} I(X; Y|\hat{H}) \tag{1.27}$$

where the supremum is over all distributions of X satisfying $\mathsf{E}\big[|X|^2\big] \le P$. The discrete time indices have been omitted because they are immaterial due to the i.i.d. assumption for all random sequences involved in the system equation (1.22).

1.8 Bounds on the Mutual Information and Capacity

Since (1.25) and (1.27) are difficult to evaluate, even if $\hat{H}$ and $\tilde{H}$ are Gaussian, it is common to assess $C(P)$ using upper and lower bounds. A widely-used lower bound on $C(P)$ is due to Médard [M00]:

$$C(P) \geq \mathsf{E}\left[\log\left(1 + \frac{|\hat{H}|^2 P}{\tilde{V}(\hat{H})P + N_0}\right)\right] \triangleq \underline{R}(P). \tag{1.28}$$

The lower bound (1.28) follows from (1.27) by choosing the input X_{G} to be zero-mean, variance-P, circularly-symmetric, complex Gaussian[1] and by upper-bounding the differential entropy of X_{G} conditioned on Y and $\hat{H}$ as

$$\begin{aligned} h(X_{\mathrm{G}}|Y,\hat{H}) &= h(X_{\mathrm{G}} - \alpha Y|Y,\hat{H}) \\ &\leq h(X_{\mathrm{G}} - \alpha Y|\hat{H}) \\ &\leq \mathsf{E}\left[\log\left(\pi e\, \mathsf{E}[|X_{\mathrm{G}} - \alpha Y|^2 \mid \hat{H}]\right)\right] \end{aligned} \tag{1.29}$$

for any $\alpha \in \mathbb{C}$. Here the first inequality follows because conditioning cannot increase entropy, and the subsequent inequality follows because the Gaussian distribution maximizes differential entropy for a given second moment [Cov91, Theorem 9.6.5]. By expressing the mutual information $I(X_{\mathrm{G}};Y|\hat{H})$ as

$$I(X_{\mathrm{G}};Y|\hat{H}) = h(X_{\mathrm{G}}) - h(X_{\mathrm{G}}|Y,\hat{H}) \tag{1.30}$$

and by choosing α in (1.29) so that αY is the linear MMSE estimate of X_{G}, the lower bound (1.28) follows. We shall refer to this bound as the *worst-case-noise bound*, following the nomenclature from [Has03].

When the receiver has perfect CSI so that $\mathsf{E}[\tilde{V}(\hat{H})] = 0$, the worst-case-noise bound $\underline{R}(P)$ is equal to the channel capacity

$$C_{\mathrm{coh}}(P) = \mathsf{E}\left[\log\left(1 + \frac{|H|^2 P}{N_0}\right)\right] \geq I(X_{\mathrm{G}};Y|\hat{H}). \tag{1.31}$$

Consequently, for perfect CSI the worst-case-noise bound (1.28) is tight. In contrast, when the receiver has imperfect CSI and the distributions of $\tilde{V}(\hat{H})$ and $\hat{H}$ do not depend on P, the worst-case-noise bound (1.28) is loose. In fact, in this case $\underline{R}(P)$ is bounded in P, whereas the capacity $C(P)$ is known to be unbounded. For instance, if the conditional entropy of $\tilde{H}$ given $\hat{H}$ is finite, then the capacity has a double-logarithmic growth in P [Lap03].[2]

[1] The subscript 'G' in X_{G} indicates a Gaussian distribution.

[2] This result can be generalized to show that if $\mathsf{E}[\log|\hat{H} + \tilde{H}|^2] > -\infty$ holds, then the capacity grows at least double-logarithmically with P.

This boundedness of $\underline{R}(P)$ is not due to the inequalities in (1.29) being loose, but is a consequence of choosing a Gaussian channel input. Indeed, if $h(\tilde{H}|\hat{H})$ is finite, then a Gaussian input X_{G} achieves [Lap02, Proposition 6.3.1 and Lemma 6.2.1] (see also [Lap03, Lemma 4.5])

$$\overline{\lim_{P\to\infty}}\, I(X_{\mathrm{G}};Y|\hat{H}) \leq \gamma + \log(\pi e\, \mathsf{E}[|\hat{H}+\tilde{H}|^2]) - h(\tilde{H}|\hat{H}) \tag{1.32}$$

where $\gamma \approx 0.577$ denotes Euler's constant and where $\overline{\lim}$ denotes the *limit superior*. Nevertheless, even if we restrict ourselves to Gaussian inputs, the worst-case-noise bound

$$I(X_{\mathrm{G}};Y|\hat{H}) \geq \underline{R}(P) \tag{1.33}$$

is not tight.

As we shall see in the next chapter, by using a rate-splitting and successive-decoding approach, the worst-case-noise bound (1.33) can be sharpened.

Besides the worst-case-noise lower bound (1.28), an upper bound on the Gaussian-input mutual information is given by Lemma 1.1 below.

Lemma 1.1. *We have the mutual information upper bound*

$$\begin{aligned} I(X_{\mathrm{G}};Y|\hat{H}) &\leq \underline{R}(P) + \mathsf{E}\left[\log\left(\frac{\tilde{V}(\hat{H})P+N_0}{\tilde{\Phi}(\hat{H})PW+N_0}\right)\right] \\ &\triangleq I_{\mathrm{upper}}(P) \end{aligned} \tag{1.34}$$

where W is independent of $\hat{H}$ and is exponentially distributed with mean 1, and where $\tilde{\Phi}(\hat{\mathsf{H}})$ denotes the conditional entropy power of $\tilde{H}$, conditioned on $\hat{H}=\hat{\mathsf{H}}$:[3]

$$\tilde{\Phi}(\hat{\mathsf{H}}) \triangleq \begin{cases} \dfrac{1}{\pi \mathrm{e}} \mathrm{e}^{h(\tilde{H}|\hat{H}=\hat{\mathsf{H}})}, & \text{if } h(\tilde{H}|\hat{H}=\hat{\mathsf{H}}) > -\infty \\ 0, & \text{otherwise.} \end{cases} \tag{1.35}$$

Proof: See Appendix A.1. ■

The upper bound (1.34) was previously used, e.g., in [Bal01, Equation 42] and [Yoo06, Lemma 2] for the special case of a Gaussian $\tilde{H}$ and mutually independent $\hat{H}$ and $\tilde{H}$, in which case the entropy power equals the variance, i.e., $\tilde{V} = \tilde{\Phi}$.

[3] We define $h(\tilde{H}|\hat{H}=\hat{\mathsf{H}}) = -\infty$ if the conditional distribution of $\tilde{H}$, conditioned on $\hat{H}=\hat{\mathsf{H}}$, is not absolutely continuous with respect to the Lebesgue measure.

2

The Rate-Splitting Approach

2.1 Single-Input Single-Output (SISO) Channels

We adopt the discrete-time memoryless channel model (1.22) derived in the previous chapter. Following the argument in Section 1.7, we restrain the analysis to i.i.d. codebooks, so we will drop the time indices throughout, since they are immaterial. The channel equation (without time indices) is thus given by

$$Y = (\hat{H} + \tilde{H})X + Z. \tag{2.1}$$

In the following, the channel's mutual information (1.25) is the quantity of interest.

2.1.1 Worst-Case-Noise Lower Bound

For future reference, we state Médard's worst-case-noise lower bound (1.28) in a slightly more general form in the following Lemma.

Lemma 2.1. *Let S be a zero-mean, circularly-symmetric, complex Gaussian random variable of variance P. Let A and B be complex-valued random variables of finite second moments, and let C be an arbitrary random variable. Assume that S is independent of (A, C), and that, conditioned on (A, C), the variables S and B are uncorrelated. Then*

$$I(S; AS + B|A, C) \geq \mathsf{E}\left[\log\left(1 + \frac{|A|^2 P}{V_B(A, C)}\right)\right] \tag{2.2}$$

where $V_B(a, c)$ denotes the conditional variance of B conditioned on $(A, C) = (a, c)$.

Proof: See Appendix B.1. ■

2.1.2 How Rate Splitting Improves the Worst-Case-Noise Lower Bound

Using Lemma 2.1, we show that, for imperfect CSI and $\mathsf{E}[|\hat{H}|^2] > 0$, rate splitting with two layers strictly improves the lower bound (1.33). Indeed, let X_1 and X_2 be independent, zero-mean, circularly-symmetric, complex Gaussian random variables with respective variances P_1 and P_2 (satisfying $P_1 + P_2 = P$) such that $X_\mathrm{G} = X_1 + X_2$. By the chain rule for mutual information, we obtain

$$\begin{aligned} I(X_\mathrm{G}; Y|\hat{H}) &= I(X_1, X_2; Y|\hat{H}) \\ &= I(X_1; Y|\hat{H}) + I(X_2; Y|\hat{H}, X_1). \end{aligned} \tag{2.3}$$

By replacing the random variables A, B, C, and S in Lemma 2.1 with

$$A \leftarrow \hat{H}, \quad B \leftarrow \hat{H}X_2 + \tilde{H}X + Z, \quad C \leftarrow 0, \quad S \leftarrow X_1$$

and by noting that these random variables satisfy the lemma's conditions, it follows that the first term on the right-hand side of (2.3) is lower-bounded as

$$I(X_1; Y|\hat{H}) \geq \mathsf{E}\left[\log\left(1 + \frac{|\hat{H}|^2 P_1}{\tilde{V}(\hat{H})P_1 + (|\hat{H}|^2 + \tilde{V}(\hat{H}))P_2 + N_0}\right)\right] \triangleq R_1(P_1, P_2). \tag{2.4}$$

Similarly, by replacing A, B, C, and S in Lemma 2.1 with

$$A \leftarrow \hat{H}, \quad B \leftarrow \hat{H}X_1 + \tilde{H}X + Z, \quad C \leftarrow X_1, \quad S \leftarrow X_2$$

and by noting that these random variables satisfy the lemma's condition, we obtain for the second term on the right-hand side of (2.3)

$$I(X_2; Y|\hat{H}, X_1) \geq \mathsf{E}\left[\log\left(1 + \frac{|\hat{H}|^2 P_2}{\tilde{V}(\hat{H})(|X_1|^2 + P_2) + N_0}\right)\right] \triangleq R_2(P_1, P_2). \tag{2.5}$$

Since for every $\alpha > 0$, the function $x \mapsto \log(1 + \alpha/x)$ is strictly convex in $x > 0$, it follows from Jensen's inequality that the right-hand side of (2.5) is lower-bounded as

$$\mathsf{E}\left[\log\left(1 + \frac{|\hat{H}|^2 P_2}{\tilde{V}(\hat{H})(|X_1|^2 + P_2) + N_0}\right)\right] \geq \mathsf{E}\left[\log\left(1 + \frac{|\hat{H}|^2 P_2}{\tilde{V}(\hat{H})(P_1 + P_2) + N_0}\right)\right] \tag{2.6}$$

with the inequality being strict except in the trivial cases where $P_1 = 0$, $P_2 = 0$, or if, with probability one, at least one of $|\hat{H}|$ and $\tilde{V}(\hat{H})$ is zero.[1] Thus, combining (2.3)–(2.6), we obtain

$$R_1(P_1, P_2) + R_2(P_1, P_2) \geq \mathsf{E}\left[\log\left(1 + \frac{|\hat{H}|^2 P}{\tilde{V}(\hat{H})P + N_0}\right)\right] \tag{2.7}$$

[1] We shall write this as $\mathsf{Pr}\left\{\hat{H} \cdot \tilde{V}(\hat{H}) = 0\right\} = 1$. For example, this occurs when the receiver has perfect CSI, in which case $\tilde{V}(\hat{H}) = 0$ almost surely.

demonstrating that, when the receiver has imperfect CSI, rate splitting with two layers strictly improves the lower bound (1.28) (except in trivial cases).

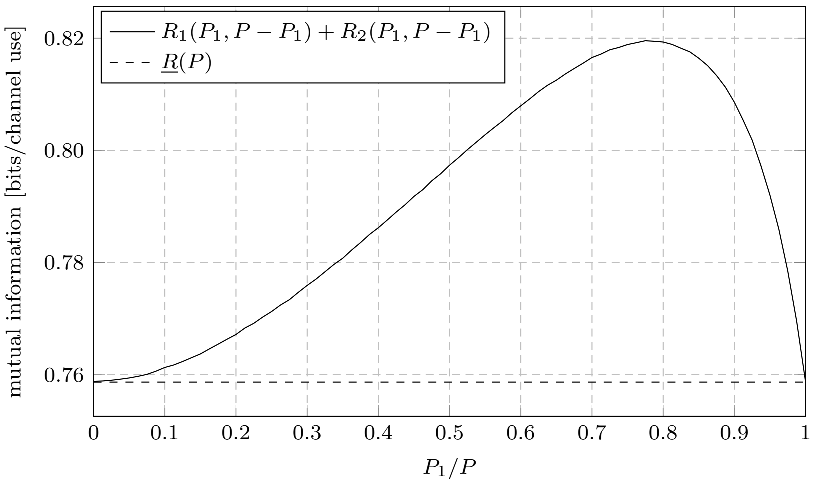


Figure 2.1 Comparison of the two-layer lower bound $R_1(P_1, P - P_1) + R_2(P_1, P - P_1)$ (continuous line) with Médard's lower bound $\underline{R}(P)$ (dashed line) as a function of the power fraction P_1/P assigned to the first layer.

Figure 2.1 compares the two-layer bound $R_1(P_1, P_2) + R_2(P_1, P_2)$ with $\underline{R}(P)$ (dashed line) as a function of P_1/P, for $\hat{H}$ and $\tilde{H}$ being mutually independent and circularly-symmetric Gaussian with parameters $\mu = 0$, $\hat{V} = \frac{1}{2}$, $\tilde{V}(\hat{\mathsf{H}}) = \frac{1}{2}$ for $\hat{\mathsf{H}} \in \mathbb{C}$, $P = 10$, and $N_0 = 1$. The figure confirms that, when the receiver has imperfect CSI and $P_1 > 0$ and $P_2 > 0$, rate splitting with two layers outperforms $\underline{R}(P)$ (1.28). In this example, the optimal power allocation is approximately at $P_1 \approx 0.78P$ and $P_2 \approx 0.22P$. In general, the optimal power allocation is difficult to compute analytically.

2.1.3 Rate Splitting with an Arbitrary Finite Number of Layers

One might wonder whether extending our approach to more than two layers can further improve the lower bound. As we shall see in the following section, it does. In fact, for every positive power $P > 0$ we show that, once that the power is optimally allocated across layers, the rate-splitting lower bound is strictly increasing in the number of layers.

Let $X_1, \ldots, X_L$ be independent, zero-mean, circularly-symmetric, complex Gaussian random variables with respective variances $P_1, \ldots, P_L$ satisfying

$$P = \sum_{\ell=1}^{L} P_\ell \tag{2.8}$$

and

$$X_{\mathrm{G}} = \sum_{\ell=1}^{L} X_\ell. \tag{2.9}$$

Let the cumulative power Q_k be given by

$$Q_k \triangleq \sum_{\ell=1}^{k} P_\ell. \tag{2.10}$$

We denote the collection of cumulative powers as

$$\mathbf{Q} \triangleq \{Q_1, \ldots, Q_L\} \tag{2.11}$$

and refer to it as an L-layering.

It follows from the chain rule for mutual information that

$$I(X^L; Y|\hat{H}) = \sum_{\ell=1}^{L} I(X_\ell; Y|X^{\ell-1}, \hat{H}) \tag{2.12}$$

where we use the shorthand A^N to denote the sequence $A_1, \ldots, A_N$, and A^0 denotes the empty sequence. Applying Lemma 2.1 by replacing the respective A, B, C, and S with

$$A \leftarrow \hat{H}, \quad B \leftarrow \hat{H} \sum_{\ell' \neq \ell} X_{\ell'} + \tilde{H}X + Z, \quad C \leftarrow X^{\ell-1}, \quad S \leftarrow X_\ell$$

and by noting that these random variables satisfy the lemma's conditions, we can lower-bound the ℓ-th summand on the right-hand side of (2.12) as

$$\begin{aligned} I(X_\ell; Y|X^{\ell-1}, \hat{H}) &\geq \mathsf{E}\left[\log(1 + \Gamma_{\ell,\mathbf{Q}}(X^{\ell-1}, \hat{H}))\right] \\ &\triangleq R_\ell[\mathbf{Q}] \end{aligned} \tag{2.13}$$

where

$$\Gamma_{\ell,\mathbf{Q}}(X^{\ell-1}, \hat{H}) \triangleq \frac{|\hat{H}|^2 P_\ell}{\tilde{V}(\hat{H})\left|\sum_{i<\ell} X_i\right|^2 + \tilde{V}(\hat{H})P_\ell + \left(|\hat{H}|^2 + \tilde{V}(\hat{H})\right)\sum_{i>\ell} P_i + N_0} \tag{2.14}$$

and where the last line in (2.13) should be viewed as the definition of $R_\ell[\mathbf{Q}]$. Defining

$$R[\mathbf{Q}] \triangleq R_1[\mathbf{Q}] + \ldots + R_L[\mathbf{Q}] \tag{2.15}$$

we obtain from (2.12) and (2.13) the lower bound

$$I(X_{\mathrm{G}}; Y|\hat{H}) = I(X^L; Y|\hat{H}) \geq R[\mathbf{Q}]. \tag{2.16}$$

Note that $Q_{\ell-1} = Q_\ell$ implies $P_\ell = 0$, which in turn implies $R_\ell[\mathbf{Q}] = 0$. Without loss of optimality, we can therefore restrict ourselves to L-layerings satisfying

$$0 < Q_1 < \ldots < Q_L = P. \tag{2.17}$$

We shall denote the set of all L-layerings satisfying (2.17) by $\mathcal{Q}(P, L)$. Note that this definition of layerings precludes $P = 0$, and we shall from now on assume that $P > 0$.

2.1.4 Rate Splitting with an Infinite Number of Layers

Let $R^\star(P, L)$ denote the lower bound $R[\mathbf{Q}]$ optimized over all $\mathbf{Q} \in \mathcal{Q}(P, L)$, i.e.,

$$R^\star(P, L) \triangleq \sup_{\mathbf{Q} \in \mathcal{Q}(P,L)} R[\mathbf{Q}]. \tag{2.18}$$

In the following, we show that $R^\star(P, L)$ is monotonically increasing in L. To this end, we need the following lemma.

Lemma 2.2. *Let $L' > L$, and let the L-layering $\mathbf{Q} \in \mathcal{Q}(P, L)$ and the L'-layering $\mathbf{Q}' \in \mathcal{Q}(P, L')$ satisfy*

$$\{Q_1, \ldots, Q_L\} \subset \{Q'_1, \ldots, Q'_{L'}\}. \tag{2.19}$$

Then

$$R[\mathbf{Q}] \leq R[\mathbf{Q}'] \tag{2.20}$$

with equality if, and only if, $\mathsf{Pr}\{\hat{H} \cdot \tilde{V}(\hat{H}) = 0\} = 1$.

Proof: See Appendix B.2. ∎

Theorem 2.1. *The rate $R^\star(P, L)$ is monotonically non-decreasing in L. Moreover, if* $\mathsf{Pr}\{\hat{H} \cdot \tilde{V}(\hat{H}) = 0\} = 1$*, then $R^\star(P, L) = \underline{R}(P)$ for every $L \in \mathbb{N}$*

Proof: See Appendix B.3. ∎

It follows from Theorem 2.1 that the best lower bound, optimized over all layerings of fixed sum-power P, namely

$$R^\star(P) \triangleq \sup_{L \in \mathbb{N}} \sup_{\mathbf{Q} \in \mathcal{Q}(P,L)} R[\mathbf{Q}] = \sup_{L \in \mathbb{N}} R^\star(P, L) \tag{2.21}$$

is approached by letting the number of layers L tend to infinity. An explicit expression for $R^\star(P)$ is provided by the following theorem.

Theorem 2.2. *For a given input power P, the supremum of all rate-splitting lower bounds $R[\mathbf{Q}]$ over $\mathbf{Q} \in \mathcal{Q}(P, L)$ and $L \in \mathbb{N}$ is given by the limit*

$$R^\star(P) = \lim_{L \to \infty} R^\star(P, L) \tag{2.22}$$

and has an analytical expression which reads as

$$\boxed{R^\star(P) = \int_0^1 \mathsf{E}\left[\frac{|\hat{H}|^2}{(\tilde{V}(\hat{H})(W-1) - |\hat{H}|^2)\iota + |\hat{H}|^2 + \tilde{V}(\hat{H}) + \frac{N_0}{P}}\right] \mathrm{d}\iota} \tag{2.23}$$

where W is independent of $\hat{H}$ and exponentially distributed with mean 1.

Proof: See Appendix B.4. ■

Remark 2.1. *The expression* (2.23) *features multiple integrations: an integration over $\iota \in [0; 1]$, an expectation over W, and an expectation over $\hat{H}$. By Fubini's Theorem, these three operations can be performed in any order.*

Remark 2.2. *By solving the integral corresponding to the expectation over W, the expression* (2.23) *reduces from a triple to a double integral*

$$\boxed{R^\star(P) = \int_0^1 \mathsf{E}\left[\frac{|\hat{H}|^2}{\tilde{V}\iota}\varsigma\left(\frac{(|\hat{H}|^2 + \tilde{V}(\hat{H}))(1-\iota) + \frac{N_0}{P}}{\tilde{V}\iota}\right)\right] \mathrm{d}\iota} \tag{2.24}$$

where the function $\varsigma(\cdot)$ is defined as

$$\varsigma(x) \triangleq -e^x \,\mathrm{Ei}(-x) \tag{2.25}$$

and where $\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t}\,\mathrm{d}t$ denotes the exponential integral function. In contrast, if we solve the integral over $\iota \in [0; 1]$, we obtain the alternative representation

$$\boxed{R^\star(P) = \mathsf{E}\left[\frac{|\hat{H}|^2}{|\hat{H}|^2 + \tilde{V}(\hat{H}) + \frac{N_0}{P}}\,\Theta\left(\frac{\tilde{V}(\hat{H})(W-1) - |\hat{H}|^2}{|\hat{H}|^2 + \tilde{V}(\hat{H}) + \frac{N_0}{P}}\right)\right]} \tag{2.26}$$

where the function $\Theta(\cdot)$ is defined as

$$\Theta(x) \triangleq \begin{cases} \dfrac{\log(1+x)}{x}, & \text{if } -1 < x < 0 \text{ or } x > 0 \\ 1, & \text{if } x = 0. \end{cases} \tag{2.27}$$

Remark 2.3. *It is easy to see in* (2.23) *that the expression inside the expectation operator is convex in W for any fixed $\hat{H}$ and ι. Therefore, by Jensen's inequality, one can lower-bound $R^\star(P)$ by moving the expectation over W into the denominator. By doing so, we recover Médard's bound $\underline{R}(P)$. The same applies to* (2.26)*, since the function Θ can be shown to be convex on $(-1, \infty)$.*

Remark 2.4. *The proof of Theorem 2.2 hinges on the observation that the supremum $R^\star(P)$ is approached by an equi-power layering*

$$\mathbf{U}(P, L) \triangleq \left\{\frac{P}{L}, 2\frac{P}{L}, \ldots, (L-1)\frac{P}{L}, P\right\} \tag{2.28}$$

when the number of layers L is taken to infinity. While this layering was chosen for mathematical convenience, any other layering would also do, provided that some regularity conditions are met. For example, one can show that for any Lipschitz-continuous monotonic bijection $F\colon [0,P] \to [0,P]$*, we have*

$$\lim_{L\to\infty} R[F(\mathbf{U}(P,L))] = \lim_{L\to\infty} R[\mathbf{U}(P,L)] = R^\star(P) \tag{2.29}$$

where $F(\mathbf{U}(P,L)) = \{F(P/L), F(2P/L), \ldots, F(P)\}$.

2.1.5 High-SNR Limit

The high-SNR limit of $R^\star(P)$ can be computed in closed form. Let

$$\mathcal{L}\{f(x)\}(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t \tag{2.30}$$

denote the Laplace transform of the function $f(x)$. We have the high-SNR limit

$$\begin{aligned}\lim_{P\to\infty} R^\star(P) &= \mathsf{E}\left[\mathcal{L}\left\{\frac{\log(x)}{x-1}\right\}(U)\right] \\ &= \mathsf{E}\left[e^{-U}\left(\frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{\mathscr{H}_n}{(n+1)!}U^n + G_{2,3}^{3,1}\left(U\middle|\begin{smallmatrix}0,1\\0,0,0\end{smallmatrix}\right)\right)\right].\end{aligned} \tag{2.31}$$

where $\mathscr{H}_n = \sum_{k=1}^n k^{-1}$ denotes the harmonic sequence, where

$$U = 1 + \frac{|\hat{H}|^2}{\tilde{V}(\hat{H})} \tag{2.32}$$

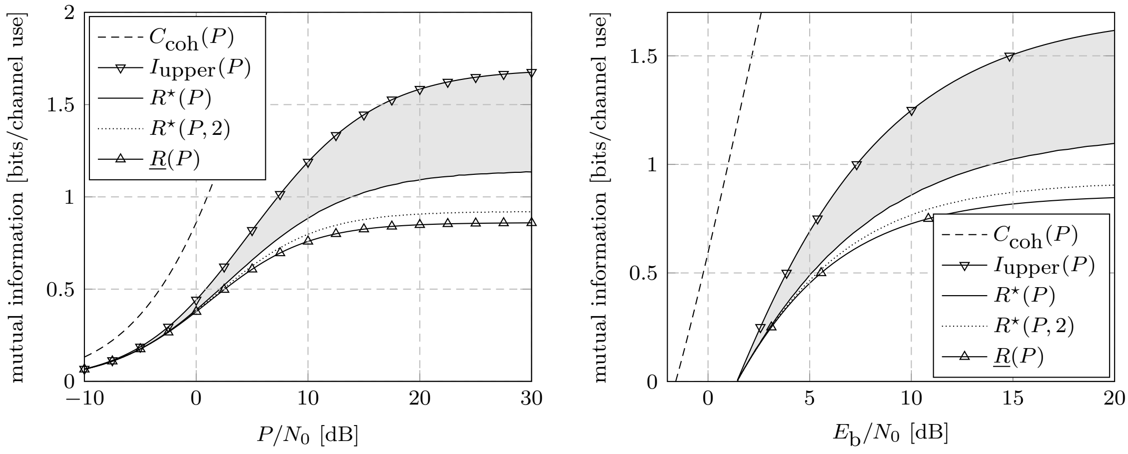
and where $G_{2,3}^{3,1}\left(x\middle|\begin{smallmatrix}0,1\\0,0,0\end{smallmatrix}\right)$ is a Meijer G-function (cf. [Gra07, Definition 9.301]).

This statement is given without proof. Nonetheless, it is worth mentioning because as a by-product, by specializing it to the case where $\hat{H}$ is equal to some $\hat{\mathsf{H}}$ with probability 1 (i.e., $\hat{H}$ is a deterministic constant that can be assimilated to the channel mean), we obtain a novel closed-form lower bound on the mutual information of channels with a Gaussian input X_G and pure multiplicative noise $\hat{\mathsf{H}} + \tilde{H}$ with a non-zero mean $\hat{\mathsf{H}}$. Namely,

$$\boxed{I(X_\mathrm{G}; (\hat{\mathsf{H}} + \tilde{H})X_\mathrm{G}) \geq e^{-\mathsf{U}}\left(\frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{\mathscr{H}_n}{(n+1)!}\mathsf{U}^n + G_{2,3}^{3,1}\left(\mathsf{U}\middle|\begin{smallmatrix}0,1\\0,0,0\end{smallmatrix}\right)\right)} \tag{2.33}$$

where

$$\mathsf{U} = 1 + \frac{|\hat{\mathsf{H}}|^2}{\tilde{V}}. \tag{2.34}$$



(a) Bounds vs. SNR.

(b) Bounds vs. energy per information bit.

Figure 2.2 Comparison of capacity and Gaussian-input mutual information bounds for fixed CSI.

2.1.6 Numerical Example

In Figure 2.2(a), several bounds on the Gaussian-input mutual information $I(X_{\mathrm{G}};Y|\hat{H})$ are plotted against the SNR on a range from $-10\,\mathrm{dB}$ to $30\,\mathrm{dB}$. From top to bottom, we have the coherent capacity (1.31); the upper bound (1.34); the supremum $R^\star(P)$ over all rate-splitting bounds (Theorem 2.2); the two-layer rate-splitting bound with optimized power allocation $R^\star(P,2)$; and Médard's lower bound $\underline{R}(P)$. The grey-shaded area indicates the region in which the curve of the exact Gaussian-input mutual information $I(X_{\mathrm{G}};Y|\hat{H})$ is located. For this simulation, we have chosen $\hat{H}$ and $\tilde{H}$ to be independent and complex circularly-symmetric Gaussian with parameters $\mu = 0$, $\hat{V} = \frac{1}{2}$, and $\tilde{V}(\hat{\mathsf{H}}) = \frac{1}{2}$, $\hat{\mathsf{H}} \in \mathbb{C}$. Observe that the proposed rate-splitting approach sharpens the bound mostly at high SNR. In this simulation, the increase $R^\star(P) - \underline{R}(P)$ is approximately 0.28 bits per channel use as P tends to infinity.

Figure 2.2(b) shows the same bounds as Figure 2.2(a), but this time with the rate plotted against the energy per information bit E_{b}/N_0. Observe that the minimum energy per bit of all bounds (except that of the coherent capacity C_{coh}) is equal to 1.41dB, thus demonstrating that the rate-splitting approach sharpens the bound only marginally at low SNR.

2.1.7 Asymptotically Perfect CSI

The numerical example considered in the previous section (see Figure 2.2(a)) assumes that $\tilde{V}(\hat{H})$ and $\hat{H}$ do not depend on the SNR P/N_0. However, in practical communication systems, the channel estimation error—as measured by the mean error variance $\mathsf{E}[\tilde{V}(\hat{H})]$—typically decreases as the SNR increases. In this section, we investigate the high-SNR behavior of the derived bounds when $\mathsf{E}[\tilde{V}(\hat{H})]$ vanishes as the SNR tends to infinity. When this condition is satisfied, we shall say that we have *asymptotically perfect CSI.*

2.1.7.1 Asymptotic Tightness

We will consider a family of joint distributions of $(\hat{H}, \tilde{H})$ parametrized by $\rho = P/N_0$. To make this dependence on ρ explicit, we shall write in this section the two channel components as $\hat{H}_\rho$ and $\tilde{H}_\rho$, and the respective variances as $\hat{V}_\rho$ and $\tilde{V}_\rho(\hat{H}_\rho)$. Similarly, we shall write the entropy power, defined in (1.35), as $\tilde{\Phi}_\rho(\hat{H}_\rho)$. We further adapt the notation to express Médard's lower bound, the rate-splitting lower bounds (2.18) and (2.21), and the upper bounds (1.31) and (1.34) as functions of ρ, namely, $\underline{R}(\rho)$, $R^\star(\rho, L)$, $R^\star(\rho)$, $C_{\text{coh}}(\rho)$, and $I_{\text{upper}}(\rho)$.

We assume that $H = \hat{H}_\rho + \tilde{H}_\rho$ does not depend on ρ and is normalized:

$$\mathsf{E}\big[|\hat{H}_\rho|^2\big] + \mathsf{E}\big[\tilde{V}_\rho(\hat{H}_\rho)\big] = 1. \tag{2.35}$$

We further assume that the variance of the estimation error $\tilde{H}_\rho$ is not larger than the variance of H, i.e., $\tilde{V}_\rho(\hat{\mathsf{H}}_\rho) \leq 1$ for every $\hat{\mathsf{H}}_\rho \in \mathbb{C}$.

Theorem 2.3. *Let $\hat{H}_\rho$, $\tilde{V}_\rho(\hat{H}_\rho)$, and $\tilde{\Phi}_\rho(\hat{H}_\rho)$ satisfy*

$$\lim_{\rho\to\infty} \mathsf{E}\big[\tilde{V}_\rho(\hat{H}_\rho)\big] = 0 \tag{2.36a}$$

$$\overline{\lim_{\rho\to\infty}} \left\{ \sup_{\xi\in\mathbb{C}} \frac{\tilde{V}_\rho(\xi)}{\tilde{\Phi}_\rho(\xi)} \right\} \leq M \tag{2.36b}$$

for some finite constant M, where we define $0/0 \triangleq 1$ and $a/0 \triangleq \infty$ for every $a > 0$. Then, we have

$$\overline{\lim_{\rho\to\infty}} \{ I(X_{\text{G}}; Y|\hat{H}_\rho) - R^\star(\rho) \} \leq \log(M) \, \mathsf{Pr}\{|H| > 0\}. \tag{2.37}$$

Proof: See Appendix B.6. ∎

Remark 2.5. *The proof of Theorem 2.3 reveals that if* $\mathsf{Pr}\{|H| > 0\} = 1$, *then one can strengthen* (2.37) *by replacing $I(X_{\text{G}}; Y|\hat{H}_\rho)$ by its upper bound $I_{\text{upper}}(\rho)$.*

If conditioned on (almost) every $\hat{H}_\rho = \hat{\mathsf{H}}_\rho$, the estimation error $\tilde{H}_\rho$ is Gaussian, then we have $\tilde{V}_\rho(\hat{\mathsf{H}}_\rho) = \tilde{\Phi}_\rho(\hat{\mathsf{H}}_\rho)$ for every $\hat{\mathsf{H}}_\rho \in \mathbb{C}$ and (2.36b) is satisfied for $M = 1$. Thus, for a conditionally Gaussian $\tilde{H}_\rho$, the lower bound $R^\star(\rho)$ is asymptotically tight.

Corollary 2.1. *Conditioned on every $\hat{H}_\rho = \hat{\mathsf{H}}_\rho$, let $\tilde{H}_\rho$ be Gaussian, and let* (2.36a) *and* (2.36b) *hold.* $\lim_{\rho\to\infty} \mathsf{E}\big[\tilde{V}_\rho(\hat{H}_\rho)\big] = 0$ *Then, we have*

$$\lim_{\rho\to\infty} \{ I_{\text{upper}}(\rho) - R^\star(\rho) \} = 0. \tag{2.38}$$

Proof: The Gaussian distribution of $\tilde{H}_\rho$ implies that the cumulative distribution function of $|H| = |\hat{H}_\rho + \tilde{H}_\rho|$ is continuous, so $\mathsf{Pr}\{H = 0\} = 0$. The result follows then from (B.77) and (B.81) in the proof of Theorem 2.3 (Appendix B.6) upon noting that, for a Gaussian distribution, (2.36b) is satisfied for $M = 1$. ■

Corollary 2.1 demonstrates that, for conditionally Gaussian $\tilde{H}_\rho$ and asymptotically perfect CSI, both bounds $I_{\text{upper}}(\rho)$ and $R^\star(\rho)$ are asymptotically tight in the sense that their difference to the Gaussian-input mutual information vanishes as ρ tends to infinity. In [Lap02], it was argued that the difference between $\underline{R}(\rho)$ and $C_{\text{coh}}(\rho)$ vanishes as ρ tends to infinity if $\tilde{V}_\rho(\hat{H}_\rho)$ decays *faster* than the reciprocal of ρ, in which case Médard's lower bound is asymptotically tight, too. Note however that, if $\tilde{H}_\rho$ is conditionally Gaussian, then the upper bound (1.34) becomes

$$I_{\text{upper}}(\rho) = \underline{R}(\rho) + \mathsf{E}\left[\log\left(\frac{1+\rho\tilde{V}_\rho(\hat{H}_\rho)}{1+\rho W\tilde{V}_\rho(\hat{H}_\rho)}\right)\right] \tag{2.39}$$

from which follows that

$$\lim_{\rho\to\infty}\{I_{\text{upper}}(\rho) - \underline{R}(\rho)\} = 0 \quad\Longleftrightarrow\quad \lim_{\rho\to\infty}\rho\,\mathsf{E}\big[\tilde{V}_\rho(\hat{H}_\rho)\big] = 0. \tag{2.40}$$

Thus, for conditionally Gaussian $\tilde{H}_\rho$ and asymptotically perfect CSI, Médard's lower bound is asymptotically tight if, and only if, $\mathsf{E}\big[\tilde{V}_\rho(\hat{H}_\rho)\big]$ decays faster than the reciprocal of ρ, whereas $R^\star(\rho)$ is asymptotically tight *irrespective* of the rate of decay.

It follows directly from (B.77)–(B.80) and Lemma B.2 used within the proof of Theorem 2.3 (Appendix B.6) that for any fading distribution satisfying (2.36b),

$$\overline{\lim_{\rho\to\infty}}\{I_{\text{upper}}(\rho) - \underline{R}(\rho)\} \le \gamma + \log(M) \tag{2.41}$$

where $\gamma \approx 0.577$ denotes Euler's constant. Consequently, at high SNR, the bounds $I_{\text{upper}}(\rho)$, $R^\star(\rho)$, and $\underline{R}(\rho)$ all have the same logarithmic slope.

2.1.7.2 Prediction- and Interpolation-Based Channel Estimation

We evaluate the lower bounds $\underline{R}(\rho)$, $R^\star(\rho, 2)$, and $R^\star(\rho)$ together with the upper bound $I_{\text{upper}}(\rho)$ for two specific channel estimation errors satisfying (2.36a). We assume that $\hat{H}_\rho$ and $\tilde{H}_\rho$ are zero-mean, circularly-symmetric, complex Gaussian random variables that are independent of each other[2] and satisfy the normalization (2.35). The former has variance $\hat{V}_\rho$ and the latter has variance $\tilde{V}_\rho$. We consider variances $\tilde{V}_\rho$ of the forms

$$\tilde{V}_\rho = \left(\frac{1}{2B} + \frac{1}{\rho}\right)^{2B}\rho^{2B-1} - \frac{1}{\rho} \tag{2.42a}$$

[2] Consequently, $\tilde{V}_\rho$ does not depend on $\hat{H}_\rho$ either.

and

$$\tilde{V}_\rho = \frac{2BT}{\rho + 2BT} \tag{2.42b}$$

for some $0 < B < \frac{1}{2}$, where $T = \lfloor 1/(2B) \rfloor$ is the largest integer not greater than $1/(2B)$.

As we shall argue next, (2.42a) corresponds to prediction-based channel estimation, whereas (2.42b) corresponds to interpolation-based channel estimation. Indeed, suppose for a moment that the fading process $\{H[k]\}_{k\in\mathbb{Z}}$ is not i.i.d. but is a zero-mean, unit-variance, stationary, circularly-symmetric, complex Gaussian process with power spectral density

$$f_H(\lambda) = \begin{cases} \frac{1}{2B}, & |\lambda| < B \\ 0, & B \le |\lambda| \le \frac{1}{2} \end{cases} \tag{2.43}$$

for some $0 < B < \frac{1}{2}$. The fading's autocovariance function is determined by $f_H(\cdot)$ through the expression

$$\mathsf{E}\Big[H[k+m]H[k]^*\Big] = \int_{-1/2}^{1/2} e^{\jmath 2\pi m\lambda} f_H(\lambda)\,\mathrm{d}\lambda \tag{2.44}$$

We obtain (2.42a) if we let $\hat{H}[k]$ be the minimum mean-square error (MMSE) predictor in predicting $H[k]$ from a noisy observation of its past

$$H[k-1]\sqrt{P} + Z[k-1], H[k-2]\sqrt{P} + Z[k-2], \ldots \tag{2.45}$$

Indeed, in this case $\hat{H}[k]$ and $\tilde{H}[k] = H[k] - \hat{H}[k]$ are zero-mean, circularly-symmetric, complex Gaussian random variables that are independent of each other, the latter with mean zero and variance [Gre84, Section 10.8, p. 181–184], [Lap05, Equation (11)]

$$\tilde{V}_\rho = \exp\left\{\int_{-1/2}^{1/2} \log\left(f_H(\lambda) + \frac{1}{\rho}\right) \mathrm{d}\lambda\right\} - \frac{1}{\rho}. \tag{2.46}$$

For the power spectral density (2.43) this gives (2.42a). Note that, even though the lower bounds $\underline{R}(\rho)$, $R^\star(\rho, L)$, and $R^\star(\rho)$ were derived for i.i.d. fading $\{\hat{H}_\rho[k], \tilde{H}_\rho[k]\}_{k\in\mathbb{Z}}$, by evaluating them for $\tilde{H}_\rho[k]$ having variance (2.42a), they can be used to derive lower bounds on the capacity of noncoherent fading channels with stationary fading having power spectral density $f_H(\cdot)$; see, e.g., [Lap05].

The variance (2.42b) corresponds to a channel-estimation scheme where the transmitter emits every T time instants (say at $k = nT$, $n \in \mathbb{Z}$) a pilot symbol $\sqrt{P}$ and where the receiver estimates the fading coefficients at the remaining time instants k (i.e., where k is not an integer multiple of T) from the noisy observations

$$H[nT]\sqrt{P} + Z[nT], \quad n \in \mathbb{Z} \tag{2.47}$$

using an MMSE interpolator; see, e.g., [Don04, Loz08, Asy11, Asy13]. When the power spectral density $f_H(\cdot)$ is bandlimited to B and when $T \le 1/(2B)$, it can be shown that the variance of the estimation error is given by [Ohn02]

$$\tilde{V}_\rho = 1 - \int_{-B}^{B} \frac{\rho f_H^2(\lambda)}{\rho f_H(\lambda) + T} \, \mathrm{d}\lambda. \tag{2.48}$$

For the power spectral density (2.43) this gives (2.42b). Again, even though the lower bounds $\underline{R}(\rho)$, $R^\star(\rho, L)$, and $R^\star(\rho)$ were derived for i.i.d. fading $\{\hat{H}_\rho[k], \tilde{H}_\rho[k]\}_{k\in\mathbb{Z}}$, by evaluating them for $\tilde{H}_\rho[k]$ having variance (2.42b), they can be directly used to derive lower bounds on the capacity of noncoherent fading channels with stationary fading having power spectral density $f_H(\cdot)$, provided that we account for the rate loss due to the transmission of pilots. In fact, it was shown that, when $1/(2B)$ is an integer, the above interpolation-based channel estimation scheme together with Médard's lower bound $\underline{R}(\rho)$ achieves the capacity pre-log [Loz08, Asy11, Asy13].[3]

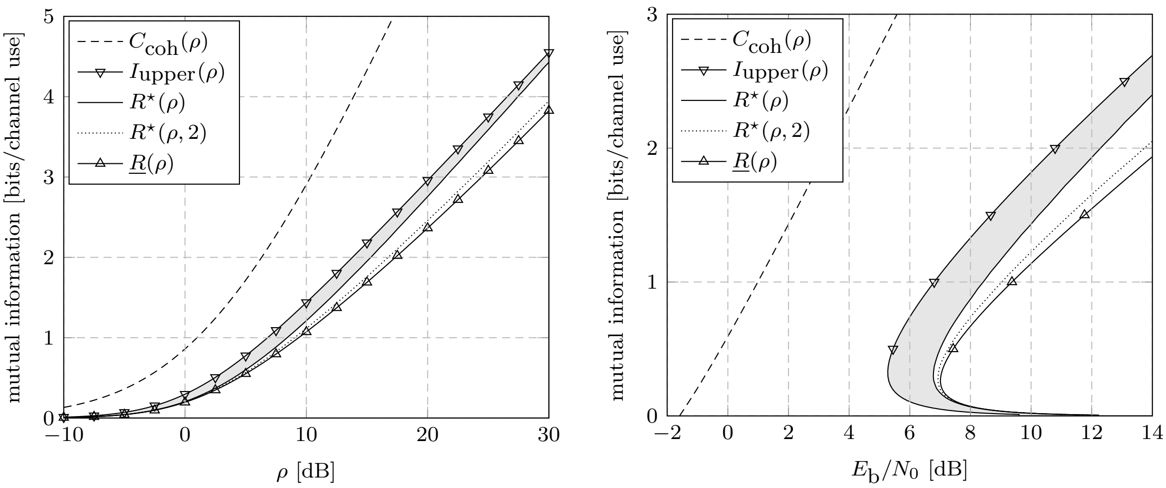
2.1.7.3 Numerical Examples

For Figures 2.3–2.5 below, we assume that $\hat{H}_\rho$ and $\tilde{H}_\rho$ are independent, zero-mean, circularly-symmetric, complex Gaussian random variables.

Figure 2.3(a) shows the lower bounds $\underline{R}(\rho)$, $R^\star(\rho, 2)$, and $R^\star(\rho)$ together with the upper bounds $I_{\text{upper}}(\rho)$ and $C_{\text{coh}}(\rho)$ as a function of ρ for $\tilde{H}_\rho$ having variance (2.42a), with $B = 1/4$. Figure 2.3(b) shows the same bounds, but as a function of the energy per information bit. The curve of the exact Gaussian-input mutual information $I(X_{\mathrm{G}}; Y|\hat{H})$ is located within the shaded area. Observe that, in contrast to the curves in Figure 2.2(a), all curves are unbounded in the SNR, which is a consequence of the fact that $\tilde{V}_\rho$ vanishes as ρ tends to infinity. Further observe that the shaded area narrows down as ρ grows. This is consistent with Corollary 2.1, which states that for (conditionally) Gaussian $\tilde{H}_\rho$ and asymptotically perfect CSI, the bounds $I_{\text{upper}}(\rho)$ and $R^\star(\rho)$ are asymptotically tight. Note that, as demonstrated by (2.41), the upper bound $I_{\text{upper}}(\rho)$ and all lower bounds have the same logarithmic slope at high SNR.

Figure 2.4(a) shows the lower bounds $\underline{R}(\rho)$, $R^\star(\rho, 2)$, and $R^\star(\rho)$ together with the upper bounds $I_{\text{upper}}(\rho)$ and $C_{\text{coh}}(\rho)$ as a function of ρ for $\tilde{H}_\rho$ having variance (2.42b), with $BT = 1/2$. Again, observe that all curves are unbounded in the SNR and that the lower bound $R^\star(\rho)$ is asymptotically tight as ρ tends to infinity. What is more, $R^\star(\rho)$ is close to $I_{\text{upper}}(\rho)$ for a large range of SNR. Further observe that, at high SNR, the upper bound $I_{\text{upper}}(\rho)$ and all lower bounds have the same logarithmic slope as $C_{\text{coh}}(\rho)$. This fact was used in [Loz08, Asy11, Asy13] to derive tight lower bounds on the capacity pre-log of noncoherent fading channels.

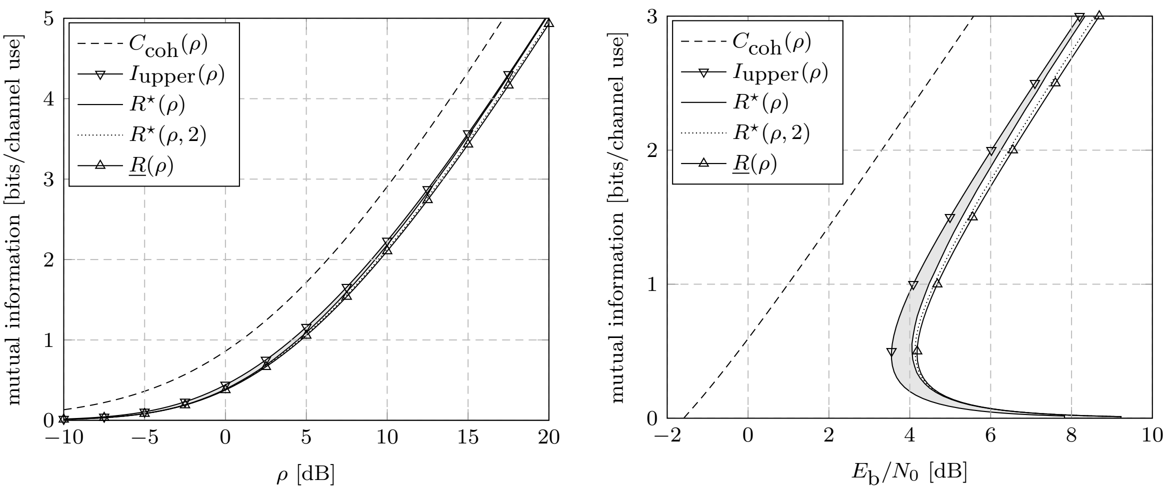
[3] The *capacity pre-log* is defined as the limiting ratio of the capacity to $\log(\rho)$ as ρ tends to infinity. In multiple-input multiple-output (MIMO) systems, it is sometimes also referred to as the *number of degrees of freedom* or the *multiplexing gain.*



(a) Bounds vs. SNR.

(b) Bounds vs. energy per information bit.

Figure 2.3 Prediction-based channel estimation.



(a) Bounds vs. SNR.

(b) Bounds vs. energy per information bit.

Figure 2.4 Interpolation-based channel estimation.

Figure 2.5 shows the same plots as Figure 2.4(a), except that all values have been divided by $\underline{R}(\rho)$ so as to visualize the *relative* improvement of the rate-splitting bounds over Médard's bound. We observe that, at low SNR, these improvements are negligible. This indicates that the rate-splitting bounds may be more interesting at moderate and high SNR than at low SNR.

2.2 Multiple-Input Multiple-Output (MIMO) Channels

We now extend the SISO model from the previous section to a MIMO model. For this purpose, we consider a multiple-antenna memoryless fading channel having n_{T} transmit antennas and n_{R}

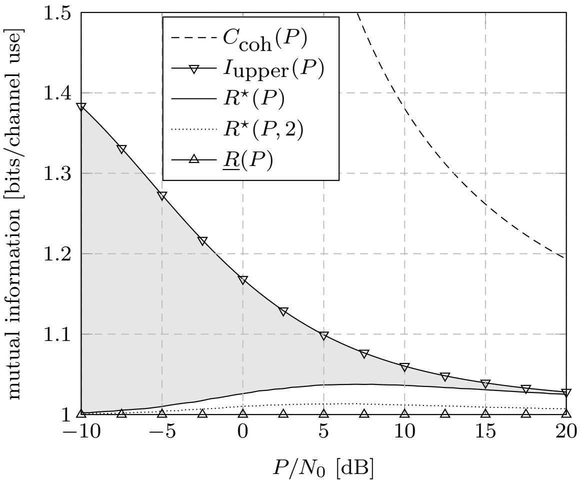


Figure 2.5 Bounds from Figure 2.4(a), divided by $\underline{R}(P)$.

receive antennas. At time instant $k \in \mathbb{Z}$, the channel output $\mathbf{y}[k] \in \mathbb{C}^{n_\mathsf{R}}$ which corresponds to the channel input $\mathbf{x}[k] = \mathsf{x} \in \mathbb{C}^{n_\mathsf{T}}$, is given by

$$\mathbf{y}[k] = \sqrt{\rho}(\hat{\mathbf{H}}[k] + \tilde{\mathbf{H}}[k])\mathsf{x} + \mathbf{n}[k]. \tag{2.49}$$

For simplicity, we assume that each of the three sequences $\{\mathbf{x}[k]\}_{k\in\mathbb{Z}}$, $\{\mathbf{n}[k]\}_{k\in\mathbb{Z}}$ and $\{(\hat{\mathbf{H}}[k], \tilde{\mathbf{H}}[k])\}_{k\in\mathbb{Z}}$ is ergodic and has independent and identically distributed (i.i.d.) elements. The noise $\mathbf{n}[k]$ and the input signal $\mathbf{x}[k]$ are assumed to have mean zero, and covariances $\mathbf{I}_{n_\mathsf{R}} = \mathsf{E}[\mathbf{n}[k]\mathbf{n}[k]^\dagger]$ and $\mathbf{Q} = \mathsf{E}[\mathbf{x}[k]\mathbf{x}[k]^\dagger]$, respectively, where $\mathbf{Q}$ fulfills the normalization $\operatorname{tr}(\mathbf{Q}) = 1$. Consequently, the scalar ρ stands for the SNR. The fading channel $\hat{\mathbf{H}}[k] + \tilde{\mathbf{H}}[k]$ is the sum of two components: the channel estimate $\hat{\mathbf{H}}[k] \in \mathbb{C}^{n_\mathsf{R}\times n_\mathsf{T}}$ and the channel estimation error $\tilde{\mathbf{H}}[k] \in \mathbb{C}^{n_\mathsf{R}\times n_\mathsf{T}}$, with respective means $\mathsf{E}[\hat{\mathbf{H}}[k]] = \mathbf{M}$ and $\mathsf{E}[\tilde{\mathbf{H}}[k]] = \mathbf{0}$. The receiver is cognizant of the joint distribution of $(\hat{\mathbf{H}}[k], \tilde{\mathbf{H}}[k])$ and of the sequence of channel estimates $\{\hat{\mathbf{H}}[k]\}_{k\in\mathbb{Z}}$, but ignores the estimation errors $\{\tilde{\mathbf{H}}[k]\}_{k\in\mathbb{Z}}$.

We assume that, conditioned on (almost) every $\hat{\mathbf{H}}[k] = \hat{\mathsf{H}}$, the channel estimate is unbiased, i.e.,[4]

$$\mathsf{E}[\tilde{\mathbf{H}}[k] \mid \hat{\mathsf{H}}] = \mathbf{0}. \tag{2.50}$$

Also, we assume that $\mathbf{x}[k]$, $\mathbf{n}[k]$, and $(\hat{\mathbf{H}}[k], \tilde{\mathbf{H}}[k])$ are mutually independent for every $k \in \mathbb{Z}$ (though $\hat{\mathbf{H}}[k]$ and $\tilde{\mathbf{H}}[k]$ may be mutually dependent). Without loss of generality, we shall assume that on average (over the $n_\mathsf{R} n_\mathsf{T}$ antennas) the channel coefficients have an expected second

[4] In systems where $\hat{\mathbf{H}}$ is a function of some channel side information Ω independent of the input $\mathbf{x}$ (e.g., a training observation in training-based channel estimation), (2.50) is tantamount to assuming $\hat{\mathbf{H}} = \mathsf{E}[\mathbf{H}|\Omega]$, i.e., $\hat{\mathbf{H}}$ is the minimum mean-square error estimator of $\mathbf{H}$ from Ω.

moment of one, that is,

$$\mathsf{E}[\|\mathbf{H}\|_{\mathsf{F}}^2] = \mathsf{E}[\|\hat{\mathbf{H}}[k]\|_{\mathsf{F}}^2] + \mathsf{E}[\|\tilde{\mathbf{H}}[k]\|_{\mathsf{F}}^2] = n_{\mathsf{R}}n_{\mathsf{T}} \tag{2.51}$$

where $\|\mathbf{A}\|_{\mathsf{F}} = \sqrt{\operatorname{tr}(\mathbf{A}^\dagger\mathbf{A})}$ denotes the Frobenius norm of $\mathbf{A}$.

The capacity of the MIMO system (2.49) is [Big98]

$$\begin{aligned} C = \sup_{\mathbf{x}} I(\mathbf{x};\mathbf{y}|\hat{\mathbf{H}}) &= h(\mathbf{x}) - h(\mathbf{x}|\mathbf{y},\hat{\mathbf{H}}) \\ &= h(\mathbf{y}|\hat{\mathbf{H}}) - h(\mathbf{y}|\mathbf{x},\hat{\mathbf{H}}) \end{aligned} \tag{2.52}$$

where the supremum is over the distribution of $\mathbf{x}$ subject to certain constraints. Due to the assumption that all sequences $\{\mathbf{x}[k]\}_{k\in\mathbb{Z}}$, $\{\mathbf{n}[k]\}_{k\in\mathbb{Z}}$ and $\{(\hat{\mathbf{H}}[k],\tilde{\mathbf{H}}[k])\}_{k\in\mathbb{Z}}$ are i.i.d. ergodic, we can again omit the time indices in (2.52) and throughout, since they are immaterial.

In the following, we will consider the *covariance-constrained* capacity, meaning that $\mathbf{x}$ is constrained to having a fixed covariance $\mathbf{Q} = \mathsf{E}[\mathbf{x}\mathbf{x}^\dagger]$. As for the SISO case, the capacity is only known in the coherent setting (i.e., when $\tilde{\mathbf{H}} = \mathbf{0}$ almost surely) [Tel99]

$$C_{\text{coh}} \triangleq \sup_{\mathbf{x}} I(\mathbf{x};\mathbf{y}|\mathbf{H}) = \mathsf{E}\left[\log\det\left(\mathbf{I}_{n_{\mathsf{T}}} + \rho\mathbf{H}^\dagger\mathbf{H}\mathbf{Q}\right)\right]. \tag{2.53}$$

In contrast, in the case of *imperfect* CSI treated here, in which the receiver knows $\hat{\mathbf{H}}$ but not $\mathbf{H}$, the capacity (2.52) is notoriously difficult to compute. Médard bound for the single-antenna case [M0́0] was extended to the MIMO case by Baltersee, Fock and Meyr [Bal01], as well as by Hassibi and Hochwald [Has03], who coined the term *worst-case noise* bound which we have adopted throughout this thesis.

By restraining the input to a Gaussian distribution ($\mathbf{x}_{\mathrm{G}}$ shall refer to a Gaussian input), and then deriving the worst-case noise lower bound on the Gaussian-input mutual information as in [M0́0], [Has03, Theorem 1], we get

$$\begin{aligned} C &\geq I(\mathbf{x}_{\mathrm{G}};\mathbf{y}|\hat{\mathbf{H}}) \\ &\geq \mathsf{E}\left[\log\det(\mathbf{I}_{n_{\mathsf{T}}} + \hat{\mathbf{H}}^\dagger\mathbf{\Gamma}^{-1}\hat{\mathbf{H}}\mathbf{Q})\right] \triangleq \underline{R} \end{aligned} \tag{2.54}$$

where $\rho\mathbf{\Gamma}$ denotes the covariance of the effective noise $\sqrt{\rho}\tilde{\mathbf{H}}\mathbf{x} + \mathbf{n}$ conditioned on $\hat{\mathbf{H}}$, i.e.,

$$\mathbf{\Gamma} = \mathsf{E}[\tilde{\mathbf{H}}\mathbf{Q}\tilde{\mathbf{H}}^\dagger \mid \hat{\mathbf{H}}] + \rho^{-1}\mathbf{I}_{n_{\mathsf{R}}}. \tag{2.55}$$

To complement this lower bound, we also state a corresponding mutual information upper bound for Gaussian inputs, which holds under the assumption that $\tilde{\mathbf{H}}$ is Gaussian conditioned on (almost) every $\hat{\mathbf{H}} = \hat{\mathsf{H}}$. It can be derived in a straightforward way along the same lines as

(1.34):

$$I(\mathbf{x}_\mathrm{G}; \mathbf{y}|\hat{\mathbf{H}}) \leq \underline{R} + \Delta \tag{2.56}$$

where

$$\Delta = \mathsf{E}\left[\log\left(\frac{\det(\mathsf{E}[\tilde{\mathbf{H}}\mathbf{Q}\tilde{\mathbf{H}}^\dagger \mid \hat{\mathbf{H}}] + \rho^{-1}\mathbf{I}_r)}{\det(\mathsf{E}[\tilde{\mathbf{H}}\mathbf{Q}^{\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}^\dagger\mathbf{Q}^{\frac{1}{2}}\tilde{\mathbf{H}}^\dagger \mid \boldsymbol{\xi}, \hat{\mathbf{H}}] + \rho^{-1}\mathbf{I}_r)}\right)\right]. \tag{2.57}$$

where $\boldsymbol{\xi} \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{I}_{n_\mathsf{T}})$.

Thanks to its simplicity, the lower bound $\underline{R}$ has become widely popular as a means to derive achievable rate expressions for the noncoherent MIMO setting (by inclusion of a training scheme as in [Has03]), or for the partially coherent setting, in which an erroneous channel estimate is assumed to be available to the receiver (e.g., [Yoo06]). This bound has been used subsequently in a large body of work on multiple-antenna systems (e.g. in [Yoo04, Mus05, Soy10, Din10], to cite only a few). In the following, we show how the lower bound (2.54) can be sharpened using rate splitting, in full analogy to the SISO case treated in the previous section.

2.2.1 The Rate-Splitting Approach

Upon decomposing the Gaussian transmit signal into a sum $\mathbf{x}_\mathrm{G} = \sum_{\ell=1}^{L} \mathbf{x}_\ell$ of mutually independent Gaussian subsignals $\mathbf{x}_\ell$, the mutual information chain rule yields

$$I(\mathbf{x}_\mathrm{G}; \mathbf{y}|\hat{\mathbf{H}}) = \sum_{\ell=1}^{L} I(\mathbf{x}_\ell; \mathbf{y}|\mathbf{x}^{\ell-1}, \hat{\mathbf{H}}) \tag{2.58}$$

where $\mathbf{x}^{\ell-1}$ denotes the collection $(\mathbf{x}_1, \ldots, \mathbf{x}_{\ell-1})$ and $\mathbf{x}^0$ is the empty collection. By lower-bounding each summand on the right-hand side of (2.58) using the worst-case-noise lower-bounding technique, one ends up with a lower bound that is sharper than the conventional single-layer bound ($L = 1$).

In analogy to the SISO case, we show that the optimal rate-splitting approach (in terms of improving the mutual information bound) consists in letting the number of layers tend to infinity, in such a way that the powers and rates associated to each layer become infinitesimally small. An analytic expression for this infinite-layering limit is proposed which constitutes an improved capacity lower bound of MIMO channels with imperfect receiver CSI.

However, in contrast to the SISO case, in the case of multiple transmit antennas there is an infinitely large family of such infinite-layering approaches, in that there are infinitely many possibilities of spatially decomposing a Gaussian transmit signal of given covariance into a sum of independent Gaussian subsignals (each associated to a layer). Therefore, the rate-splitting approach gives rise to a whole family of improved capacity bounds, out of which the best bound is not easy to determine. Nonetheless, any arbitrary layering yields an improved capacity bound and we will derive analytical bound expressions for two simple exemplary layerings in the following chapter.

Consider the transmit covariance $\mathbf{Q}$ to be fixed throughout. In a system which performs rate splitting and successive decoding with L layers, we write the Gaussian transmit signal $\mathbf{x}_\mathrm{G}$ as a sum of mutually independent $\mathbf{x}_\ell \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{Q}_\ell)$ with $\ell = 1, \dots, L$, satisfying

$$\mathbf{x}_\mathrm{G} = \sum_{\ell=1}^{L} \mathbf{x}_\ell \qquad \mathbf{Q} = \sum_{\ell=1}^{L} \mathbf{Q}_\ell \tag{2.59}$$

We further assume that all $\mathbf{x}_\ell$ are independent of $(\hat{\mathbf{H}}, \tilde{\mathbf{H}})$. The sequence of transmit covariances $(\mathbf{Q}_\ell)_{\ell=1,\dots,L}$ is thus denoted as $\mathbf{Q}^L$. Since $\mathbf{y}$ depends on $\mathbf{x}^L$ only via the sum $\mathbf{x}_\mathrm{G}$ of its elements, $\mathbf{y}$ has the same distribution whether it is conditioned on $\mathbf{x}_\mathrm{G}$ or on $\mathbf{x}^L$, so

$$I(\mathbf{x}_\mathrm{G}; \mathbf{y}|\hat{\mathbf{H}}) = I(\mathbf{x}^L; \mathbf{y}|\hat{\mathbf{H}}) = \sum_{\ell=1}^{L} I(\mathbf{x}_\ell; \mathbf{y}|\mathbf{x}^{\ell-1}, \hat{\mathbf{H}}). \tag{2.60}$$

The second equality follows from the chain rule for mutual information.

To lower-bound the mutual information terms appearing in the sum on the right-hand side of (2.60), we state a well-known result by Hassibi and Hochwald [Has03, Theorem 1] in a version that is tailored for our needs. Its proof, provided in Appendix B.11, is largely similar to that found in the original paper [Has03], and its proof goes along the same lines as that of Lemma 2.1.

Theorem 2.4. *We have the following lower bound on the ℓ-th term in the sum* (2.60)*:*

$$I(\mathbf{x}_\ell; \mathbf{y}|\mathbf{x}^{\ell-1}, \hat{\mathbf{H}}) \geq \mathsf{E}\left[\log\det\left(\mathbf{I}_{n_\mathsf{T}} + \hat{\mathbf{H}}^\dagger \mathbf{\Gamma}_\ell^{-1} \hat{\mathbf{H}} \mathbf{Q}_\ell\right)\right] \tag{2.61}$$

Here, the random matrix $\mathbf{\Gamma}_\ell$ is given by

$$\mathbf{\Gamma}_\ell \triangleq \mathsf{E}[\tilde{\mathbf{H}} \underline{\mathbf{Q}}_\ell^{\frac{1}{2}} \boldsymbol{\xi}\boldsymbol{\xi}^\dagger \underline{\mathbf{Q}}_\ell^{\frac{1}{2}} \tilde{\mathbf{H}}^\dagger \mid \boldsymbol{\xi}, \hat{\mathbf{H}}] + \mathsf{E}[\tilde{\mathbf{H}} \mathbf{Q}_\ell \tilde{\mathbf{H}}^\dagger \mid \hat{\mathbf{H}}] + \hat{\mathbf{H}} \overline{\mathbf{Q}}_\ell \hat{\mathbf{H}}^\dagger + \mathsf{E}[\tilde{\mathbf{H}} \overline{\mathbf{Q}}_\ell \tilde{\mathbf{H}}^\dagger \mid \hat{\mathbf{H}}] + \rho^{-1} \mathbf{I}_{n_\mathsf{R}} \tag{2.62}$$

where $\boldsymbol{\xi} \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{I}_{n_\mathsf{T}})$ is independent of $\hat{\mathbf{H}}$, and we have used abbreviations $\underline{\mathbf{Q}}_\ell = \sum_{i=1}^{\ell-1} \mathbf{Q}_i$ and $\overline{\mathbf{Q}}_\ell = \sum_{i=\ell+1}^{L} \mathbf{Q}_i$.

Proof: See Appendix B.11. ■

The covariance (2.62) contains four terms, each of which has a physical interpretation. When decoding the ℓ-th layer, the decoder treats the following terms as noise:

(1) $\mathsf{E}[\tilde{\mathbf{H}} \underline{\mathbf{Q}}_\ell^{\frac{1}{2}} \boldsymbol{\xi}\boldsymbol{\xi}^\dagger \underline{\mathbf{Q}}_\ell^{\frac{1}{2}} \tilde{\mathbf{H}}^\dagger \mid \boldsymbol{\xi}, \hat{\mathbf{H}}]$ is due to residual interference from previously decoded layers $1, \dots, \ell-1$, which could not be cancelled due to the imperfect CSI;

(2) $\mathsf{E}[\tilde{\mathbf{H}} \mathbf{Q}_\ell \tilde{\mathbf{H}}^\dagger \mid \hat{\mathbf{H}}]$ accounts for the imperfection of CSI impairing the decoding of the ℓ-th layer;

(3) $\hat{\mathbf{H}}\overline{\mathbf{Q}}_\ell\hat{\mathbf{H}}^\dagger + \mathsf{E}[\tilde{\mathbf{H}}\overline{\mathbf{Q}}_\ell\tilde{\mathbf{H}}^\dagger \mid \hat{\mathbf{H}}]$ can be assimilated to the interference caused by yet undecoded layers $\ell+1,\ldots,L$;

(4) $\rho^{-1}\mathbf{I}_{n_\mathsf{R}}$ reflects the independent additive white noise.

2.2.2 Layering Functions and Indexings

A rate-splitting allocation is specified by a decomposition of the transmit covariance $\mathbf{Q}$ into a sum of $\mathbf{Q}_\ell \in \mathbb{C}_+^{n_\mathsf{T}\times n_\mathsf{T}}$, each associated to (2.59). In order to facilitate the systematic treatment of such rate-splitting schemes, let us introduce so-called *layering functions* and *indexings*.

Definition 2.1 (Layering function)**.** *A function* $\mathbf{L}\colon [0;1] \to \mathbb{C}_+^{n_\mathsf{T}\times n_\mathsf{T}}, \iota \mapsto \mathbf{L}(\iota)$ *which fulfills*

(1) $\mathbf{L}(0) = \mathbf{0}$ *and* $\mathbf{L}(1) = \mathbf{Q}$ *(border values);*

(2) $\mathbf{L}(\iota)$ *for all* $\iota \in [0;1]$ *(positive semidefiniteness);*

(3) $\iota_1 < \iota_2 \to \mathbf{L}(\iota_1) \preceq \mathbf{L}(\iota_2)$ *(monotonicity);*

(4) $\mathrm{tr}(\mathbf{L}(\iota)) = \iota$ *(normalization)*

is called a layering function. The set of all layering functions is denoted as $\mathbb{L}$*.*

A further property of layering functions which follows directly from the properties listed in Definition 2.1 is Lipschitz-continuity: the entries $L_{i,j}(\iota)$ of a layering function $\mathbf{L}(\iota)$ are Lipschitz-continuous with modulus 1. This means that for any $(\iota_1,\iota_2) \in [0;1]^2$, we have

$$\big|L_{i,j}(\iota_1) - L_{i,j}(\iota_2)\big| \leq |\iota_1 - \iota_2|. \tag{2.63}$$

A proof is provided in Appendix B.13. This Lipschitz-continuity will be useful later.

Definition 2.2. *A collection of* $L+1$ *distinct real numbers* $\mathscr{I} = \{\iota_0,\iota_1,\ldots,\iota_L\} \in [0;1]^{L+1}$ *containing at least the two elements* 0 *and* 1*, is referred to as an L-indexing. The elements of an L-indexing are labelled in ascending order, i.e.,* $0 = \iota_0 < \iota_1 < \ldots < \iota_L = 1$*. If an indexing* $\mathscr{I}$ *is contained in another indexing* $\mathscr{I}'$ *such that* $\mathscr{I} \neq \mathscr{I}'$*, we shall say that* $\mathscr{I}'$ *is a refinement of* $\mathscr{I}$*. The set of all L-indexings is denoted as* $\mathbb{I}(L)$*, while the set of all indexings* $\bigcup_{L\in\mathbb{N}}\mathbb{I}(L)$ *is denoted as* $\mathbb{I}$*.*

For any such layering, we further define the following notations:

$$\mathbf{L}_\ell^{\mathscr{I}} \triangleq \mathbf{L}(\iota_\ell), \qquad \ell = 0,\ldots,L \tag{2.64}$$

$$\Delta\mathbf{L}_\ell^{\mathscr{I}} \triangleq \mathbf{L}(\iota_\ell) - \mathbf{L}(\iota_{\ell-1}), \quad \ell = 1,\ldots,L. \tag{2.65}$$

Note that as a consequence of the properties of layering functions [cf. Definition 2.1], the terms $\Delta\mathbf{L}_\ell^{\mathscr{I}}$ are non-zero and positive semidefinite, and they sum up to $\mathbf{Q}$.

Therefore, to characterize a rate-splitting scheme, instead of specifying a collection $(\mathbf{Q}_1, \ldots, \mathbf{Q}_L)$, we will find it more convenient to specify a layering function and an indexing, and relate both descriptions via

$$\mathbf{Q}_\ell = \Delta\mathbf{L}_\ell^{\mathscr{I}}. \tag{2.66}$$

Hence, from now on, a layering shall be a pair $(\mathbf{L}, \mathscr{I}) \in \mathbb{L} \times \mathbb{I}$.

Upon making appropriate replacements, we can express the rate-splitting bound (2.61) as a function of the layering $(\mathbf{L}, \mathscr{I})$, namely

$$\begin{aligned} I(\mathbf{x}_\mathrm{G}; \mathbf{y}|\hat{\mathbf{H}}) &\geq \sum_{\ell=1}^{L} \mathsf{E} \log\det\Big(\mathbf{I}_{n_\mathsf{T}} + \hat{\mathbf{H}}^\dagger (\mathbf{\Gamma}_\ell^{\mathscr{I}})^{-1} \hat{\mathbf{H}}\, \Delta\mathbf{L}_\ell^{\mathscr{I}}\Big) \\ &\triangleq R(\mathbf{L}, \mathscr{I}) \end{aligned} \tag{2.67}$$

where

$$\begin{aligned} \mathbf{\Gamma}_\ell^{\mathscr{I}} \triangleq \mathsf{E}[\tilde{\mathbf{H}}(\mathbf{L}_{\ell-1}^{\mathscr{I}})^{\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}^\dagger(\mathbf{L}_{\ell-1}^{\mathscr{I}})^{\frac{1}{2}}\tilde{\mathbf{H}}^\dagger \mid \boldsymbol{\xi}, \hat{\mathbf{H}}] + \mathsf{E}[\tilde{\mathbf{H}}\Delta\mathbf{L}_\ell^{\mathscr{I}}\tilde{\mathbf{H}}^\dagger \mid \hat{\mathbf{H}}] +& \\ + \hat{\mathbf{H}}\bar{\mathbf{L}}_\ell^{\mathscr{I}}\hat{\mathbf{H}}^\dagger + \mathsf{E}[\tilde{\mathbf{H}}\bar{\mathbf{L}}_\ell^{\mathscr{I}}\tilde{\mathbf{H}}^\dagger \mid \hat{\mathbf{H}}] + \rho^{-1}\mathbf{I}_{n_\mathsf{R}}& \end{aligned} \tag{2.68}$$

where $\bar{\mathbf{L}}_\ell^{\mathscr{I}} \triangleq \mathbf{Q} - \mathbf{L}_\ell^{\mathscr{I}}$ and where $\boldsymbol{\xi} \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{I}_{n_\mathsf{T}})$ is independent of $\hat{\mathbf{H}}$.

Note that for a single layer, i.e., for $\mathscr{I} = \{0, 1\}$, the rate-splitting bound $R(\mathbf{L}, \{0, 1\})$ coincides with the conventional worst-case-noise bound $\underline{R}$.

2.2.3 Rate Splitting with an Arbitrary Finite Number of Layers

It can be readily verified that any rate-splitting bound $R(\mathbf{L}, \mathscr{I})$ is not smaller than the conventional bound (2.54), i.e.,

$$\underline{R} \leq R(\mathbf{L}, \mathscr{I}) \leq I(\mathbf{x}_\mathrm{G}; \mathbf{y}|\hat{\mathbf{H}}). \tag{2.69}$$

In fact, by averaging $\mathbf{\Gamma}_\ell^{\mathscr{I}}$ over $\boldsymbol{\xi}$, we observe that [cf. (2.55), (2.68)]

$$\mathsf{E}[\mathbf{\Gamma}_\ell^{\mathscr{I}} \mid \hat{\mathbf{H}}] = \mathbf{\Gamma} + \hat{\mathbf{H}}\overline{\mathbf{L}}_\ell\hat{\mathbf{H}}^\dagger \tag{2.70}$$

and upon noting that the function $\mathbf{X} \mapsto \log\det(\mathbf{I} + \mathbf{A}\mathbf{X}^{-1})$ for $\mathbf{A} \succeq \mathbf{0}$ is convex over the set of positive definite $\mathbf{X} \succ \mathbf{0}$, we can apply Jensen's inequality to get

$$\begin{aligned} R(\mathbf{L}, \mathscr{I}) &\geq \sum_{\ell=1}^{L} \mathsf{E} \log\det\Big(\mathbf{I}_{n_\mathrm{T}} + \hat{\mathbf{H}}^\dagger\, \mathsf{E}[\boldsymbol{\Gamma}_\ell^{\mathscr{I}} | \hat{\mathbf{H}}]^{-1} \hat{\mathbf{H}}\, \Delta \mathbf{L}_\ell\Big) \\ &= \sum_{\ell=1}^{L} \mathsf{E}\left[\log\det\Big(\boldsymbol{\Gamma} + \hat{\mathbf{H}}\overline{\mathbf{L}}_\ell^{\mathscr{I}} \hat{\mathbf{H}}^\dagger + \hat{\mathbf{H}}\Delta\mathbf{L}_\ell\hat{\mathbf{H}}^\dagger\Big) - \log\det\Big(\boldsymbol{\Gamma} + \hat{\mathbf{H}}\overline{\mathbf{L}}_\ell^{\mathscr{I}} \hat{\mathbf{H}}^\dagger\Big)\right] \\ &= \sum_{\ell=1}^{L} \mathsf{E}\left[\log\det\Big(\boldsymbol{\Gamma} + \hat{\mathbf{H}}\overline{\mathbf{L}}_{\ell-1}^{\mathscr{I}} \hat{\mathbf{H}}^\dagger\Big) - \log\det\Big(\boldsymbol{\Gamma} + \hat{\mathbf{H}}\overline{\mathbf{L}}_\ell^{\mathscr{I}} \hat{\mathbf{H}}^\dagger\Big)\right] \\ &= \mathsf{E}\left[\log\det\Big(\boldsymbol{\Gamma} + \hat{\mathbf{H}}\overline{\mathbf{L}}_0^{\mathscr{I}} \hat{\mathbf{H}}^\dagger\Big) - \log\det(\boldsymbol{\Gamma})\right] \\ &= \underline{R} \end{aligned} \tag{2.71}$$

with equality in the single-layer case, in which $\mathscr{I} = \{0, 1\}$. The last equality follows because $\overline{\mathbf{L}}_0^{\mathscr{I}} = \mathbf{Q}$ and from comparison with (2.54) and (2.55).

The following Theorem generalizes the above finding to arbitrary indexing refinements.

Theorem 2.5. *For any fixed layering function* $\mathbf{L}$*, if the indexing* $\mathscr{I}'$ *is a refinement of* $\mathscr{I}$*, then* $\mathscr{I}'$ *yields a rate not smaller than* $\mathscr{I}$*. Stated formally,*

$$\mathscr{I} \subsetneq \mathscr{I}' \rightarrow R(\mathbf{L}, \mathscr{I}) \leq R(\mathbf{L}, \mathscr{I}'). \tag{2.72}$$

Proof: See Appendix B.12. ■

The left-hand side of (2.69) then follows by noting that $\underline{R} = R(\mathbf{L}, \{0, 1\})$ and that $\{0, 1\} \subseteq \mathscr{I}$ for any indexing $\mathscr{I} \in \mathbb{I}$. This Theorem is proved in a similar way as in the derivation above, using Jensen's inequality.

2.2.4 Rate Splitting with an Infinite Number of Layers

We are interested in finding the best among all rate-splitting bounds. That is, we seek to optimize the indexings and layering functions so as to approach the following supremum:

$$R^{\star\star} \triangleq \sup_{(\mathbf{L}, \mathscr{I}) \in \mathbb{L} \times \mathbb{I}} R(\mathbf{L}, \mathscr{I}). \tag{2.73}$$

Let us also define the supremum over the indexings alone:

$$R^{\star}(\mathbf{L}) \triangleq \sup_{\mathscr{I} \in \mathbb{I}} R(\mathbf{L}, \mathscr{I}). \tag{2.74}$$

Clearly, for any layering $(\mathbf{L}, \mathscr{I}) \in \mathbb{L} \times \mathbb{I}$ we have

$$\underline{R} \leq R(\mathbf{L}, \mathscr{I}) \leq R^{\star}(\mathbf{L}) \leq R^{\star\star} \leq I(\mathbf{x}_\mathrm{G}; \mathbf{y} | \hat{\mathbf{H}}) \tag{2.75}$$

where $R^{\star\star}$ represents the best rate-splitting bound. Unfortunately, the computation of $R^{\star\star}$ seems elusive. However, given any layering function $\mathbf{L} \in \mathbb{L}$, we can derive an analytic expression for $R^{\star}(\mathbf{L})$, which is given by Theorem 2.6 below.

For this purpose, observe that from Theorem 2.5 we can infer that for a fixed layering function $\mathbf{L}$,

$$L \mapsto \sup_{\mathscr{I} \in \mathbb{I}(L)} R(\mathbf{L}, \mathscr{I}) \tag{2.76}$$

is a non-decreasing function. Therefore, $R^{\star}(\mathbf{L})$ equals the limit of (2.76) as the number of layers tends to infinity:

$$R^{\star}(\mathbf{L}) = \lim_{L\to\infty} \left\{ \sup_{\mathscr{I} \in \mathbb{I}(L)} R(\mathbf{L}, \mathscr{I}) \right\}. \tag{2.77}$$

Theorem 2.6 below provides an analytic expression for this limit.

Definition 2.3 (Riemann-Stieltjes integral)**.** *Consider a function $f\colon [a;b] \subseteq [-\infty;+\infty] \to \mathbb{R}$ and a function $g\colon \mathbb{R} \to \mathbb{R}$. For any set $\mathcal{X} = \{x_0, \ldots, x_N\} \in [a;b]^{N+1}$ of distinct elements with $a = x_0 < \ldots < x_N = b$, let the lower and upper sum be respectively defined as*

$$\underline{S}(f, g, \mathcal{X}) = \sum_{n=1}^{N} \inf_{x \in [x_n; x_{n-1}]} f(x)\big(g(x_{n+1}) - g(x_n)\big) \tag{2.78}$$

$$\overline{S}(f, g, \mathcal{X}) = \sum_{n=1}^{N} \sup_{x \in [x_n; x_{n-1}]} f(x)\big(g(x_{n+1}) - g(x_n)\big). \tag{2.79}$$

If, over all partitions $\mathcal{X}$, the infimum of the upper sum and the supremum of the lower sum coincide, then the common value is denoted as

$$\int_a^b f(x)\,\mathrm{d}g(x) = \sup_{\mathcal{X}} \underline{S}(f, g, \mathcal{X}) = \inf_{\mathcal{X}} \overline{S}(f, g, \mathcal{X}) \tag{2.80}$$

and is called the Riemann-Stieltjes integral of the integrand function f over $[a;b]$ with integrator function g. The notation $\int_0^1 \operatorname{tr}\big[\mathbf{F}(\iota)\,\mathrm{d}\mathbf{G}(\iota)\big]$ with a matrix-valued integrand $\mathbf{F}(\iota) = [F_{i,j}(\iota)]_{i,j}$ and a matrix-valued integrator $\mathbf{G}(\iota) = [G_{i,j}(\iota)]_{i,j}$ shall be a compact notation for the Riemann-Stieltjes integral $\int_0^1 \sum_{i,j} F_{i,j}(\iota)\,\mathrm{d}G_{j,i}(\iota)$.

Theorem 2.6. *For any prescribed layering function $\mathbf{L}$, the best rate-splitting bound is given by*

$$\boxed{R^{\star}(\mathbf{L}) = \int_0^1 \mathsf{E}\operatorname{tr}\Big[\hat{\mathbf{H}}^{\dagger}\boldsymbol{\Gamma}\big(\mathbf{L}(\iota)\big)^{-1}\hat{\mathbf{H}}\,\mathrm{d}\mathbf{L}(\iota)\Big]} \tag{2.81}$$

where the function $\boldsymbol{\Gamma}\colon \mathbb{C}_{+}^{n_\mathsf{T}\times n_\mathsf{T}} \to \mathbb{C}^{n_\mathsf{R}\times n_\mathsf{R}}$ is given by

$$\boldsymbol{\Gamma}(\mathbf{X}) = \mathsf{E}\big[\tilde{\mathbf{H}}\mathbf{X}^{\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}^{\dagger}\mathbf{X}^{\frac{1}{2}}\tilde{\mathbf{H}}^{\dagger} \mid \boldsymbol{\xi}, \hat{\mathbf{H}}\big] + \hat{\mathbf{H}}(\mathbf{Q} - \mathbf{X})\hat{\mathbf{H}}^{\dagger} + \mathsf{E}\big[\tilde{\mathbf{H}}(\mathbf{Q} - \mathbf{X})\tilde{\mathbf{H}}^{\dagger} \mid \hat{\mathbf{H}}\big] + \rho^{-1}\mathbf{I}_{n_\mathsf{R}} \tag{2.82}$$

with $\boldsymbol{\xi} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I}_{n_\mathsf{T}})$ independent of $\hat{\mathbf{H}}$.

Proof: See Appendix B.14. ■

Numerical simulations of this new class of mutual information bounds are postponed to Section 3.2 in the next chapter. Meanwhile, a couple of remarks are due concerning Theorem 2.6.

Remark 2.6. *The integral notation involving the infinitesimal* $\mathrm{d}\mathbf{L}$ *inside the trace operator should be understood as a compact matrix notation for a linear combination of scalar Riemann-Stieltjes integrals. Specifically, the notation* $\int_0^1 \operatorname{tr}[\mathbf{A}(\iota)\,\mathrm{d}\mathbf{B}(\iota)]$ *with matrix-valued functions* $\mathbf{A}(\iota) = [A_{i,j}(\iota)]_{i,j}$ *and* $\mathbf{B}(\iota) = [B_{i,j}(\iota)]_{i,j}$ *stands for the sum of Riemann-Stieltjes integrals* $\sum_{i,j}\int_0^1 A_{i,j}(\iota)\,\mathrm{d}B_{j,i}(\iota)$.

Remark 2.7. *If the layering function* $\mathbf{L}$ *is (entrywise) continuously differentiable and its derivative with respect to* ι *is denoted as* $\dot{\mathbf{L}}(\iota)$*, then* (2.81) *can be written as a Riemann integral*

$$\boxed{R^\star(\mathbf{L}) = \int_0^1 \mathsf{E}\operatorname{tr}\Big[\hat{\mathbf{H}}^\dagger \mathbf{\Gamma}(\mathbf{L}(\iota))^{-1}\hat{\mathbf{H}}\dot{\mathbf{L}}(\iota)\Big]\,\mathrm{d}\iota} \tag{2.83}$$

Remark 2.8. *Assuming an entrywise continuously differentiable layering function* $\mathbf{L}$*, when one writes* $R^\star(\mathbf{L})$ *as a Riemann integral as in* (2.83)*, then the single-layer bound* $\underline{R}$ *can be promptly recovered by means of Jensen's inequality. Indeed, since* $\dot{\mathbf{L}}$ *is positive semidefinite, the mapping* $\mathbf{X} \mapsto \operatorname{tr}(\hat{\mathbf{H}}^\dagger\mathbf{X}^{-1}\hat{\mathbf{H}}\dot{\mathbf{L}})$ *is convex on the cone of positive definite matrices, so we infer that by moving the expectation over* $\boldsymbol{\xi}$ *from outside the trace operator onto the inverted matrix* $\mathbf{\Gamma}(\mathbf{L}(\iota))$*, one obtains a lower bound on* $R^\star(\mathbf{L})$*, which after solving the integral over* ι *turns out to be* $\underline{R}$.

Remark 2.9. *Notice that, using the identity*

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\log\det\left(\mathbf{A}+\hat{\mathbf{H}}\mathbf{B}(\alpha)\hat{\mathbf{H}}^\dagger\right) = \operatorname{tr}\left(\hat{\mathbf{H}}^\dagger\left(\mathbf{A}+\hat{\mathbf{H}}\mathbf{B}(\alpha)\hat{\mathbf{H}}^\dagger\right)^{-1}\hat{\mathbf{H}}\dot{\mathbf{B}}(\alpha)\right) \tag{2.84}$$

one can easily verify that, if $\hat{\mathbf{H}}$ *and* $\tilde{\mathbf{H}}$ *are mutually independent, then* (2.83) *can be rewritten as*

$$\boxed{R^\star(\mathbf{L}) = -\,\mathsf{E}\left[\int_0^1 K_\iota\left(\mathsf{E}\Big[\tilde{\mathbf{H}}(\mathbf{L}(\iota)^{\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}^\dagger\mathbf{L}(\iota)^{\frac{1}{2}} - \mathbf{L}(\iota))\tilde{\mathbf{H}}^\dagger \,\Big|\, \boldsymbol{\xi}\Big]\right)\mathrm{d}\iota\right]} \tag{2.85}$$

where

$$K_\iota(\mathbf{\Delta}) = \frac{\mathrm{d}}{\mathrm{d}\iota}\,\mathsf{E}\Big[\log\det\left(\mathbf{\Delta} + \mathsf{E}[\tilde{\mathbf{H}}\mathbf{Q}\tilde{\mathbf{H}}^\dagger] + \rho^{-1}\mathbf{I}_{n_\mathsf{R}} + \hat{\mathbf{H}}(\mathbf{Q}-\mathbf{L}(\iota))\hat{\mathbf{H}}^\dagger\right)\Big] \tag{2.86}$$

This alternative representation will be useful to prove Theorem 3.1 in Chapter 3.

2.2.5 Further Properties of Layering Functions

We provide three lemmata which shed more light on the function $R^\star(\mathbf{L})$ and on the set of layering functions.

Lemma 2.3 (Continuity in the layering function)**.** *If* $\mathsf{E}[\|\hat{\mathbf{H}}\|_F^4] < \infty$*, then the function* $R^\star(\mathbf{L})$

is uniformly continuous in $\mathbf{L}$. *This means that, for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\|\mathbf{L}_1 - \mathbf{L}_2\|_\infty \leq \delta \Rightarrow \big|R^\star(\mathbf{L}_1) - R^\star(\mathbf{L}_2)\big| \leq \epsilon. \tag{2.87}$$

Proof: See Appendix B.15. ■

The condition that the fourth moment of $\hat{\mathbf{H}}$ must be finite may not be necessary for the continuity to hold, but the proposed proof given in Appendix B.15 relies on this assumption. The next two lemmata below also require this finiteness because they rely on Lemma 2.3.

Let $\mathbb{L}_\mathcal{D}$ denote the set of (entrywise) continuously differentiable layering functions.

Lemma 2.4 (Differentiable layering functions). *If* $\mathsf{E}\big[\|\hat{\mathbf{H}}\|_F^4\big] < \infty$, *the set* $\mathbb{L}_\mathcal{D}$ *is dense in* $\mathbb{L}$ *in the sense that for every* $\mathbf{L} \in \mathbb{L}$ *and* $\epsilon > 0$, *there exists a* $\widetilde{\mathbf{L}} \in \mathbb{L}_\mathcal{D}$ *such that* $\|\mathbf{L} - \widetilde{\mathbf{L}}\|_\infty < \epsilon$.

Proof: See Appendix B.17. ■

Corollary 2.2. *The best rate-splitting bound* $R^{\star\star}$ *is the supremum of* $R^\star(\mathbf{L})$ *over* $\mathbb{L}_\mathcal{D}$, *i.e.,*

$$R^{\star\star} = \sup_{\mathbf{L}\in\mathbb{L}} R^\star(\mathbf{L}) = \sup_{\mathbf{L}\in\mathbb{L}_\mathcal{D}} R^\star(\mathbf{L}). \tag{2.88}$$

Corollary 2.2 allows to restrict the analysis to continuously differentiable layering functions, and therefore to expressions of $R^\star(\mathbf{L})$ involving a simple Riemann integral (2.83) instead of a Riemann-Stieltjes integral.

Lemma 2.5 (Optimal layering). *If* $\mathsf{E}\big[\|\hat{\mathbf{H}}\|_F^4\big] < \infty$, *there exists an optimal layering function*

$$\mathbf{L}^\star = \operatorname*{argmax}_{\mathbf{L}\in\mathbb{L}} R^\star(\mathbf{L}). \tag{2.89}$$

Proof: See Appendix C.1. ■

In other words, the supremum (2.73) can as well be written as a maximum, which is achieved by $\mathbf{L}^\star$ such that $R^{\star\star} = R^\star(\mathbf{L}^\star)$. For more than a single transmit antenna, there are many possible choices for the layering function. Determining the best layering function $\mathbf{L}^\star$ constitutes a difficult variational problem, and seems beyond reach. As a remedy, one can simply choose $\mathbf{L}(\iota) = \iota\mathbf{Q}$, although this choice is not generally optimal. To acquire some understanding on how the layering function affects the rate-splitting bound, in the next Section we particularize the fading model and define two special layering functions, to be compared in simulations. Note, however, that the optimal layering function $\mathbf{L}^\star$ need not belong to $\mathbf{L} \in \mathbb{L}_\mathcal{D}$.

3

Rate-Splitting Bounds for the MIMO IID Rayleigh Fading Channel

To gain additional insights, we dedicate this section to the study of the special case of spatially uncorrelated and mutually independent $\mathrm{vec}(\hat{\mathbf{H}}) \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \hat{V}\mathbf{I}_{n_\mathsf{R} n_\mathsf{T}})$ and $\mathrm{vec}(\tilde{\mathbf{H}}) \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \tilde{V}\mathbf{I}_{n_\mathsf{R} n_\mathsf{T}})$, with $\hat{V} + \tilde{V} = 1$. Accordingly, we will write the random matrices $\hat{\mathbf{H}}$ and $\tilde{\mathbf{H}}$ as

$$\hat{\mathbf{H}} = \sqrt{\hat{V}}\hat{\mathbf{W}} \qquad\qquad \tilde{\mathbf{H}} = \sqrt{\tilde{V}}\tilde{\mathbf{W}} \tag{3.1}$$

where $\hat{\mathbf{W}} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I}_{n_\mathsf{R} n_\mathsf{T}})$ and $\tilde{\mathbf{W}} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I}_{n_\mathsf{R} n_\mathsf{T}})$ are mutually independent[1]. In the remainder of this chapter, we shall refer to this situation as the *i.i.d. Rayleigh fading* assumption.

Under these assumptions, the worst-case-noise bound reads as [cf. (2.54)]

$$\underline{R} = \mathsf{E}\left[\log\det\left(\mathbf{I}_{n_\mathsf{T}} + \frac{\hat{V}}{\tilde{V} + \rho^{-1}}\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}\mathbf{Q}\right)\right] \tag{3.2}$$

whereas the infinite-layering bound reads as [cf. (2.81)]

$$\boxed{R^\star(\mathbf{L}) = \hat{V}\int_0^1 \mathsf{E}\,\mathrm{tr}\Big[\hat{\mathbf{W}}^\dagger\mathbf{\Gamma}(\mathbf{L}(\iota))^{-1}\hat{\mathbf{W}}\,\mathrm{d}\mathbf{L}(\iota)\Big]} \tag{3.3}$$

where [cf. (2.82)]

$$\mathbf{\Gamma}(\mathbf{L}(\iota)) = \tilde{V}\boldsymbol{\xi}^\dagger\mathbf{L}(\iota)\boldsymbol{\xi}\cdot\mathbf{I}_{n_\mathsf{R}} + \hat{V}\hat{\mathbf{W}}\bar{\mathbf{L}}(\iota)\hat{\mathbf{W}}^\dagger + (1-\iota)\tilde{V}\mathbf{I}_{n_\mathsf{R}} + \rho^{-1}\mathbf{I}_{n_\mathsf{R}} \tag{3.4}$$

where $\bar{\mathbf{L}}(\iota) \triangleq \mathbf{Q} - \mathbf{L}(\iota)$, and where $\boldsymbol{\xi} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I}_{n_\mathsf{T}})$ is independent of $\hat{\mathbf{W}}$.

For simplicity, we will only consider continuously differentiable layering functions $\mathbf{L} \in \mathbb{L}_{\mathcal{D}}$. The loss of optimality as a consequence of this restriction can be made arbitrarily small, accord-

[1] The letter 'W' is chosen in reference to the fact that such matrices are sometimes called *white*.

ing to Lemma 2.4.

Furthermore, we will usually set the transmit covariance to $\mathbf{Q} = \frac{1}{n_\mathsf{T}}\mathbf{I}_{n_\mathsf{T}}$, except for Theorem 3.4, which holds for any transmit covariance matrix.

3.1 Diagonal Layering Functions

Theorem 3.1. *Under i.i.d. Rayleigh fading assumptions* (3.1)*, for a transmit covariance* $\mathbf{Q} = \frac{1}{n_\mathsf{T}}\mathbf{I}_{n_\mathsf{T}}$ *and for continuously differentiable* $\mathbf{L} \in \mathbb{L}_\mathcal{D}$*, the rate-splitting bound* $R^\star(\mathbf{L})$ *does not depend on the eigenbasis of* $\mathbf{L}$*. Specifically, one can construct (entrywise) continuously differentiable functions*

$$\mathbf{U}_\mathbf{L} \colon [0;1] \to \mathbb{U}^{n_\mathsf{T} \times n_\mathsf{T}} \tag{3.5a}$$

$$\mathbf{\Lambda}_\mathbf{L} \colon [0;1] \to \mathbb{R}_+^{n_\mathsf{T} \times n_\mathsf{T}} \tag{3.5b}$$

where $\mathbf{\Lambda}_\mathbf{L}$ *is diagonal, such that*

$$\mathbf{L}(\iota) = \mathbf{U}_\mathbf{L}(\iota)\mathbf{\Lambda}_\mathbf{L}(\iota)\mathbf{U}_\mathbf{L}(\iota)^\dagger \tag{3.6}$$

is the eigendecomposition of $\mathbf{L}(\iota)$*, and such that* $\mathbf{\Lambda}_\mathbf{L} \in \mathbb{L}$ *is a layering function*[2] *satisfying*

$$R^\star(\mathbf{L}) = R^\star(\mathbf{\Lambda}_\mathbf{L}). \tag{3.7}$$

Proof: See Appendix C.3. ■

As a consequence of Theorem 3.1, we can restrict $\mathbf{L}(\iota)$ to be diagonal for all $\iota \in [0;1]$ without loss of generality or optimality. Of course, this is also true for the optimal layering function $\mathbf{L}^\star$. Note, however, that the proof of this property depends critically on the assumptions of i.i.d. Rayleigh fading and of a scaled-identity transmit covariance.

3.2 Two Exemplary Layering Functions

For the transmit covariance $\mathbf{Q} = \frac{1}{n_\mathsf{T}}\mathbf{I}_{n_\mathsf{T}}$, we define two exemplary layering functions, the so-called *staggered layering* and *levelled layering*, denoted respectively as $\mathbf{L}_\text{stag}$ and $\mathbf{L}_\text{lev}$.

3.2.1 Levelled Layering

Definition 3.1 (Levelled layering function)**.** *The layering function* $\mathbf{L}_\text{lev}(\iota) = \frac{1}{n_\mathsf{T}}\mathbf{I}_{n_\mathsf{T}}\iota$ *is called the* levelled *layering function.*

[2] In fact, if we do note require its diagonal entries to be ordered, $\mathbf{\Lambda}_\mathbf{L}$ can even be made a continuously differentiable layering function.

For levelled layering, the infinite-layering bound (3.3) specializes to

$$\boxed{R^\star(\mathbf{L}_{\mathrm{lev}}) = \frac{\hat{V}}{n_\mathsf{T}} \int_0^1 \mathsf{E}\,\mathrm{tr}\Big[\hat{\mathbf{W}}^\dagger \boldsymbol{\Gamma}(\mathbf{L}_{\mathrm{lev}}(\iota))^{-1} \hat{\mathbf{W}}\Big]\, \mathrm{d}\iota} \tag{3.8}$$

where [cf. (3.4)]

$$\boldsymbol{\Gamma}(\mathbf{L}_{\mathrm{lev}}(\iota)) = \frac{\Xi_{n_\mathsf{T}}}{n_\mathsf{T}} \tilde{V} \mathbf{I}_{n_\mathsf{R}} \iota + \left(\frac{\hat{V}}{n_\mathsf{T}} \hat{\mathbf{W}} \hat{\mathbf{W}}^\dagger + \tilde{V} \mathbf{I}_{n_\mathsf{R}} \right) (1-\iota) + \rho^{-1} \mathbf{I}_{n_\mathsf{R}}. \tag{3.9}$$

with Ξ_{n_T} being a gamma-distributed random variable with shape n_T and scale 1 whose probability density function is given as

$$f_{\Xi_{n_\mathsf{T}}}(\xi) = \begin{cases} 0 & \text{for } \xi < 0 \\ \dfrac{\xi^{n_\mathsf{T}-1}}{(n_\mathsf{T}-1)!} e^{-\xi} & \text{for } \xi \geq 0. \end{cases} \tag{3.10}$$

3.2.2 Staggered Layering

Definition 3.2 (Staggered layering function)**.** *The layering function*

$$\mathbf{L}_{\mathrm{stag}}(\iota) = \frac{1}{n_\mathsf{T}} \begin{bmatrix} \kappa(\iota n_\mathsf{T}) & & & 0 \\ & \kappa(\iota n_\mathsf{T} - 1) & & \\ & & \ddots & \\ 0 & & & \kappa(\iota n_\mathsf{T} - n_\mathsf{T} + 1) \end{bmatrix} \tag{3.11}$$

where $\kappa(x)$ is defined as

$$\kappa(x) = \begin{cases} 0 & \textit{for } x \leq 0 \\ x & \textit{for } 0 \leq x \leq 1 \\ 1 & \textit{for } x \geq 1 \end{cases} \tag{3.12}$$

is called the staggered *layering function.*

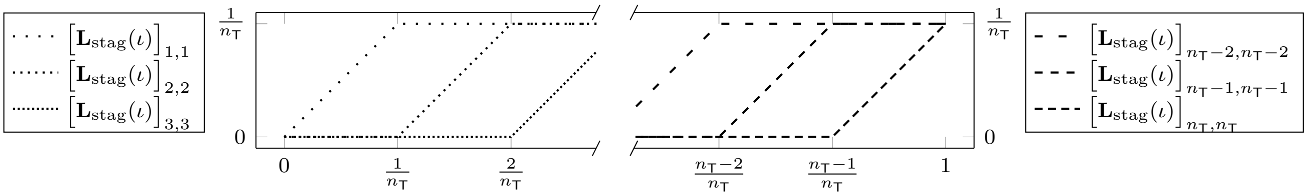


Figure 3.1 Diagonal entries $\big[\mathbf{L}_{\mathrm{stag}}(\iota)\big]_{i,i}$ of the staggered layering function

For staggered layering, we derive in Section C.2 that the infinite-layering bound (3.3) spe-

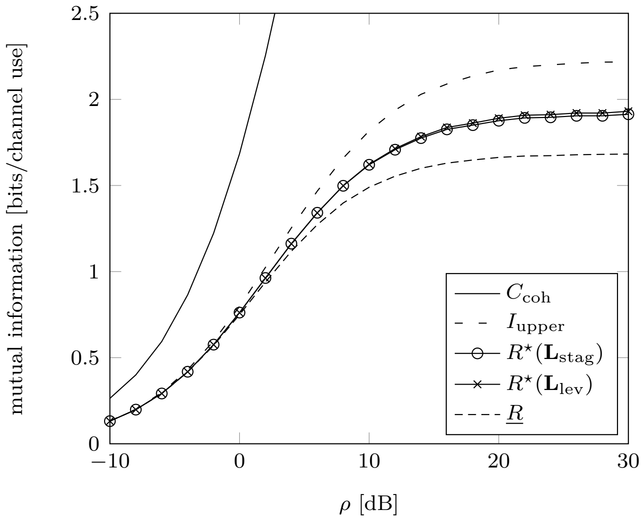


Figure 3.2 Capacity and mutual information bounds for $n_{\mathsf{R}} = n_{\mathsf{T}} = 2$ and $\hat{V} = \tilde{V} = \frac{1}{2}$

cializes to

$$\boxed{R^\star(\mathbf{L}_{\text{stag}}) = \hat{V}\,\mathsf{E}\left[\sum_{i=1}^{n_{\mathsf{T}}} \int_0^1 \frac{\hat{\mathbf{w}}_i^\dagger \mathbf{A}_i(\nu)^{-1}\hat{\mathbf{w}}_i}{1 + \hat{V}(1-\nu)\hat{\mathbf{w}}_i^\dagger \mathbf{A}_i(\nu)^{-1}\hat{\mathbf{w}}_i}\,\mathrm{d}\nu\right]} \tag{3.13}$$

with

$$\mathbf{A}_i(\nu) = \left(\tilde{V}\left(\Xi_{i-1} + \nu\tilde{\Xi}_1 + 1 - \nu + n_{\mathsf{T}} - i\right) + n_{\mathsf{T}}\rho^{-1}\right)\mathbf{I}_{n_{\mathsf{R}}} + \hat{V}\hat{\mathbf{W}}_{(i+1):n_{\mathsf{T}}}\hat{\mathbf{W}}_{(i+1):n_{\mathsf{T}}}^\dagger \tag{3.14}$$

where Ξ_{i-1} and $\tilde{\Xi}_1$ are mutually independent, gamma-distributed with scale 1 and respective shapes $i-1$ and 1 [cf. (3.10)], and where $\hat{\mathbf{W}}_{i:j}$ stands for the matrix composed of the columns i through j of the matrix $\hat{\mathbf{W}}$.

Alongside the capacity and mutual information lower bounds, Figure 3.2 also shows the mutual information upper bound (2.56), which specializes to

$$I_{\text{upper}}(\rho) = \underline{R}(\rho) + n_{\mathsf{R}}\,\mathsf{E}\left[\log \frac{\tilde{V} + \rho^{-1}}{\tilde{V}\frac{\Xi_{n_{\mathsf{T}}}}{n_{\mathsf{T}}} + \rho^{-1}}\right] \tag{3.15}$$

and is stated in general form in (2.56).

The *levelled* and *staggered* layering are chosen for illustrative purposes, since they appear as the two most natural choices of diagonal layering functions. However, we stress that nothing is known as to whether they are optimal in any sense, not even in the highly symmetric i.i.d. Rayleigh fading scenario. In general, they produce different bounds $R^\star(\mathbf{L}_{\text{lev}})$ and $R^\star(\mathbf{L}_{\text{stag}})$, as can be seen by the slight difference between their corresponding curves in Figure 3.2. It should not be inferred from Figure 3.2 that levelled layering generally outperforms staggered layering. In fact, further numerical comparisons suggest that neither of these two contenders is a candidate for optimality, since the superiority of one over the other depends on the specific values of system parameters. Clearly, the computation of the best layering $\mathbf{L}^\star$ is a difficult problem even in a comparatively simple and symmetric scenario such as i.i.d. Rayleigh fading.

3.3 Asymptotically Perfect CSI

In the numerical example considered in the previous section (cf. Figure 3.2), it is assumed that the joint distribution of $(\hat{\mathbf{H}}, \tilde{\mathbf{H}})$ does not depend on the SNR ρ, in that $\hat{V}$ and $\tilde{V}$ are constant. However, in practical communication systems, the channel estimation error—as measured by $\tilde{V}$—typically decreases as the SNR increases. This is because, with increasing SNR, transmit power becomes increasingly affordable not only for data transmission, but as well for channel estimation. In this section, we investigate the high-SNR behavior of rate-splitting bounds in circumstances where the channel estimation error vanishes as the SNR tends to infinity. When this condition is satisfied, we shall say that we have *asymptotically perfect CSI.* Note that, unlike Subsection 2.1.7 in which the SISO case was treated, here $\tilde{V}$ is not a function of the channel estimate, since we assumed (for reasons of tractability of the MIMO case) that $\mathbf{H}$ and $\tilde{\mathbf{H}}$ are mutually independent.

We will consider that $\tilde{V}_\rho$ and $\hat{V}_\rho = 1 - \tilde{V}_\rho$ are functions of ρ such that

$$\lim_{\rho\to\infty} \tilde{V}_\rho = 0. \tag{3.16}$$

We also adapt the notation of the capacity and mutual information bounds to reflect the fact that they are functions of ρ. Therefore, we will write $\underline{R}_\rho$, $R_\rho^\star(\mathbf{L})$, and $I_{\text{upper},\rho}$.

Theorem 3.2. *The bound $R_\rho^\star(\mathbf{L})$ is asymptotically tight for any layering function $\mathbf{L}$, in the sense that*

$$\lim_{\rho\to\infty} \left\{ I_{\text{upper}}(\mathbf{x}_\text{G}; \mathbf{y}|\hat{\mathbf{H}}) - R_\rho^\star(\mathbf{L}) \right\} = 0 \tag{3.17}$$

and consequently,

$$\lim_{\rho\to\infty} \left\{ I(\mathbf{x}_\text{G}; \mathbf{y}|\hat{\mathbf{H}}) - R_\rho^\star(\mathbf{L}) \right\} = 0. \tag{3.18}$$

Proof: See Appendix C.4. ■

Figure 3.3 shows the capacity and mutual information bounds corresponding to Figure 3.2, but for asymptotically perfect CSI. The estimation error variance is chosen to be $\tilde{V}_\rho = \frac{1}{\rho+1}$, as for interpolation-based channel estimation [cf. (2.42b)].

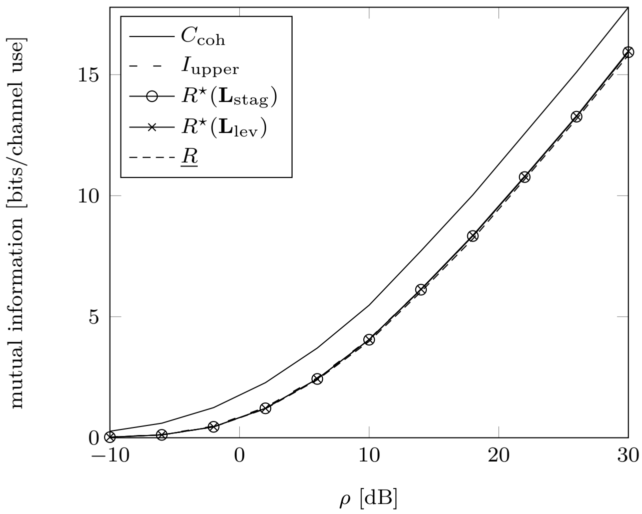


Figure 3.3 Capacity and mutual information bounds for asymptotially perfect CSI

3.4 Large MIMO Systems

3.4.1 Large Transmit Antenna Arrays

Consider a sequence of channels with increasing number of transmit antennas. To make the parameter n_T explicit in notation, we write (3.1) as

$$\hat{\mathbf{H}}_{n_\mathsf{T}} = \sqrt{\hat{V}}\hat{\mathbf{W}}_{n_\mathsf{T}} \qquad \tilde{\mathbf{H}}_{n_\mathsf{T}} = \sqrt{\tilde{V}}\tilde{\mathbf{W}}_{n_\mathsf{T}} \tag{3.19}$$

where $\mathrm{vec}(\hat{\mathbf{W}}_{n_\mathsf{T}}) \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{I}_{n_\mathsf{R}n_\mathsf{T}})$ and $\mathrm{vec}(\tilde{\mathbf{W}}_{n_\mathsf{T}}) \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{I}_{n_\mathsf{R}n_\mathsf{T}})$. On the n_T-th channel, let the transmit covariance be $\frac{1}{n_\mathsf{T}}\mathbf{I}_{n_\mathsf{T}}$ and the layering function be $\mathbf{L}_{n_\mathsf{T}}$. The sequence $\{\mathbf{L}_{n_\mathsf{T}}\}_{n_\mathsf{T}\in\mathbb{N}}$ of layering functions $\mathbf{L}_{n_\mathsf{T}} \in \mathbb{L}_\mathcal{D}(\frac{1}{n_\mathsf{T}}\mathbf{I}_{n_\mathsf{T}})$ may be arbitrary. For the n_T-th channel, the worst-case-noise bound $\underline{R}_{n_\mathsf{T}}$ and the rate-splitting bound $R^\star_{n_\mathsf{T}}(\mathbf{L}_{n_\mathsf{T}})$ are given respectively by [cf. (3.2), (3.3)]

$$\underline{R}_{n_\mathsf{T}} \triangleq \mathsf{E}\log\det\left(\mathbf{I}_{n_\mathsf{R}} + \frac{\hat{V}}{\tilde{V}+\rho^{-1}}\frac{\hat{\mathbf{W}}_{n_\mathsf{T}}\hat{\mathbf{W}}^\dagger_{n_\mathsf{T}}}{n_\mathsf{T}}\right) \tag{3.20}$$

and

$$R^\star_{n_\mathsf{T}}(\mathbf{L}_{n_\mathsf{T}}) \triangleq \hat{V}\int_0^1 \mathsf{E}\,\mathrm{tr}\Big[\hat{\mathbf{W}}^\dagger_{n_\mathsf{T}}\mathbf{\Gamma}_{n_\mathsf{T}}(\mathbf{L}_{n_\mathsf{T}}(\iota))^{-1}\hat{\mathbf{W}}_{n_\mathsf{T}}\dot{\mathbf{L}}_{n_\mathsf{T}}(\iota)\Big]\,\mathrm{d}\iota \tag{3.21}$$

respectively, where [cf. (3.4)]

$$\mathbf{\Gamma}_{n_\mathsf{T}}(\mathbf{L}_{n_\mathsf{T}}(\iota)) \triangleq \Big(\tilde{V}\boldsymbol{\xi}^\dagger\mathbf{L}_{n_\mathsf{T}}(\iota)\boldsymbol{\xi} + \tilde{V}(1-\iota) + \rho^{-1}\Big)\mathbf{I}_{n_\mathsf{R}} + \hat{V}\hat{\mathbf{W}}_{n_\mathsf{T}}\big(\tfrac{1}{n_\mathsf{T}}\mathbf{I}_{n_\mathsf{T}} - \mathbf{L}_{n_\mathsf{T}}(\iota)\big)\hat{\mathbf{W}}^\dagger_{n_\mathsf{T}}. \tag{3.22}$$

Theorem 3.3. *The difference between the rate-splitting bound and the worst-case-noise lower* *bound vanishes in the limit as the number of transmit antennas grows to infinity. Formally,*

$$\boxed{\lim_{n_\mathsf{T}\to\infty}\left\{R^\star_{n_\mathsf{T}}(\mathbf{L}_{n_\mathsf{T}}) - \underline{R}_{n_\mathsf{T}}\right\} = 0} \tag{3.23}$$

Proof: See Appendix C.5. ■

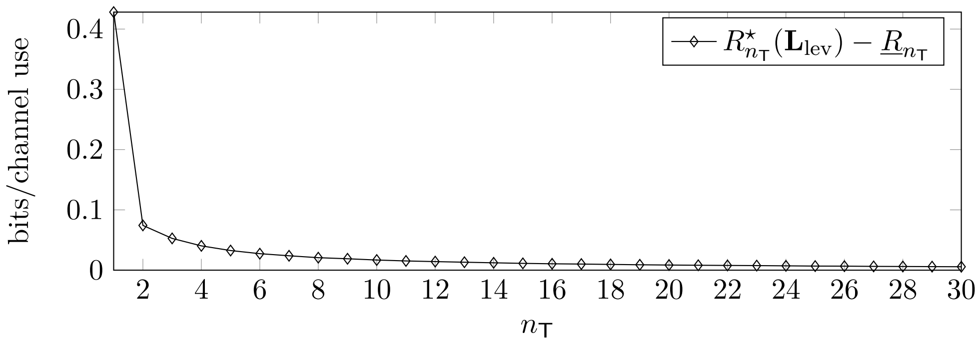


Figure 3.4 Bound difference $R^\star_{n_\mathsf{T}}(\mathbf{L}_\mathrm{lev}) - \underline{R}_{n_\mathsf{T}}$ for a MISO channel ($n_\mathsf{R} = 1$) as a function of the number of transmit antennas n_T. The chosen parameters are $\rho = 10\text{dB}$, $\hat{V} = \tilde{V} = \frac{1}{2}$.

3.4.2 Large Receive Antenna Arrays

Similarly to the previous subsection, we now consider a sequence of channels with increasing number of *receive* antennas n_R. To make the parameter n_R explicit in notation, we write (3.1) as

$$\hat{\mathbf{H}}_{n_\mathsf{R}} = \sqrt{\hat{V}}\hat{\mathbf{W}}_{n_\mathsf{R}} \qquad \tilde{\mathbf{H}}_{n_\mathsf{R}} = \sqrt{\tilde{V}}\tilde{\mathbf{W}}_{n_\mathsf{R}} \tag{3.24}$$

where $\mathrm{vec}(\hat{\mathbf{W}}_{n_\mathsf{R}}) \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{I}_{n_\mathsf{R}n_\mathsf{T}})$ and $\mathrm{vec}(\tilde{\mathbf{W}}_{n_\mathsf{R}}) \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{I}_{n_\mathsf{R}n_\mathsf{T}})$. The transmit covariance $\mathbf{Q}$ and the layering function $\mathbf{L} \in \mathbb{L}_\mathcal{D}$ shall be arbitrary and do not depend on the number of receive antennas. For the n_R-th channel, the worst-case-noise bound $\underline{R}_{n_\mathsf{R}}$ and the rate-splitting bound $R^\star_{n_\mathsf{R}}(\mathbf{L})$ are given respectively by [cf. (3.2), (3.3)]

$$\underline{R}_{n_\mathsf{R}} \triangleq \mathsf{E}\log\det\left(\mathbf{I}_{n_\mathsf{T}} + \frac{\hat{V}}{\tilde{V}+\rho^{-1}}\hat{\mathbf{W}}^\dagger_{n_\mathsf{R}}\hat{\mathbf{W}}_{n_\mathsf{R}}\mathbf{Q}\right) \tag{3.25}$$

and

$$R^\star_{n_\mathsf{R}}(\mathbf{L}) \triangleq \hat{V}\int_0^1 \mathsf{E}\,\mathrm{tr}\Big[\hat{\mathbf{W}}^\dagger_{n_\mathsf{R}}\mathbf{\Gamma}_{n_\mathsf{R}}(\mathbf{L}(\iota))^{-1}\hat{\mathbf{W}}_{n_\mathsf{R}}\dot{\mathbf{L}}(\iota)\Big]\,\mathrm{d}\iota \tag{3.26}$$

respectively, where [cf. (3.4)]

$$\mathbf{\Gamma}_{n_\mathsf{R}}(\mathbf{L}(\iota)) \triangleq \Big(\tilde{V}\boldsymbol{\xi}^\dagger\mathbf{L}(\iota)\boldsymbol{\xi} + \tilde{V}(1-\iota) + \rho^{-1}\Big)\mathbf{I}_{n_\mathsf{R}} + \hat{V}\hat{\mathbf{W}}_{n_\mathsf{R}}\bar{\mathbf{L}}(\iota)\hat{\mathbf{W}}^\dagger_{n_\mathsf{R}} \tag{3.27}$$

and where $\bar{\mathbf{L}}(\iota) \triangleq \mathbf{Q} - \mathbf{L}(\iota)$.

Theorem 3.4. *The difference between the rate-splitting bound and the worst-case-noise lower bound tends to a positive limit as the number of receive antennas grows to infinity. Formally,*

$$\boxed{\lim_{n_\mathsf{R}\to\infty}\left\{R^\star_{n_\mathsf{R}}(\mathbf{L}) - \underline{R}_{n_\mathsf{R}}\right\} = n_\mathsf{T}\,\mathsf{E}\left[\log\left(\frac{\tilde{V}+\rho^{-1}}{\tilde{V}\boldsymbol{\xi}^\dagger\mathbf{Q}\boldsymbol{\xi}+\rho^{-1}}\right)\right]} \tag{3.28}$$

Proof: See Appendix C.6. ■

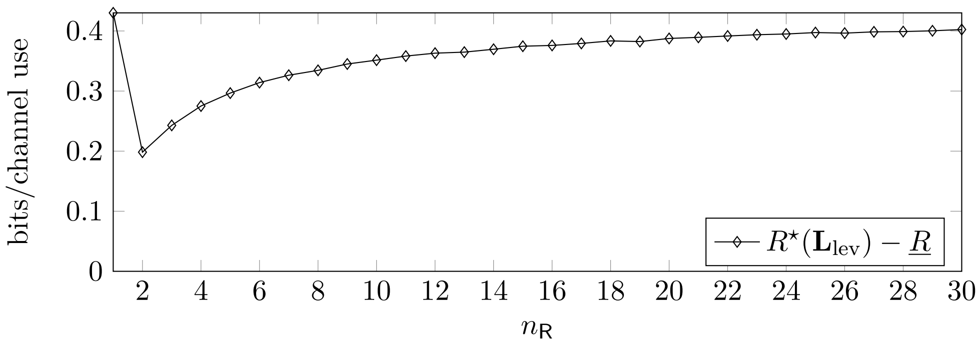


Figure 3.5 Bound difference $R^\star_{n_\mathsf{R}}(\mathbf{L}_\text{lev}) - \underline{R}_{n_\mathsf{R}}$ for a SIMO channel ($n_\mathsf{T} = 1$) as a function of the number of receive antennas n_R. The chosen parameters are $\rho = 10\text{dB}$, $\hat{V} = \tilde{V} = \frac{1}{2}$.

To make sense of the value of the large-system limit from Theorem 3.4, observe that, since the distribution of $\boldsymbol{\xi} \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{I}_{n_\mathsf{T}})$ is rotationally invariant, the quantity $\mathsf{E}\left[\log\left(\tilde{V}\boldsymbol{\xi}^\dagger\mathbf{Q}\boldsymbol{\xi}+\rho^{-1}\right)\right]$ is a concave and symmetric function of the eigenvalues of $\mathbf{Q}$. Over the set of unit-trace positive semidefinite matrices $\mathbf{Q}$, it is minimized by a rank-one matrix (i.e., $\mathbf{Q}$ has a single non-zero eigenvalue, which is equal to one) and maximized by the scaled identity matrix $\frac{1}{n_\mathsf{T}}\mathbf{I}_{n_\mathsf{T}}$. We thus obtain the following upper and lower bounds:

$$n_\mathsf{T}\,\mathsf{E}\left[\log\left(\frac{\tilde{V}+\rho^{-1}}{\tilde{V}\frac{\Xi_{n_\mathsf{T}}}{n_\mathsf{T}}+\rho^{-1}}\right)\right] \le n_\mathsf{T}\,\mathsf{E}\left[\log\left(\frac{\tilde{V}+\rho^{-1}}{\tilde{V}\boldsymbol{\xi}^\dagger\mathbf{Q}\boldsymbol{\xi}+\rho^{-1}}\right)\right] \le n_\mathsf{T}\,\mathsf{E}\left[\log\left(\frac{\tilde{V}+\rho^{-1}}{\tilde{V}|\xi_1|^2+\rho^{-1}}\right)\right]. \tag{3.29}$$

Here Ξ_{n_T} stands for a gamma-distributed variable with shape n_T and scale 1. The upper and lower bound on either side of this inequality can both be evaluated in closed form using [Gra07, (4.337), p. 568], resulting in expressions that involve the exponential integral function (or the incomplete gamma function).

Note that the upper bound in the above inequality is monotonically increasing in ρ (much like the other two quantities for that matter). Taking the limit as $\rho \to \infty$ on the right-hand side expression, we notice that a looser and simpler upper bound is given by

$$\lim_{\rho\to\infty}\left\{n_\mathsf{T}\,\mathsf{E}\left[\log\left(\frac{\tilde{V}+\rho^{-1}}{\tilde{V}|\xi_1|^2+\rho^{-1}}\right)\right]\right\} = n_\mathsf{T}\,\mathsf{E}[\log(|\xi_1|^2)] = n_\mathsf{T}\gamma \tag{3.30}$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant. That is,

$$\boxed{\lim_{n_\mathsf{R}\to\infty}\left\{R^\star_{n_\mathsf{R}}(\mathbf{L}) - \underline{R}_{n_\mathsf{R}}\right\} \le n_\mathsf{T}\gamma.} \tag{3.31}$$

Another noteworthy observation is that, if the number of receive antennas tends to infinity while the number of precoded streams (i.e., the rank of $\mathbf{Q}$) remains bounded, then the asymptotic bound difference diverges to infinity in n_T:

$$\boxed{\lim_{n_\mathsf{T}\to\infty}\lim_{n_\mathsf{R}\to\infty}\left\{R^\star_{n_\mathsf{R}}(\mathbf{L}) - \underline{R}_{n_\mathsf{R}}\right\} = +\infty \qquad \text{(for rank-limited } \mathbf{Q}\text{)}} \tag{3.32}$$

For example, in case of a rank-one transmit covariance matrix (single-beam precoding), we would have

$$\lim_{n_\mathsf{R}\to\infty}\left\{R^\star_{n_\mathsf{R}}(\mathbf{L}) - \underline{R}_{n_\mathsf{R}}\right\} = n_\mathsf{T}\,\mathsf{E}\left[\log\left(\frac{\tilde{V}+\rho^{-1}}{\tilde{V}|\xi_1|+\rho^{-1}}\right)\right] \tag{3.33}$$

which is directly proportional to n_T. Instead, for a full-multiplexing transmit covariance $\mathbf{Q} = \frac{1}{n_\mathsf{T}}\mathbf{I}_{n_\mathsf{T}}$, we have

$$\boxed{\lim_{n_\mathsf{T}\to\infty}\lim_{n_\mathsf{R}\to\infty}\left\{R^\star_{n_\mathsf{R}}(\mathbf{L}) - \underline{R}_{n_\mathsf{R}}\right\} = \frac{1}{2} \qquad \text{(for } \mathbf{Q} = \tfrac{1}{n_\mathsf{T}}\mathbf{I}_{n_\mathsf{T}}\text{)}} \tag{3.34}$$

This follows from [Yoo06, Lemma 3, Appendix III].

All this being said, the significance of these asymptotic results should not be overstated. In fact, the speed of convergence to these limits has not been studied. Besides, their explanatory power might be very limited due to the fact that we are analyzing *nested* limits. Additionally, one should bear in mind that both large-system limits given respectively by Theorem 3.3 (large transmit array) and Theorem 3.4 (large receive array) assume that the values of $\hat{V}$ and $\tilde{V}$ do not vary with the number of antennas. However, this might not reflect the realistic scaling behavior of the CSI quality in practical large systems. For example, if forward training is used in a system whose number of transmit antennas n_T is increased, the channel estimation error per channel coefficient is expected to decrease with n_T. Therefore, one should be very cautious when interpreting the above results.

Nonetheless, the comparison between the rather pessimistic result from Theorem 3.3 and the rather optimistic result from Theorem 3.4 might at least indicate, as a rule of thumb, that the superiority of the infinite-layering bound over the worst-case-noise bound is particularly pronounced when the number of receive antennas (rather than that of the transmit antennas) is large. For example, the analysis of achievable rates in a cellular uplink with a large number of base station antennas might benefit from choosing infinite-layering bounds.

3.4.3 Large Transmit and Receive Arrays

In contrast to the previous subsection, we now consider the number of transmit *and* receive antennas to grow large. That is, we consider the large-system limit $n_{\mathsf{T}}, n_{\mathsf{R}} \to \infty$ such that

$$0 < \underline{\lim} \left(\frac{n_{\mathsf{R}}}{n_{\mathsf{T}}}\right) \leq \overline{\lim} \left(\frac{n_{\mathsf{R}}}{n_{\mathsf{T}}}\right) < +\infty \tag{3.35}$$

which will be denoted as $n_{\mathsf{T}} \to \infty$ throughout this subsection (where it is assumed that n_{R} grows as a function of n_{T}). This will allow us to derive tight and easily computable approximations of the levelled layering bound (3.8). We first recall the following result which provides an asymptotically exact approximation of the coherent capacity.

Theorem 3.5. *Let* $\mathbf{Q} = \frac{1}{n_{\mathsf{T}}}\mathbf{I}_{n_{\mathsf{T}}}$ *and* $\mathbf{w} = \mathrm{vec}(\mathbf{W}) \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I}_{n_{\mathsf{R}}n_{\mathsf{T}}})$ *where* $\mathbf{W} \in \mathbb{C}^{n_{\mathsf{R}} \times n_{\mathsf{T}}}$*. The coherent capacity* C_{coh} *as defined in* (2.53) *satisfies [Hac08, Theorem 8]*

$$\begin{aligned} C_{\mathrm{coh}} &= \mathsf{E}\left[\log\det\left(\mathbf{I}_{n_{\mathsf{T}}} + \frac{\rho}{n_{\mathsf{T}}}\mathbf{W}^{\dagger}\mathbf{W}\right)\right] \\ &= \bar{C} + \mathcal{O}({n_{\mathsf{T}}}^{-1}) \end{aligned} \tag{3.36}$$

as $n_{\mathsf{T}} \to \infty$*, where*

$$\begin{aligned} \bar{C} &= \int_{\frac{1}{\rho}}^{\infty} \left(\frac{n_{\mathsf{R}}}{t} - n_{\mathsf{T}}\zeta(t)\right) \mathrm{d}t \\ &= n_{\mathsf{T}}\log(1+\zeta(\rho^{-1})) + n_{\mathsf{R}}\log\left(1 + \frac{1}{\rho(1+\zeta(\rho^{-1}))}\right) - \frac{n_{\mathsf{T}}\zeta(\rho^{-1})}{1+\zeta(\rho^{-1})} \end{aligned} \tag{3.37}$$

and the function ζ *is given by*

$$\zeta(t) = \frac{\frac{n_{\mathsf{R}}}{n_{\mathsf{T}}} - 1}{2t} - \frac{1}{2} + \frac{\sqrt{\left(1 - \frac{n_{\mathsf{R}}}{n_{\mathsf{T}}} + t\right)^2 + 4\frac{n_{\mathsf{R}}}{n_{\mathsf{T}}}t}}{2t}. \tag{3.38}$$

Note that Theorem 3.5 is a stronger result than the well-known convergence of the *per-antenna* mutual information to its asymptotic limit (see, e.g., [Ver99]) which holds for channel matrices composed of arbitrary i.i.d. entries with finite second-order moment. As a direct consequence of Theorem 3.5, we obtain the following approximation of the lower bound in (2.54):

$$\underline{R} = \bar{\underline{R}} + \mathcal{O}({n_{\mathsf{T}}}^{-1}) \tag{3.39}$$

where

$$\bar{\underline{R}} = \bar{C}\left(\frac{\hat{V}}{\tilde{V} + \rho^{-1}}\right) \tag{3.40}$$

Since one can show (cf. [Hac08]) that the $\mathcal{O}({n_{\mathsf{T}}}^{-1})$-term in Theorem 3.5 is integrable over any closed interval $[0, \rho]$, we can approximate the L-layer rate-splitting bound in (3.8) in a similar fashion via

$$R(\mathbf{L}_{\text{lev}}, \mathcal{I}) = \bar{R}(\mathbf{L}_{\text{lev}}, \mathcal{I}) + \mathcal{O}({n_\mathsf{T}}^{-1}) \tag{3.41}$$

where

$$\bar{R}(\mathbf{L}_{\text{lev}}, \mathcal{I}) = \sum_{\ell=1}^{L} \int_0^\infty \left[\bar{C}\left(\frac{1-\iota_{\ell-1}}{\sigma_\ell^2(x)} \right) - \bar{C}\left(\frac{1-\iota_\ell}{\sigma_\ell^2(x)} \right) \right] f_{\Xi_{n_\mathsf{T}}}(x)\, \mathrm{d}x \tag{3.42}$$

with $f_{\Xi_{n_\mathsf{T}}}(x)$ as given in (3.10) and

$$\sigma_\ell^2(x) = \frac{\tilde{V}\iota_{\ell-1}\frac{x}{n_\mathsf{T}} + \tilde{V}(1-\iota_{\ell-1}) + \rho^{-1}}{\hat{V}}, \quad l = 1, \ldots, L. \tag{3.43}$$

We can also derive an approximation of the levelled layering bound with an infinite number of layers $R^\star(\mathbf{L}_{\text{lev}})$. This approximation is given in the next Theorem.

Theorem 3.6. *Let*

$$\begin{aligned} \bar{R}^\star(\mathbf{L}_{\text{lev}}) &= \mathsf{E}\left[\int_{\sigma^2(\Xi_{n_\mathsf{T}})}^\infty \frac{n_\mathsf{R} - n_\mathsf{T} g(\Xi_{n_\mathsf{T}}, t)\zeta(g(\Xi_{n_\mathsf{T}}, t))}{t}\, \mathrm{d}t \right] \\ &= \int_0^\infty \int_{\sigma^2(x)}^\infty \frac{n_\mathsf{R} - n_\mathsf{T} g(x,t)\zeta(g(x,t))}{t} f_{\Xi_{n_\mathsf{T}}}(x)\, \mathrm{d}t\, \mathrm{d}x \end{aligned} \tag{3.44}$$

where $\zeta(x)$ was defined earlier in (3.38)*, where Ξ_{n_T} is a gamma-distributed variable with shape n_T and scale* 1*, whose probability density function $f_{\Xi_{n_\mathsf{T}}}$ is given in* (3.10)*, and where the functions σ^2 and g are given by*

$$\sigma^2(x) = \frac{\frac{x}{n_\mathsf{T}}\tilde{V} + \rho^{-1}}{\hat{V}} \tag{3.45a}$$

$$g(x,t) = \left(1 - \frac{x}{n_\mathsf{T}}\right)\frac{\tilde{V}}{\hat{V}} + t. \tag{3.45b}$$

Then, as $n_\mathsf{T} \to \infty$,

$$R^\star(\mathbf{L}_{\text{lev}}) = \bar{R}^\star(\mathbf{L}_{\text{lev}}) + \mathcal{O}({n_\mathsf{T}}^{-1}). \tag{3.46}$$

Proof: The proof is provided in Appendix C.9. ■

Notice that, by comparison with (3.37), the inner integral in (3.44) evaluated for $x = n_\mathsf{T} = \mathsf{E}[\Xi_{n_\mathsf{T}}]$ corresponds exactly to $\bar{C}$ evaluated at $\sigma^{-2}(n_\mathsf{T}) = \hat{V}/(\tilde{V} + \rho^{-1})$, which is the lower bound $\underline{\bar{R}}$. In fact, this is consistent with the observation that the worst-case-noise bound $\underline{R}$ can be recovered by lower-bounding $R^\star(\mathbf{L}_{\text{lev}})$ via Jensen's inequality [cf. Remark 2.8 under Theorem 2.6].

Although the computation of $\bar{R}^\star(\mathbf{L}_{\text{lev}})$ requires the numerical evaluation of a double integral, it can be computed very efficiently and, most importantly, much faster than $R^\star(\mathbf{L}_{\text{lev}})$.

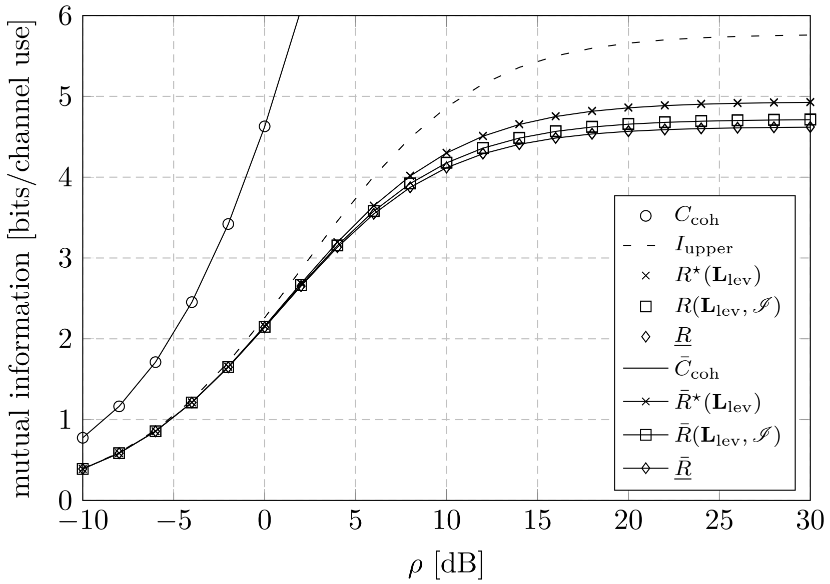


Figure 3.6 Coherent capacity, bounds, and asymptotic approximations vs. SNR for $(n_{\mathsf{R}}, n_{\mathsf{T}}) = (6, 4)$, $\hat{V} = \tilde{V} = \frac{1}{2}$. The finite-layering bounds are evaluated for an indexing $\mathscr{I} = \{0, \frac{1}{2}, 1\}$, i.e., for equi-power layering with two layers.

3.5 Semi-Closed-Form Expressions

Computing the exact values of levelled and staggered layering bounds with infinite number of layers involves several nested integrations: an expectation with respect to the distribution of the random matrix $\hat{\mathbf{H}}$ of size $n_{\mathsf{R}} \times n_{\mathsf{T}}$, an expectation over the random vector $\boldsymbol{\xi}$ of size $n_{\mathsf{T}} \times 1$ (which simplifies to a double integration for staggered layering, and a single scalar integration for levelled layering), and an integration over the layering index $\iota \in [0; 1]$.

The large random-matrix approximation from the previous section already offered a way to decrease this complexity via an approximation. Though simulations suggest that this approximation is highly accurate even for a relatively low number of antennas, no bound on the approximation error is known. Besides this, the approximation method is not extensible to staggered layering.

To reduce computation time for the *exact* value of the levelled and staggered layering bounds, we will derive closed-form expressions of the expectation over $\hat{\mathbf{H}}$, since in multiple-antenna channels the latter accounts for the highest computational burden, given that the number of integration steps scales with the number of antennas.

3.5.1 Levelled layering

A semi-closed-form expression of the levelled layering bound (3.8) is given by

$$\boxed{R^\star(\mathbf{L}_{\text{lev}}) = \int_0^\infty \int_0^1 \frac{1}{1-\iota} \,\mathsf{E}\left[\Omega_{\text{lev}}\left(\frac{\Xi_{n_\mathsf{T}}\tilde{V}\iota + \tilde{V}n_\mathsf{T}(1-\iota) + \rho^{-1}n_\mathsf{T}}{\hat{V}(1-\iota)}\right)\right] \mathrm{d}\iota\,\mathrm{d}\xi.} \tag{3.47}$$

where Ξ_{n_T} is gamma-distributed with shape n_T and scale 1, and where the function Ω_{lev} is given in closed form in Appendix (C.10) by Equation (C.136), and whose evaluation also requires (C.132). The full derivation of (3.47) is detailed in Appendix C.10.

3.5.2 Staggered layering

A semi-closed-form expression of the staggered layering bound (3.13) for tall channel matrices ($n_\mathsf{R} \geq n_\mathsf{T}$) is given by

$$\boxed{R^\star(\mathbf{L}_{\text{stag}}) = \hat{V}\sum_{i=1}^{n_\mathsf{T}} \int_0^1 \mathsf{E}\left[\frac{1}{\alpha_i(\nu)}\Omega_{\text{stag},i}\left(\frac{\alpha_i(\nu)}{\hat{V}}, \frac{\alpha_i(\nu)}{\hat{V}(1-\nu)}\right)\right] \mathrm{d}\nu} \tag{3.48}$$

where the $\alpha_i(\nu)$ are random functions given by

$$\alpha_i(\nu) = \tilde{V}\left(\frac{\Xi_{i-1}}{n_\mathsf{T}} + \frac{\nu}{n_\mathsf{T}}\tilde{\Xi}_1 + 1 - \nu + n_\mathsf{T} - i\right) + \rho^{-1}. \tag{3.49}$$

where Ξ_i denotes a gamma-distributed variable of shape i and scale 1, and where $\tilde{\Xi}_1$ is independent and exponentially distributed. A derivation of this formula is provided in Appendix C.11, and the functions $\Omega_{\text{stag},i}$ are given therein under Equation (C.154) for tall or square channel matrices (i.e., for $n_\mathsf{T} \leq n_\mathsf{R}$) and under Equation (C.170) for broad channel matrices (i.e., for $n_\mathsf{T} > n_\mathsf{R}$).

4

Pilot-Assisted Communication

We consider a MIMO link and assume a highly scattering environment at the receiver—as is the case in many downlink scenarios—so that the fading is correlated only at the transmitter side. This encompasses the important special case of an arbitrarily correlated MISO link. In this setting with antenna correlations, we propose to revisit the problem treated in [Has03] and extend it to spatially *correlated* fading. In [Has03], the concept of *effective SNR* was introduced to designate an SNR that accounts for the imperfection of CSI at the receiver and serves as the utility to be maximized. Due to the presence of antenna correlation, the effective SNR needs to be extended from a scalar to a matrix-valued quantity, and the entire spatial structures of the pilot sequence and the linear precoder need to be jointly optimized. Thus, we are facing a high-dimensional optimization problem which entails a non-trivial extension of the approach from [Has03].

Most performance measures of pilot-assisted MIMO systems are functions that depend on both the linear precoder and the pilot sequence. A framework for the optimization of these two parameters is proposed, based on a matrix-valued generalization of the concept of effective SNR from [Has03]. Our framework aims to extend their results by allowing for transmit-side fading correlations, and by considering a class of utility functions of said effective SNR matrix, most notably including the well-known worst-case-noise capacity lower bound also used in [Has03]. We tackle the joint optimization problem by recasting the optimization of the precoder (resp. pilot sequence) subject to a fixed pilot sequence (resp. precoder) into a convex problem. Furthermore, we prove that joint optimality requires that the eigenbases of the precoder and pilot sequence be both aligned along the eigenbasis of the channel correlation matrix. We finally describe how to wrap all studied subproblems into an iteration that converges to a local optimum of the joint optimization.

4.1 System Model

Our discrete-time system consists of a standard single-user MIMO link with an $n_\mathsf{R} \times n_\mathsf{T}$ random channel matrix $\mathbf{H}$ expressible as

$$\mathbf{H} = \mathbf{W}\mathbf{R}^{\frac{1}{2}}, \tag{4.1}$$

where the entries of $\mathbf{W} \in \mathbb{C}^{n_\mathsf{R} \times n_\mathsf{T}}$ are independent and identically distributed (i.i.d.) zero-mean circularly-symmetric unit-variance complex Gaussian, i.e., $\mathrm{vec}(\mathbf{W}) \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{I})$. The deterministic matrix $\mathbf{R} = \frac{1}{n_\mathsf{R}} \mathsf{E}[\mathbf{H}^\dagger \mathbf{H}]$ characterizes the transmit-side correlation, and is assumed to be full-rank, since we shall ignore keyhole effects. Equivalently, we can write the distribution of $\mathbf{H}$ as

$$\mathbf{h} = \mathrm{vec}(\mathbf{H}) \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{R}^\mathrm{T} \otimes \mathbf{I}). \tag{4.2}$$

This correlation model is valid in setups where numerous scatterers are located in the vicinity of the receiver, and notably subsumes the case of arbitrarily correlated multiple-input single-output (MISO) channels, which are especially relevant in wireless downlinks.

The channel $\mathbf{H}$ remains constant for a duration T called the channel coherence time, after which it changes to a new realization that is independent of all previous ones (block-fading). Within every such fading block, we reserve T_p time slots to transmit a sequence of pilot symbols known at the receiver, while the data is transmitted during the remaining $T_\mathsf{d} = T - T_\mathsf{p}$ time slots. For example (and without loss of generality), we can accommodate the pilot symbols into the first T_p time slots of each fading block. The resulting training observation is

$$\mathbf{Y}_\mathsf{p} = \mathbf{H}\mathbf{X}_\mathsf{p} + \mathbf{Z}_\mathsf{p}, \tag{4.3}$$

where $\mathbf{X}_\mathsf{p} \in \mathbb{C}^{n_\mathsf{T} \times T_\mathsf{p}}$ is a matrix whose columns are the pilot symbols, and the noise matrix $\mathbf{Z}_\mathsf{p} \in \mathbb{C}^{n_\mathsf{R} \times T_\mathsf{p}}$ is distributed as $\mathrm{vec}(\mathbf{Z}_\mathsf{p}) \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{I})$ and independent of $\mathbf{H}$. During the data transmission phase of one fading block, the received signal is

$$\mathbf{Y}_\mathsf{d} = \mathbf{H}\mathbf{F}\mathbf{X}_\mathsf{d} + \mathbf{Z}_\mathsf{d} \tag{4.4}$$

where $\mathbf{X}_\mathsf{d} \in \mathbb{C}^{r \times T_\mathsf{d}}$ (with $r \leq n_\mathsf{T}$) and $\mathbf{Z}_\mathsf{d} \in \mathbb{C}^{n_\mathsf{R} \times T_\mathsf{p}}$ are respectively the data symbol matrix and the additive noise matrix, with respective distributions $\mathrm{vec}(\mathbf{X}_\mathsf{d}) \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{I})$ and $\mathrm{vec}(\mathbf{Z}_\mathsf{d}) \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{I})$, whereas $\mathbf{F} \in \mathbb{C}^{n_\mathsf{T} \times r}$ is the linear precoder, and is assumed to have full column rank. The number r of rows of $\mathbf{X}_\mathsf{d}$ (and of columns of $\mathbf{F}$) represents the number of streams into which the channel input is multiplexed.

It can be shown that the minimum mean-square error (MMSE) channel estimate $\hat{\mathbf{H}} = \mathsf{E}[\mathbf{H}|\mathbf{Y}_\mathsf{p}]$ is obtained from the training observation $\mathbf{Y}_\mathsf{p}$ by right-multiplying it with the estimator matrix $\mathbf{G} = (\mathbf{X}_\mathsf{p}^\dagger \mathbf{R} \mathbf{X}_\mathsf{p} + \mathbf{I})^{-1} \mathbf{X}_\mathsf{p}^\dagger \mathbf{R}$ as

$$\hat{\mathbf{H}} = \mathbf{Y}_\mathsf{p}\mathbf{G}. \tag{4.5}$$

As a consequence of the distributions and correlation models for $\mathbf{H}$ and $\mathbf{Z}_\mathsf{p}$, the respective marginal distributions of the estimate $\hat{\mathbf{H}}$ and of the estimation error $\tilde{\mathbf{H}} = \mathbf{H} - \hat{\mathbf{H}}$ turn out to be

$$\hat{\mathbf{h}} = \mathrm{vec}(\hat{\mathbf{H}}) \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \hat{\mathbf{R}}^\mathrm{T} \otimes \mathbf{I}) \qquad \tilde{\mathbf{h}} = \mathrm{vec}(\tilde{\mathbf{H}}) \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \tilde{\mathbf{R}}^\mathrm{T} \otimes \mathbf{I}) \tag{4.6}$$

with transmit-side covariances

$$\hat{\mathbf{R}} = \tfrac{1}{n_\mathsf{R}} \mathsf{E}[\hat{\mathbf{H}}^\dagger \hat{\mathbf{H}}] = \mathbf{R} - \tilde{\mathbf{R}} \tag{4.7a}$$

$$\tilde{\mathbf{R}} = \tfrac{1}{n_\mathsf{R}} \mathsf{E}[\tilde{\mathbf{H}}^\dagger \tilde{\mathbf{H}}] = (\mathbf{R}^{-1} + \mathbf{X}_\mathsf{d}\mathbf{X}_\mathsf{d}^\dagger)^{-1}. \tag{4.7b}$$

We further define the two $n_\mathsf{T} \times n_\mathsf{T}$ Gram matrices

$$\mathbf{Q} = \mathbf{F}\mathbf{F}^\dagger \qquad \mathbf{P} = \mathbf{X}_\mathsf{p}\mathbf{X}_\mathsf{p}^\dagger \tag{4.8}$$

which are called the *transmit covariance*[1] and the *pilot Gram*, respectively. These two matrices will be subject to optimization.

4.2 Problem Statement

A capacity lower bound (in bits per channel use) of the communication system is given by $\frac{1}{T} I(\mathbf{X}_\mathsf{d}; \mathbf{Y}_\mathsf{d} | \mathbf{Y}_\mathsf{p})$, where $I(\mathbf{X}_\mathsf{d}; \mathbf{Y}_\mathsf{d} | \mathbf{Y}_\mathsf{p})$ stands for the mutual information between the block of input symbols $\mathbf{X}_\mathsf{d}$ and their corresponding outputs $\mathbf{Y}_\mathsf{d}$, conditioned on the side information $\mathbf{Y}_\mathsf{p}$ (training observation). Using a well-known lower bound on this mutual information, we get the achievable rate expression

$$\frac{T_\mathsf{d}}{T} \mathsf{E}\left[\log\det\left(\mathbf{I} + \frac{\hat{\mathbf{H}}\mathbf{Q}\hat{\mathbf{H}}^\dagger}{1 + \mathrm{tr}(\tilde{\mathbf{R}}\mathbf{Q})}\right)\right] \le \frac{1}{T} I(\mathbf{X}_\mathsf{d}; \mathbf{Y}_\mathsf{d} | \mathbf{Y}_\mathsf{p}), \tag{4.9}$$

where $\hat{\mathbf{H}}$ is given in (4.5). This bound is based on a worst-case noise approach [Has03] and has been widely used and studied in the literature, e.g., [M0́0], [Has03], [Yoo06], [Aub13] and many more. In [Soy10], the bound (4.9) is used in the exact same system setup as here. Using (4.6), we can write $\hat{\mathbf{H}} = \hat{\mathbf{W}}\hat{\mathbf{R}}^{\frac{1}{2}}$ with $\mathrm{vec}(\hat{\mathbf{W}}) \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{I})$ so that the expectation in (4.9) reads simply

$$I(\mathbf{S}) \triangleq \mathsf{E}\left[\log\det\left(\mathbf{I} + \hat{\mathbf{W}}\mathbf{S}\hat{\mathbf{W}}^\dagger\right)\right]. \tag{4.10}$$

The latter is a function of the matrix argument

$$\mathbf{S} = \mathbf{S}(\mathbf{P}, \mathbf{Q}) = \frac{\hat{\mathbf{R}}^{\frac{1}{2}}\mathbf{Q}\hat{\mathbf{R}}^{\frac{1}{2}}}{1 + \mathrm{tr}(\tilde{\mathbf{R}}\mathbf{Q})}, \tag{4.11}$$

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[1] in reference to the fact that $\mathbf{F}\mathbf{F}^\dagger$ is the covariance of the transmitted data signal

which constitutes a matrix-valued *effective SNR* generalization of the scalar effective SNR introduced by Hassibi and Hochwald in [Has03]. We write $\mathbf{S}(\mathbf{P},\mathbf{Q})$ instead of $\mathbf{S}$ whenever we wish to emphasize the dependency on $(\mathbf{P},\mathbf{Q})$. Let $\mathbf{s}$ denote the *(eigenvalue) profile* of $\mathbf{S}$, i.e., the vector of non-increasingly ordered eigenvalues of $\mathbf{S}$. Since $\hat{\mathbf{W}}$ and $\hat{\mathbf{W}}\mathbf{U}$ have the same marginal distribution for any unitary $\mathbf{U}$, the function $I(\mathbf{S})$ is in fact a function of the unordered eigenvalues of $\mathbf{S}$, i.e., a symmetric function of $\mathbf{s}$. We denote the latter as $\breve{I}(\mathbf{s})$. We thus have

$$\breve{I}(\mathbf{s}) \triangleq \mathsf{E} \log\det\left(\mathbf{I} + \hat{\mathbf{W}}\operatorname{diag}(\mathbf{s})\hat{\mathbf{W}}^\dagger\right) \tag{4.12}$$

with $I(\mathbf{S}) = \breve{I}(\mathbf{s})$. We will prefer one notation over the other depending on the situation.

To keep derivations as general as possible, we consider a general class $\mathcal{F}$ of utility functions sharing similar properties with I (resp. $\breve{I}$). The class $\mathcal{F}$ (resp. $\bar{\mathcal{F}}$) shall be the set of functions which, like I (resp. $\breve{I}$), are unitarily invariant and matrix-monotonic (resp. symmetric and vector-monotonic). That is, for $f \in \mathcal{F}$ (resp. $\bar{f} \in \bar{\mathcal{F}}$) we have

- $f(\mathbf{S}) = f(\mathbf{U}\mathbf{S}\mathbf{U}^\dagger)$ for any unitary matrix $\mathbf{U}$
- $\mathbf{0} \preceq \mathbf{S} \preceq \breve{\mathbf{S}} \Rightarrow f(\mathbf{S}) \le f(\breve{\mathbf{S}})$

and

- $\bar{f}(\mathbf{\Pi}\mathbf{s}) = \bar{f}(\mathbf{s})$ for any permutation $\mathbf{\Pi}$
- $\mathbf{0} \le \mathbf{s} \le \mathbf{s}' \Rightarrow \bar{f}(\mathbf{s}) \le \bar{f}(\mathbf{s}')$

We define the trace-constrained sets

$$\mathcal{P}(\mu_{\mathcal{P}}) = \left\{\mathbf{P} \in \mathbb{C}_+^{n_\mathsf{T}\times n_\mathsf{T}} :\ \operatorname{tr}(\mathbf{P}) \le \mu_{\mathcal{P}}\right\} \tag{4.13a}$$

$$\mathcal{Q}(\mu_{\mathcal{Q}}) = \left\{\mathbf{Q} \in \mathbb{C}_+^{n_\mathsf{T}\times n_\mathsf{T}} :\ \operatorname{tr}(\mathbf{Q}) \le \mu_{\mathcal{Q}}\right\} \tag{4.13b}$$

where $\mu_{\mathcal{P}}$ and $\mu_{\mathcal{Q}}$ stand for the maximum overall pilot symbol energy, and the average data symbol power (energy per time unit), respectively.

The pilot-assisted system shall be constrained by a maximum average energy consumption per time unit, denoted as μ. Consequently, the set of admissible values of the pilot-precoder pair $(\mathbf{P},\mathbf{Q})$ is

$$\mathcal{PQ}(T_\mathsf{p}) \triangleq \bigcup_{\substack{\mu_{\mathcal{P}},\mu_{\mathcal{Q}}\ge 0\\ \mu_{\mathcal{P}}+(T-T_\mathsf{p})\mu_{\mathcal{Q}}=T\mu}} \mathcal{P}(\mu_{\mathcal{P}}) \times \mathcal{Q}(\mu_{\mathcal{Q}}). \tag{4.14}$$

For future reference, we define five different pilot/precoder optimization problems (of increasing number of variables to be optimized) for some given utility $F \in \mathcal{F}$: [2]

$$\text{(P.1.a)} \qquad \max_{\mathbf{Q} \in \mathcal{Q}(\mu_{\mathcal{Q}})} F(\mathbf{S}(\mathbf{P}, \mathbf{Q})) \tag{4.15a}$$

$$\text{(P.1.b)} \qquad \max_{\mathbf{P} \in \mathcal{P}(\mu_{\mathcal{P}})} F(\mathbf{S}(\mathbf{P}, \mathbf{Q})) \tag{4.15b}$$

$$\text{(P.2)} \qquad \max_{(\mathbf{P},\mathbf{Q}) \in \mathcal{P}(\mu_{\mathcal{P}}) \times \mathcal{Q}(\mu_{\mathcal{Q}})} F(\mathbf{S}(\mathbf{P}, \mathbf{Q})) \tag{4.15c}$$

$$\text{(P.3)} \qquad \max_{(\mathbf{P},\mathbf{Q}) \in \mathcal{PQ}(T_{\mathsf{p}})} F(\mathbf{S}(\mathbf{P}, \mathbf{Q})) \tag{4.15d}$$

$$\text{(P.4)} \qquad \max_{T_{\mathsf{p}} \in \{1,\ldots,T-1\}} \max_{(\mathbf{P},\mathbf{Q}) \in \mathcal{PQ}(T_{\mathsf{p}})} F(\mathbf{S}(\mathbf{P}, \mathbf{Q})) \tag{4.15e}$$

Problems (P.1.a) and (P.1.b) are the partial problems that consist in optimizing one among the two variables $\mathbf{P} \in \mathcal{P}(\mu_{\mathcal{P}})$ and $\mathbf{Q} \in \mathcal{Q}(\mu_{\mathcal{Q}})$, while the other variable has a fixed value. The parameters $\mu_{\mathcal{P}}$ and $\mu_{\mathcal{Q}}$ can be considered as arbitrary constants. Problem (P.2) is the simplest joint optimization problem, having two independent trace constraints on $\mathbf{P}$ and $\mathbf{Q}$. Adding on (P.2) an outer optimization over pairs $(\mu_{\mathcal{P}}, \mu_{\mathcal{Q}})$ fulfilling the weighted-sum constraint $\mu_{\mathcal{P}} + (T - T_{\mathsf{p}})\mu_{\mathcal{Q}} = T\mu$, we obtain Problem (P.3). This balancing of pilot symbol and data symbol energies is what we call *energy boost*[3]. Adding yet another optimization over the training duration T_{p}, and incorporating an overhead factor $(T - T_{\mathsf{p}})/T$ which accounts for the loss of spectral efficiency due to the pilot symbols, we obtain the full-fledged Problem (P.4). Step by step, the present Section builds up a procedure for tackling (P.4).

The optimization of T_{p} in (P.4) is over a finite set and is solved by an exhaustive search, thus we will leave it aside until Section 4.6. Notice that in [Has03] the authors posit for the same (though uncorrelated) channel model (and utility $f = I$) that the receiver should have a representative estimate of the *complete* channel state, described by $n_{\mathsf{T}} n_{\mathsf{R}}$ fading coefficients. Therefore, they assume that the training duration T_{p} should be at least the number of transmit antennas n_{T}, so as to generate at least as many observables as there are coefficients to estimate. However, in the case where only a limited number of data streams are to be precoded, it might be more economic to only estimate a properly chosen subspace of the channel covariance spanned by the stronger eigenmodes. In fact, since T_{p} is defined as the number of columns of the pilot matrix $\mathbf{X}_{\mathsf{p}}$, and given that the utility function and power constraint [cf. (4.14)–(4.15e)] depend on $\mathbf{X}_{\mathsf{p}}$ only via its Gram matrix $\mathbf{P} = \mathbf{X}_{\mathsf{p}} \mathbf{X}_{\mathsf{p}}^{\dagger}$, we can assume that $\mathbf{X}_{\mathsf{p}}$ has full column rank and set the training duration equal to the rank of $\mathbf{P}$, i.e., $T_{\mathsf{p}} = \operatorname{rank}(\mathbf{P}) \leq n_{\mathsf{T}}$, and accordingly reduce the search interval in (4.15e) from $\{1, \ldots, T-1\}$ down to $\{1, \ldots, \min(T-1, n_{\mathsf{T}})\}$.

We will start by studying Problems (P.1.a) and (P.1.b) in the next two sections. These

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[2] Obviously, these problems could be equivalently stated in terms of $\bar{f} \in \bar{\mathcal{F}}$.

[3] In the literature, the balancing between pilot/data symbol *powers* under an overall average power constraint and for fixed time fractions assigned to training and data transmission, is sometimes referred to as *power boost* (e.g., [Loz08]). Since in our setup, $\mu_{\mathcal{P}}$ represents a pilot *energy* budget, and given that the training duration T_{p} is not fixed (but subject to an outer optimization), we prefer the term *energy boost*.

individual optimizations will form two building blocks of an algorithmic approach that aims to solve (P.3), and eventually (P.4). However, they may also be considered as two stand-alone problems in their own right.

4.3 Precoder Design for Prescribed Pilots

In this section, we consider the optimization of the transmit covariance $\mathbf{Q}$ alone, while the pilot Gram $\mathbf{P}$ has a fixed value [Problem (P.1.a)]. The optimal transmit covariance is

$$\mathbf{Q}^\star(\mathbf{P}) = \underset{\mathbf{Q}\in\mathcal{Q}(\mu_\mathcal{Q})}{\operatorname{argmax}}\, F(\mathbf{S}(\mathbf{P},\mathbf{Q})). \tag{4.16}$$

4.3.1 Number of Streams and Pilot Symbols

Recall that the ranks of $\mathbf{P}$ and $\mathbf{Q}$ represent the number of pilot symbols and precoded streams, respectively. To establish a relation between them, we first need to uncover an important property of the range space of $\mathbf{Q}^\star(\mathbf{P})$.

Theorem 4.1. *For any utility $F \in \mathcal{F}$ and a prescribed pilot Gram $\mathbf{P}$, the range space of the optimal transmit covariance $\mathbf{Q}^\star(\mathbf{P})$ must be contained in the range space of the channel estimate covariance $\hat{\mathbf{R}}$:*

$$\operatorname{range}(\mathbf{Q}^\star(\mathbf{P})) \subseteq \operatorname{range}(\hat{\mathbf{R}}). \tag{4.17}$$

Proof: See Appendix D.2. ■

Complementing Theorem 4.1, notice that the rank equality

$$\operatorname{rank}(\hat{\mathbf{R}}) = \operatorname{rank}(\mathbf{P}) \tag{4.18}$$

always holds. This is easily seen by application of the matrix inversion lemma:

$$\begin{aligned}\hat{\mathbf{R}} &= \mathbf{R} - (\mathbf{R}^{-1} + \mathbf{X}_\mathsf{p}\mathbf{X}_\mathsf{p}^\dagger)^{-1} \\ &= \mathbf{R}\mathbf{X}_\mathsf{p}(\mathbf{I} + \mathbf{X}_\mathsf{p}^\dagger\mathbf{R}\mathbf{X}_\mathsf{p})^{-1}\mathbf{X}_\mathsf{p}^\dagger\mathbf{R}.\end{aligned} \tag{4.19}$$

Since $\mathbf{X}_\mathsf{p}$ has full column rank by assumption [cf. Section 4.2], it becomes manifest that the rank of $\hat{\mathbf{R}}$ equals the number of columns of $\mathbf{X}_\mathsf{p}$, which is equal to $T_\mathsf{p} = \operatorname{rank}(\mathbf{P})$, hence (4.18).

Combining (4.18) with Theorem 4.1 directly implies the rank inequality

$$\operatorname{rank}(\mathbf{Q}^\star(\mathbf{P})) \leq \operatorname{rank}(\hat{\mathbf{R}}) = \operatorname{rank}(\mathbf{P}), \tag{4.20}$$

or in words,

$$\textbf{number of streams} \leq \textbf{number of pilot symbols} \tag{4.21}$$

The idea behind the proof of Theorem 4.1 is that, if $\mathbf{Q}^\star(\mathbf{P})$ had eigenvectors (transmit directions) lying outside the range space of the estimate covariance $\hat{\mathbf{R}}$, then the transmitter would be radiating some of its transmit power into channel directions of which the receiver has no estimate (and thus cannot detect coherently), thus incurring a waste of power. As a particular consequence, (4.20) tells us that the number of precoded streams should never exceed the number of training symbols.

4.3.2 Feasible Effective SNR Matrices

It is convenient to reformulate Problem (P.1.a) as

$$\mathbf{S}^\star(\mathbf{P}) = \underset{\mathbf{S}\in\mathbf{S}(\mathbf{P},\mathcal{Q}(\mu_{\mathcal{Q}}))}{\operatorname{argmax}} \; F(\mathbf{S}) \tag{4.22}$$

in terms of effective SNR matrices, which belong to a feasible set

$$\mathbf{S}(\mathbf{P},\mathcal{Q}(\mu_{\mathcal{Q}})) = \{\mathbf{S}(\mathbf{P},\mathbf{Q}) \mid \mathbf{Q}\in\mathcal{Q}(\mu_{\mathcal{Q}})\}. \tag{4.23}$$

We will show that $\mathbf{S}(\mathbf{P},\mathcal{Q}(\mu_{\mathcal{Q}}))$ is a convex set. For this purpose, observe that in the expression of the function

$$\mathbf{Q} \mapsto \mathbf{S}(\mathbf{P},\mathbf{Q}) = \frac{\hat{\mathbf{R}}^{\frac{1}{2}}\mathbf{Q}\hat{\mathbf{R}}^{\frac{1}{2}}}{1+\operatorname{tr}(\mathbf{Q}\tilde{\mathbf{R}})}, \tag{4.24}$$

the argument $\mathbf{Q}$ appears in the matrix-valued numerator, and inside a trace operator in the denominator. This function $\mathbf{Q} \mapsto \mathbf{S}(\mathbf{P},\mathbf{Q})$ is thus reminiscent of fractions of monomials such as $q \mapsto \frac{aq}{1+bq}$, except that it is defined for matrices. In fact, the function $\mathbf{Q} \mapsto \mathbf{S}(\mathbf{P},\mathbf{Q})$ pertains to what can be defined in the following Definition 4.1 as a generalization of linear fractional functions. The latter are commonly defined for the scalar case (e.g., [Boy04, Sec. 2.3.3]).

Definition 4.1. *Let $\mathcal{X} \subset \mathbb{C}^{n\times n}$ denote a set of Hermitian matrices of size $n\times n$ whose elements $\mathbf{X}\in\mathcal{X}$ satisfy $\operatorname{tr}(\mathbf{B}\mathbf{X}) \neq -1$ with some given Hermitian matrix $\mathbf{B}\in\mathbb{C}^{n\times n}$. A function $\mathbf{X}\mapsto \phi(\mathbf{X};\mathbf{A},\mathbf{B})$ that is defined as*

$$\mathcal{X} \;\to\; \mathbb{C}^{m\times m},\;\; \mathbf{X} \;\mapsto\; \phi(\mathbf{X};\mathbf{A},\mathbf{B}) = \frac{\mathbf{A}\mathbf{X}\mathbf{A}^\dagger}{1+\operatorname{tr}(\mathbf{B}\mathbf{X})} \tag{4.25}$$

shall be called a linear fractional *function with parameters $\mathbf{A}\in\mathbb{C}^{m\times n}$ and $\mathbf{B}\in\mathbb{C}^{n\times n}$.*

Note that the Hermitianity of $\mathbf{B}$ and of the argument $\mathbf{X}$ ensures the Hermitianity of the image $\phi(\mathbf{X};\mathbf{A},\mathbf{B})$. Linear fractional functions may or may not be injective functions, depending on the properties of the parameter $\mathbf{A}$. Let $\mathbf{A}^\sharp = (\mathbf{A}^\dagger\mathbf{A})^{-1}\mathbf{A}^\dagger$ denote the left pseudoinverse of $\mathbf{A}$, and $\mathbf{A}^\flat = \mathbf{A}^\dagger(\mathbf{A}\mathbf{A}^\dagger)^{-1}$ denote the right pseudoinverse of $\mathbf{A}$.

Lemma 4.1. *The linear fractional function $\mathbf{X}\mapsto\phi(\mathbf{X};\mathbf{A},\mathbf{B})$ from Definition 4.1 is injective* *(one-to-one) if one at least of the following two conditions apply:*

(1) *The parameter* $\mathbf{A}$ *has full column rank*

(2) *The parameter* $\mathbf{A}$ *has full row rank and the domain* $\mathcal{X}$ *is such that* $\forall \mathbf{X} \in \mathcal{X}\colon \operatorname{range}(\mathbf{X}) = \operatorname{range}(\mathbf{A}^\dagger)$.[4]

In these two respective cases, its inverse function $\phi^{-1}\colon \phi(\mathcal{X};\mathbf{A},\mathbf{B}) \to \mathcal{X}, \mathbf{Y} \mapsto \phi^{-1}(\mathbf{Y};\mathbf{A},\mathbf{B})$ *is*

(1) *linear fractional with parameters* $\mathbf{A}^\sharp$ *and* $-\mathbf{A}^{\sharp\dagger}\mathbf{B}\mathbf{A}^\sharp$, *i.e.,* $\phi^{-1}(\bullet;\mathbf{A},\mathbf{B}) = \phi(\bullet;\mathbf{A}^\sharp, -\mathbf{A}^{\sharp\dagger}\mathbf{B}\mathbf{A}^\sharp)$.

(2) *linear fractional with parameters* $\mathbf{A}^\flat$ *and* $-\mathbf{A}^{\flat\dagger}\mathbf{B}\mathbf{A}^\flat$, *i.e.,* $\phi^{-1}(\bullet;\mathbf{A},\mathbf{B}) = \phi(\bullet;\mathbf{A}^\flat, -\mathbf{A}^{\flat\dagger}\mathbf{B}\mathbf{A}^\flat)$.

Proof: See Appendix D.3. ■

In the following we will optimize $\mathbf{S}(\mathbf{P},\mathbf{Q})$ rather than $\mathbf{Q}$, and consider that Lemma 4.1 can be used to recover the optimal transmit covariance $\mathbf{Q}^\star(\mathbf{P})$ from the optimal $\mathbf{S}$, by means of the appropriate inverse linear fractional function.

Prescribing the pilot Gram $\mathbf{P}$ means that the matrices $\tilde{\mathbf{R}} = (\mathbf{R}^{-1} + \mathbf{P})^{-1}$ and $\hat{\mathbf{R}} = \mathbf{R} - \tilde{\mathbf{R}}$ are prescribed. Therefore, the function $\mathbf{Q} \mapsto \mathbf{S}(\mathbf{P},\mathbf{Q})$ as given in (4.24) is linear fractional with parameters $\mathbf{A} = \hat{\mathbf{R}}^{\frac{1}{2}}$ and $\mathbf{B} = \tilde{\mathbf{R}}$, i.e.,

$$\mathbf{S}(\mathbf{P},\mathbf{Q}) = \phi(\mathbf{Q};\hat{\mathbf{R}}^{\frac{1}{2}},\tilde{\mathbf{R}}). \tag{4.26}$$

The key property of linear fractional functions that we need for understanding Problem (P.1.a) is that they preserve the linearity of segments.

Lemma 4.2. *An injective linear fractional function* $\varphi(\bullet) = \phi(\bullet;\mathbf{A},\mathbf{B})$ *with some given parameters* $\mathbf{A}$ *and* $\mathbf{B}$ *uniquely maps linear segments onto linear segments in a one-to-one manner, i.e.,*

$$\forall(\mathbf{X}_1,\mathbf{X}_2,\alpha) \in \mathcal{X}^2 \times [0;1], \exists\beta \in [0;1]\colon \varphi(\alpha\mathbf{X}_1 + (1-\alpha)\mathbf{X}_2) = \beta\varphi(\mathbf{X}_1) + (1-\beta)\varphi(\mathbf{X}_2). \tag{4.27}$$

Proof: This is readily verified by inserting the explicit value

$$\beta = \frac{\alpha(1 + \operatorname{tr}(\mathbf{B}\mathbf{X}_1))}{1 + \alpha\operatorname{tr}(\mathbf{B}\mathbf{X}_1) + (1-\alpha)\operatorname{tr}(\mathbf{B}\mathbf{X}_2)} \tag{4.28}$$

into the equality (4.27). ■

Figure 4.1 symbolically depicts the behavior of linear fractional functions: a convex combination of two points is mapped onto a convex combination of the respective images of said

[4] In case $\mathbf{A}$ has neither full column nor full row rank, one can bring the problem back to one of the two considered cases by an appropriate rank reduction.

points, thus preserving segments. They are not linear functions though, because α and β can be different.

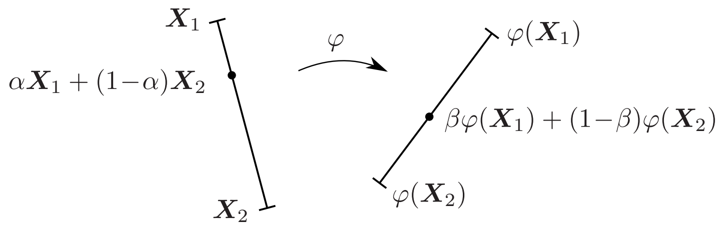


Figure 4.1 Linear fractional functions preserve segments

Corollary 4.1. *Linear fractional mappings preserve set convexity.*

Proof: Take a pair $(\mathbf{X}_1, \mathbf{X}_2) \in \mathcal{X}^2$ with a convex $\mathcal{X}$. According to Lemma 4.2, any convex combination of $\mathbf{X}_1$ and $\mathbf{X}_2$ is mapped onto a convex combination of $\varphi(\mathbf{X}_1)$ and $\varphi(\mathbf{X}_2)$. Therefore, the codomain $\varphi(\mathcal{X})$ is convex. ■

As a consequence, $\mathbf{S}(\mathbf{P}, \mathcal{Q}(\mu_\mathcal{Q}))$ is a convex set because $\mathcal{Q}(\mu_\mathcal{Q})$ is convex [cf. (4.13b)]. So if a utility F is concave in $\mathbf{S}$, then Problem (P.1.a) in its formulation (4.22) is convex. The optimal transmit covariance $\mathbf{Q}^\star(\mathbf{P})$ is then computed from $\mathbf{S}^\star(\mathbf{P})$ by means of the appropriate inverse linear fractional function (cf. Lemma 4.1). More generally speaking, if F is quasi-concave in $\mathbf{S}$, then the problem (4.16) can be recast into a convex problem by an appropriate transformation. Even if F is only unimodal on $\mathbf{S}(\mathbf{P}, \mathcal{Q}(\mu_\mathcal{Q}))$—that is, it has a single local maximum on the convex compact $\mathbf{S}(\mathbf{P}, \mathcal{Q}(\mu_\mathcal{Q}))$—one can still optimize it efficiently via bisection. The mutual information I is one example of a concave utility. Other examples of concave or log-concave (quasi-concave) utilities are given in Table D.1 in Appendix D.1.

4.3.3 Feasible Effective SNR Eigenvalues

It is convenient to rewrite Problem (P.1.a) as

$$\mathbf{s}^\star(\mathbf{P}) = \underset{\mathbf{s} \in \mathbf{s}(\mathbf{P}, \mathcal{Q}(\mu_\mathcal{Q}))}{\operatorname{argmax}} \; \bar{f}(\mathbf{s}), \tag{4.29}$$

in terms of effective SNR eigenvalue profiles, which are to be searched in a feasible set

$$\mathbf{s}(\mathbf{P}, \mathcal{Q}(\mu_\mathcal{Q})) = \{\mathbf{s}(\mathbf{P}, \mathbf{Q}) \mid \mathbf{Q} \in \mathcal{Q}(\mu_\mathcal{Q})\}. \tag{4.30}$$

In the previous subsection, we have shown that the set $\mathbf{S}(\mathbf{P}, \mathcal{Q}(\mu_\mathcal{Q}))$ is convex. Note that this convexity, however, does not generally imply (nor is implied by) the convexity of the set $\mathbf{s}(\mathbf{P}, \mathcal{Q}(\mu_\mathcal{Q}))$, hence establishing the convexity of $\mathbf{s}(\mathbf{P}, \mathcal{Q}(\mu_\mathcal{Q}))$ requires a separate proof. Indeed, we show in the following that $\mathbf{s}(\mathbf{P}, \mathcal{Q}(\mu_{\mathcal{Q}}))$ is also convex and has a simplex shape, whose vertices are characterized by Theorem 4.2 below.

Let ω_i denote the non-increasingly ordered eigenvalues of the generalized eigenvalue problem

$$\hat{\mathbf{R}}\mathbf{v}_i = \omega_i(\mu_{\mathcal{Q}}^{-1}\mathbf{I} + \tilde{\mathbf{R}})\mathbf{v}_i. \tag{4.31}$$

Due to $\mathrm{rank}(\hat{\mathbf{R}}) = \mathrm{rank}(\mathbf{P})$ [cf. (4.18)], only the first $r_{\mathbf{P}} = \mathrm{rank}(\mathbf{P})$ eigenvalues ω_i are different from zero. Let $\boldsymbol{\lambda}(\mathbf{A})$ denote the vector of non-increasingly ordered eigenvalues of a Hermitian matrix $\mathbf{A}$.

Theorem 4.2. *The set [cf.* (4.13b), (4.24)*]*

$$\mathbf{s}(\mathbf{P}, \mathcal{Q}(\mu_{\mathcal{Q}})) = \left\{ \boldsymbol{\lambda}\left(\frac{\hat{\mathbf{R}}^{\frac{1}{2}}\mathbf{Q}\hat{\mathbf{R}}^{\frac{1}{2}}}{1 + \mathrm{tr}(\mathbf{Q}\tilde{\mathbf{R}})} \right) \,\middle|\, \mathbf{Q} \in \mathcal{Q}(\mu_{\mathcal{Q}}) \right\} \tag{4.32}$$

is a simplex given by the convex hull of the origin $\boldsymbol{\sigma}^{(0)} \triangleq \mathbf{0}$ and of the $r_{\mathbf{P}}$ linearly independent points

$$\boldsymbol{\sigma}^{(n)} = \mathcal{H}(\omega_1, \ldots, \omega_n) \sum_{j=1}^{n} \mathbf{e}_j, \quad n \in \{1, \ldots, r_{\mathbf{P}}\} \tag{4.33}$$

where $[\mathbf{e}_1, \ldots, \mathbf{e}_{n_\mathrm{T}}] = \mathbf{I}$ is the canonical basis, and $\mathcal{H}(x_1, \ldots, x_n) = (\sum_{i=1}^{n} x_i^{-1})^{-1}$ with n arguments $x_1, \ldots, x_n$ denotes the harmonic mean thereof, divided by n.

Proof: See Appendix D.4. ■

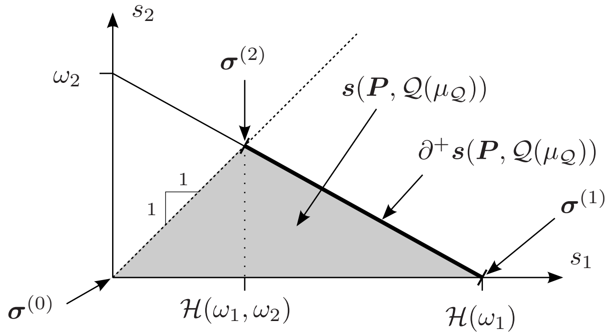


Figure 4.2 Sketch of a simplex set $\mathbf{s}(\mathbf{P}, \mathcal{Q}(\mu_{\mathcal{Q}}))$ (shaded region). The so-called Pareto border $\partial^+\mathbf{s}(\mathbf{P}, \mathcal{Q}(\mu_{\mathcal{Q}}))$ contains those points from $\mathbf{s}(\mathbf{P}, \mathcal{Q}(\mu_{\mathcal{Q}}))$ that are not dominated by any other point from $\mathbf{s}(\mathbf{P}, \mathcal{Q}(\mu_{\mathcal{Q}}))$, and is the convex hull of $\boldsymbol{\sigma}^{(n)}$ for $n \in \{1, \ldots, r_{\mathbf{P}}\}$ (excluding the origin).

As a byproduct, the proof of Theorem 4.2 reveals that if the set of eigenvectors of $\hat{\mathbf{R}}$ is contained in the set of eigenvectors of $\tilde{\mathbf{R}}$, i.e., $\mathrm{col}(\mathbf{U}_{\hat{\mathbf{R}}}) \subseteq \mathrm{col}(\mathbf{U}_{\tilde{\mathbf{R}}})$, then it is optimal with respect to any utility $F \in \mathcal{F}$ that the eigenbasis $\mathbf{U}_{\mathbf{Q}^\star(\mathbf{P})}$ of the optimal matrix $\mathbf{Q}^\star(\mathbf{P})$ be chosen such that as

$$\operatorname{col}(\mathbf{U}_{\mathbf{Q}}) \subseteq \operatorname{col}(\mathbf{U}_{\hat{\mathbf{R}}}). \tag{4.34}$$

Note that this requirement is stronger than the range space inclusion property of Theorem 4.1 [cf. (4.17)]. This particular situation of eigenbasis alignment $\operatorname{col}(\mathbf{U}_{\hat{\mathbf{R}}}) \subseteq \operatorname{col}(\mathbf{U}_{\tilde{\mathbf{R}}})$ occurs, for example, when

- using n_{T} unitary pilots (i.e., $\mathbf{P} = \frac{\operatorname{tr}(\mathbf{P})}{n_{\mathsf{T}}}\mathbf{I}_{n_{\mathsf{T}}}$ is a scaled identity matrix)
- the channel gains are independently and identically distributed ($\mathbf{R} = \mathbf{I}$)
- the channel estimation error vanishes ($\tilde{\mathbf{R}} = \mathbf{0}$, $\hat{\mathbf{R}} = \mathbf{R}$)
- the pilots are aligned with the channel covariance, i.e., $\operatorname{col}(\mathbf{U}_{\mathbf{P}}) \subseteq \operatorname{col}(\mathbf{U}_{\mathbf{R}})$.

As we shall see later in Section 4.5, the latter condition $\operatorname{col}(\mathbf{U}_{\mathbf{P}}) \subseteq \operatorname{col}(\mathbf{U}_{\mathbf{R}})$ is in fact necessary for joint optimality of $\mathbf{P}$ and $\mathbf{Q}$.

4.4 Pilot Design for a Prescribed Precoder

To complement the previous Section 4.3, we will now swap the roles of $\mathbf{P}$ and $\mathbf{Q}$ so as to consider the optimization of the pilot Gram $\mathbf{P}$ under a trace constraint, while the transmit covariance $\mathbf{Q}$ has a fixed value [Problem (P.1.b)]. The optimal pilot sequence reads as

$$\mathbf{P}^{\star}(\mathbf{Q}) = \operatorname*{argmax}_{\mathbf{P} \in \mathcal{P}(\mu_{\mathcal{P}})} F(\mathbf{S}(\mathbf{P}, \mathbf{Q})). \tag{4.35}$$

4.4.1 Number of Streams and Pilot Symbols

In analogy to the inequality (4.20) relating the ranks of $\mathbf{P}$ and $\mathbf{Q}^{\star}(\mathbf{P})$, we have a similar rank inequality for Problem (P.1.b) too.

Theorem 4.3. *For any utility $f \in \mathcal{F}$ and a prescribed transmit covariance $\mathbf{Q}$, the rank of the optimal pilot Gram $\mathbf{P}^{\star}(\mathbf{Q})$ under a trace constraint is not larger than the rank of $\mathbf{Q}$:*

$$\operatorname{rank}(\mathbf{P}^{\star}(\mathbf{Q})) \leq \operatorname{rank}(\mathbf{Q}). \tag{4.36}$$

Proof: See Appendix D.5. ■

In words, we can state this as [compare with (4.21)]

$$\textbf{number of pilot symbols} \leq \textbf{number of streams} \tag{4.37}$$

The interpretation behind this rank inequality is that, if there were more orthogonal training directions than there are data streams precoded, we would necessarily be wasting some pilot energy into directions that are not used for transmission anyway.

4.4.2 Feasible Set of Effective SNR Matrices

Let us rewrite Problem (P.1.b) as

$$\mathbf{S}^\star(\mathbf{Q}) = \underset{\mathbf{S}\in\mathbf{S}(\mathcal{P}(\mu_\mathcal{P}),\mathbf{Q})}{\operatorname{argmax}} \; F(\mathbf{S}) \tag{4.38}$$

in terms of effective SNR matrices, which belong to a feasible set

$$\mathbf{S}(\mathcal{P}(\mu_\mathcal{P}),\mathbf{Q}) = \{\mathbf{S}(\mathbf{P},\mathbf{Q}) \mid \mathbf{P}\in\mathcal{P}(\mu_\mathcal{P})\}. \tag{4.39}$$

We will show that this set is convex. To this end, we write out $\tilde{\mathbf{R}}$ as $\mathbf{R}-\hat{\mathbf{R}}$, then $\mathbf{S}(\mathbf{P},\mathbf{Q})$ reads as [cf. (4.11)]

$$\mathbf{S}(\mathbf{P},\mathbf{Q}) = \frac{\hat{\mathbf{R}}^{\frac{1}{2}}\mathbf{Q}\hat{\mathbf{R}}^{\frac{1}{2}}}{1+\operatorname{tr}(\mathbf{Q}\mathbf{R})-\operatorname{tr}(\mathbf{Q}\hat{\mathbf{R}})}, \tag{4.40}$$

which is unitarily equivalent to

$$\begin{aligned}\breve{\mathbf{S}}(\mathbf{P},\mathbf{Q}) &= \frac{\mathbf{Q}^{\frac{1}{2}}\hat{\mathbf{R}}\mathbf{Q}^{\frac{1}{2}}}{1+\operatorname{tr}(\mathbf{Q}\mathbf{R})-\operatorname{tr}(\mathbf{Q}\hat{\mathbf{R}})}\\ &= \frac{\mathbf{Q}^{\frac{1}{2}}\hat{\mathbf{R}}\mathbf{Q}^{\frac{1}{2}}}{\tau-\operatorname{tr}(\mathbf{Q}\hat{\mathbf{R}})},\end{aligned} \tag{4.41}$$

where $\tau = 1+\operatorname{tr}(\mathbf{Q}\mathbf{R})$. The unitary equivalence is due the Hermitian matrices $\hat{\mathbf{R}}^{\frac{1}{2}}\mathbf{Q}\hat{\mathbf{R}}^{\frac{1}{2}}$ and $\mathbf{Q}^{\frac{1}{2}}\hat{\mathbf{R}}\mathbf{Q}^{\frac{1}{2}}$ having the same eigenvalues because of the identity $\boldsymbol{\lambda}(\mathbf{A}\mathbf{B}) = \boldsymbol{\lambda}(\mathbf{B}\mathbf{A})$. As a consequence, $F(\mathbf{S}(\mathbf{P},\mathbf{Q})) = F(\breve{\mathbf{S}}(\mathbf{P},\mathbf{Q}))$ for any $F\in\mathcal{F}$, so $\mathbf{S}(\mathbf{P},\mathbf{Q})$ and $\breve{\mathbf{S}}(\mathbf{P},\mathbf{Q})$ can be used interchangeably. By comparing Expression (4.41) with the definition of linear fractional functions (cf. Definition 4.1), we identify $\hat{\mathbf{R}}\mapsto\breve{\mathbf{S}}(\mathbf{P},\mathbf{Q})$ as a linear fractional function with parameters $\mathbf{A} = \frac{1}{\sqrt{\tau}}\mathbf{Q}^{\frac{1}{2}}$ and $\mathbf{B} = -\frac{1}{\tau}\mathbf{Q}$, i.e.,

$$\breve{\mathbf{S}}(\mathbf{P},\mathbf{Q}) = \phi\Big(\hat{\mathbf{R}}; \tfrac{1}{\sqrt{\tau}}\mathbf{Q}^{\frac{1}{2}}, -\tfrac{1}{\tau}\mathbf{Q}\Big). \tag{4.42}$$

In Appendix D.6, we show that the set of feasible $\hat{\mathbf{R}}$, namely

$$\{\mathbf{R}-(\mathbf{R}^{-1}+\mathbf{P})^{-1} \mid \mathbf{P}\in\mathcal{P}(\mu_\mathcal{P})\}, \tag{4.43}$$

is convex, from which follows immediately with Corollary 4.1 that $\breve{\mathbf{S}}(\mathcal{P}(\mu_\mathcal{P}),\mathbf{Q})$ is a convex set. For solving Problem (P.1.b), it now suffices to replace $\mathbf{S}(\mathcal{P}(\mu_\mathcal{P}),\mathbf{Q})$ with $\breve{\mathbf{S}}(\mathcal{P}(\mu_\mathcal{P}),\mathbf{Q})$ in formulation (4.38) of Problem (P.1.b). Granted that the utility F is concave, quasi-concave or unimodal, Problem (P.1.b) can be solved efficiently.

4.5 Jointly Pareto Optimal Pilot-Precoder Pairs

4.5.1 Problem Statement

We move on to study a subproblem of (P.3). For this purpose, we restate Problem (P.3) in the vector domain of feasible profiles $\mathbf{s}$ so that it reads

$$\max_{\mathbf{s}\in\mathbf{s}(\mathcal{PQ}(T_{\mathsf{p}}))} \bar{f}(\mathbf{s}), \tag{4.44}$$

where the search set is

$$\mathbf{s}(\mathcal{PQ}(T_{\mathsf{p}})) = \left\{\mathbf{s}(\mathbf{P},\mathbf{Q}) \,\middle|\, (\mathbf{P},\mathbf{Q}) \in \mathcal{PQ}(T_{\mathsf{p}})\right\} \tag{4.45}$$

and $\mathcal{PQ}(T_{\mathsf{p}})$ was defined in (4.14). Exploiting the monotonicity of utilities $\bar{f} \in \bar{\mathcal{F}}$, we can restrict the search set $\mathbf{s}(\mathcal{PQ}(T_{\mathsf{p}}))$ to its Pareto border $\partial^{+}\mathbf{s}(\mathcal{PQ}(T_{\mathsf{p}}))$. To be precise, the Pareto border of a set $\mathcal{A} \subset \mathbb{R}^N$ is the subset

$$\partial^{+}\mathcal{A} = \left\{\mathbf{a} \in \mathcal{A} \,\middle|\, \nexists \mathbf{a}' \in \mathcal{A} \colon \mathbf{a}' \geq \mathbf{a} \text{ with } \mathbf{a}' \neq \mathbf{a}\right\}. \tag{4.46}$$

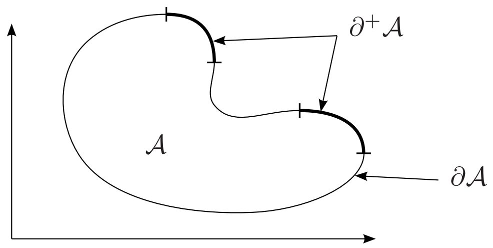


Figure 4.3 Pareto border $\partial^{+}\mathcal{A}$ of a compact set $\mathcal{A} \subset \mathbb{R}^2$

Regardless of which utility function $\bar{f} \in \bar{\mathcal{F}}$ we are considering, there exists an important subproblem of (P.3) that is common to all utility functions: the characterization of the search set $\partial^{+}\mathbf{s}(\mathcal{PQ}(T_{\mathsf{p}}))$. The following subsections will build up this characterization in several steps.

4.5.2 Number of Streams and Pilot Symbols

Recall that (P.3) can be expressed in terms of (P.2) by adding an outer optimization of the energy boost [cf. (4.15)]:

$$\max_{\substack{\mu_{\mathcal{P}},\mu_{\mathcal{Q}}\geq 0\\ \mu_{\mathcal{P}}+(T-T_{\mathsf{p}})\mu_{\mathcal{Q}}=T\mu}} \left\{ \max_{(\mathbf{P},\mathbf{Q})\in\mathcal{P}(\mu_{\mathcal{P}})\times\mathcal{Q}(\mu_{\mathcal{Q}})} F(\mathbf{S}(\mathbf{P},\mathbf{Q}))\right\}. \tag{4.47}$$

Inside the curly braces is Problem (P.2), which can be equivalently written as

$$\max_{\mathbf{P}\in\mathcal{P}(\mu_{\mathcal{P}})} F\big(\mathbf{S}(\mathbf{P},\mathbf{Q}^\star(\mathbf{P}))\big) = \max_{\mathbf{Q}\in\mathcal{Q}(\mu_{\mathcal{Q}})} F\big(\mathbf{S}(\mathbf{P}^\star(\mathbf{Q}),\mathbf{Q})\big) \tag{4.48}$$

with $\mathbf{Q}^\star(\mathbf{P})$ and $\mathbf{P}^\star(\mathbf{Q})$ defined in (4.16) and (4.35), respectively. We infer that the jointly optimal pilot-precoder pair for Problem (P.2) must simultaneously fulfill the rank inequalities (4.20) and (4.36). This means that the pilot Gram and transmit covariance have equal ranks. Since this rank equality holds regardless of the value of $(\mu_{\mathcal{P}}, \mu_{\mathcal{Q}})$, it also holds for Problem (P.3). Since it also holds regardless of the value of T_{p}, it also holds for Problem (P.4). Hence, we can state generally that, for Problems (P.2), (P.3), (P.4),

$$r^\star \triangleq \textbf{number of streams} = \textbf{number of pilot symbols} \tag{4.49}$$

is a necessary condition for a pilot-precoder pair $(\mathbf{P},\mathbf{Q})$ to be jointly optimal. Note that, in addition, Theorem 4.1 requires the optimal transmit covariance to lie in the range space of $\hat{\mathbf{R}}$ and as a consequence, the rank of $\mathbf{S}(\mathbf{P},\mathbf{Q})$, which is the number of non-zero entries in $\mathbf{s}(\mathbf{P},\mathbf{Q})$, is equal to $r^\star$ as well.

4.5.3 Jointly Optimal Transmit and Training Directions

A fortunate circumstance when treating the joint problems (P.2), (P.3), (P.4), is that the jointly optimal transmit and training directions have a very simple and intuitive characterization, enunciated in Theorem 4.4 below. As in the previous Section, we set our focus on Problem (P.2), since the property will extend immediately to Problems (P.3) and (P.4).

Let the channel covariance $\mathbf{R}$, the pilot Gram $\mathbf{P}$ and the transmit covariance $\mathbf{Q}$ have the following (reduced) eigendecompositions:

$$\mathbf{R} = \mathbf{U_R}\mathbf{\Lambda_R}\mathbf{U_R}^\dagger, \qquad \mathbf{P} = \mathbf{U_P}\mathbf{\Lambda_P}\mathbf{U_P}^\dagger, \qquad \mathbf{Q} = \mathbf{U_Q}\mathbf{\Lambda_Q}\mathbf{U_Q}^\dagger.$$

Without loss of generality, we assume that the eigenvalues of $\mathbf{R}$ are arranged in non-increasing order on the diagonal positions of $\mathbf{\Lambda_R}$, whereas the eigenvalues of $\mathbf{\Lambda_P}$ and $\mathbf{\Lambda_Q}$ are not sorted in any specific order. Let the set of columns of a matrix $\mathbf{A}$ be denoted as $\mathrm{col}(\mathbf{A})$.

Theorem 4.4. *For any utility $\bar{f} \in \bar{\mathcal{F}}$, in the joint optimization problem (P.2), there is no loss of optimality in restricting the eigenvectors of the pilot Gram $\mathbf{P}$ (i.e., the left singular vectors of the pilot sequence $\mathbf{X}_{\mathsf{p}}$) and the eigenvectors of the transmit covariance $\mathbf{Q}$ (i.e., the left singular vectors of the precoder $\mathbf{F}$) to be a common subset of the eigenvectors of the channel covariance $\mathbf{R}$ corresponding to the largest eigenvalues of $\mathbf{R}$. Formally, this is to say that the (reduced)* *eigenbases* $\mathbf{U}_{\mathbf{P}}$ *and* $\mathbf{U}_{\mathbf{Q}}$ *should satisfy*

$$\mathrm{col}(\mathbf{U}_{\mathbf{P}}) = \mathrm{col}(\mathbf{U}_{\mathbf{Q}}) = \{\mathbf{u}_{\mathbf{R},1}, \ldots, \mathbf{u}_{\mathbf{R},r^\star}\} \subseteq \mathrm{col}(\mathbf{U}_{\mathbf{R}}), \tag{4.50}$$

where $\mathbf{U}_{\mathbf{R}} \triangleq [\mathbf{u}_{\mathbf{R},1}, \ldots, \mathbf{u}_{\mathbf{R},n_\mathrm{T}}]$, *and* $r^\star = \mathrm{rank}(\mathbf{P}^\star) = \mathrm{rank}(\mathbf{Q}^\star)$ *denotes the pilot/precoder rank at the joint optimum* $(\mathbf{P}^\star, \mathbf{Q}^\star)$ *of Problem (P.2).* [5]

Proof: See Appendix D.7. ■

Since Theorem 4.4 holds irrespective of the values of $(\mu_\mathcal{P}, \mu_\mathcal{Q})$ and of T_p, we infer that it also holds for Problems (P.3) and (P.4).

This Theorem echoes similar results from previous publications. For example, in [Soy10], the authors find that the optimal pilot symbols are, as in the present case, scaled eigenvectors of the channel's transmit correlation matrix. However, they optimize the pilot sequence with respect to the Frobenius norm of the channel estimation's mean-square error matrix instead of the achievable rate. Similarly, in [Aub13] it is proven for a multiple-access setup that, under the assumption that the channel estimate follows an UIU model (in the terminology of [Tul06]) slightly more general than ours, the transmit covariances are aligned with the channel correlation as well. As to the pilot sequence, it is optimized with respect to different objectives: the trace and the determinant of the mean-square error matrix of the channel estimation. The corresponding optimal eigenbases for the pilot sequences are similarly aligned. The main contribution of Theorem 4.4 is that of establishing the jointly optimal eigenbases of pilot Gram and transmit covariance with respect to a common utility function.

4.5.4 Pareto Optimal Allocation

Consequently, and without loss of optimality, we will align the eigenbases of $\mathbf{P}$ and $\mathbf{Q}$ in conformity with (4.50). The scalars $[\mathbf{r}]_i = r_i$, $[\mathbf{p}]_i = p_i$, and $[\mathbf{q}]_i = q_i$ shall denote the eigenvalues of $\mathbf{R}$, $\mathbf{P}$, and $\mathbf{Q}$, respectively. Under such assumptions, all matrices involved in the expression of the effective SNR (4.11), namely $\hat{\mathbf{R}}$ and $\tilde{\mathbf{R}}$ [cf. (4.7)], as well as $\mathbf{Q}$, acquire the same eigenbasis $\mathbf{U}_{\mathbf{R}}$. We can readily see from Expression (4.11) that $\mathbf{S}$ then inherits the (common) eigenvectors of $\mathbf{P}$ and $\mathbf{Q}$, i.e., $\mathrm{col}(\mathbf{U}_{\mathbf{S}}) = \mathrm{col}(\mathbf{U}_{\mathbf{P}}) = \mathrm{col}(\mathbf{U}_{\mathbf{Q}}) \subseteq \mathrm{col}(\mathbf{U}_{\mathbf{R}})$, so that the profile $\mathbf{s}$ is given by [cf. (4.11)]

$$\mathbf{s} = \frac{\hat{\mathbf{r}} \odot \mathbf{q}}{1 + \mathbf{q}^\mathrm{T}\tilde{\mathbf{r}}} \tag{4.51}$$

---

[5] Obviously, the rank $r^\star$ is not known *a priori* before solving the problem. The notation in (4.50) is merely to indicate that $\mathrm{col}(\mathbf{U}_{\mathbf{P}})$ and $\mathrm{col}(\mathbf{U}_{\mathbf{Q}})$ should contain eigenvectors of $\mathbf{R}$ corresponding to the *largest* eigenvalues of $\mathbf{R}$.

and '$\odot$' denotes the componentwise product. Here, the eigenvalue vectors $\tilde{\mathbf{r}} = \tilde{\mathbf{r}}(\mathbf{p})$ and $\hat{\mathbf{r}} = \hat{\mathbf{r}}(\mathbf{p}) = \mathbf{r} - \tilde{\mathbf{r}}(\mathbf{p})$ are functions of $\mathbf{p}$ and respectively have entries

$$\tilde{r}_i(p_i) = \frac{r_i}{1 + r_i p_i}, \qquad \hat{r}_i(p_i) = \frac{r_i^2 p_i}{1 + r_i p_i}. \tag{4.52}$$

Hereinforth, we will write $\mathbf{s}(\mathbf{p}, \mathbf{q})$ instead of $\mathbf{s}(\mathbf{P}, \mathbf{Q})$ whenever we implicitly assume that the eigenbases are optimally aligned according to (4.50). We do not impose any ordering of the eigenvalues p_i, q_i, and r_i. Instead we assume, without loss of generality, that they are arranged in such way that the s_i are non-increasingly ordered.

Upon optimally aligning the eigenbases as according to Theorem 4.4, we now consider the remaining problem that consists in jointly optimizing the allocation vector pair $(\mathbf{p}, \mathbf{q})$, which belongs to a set that constrains the average power radiated by the transmitter array:

$$\Gamma = \left\{ (\mathbf{p}, \mathbf{q}) \in \mathbb{R}_+^{2n_\mathsf{T}} \middle| \mathbf{1}^\mathrm{T}\mathbf{p} + (T - T_\mathsf{p})\mathbf{1}^\mathrm{T}\mathbf{q} \leq T\mu \right\}. \tag{4.53}$$

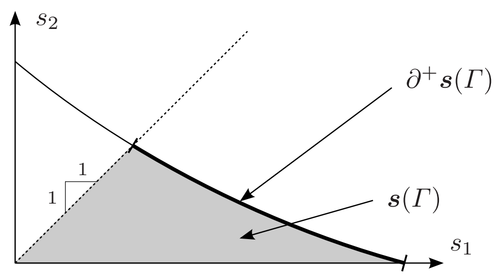


Figure 4.4 Sketch of the typical shape of a set $\mathbf{s}(\Gamma)$ and its Pareto border $\partial^+\mathbf{s}(\Gamma)$ for $n_\mathsf{T} = 2$.

By virtue of Theorem 4.4, we have $\mathbf{s}(\mathcal{PQ}(T_\mathsf{p})) = \mathbf{s}(\Gamma)$. In the following, we will devise a procedure for computing the set of all allocations $(\mathbf{p}, \mathbf{q})$ that yield points located on the Pareto border $\partial^+\mathbf{s}(\mathcal{PQ}(T_\mathsf{p})) = \partial^+\mathbf{s}(\Gamma)$. Given the monotonicity of the entries of $\mathbf{s}(\mathbf{p}, \mathbf{q})$ in p_i and q_i, we are certain that any Pareto optimal allocation $(\mathbf{p}, \mathbf{q})$ will expend the full power budget, and thus belong to

$$\partial^+\Gamma = \left\{ (\mathbf{p}, \mathbf{q}) \in \Gamma \middle| \mathbf{1}^\mathrm{T}\mathbf{p} + (T - T_\mathsf{p})\mathbf{1}^\mathrm{T}\mathbf{q} = T\mu \right\}. \tag{4.54}$$

Hence, $\partial^+\mathbf{s}(\Gamma) = \partial^+\mathbf{s}(\partial^+\Gamma)$. Now note that the search set $\partial^+\mathbf{s}(\partial^+\Gamma)$ is *not* equal to the set $\mathbf{s}(\partial^+\Gamma)$, meaning that it is not sufficient to simply choose some full-power allocation $(\mathbf{p}, \mathbf{q}) \in \partial^+\Gamma$ in order to obtain a *Pareto optimal* allocation. Instead, we have the proper inclusion

$$\partial^+\mathbf{s}(\Gamma) = \partial^+\mathbf{s}(\partial^+\Gamma) \subsetneq \mathbf{s}(\partial^+\Gamma). \tag{4.55}$$

In fact, any Pareto optimal allocation is a full-power allocation, but the converse is not true. This becomes clear when counting dimensions: the vector $\mathbf{s}$ has n_T real entries, so any parametrization of the feasible set $\mathbf{s}(\Gamma)$ with minimal number of parameters will require at most n_T real parameters. A parametrization of the Pareto border $\partial^+\mathbf{s}(\Gamma)$ will require $n_\mathsf{T}-1$ parameters. However, the entries of the vector pair $(\mathbf{p},\mathbf{q})$ represent $2n_\mathsf{T}$ parameters. Thus, to obtain a minimal parametrization of $\partial^+\mathbf{s}(\Gamma)$ there are at least $n_\mathsf{T}-1$ excess parameters to be eliminated. A direct elimination by working off the explicit expression of $\mathbf{s}(\mathbf{p},\mathbf{q})$ in (4.51) does not seem possible. Even replacing Γ with $\partial^+\Gamma$ only saves one parameter.

The idea for reducing the parameter set so as to efficiently compute Pareto optimal allocations $(\mathbf{p},\mathbf{q})$ will be as follows: we choose some vector norm $\|\cdot\|$, then fix a non-negative direction vector $\mathbf{e} \geq \mathbf{0}$ that is normalized as $\|\mathbf{e}\| = 1$. This normalized vector points into the positive orthant of the $\mathbf{s}$ domain and defines a half-line departing from the origin. We then maximize the norm $\|\mathbf{s}(\mathbf{p},\mathbf{q})\|$ with respect to the allocation $(\mathbf{p},\mathbf{q})$ under the constraint that $\mathbf{s}(\mathbf{p},\mathbf{q})$ points into the direction of $\mathbf{e}$. In other terms, we determine the point from the set $\mathbf{s}(\Gamma)$ which lies farthest away from the origin, and is located on the line running along $\mathbf{e}$.

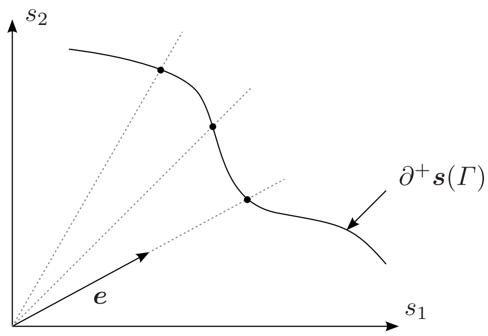


Figure 4.5 Symbolic sketch of the procedure for computing Pareto border points from $\partial^+\mathbf{s}(\Gamma)$. Said points are parametrized by a unit-norm direction vector $\mathbf{e}$

Formally, the problem at hand can be stated as:

$$\max_{(\mathbf{p},\mathbf{q})\in\Gamma} \nu \qquad \text{s.t.} \quad \mathbf{s}(\mathbf{p},\mathbf{q}) = \nu\mathbf{e} \tag{4.56}$$

where $\nu = \|\mathbf{s}(\mathbf{p},\mathbf{q})\|$ stands for the norm of $\mathbf{s}$, while the function $\mathbf{s}(\mathbf{p},\mathbf{q})$ is given by (4.51) as

$$\mathbf{s}(\mathbf{p},\mathbf{q}) = \frac{\hat{\mathbf{r}} \odot \mathbf{q}}{1 + \tilde{\mathbf{r}}^\mathrm{T}\mathbf{q}} = \frac{\hat{\mathbf{r}} \odot \mathbf{q}}{1 + \mathbf{r}^\mathrm{T}\mathbf{q} - \hat{\mathbf{r}}^\mathrm{T}\mathbf{q}} \tag{4.57}$$

and $\mathbf{e}$ is some normalized direction vector pointing into the positive orthant, i.e., $\mathbf{e} \geq \mathbf{0}$ and $\|\mathbf{e}\| = 1$. As usual, the search set Γ can be reduced to $\partial^+\Gamma$.

When we vary $\mathbf{e}$, the set of all points $\nu_{\mathsf{max}}\mathbf{e}$ that are determined by this maximization procedure constitute what we shall call a *front border*.

The front border of a compact set $\mathcal{A} \subseteq \mathbb{R}^n_+$ shall be denoted as $\partial^\mathrm{f}\mathcal{A}$ and be formally defined

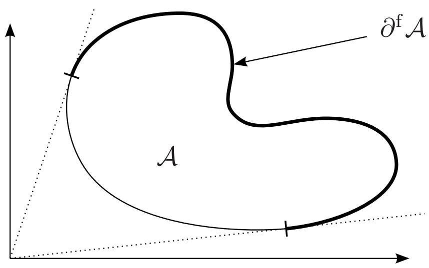


Figure 4.6 Front border $\partial^{\mathrm{f}}\mathcal{A}$ of a closed set $\mathcal{A} \subset \mathbb{R}^2$

as

$$\partial^{\mathrm{f}}\mathcal{A} = \bigcup_{\substack{\mathbf{e} \geq \mathbf{0} \\ \|\mathbf{e}\|=1}} \operatorname*{argmax}_{\substack{\mathbf{a}\in\mathcal{A} \\ \mathbf{a}=\nu\mathbf{e}}} \nu. \tag{4.58}$$

Note that certain directions $\mathbf{e}$ may yield empty sets $\{\mathbf{a} \in \mathcal{A} | \mathbf{a} = \nu\mathbf{e}\} = \emptyset$, so only non-trivial contributions (non-empty sets) should be retained when taking the union (4.58). As we easily intuit from comparing Figures 4.3 and 4.6, the *Pareto border* and *front border* of a compact set are not generally identical. However, according to the next Lemma, identity holds for the set $\mathbf{s}(\Gamma)$.

Lemma 4.3. *The Pareto border and the front border of the set* $\mathbf{s}(\Gamma)$ *coincide.*

Proof: See Appendix D.8. ■

As a consequence, we can compute the Pareto border by the above-mentioned technique. Let us choose the norm $\|\cdot\|$ to be the 1-norm $\|\mathbf{s}\|_1 = \sum_i s_i$, as this will turn out to be a convenient choice. The quantity ν that is maximized in (4.56) is the 1-norm of the vector $\mathbf{s}(\mathbf{p},\mathbf{q})$, constrained to being colinear with $\mathbf{e}$, i.e.,

$$\mathbf{s}(\mathbf{p},\mathbf{q}) = \nu\mathbf{e} \qquad\qquad \|\mathbf{s}(\mathbf{p},\mathbf{q})\|_1 = \frac{\eta}{1+\mathbf{r}^{\mathrm{T}}\mathbf{q}-\eta} = \nu \tag{4.59}$$

where η stands for [cf. (4.57)]

$$\eta = \|\hat{\mathbf{r}} \odot \mathbf{q}\|_1 = \hat{\mathbf{r}}^{\mathrm{T}}\mathbf{q}. \tag{4.60}$$

Note that the colinearity constraint $\mathbf{s}(\mathbf{p},\mathbf{q}) = \nu\mathbf{e}$ implies the colinearity $\hat{\mathbf{r}} \odot \mathbf{q} = \eta\mathbf{e}$. Componentwise, the latter reads as [cf. (4.52)]

$$\frac{r_i^2 p_i q_i}{1 + r_i p_i} = \eta e_i \tag{4.61}$$

Consider $\mathbf{e}$ to be fixed. Then we see from (4.61) that, once η is given, p_i and q_i are entirely determined from one another: given any value of $q_i \geq 0$, the corresponding value of $p_i \geq 0$ is uniquely determined (as long as $\frac{\eta e_i}{q_i} < r_i$), and conversely, given any value of $p_i \geq 0$, the value of $q_i \geq 0$ is uniquely determined. This allows us to effectuate a (one-to-one) change of parameters: we drop the q_i and replace them by e_i, thus effectively replacing the parameter pair $(\mathbf{p}, \mathbf{q}) \in \Gamma$ by the new pair $(\mathbf{p}, \mathbf{e}) \in \mathcal{D}(T\mu) \times \mathcal{D}(1)$, where $\mathcal{D}(\cdot)$ is defined as

$$\mathcal{D}(\alpha) = \{\mathbf{d} \in \mathbb{R}_+^{n_\mathsf{T}} \mid \mathbf{1}^\mathrm{T}\mathbf{d} \leq \alpha\}. \tag{4.62}$$

From (4.61), the q_i can now be expressed in terms of p_i and e_i as

$$q_i(p_i, e_i) = \eta e_i \frac{1 + r_i p_i}{r_i^2 p_i}. \tag{4.63}$$

By summing (4.63) up over i, and taking into account the energy conservation $\sum_i p_i + (T - T_\mathsf{p}) \sum_i q_i = T\mu$, we obtain expressions of η and of q_i which are functions of $(\mathbf{p}, \mathbf{e})$:

$$\eta(\mathbf{p}, \mathbf{e}) = \frac{T\mu - \mathbf{1}^\mathrm{T}\mathbf{p}}{T - T_\mathsf{p}} \left(\sum_{i=1}^{n_\mathsf{T}} e_i \frac{1 + r_i p_i}{r_i^2 p_i} \right)^{-1} \tag{4.64}$$

$$q_i(\mathbf{p}, \mathbf{e}) = \frac{T\mu - \mathbf{1}^\mathrm{T}\mathbf{p}}{T - T_\mathsf{p}} \frac{e_i \frac{1+r_i p_i}{r_i^2 p_i}}{\sum_j e_j \frac{1+r_j p_j}{r_j^2 p_j}}. \tag{4.65}$$

Consequently, ν can itself be expressed as a function of $(\mathbf{p}, \mathbf{e})$ too [cf. (4.56)]:

$$\nu(\mathbf{p}, \mathbf{e}) = \frac{\eta(\mathbf{p}, \mathbf{e})}{1 + \mathbf{r}^\mathrm{T}\mathbf{q}(\mathbf{p}, \mathbf{e}) - \eta(\mathbf{p}, \mathbf{e})}. \tag{4.66}$$

We can now dismiss the initial problem formulation (4.56) in favor of the equivalent formulation

$$\mathbf{p}^\star(\mathbf{e}) = \operatorname*{argmax}_{\mathbf{p} \in \mathcal{D}(T\mu)} \nu(\mathbf{p}, \mathbf{e}) \tag{4.67}$$

with $\nu(\mathbf{p}, \mathbf{e})$ as given in (4.66). Once the maximizer $\mathbf{p}^\star(\mathbf{e})$ is determined, we compute the corresponding $\mathbf{q}^\star(\mathbf{e})$ via (4.65) as

$$\mathbf{q}^\star(\mathbf{e}) = \begin{bmatrix} q_1(\mathbf{p}^\star(\mathbf{e}), \mathbf{e}) \\ \vdots \\ q_{n_\mathsf{T}}(\mathbf{p}^\star(\mathbf{e}), \mathbf{e}) \end{bmatrix}. \tag{4.68}$$

The Pareto border $\partial^+\mathbf{s}(\mathcal{PQ}(T_\mathsf{p})) = \partial^+\mathbf{s}(\Gamma)$ is described in its entirety by the union (see Figure 4.5)

$$\partial^+\mathbf{s}(\Gamma) = \bigcup_{\substack{\mathbf{e} \geq \mathbf{0} \\ \|\mathbf{e}\|_1 = 1}} \mathbf{s}(\mathbf{p}^\star(\mathbf{e}), \mathbf{q}^\star(\mathbf{e})). \tag{4.69}$$

Definition 4.2. *A function $f \colon \mathcal{X} \mapsto \mathbb{R}$ is quasi-concave (resp. quasi-convex) on a convex and* *compact set $\mathcal{X} \subset \mathbb{R}^n$ if it can be represented as a concatenation*

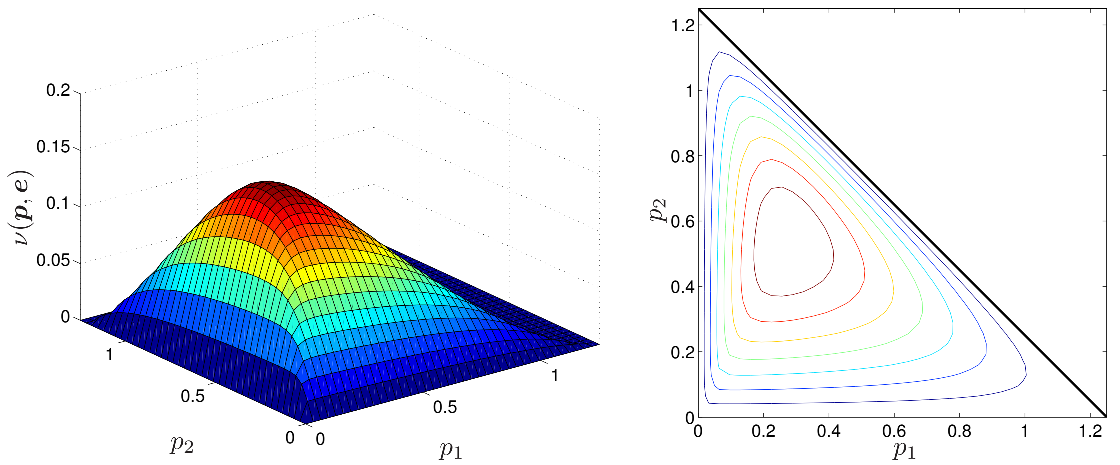
$$f(\mathbf{x}) = (g \circ h)(\mathbf{x}) \tag{4.70}$$

of a concave (resp. convex) function $h\colon \mathcal{X} \to \mathbb{R}$ and a non-decreasing function $g\colon \mathbb{R} \to \mathbb{R}$.

Lemma 4.4. *The function $\nu(\mathbf{p}, \mathbf{e})$ is quasi-concave in $\mathbf{p}$.*

Proof: See Appendix D.9. ■

This lemma renders (4.67) a quasi-convex problem, which can be solved efficiently.



(a) $\nu(\mathbf{p}, \mathbf{e})$ as a function of p_1 and p_2

(b) Contour plot of the function $\nu(\mathbf{p}, \mathbf{e})$ from Figure 4.7(a)

Figure 4.7 Three-dimensional representation and corresponding contour plot of a function $\nu(\mathbf{p}, \mathbf{e})$

Figures 4.7(a) and 4.7(b) illustrate an example of a function $\nu(\mathbf{p}, \mathbf{e})$ for $n_\mathrm{T} = 2$ transmit antennas, channel coherence $T = 10$ and SNR $\mu = 1$, $\mathbf{r} = [2/3 \;\; 1/3]^\mathrm{T}$ and $\mathbf{e} = [1/2 \;\; 1/2]^\mathrm{T}$. The quasi-concavity (but non-concavity) can be well appreciated in said plot, since $\nu(\mathbf{p}, \mathbf{e})$ appears to be convex in $\mathbf{p}$ near the borders of its triangular domain $\mathcal{D}(T\mu)$, while it is concave in an inner region. Notwithstanding this change of curvature, the function is globally quasi-concave in $\mathbf{p}$, since all upper contour sets, as illustrated in Figure 4.7(b), are convex.

4.6 Assembling It All: Iterative Joint Design

Having studied Problems (P.1.a) and (P.1.b) as well as a subproblem (Pareto optimal allocations) of Problem (P.3), we now propose an iterative approach to solving Problems (P.3) and (P.4).

Using Theorem 4.4, we align the eigenbases as $\operatorname{col}(\mathbf{U_P}) = \operatorname{col}(\mathbf{U_Q}) \subseteq \operatorname{col}(\mathbf{U_R})$. Since we know from Sections 4.3 and 4.4 how to efficiently solve Problems (P.1.a) and (P.1.b) for (quasi-

)concave utilities F, a natural way of tackling the joint problem (P.2) is by alternating between the two problems in the fashion of a block gradient ascent:

$$\begin{cases} \mathbf{p}_{n+1} = \mathbf{p}^\star(\mathbf{q}_n) \\ \mathbf{q}_{n+1} = \mathbf{q}^\star(\mathbf{p}_{n+1}) \end{cases} \quad \text{or} \quad \begin{cases} \mathbf{q}_{n+1} = \mathbf{q}^\star(\mathbf{p}_n) \\ \mathbf{p}_{n+1} = \mathbf{p}^\star(\mathbf{q}_{n+1}) \end{cases} \tag{4.71}$$

This procedure converges monotonically towards a fixed-point of the iteration $(\mathbf{p}_{n+1}, \mathbf{q}_{n+1}) = (\mathbf{p}^\star(\mathbf{q}_n), \mathbf{q}^\star(\mathbf{p}_n))$, which is a local optimum for Problem (P.2). However, it yields no local optimum of Problem (P.3), the reason being that no step in the above iteration ever changes the balance between pilot energy $\mathbf{1}^{\mathrm{T}}\mathbf{p}$ and transmit power $\mathbf{1}^{\mathrm{T}}\mathbf{q}$ (energy boost). To accomodate the energy boost, an additional step needs to be inserted in the above iteration in order to readjust the allocation $(\mathbf{p}, \mathbf{q})$ so as to remain Pareto optimal. This step can be performed with the methods for computing the Pareto border, developed in Subsection 4.5.4. Therefore, the proposed algorithm for solving (P.3) should cycle through the following three steps:

(1) Optimize $\mathbf{p}$ for a prescribed $\mathbf{q}$
(2) Optimize $\mathbf{q}$ for a prescribed $\mathbf{p}$
(3) Adjust $(\mathbf{p}, \mathbf{q})$ to be Pareto optimal

Algorithm 1 Iteration for solving (P.3)

1: $\mathbf{p}_0 \leftarrow \frac{T_{\mathsf{p}}\mu}{n_{\mathsf{T}}}\mathbf{1}_{n_{\mathsf{T}}}$
2: $\mathbf{q}_0 \leftarrow \frac{(T-T_{\mathsf{p}})\mu}{n_{\mathsf{T}}}\mathbf{1}_{n_{\mathsf{T}}}$
3: $n \leftarrow 0$
4: **repeat**
5: $\quad \mathbf{p}' \leftarrow \operatorname{argmax}_{\mathbf{p}\in\mathcal{D}(\mathbf{1}^{\mathrm{T}}\mathbf{p}_n)} f(\mathbf{s}(\mathbf{p},\mathbf{q}))$
6: $\quad \mathbf{q}' \leftarrow \operatorname{argmax}_{\mathbf{q}\in\mathcal{D}(\mathbf{1}^{\mathrm{T}}\mathbf{q}_n)} f(\mathbf{s}(\mathbf{p},\mathbf{q}))$
7: $\quad \mathbf{e}_{n+1} \leftarrow \frac{\mathbf{s}(\mathbf{p}',\mathbf{q}')}{\|\mathbf{s}(\mathbf{p}',\mathbf{q}')\|_1}$
8: $\quad \mathbf{p}_{n+1} \leftarrow \operatorname{argmax}_{\mathbf{p}\in\mathcal{D}(T\mu)} \nu(\mathbf{p}, \mathbf{e}_{n+1})$
9: $\quad \mathbf{q}_{n+1} \leftarrow \mathbf{q}(\mathbf{p}_{n+1}, \mathbf{e}_{n+1})$
10: $\quad \mathbf{s}_{n+1} \leftarrow \mathbf{e}_{n+1}\nu(\mathbf{p}_{n+1}, \mathbf{e}_{n+1})$
11: $\quad n \leftarrow n+1$
12: **until** $f(\mathbf{s}_n) - f(\mathbf{s}_{n-1}) \leq \epsilon$

In pseudocode, the algorithm is written out in Algorithm 1. For (quasi-)concave utilities $\bar{f} \in \bar{\mathcal{F}}$, Steps 5 and 6 were shown to be (quasi-)convex optimizations in Sections 4.3 and 4.4, respectively. The computation of Steps 7 through 10 has been explained in detail in Subsection 4.5.4, wherein Step 8 was shown to be a quasi-convex optimization.

We already mentioned in Section 4.2 that the problem of optimally tuning the training duration length T_{p} could be tackled by an exhaustive search over the set $\{1, \ldots, \min(T-1, n_{\mathsf{T}})\}$. We thus simply need to wrap Algorithm 1 into an outer loop. Note that the function $\nu(\cdot,\cdot)$

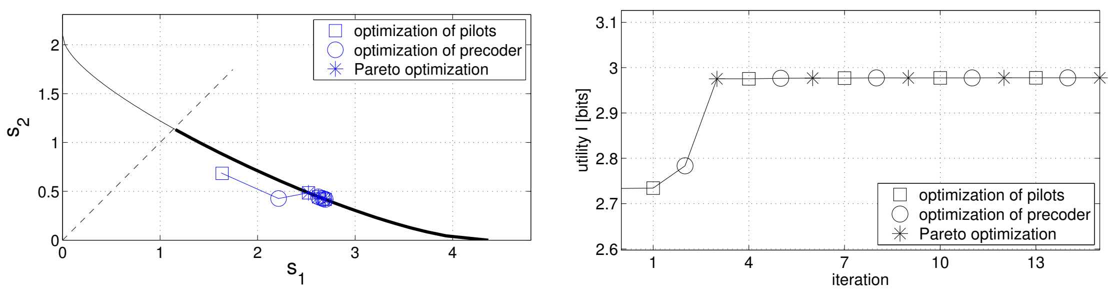
is dependent on the parameter T_p (though this is not reflected in notation), and should be updated accordingly with the loop count. The algorithm for (P.4) in pseudocode is written out in Algorithm 2 below:

Algorithm 2 Iteration for solving (P.4)

$T_\mathsf{p} \leftarrow 1$
repeat
  . . . Algorithm 1 . . .
  $(\mathbf{p}^\star(T_\mathsf{p}), \mathbf{q}^\star(T_\mathsf{p})) \leftarrow (\mathbf{p}_n, \mathbf{q}_n)$
  $T_\mathsf{p} \leftarrow T_\mathsf{p} + 1$
until $T_\mathsf{p} = \min(T-1, n_\mathsf{T})$
$(\mathbf{p}^\star, \mathbf{q}^\star) = \max_i f(\mathbf{s}(\mathbf{p}^\star(i), \mathbf{q}^\star(i)))$

The proposed algorithm may still suffer from efficiency issues due, for instance, to the precision parameter ϵ [Step 12 in Algorithm 1] being left as an arbitrary choice. A fast single-loop iteration in the spirit of Algorithm 1 in [Tul06] would be preferable, but such a reformulation of our algorithm following the ideas of [Tul06] does not seem straightforward. For one thing, because the constraints on the feasible values of the effective SNR eigenvalues $\mathbf{s}$ are far more intricate as is the simple trace constraint on the transmit covariance in [Tul06].

4.7 Numerical Simulations

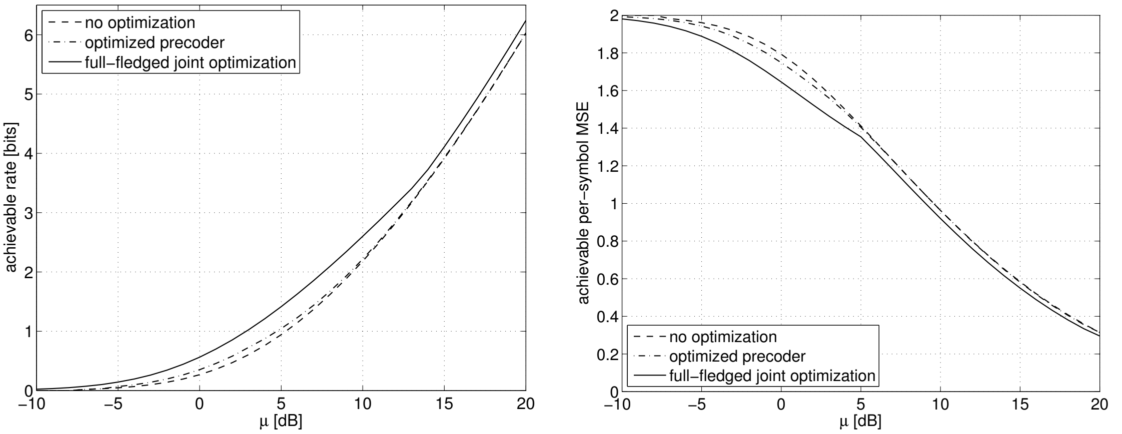


(a) convergence of the profile $\mathbf{s}$ towards the optimizer, located on the Pareto border (thick black line) of the feasible set $\mathbf{s}(\Gamma)$

(b) convergence of the utility function $I(\mathbf{s})$ towards the optimum

Figure 4.8 Convergence of Algorithm 1 at an SNR of 10dB ($\mu = 10$) for an exemplary 2×2 MIMO channel, both in the $\mathbf{s}$-domain [Fig. 4.8(a)] and in terms of the utility value $I(\mathbf{s})$ [Fig. 4.8(b)]

Figure 4.8 shows how Algorithm 1 (for fixed $T_\mathsf{p} = 2$) converges to the jointly optimal solution for the utility function $f(\mathbf{s}) = I(\mathbf{s})$. The parameters chosen in this simulation are $T = 10$, $\mu = 10$ (i.e., 10dB), $(n_\mathsf{T}, n_\mathsf{R}) = (2, 2)$, and $(r_1, r_2) = (\frac{2}{3}, \frac{1}{3})$.



(a) achievable rate as a function of the SNR (b) achievable mean-square error as a function of the SNR

Figure 4.9 Two different utility functions against the SNR parameter μ for an examplary 2×2 MIMO system. For both utilities, the performance of full-fledged optimization is compared to a partial optimization (precoder only) and no optimization at all.

Figures 4.9(a) and 4.9(b) respectively show the quantities

$$\bar{f}(\mathbf{s}) = \frac{T - T_{\mathsf{p}}}{T} I(\mathbf{s}) \qquad\qquad -\bar{f}(\mathbf{s}) = \operatorname{tr} \mathsf{E}\big[(\mathbf{I} + \hat{\mathbf{W}}\mathbf{S}\hat{\mathbf{W}}^{\dagger})^{-1}\big] \tag{4.72}$$

optimized using Algorithm 2 and plotted against the SNR μ (in decibels) for the same 2×2 system as for Figure 4.8, i.e., $(r_1, r_2) = (\frac{2}{3}, \frac{1}{3})$ and $T = 10$. The former utility represents an achievable rate [including an overhead factor as in (4.9)]. The latter utility is the negative of the per-symbol mean-square error achieved by a linear minimum mean-square error symbol estimator [see also Utility 9 in Table D.1]. In each of the two figures, three curves are plotted for comparison: the utility achieved with full-fledged joint optimization (P.4) using Algorithm 2; the utility achieved for precoder optimization (P.1.a) alone; the utility achieved when no optimization is performed, i.e., $\mathbf{P}_0 = \frac{T_{\mathsf{p}}\mu}{n_{\mathsf{T}}}\mathbf{I}_{n_{\mathsf{T}}}$ and $\mathbf{Q}_0 = \frac{(T-T_{\mathsf{p}})\mu}{n_{\mathsf{T}}}\mathbf{I}_{n_{\mathsf{T}}}$. The relative gains in mutual information are well noticeable especially for low and moderate SNR values. For higher SNR instead, these gains are far less significant.

Conclusion

The present thesis has developed new results on achievable communication rates over fading channels under imperfect CSI. The results run along two lines of research.

In the first part (Chapter 2 and 3), we have demonstrated that a rate-splitting approach improves the well-known worst-case-noise capacity lower bound, and we have developed the rate-splitting bounds to cover the general MIMO case.

By computing the supremum of these bounds over all possible rate-splitting strategies, we have established a novel capacity lower bound for the SISO case which is larger than the worst-case noise bound, and a family of novel capacity lower bounds for the MIMO case, which are parametrized by so-called *layering functions*.

We have further studied the high-SNR behavior of the novel bound under the assumption that the variance of the channel estimation error tends to zero with the SNR. We have shown that, for a Gaussian estimation error, all rate-splitting bounds (both SISO and MIMO) are asymptotically tight in the sense that their difference to the Gaussian-input mutual information vanishes as the SNR tends to infinity. This is in contrast to the worst-case-noise lower bound, which is asymptotically tight only if the variance of the estimation error decays faster than the reciprocal of the SNR. The rate-splitting bounds are asymptotically tight *irrespective* of the rate at which this variance decays.

While rate-splitting bounds outperform the worst-case-noise bound, one may argue that they are less practical due to the successive-decoding strategy, which is more susceptible to error propagation. Nevertheless, rate-splitting bounds are of theoretical importance, since their discovery and better understanding may be a useful step in the quest for a more accurate characterization of the capacity of noncoherent fading channels. For example, the rate-splitting bound converges to the Gaussian-input mutual information as the SNR tends to infinity. Consequently, at high SNR, any gap to capacity is merely due to the (potentially suboptimal) Gaussian input distribution and not due to the bounding techniques used to evaluate mutual information. In order to find the high-SNR capacity of this channel, it thus remains to assess the optimality of Gaussian inputs. While such inputs are highly suboptimal for imperfect CSI, they may in fact be optimal when the CSI is asymptotically perfect.

In a second part (Chapter 4), we have studied a joint pilot-precoder optimization problem which involves achievable rate expressions under imperfect CSI. We have presented an in-depth study of a joint pilot-precoder optimization for a general class of utility functions of the effective SNR matrix.

Upon analyzing the two separate problems of pilot and precoder optimization, both of which can be cast into (quasi-)convex problems in the effective SNR domain provided that the utility is itself (quasi-)concave, we have shown that joint optimization requires the eigenbases of both the pilot Gram and the transmit covariance to be aligned with the channel's transmit correlation matrix. This allows to simplify the joint problem significantly. Furthermore, by deriving a method for computing energy allocations which are Pareto optimal in terms of the effective SNR eigenvalues, we managed to further reduce the dimensionality of the problem. By combining these approaches into a single iteration, we have proposed an algorithm for computing local optima of the joint optimization problem.

It should be noted that the main contributions of this chapter consist in offering insights and theorems that allow us to reduce a high-dimensional optimization problem so as to render it tractable using common optimization tools. As pointed out in Section 4.6, the focus has not been on an efficient and practical implementation, though there may be room for further improvement. Future work may thus look at possible refinements of the iterative algorithm, or extensions of the system model, for instance to include more general fading models.

List of Related Publications

Journal Papers

[Pas12b] A. Pastore, M. Joham, and J.R. Fonollosa, “A framework for the joint design of pilot sequence and linear precoder”, *IEEE Transactions on Information Theory*, 2012, submitted.

[Pas13b] A. Pastore, T. Koch, and J.R. Fonollosa, “A rate-splitting approach to fading channels with imperfect channel-state information”, *IEEE Transactions on Information Theory*, 2013, accepted for publication.

Conference Papers

[Pas09b] A. Pastore, and M. Joham, “Mutual information bounds for MIMO” channels under imperfect receiver CSI, *Proc. 43rd Asilomar Conference on Signals, Systems and Computers*, pp. 1456–1460, November 2009.

[Pas11c] A. Pastore, M. Joham, and J.R. Fonollosa, “On a rate region approximation of MIMO” channels under partial CSI, *Proc. International ITG Workshop on Smart Antennas*, pp. 1–8, February 2011.

[Pas11b] A. Pastore, M. Joham, and J.R. Fonollosa, “On a mutual information and a capacity bound gap of pilot-aided MIMO” channels, *Proc. IEEE International Conference on Communications*, pp. 1–6, June 2011.

[Pas11a] A. Pastore, M. Joham, and J.R. Fonollosa, “Joint pilot and precoder design for optimal throughput”, *Proc. IEEE International Symposium on Information Theory*, pp. 371–375, August 2011.

[Pas12a] A. Pastore, and J.R. Fonollosa, “Optimal pilot design and power control in correlated MISO” links, *Proc. IEEE International Conference on Communications*, June 2012.

[Pas12c] A. Pastore, T. Koch, and J.R. Fonollosa, “Improved capacity lower bounds for fading channels with imperfect CSI” using rate splitting, *Proc. IEEE 27th Convention of Electrical Electronics Engineers in Israel (IEEEI)*, pp. 1–5, November 2012.

[Joh13] M. Joham, A. Gründinger, A. Pastore, J.R. Fonollosa, and W. Utschick, “Rate balancing in the vector BC” with erroneous CSI at the receivers, *47th Annual Conference on Information Sciences and Systems*, pp. 1–6, March 2013.

[Pas13a] A. Pastore, J. Hoydis, and J.R. Fonollosa, "Sharpened capacity lower bounds of fading MIMO" channels with imperfect CSI, *Proc. IEEE International Symposium on Information Theory*, pp. 2089–2093, July 2013.

[Pas14b] A. Pastore, T. Koch, and J.R. Fonollosa, "A rate-splitting approach to fading multiple-access channels with imperfect channel-state information", *Proc. IEEE International Zurich Seminar on Communications*, pp. 9–12, February 2014.

A

Appendix to Chapter 1

A.1 Proof of Lemma 1.1

We first upper-bound the mutual information as

$$\begin{aligned} I(X_{\rm G};Y|\hat{H}) &= h(Y|\hat{H}) - h(Y|X_{\rm G},\hat{H}) \\ &\le \mathsf{E}\Big[\log\Big(\pi e\big(|\hat{H}|^2 P + \tilde{V}P + N_0\big)\Big)\Big] - h(Y|X_{\rm G},\hat{H}). \end{aligned} \tag{A.1}$$

The inequality follows because the Gaussian distribution is entropy-maximizing under a fixed-variance constraint. Next, we lower-bound the remaining entropy term. Denote as $P_{X_{\rm G}}$ and $P_{\hat{H}}$ the probability measures of $X_{\rm G}$ and $\hat{H}$, respectively. By means of the entropy-power inequality, we have

$$\begin{aligned} h(Y|X_{\rm G},\hat{H}) &= h(\tilde{H}X_{\rm G} + Z \mid X_{\rm G},\hat{H}) \\ &= \iint h(\tilde{H}\mathsf{X} + Z \mid \hat{H} = \hat{\mathsf{H}})\,\mathrm{d}P_{X_{\rm G}}(\mathsf{X})\,\mathrm{d}P_{\hat{H}}(\hat{\mathsf{H}}) \\ &\ge \iint h(\tilde{H}'\mathsf{X} + Z \mid \hat{H} = \hat{\mathsf{H}})\,\mathrm{d}P_{X_{\rm G}}(\mathsf{X})\,\mathrm{d}P_{\hat{H}}(\hat{\mathsf{H}}), \end{aligned} \tag{A.2}$$

where $\tilde{H}'$, conditioned on $\hat{H} = \hat{\mathsf{H}}$, is a zero-mean Gaussian variable independent of Z, having variance $\tilde{\Phi}(\hat{\mathsf{H}})$ and an entropy equal to the entropy of $\tilde{H}$ conditioned on $\hat{H} = \hat{\mathsf{H}}$, i.e.,

$$\tilde{\Phi}(\hat{\mathsf{H}}) = \frac{1}{\pi e} e^{h(\tilde{H}|\hat{H}=\hat{\mathsf{H}})}. \tag{A.3}$$

We have by the entropy-power inequality

$$\tilde{\Phi}(\hat{\mathsf{H}}) \le \tilde{V}(\hat{\mathsf{H}}), \quad \hat{\mathsf{H}} \in \mathbb{C}. \tag{A.4}$$

Hence

$$h(Y|X_{\mathrm{G}},\hat{H}) \geq \mathsf{E}\Big[\log\Big(\pi e\big(\tilde{\Phi}(\hat{H})WP + N_0\big)\Big)\Big] \tag{A.5}$$

with W an independent unit-mean exponential variable. By inserting (A.5) into (A.1), we finally obtain

$$I(X_{\mathrm{G}};Y|\hat{H}) \leq \mathsf{E}\left[\log\left(1+\frac{|\hat{H}|^2P}{\tilde{V}(|\hat{H}|^2)P+N_0}\right)\right] + \mathsf{E}\left[\log\left(\frac{\tilde{V}(\hat{H})P+N_0}{\tilde{\Phi}(\hat{H})PW+N_0}\right)\right]. \tag{A.6}$$

B

Appendices to Chapter 2

B.1 Proof of Lemma 2.1

We expand the mutual information as

$$I(S; AS + B|A, C) = h(S|A, C) - h(S|AS + B, A, C). \tag{B.1}$$

Since, by assumption, S is zero-mean, variance-P, circularly-symmetric, complex Gaussian and independent of (A, C), the first entropy on the RHS of (B.1) is readily evaluated as

$$h(S|A, C) = h(S) = \log(\pi e P). \tag{B.2}$$

Conditioned on $(A, C) = (a, c)$, the second entropy can be upper-bounded as follows:

$$\begin{aligned} h(S|AS + B, A = a, C = c) &= h\big(S - \alpha_{A,C}(AS + B - \mu_{B|A,C}) \mid AS + B, A = a, C = c\big) \\ &\leq h\big(S - \alpha_{A,C}(AS + B - \mu_{B|A,C}) \mid A = a, C = c\big) \\ &\leq \log\left(\pi e\, \mathsf{E}\left[\left|S - \alpha_{A,C}(AS + B - \mu_{B|A,C})\right|^2 \,\middle|\, A = a, C = c\right]\right) \end{aligned} \tag{B.3}$$

for any arbitrary $\alpha_{a,c} \in \mathbb{C}$, where $\mu_{B|a,c} \triangleq \mathsf{E}[B|A = a, C = c]$. Here the first inequality follows because conditioning cannot increase entropy, and the second inequality follows from the entropy-maximizing property of the Gaussian distribution. Combining (B.3) with (B.2) and (B.1) thus yields for every $(A, C) = (a, c)$ and $\alpha_{a,c}$

$$I(S; AS + B|A = a, C = c) \geq \log \frac{P}{\mathsf{E}\left[|S - \alpha_{A,C}(AS + B - \mu_{B|A,C})|^2 \,\middle|\, A = a, C = c\right]}. \tag{B.4}$$

We choose $\alpha_{a,c}$ so that $\alpha_{a,c}(aS + B - \mu_{B|a,c})$ is the linear MMSE estimate of S from the observation $AS+B$ given $(A,C)=(a,c)$, namely,

$$\alpha_{a,c}=\frac{\mathsf{E}\big[S(AS+B-\mu_{B|A,C})^* \mid A=a, C=c\big]}{\mathsf{E}\big[|AS+B-\mu_{B|A,C}|^2 \mid A=a, C=c\big]}=\frac{a^*P}{|a|^2P+V_B(a,c)} \tag{B.5}$$

where $V_B(a,c)$ denotes the conditional variance of B conditioned on $(A,C)=(a,c)$. Here we have used that, conditioned on $(A,C)=(a,c)$, the random variables S and B are uncorrelated and that S has zero mean and variance P and is independent of (A,C). Combining these conditions with (B.5), we obtain

$$\mathsf{E}\Big[\big|S-\alpha_{A,C}(AS+B-\mu_{B|A,C})\big|^2 \;\Big|\; A=a, C=c\Big]=P\frac{V_B(a,c)}{|a|^2P+V_B(a,c)}. \tag{B.6}$$

Consequently, (B.6) and (B.4) give for every $(A,C)=(a,c)$

$$I(S;AS+B|A=a,C=c)\geq\log\left(1+\frac{|a|^2P}{V_B(a,c)}\right). \tag{B.7}$$

Lemma 2.1 follows then by averaging over (A,C).

B.2 Proof of Lemma 2.2

To prove Lemma 2.2, we shall demonstrate for every $L\in\mathbb{N}$ that, if the layerings $\mathbf{Q}\in\mathcal{Q}(P,L)$ and $\mathbf{Q}'\in\mathcal{Q}(P,L+1)$ satisfy

$$\{Q_1,\ldots,Q_L\}\subset\{Q'_1,\ldots,Q'_{L+1}\} \tag{B.8}$$

then $R[\mathbf{Q}]\leq R[\mathbf{Q}']$ with equality if, and only if, $\mathsf{Pr}\{\hat{H}\cdot\tilde{V}(\hat{H})=0\}=1$. The general case where $\mathbf{Q}'\in\mathcal{Q}(P,L')$ for some arbitrary $L'>L$ follows directly from the case $L'=L+1$ by applying the above result $(L'-L)$ times.

Let the element in $\mathbf{Q}'$ that is not contained in $\mathbf{Q}$ be at position $\tau\in\{1,\ldots,L\}$, i.e.,[1]

$$Q_\ell=Q'_\ell,\quad \ell=1,\ldots,\tau-1 \tag{B.9a}$$

and

$$Q_\ell=Q'_{\ell+1},\quad \ell=\tau,\ldots,L. \tag{B.9b}$$

We next express $\Gamma_{\ell,\mathbf{A}}(X^{\ell-1},\hat{H})$ in (2.14) for some general layering $\mathbf{A}$ as

$$\Gamma_{\ell,\mathbf{A}}(X^{\ell-1},\hat{H})=\frac{|\hat{H}|^2(A_\ell-A_{\ell-1})}{\tilde{V}(\hat{H})\,|\sum_{i<\ell}X_i|^2+\big(|\hat{H}|^2+\tilde{V}(\hat{H})\big)P-|\hat{H}|^2A_\ell-\tilde{V}(\hat{H})A_{\ell-1}+N_0}. \tag{B.10}$$

[1] By the definition of a layering, we have $Q'_{L+1}=Q_L=P$, so the element in $\mathbf{Q}'$ not contained in $\mathbf{Q}$ cannot be at position $\tau=L+1$.

Noting that for the layering $\mathbf{Q}$ the term $|\sum_{i<\ell} X_i|^2$ has an exponential distribution with mean $Q_{\ell-1}$, whereas for the layering $\mathbf{Q}'$ it has an exponential distribution with mean $Q'_{\ell-1}$, and using (B.9a) and (B.9b), it can be easily verified that

$$\mathsf{E}\left[\log(1+\Gamma_{\ell,\mathbf{Q}}(X^{\ell-1},\hat{H}))\right] = \mathsf{E}\left[\log(1+\Gamma_{\ell,\mathbf{Q}'}(X^{\ell-1},\hat{H}))\right], \quad \ell = 1,\ldots,\tau-1 \tag{B.11}$$

and

$$\mathsf{E}\left[\log(1+\Gamma_{\ell,\mathbf{Q}}(X^{\ell-1},\hat{H}))\right] = \mathsf{E}\left[\log(1+\Gamma_{\ell+1,\mathbf{Q}'}(X^{\ell},\hat{H}))\right], \quad \ell = \tau+1,\ldots,L. \tag{B.12}$$

Subtracting $R[\mathbf{Q}]$ from $R[\mathbf{Q}']$ yields

$$\begin{aligned} R[\mathbf{Q}'] - R[\mathbf{Q}] = \mathsf{E}&\left[\log(1+\Gamma_{\tau,\mathbf{Q}'}(X^{\tau-1},\hat{H}))\right] \\ &+ \mathsf{E}\left[\log(1+\Gamma_{\tau+1,\mathbf{Q}'}(X^{\tau},\hat{H}))\right] - \mathsf{E}\left[\log(1+\Gamma_{\tau,\mathbf{Q}}(X^{\tau-1},\hat{H}))\right]. \end{aligned} \tag{B.13}$$

Since the random variables $X_1,\ldots,X_\tau,\hat{H}$ are independent, we can express the second expectation as

$$\mathsf{E}\left[\mathsf{E}_{X_\tau}\left[\log(1+\Gamma_{\tau+1,\mathbf{Q}'}(X^{\tau},\hat{H}))\,\middle|\, X^{\tau-1},\hat{H}\right]\right] \tag{B.14}$$

where the subscript indicates that the inner expected value is computed with respect to X_τ. Using that, for every $\alpha > 0$, the function $x \mapsto \log(1+\alpha/x)$ is strictly convex in $x > 0$, it follows from Jensen's inequality that, for every $X^{\tau-1} = x^{\tau-1}$ and $\hat{H} = \hat{h}$, the inner expectation is lower-bounded by[2]

$$\mathsf{E}_{X_\tau}\left[\log(1+\Gamma_{\tau+1,\mathbf{Q}'}(x^{\tau-1},X_\tau,\hat{h}))\right] \geq \log(1+\bar{\Gamma}_{\tau+1,\mathbf{Q}'}(x^{\tau-1},\hat{h})) \tag{B.15}$$

where we define

$$\bar{\Gamma}_{\tau+1,\mathbf{Q}'}(x^{\tau-1},\hat{h}) \triangleq \frac{|\hat{h}|^2(Q'_{\tau+1}-Q'_\tau)}{\tilde{V}(\hat{h})\,|\sum_{i<\tau} x_i|^2 + (|\hat{h}|^2+\tilde{V}(\hat{h}))P - |\hat{h}|^2 Q'_{\tau+1} - \tilde{V}(\hat{h})Q'_{\tau-1} + N_0}. \tag{B.16}$$

The denominator of $\bar{\Gamma}_{\tau+1,\mathbf{Q}'}(x^{\tau-1},\hat{h})$ is obtained by noting that X_τ has zero mean, so

$$\mathsf{E}_{X_\tau}\left[\left|\sum_{i<\tau} x_i + X_\tau\right|^2\right] = \left|\sum_{i<\tau} x_i\right|^2 + Q'_\tau - Q'_{\tau-1}. \tag{B.17}$$

Since $\mathbf{Q}' \in \mathcal{Q}(P, L+1)$ implies that $\mathsf{E}\left[|X_\tau|^2\right] > 0$, the inequality in (B.15) is strict except in the trivial cases $\tilde{V}(\hat{h}) = 0$ or $\hat{h} = 0$. Combining (B.14) and (B.15) yields

$$\mathsf{E}\left[\log(1+\Gamma_{\tau+1,\mathbf{Q}'}(X^{\tau},\hat{H}))\right] \geq \mathsf{E}\left[\log(1+\bar{\Gamma}_{\tau+1,\mathbf{Q}'}(X^{\tau-1},\hat{H}))\right] \tag{B.18}$$

[2] With a slight abuse of notation, we write $\Gamma_{\tau+1,\mathbf{Q}'}(x^\tau,\hat{h})$ as $\Gamma_{\tau+1,\mathbf{Q}'}(x^{\tau-1},x_\tau,\hat{h})$.

which together with (B.13) gives

$$R[\mathbf{Q}'] - R[\mathbf{Q}] \geq \mathsf{E}\left[\log(1+\Gamma_{\tau,\mathbf{Q}'}(X^{\tau-1},\hat{H}))\right] + \mathsf{E}\left[\log\left(\frac{1+\bar{\Gamma}_{\tau+1,\mathbf{Q}'}(X^{\tau-1},\hat{H})}{1+\Gamma_{\tau,\mathbf{Q}}(X^{\tau-1},\hat{H})}\right)\right] \tag{B.19}$$

with the inequality being strict except if $\mathsf{Pr}\{\hat{H}\cdot\tilde{V}(\hat{H})=0\}=1$.

We next use (B.9a) and (B.9b) and the fact that $|\sum_{i<\tau} X_i|^2$ has an exponential distribution with mean $Q'_{\tau-1}$ under both layerings $\mathbf{Q}$ and $\mathbf{Q}'$ to evaluate the second expected value on the RHS of (B.19):

$$\begin{aligned}\mathsf{E}\left[\log\left(\frac{1+\bar{\Gamma}_{\tau+1,\mathbf{Q}'}(X^{\tau-1},\hat{H})}{1+\Gamma_{\tau,\mathbf{Q}}(X^{\tau-1},\hat{H})}\right)\right] &= \mathsf{E}\left[\log\left(\frac{T-|\hat{H}|^2Q'_\tau-\tilde{V}(\hat{H})Q'_{\tau-1}+N_0}{T-|\hat{H}|^2Q'_{\tau-1}-\tilde{V}(\hat{H})Q'_{\tau-1}+N_0}\right)\right]\\ &= -\,\mathsf{E}\left[\log\left(1+\frac{|\hat{H}|^2(Q'_\tau-Q'_{\tau-1})}{T-|\hat{H}|^2Q'_\tau-\tilde{V}(\hat{H})Q'_{\tau-1}+N_0}\right)\right]\end{aligned} \tag{B.20}$$

where we introduce

$$T \triangleq \tilde{V}(\hat{H})\left|\sum_{i<\tau} X_i\right|^2 + (|\hat{H}|^2+\tilde{V}(\hat{H}))P \tag{B.21}$$

for ease of exposition. By noting that

$$\frac{|\hat{H}|^2(Q'_\tau-Q'_{\tau-1})}{T-|\hat{H}|^2Q'_\tau-\tilde{V}(\hat{H})Q'_{\tau-1}+N_0} = \Gamma_{\tau,\mathbf{Q}'}(X^{\tau-1},\hat{H}) \tag{B.22}$$

it follows that the RHS of (B.19) is zero, thus demonstrating that

$$R[\mathbf{Q}] \leq R[\mathbf{Q}'] \tag{B.23}$$

with equality if, and only if, $\mathsf{Pr}\{\hat{H}\cdot\tilde{V}(\hat{H})=0\}=1$. This proves Lemma 2.2.

B.3 Proof of Theorem 2.1

For every L-layering $\mathbf{Q}\in\mathcal{Q}(P,L)$, we can construct an $(L+1)$-layering $\mathbf{Q}'\in\mathcal{Q}(P,L+1)$ satisfying $\mathbf{Q}\subset\mathbf{Q}'$ by adding $(Q_1+Q_2)/2$ to $\mathbf{Q}$. Together with Lemma 2.2, this implies that for every $\mathbf{Q}\in\mathcal{Q}(P,L)$ there exists a $\mathbf{Q}'\in\mathcal{Q}(P,L+1)$ such that $R[\mathbf{Q}]\leq R[\mathbf{Q}']$, from which we obtain that $R^\star(P,L)$ is monotonically nondecreasing upon maximizing both sides of the inequality over all layerings $\mathbf{Q}\in\mathcal{Q}(P,L)$ and $\mathbf{Q}'\in\mathcal{Q}(P,L+1)$, respectively.

To show that if $\mathsf{Pr}\{\hat{H}\cdot\tilde{V}(\hat{H})=0\}=1$ then $R^\star(P,L)=\underline{R}(P)$, $L\in\mathbb{N}$ (where $\mathbb{N}$ denotes the set of positive integers), we first note that Médard's lower bound (1.28) corresponds to $R[\mathbf{Q}]$ with $\mathbf{Q}\in\mathcal{Q}(P,1)$. Since the only 1-layering is $\{P\}$, it follows that $R[\mathbf{Q}]=R^\star(P,1)=\underline{R}(P)$. Furthermore, every L-layering $\mathbf{Q}'\in\mathcal{Q}(P,L)$, $L>1$ satisfies $\mathbf{Q}\subset\mathbf{Q}'$, so applying Lemma 2.2 with the condition $\mathsf{Pr}\{\hat{H}\cdot\tilde{V}(\hat{H})=0\}=1$ yields $R[\mathbf{Q}']=R[\mathbf{Q}]=\underline{R}(P)$ for every $\mathbf{Q}'\in\mathcal{Q}(P,L)$ and $L\in\mathbb{N}$. The claim follows then by maximizing $R[\mathbf{Q}']$ over all L-layerings $\mathcal{Q}(P,L)$.

B.4 Proof of Theorem 2.2

To prove theorem 2.2, we first note that for $\hat{h} = 0$

$$\log\left(1 + \frac{|\hat{h}|^2 P}{\tilde{V}(\hat{h})P + N_0}\right) = 0 \tag{B.24}$$

whereas for $\tilde{V}(\hat{h}) = 0$

$$\log\left(1 + \frac{|\hat{h}|^2 P}{\tilde{V}(\hat{h})P + N_0}\right) = \log\left(1 + \frac{|\hat{h}|^2 P}{N_0}\right) \tag{B.25}$$

which in both cases is equal to

$$\frac{|\hat{h}|}{|\hat{h}| + \tilde{V}(\hat{h}) + \frac{N_0}{P}} \Theta\left(\frac{\tilde{V}(\hat{h})(W-1) - |\hat{h}|^2}{|\hat{h}|^2 + \tilde{V}(\hat{h}) + \frac{N_0}{P}}\right) \tag{B.26}$$

where W is as in Theorem 2.2. This implies that, if $\mathsf{Pr}\{\hat{H} \cdot \tilde{V}(\hat{H}) = 0\} = 1$, then

$$\underline{R}(P) = \mathsf{E}\left[\frac{|\hat{H}|}{|\hat{H}| + \tilde{V}(\hat{H}) + \frac{N_0}{P}} \Theta\left(\frac{\tilde{V}(\hat{H})(W-1) - |\hat{H}|^2}{|\hat{H}|^2 + \tilde{V}(\hat{H}) + \frac{N_0}{P}}\right)\right] \tag{B.27}$$

from which Theorem 2.2 follows because, by Theorem 2.1, $R^\star(P, L) = \underline{R}(P)$, $L \in \mathbb{N}$.

In the following, we consider the case where $\mathsf{Pr}\{\hat{H} \cdot \tilde{V}(\hat{H}) = 0\} < 1$. To this end, we first show that it suffices to consider equi-power layerings

$$\mathbf{U}(P, K) \triangleq \left\{\frac{P}{K}, 2\frac{P}{K}, \ldots, (K-1)\frac{P}{K}, P\right\}. \tag{B.28}$$

More precisely, we shall show that for every L-layering $\mathbf{Q} \in \mathcal{Q}(P, L)$ there exists some sufficiently large K such that $\mathbf{U}(P, K)$ outperforms $\mathbf{Q}$, i.e.,

$$R[\mathbf{U}(P, K)] > R[\mathbf{Q}]. \tag{B.29}$$

This then implies that

$$R^\star(P) = \sup_{L \in \mathbb{N}} \left\{\sup_{\mathbf{Q} \in \mathcal{Q}(P,L)} R[\mathbf{Q}]\right\} = \sup_{K \in \mathbb{N}} R[\mathbf{U}(P, K)] \tag{B.30}$$

from which we obtain, by Lemma 2.2, that

$$R^\star(P) = \varlimsup_{K \to \infty} R[\mathbf{U}(P, K)] \tag{B.31}$$

upon noting that $\mathbf{U}(P, K) \subset \mathbf{U}(P, 2K)$ for every $K \in \mathbb{N}$.

To show that for every L-layering $\mathbf{Q} \in \mathcal{Q}(P, L)$ there exists some $\mathbf{U}(P, K)$ (with K sufficiently large) outperforming $\mathbf{Q}$, we first note that for every $\epsilon > 0$ one can find a sufficiently large K and two $(L+1)$-layerings $\mathbf{S} \in \mathcal{Q}(P, L+1)$ and $\mathbf{T} \in \mathcal{Q}(P, L+1)$ satisfying $\mathbf{Q} \subset \mathbf{S}$ and $\mathbf{T} \subset \mathbf{U}(P, K)$

such that

$$\max_{1\le \ell \le L+1} |T_\ell - S_\ell| \le \epsilon. \tag{B.32}$$

Indeed, $\mathbf{S}$ may be obtained by including $(Q_1 + Q_2)/2$ into $\mathbf{Q}$, i.e., $\mathbf{S} = \mathbf{Q} \cup \{(Q_1 + Q_2)/2\}$. Furthermore, for K larger than $P/(\min_{0\le\ell\le L} |S_{\ell+1} - S_\ell|)$ (where $S_0 = 0$ by convention), choosing

$$T_\ell = \left\lceil \frac{S_\ell K}{P} \right\rceil \frac{P}{K}, \quad \ell = 1, \ldots, L+1$$

yields $\mathbf{T} \subset \mathbf{U}(P, K)$ and

$$\max_{1\le \ell \le L+1} |T_\ell - S_\ell| \le \frac{P}{K} \tag{B.33}$$

from which (B.32) follows. To prove (B.29), we then need the following lemma.

Lemma B.1. *The function* $R[\mathbf{Q}]$ *satisfies*

$$\lim_{\mathbf{Q}\to\mathbf{Q}'} R[\mathbf{Q}] = R[\mathbf{Q}'] \tag{B.34}$$

where $\mathbf{Q} \to \mathbf{Q}'$ *is to be understood as* $\max_\ell |Q_\ell - Q'_\ell| \to 0$ *with* $\mathbf{Q}$ *and* $\mathbf{Q}'$ *having an equal number of layers.*

Proof: See Appendix B.5. ■

From Lemma B.1 and from the observation (B.32), it follows that for every $\delta > 0$ there exists a sufficiently large K such that

$$\big|R[\mathbf{T}] - R[\mathbf{S}]\big| \le \delta. \tag{B.35}$$

Since by Lemma 2.2 and the assumption $\mathsf{Pr}\big\{\hat{H} \cdot \tilde{V}(\hat{H}) = 0\big\} < 1$ we have

$$R[\mathbf{Q}] < R[\mathbf{S}] \quad \text{and} \quad R[\mathbf{T}] < R[\mathbf{U}(P, K)] \tag{B.36}$$

this yields

$$R[\mathbf{Q}] < R[\mathbf{S}] \le R[\mathbf{T}] + \delta \tag{B.37}$$

which for a sufficiently small δ is strictly smaller than $R[\mathbf{U}(P, K)]$ due to $\mathbf{T} \subset \mathbf{U}(P, K)$. This proves (B.29).

Recalling that (B.29) implies (B.31), we continue by evaluating $R[\mathbf{U}(P, K)]$ in the limit as K tends to infinity. To this end, we write $R[\mathbf{U}(P, K)]$ as

$$R[\mathbf{U}(P, K)] = \sum_{\ell=1}^{K} \mathsf{E}\Big[\log\big(1 + \Gamma_{\ell,\mathbf{U}}(W_\ell, \hat{H})\big)\Big] \tag{B.38}$$

with [cf. (2.14)]

$$\Gamma_{\ell,\mathbf{U}}(W_\ell, \hat{H}) = \frac{|\hat{H}|^2}{\tilde{V}(\hat{H})(\ell - 1)W_\ell + \tilde{V}(\hat{H}) + (|\hat{H}|^2 + \tilde{V}(\hat{H}))(K - \ell) + N_0 \frac{K}{P}} \tag{B.39}$$

and

$$W_\ell \triangleq \begin{cases} 0, & \ell = 1 \\ \dfrac{1}{(\ell-1)\frac{P}{K}} \left|\sum_{i<\ell} X_i\right|^2, & \ell = 2,\dots,K. \end{cases} \tag{B.40}$$

The random variables $(W_1,\dots,W_K)$ are dependent but have equal marginals. (Each marginal has a unit-mean exponential distribution.) Since the RHS of (B.38) depends on $(W_1,\dots,W_K)$ only via their marginal distributions, we can thus express $R[\mathbf{U}(P,K)]$ as

$$R[\mathbf{U}(P,K)] = \mathsf{E}\left[\sum_{\ell=1}^{K} \log\bigl(1+\Gamma_{\ell,\mathbf{U}}(W,\hat{H})\bigr)\right] \tag{B.41}$$

where W is independent of $\hat{H}$ and has a unit-mean exponential distribution.

Combining (B.41) with (B.31) yields

$$R^\star(P) = \varlimsup_{K\to\infty} \mathsf{E}\left[\sum_{\ell=1}^{K} \log\bigl(1+\Gamma_{\ell,\mathbf{U}}(W,\hat{H})\bigr)\right]. \tag{B.42}$$

We next show that

$$R^\star(P) = \mathsf{E}\left[\lim_{K\to\infty} \sum_{\ell=1}^{K} \Gamma_{\ell,\mathbf{U}}(W,\hat{H})\right] \tag{B.43}$$

and evaluate $\sum_{\ell=1}^{K}\Gamma_{\ell,\mathbf{U}}(W,\hat{H})$ for every $(W,\hat{H})=(w,\hat{h})$ in the limit as K tends to infinity. To this end, we first lower-bound $R^\star(P)$ using Fatou's Lemma [Ash00, (1.6.8), p. 50] and the lower bound $\log(1+x) \geq x - x^2/2$, $x \geq 0$:

$$\begin{aligned} R^\star(P) &= \varlimsup_{K\to\infty} \mathsf{E}\left[\sum_{\ell=1}^{K} \log\bigl(1+\Gamma_{\ell,\mathbf{U}}(W,\hat{H})\bigr)\right] \\ &\geq \mathsf{E}\left[\varliminf_{K\to\infty} \sum_{\ell=1}^{K} \log\bigl(1+\Gamma_{\ell,\mathbf{U}}(W,\hat{H})\bigr)\right] \\ &\geq \mathsf{E}\left[\varliminf_{K\to\infty} \sum_{\ell=1}^{K} \Gamma_{\ell,\mathbf{U}}(W,\hat{H})\right] - \frac{1}{2}\mathsf{E}\left[\varlimsup_{K\to\infty} \sum_{\ell=1}^{K} \Gamma_{\ell,\mathbf{U}}^2(W,\hat{H})\right] \end{aligned} \tag{B.44}$$

where $\varliminf$ denotes the *limit inferior*. We next argue that the second term on the RHS of (B.44) is zero. Indeed, we have for every $(W,\hat{H})=(w,\hat{h})$ [cf. (B.39)]

$$\begin{aligned} \Gamma_{\ell,\mathbf{U}}^2(w,\hat{h}) &= \frac{|\hat{h}|^4}{\left[\tilde{V}(\hat{h})(\ell-1)w + \tilde{V}(\hat{h}) + (|\hat{h}|^2+\tilde{V}(\hat{h}))(K-\ell) + N_0\frac{K}{P}\right]^2} \\ &\leq \frac{|\hat{h}|^4}{\left[\min\left\{\tilde{V}(\hat{h})w, (|\hat{h}|^2+\tilde{V}(\hat{h}))\right\}(K-1) + \tilde{V}(\hat{h}) + N_0\frac{K}{P}\right]^2} \end{aligned} \tag{B.45}$$

where the inequality follows from observing that the denominator of $\Gamma_{\ell,\mathbf{U}}^2(w,\hat{h})$ is the square of a positive affine linear function of $\ell \in \{1,\dots,K\}$ and is therefore minimized for either $\ell = 1$ or $\ell = K$. This yields

$$\sum_{\ell=1}^{K} \Gamma_{\ell,\mathbf{U}}^2(w,\hat{h}) \leq \frac{K|\hat{h}|^4}{\left[\min\left\{\tilde{V}(\hat{h})w, (|\hat{h}|^2+\tilde{V}(\hat{h}))\right\}(K-1)+\tilde{V}(\hat{h})+N_0\frac{K}{P}\right]^2}. \tag{B.46}$$

Since $\sum_{\ell=1}^{K} \Gamma_{\ell,\mathbf{U}}^2(w,\hat{h})$ is nonnegative, and since the RHS of (B.46) vanishes as K tends to infinity, it follows that, for every $(W,\hat{H}) = (w,\hat{h})$,

$$\overline{\lim_{K\to\infty}} \sum_{\ell=1}^{K} \Gamma_{\ell,\mathbf{U}}^2(w,\hat{h}) = 0. \tag{B.47}$$

Combining (B.47) with (B.44) yields

$$R^{\star}(P) \geq \mathsf{E}\left[\underline{\lim}_{K\to\infty} \sum_{\ell=1}^{K} \Gamma_{\ell,\mathbf{U}}(W,\hat{H})\right]. \tag{B.48}$$

We next show that

$$R^{\star}(P) \leq \mathsf{E}\left[\overline{\lim_{K\to\infty}} \sum_{\ell=1}^{K} \Gamma_{\ell,\mathbf{U}}(W,\hat{H})\right]. \tag{B.49}$$

To this end, we first use the upper bound $\log(1+x) \leq x$, $x \geq 0$ to obtain

$$\begin{aligned} R^{\star}(P) &= \overline{\lim_{K\to\infty}}\, \mathsf{E}\left[\sum_{\ell=1}^{K} \log(1+\Gamma_{\ell,\mathbf{U}}(W,\hat{H}))\right] \\ &\leq \overline{\lim_{K\to\infty}}\, \mathsf{E}\left[\sum_{\ell=1}^{K} \Gamma_{\ell,\mathbf{U}}(W,\hat{H})\right]. \end{aligned} \tag{B.50}$$

Noting that, for every $(W,\hat{H}) = (w,\hat{h})$, the sum inside the expectation is upper-bounded by

$$\sum_{\ell=1}^{K} \Gamma_{\ell,\mathbf{U}}(w,\hat{h}) \leq \sum_{\ell=1}^{K} \frac{|\hat{h}|^2}{N_0\frac{K}{P}} = \frac{P|\hat{h}|^2}{N_0} \triangleq \zeta(\hat{h}) \tag{B.51}$$

and noting that, since $\hat{H}$ has a finite second moment, we have that $0 < \mathsf{E}[\zeta(\hat{H})] < \infty$, we obtain (B.49) upon applying Fatou's Lemma to the nonnegative function $(w,\hat{h}) \mapsto \zeta(\hat{h}) - \sum_{\ell=1}^{K} \Gamma_{\ell,\mathbf{U}}(w,\hat{h})$.

It remains to show that, for every $(W,\hat{H}) = (w,\hat{h})$,

$$\lim_{K\to\infty} \sum_{\ell=1}^{K} \Gamma_{\ell,\mathbf{U}}(w,\hat{h}) = \frac{|\hat{h}|^2}{|\hat{h}|^2+\tilde{V}(\hat{h})+\frac{N_0}{P}} \Theta\left(\frac{\tilde{V}(\hat{h})(w-1)-|\hat{h}|^2}{|\hat{h}|^2+\tilde{V}(\hat{h})+\frac{N_0}{P}}\right) \tag{B.52}$$

where $\Theta(\cdot)$ is defined in (2.27). This then implies that the bounds (B.48) and (B.49) coincide and

$$R^{\star}(P) = \mathsf{E}\left[\frac{|\hat{H}|^2}{|\hat{H}|^2+\tilde{V}(\hat{H})+\frac{N_0}{P}} \Theta\left(\frac{\tilde{V}(\hat{H})(W-1)-|\hat{H}|^2}{|\hat{H}|^2+\tilde{V}(\hat{H})+\frac{N_0}{P}}\right)\right] \tag{B.53}$$

which proves Theorem 2.2.

To show (B.52), we express the denominator in $\Gamma_{\ell,\mathbf{U}}(w,\hat{h})$ as $a\ell + bK + c$ with

$$a = \tilde{V}(\hat{h})(w-1) - |\hat{h}|^2 \tag{B.54a}$$

$$b = |\hat{h}|^2 + \tilde{V}(\hat{h}) + \frac{N_0}{P} \tag{B.54b}$$

$$c = \tilde{V}(\hat{h})(1-w) \tag{B.54c}$$

allowing us to write

$$\sum_{\ell=1}^{K} \Gamma_{\ell,\mathbf{U}}(w,\hat{h}) = \sum_{\ell=1}^{K} \frac{|\hat{h}|^2}{a\ell + bK + c}. \tag{B.55}$$

Observe that, for every $(w,\hat{h})$, $a+b$ and a are strictly positive.

If $a = 0$, then we get the limit

$$\lim_{K\to\infty} \sum_{\ell=1}^{K} \Gamma_{\ell,\mathbf{U}}(w,\hat{h}) = \frac{|\hat{h}|^2}{b}. \tag{B.56}$$

We next consider the case $a \neq 0$. Note that

$$\lim_{K\to\infty} \left(\sum_{\ell=1}^{K} \frac{|\hat{h}|^2}{a\ell + bK + c} - \sum_{\ell=1}^{K} \frac{|\hat{h}|^2}{a\ell + bK} \right) = 0. \tag{B.57}$$

Indeed, by the triangle inequality, we have

$$\left| \sum_{\ell=1}^{K} \frac{|\hat{h}|^2}{a\ell + bK + c} - \sum_{\ell=1}^{K} \frac{|\hat{h}|^2}{a\ell + bK} \right| \leq \sum_{\ell=1}^{K} \frac{|c||\hat{h}|^2}{(a\ell + bK + c)(a\ell + bK)}. \tag{B.58}$$

Since the two factors $(a\ell+bK+c)$ and $(a\ell+bK)$ appearing in the denominator are both positive affine functions of ℓ with equal coefficient a, their product takes its extremal values at $\ell = 1$ or $\ell = K$, depending on the sign of a. If $a > 0$, then

$$\left| \sum_{\ell=1}^{K} \frac{|\hat{h}|^2}{a\ell + bK + c} - \sum_{\ell=1}^{K} \frac{|\hat{h}|^2}{a\ell + bK} \right| \leq \frac{K|c||\hat{h}|^2}{(a + bK + c)(a + bK)}. \tag{B.59}$$

If $a \leq 0$, then

$$\left| \sum_{\ell=1}^{K} \frac{|\hat{h}|^2}{a\ell + bK + c} - \sum_{\ell=1}^{K} \frac{|\hat{h}|^2}{a\ell + bK} \right| \leq \frac{K|c||\hat{h}|^2}{((a+b)K + c)(a+b)K}. \tag{B.60}$$

Since the RHS of (B.59) and of (B.60) vanish as K tends to infinity, this yields (B.57). Consequently,

$$\begin{aligned}\lim_{K\to\infty}\sum_{\ell=1}^{K}\frac{|\hat{h}|^2}{a\ell+bK+c} &= \lim_{K\to\infty}\sum_{\ell=1}^{K}\frac{|\hat{h}|^2}{a\ell+bK}\\ &= \lim_{K\to\infty}\frac{1}{K}\sum_{\ell=1}^{K}\frac{|\hat{h}|^2}{a\frac{\ell}{K}+b}\\ &= \int_0^1\frac{|\hat{h}|^2}{ax+b}\,\mathrm{d}x\\ &= \frac{|\hat{h}|^2}{a}\log\left(1+\frac{a}{b}\right)\end{aligned} \tag{B.61}$$

where the third step follows by noting that the function $x \mapsto \frac{1}{ax+b}$ is Riemann integrable, so the Riemann sum converges to the integral.

Using the definition of $\Theta(\cdot)$ [cf. (2.27)], it follows from (B.56) and (B.61) that

$$\lim_{K\to\infty}\sum_{\ell=1}^{K}\Gamma_{\ell,\mathbf{U}}(w,\hat{h}) = \frac{|\hat{h}|^2}{b}\Theta\left(\frac{a}{b}\right) = \frac{|\hat{h}|^2}{|\hat{h}|^2+\tilde{V}(\hat{h})+\frac{N_0}{P}}\Theta\left(\frac{\tilde{V}(\hat{h})(w-1)-|\hat{h}|^2}{|\hat{h}|^2+\tilde{V}(\hat{h})+\frac{N_0}{P}}\right) \tag{B.62}$$

thus proving (B.52), which in turn proves Theorem 2.2.

B.5 Proof of Lemma B.1

We show that

$$\lim_{\mathbf{Q}\to\mathbf{Q}'} R[\mathbf{Q}] = R[\mathbf{Q}'] \tag{B.63}$$

where $\mathbf{Q}\to\mathbf{Q}'$ should be read as

$$\max_{\ell}|Q_\ell - Q'_\ell| \to 0. \tag{B.64}$$

To this end, we write $R[\mathbf{Q}]$ as

$$R[\mathbf{Q}] = \sum_{\ell=1}^{L}\mathsf{E}\left[\log(1+\Gamma_{\ell,\mathbf{Q}}(W_\ell,\hat{H}))\right] \tag{B.65}$$

with

$$\Gamma_{\ell,\mathbf{Q}}(W_\ell,\hat{H}) \triangleq \frac{|\hat{H}|^2(Q_\ell-Q_{\ell-1})}{\tilde{V}(\hat{H})W_\ell Q_{\ell-1}+\tilde{V}(\hat{H})(Q_\ell-Q_{\ell-1})+\big(|\hat{H}|^2+\tilde{V}(\hat{H})\big)(P-Q_\ell)+N_0} \tag{B.66}$$

(assuming that $Q_0 = 0$) and

$$W_\ell \triangleq \begin{cases} 0, & \ell = 0\\ \dfrac{1}{Q_{\ell-1}}\left|\displaystyle\sum_{i<\ell}X_i\right|^2, & \ell = 2,\dots,L.\end{cases} \tag{B.67}$$

Using that, with probability one,

$$0 \leq \log(1 + \Gamma_{\ell,\mathbf{Q}}(W_\ell, \hat{H})) \leq \frac{|\hat{H}|^2 P}{N_0} \tag{B.68}$$

and that $\hat{H}$ has finite variance, it follows from the Dominated Convergence Theorem [Ash00, (1.6.9), p. 50] that

$$\begin{aligned}\lim_{\mathbf{Q}\to\mathbf{Q}'} \mathsf{E}\left[\log(1 + \Gamma_{\ell,\mathbf{Q}}(W_\ell, \hat{H}))\right] &= \mathsf{E}\left[\lim_{\mathbf{Q}\to\mathbf{Q}'} \log(1 + \Gamma_{\ell,\mathbf{Q}}(W_\ell, \hat{H}))\right] \\ &= \mathsf{E}\left[\log(1 + \Gamma_{\ell,\mathbf{Q}'}(W_\ell, \hat{H}))\right]\end{aligned} \tag{B.69}$$

where the last step follows by noting that, for every $(w_\ell, \hat{h})$, the function $\mathbf{Q} \mapsto \log(1+\Gamma_{\ell,\mathbf{Q}}(w_\ell, \hat{h}))$ is continuous. Combining (B.69) with (B.65) proves (B.63) and, hence, Lemma B.1.

B.6 Proof of Theorem 2.3

To prove theorem 2.3, we show that, in the limit as the SNR tends to infinity, the difference

$$I(X_{\mathrm{G}}; Y|\hat{H}_\rho) - R^\star(\rho) \tag{B.70}$$

is upper-bounded by $\log(M)\,\mathsf{Pr}\{|H| > 0\}$ provided that (2.36a)–(2.36b) are satisfied. To this end, we introduce the random variable

$$D \triangleq \begin{cases} 0 & \text{if } H = 0 \\ 1 & \text{if } |H| > 0 \end{cases} \tag{B.71}$$

and upper-bound the mutual information in (B.70) as

$$I(X_{\mathrm{G}}; Y|\hat{H}_\rho) \leq I(X_{\mathrm{G}}; Y|\hat{H}_\rho, D) \tag{B.72}$$

which follows because X_{G} is independent of $\hat{H}_\rho$ and D. We next note that

$$I(X_{\mathrm{G}}; Y|\hat{H}_\rho, D = 0) = I(X_{\mathrm{G}}; Z) = 0 \tag{B.73}$$

since X_{G}, Z, and $(\hat{H}_\rho, H)$ are independent. If $\mathsf{Pr}\{H = 0\} = 1$, then Theorem 2.3 follows directly from (B.72), (B.73), and the nonnegativity of $R^\star(\rho)$. In the following, we assume that $\mathsf{Pr}\{H = 0\} < 1$.

We express $R^\star(\rho)$ in (2.26) as $\mathsf{E}[R^\star(\rho, W, \hat{H}_\rho)]$ with[3]

$$R^\star(\rho, w, \xi) \triangleq \frac{|\xi|^2}{|\xi|^2 + \tilde{V}_\rho(\xi) + \rho^{-1}} \Theta\left(\frac{(w-1)\tilde{V}_\rho(\xi) - |\xi|^2}{|\xi|^2 + \tilde{V}_\rho(\xi) + \rho^{-1}}\right), \quad (\rho > 0, w \geq 0, \xi \in \mathbb{C}). \tag{B.74}$$

[3] Recall W is independent of $\hat{H}_\rho$ and has a unit-mean exponential distribution.

Note that, by the definition of $\Theta(\cdot)$ in (2.27), $R^\star(\rho,w,\xi)\geq 0$ for every $(\rho>0, w\geq 0, \xi\in\mathbb{C})$. Using this result together with (B.72) and (B.73), we obtain

$$I(X_{\mathrm{G}};Y|\hat{H}_\rho)-R^\star(\rho)\leq I(X_{\mathrm{G}};Y|\hat{H}_\rho,D=1)\,\mathsf{Pr}\{|H|>0\}-\mathsf{E}\big[R^\star(\rho,W,\hat{H}_\rho)\mid D=1\big]\,\mathsf{Pr}\{|H|>0\}. \tag{B.75}$$

To prove Theorem 2.3, it remains to show that if (2.36a)–(2.36b) hold, then

$$\varlimsup_{\rho\to\infty}\Big\{I(X_{\mathrm{G}};Y|\hat{H}_\rho,D=1)-\mathsf{E}\big[R^\star(\rho,W,\hat{H}_\rho)\mid D=1\big]\Big\}\leq\log(M). \tag{B.76}$$

For ease of exposition, we will omit in the remainder of the proof the conditioning on the event $|H|>0$ and replace tacitly the joint distribution of $(\hat{H}_\rho,H)$ by its conditional distribution, conditioned on $|H|>0$. This change will not affect the bounds (1.28), (2.26), and (1.34), since they hold irrespective of the distribution of $(\hat{H}_\rho,H)$ (provided that H and $\hat{H}_\rho$ satisfy the conditions indicated in the Introduction). Note that, under this new distribution, we have $\mathsf{Pr}\{H=0\}=0$.

To prove (B.76), we upper-bound $I(X_{\mathrm{G}};Y|\hat{H}_\rho)$ by $I_{\text{upper}}(\rho)$ using (1.34) and express $I_{\text{upper}}(\rho)-R^\star(\rho)$ as

$$\begin{aligned}I_{\text{upper}}(\rho)-R^\star(\rho)&=\mathsf{E}\big[\underline{R}(\rho,\hat{H}_\rho)\big]+\mathsf{E}\big[\Delta(\rho,W,\hat{H}_\rho)\big]-\mathsf{E}\big[R^\star(\rho,W,\hat{H}_\rho)\big]\\&=\mathsf{E}\big[\Sigma(\rho,\hat{H}_\rho)\big]\end{aligned} \tag{B.77}$$

where we have defined [cf. (1.28), (1.34)]

$$\underline{R}(\rho,\xi)\triangleq\log\left(1+\frac{|\xi|^2}{\tilde{V}_\rho(\xi)+\rho^{-1}}\right),\quad(\rho>0,w\geq 0,\xi\in\mathbb{C}) \tag{B.78}$$

$$\Delta(\rho,w,\xi)\triangleq\log\left(\frac{\tilde{V}_\rho(\xi)+\rho^{-1}}{\tilde{\Phi}_\rho(\xi)w+\rho^{-1}}\right),\qquad(\rho>0,w\geq 0,\xi\in\mathbb{C}) \tag{B.79}$$

and

$$\Sigma(\rho,\xi)\triangleq\underline{R}(\rho,\xi)+\mathsf{E}\big[\Delta(\rho,W,\xi)\big]-\mathsf{E}\big[R^\star(\rho,W,\xi)\big],\quad(\rho>0,\ \xi\in\mathbb{C}). \tag{B.80}$$

Note that $\Sigma(\rho,\xi)\geq 0$, $\xi\in\mathbb{C}$ since $I_{\text{upper}}(\rho)-R^\star(\rho)$ is nonnegative for any distribution of $\hat{H}_\rho$, hence it is also nonnegative if $\hat{H}_\rho=\xi$ with probability one.

We next show that

$$\varlimsup_{\rho\to\infty}\mathsf{E}\big[\Sigma(\rho,\hat{H}_\rho)\big]\leq\log(M). \tag{B.81}$$

To this end, we write the RHS of (B.77) as

$$\mathsf{E}\big[\Sigma(\rho,\hat{H}_\rho)\big]=\mathsf{E}\Big[\Sigma(\rho,\hat{H}_\rho)\,\mathrm{I}\{|\hat{H}_\rho|\leq\xi_0\}\Big]+\mathsf{E}\Big[\Sigma(\rho,\hat{H}_\rho)\,\mathrm{I}\{|\hat{H}_\rho|>\xi_0\}\Big] \tag{B.82}$$

for some arbitrary $0 < \xi_0 < 1$, where $\mathrm{I}\{\cdot\}$ denotes the indicator function. We then show that

$$\lim_{\xi_0 \downarrow 0} \varlimsup_{\rho\to\infty} \mathsf{E}\Big[\Sigma(\rho, \hat{H}_\rho)\, \mathrm{I}\{|\hat{H}_\rho| \le \xi_0\}\Big] = 0 \tag{B.83a}$$

and

$$\varlimsup_{\xi_0 \downarrow 0} \varliminf_{\rho\to\infty} \mathsf{E}\Big[\Sigma(\rho, \hat{H}_\rho)\, \mathrm{I}\{|\hat{H}_\rho| > \xi_0\}\Big] \le \log(M). \tag{B.83b}$$

To prove (B.83a), we need the following two lemmas.

Lemma B.2. *We have*

$$\varlimsup_{\rho\to\infty} \sup_{\xi\in\mathbb{C}} \Sigma(\rho, \xi) \le \gamma + \log(M). \tag{B.84}$$

where $\gamma \approx 0.577$ *denotes Euler's constant.*

Proof: See Appendix B.7. ■

Lemma B.3. *Let* $\tilde{V}_\rho(\hat{H}_\rho)$ *and* $\tilde{H}_\rho$ *satisfy* (2.36a) *and* (2.36b)*, and assume that* $\mathsf{Pr}\{H = 0\} = 0$*. Then*

$$\lim_{\xi_0 \downarrow 0} \varliminf_{\rho\to\infty} \mathsf{Pr}\{|\hat{H}_\rho| > \xi_0\} = 1. \tag{B.85}$$

Proof: See Appendix B.8. ■

Lemma B.2 implies that for every $\epsilon > 0$ there exists a $\rho_0 > 0$ such that

$$\sup_{\xi\in\mathbb{C}} \Sigma(\rho, \xi) \le \gamma + \log(M) + \epsilon, \quad \rho \ge \rho_0. \tag{B.86}$$

Consequently, for $\rho \ge \rho_0$ we have

$$\mathsf{E}\Big[\Sigma(\rho, \hat{H}_\rho)\, \mathrm{I}\{|\hat{H}_\rho| \le \xi_0\}\Big] \le (\gamma + \log(M) + \epsilon)\, \mathsf{Pr}\{|\hat{H}_\rho| \le \xi_0\}. \tag{B.87}$$

Together with Lemma B.3, this yields (B.83a) upon taking limits on both sides of (B.87):

$$\begin{aligned} &\lim_{\xi_0 \downarrow 0} \varlimsup_{\rho\to\infty} \Big\{\mathsf{E}\Big[\Sigma(\rho, \hat{H}_\rho)\, \mathrm{I}\{|\hat{H}_\rho| \le \xi_0\}\Big]\Big\} \\ &\quad \le (\gamma + \log(M) + \epsilon) \Big\{\lim_{\xi_0 \downarrow 0} \varlimsup_{\rho\to\infty} \mathsf{Pr}\{|\hat{H}_\rho| \le \xi_0\}\Big\} \\ &\quad = 0. \end{aligned} \tag{B.88}$$

To prove (B.83b), we first upper-bound $\Sigma(\rho, \xi)$ by lower-bounding $\mathsf{E}[R^\star(\rho, W, \xi)]$ for $\rho > 0$ and $|\xi| > \xi_0$ using that $R^\star(\rho, w, \xi)$ is nonnegative and recalling that W is unit-mean exponentially distributed:

$$\mathsf{E}[R^\star(\rho, W, \xi)] \ge \int_0^{\kappa(\rho,\xi)} R^\star(\rho, w, \xi) e^{-w}\, \mathrm{d}w, \quad (\rho > 0,\ |\xi| > \xi_0) \tag{B.89}$$

where

$$\kappa(\rho,\xi) \triangleq \frac{\xi_0^2}{\sqrt{\tilde{V}_\rho(\xi)+\rho^{-1}}}. \tag{B.90}$$

This choice for $\kappa(\rho,\xi)$ together with the assumption $\tilde{V}_\rho(\xi) \leq 1$ ensures that $(1-w)\tilde{V}_\rho(\xi)+|\xi|^2$ is strictly positive for all values of the integration variable w and for all $|\xi| > \xi_0$. Using (B.74) and the definition (2.27) of the function $\Theta(\cdot)$, the lower bound (B.89) reads as

$$\mathsf{E}\big[R^\star(\rho,W,\xi)\big] \geq \int_0^{\kappa(\rho,\xi)} \frac{|\xi|^2}{(w-1)\tilde{V}_\rho(\xi)-|\xi|^2} \log\left(1+\frac{(w-1)\tilde{V}_\rho(\xi)-|\xi|^2}{|\xi|^2+\tilde{V}_\rho(\xi)+\rho^{-1}}\right) e^{-w}\,\mathrm{d}w, \qquad (\rho>0,\ |\xi|>\xi_0). \tag{B.91}$$

Combining (B.91) with (B.78)–(B.80) yields

$$\begin{aligned}\Sigma(\rho,\xi) \leq{}& \log\left(1+\frac{|\xi|^2}{\tilde{V}_\rho(\xi)+\rho^{-1}}\right) + \int_0^\infty \log\left(\frac{\tilde{V}_\rho(\xi)+\rho^{-1}}{\tilde{\Phi}_\rho(\xi)w+\rho^{-1}}\right) e^{-w}\,\mathrm{d}w \\ &+ \int_0^{\kappa(\rho,\xi)} \frac{|\xi|^2}{(1-w)\tilde{V}_\rho(\xi)+|\xi|^2} \log\left(1+\frac{(w-1)\tilde{V}_\rho(\xi)-|\xi|^2}{|\xi|^2+\tilde{V}_\rho(\xi)+\rho^{-1}}\right) e^{-w}\,\mathrm{d}w, \\ &(\rho>0,\ |\xi|>\xi_0).\end{aligned} \tag{B.92}$$

This upper bound has the form $\Sigma(\rho,\xi) \leq \Sigma_1(\rho,\xi)+\Sigma_2(\rho,\xi)+\Sigma_3(\rho,\xi)$ where the terms can be expanded as

$$\begin{aligned}\Sigma_1(\rho,\xi) ={}& \log\left(|\xi|^2+\tilde{V}_\rho(\xi)+\rho^{-1}\right)\int_0^{\kappa(\rho,\xi)} e^{-w}\,\mathrm{d}w - \int_0^{\kappa(\rho,\xi)} \log\left(\tilde{V}_\rho(\xi)+\rho^{-1}\right) e^{-w}\,\mathrm{d}w \\ &+ \int_{\kappa(\rho,\xi)}^\infty \log\left(1+\frac{|\xi|^2}{\tilde{V}_\rho(\xi)+\rho^{-1}}\right) e^{-w}\,\mathrm{d}w \qquad\qquad \text{(B.93a)} \\ \Sigma_2(\rho,\xi) ={}& \int_0^{\kappa(\rho,\xi)} \log\left(\tilde{V}_\rho(\xi)+\rho^{-1}\right) e^{-w}\,\mathrm{d}w - \int_0^{\kappa(\rho,\xi)} \log\left(\tilde{\Phi}_\rho(\xi)w+\rho^{-1}\right) e^{-w}\,\mathrm{d}w \\ &+ \int_{\kappa(\rho,\xi)}^\infty \log\left(\frac{\tilde{V}_\rho(\xi)+\rho^{-1}}{\tilde{\Phi}_\rho(\xi)w+\rho^{-1}}\right) e^{-w}\,\mathrm{d}w \qquad\qquad \text{(B.93b)} \\ \Sigma_3(\rho,\xi) ={}& \int_0^{\kappa(\rho,\xi)} \frac{|\xi|^2}{(1-w)\tilde{V}_\rho(\xi)+|\xi|^2} \log\left(\tilde{V}_\rho(\xi)w+\rho^{-1}\right) e^{-w}\,\mathrm{d}w \\ &- \log\left(|\xi|^2+\tilde{V}_\rho(\xi)+\rho^{-1}\right)\int_0^{\kappa(\rho,\xi)} \frac{|\xi|^2}{(1-w)\tilde{V}_\rho(\xi)+|\xi|^2} e^{-w}\,\mathrm{d}w. \qquad\qquad \text{(B.93c)}\end{aligned}$$

Upon reordering terms in (B.93a)–(B.93c), the upper bound (B.92) can be further rewritten as

$$\Sigma(\rho,\xi) \leq \sum_{i=1}^{5} J_i(\rho,\xi) \tag{B.94}$$

with the five terms

$$J_1(\rho,\xi) \triangleq \int_0^{\kappa(\rho,\xi)} \log\left(\frac{\tilde{V}_\rho(\xi)w+\rho^{-1}}{\tilde{\Phi}_\rho(\xi)w+\rho^{-1}}\right) e^{-w}\,\mathrm{d}w \tag{B.95a}$$

$$J_2(\rho,\xi) \triangleq -\int_0^{\kappa(\rho,\xi)} \frac{(1-w)\tilde{V}_\rho(\xi)}{(1-w)\tilde{V}_\rho(\xi)+|\xi|^2}\log(\tilde{V}_\rho(\xi)w+\rho^{-1})e^{-w}\,\mathrm{d}w \tag{B.95b}$$

$$J_3(\rho,\xi) \triangleq \log(|\xi|^2+\tilde{V}_\rho(\xi)+\rho^{-1})\int_0^{\kappa(\rho,\xi)} \frac{(1-w)\tilde{V}_\rho(\xi)}{(1-w)\tilde{V}_\rho(\xi)+|\xi|^2}e^{-w}\,\mathrm{d}w \tag{B.95c}$$

$$J_4(\rho,\xi) \triangleq \int_{\kappa(\rho,\xi)}^{\infty} \log\left(\frac{\tilde{V}_\rho(\xi)+\rho^{-1}}{\tilde{\Phi}_\rho(\xi)w+\rho^{-1}}\right) e^{-w}\,\mathrm{d}w \tag{B.95d}$$

$$J_5(\rho,\xi) \triangleq \log\left(1+\frac{|\xi|^2}{\tilde{V}_\rho(\xi)+\rho^{-1}}\right) e^{-\kappa(\rho,\xi)}. \tag{B.95e}$$

Here, the term $J_1(\rho,\xi)$ is the second term of (B.93b) to which we add $\int_0^{\kappa(\rho,\xi)}\log(\tilde{V}_\rho(\xi)w+\rho^{-1})e^{-w}\,\mathrm{d}w$; the term $J_2(\rho,\xi)$ is the first term in (B.93c) from which we subtract $\int_0^{\kappa(\rho,\xi)}\log(\tilde{V}_\rho(\xi)w+\rho^{-1})e^{-w}\,\mathrm{d}w$; the term $J_3(\rho,\xi)$ follows from adding the first term in (B.93a) to the second term in (B.93c); the term $J_4(\rho,\xi)$ is the third term in (B.93b); the term $J_5(\rho,\xi)$ is the third term in (B.93a). The second term of (B.93a) and the first term of (B.93b) cancel out.

We proceed by showing that, for every $\xi_0 > 0$,

$$\overline{\lim_{\rho\to\infty}}\, \mathsf{E}\big[J_1(\rho,\hat{H}_\rho)\,\mathrm{I}\{|\hat{H}_\rho|>\xi_0\}\big] \le \log(M) \tag{B.96a}$$

$$\overline{\lim_{\rho\to\infty}}\, \mathsf{E}\big[J_i(\rho,\hat{H}_\rho)\,\mathrm{I}\{|\hat{H}_\rho|>\xi_0\}\big] \le 0, \qquad i=2,3,4,5. \tag{B.96b}$$

The claim (B.83b) then follows by combining (B.96a) and (B.96b) with (B.94) and by letting ξ_0 tend to zero from above. The following lemma will be useful.

Lemma B.4. *Consider the family of random variables Υ_ρ parametrized by $\rho > 0$ taking values on $(0,\eta)$ and satisfying $\lim_{\rho\to\infty}\mathsf{E}[\Upsilon_\rho]=0$, where η belongs to the extended positive reals, i.e., $\eta\in(0,\infty]$. Let $f(\cdot)$ be a continuous bounded function on the interval $(0,\eta)$ with limit $\lim_{t\downarrow 0} f(t)=f_0$. Then*

$$\lim_{\rho\to\infty}\mathsf{E}\big[f(\Upsilon_\rho)\big]=f_0. \tag{B.97}$$

Proof: See Appendix B.9. ∎

B.6.1 Limit related to $J_1(\rho,\xi)$

Noting that $\tilde{\Phi}_\rho(\xi)\le\tilde{V}_\rho(\xi)$, we have that

$$w\mapsto\frac{\tilde{V}_\rho(\xi)w+\rho^{-1}}{\tilde{\Phi}_\rho(\xi)w+\rho^{-1}}$$

is monotonically increasing in w. Consequently, $J_1(\rho,\xi)$ is upper-bounded by

$$\begin{aligned} J_1(\rho,\xi) &\leq \int_0^{\kappa(\rho,\xi)} \log \frac{\tilde{V}_\rho(\xi)}{\tilde{\Phi}_\rho(\xi)} e^{-w}\,\mathrm{d}w \\ &\leq \left[1 - e^{-\frac{\xi_0^2}{\sqrt{\tilde{V}_\rho(\xi)+\rho^{-1}}}}\right] \log\left(\sup_{\xi\in\mathbb{C}} \frac{\tilde{V}_\rho(\xi)}{\tilde{\Phi}_\rho(\xi)}\right), \quad (\rho > 0,\ |\xi| > \xi_0) \end{aligned} \tag{B.98}$$

where in the last step we have used (B.90). Setting ξ to $\hat{H}_\rho$, averaging (B.98) over $\hat{H}_\rho$, and upper-bounding

$$\mathrm{I}\{|\hat{H}_\rho| > \xi_0\} \leq 1 \tag{B.99}$$

we obtain

$$\mathsf{E}\big[J_1(\rho,\hat{H}_\rho)\,\mathrm{I}\{|\hat{H}_\rho| > \xi_0\}\big] \leq \left(1 - \mathsf{E}\left[e^{-\frac{\xi_0^2}{\sqrt{\tilde{V}_\rho(\hat{H}_\rho)+\rho^{-1}}}}\right]\right) \log\left(\sup_{\xi\in\mathbb{C}} \frac{\tilde{V}_\rho(\xi)}{\tilde{\Phi}_\rho(\xi)}\right), \quad \rho > 0. \tag{B.100}$$

Noting that the function $t \mapsto \exp\left(-\xi_0^2/\sqrt{t}\right)$ is continuous and bounded on $(0,\infty)$ and vanishes as n_{T} tends to zero, it follows from (2.36a) and Lemma B.4 that

$$\lim_{\rho\to\infty} \mathsf{E}\left[\exp\left(-\frac{\xi_0^2}{\sqrt{\tilde{V}_\rho(\hat{H}_\rho) + \rho^{-1}}}\right)\right] = 0. \tag{B.101}$$

We further have by (2.36b) as well as the continuity and monotonicity of $x \mapsto \log(x)$ that

$$\overline{\lim_{\rho\to\infty}} \log\left(\sup_{\xi\in\mathbb{C}} \frac{\tilde{V}_\rho(\xi)}{\tilde{\Phi}_\rho(\xi)}\right) \leq \log(M). \tag{B.102}$$

Combining (B.101) and (B.102) with (B.100) proves (B.96a).

B.6.2 Limit related to $J_2(\rho,\xi)$

To prove (B.96b) for $i = 2$, first note that $0 < \xi_0 < 1$ implies that, for sufficiently large ρ, we have

$$\tilde{V}_\rho(\xi)w + \rho^{-1} \leq 1, \quad 0 \leq w \leq \kappa(\rho,\xi) \tag{B.103}$$

and

$$(1-w)\tilde{V}_\rho(\xi) \geq -|\xi|^2, \quad 0 \leq w \leq \kappa(\rho,\xi). \tag{B.104}$$

Further note that $t \mapsto t/(t+|\xi|^2)$ is monotonically increasing on $(-|\xi|^2,\infty)$. Consequently, for sufficiently large ρ, (B.95b) is upper-bounded by

$$\begin{aligned} J_2(\rho,\xi) &\leq -\frac{\tilde{V}_\rho(\xi)}{\tilde{V}_\rho(\xi)+|\xi|^2}\int_0^{\kappa(\rho,\xi)} \log\big(\tilde{V}_\rho(\xi)w+\rho^{-1}\big)e^{-w}\,\mathrm{d}w \\ &\leq \frac{\tilde{V}_\rho(\xi)}{\tilde{V}_\rho(\xi)+|\xi|^2}\left[\left(1-e^{-\kappa(\xi,\rho)}\right)\log\frac{1}{\tilde{V}_\rho(\xi)}+\int_0^{\kappa(\rho,\xi)}|\log(w)|\,e^{-w}\,\mathrm{d}w\right] \end{aligned} \tag{B.105}$$

where the second inequality follows by lower-bounding $\log\big(\tilde{V}_\rho(\xi)w+\rho^{-1}\big) \geq \log\big(\tilde{V}_\rho(\xi)\big)+\log(w)$ and from the triangle inequality.

By using that the exponential function is nonnegative, by upper-bounding the integral by integrating to infinity, and by using that $|\xi|>\xi_0$, we can further upper-bound (B.105), for sufficiently large ρ, by

$$J_2(\rho,\xi) \leq \frac{\tilde{V}_\rho(\xi)}{\tilde{V}_\rho(\xi)+\xi_0^2}\left[\log\frac{1}{\tilde{V}_\rho(\xi)}+K\right] \tag{B.106}$$

where we define

$$K \triangleq \int_0^\infty |\log(w)|\,e^{-w}\,\mathrm{d}w = \gamma - 2\,\mathrm{Ei}(-1) \tag{B.107}$$

and where $\mathrm{Ei}(\cdot)$ denotes the exponential integral function, i.e.,

$$\mathrm{Ei}(x) \triangleq -\int_{-x}^\infty \frac{e^{-u}}{u}\mathrm{d}u. \tag{B.108}$$

Noting that the RHS of (B.106) is a continuous and bounded function of $0<\tilde{V}_\rho(\xi)\leq 1$ that vanishes as $\tilde{V}_\rho(\xi)$ tends to zero, it follows from (B.106), (B.99), (2.36a), and Lemma B.4 that

$$\varlimsup_{\rho\to\infty}\mathsf{E}\big[J_2(\rho,\hat{H}_\rho)\,\mathrm{I}\{|\hat{H}_\rho|>\xi_0\}\big] \leq \varlimsup_{\rho\to\infty}\mathsf{E}\left[\frac{\tilde{V}_\rho(\hat{H}_\rho)}{\tilde{V}_\rho(\hat{H}_\rho)+\xi_0^2}\left(\log\frac{1}{\tilde{V}_\rho(\hat{H}_\rho)}+K\right)\right] \leq 0 \tag{B.109}$$

thus proving (B.96b) for $i=2$.

B.6.3 Limit related to $J_3(\rho,\xi)$

To prove (B.96b) for $i=3$, we shall prove the stronger statement

$$\lim_{\rho\to\infty}\mathsf{E}\left[\big|J_3(\rho,\hat{H}_\rho)\big|\,\mathrm{I}\{|\hat{H}_\rho|>\xi_0\}\right]=0. \tag{B.110}$$

To this end, note that by the triangle inequality

$$\left|\frac{(1-w)\tilde{V}_\rho(\xi)}{(1-w)\tilde{V}_\rho(\xi)+|\xi|^2}\right| \leq \frac{(1+\kappa(\rho,\xi))\tilde{V}_\rho(\xi)}{(1-\kappa(\rho,\xi))\tilde{V}_\rho(\xi)+\xi_0^2}, \quad 0\leq w\leq \kappa(\rho,\xi). \tag{B.111}$$

In (B.111) we have used that, for $0\leq w\leq\kappa(\rho,\xi)$ and $|\xi|\geq\xi_0$, the denominator is lower-bounded by $(1-\kappa(\rho,\xi))\tilde{V}_\rho(\xi)+\xi_0^2>0$. It follows from (B.111) and the triangle inequality that the absolute value of the integral in (B.95c) is upper-bounded by

$$\left|\int_0^{\kappa(\rho,\xi)} \frac{(1-w)\tilde{V}_\rho(\xi)}{(1-w)\tilde{V}_\rho(\xi)+|\xi|^2} e^{-w}\,\mathrm{d}w\right| \le \left(1-e^{-\kappa(\rho,\xi)}\right)\frac{(1+\kappa(\rho,\xi))\tilde{V}_\rho(\xi)}{(1-\kappa(\rho,\xi))\tilde{V}_\rho(\xi)+\xi_0^2}. \quad \text{(B.112)}$$

Consequently,

$$\begin{aligned} |J_3(\rho,\xi)| &\le \left(1-e^{-\kappa(\rho,\xi)}\right)\frac{(1+\kappa(\rho,\xi))\tilde{V}_\rho(\xi)}{(1-\kappa(\rho,\xi))\tilde{V}_\rho(\xi)+\xi_0^2}\left|\log\left(|\xi|^2+\tilde{V}_\rho(\xi)+\rho^{-1}\right)\right| \\ &\le \frac{(1+\kappa(\rho,\xi))\left(\tilde{V}_\rho(\xi)+\rho^{-1}\right)}{\xi_0^2-\left(\kappa(\rho,\xi)-1\right)^+\left(\tilde{V}_\rho(\xi)+\rho^{-1}\right)}\left|\log\left(|\xi|^2+\tilde{V}_\rho(\xi)+\rho^{-1}\right)\right|, \quad (\rho>0,\ |\xi|\ge\xi_0) \end{aligned} \quad \text{(B.113)}$$

where we define $(a)^+ \triangleq \max(a,0)$.[4] Here the last step follows by upper-bounding $\tilde{V}_\rho(\xi) \le \tilde{V}_\rho(\xi)+\rho^{-1}$ and by lower-bounding $(1-\kappa(\rho,\xi))\tilde{V}_\rho(\xi) \ge -(\kappa(\rho,\xi)-1)^+\left(\tilde{V}_\rho(\xi)+\rho^{-1}\right)$ and $e^{-\kappa(\rho,\xi)} \ge 0$.

Using the definition (B.90) of $\kappa(\rho,\xi)$, and defining $\Upsilon_\rho(\xi) \triangleq \tilde{V}_\rho(\xi)+\rho^{-1}$, the RHS of (B.113) reads as

$$\frac{\Upsilon_\rho(\xi)+\xi_0^2\sqrt{\Upsilon_\rho(\xi)}}{\xi_0^2-\left(\xi_0^2-\sqrt{\Upsilon_\rho(\xi)}\right)^+\sqrt{\Upsilon_\rho(\xi)}}\left|\log\left(|\xi|^2+\Upsilon_\rho(\xi)\right)\right|. \quad \text{(B.114)}$$

Since $\tilde{V}_\rho(\xi) \le 1$ and $x \mapsto \log(x)$ is a monotonically increasing function, we have

$$\log\left(\Upsilon_\rho(\xi)\right) \le \log\left(|\xi|^2+\Upsilon_\rho(\xi)\right) \le \log\left(1+\rho^{-1}+|\xi|^2\right). \quad \text{(B.115)}$$

The absolute value of the logarithm on the RHS of (B.114) is thus upper-bounded by

$$\left|\log\left(|\xi|^2+\Upsilon_\rho(\xi)\right)\right| \le \left|\log\left(\Upsilon_\rho(\xi)\right)\right| + \log\left(1+\rho^{-1}+|\xi|^2\right). \quad \text{(B.116)}$$

By noting that

$$\Upsilon_\rho(\xi) \mapsto \frac{\Upsilon_\rho(\xi)+\xi_0^2\sqrt{\Upsilon_\rho(\xi)}}{\xi_0^2-\left(\xi_0^2-\sqrt{\Upsilon_\rho(\xi)}\right)^+\sqrt{\Upsilon_\rho(\xi)}}\left|\log\left(\Upsilon_\rho(\xi)\right)\right| \quad \text{(B.117)}$$

is a continuous and bounded function of $0 < \Upsilon_\rho(\xi) \le 1+\rho^{-1}$ that vanishes as $\Upsilon_\rho(\xi)$ tends to zero, we obtain from (2.36a), (B.99), and Lemma B.4 that

$$\begin{aligned} &\lim_{\rho\to\infty} \mathsf{E}\left[\frac{\Upsilon_\rho(\hat{H}_\rho)+\xi_0^2\sqrt{\Upsilon_\rho(\hat{H}_\rho)}}{\xi_0^2-\left(\xi_0^2-\sqrt{\Upsilon_\rho(\hat{H}_\rho)}\right)^+\sqrt{\Upsilon_\rho(\hat{H}_\rho)}}\left|\log\left(\Upsilon_\rho(\hat{H}_\rho)\right)\right| \mathrm{I}\{|\hat{H}_\rho|>\xi_0\}\right] \\ &\le \lim_{\rho\to\infty} \mathsf{E}\left[\frac{\Upsilon_\rho(\hat{H}_\rho)+\xi_0^2\sqrt{\Upsilon_\rho(\hat{H}_\rho)}}{\xi_0^2-\left(\xi_0^2-\sqrt{\Upsilon_\rho(\hat{H}_\rho)}\right)^+\sqrt{\Upsilon_\rho(\hat{H}_\rho)}}\left|\log\left(\Upsilon_\rho(\hat{H}_\rho)\right)\right|\right] \\ &= 0. \end{aligned} \quad \text{(B.118)}$$

[4] The condition $\xi_0 < 1$ ensures that the denominator remains positive.

Furthermore, (B.99) together with the Cauchy-Schwarz inequality yields

$$\begin{aligned}
&\mathsf{E}\left[\frac{\Upsilon_\rho(\hat{H}_\rho)+\xi_0^2\sqrt{\Upsilon_\rho(\hat{H}_\rho)}}{\xi_0^2-(\xi_0^2-\sqrt{\Upsilon_\rho(\hat{H}_\rho)})^+\sqrt{\Upsilon_\rho(\hat{H}_\rho)}}\log\left(1+\rho^{-1}+|\hat{H}_\rho|^2\right)\mathrm{I}\{|\hat{H}_\rho|>\xi_0\}\right]\\
&\quad\le \mathsf{E}\left[\frac{\Upsilon_\rho(\hat{H}_\rho)+\xi_0^2\sqrt{\Upsilon_\rho(\hat{H}_\rho)}}{\xi_0^2-(\xi_0^2-\sqrt{\Upsilon_\rho(\hat{H}_\rho)})^+\sqrt{\Upsilon_\rho(\hat{H}_\rho)}}\log\left(1+\rho^{-1}+|\hat{H}_\rho|^2\right)\right]\\
&\quad\le \sqrt{\mathsf{E}\left[\left(\frac{\Upsilon_\rho(\hat{H}_\rho)+\xi_0^2\sqrt{\Upsilon_\rho(\hat{H}_\rho)}}{\xi_0^2-(\xi_0^2-\sqrt{\Upsilon_\rho(\hat{H}_\rho)})^+\sqrt{\Upsilon_\rho(\hat{H}_\rho)}}\right)^2\right]}\sqrt{\mathsf{E}\left[\log^2\left(1+\rho^{-1}+|\hat{H}_\rho|^2\right)\right]}. \qquad \text{(B.119)}
\end{aligned}$$

Note that the term inside the first expected value is a continuous and bounded function of $0<\Upsilon_\rho(\hat{H}_\rho)\le 1+\rho^{-1}$ that vanishes as $\Upsilon_\rho(\hat{H}_\rho)$ tends to zero, so it follows from (2.36a) and Lemma B.4 that the first expected value on the RHS of (B.119) vanishes as ρ tends to infinity. We further show in Appendix B.10 that

$$\varlimsup_{\rho\to\infty}\mathsf{E}\left[\log^2\left(1+\rho^{-1}+|\hat{H}_\rho|^2\right)\right]<\infty. \qquad \text{(B.120)}$$

The above arguments combine to demonstrate that

$$\lim_{\rho\to\infty}\mathsf{E}\left[\frac{\Upsilon_\rho(\hat{H}_\rho)+\xi_0^2\sqrt{\Upsilon_\rho(\hat{H}_\rho)}}{\xi_0^2-(\xi_0^2-\sqrt{\Upsilon_\rho(\hat{H}_\rho)})^+\sqrt{\Upsilon_\rho(\hat{H}_\rho)}}\log\left(1+\rho^{-1}+|\hat{H}_\rho|^2\right)\mathrm{I}\{|\hat{H}_\rho|>\xi_0\}\right]=0. \qquad \text{(B.121)}$$

Combining (B.121), (B.118), (B.116), and (B.113) proves (B.110).

B.6.4 Limit related to $J_4(\rho,\xi)$

To upper-bound $J_4(\rho,\xi)$, we use that, for $w\ge\kappa(\rho,\xi)$,

$$\begin{aligned}
\frac{\tilde{V}_\rho(\xi)+\rho^{-1}}{\tilde{\Phi}_\rho(\xi)w+\rho^{-1}} &= \frac{\tilde{V}_\rho(\xi)}{\tilde{\Phi}_\rho(\xi)w+\rho^{-1}}+\frac{\rho^{-1}}{\tilde{\Phi}_\rho(\xi)w+\rho^{-1}}\\
&\le \frac{\tilde{V}_\rho(\xi)}{\tilde{\Phi}_\rho(\xi)\kappa(\rho,\xi)}+1\\
&\le \sup_{\xi\in\mathbb{C}}\left\{\frac{\tilde{V}_\rho(\xi)}{\tilde{\Phi}_\rho(\xi)}\right\}\frac{\sqrt{1+\rho^{-1}}}{\xi_0^2}+1 \qquad \text{(B.122)}
\end{aligned}$$

where the first inequality follows by lower-bounding $\rho^{-1}\ge 0$ and $w\ge\kappa(\rho,\xi)$ in the denominator of the first fraction and by lower-bounding $\tilde{\Phi}_\rho(\xi)w\ge 0$ in the denominator of the second fraction; and where the second inequality follows by lower-bounding $\kappa(\rho,\xi)\ge\xi_0^2/\sqrt{1+\rho^{-1}}$ using $\tilde{V}_\rho(\xi)\le$

1 and by maximizing over ξ. Combining (B.122) with (B.95d) yields

$$J_4(\rho,\xi) \le \log\left(1+\sup_{\xi\in\mathbb{C}}\left\{\frac{\tilde{V}_\rho(\xi)}{\tilde{\Phi}_\rho(\xi)}\right\}\frac{\sqrt{1+\rho^{-1}}}{\xi_0^2}\right)\int_{\kappa(\rho,\xi)}^{\infty} e^{-w}\,\mathrm{d}w$$
$$= \exp\left(-\frac{\xi_0^2}{\sqrt{\tilde{V}_\rho(\xi)+\rho^{-1}}}\right)\log\left(1+\sup_{\xi\in\mathbb{C}}\left\{\frac{\tilde{V}_\rho(\xi)}{\tilde{\Phi}_\rho(\xi)}\right\}\frac{\sqrt{1+\rho^{-1}}}{\xi_0^2}\right). \tag{B.123}$$

Setting ξ to $\hat{H}_\rho$, averaging (B.123) over $\hat{H}_\rho$, and using (B.99), we obtain

$$\mathsf{E}\big[J_4(\rho,\hat{H}_\rho)\,\mathrm{I}\{|\hat{H}_\rho|>\xi_0\}\big]$$
$$\le \mathsf{E}\left[\exp\left(-\frac{\xi_0^2}{\sqrt{\tilde{V}_\rho(\hat{H}_\rho)+\rho^{-1}}}\right)\right]\log\left(1+\sup_{\xi\in\mathbb{C}}\left\{\frac{\tilde{V}_\rho(\xi)}{\tilde{\Phi}_\rho(\xi)}\right\}\frac{\sqrt{1+\rho^{-1}}}{\xi_0^2}\right). \tag{B.124}$$

Since, by (2.36b), the term inside the logarithm is bounded for sufficiently large ρ, (B.96b) for $i=4$ follows by combining (B.124) with (B.101).

B.6.5 Limit related to $J_5(\rho,\xi)$

Using (B.90) and defining $\Upsilon_\rho(\xi) \triangleq \tilde{V}_\rho(\xi)+\rho^{-1}$, the term $J_5(\rho,\xi)$ reads as

$$J_5(\rho,\xi) = e^{-\frac{\xi_0^2}{\sqrt{\Upsilon_\rho(\xi)}}}\log\left(1+\frac{|\xi|^2}{\Upsilon_\rho(\xi)}\right). \tag{B.125}$$

Since $\tilde{V}_\rho(\xi)\le 1$ and $x\mapsto\log(x)$ is a monotonically increasing function, this can be upper-bounded as

$$J_5(\rho,\xi) \le e^{-\frac{\xi_0^2}{\sqrt{\Upsilon_\rho(\xi)}}}\log\left(1+\rho^{-1}+|\xi|^2\right)+e^{-\frac{\xi_0^2}{\sqrt{\Upsilon_\rho(\xi)}}}\big|\log(\Upsilon_\rho(\xi))\big|. \tag{B.126}$$

We next note that the function $t\mapsto e^{-\xi_0^2/\sqrt{t}}|\log(t)|$ is continuous and bounded on $(0,1+\rho^{-1})$ and tends to zero as n_T tends to zero. Consequently, (B.99) and Lemma B.4 yield

$$\lim_{\rho\to\infty}\mathsf{E}\left[e^{-\frac{\xi_0^2}{\sqrt{\Upsilon_\rho(\hat{H}_\rho)}}}\big|\log(\Upsilon_\rho(\hat{H}_\rho))\big|\cdot\mathrm{I}\{|\hat{H}_\rho|>\xi_0\}\right]=0. \tag{B.127}$$

Furthermore, by (B.99) and the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\mathsf{E}\left[e^{-\frac{\xi_0^2}{\sqrt{\Upsilon_\rho(\hat{H}_\rho)}}}\log\left(1+\rho^{-1}+|\hat{H}_\rho|^2\right)\mathrm{I}\{|\hat{H}_\rho|>\xi_0\}\right]\\
&\qquad\le \mathsf{E}\left[e^{-\frac{\xi_0^2}{\sqrt{\Upsilon_\rho(\hat{H}_\rho)}}}\log\left(1+\rho^{-1}+|\hat{H}_\rho|^2\right)\right]\\
&\qquad\le \sqrt{\mathsf{E}\left[e^{-\frac{2\xi_0^2}{\sqrt{\Upsilon_\rho(\hat{H}_\rho)}}}\right]}\sqrt{\mathsf{E}\left[\log^2\left(1+\rho^{-1}+|\hat{H}_\rho|^2\right)\right]}. \qquad \text{(B.128)}
\end{aligned}$$

Since the function $t\mapsto\exp\left(-2\xi_0^2/\sqrt{t}\right)$ is continuous and bounded on $(0,\infty)$ and vanishes as n_T tends to zero, it follows from (2.36b) and Lemma B.4 that the first expected value on the RHS of (B.128) vanishes as ρ tends to infinity. Furthermore, by (B.120), the second expected value on the RHS of (B.128) is bounded for sufficiently small ρ. The above arguments combine to demonstrate that

$$\lim_{\rho\to\infty}\mathsf{E}\left[e^{-\frac{\xi_0^2}{\sqrt{\Upsilon_\rho(\hat{H}_\rho)}}}\log\left(1+\rho^{-1}+|\hat{H}_\rho|^2\right)\mathrm{I}\{|\hat{H}_\rho|>\xi_0\}\right]=0 \qquad \text{(B.129)}$$

which together with (B.126) and (B.127) proves (B.96b) for $i=5$.

B.7 Proof of Lemma B.2

We first note that, by specializing Theorem 2.1 to the case where $\hat{H}=\xi$ with probability one, it follows that

$$\underline{R}(\rho,\xi)\le\mathsf{E}\big[R^\star(\rho,W,\xi)\big],\quad(\rho>0,\ \xi\in\mathbb{C}). \qquad \text{(B.130)}$$

Combining (B.130) with (B.79) and (B.80), we obtain

$$\begin{aligned}
\Sigma(\rho,\xi)&\le\mathsf{E}\big[\Delta(\rho,W,\xi)\big]\\
&=\log\left(\frac{\tilde{V}_\rho(\xi)+\rho^{-1}}{\tilde{\Phi}_\rho(\xi)}\right)-\mathsf{E}\left[\log\left(W+\frac{1}{\rho\tilde{\Phi}_\rho(\xi)}\right)\right]. \qquad \text{(B.131)}
\end{aligned}$$

The expected value on the RHS of (B.131) can be evaluated as [Gra07, (4.337), p. 568]

$$\mathsf{E}\left[\log\left(W+\frac{1}{\rho\tilde{\Phi}_\rho(\xi)}\right)\right]=\log\left(\frac{1}{\rho\tilde{\Phi}_\rho(\xi)}\right)-e^{\frac{1}{\rho\tilde{\Phi}_\rho(\xi)}}\,\mathrm{Ei}\left(-\frac{1}{\rho\tilde{\Phi}_\rho(\xi)}\right) \qquad \text{(B.132)}$$

where Ei(·) denotes the exponential integral as defined in (B.108). This yields

$$\begin{aligned}\mathsf{E}[\Delta(\rho,W,\xi)] &= \log(1+\rho\tilde{V}_\rho(\xi)) + e^{\frac{1}{\rho\tilde{\Phi}_\rho(\xi)}}\,\mathrm{Ei}\left(-\frac{1}{\rho\tilde{\Phi}_\rho(\xi)}\right)\\ &= \log\left(1+\rho\tilde{\Phi}_\rho(\xi)\frac{\tilde{V}_\rho(\xi)}{\tilde{\Phi}_\rho(\xi)}\right) + e^{\frac{1}{\rho\tilde{\Phi}_\rho(\xi)}}\,\mathrm{Ei}\left(-\frac{1}{\rho\tilde{\Phi}_\rho(\xi)}\right)\\ &\le \log\left(1+\rho\tilde{\Phi}_\rho(\xi)\frac{\tilde{V}_\rho(\xi)}{\tilde{\Phi}_\rho(\xi)}\right) + \mathrm{Ei}\left(-\frac{1}{\rho\tilde{\Phi}_\rho(\xi)}\right)\\ &= g\left(\rho\tilde{\Phi}_\rho(\xi);\frac{\tilde{V}_\rho(\xi)}{\tilde{\Phi}_\rho(\xi)}\right) \end{aligned}\tag{B.133}$$

where we define

$$g(t;a) \triangleq \log(1+at) + \mathrm{Ei}\left(-\frac{1}{t}\right). \tag{B.134}$$

The inequality in (B.133) follows because $\mathrm{Ei}(-x)$ is negative for $x>0$ and $e^x \ge 1$, $x \ge 0$.

For a fixed a, the function $t \mapsto g(t;a)$ satisfies [Lap03, Section VI-A][5]

$$\lim_{n_\mathsf{T}\to\infty} g(t;a) = \gamma + \log(a). \tag{B.135}$$

We next show that, for every $a \ge 1$, the function $t \mapsto g(t;a)$ is monotonically increasing. Indeed, using $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Ei}(-x) = e^{-x}/x$, we have

$$\begin{aligned}\frac{\partial}{\partial t} g(t;a) &= \frac{e^{-\frac{1}{t}}}{(1+at)t}\left[e^{\frac{1}{t}}at - 1 - at\right]\\ &\ge \frac{e^{-\frac{1}{t}}}{(1+at)t}[a-1]\\ &\ge 0, \quad a \ge 1\end{aligned}\tag{B.136}$$

where the second step follows from the lower bound $e^{\frac{1}{t}} \ge 1 + \frac{1}{t}$, $t \ge 0$.

By (A.4), we have that $\tilde{V}_\rho(\xi)/\tilde{\Phi}_\rho(\xi) \ge 1$. It thus follows from (B.131)–(B.136) that

$$\begin{aligned}\Sigma(\rho,\xi) &\le g\left(\rho\tilde{\Phi}_\rho(\xi);\frac{\tilde{V}_\rho(\xi)}{\tilde{\Phi}_\rho(\xi)}\right)\\ &\le \lim_{t\to\infty} g\left(t;\frac{\tilde{V}_\rho(\xi)}{\tilde{\Phi}_\rho(\xi)}\right)\\ &= \gamma + \log\left(\frac{\tilde{V}_\rho(\xi)}{\tilde{\Phi}_\rho(\xi)}\right), \quad (\rho>0,\ \xi\in\mathbb{C}).\end{aligned}\tag{B.137}$$

[5] The function $g(\cdot;\cdot)$ corresponds to $g_0(\cdot)$ in [Lap03, Equation (210)] via $g(t;a) = \log(a) + \log\left(1+\frac{1}{at}\right) - g_0\left(\frac{1}{t}\right)$. The result (B.135) follows by noting that $g_0(0) = -\gamma$; cf. [Lap03, Equations (212) and (213)].

Maximizing the RHS of (B.137) over $\xi \in \mathbb{C}$, and computing the limit as ρ tends to infinity, gives

$$\begin{aligned}\varlimsup_{\rho\to\infty} \sup_{\xi\in\mathbb{C}} \Sigma(\rho,\xi) &\le \gamma + \varlimsup_{\rho\to\infty} \log\left(\sup_{\xi\in\mathbb{C}} \frac{\tilde{V}_\rho(\xi)}{\tilde{\Phi}_\rho(\xi)}\right) \\ &\le \gamma + \log(M) \end{aligned} \tag{B.138}$$

where the last step follows from the continuity and monotonicity of $x \mapsto \log(x)$ and from (2.36b). This proves Lemma B.2.

B.8 Proof of Lemma B.3

By the law of total probability, we have

$$\begin{aligned}\mathsf{Pr}\{|H| > 2\xi_0\} &= \mathsf{Pr}\{|H| > 2\xi_0, |\hat{H}_\rho| \le \xi_0\} + \mathsf{Pr}\{|H| > 2\xi_0, |\hat{H}_\rho| > \xi_0\} \\ &\le \mathsf{Pr}\{|H - \hat{H}_\rho| > \xi_0\} + \mathsf{Pr}\{|\hat{H}_\rho| > \xi_0\} \end{aligned} \tag{B.139}$$

using the fact that $|H| > 2\xi_0$ and $|\hat{H}_\rho| \le \xi_0$ together imply that $|H - \hat{H}_\rho| > \xi_0$ due to the triangle inequality, and that $|H| > 2\xi_0$ and $|\hat{H}_\rho| > \xi_0$ together imply $|\hat{H}_\rho| > \xi_0$. Using Chebyshev's inequality [Ash00, (4.10.7), p. 192], the first term on the RHS of (B.139) can be further upper-bounded by

$$\mathsf{Pr}\{|H - \hat{H}_\rho| > \xi_0\} \le \frac{\mathsf{E}[\tilde{V}_\rho(\hat{H}_\rho)]}{\xi_0^2}. \tag{B.140}$$

Combining (B.140) with (B.139) gives

$$\mathsf{Pr}\{|H| > 2\xi_0\} \le \frac{\mathsf{E}[\tilde{V}_\rho(\hat{H}_\rho)]}{\xi_0^2} + \mathsf{Pr}\{|\hat{H}_\rho| > \xi_0\}. \tag{B.141}$$

By (2.36a), taking the limit inferior for $\rho \to \infty$ on either side of (B.141) yields

$$\mathsf{Pr}\{|H| > 2\xi_0\} \le \varliminf_{\rho\to\infty} \mathsf{Pr}\{|\hat{H}_\rho| > \xi_0\}. \tag{B.142}$$

Furthermore, the assumption that $\mathsf{Pr}\{H = 0\} = 0$, we have

$$\lim_{\xi_0\downarrow 0} \mathsf{Pr}\{|H| > 2\xi_0\} = \mathsf{Pr}\{|H| > 0\} = 1. \tag{B.143}$$

Lemma B.3 follows therefore by taking limits as $\xi_0 \downarrow 0$ on both sides of (B.142).

B.9 Proof of Lemma B.4

For every family of random variables Υ_ρ parametrized by $\rho > 0$ and taking values on $(0, \eta)$ with $\eta \in (0, \infty]$, we have by Markov's inequality

$$\Pr\{\Upsilon_\rho > \nu\} \le \frac{\mathsf{E}[\Upsilon_\rho]}{\nu}, \quad \text{for every } \nu \in (0, \eta). \tag{B.144}$$

Using that $\lim_{\rho\to\infty} \mathsf{E}[\Upsilon_\rho] = 0$, we thus have

$$\lim_{\rho\to\infty} \Pr\{\Upsilon_\rho > \nu\} = 0, \quad \text{for every } \nu \in (0, \eta) \tag{B.145}$$

or equivalently, $\lim_{\rho\to\infty} \Pr\{\Upsilon_\rho \le \nu\} = 1$. We upper-bound $\mathsf{E}[f(\Upsilon_\rho)]$ for any $\nu \in (0, \eta)$ as

$$\begin{aligned}\mathsf{E}[f(\Upsilon_\rho)] &= \mathsf{E}\left[f(\Upsilon_\rho)\ \mathrm{I}\{\Upsilon_\rho \le \nu\}\right] + \mathsf{E}\left[f(\Upsilon_\rho)\ \mathrm{I}\{\Upsilon_\rho > \nu\}\right] \\ &\le \sup_{0<t\le\nu} f(t) \Pr\{\Upsilon_\rho \le \nu\} + \sup_{\nu<t<\eta} f(t) \Pr\{\Upsilon_\rho > \nu\}. \end{aligned} \tag{B.146a}$$

Similarly, we lower-bound $\mathsf{E}[f(\Upsilon_\rho)]$ for any $\nu \in (0, \eta)$ as

$$\mathsf{E}[f(\Upsilon_\rho)] \ge \inf_{0<t\le\nu} f(t) \Pr\{\Upsilon_\rho \le \nu\} + \inf_{\nu<t<\eta} f(t) \Pr\{\Upsilon_\rho > \nu\}. \tag{B.146b}$$

Since $f(\cdot)$ is bounded, and by (B.145), taking limits for $\rho \to \infty$ in (B.146a) and (B.146b) gives

$$\inf_{0<t\le\nu} f(t) \le \varliminf_{\rho\to\infty} \mathsf{E}[f(\Upsilon_\rho)] \le \varlimsup_{\rho\to\infty} \mathsf{E}[f(\Upsilon_\rho)] \le \sup_{0<t\le\nu} f(t). \tag{B.147}$$

Taking the limit as ν tends to zero from above and using the continuity of f, we finally obtain

$$\lim_{\rho\to\infty} \mathsf{E}[f(\Upsilon_\rho)] = \lim_{t\downarrow 0} f(t) = f_0 \tag{B.148}$$

which proves Lemma B.4.

B.10 Proof of (B.120)

To prove (B.120), we first note that the function $x \mapsto \log^2(1+x)$ is concave for $x \ge e-1$. We thus have for an arbitrary $\delta \ge e - 1$ and for $\rho \ge 1$

$$\begin{aligned}&\mathsf{E}\left[\log^2\left(1+\rho^{-1}+|\hat{H}_\rho|^2\right)\right] \\ &\quad = \mathsf{E}\left[\log^2\left(1+\rho^{-1}+|\hat{H}_\rho|^2\right)\mathrm{I}\left\{|\hat{H}_\rho|^2 \le \delta\right\}\right] + \mathsf{E}\left[\log^2\left(1+\rho^{-1}+|\hat{H}_\rho|^2\right)\mathrm{I}\left\{|\hat{H}_\rho|^2 > \delta\right\}\right] \\ &\quad \le \log^2(2+\delta) + \mathsf{E}\left[\log^2\left(2+|\hat{H}_\rho|^2\right)\mathrm{I}\left\{|\hat{H}_\rho|^2 > \delta\right\}\right] \\ &\quad \le \log^2(2+\delta) + \Pr\left\{|\hat{H}_\rho|^2 > \delta\right\}\log^2\left(2 + \frac{\mathsf{E}\left[|\hat{H}_\rho|^2\,\mathrm{I}\left\{|\hat{H}_\rho|^2 > \delta\right\}\right]}{\Pr\left\{|\hat{H}_\rho|^2 > \delta\right\}}\right)\end{aligned} \tag{B.149}$$

where we define $0 \log^2(1 + a/0) \triangleq 0$ for every $a \geq 0$. Here the first inequality follows by upper-bounding $\rho^{-1} \leq 1$ in both expected values and by upper-bounding $|\hat{H}_\rho|^2 \leq \delta$ in the first expected value, and the second inequality follows by upper-bounding the second expected value using Jensen's inequality.

We next use (B.99) and (2.35) to upper-bound

$$\mathsf{E}\left[|\hat{H}_\rho|^2 \, \mathrm{I}\left\{|\hat{H}_\rho|^2 > \delta\right\}\right] \leq \mathsf{E}\left[|\hat{H}_\rho|^2\right] \leq 1. \tag{B.150}$$

This yields

$$\begin{aligned}\mathsf{E}\left[\log^2\left(1 + \rho^{-1} + |\hat{H}_\rho|^2\right)\right] &\leq \log^2(2+\delta) + \mathsf{Pr}\left\{|\hat{H}_\rho|^2 > \delta\right\} \log^2\left(2 + \frac{1}{\mathsf{Pr}\left\{|\hat{H}_\rho|^2 > \delta\right\}}\right) \\ &\leq \log^2(2+\delta) + \sup_{0<x\leq 1}\left\{x \log^2\left(2 + \frac{1}{x}\right)\right\}, \quad \rho \geq 1\end{aligned} \tag{B.151}$$

where the second step follows by maximising the second term over $\mathsf{Pr}\{|\hat{H}_\rho|^2 > \delta\}$. Note that the supremum on the RHS of (B.151) is finite since the function $x \mapsto x\log^2(2 + 1/x)$ is continuous on $0 < x \leq 1$ and tends to zero as x tends to zero. Consequently, we have

$$\varlimsup_{\rho\to\infty} \mathsf{E}\left[\log^2\left(1 + \rho^{-1} + |\hat{H}_\rho|^2\right)\right] \leq \log^2(2+\delta) + \sup_{0<x\leq 1}\left\{x \log^2\left(2 + \frac{1}{x}\right)\right\} < \infty \tag{B.152}$$

which proves (B.120).

B.11 Proof of Theorem 2.4

Let us assume without loss of generality that the covariance matrix of $\mathbf{x}_\ell$ is full-rank[6]. We expand the mutual information as

$$I(\mathbf{x}_\ell; \mathbf{y} \mid \mathbf{x}^{\ell-1}, \hat{\mathbf{H}}) = h(\mathbf{x}_\ell \mid \mathbf{x}^{\ell-1}, \hat{\mathbf{H}}) - h(\mathbf{x}_\ell \mid \mathbf{x}^{\ell-1}, \hat{\mathbf{H}}, \mathbf{y}). \tag{B.153}$$

Since, by assumption, $\mathbf{x}_\ell \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{Q}_\ell)$ is independent of $(\mathbf{x}^{\ell-1}, \hat{\mathbf{H}})$ and has a full-rank covariance matrix, the first entropy on the right-hand side of (B.153) is readily evaluated as

$$h(\mathbf{x}_\ell \mid \mathbf{x}^{\ell-1}, \hat{\mathbf{H}}) = h(\mathbf{x}_\ell) = \log\det(\pi e \mathbf{Q}_\ell). \tag{B.154}$$

Let

$$\mu_{\mathsf{y}|\mathsf{x}^{\ell-1},\hat{\mathsf{H}}} \triangleq \mathsf{E}[\mathbf{y}|\mathsf{x}^{\ell-1}, \hat{\mathsf{H}}] = \hat{\mathsf{H}} \sum_{i=1}^{\ell-1} \mathsf{x}_i \tag{B.155}$$

denote the expectation of $\mathbf{y}$ conditioned on $(\mathbf{x}^{\ell-1}, \hat{\mathbf{H}}) = (\mathsf{x}^{\ell-1}, \hat{\mathsf{H}})$ and let $f_{\mathsf{x}^{\ell-1},\hat{\mathsf{H}}}(\cdot)$ denote some arbitrary complex-valued function. Conditioned on $(\mathbf{x}^{\ell-1}, \hat{\mathbf{H}}) = (\mathsf{x}^{\ell-1}, \hat{\mathsf{H}})$, the second entropy

[6] the general case of a possibly rank-deficient covariance may be treated by an appropriate rank reduction

can be upper-bounded as follows:

$$\begin{aligned} h(\mathbf{x}_\ell \mid \mathbf{x}^{\ell-1} = \mathsf{x}^{\ell-1}, \hat{\mathbf{H}} = \hat{\mathsf{H}}, \mathbf{y}) &= h\Big(\mathbf{x}_\ell - f_{\mathsf{x}^{\ell-1},\hat{\mathsf{H}}}(\mathbf{y} - \mu_{\mathbf{y}|\mathsf{x}^{\ell-1},\hat{\mathsf{H}}}) \;\Big|\; \mathbf{x}^{\ell-1} = \mathsf{x}^{\ell-1}, \hat{\mathbf{H}} = \hat{\mathsf{H}}, \mathbf{y}\Big) \\ &\le h\Big(\mathbf{x}_\ell - f_{\mathsf{x}^{\ell-1},\hat{\mathsf{H}}}(\mathbf{y} - \mu_{\mathbf{y}|\mathsf{x}^{\ell-1},\hat{\mathsf{H}}}) \;\Big|\; \mathbf{x}^{\ell-1} = \mathsf{x}^{\ell-1}, \hat{\mathbf{H}} = \hat{\mathsf{H}}\Big) \\ &\le \log\det\Big(\pi e \operatorname{cov}\Big(\tilde{\mathbf{x}}_\ell \;\Big|\; \mathsf{x}^{\ell-1}, \hat{\mathsf{H}}\Big)\Big) \\ &\le \log\det\Big(\pi e\, \mathsf{E}\Big[\tilde{\mathbf{x}}_\ell \tilde{\mathbf{x}}_\ell^\dagger \;\Big|\; \mathsf{x}^{\ell-1}, \hat{\mathsf{H}}\Big]\Big) \end{aligned} \tag{B.156}$$

where

$$\tilde{\mathbf{x}}_\ell \triangleq \mathbf{x}_\ell - f_{\mathsf{x}^{\ell-1},\hat{\mathsf{H}}}(\mathbf{y} - \mu_{\mathbf{y}|\mathsf{x}^{\ell-1},\hat{\mathsf{H}}}). \tag{B.157}$$

In (B.156) the first inequality follows because conditioning cannot increase entropy; the second inequality follows from the entropy-maximizing property of the Gaussian distribution; the third inequality follows because the covariance is not larger than the second moment (in a matrix-monotone sense).

Combining (B.156) with (B.154) and (B.153) thus yields for every $(\mathbf{x}^{\ell-1}, \hat{\mathbf{H}}) = (\mathsf{x}^{\ell-1}, \hat{\mathsf{H}})$ and $f_{\mathsf{x}^{\ell-1},\hat{\mathsf{H}}}(\cdot)$ the inequality

$$I(\mathbf{x}_\ell; \mathbf{y} \mid \mathbf{x}^{\ell-1} = \mathsf{x}^{\ell-1}, \hat{\mathbf{H}} = \hat{\mathsf{H}}) \ge \log\det(\mathbf{Q}_\ell) - \log\det\Big(\mathsf{E}\Big[\tilde{\mathbf{x}}_\ell \tilde{\mathbf{x}}_\ell^\dagger \;\Big|\; \mathsf{x}^{\ell-1}, \hat{\mathsf{H}}\Big]\Big). \tag{B.158}$$

We choose the function $f_{\mathsf{x}^{\ell-1},\hat{\mathsf{H}}}(\cdot)$ so that $f_{\mathsf{x}^{\ell-1},\hat{\mathsf{H}}}(\mathbf{y} - \mu_{\mathbf{y}|\mathsf{x}^{\ell-1},\hat{\mathsf{H}}})$ is the linear MMSE estimate of $\mathbf{x}_\ell$ from $(\mathbf{x}^{\ell-1}, \hat{\mathbf{H}}, \mathbf{y}) = (\mathsf{x}^{\ell-1}, \hat{\mathsf{H}}, \mathbf{y})$, namely,

$$f_{\mathsf{x}^{\ell-1},\hat{\mathsf{H}}}(\mathbf{y} - \mu_{\mathbf{y}|\mathsf{x}^{\ell-1},\hat{\mathsf{H}}}) = \mathbf{F}\mathbf{G}^{-1}(\mathbf{y} - \mu_{\mathbf{y}|\mathsf{x}^{\ell-1},\hat{\mathsf{H}}}) \tag{B.159}$$

with matrices $\mathbf{F}$ and $\mathbf{G}$ given by

$$\begin{aligned} \mathbf{F} &= \mathsf{E}\Big[\mathbf{x}_\ell \Big(\mathbf{y} - \mu_{\mathbf{y}|\mathsf{x}^{\ell-1},\hat{\mathsf{H}}}\Big)^\dagger \;\Big|\; \mathsf{x}^{\ell-1}, \hat{\mathsf{H}}\Big] \\ &= \sqrt{\rho}\hat{\mathsf{H}}\mathbf{Q}_\ell \end{aligned} \tag{B.160a}$$

$$\begin{aligned} \mathbf{G} &= \operatorname{cov}(\mathbf{y} \mid \mathsf{x}^{\ell-1}, \hat{\mathsf{H}}) \\ &= \mathsf{E}\Big[\Big(\mathbf{y} - \mu_{\mathbf{y}|\mathsf{x}^{\ell-1},\hat{\mathsf{H}}}\Big)\Big(\mathbf{y} - \mu_{\mathbf{y}|\mathsf{x}^{\ell-1},\hat{\mathsf{H}}}\Big)^\dagger \;\Big|\; \mathsf{x}^{\ell-1}, \hat{\mathsf{H}}\Big] \\ &= \rho\hat{\mathsf{H}}\mathbf{Q}_\ell\hat{\mathsf{H}}^\dagger + \operatorname{cov}(\tilde{\mathbf{z}} \mid \mathsf{x}^{\ell-1}, \hat{\mathsf{H}}) \end{aligned} \tag{B.160b}$$

respectively, and where $\tilde{\mathbf{z}} = \mathbf{y} - \sqrt{\rho}\hat{\mathsf{H}}\mathbf{x}_\ell$.

Inserting (B.159)–(B.160) into the expression of $\mathsf{E}[\tilde{\mathbf{x}}_\ell \tilde{\mathbf{x}}_\ell^\dagger \mid \mathsf{x}^{\ell-1}, \hat{\mathsf{H}}]$ yields

$$\begin{aligned} \mathsf{E}\Big[\tilde{\mathbf{x}}_\ell \tilde{\mathbf{x}}_\ell^\dagger \;\Big|\; \mathsf{x}^{\ell-1}, \hat{\mathsf{H}}\Big] &= \mathbf{Q}_\ell - \rho\hat{\mathsf{H}}\mathbf{Q}_\ell\Big(\rho\hat{\mathsf{H}}\mathbf{Q}_\ell\hat{\mathsf{H}}^\dagger + \operatorname{cov}(\tilde{\mathbf{z}} \mid \mathsf{x}^{\ell-1}, \hat{\mathsf{H}})\Big)\mathbf{Q}_\ell\hat{\mathsf{H}}^\dagger \\ &= \Big(\rho\hat{\mathsf{H}}^\dagger \operatorname{cov}(\tilde{\mathbf{z}} \mid \mathsf{x}^{\ell-1}, \hat{\mathsf{H}})^{-1}\hat{\mathsf{H}} + \mathbf{Q}_\ell^{-1}\Big)^{-1} \end{aligned} \tag{B.161}$$

where in the last step we have used the Matrix Inversion Lemma. Finally, combining (B.161) with (B.158) yields

$$I(\mathbf{x}_\ell; \mathbf{y} \mid \mathsf{x}^{\ell-1} = \mathbf{x}^{\ell-1}, \hat{\mathsf{H}} = \hat{\mathbf{H}}) \geq \log\det\left(\mathbf{I}_{n_\mathrm{T}} + \rho\hat{\mathbf{H}}^\dagger \operatorname{cov}(\tilde{\mathbf{z}} \mid \mathbf{x}^{\ell-1}, \hat{\mathbf{H}})^{-1}\hat{\mathbf{H}}\mathbf{Q}_\ell\right). \tag{B.162}$$

Theorem 2.4 follows by averaging (B.162) over $(\mathbf{x}^{\ell-1}, \hat{\mathbf{H}})$.

B.12 Proof of Theorem 2.5

It suffices to prove that if the $(L+1)$-indexing $\mathscr{I}'$ is a refinement of the L-indexing $\mathscr{I}$, then $R(\mathbf{L}, \mathscr{I}) \leq R(\mathbf{L}, \mathscr{I}')$. The general case where the refinement $\mathscr{I}'$ contains *any* number of elements $L' > L$ then follows by applying the result $(L' - L)$ times.

Let us write the indexings $\mathscr{I}$ and $\mathscr{I}'$ as

$$\mathscr{I} = \{0 = \iota_0, \ldots, \iota_L = 1\} \tag{B.163a}$$
$$\mathscr{I}' = \{0 = \iota'_0, \ldots, \iota'_\tau, \ldots, \iota'_{L+1} = 1\} \tag{B.163b}$$

with labels in ascending order, i.e., $\iota_0 < \ldots < \iota_L$ and $\iota'_0 < \ldots < \iota'_\tau < \ldots < \iota'_{L+1}$ respectively, such that the single element of $\mathscr{I}' \setminus \mathscr{I} = \{\iota'_\tau\}$ is located at position $\tau \in \{1, \ldots, L\}$. Let us further define

$$\mathbf{L}_\ell^{\mathscr{I}} \triangleq \mathbf{L}(\iota_\ell), \quad \ell = 0, \ldots, L \tag{B.164a}$$
$$\mathbf{L}_\ell^{\mathscr{I}'} \triangleq \mathbf{L}(\iota'_\ell), \quad \ell = 0, \ldots, L+1 \tag{B.164b}$$

as well as the differences

$$\Delta\mathbf{L}_\ell^{\mathscr{I}} \triangleq \mathbf{L}(\iota_\ell) - \mathbf{L}(\iota_{\ell-1}), \quad \ell = 1, \ldots, L \tag{B.165a}$$
$$\Delta\mathbf{L}_\ell^{\mathscr{I}'} \triangleq \mathbf{L}(\iota'_\ell) - \mathbf{L}(\iota'_{\ell-1}), \quad \ell = 1, \ldots, L+1. \tag{B.165b}$$

Since $\mathscr{I}'$ matches $\mathscr{I}$ except for the single element ι_τ, we have that

$$\mathbf{L}_\ell^{\mathscr{I}} = \mathbf{L}_\ell^{\mathscr{I}'}, \quad \ell = 0, \ldots, \tau - 1 \tag{B.166a}$$
$$\mathbf{L}_\ell^{\mathscr{I}} = \mathbf{L}_{\ell+1}^{\mathscr{I}'}, \quad \ell = \tau, \ldots, L \tag{B.166b}$$

as well as

$$\Delta\mathbf{L}_\ell^{\mathscr{I}} = \Delta\mathbf{L}_\ell^{\mathscr{I}'}, \quad \ell = 1, \ldots, \tau - 1 \tag{B.167a}$$
$$\Delta\mathbf{L}_\ell^{\mathscr{I}} = \Delta\mathbf{L}_{\ell+1}^{\mathscr{I}'}, \quad \ell = \tau + 1, \ldots, L. \tag{B.167b}$$

In particular, it follows from [cf. (2.59)]

$$\sum_{\ell=1}^{L} \Delta\mathbf{L}_\ell^{\mathscr{I}} = \sum_{\ell=1}^{L+1} \Delta\mathbf{L}_\ell^{\mathscr{I}'} = \mathbf{Q}$$

together with (B.165) that

$$\Delta\mathbf{L}_\tau^{\mathscr{I}} = \Delta\mathbf{L}_\tau^{\mathscr{I}'} + \Delta\mathbf{L}_{\tau+1}^{\mathscr{I}'}. \tag{B.168}$$

The rate-splitting bounds $R(\mathbf{L},\mathscr{I})$ and $R(\mathbf{L},\mathscr{I}')$ are respectively given by [cf. (2.67)]

$$R(\mathbf{L},\mathscr{I}) = \sum_{\ell=1}^{L} \mathsf{E}\log\det\Big(\mathbf{I}_{n_\mathsf{T}} + \hat{\mathbf{H}}^\dagger(\boldsymbol{\Gamma}_\ell^{\mathscr{I}})^{-1}\hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{\mathscr{I}}\Big) \tag{B.169a}$$

$$R(\mathbf{L},\mathscr{I}') = \sum_{\ell=1}^{L+1} \mathsf{E}\log\det\Big(\mathbf{I}_{n_\mathsf{T}} + \hat{\mathbf{H}}^\dagger(\boldsymbol{\Gamma}_\ell^{\mathscr{I}'})^{-1}\hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{\mathscr{I}}\Big) \tag{B.169b}$$

with random matrices $\boldsymbol{\Gamma}_\ell$ and $\boldsymbol{\Gamma}'_\ell$ defined respectively as [cf. (2.68)]

$$\begin{aligned}\boldsymbol{\Gamma}_\ell^{\mathscr{I}} &\triangleq \mathsf{E}[\tilde{\mathbf{H}}(\mathbf{L}_{\ell-1}^{\mathscr{I}})^{\frac12}\boldsymbol{\xi}\boldsymbol{\xi}^\dagger(\mathbf{L}_{\ell-1}^{\mathscr{I}})^{\frac12}\tilde{\mathbf{H}}^\dagger \mid \boldsymbol{\xi},\hat{\mathbf{H}}] + \\ &\quad + \hat{\mathbf{H}}(\mathbf{Q}-\mathbf{L}_\ell^{\mathscr{I}})\hat{\mathbf{H}}^\dagger + \mathsf{E}[\tilde{\mathbf{H}}(\mathbf{Q}-\mathbf{L}_{\ell-1}^{\mathscr{I}})\tilde{\mathbf{H}}^\dagger \mid \hat{\mathbf{H}}] + \rho^{-1}\mathbf{I}_{n_\mathsf{R}}, \quad \ell=1,\ldots,L\end{aligned} \tag{B.170a}$$

$$\begin{aligned}\boldsymbol{\Gamma}_\ell^{\mathscr{I}'} &\triangleq \mathsf{E}[\tilde{\mathbf{H}}(\mathbf{L}_{\ell-1}^{\mathscr{I}'})^{\frac12}\boldsymbol{\xi}\boldsymbol{\xi}^\dagger(\mathbf{L}_{\ell-1}^{\mathscr{I}'})^{\frac12}\tilde{\mathbf{H}}^\dagger \mid \boldsymbol{\xi},\hat{\mathbf{H}}] + \\ &\quad + \hat{\mathbf{H}}(\mathbf{Q}-\mathbf{L}_\ell^{\mathscr{I}'})\hat{\mathbf{H}}^\dagger + \mathsf{E}[\tilde{\mathbf{H}}(\mathbf{Q}-\mathbf{L}_{\ell-1}^{\mathscr{I}'})\tilde{\mathbf{H}}^\dagger \mid \hat{\mathbf{H}}] + \rho^{-1}\mathbf{I}_{n_\mathsf{R}}, \quad \ell=1,\ldots,L+1\end{aligned} \tag{B.170b}$$

with $\boldsymbol{\xi}\sim\mathcal{N}_\mathbb{C}(\mathbf{0},\mathbf{I}_{n_\mathsf{T}})$ being independent of $\hat{\mathbf{H}}$.

We see from (B.169)–(B.170) that the ℓ-th term of the sum (B.169a) depends on the layering $\mathbf{L},\mathscr{I}$ via $\mathbf{L}_{\ell-1}^{\mathscr{I}}$ and $\Delta\mathbf{L}_\ell^{\mathscr{I}}$, and similarly, that the ℓ-th term of the sum (B.169b) depends on the layering $(\mathbf{L},\mathscr{I}')$ via $\mathbf{L}_{\ell-1}^{\mathscr{I}'}$ and $\Delta\mathbf{L}_\ell^{\mathscr{I}'}$. As a consequence, by comparing the terms of the respective sums (B.169a) and (B.169b), we infer that the following terms coincide:

$$\mathsf{E}\log\det\Big(\mathbf{I}_{n_\mathsf{T}} + \hat{\mathbf{H}}^\dagger(\boldsymbol{\Gamma}_\ell^{\mathscr{I}})^{-1}\hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{\mathscr{I}}\Big) = \mathsf{E}\log\det\Big(\mathbf{I}_{n_\mathsf{T}} + \hat{\mathbf{H}}^\dagger(\boldsymbol{\Gamma}_\ell^{\mathscr{I}'})^{-1}\hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{\mathscr{I}'}\Big), \quad \ell=1,\ldots,\tau-1 \tag{B.171}$$

$$\mathsf{E}\log\det\Big(\mathbf{I}_{n_\mathsf{T}} + \hat{\mathbf{H}}^\dagger(\boldsymbol{\Gamma}_\ell^{\mathscr{I}})^{-1}\hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{\mathscr{I}}\Big) = \mathsf{E}\log\det\Big(\mathbf{I}_{n_\mathsf{T}} + \hat{\mathbf{H}}^\dagger(\boldsymbol{\Gamma}_{\ell+1}^{\mathscr{I}'})^{-1}\hat{\mathbf{H}}\,\Delta\mathbf{L}_{\ell+1}^{\mathscr{I}'}\Big), \quad \ell=\tau+1,\ldots,L. \tag{B.172}$$

Subtracting $R(\mathbf{L},\mathscr{I})$ from $R(\mathbf{L},\mathscr{I}')$, these identical terms cancel out, leaving us with

$$\begin{aligned}R(\mathbf{L},\mathscr{I}') - R(\mathbf{L},\mathscr{I}) &= \mathsf{E}\log\det\Big(\mathbf{I}_{n_\mathsf{T}} + \hat{\mathbf{H}}^\dagger(\boldsymbol{\Gamma}_\tau^{\mathscr{I}'})^{-1}\hat{\mathbf{H}}\,\Delta\mathbf{L}_\tau^{\mathscr{I}'}\Big) \\ &+ \mathsf{E}\log\det\Big(\mathbf{I}_{n_\mathsf{T}} + \hat{\mathbf{H}}^\dagger(\boldsymbol{\Gamma}_{\tau+1}^{\mathscr{I}'})^{-1}\hat{\mathbf{H}}\,\Delta\mathbf{L}_{\tau+1}^{\mathscr{I}'}\Big) - \mathsf{E}\log\det\Big(\mathbf{I}_{n_\mathsf{T}} + \hat{\mathbf{H}}^\dagger(\boldsymbol{\Gamma}_\tau^{\mathscr{I}})^{-1}\hat{\mathbf{H}}\,\Delta\mathbf{L}_\tau^{\mathscr{I}}\Big).\end{aligned} \tag{B.173}$$

To show that this quantity is non-negative, we will lower-bound the second term on the right-hand side of (B.173), which involves the random matrix $\boldsymbol{\Gamma}_{\tau+1}^{\mathscr{I}'}$ [cf. (B.170)], which itself involves a Gaussian random vector $\boldsymbol{\xi}\sim\mathcal{N}_\mathbb{C}(\mathbf{0},\mathbf{I}_{n_\mathsf{T}})$. Upon observing that, due to $\mathbf{L}_\tau^{\mathscr{I}'} = \mathbf{L}_{\tau-1}^{\mathscr{I}'} + \Delta\mathbf{L}_\tau^{\mathscr{I}'}$, the random matrix $(\mathbf{L}_\tau^{\mathscr{I}'})^{\frac12}\boldsymbol{\xi}\sim\mathcal{N}_\mathbb{C}(\mathbf{0},\mathbf{L}_\tau^{\mathscr{I}'})$ has the same marginal distribution as

$$(\mathbf{L}_{\tau-1}^{\mathscr{I}'})^{\frac12}\boldsymbol{\xi} + (\Delta\mathbf{L}_\tau^{\mathscr{I}'})^{\frac12}\boldsymbol{\eta} \;\sim\; \mathcal{N}_\mathbb{C}(\mathbf{0},\mathbf{L}_\tau^{\mathscr{I}'})$$

we can replace $\boldsymbol{\Gamma}_{\tau+1}^{\mathscr{I}'}$ on the right-hand side of Expression (B.173) with a matrix of same distribution

$$\bar{\boldsymbol{\Gamma}}_{\tau+1}^{\mathscr{I}'} \triangleq \mathsf{E}\Big[\tilde{\mathbf{H}}\Big((\mathbf{L}_{\tau-1}^{\mathscr{I}'})^{\frac{1}{2}}\boldsymbol{\xi} + (\Delta\mathbf{L}_{\tau}^{\mathscr{I}'})^{\frac{1}{2}}\boldsymbol{\eta}\Big)\Big((\mathbf{L}_{\tau-1}^{\mathscr{I}'})^{\frac{1}{2}}\boldsymbol{\xi} + (\Delta\mathbf{L}_{\tau}^{\mathscr{I}'})^{\frac{1}{2}}\boldsymbol{\eta}\Big)^{\dagger}\tilde{\mathbf{H}}^{\dagger} \;\Big|\; \boldsymbol{\xi}, \hat{\mathbf{H}}\Big] + \\ + \hat{\mathbf{H}}(\mathbf{Q} - \mathbf{L}_{\tau-1}^{\mathscr{I}'} - \Delta\mathbf{L}_{\tau}^{\mathscr{I}'} - \Delta\mathbf{L}_{\tau+1}^{\mathscr{I}'})\hat{\mathbf{H}}^{\dagger} + \mathsf{E}\Big[\tilde{\mathbf{H}}(\mathbf{Q} - \mathbf{L}_{\tau-1}^{\mathscr{I}'} - \Delta\mathbf{L}_{\tau}^{\mathscr{I}'})\tilde{\mathbf{H}}^{\dagger} \;\Big|\; \hat{\mathbf{H}}\Big] + \rho^{-1}\mathbf{I}_{n_{\mathsf{R}}}. \quad \text{(B.174)}$$

wherein $\boldsymbol{\eta} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I}_{n_{\mathsf{T}}})$ is independent of $(\boldsymbol{\xi}, \hat{\mathbf{H}})$. Using the fact that, for every $\mathbf{A} \succeq \mathbf{0}$, the function $\mathbf{X} \mapsto \log\det(\mathbf{I} + \mathbf{A}\mathbf{X}^{-1})$ is strictly convex on the set of positive definite matrices $\mathbf{X} \succ \mathbf{0}$, it follows from Jensen's inequality that the second term on the right-hand side of (B.173) is lower-bounded as

$$\begin{aligned} &\mathsf{E}\log\det\Big(\mathbf{I}_{n_{\mathsf{T}}} + \hat{\mathbf{H}}^{\dagger}(\boldsymbol{\Gamma}_{\tau+1}^{\mathscr{I}'})^{-1}\hat{\mathbf{H}}\,\Delta\mathbf{L}_{\tau+1}^{\mathscr{I}'}\Big) \\ &\qquad = \mathsf{E}\log\det\Big(\mathbf{I}_{n_{\mathsf{T}}} + \hat{\mathbf{H}}^{\dagger}(\bar{\boldsymbol{\Gamma}}_{\tau+1}^{\mathscr{I}'})^{-1}\hat{\mathbf{H}}\,\Delta\mathbf{L}_{\tau+1}^{\mathscr{I}'}\Big) \\ &\qquad \geq \mathsf{E}\log\det\Big(\mathbf{I}_{n_{\mathsf{T}}} + \hat{\mathbf{H}}^{\dagger}\,\mathsf{E}[\bar{\boldsymbol{\Gamma}}_{\tau+1}^{\mathscr{I}'} \mid \boldsymbol{\xi}, \hat{\mathbf{H}}]^{-1}\hat{\mathbf{H}}\,\Delta\mathbf{L}_{\tau+1}^{\mathscr{I}'}\Big). \end{aligned} \quad \text{(B.175)}$$

Using the identities $\mathbf{L}_{\tau-1}^{\mathscr{I}} = \mathbf{L}_{\tau-1}^{\mathscr{I}'}$ [cf. (B.166a)] and (B.168), we find by comparison of (B.170a) and (B.174) that

$$\mathsf{E}[\bar{\boldsymbol{\Gamma}}_{\tau+1}^{\mathscr{I}'} \mid \boldsymbol{\xi}, \hat{\mathbf{H}}] = \boldsymbol{\Gamma}_{\tau}^{\mathscr{I}} \quad \text{(B.176)}$$

and by comparison of (B.170a) and (B.170b) that

$$\boldsymbol{\Gamma}_{\tau}^{\mathscr{I}'} = \boldsymbol{\Gamma}_{\tau}^{\mathscr{I}} + \hat{\mathbf{H}}\Delta\mathbf{L}_{\tau+1}^{\mathscr{I}'}\hat{\mathbf{H}}^{\dagger}. \quad \text{(B.177)}$$

Hence, combining (B.173), (B.175), (B.176) and (B.177), we have

$$\begin{aligned} R(\mathbf{L}, \mathscr{I}') - R(\mathbf{L}, \mathscr{I}) &\geq \mathsf{E}\log\det\Big(\mathbf{I}_{n_{\mathsf{T}}} + \hat{\mathbf{H}}^{\dagger}(\boldsymbol{\Gamma}_{\tau}^{\mathscr{I}} + \hat{\mathbf{H}}\Delta\mathbf{L}_{\tau+1}^{\mathscr{I}'}\hat{\mathbf{H}}^{\dagger})^{-1}\hat{\mathbf{H}}\,\Delta\mathbf{L}_{\tau}^{\mathscr{I}'}\Big) \\ &\qquad + \mathsf{E}\log\det\Big(\mathbf{I}_{n_{\mathsf{T}}} + \hat{\mathbf{H}}^{\dagger}(\boldsymbol{\Gamma}_{\tau}^{\mathscr{I}})^{-1}\hat{\mathbf{H}}\,\Delta\mathbf{L}_{\tau+1}^{\mathscr{I}'}\Big) \\ &\quad - \mathsf{E}\log\det\Big(\mathbf{I}_{n_{\mathsf{T}}} + \hat{\mathbf{H}}^{\dagger}(\boldsymbol{\Gamma}_{\tau}^{\mathscr{I}})^{-1}\hat{\mathbf{H}}\,\Delta\mathbf{L}_{\tau}^{\mathscr{I}}\Big) \\ &= 0 \end{aligned} \quad \text{(B.178)}$$

where the last equality follows again from the identity (B.168). This concludes the proof of the inequality (2.72).

B.13 Proof of the Lipschitz Property of Layering Functions (2.63)

Due to $\operatorname{tr}(\mathbf{L}(\iota)) = \sum_{i=1}^{n_{\mathsf{T}}} L_{i,i}(\iota) = \iota$ [cf. Definition 2.1], it follows that

$$\begin{aligned} \operatorname{tr}(\mathbf{L}(\iota_2)) - \operatorname{tr}(\mathbf{L}(\iota_1)) &= \sum_{i=1}^{n_{\mathsf{T}}}(L_{i,i}(\iota_2) - L_{i,i}(\iota_1)) \\ &= \iota_2 - \iota_1. \end{aligned} \quad \text{(B.179)}$$

The positive semidefiniteness of $\mathbf{L}(\iota)$ [cf. Definition 2.1] warrants that its diagonal entries $L_{i,i}(\iota)$ are real-valued and non-negative, and the matrix-monotonicity of layering functions [cf. (3)] implies that $L_{i,i}(\iota)$ is non-decreasing. Therefore, for any $0 \leq \iota_1 \leq \iota_2 \leq 1$ we have

$$0 \leq L_{i,i}(\iota_2) - L_{i,i}(\iota_1) \leq \iota_2 - \iota_1 \tag{B.180}$$

from which the claim follows for the diagonal entries $L_{i,i}(\iota)$.

As regards the off-diagonal entries $L_{j,i}(\iota)$ (with $i \neq j$), we proceed as follows: given that $\mathbf{L}(\iota_2) - \mathbf{L}(\iota_1)$ is positive semidefinite, all its principal minors are non-negative [Hor90, Corollary 7.1.5], i.e.,

$$\det\left(\begin{bmatrix} L_{i,i}(\iota_2) - L_{i,i}(\iota_1) & L^*_{j,i}(\iota_2) - L^*_{j,i}(\iota_1) \\ L_{j,i}(\iota_2) - L_{j,i}(\iota_1) & L_{j,j}(\iota_2) - L_{j,j}(\iota_1) \end{bmatrix}\right) \geq 0. \tag{B.181}$$

This translates to

$$\begin{aligned} \left|L_{j,i}(\iota_2) - L_{j,i}(\iota_1)\right| &\leq \sqrt{\left(L_{i,i}(\iota_2) - L_{i,i}(\iota_1)\right)\left(L_{j,j}(\iota_2) - L_{j,j}(\iota_1)\right)} \\ &\leq \frac{\left(L_{i,i}(\iota_2) - L_{i,i}(\iota_1)\right) + \left(L_{j,j}(\iota_2) - L_{j,j}(\iota_1)\right)}{2} \\ &\leq \iota_2 - \iota_1 \end{aligned} \tag{B.182}$$

where the second bounding step is due to the inequality between the geometric and the arithmetic mean, whereas the last bounding step follows from (B.180). In particular, it follows that

$$\max\Big\{\left|\Re L_{j,i}(\iota_2) - \Re L_{j,i}(\iota_1)\right|, \left|\Im L_{j,i}(\iota_2) - \Im L_{j,i}(\iota_1)\right|\Big\} \leq \iota_2 - \iota_1. \tag{B.183}$$

Hence, both the real and the imaginary part of any entry $L_{j,i}(\iota)$ of the layering function $\mathbf{L}(\iota)$ is Lipschitz-continuous with Lipschitz constant 1. This concludes the proof.

B.14 Proof of Theorem 2.6

Let the transmit covariance $\mathbf{Q}$ and the layering function $\mathbf{L}$ be fixed throughout. Given some L-indexing $\mathscr{I} = (0, \iota_1, \ldots, \iota_{L-1}, 1)$, consider the uniquely defined, piecewise linear function $\Phi_{\mathscr{I}} \colon [0;1] \to [0;1]$ with support points $(\frac{\ell}{L}, \iota_\ell), \ell = 0, \ldots, L$. This is to say that the function $\Phi_{\mathscr{I}}$ is defined as

$$\Phi_{\mathscr{I}}(x) = (\iota_\ell - \iota_{\ell-1})\,(Lx - \ell + 1) + \iota_{\ell-1}, \quad \text{for } \frac{\ell-1}{L} \leq x \leq \frac{\ell}{L},\ \ell = 0, \ldots, L. \tag{B.184}$$

We define the equi-power N-indexing as

$$\mathscr{E}_N \triangleq \left(0, \frac{1}{N}, \ldots, \frac{N-1}{N}, 1\right) \tag{B.185}$$

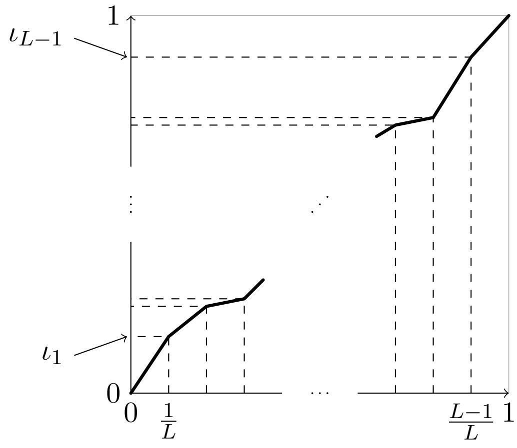


Figure B.1 Example of a piecewise linear function $\Phi_{\mathscr{I}}$

with the help of which we can express the indexing $\mathscr{I}$ as

$$\mathscr{I} = \Phi_{\mathscr{I}}(\mathscr{E}_L) \tag{B.186}$$

where the notation should be understood as an elementwise application of the function $\Phi_{\mathscr{I}}$, i.e.,

$$\Phi_{\mathscr{I}}(\mathscr{E}_L) = \left(\Phi_{\mathscr{I}}(0), \Phi_{\mathscr{I}}\left(\frac{1}{L}\right), \ldots, \Phi_{\mathscr{I}}\left(\frac{L-1}{L}\right), \Phi_{\mathscr{I}}(1)\right). \tag{B.187}$$

This way, we can express the rate-splitting bound $R(\mathbf{L}, \mathscr{I})$ in terms of an equi-power indexing, namely,

$$R(\mathbf{L}, \mathscr{I}) = R\left(\mathbf{L}, \Phi_{\mathscr{I}}(\mathscr{E}_L)\right). \tag{B.188}$$

Clearly, the function $\Phi_{\mathscr{I}}$, like any increasing bijection of the unit interval onto itself, preserves indexings. Specifically, for any indexings $(\mathscr{I}_1, \mathscr{I}_2) \in \mathbb{I}^2$, we have that $\Phi_{\mathscr{I}_1}(\mathscr{I}_2) \in \mathbb{I}$. Noting that $\mathscr{E}_N \subset \mathscr{E}_{2N}$ for any $N \in \mathbb{N}$, by virtue of Theorem 2.5, the sequence $R(\mathbf{L}, \mathscr{E}_{2^n L})$ is non-decreasing in n. Therefore, $R(\mathbf{L}, \mathscr{I})$ can be upper-bounded as

$$\begin{aligned} R(\mathbf{L}, \mathscr{I}) &= R\left(\mathbf{L}, \Phi_{\mathscr{I}}(\mathscr{E}_L)\right) \\ &\leq \lim_{n\to\infty} R\left(\mathbf{L}, \Phi_{\mathscr{I}}(\mathscr{E}_{2^n L})\right) \\ &\triangleq R^{\infty}(\mathbf{L}, \mathscr{I}). \end{aligned} \tag{B.189}$$

Since $R^{\infty}(\mathbf{L}, \mathscr{I})$ is the limit of a sequence of rate-splitting bounds with fixed layering function and varying indexing, it must therefore be upper-bounded by the supremum

$$R^{\infty}(\mathbf{L}, \mathscr{I}) \leq \sup_{\mathscr{I}' \in \mathbb{I}} R(\mathbf{L}, \mathscr{I}') = R^{\star}(\mathbf{L}). \tag{B.190}$$

In the following, we evaluate the limit $R^{\infty}(\mathbf{L}, \mathscr{I})$ and then show that it is not a function of the indexing $\mathscr{I}$. Therefore, it eventually follows that

$$R^{\star}(\mathbf{L}) = R^{\infty}(\mathbf{L}, \mathscr{I}). \tag{B.191}$$

To prove this claim, it will thus suffice to compute an analytical expression of the limit $R^{\infty}(\mathbf{L}, \mathscr{I})$ as per (B.189), and then verify that it is not a function of the indexing $\mathscr{I}$.

For notational brevity, let us denote

$$\tilde{\mathscr{I}}_{2^n L} \triangleq \Phi_{\mathscr{I}}(\mathscr{E}_{2^n L}) \tag{B.192}$$

so we have

$$\mathscr{I} = \tilde{\mathscr{I}}_{L} \subset \tilde{\mathscr{I}}_{2L} \subset \tilde{\mathscr{I}}_{4L} \subset \tilde{\mathscr{I}}_{8L} \subset \dots \tag{B.193}$$

Note as well that the piecewise linear functions are constructed in such way that the following holds:

$$\Phi_{\mathscr{I}} = \Phi_{\tilde{\mathscr{I}}_{2^n L}} \quad n \in \mathbb{N}. \tag{B.194}$$

By writing the channel input as a sum

$$\mathbf{x}_{\mathrm{G}} = \sum_{\ell=1}^{2^n L} \mathbf{x}_{\ell}$$

of $2^n L$ mutually independent complex circularly-symmetric zero-mean Gaussian variables $\mathbf{x}_{\ell}$ of covariance [cf. (2.65)]

$$\mathsf{E}\big[\mathbf{x}_{\ell}\mathbf{x}_{\ell}^{\dagger}\big] = \Delta\mathbf{L}_{\ell}^{\tilde{\mathscr{I}}_{2^n L}}, \quad \ell = 1, \dots, 2^n L \tag{B.195}$$

and using Expression (2.67), we can write out $R^{\infty}(\mathbf{L}, \mathscr{I})$ as

$$R^{\infty}(\mathbf{L}, \mathscr{I}) = \lim_{n\to\infty} \sum_{\ell=1}^{2^n L} \mathsf{E}\left[\log\det\left(\mathbf{I}_{n_{\mathsf{T}}} + \hat{\mathbf{H}}^{\dagger}\left(\boldsymbol{\Upsilon}_{\ell}^{\tilde{\mathscr{I}}_{2^n L}}\right)^{-1} \hat{\mathbf{H}}\,\Delta\mathbf{L}_{\ell}^{\tilde{\mathscr{I}}_{2^n L}}\right)\right] \tag{B.196}$$

with the random matrix $\boldsymbol{\Upsilon}_{\ell}^{\mathscr{A}}$ being defined for an indexing $\mathscr{A}$ as in (2.68), i.e.,

$$\boldsymbol{\Upsilon}_{\ell}^{\mathscr{A}} \triangleq \mathsf{E}\left[\tilde{\mathbf{H}}(\mathbf{L}_{\ell-1}^{\mathscr{A}})^{\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}^{\dagger}(\mathbf{L}_{\ell-1}^{\mathscr{A}})^{\frac{1}{2}}\tilde{\mathbf{H}}^{\dagger}\middle|\boldsymbol{\xi}, \hat{\mathbf{H}}\right] + \hat{\mathbf{H}}\bar{\mathbf{L}}_{\ell}^{\mathscr{A}}\hat{\mathbf{H}}^{\dagger} + \mathsf{E}\left[\tilde{\mathbf{H}}\bar{\mathbf{L}}_{\ell-1}^{\mathscr{A}}\tilde{\mathbf{H}}^{\dagger} \,\middle|\, \hat{\mathbf{H}}\right] + \rho^{-1}\mathbf{I}_{n_{\mathsf{R}}} \tag{B.197}$$

with $\boldsymbol{\xi} \sim \mathscr{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I}_{n_{\mathsf{T}}})$ independent of $\hat{\mathbf{H}}$.

Given that Theorem 2.5 holds irrespective of the distribution of $\hat{\mathbf{H}}$, we have that for any $\ell = 1, \dots, 2^n L$, the conditional expectation

$$\mathsf{E}\left[\log\det\left(\mathbf{I}_{n_{\mathsf{T}}} + \hat{\mathsf{H}}^{\dagger}\left(\boldsymbol{\Upsilon}_{\ell}^{\tilde{\mathscr{I}}_{2^n L}}\right)^{-1} \hat{\mathsf{H}}\,\Delta\mathbf{L}_{\ell}^{\tilde{\mathscr{I}}_{2^n L}}\right)\middle|\hat{\mathsf{H}}\right]$$

is non-decreasing in n (for any $\hat{\mathbf{H}} = \hat{\mathsf{H}}$), and therefore, by the Monotone Convergence Theorem [Rud87], we can exchange the limit and expectation over $\hat{\mathbf{H}}$ in Expression (B.196) to get

$$R^{\infty}(\mathbf{L}, \mathscr{I}) = \mathsf{E}\left[\lim_{n\to\infty} \mathsf{E}\left[\sum_{\ell=1}^{2^n L} \log\det\left(\mathbf{I}_{n_\mathsf{T}} + \hat{\mathbf{H}}^\dagger \left(\boldsymbol{\Upsilon}_\ell^{\tilde{\mathscr{I}}_{2^n L}}\right)^{-1} \hat{\mathbf{H}}\, \Delta\mathbf{L}_\ell^{\tilde{\mathscr{I}}_{2^n L}}\right) \middle| \hat{\mathbf{H}}\right]\right] \tag{B.198}$$

if this pointwise limit exists. Using the fact that $\boldsymbol{\Upsilon}_\ell^{\tilde{\mathscr{I}}_{2^n L}} \succeq \rho^{-1}\mathbf{I}_{n_\mathsf{R}}$, and that $\log\det(\mathbf{I}+\mathbf{A}) \le \operatorname{tr}(\mathbf{A})$ for a positive semidefinite matrix $\mathbf{A}$, we can upper-bound the sum of log-determinants in (B.198) by means of[7]

$$\begin{aligned}\sum_{\ell=1}^{2^n L} \log\det\left(\mathbf{I}_{n_\mathsf{T}} + \hat{\mathbf{H}}^\dagger \left(\boldsymbol{\Upsilon}_\ell^{\tilde{\mathscr{I}}_{2^n L}}\right)^{-1} \hat{\mathbf{H}}\, \Delta\mathbf{L}_\ell^{\tilde{\mathscr{I}}_{2^n L}}\right) &\le \rho \sum_{\ell=1}^{2^n L} \operatorname{tr}\left(\hat{\mathbf{H}}^\dagger \hat{\mathbf{H}}\, \Delta\mathbf{L}_\ell^{\tilde{\mathscr{I}}_{2^n L}}\right) \\ &= \rho \operatorname{tr}\left(\hat{\mathbf{H}}^\dagger \hat{\mathbf{H}} \mathbf{Q}\right).\end{aligned} \tag{B.199}$$

Thus, the pointwise limit in (B.198) exists. Furthermore, we can exploit this boundedness to apply the Dominated Convergence Theorem [Rud87], whereby we can interchange the limit and inner expectation (over $\boldsymbol{\xi}$) in (B.198) so as to obtain

$$R^{\infty}(\mathbf{L}, \mathscr{I}) = \mathsf{E}\left[\lim_{n\to\infty} \sum_{\ell=1}^{2^n L} \log\det\left(\mathbf{I}_{n_\mathsf{T}} + \hat{\mathbf{H}}^\dagger \left(\boldsymbol{\Upsilon}_\ell^{\tilde{\mathscr{I}}_{2^n L}}\right)^{-1} \hat{\mathbf{H}}\, \Delta\mathbf{L}_\ell^{\tilde{\mathscr{I}}_{2^n L}}\right)\right] \tag{B.200}$$

provided that this limit exists. As we shall see in the following, it does exist and can be represented as a Riemann integral. To prove its existence, we will lower-bound the corresponding *limit inferior* and upper-bound the corresponding *limit superior*, and eventually show that both limits coincide.

Let us define

$$\boldsymbol{\Upsilon}_\ell^{(n)} \triangleq \boldsymbol{\Upsilon}_\ell^{\tilde{\mathscr{I}}_{2^n L}} \qquad \Delta\mathbf{L}_\ell^{(n)} \triangleq \Delta\mathbf{L}_\ell^{\tilde{\mathscr{I}}_{2^n L}} \qquad \mathbf{L}_\ell^{(n)} \triangleq \mathbf{L}_\ell^{\tilde{\mathscr{I}}_{2^n L}} \tag{B.201}$$

to alleviate notation. With this notation, the problem at hand is to compute the two limits

$$\varliminf_{n\to\infty} \sum_{\ell=1}^{2^n L} \log\det\left(\mathbf{I}_{n_\mathsf{T}} + \hat{\mathbf{H}}^\dagger \left(\boldsymbol{\Upsilon}_\ell^{(n)}\right)^{-1} \hat{\mathbf{H}}\, \Delta\mathbf{L}_\ell^{(n)}\right) \tag{B.202a}$$

$$\varlimsup_{n\to\infty} \sum_{\ell=1}^{2^n L} \log\det\left(\mathbf{I}_{n_\mathsf{T}} + \hat{\mathbf{H}}^\dagger \left(\boldsymbol{\Upsilon}_\ell^{(n)}\right)^{-1} \hat{\mathbf{H}}\, \Delta\mathbf{L}_\ell^{(n)}\right). \tag{B.202b}$$

and show that they coincide.

An upper bound on the limit superior (B.202b) is obtained by using the inequality $\log\det(\mathbf{I}+$

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[7] Note that for same-sized positive semidefinite matrices $\mathbf{A}$ and $\mathbf{B}$, we have $\log\det(\mathbf{I}+\mathbf{AB}) = \log\det\left(\mathbf{I}+\mathbf{A}^{\frac{1}{2}}\mathbf{B}\mathbf{A}^{\frac{1}{2}}\right) \le \operatorname{tr}\left(\mathbf{A}^{\frac{1}{2}}\mathbf{B}\mathbf{A}^{\frac{1}{2}}\right) = \operatorname{tr}(\mathbf{AB})$

$\mathbf{A}) \le \operatorname{tr}(\mathbf{A})$, valid for $\mathbf{A} \succeq \mathbf{0}$, leading to

$$\begin{aligned}\varlimsup_{n\to\infty} \sum_{\ell=1}^{2^n L} \log\det\Big(\mathbf{I}_{n_\mathsf{T}} + \hat{\mathbf{H}}^\dagger \left(\boldsymbol{\Upsilon}_\ell^{(n)}\right)^{-1} \hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big) \\ \le \varlimsup_{n\to\infty} \sum_{\ell=1}^{2^n L} \operatorname{tr}\Big(\hat{\mathbf{H}}^\dagger \left(\boldsymbol{\Upsilon}_\ell^{(n)}\right)^{-1} \hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big).\end{aligned} \tag{B.203}$$

Similarly, a lower bound on the limit inferior (B.202a) is obtained by using the inequality $\operatorname{tr}(\mathbf{A}) - \frac{1}{2}\operatorname{tr}(\mathbf{A}^2) \le \log\det(\mathbf{I} + \mathbf{A})$, valid for $\mathbf{A} \succeq \mathbf{0}$, and the superadditivity of the limit inferior. It reads as[8]

$$\begin{aligned}\varliminf_{n\to\infty} \sum_{\ell=1}^{2^n L} \log\det\Big(\mathbf{I}_{n_\mathsf{T}} + \hat{\mathbf{H}}^\dagger \left(\boldsymbol{\Upsilon}_\ell^{(n)}\right)^{-1} \hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big) \\ \ge \varliminf_{n\to\infty} \sum_{\ell=1}^{2^n L} \operatorname{tr}\Big(\hat{\mathbf{H}}^\dagger \left(\boldsymbol{\Upsilon}_\ell^{(n)}\right)^{-1} \hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big) \\ - \frac{1}{2}\varlimsup_{n\to\infty} \sum_{\ell=1}^{2^n L} \operatorname{tr}\Big(\hat{\mathbf{H}}^\dagger \left(\boldsymbol{\Upsilon}_\ell^{(n)}\right)^{-1} \hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{(n)} \hat{\mathbf{H}}^\dagger \left(\boldsymbol{\Upsilon}_\ell^{(n)}\right)^{-1} \hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big).\end{aligned} \tag{B.204}$$

Here, the limit superior [second term on the right-hand side of (B.204)] vanishes. This is best seen if, using $\boldsymbol{\Upsilon}_\ell^{(n)} \succeq \rho^{-1}\mathbf{I}_{n_\mathsf{R}}$ and the inequality $\operatorname{tr}(\mathbf{AB}) \le \operatorname{tr}(\mathbf{A})\operatorname{tr}(\mathbf{B})$ for same-sized positive semidefinite $\mathbf{A}$ and $\mathbf{B}$, we bound it via

$$\begin{aligned}0 &\le \operatorname{tr}\Big(\hat{\mathbf{H}}^\dagger \left(\boldsymbol{\Upsilon}_\ell^{(n)}\right)^{-1} \hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{(n)} \hat{\mathbf{H}}^\dagger \left(\boldsymbol{\Upsilon}_\ell^{(n)}\right)^{-1} \hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big) \\ &\le \rho^2 \operatorname{tr}\Big(\hat{\mathbf{H}}^\dagger\hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\hat{\mathbf{H}}^\dagger\hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big) \\ &\le \rho^2 \operatorname{tr}\Big(\hat{\mathbf{H}}^\dagger\hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big)^2.\end{aligned} \tag{B.205}$$

By the Heine-Cantor Theorem [Rud76, Theorem 4.19], the continuous mapping

$$[0;1] \to \mathbb{R},\ \iota \mapsto \operatorname{tr}\Big(\hat{\mathbf{H}}^\dagger\hat{\mathbf{H}}\mathbf{L}(\iota)\Big)$$

must be uniformly continuous due to $[0;1]$ being a compact set. It follows that for every $\epsilon > 0$, there exists a large enough n such that for any $\ell \in \{1, \dots, 2^n L\}$, we have $\operatorname{tr}\Big(\hat{\mathbf{H}}^\dagger\hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big) < \epsilon$, hence

$$\begin{aligned}\rho^2 \operatorname{tr}\Big(\hat{\mathbf{H}}^\dagger\hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big)^2 &\le \rho^2\epsilon \sum_{\ell=1}^{2^n L} \operatorname{tr}\Big(\hat{\mathbf{H}}^\dagger\hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big) \\ &= \rho^2\epsilon \operatorname{tr}\Big(\hat{\mathbf{H}}^\dagger\hat{\mathbf{H}}\mathbf{Q}\Big).\end{aligned} \tag{B.206}$$

Since $\epsilon > 0$ can be arbitrarily small, we conclude from (B.205)–(B.206) that the limit superior

[8] Note that for same-sized positive semidefinite $\mathbf{A}$ and $\mathbf{B}$, we have $\log\det(\mathbf{I}+\mathbf{AB}) = \log\det\Big(\mathbf{I}+\mathbf{A}^{\frac{1}{2}}\mathbf{B}\mathbf{A}^{\frac{1}{2}}\Big) \ge \operatorname{tr}\Big(\mathbf{A}^{\frac{1}{2}}\mathbf{B}\mathbf{A}^{\frac{1}{2}}\Big) - \operatorname{tr}\Big(\mathbf{A}^{\frac{1}{2}}\mathbf{B}\mathbf{A}^{\frac{1}{2}} \cdot \mathbf{A}^{\frac{1}{2}}\mathbf{B}\mathbf{A}^{\frac{1}{2}}\Big) = \operatorname{tr}(\mathbf{AB}) - \operatorname{tr}(\mathbf{ABAB})$.

in (B.204) vanishes indeed. Hence (B.204) reduces to

$$\varliminf_{n\to\infty}\sum_{\ell=1}^{2^n L}\log\det\Big(\mathbf{I}_{n_\mathsf{T}}+\hat{\mathsf{H}}^\dagger\left(\boldsymbol{\Upsilon}_\ell^{(n)}\right)^{-1}\hat{\mathsf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big)\geq\varliminf_{n\to\infty}\sum_{\ell=1}^{2^n L}\mathrm{tr}\Big(\hat{\mathsf{H}}^\dagger\left(\boldsymbol{\Upsilon}_\ell^{(n)}\right)^{-1}\hat{\mathsf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big).\tag{B.207}$$

Next, we focus on computing the right-hand sides of (B.203) and (B.207), which will turn out to coincide. For this purpose, let us define, for any indexing $\mathscr{A}\in\mathbb{I}(L_\mathscr{A})$, the random matrix [cf. (B.197)]

$$\begin{aligned}\widetilde{\boldsymbol{\Upsilon}}_\ell^{\mathscr{A}}&\triangleq\boldsymbol{\Upsilon}_\ell^{\mathscr{A}}+\hat{\mathbf{H}}\Delta\mathbf{L}_\ell\hat{\mathbf{H}}^\dagger\\&=\mathsf{E}\left[\tilde{\mathbf{H}}(\mathbf{L}_{\ell-1}^{\mathscr{A}})^{\frac12}\boldsymbol{\xi}\boldsymbol{\xi}^\dagger(\mathbf{L}_{\ell-1}^{\mathscr{A}})^{\frac12}\tilde{\mathbf{H}}^\dagger\middle|\boldsymbol{\xi},\hat{\mathbf{H}}\right]+\hat{\mathbf{H}}\bar{\mathbf{L}}_{\ell-1}^{\mathscr{A}}\hat{\mathbf{H}}^\dagger+\mathsf{E}\Big[\tilde{\mathbf{H}}\bar{\mathbf{L}}_{\ell-1}^{\mathscr{A}}\tilde{\mathbf{H}}^\dagger\;\Big|\;\hat{\mathsf{H}}\Big]+\rho^{-1}\mathbf{I}_{n_\mathsf{R}}.\end{aligned}\tag{B.208}$$

Notice that $\widetilde{\boldsymbol{\Upsilon}}_\ell^{\mathscr{A}}$ was constructed so as to fulfill [cf. (2.82)]

$$\widetilde{\boldsymbol{\Upsilon}}_\ell^{\mathscr{A}}=\boldsymbol{\Gamma}\left(\mathbf{L}_{\ell-1}^{\mathscr{A}}\right)=\boldsymbol{\Gamma}\left(\mathbf{L}\left(\Phi_\mathscr{A}\left(\frac{\ell-1}{L_\mathscr{A}}\right)\right)\right)\tag{B.209}$$

and we will need this identity later on. In analogy to the concise notations (B.201), we define

$$\widetilde{\boldsymbol{\Upsilon}}_\ell^{(n)}\triangleq\widetilde{\boldsymbol{\Upsilon}}_\ell^{\mathscr{I}_{2^n L}}.\tag{B.210}$$

We now show that, in the limit as $n\to\infty$, the right-hand sides of (B.203) and (B.207) do not change if we replace $\boldsymbol{\Upsilon}_\ell^{(n)}$ by $\widetilde{\boldsymbol{\Upsilon}}_\ell^{(n)}$, that is,

$$\varliminf_{n\to\infty}\sum_{\ell=1}^{2^n L}\mathrm{tr}\Big(\hat{\mathsf{H}}^\dagger\left(\boldsymbol{\Upsilon}_\ell^{(n)}\right)^{-1}\hat{\mathsf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big)=\varliminf_{n\to\infty}\sum_{\ell=1}^{2^n L}\mathrm{tr}\Big(\hat{\mathsf{H}}^\dagger\left(\widetilde{\boldsymbol{\Upsilon}}_\ell^{(n)}\right)^{-1}\hat{\mathsf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big)\tag{B.211a}$$

$$\varlimsup_{n\to\infty}\sum_{\ell=1}^{2^n L}\mathrm{tr}\Big(\hat{\mathsf{H}}^\dagger\left(\boldsymbol{\Upsilon}_\ell^{(n)}\right)^{-1}\hat{\mathsf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big)=\varlimsup_{n\to\infty}\sum_{\ell=1}^{2^n L}\mathrm{tr}\Big(\hat{\mathsf{H}}^\dagger\left(\widetilde{\boldsymbol{\Upsilon}}_\ell^{(n)}\right)^{-1}\hat{\mathsf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big).\tag{B.211b}$$

To prove this, consider the sequence of inequalities

$$\begin{aligned}0&\leq\mathrm{tr}\Big(\hat{\mathsf{H}}^\dagger\left(\boldsymbol{\Upsilon}_\ell^{(n)}\right)^{-1}\hat{\mathsf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big)-\mathrm{tr}\Big(\hat{\mathsf{H}}^\dagger\left(\widetilde{\boldsymbol{\Upsilon}}_\ell^{(n)}\right)^{-1}\hat{\mathsf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big)\\&\leq\mathrm{tr}\Big(\left(\boldsymbol{\Upsilon}_\ell^{(n)}\right)^{-1}-\left(\widetilde{\boldsymbol{\Upsilon}}_\ell^{(n)}\right)^{-1}\Big)\,\mathrm{tr}\Big(\hat{\mathsf{H}}^\dagger\hat{\mathsf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big)\\&\leq\mathrm{tr}\Big(\left(\rho^{-1}\mathbf{I}_{n_\mathsf{R}}+\hat{\mathsf{H}}\bar{\mathbf{L}}_\ell^{(n)}\hat{\mathsf{H}}^\dagger\right)^{-1}-\left(\rho^{-1}\mathbf{I}_{n_\mathsf{R}}+\hat{\mathsf{H}}\bar{\mathbf{L}}_{\ell-1}^{(n)}\hat{\mathsf{H}}^\dagger\right)^{-1}\Big)\,\mathrm{tr}\Big(\hat{\mathsf{H}}^\dagger\hat{\mathsf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big)\end{aligned}\tag{B.212}$$

In (B.212), the first inequality is due to $\widetilde{\boldsymbol{\Upsilon}}_\ell^{(n)}\succeq\boldsymbol{\Upsilon}_\ell^{(n)}$; the second inequality is due to $\mathrm{tr}(\mathbf{AB})\leq\mathrm{tr}(\mathbf{A})\,\mathrm{tr}(\mathbf{B})$ for same-sized positive semidefinite matrices $\mathbf{A}$ and $\mathbf{B}$; the third and last inequality is obtained by subtracting a common quantity

$$\mathsf{E}\left[\tilde{\mathbf{H}}(\mathbf{L}_{\ell-1}^{\mathscr{A}})^{\frac12}\boldsymbol{\xi}\boldsymbol{\xi}^\dagger(\mathbf{L}_{\ell-1}^{\mathscr{A}})^{\frac12}\tilde{\mathbf{H}}^\dagger\middle|\boldsymbol{\xi},\hat{\mathsf{H}}\right]+\mathsf{E}\Big[\tilde{\mathbf{H}}\bar{\mathbf{L}}_{\ell-1}^{\mathscr{A}}\tilde{\mathbf{H}}^\dagger\;\Big|\;\hat{\mathsf{H}}\Big]$$

from $\boldsymbol{\Upsilon}_\ell^{(n)}$ and $\widetilde{\boldsymbol{\Upsilon}}_\ell^{(n)}$ alike. The inequality holds because for $n_\mathsf{R}\times n_\mathsf{R}$ matrices $\mathbf{0}\prec\mathbf{A}\preceq\mathbf{A}'$, the function

$$\begin{aligned} F\colon \mathbb{C}_+^{n_\mathsf{R}\times n_\mathsf{R}} &\to \mathbb{R}_+ \\ \mathbf{X} &\mapsto \operatorname{tr}\Big((\mathbf{X}+\mathbf{A})^{-1} - (\mathbf{X}+\mathbf{A}')^{-1}\Big) \end{aligned} \tag{B.213}$$

is non-increasing in the sense that $\mathbf{0} \preceq \mathbf{X}_1 \preceq \mathbf{X}_2$ implies $F(\mathbf{X}_1) \succeq F(\mathbf{X}_2)$. By virtue of the Heine-Cantor Theorem [Rud76, Theorem 4.19], the continuous mapping

$$[0;1] \to \mathbb{R},\ \iota \mapsto \operatorname{tr}\Big(\Big(\rho^{-1}\mathbf{I}_{n_\mathsf{R}} + \hat{\mathbf{H}}\bar{\mathbf{L}}(\iota)\hat{\mathbf{H}}^\dagger\Big)^{-1}\Big)$$

is uniformly continuous because $[0;1]$ is compact. As a consequence, for every ϵ, there is a sufficiently large n such that for all $\ell \in \{1,\dots,2^n L\}$ we have

$$\operatorname{tr}\Big(\Big(\rho^{-1}\mathbf{I}_{n_\mathsf{R}} + \hat{\mathbf{H}}\bar{\mathbf{L}}_\ell^{(n)}\hat{\mathbf{H}}^\dagger\Big)^{-1} - \Big(\rho^{-1}\mathbf{I}_{n_\mathsf{R}} + \hat{\mathbf{H}}\bar{\mathbf{L}}_{\ell-1}^{(n)}\hat{\mathbf{H}}^\dagger\Big)^{-1}\Big) \le \epsilon \tag{B.214}$$

whence

$$\begin{aligned} 0 &\le \sum_{\ell=1}^{2^n L}\Big[\operatorname{tr}\Big(\hat{\mathbf{H}}^\dagger\left(\boldsymbol{\Upsilon}_\ell^{(n)}\right)^{-1}\hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big) - \operatorname{tr}\Big(\hat{\mathbf{H}}^\dagger\left(\widetilde{\boldsymbol{\Upsilon}}_\ell^{(n)}\right)^{-1}\hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big)\Big] \\ &\le \epsilon\sum_{\ell=1}^{2^n L}\operatorname{tr}\Big(\hat{\mathbf{H}}^\dagger\hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big) \\ &\le \epsilon\operatorname{tr}\Big(\hat{\mathbf{H}}^\dagger\hat{\mathbf{H}}\mathbf{Q}\Big). \end{aligned} \tag{B.215}$$

Since $\epsilon > 0$ may be arbitrarily small, (B.211) follows. Writing the trace expression from the right-hand side of (B.211) as a double sum

$$\begin{aligned} &\operatorname{tr}\Big(\hat{\mathbf{H}}^\dagger\left(\widetilde{\boldsymbol{\Upsilon}}_\ell^{(n)}\right)^{-1}\hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\Big) \\ &\quad= \sum_{i=1}^{n_\mathsf{T}}\sum_{j=1}^{n_\mathsf{T}}\Big[\hat{\mathbf{H}}^\dagger\left(\widetilde{\boldsymbol{\Upsilon}}_\ell^{(n)}\right)^{-1}\hat{\mathbf{H}}\Big]_{i,j}\Big[\Delta\mathbf{L}_\ell^{(n)}\Big]_{j,i} \\ &\quad= \sum_{i=1}^{n_\mathsf{T}}\sum_{j=1}^{n_\mathsf{T}} G_{i,j}\left(\Phi_{\mathscr{I}}\left(\frac{\ell-1}{2^n L}\right)\right)\left[L_{j,i}\left(\Phi_{\mathscr{I}}\left(\frac{\ell}{2^n L}\right)\right) - L_{j,i}\left(\Phi_{\mathscr{I}}\left(\frac{\ell-1}{2^n L}\right)\right)\right] \end{aligned} \tag{B.216}$$

where the function $G_{i,j}$ is defined as

$$G_{i,j}(\iota) \triangleq \Big[\hat{\mathbf{H}}^\dagger\left(\boldsymbol{\Gamma}\left(\mathbf{L}(\iota)\right)\right)^{-1}\hat{\mathbf{H}}\Big]_{i,j}. \tag{B.217}$$

With this definition of $G_{i,j}$ we can easily convince ourselves of the validity of the last equality in (B.216), because using that $\Phi_{\mathscr{I}} = \Phi_{\tilde{\mathscr{I}}_n}$ and (B.209), we have

$$\begin{aligned} G_{i,j}\left(\Phi_{\mathscr{I}}\left(\frac{\ell-1}{2^n L}\right)\right) &= \left[\hat{\mathbf{H}}^\dagger\left(\mathbf{\Gamma}\left(\mathbf{L}\left(\Phi_{\mathscr{I}}\left(\frac{\ell-1}{2^n L}\right)\right)\right)\right)^{-1}\hat{\mathbf{H}}\right]_{i,j} \\ &= \left[\hat{\mathbf{H}}^\dagger\left(\mathbf{\Gamma}\left(\mathbf{L}\left(\Phi_{\tilde{\mathscr{I}}_{2^n L}}\left(\frac{\ell-1}{2^n L}\right)\right)\right)\right)^{-1}\hat{\mathbf{H}}\right]_{i,j} \\ &= \left[\hat{\mathbf{H}}^\dagger\left(\mathbf{\Gamma}\left(\mathbf{L}_{\ell-1}^{\tilde{\mathscr{I}}_{2^n L}}\right)\right)^{-1}\hat{\mathbf{H}}\right]_{i,j} \\ &= \left[\hat{\mathbf{H}}^\dagger\left(\widetilde{\mathbf{\Upsilon}}_\ell^{(n)}\right)^{-1}\hat{\mathbf{H}}\right]_{i,j}. \end{aligned} \tag{B.218}$$

Here, the second equality is due to (B.194); the third equality holds by construction of $\Phi_{\tilde{\mathscr{I}}_{2^n L}}$; the last equality follows from (B.209)–(B.210).

Though $G_{i,j}$ and $L_{j,i}$ may be complex-valued for $i \neq j$, we know that the trace of a product of two Hermitian matrices [as is the left-hand side of (B.216)] is real, hence the double sum on the right-hand side of (B.216) must be real. We thus only need to retain the real-valued contributions to the double sum. We split these contributions as

$$\operatorname{tr}\left(\hat{\mathbf{H}}^\dagger\left(\widetilde{\mathbf{\Upsilon}}_\ell^{(n)}\right)^{-1}\hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\right) = \sum_{i=1}^{n_\mathsf{T}}\sum_{j=1}^{n_\mathsf{T}}\left(T_{\Re,i,j}^{(\ell,n)} - T_{\Im,i,j}^{(\ell,n)}\right) \tag{B.219}$$

with the real-valued functions

$$T_{\Re,i,j}^{(\ell,n)} = \Re G_{i,j}\left(\Phi_{\mathscr{I}}\left(\frac{\ell}{n}\right)\right)\left[\Re L_{j,i}\left(\Phi_{\mathscr{I}}\left(\frac{\ell}{n}\right)\right) - \Re L_{j,i}\left(\Phi_{\mathscr{I}}\left(\frac{\ell-1}{n}\right)\right)\right] \tag{B.220a}$$

$$T_{\Im,i,j}^{(\ell,n)} = \Im G_{i,j}\left(\Phi_{\mathscr{I}}\left(\frac{\ell}{n}\right)\right)\left[\Im L_{j,i}\left(\Phi_{\mathscr{I}}\left(\frac{\ell}{n}\right)\right) - \Im L_{j,i}\left(\Phi_{\mathscr{I}}\left(\frac{\ell-1}{n}\right)\right)\right] \tag{B.220b}$$

where $\Re f$ and $\Im f$ denote the real and imaginary part of f, respectively. The superadditivity (resp. subadditivity) of the limit inferior (resp. limit superior) imply

$$\varliminf_{n\to\infty}\sum_{\ell=1}^{2^n L}\operatorname{tr}\left(\hat{\mathbf{H}}^\dagger\left(\widetilde{\mathbf{\Upsilon}}_\ell^{(n)}\right)^{-1}\hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\right) \geq \sum_{i=1}^{n_\mathsf{T}}\sum_{j=1}^{n_\mathsf{T}}\varliminf_{n\to\infty}\sum_{\ell=1}^{2^n L}\left(T_{\Re,i,j}^{(\ell,n)} - T_{\Im,i,j}^{(\ell,n)}\right) \tag{B.221a}$$

$$\varlimsup_{n\to\infty}\sum_{\ell=1}^{2^n L}\operatorname{tr}\left(\hat{\mathbf{H}}^\dagger\left(\widetilde{\mathbf{\Upsilon}}_\ell^{(n)}\right)^{-1}\hat{\mathbf{H}}\,\Delta\mathbf{L}_\ell^{(n)}\right) \leq \sum_{i=1}^{n_\mathsf{T}}\sum_{j=1}^{n_\mathsf{T}}\varlimsup_{n\to\infty}\sum_{\ell=1}^{2^n L}\left(T_{\Re,i,j}^{(\ell,n)} - T_{\Im,i,j}^{(\ell,n)}\right) \tag{B.221b}$$

On the right-hand side of (B.221), the sums of $T_{\Re,i,j}^{(\ell,n)}$ and $T_{\Im,i,j}^{(\ell,n)}$ over ℓ are Riemann-Stieltjes sums. A sufficient condition for the existence of the Riemann-Stieltjes integral as the limit of its corresponding Riemann-Stieltjes sum is that the integrand is continuous and the integrator function is of bounded variation. In our specific case, the integrand functions are $\Re G_{i,j}$ and $\Im G_{i,j}$, whereas the integrator functions are $\Re L_{j,i}$ and $\Im L_{j,i}$. The continuity of $G_{i,j}$ is a consequence of the (entrywise) continuity of the layering function $\mathbf{L}(\iota)$, whereas the variation-boundedness of $L_{j,i}$ is a consequence of the (entrywise) Lipschitz-continuity of the layering function $\mathbf{L}$, which was established in Lemma B.13. As a consequence, the following Riemann-Stieltjes integrals exist:

$$\lim_{n\to\infty}\sum_{\ell=1}^{2^n L} T_{\Re,i,j}^{(\ell,n)} = \int_0^1 \Re G_{i,j}(\Phi_{\mathscr{I}}(\iota))\,\mathrm{d}\Re L_{j,i}(\Phi_{\mathscr{I}}(\iota)) \tag{B.222a}$$

$$\lim_{n\to\infty}\sum_{\ell=1}^{2^n L} T_{\Im,i,j}^{(\ell,n)} = \int_0^1 \Im G_{i,j}(\Phi_{\mathscr{I}}(\iota))\,\mathrm{d}\Im L_{j,i}(\Phi_{\mathscr{I}}(\iota)). \tag{B.222b}$$

From the existence of these two Riemann-Stieltjes integrals, we can infer that the right-hand sides of (B.221a) and (B.221b) coincide, and both are equal to the double sum of (complex-valued) Riemann-Stieltjes integrals:

$$\begin{aligned}\sum_{i=1}^{n_\mathsf{T}}\sum_{j=1}^{n_\mathsf{T}}\int_0^1 G_{i,j}(\Phi_{\mathscr{I}}(\iota))\,\mathrm{d}L_{j,i}(\Phi_{\mathscr{I}}(\iota)) &= \int_0^1 \sum_{i=1}^{n_\mathsf{T}}\sum_{j=1}^{n_\mathsf{T}} G_{i,j}(\Phi_{\mathscr{I}}(\iota))\,\mathrm{d}L_{j,i}(\Phi_{\mathscr{I}}(\iota))\\ &\triangleq \int_0^1 \mathrm{tr}\Big[\hat{\mathbf{H}}^\dagger\Big(\mathbf{\Gamma}\Big(\mathbf{L}(\Phi_{\mathscr{I}}(\iota))\Big)\Big)^{-1}\hat{\mathbf{H}}\,\mathrm{d}\mathbf{L}(\Phi_{\mathscr{I}}(\iota))\Big].\end{aligned} \tag{B.223}$$

Here, the right-hand side, which involves the infinitesimal quantity $\mathrm{d}\mathbf{L}$ sitting inside the trace operator, is nothing but a compact notation for the double sum of Riemann-Stieltjes integrals on the left-hand side of the last equation. After a change of variable $\iota' = \Phi_{\mathscr{I}}(\iota)$ using the increasing bijective map $\Phi_{\mathscr{I}}\colon [0;1] \to [0,1]$, the Riemann-Stieltjes integrals simplify to

$$\int_0^1 G_{i,j}(\Phi_{\mathscr{I}}(\iota))\,\mathrm{d}L_{j,i}(\Phi_{\mathscr{I}}(\iota)) = \int_0^1 G_{i,j}(\iota)\,\mathrm{d}L_{j,i}(\iota) \tag{B.224}$$

and we thus end up with [Rud76, Theorem 6.19]

$$\sum_{i=1}^{n_\mathsf{T}}\sum_{j=1}^{n_\mathsf{T}}\int_0^1 G_{i,j}(\Phi_{\mathscr{I}}(\iota))\,\mathrm{d}L_{j,i}(\Phi_{\mathscr{I}}(\iota)) = \int_0^1 \mathrm{tr}\Big[\hat{\mathbf{H}}^\dagger\Big(\mathbf{\Gamma}(\mathbf{L}(\iota'))\Big)^{-1}\hat{\mathbf{H}}\,\mathrm{d}\mathbf{L}(\iota')\Big]. \tag{B.225}$$

We deduce that the right-hand sides of (B.202b) and (B.202a) coincide with the Riemann-Stieltjes integral (B.225), and thus, that the limit involved in (B.200) exists. Therefore, by inserting (B.225) into (B.200), we obtain

$$R^\infty(\mathbf{L},\mathscr{I}) = \mathsf{E}\left[\int_0^1 \mathrm{tr}\Big(\hat{\mathbf{H}}^\dagger\mathbf{\Gamma}(\mathbf{L}(\iota))^{-1}\hat{\mathbf{H}}\,\mathrm{d}\mathbf{L}(\iota)\Big)\right]. \tag{B.226}$$

Note that in this expression, the expectation and integration operators can be interchanged, owing to the Fubini Theorem [Rud87, Theorem 8.8]. Finally, since the right-hand side of (B.226) does not depend on the indexing $\mathscr{I}$, we infer that (B.191) holds. This concludes the proof of Theorem (2.6).

B.15 Proof of Lemma 2.3

Let $\mathbf{L} = [L_{i,j}]_{i,j}$ and $\widetilde{\mathbf{L}} = [L_{i,j}]_{i,j}$ denote two layering functions. We prove that for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\sup_{\iota\in[0;1]} \big\|\mathbf{L}(\iota) - \widetilde{\mathbf{L}}(\iota)\big\|_{\mathsf{F}} < \delta \quad \Rightarrow \quad \big|R^\star(\mathbf{L}) - R^\star(\widetilde{\mathbf{L}})\big| < \epsilon. \tag{B.227}$$

where $\|\cdot\|_{\mathsf{F}}$ stands for the Frobenius norm.[9] Besides the Frobenius norm $\|\mathbf{A}\|_{\mathsf{F}}$, let us also introduce the one-norm $\|\mathbf{A}\|_{1}$ of an $n \times m$ matrix $\mathbf{A} = [a_{i,j}]_{i,j}$, these two matrix norms being respectively defined as

$$\|\mathbf{A}\|_{\mathsf{F}} = \sqrt{\operatorname{tr}(\mathbf{A}^\dagger\mathbf{A})} = \sqrt{\sum_{i,j}|a_{i,j}|^2} \qquad\qquad \|\mathbf{A}\|_{1} = \sum_{i,j}|a_{i,j}|. \tag{B.228}$$

For future reference, let us state the bounds

$$\|\mathbf{A}\|_{\mathsf{F}} \le \|\mathbf{A}\|_{1} \le \sqrt{nm}\|\mathbf{A}\|_{\mathsf{F}} \tag{B.229}$$

which are a consequence of the fact that the ratio

$$\frac{\|\mathbf{A}\|_{1}}{\|\mathbf{A}\|_{\mathsf{F}}} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{m}|a_{i,j}|}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}|a_{i,j}|^2}} \tag{B.230}$$

is maximal when all $|a_{i,j}|$ are equal, and minimal when all terms $|a_{i,j}|$ vanish except one.

To begin with, let us write out the trace involved in the expression (2.81) of the rate-splitting bound, so as to represent $R^\star(\mathbf{L})$ and $R^\star(\widetilde{\mathbf{L}})$ as sums of scalar Riemann-Stieltjes integrals

$$R^\star(\mathbf{L}) = \sum_{i,j=1}^{n_{\mathsf{T}}} \int_0^1 e_{i,j}(\iota)\,\mathrm{d}L_{j,i}(\iota) \tag{B.231a}$$

$$R^\star(\widetilde{\mathbf{L}}) = \sum_{i,j=1}^{n_{\mathsf{T}}} \int_0^1 \widetilde{e}_{i,j}(\iota)\,\mathrm{d}\widetilde{K}_{j,i}(\iota), \tag{B.231b}$$

where $e_{i,j}(\iota) = \big[\mathbf{E}(\mathbf{L}(\iota))\big]_{i,j}$ and $\widetilde{e}_{i,j}(\iota) = \big[\mathbf{E}(\widetilde{\mathbf{L}}(\iota))\big]_{i,j}$ are the (i,j)-th entries of the matrices $\mathbf{E}(\mathbf{L}(\iota))$ and $\mathbf{E}(\widetilde{\mathbf{L}}(\iota))$, respectively, where $\mathbf{E}(\mathbf{X})$ is defined for arguments $\mathbf{0} \preceq \mathbf{X} \preceq \mathbf{Q}$ as

$$\mathbf{E}(\mathbf{X}) = \mathsf{E}\big[\hat{\mathbf{H}}^\dagger\mathbf{\Gamma}(\mathbf{X})^{-1}\hat{\mathbf{H}}\big]. \tag{B.232}$$

[9] We choose the Frobenius norm for convenience, but due to the norm equivalence property, any other matrix norm would also do.

Using integration by parts, the Riemann-Stieltjes integrals in (B.231) can be written as

$$\int_0^1 e_{i,j}(\iota)\,\mathrm{d}L_{j,i}(\iota) = e_{i,j}(1)L_{j,i}(1) - \int_0^1 L_{j,i}(\iota)\,\mathrm{d}e_{i,j}(\iota) \tag{B.233a}$$

$$\int_0^1 \widetilde{e}_{i,j}(\iota)\,\mathrm{d}\widetilde{K}_{j,i}(\iota) = \widetilde{e}_{i,j}(1)\widetilde{K}_{j,i}(1) - \int_0^1 \widetilde{K}_{j,i}(\iota)\,\mathrm{d}\widetilde{e}_{i,j}(\iota) \tag{B.233b}$$

because $L_{j,i}(0) = \widetilde{K}_{j,i}(0) = 0$. Note that both integrals on the right-hand side of (B.233) are well-defined, since the existence of either integral [on the left-hand or right-hand side of (B.233)] implies the existence of the other. Also note that, due to $\mathbf{L}$ and $\widetilde{\mathbf{L}}$ being layering functions for the same transmit covariance $\mathbf{Q}$, we have that $e_{i,j}(1) = \widetilde{e}_{i,j}(1)$ and $L_{j,i}(1) = \widetilde{K}_{j,i}(1)$. Denoting $\kappa_{i,j}(\iota) = L_{i,j}(\iota) - \widetilde{K}_{i,j}(\iota)$ and $\eta_{i,j}(\iota) = e_{i,j}(\iota) - \widetilde{e}_{i,j}(\iota)$, the absolute value of the difference $R^\star(\mathbf{L}) - R^\star(\widetilde{\mathbf{L}})$ can therefore be bounded as follows:

$$\begin{aligned}
\left|R^\star(\mathbf{L}) - R^\star(\widetilde{\mathbf{L}})\right| &= \left|\sum_{i,j=1}^{n_\mathsf{T}} \int_0^1 \Big[L_{i,j}(\iota)\,\mathrm{d}e_{j,i}(\iota) - \widetilde{K}_{i,j}(\iota)\,\mathrm{d}\widetilde{e}_{j,i}(\iota)\Big]\right| \\
&= \left|\sum_{i,j=1}^{n_\mathsf{T}} \int_0^1 \Big[\kappa_{i,j}(\iota)\,\mathrm{d}e_{i,j}(\iota) + \widetilde{K}_{j,i}(\iota)\,\mathrm{d}\eta_{i,j}(\iota)\Big]\right| \\
&\leq \sum_{i,j=1}^{n_\mathsf{T}} \left[\left|\int_0^1 \kappa_{i,j}(\iota)\,\mathrm{d}e_{i,j}(\iota)\right| + \left|\int_0^1 \widetilde{K}_{j,i}(\iota)\,\mathrm{d}\eta_{i,j}(\iota)\right|\right]
\end{aligned} \tag{B.234}$$

The bounding step follows from the triangle inequality.

Upon writing out the Riemann-Stieltjes integral as the limit of its corresponding Riemann-Stieltjes sum, the first term in (B.234) can be upper-bounded via

$$\begin{aligned}
\left|\int_0^1 \kappa_{i,j}(\iota)\,\mathrm{d}e_{i,j}(\iota)\right| &= \left|\lim_{N\to\infty} \sum_{n=1}^{N} \kappa_{i,j}\left(\tfrac{n}{N}\right)\left[e_{j,i}\left(\tfrac{n}{N}\right) - e_{j,i}\left(\tfrac{n-1}{N}\right)\right]\right| \\
&\leq \limsup_{N\to\infty} \sum_{n=1}^{N} \left|\kappa_{i,j}\left(\tfrac{n}{N}\right)\right| \cdot \left|e_{j,i}\left(\tfrac{n}{N}\right) - e_{j,i}\left(\tfrac{n-1}{N}\right)\right| \\
&\leq \sup_{0\leq\iota_1<\iota_2\leq 1} \left|\frac{e_{j,i}(\iota_1) - e_{j,i}(\iota_2)}{\iota_1 - \iota_2}\right| \cdot \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left|\kappa_{i,j}\left(\tfrac{n}{N}\right)\right| \\
&\leq \sup_{0\leq\iota_1<\iota_2\leq 1} \left|\frac{e_{j,i}(\iota_1) - e_{j,i}(\iota_2)}{\iota_1 - \iota_2}\right| \cdot \sup_{\iota\in[0;1]} \left|\kappa_{i,j}(\iota)\right|.
\end{aligned} \tag{B.235}$$

Here, the first inequality is the triangle inequality; the second inequality results from upper-bounding

$$N \cdot \left|e_{j,i}\left(\tfrac{n}{N}\right) - e_{j,i}\left(\tfrac{n-1}{N}\right)\right|$$

by the Lipschitz constant of $e_{j,i}$; the third and last inequality results from upper-bounding $|\kappa_{i,j}(\iota)|$ by its supremum over the integration interval $\iota \in [0; 1]$. This supremum is itself upper-bounded as

$$\begin{aligned}
\sup_{\iota\in[0;1]} |\kappa_{i,j}(\iota)| &\leq \sup_{\iota\in[0;1]} \sqrt{\sum_{i,j=1}^{n_\mathsf{T}} |\kappa_{i,j}(\iota)|^2} \\
&= \sup_{\iota\in[0;1]} \|\mathbf{L}(\iota) - \widetilde{\mathbf{L}}(\iota)\|_\mathsf{F} \\
&< \delta
\end{aligned} \tag{B.236}$$

and therefore, by plugging (B.236) into (B.235),

$$\left|\int_0^1 \kappa_{i,j}(\iota)\,\mathrm{d}e_{i,j}(\iota)\right| \leq \sup_{0\leq\iota_1<\iota_2\leq 1} \left|\frac{e_{j,i}(\iota_1) - e_{j,i}(\iota_2)}{\iota_1 - \iota_2}\right| \cdot \delta. \tag{B.237}$$

We now focus on the second term in (B.234). Using integration by parts and the fact that $\eta_{i,j}(0) = \eta_{i,j}(1) = 0$, we have

$$\int_0^1 \widetilde{K}_{j,i}(\iota)\,\mathrm{d}\eta_{i,j}(\iota) = -\int_0^1 \eta_{i,j}(\iota)\,\mathrm{d}\widetilde{K}_{j,i}(\iota). \tag{B.238}$$

Upon taking absolute values on either side of the last expression, and writing out the Riemann-Stieltjes integral as the limit of its corresponding Riemann-Stieltjes sum, the second term in (B.234) can be upper-bounded in a similar way as was done previously for the first term:

$$\begin{aligned}
\left|\int_0^1 \widetilde{K}_{j,i}(\iota)\,\mathrm{d}\eta_{i,j}(\iota)\right| &= \left|\lim_{N\to\infty} \sum_{n=1}^{N} \eta_{i,j}\left(\tfrac{n}{N}\right)\left[\widetilde{K}_{j,i}\left(\tfrac{n}{N}\right) - \widetilde{K}_{j,i}\left(\tfrac{n-1}{N}\right)\right]\right| \\
&\leq \lim_{N\to\infty} \sum_{n=1}^{N} \left|\eta_{i,j}\left(\tfrac{n}{N}\right)\right| \cdot \left|\widetilde{K}_{j,i}\left(\tfrac{n}{N}\right) - \widetilde{K}_{j,i}\left(\tfrac{n-1}{N}\right)\right| \\
&\leq \sup_{0\leq\iota_1<\iota_2\leq 1} \left|\frac{\widetilde{K}_{j,i}(\iota_1) - \widetilde{K}_{j,i}(\iota_2)}{\iota_1 - \iota_2}\right| \cdot \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{N} \left|\eta_{i,j}\left(\tfrac{n}{N}\right)\right| \\
&\leq \sup_{\iota\in[0;1]} |\eta_{i,j}(\iota)|
\end{aligned} \tag{B.239}$$

The first inequality is the triangle inequality; the second inequality results from upper-bounding

$$N \cdot \left|\widetilde{K}_{j,i}\left(\tfrac{n}{N}\right) - \widetilde{K}_{j,i}\left(\tfrac{n-1}{N}\right)\right|$$

by the Lipschitz constant of $\widetilde{K}_{j,i}$; the third and last inequality results from the Lipschitz-continuity of the layering function $\widetilde{\mathbf{L}}$ [cf. (2.63)] and from upper-bounding $|\eta_{i,j}(\iota)|$ by its supremum on the integration interval.

Combining (B.234), (B.237) and (B.239), we get

$$|R^\star(\mathbf{L}) - R^\star(\widetilde{\mathbf{L}})| \leq \sum_{i,j=1}^{n_\mathsf{T}} \left[\sup_{0\leq\iota_1<\iota_2\leq 1} \left|\frac{e_{j,i}(\iota_1) - e_{j,i}(\iota_2)}{\iota_1 - \iota_2}\right| \cdot \delta + \sup_{\iota\in[0;1]} |\eta_{i,j}(\iota)|\right]. \tag{B.240}$$

To prove Lemma 2.3, we wish to prove that the right-hand side of (B.240) can be made arbitrarily small by an appropriate choice of $\delta > 0$. We will now argue that to complete this proof, it suffices to show that $\mathbf{X} \mapsto \mathbf{E}(\mathbf{X})$ as previously defined in (B.232) is Lipschitz-continuous in the sense that for any two distinct $\mathbf{X}_1$ and $\mathbf{X}_2$ from $\mathbb{C}_+^{n_\mathsf{T} \times n_\mathsf{T}}$ fulfilling $\mathbf{0} \preceq \mathbf{X}_1 \preceq \mathbf{Q}$ and $\mathbf{0} \preceq \mathbf{X}_2 \preceq \mathbf{Q}$, the quotient

$$\frac{\|\mathbf{E}(\mathbf{X}_1) - \mathbf{E}(\mathbf{X}_2)\|_\mathsf{F}}{\|\mathbf{X}_1 - \mathbf{X}_2\|_\mathsf{F}} \leq C < +\infty \tag{B.241}$$

has a finite upper bound C which is independent of $(\mathbf{X}_1, \mathbf{X}_2)$.

In fact, if this Lipschitz property holds, this has two consequences for the terms on the right-hand side in (B.240): on the one hand, the last supremum term can then be upper-bounded as

$$\begin{aligned}
\sup_{\iota \in [0;1]} |\eta_{i,j}(\iota)| &\leq \sup_{\iota \in [0;1]} \sqrt{\sum_{i,j=1}^{n_\mathsf{T}} |\eta_{i,j}(\iota)|^2} \\
&= \sup_{\iota \in [0;1]} \big\|\mathbf{E}(\mathbf{L}(\iota)) - \mathbf{E}(\widetilde{\mathbf{L}}(\iota))\big\|_\mathsf{F} \\
&\leq \sup_{\iota \in [0;1]} C\big\|\mathbf{L}(\iota) - \widetilde{\mathbf{L}}(\iota)\big\|_\mathsf{F} \\
&\leq C\delta.
\end{aligned} \tag{B.242}$$

On the other hand, the other supremum term in (B.240) would be finite for all i, j, because for any $0 \leq \iota_1 < \iota_2 \leq 1$, one can upper-bound the ratio $|e_{j,i}(\iota_1) - e_{j,i}(\iota_2)| \,/\, |\iota_1 - \iota_2|$ by upper-bounding the numerator as

$$\begin{aligned}
|e_{j,i}(\iota_1) - e_{j,i}(\iota_2)| &\leq \sqrt{\sum_{i,j=1}^{n_\mathsf{T}} |e_{j,i}(\iota_1) - e_{j,i}(\iota_2)|^2} \\
&= \big\|\mathbf{E}(\mathbf{L}(\iota_1)) - \mathbf{E}(\mathbf{L}(\iota_2))\big\|_\mathsf{F}
\end{aligned} \tag{B.243}$$

and lower-bounding the denominator as

$$\begin{aligned}
|\iota_1 - \iota_2| &= \operatorname{tr}\big(\mathbf{L}(\iota_2) - \mathbf{L}(\iota_1)\big) \\
&\geq \frac{1}{n_\mathsf{T}} \big\|\mathbf{L}(\iota_2) - \mathbf{L}(\iota_1)\big\|_1 \\
&\geq \frac{1}{n_\mathsf{T}} \big\|\mathbf{L}(\iota_2) - \mathbf{L}(\iota_1)\big\|_\mathsf{F}.
\end{aligned} \tag{B.244}$$

Here, the equality follows from $\iota_1 < \iota_2$ and from $\operatorname{tr}(\mathbf{L}(\iota)) = \iota$ by the definition of layering functions [cf. Definition 2.1]; the first inequality follows from the fact that $\mathbf{L}(\iota_2) - \mathbf{L}(\iota_1)$ is positive semidefinite as a consequence of the definition of layering functions [cf. Definition 2.1] and by applying Lemma B.5 stated below, whose proof is relegated to the next Appendix Section; the second and last inequality follows from (B.229).

Lemma B.5. *For any $n_\mathsf{T} \times n_\mathsf{T}$ positive semidefinite matrix $\mathbf{A}$, the following inequality holds:*

$$\|\mathbf{A}\|_1 \leq n_\mathsf{T} \cdot \operatorname{tr}(\mathbf{A}). \tag{B.245}$$

Proof: The proof is relegated to Appendix B.16. ■

Combining (B.243), (B.244), together with (B.241) we get

$$\sup_{0\leq\iota_1<\iota_2\leq 1}\left|\frac{e_{j,i}(\iota_1)-e_{j,i}(\iota_2)}{\iota_1-\iota_2}\right|\leq n_{\mathsf{T}}C \tag{B.246}$$

and finally, combining (B.240), (B.242) and (B.246), we arrive at

$$\left|R^{\star}(\mathbf{L})-R^{\star}(\widetilde{\mathbf{L}})\right|\leq {n_{\mathsf{T}}}^2(n_{\mathsf{T}}+1)C\delta \tag{B.247}$$

which establishes Lemma 2.3.

To complete the proof, the Lipschitz property (B.241) remains to be proven. For the sake of notational concision, we will write $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ instead of $\mathbf{\Gamma}(\mathbf{X}_1)$ and $\mathbf{\Gamma}(\mathbf{X}_2)$, respectively. We start by upper-bounding the numerator of (B.241) as

$$\begin{aligned}\left\|\mathbf{E}(\mathbf{X}_1)-\mathbf{E}(\mathbf{X}_2)\right\|_{\mathsf{F}}&\leq\mathsf{E}\left[\left\|\hat{\mathbf{H}}^{\dagger}(\mathbf{\Gamma}_1^{-1}-\mathbf{\Gamma}_2^{-1})\hat{\mathbf{H}}\right\|_{\mathsf{F}}\right]\\&\leq\mathsf{E}\left[\|\hat{\mathbf{H}}\|_{\mathsf{F}}^2\cdot\left\|\mathbf{\Gamma}_1^{-1}-\mathbf{\Gamma}_2^{-1}\right\|_{\mathsf{F}}\right].\end{aligned} \tag{B.248}$$

Here, the first inequality follows because the absolute value $|\cdot|$ is a convex function on the complex numbers, which allows us by Jensen's inequality to upper-bound the one-norm of an expectation by the expectation of the one-norm; the second inequality results from the sub-multiplicativity of the Frobenius norm, i.e., the property $\|\mathbf{AB}\|_{\mathsf{F}}\leq\|\mathbf{A}\|_{\mathsf{F}}\cdot\|\mathbf{B}\|_{\mathsf{F}}$. Next, the factor in (B.248) involving $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ can be upper-bounded as

$$\begin{aligned}\left\|\mathbf{\Gamma}_1^{-1}-\mathbf{\Gamma}_2^{-1}\right\|_{\mathsf{F}}&=\left\|\mathbf{\Gamma}_1^{-1}(\mathbf{\Gamma}_2-\mathbf{\Gamma}_1)\mathbf{\Gamma}_2^{-1}\right\|_{\mathsf{F}}\\&\leq\|\mathbf{\Gamma}_1^{-1}\|_{\mathsf{F}}\cdot\|\mathbf{\Gamma}_2^{-1}\|_{\mathsf{F}}\cdot\left\|\mathbf{\Gamma}_2-\mathbf{\Gamma}_1\right\|_{\mathsf{F}}\\&\leq n_{\mathsf{R}}\rho^2\left\|\mathbf{\Gamma}_2-\mathbf{\Gamma}_1\right\|_{\mathsf{F}}.\end{aligned} \tag{B.249}$$

Here, the first inequality is again due to the sub-multiplicativity of the Frobenius norm, whereas the second inequality is due to $\mathbf{\Gamma}_1\succeq\rho^{-1}\mathbf{I}_{n_{\mathsf{R}}}$ and $\mathbf{\Gamma}_2\succeq\rho^{-1}\mathbf{I}_{n_{\mathsf{R}}}$, and the fact that the Frobenius norm is matrix-monotone in the sense that $\mathbf{0}\preceq\mathbf{A}\preceq\mathbf{B}$ implies $\|\mathbf{A}\|_{\mathsf{F}}\leq\|\mathbf{B}\|_{\mathsf{F}}$. The difference $\mathbf{\Gamma}_2-\mathbf{\Gamma}_1$ reads as

$$\mathbf{\Gamma}_2-\mathbf{\Gamma}_1=\mathsf{E}\left[\tilde{\mathbf{H}}(\mathbf{X}_2^{\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}^{\dagger}\mathbf{X}_2^{\frac{1}{2}}-\mathbf{X}_1^{\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}^{\dagger}\mathbf{X}_1^{\frac{1}{2}})\tilde{\mathbf{H}}^{\dagger}\;\middle|\;\boldsymbol{\xi},\hat{\mathbf{H}}\right]-\mathsf{E}[\tilde{\mathbf{H}}(\mathbf{X}_2-\mathbf{X}_1)\tilde{\mathbf{H}}^{\dagger}\mid\hat{\mathbf{H}}]-\hat{\mathbf{H}}(\mathbf{X}_2-\mathbf{X}_1)\hat{\mathbf{H}}^{\dagger} \tag{B.250}$$

so by applying the triangle inequality onto the last expression, we can upper-bound $\left\|\mathbf{\Gamma}_2-\mathbf{\Gamma}_1\right\|_{\mathsf{F}}$ by the sum of three positive terms, each of which we shall upper-bound once more. The first term in (B.250) is upper-bounded as

$$\begin{aligned}\left\|\mathsf{E}\Big[\tilde{\mathbf{H}}(\mathbf{X}_2^{\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}^\dagger\mathbf{X}_2^{\frac{1}{2}}-\mathbf{X}_1^{\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}^\dagger\mathbf{X}_1^{\frac{1}{2}})\tilde{\mathbf{H}}^\dagger \,\Big|\, \boldsymbol{\xi},\hat{\mathbf{H}}\Big]\right\|_{\mathsf{F}} &\le \left\|\mathsf{E}\Big[\tilde{\mathbf{H}}(\mathbf{X}_2^{\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}^\dagger\mathbf{X}_2^{\frac{1}{2}}-\mathbf{X}_1^{\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}^\dagger\mathbf{X}_1^{\frac{1}{2}})\tilde{\mathbf{H}}^\dagger \,\Big|\, \boldsymbol{\xi},\hat{\mathbf{H}}\Big]\right\|_1\\
&\le \mathsf{E}\Big[\left\|\tilde{\mathbf{H}}(\mathbf{X}_2^{\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}^\dagger\mathbf{X}_2^{\frac{1}{2}}-\mathbf{X}_1^{\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}^\dagger\mathbf{X}_1^{\frac{1}{2}})\tilde{\mathbf{H}}^\dagger\right\|_1 \,\Big|\, \boldsymbol{\xi},\hat{\mathbf{H}}\Big]\\
&\le \mathsf{E}[\|\tilde{\mathbf{H}}\|_1^2 \mid \hat{\mathbf{H}}]\cdot\|\mathbf{X}_2^{\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}^\dagger\mathbf{X}_2^{\frac{1}{2}}-\mathbf{X}_1^{\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}^\dagger\mathbf{X}_1^{\frac{1}{2}}\|_1\\
&\le n_{\mathsf{T}}\,\mathsf{E}[\|\tilde{\mathbf{H}}\|_1^2 \mid \hat{\mathbf{H}}]\cdot\|\mathbf{X}_2-\mathbf{X}_1\|_{\mathsf{F}}\cdot\|\boldsymbol{\xi}\|_2^2. \qquad \text{(B.251)}\end{aligned}$$

Here, the first bounding step results from (B.229); the second bounding step is due to Jensen's inequality; the third step uses that the one-norm is sub-multiplicative; the fourth step is detailed as follows:

$$\begin{aligned}\|\mathbf{X}_2^{\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}^\dagger\mathbf{X}_2^{\frac{1}{2}}-\mathbf{X}_1^{\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}^\dagger\mathbf{X}_1^{\frac{1}{2}}\|_1 &\le n_{\mathsf{T}}\|\mathbf{X}_2^{\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}^\dagger\mathbf{X}_2^{\frac{1}{2}}-\mathbf{X}_1^{\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}^\dagger\mathbf{X}_1^{\frac{1}{2}}\|_{\mathsf{F}}\\
&\le n_{\mathsf{T}}\Big|\|\mathbf{X}_2^{\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}^\dagger\mathbf{X}_2^{\frac{1}{2}}\|_{\mathsf{F}}-\|\mathbf{X}_1^{\frac{1}{2}}\boldsymbol{\xi}\boldsymbol{\xi}^\dagger\mathbf{X}_1^{\frac{1}{2}}\|_{\mathsf{F}}\Big|\\
&= n_{\mathsf{T}}|\boldsymbol{\xi}^\dagger(\mathbf{X}_2-\mathbf{X}_1)\boldsymbol{\xi}|\\
&\le n_{\mathsf{T}}\|\mathbf{X}_2-\mathbf{X}_1\|_{\mathsf{S}}\cdot\|\boldsymbol{\xi}\|_2^2\\
&\le n_{\mathsf{T}}\|\mathbf{X}_2-\mathbf{X}_1\|_{\mathsf{F}}\cdot\|\boldsymbol{\xi}\|_2^2. \qquad \text{(B.252)}\end{aligned}$$

where $\|\cdot\|_{\mathsf{S}}$ denotes the spectral radius norm, defined for an $n\times n$ matrix $\mathbf{A}$ as

$$\|\mathbf{A}\|_{\mathsf{S}} = \max_{i=1,\dots,n}|\lambda_i(\mathbf{A})|.$$

where $\lambda_i(\mathbf{A})$ denote the eigenvalues of $\mathbf{A}$. In the above chain of inequalities, the first step follows from (B.229); the second is the triangle inequality in the form $\|\mathbf{A}-\mathbf{B}\|_{\mathsf{F}} \le \big|\|\mathbf{A}\|_{\mathsf{F}}-\|\mathbf{B}\|_{\mathsf{F}}\big|$; the third inequality follows from $|\mathbf{u}^\dagger\mathbf{A}\mathbf{u}| \le \|\mathbf{u}\|_2^2\|\mathbf{A}\|_{\mathsf{S}}$; the fourth and last step follows because for a Hermitian $n\times n$ matrix $\mathbf{A}$, we have the norm inequality

$$\|\mathbf{A}\|_{\mathsf{S}} \le \sum_{i=1}^{n}|\lambda_i(\mathbf{A})| \le \sqrt{\sum_{i=1}^{n}|\lambda_i(\mathbf{A})|^2} = \|\mathbf{A}\|_{\mathsf{F}}. \qquad \text{(B.253)}$$

As to the remaining two terms in (B.250), their Frobenius norm is upper-bounded via the triangle inequality and sub-multiplicativity of the Frobenius norm as

$$\|\mathsf{E}[\tilde{\mathbf{H}}(\mathbf{X}_2-\mathbf{X}_1)\tilde{\mathbf{H}}^\dagger \mid \hat{\mathbf{H}}] + \hat{\mathbf{H}}(\mathbf{X}_2-\mathbf{X}_1)\hat{\mathbf{H}}^\dagger\|_{\mathsf{F}} \le (\|\hat{\mathbf{H}}\|_{\mathsf{F}}^2 + \mathsf{E}[\|\tilde{\mathbf{H}}\|_{\mathsf{F}}^2 \mid \hat{\mathbf{H}}])\cdot\|\mathbf{X}_2-\mathbf{X}_1\|_{\mathsf{F}}. \qquad \text{(B.254)}$$

All in all, combining (B.250), (B.251) and (B.254), we get

$$\|\boldsymbol{\Gamma}_2-\boldsymbol{\Gamma}_1\|_{\mathsf{F}} \le \Big(n_{\mathsf{T}}\,\mathsf{E}[\|\tilde{\mathbf{H}}\|_{\mathsf{F}}^2 \mid \hat{\mathbf{H}}]\cdot\|\boldsymbol{\xi}\|_2^2 + \|\hat{\mathbf{H}}\|_{\mathsf{F}}^2 + \mathsf{E}[\|\tilde{\mathbf{H}}\|_{\mathsf{F}}^2 \mid \hat{\mathbf{H}}]\Big)\cdot\|\mathbf{X}_2-\mathbf{X}_1\|_{\mathsf{F}} \qquad \text{(B.255)}$$

so that plugging the latter expression into (B.249) combined with (B.248), and recalling that

$\mathsf{E}\big[\|\boldsymbol{\xi}\|_2^2\big] = n_\mathsf{T}$, we obtain

$$\begin{aligned}
\|\mathbf{E}(\mathbf{X}_1) - \mathbf{E}(\mathbf{X}_2)\|_\mathsf{F} &\le n_\mathsf{R}\rho^2\,\mathsf{E}\Big[\|\hat{\mathbf{H}}\|_\mathsf{F}^2\left((n_\mathsf{T}{}^2+1)\,\mathsf{E}[\|\tilde{\mathbf{H}}\|_\mathsf{F}^2 \mid \hat{\mathbf{H}}] + \|\hat{\mathbf{H}}\|_\mathsf{F}^2\right)\Big]\cdot\|\mathbf{X}_2-\mathbf{X}_1\|_\mathsf{F} \\
&\le n_\mathsf{R}\rho^2(n_\mathsf{T}{}^2+1)\,\mathsf{E}\Big[\left(\mathsf{E}[\|\tilde{\mathbf{H}}\|_\mathsf{F}^2 \mid \hat{\mathbf{H}}] + \|\hat{\mathbf{H}}\|_\mathsf{F}^2\right)^2\Big]\cdot\|\mathbf{X}_2-\mathbf{X}_1\|_\mathsf{F} \\
&\le n_\mathsf{R}\rho^2(n_\mathsf{T}{}^2+1)\,\mathsf{E}\Big[\left(\mathsf{E}[\|\mathbf{H}\|_\mathsf{F}^2 \mid \hat{\mathbf{H}}]\right)^2\Big]\cdot\|\mathbf{X}_2-\mathbf{X}_1\|_\mathsf{F} \\
&\le n_\mathsf{R}\rho^2(n_\mathsf{T}{}^2+1)\,\mathsf{E}[\|\mathbf{H}\|_\mathsf{F}^4]\cdot\|\mathbf{X}_2-\mathbf{X}_1\|_\mathsf{F}.
\end{aligned} \tag{B.256}$$

Hence, if $\mathsf{E}\big[\|\mathbf{H}\|_\mathsf{F}^4\big]$ is finite, the Lipschitz property (B.241) follows. This concludes the proof of Lemma (2.3).

B.16 Proof of Lemma B.5

Since the matrix $\mathbf{A}$, whose (i,j)-th entry we shall denote as $a_{i,j}$, is assumed to be positive semidefinite, we have that the diagonal entries $a_{i,i}$ are non-negative, and that $a_{i,j} = a_{j,i}^*$. A further consequence is that all the principal submatrices of $\mathbf{A}$ are positive semidefinite. In particular, any 2×2 principal submatrix is positive semidefinite, and therefore

$$a_{i,i}a_{j,j} - |a_{i,j}|^2 \ge 0. \tag{B.257}$$

Written differently,

$$\begin{aligned}
|a_{i,j}| &\le \sqrt{a_{i,i}a_{j,j}} \\
&\le \frac{a_{i,i}+a_{j,j}}{2},
\end{aligned} \tag{B.258}$$

where the last step follows from the inequality between the geometric mean and the arithmetic mean. An upper bound on the one-norm $\|\mathbf{A}\|_1$ is therefore given by the one-norm of a matrix whose (i,j)-th entry is $(a_{i,i}+a_{j,j})/2$, i.e.,

$$\begin{aligned}
\|\mathbf{A}\|_1 &= \sum_{i,j}|a_{i,j}| \\
&\le \sum_{i,j}\frac{a_{i,i}+a_{j,j}}{2} \\
&= n_\mathsf{T}\cdot\mathrm{tr}(\mathbf{A}).
\end{aligned} \tag{B.259}$$

This upper bound is achieved when all entries of $\mathbf{A}$ are equal to a common non-negative number.

B.17 Proof of Lemma 2.4

For an arbitrary layering $\mathbf{L}\in\mathbb{L}$ and $\epsilon > 0$, we provide an explicit construction of a continuously differentiable layering $\widetilde{\mathbf{L}} \in \mathbb{L}_\mathcal{D}$ such that $\|\mathbf{L} - \widetilde{\mathbf{L}}\|_\infty < \epsilon$. The proposed construction is in two steps: we first construct a function $\mathbf{F}\colon [0;1] \to \mathbb{C}_+^{n_\mathsf{T}\times n_\mathsf{T}}$ (not necessarily a layering function) which is continuously differentiable and arbitrarily close to $\mathbf{L}$, and then another function $\widetilde{\mathbf{L}} \in \mathbb{L}$ which is a layering function arbitrarily close to $\mathbf{F}$.

Let us continuously extend the layering function $\mathbf{L}$ to negative arguments by setting $\mathbf{L}(\iota) = \mathbf{0}$ for $\iota < 0$. Let $\Phi\colon \mathbb{R} \to \mathbb{R}_+$ be an arbitrary non-negative, continuously differentiable function with finite derivative, support set[10] $(0;1)$ and which fulfills $\int_0^1 \Phi(t)\,\mathrm{d}t = 1$. Additionally, we require that there exist $t_0 > 0$ and $\alpha > 0$ such that $\Phi(t) \geq \alpha t^2$ for all $0 \leq t \leq t_0$.[11] For example, one can choose a raised-cosine

$$\Phi(\iota) = \begin{cases} 1-\cos(2\pi\iota), & 0 \leq \iota \leq 1 \\ 0, & \iota \leq 0 \text{ or } \iota \geq 1 \end{cases} \tag{B.260}$$

for which $\alpha = 1$ and $t_0 = \frac{1}{8}$ can be chosen. We define $\mathbf{F}(\iota)$ as a convolution-type integral

$$\mathbf{F}(\iota) \triangleq \int_0^1 \mathbf{L}((1+t\epsilon)\iota - t\epsilon)\Phi(t)\,\mathrm{d}t. \tag{B.261}$$

This integral is always defined, because Φ is differentiable with finite derivative, and the entries of $\mathbf{L}$ are Lipschitz-continuous (and thus integrable). In order to show that $\mathbf{F}$ is differentiable, we perform the change of variable $u = (1+t\epsilon)\iota - t\epsilon$ (valid only for $\iota \neq 1$) so as to get

$$\begin{aligned} \mathbf{F}(\iota) &= \int_{\iota-\epsilon(1-\iota)}^{\iota} \frac{\mathbf{L}(u)}{\epsilon(1-\iota)}\Phi\left(\frac{\iota-u}{\epsilon(1-\iota)}\right)\mathrm{d}u \\ &\triangleq \mathbf{G}(\iota,\iota) - \mathbf{G}(\iota-\epsilon(1-\iota),\iota), \quad (0 \leq \iota < 1). \end{aligned} \tag{B.262}$$

where we have defined $\mathbf{G}(u,\iota)$ as an antiderivative (with respect to u) of the integrand from the previous expression, i.e.,

$$\frac{\partial \mathbf{G}}{\partial u}(u,\iota) \triangleq \frac{\mathbf{L}(u)}{\epsilon(1-\iota)}\Phi\left(\frac{\iota-u}{\epsilon(1-\iota)}\right), \quad (0 \leq \iota < 1). \tag{B.263}$$

This antiderivative exists because $\mathbf{L}(\cdot)$ and $\Phi(\cdot)$ are both integrable functions. Also note that the partial derivative of $\mathbf{G}(u,\iota)$ with respect to ι also exists because $\frac{\partial \mathbf{G}}{\partial u}$ is partially differentiable with respect to ι. Therefore, we infer that $\mathbf{F}$ is continuously differentiable on $[0;1)$. Since it appears from (B.261) that $\mathbf{F}$ is left-continuous around $\iota \uparrow 1$ with finite limit $\lim_{\iota\uparrow 1}\mathbf{F}(\iota) = \mathbf{Q}$, it further follows that $\mathbf{F}$ is continuously differentiable on the entire interval $\iota \in [0;1]$.

We next prove that $\mathbf{F}$ is arbitrarily close to $\mathbf{L}$. Let us define $\Re\mathbf{F}$ and $\Re\mathbf{L}$ as the real parts,

[10] The term *support set* shall refer to the set on which Φ is non-zero.

[11] Any less restrictive condition will also do, but since Φ can be chosen freely, there is no loss of generality in making this assumption.

and $\Im\mathbf{F}$ and $\Im\mathbf{L}$ as the imaginary parts, respectively, of $\mathbf{F}$ and $\mathbf{L}$.

$$\begin{aligned}\left\|\mathbf{L}-\mathbf{F}\right\|_\infty &= \max_{i,j}\sup_{\iota\in[0;1]}\left|L_{i,j}(\iota)-F_{i,j}(\iota)\right| \\ &\le \max_{i,j}\sup_{\iota\in[0;1]}\left\{\left|\Re L_{i,j}(\iota)-\Re F_{i,j}(\iota)\right|\right\}+\max_{i,j}\sup_{\iota\in[0;1]}\left\{\left|\Im L_{i,j}(\iota)-\Im F_{i,j}(\iota)\right|\right\}. \qquad \text{(B.264)}\end{aligned}$$

Let us focus on the first term, corresponding to the real parts (the other term, corresponding to the imaginary parts, can be treated in the exact same way). According to (2.63), $L_{i,j}$ is Lipschitz-continuous with Lipschitz constant 1, so it follows that [cf. (B.261)]

$$\Re F_{i,j}(\iota)=\int_0^1 \Re L_{i,j}(\iota-\epsilon(1-\iota)t)\Phi(t)\,\mathrm{d}t \qquad \text{(B.265)}$$

can be upper- and lower-bounded by observing that for $0\le t\le 1$,

$$\begin{aligned}\left|\Re L_{i,j}(\iota-\epsilon(1-\iota)t)-\Re L_{i,j}(\iota)\right| &\le \left|L_{i,j}(\iota-\epsilon(1-\iota)n_{\mathsf{T}})-L_{i,j}(\iota)\right| \\ &\le \epsilon(1-\iota) \\ &\le \epsilon. \qquad \text{(B.266)}\end{aligned}$$

Hence, considering that the same inequality holds for the imaginary part, we get by combining (B.265) and (B.266)

$$\left|\Re L_{i,j}(\iota)-\Re F_{i,j}(\iota)\right|\le\epsilon, \qquad \left|\Im L_{i,j}(\iota)-\Im F_{i,j}(\iota)\right|\le\epsilon \qquad \text{(B.267)}$$

and thus

$$\left\|\mathbf{L}-\mathbf{F}\right\|_\infty\le 2\epsilon. \qquad \text{(B.268)}$$

Now that we have shown an example of a continuously differentiable function $\mathbf{F}$ which is arbitrarily close to $\mathbf{L}$, we will construct a layering function $\widetilde{\mathbf{L}}$ which is arbitrarily close to $\mathbf{F}$.

For this purpose, let us first study the trace of $\mathbf{F}(\iota)$. Recalling that $\mathrm{tr}(\mathbf{L}(\iota))=\iota$ for $\iota\in[0;1]$ and that $\mathrm{tr}(\mathbf{L}(\iota))=0$ for $\iota\le 0$ (due to the convention $\mathbf{L}(\iota)=\mathbf{0}$ for $\iota<0$), we observe that the trace of $\mathbf{F}(\iota)$, which we shall denote as $\tau(\iota)$, is given by

$$\begin{aligned}\tau(\iota) &\triangleq \mathrm{tr}(\mathbf{F}(\iota)) \\ &= \int_0^1 \left[\iota-\epsilon t(1-\iota)\right]^+\Phi(t)\,\mathrm{d}t \\ &= \int_0^{\min\left(\frac{\iota}{\epsilon(1-\iota)},1\right)} \left[\iota-\epsilon t(1-\iota)\right]\Phi(t)\,\mathrm{d}t. \qquad \text{(B.269)}\end{aligned}$$

Here, $[\cdot]^+$ stands for $\max(\cdot,0)$. For any $0\le\iota_1<\iota_2\le 1$, we denote $m(\iota)=\min\left(\frac{\iota}{\epsilon(1-\iota)},1\right)$ and notice that $m(\iota_1) < m(\iota_2)$, and compute the difference

$$\begin{aligned}\tau(\iota_2) - \tau(\iota_1) &= \int_0^{m(\iota_2)} [\iota_2 - \epsilon t(1-\iota_2)]\,\Phi(t)\,\mathrm{d}t - \int_0^{m(\iota_1)} [\iota_1 - \epsilon t(1-\iota_1)]\,\Phi(t)\,\mathrm{d}t \\ &= (\iota_2 - \iota_1)\int_0^{m(\iota_1)} (1+t\epsilon)\Phi(t)\,\mathrm{d}t + \int_{m(\iota_1)}^{m(\iota_2)} [\iota_2 - \epsilon t(1-\iota_2)]\,\Phi(t)\,\mathrm{d}t. \qquad \text{(B.270)}\end{aligned}$$

From inspecting the last line of (B.270), we draw a few conclusions on the properties of the function τ:

1) Continuity: Both integrands in (B.270) are finite, and since $\iota \mapsto m(\iota)$ is continuous, it is easily seen that τ is continuous.

2) Monotonicity: Both integrals in (B.270) are non-negative. If $\iota_1 = 0$, then $m(\iota_1) = 0$ and $m(\iota_2) > 0$ so the second integral is positive because its integrand is positive on $[m(\iota_1); m(\iota_2))$. Else, if $\iota_1 > 0$, then the first integral is positive. Therefore, τ is strictly increasing.

3) Border values: The function τ has endpoints $\tau(0) = 0$ and $\tau(1) = 1$.

4) Lower bound on the secant slope: The secant slope is lower-bounded as

$$\frac{\tau(\iota_2) - \tau(\iota_1)}{\iota_2 - \iota_1} \geq \int_0^{m(\iota_1)} (1+t\epsilon)\Phi(t)\,\mathrm{d}t \geq \int_0^{m(\iota_1)} \Phi(t)\,\mathrm{d}t. \qquad \text{(B.271)}$$

If $\iota_1 > \frac{\epsilon}{1+\epsilon}$, then $m(\iota_1) = 1$ and due to $\int_0^1 \Phi(t)\,\mathrm{d}t = 1$ the above lower bound becomes 1. Otherwise, let us assume that ϵ is sufficiently small (without loss of generality) to satisfy $\frac{\epsilon}{1+\epsilon} < t_0$. Now if $\iota_1 \leq \frac{\epsilon}{1+\epsilon} < t_0$, we can lower-bound the secant slope as

$$\begin{aligned}\frac{\tau(\iota_2) - \tau(\iota_1)}{\iota_2 - \iota_1} &\geq \alpha \int_0^{\frac{\iota_1}{\epsilon(1-\iota_1)}} t^2\,\mathrm{d}t \\ &\geq \frac{\alpha}{3\epsilon^3}\left(\frac{\iota_1}{1-\iota_1}\right)^3. \qquad \text{(B.272)}\end{aligned}$$

Summing up, we have for sufficiently small $\epsilon > 0$ the lower bound

$$\left|\frac{\tau(\iota_2) - \tau(\iota_1)}{\iota_2 - \iota_1}\right| \geq \min\left\{1, \frac{\alpha}{3\epsilon^3}\left(\frac{\iota_1}{1-\iota_1}\right)^3\right\}. \qquad \text{(B.273)}$$

5) Differentiability: Since $\mathbf{F}$ has been shown above to be continuously differentiable, its trace $\tau(\iota) = \mathrm{tr}(\mathbf{F}(\iota))$ is too.

Due to its strict monotonicity, we infer that the function τ has an inverse $\tau^{-1}\colon [0;1] \to [0;1]$. Said inverse

- is continuous due to the continuity of τ,
- is strictly monotone due to the continuity and strict monotonicity of τ,
- has border values $\tau^{-1}(0) = 0$ and $\tau^{-1}(1) = 1$,
- is continuously differentiable on $(0; 1]$ due to τ being continuous, continuously differentiable and monotone, and due to the secant slope of τ being lower-bounded by a positive constant on any interval comprised in $(0; 1]$.

Let us define the function

$$\widetilde{\mathbf{L}}(\iota) = \mathbf{F}(\tau^{-1}(\iota)) \tag{B.274}$$

and verify in the following, using the above findings on τ and τ^{-1}, that it is a continuously differentiable layering function, and that it is arbitrarily close to $\mathbf{L}$.

Since τ^{-1} and $\mathbf{F}$ are continuously differentiable on $(0; 1]$ and $[0; 1]$, respectively, we infer that $\widetilde{\mathbf{L}}(\iota)$, which results from the concatenation $\iota \mapsto \tau^{-1}(\iota) \mapsto \mathbf{F}(\tau^{-1}(\iota))$, is continuously differentiable on $(0; 1]$. Additionally, we know that $\lim_{\iota \downarrow 0} \widetilde{\mathbf{L}}(\iota) = \widetilde{\mathbf{L}}(0) = \mathbf{0}$ due to the continuity of $\mathbf{F}$ and τ^{-1}, so it follows that $\widetilde{\mathbf{L}}(\iota)$ is continuously differentiable on the closed interval $[0; 1]$.

The function $\widetilde{\mathbf{L}}(\iota)$ further satisfies all defining properties of layering functions:

1) Continuity: The function $\widetilde{\mathbf{L}}$ inherits the continuity of τ^{-1} and $\mathbf{F}$.

2) Positive semidefiniteness: Since $\mathbf{L}(\cdot)$ is positive semidefinite and $\Phi(\cdot)$ is non-negative, it follows from its definition (B.261) that $\mathbf{F}(\iota)$ is positive semidefinite, and so is $\mathbf{F}(\tau^{-1}(\iota)) = \widetilde{\mathbf{L}}(\iota)$.

3) Matrix-monotonicity: For any $0 \leq \iota_1 < \iota_2 \leq 1$,

$$\begin{aligned}\widetilde{\mathbf{L}}(\iota_2) - \widetilde{\mathbf{L}}(\iota_1) &= \mathbf{F}(\tau^{-1}(\iota_2)) - \mathbf{F}(\tau^{-1}(\iota_1)) \\ &= \int_0^1 \left[\mathbf{L}((1 + t\epsilon)\tau^{-1}(\iota_2) - t\epsilon) - \mathbf{L}((1 + t\epsilon)\tau^{-1}(\iota_1) - t\epsilon)\right] \Phi(t)\, \mathrm{d}t \\ &\succeq \mathbf{0}.\end{aligned} \tag{B.275}$$

The last step follows from the monotoniticity of τ^{-1} and from the matrix-monotonicity of $\mathbf{L}$.

4) Border values: We have $\widetilde{\mathbf{L}}(0) = \mathbf{0}$ and $\widetilde{\mathbf{L}}(1) = \mathbf{Q}$.

5) Trace normalization: Using the definition of τ, we have $\mathrm{tr}(\widetilde{\mathbf{L}}(\iota)) = \mathrm{tr}(\mathbf{F}(\tau^{-1}(\iota))) = \tau(\tau^{-1}(\iota)) = \iota$.

Now that we have shown that $\widetilde{\mathbf{L}}$ is a continuously differentiable layering function, it only remains to show that $\widetilde{\mathbf{L}}$ is arbitrarily close to $\mathbf{F}$ (and thus to $\mathbf{L}$). From (B.268) it follows that for any $\iota \in [0;1]$,

$$\left|\mathrm{tr}(\mathbf{F}(\iota)) - \mathrm{tr}(\mathbf{L}(\iota))\right| = |\tau(\iota) - \iota| \leq 2n_{\mathsf{T}}\epsilon. \tag{B.276}$$

Here, exploiting the bijectivity of τ, we perform the substitution $\iota' = \tau(\iota)$ to obtain, for any $\iota' \in [0;1]$,

$$\left|\iota' - \tau^{-1}(\iota')\right| \leq 2n_{\mathsf{T}}\epsilon. \tag{B.277}$$

It follows that

$$\begin{aligned}\|\widetilde{\mathbf{L}} - \mathbf{F}\|_\infty &= \max_{i,j} \sup_{\iota\in[0;1]} \left|F_{i,j}(\tau^{-1}(\iota)) - F_{i,j}(\iota)\right| \\ &\leq \sup_{\iota\in[0;1]} \left|\tau^{-1}(\iota) - \iota\right| \\ &\leq 2n_{\mathsf{T}}\epsilon.\end{aligned} \tag{B.278}$$

The first bounding step is due to Lemma B.5, stated within Appendix B.15 [cf. Equation B.245] and proven in Appendix B.16. This concludes the proof of Lemma 2.4.

C

Appendices to Chapter 3

C.1 Proof of Lemma 2.5

Let us endow the set of layering functions $\mathbb{L}$ with the metric

$$d(\mathbf{L}_1, \mathbf{L}_2) = \|\mathbf{L}_1 - \mathbf{L}_2\|_\infty = \max_{i,j} \sup_{0 \le \iota \le 1} \left| \left[\mathbf{L}_1(\iota)\right]_{i,j} - \left[\mathbf{L}_2(\iota)\right]_{i,j} \right|. \tag{C.1}$$

to form the metric space $(\mathbb{L}, d)$. This metric space is totally bounded, since for any $\mathbf{L} \in \mathbb{L}$ and any $\iota \in [0;1]$ we have $\mathbf{0} \preceq \mathbf{L}(\iota) \preceq \mathbf{Q}$ and therefore

$$d(\mathbf{L}_1, \mathbf{L}_2) \le \|\mathbf{Q}\|_\infty = \max_i \left[\mathbf{Q}\right]_{i,i} \le \|\mathbf{Q}\|_{\mathrm{tr}} = 1. \tag{C.2}$$

Since all layering functions are Lipschitz-continuous [cf. (2.63)] with a modulus of continuity not larger than 1, we have that any sequence of layering functions $\{\mathbf{L}_n\}_{n\in\mathbb{N}}$ is uniformly equicontinuous in $(\mathbb{L}, d)$. By the Arzelà-Ascoli Theorem, it follows that every sequence $\{\mathbf{L}_n\}_{n\in\mathbb{N}}$ has a convergent subsequence, so $(\mathbb{L}, d)$ is *relatively compact*.

Next, we argue that $(\mathbb{L}, d)$ is complete. Consider a Cauchy sequence $\{\mathbf{L}_n\}_{n\in\mathbb{N}}$. Since this sequence converges uniformly, it also converges pointwise, in the sense that for any given $\iota \in [0,1]$, $\{\mathbf{L}_n(\iota)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for the infinite matrix norm $\|\mathbf{A}\|_\infty = \max_{i,j}|[\mathbf{A}]_{i,j}|$. Let us define the pointwise limit

$$\mathbf{L}_\infty(\iota) \triangleq \lim_{n\to\infty} \mathbf{L}_n(\iota). \tag{C.3}$$

Since the terms of the sequences $\{\mathbf{L}_n(\iota)\}_{n\in\mathbb{N}}$ and $\{\mathbf{L}_n(\iota_2) - \mathbf{L}_n(\iota_1)\}_{n\in\mathbb{N}}$ (for any $0 \le \iota_1 < \iota_2 \le 1$) all belong to the complete metric space $(\mathbb{C}_+^{n_\mathsf{T}\times n_\mathsf{T}}, d)$, it follows that their pointwise limits $\mathbf{L}_\infty(\iota)$ and $\mathbf{L}_\infty(\iota_2) - \mathbf{L}_\infty(\iota_1)$ also belong to $\mathbb{C}_+^{n_\mathsf{T}\times n_\mathsf{T}}$. Furthermore, we have $\mathbf{L}_\infty(0) = \mathbf{0}$ and $\mathbf{L}_\infty(1) = \mathbf{Q}$

and

$$\operatorname{tr}(\mathbf{L}_\infty) = \operatorname{tr}\left(\lim_{n\to\infty} \mathbf{L}_n(\iota)\right) = \lim_{n\to\infty} \operatorname{tr}(\mathbf{L}_n(\iota)) = \iota. \tag{C.4}$$

Therefore, any pointwise limit $\mathbf{L}_\infty$ belongs to $\mathbb{L}$ and it thus follows that the metric space $(\mathbb{L}, d)$ is complete.

Since $(\mathbb{L}, d)$ is relatively compact and complete, it is compact. Due to $\mathbf{L} \mapsto R^\star(\mathbf{L})$ being a continuous mapping from $\mathbb{L}$ to $\mathbb{R}$ with respect to the metric d provided that $\mathsf{E}\left[\|\mathbf{H}\|_\mathsf{F}^4\right]$ is finite (cf. Theorem 2.3), it follows that the image set $R^\star(\mathbb{L})$ is a closed subset of the real numbers. Hence, $R^\star$ possesses a maximizer in $\mathbb{L}$.

C.2 Derivation of (3.13)

Setting the layering function to $\mathbf{L}_{\text{stag}}$ in the generic expression (3.3) and splitting the integration domain into n_T equal intervals, we get

$$R^\star(\mathbf{L}_{\text{stag}}) = \hat{V} \sum_{i=1}^{n_\mathsf{T}} \int_{\frac{i-1}{n_\mathsf{T}}}^{\frac{i}{n_\mathsf{T}}} \mathsf{E} \operatorname{tr}\left[\hat{\mathbf{W}}^\dagger \boldsymbol{\Gamma}(\mathbf{L}_{\text{stag}}(\iota))^{-1} \hat{\mathbf{W}} \, \mathrm{d}\mathbf{L}_{\text{stag}}(\iota)\right]. \tag{C.5}$$

On each closed integration interval $\left[\frac{i-1}{n_\mathsf{T}}; \frac{i}{n_\mathsf{T}}\right]$, the function $\mathbf{L}_{\text{stag}}(\iota)$ is continuously differentiable and reads as

$$\mathbf{L}_{\text{stag}}(\iota) \triangleq \frac{1}{n_\mathsf{T}} \begin{bmatrix} \mathbf{I}_{(i-1)\times(i-1)} & & \mathbf{0} \\ & \iota n_\mathsf{T} - i + 1 & \\ \mathbf{0} & & \mathbf{0}_{(n_\mathsf{T}-i)\times(n_\mathsf{T}-i)} \end{bmatrix}, \qquad \frac{i-1}{n_\mathsf{T}} \le \iota \le \frac{i}{n_\mathsf{T}} \tag{C.6}$$

or in more concise notation

$$\mathbf{L}_{\text{stag}}(\iota) = \frac{1}{n_\mathsf{T}} \sum_{j=1}^{i-1} \mathbf{e}_j \mathbf{e}_j^\mathrm{T} + \left(\iota + \frac{1-i}{n_\mathsf{T}}\right) \mathbf{e}_i \mathbf{e}_i^\mathrm{T}, \qquad \frac{i-1}{n_\mathsf{T}} \le \iota \le \frac{i}{n_\mathsf{T}} \tag{C.7}$$

where $[\mathbf{e}_1, \ldots, \mathbf{e}_{n_\mathsf{T}}] = \mathbf{I}_{n_\mathsf{T}}$ denotes the canonical basis of the $\mathbb{C}^{n_\mathsf{T}\times n_\mathsf{T}}$. Its derivative is

$$\dot{\mathbf{L}}_{\text{stag}}(\iota) = \mathbf{e}_i \mathbf{e}_i^\mathrm{T}, \qquad \frac{i-1}{n_\mathsf{T}} \le \iota \le \frac{i}{n_\mathsf{T}}. \tag{C.8}$$

Thus, the Riemann-Stieltjes integrals in the sum in (C.5) can be written as Riemann integrals [cf. Remark 2.7 under Theorem 2.6], which gives us

$$R^\star(\mathbf{L}_{\text{stag}}) = \hat{V} \sum_{i=1}^{n_\mathsf{T}} \int_{\frac{i-1}{n_\mathsf{T}}}^{\frac{i}{n_\mathsf{T}}} \mathsf{E}\left[\mathbf{e}_i^\mathrm{T} \hat{\mathbf{W}}^\dagger \boldsymbol{\Gamma}(\mathbf{L}_{\text{stag}}(\iota))^{-1} \hat{\mathbf{W}} \mathbf{e}_i\right] \mathrm{d}\iota. \tag{C.9}$$

By the Fubini Theorem [Rud87, Theorem 8.8], the expectation operator and the integral over ι can be exchanged. After a change of variable $\nu = \iota n_\mathsf{T} - i + 1$, we get

$$R^\star(\mathbf{L}_{\text{stag}}) = \frac{\hat{V}}{n_\mathsf{T}} \mathsf{E}\left[\sum_{i=1}^{n_\mathsf{T}} \int_0^1 \mathbf{e}_i^\mathrm{T} \hat{\mathbf{W}}^\dagger \mathbf{T}_i(\nu)^{-1} \hat{\mathbf{W}} \mathbf{e}_i \, \mathrm{d}\nu\right] \tag{C.10}$$

where

$$\mathbf{T}_i(\nu) = \mathbf{\Gamma}\left(\mathbf{L}_{\text{stag}}\left(\frac{\nu + i - 1}{n_\mathsf{T}}\right)\right) = \mathbf{\Gamma}\left(\frac{1}{n_\mathsf{T}} \sum_{j=1}^{i-1} \mathbf{e}_j \mathbf{e}_j^\mathrm{T} + \frac{\nu}{n_\mathsf{T}} \mathbf{e}_i \mathbf{e}_i^\mathrm{T}\right). \tag{C.11}$$

Let $\hat{\mathbf{w}}_i$ denote the i-th column of $\hat{\mathbf{W}}$ and let $\hat{\mathbf{W}}_{n:m}$ denote the matrix formed by the columns n through m of $\hat{\mathbf{W}}$. Writing out $\mathbf{T}_i(\nu)$ by means of (3.4), one can show after some algebra that

$$n_\mathsf{T} \mathbf{T}_i(\nu) = \alpha_i(\nu) \mathbf{I}_{n_\mathsf{R}} + \hat{V} \hat{\mathbf{W}}_{(i+1):n_\mathsf{T}} \hat{\mathbf{W}}_{(i+1):n_\mathsf{T}}^\dagger + \hat{V}(1-\nu) \hat{\mathbf{w}}_i \hat{\mathbf{w}}_i^\dagger \tag{C.12}$$

with

$$\alpha_i(\nu) \triangleq \tilde{V}\left(\frac{\Xi_{i-1}}{n_\mathsf{T}} + \frac{\nu}{n_\mathsf{T}} \tilde{\Xi}_1 + 1 - \nu + n_\mathsf{T} - i\right) + \rho^{-1}. \tag{C.13}$$

and where Ξ_{i-1} and $\tilde{\Xi}_1$ are gamma-distributed with shape $i-1$ and scale 1, and unit-mean exponentially distributed, respectively. Expression (C.10) becomes

$$\begin{aligned} R^\star(\mathbf{L}_{\text{stag}}) &= \hat{V}\, \mathsf{E}\left[\sum_{i=1}^{n_\mathsf{T}} \int_0^1 \hat{\mathbf{w}}_i^\dagger \left(\alpha_i(\nu) \mathbf{I}_{n_\mathsf{R}} + \hat{V} \hat{\mathbf{W}}_{(i+1):n_\mathsf{T}} \hat{\mathbf{W}}_{(i+1):n_\mathsf{T}}^\dagger + \hat{V}(1-\nu) \hat{\mathbf{w}}_i \hat{\mathbf{w}}_i^\dagger\right)^{-1} \hat{\mathbf{w}}_i \, \mathrm{d}\nu\right] \\ &= \hat{V}\, \mathsf{E}\left[\sum_{i=1}^{n_\mathsf{T}} \int_0^1 \frac{\hat{\mathbf{w}}_i^\dagger \left(\alpha_i(\nu) \mathbf{I}_{n_\mathsf{R}} + \hat{V} \hat{\mathbf{W}}_{(i+1):n_\mathsf{T}} \hat{\mathbf{W}}_{(i+1):n_\mathsf{T}}^\dagger\right)^{-1} \hat{\mathbf{w}}_i}{1 + \hat{V}(1-\nu) \hat{\mathbf{w}}_i^\dagger \left(\alpha_i(\nu) \mathbf{I}_{n_\mathsf{R}} + \hat{V} \hat{\mathbf{W}}_{(i+1):n_\mathsf{T}} \hat{\mathbf{W}}_{(i+1):n_\mathsf{T}}^\dagger\right)^{-1} \hat{\mathbf{w}}_i} \, \mathrm{d}\nu\right] \end{aligned} \tag{C.14}$$

where the last equality follows from applying the Matrix Inversion Lemma.

C.3 Proof of Theorem 3.1

Since $\mathbf{L}$ is continuously differentiable and $\hat{\mathbf{H}}$ and $\tilde{\mathbf{H}}$ are assumed to be independent, we can use the representation (2.85) for the infinite-layering bound $R^\star(\mathbf{L})$. Under the i.i.d. Rayleigh fading assumption (3.1), this specializes to

$$R^\star(\mathbf{L}) = -\int_0^1 \mathsf{E}\left[K_\iota\left(\tilde{V}(\boldsymbol{\xi}^\dagger \mathbf{L}(\iota) \boldsymbol{\xi} - \iota) \mathbf{I}_{n_\mathsf{R}}\right)\right] \mathrm{d}\iota \tag{C.15}$$

where

$$K_\iota(\mathbf{\Delta}) = \frac{\mathrm{d}}{\mathrm{d}\iota} \mathsf{E}\left[\log\det\left(\mathbf{\Delta} + (\tilde{V} + \rho^{-1}) \mathbf{I}_{n_\mathsf{R}} + \hat{V} \hat{\mathbf{W}}\left(\tfrac{1}{n_\mathsf{T}} \mathbf{I}_{n_\mathsf{T}} - \mathbf{L}(\iota)\right) \hat{\mathbf{W}}^\dagger\right)\right]. \tag{C.16}$$

Since the distribution of $\hat{\mathbf{W}}$ is rotationally invariant, we see that the function K_ι does not depend on the eigenbasis of $\mathbf{L}(\iota)$. Similarly, since the distribution of $\boldsymbol{\xi}$ is rotationally invariant, it follows that the integrand (C.15) does not depend on the eigenbasis of $\mathbf{L}(\iota)$.

If $\mathbf{L}(\iota) = \mathbf{U_L}(\iota)\mathbf{\Lambda_L}(\iota)\mathbf{U_L}(\iota)^\dagger$ denotes the eigendecomposition of $\mathbf{L}(\iota)$, then $\mathbf{\Lambda_L} \in \mathbb{L}$ is also a layering function, and we have

$$R^\star(\mathbf{L}) = R^\star(\mathbf{\Lambda_L}) \tag{C.17}$$

which concludes the proof.

C.4 Proof of Theorem 3.2

The difference $I(\mathbf{x}_\mathrm{G};\mathbf{y}|\hat{\mathbf{H}}) - R^\star(\mathbf{L})$ is clearly non-negative, and it is upper-bounded by

$$\Sigma = \underline{R} + \Delta - R^\star(\mathbf{L}) \tag{C.18}$$

due to $\underline{R} + \Delta$ being an upper bound on $I(\mathbf{x}_\mathrm{G};\mathbf{y}|\hat{\mathbf{H}})$ [cf. (2.56)]. In the following, we will write $\hat{\mathbf{H}} = \sqrt{\hat{V}}\hat{\mathbf{W}}$ and $\tilde{\mathbf{H}} = \sqrt{\tilde{V}}\tilde{\mathbf{W}}$, with $\hat{\mathbf{W}}$ and $\tilde{\mathbf{W}}$ having i.i.d. entries distributed as $\mathcal{N}_\mathbb{C}(0,1)$. Here, the three quantities involved are respectively written out as [cf. (2.54),(2.57),(2.81)]

$$\underline{R} = \mathsf{E}\log\det\Big(\tfrac{1}{n_\mathsf{T}}\hat{V}\hat{\mathbf{W}}\hat{\mathbf{W}} + (\tilde{V}+\rho^{-1})\mathbf{I}_{n_\mathsf{R}}\Big) - n_\mathsf{R}\log(\tilde{V}+\rho^{-1}) \tag{C.19a}$$

$$\Delta = n_\mathsf{R}\,\mathsf{E}\log\frac{\tilde{V}+\rho^{-1}}{\tilde{V}\Xi_{n_\mathsf{T}}+\rho^{-1}} \tag{C.19b}$$

$$R^\star(\mathbf{L}) = \hat{V}\,\mathsf{E}\int_0^1 \mathrm{tr}\left[\mathbf{\Gamma}(\mathbf{L}(\iota))^{-1}\hat{\mathbf{W}}\dot{\mathbf{L}}(\iota)\hat{\mathbf{W}}^\dagger\right]\mathrm{d}\iota \tag{C.19c}$$

where [cf. (2.55)]

$$\mathbf{\Gamma}(\mathbf{X}) = \Big(\tilde{V}\boldsymbol{\xi}^\dagger\mathbf{X}\boldsymbol{\xi} + \tilde{V}(1-\iota) + \rho^{-1}\Big)\mathbf{I}_{n_\mathsf{R}} + \hat{V}\hat{\mathbf{W}}(\tfrac{1}{n_\mathsf{T}}\mathbf{I}_{n_\mathsf{T}} - \mathbf{X})\hat{\mathbf{W}}^\dagger. \tag{C.20}$$

To prove Theorem 3.2, we will prove that Σ tends to zero as ρ tends to infinity. First note that due to

$$\frac{\mathrm{d}\mathbf{\Gamma}(\mathbf{L}(\iota))}{\mathrm{d}\iota} = \tilde{V}(\boldsymbol{\xi}^\dagger\dot{\mathbf{L}}(\iota)\boldsymbol{\xi} - 1)\mathbf{I}_{n_\mathsf{R}} - \hat{V}\hat{\mathbf{W}}\dot{\mathbf{L}}(\iota)\hat{\mathbf{W}}^\dagger \tag{C.21}$$

we have

$$R^\star = -\,\mathsf{E}\int_0^1 \mathrm{tr}\left[\mathbf{\Gamma}(\mathbf{L}(\iota))^{-1}\frac{\mathrm{d}\mathbf{\Gamma}(\mathbf{L}(\iota))}{\mathrm{d}\iota}\right]\mathrm{d}\iota + \tilde{V}\,\mathsf{E}\int_0^1 \mathrm{tr}\left[\mathbf{\Gamma}(\mathbf{L}(\iota))^{-1}(\boldsymbol{\xi}^\dagger\dot{\mathbf{L}}(\iota)\boldsymbol{\xi}-1)\right]\mathrm{d}\iota \tag{C.22}$$

wherein the first term is

$$\begin{aligned}-\,\mathsf{E}\int_0^1 \mathrm{tr}\left[\mathbf{\Gamma}(\mathbf{L}(\iota))^{-1}\frac{\mathrm{d}\mathbf{\Gamma}(\mathbf{L}(\iota))}{\mathrm{d}\iota}\right] &= -\,\mathsf{E}\left[\log\det\Big(\mathbf{\Gamma}(\mathbf{L}(1))\Big) - \log\det\Big(\mathbf{\Gamma}(\mathbf{L}(0))\Big)\right]\\ &= -n_\mathsf{R}\,\mathsf{E}\log\det(\tilde{V}\Xi_{n_\mathsf{T}}+\rho^{-1}) +\\ &\qquad + \mathsf{E}\log\det\Big(\tfrac{1}{n_\mathsf{T}}\hat{V}\hat{\mathbf{W}}\hat{\mathbf{W}} + (\tilde{V}+\rho^{-1})\mathbf{I}_{n_\mathsf{R}}\Big).\end{aligned} \tag{C.23}$$

In the expression of Σ, six terms cancel out to yield

$$\Sigma = \tilde{V}\,\mathsf{E}\int_0^1 (1-\boldsymbol{\xi}^\dagger\dot{\mathbf{L}}(\iota)\boldsymbol{\xi})\,\mathrm{tr}\left[\mathbf{\Gamma}(\mathbf{L}(\iota))^{-1}\right]\mathrm{d}\iota. \tag{C.24}$$

Since $\mathbf{\Gamma}(\mathbf{L}(\iota))$ is positive semidefinite, the trace of its inverse is positive, so that Σ can be upper-bounded using

$$\begin{aligned}1 - \boldsymbol{\xi}^\dagger \dot{\mathbf{L}}(\iota)\boldsymbol{\xi} &\leq 1 + \boldsymbol{\xi}^\dagger \dot{\mathbf{L}}(\iota)\boldsymbol{\xi} \\ &\leq 1 + \mathrm{tr}(\dot{\mathbf{L}}(\iota))\|\boldsymbol{\xi}\|_2^2 \\ &= 1 + \|\boldsymbol{\xi}\|_2^2 \end{aligned} \tag{C.25}$$

where the last equality is due to the property $\mathrm{tr}(\mathbf{L}(\iota)) = \iota$ [cf. (4)]. This yields an upper bound

$$\Sigma \leq \tilde{V}\,\mathsf{E}\left[(1 + \|\boldsymbol{\xi}\|_2^2)\int_0^1 \mathrm{tr}\left[\mathbf{\Gamma}(\mathbf{L}(\iota))^{-1}\right]\,\mathrm{d}\iota\right]. \tag{C.26}$$

Let $0 < \nu < 1$ be some arbitrary number. We now decompose $\Sigma = \Sigma_1 + \Sigma_2$ into a sum of two terms by partitioning the integration interval $\iota \in [0;1]$ into $[0;\nu) \cup [\nu;1]$. The first term Σ_1 shall correspond to the interval $[0;\nu)$, while the second term Σ_2 shall correspond to the interval $[\nu;1]$. Both terms Σ_1 and Σ_2 will be upper-bounded in the following. To begin with, consider that for $0 \leq \iota < \nu$, we have the matrix lower bound

$$\mathbf{\Gamma}(\mathbf{L}(\iota)) \succeq \tilde{V}(1-\nu)\mathbf{I}_{n_\mathsf{R}} + \hat{V}\hat{\mathbf{W}}(\tfrac{1}{n_\mathsf{T}}\mathbf{I}_{n_\mathsf{T}} - \mathbf{L}(\nu))\hat{\mathbf{W}}^\dagger \tag{C.27}$$

which, when plugged into Σ_1, yields an upper bound

$$\begin{aligned}\Sigma_1 &= \tilde{V}\,\mathsf{E}\left[(1 + \|\boldsymbol{\xi}\|_2^2)\int_0^\nu \mathrm{tr}\left[\mathbf{\Gamma}(\mathbf{L}(\iota))^{-1}\right]\,\mathrm{d}\iota\right] \\ &\leq \tilde{V}(1+n_\mathsf{T})\,\mathsf{E}\int_0^\nu \mathrm{tr}\left[\left(\tilde{V}(1-\nu)\mathbf{I}_{n_\mathsf{R}} + \hat{V}\hat{\mathbf{W}}(\tfrac{1}{n_\mathsf{T}}\mathbf{I}_{n_\mathsf{T}} - \mathbf{L}(\nu))\hat{\mathbf{W}}^\dagger\right)^{-1}\right]\,\mathrm{d}\iota \\ &= n_\mathsf{R}(1+n_\mathsf{T})\nu\,\mathsf{E}\left[\frac{\tilde{V}}{\tilde{V}(1-\nu) + \hat{V}\hat{\lambda}_\nu}\right] \end{aligned} \tag{C.28}$$

where $\hat{\lambda}_\nu$ is a random variable following the empirical eigenvalue distribution of the matrix $\hat{\mathbf{W}}(\frac{1}{n_\mathsf{T}}\mathbf{I}_{n_\mathsf{T}} - \mathbf{L}(\nu))\hat{\mathbf{W}}^\dagger$. Since $\hat{\lambda}_\nu$ is strictly positive almost surely, we can condition the expectation appearing in the last line of (C.28) on $\hat{\lambda}_\nu > 0$. Recalling that $\hat{V} = 1 - \tilde{V}$, and noting that said expectation is upper-bounded by $1/(1-\nu)$, we can interchange the expectation operator and the limit as $\rho \to \infty$ by means of the Dominated Convergence Theorem, to get

$$\begin{aligned}\limsup_{\rho\to\infty}\Sigma_1 &\leq n_\mathsf{R}(1+n_\mathsf{T})\nu \cdot \mathsf{E}\left[\limsup_{\rho\to\infty}\left\{\frac{\tilde{V}}{\tilde{V}(1-\nu-\hat{\lambda}_\nu) + \hat{\lambda}_\nu}\right\}\middle|\hat{\lambda}_\nu > 0\right] \\ &= 0 \end{aligned} \tag{C.29}$$

due to $\lim_{\rho\to\infty}\tilde{V} = 0$. Similarly, for $\nu \leq \iota \leq 1$, we can use the lower bound

$$\mathbf{\Gamma}(\mathbf{L}(\iota)) \succeq (\tilde{V}\boldsymbol{\xi}^\dagger\mathbf{L}(\nu)\boldsymbol{\xi} + \rho^{-1})\mathbf{I}_{n_\mathsf{R}} \tag{C.30}$$

by means of which we construct an upper bound on Σ_2 as

$$\begin{aligned}\Sigma_2 &= \tilde{V}\,\mathsf{E}\left[\left(1+\|\boldsymbol{\xi}\|_2^2\right)\int_{\nu}^{1}\operatorname{tr}\left[\boldsymbol{\Gamma}(\mathbf{L}(\iota))^{-1}\right]\mathrm{d}\iota\right] \\ &\leq n_{\mathsf{R}}(1-\nu)\,\mathsf{E}\left[\frac{\tilde{V}(1+\|\boldsymbol{\xi}\|_2^2)}{\tilde{V}\boldsymbol{\xi}^\dagger\mathbf{L}(\nu)\boldsymbol{\xi}+\rho^{-1}}\right].\end{aligned} \tag{C.31}$$

Since $\nu \in (0;1)$ is arbitrary, let us choose a constant $\nu_0 \in (\frac{n_{\mathsf{T}}-1}{n_{\mathsf{T}}};1)$ and constrain ν to belong to the interval $(\nu_0;1)$. This choice ensures that $\mathbf{L}(\nu)$ is full rank, because each eigenvalue of $\mathbf{L}(\nu)$ is upper-bounded by $1/n_{\mathsf{T}}$ (the multiple eigenvalue of the transmit covariance), while as a consequence of (4), the eigenvalues of $\mathbf{L}(\nu)$ sum up to $\operatorname{tr}(\mathbf{L}(\nu)) = \nu \geq \nu_0 > \frac{n_{\mathsf{T}}-1}{n_{\mathsf{T}}}$. Hence, the smallest eigenvalue of $\mathbf{L}(\nu)$ is

$$\lambda_{\min}(\mathbf{L}(\nu)) \geq \nu_0 - \frac{n_{\mathsf{T}}-1}{n_{\mathsf{T}}} > 0 \tag{C.32}$$

so that Σ_2 can be further upper-bounded as

$$\Sigma_2 \leq n_{\mathsf{R}}(1-\nu)\,\mathsf{E}\left[\frac{\tilde{V}(1+\|\boldsymbol{\xi}\|_2^2)}{\tilde{V}(\nu_0-\frac{n_{\mathsf{T}}-1}{n_{\mathsf{T}}})\|\boldsymbol{\xi}\|_2^2+\rho^{-1}}\right]. \tag{C.33}$$

Since the fraction inside the expectation operator is finite for $\|\boldsymbol{\xi}\|_2 = 0$, and since the random variable $\|\boldsymbol{\xi}\|_2^2$ is strictly positive almost surely, we can condition said expectation on $\|\boldsymbol{\xi}\|_2 > 0$ without affecting its value. Then, by removing ρ^{-1} from the denominator, we obtain the upper bound

$$\Sigma_2 \leq \frac{n_{\mathsf{R}}(1-\nu)}{\nu_0-\frac{n_{\mathsf{T}}-1}{n_{\mathsf{T}}}}\,\mathsf{E}\left[\frac{1+\|\boldsymbol{\xi}\|_2^2}{\|\boldsymbol{\xi}\|_2^2}\middle|\|\boldsymbol{\xi}\|_2>0\right]. \tag{C.34}$$

Without loss of generality, we can assume that $n_{\mathsf{T}} \geq 2$, since for a single transmit antenna, the rate-splitting bound $R^\star(\mathbf{L})$ does not depend on the layering function. For $n_{\mathsf{T}} \geq 2$, the expected value of $\|\boldsymbol{\xi}\|_2^{-2}$ is finite. Therefore, the upper bound (C.34) is finite and reads as

$$\Sigma_2 \leq \frac{n_{\mathsf{R}}(1-\nu)}{\nu_0-\frac{n_{\mathsf{T}}-1}{n_{\mathsf{T}}}}\left(1+\mathsf{E}\left[\|\boldsymbol{\xi}\|_2^{-2}\right]\right). \tag{C.35}$$

Since ν can be chosen freely in the interval $(\nu_0;1)$, the upper bound (C.35) can be made arbitrarily close to zero by choosing ν arbitrarily close to one. We conclude that upper bounds on $\Sigma = \Sigma_1 + \Sigma_2$ can be found whose limit as $\rho \to \infty$ is arbitrarily close to zero. Consequently, Σ tends to zero as $\rho \to \infty$, which finalizes the proof of Theorem 3.2.

C.5 Proof of Theorem 3.3

The matrix $\frac{1}{n_{\mathsf{T}}}\hat{\mathbf{W}}_{n_{\mathsf{T}}}^\dagger\hat{\mathbf{W}}_{n_{\mathsf{T}}}$ converges to $\mathbf{I}_{n_{\mathsf{R}}}$ almost surely as $n_{\mathsf{T}} \to \infty$. Since the function

$$\mathbf{X} \mapsto \log\left(\mathbf{I}_{n_{\mathsf{R}}} + \frac{\hat{V}}{\tilde{V}+\rho^{-1}}\mathbf{X}\right) \tag{C.36}$$

is continuous in the vicinity of $\mathbf{X} \to \mathbf{I}_{n_\mathsf{R}}$, it follows from the Continuous Mapping Theorem that $\underline{R}_{n_\mathsf{T}}$ converges to

$$\lim_{n_\mathsf{T}\to\infty} \underline{R}_{n_\mathsf{T}} = n_\mathsf{R} \log\left(1 + \frac{\hat{V}}{\tilde{V} + \rho^{-1}}\right). \tag{C.37}$$

It now suffices to prove that

$$\lim_{n_\mathsf{T}\to\infty} \underline{R}^\star_{n_\mathsf{T}}(\mathbf{L}_{n_\mathsf{T}}) = n_\mathsf{R} \log\left(1 + \frac{\hat{V}}{\tilde{V} + \rho^{-1}}\right). \tag{C.38}$$

For this purpose, let us fix ι to an arbitrary value from $[0; 1]$, and consider the functions

$$f_{n_\mathsf{T},\iota} \colon X \mapsto \mathsf{E}\,\mathrm{tr}\Big[\hat{\mathbf{W}}^\dagger_{n_\mathsf{T}} \mathbf{\Gamma}_{n_\mathsf{T}}(X)^{-1} \hat{\mathbf{W}}_{n_\mathsf{T}} \dot{\mathbf{L}}_{n_\mathsf{T}}(\iota)\Big] \tag{C.39a}$$

$$\tilde{f}_{n_\mathsf{T},\iota} \colon X \mapsto \mathsf{E}\,\mathrm{tr}\left[\frac{\hat{\mathbf{W}}_{n_\mathsf{T}} \dot{\mathbf{L}}_{n_\mathsf{T}}(\iota) \hat{\mathbf{W}}^\dagger_{n_\mathsf{T}}}{\tilde{V}X + 1 - \iota + \rho^{-1}}\right] = \frac{n_\mathsf{R}}{\tilde{V}X + 1 - \iota + \rho^{-1}} \tag{C.39b}$$

in which [cf. (3.22)]

$$\mathbf{\Gamma}_{n_\mathsf{T}}(X) = \Big(\tilde{V}X + \tilde{V}(1-\iota) + \rho^{-1}\Big)\mathbf{I}_{n_\mathsf{R}} + \hat{V}\hat{\mathbf{W}}_{n_\mathsf{T}} \bar{\mathbf{L}}_{n_\mathsf{T}}(\iota) \hat{\mathbf{W}}^\dagger_{n_\mathsf{T}}. \tag{C.40}$$

where we have used the abbreviation $\bar{\mathbf{L}}_{n_\mathsf{T}}(\iota) \triangleq \frac{1}{n_\mathsf{T}}\mathbf{I}_{n_\mathsf{T}} - \mathbf{L}_{n_\mathsf{T}}(\iota)$. By comparing (C.39) with (3.21)–(3.22), we notice that the rate-splitting bound $\underline{R}^\star_{n_\mathsf{T}}(\mathbf{L}_{n_\mathsf{T}})$ can be written as

$$\underline{R}^\star_{n_\mathsf{T}}(\mathbf{L}_{n_\mathsf{T}}) = \hat{V} \int_0^1 \mathsf{E}\left[f_{n_\mathsf{T},\iota}(\boldsymbol{\xi}^\dagger \mathbf{L}_{n_\mathsf{T}}(\iota)\boldsymbol{\xi})\right] \mathrm{d}\iota \tag{C.41}$$

where the expectation is over the distribution of $\boldsymbol{\xi} \in \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{I})$.

Notice that the expectation in (C.39a) is taken only over the distribution of $\hat{\mathbf{W}}$, and that $\tilde{f}_{n_\mathsf{T},\iota}$ results from replacing the term $\hat{\mathbf{W}}_{n_\mathsf{T}} \bar{\mathbf{L}}_{n_\mathsf{T}}(\iota) \hat{\mathbf{W}}^\dagger_{n_\mathsf{T}}$ inside the expression of $\mathbf{\Gamma}_{n_\mathsf{T}}(X)$ by its expected value

$$\mathsf{E}\left[\hat{\mathbf{W}}_{n_\mathsf{T}} \bar{\mathbf{L}}_{n_\mathsf{T}}(\iota) \hat{\mathbf{W}}^\dagger_{n_\mathsf{T}}\right] = (1-\iota)\mathbf{I}_{n_\mathsf{R}} \tag{C.42}$$

and by recalling that $\hat{V} + \tilde{V} = 1$.

Next, we show that the difference $f_{n_\mathsf{T},\iota} - \tilde{f}_{n_\mathsf{T},\iota}$ tends pointwise to zero as $n_\mathsf{T} \to \infty$. For this purpose, we upper-bound the magnitude of this difference by means of

$$\begin{aligned}
|f_{n_\mathsf{T},\iota}(X) - \tilde{f}_{n_\mathsf{T},\iota}(X)| &= \left|\mathsf{E}\,\mathrm{tr}\left[\left(\mathbf{\Gamma}_{n_\mathsf{T}}(X)^{-1} - \frac{\mathbf{I}_{n_\mathsf{R}}}{\tilde{V}X + 1 - \iota + \rho^{-1}}\right) \hat{\mathbf{W}}_{n_\mathsf{T}} \dot{\mathbf{L}}_{n_\mathsf{T}}(\iota) \hat{\mathbf{W}}^\dagger_{n_\mathsf{T}}\right]\right| \\
&= \left|\mathsf{E}\,\mathrm{tr}\left[\left(\frac{\mathbf{\Gamma}_{n_\mathsf{T}}(X)^{-1}\left(\hat{\mathbf{W}}_{n_\mathsf{T}} \bar{\mathbf{L}}_{n_\mathsf{T}}(\iota) \hat{\mathbf{W}}^\dagger_{n_\mathsf{T}} - (1-\iota)\mathbf{I}_{n_\mathsf{R}}\right)}{\tilde{V}X + 1 - \iota + \rho^{-1}}\right) \hat{\mathbf{W}}_{n_\mathsf{T}} \dot{\mathbf{L}}_{n_\mathsf{T}}(\iota) \hat{\mathbf{W}}^\dagger_{n_\mathsf{T}}\right]\right| \\
&\le \rho\,\mathsf{E}\left|\mathrm{tr}\left[\mathbf{\Gamma}_{n_\mathsf{T}}(X)^{-1}\left(\hat{\mathbf{W}}_{n_\mathsf{T}} \bar{\mathbf{L}}_{n_\mathsf{T}}(\iota) \hat{\mathbf{W}}^\dagger_{n_\mathsf{T}} - (1-\iota)\mathbf{I}_{n_\mathsf{R}}\right) \hat{\mathbf{W}}_{n_\mathsf{T}} \dot{\mathbf{L}}_{n_\mathsf{T}}(\iota) \hat{\mathbf{W}}^\dagger_{n_\mathsf{T}}\right]\right| \\
&\le \rho\,\mathsf{E}\left[\left\|\mathbf{\Gamma}_{n_\mathsf{T}}(X)^{-1}\left(\hat{\mathbf{W}}_{n_\mathsf{T}} \bar{\mathbf{L}}_{n_\mathsf{T}}(\iota) \hat{\mathbf{W}}^\dagger_{n_\mathsf{T}} - (1-\iota)\mathbf{I}_{n_\mathsf{R}}\right)\right\|_\mathrm{tr} \left\|\hat{\mathbf{W}}_{n_\mathsf{T}} \dot{\mathbf{L}}_{n_\mathsf{T}}(\iota) \hat{\mathbf{W}}^\dagger_{n_\mathsf{T}}\right\|_\mathrm{tr}\right]
\end{aligned} \tag{C.43}$$

where $\|\mathbf{A}\|_{\text{tr}} \triangleq \operatorname{tr}\big((\mathbf{A}^\dagger\mathbf{A})^{\frac{1}{2}}\big)$ stands for the trace norm. Here, the second equality is obtained using the identity $\mathbf{A}^{-1} - \mathbf{B}^{-1} = \mathbf{A}^{-1}(\mathbf{B}-\mathbf{A})\mathbf{B}^{-1}$; the first inequality is obtained by Jensen's inequality applied on the convex function $|\cdot|$, and by lower-bounding the denominator $\tilde{V}X + 1 - \iota + \rho^{-1}$ by ρ^{-1}; the second inequality is a consequence of von Neumann's trace inequality.[1] Note that for a fixed trace norm $\|\mathbf{A}\|_{\text{tr}} = \sum_{i=1}^n \sigma_i(\mathbf{A})$, the Frobenius norm $\|\mathbf{A}\|_{\mathsf{F}} = \big(\sum_{i=1}^n \sigma_i(\mathbf{A})^2\big)^{\frac{1}{2}}$ is minimal when all singular values $\sigma_i(\mathbf{A})$ are equal to $\|\mathbf{A}\|_{\text{tr}}/n$. Therefore the trace norm and Frobenius norm are related via an inequality

$$\|\mathbf{A}\|_{\text{tr}} \leq \sqrt{n}\|\mathbf{A}\|_{\mathsf{F}}. \tag{C.44}$$

We apply the latter onto (C.43), and exploit the submultiplicative property $\|\mathbf{AB}\|_{\mathsf{F}} \leq \|\mathbf{A}\|_{\mathsf{F}}\|\mathbf{B}\|_{\mathsf{F}}$ to get

$$\begin{aligned}
&\left|f_{n_{\mathsf{T}},\iota}(X) - \tilde{f}_{n_{\mathsf{T}},\iota}(X)\right| \\
&\quad\leq \rho n_{\mathsf{R}}\, \mathsf{E}\left[\left\|\mathbf{\Gamma}_{n_{\mathsf{T}}}(X)^{-1}\right\|_{\mathsf{F}} \left\|\hat{\mathbf{W}}_{n_{\mathsf{T}}}\bar{\mathbf{L}}_{n_{\mathsf{T}}}(\iota)\hat{\mathbf{W}}_{n_{\mathsf{T}}}^\dagger - (1-\iota)\mathbf{I}_{n_{\mathsf{R}}}\right\|_{\mathsf{F}} \|\hat{\mathbf{W}}_{n_{\mathsf{T}}}\dot{\mathbf{L}}_{n_{\mathsf{T}}}(\iota)\hat{\mathbf{W}}_{n_{\mathsf{T}}}^\dagger\|_{\mathsf{F}}\right] \\
&\quad\leq \rho^2 n_{\mathsf{R}}\sqrt{n_{\mathsf{R}}}\, \mathsf{E}\left[\left\|\hat{\mathbf{W}}_{n_{\mathsf{T}}}\bar{\mathbf{L}}_{n_{\mathsf{T}}}(\iota)\hat{\mathbf{W}}_{n_{\mathsf{T}}}^\dagger - (1-\iota)\mathbf{I}_{n_{\mathsf{R}}}\right\|_{\mathsf{F}} \|\hat{\mathbf{W}}_{n_{\mathsf{T}}}\dot{\mathbf{L}}_{n_{\mathsf{T}}}(\iota)\hat{\mathbf{W}}_{n_{\mathsf{T}}}^\dagger\|_{\mathsf{F}}\right] \\
&\quad\leq \rho^2 n_{\mathsf{R}}\sqrt{n_{\mathsf{R}}}\sqrt{\mathsf{E}\left[\left\|\hat{\mathbf{W}}_{n_{\mathsf{T}}}\bar{\mathbf{L}}_{n_{\mathsf{T}}}(\iota)\hat{\mathbf{W}}_{n_{\mathsf{T}}}^\dagger - (1-\iota)\mathbf{I}_{n_{\mathsf{R}}}\right\|_{\mathsf{F}}^2\right] \mathsf{E}\left[\|\hat{\mathbf{W}}_{n_{\mathsf{T}}}\dot{\mathbf{L}}_{n_{\mathsf{T}}}(\iota)\hat{\mathbf{W}}_{n_{\mathsf{T}}}^\dagger\|_{\mathsf{F}}^2\right]}.
\end{aligned} \tag{C.45}$$

Here, the second inequality is obtained by noting that $\mathbf{\Gamma}_{n_{\mathsf{T}}}(X)^{-1} \preceq \rho\mathbf{I}_{n_{\mathsf{R}}}$, hence $\|\mathbf{\Gamma}_{n_{\mathsf{T}}}(X)^{-1}\|_{\mathsf{F}} \leq \rho\sqrt{n_{\mathsf{R}}}$; the third bounding step is due to the Cauchy-Schwarz inequality.

Next, we determine an upper bound on the expectation $\mathsf{E}\big[\|\hat{\mathbf{W}}_{n_{\mathsf{T}}}\dot{\mathbf{L}}_{n_{\mathsf{T}}}(\iota)\hat{\mathbf{W}}_{n_{\mathsf{T}}}^\dagger\|_{\mathsf{F}}^2\big]$. Note that for any $\hat{\mathbf{W}}$, the expression $\|\hat{\mathbf{W}}\dot{\mathbf{L}}_{n_{\mathsf{T}}}(\iota)\hat{\mathbf{W}}^\dagger\|_{\mathsf{F}}^2 = \operatorname{tr}\big(\hat{\mathbf{W}}\dot{\mathbf{L}}_{n_{\mathsf{T}}}(\iota)\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}\dot{\mathbf{L}}_{n_{\mathsf{T}}}(\iota)\hat{\mathbf{W}}^\dagger\big)$ is convex in $\dot{\mathbf{L}}_{n_{\mathsf{T}}}(\iota)$ on the cone of positive semidefinite matrices, and thus convex in the diagonal entries of $\dot{\mathbf{L}}_{n_{\mathsf{T}}}(\iota)$. Therefore, if we represent $\dot{\mathbf{L}}_{n_{\mathsf{T}}}(\iota) \triangleq \operatorname{diag}\big(\dot{L}_{n_{\mathsf{T}},1}(\iota), \ldots, \dot{L}_{n_{\mathsf{T}},n_{\mathsf{T}}}(\iota)\big)$ as a convex combination of matrices $\mathbf{e}_i\mathbf{e}_i^{\mathrm{T}}$, where $\mathbf{e}_i, i = 1, \ldots, n_{\mathsf{T}}$ denote the canonical unit vectors, i.e.,

$$\dot{\mathbf{L}}_{n_{\mathsf{T}}}(\iota) = \sum_{i=1}^{n_{\mathsf{T}}} \dot{L}_{n_{\mathsf{T}},i}(\iota)\mathbf{e}_i\mathbf{e}_i^{\mathrm{T}} \tag{C.46}$$

then we can use said convexity to upper-bound the expectation $\mathsf{E}\big[\|\hat{\mathbf{W}}_{n_{\mathsf{T}}}\dot{\mathbf{L}}_{n_{\mathsf{T}}}(\iota)\hat{\mathbf{W}}_{n_{\mathsf{T}}}^\dagger\|_{\mathsf{F}}^2\big]$ as

$$\begin{aligned}
\mathsf{E}\left[\|\hat{\mathbf{W}}_{n_{\mathsf{T}}}\dot{\mathbf{L}}_{n_{\mathsf{T}}}(\iota)\hat{\mathbf{W}}_{n_{\mathsf{T}}}^\dagger\|_{\mathsf{F}}^2\right] &= \mathsf{E}\left[\left\|\hat{\mathbf{W}}_{n_{\mathsf{T}}}\left(\sum_{i=1}^{n_{\mathsf{T}}} \dot{L}_{n_{\mathsf{T}},i}(\iota)\mathbf{e}_i\mathbf{e}_i^{\mathrm{T}}\right)\hat{\mathbf{W}}_{n_{\mathsf{T}}}^\dagger\right\|_{\mathsf{F}}^2\right] \\
&\leq \max_{i\in\{1,\ldots,n_{\mathsf{T}}\}} \mathsf{E}\left[\left\|\hat{\mathbf{W}}_{n_{\mathsf{T}}}\mathbf{e}_i\mathbf{e}_i^{\mathrm{T}}\hat{\mathbf{W}}_{n_{\mathsf{T}}}^\dagger\right\|_{\mathsf{F}}^2\right] \\
&= \mathsf{E}\left[\|\hat{\mathbf{w}}_{n_{\mathsf{T}},1}\|_2^4\right]
\end{aligned} \tag{C.47}$$

[1] Von Neumann's trace inequality states that for $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, we have $|\operatorname{tr}(\mathbf{AB})| \leq \sum_{i=1}^n \sigma_i(\mathbf{A})\sigma_i(\mathbf{B})$ where $\sigma_i(\mathbf{X}), i = 1, \ldots, n$ denote the non-increasingly ordered singular values of $\mathbf{X}$. Clearly, this implies the weaker result $|\operatorname{tr}(\mathbf{AB})| \leq \big(\sum_{i=1}^n \sigma_i(\mathbf{A})\big)\big(\sum_{k=1}^n \sigma_k(\mathbf{B})\big) = \|\mathbf{A}\|_{\text{tr}}\|\mathbf{B}\|_{\text{tr}}$.

where $\hat{\mathbf{w}}_{n_\mathsf{T},i} \triangleq \hat{\mathbf{W}}_{n_\mathsf{T}}\mathbf{e}_i, i = 1,\dots,n_\mathsf{T}$ denote the columns of $\hat{\mathbf{W}}_{n_\mathsf{T}}$. The last equality follows because said columns are identically distributed. The vector $\hat{\mathbf{w}}_{n_\mathsf{T},1} \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{I}_{n_\mathsf{R}})$ has a squared Euclidian norm which is gamma-distributed with shape n_R and scale 1. Hence, we can compute the right-hand side of (C.47) as

$$\mathsf{E}\left[\|\hat{\mathbf{w}}_{n_\mathsf{T},1}\|_2^4\right] = \frac{1}{(n_\mathsf{R}-1)!}\int_0^\infty x^{n_\mathsf{R}+1}e^{-x}\,\mathrm{d}x = n_\mathsf{R}(n_\mathsf{R}+1). \tag{C.48}$$

Combining (C.45), (C.47) and (C.48), we obtain

$$|f_{n_\mathsf{T},\iota}(X) - \tilde{f}_{n_\mathsf{T},\iota}(X)| \le \rho^2 n_\mathsf{R}{}^2\sqrt{n_\mathsf{R}+1}\sqrt{\mathsf{E}\left[\left\|\hat{\mathbf{W}}_{n_\mathsf{T}}\bar{\mathbf{L}}_{n_\mathsf{T}}(\iota)\hat{\mathbf{W}}_{n_\mathsf{T}}^\dagger - (1-\iota)\mathbf{I}_{n_\mathsf{R}}\right\|_\mathsf{F}^2\right]}. \tag{C.49}$$

Next, we concentrate on the remaining factor on the right-hand side of (C.49). Note that $\bar{\mathbf{L}}_{n_\mathsf{T}}(\iota) = \frac{1}{n_\mathsf{T}}\mathbf{I}_{n_\mathsf{T}} - \mathbf{L}_{n_\mathsf{T}}(\iota)$ is a diagonal, positive semidefinite matrix whose diagonal entries belong to the interval $\left[0; \frac{1}{n_\mathsf{T}}\right]$ and sum up to $1-\iota \in [0;1]$. Let us define the function

$$\begin{aligned} g\colon \mathbb{R}_+^{n_\mathsf{T}} &\to \mathbb{R} \\ \boldsymbol{\lambda} &\mapsto \mathsf{E}\left[\left\|\hat{\mathbf{W}}_{n_\mathsf{T}}\operatorname{diag}(\boldsymbol{\lambda})\hat{\mathbf{W}}_{n_\mathsf{T}}^\dagger - (1-\iota)\mathbf{I}_{n_\mathsf{R}}\right\|_\mathsf{F}^2\right]. \end{aligned} \tag{C.50}$$

The expectation on the right-hand side of (C.49) can be upper-bounded by means of the maximum value

$$\max_{(\lambda_1,\dots,\lambda_{n_\mathsf{T}})\in\left[0;\frac{1}{n_\mathsf{T}}\right]^{n_\mathsf{T}}} g(\boldsymbol{\lambda}) \quad \text{subject to} \quad \sum_{i=1}^{n_\mathsf{T}}\lambda_i = 1-\iota. \tag{C.51}$$

Due to the symmetry of the constraint and of the objective function g in the entries of $\boldsymbol{\lambda}$, we can restrict the search set to one of non-increasingly ordered entries, that is, we can assume without loss of generality that $\lambda_1 \ge \dots \ge \lambda_{n_\mathsf{T}}$. The resulting search set thus reduces to

$$\left\{\boldsymbol{\lambda} \in \left[0;\frac{1}{n_\mathsf{T}}\right]^{n_\mathsf{T}} \middle| \lambda_1 \ge \dots \ge \lambda_{n_\mathsf{T}}, \sum_{i=1}^{n_\mathsf{T}}\lambda_i = 1-\iota\right\}. \tag{C.52}$$

Definition C.1 (Schur convexity). *A symmetric[2] function $g\colon \mathbb{R}_+^n \to \mathbb{R}$ is said to be Schur convex if for any two vectors $\mathbf{p} = [p_1 \ \dots \ p_n]^\mathrm{T}$ and $\mathbf{q} = [q_1 \ \dots \ q_n]^\mathrm{T}$ with non-negative entries fulfilling*

(1) $p_1 \ge \dots \ge p_n \ge 0$

(2) $q_1 \ge \dots \ge q_n \ge 0$

(3) $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i$

(4) $\sum_{i=1}^j p_i \ge \sum_{i=1}^j q_i$ *for all* $j = 1,\dots,n-1$

[2] A symmetric function is a function whose image is invariant against permutations of its arguments

we have that $g(\mathbf{p}) \geq g(\mathbf{q})$.

A necessary and sufficient condition for a differentiable function g to be Schur convex is that it is symmetric and fulfills

$$(x_i - x_j)\left(\frac{\partial g}{\partial x_i} - \frac{\partial g}{\partial x_j}\right) \geq 0. \tag{C.53}$$

We now verify that the objective function of (C.51), which is symmetric and differentiable, fulfills the condition (C.53) for Schur convexity. Its partial derivatives read as

$$\frac{\partial g}{\partial \lambda_i} = 2\,\mathsf{E}\left[\hat{\mathbf{w}}_{n_\mathsf{T},i}^\dagger \left(\hat{\mathbf{W}}_{n_\mathsf{T}} \operatorname{diag}(\boldsymbol{\lambda})\hat{\mathbf{W}}_{n_\mathsf{T}}^\dagger - (1-\iota)\mathbf{I}_{n_\mathsf{R}}\right)\hat{\mathbf{w}}_{n_\mathsf{T},i}\right]. \tag{C.54}$$

Hence,

$$\begin{aligned}\frac{1}{2}\left(\frac{\partial g}{\partial \lambda_i} - \frac{\partial g}{\partial \lambda_j}\right) &= \mathsf{E}\operatorname{tr}\left[\hat{\mathbf{W}}_{n_\mathsf{T}} \operatorname{diag}(\boldsymbol{\lambda})\hat{\mathbf{W}}_{n_\mathsf{T}}^\dagger \left(\hat{\mathbf{w}}_{n_\mathsf{T},i}\hat{\mathbf{w}}_{n_\mathsf{T},i}^\dagger - \hat{\mathbf{w}}_{n_\mathsf{T},j}\hat{\mathbf{w}}_{n_\mathsf{T},j}^\dagger\right)\right] \\ &= \mathsf{E}\operatorname{tr}\left[\left(\sum_{k=1}^{n_\mathsf{T}} \lambda_k \hat{\mathbf{w}}_{n_\mathsf{T},k}\hat{\mathbf{w}}_{n_\mathsf{T},k}^\dagger\right)\left(\hat{\mathbf{w}}_{n_\mathsf{T},i}\hat{\mathbf{w}}_{n_\mathsf{T},i}^\dagger - \hat{\mathbf{w}}_{n_\mathsf{T},j}\hat{\mathbf{w}}_{n_\mathsf{T},j}^\dagger\right)\right] \\ &= \mathsf{E}\left[\lambda_i \|\hat{\mathbf{w}}_{n_\mathsf{T},i}\|_2^4 - \lambda_j \|\hat{\mathbf{w}}_{n_\mathsf{T},j}\|_2^4\right] \\ &= (\lambda_i - \lambda_j)\,\mathsf{E}\left[\|\hat{\mathbf{w}}_{n_\mathsf{T},1}\|_2^4\right]. \end{aligned} \tag{C.55}$$

We infer that the objective function g is Schur convex. Consider the vector

$$\boldsymbol{\lambda}^\star \triangleq \Bigg[\underbrace{\frac{1}{n_\mathsf{T}} \quad \frac{1}{n_\mathsf{T}} \quad \ldots \quad \frac{1}{n_\mathsf{T}}}_{\left\lfloor \frac{1-\iota}{1/n_\mathsf{T}} \right\rfloor \text{ elements}} \quad (1-\iota) - \left\lfloor \frac{1-\iota}{1/n_\mathsf{T}} \right\rfloor \frac{1}{n_\mathsf{T}} \quad 0 \quad \ldots \quad 0\Bigg]^\mathrm{T} \tag{C.56}$$

We can readily verify that $\boldsymbol{\lambda}^\star$ belongs to the search set (C.52) and that for any vector $\boldsymbol{\lambda}$ from said search set, we have [cf. Definition C.1, Condition 4]

$$\sum_{i=1}^{j} \lambda_i^\star \geq \sum_{i=1}^{j} \lambda_i, \qquad j = 1, \ldots, n-1. \tag{C.57}$$

We conclude from Definition C.1 that $g(\boldsymbol{\lambda}) \leq g(\boldsymbol{\lambda}^\star)$, so $\boldsymbol{\lambda}^\star$ is the maximizer of (C.51). Denoting $\tau \triangleq \lfloor n_\mathsf{T}(1-\iota)\rfloor$, the corresponding maximum $g(\boldsymbol{\lambda}^\star)$ can be bounded as follows:

$$\begin{aligned} g(\boldsymbol{\lambda}^\star) &= \mathsf{E}\left[\left\|\hat{\mathbf{W}}_{n_\mathsf{T}} \operatorname{diag}(\boldsymbol{\lambda}^\star)\hat{\mathbf{W}}_{n_\mathsf{T}}^\dagger - (1-\iota)\mathbf{I}_{n_\mathsf{R}}\right\|_\mathsf{F}^2\right] \\ &= \mathsf{E}\left[\left\|\frac{1}{n_\mathsf{T}}\sum_{i=1}^{\tau} \hat{\mathbf{w}}_{n_\mathsf{T},i}\hat{\mathbf{w}}_{n_\mathsf{T},i}^\dagger + \left(1-\iota-\frac{\tau}{n_\mathsf{T}}\right)\hat{\mathbf{w}}_{n_\mathsf{T},\tau+1}\hat{\mathbf{w}}_{n_\mathsf{T},\tau+1}^\dagger - (1-\iota)\mathbf{I}_{n_\mathsf{R}}\right\|_\mathsf{F}^2\right] \\ &\leq \mathsf{E}\left[\left\|\frac{1}{n_\mathsf{T}}\sum_{i=1}^{\tau+1} \hat{\mathbf{w}}_{n_\mathsf{T},i}\hat{\mathbf{w}}_{n_\mathsf{T},i}^\dagger - (1-\iota)\mathbf{I}_{n_\mathsf{R}}\right\|_\mathsf{F}^2\right] \end{aligned}$$

$$= \mathsf{E}\left[\frac{1}{n_{\mathsf{T}}{}^2}\sum_{i=1}^{\tau+1}\sum_{j=1}^{\tau+1}\left|\hat{\mathbf{w}}_{n_{\mathsf{T}},i}^{\dagger}\hat{\mathbf{w}}_{n_{\mathsf{T}},j}\right|^2 - 2\frac{1-\iota}{n_{\mathsf{T}}}\sum_{k=1}^{\tau+1}\|\hat{\mathbf{w}}_{n_{\mathsf{T}},k}\|_2^2 + n_{\mathsf{R}}(1-\iota)^2\right]$$
$$= \frac{1}{n_{\mathsf{T}}{}^2}\sum_{\substack{i,j=1\\ i\neq j}}^{\tau+1}\operatorname{tr}\mathsf{E}\left[\hat{\mathbf{w}}_{n_{\mathsf{T}},i}\hat{\mathbf{w}}_{n_{\mathsf{T}},i}^{\dagger}\hat{\mathbf{w}}_{n_{\mathsf{T}},j}\hat{\mathbf{w}}_{n_{\mathsf{T}},j}^{\dagger}\right] + \frac{1}{n_{\mathsf{T}}{}^2}\sum_{\ell=1}^{\tau+1}\mathsf{E}\left[\|\hat{\mathbf{w}}_{n_{\mathsf{T}},\ell}\|_2^4\right]$$
$$- 2\frac{1-\iota}{n_{\mathsf{T}}}\sum_{k=1}^{\tau+1}\mathsf{E}\left[\|\hat{\mathbf{w}}_{n_{\mathsf{T}},k}\|_2^2\right] + n_{\mathsf{R}}(1-\iota)^2$$
$$= \frac{\tau(\tau+1)}{n_{\mathsf{T}}{}^2}n_{\mathsf{R}} + \frac{\tau+1}{n_{\mathsf{T}}{}^2}n_{\mathsf{R}}(n_{\mathsf{R}}+1) - 2\frac{1-\iota}{n_{\mathsf{T}}}(\tau+1)n_{\mathsf{R}} + n_{\mathsf{R}}(1-\iota)^2. \tag{C.58}$$

Here, the inequality follows from upper-bounding the factor $\left(1-\iota-\frac{\tau}{n_{\mathsf{T}}}\right)$ by $\frac{1}{n_{\mathsf{T}}}$. For the last equality, we have used $\mathsf{E}\left[\|\hat{\mathbf{w}}_{n_{\mathsf{T}},\ell}\|_2^4\right] = n_{\mathsf{R}}(n_{\mathsf{R}}+1)$, which was already evaluated in (C.48). Using

$$n_{\mathsf{T}}(1-\iota) - 1 \leq \tau \leq n_{\mathsf{T}}(1-\iota) \tag{C.59}$$

we can further upper-bound (C.58) as

$$g(\boldsymbol{\lambda}^{\star})$$
$$\leq \frac{n_{\mathsf{T}}(1-\iota)(n_{\mathsf{T}}(1-\iota)+1)}{n_{\mathsf{T}}{}^2}n_{\mathsf{R}} + \frac{n_{\mathsf{T}}(1-\iota)+1}{n_{\mathsf{T}}{}^2}n_{\mathsf{R}}(n_{\mathsf{R}}+1) - 2\frac{1-\iota}{n_{\mathsf{T}}}n_{\mathsf{T}}(1-\iota)n_{\mathsf{R}} + n_{\mathsf{R}}(1-\iota)^2$$
$$= n_{\mathsf{R}}\left(\frac{n_{\mathsf{T}}(1-\iota)}{n_{\mathsf{T}}{}^2} + \frac{n_{\mathsf{T}}(1-\iota)+1}{n_{\mathsf{T}}{}^2}(n_{\mathsf{R}}+1)\right)$$
$$\leq n_{\mathsf{R}}(n_{\mathsf{R}}+2)\frac{n_{\mathsf{T}}(1-\iota)+1}{n_{\mathsf{T}}{}^2}. \tag{C.60}$$

Finally, plugging (C.60) into (C.49), we end up with the bound

$$\left|f_{n_{\mathsf{T}},\iota}(X) - \tilde{f}_{n_{\mathsf{T}},\iota}(X)\right| \leq \rho^2 n_{\mathsf{R}}{}^2\sqrt{n_{\mathsf{R}}(n_{\mathsf{R}}+1)(n_{\mathsf{R}}+2)}\frac{\sqrt{n_{\mathsf{T}}(1-\iota)+1}}{n_{\mathsf{T}}}. \tag{C.61}$$

Next, by a similar reasoning we will determine an upper bound on the difference [cf. (C.39b)]

$$\left|\mathsf{E}\left[\tilde{f}_{n_{\mathsf{T}},\iota}(\boldsymbol{\xi}^{\dagger}\mathbf{L}_{n_{\mathsf{T}}}(\iota)\boldsymbol{\xi})\right] - \tilde{f}_{n_{\mathsf{T}},\iota}(\iota)\right| = \left|\mathsf{E}\left[\frac{n_{\mathsf{R}}}{\tilde{V}\boldsymbol{\xi}^{\dagger}\mathbf{L}_{n_{\mathsf{T}}}(\iota)\boldsymbol{\xi}+1-\iota+\rho^{-1}}\right] - \frac{n_{\mathsf{R}}}{\tilde{V}X+1-\iota+\rho^{-1}}\right|$$
$$= n_{\mathsf{R}}\tilde{V}\left|\mathsf{E}\left[\frac{\boldsymbol{\xi}^{\dagger}\mathbf{L}_{n_{\mathsf{T}}}(\iota)\boldsymbol{\xi}-\iota}{(\tilde{V}\boldsymbol{\xi}^{\dagger}\mathbf{L}_{n_{\mathsf{T}}}(\iota)\boldsymbol{\xi}+1-\iota+\rho^{-1})(\tilde{V}X+1-\iota+\rho^{-1})}\right]\right|$$
$$\leq n_{\mathsf{R}}\tilde{V}\,\mathsf{E}\left[\frac{\left|\boldsymbol{\xi}^{\dagger}\mathbf{L}_{n_{\mathsf{T}}}(\iota)\boldsymbol{\xi}-\iota\right|}{(\tilde{V}\boldsymbol{\xi}^{\dagger}\mathbf{L}_{n_{\mathsf{T}}}(\iota)\boldsymbol{\xi}+1-\iota+\rho^{-1})(\tilde{V}X+1-\iota+\rho^{-1})}\right]$$
$$\leq n_{\mathsf{R}}\rho^2\tilde{V}\,\mathsf{E}\left[\left|\boldsymbol{\xi}^{\dagger}\mathbf{L}_{n_{\mathsf{T}}}(\iota)\boldsymbol{\xi}-\iota\right|\right]$$
$$\leq n_{\mathsf{R}}\rho^2\tilde{V}\sqrt{\mathsf{E}\left[\left(\boldsymbol{\xi}^{\dagger}\mathbf{L}_{n_{\mathsf{T}}}(\iota)\boldsymbol{\xi}-\iota\right)^2\right]}. \tag{C.62}$$

Here, the first inequality is Jensen's inequality applied onto the convex function $|\cdot|$; the second inequality is obtained by lower-bounding the denominator by ρ^{-2}; the last inequality is again

Jensen's inequality applied on the convex square function. We will now continue upper-bounding the expectation on the right-hand side of (C.62). This expectation reads as

$$\mathsf{E}\left[\left(\boldsymbol{\xi}^\dagger \mathbf{L}_{n_\mathsf{T}}(\iota)\boldsymbol{\xi} - \iota\right)^2\right] = \mathsf{E}\left[\left(\boldsymbol{\xi}^\dagger \mathbf{L}_{n_\mathsf{T}}(\iota)\boldsymbol{\xi}\right)^2\right] - \iota^2 \tag{C.63}$$

because $\mathsf{E}[\boldsymbol{\xi}^\dagger \mathbf{L}_{n_\mathsf{T}}(\iota)\boldsymbol{\xi}] = \mathrm{tr}(\mathbf{L}_{n_\mathsf{T}}(\iota)) = \iota$. Note that $\mathbf{L}_{n_\mathsf{T}}(\iota)$ is a diagonal, positive semidefinite matrix whose diagonal entries belong to the interval $[0; \frac{1}{n_\mathsf{T}}]$ and sum up to $\iota \in [0; 1]$. Let us define the function

$$\begin{aligned} \tilde{g} \colon \mathbb{R}_+^{n_\mathsf{T}} &\to \mathbb{R} \\ \tilde{\boldsymbol{\lambda}} &\mapsto \mathsf{E}\left[\left(\boldsymbol{\xi}^\dagger \operatorname{diag}(\tilde{\boldsymbol{\lambda}})\boldsymbol{\xi}\right)^2\right]. \end{aligned} \tag{C.64}$$

The expectation on the right-hand side of (C.63) can be upper-bounded by the maximum

$$\max_{(\tilde{\lambda}_1,\ldots,\tilde{\lambda}_{n_\mathsf{T}}) \in \left[0;\frac{1}{n_\mathsf{T}}\right]^{n_\mathsf{T}}} \tilde{g}(\tilde{\boldsymbol{\lambda}}) \quad \text{subject to} \quad \sum_{i=1}^{n_\mathsf{T}} \tilde{\lambda}_i = \iota. \tag{C.65}$$

Due to the symmetry of the constraint and of the objective function $\tilde{g}$ in the entries of $\tilde{\boldsymbol{\lambda}}$, we can restrict the search set to one of non-increasingly ordered entries, i.e.,

$$\left\{ \tilde{\boldsymbol{\lambda}} \in \left[0; \frac{1}{n_\mathsf{T}}\right]^{n_\mathsf{T}} \middle| \tilde{\lambda}_1 \geq \ldots \geq \tilde{\lambda}_{n_\mathsf{T}}, \sum_{i=1}^{n_\mathsf{T}} \tilde{\lambda}_i = \iota \right\}. \tag{C.66}$$

The function $\tilde{g}$ is symmetric, differentiable, and Schur convex since

$$\frac{\partial \tilde{g}}{\partial \lambda_i} - \frac{\partial \tilde{g}}{\partial \lambda_j} = 2\,\mathsf{E}\left[\boldsymbol{\xi}^\dagger \operatorname{diag}(\tilde{\boldsymbol{\lambda}})\boldsymbol{\xi} \cdot (|\xi_i|^2 - |\xi_j|^2)\right] \tag{C.67}$$

$$= 2(\lambda_i - \lambda_j)\,\mathsf{E}\left[|\xi_1|^4\right] \tag{C.68}$$

where the last equality is due to the components $\xi_i, i = 1, \ldots, n_\mathsf{T}$ of the vector $\boldsymbol{\xi}$ being identically distributed. The vector

$$\tilde{\boldsymbol{\lambda}}^\star \triangleq \Bigg[\underbrace{\frac{1}{n_\mathsf{T}} \quad \frac{1}{n_\mathsf{T}} \quad \ldots \quad \frac{1}{n_\mathsf{T}}}_{\lfloor \iota n_\mathsf{T} \rfloor \text{ elements}} \quad \iota - \lfloor \iota n_\mathsf{T} \rfloor \frac{1}{n_\mathsf{T}} \quad 0 \quad \ldots \quad 0\Bigg]^\mathrm{T} \tag{C.69}$$

belongs to the search set (C.66) and is such that for any vector $\tilde{\boldsymbol{\lambda}} = [\tilde{\lambda}_1 \ \ldots \ \tilde{\lambda}_{n_\mathsf{T}}]^\mathrm{T}$ belonging to this search set, we have [cf. Definition C.1, Condition 4]

$$\sum_{i=1}^{j} \tilde{\lambda}_i^\star \geq \sum_{i=1}^{j} \tilde{\lambda}_i, \qquad j = 1, \ldots, n-1. \tag{C.70}$$

We conclude from Definition C.1 that $\tilde{g}(\tilde{\boldsymbol{\lambda}}) \leq \tilde{g}(\tilde{\boldsymbol{\lambda}}^\star)$, hence $\tilde{\boldsymbol{\lambda}}^\star$ is the maximizer of (C.51). The resulting maximum $\tilde{g}(\tilde{\boldsymbol{\lambda}}^\star)$ can be bounded as follows:

$$\begin{aligned}
\tilde{g}(\tilde{\boldsymbol{\lambda}}^\star) &= \mathsf{E}\left[\left(\boldsymbol{\xi}^\dagger \operatorname{diag}(\tilde{\boldsymbol{\lambda}}^\star)\boldsymbol{\xi}\right)^2\right] \\
&= \mathsf{E}\left[\left(\frac{1}{n_\mathsf{T}}\sum_{i=1}^{\lfloor \iota n_\mathsf{T}\rfloor}|\xi_i|^2 + \left(\iota - \lfloor \iota n_\mathsf{T}\rfloor \frac{1}{n_\mathsf{T}}\right)|\xi_{\lfloor \iota n_\mathsf{T}\rfloor+1}|^2\right)^2\right] \\
&\le \mathsf{E}\left[\left(\frac{1}{n_\mathsf{T}}\sum_{i=1}^{\lfloor \iota n_\mathsf{T}\rfloor+1}|\xi_i|^2\right)^2\right] \\
&= \frac{1}{{n_\mathsf{T}}^2}\left(2\sum_{1\le i<j\le \lfloor \iota n_\mathsf{T}\rfloor+1}\mathsf{E}\left[|\xi_i|^2|\xi_j|^2\right] + \sum_{k=1}^{\lfloor \iota n_\mathsf{T}\rfloor+1}\mathsf{E}\left[|\xi_k|^4\right]\right) \\
&= \frac{1}{{n_\mathsf{T}}^2}\left(\lfloor \iota n_\mathsf{T}\rfloor(\lfloor \iota n_\mathsf{T}\rfloor+1) + 2(\lfloor \iota n_\mathsf{T}\rfloor+1)\right) \\
&\le \left(\iota + \frac{2}{n_\mathsf{T}}\right)^2.
\end{aligned} \tag{C.71}$$

Here, the first inequality is obtained by upper-bounding the factor $(\iota - \lfloor \iota n_\mathsf{T}\rfloor \frac{1}{n_\mathsf{T}})$ by $\frac{1}{n_\mathsf{T}}$; the last inequality results from upper-bounding the factors $(\lfloor \iota n_\mathsf{T}\rfloor + 1)$ by $(\lfloor \iota n_\mathsf{T}\rfloor + 2)$. In between these inequalities, we have also used

$$\mathsf{E}\left[|\xi_k|^2\right] = \int_0^\infty x\, e^{-x}\,\mathrm{d}x = 1 \qquad\qquad \mathsf{E}\left[|\xi_k|^4\right] = \int_0^\infty x^2 e^{-x}\,\mathrm{d}x = 2. \tag{C.72}$$

By plugging (C.71) into (C.63) to upper-bound the right-hand side of (C.62), we finally obtain

$$\begin{aligned}
\left|\mathsf{E}\left[\tilde{f}_{n_\mathsf{T},\iota}(\boldsymbol{\xi}^\dagger \mathbf{L}_{n_\mathsf{T}}(\iota)\boldsymbol{\xi})\right] - \tilde{f}_{n_\mathsf{T},\iota}(\iota)\right| &\le n_\mathsf{R}\rho^2\tilde{V}\sqrt{\left(\iota + \frac{2}{n_\mathsf{T}}\right)^2 - \iota^2} \\
&= 2r\rho^2\tilde{V}\sqrt{\frac{1}{n_\mathsf{T}}\left(1 + \frac{1}{n_\mathsf{T}}\right)}.
\end{aligned} \tag{C.73}$$

Combining the bounds (C.61) and (C.73) by means of the triangle inequality, we obtain a bound on the difference between the expectation $\mathsf{E}\left[f_{n_\mathsf{T},\iota}(\boldsymbol{\xi}^\dagger \mathbf{L}_{n_\mathsf{T}}(\iota)\boldsymbol{\xi})\right]$ and $\tilde{f}_{n_\mathsf{T},\iota}(\iota)$ which reads as

$$\begin{aligned}
&\left|\mathsf{E}\left[f_{n_\mathsf{T},\iota}(\boldsymbol{\xi}^\dagger \mathbf{L}_{n_\mathsf{T}}(\iota)\boldsymbol{\xi})\right] - \tilde{f}_{n_\mathsf{T},\iota}(\iota)\right| \\
&\qquad \le 2r\rho^2\tilde{V}\sqrt{\frac{1}{n_\mathsf{T}}\left(1 + \frac{1}{n_\mathsf{T}}\right)} + \rho^2{n_\mathsf{R}}^2\sqrt{n_\mathsf{R}(n_\mathsf{R}+1)(n_\mathsf{R}+2)}\frac{\sqrt{n_\mathsf{T}(1-\iota)+1}}{n_\mathsf{T}}.
\end{aligned} \tag{C.74}$$

Since this upper bound tends to zero as n_T tends to infinity, we conclude that the limit of

$\underline{R}^{\star}_{n_\mathsf{T}}(\mathbf{L}_{n_\mathsf{T}})$ as n_T tends to infinity is equal to [cf. (C.41)]

$$\begin{aligned}\lim_{n_\mathsf{T}\to\infty} \underline{R}^{\star}_{n_\mathsf{T}}(\mathbf{L}_{n_\mathsf{T}}) &= \hat{V} \lim_{n_\mathsf{T}\to\infty} \left\{ \int_0^1 \mathsf{E}\left[f_{n_\mathsf{T},\iota}(\boldsymbol{\xi}^\dagger \mathbf{L}_{n_\mathsf{T}}(\iota)\boldsymbol{\xi})\right] \mathrm{d}\iota \right\} \\ &= \hat{V} \lim_{n_\mathsf{T}\to\infty} \left\{ \int_0^1 \tilde{f}_{n_\mathsf{T},\iota}(\iota)\, \mathrm{d}\iota \right\} \\ &= \hat{V} \lim_{n_\mathsf{T}\to\infty} \left\{ \int_0^1 \frac{n_\mathsf{R}}{\tilde{V}\iota + 1 - \iota + \rho^{-1}}\, \mathrm{d}\iota \right\} \\ &= n_\mathsf{R} \log\left(1 + \frac{\hat{V}}{\tilde{V} + \rho^{-1}}\right). \end{aligned} \tag{C.75}$$

In the last step, we have integrated over $\iota \in [0;1]$ and made use of the fact that $\hat{V} + \tilde{V} = 1$. This establishes (C.38) and concludes the proof.

C.6 Proof of Theorem 3.4

Since we are interested in the large-system limit as $n_\mathsf{R} \to \infty$, we will assume throughout that $n_\mathsf{R} \geq n_\mathsf{T}$, without loss of generality.

We start by observing that the worst-case-noise bound under i.i.d. Rayleigh fading (3.2) [or (3.25)] is expressible as the expected value of a function of the Gram form $\hat{V}\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}$ [cf. (3.3)]

$$\begin{aligned}\underline{R} &= \mathsf{E}\left[\log\det\left(\mathbf{I}_{n_\mathsf{T}} + \frac{\hat{V}\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}}{\tilde{V} + \rho^{-1}}\mathbf{Q}\right)\right] \\ &\triangleq \mathsf{E}\left[\underline{R}_{n_\mathsf{R}\times n_\mathsf{T}}\left(\hat{V}\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}, \tilde{V}, \rho\right)\right]\end{aligned} \tag{C.76}$$

where $\underline{R}_{n_\mathsf{R}\times n_\mathsf{T}}(\cdot,\cdot,\cdot)$ is a function of three arguments defined as

$$\begin{aligned}\underline{R}_{n_\mathsf{R}\times n_\mathsf{T}}:\quad \mathbb{C}_+^{n_\mathsf{T}\times n_\mathsf{T}} \times \mathbb{R}_+ \times \mathbb{R}_{++} \quad &\to \quad \mathbb{R}_+ \\ (\mathbf{X}, \tilde{V}, \rho) \quad &\mapsto \quad \log\det\left(\mathbf{I}_{n_\mathsf{T}} + \frac{\mathbf{X}}{\tilde{V} + \rho^{-1}}\mathbf{Q}\right)\end{aligned} \tag{C.77}$$

Here, $\hat{V}$ and the transmit covariance $\mathbf{Q}$ are assumed to have fixed values and are not included as arguments of the function $\underline{R}_{n_\mathsf{R}\times n_\mathsf{T}}$, since their influence on the worst-case-noise bound is not relevant in this proof.

A first observation is that we have the identity

$$\underline{R}_{n_\mathsf{R}\times n_\mathsf{T}}\left(\hat{V}\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}, \tilde{V}, \rho\right) = \underline{R}_{n_\mathsf{T}\times n_\mathsf{T}}\left(\hat{V}\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}, \tilde{V}, \rho\right) = \underline{R}_{n_\mathsf{T}\times n_\mathsf{T}}\left(\hat{\mathbf{C}}^\dagger\hat{\mathbf{C}}, \tilde{V}, \rho\right) \tag{C.78}$$

where $\hat{V}\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}} = \hat{\mathbf{C}}\hat{\mathbf{C}}^\dagger$ is the Cholesky decomposition of the matrix $\hat{V}\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}$, that is, $\hat{\mathbf{C}}$ is a $n_\mathsf{T} \times n_\mathsf{T}$ square matrix. We infer that the worst-case-noise lower bound $\underline{R}(\mathbf{L})$ is not only a mutual information lower bound for a channel with a $n_\mathsf{R} \times n_\mathsf{T}$ tall channel matrix $\sqrt{\hat{V}}\hat{\mathbf{W}}$, but also a mutual information lower bound for a channel with a $n_\mathsf{T} \times n_\mathsf{T}$ square channel matrix $\hat{\mathbf{C}}$.

We call this channel the *associated square channel.*

To make this statement precise, we have that $\underline{R}$ is simultaneously a lower bound on two mutual informations, namely,

$$\underline{R} \leq \min\left\{ I\left(\mathbf{x}_\mathrm{G}; \left(\sqrt{\hat{V}}\hat{\mathbf{W}} + \sqrt{\tilde{V}}\tilde{\mathbf{W}}\right)\mathbf{x}_\mathrm{G} + \mathbf{z}\middle|\hat{\mathbf{W}}\right), I\left(\mathbf{x}_\mathrm{G}; \left(\hat{\mathbf{C}} + \tilde{\mathbf{C}}\right)\mathbf{x}_\mathrm{G} + \mathbf{z}'\middle|\hat{\mathbf{C}}\right)\right\} \tag{C.79}$$

where $\mathrm{vec}(\tilde{\mathbf{C}}) \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \tilde{V}\mathbf{I}_{n_\mathsf{T}^2})$ is independent of $\hat{\mathbf{C}}$ and where $\mathbf{z}' \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{I}_{n_\mathsf{T}})$.

A second observation is that for any $k > 0$, we have

$$\underline{R}_{n_\mathsf{R}\times n_\mathsf{T}}\left(k\mathbf{X}, \tilde{V}, \rho\right) = \underline{R}_{n_\mathsf{R}\times n_\mathsf{T}}\left(\mathbf{X}, \frac{\tilde{V}}{k}, k\rho\right). \tag{C.80}$$

It means that scaling the channel component $\hat{\mathbf{H}} = \sqrt{\hat{V}}\hat{\mathbf{W}}$ by a factor $\sqrt{k}$ amounts to both multiplying the SNR ρ and dividing the channel error variance $\tilde{V}$ by a factor k. Obviously, this property also holds for the associated square channel. It is important to note that here, we do not assume that $\hat{V}$ and $\tilde{V}$ are related via $\hat{V} + \tilde{V} = 1$. Instead, $\hat{V}$ has some arbitrary, constant value throughout. Hence, changing the second argument of the function $\underline{R}_{n_\mathsf{R}\times n_\mathsf{T}}$ does not change the value of $\hat{V}$.

The same two observations apply to the infinite-layering bounds as well. In fact, the infinite-layering bound (3.3) [or (3.26)] reads as

$$R^\star(\mathbf{L}) = \hat{V}\int_0^1 \mathsf{E}\,\mathrm{tr}\left[\hat{\mathbf{W}}^\dagger(\alpha\mathbf{I}_{n_\mathsf{R}} + \hat{V}\hat{\mathbf{W}}\bar{\mathbf{L}}(\iota)\hat{\mathbf{W}}^\dagger)^{-1}\hat{\mathbf{W}}\dot{\mathbf{L}}(\iota)\right]\mathrm{d}\iota \tag{C.81}$$

where α is the random variable

$$\alpha = \tilde{V}\boldsymbol{\xi}^\dagger\mathbf{L}(\iota)\boldsymbol{\xi} + \tilde{V}(1-\iota) + \rho^{-1}. \tag{C.82}$$

Using the reduced factorization $\bar{\mathbf{L}}(\iota) = \mathbf{A}\mathbf{A}^\dagger$ such that $\mathbf{A} \in \mathbb{C}^{n_\mathsf{T}\times\mathrm{rank}(\bar{\mathbf{L}}(\iota))}$, and applying the Matrix Inversion Lemma to the matrix $\hat{V}\hat{\mathbf{W}}^\dagger\boldsymbol{\Gamma}(\mathbf{L}(\iota))^{-1}\hat{\mathbf{W}}$ we obtain

$$\begin{aligned}\hat{V}\hat{\mathbf{W}}^\dagger\boldsymbol{\Gamma}(\mathbf{L}(\iota))^{-1}\hat{\mathbf{W}} &= \hat{V}\hat{\mathbf{W}}^\dagger\left(\alpha\mathbf{I} + \hat{V}\hat{\mathbf{W}}\mathbf{A}\mathbf{A}^\dagger\hat{\mathbf{W}}^\dagger\right)^{-1}\hat{\mathbf{W}} \\ &= \frac{\hat{V}}{\alpha}\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}} - \frac{\hat{V}^2}{\alpha^2}\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}\mathbf{A}\left(\mathbf{I}_{\mathrm{rank}(\mathbf{A})} + \frac{\hat{V}}{\alpha}\mathbf{A}^\dagger\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}\mathbf{A}\right)^{-1}\mathbf{A}^\dagger\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}.\end{aligned} \tag{C.83}$$

This expression reveals that the expression inside the trace operator of (C.81) actually depends on the random matrix $\hat{\mathbf{W}}$ via its Gram form $\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}$. Therefore, similarly to the worst-case-noise bound expressed as the expected value of a function $\underline{R}_{n_\mathsf{R}\times n_\mathsf{T}}$, we can express the infinite-layering bound as an expected value

$$R^\star(\mathbf{L}) = \mathsf{E}\left[R^\star_{n_\mathsf{R}\times n_\mathsf{T}}\left(\hat{V}\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}, \tilde{V}, \rho, \mathbf{L}\right)\right] \tag{C.84}$$

where $R^\star_{n_\mathsf{R}\times n_\mathsf{T}}(\cdot,\cdot,\cdot,\cdot)$ denotes a function of four arguments defined as

$$\begin{aligned} R^\star_{n_\mathsf{R}\times n_\mathsf{T}}: \quad \mathbb{C}_+^{n_\mathsf{T}\times n_\mathsf{T}} \times \mathbb{R}_+ \times \mathbb{R}_{++} \times \mathbb{L}_\mathcal{D} &\to \mathbb{R}_+ \\ (\mathbf{X}, \tilde{V}, \rho, \mathbf{L}) &\mapsto R^\star_{n_\mathsf{R}\times n_\mathsf{T}}(\mathbf{X}, \tilde{V}, \rho, \mathbf{L}) \end{aligned} \tag{C.85}$$

where

$$R^\star_{n_\mathsf{R}\times n_\mathsf{T}}(\mathbf{X}, \tilde{V}, \rho, \mathbf{L}) = \int_0^1 \mathsf{E}\,\mathrm{tr}\left[\left(\frac{\mathbf{X}}{\alpha} - \frac{1}{\alpha^2}\mathbf{X}\mathbf{A}\left(\mathbf{I}_{\mathrm{rank}(\mathbf{A})} + \frac{1}{\alpha}\mathbf{A}^\dagger\mathbf{X}\mathbf{A}\right)^{-1}\mathbf{A}^\dagger\mathbf{X}\right)\dot{\mathbf{L}}(\iota)\right]\mathrm{d}\iota. \tag{C.86}$$

Here, the expectation is over the distribution of α.

Using the same Cholesky decomposition $\hat{V}\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}} = \hat{\mathbf{C}}\hat{\mathbf{C}}^\dagger$ as before, we observe firstly that

$$R^\star_{n_\mathsf{R}\times n_\mathsf{T}}\left(\hat{V}\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}, \tilde{V}, \rho, \mathbf{L}\right) = R^\star_{n_\mathsf{T}\times n_\mathsf{T}}\left(\hat{V}\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}, \tilde{V}, \rho, \mathbf{L}\right) = R^\star_{n_\mathsf{T}\times n_\mathsf{T}}\left(\hat{\mathbf{C}}^\dagger\hat{\mathbf{C}}, \tilde{V}, \rho, \mathbf{L}\right) \tag{C.87}$$

so it can be inferred that the infinite-layering bound $R^\star(\mathbf{L})$ is not only a mutual information lower bound for a channel with a $n_\mathsf{R}\times n_\mathsf{T}$ tall random channel matrix $\sqrt{\hat{V}}\hat{\mathbf{W}}$, but also a mutual information lower bound for its associated $n_\mathsf{T}\times n_\mathsf{T}$ square channel with random channel matrix $\hat{\mathbf{C}}$.

More precisely, we have that $R^\star(\mathbf{L})$ is a common lower bound for two mutual informations, namely

$$R^\star(\mathbf{L}) \le \min\left\{ I\left(\mathbf{x}_\mathrm{G}; \left(\sqrt{\hat{V}}\hat{\mathbf{W}} + \sqrt{\tilde{V}}\tilde{\mathbf{W}}\right)\mathbf{x}_\mathrm{G} + \mathbf{z}\middle|\hat{\mathbf{W}}\right), I\left(\mathbf{x}_\mathrm{G}; \left(\hat{\mathbf{C}} + \tilde{\mathbf{C}}\right)\mathbf{x}_\mathrm{G} + \mathbf{z}'\middle|\hat{\mathbf{C}}\right)\right\} \tag{C.88}$$

where $\mathrm{vec}(\tilde{\mathbf{C}}) \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \tilde{V}\mathbf{I}_{n_\mathsf{T}^2})$ is independent of $\hat{\mathbf{C}}$ and where $\mathbf{z}' \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{I}_{n_\mathsf{T}})$.

We observe secondly that for any $k > 0$, we have the identity

$$R^\star_{n_\mathsf{R}\times n_\mathsf{T}}\left(k\mathbf{X}, \tilde{V}, \rho, \mathbf{L}\right) = R^\star_{n_\mathsf{R}\times n_\mathsf{T}}\left(\mathbf{X}, \frac{\tilde{V}}{k}, k\rho, \mathbf{L}\right). \tag{C.89}$$

Let us now proceed to bounding the difference $R^\star(\mathbf{L}) - \underline{R}$ from above and from below in order to derive tight bounds that will allow to compute its limit as $n_\mathsf{R}\to\infty$. We start by switching to the associated square channel representation:

$$\begin{aligned} R^\star(\mathbf{L}) - \underline{R} &= \mathsf{E}\left[R^\star_{n_\mathsf{R}\times n_\mathsf{T}}\left(\hat{V}\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}, \tilde{V}, \rho, \mathbf{L}\right)\right] - \mathsf{E}\left[\underline{R}_{n_\mathsf{R}\times n_\mathsf{T}}\left(\hat{V}\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}, \tilde{V}, \rho\right)\right] \\ &= \mathsf{E}\left[R^\star_{n_\mathsf{T}\times n_\mathsf{T}}\left(\hat{\mathbf{C}}^\dagger\hat{\mathbf{C}}, \tilde{V}, \rho, \mathbf{L}\right)\right] - \mathsf{E}\left[\underline{R}_{n_\mathsf{T}\times n_\mathsf{T}}\left(\hat{\mathbf{C}}^\dagger\hat{\mathbf{C}}, \tilde{V}, \rho\right)\right]. \end{aligned} \tag{C.90}$$

Since the minuend on the right-hand side of the last expression is a lower bound on the mutual information of the associated square channel, this minuend is upper-bounded by the mutual information upper bound (2.56) corresponding to said square channel. For the associated square channel at hand, the general expression of the upper bound (2.56)–(2.57) specializes to

$$I\left(\mathbf{x}_{\mathrm{G}};\left(\hat{\mathbf{C}}+\tilde{\mathbf{C}}\right)\mathbf{x}_{\mathrm{G}}+\mathbf{z}'\middle|\hat{\mathbf{C}}\right)\leq\mathsf{E}\left[\underline{R}_{n_{\mathsf{T}}\times n_{\mathsf{T}}}\left(\hat{\mathbf{C}}^{\dagger}\hat{\mathbf{C}},\tilde{V},\rho\right)\right]+\Delta_{n_{\mathsf{T}}\times n_{\mathsf{T}}}(\tilde{V},\rho) \tag{C.91}$$

where we have implicitly defined the bound gap $\Delta_{n_{\mathsf{T}}\times n_{\mathsf{T}}}(\tilde{V},\rho)$ to stand for

$$\Delta_{n_{\mathsf{T}}\times n_{\mathsf{T}}}(\tilde{V},\rho)=n_{\mathsf{T}}\,\mathsf{E}\left[\log\left(\frac{\tilde{V}+\rho^{-1}}{\tilde{V}\boldsymbol{\xi}^{\dagger}\mathbf{Q}\boldsymbol{\xi}+\rho^{-1}}\right)\right]. \tag{C.92}$$

Since the latter expression does not depend on n_{R}, it follows from combining (C.90) and (C.91) that the large system limit is upper-bounded as

$$\varlimsup_{n_{\mathsf{R}}\to\infty}\left\{R^{\star}(\mathbf{L})-\underline{R}\right\}\leq\Delta_{n_{\mathsf{T}}\times n_{\mathsf{T}}}(\tilde{V},\rho). \tag{C.93}$$

Next, we will derive a matching lower bound. We start again with the expression of the bound gap in terms of the associated square channel (C.90). The subtrahend can be upper-bounded by Jensen's inequality as

$$\begin{aligned}\underline{R}&=\mathsf{E}\left[\underline{R}_{n_{\mathsf{T}}\times n_{\mathsf{T}}}\left(\hat{\mathbf{C}}^{\dagger}\hat{\mathbf{C}},\tilde{V},\rho\right)\right]\\&=\mathsf{E}\left[\log\det\left(\mathbf{I}_{n_{\mathsf{T}}}+\frac{\hat{\mathbf{C}}^{\dagger}\hat{\mathbf{C}}}{\tilde{V}+\rho^{-1}}\right)\right]\\&\geq n_{\mathsf{T}}\log\left(1+n_{\mathsf{R}}\frac{\hat{V}}{\tilde{V}+\rho^{-1}}\right)\end{aligned} \tag{C.94}$$

because $\mathsf{E}[\hat{\mathbf{C}}^{\dagger}\hat{\mathbf{C}}]=\hat{V}\,\mathsf{E}[\hat{\mathbf{W}}^{\dagger}\hat{\mathbf{W}}]=n_{\mathsf{R}}\mathbf{I}_{n_{\mathsf{T}}}$.

As to the minuend of (C.90), we fix an arbitrary $\epsilon\in(0;1)$ and lower-bound the minuend as follows:

$$\begin{aligned}R^{\star}(\mathbf{L})&=\mathsf{E}\left[R^{\star}_{n_{\mathsf{T}}\times n_{\mathsf{T}}}\left(\hat{\mathbf{C}}^{\dagger}\hat{\mathbf{C}},\tilde{V},\rho,\mathbf{L}\right)\right]\\&\geq\mathsf{E}\left[R^{\star}_{n_{\mathsf{T}}\times n_{\mathsf{T}}}\left(\hat{\mathbf{C}}^{\dagger}\hat{\mathbf{C}},\tilde{V},\rho,\mathbf{L}\right)\middle|\hat{\mathbf{C}}^{\dagger}\hat{\mathbf{C}}\succeq n_{\mathsf{R}}\hat{V}(1-\epsilon)\mathbf{I}_{n_{\mathsf{T}}}\right]\mathsf{Pr}\left\{\hat{\mathbf{C}}^{\dagger}\hat{\mathbf{C}}\succeq n_{\mathsf{R}}\hat{V}(1-\epsilon)\mathbf{I}_{n_{\mathsf{T}}}\right\}.\end{aligned} \tag{C.95}$$

We next lower-bound the first factor on the right-hand side of the last expression using the following lemma.

Lemma C.1. *The function*

$$\mathsf{X}\mapsto R^{\star}_{n_{\mathsf{T}}\times n_{\mathsf{T}}}\left(\mathsf{X},\tilde{V},\rho,\mathbf{L}\right) \tag{C.96}$$

is matrix-monotone, i.e.,

$$\mathsf{X}_1\preceq\mathsf{X}_2\quad\Rightarrow\quad R^{\star}_{n_{\mathsf{T}}\times n_{\mathsf{T}}}\left(\mathsf{X}_1,\tilde{V},\rho,\mathbf{L}\right)\leq R^{\star}_{n_{\mathsf{T}}\times n_{\mathsf{T}}}\left(\mathsf{X}_2,\tilde{V},\rho,\mathbf{L}\right) \tag{C.97}$$

Proof: See Section C.7. ■

Hence,

$$
\begin{aligned}
R^\star(\mathbf{L}) &\geq R^\star_{n_\mathsf{T}\times n_\mathsf{T}}\left(n_\mathsf{R}\hat{V}(1-\epsilon)\mathbf{I}_{n_\mathsf{T}}, \tilde{V}, \rho, \mathbf{L}\right)\mathsf{Pr}\left\{\hat{\mathbf{C}}^\dagger\hat{\mathbf{C}} \succeq n_\mathsf{R}\hat{V}(1-\epsilon)\mathbf{I}_{n_\mathsf{T}}\right\} \\
&= R^\star_{n_\mathsf{T}\times n_\mathsf{T}}\left(\hat{V}(1-\epsilon)\mathbf{I}_{n_\mathsf{T}}, \tfrac{\tilde{V}}{n_\mathsf{R}}, n_\mathsf{R}\rho, \mathbf{L}\right)\mathsf{Pr}\left\{\hat{\mathbf{C}}^\dagger\hat{\mathbf{C}} \succeq n_\mathsf{R}\hat{V}(1-\epsilon)\mathbf{I}_{n_\mathsf{T}}\right\} \qquad \text{(C.98)}
\end{aligned}
$$

where for the last inequality we have used the identity (C.89).

Denoting the eigenvalues of $\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}$ as $\lambda_i(\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}),\ i = 1,\dots,n_\mathsf{T}$, we can lower-bound the last factor in (C.98) as follows:

$$
\begin{aligned}
\mathsf{Pr}\left\{\hat{\mathbf{C}}^\dagger\hat{\mathbf{C}} \succeq n_\mathsf{R}\hat{V}(1-\epsilon)\mathbf{I}_{n_\mathsf{T}}\right\} &= \mathsf{Pr}\left\{\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}} \succeq n_\mathsf{R}(1-\epsilon)\mathbf{I}_{n_\mathsf{T}}\right\} \\
&= \mathsf{Pr}\left\{\bigcup_{i=1}^{n_\mathsf{T}}\left\{\lambda_i(\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}) \geq n_\mathsf{R}(1-\epsilon)\right\}\right\} \\
&= 1 - \mathsf{Pr}\left\{\bigcap_{i=1}^{n_\mathsf{T}}\left\{\lambda_i(\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}) < n_\mathsf{R}(1-\epsilon)\right\}\right\} \\
&\geq 1 - \mathsf{Pr}\left\{\bigcup_{i=1}^{n_\mathsf{T}}\left\{\lambda_i(\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}) < n_\mathsf{R}(1-\epsilon)\right\}\right\} \\
&\geq 1 - \sum_{i=1}^{n_\mathsf{T}}\mathsf{Pr}\left\{\lambda_i(\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}) < n_\mathsf{R}(1-\epsilon)\right\} \\
&= 1 - n_\mathsf{T}\,\mathsf{Pr}\left\{\lambda_1(\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}) < n_\mathsf{R}(1-\epsilon)\right\}. \qquad \text{(C.99)}
\end{aligned}
$$

Here, the first inequality is due to $\mathsf{Pr}\{A \cup B\} \geq \mathsf{Pr}\{A \cap B\}$; for the second inequality we have used the union bound; the last equality holds because the eigenvalues of $\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}$ have a symmetric distribution, so the marginal distribution of each eigenvalue $\lambda_i(\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}})$ is equal to the empirical eigenvalue distribution of $\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}$. To further pursue this lower-bounding, we need the following lemma.

Lemma C.2. *For a real-valued random variable X with mean $\mathsf{E}[X] = \mu$ and variance $\mathsf{E}[X^2] - \mu^2 = \sigma^2$, we have for any $\eta \leq \mu$ the inequality*

$$
\mathsf{Pr}\{X \leq \eta\} \leq \frac{\sigma^2}{(\mu-\eta)^2 + \sigma^2}. \qquad \text{(C.100)}
$$

Proof: See Section C.8. ■

Noting that

$$
\begin{aligned}
\mathsf{E}[\lambda_1(\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}})] &= \frac{1}{n_\mathsf{T}}\sum_{i=1}^{n_\mathsf{T}}\mathsf{E}\left[\lambda_i(\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}})\right] \\
&= \frac{1}{n_\mathsf{T}}\,\mathsf{E}\,\mathrm{tr}(\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}) \\
&= n_\mathsf{R} \qquad \text{(C.101)}
\end{aligned}
$$

is larger than $n_\mathsf{R}(1-\epsilon)$, we can apply Lemma C.2 on (C.99) to obtain the lower bound

$$\begin{aligned}\mathsf{Pr}\Big\{\hat{\mathbf{C}}^\dagger\hat{\mathbf{C}} \succeq n_\mathsf{R}\hat{V}(1-\epsilon)\mathbf{I}_{n_\mathsf{T}}\Big\} &\geq 1 - n_\mathsf{T}\,\mathsf{Pr}\Big\{\lambda_1(\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}) < n_\mathsf{R}(1-\epsilon)\Big\}\\ &\geq 1 - n_\mathsf{T}\,\mathsf{Pr}\Big\{\lambda_1(\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}) \leq n_\mathsf{R}(1-\epsilon)\Big\}\\ &\geq 1 - n_\mathsf{T}\frac{\mathsf{var}\Big(\lambda_1(\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}})\Big)}{(n_\mathsf{R}\epsilon)^2 + \mathsf{var}\Big(\lambda_1(\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}})\Big)}.\end{aligned} \tag{C.102}$$

Here, the second inequality is because $\lambda_1(\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}) < n_\mathsf{R}(1-\epsilon)$ implies $\lambda_1(\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}) \leq n_\mathsf{R}(1-\epsilon)$. Denoting as $\hat{\mathbf{w}}_i$ the i-th column of $\hat{\mathbf{W}}$, and $\hat{W}_{i,j}$ the (i,j)-th entry of $\hat{\mathbf{W}}$, the variance in the last expression can be computed as

$$\begin{aligned}\mathsf{var}\Big(\lambda_1(\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}})\Big) &= \frac{1}{n_\mathsf{T}}\,\mathsf{E}\left[\sum_{i=1}^{n_\mathsf{T}}\lambda_i(\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}})^2\right] - \mathsf{E}[\lambda_1(\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}})]^2\\ &= \frac{1}{n_\mathsf{T}}\,\mathsf{E}\left[\mathrm{tr}(\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}})\right] - {n_\mathsf{R}}^2\\ &= \mathsf{E}\left[\hat{\mathbf{w}}_1^\dagger\hat{\mathbf{W}}\hat{\mathbf{W}}^\dagger\hat{\mathbf{w}}_1\right] - {n_\mathsf{R}}^2\\ &= \mathsf{E}\left[\|\hat{\mathbf{w}}_1\|_2^4\right] + (n_\mathsf{T}-1)\,\mathsf{E}\left[|\hat{\mathbf{w}}_1^\dagger\hat{\mathbf{w}}_2|^2\right] - {n_\mathsf{R}}^2\\ &= n_\mathsf{R}(n_\mathsf{R}+1) + (n_\mathsf{T}-1)\,\mathsf{E}\left[\left|\sum_{j=1}^{n_\mathsf{R}}\hat{W}_{j,1}^*\hat{W}_{j,2}\right|^2\right] - {n_\mathsf{R}}^2\\ &= n_\mathsf{R}(n_\mathsf{R}+1) + (n_\mathsf{T}-1)\sum_{j=1}^{n_\mathsf{R}}\mathsf{E}\left[\left|\hat{W}_{j,1}\right|^2\right]\mathsf{E}\left[\left|\hat{W}_{j,2}\right|^2\right] - {n_\mathsf{R}}^2\\ &= n_\mathsf{R}(n_\mathsf{R}+1) + (n_\mathsf{T}-1)n_\mathsf{R} - {n_\mathsf{R}}^2\\ &= n_\mathsf{T}n_\mathsf{R}.\end{aligned} \tag{C.103}$$

Combining this with (C.102), we end up with the lower bound

$$\mathsf{Pr}\Big\{\hat{\mathbf{C}}^\dagger\hat{\mathbf{C}} \succeq n_\mathsf{R}\hat{V}(1-\epsilon)\mathbf{I}_{n_\mathsf{T}}\Big\} \geq 1 - \frac{{n_\mathsf{T}}^2}{n_\mathsf{R}\epsilon^2 + n_\mathsf{T}} \tag{C.104}$$

which we insert into (C.98) to obtain

$$R^\star(\mathbf{L}) \geq \left(1 - \frac{{n_\mathsf{T}}^2}{n_\mathsf{R}\epsilon^2 + n_\mathsf{T}}\right)\cdot R^\star_{n_\mathsf{T}\times n_\mathsf{T}}\left(\hat{V}(1-\epsilon)\mathbf{I}_{n_\mathsf{T}}, \tfrac{\tilde{V}}{n_\mathsf{R}}, n_\mathsf{R}\rho, \mathbf{L}\right). \tag{C.105}$$

Applying Theorem 3.2 to the associated square channel, we have

$$\lim_{n_\mathsf{R}\to\infty}\Bigg\{\underline{R}_{n_\mathsf{T}\times n_\mathsf{T}}\left(\hat{V}(1-\epsilon)\mathbf{I}_{n_\mathsf{T}}, \tfrac{\tilde{V}}{n_\mathsf{R}}, n_\mathsf{R}\rho\right) + \Delta_{n_\mathsf{T}\times n_\mathsf{T}}(\tilde{V},\rho) \\ - R^\star_{n_\mathsf{T}\times n_\mathsf{T}}\left(\hat{V}(1-\epsilon)\mathbf{I}_{n_\mathsf{T}}, \tfrac{\tilde{V}}{n_\mathsf{R}}, n_\mathsf{R}\rho, \mathbf{L}\right)\Bigg\} = 0. \tag{C.106}$$

Therefore, we can write

$$\begin{aligned}
&R^{\star}_{n_\mathsf{T}\times n_\mathsf{T}}\left(\hat{V}(1-\epsilon)\mathbf{I}_{n_\mathsf{T}}, \tfrac{\tilde{V}}{n_\mathsf{R}}, n_\mathsf{R}\rho, \mathbf{L}\right)\\
&\quad= \underline{R}_{n_\mathsf{T}\times n_\mathsf{T}}\left(\hat{V}(1-\epsilon)\mathbf{I}_{n_\mathsf{T}}, \tfrac{\tilde{V}}{n_\mathsf{R}}, n_\mathsf{R}\rho\right) + \Delta_{n_\mathsf{T}\times n_\mathsf{T}}(\tilde{V},\rho) - \delta(n_\mathsf{R})\\
&\quad= n_\mathsf{T}\log\left(1+n_\mathsf{R}\frac{\hat{V}(1-\epsilon)}{\tilde{V}+\rho^{-1}}\right) + \Delta_{n_\mathsf{T}\times n_\mathsf{T}}(\tilde{V},\rho) - \delta(n_\mathsf{R})
\end{aligned} \tag{C.107}$$

with a non-negative term $\delta(n_\mathsf{R})$ fulfilling

$$\lim_{n_\mathsf{R}\to\infty} \delta(n_\mathsf{R}) = 0. \tag{C.108}$$

Plugging (C.107) into (C.105), we get

$$R^{\star}(\mathbf{L}) \geq \left(1 - \frac{{n_\mathsf{T}}^2}{n_\mathsf{R}\epsilon^2 + n_\mathsf{T}}\right)\left(n_\mathsf{T}\log\left(1+n_\mathsf{R}\frac{\hat{V}(1-\epsilon)}{\tilde{V}+\rho^{-1}}\right) + \Delta_{n_\mathsf{T}\times n_\mathsf{T}}(\tilde{V},\rho) - \delta(n_\mathsf{R})\right). \tag{C.109}$$

We can now combine this lower bound on $R^{\star}(\mathbf{L})$ with the upper bound (C.94) on $\underline{R}$ to compute a lower bound on the limit of the bound difference $R^{\star}(\mathbf{L}) - \underline{R}$ as $n_\mathsf{R}\to\infty$. Note that

$$\lim_{n_\mathsf{R}\to\infty}\left\{\frac{n_\mathsf{T}\log\left(1+n_\mathsf{R}\frac{\hat{V}(1-\epsilon)}{\tilde{V}+\rho^{-1}}\right)}{n_\mathsf{R}\epsilon^2 + n_\mathsf{T}}\right\} = 0 \tag{C.110}$$

and that

$$\lim_{n_\mathsf{R}\to\infty}\left\{n_\mathsf{T}\log\left(1+n_\mathsf{R}\frac{\hat{V}(1-\epsilon)}{\tilde{V}+\rho^{-1}}\right) - n_\mathsf{T}\log\left(1+n_\mathsf{R}\frac{\hat{V}}{\tilde{V}+\rho^{-1}}\right)\right\} = n_\mathsf{T}\log(1-\epsilon). \tag{C.111}$$

Combining (C.110)–(C.111) with (C.94) and (C.109), we finally obtain

$$\varliminf_{n_\mathsf{R}\to\infty}\left\{R^{\star}(\mathbf{L}) - \underline{R}\right\} \geq \Delta_{n_\mathsf{T}\times n_\mathsf{T}}(\tilde{V},\rho) + n_\mathsf{T}\log(1-\epsilon). \tag{C.112}$$

Since ϵ can be chosen arbitrarily small, we conlude by comparison of (C.93) and (C.112) that

$$\lim_{n_\mathsf{R}\to\infty}\left\{R^{\star}(\mathbf{L}) - \underline{R}\right\} = \Delta_{n_\mathsf{T}\times n_\mathsf{T}}(\tilde{V},\rho). \tag{C.113}$$

This concludes the proof of Theorem 3.4.

C.7 Proof of Lemma C.1

By comparison with (C.86), it suffices to prove that the function

$$f\colon \mathbb{C}_+^{n_\mathsf{T}\times n_\mathsf{T}} \to \mathbb{R}_+, \mathsf{X} \mapsto \operatorname{tr}\left[\left(\frac{\mathsf{X}}{\alpha} - \frac{1}{\alpha^2}\mathsf{X}\mathbf{A}\left(\mathbf{I}_{\operatorname{rank}(\mathbf{A})} + \frac{1}{\alpha}\mathbf{A}^{\dagger}\mathsf{X}\mathbf{A}\right)^{-1}\mathbf{A}^{\dagger}\mathsf{X}\right)\dot{\mathbf{L}}(\iota)\right] \tag{C.114}$$

is matrix-monotone for any given $\alpha > 0$. For this purpose, we will prove that its derivative

$$\frac{\mathrm{d}}{\mathrm{d}\nu} f(\mathbf{X} + \nu \mathbf{X}_0)\Big|_{\nu=0} \tag{C.115}$$

is non-negative. Using the abbreviation

$$\mathbf{B} = \mathbf{A}\left(\mathbf{I}_{\mathrm{rank}(\mathbf{A})} + \frac{1}{\alpha}\mathbf{A}^\dagger \mathbf{X} \mathbf{A}\right)^{-1} \mathbf{A}^\dagger \tag{C.116}$$

this derivative reads as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\nu} f(\mathbf{X} + \nu \mathbf{X}_0)\Big|_{\nu=0} &= \frac{1}{\alpha^3} \operatorname{tr}\left[\left(\alpha^2 \mathbf{X}_0 - \alpha \mathbf{X}_0 \mathbf{B} \mathbf{X} - \alpha \mathbf{X} \mathbf{B} \mathbf{X}_0 + \mathbf{X} \mathbf{B} \mathbf{X}_0 \mathbf{B} \mathbf{X}\right) \dot{\mathbf{L}}(\iota)\right] \\ &= \frac{1}{\alpha^3} \operatorname{tr}\left[\left(\alpha \mathbf{I}_{n_\mathsf{T}} - \mathbf{X}\mathbf{B}\right) \mathbf{X}_0 \left(\alpha \mathbf{I}_{n_\mathsf{T}} - \mathbf{B}\mathbf{X}\right) \dot{\mathbf{L}}(\iota)\right] \\ &\geq 0. \end{aligned} \tag{C.117}$$

The last inequality follows because $\mathbf{X}_0$ and $\dot{\mathbf{L}}(\iota)$ are positive semidefinite, and the trace of a product of positive semidefinite matrices is non-negative.

C.8 Proof of Lemma C.2

By the law of total probability, we have

$$\mu = \mathsf{E}[X | X \leq \eta] \, \mathsf{Pr}\{X \leq \eta\} + \mathsf{E}[X | X > \eta] \, \mathsf{Pr}\{X > \eta\} \tag{C.118a}$$

$$\mu^2 + \sigma^2 = \mathsf{E}[X^2 | X \leq \eta] \, \mathsf{Pr}\{X \leq \eta\} + \mathsf{E}[X^2 | X > \eta] \, \mathsf{Pr}\{X > \eta\}. \tag{C.118b}$$

Since the square function is convex, by Jensen's inequality we have

$$\mu^2 + \sigma^2 \geq \mathsf{E}[X^2 | X \leq \eta] \, \mathsf{Pr}\{X \leq \eta\} + \mathsf{E}[X | X > \eta]^2 \, \mathsf{Pr}\{X > \eta\}. \tag{C.119}$$

Using (C.118a), we can eliminate the expectation $\mathsf{E}[X | X > \eta]$ from (C.119) and obtain

$$\mu^2 + \sigma^2 \geq \mathsf{E}[X^2 | X \leq \eta] \, \mathsf{Pr}\{X \leq \eta\} + \left(\frac{\mu - \mathsf{E}[X | X \leq \eta] \, \mathsf{Pr}\{X \leq \eta\}}{\mathsf{Pr}\{X > \eta\}}\right)^2 \mathsf{Pr}\{X > \eta\}. \tag{C.120}$$

Using that $\mathsf{Pr}\{X > \eta\} = 1 - \mathsf{Pr}\{X \leq \eta\}$ and upon rearranging terms, we get

$$\begin{aligned}\sigma^2 &\geq \left(\left(\mu - \mathsf{E}[X | X \leq \eta]\right)^2 + \sigma^2\right) \mathsf{Pr}\{X \leq \eta\} + \mathsf{E}[X^2 | X \leq \eta] \, \mathsf{Pr}\{X \leq \eta\} \, \mathsf{Pr}\{X > \eta\} \\ &\geq \left(\left(\mu - \mathsf{E}[X | X \leq \eta]\right)^2 + \sigma^2\right) \mathsf{Pr}\{X \leq \eta\} \\ &\geq \left((\mu - \eta)^2 + \sigma^2\right) \mathsf{Pr}\{X \leq \eta\} \end{aligned} \tag{C.121}$$

where the last inequality follows because $\mu \geq \eta$ by assumption. This proves Lemma C.2.

C.9 Proof of Theorem 3.6

Denote $\Xi_{n_\mathsf{T}} = \|\boldsymbol{\xi}\|_2^2$ which is a gamma-distributed random variable with shape n_T and scale 1 whose probability density function is given by $f_{\Xi_{n_\mathsf{T}}}(x)$ for $x \geq 0$ [cf. (3.10)]. Upon performing the variable change $u = \frac{\sigma^2(x)}{1-\iota}$, applying the identity $\mathbf{A}(\mathbf{A}+z\mathbf{I})^{-1} = \mathbf{I} - z(\mathbf{A}+z\mathbf{I})^{-1}$ for some matrix $\mathbf{A}$ and scalar z, and using the Fubini theorem, one can express the rate-splitting bound as

$$R^\star(\mathbf{L}_{\text{lev}}) = \int_0^\infty \int_{\sigma^2(x)}^\infty \left(\frac{n_\mathsf{R}}{u} - \frac{g(x,u)}{u} \mathsf{E}\left[\operatorname{tr}\left(\frac{\mathbf{W}\mathbf{W}^\dagger}{n_\mathsf{T}} + g(u,x)\mathbf{I}_{n_\mathsf{R}} \right)^{-1} \right] \right) f_{\Xi_{n_\mathsf{T}}}(x)\,\mathrm{d}u\,\mathrm{d}x. \quad \text{(C.122)}$$

Lemma C.3. *Let* $\operatorname{vec}(\mathbf{W}) \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{I}_{n_\mathsf{R}n_\mathsf{T}})$ *where* $\mathbf{W} \in \mathbb{C}^{n_\mathsf{R}\times n_\mathsf{T}}$*. Then, for any* $x > 0$*,*

$$\left| \mathsf{E}\left[\operatorname{tr}\left(\frac{\mathbf{W}\mathbf{W}^\dagger}{n_\mathsf{T}} + x\mathbf{I}_{n_\mathsf{R}} \right)^{-1} \right] - n_\mathsf{T}\zeta(x) \right| \leq \frac{2n_\mathsf{R}}{x^4 {n_\mathsf{T}}^2} \quad \text{(C.123)}$$

where ζ *is given in* (3.38)*.*

Sketch: The result can be easily proved along the lines of the proofs of [Hac08, Theorem 3, Proposition 3 and 5]. ■

Applying Lemma C.3 to the trace term on the right-hand side of the last equation yields

$$R^\star(\mathbf{L}_{\text{lev}}) = \int_0^\infty \int_{\sigma^2(x)}^\infty \left(\frac{n_\mathsf{R} - n_\mathsf{R} g(x,u)\zeta(g(x,u))}{u} + \varepsilon \right) f_{\Xi_{n_\mathsf{T}}}(x)\,\mathrm{d}u\,\mathrm{d}x \quad \text{(C.124)}$$

for some ε, satisfying $|\varepsilon| \leq \frac{2n_\mathsf{R}}{ug(x,u)^3{n_\mathsf{T}}^2}$. One can verify that $n_\mathsf{R} g(x,u)\zeta(g(x,u)) \leq n_\mathsf{R}$ and continue by bounding the error term:

$$\begin{aligned}
\left| \int_0^\infty \int_{\sigma^2(x)}^\infty \varepsilon f_{\Xi_{n_\mathsf{T}}}(x)\,\mathrm{d}u\,\mathrm{d}x \right| &\leq \int_0^\infty \int_{\sigma^2(x)}^\infty \frac{2n_\mathsf{R}}{ug(x,u)^3{n_\mathsf{T}}^2} f_{\Xi_{n_\mathsf{T}}}(x)\,\mathrm{d}u\,\mathrm{d}x \\
&\leq \int_0^\infty \int_{\sigma^2(x)}^\infty \frac{2n_\mathsf{R}}{\sigma^2(x)g(x,u)^3{n_\mathsf{T}}^2} f_{\Xi_{n_\mathsf{T}}}(x)\,\mathrm{d}u\,\mathrm{d}x \\
&= \int_0^\infty \frac{2n_\mathsf{R}}{\sigma^2(x)(g(x,\sigma^2(x)))^2{n_\mathsf{T}}^2} f_{\Xi_{n_\mathsf{T}}}(x)\,\mathrm{d}x \\
&\leq \frac{2n_\mathsf{R}\rho\hat{V}^3}{(\tilde{V}+\rho^{-1})^2 {n_\mathsf{T}}^2} = \mathcal{O}({n_\mathsf{T}}^{-1}). \quad \text{(C.125)}
\end{aligned}$$

Since this integral is finite, the integral over the first two integrands in (C.124) must also exist. This concludes the proof.

C.10 Derivation of (3.47)

Let us define

$$n_{\mathsf{min}} \triangleq \min(n_{\mathsf{R}}, n_{\mathsf{T}}) \qquad n_{\mathsf{max}} \triangleq \max(n_{\mathsf{R}}, n_{\mathsf{T}}) \qquad n_{\Delta} \triangleq n_{\mathsf{max}} - n_{\mathsf{min}}. \tag{C.126}$$

In [Tel99], Telatar proved that for $\operatorname{vec}(\hat{\mathbf{W}}) \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I}_{n_{\mathsf{R}} n_{\mathsf{T}}})$, we have the closed-form expression

$$\begin{aligned} &\mathsf{E}\left[\log\det\left(\mathbf{I}_{n_{\mathsf{R}}} + \alpha^{-1}\hat{\mathbf{W}}\hat{\mathbf{W}}^{\dagger}\right)\right] \\ &\qquad = \int_0^{\infty} \log\left(1 + \frac{\nu}{\alpha}\right) \sum_{k=0}^{n_{\mathsf{min}}-1} \frac{k!}{(k+n_{\Delta})!} \left[\Lambda_k^{n_{\Delta}}(\nu)\right]^2 \nu^{n_{\Delta}} e^{-\nu} \,\mathrm{d}\nu \end{aligned} \tag{C.127}$$

in which Λ_j^i denote Laguerre polynomials of order j defined as

$$\Lambda_j^i(x) = \frac{1}{j!} e^x x^{-i} \frac{\mathrm{d}^j}{\mathrm{d}x^j}(e^{-x} x^{i+k}). \tag{C.128}$$

Using the function $\varsigma(x) = -e^x \operatorname{Ei}(-x)$ defined earlier in (2.25), we can evaluate the definite integral from (C.127) as [Alo99, Equation (78)]

$$\begin{aligned} &\int_0^{\infty} \log\left(1 + \frac{x}{\alpha}\right) x^n e^{-x} \,\mathrm{d}x \\ &\qquad = \sum_{i=0}^{n} \frac{n!}{(n-i)!} \left[\sum_{k=1}^{n-i} (k-1)!(-\alpha)^{n-i-k} - (n-i-1)!\alpha^{n-i}\varsigma(\alpha)\right] \end{aligned} \tag{C.129}$$

or alternatively, in terms of the incomplete gamma function [Gra07, Formula 4.337.5]

$$\int_0^{\infty} \log\left(1 + \frac{x}{\alpha}\right) x^n e^{-x} \,\mathrm{d}x = n! e^{\alpha} \sum_{i=1}^{n+1} \alpha^{n-i+1} \Gamma(-n-i-1, \alpha) \tag{C.130}$$

where said incomplete gamma function is defined as [Gra07, Formula 8.350.2]

$$\Gamma(u, v) = \int_v^{\infty} e^{-t} t^{u-1} \,\mathrm{d}t. \tag{C.131}$$

The resulting closed-form expression of (C.127) is given in [Kan06, Equation (34)]. It reads as

$$\begin{aligned} \mathsf{E}\left[\log\det\left(\mathbf{I}_{n_{\mathsf{R}}} + \alpha^{-1}\hat{\mathbf{W}}\hat{\mathbf{W}}^{\dagger}\right)\right] &= e^{\alpha} \sum_{i=0}^{n_{\mathsf{min}}-1} \frac{(n_{\Delta}+i)!}{(n_{\Delta})!^2 i!} \times \\ &\times \sum_{j=0}^{2i} \left(\sum_{k=0}^{j} \frac{(-i)_k (-i)_{j-k} (n_{\Delta}+j)!}{(n_{\Delta}+1)_k k! (n_{\Delta}+1)_{j-k} (j-k)!} \right) \times \\ &\times \sum_{\ell=1}^{n_{\Delta}+j+1} \Gamma(-n_{\Delta} - j + \ell - 1, \alpha) \alpha^{n_{\Delta}+j-\ell+1} \end{aligned} \tag{C.132}$$

where $(a)_q = a \cdot (a+1) \cdot \ldots \cdot (a+q-1) = \Gamma(a+q)/\Gamma(a)$ is the Pochhammer symbol.

A similar closed-form expression for the function

$$\Omega_{\text{lev}}(\alpha) \triangleq \frac{1}{\alpha} \mathsf{E}\,\mathrm{tr}\left[\hat{\mathbf{W}}^\dagger \left(\mathbf{I}_{n_\mathsf{R}} + \alpha^{-1}\hat{\mathbf{W}}\hat{\mathbf{W}}^\dagger\right)^{-1} \hat{\mathbf{W}}\right] \tag{C.133}$$

can be derived by observing that

$$\Omega_{\text{lev}}(\alpha) = -\alpha \frac{\mathrm{d}}{\mathrm{d}\alpha} \mathsf{E}\left[\log\det\left(\mathbf{I}_{n_\mathsf{R}} + \alpha^{-1}\hat{\mathbf{W}}\hat{\mathbf{W}}^\dagger\right)\right]. \tag{C.134}$$

Using that for non-negative exponents $N \geq 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\Gamma(-N,\alpha)\alpha^N = \frac{1}{\alpha}\left(N\Gamma(-N,\alpha)\alpha^N - e^{-\alpha}\right) \tag{C.135}$$

we compute the derivative on the right-hand side of (C.134) by differentiation of (C.132) with respect to α, which gives us

$$\begin{aligned}\Omega_{\text{lev}}(\alpha) = &-\alpha\,\mathsf{E}\left[\log\det\left(\mathbf{I}_{n_\mathsf{R}} + \alpha^{-1}\hat{\mathbf{W}}\hat{\mathbf{W}}^\dagger\right)\right] + \\ &- e^{\alpha}\sum_{i=0}^{n_{\mathsf{min}}-1}\frac{(n_\Delta+i)!}{(n_\Delta)!^2 i!}\sum_{j=0}^{2i}\left(\sum_{k=0}^{j}\frac{(-i)_k(-i)_{j-k}(n_\Delta+j)!}{(n_\Delta+1)_k k!(n_\Delta+1)_{j-k}(j-k)!}\right)\times \\ &\times \sum_{\ell=1}^{n_\Delta+j+1}\left[(n_\Delta+j-\ell+1)\Gamma(-n_\Delta-j+\ell-1,\alpha)\alpha^{n_\Delta+j-\ell+1} - e^{-\alpha}\right].\end{aligned} \tag{C.136}$$

wherein the log-determinant term can be evaluated with the help of (C.132). The semi-closed form of $R^\star(\mathbf{L}_{\text{lev}})$ is now obtained by observing that (3.8) can be written as

$$R^\star(\mathbf{L}_{\text{lev}}) = \int_0^1 \mathsf{E}\left[\frac{1}{1-\iota}\Omega_{\text{lev}}\left(\frac{\Xi_{n_\mathsf{T}}\tilde{V} + (1-\iota)\tilde{V}n_\mathsf{T} + \rho^{-1}n_\mathsf{T}}{(1-\iota)\hat{V}}\right)\right]\mathrm{d}\iota \tag{C.137}$$

in which the random variable Ξ_{n_T} has a probability density function given by (3.10).

C.11 Derivation of (3.48)

We start with the first line of Equation (C.14) from Appendix C.2, which is an expression for $R^\star(\mathbf{L}_{\text{stag}})$ reading as

$$R^\star(\mathbf{L}_{\text{stag}}) = \hat{V}\sum_{i=1}^{n_\mathsf{T}}\int_0^1 \mathsf{E}\left[\hat{\mathbf{w}}_i^\dagger\left(\alpha_i(\nu)\mathbf{I}_{n_\mathsf{R}} + \hat{V}(1-\nu)\hat{\mathbf{w}}_i\hat{\mathbf{w}}_i^\dagger + \hat{V}\hat{\mathbf{W}}_{(i+1):n_\mathsf{T}}\hat{\mathbf{W}}_{(i+1):n_\mathsf{T}}^\dagger\right)^{-1}\hat{\mathbf{w}}_i\right]\mathrm{d}\nu. \tag{C.138}$$

Recall that $\alpha_i(\nu)$ is given by [cf. (C.13)]

$$\alpha_i(\nu) \triangleq \tilde{V}\left(\frac{\Xi_{i-1}}{n_\mathsf{T}} + \frac{\nu}{n_\mathsf{T}}\tilde{\Xi}_1 + 1 - \nu + n_\mathsf{T} - i\right) + \rho^{-1} \tag{C.139}$$

where Ξ_{i-1} and $\tilde{\Xi}_1$ are mutually independent, gamma-distributed with scale 1 and respective shapes $i-1$ and 1 [cf. (3.10)].

C.11.1 Tall or Square Channel Matrices ($n_\mathsf{T} \leq n_\mathsf{R}$)

Assume that $n_\mathsf{T} \leq n_\mathsf{R}$. We will first derive a closed-form expression for an expectation of the form

$$\mathsf{E}\Big[\log\det\Big(\mathbf{I}_m + \beta^{-1}\hat{\mathbf{W}}_{1:(n-1)}\hat{\mathbf{W}}_{1:(n-1)}^\dagger + \delta^{-1}\hat{\mathbf{w}}_n\hat{\mathbf{w}}_n^\dagger\Big)\Big] = \mathsf{E}\left[\log\det\left(\mathbf{I}_m + \hat{\mathbf{W}}\begin{bmatrix}\beta^{-1}\mathbf{I}_{n-1} & \mathbf{0}\\ \mathbf{0} & \delta^{-1}\end{bmatrix}\hat{\mathbf{W}}^\dagger\right)\right] \quad \text{(C.140)}$$

in which $0 < \beta^{-1} < \delta^{-1}$, and where $\hat{\mathbf{W}} \in \mathbb{C}^{m\times n}$ is tall or square (i.e., $m \geq n$) and follows a distribution $\mathrm{vec}(\hat{\mathbf{W}}) \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{I}_{n_\mathsf{R}n_\mathsf{T}})$.

To this end, we start with a known closed-form expression for

$$\mathsf{E}\Big[\log\det\Big(\mathbf{I}_m + \hat{\mathbf{W}}\mathbf{\Lambda}^{-1}\hat{\mathbf{W}}^\dagger\Big)\Big] \quad \text{(C.141)}$$

where $\mathbf{\Lambda}^{-1}$ is a $n\times n$ diagonal matrix having positive, distinct and increasingly ordered entries $\Lambda_j^{-1},\ j = 1,\dots,n$, i.e., $0 < \Lambda_1^{-1} < \dots < \Lambda_n^{-1}$. Said closed-form expression reads as [Kan03, Theorem 2, Case 1]

$$\mathsf{E}\Big[\log\det\Big(\mathbf{I}_m + \hat{\mathbf{W}}\mathbf{\Lambda}^{-1}\hat{\mathbf{W}}^\dagger\Big)\Big] = \frac{\det(\mathbf{\Lambda})^n \sum_{k=1}^n \det(\mathbf{\Psi}_k(\mathbf{\Lambda}))}{\mathcal{V}(\mathbf{\Lambda})\prod_{j=1}^n (m-j)!} \quad \text{(C.142)}$$

where

$$\mathcal{V}(\mathbf{\Lambda}) = \prod_{p<q}(\Lambda_q - \Lambda_p) \quad \text{(C.143)}$$

denotes the determinant of the Vandermonde matrix[3]

$$\begin{bmatrix} 1 & \Lambda_1 & \Lambda_1^2 & \dots & \Lambda_1^n\\ 1 & \Lambda_2 & \Lambda_2^2 & \dots & \Lambda_2^n\\ \vdots & & & \ddots & \\ 1 & \Lambda_n & \Lambda_n^2 & \dots & \Lambda_n^n \end{bmatrix} \quad \text{(C.144)}$$

and where $\mathbf{\Psi}_k(\mathbf{\Lambda}),\ k = 1,\dots,n$ are $n\times n$ matrices whose (p,q)-th entry is given by

$$\big[\mathbf{\Psi}_k(\mathbf{\Lambda})\big]_{p,q} = \begin{cases} \displaystyle\int_0^\infty \log(1+x)x^{m-p}e^{-\Lambda_q x}\,\mathrm{d}x & \text{if } p = k\\ (m-p)!\Lambda_q^{p-m-1} & \text{if } p \neq k. \end{cases} \quad \text{(C.145)}$$

Notice that the p-th column of $\mathbf{\Psi}_k(\mathbf{\Lambda})$ is a function of Λ_p only, so $\mathbf{\Psi}_k(\mathbf{\Lambda})$ can be represented as

$$\mathbf{\Psi}_k(\mathbf{\Lambda}) = \Big[\boldsymbol{\psi}_{k,1}(\Lambda_1) \quad \dots \quad \boldsymbol{\psi}_{k,n}(\Lambda_n)\Big] \quad \text{(C.146)}$$

[3] Note that for $n = 1$, the Vandermonde determinant equals 1

where $\boldsymbol{\psi}_{k,p}(\Lambda_p)$ denotes the p-th column of $\boldsymbol{\Psi}_k(\boldsymbol{\Lambda})$.

The closed-form expression on the right-hand side of (C.142) is only valid when the values $\Lambda_p,\ p = 1,\ldots,n$ are distinct. In fact, whenever two values are equal, the determinants $\det(\boldsymbol{\Psi}_k(\boldsymbol{\Lambda}))$ and $\mathcal{V}(\boldsymbol{\Lambda})$ become zero, leading to an indeterminate form of the kind '0/0'. Nevertheless, it can be easily shown that the left-hand side of (C.142) is continuous in the vector $(\Lambda_1,\ldots,\Lambda_n)$ and well-defined even when the entries of the latter are not all distinct. Therefore, we can extend the above closed-form expression to cases of eigenvalue multiplicities by taking appropriate limits so as to derive an expression for (C.140). Specifically, we are interested in the limiting expression for all Λ_p tending to some value $\beta > 0$ except one value that is equal to $\delta > 0$, where it holds that $\delta < \beta$. That is, we want to compute the limit

$$\begin{aligned}\mathsf{E}\left[\log\det\left(\mathbf{I}_m + \hat{\mathbf{W}}\begin{bmatrix}\beta^{-1}\mathbf{I}_{n-1} & \mathbf{0}\\ \mathbf{0} & \delta^{-1}\end{bmatrix}\hat{\mathbf{W}}^\dagger\right)\right] &= \lim_{\substack{(\Lambda_1,\ldots,\Lambda_{n-1})\to(\beta,\ldots,\beta)\\ \Lambda_n\to\delta}} \mathsf{E}\Big[\log\det\Big(\mathbf{I}_m + \hat{\mathbf{W}}\boldsymbol{\Lambda}^{-1}\hat{\mathbf{W}}^\dagger\Big)\Big]\\ &= \lim_{\substack{(\Lambda_1,\ldots,\Lambda_{n-1})\to(\beta,\ldots,\beta)\\ \Lambda_n\to\delta}} \frac{\det(\boldsymbol{\Lambda})^n\sum_{k=1}^n \det(\boldsymbol{\Psi}_k(\boldsymbol{\Lambda}))}{\mathcal{V}(\boldsymbol{\Lambda})\prod_{j=1}^n (m-j)!}\end{aligned} \tag{C.147}$$

where $0 < \beta^{-1} < \delta^{-1}$. Using [Sim06, Lemma 6], we have for $n \geq 3$ that[4]

$$\lim_{\substack{(\Lambda_1,\ldots,\Lambda_{n-1})\to(\beta,\ldots,\beta)\\ \Lambda_n\to\delta}} \frac{\det(\boldsymbol{\Psi}_k(\boldsymbol{\Lambda}))}{\mathcal{V}(\boldsymbol{\Lambda})} = \frac{\det\left(\left[\boldsymbol{\Phi}_k(\beta)\quad \boldsymbol{\psi}_{k,n}(\delta)\right]\right)}{(\delta-\beta)^{n-1}\prod_{j=1}^{n-2} j!} \tag{C.148}$$

Here, $\boldsymbol{\Phi}_k(\beta)$ stands for the $n \times (n-1)$ matrix

$$\boldsymbol{\Phi}_k(\beta) = \left[\boldsymbol{\psi}_{k,1}(\beta)\quad \boldsymbol{\psi}_{k,2}^{(1)}(\beta)\quad \boldsymbol{\psi}_{k,3}^{(2)}(\beta)\quad \ldots\quad \boldsymbol{\psi}_{k,n-1}^{(n-2)}(\beta)\right] \tag{C.149}$$

where $\boldsymbol{\psi}_{k,p}(\Lambda_p)$ denotes the p-th column of $\boldsymbol{\Psi}_k(\boldsymbol{\Lambda})$, and where $\boldsymbol{\psi}_{k,p}^{(\nu)}(x)$ denotes its ν-th derivative function $\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}\boldsymbol{\psi}_{k,p}(x)$. By differentiation of (C.145), we can express the entries of $\boldsymbol{\Phi}_k(\beta)$ as

$$\left[\boldsymbol{\Phi}_k(\beta)\right]_{p,q} = \begin{cases}(-\beta)^{q-1}\displaystyle\int_0^\infty \log(1+x)x^{m-p}e^{-\beta x}\,\mathrm{d}x & \text{if } p = k \text{ and } q < n\\ (-1)^{q-1}(m-p+q-1)!\beta^{p-m-q} & \text{if } p \neq k \text{ and } q < n.\end{cases} \tag{C.150}$$

Upon setting $m = n_\mathsf{R}$ and $n = n_\mathsf{T} - i + 1$, and noticing that $(\hat{\mathbf{W}}_{1:(n-1)}, \hat{\mathbf{w}}_n)$ and $(\hat{\mathbf{W}}_{(i+1):n_\mathsf{T}}, \hat{\mathbf{w}}_i)$

[4] The case $n = 2$ entails no eigenvalue multiplicities and should be handled directly using (C.142) without taking limits.

have the same distribution, by combining (C.140), (C.147) and (C.148), we obtain

$$\mathsf{E}\Big[\log\det\Big(\mathbf{I}_{n_\mathsf{R}} + \beta^{-1}\hat{\mathbf{W}}_{(i+1):n_\mathsf{T}}\hat{\mathbf{W}}^\dagger_{(i+1):n_\mathsf{T}} + \delta^{-1}\hat{\mathbf{w}}_i\hat{\mathbf{w}}_i^\dagger\Big)\Big]$$
$$= \frac{\beta^{(n_\mathsf{T}-i)(n_\mathsf{T}-i+1)}}{\prod_{j=1}^{n_\mathsf{T}-i+1}(n_\mathsf{R}-j)!\prod_{\ell=1}^{n_\mathsf{T}-i-1}\ell!}\sum_{k=1}^{n_\mathsf{T}-i+1}\delta\left(\frac{\delta}{\delta-\beta}\right)^{n_\mathsf{T}-i}\det\Big(\Big[\mathbf{\Phi}_k(\beta)\quad \boldsymbol{\psi}_{k,n_\mathsf{T}-i+1}(\delta)\Big]\Big). \quad \text{(C.151)}$$

Let us denote

$$\Omega_{\text{stag},i}(\beta,\delta) \triangleq \mathsf{E}\Big[\hat{\mathbf{w}}_i\Big(\mathbf{I}_{n_\mathsf{R}} + \beta^{-1}\hat{\mathbf{W}}_{(i+1):n_\mathsf{T}}\hat{\mathbf{W}}^\dagger_{(i+1):n_\mathsf{T}} + \delta^{-1}\hat{\mathbf{w}}_i\hat{\mathbf{w}}_i^\dagger\Big)^{-1}\hat{\mathbf{w}}_i\Big]. \quad \text{(C.152)}$$

Observe that

$$\Omega_{\text{stag},i}(\beta,\delta) = -\delta^2\frac{\mathrm{d}}{\mathrm{d}\delta}\,\mathsf{E}\Big[\log\det\Big(\mathbf{I}_{n_\mathsf{R}} + \beta^{-1}\hat{\mathbf{W}}_{(i+1):n_\mathsf{T}}\hat{\mathbf{W}}^\dagger_{(i+1):n_\mathsf{T}} + \delta^{-1}\hat{\mathbf{w}}_i\hat{\mathbf{w}}_i^\dagger\Big)\Big]. \quad \text{(C.153)}$$

Therefore, upon differentiating (C.151) with respect to δ, we get a closed-form expression for $\Omega_{\text{stag},i}(\beta,\delta)$ which reads as

$$\Omega_{\text{stag},i}(\beta,\delta) = \frac{\beta^{(n_\mathsf{T}-i)(n_\mathsf{T}-i+1)}}{\prod_{j=1}^{n_\mathsf{T}-i+1}(n_\mathsf{R}-j)!\prod_{\ell=1}^{n_\mathsf{T}-i-1}\ell!}\sum_{k=1}^{n_\mathsf{T}-i+1}\left(\frac{\delta}{\delta-\beta}\right)^{n_\mathsf{T}-i}\times$$
$$\times\left[\left(1-\frac{\beta(n_\mathsf{T}-i)}{\delta-\beta}\right)\det\Big(\Big[\mathbf{\Phi}_k(\beta)\quad \boldsymbol{\psi}_{k,n_\mathsf{T}-i+1}(\delta)\Big]\Big) + \delta\det\Big(\Big[\mathbf{\Phi}_k(\beta)\quad \boldsymbol{\psi}^{(1)}_{k,n_\mathsf{T}-i+1}(\delta)\Big]\Big)\right]. \quad \text{(C.154)}$$

By comparison with (C.138) and (C.139), we infer that

$$R^\star(\mathbf{L}_{\text{stag}}) = \hat{V}\sum_{i=1}^{n_\mathsf{T}}\int_0^1\mathsf{E}\left[\frac{1}{\alpha_i(\nu)}\Omega_{\text{stag},i}\left(\frac{\alpha_i(\nu)}{\hat{V}},\frac{\alpha_i(\nu)}{\hat{V}(1-\nu)}\right)\right]\mathrm{d}\nu. \quad \text{(C.155)}$$

C.11.2 Broad Channel Matrices ($n_\mathsf{R} < n_\mathsf{T}$)

Assume that $n_\mathsf{R} < n_\mathsf{T}$. We will follow the same proof steps as in Subsection C.11.1. We first derive a closed-form expression for

$$\mathsf{E}\Big[\log\det\Big(\mathbf{I}_m + \beta^{-1}\hat{\mathbf{W}}_{1:(n-1)}\hat{\mathbf{W}}^\dagger_{1:(n-1)} + \delta^{-1}\hat{\mathbf{w}}_n\hat{\mathbf{w}}_n^\dagger\Big)\Big]$$
$$= \mathsf{E}\left[\log\det\left(\mathbf{I}_m + \hat{\mathbf{W}}\begin{bmatrix}\beta^{-1}\mathbf{I}_{n-1} & \mathbf{0}\\ \mathbf{0} & \delta^{-1}\end{bmatrix}\hat{\mathbf{W}}^\dagger\right)\right] \quad \text{(C.156)}$$

in which $0 < \beta^{-1} < \delta^{-1}$, and where $\hat{\mathbf{W}} \in \mathbb{C}^{m\times n}$ is broad (i.e., $m < n$) and follows a distribution $\mathrm{vec}(\hat{\mathbf{W}}) \sim \mathcal{N}_\mathbb{C}(\mathbf{0}, \mathbf{I}_{n_\mathsf{R}n_\mathsf{T}})$.

We start with a known closed-form expression for an expectation

$$\mathsf{E}\Big[\log\det\Big(\mathbf{I}_m + \hat{\mathbf{W}}\mathbf{\Lambda}^{-1}\hat{\mathbf{W}}^\dagger\Big)\Big] \quad \text{(C.157)}$$

where $\mathbf{\Lambda}^{-1}$ is a $n \times n$ diagonal matrix having positive, distinct and increasingly ordered entries $\Lambda_j^{-1},\ j = 1, \dots, n$, i.e., $0 < \Lambda_1^{-1} < \dots < \Lambda_n^{-1}$. Said closed-form expression reads as [Kan03, Theorem 2, Case 2]

$$\mathsf{E}\left[\log\det\left(\mathbf{I}_m + \hat{\mathbf{W}}\mathbf{\Lambda}^{-1}\hat{\mathbf{W}}^\dagger\right)\right] = (-1)^{m(n-m)} \frac{\det(\mathbf{\Lambda})^m \sum_{k=1}^m \det(\tilde{\mathbf{\Psi}}_k(\mathbf{\Lambda}))}{\mathcal{V}(\mathbf{\Lambda}) \prod_{j=1}^m (j-1)!} \tag{C.158}$$

where the $n \times n$ matrix $\tilde{\mathbf{\Psi}}_k(\mathbf{\Lambda})$ can be partitioned in an upper and a lower part as

$$\tilde{\mathbf{\Psi}}_k(\mathbf{\Lambda}) = \begin{bmatrix} \tilde{\mathbf{\Psi}}_{\text{upper}}(\mathbf{\Lambda}) \\ \tilde{\mathbf{\Psi}}_{\text{lower},k}(\mathbf{\Lambda}) \end{bmatrix}, \quad k = 1, \dots, m \tag{C.159}$$

whose (p, q)-th entries are given respectively by

$$\left[\tilde{\mathbf{\Psi}}_{\text{upper}}(\mathbf{\Lambda})\right]_{p,q} = (-\Lambda_q)^{n-m-p} \qquad (p = 1, \dots, n-m,\ q = 1, \dots, n) \tag{C.160a}$$

$$\left[\tilde{\mathbf{\Psi}}_{\text{lower},k}(\mathbf{\Lambda})\right]_{p,q} = \begin{cases} \int_0^\infty \log(1+x) x^{m-p} e^{-\Lambda_q x}\, \mathrm{d}x & \text{if } p = k \\ (m-p)!\Lambda_q^{p-m-1} & \text{if } p \neq k \end{cases} \qquad (p = 1, \dots, m,\ q = 1, \dots, n) \tag{C.160b}$$

The p-th column of $\tilde{\mathbf{\Psi}}_{\text{upper}}(\mathbf{\Lambda})$ and of $\tilde{\mathbf{\Psi}}_{\text{lower},k}(\mathbf{\Lambda})$ are functions of Λ_p only, so these matrices can be represented as

$$\tilde{\mathbf{\Psi}}_{\text{upper}}(\mathbf{\Lambda}) = \left[\tilde{\boldsymbol{\psi}}_{\text{upper},1}(\Lambda_1) \quad \dots \quad \tilde{\boldsymbol{\psi}}_{\text{upper},n}(\Lambda_n)\right] \tag{C.161a}$$

$$\tilde{\mathbf{\Psi}}_{\text{lower},k}(\mathbf{\Lambda}) = \left[\tilde{\boldsymbol{\psi}}_{\text{lower},k,1}(\Lambda_1) \quad \dots \quad \tilde{\boldsymbol{\psi}}_{\text{lower},k,n}(\Lambda_n)\right] \tag{C.161b}$$

where $\tilde{\boldsymbol{\psi}}_{\text{upper},p}(\Lambda_p)$ and $\tilde{\boldsymbol{\psi}}_{\text{lower},k,p}(\Lambda_p)$ denote the p-th column of $\tilde{\mathbf{\Psi}}_{\text{upper}}(\mathbf{\Lambda})$ and $\tilde{\mathbf{\Psi}}_{\text{lower},k}(\mathbf{\Lambda})$, respectively.

As in Subsection C.11.1, we are interested in the limiting expression for all Λ_p tending to a common value $\beta > 0$ except for one value, which tends to $\delta > 0$, where it holds that $\delta < \beta$. That is, we want to compute the limit

$$\begin{aligned} &\mathsf{E}\left[\log\det\left(\mathbf{I}_m + \hat{\mathbf{W}} \begin{bmatrix} \beta^{-1}\mathbf{I}_{n-1} & \mathbf{0} \\ \mathbf{0} & \delta^{-1} \end{bmatrix} \hat{\mathbf{W}}^\dagger\right)\right] \\ &= \lim_{\substack{(\Lambda_1,\dots,\Lambda_{n-1})\to(\beta,\dots,\beta) \\ \Lambda_n\to\delta}} \mathsf{E}\left[\log\det\left(\mathbf{I}_m + \hat{\mathbf{W}}\mathbf{\Lambda}^{-1}\hat{\mathbf{W}}^\dagger\right)\right] \\ &= \lim_{\substack{(\Lambda_1,\dots,\Lambda_{n-1})\to(\beta,\dots,\beta) \\ \Lambda_n\to\delta}} (-1)^{m(n-m)} \frac{\det(\mathbf{\Lambda})^m \sum_{k=1}^m \det(\tilde{\mathbf{\Psi}}_k(\mathbf{\Lambda}))}{\mathcal{V}(\mathbf{\Lambda}) \prod_{j=1}^m (j-1)!}. \end{aligned} \tag{C.162}$$

Using [Sim06, Lemma 6], we have for $n \geq 3$ that

$$\lim_{\substack{(\Lambda_1,\dots,\Lambda_{n-1})\to(\beta,\dots,\beta)\\ \Lambda_n\to\delta}} \frac{\det(\tilde{\boldsymbol{\Psi}}_k(\boldsymbol{\Lambda}))}{\mathcal{V}(\boldsymbol{\Lambda})} = \frac{\det\left(\begin{bmatrix}\tilde{\boldsymbol{\Phi}}_k(\beta) & \tilde{\boldsymbol{\psi}}_{k,n}(\delta)\end{bmatrix}\right)}{(\delta-\beta)^{n-1}\prod_{j=1}^{n-2} j!}. \tag{C.163}$$

Here, $\tilde{\boldsymbol{\Phi}}_k(\beta)$ stands for the $n \times (n-1)$ matrix

$$\tilde{\boldsymbol{\Phi}}_k(\beta) = \begin{bmatrix}\tilde{\boldsymbol{\psi}}_{k,1}(\beta) & \tilde{\boldsymbol{\psi}}_{k,2}^{(1)}(\beta) & \tilde{\boldsymbol{\psi}}_{k,3}^{(2)}(\beta) & \dots & \tilde{\boldsymbol{\psi}}_{k,n-1}^{(n-2)}(\beta)\end{bmatrix} \tag{C.164}$$

where $\tilde{\boldsymbol{\psi}}_{k,p}(\Lambda_p)$ denotes the p-th column of $\tilde{\boldsymbol{\Psi}}_k(\boldsymbol{\Lambda})$, and where $\tilde{\boldsymbol{\psi}}_{k,p}^{(\nu)}(x)$ denotes its ν-th derivative function $\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}\tilde{\boldsymbol{\psi}}_{k,p}(x)$. Much like the matrix $\tilde{\boldsymbol{\Psi}}_k(\boldsymbol{\Lambda})$, we can partition the matrix $\tilde{\boldsymbol{\Phi}}_k(\beta)$ into an upper and a lower part as

$$\tilde{\boldsymbol{\Phi}}_k(\beta) = \begin{bmatrix}\tilde{\boldsymbol{\Phi}}_{\text{upper}}(\beta)\\ \tilde{\boldsymbol{\Phi}}_{\text{lower},k}(\beta)\end{bmatrix}. \tag{C.165}$$

By differentiation of (C.160), we can express the entries of $\tilde{\boldsymbol{\Phi}}_k(\beta)$ as

$$\left[\tilde{\boldsymbol{\Phi}}_{\text{upper}}(\beta)\right]_{p,q} = \begin{cases}(n-m-p)_{q-1}(-\beta)^{n-m-p-q+1} & \text{if } p+q \leq n-m+1\\ 0 & \text{if } p+q > n-m+1\end{cases} \tag{C.166a}$$

$$(p = 1,\dots,n-m,\ q = 1,\dots,n)$$

$$\left[\tilde{\boldsymbol{\Phi}}_{\text{lower},k}(\beta)\right]_{p,q} = \begin{cases}(-\beta)^{q-1}\int_0^\infty \log(1+x)x^{m-p}e^{-\beta x}\,\mathrm{d}x & \text{if } p = k\\ (-1)^{q-1}(m-p+q-1)!\beta^{p-m-q} & \text{if } p \neq k\end{cases} \tag{C.166b}$$

$$(p = 1,\dots,m,\ q = 1,\dots,n)$$

Upon setting $m = n_{\mathsf{R}}$ and $n = n_{\mathsf{T}} - i + 1$, and noticing that $(\hat{\mathbf{W}}_{1:(n-1)}, \hat{\mathbf{w}}_n)$ and $(\hat{\mathbf{W}}_{(i+1):n_{\mathsf{T}}}, \hat{\mathbf{w}}_i)$ have the same distribution, by combining (C.156), (C.162) and (C.163), we obtain

$$\tag{C.167}$$

$$\begin{aligned}&\mathsf{E}\Big[\log\det\Big(\mathbf{I}_{n_{\mathsf{R}}} + \beta^{-1}\hat{\mathbf{W}}_{(i+1):n_{\mathsf{T}}}\hat{\mathbf{W}}_{(i+1):n_{\mathsf{T}}}^\dagger + \delta^{-1}\hat{\mathbf{w}}_i\hat{\mathbf{w}}_i^\dagger\Big)\Big]\\ &\quad= \frac{(-1)^{n_{\mathsf{R}}(n_{\mathsf{T}}-n_{\mathsf{R}}-i+1)}\beta^{n_{\mathsf{R}}(n_{\mathsf{T}}-i)}}{\prod_{j=1}^{n_{\mathsf{R}}}(j-1)!\prod_{\ell=1}^{n_{\mathsf{T}}-i-1}\ell!}\cdot\frac{\delta^{n_{\mathsf{R}}}}{(\delta-\beta)^{n_{\mathsf{T}}-i}}\sum_{k=1}^{n_{\mathsf{R}}}\det\left(\begin{bmatrix}\tilde{\boldsymbol{\Phi}}_k(\beta) & \tilde{\boldsymbol{\psi}}_{k,n_{\mathsf{T}}-i+1}(\delta)\end{bmatrix}\right).\end{aligned} \tag{C.168}$$

Let us denote

$$\begin{aligned}\Omega_{\text{stag},i}(\beta,\delta) &\triangleq \mathsf{E}\Big[\hat{\mathbf{w}}_i\Big(\mathbf{I}_{n_{\mathsf{R}}} + \beta^{-1}\hat{\mathbf{W}}_{(i+1):n_{\mathsf{T}}}\hat{\mathbf{W}}_{(i+1):n_{\mathsf{T}}}^\dagger + \delta^{-1}\hat{\mathbf{w}}_i\hat{\mathbf{w}}_i^\dagger\Big)^{-1}\hat{\mathbf{w}}_i\Big]\\ &= -\delta^2\frac{\mathrm{d}}{\mathrm{d}\delta}\,\mathsf{E}\Big[\log\det\Big(\mathbf{I}_{n_{\mathsf{R}}} + \beta^{-1}\hat{\mathbf{W}}_{(i+1):n_{\mathsf{T}}}\hat{\mathbf{W}}_{(i+1):n_{\mathsf{T}}}^\dagger + \delta^{-1}\hat{\mathbf{w}}_i\hat{\mathbf{w}}_i^\dagger\Big)\Big].\end{aligned} \tag{C.169}$$

Upon differentiating (C.168) with respect to δ, we get

$$\Omega_{\text{stag},i}(\beta,\delta) = -\delta^2 \frac{(-1)^{n_\mathsf{R}(n_\mathsf{T}-n_\mathsf{R}-i+1)}\beta^{n_\mathsf{R}(n_\mathsf{T}-i)}}{\prod_{j=1}^{n_\mathsf{R}}(j-1)!\prod_{\ell=1}^{n_\mathsf{T}-i-1}\ell!} \cdot \frac{\delta^{n_\mathsf{R}}}{(\delta-\beta)^{n_\mathsf{T}-i}} \times$$
$$\times \sum_{k=1}^{n_\mathsf{R}} \left[\left(\frac{n_\mathsf{R}}{\delta} - \frac{n_\mathsf{T}-i}{\delta-\beta} \right) \det\left(\begin{bmatrix} \tilde{\boldsymbol{\Phi}}_k(\beta) & \tilde{\boldsymbol{\psi}}_{k,n_\mathsf{T}-i+1}(\delta) \end{bmatrix} \right) + \det\left(\begin{bmatrix} \tilde{\boldsymbol{\Phi}}_k(\beta) & \tilde{\boldsymbol{\psi}}^{(1)}_{k,n_\mathsf{T}-i+1}(\delta) \end{bmatrix} \right) \right]. \tag{C.170}$$

By comparison with (C.138) and (C.139), we infer that

$$R^\star(\mathbf{L}_{\text{stag}}) = \hat{V} \sum_{i=1}^{n_\mathsf{T}} \int_0^1 \mathsf{E}\left[\frac{1}{\alpha_i(\nu)} \Omega_{\text{stag},i}\left(\frac{\alpha_i(\nu)}{\hat{V}}, \frac{\alpha_i(\nu)}{\hat{V}(1-\nu)} \right) \right] \mathrm{d}\nu. \tag{C.171}$$

D

Appendices to Chapter 4

D.1 Examples of utilities

Table D.1 Examples of utilities from the class $\mathcal{F}$

	Utility	Curvature in $\mathbf{S}$
1	$I(\mathbf{S})$	concave
2	$\mathrm{tr}(\mathbf{S})$	linear
3	$\det(\mathbf{S})$	log-concave
4	$\mathrm{tr}(\mathbf{S}^{-1})^{-1}$	log-concave
5	$\det(\mathbf{I}+\nu\mathbf{S})$ with $\nu \geq 0$	log-concave
6	$\mathsf{E}\det(\mathbf{I}+\hat{\mathbf{W}}\mathbf{S}\hat{\mathbf{W}}^\dagger)$	log-concave
7	$\mathsf{E}\log\det(\hat{\mathbf{W}}\mathbf{S}\hat{\mathbf{W}}^\dagger)$ for $n_\mathsf{T} \geq n_\mathsf{R}$	concave
8	$\mathsf{E}\det(\hat{\mathbf{W}}\mathbf{S}\hat{\mathbf{W}}^\dagger)$ for $n_\mathsf{T} \geq n_\mathsf{R}$	log-concave
9	$-\mathrm{tr}\,\mathsf{E}\left\{(\mathbf{I}+\hat{\mathbf{W}}\mathbf{S}\hat{\mathbf{W}}^\dagger)^{-1}\right\}$	concave
10	$\mathrm{tr}\,\mathsf{E}\left\{(\mathbf{S}^{-1}+\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}})^{-1}\right\}$ for $\det(\mathbf{S}) \neq 0$	concave
11	$\mathsf{Pr}\big(\det(\mathbf{I}+\hat{\mathbf{W}}\mathbf{S}\hat{\mathbf{W}}^\dagger) \geq \eta\big)$	–/–
12	$\mathsf{Pr}\big(\log\det(\hat{\mathbf{W}}\mathbf{S}\hat{\mathbf{W}}^\dagger) \geq \eta\big)$ for $n_\mathsf{T} \geq n_\mathsf{R}$	–/–
13	$\mathsf{Pr}\big(\det(\hat{\mathbf{W}}\mathbf{S}\hat{\mathbf{W}}^\dagger) \geq \eta\big)$ for $n_\mathsf{T} \geq n_\mathsf{R}$	–/–
14	$\mathsf{Pr}\big(-\mathrm{tr}\left\{(\mathbf{I}+\hat{\mathbf{W}}\mathbf{S}\hat{\mathbf{W}}^\dagger)^{-1}\right\} \geq \eta\big)$	–/–
15	$\mathsf{Pr}\big(\mathrm{tr}\left\{(\mathbf{S}^{-1}+\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}})^{-1}\right\} \geq \eta\big)$	–/–
16	$\lVert\mathbf{S}\rVert_\mathsf{F}^2$	convex
17	$\lambda_{\mathsf{max}}(\mathbf{S})$	convex

In the following, we provide a few examples illustrating from which bounds or approximations of the mutual information $I(\mathbf{S})$ the above utilities may arise.

Utility 2 A simple upper bound on $I(\mathbf{S})$ is obtained using the fact that $\mathsf{E}[\hat{\mathbf{W}}\mathbf{S}\hat{\mathbf{W}}^\dagger] = \operatorname{tr}(\mathbf{S})n_\mathsf{T}\mathbf{I}$ and by applying Jensen's inequality to the concave log-determinant:

$$I(\mathbf{S}) \leq \sum_{i=1}^{n_\mathsf{R}} \log(1 + \operatorname{tr}(\mathbf{S})n_\mathsf{T}). \tag{D.1}$$

Utility 5 with $\nu = n_\mathsf{T} n_\mathsf{R}$ Using the determinant identity $\det(\mathbf{I}+\mathbf{AB}) = \det(\mathbf{I}+\mathbf{BA})$ to write $I(\mathbf{S}) = \mathsf{E}\log\det(\mathbf{I} + \mathbf{S}\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}})$, and applying Jensen's inequality, we get another upper bound:

$$I(\mathbf{S}) \leq \log\det\left(\mathbf{I} + n_\mathsf{T} n_\mathsf{R}\mathbf{S}\right). \tag{D.2}$$

Here, we have used $\mathsf{E}[\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}] = n_\mathsf{T} n_\mathsf{R}$.

Utilities 6 and 11 By applying Jensen's inequality to the concave log function, we get the upper bound

$$I(\mathbf{S}) \leq \log \mathsf{E}[\det(\mathbf{I} + \hat{\mathbf{W}}\mathbf{S}\hat{\mathbf{W}}^\dagger)]. \tag{D.3}$$

Utilities 3, 7 and 12 We can lower bound $I(\mathbf{S})$ by removing the identity matrix inside the log-determinant. Depending on the sizes of antenna arrays, this gives us a bound $I(\mathbf{S}) \geq \underline{I}(\mathbf{S})$ with

$$\underline{I}(\mathbf{S}) = \begin{cases} \mathsf{E}\log\det(\hat{\mathbf{W}}\mathbf{S}\hat{\mathbf{W}}^\dagger) & \text{for } n_\mathsf{T} \geq n_\mathsf{R} \\ \mathsf{E}\log\det(\mathbf{S}\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}) & \text{for } n_\mathsf{T} \leq n_\mathsf{R}, \end{cases} \tag{D.4}$$

The former case (i.e., $n_\mathsf{T} \geq n_\mathsf{R}$) justifies utilities 7 and 12. In the latter case (i.e., $n_\mathsf{T} \leq n_\mathsf{R}$), note that

$$\underline{I}(\mathbf{S}) = \log\det(\mathbf{S}) + \mathsf{E}\log\det(\hat{\mathbf{W}}^\dagger\hat{\mathbf{W}}) \tag{D.5}$$

leads to utility 3. Clearly, $\underline{I}(\mathbf{S})$ is good as an approximation of $I(\mathbf{S})$ at high SNR, and was used as such in [Gau00], [Gra02]. Let us also mention the tighter lower bound [Oym02]

$$I(\mathbf{S}) \geq n_\mathsf{R} \log\left(1 + \exp\left(\frac{\log(\mathrm{e})}{n_\mathsf{R}}\underline{I}(\mathbf{S})\right)\right), \tag{D.6}$$

the derivation of which makes use of the Minkowski inequality for determinants.

D.2 Proof of Theorem 4.1

Let $\mathbf{P}$ and $\mathbf{Q}$ have $\operatorname{rank}(\mathbf{P}) = r_\mathbf{P}$ and $\operatorname{rank}(\mathbf{Q}) = r_\mathbf{Q}$ respectively, with $r_\mathbf{P}, r_\mathbf{Q} \in \{1, \ldots, n_\mathsf{T}\}$. The absolute difference of ranks be $d = |r_\mathbf{P} - r_\mathbf{Q}|$. The pilot matrix and precoder have reduced spectral decompositions $\mathbf{P} = \mathbf{U_P}\mathbf{\Lambda_P}\mathbf{U_P}^\dagger$ and $\mathbf{Q} = \mathbf{U_Q}\mathbf{\Lambda_Q}\mathbf{U_Q}^\dagger$, respectively, where the eigenbases $\mathbf{U_P} \in \mathbb{U}^{n_\mathsf{T} \times r_\mathbf{P}}$ and $\mathbf{U_Q} \in \mathbb{U}^{n_\mathsf{T} \times r_\mathbf{Q}}$ are tall or square, whereas $\mathbf{\Lambda_P}$ and $\mathbf{\Lambda_Q}$ are diagonal and positive definite. Let $\mathbf{U_{P^\perp}} \in \mathbb{U}^{n_\mathsf{T} \times (n_\mathsf{T} - r_\mathbf{P})}$ and $\mathbf{U_{Q^\perp}} \in \mathbb{U}^{n_\mathsf{T} \times (n_\mathsf{T} - r_\mathbf{Q})}$ denote orthonormal bases of the nullspaces of $\mathbf{P}$ and $\mathbf{Q}$, respectively, so that $\mathbf{U_P}^\dagger \mathbf{U_{P^\perp}} = \mathbf{0}$ and $\mathbf{U_Q}^\dagger \mathbf{U_{Q^\perp}} = \mathbf{0}$.

The reduced eigendecomposition of $\hat{\mathbf{R}}$ is consistently denoted as $\mathbf{U_{\hat{R}}}\mathbf{\Lambda_{\hat{R}}}\mathbf{U_{\hat{R}}}^\dagger$, where $\mathbf{\Lambda_{\hat{R}}} \in \mathbb{R}_+^{r_\mathbf{P} \times r_\mathbf{P}}$ is diagonal positive definite and of size $r_\mathbf{P} \times r_\mathbf{P}$, due to the rank equality (4.18) which states that $\operatorname{rank}(\hat{\mathbf{R}}) = \operatorname{rank}(\mathbf{P})$. The orthonormal nullspace of $\hat{\mathbf{R}}$ is denoted as $\mathbf{U_{\hat{R}^\perp}} \in \mathbb{U}^{n_\mathsf{T} \times (n_\mathsf{T} - r_\mathbf{P})}$. We introduce the notation $\mathbf{L_{A \cap B}}$ to designate an orthonormal basis of the intersection of range spaces $\operatorname{range}(\mathbf{A})$ and $\operatorname{range}(\mathbf{B})$. If it exists, $\mathbf{L_{A \cap B}}$ is a matrix with the maximum number of columns defined as

$$\mathbf{L_{A\cap B}} = \left\{ \mathbf{L} \;\middle|\; \begin{array}{l} \mathbf{L}^\dagger \mathbf{L} = \mathbf{I}, \\ \forall \mathbf{x} \neq \mathbf{0} \colon \mathbf{ALx} \neq \mathbf{0} \text{ and } \mathbf{BLx} \neq \mathbf{0} \end{array} \right\}. \tag{D.7}$$

Assume that $\operatorname{range}(\mathbf{Q}) \not\subseteq \operatorname{range}(\hat{\mathbf{R}})$, so the matrix $\mathbf{L_{Q \cap \hat{R}^\perp}}$ is defined and has at least one column. We define a new precoder $\mathbf{Q}' \in \mathcal{Q}(\mu_\mathcal{Q})$ as

$$\mathbf{Q}' = \mathbf{Q} - \lambda_{r_\mathbf{Q}}(\mathbf{Q}) \mathbf{L_{Q\cap\hat{R}^\perp}} \mathbf{L}^\dagger_{\mathbf{Q\cap\hat{R}^\perp}}, \tag{D.8}$$

where $\lambda_{r_\mathbf{Q}}(\mathbf{Q})$ is the smallest non-zero eigenvalue of $\mathbf{Q}$. First, we verify that $\mathbf{Q}' \in \mathcal{Q}(\mu_\mathcal{Q})$. Clearly, since $\lambda_{r_\mathbf{Q}}(\mathbf{Q}) \geq 0$, we have $\mathbf{Q}' \preceq \mathbf{Q}$, and therefore $\operatorname{tr}(\mathbf{Q}') \leq \operatorname{tr}(\mathbf{Q})$. What remains to prove is that $\mathbf{Q}' \succeq \mathbf{0}$. The smallest eigenvalue of $\mathbf{Q}'$ is

$$\lambda_{\mathsf{min}}(\mathbf{Q}') = \min_{\|\mathbf{w}\|=1} \mathbf{w}^\dagger \mathbf{Q}' \mathbf{w}.$$

But since by definition of $\mathbf{L_{Q\cap\hat{R}}}$, the range space of $\mathbf{Q}$ contains the range space of $\mathbf{L_{Q\cap\hat{R}}}\mathbf{L}^\dagger_{\mathbf{Q\cap\hat{R}}}$, we have that $\mathbf{w}^\dagger\mathbf{Q}'\mathbf{w}$ is equal to $\mathbf{w}^\dagger \mathbf{\Pi_Q}^\dagger \mathbf{Q}' \mathbf{\Pi_Q} \mathbf{w}$, where $\mathbf{\Pi_Q} = \mathbf{U_Q}(\mathbf{U_Q}^\dagger \mathbf{U_Q})^{-1}\mathbf{U_Q}^\dagger$ is the projector from $\mathbb{C}^{n_\mathsf{T} \times n_\mathsf{T}}$ onto the basis $\mathbf{U_Q}$. We thus have

$$\begin{aligned} \lambda_{\mathsf{min}}(\mathbf{Q}') &= \lambda_{\mathsf{min}}\Big(\mathbf{\Pi_Q}^\dagger\Big(\mathbf{Q} - \lambda_{r_\mathbf{Q}}(\mathbf{Q})\mathbf{L_{Q\cap\hat{R}^\perp}}\mathbf{L}^\dagger_{\mathbf{Q\cap\hat{R}^\perp}}\Big)\mathbf{\Pi_Q}\Big) \\ &\geq \lambda_{\mathsf{min}}\Big(\mathbf{\Pi_Q}^\dagger\mathbf{Q}\mathbf{\Pi_Q}\Big) - \lambda_{r_\mathbf{Q}}(\mathbf{Q})\lambda_{\mathsf{max}}\Big(\mathbf{\Pi_Q}^\dagger\mathbf{L_{Q\cap\hat{R}^\perp}}\mathbf{L}^\dagger_{\mathbf{Q\cap\hat{R}^\perp}}\mathbf{\Pi_Q}\Big) \\ &\geq \lambda_{r_\mathbf{Q}}(\mathbf{Q})\Big(1 - \lambda_{\mathsf{max}}(\mathbf{\Pi_Q}^\dagger\mathbf{\Pi_Q})\lambda_{\mathsf{max}}(\mathbf{L}^\dagger_{\mathbf{Q\cap\hat{R}^\perp}}\mathbf{L_{Q\cap\hat{R}^\perp}})\Big) \\ &= 0. \end{aligned} \tag{D.9}$$

The second inequality holds because the spectral radius norm $\lambda_{\mathsf{max}}(\cdot)$ is sub-multiplicative, while the last equality holds because the projector $\mathbf{\Pi_Q}$ and the (sub)unitary $\mathbf{L_{Q\cap\hat{R}^\perp}}$ have a largest singular value of at most 1. We infer that $\mathbf{Q}' \in \mathcal{Q}(\mu_\mathcal{Q})$.

Notice that $\mathbf{Q}'$ is purposely constructed so that $\mathbf{Q}'\hat{\mathbf{R}} = \mathbf{Q}\hat{\mathbf{R}}$. As compared to the matrix

$\mathbf{S} = \mathbf{S}(\mathbf{P}, \mathbf{Q})$ obtained with the precoder $\mathbf{Q}$, the new matrix $\mathbf{S}' = \mathbf{S}(\mathbf{P}, \mathbf{Q}')$ thus reads as

$$\begin{aligned}\mathbf{S}' &= \frac{\hat{\mathbf{R}}^{\frac{1}{2}}\mathbf{Q}'\hat{\mathbf{R}}^{\frac{1}{2}}}{1+\operatorname{tr}(\mathbf{Q}'\tilde{\mathbf{R}})} \\ &= \frac{\hat{\mathbf{R}}^{\frac{1}{2}}\mathbf{Q}\hat{\mathbf{R}}^{\frac{1}{2}}}{1+\operatorname{tr}(\mathbf{Q}\tilde{\mathbf{R}}) - \lambda_{r_{\mathbf{Q}}}(\mathbf{Q})\operatorname{tr}(\mathbf{L}^{\dagger}_{\mathbf{Q}\cap\hat{\mathbf{R}}^{\perp}}\tilde{\mathbf{R}}\mathbf{L}_{\mathbf{Q}\cap\hat{\mathbf{R}}^{\perp}})} \\ &= k\mathbf{S} \end{aligned} \tag{D.10}$$

and thus turns out to be a scaled version of $\mathbf{S}$, where the positive scalar k is

$$k = \frac{1+\operatorname{tr}(\mathbf{Q}\tilde{\mathbf{R}})}{1+\operatorname{tr}(\mathbf{Q}\tilde{\mathbf{R}}) - \lambda_{r_{\mathbf{Q}}}(\mathbf{Q})\operatorname{tr}(\mathbf{L}^{\dagger}_{\mathbf{Q}\cap\hat{\mathbf{R}}^{\perp}}\tilde{\mathbf{R}}\mathbf{L}_{\mathbf{Q}\cap\hat{\mathbf{R}}^{\perp}})} > 1.$$

Therefore, we have $\mathbf{S}' \succ \mathbf{S}$, so the precoder $\mathbf{Q}$ is necessarily suboptimal, which means that $\operatorname{range}(\mathbf{Q}) \nsubseteq \operatorname{range}(\hat{\mathbf{R}})$ cannot hold for optimal $\mathbf{Q}$. Instead, we must have $\operatorname{range}(\mathbf{Q}) \subseteq \operatorname{range}(\hat{\mathbf{R}})$ for optimality, which concludes the proof.

D.3 Proof of Lemma 4.1

Supposing we are in the first situation, i.e., $\mathbf{A}$ has full column rank, then $\mathbf{A}$ has a left pseudoinverse $\mathbf{A}^{\sharp} = (\mathbf{A}^{\dagger}\mathbf{A})^{-1}\mathbf{A}^{\dagger}$ which can be used to define the inverse function ϱ^{-1}. Let $\mathbf{Z} = \varrho(\mathbf{X})$ be the image of $\mathbf{X}$. Given $\mathbf{Z}$, one obtains $\mathbf{X}$ by insulating it via left-multiplication with $\mathbf{A}^{\sharp}$ and right-multiplication with $\mathbf{A}^{\sharp\dagger}$, and appropriate scaling:

$$\mathbf{A}^{\sharp}\mathbf{Z}\mathbf{A}^{\sharp\dagger}(1+\operatorname{tr}(\mathbf{B}\mathbf{X})) = \mathbf{X} \tag{D.11}$$

Left-multiplying (D.11) with $\mathbf{B}$ and taking the trace yields

$$\operatorname{tr}(\mathbf{B}\mathbf{A}^{\sharp}\mathbf{Z}\mathbf{A}^{\sharp\dagger}) = \frac{\operatorname{tr}(\mathbf{B}\mathbf{X})}{1+\operatorname{tr}(\mathbf{B}\mathbf{X})}, \tag{D.12}$$

or equivalently,

$$1+\operatorname{tr}(\mathbf{B}\mathbf{X}) = \frac{1}{1-\operatorname{tr}(\mathbf{B}\mathbf{A}^{\sharp}\mathbf{Z}\mathbf{A}^{\sharp\dagger})} \tag{D.13}$$

By combining (D.11) with (D.13), we recover the pre-image $\mathbf{X} = \varrho^{-1}(\mathbf{Z})$, and see that the inverse function ϱ^{-1} is linear fractional with parameters $\mathbf{A}^{\sharp}$ and $-\mathbf{A}^{\sharp\dagger}\mathbf{B}\mathbf{A}^{\sharp}$.

We now suppose that we are in the second situation, i.e., $\mathbf{A}$ has full row rank and $\mathcal{X}$ is such that $\operatorname{range}(\mathbf{X}) = \operatorname{range}(\mathbf{A}^{\dagger})$ for every element of $\mathcal{X}$. Due to the latter constraint on the span of $\mathbf{X}$, we can write any $\mathbf{X} \in \mathcal{X}$ as $\mathbf{X} = \mathbf{A}^{\dagger}\hat{\mathbf{X}}\mathbf{A}$, with $\hat{\mathbf{X}}$ given by the inverse relation $\hat{\mathbf{X}} = \mathbf{A}^{\flat\dagger}\mathbf{X}\mathbf{A}^{\flat}$, where $\mathbf{A}^{\flat} = \mathbf{A}^{\dagger}(\mathbf{A}\mathbf{A}^{\dagger})^{-1}$ denotes the right pseudoinverse of $\mathbf{A}$. The function ϱ can be represented as

$$\varrho\colon\ \mathbf{X} \ \mapsto\ \frac{\mathbf{A}\mathbf{X}\mathbf{A}^\dagger}{1+\operatorname{tr}(\mathbf{B}\mathbf{X})} = \frac{\hat{\mathbf{A}}\hat{\mathbf{X}}\hat{\mathbf{A}}^\dagger}{1+\operatorname{tr}(\hat{\mathbf{B}}\hat{\mathbf{X}})} \tag{D.14}$$

with abbreviations $\mathbf{A}\mathbf{A}^\dagger \triangleq \hat{\mathbf{A}}$ and $\hat{\mathbf{B}} \triangleq \mathbf{A}\mathbf{B}\mathbf{A}^\dagger$. Since $\hat{\mathbf{A}} = \mathbf{A}\mathbf{A}^\dagger$ has full rank (because $\mathbf{A}$ has full row rank), the function ϱ appears as an injective linear fractional function of $\hat{\mathbf{X}}$ with parameters $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$, whose inverse, according to findings above, is linear fractional with parameters $\hat{\mathbf{A}}^\sharp$ and $-\hat{\mathbf{A}}^{\sharp\dagger}\hat{\mathbf{B}}\hat{\mathbf{A}}^\sharp$. Denoting as $\mathbf{Z} = \varrho(\mathbf{X})$ the image of $\mathbf{X}$ under the function ϱ, we can thus recover the pre-image $\mathbf{X}$ from $\mathbf{Z}$ as

$$\begin{aligned}\mathbf{X} = \mathbf{A}^\dagger\hat{\mathbf{X}}\mathbf{A} &= \mathbf{A}^\dagger\frac{\hat{\mathbf{A}}^\sharp\mathbf{Z}\hat{\mathbf{A}}^{\sharp\dagger}}{1-\operatorname{tr}(\hat{\mathbf{A}}^{\sharp\dagger}\mathbf{B}\hat{\mathbf{A}}^\sharp\mathbf{Z})}\mathbf{A}\\ &= \frac{\mathbf{A}^\flat\mathbf{Z}\mathbf{A}^{\flat\dagger}}{1-\operatorname{tr}(\mathbf{A}^{\flat\dagger}\mathbf{B}\mathbf{A}^\flat\mathbf{Z})}.\end{aligned} \tag{D.15}$$

Consequently, the inverse ϱ^{-1} is linear fractional with parameters $\mathbf{A}^\flat$ and $-\mathbf{A}^{\flat\dagger}\mathbf{B}\mathbf{A}^\flat$.

D.4 Proof of Lemma 4.2

We first establish that $\mathbf{S}(\mathbf{P},\mathbf{Q})$ is monotonic in the transmit power, in the sense that

$$0 \le k < k' \Rightarrow \mathbf{S}(\mathbf{P},k\mathbf{Q}) \prec \mathbf{S}(\mathbf{P},k'\mathbf{Q}). \tag{D.16}$$

This monotonicity holds because

$$\mathbf{S}(\mathbf{P},k\mathbf{Q}) = \frac{k}{1+k\operatorname{tr}(\mathbf{Q}\tilde{\mathbf{R}})}\hat{\mathbf{R}}^{\frac{1}{2}}\mathbf{Q}\hat{\mathbf{R}}^{\frac{1}{2}} \tag{D.17}$$

$$< \frac{k'}{1+k'\operatorname{tr}(\mathbf{Q}\tilde{\mathbf{R}})}\hat{\mathbf{R}}^{\frac{1}{2}}\mathbf{Q}\hat{\mathbf{R}}^{\frac{1}{2}} = \mathbf{S}(\mathbf{P},k'\mathbf{Q}) \tag{D.18}$$

owing to the fact that $k \mapsto \frac{k}{1+k\operatorname{tr}(\mathbf{Q}\tilde{\mathbf{R}})}$ is monotonically increasing in k.

Now, as a consequence of Theorem 4.1 and of the power monotonicity (D.16), optimal precoders $\mathbf{Q}$ will be elements of $\partial^+\mathcal{Q}(\mu_\mathcal{Q})\cap\operatorname{range}(\hat{\mathbf{R}})$. By definition, this set can be parametrized by $r_\mathbf{P}$ non-negative coefficients $\boldsymbol{\psi} = [\psi_1,\ldots,\psi_{r_\mathbf{P}}]^\mathrm{T} \in \mathbb{R}_+^{r_\mathbf{P}}$ stored in a diagonal matrix $\boldsymbol{\Psi} = \operatorname{diag}(\boldsymbol{\psi})$, and a tall or square (sub)unitary basis $\boldsymbol{\Upsilon} \in \mathbb{U}^{n_\mathrm{T}\times r_\mathbf{P}}$ as follows:

$$\mathbf{Q}_{\boldsymbol{\Psi},\boldsymbol{\Upsilon}} = (\hat{\mathbf{R}}^{\frac{1}{2}})^+\boldsymbol{\Upsilon}\boldsymbol{\Psi}\boldsymbol{\Upsilon}^\dagger(\hat{\mathbf{R}}^{\frac{1}{2}})^+, \tag{D.19}$$

where $(\bullet)^+$ denotes the Moore-Penrose pseudoinverse, and where the parameter pair $(\boldsymbol{\Psi},\boldsymbol{\Upsilon})$

shall be subject to the four constraints

$$\boldsymbol{\psi} \geq \mathbf{0}, \tag{D.20a}$$

$$\operatorname{tr}(\mathbf{Q}_{\boldsymbol{\Psi},\boldsymbol{\Upsilon}}) = \mu_{\mathcal{Q}}, \tag{D.20b}$$

$$\boldsymbol{\Upsilon}^{\dagger}\boldsymbol{\Upsilon} = \mathbf{I}, \tag{D.20c}$$

$$\operatorname{range}(\boldsymbol{\Upsilon}) = \operatorname{range}(\hat{\mathbf{R}}). \tag{D.20d}$$

The first two constraints ensure that $\mathbf{Q}_{\boldsymbol{\Psi},\boldsymbol{\Upsilon}}$ belongs to $\partial^{+}\mathcal{Q}(\mu_{\mathcal{Q}})$, while the structure of Expression (D.19) ensures that $\mathbf{Q}_{\boldsymbol{\Psi},\boldsymbol{\Upsilon}}$ belongs to range($\hat{\mathbf{R}}$). The third and fourth constraints (D.20c)–(D.20d) are clearly not necessary to fulfill $\mathbf{Q}_{\boldsymbol{\Psi},\boldsymbol{\Upsilon}} \in \operatorname{range}(\hat{\mathbf{R}}) \cap \partial^{+}\mathcal{Q}(\mu_{\mathcal{Q}})$, but they induce no loss of generality either and will turn out helpful later. The set $\partial^{+}\mathcal{Q}(\mu_{\mathcal{Q}})$ is thus entirely parametrized by the parameter pair $(\boldsymbol{\Psi}, \boldsymbol{\Upsilon})$ subject to the constraints (D.20). Consider now the feasible vectors

$$\begin{aligned}\mathbf{s}(\mathbf{P}, \mathbf{Q}_{\boldsymbol{\Psi},\boldsymbol{\Upsilon}}) &= \boldsymbol{\lambda}\left(\frac{\hat{\mathbf{R}}^{\frac{1}{2}}\mathbf{Q}_{\boldsymbol{\Psi},\boldsymbol{\Upsilon}}\hat{\mathbf{R}}^{\frac{1}{2}}}{1+\operatorname{tr}(\mathbf{Q}_{\boldsymbol{\Psi},\boldsymbol{\Upsilon}}\tilde{\mathbf{R}})}\right) \\ &= \boldsymbol{\lambda}\left(\frac{\mathbf{U}_{\hat{\mathbf{R}}}\mathbf{U}_{\hat{\mathbf{R}}}^{\dagger}\boldsymbol{\Upsilon}\boldsymbol{\Psi}\boldsymbol{\Upsilon}^{\dagger}\mathbf{U}_{\hat{\mathbf{R}}}\mathbf{U}_{\hat{\mathbf{R}}}^{\dagger}}{1+\operatorname{tr}(\mathbf{Q}_{\boldsymbol{\Psi},\boldsymbol{\Upsilon}}\tilde{\mathbf{R}})}\right) \in \mathbb{R}_{+}^{n_{\mathrm{T}}}.\end{aligned} \tag{D.21}$$

This vector has at most $r_{\mathbf{P}}$ non-zero entries because $\boldsymbol{\Psi}$ is $r_{\mathbf{P}} \times r_{\mathbf{P}}$. Therefore, we define a vector $\bar{\mathbf{s}} \in \mathbb{R}_{+}^{r_{\mathbf{P}}}$ of reduced dimension, which contains the $r_{\mathbf{P}}$ topmost (i.e., largest) entries of $\mathbf{s}$. Since range($\boldsymbol{\Upsilon}$) = range($\hat{\mathbf{R}}$) [cf. (D.20d)], the matrix $\mathbf{U}_{\hat{\mathbf{R}}}^{\dagger}\boldsymbol{\Upsilon}$ is unitary, so we have

$$\begin{aligned}\bar{\mathbf{s}}(\mathbf{P}, \mathbf{Q}_{\boldsymbol{\Psi},\boldsymbol{\Upsilon}}) &= \frac{\boldsymbol{\psi}}{1+\operatorname{tr}(\mathbf{Q}_{\boldsymbol{\Psi},\boldsymbol{\Upsilon}}\tilde{\mathbf{R}})} \\ &= \frac{\boldsymbol{\psi}}{1+\operatorname{tr}\left(\boldsymbol{\Upsilon}^{\dagger}(\hat{\mathbf{R}}^{\frac{1}{2}})^{+}\tilde{\mathbf{R}}(\hat{\mathbf{R}}^{\frac{1}{2}})^{+}\boldsymbol{\Upsilon}\boldsymbol{\Psi}\right)}.\end{aligned} \tag{D.22}$$

Note that we have not assumed so far that the entries of $\boldsymbol{\psi}$ or $\bar{\mathbf{s}}$ are sorted in any specific order. For notational brevity, call $\boldsymbol{\alpha}$ the vector of entries $\alpha_i = \left[\boldsymbol{\Upsilon}^{\dagger}(\hat{\mathbf{R}}^{\frac{1}{2}})^{+}\tilde{\mathbf{R}}(\hat{\mathbf{R}}^{\frac{1}{2}})^{+}\boldsymbol{\Upsilon}\right]_{i,i}$, then

$$\bar{\mathbf{s}}(\mathbf{P}, \mathbf{Q}_{\boldsymbol{\Psi},\boldsymbol{\Upsilon}}) = \frac{\boldsymbol{\psi}}{1+\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\psi}}. \tag{D.23}$$

On the other hand, the second constraint (D.20b) translates to $\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\psi} = \mu_{\mathcal{Q}}$, where $\boldsymbol{\beta}$ denotes the vector of diagonal entries of $\boldsymbol{\Upsilon}^{\dagger}\hat{\mathbf{R}}^{+}\boldsymbol{\Upsilon}$, i.e., $\beta_i = [\boldsymbol{\Upsilon}^{\dagger}\hat{\mathbf{R}}^{+}\boldsymbol{\Upsilon}]_{i,i}$. Together, this constraint and equation (D.23) describe an affine plane of dimension $r_{\mathbf{P}} - 1$, because left-multiplying (D.23) with $\frac{1}{\mu_{\mathcal{Q}}}\boldsymbol{\beta}^{\mathrm{T}} + \boldsymbol{\alpha}^{\mathrm{T}}$ leads to the affine equation

$$\left(\frac{1}{\mu_{\mathcal{Q}}}\boldsymbol{\beta} + \boldsymbol{\alpha}\right)^{\mathrm{T}}\bar{\mathbf{s}}(\mathbf{P}, \mathbf{Q}_{\boldsymbol{\Psi},\boldsymbol{\Upsilon}}) = 1. \tag{D.24}$$

This affine equation, together with the non-negativity constraint $\boldsymbol{\psi} \geq \mathbf{0}$ [cf. (D.20a)], thus delimit a $(r_{\mathbf{P}} - 1)$-dimensional simplex, whose elements fulfill

$$\begin{cases} \sum_i \frac{\bar{s}_i}{\omega_i} = 1 \\ \bar{\mathbf{s}} \geq \mathbf{0} \end{cases} \tag{D.25}$$

where

$$\omega_i = \frac{1}{\frac{1}{\mu_Q}\beta_i + \alpha_i}. \tag{D.26}$$

The $r_{\mathbf{P}}$ vertices of the simplex described by (D.25) are the axis points $\omega_i \mathbf{e}_i$.

Due to the symmetry property of utilities from the class $\mathcal{F}$, the ordering of the $\bar{s}_i$ does not influence the utility value. Assume that, for a given $\bar{\mathbf{s}} \geq \mathbf{0}$ fulfilling (D.25), there exists an index permutation π such that $\sum_i \frac{\bar{s}_{\pi(i)}}{\omega_i} < 1$, then $\bar{\mathbf{s}}$ is suboptimal, since there exists an $\bar{\mathbf{s}}' = k\mathbf{\Pi}\bar{\mathbf{s}}$ with $k > 1$ which also fulfills (D.25) and yields a larger utility value $f(\bar{\mathbf{s}}') = f(k\mathbf{\Pi}\bar{\mathbf{s}}) = f(k\bar{\mathbf{s}}) > f(\bar{\mathbf{s}})$. Therefore, we can discard all $\bar{\mathbf{s}}$ for which some permutation π yields $\sum_i \frac{\bar{s}_{\pi(i)}}{\omega_i} < 1$. This is equivalent to the requirement that the $\bar{s}_i$ and ω_i be ordered in the same way, i.e.,

$$\omega_i \leq \omega_j \Rightarrow \bar{s}_i \leq \bar{s}_j. \tag{D.27}$$

Hence, without loss of optimality, we will restrain the set of admissible $\bar{\mathbf{s}}$ to the following convex set, called $\mathcal{S}$:

$$\mathcal{S} = \left\{ \bar{\mathbf{s}} \in \mathbb{R}_+^{r_{\mathbf{P}}} \;\middle|\; \sum_{i=1}^{r_{\mathbf{P}}} \frac{\bar{s}_i}{\bar{\omega}_i} = 1, \forall j\colon \bar{s}_j \geq \bar{s}_{j+1} \right\}, \tag{D.28}$$

where $\bar{\boldsymbol{\omega}} = [\bar{\omega}_1, \ldots, \bar{\omega}_{r_{\mathbf{P}}}]^{\mathrm{T}}$ contains the entries of $\boldsymbol{\omega}$ arranged in non-increasing order, i.e., $\bar{\omega}_1 \geq \ldots \geq \bar{\omega}_{r_{\mathbf{P}}}$. Let us define $r_{\mathbf{P}}$ special points pertaining to $\mathcal{S}$, which we shall denote as $\boldsymbol{\sigma}^{(n)}$, and define as

$$\forall n = 1, \ldots, r_{\mathbf{P}}\colon \quad \boldsymbol{\sigma}^{(n)} = \mathcal{H}(\bar{\omega}_1, \ldots, \bar{\omega}_n) \sum_{j=1}^{n} \mathbf{e}_j, \tag{D.29}$$

where $\mathcal{H}(\cdot, \ldots, \cdot)$ and $\mathbf{e}_j$ are defined in the statement of Lemma 4.2. In fact, it is easy to see that the $\boldsymbol{\sigma}^{(n)}$ have non-increasing entries and fulfill $\sum_{i=1}^{r_{\mathbf{P}}} \frac{\sigma_i^{(n)}}{\bar{\omega}_i} = 1$, and thus belong to $\mathcal{S}$. Now, we will show that the set $\mathcal{S}'$ of all convex combinations of the $\boldsymbol{\sigma}^{(n)}$, i.e,

$$\mathcal{S}' = \left\{ \sum_{n=1}^{r_{\mathbf{P}}} \nu_n \boldsymbol{\sigma}^{(n)} \;\middle|\; \sum_n \nu_n = 1, \forall n\colon \nu_n \geq 0 \right\}, \tag{D.30}$$

is the same as the set $\mathcal{S}$. We know that $\mathcal{S}'$ is a subset of the convex set $\mathcal{S}$, for being a convex combination of a collection of points from $\mathcal{S}$, hence $\mathcal{S}' \subseteq \mathcal{S}$. Next, we argue that, if we assume that some particular point $\tilde{\boldsymbol{\sigma}}$ belongs to $\mathcal{S} \setminus \mathcal{S}'$, this implies that $\tilde{\boldsymbol{\sigma}}$ does not lie in $\mathcal{S}$ because it would fail to comply with some constraint from the definition (D.28) of $\mathcal{S}$. Therefrom, it will follow that $\mathcal{S} = \mathcal{S}'$.

Since the $\boldsymbol{\sigma}^{(n)}$ pertain to $\mathcal{S}$ and are $r_{\mathbf{P}}$ linearly independent vectors, they define the $(r_{\mathbf{P}}-1)$-dimensional affine plane described by $\sum_{i=1}^{r_{\mathbf{P}}} \frac{\bar{s}_i}{\bar{\omega}_i} = 1$. Therefore, to prove the equality $\mathcal{S} = \mathcal{S}'$, it will be sufficient to take some point $\tilde{\boldsymbol{\sigma}} = [\tilde{\sigma}_1, \ldots, \tilde{\sigma}_{r_{\mathbf{P}}}]^{\mathrm{T}}$ to lie on said plane, and show that an infringement of an inequality $\tilde{\sigma}_i \geq \tilde{\sigma}_{i+1}$ implies that $\tilde{\boldsymbol{\sigma}} = \sum_{n=1}^{r_{\mathbf{P}}} \tilde{\nu}_n \boldsymbol{\sigma}^{(n)}$ with coefficients $\tilde{\nu}_n$ such that either $\sum_n \tilde{\nu}_n \neq 1$ or $\tilde{\nu}_n < 0$ for some index n. So, assume that $\tilde{\sigma}_i < \tilde{\sigma}_{i+1}$ for a given i. There exist unique coefficients $\tilde{\nu}_n$ such that $\tilde{\boldsymbol{\sigma}} = \sum_{n=1}^{r_{\mathbf{P}}} \tilde{\nu}_n \boldsymbol{\sigma}^{(n)}$. The inequality $\tilde{\sigma}_i < \tilde{\sigma}_{i+1}$ can thus be written as

$$\sum_{n=1}^{r_{\mathbf{P}}} \tilde{\nu}_n \mathbf{e}_i^{\mathrm{T}} \boldsymbol{\sigma}^{(n)} < \sum_{n=1}^{r_{\mathbf{P}}} \tilde{\nu}_n \mathbf{e}_{i+1}^{\mathrm{T}} \boldsymbol{\sigma}^{(n)}. \tag{D.31}$$

By inserting (D.29) into the latter inequality, we get

$$\sum_{n=i}^{r_{\mathbf{P}}} \tilde{\nu}_n \mathcal{H}(\bar{\omega}_1, \ldots, \bar{\omega}_n) < \sum_{n=i+1}^{r_{\mathbf{P}}} \tilde{\nu}_n \mathcal{H}(\bar{\omega}_1, \ldots, \bar{\omega}_n), \tag{D.32}$$

which boils down to $\tilde{\nu}_i < 0$. This concludes the proof that $\mathcal{S} = \mathcal{S}'$. Also, this simplex $\mathcal{S}$ contains only Pareto border points, in the sense that $\mathcal{S} = \partial^+\mathcal{S}$. In fact, any point $\bar{\mathbf{s}}''$ dominating some point $\bar{\mathbf{s}}' \in \mathcal{S}$ would fulfill $\sum_{i=1}^{r_{\mathbf{P}}} \bar{s}_i''/\bar{\omega}_i > 1$ and thus lie outside $\mathcal{S}$.

Now that we have fully characterized the set of Pareto border points $\bar{\mathbf{s}} = \bar{\mathbf{s}}(\mathbf{P}, \mathbf{Q}_{\boldsymbol{\Psi},\boldsymbol{\Upsilon}})$ for a fixed $\boldsymbol{\Upsilon}$ as a simplex set $\mathcal{S}$, we ask what the best choice for $\boldsymbol{\Upsilon}$ is under the constraints (D.20c)–(D.20d). Clearly, if there exists one single $\boldsymbol{\Upsilon}^\star$ that simultaneously maximizes all vertices $\boldsymbol{\sigma}^{(n)}$ in the sense that for any $\boldsymbol{\Upsilon}$, we have

$$\boldsymbol{\sigma}^{(n)}(\boldsymbol{\Upsilon}^\star) \geq \boldsymbol{\sigma}^{(n)}(\boldsymbol{\Upsilon}), \qquad n = 1, \ldots, r_{\mathbf{P}} \tag{D.33}$$

then this $\boldsymbol{\Upsilon}^\star$ is optimal. Here, $\boldsymbol{\sigma}^{(n)}(\boldsymbol{\Upsilon})$ denotes the value of $\boldsymbol{\sigma}^{(n)}$, as defined in (D.29), with the $\bar{\omega}_i$ interpreted as functions of $\boldsymbol{\Upsilon}$. Next, we show that such $\boldsymbol{\Upsilon}^\star$ is well-defined and characterize it.

We state the multiobjective optimization problem

$$\forall n = 1, \ldots, r_{\mathbf{P}}\colon \quad \max_{\substack{\boldsymbol{\Upsilon} \in \mathbb{U}^{n_{\mathrm{T}} \times r_{\mathbf{P}}} \\ \mathrm{range}(\boldsymbol{\Upsilon}) = \mathrm{range}(\hat{\mathbf{R}})}} \quad \mathcal{H}(\bar{\omega}_1, \ldots, \bar{\omega}_n). \tag{D.34}$$

Omitting the range space constraint on $\boldsymbol{\Upsilon}$, we have that, with the definition (D.26) of the coefficients ω_i together with the definitions of α_i and β_i, this multiobjective problem reads as

$$\forall n\colon \min_{\boldsymbol{\Upsilon} \in \mathbb{U}^{n_{\mathrm{T}} \times r_{\mathbf{P}}}} \sum_{i=1}^{n} \Big[\boldsymbol{\Upsilon}^\dagger (\hat{\mathbf{R}}^{\frac{1}{2}})^+ (\tfrac{\mathbf{I}}{\mu_Q} + \tilde{\mathbf{R}}) (\hat{\mathbf{R}}^{\frac{1}{2}})^+ \boldsymbol{\Upsilon} \Big]_{\pi(i),\pi(i)}, \tag{D.35}$$

where π denotes the permutation which orders the diagonal entries of the matrix between square brackets so as to be non-decreasingly ordered. If $\mathbf{W}\mathbf{D}\mathbf{W}^\dagger$ denotes the spectral decomposition of $(\hat{\mathbf{R}}^{\frac{1}{2}})^+(\frac{\mathbf{I}}{\mu_Q}+\tilde{\mathbf{R}})(\hat{\mathbf{R}}^{\frac{1}{2}})^+$ where $\mathbf{W}\in\mathbb{U}^{n_\mathrm{T}\times r_\mathbf{P}}$, and $\mathbf{D}$ has non-increasingly ordered, positive diagonal entries, then it is well known from majorization theory that the solution of (D.35) is $\boldsymbol{\Upsilon}^\star=\mathbf{W}$, up to a column permutation (e.g., [Hor90, Theorem 4.3.26]). It turns out as well that $\mathrm{range}(\boldsymbol{\Upsilon}^\star)=\mathrm{range}(\mathbf{W})=\mathrm{range}(\hat{\mathbf{R}})$, so the range space constraint is systematically fulfilled. The columns of $\boldsymbol{\Upsilon}^\star$ contain the eigenvectors $\boldsymbol{v}_i$ of the generalized eigenvalue problem

$$\hat{\mathbf{R}}\boldsymbol{v}_i=\omega_i(\tfrac{1}{\mu_Q}\mathbf{I}+\tilde{\mathbf{R}})\boldsymbol{v}_i, \tag{D.36}$$

corresponding to the $r_\mathbf{P}$ largest generalized eigenvalues ω_i.

D.5 Proof of Theorem 4.3

Assume that $r_\mathbf{P}>r_\mathbf{Q}$. Similarly as for the proof of Theorem 4.1 in Appendix D.2, we will proceed by constructing another pilot matrix in $\mathcal{P}(\mu_\mathcal{P})$ which strictly outperforms $\mathbf{P}$. Recall that the covariance of the channel estimate is $\hat{\mathbf{R}}=\mathbf{R}-\tilde{\mathbf{R}}$, as usual, and $\tilde{\mathbf{R}}=(\mathbf{R}^{-1}+\mathbf{P})^{-1}$ is the estimation error covariance. We construct $\mathbf{P}'$ as

$$\mathbf{P}'=\left[\tilde{\mathbf{R}}+\lambda_{r_\mathbf{P}}(\hat{\mathbf{R}})\mathbf{L}_{\mathbf{Q}^\perp\cap\hat{\mathbf{R}}}\mathbf{L}_{\mathbf{Q}^\perp\cap\hat{\mathbf{R}}}^\dagger\right]^{-1}-\mathbf{R}^{-1}. \tag{D.37}$$

The subunitary matrix $\mathbf{L}_{\mathbf{Q}^\perp\cap\hat{\mathbf{R}}}$ is defined as in the proof of Theorem 4.1. It exists and has at least $d=r_\mathbf{P}-r_\mathbf{Q}$ columns. First, we verify that $\mathbf{P}'\in\mathcal{P}(\mu_\mathcal{P})$. In fact, $\mathbf{P}'$ can be written out as

$$\mathbf{P}'=\left[\mathbf{R}-\hat{\mathbf{R}}+\lambda_{r_\mathbf{P}}(\hat{\mathbf{R}})\mathbf{L}_{\mathbf{Q}^\perp\cap\hat{\mathbf{R}}}\mathbf{L}_{\mathbf{Q}^\perp\cap\hat{\mathbf{R}}}^\dagger\right]^{-1}-\mathbf{R}^{-1}, \tag{D.38}$$

where it becomes clear that $\mathbf{P}'\succeq\mathbf{0}$, because $\hat{\mathbf{R}}-\lambda_{r_\mathbf{P}}(\hat{\mathbf{R}})\mathbf{L}_{\mathbf{Q}^\perp\cap\hat{\mathbf{R}}}\mathbf{L}_{\mathbf{Q}^\perp\cap\hat{\mathbf{R}}}^\dagger\succeq\mathbf{0}$. On the other hand, the trace of $\mathbf{P}'$ is upper-bounded as

$$\begin{aligned}\mathrm{tr}(\mathbf{P}')&=\mathrm{tr}\left(\left[\tilde{\mathbf{R}}+\lambda_{r_\mathbf{P}}(\hat{\mathbf{R}})\mathbf{L}_{\mathbf{Q}^\perp\cap\hat{\mathbf{R}}}\mathbf{L}_{\mathbf{Q}^\perp\cap\hat{\mathbf{R}}}^\dagger\right]^{-1}\right)-\mathrm{tr}(\mathbf{R}^{-1})\\&<\mathrm{tr}(\tilde{\mathbf{R}}^{-1})-\mathrm{tr}(\mathbf{R}^{-1})\\&=\mathrm{tr}(\mathbf{P}).\end{aligned} \tag{D.39}$$

If we write $\tilde{\mathbf{R}}=\tilde{\mathbf{R}}(\mathbf{P})$ to stress that it is essentially a function of $\mathbf{P}$, then we notice that $\mathbf{P}'$ is designed so as to leave the product

$$\begin{aligned}\mathbf{Q}\tilde{\mathbf{R}}(\mathbf{P}')&=\mathbf{Q}(\tilde{\mathbf{R}}+\lambda_{r_\mathbf{P}}(\hat{\mathbf{R}})\mathbf{L}_{\mathbf{Q}^\perp\cap\hat{\mathbf{R}}}\mathbf{L}_{\mathbf{Q}^\perp\cap\hat{\mathbf{R}}}^\dagger)\\&=\mathbf{Q}\tilde{\mathbf{R}}(\mathbf{P})\end{aligned} \tag{D.40}$$

unchanged, irrespective of whether the pilots are $\mathbf{P}$ or $\mathbf{P}'$. The same is true for $\mathbf{s}=\boldsymbol{\lambda}(\mathbf{S})$, which is left unchanged when replacing $\mathbf{P}$ by $\mathbf{P}'$, because $\mathbf{s}$ depends on $\mathbf{P}$ only via the product $\mathbf{Q}\tilde{\mathbf{R}}(\mathbf{P})$, as seen from the relationship

$$\mathbf{s} = \frac{\boldsymbol{\lambda}(\hat{\mathbf{R}}^{\frac{1}{2}}\mathbf{Q}\hat{\mathbf{R}}^{\frac{1}{2}})}{1+\operatorname{tr}(\mathbf{Q}\tilde{\mathbf{R}})} = \frac{\boldsymbol{\lambda}(\mathbf{QR}-\mathbf{Q}\tilde{\mathbf{R}})}{1+\operatorname{tr}(\mathbf{Q}\tilde{\mathbf{R}})}. \tag{D.41}$$

We have thus constructed alternative pilots $\mathbf{P}'$ which yield the same utility value $f(\mathbf{s})$, yet saving on the training energy, since $\operatorname{tr}(\mathbf{P}') < \operatorname{tr}(\mathbf{P})$. We generate another pilot matrix $\mathbf{P}'' = \kappa\mathbf{P}'$ with $\kappa = \operatorname{tr}(\mathbf{P})/\operatorname{tr}(\mathbf{P}')$. The new pilots $\mathbf{P}''$ spend the same amount of training energy as $\mathbf{P}$, but yield a strictly larger $\mathbf{S}'' = \mathbf{S}(\mathbf{P}'', \mathbf{Q}) \succ \mathbf{S}(\mathbf{P}, \mathbf{Q})$. Hence, $\mathbf{P}$ is suboptimal.

D.6 Convexity of the set of feasible $\hat{\mathrm{R}}$

Showing the convexity of the set of feasible $\hat{\mathbf{R}}$ is equivalent to showing the convexity of the set of feasible $\tilde{\mathbf{R}}$, because $\hat{\mathbf{R}} = \mathbf{R} - \tilde{\mathbf{R}}$ is merely $\tilde{\mathbf{R}}$ scaled with -1 and summed with a constant matrix $\mathbf{R}$. Therefore, we show that the set

$$\{\tilde{\mathbf{R}} = (\mathbf{R}^{-1} + \mathbf{P})^{-1} | \mathbf{P} \in \mathcal{P}(\mu_{\mathcal{P}})\} \tag{D.42}$$

is convex. For any pair $(\mathbf{P}_1, \mathbf{P}_2) \in \mathcal{P}(\mu_{\mathcal{P}})^2$, there exists a $\mathbf{P}_3 \in \mathcal{P}(\mu_{\mathcal{P}})$ and a $\alpha \in [0;1]$ such that

$$\alpha\tilde{\mathbf{R}}_1 + (1-\alpha)\tilde{\mathbf{R}}_2 = \tilde{\mathbf{R}}_3, \tag{D.43}$$

where $\tilde{\mathbf{R}}_i = (\mathbf{R}^{-1} + \mathbf{P}_i)^{-1}$ for $i = 1, 2, 3$. By isolating $\mathbf{P}_3$ in (D.43), the pilot Gram $\mathbf{P}_3$ is given by

$$\mathbf{P}_3 = \left[\alpha\tilde{\mathbf{R}}_1 + (1-\alpha)\tilde{\mathbf{R}}_2\right]^{-1} - \mathbf{R}^{-1}. \tag{D.44}$$

Obviously, since $\tilde{\mathbf{R}}_i \preceq \mathbf{R}$ for $i = 1, 2$, we have $\mathbf{P}_3 \succeq \mathbf{0}$. What remains to prove is that $\operatorname{tr}(\mathbf{P}_3) \le \mu_{\mathcal{P}}$. Knowing that the function $\mathbf{X} \mapsto \operatorname{tr}(\mathbf{X}^{-1})$ is convex on the positive cone $\mathbf{X} \succ \mathbf{0}$, we have

$$\begin{aligned}\operatorname{tr}(\mathbf{P}_3) &\le \alpha\operatorname{tr}(\tilde{\mathbf{R}}_1^{-1}) + (1-\alpha)\operatorname{tr}(\tilde{\mathbf{R}}_2^{-1}) - \operatorname{tr}(\mathbf{R}^{-1}) \\ &= \alpha\operatorname{tr}(\mathbf{P}_1) + (1-\alpha)\operatorname{tr}(\mathbf{P}_2) \\ &\le \mu_{\mathcal{P}}.\end{aligned} \tag{D.45}$$

Hence, the set of feasible $\hat{\mathbf{R}}$ is convex, and so is Problem (P.1.b).

D.7 Proof of Theorem 4.4

We will proceed by showing that, in Problem (P.2), for any given value of the pair $(\mu_{\mathcal{P}}, \mu_{\mathcal{Q}})$, the search set $\mathbf{s}(\mathcal{P}(\mu_{\mathcal{P}}), \mathcal{Q}(\mu_{\mathcal{Q}}))$—and thus its Pareto border $\partial^+\mathbf{s}(\mathcal{P}(\mu_{\mathcal{P}}), \mathcal{Q}(\mu_{\mathcal{Q}}))$—is left unchanged whether we allow $(\mathbf{P}, \mathbf{Q})$ to take *any* value within $\mathcal{P}(\mu_{\mathcal{P}}) \times \mathcal{Q}(\mu_{\mathcal{Q}})$, or whether we restrict the choice of the basis $\mathbf{U}_\mathbf{P}$ such that $\operatorname{col}(\mathbf{U}_\mathbf{P}) = \{\mathbf{u}_{\mathbf{R},1}, \ldots, \mathbf{u}_{\mathbf{R},r^\star}\}$, where $r^\star$ denotes the number of non-zero entries of the $\mathbf{s}^\star$. With a consequence of Theorem 4.2, we will eventually conclude on the desired result $\mathrm{col}(\mathbf{U_P}) = \mathrm{col}(\mathbf{U_Q}) = \{\mathbf{u}_{\mathbf{R},1}, \dots, \mathbf{u}_{\mathbf{R},r^\star}\}$.

To begin with, note that the set $\mathbf{s}(\mathcal{P}(\mu_\mathcal{P}), \mathcal{Q}(\mu_\mathcal{Q}))$ can be represented as the union

$$\mathbf{s}(\mathcal{P}(\mu_\mathcal{P}), \mathcal{Q}(\mu_\mathcal{Q})) = \bigcup_{\mathbf{P}\in\mathcal{P}(\mu_\mathcal{P})} \mathbf{s}(\mathbf{P}, \mathcal{Q}(\mu_\mathcal{Q})). \tag{D.46}$$

As a consequence of the rank equality (4.18), the elements of $\mathbf{s}(\mathbf{P}, \mathcal{Q}(\mu_\mathcal{Q}))$ have at most $r_\mathbf{P}$ non-zero entries, because $\mathrm{rank}(\mathbf{S}) = \mathrm{rank}(\hat{\mathbf{R}}^{\frac{1}{2}}\mathbf{Q}\hat{\mathbf{R}}^{\frac{1}{2}}) \le \mathrm{rank}(\hat{\mathbf{R}}) = \mathrm{rank}(\mathbf{P}) = r_\mathbf{P}$. They can thus be written as $\mathbf{s}(\mathbf{P},\mathbf{Q}) = \begin{bmatrix}\bar{\mathbf{s}}(\mathbf{P},\mathbf{Q})^\mathrm{T} & \mathbf{0}^\mathrm{T}\end{bmatrix}^\mathrm{T}$ with $\bar{\mathbf{s}}(\mathbf{P},\mathbf{Q}) \in \mathbb{R}_+^{r_\mathbf{P}}$ of reduced size. According to Theorem 4.2, this set of reduced-size vectors $\bar{\mathbf{s}}(\mathbf{P}, \mathcal{Q}(\mu_\mathcal{Q}))$ is the simplex given by the convex hull of the points

$$\boldsymbol{\sigma}^{(0)} = \mathbf{0} \qquad\qquad \boldsymbol{\sigma}^{(n)} = \mathcal{H}(\omega_1, \dots, \omega_n)\sum_{j=1}^{n}\mathbf{e}_j, \quad n \in \{1, \dots, r_\mathbf{P}\}. \tag{D.47}$$

Every such simplex is entirely described by $\boldsymbol{\omega} = [\omega_1, \dots, \omega_{r_\mathbf{P}}]^\mathrm{T}$, the vector non-increasingly ordered eigenvalues of the matrix $\hat{\mathbf{R}}(\mu_\mathcal{Q}^{-1}\mathbf{I} + \tilde{\mathbf{R}})^{-1}$, which is a function of $\mathbf{P}$ alone (not of $\mathbf{Q}$). Consistently with the notation used so far, $\boldsymbol{\omega}(\mathcal{P}(\mu_\mathcal{P}))$ shall denote the set of feasible $\boldsymbol{\omega}$ given that $\mathbf{P}$ belongs to $\mathcal{P}(\mu_\mathcal{P})$. To prove Theorem 4.4, we will first show that the set of Pareto border points $\partial^+\boldsymbol{\omega}(\mathcal{P}(\mu_\mathcal{P}))$ is still achievable under the restriction $\mathrm{col}(\mathbf{U_P}) \subseteq \{\mathbf{u}_{\mathbf{R},1}, \dots, \mathbf{u}_{\mathbf{R},r^\star}\}$. Recalling that $\mathbf{R} = \hat{\mathbf{R}} + \tilde{\mathbf{R}}$ and $\tilde{\mathbf{R}} = (\mathbf{R}^{-1} + \mathbf{P})^{-1}$, we write out $\boldsymbol{\omega}$ as

$$\begin{aligned}\boldsymbol{\omega} &= \boldsymbol{\lambda}\left(\hat{\mathbf{R}}(\tfrac{1}{\mu_\mathcal{Q}}\mathbf{I} + \tilde{\mathbf{R}})^{-1}\right)\\ &= \boldsymbol{\lambda}\left(\left(\mathbf{R} - (\mathbf{R}^{-1} + \mathbf{P})^{-1}\right)\left(\tfrac{1}{\mu_\mathcal{Q}}\mathbf{I} + (\mathbf{R}^{-1} + \mathbf{P})^{-1}\right)^{-1}\right).\end{aligned}$$

Let us denote $\mathbf{P}' = \mathbf{R}^{\frac{1}{2}}\mathbf{P}\mathbf{R}^{\frac{1}{2}}$, then using the property $\boldsymbol{\lambda}(\mathbf{AB}) = \boldsymbol{\lambda}(\mathbf{BA})$, the last expression can be rewritten as

$$\boldsymbol{\omega} = \boldsymbol{\lambda}\left(\left(\mathbf{I} - (\mathbf{I} + \mathbf{P}')^{-1}\right)\left((\mu_\mathcal{Q}\mathbf{R})^{-1} + (\mathbf{I} + \mathbf{P}')^{-1}\right)^{-1}\right).$$

Let us now denote $\mathbf{P}'' = \mathbf{U}_\mathbf{R}^\dagger\mathbf{P}'\mathbf{U}_\mathbf{R}$, so that the last expression becomes

$$\boldsymbol{\omega} = \boldsymbol{\lambda}\left(\left(\mathbf{I} - (\mathbf{I} + \mathbf{P}'')^{-1}\right)\left((\mu_\mathcal{Q}\boldsymbol{\Lambda}_\mathbf{R})^{-1} + (\mathbf{I} + \mathbf{P}'')^{-1}\right)^{-1}\right). \tag{D.48}$$

Let us write out the mutual relations linking $\mathbf{P}$ and $\mathbf{P}''$ in full:

$$\mathbf{P}'' = \boldsymbol{\Lambda}_\mathbf{R}^{\frac{1}{2}}\mathbf{U}_\mathbf{R}^\dagger\mathbf{U}_\mathbf{P}\,\mathrm{diag}(\mathbf{p})\mathbf{U}_\mathbf{P}^\dagger\mathbf{U}_\mathbf{R}\boldsymbol{\Lambda}_\mathbf{R}^{\frac{1}{2}} \tag{D.49a}$$

$$\mathbf{P} = \mathbf{U}_\mathbf{R}\boldsymbol{\Lambda}_\mathbf{R}^{-\frac{1}{2}}\mathbf{U}_{\mathbf{P}''}\,\mathrm{diag}(\mathbf{p}'')\mathbf{U}_{\mathbf{P}''}^\dagger\boldsymbol{\Lambda}_\mathbf{R}^{-\frac{1}{2}}\mathbf{U}_\mathbf{R}^\dagger. \tag{D.49b}$$

Regarding the (non-reduced) eigendecomposition $\mathbf{P}'' = \mathbf{U}_{\mathbf{P}''}\boldsymbol{\Lambda}_{\mathbf{P}''}\mathbf{U}_{\mathbf{P}''}^\dagger$ with $\mathbf{U}_{\mathbf{P}''} \in \mathbb{U}^{n_\mathrm{T}\times n_\mathrm{T}}$ and

$\boldsymbol{\Lambda}_{\mathbf{P}''} = \operatorname{diag}(\mathbf{p}'') = \operatorname{diag}(p_1'', \ldots, p_{n_\mathsf{T}}'')$, one can say that, if $\mathbf{P}$ is drawn from $\mathcal{P}(\mu_\mathcal{P})$, then the corresponding eigenvalue profile $\mathbf{p}'' = \boldsymbol{\lambda}(\mathbf{P}'') = \boldsymbol{\lambda}(\boldsymbol{\Lambda}_\mathbf{R}^{\frac{1}{2}} \mathbf{U}_\mathbf{R}^\dagger \mathbf{P} \mathbf{U}_\mathbf{R} \boldsymbol{\Lambda}_\mathbf{R}^{\frac{1}{2}}) = \boldsymbol{\lambda}(\mathbf{PR})$ [cf. (D.49a)] is drawn from a feasible set which we shall call $\mathbf{p}''(\mathcal{P}(\mu_\mathcal{P}))$, a notation which emphasizes its direct dependence on the domain $\mathcal{P}(\mu_\mathcal{P})$. As to the eigenbasis $\mathbf{U}_{\mathbf{P}''}$, it obviously belongs to $\mathbb{U}^{n_\mathsf{T} \times n_\mathsf{T}}$ by definition, yet in general, we must presume that not all pairs $(\mathbf{p}'', \mathbf{U}_{\mathbf{P}''}) \in \mathbf{p}''(\mathcal{P}(\mu_\mathcal{P})) \times \mathbb{U}^{n_\mathsf{T} \times n_\mathsf{T}}$ are jointly feasible, since the eigenbasis $\mathbf{U}_{\mathbf{P}''}$ and the eigenvalues $\mathbf{p}''$ cannot be chosen independently of each other, due to the special structure of Expression (D.49a). Instead, $\mathbf{U}_\mathbf{P}''$ belongs to a feasible set $\mathbf{U}_{\mathbf{P}''}(\mathbf{p}'') \subseteq \mathbb{U}^{n_\mathsf{T} \times n_\mathsf{T}}$ (which depends on $\mathbf{p}''$), so the overall set of feasible pairs $(\mathbf{p}'', \mathbf{U}_{\mathbf{P}''})$ forms a *subset* of the Cartesian product $\mathbf{p}''(\mathcal{P}(\mu_\mathcal{P})) \times \mathbb{U}^{n_\mathsf{T} \times n_\mathsf{T}}$.

However, suppose for a while that $\mathbf{p}''$ and $\mathbf{U}_{\mathbf{P}''}$ can be drawn *independently* of each other from their respective domains $\mathbf{p}''(\mathcal{P}(\mu_\mathcal{P}))$ and $\mathbb{U}^{n_\mathsf{T} \times n_\mathsf{T}}$. This assumption then corresponds to a *relaxation* of the original problem, as it possibly extends the overall set of feasible $\mathbf{P}''$, and consequently, of feasible $\boldsymbol{\omega}$. The resulting set of achievable vectors $\boldsymbol{\omega}$ under this relaxation shall be denoted $\bar{\boldsymbol{\omega}}(\mathcal{P}(\mu_\mathcal{P})) \supseteq \boldsymbol{\omega}(\mathcal{P}(\mu_\mathcal{P}))$ and is formally defined as

$$\bar{\boldsymbol{\omega}}(\mathcal{P}(\mu_\mathcal{P})) = \left\{ \boldsymbol{\omega}(\mathbf{p}'', \mathbf{U}_{\mathbf{P}''}) \middle| (\mathbf{p}'', \mathbf{U}_{\mathbf{P}''}) \in \mathbf{p}''(\mathcal{P}(\mu_\mathcal{P})) \times \mathbb{U}^{n_\mathsf{T} \times n_\mathsf{T}} \right\}, \tag{D.50}$$

wherein the two-argument notation $\boldsymbol{\omega}(\bullet, \bullet)$ is defined as [cf. (D.49b)]

$$\boldsymbol{\omega}(\mathbf{p}'', \mathbf{U}_{\mathbf{P}''}) \triangleq \boldsymbol{\omega}(\mathbf{U}_\mathbf{R} \boldsymbol{\Lambda}_\mathbf{R}^{-\frac{1}{2}} \mathbf{U}_{\mathbf{P}''} \operatorname{diag}(\mathbf{p}'') \mathbf{U}_{\mathbf{P}''}^\dagger \boldsymbol{\Lambda}_\mathbf{R}^{-\frac{1}{2}} \mathbf{U}_\mathbf{R}^\dagger). \tag{D.51}$$

The set $\bar{\boldsymbol{\omega}}(\mathcal{P}(\mu_\mathcal{P}))$ can be represented as a double union

$$\begin{aligned} \bar{\boldsymbol{\omega}}(\mathcal{P}(\mu_\mathcal{P})) &= \bigcup_{\mathbf{p}'' \in \mathbf{p}''(\mathcal{P}(\mu_\mathcal{P}))} \; \bigcup_{\mathbf{U}_{\mathbf{P}''} \in \mathbb{U}^{n_\mathsf{T} \times n_\mathsf{T}}} \boldsymbol{\omega}(\mathbf{p}'', \mathbf{U}_{\mathbf{P}''}) \\ &= \bigcup_{\mathbf{p}'' \in \mathbf{p}''(\mathcal{P}(\mu_\mathcal{P}))} \boldsymbol{\omega}(\mathbf{p}'', \mathbb{U}^{n_\mathsf{T} \times n_\mathsf{T}}). \end{aligned} \tag{D.52}$$

As seen from expression (D.48), $\boldsymbol{\omega}(\mathbf{p}'', \mathbf{U}_{\mathbf{P}''})$ is monotonic in the eigenvalues p_i'', meaning that

$$\forall \mathbf{d} \geq \mathbf{0} \colon \boldsymbol{\omega}(\mathbf{p}'' + \mathbf{d}, \mathbf{U}_{\mathbf{P}''}) \geq \boldsymbol{\omega}(\mathbf{p}'', \mathbf{U}_{\mathbf{P}''}).$$

Hence, since we are essentially interested in the Pareto border $\partial^+ \bar{\boldsymbol{\omega}}(\mathcal{P}(\mu_\mathcal{P}))$ of the set $\bar{\boldsymbol{\omega}}(\mathcal{P}(\mu_\mathcal{P}))$, we can restrict our further analysis to the set[1]

$$\bar{\boldsymbol{\omega}}^+(\mathcal{P}(\mu_\mathcal{P})) = \bigcup_{\mathbf{p}'' \in \partial^+ \mathbf{p}''(\mathcal{P}(\mu_\mathcal{P}))} \boldsymbol{\omega}(\mathbf{p}'', \mathbb{U}^{n_\mathsf{T} \times n_\mathsf{T}}). \tag{D.53}$$

The remainder of the proof of Theorem 4.4 is completed in four successive steps, each of which is detailed in a separate paragraph, for the sake of a clearer structure: first, we specify

[1] Note that $\bar{\boldsymbol{\omega}}^+(\mathcal{P}(\mu_\mathcal{P}))$ is generally not the Pareto border of $\bar{\boldsymbol{\omega}}(\mathcal{P}(\mu_\mathcal{P}))$, but rather a superset thereof.

a method for constructing a particular Pareto border point of the set $\boldsymbol{\omega}(\mathbf{p}'', \mathbb{U}^{n_\mathsf{T} \times n_\mathsf{T}})$ given a particular value of the vector $\mathbf{p}''$, where we show that this construction requires the alignment $\mathrm{col}(\mathbf{U_P}) \subseteq \mathrm{col}(\mathbf{U_R})$; second, we show that the point constructed this way, besides yielding a Pareto border point of the relaxed set $\bar{\boldsymbol{\omega}}(\mathcal{P}(\mu_\mathcal{P}))$, is also contained in the smaller (non-relaxed) set $\boldsymbol{\omega}(\mathcal{P}(\mu_\mathcal{P}))$, so it must be a Pareto border point of $\boldsymbol{\omega}(\mathcal{P}(\mu_\mathcal{P}))$ as well; third, we show that, by varying the eigenvalues $\mathbf{p}''$ over the feasible set $\mathbf{p}''(\mathcal{P}(\mu_\mathcal{P}))$, with the aforementioned method of constructing particular Pareto border points, we reach the whole Pareto border of $\boldsymbol{\omega}(\mathcal{P}(\mu_\mathcal{P}))$; fourth, we show that the alignment $\mathrm{col}(\mathbf{U_P}) \subseteq \mathrm{col}(\mathbf{U_R})$ implies that $\mathbf{U_Q}$ must as well be aligned such that $\mathrm{col}(\mathbf{U_Q}) = \mathrm{col}(\mathbf{U_P})$ to reach the whole feasible set $\mathbf{s}(\mathcal{P}(\mu_\mathcal{P}), \mathcal{Q}(\mu_\mathcal{Q}))$, and conclude.

1) Let the orthonormal eigenbasis $\mathbf{U}_{\mathbf{P}''} = [\mathbf{u}_1, \dots, \mathbf{u}_{n_\mathsf{T}}]$ be spanned by unit vectors $\mathbf{u}_i$, where the i-th vector $\mathbf{u}_i$ is associated to the i-th largest eigenvalue p''_i. Given a fixed value of $\mathbf{p}'' \in \mathbf{p}''(\mathcal{P}(\mu_\mathcal{P}))$, we construct a particular point of the Pareto border $\partial^+\boldsymbol{\omega}(\mathbf{p}'', \mathbb{U}^{n_\mathsf{T} \times n_\mathsf{T}})$ by solving the sequence of optimization problems:

$$\begin{aligned} \forall i \in \{1, \dots, n_\mathsf{T}\}: \qquad & \mathbf{U}^{(i)} = \operatorname*{argmax}_{\mathbf{U} \in \mathbb{U}^{n_\mathsf{T} \times n_\mathsf{T}}} \ \omega_i(\mathbf{p}'', \mathbf{U}) \\ \text{s.t.} \ \ \forall \ell \in \{1, \dots, i-1\}: \qquad & \omega_\ell = \omega_\ell(\mathbf{p}'', \mathbf{U}^{(i-1)}). \end{aligned} \tag{D.54}$$

Clearly, $\mathbf{U}^{(n_\mathsf{T})}$ will yield a Pareto optimal point, that is,

$$\boldsymbol{\omega}(\mathbf{p}'', \mathbf{U}^{(n_\mathsf{T})}) \in \partial^+\boldsymbol{\omega}(\mathbf{p}'', \mathbb{U}^{n_\mathsf{T} \times n_\mathsf{T}}). \tag{D.55}$$

Next, we will show by induction that $\mathbf{U}^{(n_\mathsf{T})} = \mathbf{I}$. For this purpose, let us explicitly solve the first problem ($i = 1$) of (D.54), i.e.,

$$\mathbf{U}^{(1)} = \operatorname*{argmax}_{\mathbf{U} \in \mathbb{U}^{n_\mathsf{T} \times n_\mathsf{T}}} \ \omega_1(\mathbf{p}'', \mathbf{U}) \tag{D.56}$$

With Expression (D.48), this reads as

$$\begin{aligned} & \max_{\mathbf{U} \in \mathbb{U}^{n_\mathsf{T} \times n_\mathsf{T}}} \max_{\|\mathbf{v}_1\| = 1} \left[\frac{\mathbf{v}_1^\dagger (\mathbf{I} - (\mathbf{I} + \mathbf{U}\boldsymbol{\Lambda}_{\mathbf{P}''}\mathbf{U}^\dagger)^{-1})\mathbf{v}_1}{\mathbf{v}_1^\dagger ((\mu_\mathcal{Q}\boldsymbol{\Lambda}_\mathbf{R})^{-1} + (\mathbf{I} + \mathbf{U}\boldsymbol{\Lambda}_{\mathbf{P}''}\mathbf{U}^\dagger)^{-1})\mathbf{v}_1} \right] \\ & \leq \max_{\mathbf{U} \in \mathbb{U}^{n_\mathsf{T} \times n_\mathsf{T}}} \left[\frac{1 - \lambda_{\mathsf{min}}((\mathbf{I} + \mathbf{U}\boldsymbol{\Lambda}_{\mathbf{P}''}\mathbf{U}^\dagger)^{-1})}{\lambda_{\mathsf{min}}((\mu_\mathcal{Q}\boldsymbol{\Lambda}_\mathbf{R})^{-1}) + \lambda_{\mathsf{min}}((\mathbf{I} + \mathbf{U}\boldsymbol{\Lambda}_{\mathbf{P}''}\mathbf{U}^\dagger)^{-1})} \right] \\ & = \frac{1 - \frac{1}{1 + \lambda_{\mathsf{max}}(\boldsymbol{\Lambda}_{\mathbf{P}''})}}{(\mu_\mathcal{Q}\lambda_{\mathsf{max}}(\boldsymbol{\Lambda}_\mathbf{R}))^{-1} + \frac{1}{1 + \lambda_{\mathsf{max}}(\boldsymbol{\Lambda}_{\mathbf{P}''})}} \\ & = \frac{1 - \frac{1}{1 + p''_1}}{(\mu_\mathcal{Q} r_1)^{-1} + \frac{1}{1 + p''_1}}. \end{aligned} \tag{D.57}$$

This upper bound is tight and achieved if and only if $\mathbf{v}_1 = \mathbf{e}_1$, and when $\mathbf{U}$ is of the form

$$\mathbf{U}^{(1)} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{W}^{(1)} \end{bmatrix} \tag{D.58}$$

with some arbitrary $\mathbf{W}^{(1)} \in \mathbb{U}^{(n_\mathrm{T}-1)\times(n_\mathrm{T}-1)}$. To prove the induction step, we will show that if for a certain $i \geq 1$, all maximizers $\mathbf{U}^{(i)}$ are of the form

$$\mathbf{U}^{(i)} = \begin{bmatrix} \mathbf{I}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{W}^{(i)} \end{bmatrix} \tag{D.59}$$

with some arbitrary $\mathbf{W}^{(i)} \in \mathbb{U}^{(n_\mathrm{T}-i)\times(n_\mathrm{T}-i)}$, and $\forall \ell = 1, \ldots, i\colon \mathbf{v}_\ell = \mathbf{e}_\ell$, then $\mathbf{U}^{(i+1)}$ also has the above block structure (D.59), with an identity matrix $\mathbf{I}_{i+1}$ top left and an arbitrary rotation matrix $\mathbf{W}^{(i+1)}$ bottom right. After solving the i-th problem, we know that all solutions thereof are of the form (D.59), which implies that the equality constraints for the $(i+1)$-th problem [cf. (D.54)] can only be fulfilled if $\mathbf{U}^{(i+1)}$ has the same structure as $\mathbf{U}^{(i)}$, i.e.,

$$\mathbf{U}^{(i+1)} = \begin{bmatrix} \mathbf{I}_i & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{W}}^{(i)} \end{bmatrix} \tag{D.60}$$

with some unitary matrix $\tilde{\mathbf{W}}^{(i)} \in \mathbb{U}^{(n_\mathrm{T}-i)\times(n_\mathrm{T}-i)}$ to be determined. According to a straightforward adaptation of the Courant-Fisher Theorem [Hor90, Theorem 4.2.11], the non-increasingly ordered eigenvalues $\lambda_i(\mathbf{A}\mathbf{B}^{-1})$ with corresponding eigenvectors $\mathbf{v}_i$ of a product of two Hermitian matrices $\mathbf{A}$ and $\mathbf{B}^{-1}$ can be expressed as

$$\lambda_i(\mathbf{A}\mathbf{B}^{-1}) = \max_{\mathbf{v}\perp\mathbf{B}\mathbf{v}_{i-1},\ldots,\mathbf{B}\mathbf{v}_1} \frac{\mathbf{v}^\dagger\mathbf{A}\mathbf{v}}{\mathbf{v}^\dagger\mathbf{B}\mathbf{v}}. \tag{D.61}$$

The $(i+1)$-th optimization problem reads as

$$\mathbf{U}^{(i+1)} = \operatorname*{argmax}_{\mathbf{U}\in\mathbb{U}^{n_\mathrm{T}\times n_\mathrm{T}}} \left\{ \max_{\mathbf{v}_{i+1}\perp\mathbf{B}\mathbf{v}_i,\ldots,\mathbf{B}\mathbf{v}_1} \frac{\mathbf{v}_{i+1}^\dagger\mathbf{A}\mathbf{v}_{i+1}}{\mathbf{v}_{i+1}^\dagger\mathbf{B}\mathbf{v}_{i+1}} \right\} \qquad \text{s.t.}\quad \mathbf{U} = \begin{bmatrix} \mathbf{I}_i & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{W}}^{(i)} \end{bmatrix} \tag{D.62}$$

with $\mathbf{A} = \mathbf{I} - (\mathbf{I} + \mathbf{U}\mathbf{\Lambda}_{\mathbf{P}''}\mathbf{U}^\dagger)^{-1}$ and $\mathbf{B} = (\mu_Q\mathbf{\Lambda}_\mathbf{R})^{-1} + (\mathbf{I} + \mathbf{U}\mathbf{\Lambda}_{\mathbf{P}''}\mathbf{U}^\dagger)^{-1}$ [cf. (D.48)], and $\forall \ell = 1, \ldots, i\colon \mathbf{v}_\ell = \mathbf{e}_\ell$. When writing out $\mathbf{B}$, the vectors involved in the orthogonality constraints $\mathbf{v}_i \perp \mathbf{B}\mathbf{e}_{i-1}, \ldots, \mathbf{B}\mathbf{e}_1$ read as

$$\begin{aligned} \mathbf{B}\mathbf{e}_\ell &= \Big[(\mu_Q\mathbf{\Lambda}_\mathbf{R})^{-1} + (\mathbf{I} + \mathbf{U}\mathbf{\Lambda}_{\mathbf{P}''}\mathbf{U}^\dagger)^{-1}\Big]\mathbf{e}_\ell \\ &= \frac{\mathbf{e}_\ell}{\mu_Q r_\ell} + \begin{bmatrix} (\mathbf{I}+\mathbf{\Lambda}_{\mathbf{P}''}^{[i]})^{-1} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{W}}^{(i)}(\mathbf{I}+\bar{\mathbf{\Lambda}}_{\mathbf{P}''}^{[i]})^{-1}(\tilde{\mathbf{W}}^{(i)})^\dagger \end{bmatrix}\mathbf{e}_\ell \\ &= \Big[\frac{1}{\mu_Q r_\ell} + \frac{1}{1+p''_\ell}\Big]\mathbf{e}_\ell \qquad \forall \ell = 1, \ldots, i \end{aligned} \tag{D.63}$$

where $\mathbf{\Lambda}_{\mathbf{P}''}^{[i]} = \operatorname{diag}(p_1'', \dots, p_i'')$ and $\bar{\mathbf{\Lambda}}_{\mathbf{P}''}^{[i]} = \operatorname{diag}(p_{i+1}'', \dots, p_{n_\mathsf{T}}'')$. Thus, the orthogonality constraints simply translate into $\mathbf{v}_{i+1} \perp \mathbf{e}_i, \dots, \mathbf{e}_1$. In other terms, the first i entries of $\mathbf{v}_{i+1}$ must be zero. Thus, we can define matrices $(n_\mathsf{T} - i) \times (n_\mathsf{T} - i)$ matrices $\breve{\mathbf{A}}$ and $\breve{\mathbf{B}}$ as

$$\begin{aligned} \breve{\mathbf{A}} &= \mathbf{I} - \left(\mathbf{I} + \tilde{\mathbf{W}}^{(i)} \bar{\mathbf{\Lambda}}_{\mathbf{P}''}^{[i]} (\tilde{\mathbf{W}}^{(i)})^\dagger\right)^{-1} \\ \breve{\mathbf{B}} &= (\mu_\mathcal{Q} \bar{\mathbf{\Lambda}}_{\mathbf{R}}^{(i)})^{-1} + \left(\mathbf{I} + \tilde{\mathbf{W}}^{(i)} \bar{\mathbf{\Lambda}}_{\mathbf{P}''}^{[i]} (\tilde{\mathbf{W}}^{(i)})^\dagger\right)^{-1}, \end{aligned} \tag{D.64}$$

so that the optimization problem (D.62) boils down to solving

$$\tilde{\mathbf{W}}^{(i)} = \underset{\mathbf{W} \in \mathbb{U}^{(n_\mathsf{T}-i) \times (n_\mathsf{T}-i)}}{\operatorname{argmax}} \left\{ \max_{\breve{\mathbf{v}}_{i+1}} \frac{\breve{\mathbf{v}}_{i+1}^\dagger \breve{\mathbf{A}} \breve{\mathbf{v}}_{i+1}}{\breve{\mathbf{v}}_{i+1}^\dagger \breve{\mathbf{B}} \breve{\mathbf{v}}_{i+1}} \right\}. \tag{D.65}$$

This problem is fully equivalent in structure to the first optimization problem ($i = 1$) as written out in Equation (D.57) and has the same solution, i.e., [cf. (D.58)]

$$\tilde{\mathbf{W}}^{(i)} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{W}^{(i+1)} \end{bmatrix}. \tag{D.66}$$

Consequently, $\mathbf{U}^{(i+1)}$ has indeed the structure (D.59), which concludes the induction proof. We infer that $\mathbf{U}^{(n_\mathsf{T})} = \mathbf{I}$, and thus

$$\boldsymbol{\omega}(\mathbf{p}'', \mathbf{I}) \in \partial^+ \boldsymbol{\omega}(\mathbf{p}'', \mathbb{U}^{n_\mathsf{T} \times n_\mathsf{T}}) \subset \bar{\boldsymbol{\omega}}^+(\mathcal{P}(\mu_\mathcal{P})). \tag{D.67}$$

Thus, we have specified a method to construct specific Pareto optimal points of the inner union in (D.52).

2) Recalling how $\mathbf{P}''$ is obtained from $\mathbf{P} \in \mathcal{P}(\mu_\mathcal{P})$, namely [cf. (D.49a)]

$$\mathbf{P}'' = \mathbf{\Lambda}_{\mathbf{R}}^{\frac{1}{2}} \mathbf{U}_{\mathbf{R}}^\dagger \mathbf{U}_{\mathbf{P}} \operatorname{diag}(\mathbf{p}) \mathbf{U}_{\mathbf{P}}^\dagger \mathbf{U}_{\mathbf{R}} \mathbf{\Lambda}_{\mathbf{R}}^{\frac{1}{2}},$$

we can leverage Theorem 4.2 (although with other variables) to characterize the set $\mathbf{p}''(\mathcal{P}(\mu_\mathcal{P}))$ of vectors of feasible, non-increasingly sorted eigenvalues of the above matrix. First note that $\mathbf{P}''$ has the same eigenvalues as $\mathbf{U}_{\mathbf{R}} \mathbf{P}'' \mathbf{U}_{\mathbf{R}}^\dagger$, so that the set $\mathbf{p}''(\mathcal{P}(\mu_\mathcal{P}))$ may be defined as [compare with (4.32)]

$$\mathbf{p}''(\mathcal{P}(\mu_\mathcal{P})) = \left\{ \boldsymbol{\lambda}\left(\mathbf{R}^{\frac{1}{2}} \mathbf{P} \mathbf{R}^{\frac{1}{2}}\right) \,\middle|\, \mathbf{P} \in \mathbb{C}_+^{n_\mathsf{T} \times n_\mathsf{T}},\ \operatorname{tr}(\mathbf{P}) \le \mu_\mathcal{P} \right\} \tag{D.68}$$

Now, Theorem 4.2 can be applied upon replacing $\hat{\mathbf{R}}$, $\tilde{\mathbf{R}}$, $\mathbf{Q}$, $\mathcal{Q}(\mu_\mathcal{Q})$ and $\mu_\mathcal{Q}$ (as they appear in the formulation of said theorem) with $\mathbf{R}$, $\mathbf{0}$, $\mathbf{P}$, $\mathcal{P}(\mu_\mathcal{P})$ and $\mu_\mathcal{P}$ respectively. This leads to $\mathbf{p}''(\mathcal{P}(\mu_\mathcal{P}))$ being characterized as the convex hull of the points $\boldsymbol{\sigma}''^{(n)}, n = 0, \dots, n_\mathsf{T}$ defined as

$$\boldsymbol{\sigma}''^{(0)} = \mathbf{0} \qquad \boldsymbol{\sigma}''^{(n)} = \mu_\mathcal{P} \cdot \mathcal{H}(r_1, \dots, r_n) \sum_{\ell=1}^{n} \mathbf{e}_\ell, \quad n = 1, \dots, n_\mathsf{T}. \tag{D.69}$$

It can be readily verified that all points of this convex hull can be reached when setting $\mathrm{col}(\mathbf{U_P}) \subseteq \mathrm{col}(\mathbf{U_R})$. When doing so, the eigenbasis of $\mathbf{P}''$ is precisely $\mathbf{U}_{\mathbf{P}''} = \mathbf{I}$. But remember that the choice $\mathbf{U}_{\mathbf{P}''} = \mathbf{I}$ was required in the previous paragraph for constructing a Pareto optimal point of $\partial^+\boldsymbol{\omega}(\mathbf{p}'', \mathbb{U}^{n_\mathsf{T}\times n_\mathsf{T}})$. Consequently, this Pareto optimal point is also contained in the subset $\boldsymbol{\omega}(\mathbf{p}'', \mathbf{U}_{\mathbf{P}''}(\mathbf{p}'')) \subseteq \boldsymbol{\omega}(\mathbf{p}'', \mathbb{U}^{n_\mathsf{T}\times n_\mathsf{T}})$, and is thus necessarily a Pareto optimal point of $\boldsymbol{\omega}(\mathbf{p}'', \mathbf{U}_{\mathbf{P}''}(\mathbf{p}''))$, too.

3) We now ask whether all points of the overall Pareto border $\partial^+\boldsymbol{\omega}(\mathcal{P}(\mu_\mathcal{P}))$ are attained by the construction method specified above, i.e., whether

$$\partial^+\boldsymbol{\omega}(\mathcal{P}(\mu_\mathcal{P})) \subseteq \bigcup_{\mathbf{p}''\in\partial^+\mathbf{p}''(\mathcal{P}(\mu_\mathcal{P}))} \boldsymbol{\omega}(\mathbf{p}'', \mathbf{I}). \tag{D.70}$$

Let us write out $\boldsymbol{\omega}(\mathbf{p}'', \mathbf{I})$ by means of (D.48) as

$$\boldsymbol{\Pi}\boldsymbol{\omega}(\mathbf{p}'', \mathbf{I}) = \mathbf{p}'' \odot \left((\mu_\mathcal{Q}\mathbf{r})^{-1}\mathbf{p}'' + \boldsymbol{\xi}\right)^{-1}, \tag{D.71}$$

where $\boldsymbol{\Pi} \in \mathbb{P}^{n_\mathsf{T}}$ is a sorting permutation, '$\odot$' denotes componentwise multiplication, $\mathbf{r}^{-1}$ denotes the vector of entries r_i^{-1} (i.e., componentwise reciprocal), and $\mathrm{diag}(\boldsymbol{\xi}) = \boldsymbol{\Xi} = \mathbf{I} + (\mu_\mathcal{Q}\boldsymbol{\Lambda}_\mathbf{R})^{-1}$. The mapping $\mathbf{p}'' \mapsto \mathbf{p}'' \odot \left((\mu_\mathcal{Q}\mathbf{r})^{-1}\mathbf{p}'' + \boldsymbol{\xi}\right)^{-1}$ is clearly injective, since $\boldsymbol{\xi} > \mathbf{0}$. Additionally, it has the property that for any real unit-norm vector $\mathbf{e} \geq \mathbf{0}$, there exists a scalar $\epsilon > 0$ and a single feasible vector $\mathbf{p}'' \in \partial^+\mathbf{p}''(\mathcal{P}(\mu_\mathcal{P}))$ such that

$$\mathbf{p}'' \odot \left((\mu_\mathcal{Q}\mathbf{r})^{-1}\mathbf{p}'' + \boldsymbol{\xi}\right)^{-1} = \epsilon\mathbf{e}. \tag{D.72}$$

To see this, we first rewrite Expression (D.72) as

$$\mathbf{p}'' = \epsilon\mathbf{e} \odot \boldsymbol{\xi} \odot \left(\mathbf{1} - \epsilon\mathbf{e} \odot (\mu_\mathcal{Q}\mathbf{r})^{-1}\right)^{-1}. \tag{D.73}$$

Since $\mathbf{p}'' \geq \mathbf{0}$, the scalar ϵ must lie in the semi-open interval $\epsilon \in [0; \min_i \mu_\mathcal{Q} r_i/e_i[$. From taking the Euclidian norm of Expression (D.73), we obtain a function $\epsilon \mapsto \|\mathbf{p}''\|_2$ which bijectively maps $[0; \min_i \mu_\mathcal{Q} r_i/e_i[$ onto $\mathbb{R}_+$. Since any $\mathbf{p}'' \in \partial^+\mathbf{p}''(\mathcal{P}(\mu_\mathcal{P}))$ has finite norm, there must necessarily exist one single value of ϵ fulfilling

$$\epsilon\mathbf{e} \odot \boldsymbol{\xi} \odot \left(\mathbf{1} - \epsilon\mathbf{e} \odot (\mu_\mathcal{Q}\mathbf{r})^{-1}\right)^{-1} \in \partial^+\mathbf{p}''(\mathcal{P}(\mu_\mathcal{P})). \tag{D.74}$$

Consequently, all Pareto optimal points $\partial^+\boldsymbol{\omega}(\mathcal{P}(\mu_\mathcal{P}))$ can be reached by the construction method from paragraphs 2) and 3), so that we may write

$$\partial^+\boldsymbol{\omega}(\mathcal{P}(\mu_\mathcal{P})) = \boldsymbol{\omega}(\partial^+\mathbf{p}''(\mathcal{P}(\mu_\mathcal{P})), \mathbf{I}). \tag{D.75}$$

4) Now that we have established that the Pareto border $\partial^+\boldsymbol{\omega}(\mathcal{P}(\mu_\mathcal{P}))$ can be reached by setting $\mathrm{col}(\mathbf{U_P}) \subseteq \mathrm{col}(\mathbf{U_R})$, we have that $\tilde{\mathbf{R}} = (\mathbf{R}^{-1} + \mathbf{P})^{-1}$ and $\hat{\mathbf{R}} = \mathbf{R} - \tilde{\mathbf{R}}$ acquire the same eigenbasis, up to a column permutation. Specifically, we have that the alignment $\mathrm{col}(\mathbf{U_P}) \subseteq \mathrm{col}(\mathbf{U_R})$ implies $\mathrm{col}(\mathbf{U_P}) = \mathrm{col}(\mathbf{U}_{\hat{\mathbf{R}}}) \subseteq \mathrm{col}(\mathbf{U}_{\tilde{\mathbf{R}}})$. But as a consequence of Theorem 4.2, the alignment $\mathrm{col}(\mathbf{U}_{\hat{\mathbf{R}}}) \subseteq \mathrm{col}(\mathbf{U}_{\tilde{\mathbf{R}}})$ leads to [cf. (4.34)]

$$\mathrm{col}(\mathbf{U_Q}) \subseteq \mathrm{col}(\mathbf{U}_{\hat{\mathbf{R}}}). \tag{D.76}$$

Hence, we obtain $\mathrm{col}(\mathbf{U_Q}) \subseteq \mathrm{col}(\mathbf{U_P}) \subseteq \mathrm{col}(\mathbf{U_R})$. Since we know from Section 4.5.2 that $\mathrm{rank}(\mathbf{P}^\star) = \mathrm{rank}(\mathbf{Q}^\star)$ at any joint optimum $(\mathbf{P}^\star, \mathbf{Q}^\star)$, we get the desired alignment property

$$\mathrm{col}(\mathbf{U_P}) = \mathrm{col}(\mathbf{U_Q}) \subseteq \mathrm{col}(\mathbf{U_R}). \tag{D.77}$$

Obviously, in case (D.77) is a strict inclusion, the eigenbases of $\mathbf{P}$ and $\mathbf{Q}$ should contain the eigenvectors of $\mathbf{R}$ associated to the largest eigenvalues of $\mathbf{R}$, hence

$$\mathrm{col}(\mathbf{U_P}) = \mathrm{col}(\mathbf{U_Q}) = \{\mathbf{u}_{\mathbf{R},1}, \ldots, \mathbf{u}_{\mathbf{R},r^\star}\} \subseteq \mathrm{col}(\mathbf{U_R}), \tag{D.78}$$

which concludes the proof of Theorem 4.4.

D.8 Proof of Lemma 4.3

For any set $\mathcal{A} \subseteq \mathbb{R}_+^n$, the Pareto border $\partial^+\mathcal{A}$ is a subset of the front border $\partial^{\mathrm{f}}\mathcal{A}$. In fact, if it were not so, then there would exist a Pareto optimal point, say $\mathbf{a}' \in \partial^+\mathcal{A}$, which would not be the solution to

$$\max_{\substack{\mathbf{a}\in\mathcal{A} \\ \mathbf{a}=\nu\mathbf{a}'}} \nu \tag{D.79}$$

in that another $\mathbf{a}'' \in \mathcal{A}$ colinear with $\mathbf{a}'$ would exist that would have larger norm, i.e., $\|\mathbf{a}''\| > \|\mathbf{a}'\|$. Yet this is impossible by the definition of $\partial^+\mathcal{A}$, because $\mathbf{a}''$ would dominate $\mathbf{a}'$ in the sense $\mathbf{a}'' \geq \mathbf{a}'$, hence the contradiction.

It thus suffices to prove that $\partial^{\mathrm{f}}\mathbf{s}(\Gamma) \subseteq \partial^+\mathbf{s}(\Gamma)$ in order to conclude on set equality $\partial^{\mathrm{f}}\mathbf{s}(\Gamma) = \partial^+\mathbf{s}(\Gamma)$. For this purpose, take $\mathbf{s}'$ to be some point of the front border $\partial^{\mathrm{f}}\mathbf{s}(\Gamma)$. Assume that there would exist another point $\mathbf{s}'' \in \mathbf{s}(\Gamma)$ different from $\mathbf{s}'$ that dominates $\mathbf{s}'$, that is, $\mathbf{s}'' \geq \mathbf{s}'$. For belonging to the set $\mathbf{s}(\Gamma)$, which is the union

$$\mathbf{s}(\Gamma) = \bigcup_{\substack{(\mu_{\mathcal{P}},\mu_{\mathcal{Q}}) \\ \mu_{\mathcal{P}}+(T-T_\tau)\mu_{\mathcal{Q}}\leq T\mu}} \ \bigcup_{\mathbf{P}\in\mathcal{P}(\mu_{\mathcal{P}})} \mathbf{s}(\mathbf{P}, \mathcal{Q}(\mu_{\mathcal{Q}})), \tag{D.80}$$

the point $\mathbf{s}''$ would be contained in at least one of the sets $\mathbf{s}(\mathbf{P}, \mathcal{Q}(\mu_{\mathcal{Q}}))$. Call $\mathbf{P}'' \in \mathcal{P}(\mu_{\mathcal{P}})$ a pilot Gram of rank $r_{\mathbf{P}''}$ such that $\mathbf{s}''$ lies in $\mathbf{s}(\mathbf{P}'', \mathcal{Q}(\mu_{\mathcal{Q}}))$. According to Theorem 4.2, the set $\mathbf{s}(\mathbf{P}'', \mathcal{Q}(\mu_{\mathcal{Q}}))$ is a simplex consisting of all convex combinations of $r_{\mathbf{P}''} + 1$ points $\boldsymbol{\sigma}^{(n)}, n =$

$0, \dots, r_{\mathbf{P}''}$, with $\boldsymbol{\sigma}^{(0)} = \mathbf{0}$ and [cf. (4.33)]

$$\boldsymbol{\sigma}^{(n)} = \mathcal{H}(\omega_1, \dots, \omega_n) \sum_{j=1}^{n} \mathbf{e}_j, \quad n \in \{1, \dots, r_{\mathbf{P}''}\}, \tag{D.81}$$

where ω_i are the non-increasingly ordered eigenvalues of the generalized eigenvalue problem [cf. (4.31)]

$$\hat{\mathbf{R}}''\mathbf{v}_i = \omega_i(\mu_{\mathcal{Q}}^{-1}\mathbf{I} + \tilde{\mathbf{R}}'')\mathbf{v}_i \tag{D.82}$$

with $\tilde{\mathbf{R}}'' = (\mathbf{R}^{-1} + \mathbf{P}'')^{-1}$ and $\hat{\mathbf{R}}'' = \mathbf{R} - \tilde{\mathbf{R}}''$. Notice that the linearly independent vectors $\boldsymbol{\sigma}^{(n)}, n = 1, \dots, r_{\mathbf{P}''}$, when linearly combined with non-negative coefficients, span the linear subspace of $\mathbb{R}_+^{n_\mathrm{T}}$ of vectors having non-increasingly sorted entries on positions 1 through $r_{\mathbf{P}''}$, and zero entries on positions $r_{\mathbf{P}''} + 1$ through n_T. Consequently, both $\mathbf{s}'$ and $\mathbf{s}''$, which by definition have non-increasing non-negative entries, can be written as linear combinations

$$\mathbf{s}' = \sum_{n=1}^{r_{\mathbf{P}''}} \nu_n' \boldsymbol{\sigma}^{(n)} \qquad \mathbf{s}'' = \sum_{n=1}^{r_{\mathbf{P}''}} \nu_n'' \boldsymbol{\sigma}^{(n)} \tag{D.83}$$

with unique non-negative coefficients ν_n' and ν_n''. Since $\mathbf{s}'' \in \mathbf{s}(\mathbf{P}'', \mathcal{Q}(\mu_{\mathcal{Q}}))$, the coefficients ν_n'' sum up to $\sum_n \nu_n'' \leq 1$. Now, since $\mathbf{s}'$ and $\mathbf{s}''$ are distinct, and $\mathbf{s}' \leq \mathbf{s}''$ by assumption, we must have

$$\sum_{n=1}^{r_{\mathbf{P}''}} \nu_n' < \sum_{n=1}^{r_{\mathbf{P}''}} \nu_n'' \leq 1. \tag{D.84}$$

Therefore $\mathbf{s}'$ lies in the interior of $\mathbf{s}(\mathbf{P}'', \mathcal{Q}(\mu_{\mathcal{Q}}))$. Consequently, for a small enough $\epsilon > 0$, the point $(1 + \epsilon)\mathbf{s}'$ is element of $\mathbf{s}(\mathbf{P}'', \mathcal{Q}(\mu_{\mathcal{Q}}))$, and thus of $\mathbf{s}(\Gamma)$, which contradicts the initial assumption that $\mathbf{s}' \in \partial^\mathrm{f}\mathbf{s}(\Gamma)$. Hence $\partial^\mathrm{f}\mathbf{s}(\Gamma) = \partial^+\mathbf{s}(\Gamma)$.

D.9 Proof of Lemma 4.4

Clearly, maximizing $\nu(\mathbf{p}, \mathbf{e})$ as defined in (4.66) is equivalent to minimizing the function

$$\breve{\nu}(\mathbf{p}) = \frac{1}{\nu(\mathbf{p}, \mathbf{e})} + 1 = \frac{1 + \mathbf{r}^\mathrm{T}\mathbf{q}(\mathbf{p}, \mathbf{e})}{\eta(\mathbf{p}, \mathbf{e})}, \tag{D.85}$$

where contrary to $\nu(\mathbf{p},\mathbf{e})$, the direction vector $\mathbf{e}$ is omitted in the notation of the function $\breve{\nu}(\mathbf{p})$. Writing the latter function out in full with help of definitions (4.64) and (4.65) yields

$$\begin{aligned}\breve{\nu}(\mathbf{p}) &= \frac{1+\frac{T\mu-\sum_i p_i}{T-T_\tau}\left(\sum_i e_i\frac{1+r_ip_i}{r_i^2p_i}\right)^{-1}\left(\sum_i e_i\frac{1+r_ip_i}{r_ip_i}\right)}{\frac{T\mu-\sum_i p_i}{T-T_\tau}\left(\sum_i e_i\frac{1+r_ip_i}{r_i^2p_i}\right)^{-1}} \\ &= \frac{T-T_\tau}{T\mu-\sum_i p_i}\left(\sum_i e_i\frac{1+r_ip_i}{r_i^2p_i}\right)+\sum_i e_i\frac{1+r_ip_i}{r_ip_i} \\ &\triangleq (T-T_\tau)\big(\breve{\nu}_1(\mathbf{p})+\breve{\nu}_2(\mathbf{p})\big)+\breve{\nu}_3(\mathbf{p}),\end{aligned} \tag{D.86}$$

the three functions $\breve{\nu}_1(\mathbf{p})$, $\breve{\nu}_2(\mathbf{p})$, $\breve{\nu}_3(\mathbf{p})$ in the last line being

$$\breve{\nu}_1(\mathbf{p}) = \sum_j e_jr_j^{-2}\frac{1}{p_j(T\mu-\sum_i p_i)} \tag{D.87a}$$

$$\breve{\nu}_2(\mathbf{p}) = \sum_j e_jr_j^{-1}\frac{1}{T\mu-\sum_i p_i} \tag{D.87b}$$

$$\breve{\nu}_3(\mathbf{p}) = \sum_i e_i\frac{1+r_ip_i}{r_ip_i}. \tag{D.87c}$$

We will now show that the three functions $\breve{\nu}_1(\mathbf{p})$, $\breve{\nu}_2(\mathbf{p})$ and $\breve{\nu}_3(\mathbf{p})$ are all convex functions of $\mathbf{p}$ on the interior of $\mathcal{D}(T\mu)\subset(0;\infty)^{n_\mathsf{T}}$, which we shall denote as $\mathrm{int}(\mathcal{D}(T\mu))$. It is easy to see that $\breve{\nu}_3$ is essentially a linear combination (plus a constant) of functions $1/p_i$ that are convex on the entire open orthant $(0,\infty)^{n_\mathsf{T}}$, and thus on $\mathrm{int}(\mathcal{D}(T\mu))\subset(0;\infty)^{n_\mathsf{T}}$. Similarly, $\breve{\nu}_2$ is convex on the open half-space $\sum_i p_i<T\mu$, and thus on the subset $\mathrm{int}(\mathcal{D}(T\mu))$ thereof. Finally, $\breve{\nu}_1$ is a linear combination of functions $\frac{1}{p_j}\frac{1}{T\mu-\sum_i p_i}$, each of which is convex in $\mathbf{p}$ on $\mathrm{int}(\mathcal{D}(T\mu))$. This can be shown as follows: take a pair of points $(\mathbf{p}^{(1)},\mathbf{p}^{(2)})\in\mathrm{int}(\mathcal{D}(T\mu))^2$, then for any $\theta\in[0;1]$,

$$\frac{1}{\theta p_j^{(1)}+(1-\theta)p_j^{(2)}}\frac{1}{T\mu-\sum_i(\theta p_i^{(1)}+(1-\theta)p_i^{(2)})}\le\theta\frac{1}{p_j^{(1)}}\frac{1}{T\mu-\sum_i p_i^{(1)}}+(1-\theta)\frac{1}{p_j^{(2)}}\frac{1}{T\mu-\sum_i p_i^{(2)}}, \tag{D.88}$$

because the left-hand side of the latter inequality is convex in $\theta\in[0;1]$, since it is of the form

$$A\frac{1}{1+B\theta}\frac{1}{1+C\theta} \tag{D.89}$$

with constants $A=\frac{1}{p_j^{(2)}(T\mu-\sum_i p_i^{(2)})}\ge 0$, $B=\frac{p_j^{(1)}-p_j^{(2)}}{p_j^{(2)}}$, $C=\frac{\sum_i(p_i^{(2)}-p_i^{(1)})}{T\mu-\sum_i p_i^{(2)}}$, and $1+B\theta\ge 0$ and $1+C\theta\ge 0$ by construction. The convexity of (D.89) is best seen by differentiating twice:

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left[\frac{1}{1+B\theta}\frac{1}{1+C\theta}\right]=\frac{2}{(1+B\theta)(1+C\theta)}\left(\frac{B^2}{(1+B\theta)^2}+\frac{C^2}{(1+C\theta)^2}+\frac{BC}{(1+B\theta)(1+C\theta)}\right). \tag{D.90}$$

The above expression is obviously positive if $BC \geq 0$. Otherwise, if $BC \leq 0$, then the expression between square brackets on the right-hand side of the last equality is lower bounded by

$$\frac{B^2}{(1+B\theta)^2} + \frac{C^2}{(1+C\theta)^2} + \frac{2BC}{(1+B\theta)(1+C\theta)} = \left[\frac{B}{(1+B\theta)} + \frac{C}{(1+C\theta)}\right]^2 \geq 0. \tag{D.91}$$

Hence (D.88), and all the three functions $\breve{\nu}_1$, $\breve{\nu}_2$ and $\breve{\nu}_3$ are convex in $\mathbf{p}$ on the open set $\mathrm{int}(\mathcal{D}(T\mu))$. Thus, $\breve{\nu}(\mathbf{p})$ is convex on $\mathrm{int}(\mathcal{D}(T\mu))$. Therefore $\nu(\mathbf{p},\mathbf{e}) = 1/(\breve{\nu}(\mathbf{p}) - 1)$, which is a decreasing function of $\breve{\nu}(\mathbf{p}) > 1$, is quasi-concave in $\mathbf{p}$ on $\mathrm{int}(\mathcal{D}(T\mu))$, according to Definition 4.2. Since $\nu(\mathbf{p},\mathbf{e})$ vanishes on the boundary of $\mathcal{D}(T\mu)$ and is continuous in the vicinity of this boundary, we conclude that $\mathbf{p} \mapsto \nu(\mathbf{p},\mathbf{e})$ is quasi-concave on the closure $\mathcal{D}(T\mu)$.

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